This is to certify that the

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MODULAR MODELING OF ENGINEERING SYSTEMS
USING FIXED INPUT-OUTPUT STRUCTURE

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Brooks Philip Byam

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Clark J. Radcliffe
Major professor

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MODULAR MODELING OF ENGINEERING SYSTEMS USING FIXED INPUT-OUTPUT STRUCTURE

By

Brooks Philip Byam

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ABSTRACT

MODULAR MODELING OF ENGINEERING SYSTEMS
USING FIXED INPUT-OUTPUT STRUCTURE

By

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Computer modeling is common in the design and development of complex engineering systems. A system model is built up by connecting the inputs and outputs of several subsystem models. This process requires flexible modeling tools. Models with arbitrary input-output structure have this flexibility but must have their internal equations reformulated to agree with the inputs and outputs used. The flexibility achieved with arbitrary input-output structure occurs at the cost of globally reformulating the equations of each subsystem and component model with every change. Each model equation formulation requires performance verification because every formulation does not have the same guaranteed performance. This can be particularly cumbersome in large models. A fixed input-output structure allows elements to be used without modification of their internal equations and eliminates equation reformulation. Modular modeling models have the property that their elements have fixed input-output structure. The cost is a connector to assemble elements adding complexity to the global model. Modular modeling is a systematic realization of compatible standardized modular elements and connectors that maintain modularity with the flexible assembly required in today's large complex modeling environments. Structural and automotive examples are given.

In memory of a dear friend Martha Somers

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INTRODUCTION

Increasingly large and complex computer models are becoming standard practice in design and development of engineering systems. Models should have sufficient complexity to predict actual behavior of complex engineering systems (Ferris etal, 1994). Chrysler used a large computer model to completely design and analyze the geometry of their 1998 Chrysler Concorde and 1998 Dodge Intrepid. The Chrysler large car models were limited to mechanical geometry studies and still contained representations of over 5500 interconnected physical subsystems ("Computers In Engineering: Chrysler designs paperless cars", 1998). Chrysler engineers resolved design issues by developing this computer model instead of physical prototypes reducing the cycle time from 39 to 31 months saving the company more than $75 million (Jost, 1998). Efficient design, development, and refinement of large computer models of complex engineering systems are critical to engineers and their companies. A systematic approach to design and development is the most efficient (Shigley and Mischke, 1989).

Several existing modeling methodologies use systematic techniques for the design and development of engineering system computer models. Kinematic and dynamic models of mechanical systems are developed using the systematic method of generalized Cartesian coordinates (Haug, 1989 and Nikravesh, 1988). Electrical system models are developed using the systematic method of applying Kirchhoff Laws from a network topology (Chua and Lin, 1975, Vlach and Singhal, 1983, and Calahan, 1972). These methods are useful but are limited to their respective energy domains.

A systematic method that includes different energy domains is Finite Element Analysis (FEA). FEA methods systematically construct grids of similar elements to model engineering components and systems in mechanical and thermal energy domains (Zienkiewiez, 1977). These methods are useful for systematic generation of model equations and models grow to be quite large and complex. However, FEA models are not generated with an input-output structure that allows them to be easily interconnected. Therefore, assembling several of these independent models requires reformulating an entirely new model or using special purpose software like PDESolve (PDESolve, BEAM Technologies) to "connect" them, which is both cumbersome and expensive.

Another systematic inter-energy domain modeling method is bond graphs. Bond graphs have a systematic approach of using graphical multi-port elements and junctions to develop component and system models in mechanical, electrical, hydraulic, and thermal energy domains (Karnopp etal, 1990). System model equations are systematically generated with some hierarchical design but the assembly reformulation issue still exists due to arbitrary input-output definitions (Karnopp etal, 1990). Recent research has further enhanced the hierarchical design of bond graph models such that some reformulation can be avoided (Hales, 1995). Reformulation of model equations is the practice that prevents efficient development of large, complex, models.

A new approach to systematic modeling across multiple energy domains provides a modular fixed input-output structure modeling method. Modular in the sense that the physics that describes each subsystem model remains the same whether the subsystems are separate or assembled into a system (Hogan, 1987). Fixed input-output structure in the sense that the inputs and outputs are standardized so the internal equations of mathematical subsystem models of engineering systems have the same modularity as the engineering system. Modular modeling with fixed input-output structure is a power-based physically intuitive top-down methodology systematic modeling method that eliminates equation reformulation from large model development across multiple energy domains (Byam and Radcliffe, 1999).

Modular modeling is a top-down systematic equation assembly scheme well suited to multi DOF components. Modeling efficiency is diminished for development and assembly of idealized single DOF model components. This method has a standardized input-output definition for all multi port multi degree of freedom modular modeling elements resulting in a single standardized element formulation. A single standardized formulation allows modelers to gain performance verification experience enhancing the verification process. The single standardized modular modeling element combined with the compatible modular connector makes modular model realization a simple systematic process. Modular modeling makes the large model design, development, and refinement process systematic, functional, and physically intuitive.

Chapter 1

MODULAR MODELING: THE IMPACT OF CAUSALITY

Modular modeling is defined by a fixed input-output structure. This strict input-output approach standardizes the internal equation formulation of multi degree of freedom (DOF) subsystem models. The objective is mathematical models of engineering systems with equations that have the same modularity as the engineering system. This method is not intended for single DOF model elements or single component modeling. It is intended for large system models with many multi DOF subsystems. Modular modeling is particularly advantageous for large system models because the multi DOF subsystem models fixed formulation simplifies large model design, development, and refinement.

Large complex models of engineering systems contain a large number of multi DOF subsystem models (5500 in the Chrysler large car models). Each physical multi DOF subsystem model has one or more connections through which it is attached to other subsystem models. For example, the transmission subsystem model of a pick-up truck has connections to the engine model, the frame model, the driver controls model, and the drive shaft model. Each physical subsystem model connection has an input-output definition, conveys input-output variables to interconnected subsystem models, and hence defines the internal formulations of the subsystem models. Controlling the number of internal formulations of interconnected multi DOF subsystem models is key for design, development, and refinement of large models.

Subsystem model internal formulations result from input-output connections defined here in two general forms: signal-type and natural-type. Signal-type model connections contain single variables and can only be defined as input or an output. For example, the driver controls-to-transmission model connection is a signal-type model input connection. The only reasonable model is a selected-gear control signal input to the transmission model. Indicator lights are an example of a signal-type model connection only reasonably defined as a model output. Once defined, signal-type connections have only one possible definition and their effects on the model's internal equations is fixed.

Natural-type model connections may have many variables and many valid input-output definitions resulting in many useful subsystem model equation formulations. For example, a transmission-to-drive shaft model connection may have mechanical rotation and mechanical translation variables. Each model must have an input-output definition and hence internal formulation that provides the appropriate input and output variables at the connection. An output of one subsystem must be an input to the other. There are many possible useful input-output definitions, which could be used to assemble these elements. Each definition requires a different, useful, well-posed internal formulation of the connected multi DOF subsystems' internal equations.

Power-based models represent natural-type physical model connections with power ports. A power port is a place where physical systems are connected and exchange power. Power is commonly modeled as the product of two variables such as force and velocity, pressure and volume flow rate, or voltage and current. The variable pairs are often referred to as the effort-flow variable pair, e_i-f_i, at each power port (Karnopp etal, 1990). Power-based simulations pair these power variables at each port. Each port has causality, an input-output definition (Karnopp etal, 1990). Causality manages connected power port's physical cause and effect between the variable pairs by defining one variable as a port input and the other variable as a port output. Connected ports must have reciprocal causality (Rosenberg and Karnopp, 1983). In other words, the input of one port is the output of its connected port, and vice versa. Each port affects the model's internal formulation through its input-output causality definition.

1.1 The Impact of Causality: Model Internal Formulation

Causality has a great impact on the number of possible different, useful, internal equation formulations of a computer model. Power port causality allows two possible, different, reciprocal, input-output definitions. Let the two variables, e_i and f_i; be the input-output variables at the multi DOF element model port *i*. There are two possible different input-output definitions at a multi DOF element port *i* (Fig. 1.1). For example, the mechanical shaft of a pump can be modeled with a torque input and an angular velocity output or with the reciprocal causality. Useful pump models can be formulated with either causality.

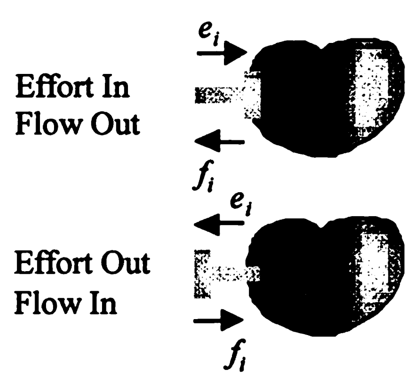

Figure 1.1: Two Possible Different Input-Output Causal Definitions At A Multi DOF Element Port *i*

A multi DOF element model with *n* power ports with arbitrary causality permitting 2 possible reciprocal input-output definitions at each port will have N_a possible different internal equation formulations (Fig. 1.2).

$$N_a = 2^n \tag{1.1}$$

Typical multi DOF subsystem models have 1 or more power ports. For example, a simple hydraulic pump model may have 3 power ports (1 for the mechanical shaft, and 2 for the hydraulic high and low pressure connections). This leads to

$$2^3 = 8 \tag{1.2}$$

possible specific causal formulations.

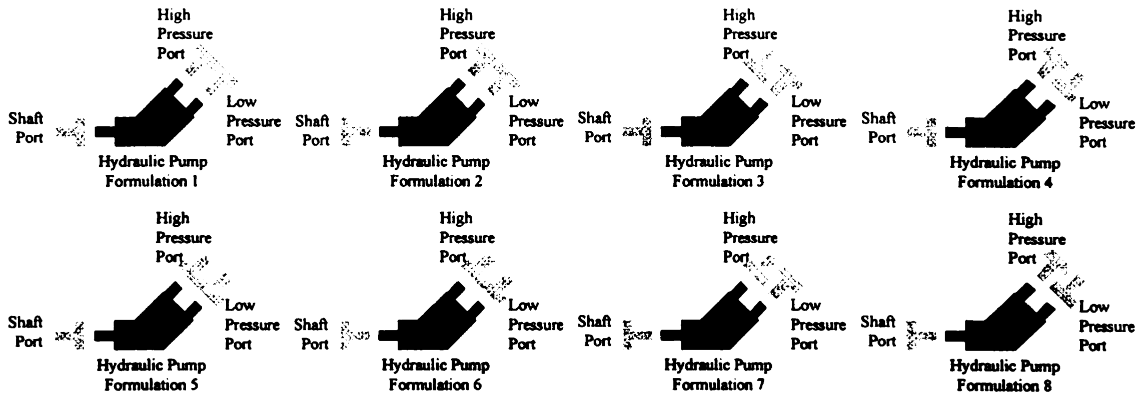

Figure 1.2: 2 Possible Different Input-Output Causal Definitions At *n* Power Ports Yields 2^n Possible Different Useful Multi DOF Element Formulations – Three Port Hydraulic Pump Example

2^n possible useful different element formulations requires 2^n different model verifications. Verifying 2^n possible correct element formulations is a staggering task. Consider interconnecting 5500 multi DOF model elements (e.g. 1998 Chrysler large car models) with reciprocal causality. In the most simplistic interconnected form, 2 power ports per element, there are 5500×2 = 11,000 power ports. There is a 50% probability of a perfect input-output causal match but the number of possible useful system model formulations to verify is an impossible task.

$$50\% \text{ of } 2^{11000} = 1.06885 \times 10^{3310} \qquad (1.3)$$

For this reason users of models with arbitrary causality have made the choice to verify at the component level and reformulate the system equations after every change.

Fixing power port causality at every multi DOF element port selects a single element internal equation formulation (Fig. 1.3). This standardized functionality simplifies the design, development, and refinement of large models. There is only one system configuration, which enables verification at the subsystem level. The key concept of modular modeling is a physically intuitive fixed causality at every port of every modular modeling element.

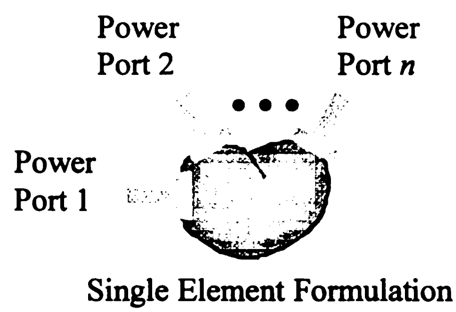

Figure 1.3: 1 Possible Input-Output Causal Definition At *n* Power Ports Yields A Single Multi DOF Element Formulation

1.2 Modular Modeling Elements

Modular modeling elements are multi-port multi DOF subsystem models with a fixed causality at every power port and a single fixed formulation. The element internal equation formulation fits the fixed causality and the number of ports. Since the causality is fixed, the formulation of element's equations does not change. Once formulated modelers can gain validation experience with each specific element. The fixed causality at every port leads to a standardized element functional form.

$$\underset{\sim}{y}_{n\times 1} = element(\underset{\sim}{u}_{n\times 1}) \qquad (1.4)$$

n is the number of element power ports, $\underset{\sim}{u}_{n\times 1}$ is a vector of inputs and $\underset{\sim}{y}_{n\times 1}$ is a vector of outputs. The multi port multi DOF element calculates a vector of outputs, $\underset{\sim}{y}_{n\times 1}$, given the vector of inputs, $\underset{\sim}{u}_{n\times 1}$, and some internal element parameters. This functional form can be generated for most engineering elements with any number of power ports.

Modular modeling standardizes the choice of causality for every port per energy domain to realize the objective of an internal equation formulation with the same modularity as physical systems. The choice of the modular modeling element fixed port causality is motivated by physical measurements. Ideal physical measurements occur at a natural power port with a specific physical location and zero power flow such that there is no effect on the response of the system. In other words, physical measurements have a specific physical location, an externally sensed output at that location, and zero input at that location to zero the power flow. This measurement perspective defines the fixed port causality of modular modeling elements. The port output is the variable related to the externally sensed physical quantity. The port input is the variable related to the internal physical quantity typically assumed zero to attain zero power flow.

The measurement perspective modular modeling element fixed port causality for engineering systems across multiple energy domains are shown in Table 1. Externally sensed physical quantities are electrical potential, curvilinear mechanical motion, angular mechanical motion, hydraulic pressure, acoustic sound pressure, and temperature. Internal physical quantities typically assumed to be zero are electrical current, mechanical force, mechanical torque, hydraulic volume flow rate, acoustic volume velocity, and thermal heat flux. The resulting measurement perspective modular modeling element fixed port causality of electrical, mechanical translation, mechanical rotation, hydraulic, acoustic, and heat transfer systems are current input-potential output, force input-velocity output, torque input-angular velocity output, flow input-pressure output, flow input-pressure output, and heat flux input-temperature output respectively.

ENGINEERING SYSTEM	MEASUREMENT PERSPECTIVE MODULAR MODELING ELEMENT FIXED PORT CAUSALITY	
Electrical	Current Input – Potential Output	u = Current, y = Potential
Mechanical Translation	Force Input – Velocity Output	u = Force, y = Velocity
Mechanical Rotation	Torque Input – Angular Velocity Output	u = Torque , y = Angular Velocity
Hydraulic	Volume Flow Rate Input – Pressure Output	u = Volume Flow Rate, y = Pressure
Acoustic	Volume Velocity Input – Sound Pressure Output	u = Volume Velocity, y = Sound Pressure
Heat Transfer	Heat Flux Input – Temperature Output	u = Heat Flux, y = Temperature

Table 1.1: Measurement Perspective Modular Modeling Element Fixed Port Causality Across Multiple Energy Domains

Measurement perspective causality is equivalent to implementing nonessential or natural boundary conditions at every port (Meirovitch, 1967). This ensures an internal formulation with a mathematical eigen-structure that does not change whether the model is separate or assembled into a system model. Essential boundary conditions change the model's differential operators and impose specific geometric constraints. Modular modeling implements essential boundary conditions with modeling elements. For example, a mechanical fixed-point element would output a zero velocity regardless of the force input. Measurement perspective fixed port causality enables modular modeling elements with physical system modularity

A benefit of measurement perspective fixed port causality is the flexibility to maintain any number of physically "open" power ports at discretionary physical locations without affecting the element internal formulation. A physically "open" power port i is physically disconnected with zero input.

$$u_i = 0 \tag{1.5}$$

The input, u_i, is zero to achieve no power flow but the output, y_i, is open to be defined by the element. For example, a modular (multi DOF) mechanical beam model element whose ports are defined can maintain any number of "open" zero force-input ports without changing the model's mathematical formulation. A modular modeling beam element will have the same internal formulation and the same performance regardless of the modeling task. The response of the system can be "measured" at any "open" port with the power port's output variable. An element formulated from velocity input-force output "fixed" causality does not have this flexibility. This flexibility enables the single formulation of modular modeling elements to maintain a large number of power ports with no reformulation or revalidation cost.

The modular modeling elements graphical notation represents a multi port multi DOF physical subsystem model with a rectangle (Fig. 1.4). The bold lines represent the n power ports with fixed measurement perspective causality (Table 1.1). Arrows on the port lines define the standardized direction of positive power for modular modeling elements, when u_1 and y_1 are both positive, power flows into port 1. Port power orientation has a similar 2^n effect on model formulation but does not affect the eigen-structure of the internal formulation. The fixed causality input variable, u_i, is always shown on the top or to the left of the port. The fixed causality output variable, y_i, is always shown on the bottom or to the right of the port.

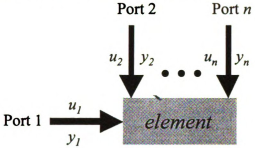

Figure 1.4: Modular Modeling Element Graphical Notation

1.3 Connector for Element Interconnections

Modular modeling requires a connector to join incompatible modular modeling element power ports. The connector provides the compatible port causality. Modular modeling connectors provide the proper physical connection constraints for connecting the ports of engineering subsystems. The connector is not a model of a physical connection subsystem. It is a power transmission mechanism that enforces constraints between subsystems.

The connector graphical notation represents the connector with a circle (Fig. 1.5). The bold lines represent the connected modular modeling element ports, *i* and *j*. Arrows on the port lines define the standardized direction of positive power for connectors, when u_i and y_i are both positive, power flows out of port i. The input variable, u, is always shown on the top or to the left of the port. The output variable, y, is always shown on the bottom or to the right of the port.

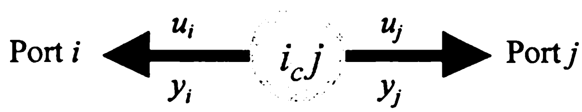

Figure 1.5: Modular Modeling Connector Graphical Notation

Modular modeling connectors provide a power constraint. Power is conserved across modular connectors because modular connectors are power transfer mechanisms. The power at the connected modular modeling element ports sum to zero.

$$\sum_{k=1,2} P_k = 0 \tag{1.6}$$

Recall that the product of port variables, u_i and y_i, is power.

$$P_i = u_i y_i \tag{1.7}$$

An additional constraint is required for the connected modular modeling element ports. Connectors are defined to constrain connected modular modeling element ports' outputs to be the same implementing connections of elements.

$$y_i = y_j \tag{1.8}$$

From (1.6) - (1.8) the inputs at the connected ports must be equal and opposite.

$$u_i = -u_j \tag{1.9}$$

The functional definition of the connector is a 2-port power constraint with port causality compatible with modular modeling element port causality (1.4). In modular modeling, (1.8) and (1.9) are the defining equations for all modular connectors.

There are two important characteristics of modular modeling elements and connectors to aid in the assembly of a modular modeling system models. First, properly connecting any number of power ports of modular modeling elements with the 2-port connector requires a junction structure inside modular modeling elements for each port. The internal junction structure has the affect of summing port inputs and constraining the port outputs to have the same output. This is consistent with the measurement perspective that defines the fixed modular modeling causality. For example, measuring a voltage at a point on an electrical circuit will have one voltage but there may be several currents input to that point. Second, the explicit difference between modular modeling elements and connectors is that modeler defined equations are in modular modeling elements. Modular connector equations are fixed (1.8) and (1.9). All modeling is done in modular modeling elements, none in modular connectors. Modeling a physical connection requires modular modeling elements.

Structural models of 3 bars connected with a shear pin will provide a demonstration (Fig. 1.6). The simplest model (Fig. 1.6 b) neglects deflection of the pin. In this simple case, no pin deflection model is required and the bars are connected with displacement and force constraints. When the pin deflection is large (Fig. 1.6 c), a model for the physical deflection of the pin is required and a pin modeling element is used.

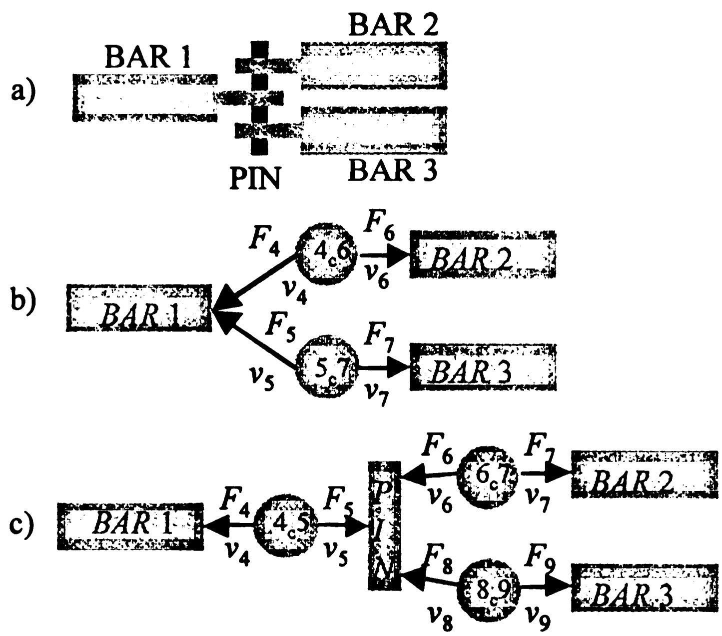

Figure 1.6: Structural Models Of 3 Bars Connected With A Shear Pin: a) Physical System b) Negligible Pin Deflection c) Large Pin Deflection

The first rigid pin modular model is a demonstration of the choice to not model the pin. The pin is ignored and 2 modular connectors join the 3 bars. The modeler defines the multi-port multi-DOF modular modeling elements *BAR* 1, *BAR* 2, and *BAR* 3. The modular modeling element *BAR* 1 has a port 45 with two input-output pairs 4 and 5 where power is transferred from *BAR* 2 and *BAR* 3 respectively. Each modular connector is associated with a power transfer mechanism between subsystem models. More than one power transfer mechanism can occur at a single port of a model element because of the internal junction structure. Port 45 power pair variables 4 and 5 are input and output from the same apparent geometric point on *BAR* 1. The internal junction structure has the affect of summing the 4 and 5 inputs and constraining the 4 and 5 to have the same output at that port. The right end of *BAR* 1 has the input force F_{45} and the output velocity v_{45}.

$$\begin{aligned} F_{45} &= F_4 + F_5 \\ v_{45} &= v_4 = v_5 \end{aligned} \tag{1.10}$$

Modular connectors 4_c6 and 5_c7 are the power transfer mechanisms that join port 45 of *BAR* 1 to port 6 of *BAR* 2 and port 7 of *BAR* 3 respectively. The connectors enforce the fixed power constraints (1.8) and (1.9) to the power transfer such that the respective outputs are the same and the respective inputs are equal and opposite. Two 2-port modular connectors properly constrain the power transfer to join 3 modular modeling element ports at the apparent same geometric point. This can be extended to any number of connected modular modeling element ports. Multiple ports at one-point increases the number of ports in the model but modular modeling elements can maintain any number of "open" ports with no internal formulation change.

The second flexible pin modular model is a demonstration of the choice to model the pin. The modeler defines the multi-port multi-DOF modular modeling elements *PIN*, *BAR* 1, *BAR* 2 and *BAR* 3. *PIN* has three power ports 5, 6, and 8 at the connection or power transfer points with *BAR* 1, *BAR* 2 and *BAR* 3 respectively. Modular connectors 4_c5, 6_c7, and 8_c9 are the power transfer mechanisms that join the power ports 4, 7, and 9 of *BAR* 1, *BAR* 2 and *BAR* 3 to the *PIN* power ports 5, 6, and 8 respectively. The connectors facilitate the power transfer between power ports implementing (1.8) and (1.9). For example, connector 4_c5 enforces power conservation constraints on the power transfer between power ports 4 and 5 such that v_4 and v_5 are equal and F_4 and F_5 are equal and opposite. Connectors 6_c7 and 8_c9 facilitate power transfer on their respective ports enforcing the same constraints.

An automotive model of a conceptual rear-wheel drive power train (Fig. 1.7) is another modular modeling example. The physical system (Fig. 1.7a) consists of an engine, a transmission, three mounts, a flex plate, and a frame. The engine has physical connections to the transmission through a flex plate and to the frame through mounts on the either side of the engine. The transmission has physical connections to the engine through a flex plate and to the frame through a transmission mount. The frame has physical connections to the engine and the transmission through the mounts. The modular model of the drive train (Fig. 1.7 b) shows the modeling choice not to model the flex plate deflection. The deflection of the flex plate is small relative to the deflection of three mounts. The rigid flex plate is not considered an element, so the engine and transmission transfer power to one another through modular connectors. The flexible mounts require modeler-defined equations to describe their deflections and are considered modular modeling elements. Modular connectors are power transfer mechanisms composed of standard constraints to conserve power. This is quite different from multi DOF modular subsystem elements, which implement all modeling analysis.

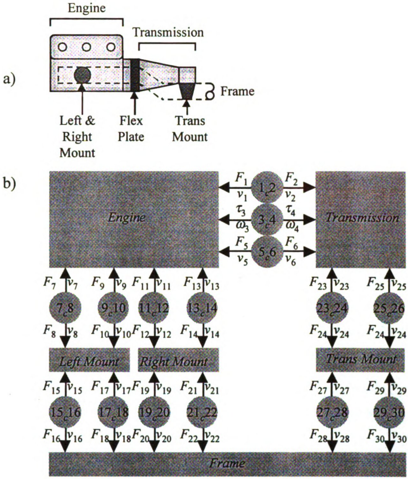

Figure 1.7: A Conceptual Rear-Wheel Drive Power Train Model: a) Physical System b) Modular Model

The modular connector can not implement the constraint (1.8) and (1.9) on ports of different energy domains. For example, from Table 1.1 and constraint (1.8), an electrical potential output and a mechanical velocity output can not be equal. The solution is to define an appropriate modular transducer model element with fixed measurement perspective port causality, then use modular connectors to join the modular transducer element ports and the element ports in different energy domains. Modeler-defined multi-port modular transducer elements have the same form and function as any modular modeling elements and are simply another element in the modular model.

The standardized port causality and sign conventions of modular modeling give modular connectors the exact same appearance (Fig. 1.7). The modular connector notation (Fig. 1.5) can be replaced with a single line. This simplified modular modeling notation for the conceptual power train example has a traditional block diagram appearance (Fig. 1.8). This simpler notation aids only in graphically communicating the model. The power-based fixed measurement perspective port causality and sign convention input-output structure implicit in every port line is critically important. For example, implicit in the port line connecting the *Engine* and *Transmission* modeling elements are two power variables F_{12}^{c} and v_{12}^{c} following the input-output causality in Table 1.1 constrained by (1.8) and (1.9).

$$Input \Rightarrow \begin{cases} F_1 = F_{12}^{c} \\ F_2 = -F_{12}^{c} \end{cases} \qquad Output \Rightarrow \{ \; v_1 = v_2 = v_{12}^{c} \qquad (1.11)$$

The dogmatic input-output structure of modeling elements is the key concept of modular modeling.

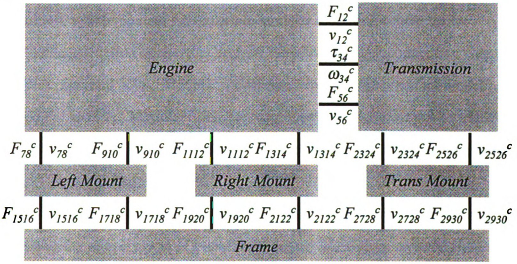

Figure 1.8: Simplified Modular Modeling Notation of the Conceptual Power Train Model in Figure 1.7

Chapter 2

ASSEMBLY AND SOLUTION OF LINEAR MODULAR MODELS USING FIXED INPUT-OUTPUT STRUCTURE

2.1 Modular Modeling

Modular modeling is a modeling method designed to eliminate model equation reformulation and enhance model performance verification. Modular models are constrained assemblies of power-based multi degree of freedom modular modeling elements of physical systems with a *single* standardized equation formulation. Standardized modular modeling elements enhance performance verification because their formulation does not change. Modular modeling elements have incompatible power ports by definition, which requires a compatible modular modeling connector for assembly. Modular modeling connectors do *NOT* describe physical connection elements like welds, shear pins, and bolts. Modular modeling elements implement all modeling activity in modular models. Modular modeling connectors are two-port power transfer mechanisms that implement standardized power constraints between connected modular element ports (Byam and Radcliffe, 1999).

The key concept of modular modeling is the fixed measurement perspective input-output causality at every port of every multi degree of freedom modular modeling element per energy domain. This causality is based on physical measurements. The port output variables are defined as the typically sensed physical system response. Development of the modular modeling method reveals that the port input variable can be assumed zero for zero power transfer and zero effect on system performance. This causality choice (Table 1.1) standardizes modular element formulations and allows modular elements to maintain any number of "open" or physically disconnected power ports. Modular modeling elements with fixed measurement perspective causality have physical system modularity.

Modular modeling elements have a standardized user-defined functional form that does not change.

$$\underset{\sim p\times 1}{y} = element(\underset{\sim p\times 1}{u}) \tag{2.1}$$

Where p is the number of element power ports, $\underset{\sim p\times 1}{u}$ is a vector of inputs at those ports and $\underset{\sim p\times 1}{y}$ is a vector of outputs at those ports defined by the fixed measurement perspective input-output power port causality. This functional form can be generated for any energy domain in Table 1.1 with any number of power ports.

Modular modeling element graphical notation represents user-defined multi-port multi-DOF subsystem models with a rectangle (Fig 2.1). The bold lines represent the *n* power ports with implicit standardized direction of positive power into the element and standardized input-output port causality. Standardization of positive power direction and input-output causality standardizes the modular modeling elements internal formulation, which is the essence of modular modeling. All modeling equations in modular models are in modular modeling elements by definition. Modeling the physical behavior of a connection like an engine mount or a shear pin requires a modeling element.

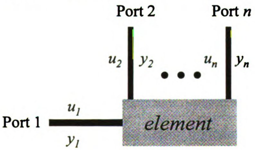

Figure 2.1: Modular Modeling Element Graphical Notation. Power flows into port 1 if u_1 and y_1 are both positive. The fixed causality input variable, u_i, is always shown on the top or to the left of the port. The fixed causality output variable, y_i, is always shown on the bottom or to the right of the port.

Modular modeling connectors implement standard output and power constraints. There are no physical model equations in connectors. They simply connect modular element ports by constraining their outputs to match and conserve power flow. Connectors are used exclusively to implement output and power constraints between ports of modular modeling elements.

The connector forces two connected modular element ports', *i* and *j*, outputs to be equal to connect the ports and their power flow to sum to zero to conserve power. The power flow constraint is translated to equal and opposite inputs at connected modular element ports since the product of power port variables is power. Connected modular element ports, *i* and *j*, have the same output and equal and opposite inputs.

$$y_i = y_j = y^c \tag{2.2a}$$

$$\begin{aligned} u_i &= u^c \\ u_j &= -u^c \end{aligned} \tag{2.2b}$$

In modular modeling, the defining relationship for all connectors (2.2) does not change.

The modular modeling connector graphical notation represents connectors with a bold port line between modular elements (Fig 2.2). By definition, connectors have the compatible input-output structure to modular elements. The bold lines have implicit standardized direction of positive power into the connected modular element ports, *i* and *j* and modular connector constraints (2.2). The modular connector has the flexibility to assemble by pairs any number of modular element power ports because modular modeling elements have an internal junction structure at each port (Byam and Radcliffe, 1999). The only function of modular modeling connectors is to constrain connected modular element ports.

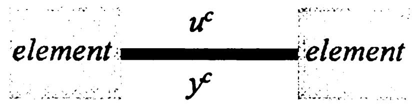

Figure 2.2: Modular Connector Graphical Notation. Power flows into the connected elements if y^c and u^c are both positive. The input variable, u, is always shown on the top or to the left of the port. The output variable, y, is always shown on the bottom or to the right of the port.

The objective of modular modeling is single standardized modular formulations for all user-defined multi-port multi-degree of freedom modeling elements. This requires separate connector constraints, which adds complexity to models. However, each standardized modular modeling element has a single formulation. A single fixed formulation allows modelers to gain experience verifying their formulation's performance. Modular models, which are constrained assemblies of n modular modeling elements, are also used with no reformulation and no reverification. Modular models in pre-assembled form are simply a concatenation of n modular modeling elements.

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_n \end{bmatrix} = \begin{bmatrix} element_1(\underset{\sim}{u}_1) \\ element_2(\underset{\sim}{u}_2) \\ \vdots \\ element_n(\underset{\sim}{u}_n) \end{bmatrix} \tag{2.3}$$

The standardized input-output structure enables a systematic direct-insertion realization of modular models from modular elements and connectors.

2.2 Linear Modular Modeling Analysis

Linear modular modeling analysis is a systematic direct-insertion realization of linear modular models of the form (2.3). Using a known input-output topology constrained by the output and power constraints of the modular modeling connector (2.2) the constrained modular model is realized. Linear modular modeling elements are independently user-formulated power-based multi-port multi-degree of freedom linear modeling equations with a standardized input-output causality and sign convention. Connectors are two-port output and conservative power constraints between modular modeling element ports. Modular connector constraints are known. Modular elements have a known form but their equations are user-defined.

The possible user-defined linear modular element equations are linear algebraic differential equations and algebraic equations. The linear algebraic differential equations can be represented in a form convenient for the application of the modular modeling fixed input-output functional form. A state-space form is the best fit because inputs and outputs are explicit. Any algebraic differential equation model can be written in state-space form. Algebraic equation models are typically expressed in an explicit input-output form.

2.2.1 Linear Algebraic Differential Equations

Modular modeling elements with user-defined linear algebraic differential equations have a traditional state-space form where the inputs and outputs are explicit.

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underline{A}\,\underset{\sim}{x} + \underline{B}\,\underset{\sim}{u} \\ \underset{\sim}{y} &= \underline{C}\,\underset{\sim}{x} + \underline{D}\,\underset{\sim}{u} \end{aligned} \tag{2.4}$$

The fixed input-output functional form (2.1) of modular elements is seen in (4), where $\underset{\sim}{u}$ is a vector of port inputs and $\underset{\sim}{y}$ is a vector of port outputs ordered in port pairs. $\underset{\sim}{x}$ is a vector of states. $\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ are matrices with time invariant coefficients independent of the x-variables and u-variables.

Consider a modular model in unconnected form as a concatenation of n independently formulated user-defined linear modular elements of the form (2.4). Let this model have s total states and p total input-output power ports.

$$\begin{bmatrix} \dot{\underset{\sim}{x}}_1 \\ \dot{\underset{\sim}{x}}_2 \\ \vdots \\ \dot{\underset{\sim}{x}}_n \end{bmatrix}_{s\times 1} = \begin{bmatrix} \underline{A}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{A}_n \end{bmatrix}_{s\times s} \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_n \end{bmatrix}_{s\times 1} + \begin{bmatrix} \underline{B}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{B}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{B}_n \end{bmatrix}_{s\times p} \begin{bmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \vdots \\ \underset{\sim}{u}_n \end{bmatrix}_{p\times 1} \tag{2.5a}$$

$$\dot{X} = \mathcal{A}X + \mathcal{B}U$$

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_n \end{bmatrix}_{p\times 1} = \begin{bmatrix} \underline{C}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{C}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{C}_n \end{bmatrix}_{p\times s} \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_n \end{bmatrix}_{s\times 1} + \begin{bmatrix} \underline{D}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{D}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{D}_n \end{bmatrix}_{p\times p} \begin{bmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \vdots \\ \underset{\sim}{u}_n \end{bmatrix}_{p\times 1} \tag{2.5b}$$

$$Y = \mathcal{C}X + \mathcal{D}U$$

For example, the modular model vector X is the concatenation of the n modular element state vectors with a total size $s\times 1$ and U is the concatenation of the n modular element input vectors with a total size $p\times 1$. The fixed input-output functional form (2.3) of a modular model is seen in (2.5). The modular model (2.5) has an equal number of inputs and outputs because all modular elements power ports always has one input and one output. Each modular element of (2.5) is completely independent and uncoupled from the other modular elements. Given the element's input-output topology the modular connector constraints (2.2a-2.2b) provide the coupling between the modular element equations.

The key concept of modular model analysis is isolating the internal element input-output power ports from the external element input-output power ports in a known input-output topology. External input-output power ports have known inputs. Internal input-output power ports are ports joined to other element ports through connectors (Fig. 2.2). The input-output topology of (2.5) has a total of p input-output power ports from the n modular elements. Let m be the number of connectors, hence there are $2m$ internal element input-output power ports, which leaves $q = p - 2m$ external element input-output power ports. The standardized form of modular elements (2.1) and modular connectors (2.2) makes isolating external and internal element ports in the unconnected modular model (2.5) a simple reordering of the systems' concatenated input and output vectors U and Y.

A transformation matrix reorders the concatenated input and output vectors of (2.5). The vectors are reordered so all external ports' variables appear first in the vectors followed by all the internal ports' variables. The internal ports' variables are ordered such that connected port pairs' appear together. For example, if port *i* and port *j* are connected u_i should be followed immediately by u_j, similarly for the outputs y_i and y_j. The reordered input and output vectors are the input and output vectors of (2.5) pre-multiplied by the transformation matrix $\underline{T}$.

$$\underline{T}U = \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix} \tag{2.6a}$$

$$\underline{T}Y = \begin{bmatrix} Y_{ext} \\ Y_{int} \end{bmatrix} \tag{2.6b}$$

The transformation matrix $\underline{T}$ does not add, remove, or combine variables of the original vectors; it only changes the order in which variables appear. This makes $\underline{T}$ a linear and nonsingular reordered $p \times p$ identity matrix $\underline{I}$.

$$\underline{T}\underline{T}^T = \underline{I} \tag{2.7}$$

The external input U_{ext} is the $q \times 1$ vector of external port inputs. The internal input U_{int} is the $2m \times 1$ vector of internal port inputs. The external output Y_{ext} is the $q \times 1$ vector of external port outputs. The internal output Y_{int} is the $2m \times 1$ vector of internal port outputs. The mechanism for reorganizing the input-output topology of (2.5) to isolate the external and internal ports is the matrix $(\underline{T})_{p \times p}$.

The reordered modular model equations are written in terms of the external and internal inputs and outputs where $U = T^T[U_{ext}, U_{int}]^T$.

$$\dot{X} = \mathcal{A}X + \mathcal{B}\underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix} \tag{2,8a}$$

$$\underline{T}Y = \begin{bmatrix} Y_{ext} \\ Y_{int} \end{bmatrix} = \underline{T}\mathcal{C}X + \underline{T}\mathcal{D}\underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix} \tag{2.8b}$$

Using the transformation matrix $\underline{T}$ the matrices in (2.8) are partitioned to further isolate internal and external equations. The $\mathcal{B}$ matrix is partitioned by $\underline{T}^T$ into coefficients of the external and internal inputs. The $\mathcal{C}$ matrix is partitioned by $\underline{T}$ into the states' coefficients of the external and internal outputs. The $\mathcal{D}$ matrix is partitioned by $\underline{T}^T$ and $\underline{T}$ into external and internal inputs' coefficients of the external and internal outputs.

$$\mathcal{B}\underline{T}^T = \left[\mathcal{B}_{ext}\ \mathcal{B}_{int}\right] \tag{2.9a}$$

$$\underline{T}\mathcal{C} = \begin{bmatrix} \mathcal{C}_{ext} \\ \mathcal{C}_{int} \end{bmatrix} \tag{2.9b}$$

$$\underline{T}\mathcal{D}\underline{T}^T = \begin{bmatrix} \mathcal{D}_{extext} & \mathcal{D}_{extint} \\ \mathcal{D}_{intext} & \mathcal{D}_{intint} \end{bmatrix} \tag{2.9c}$$

The external partition of the $\mathcal{B}$ matrix $\mathcal{B}_{ext}$ is a $s \times q$ matrix. The internal partition of the $\mathcal{B}$ matrix $\mathcal{B}_{int}$ is a $s \times 2m$ matrix. The external partition of the $\mathcal{C}$ matrix $\mathcal{C}_{ext}$ is a $q \times s$ matrix. The internal partition of the $\mathcal{C}$ matrix $\mathcal{C}_{int}$ is a $2m \times s$ matrix. The sizes of the partitions of the $\mathcal{D}$ matrix $\mathcal{D}_{extext}$, $\mathcal{D}_{extint}$, $\mathcal{D}_{intext}$, and $\mathcal{D}_{intint}$ are $q \times q$, $q \times 2m$, $2m \times q$, and $2m \times 2m$ respectively. The reordered unconnected modular model equations are rewritten in terms of the external and internal inputs and outputs and the partitioned matrices.

$$\dot{X} = \mathcal{A}X + \mathcal{B}_{ext}U_{ext} + \mathcal{B}_{int}U_{int} \tag{2.10a}$$

$$Y_{ext} = \mathcal{C}_{ext}X + \mathcal{D}_{extext}U_{ext} + \mathcal{D}_{extint}U_{int} \tag{2.10b}$$

$$Y_{int} = \mathcal{C}_{int}X + \mathcal{D}_{intext}U_{ext} + \mathcal{D}_{intint}U_{int} \tag{2.10c}$$

The internal outputs Y_{int} of (2.10c) are ordered in connected internal port pairs. At each connected pair, the two output values are constrained by (2.2a) to have equal values. In order to apply the outputs of (2.10c) to the constraint (2.2a) the connected output pairs need to be selected from the matrix equation (2.10c). Define two $m \times 2m$ equation selector matrices $\underline{S}_o$ and $\underline{S}_e$ to select the odd and even equations of (2.10c) respectively.

$$\underline{S}_o = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & \vdots & 1 & \vdots & & \vdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix}_{m\times 2m} \qquad \underline{S}_e = \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ \vdots & 0 & \vdots & 1 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \end{bmatrix}_{m\times 2m} \tag{2.11}$$

Pre-multiplying (2.10c) by $\underline{S}_o$ selects the odd (1^{st}, 3^{rd}, 5^{th}, … , $m-1^{st}$) internal port output equations. Pre-multiplying (2.10c) by $\underline{S}_e$ selects the even (2^{nd}, 4^{th}, 6^{th}, … , m^{th}) internal port output equations. Substitute selected equations in the output constraint (2.2a).

$$\underline{S}_o Y_{int} = Y^c \tag{2.12a}$$

$$\underline{S}_e Y_{int} = Y^c \tag{2.12b}$$

The constrained internal ports' output Y^c is a $m\times 1$ vector of the m constrained inputs at the $2m$ connected internal ports (Fig. 2.2) of the system (2.5a-2.5b). Rewrite the output constraint (2.12a-2.12b) in terms of a difference to eliminate the constrained internal port's output.

$$(\underline{S}_o - \underline{S}_e) Y_{int} = \underline{0}_{m\times 1} \tag{2.13}$$

Substituting the internal output equation (2.10c) into the internal output constraint (13) results in the output constraints of the modular model.

$$(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int} X + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext} U_{ext} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint} U_{int} = \underline{0} \tag{2.14}$$

The three output constrained modular system equations are written in terms of the states, X, the external inputs, U_{ext}, the external outputs, Y_{ext}, and the internal inputs, U_{int}.

$$\dot{X} = \mathcal{A}X + \mathcal{B}_{ext} U_{ext} + \mathcal{B}_{int} U_{int} \tag{2.15a}$$

$$Y_{ext} = \mathcal{C}_{ext} X + \mathcal{D}_{extext} U_{ext} + \mathcal{D}_{extint} U_{int} \tag{2.15b}$$

$$(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int} X + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext} U_{ext} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint} U_{int} = \underline{0} \tag{2.15c}$$

The three output constrained modular model equations (2.15a-2.15c) have three unknowns, the states, X, the external outputs, Y_{ext}, and the internal inputs, U_{int}. The state equation (2.15a) and the output constraint equation (2.15c) are independent of the external outputs, Y_{ext}, and can be solved for the states, X, and the internal inputs, U_{int}, then substituted in the external output equation to find the external outputs, Y_{ext}. Rewrite (2.15a) and (2.15c) as a system of two equations with the two unknowns the states, X, and the internal inputs, U_{int}.

$$\dot{X} - \mathcal{A}X - \mathcal{B}_{int}U_{int} = \mathcal{B}_{ext}U_{ext} \tag{2.16a}$$

$$-(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}X - (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}U_{int} = (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext} \tag{2.16b}$$

The objective is to find the internal input, U_{int}, in terms of the states, X, and external inputs, U_{ext}, that satisfies (2.16b).

The internal input, U_{int}, of the output constrained modular system model is dependent on the state response of the system.

$$X(t) = \Phi(t,t_0)X_0 + \int_{t_0}^{t} \Phi(t,\sigma)\left(\mathcal{B}_{int}U_{int}(\sigma) + \mathcal{B}_{ext}U_{ext}(\sigma)\right)d\sigma \tag{2.17}$$

The state response of the output constrained modular system (2.17) at some time t involves the state transition matrix $\Phi(t,t_0)$ from some initial time t_0 and an initial state X_0. Substitute the state response of the system (2.17) into the output constraint equation (2.16b).

$$\begin{aligned}-(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}\left(\Phi(t,t_0)X_0 + \int_{t_0}^{t} \Phi(t,\sigma)\left(\mathcal{B}_{int}U_{int}(\sigma) + \mathcal{B}_{ext}U_{ext}(\sigma)\right)d\sigma\right) \\ = (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}U_{int} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext}\end{aligned} \tag{2.18}$$

Group the internal inputs, U_{int}, and the external inputs, U_{ext} together using the screening property of the Dirac delta.

$$\begin{aligned}\int_{t_0}^{t} (\underline{S}_o - \underline{S}_e)\left(\mathcal{C}_{int}\Phi(t,\sigma)\mathcal{B}_{int} + \mathcal{D}_{intint}\delta(t-\sigma)\right)U_{int}(\sigma)d\sigma = \\ -(\underline{S}_o - \underline{S}_e)\left(\mathcal{C}_{int}\Phi(t,t_0)X_0 + \int_{t_0}^{t}\left(\mathcal{C}_{int}\Phi(t,\sigma)\mathcal{B}_{ext} + \mathcal{D}_{intext}\delta(t-\sigma)\right)U_{ext}(\sigma)d\sigma\right)\end{aligned} \tag{2.19}$$

The internal input, U_{int}, that satisfies the constrained output equation (2.19) requires the definition of two matrices.

$$\underline{G}(\sigma) = (\underline{S}_o - \underline{S}_e)\left(\mathcal{C}_{int}\Phi(t,\sigma)\mathcal{B}_{int} + \mathcal{D}_{intint}\delta(t-\sigma)\right) \tag{2.20a}$$

$$\underline{Y}(t) = \int_{t_0}^{t} \underline{G}(\xi)\underline{G}^*(\xi)d\xi \tag{2.20b}$$

Rewrite $\underline{Y}(t)$ using the screening property of the Dirac delta and the identity $(\underline{HG})^* = \underline{G}^*\underline{H}^*$.

$$\underline{Y}(t) = (\underline{S}_o - \underline{S}_e)\left(\mathcal{D}_{intint}\mathcal{D}_{intint}^*\frac{1}{2\varepsilon} + \mathcal{D}_{intint}\mathcal{B}_{int}^*\mathcal{C}_{int}^* + \mathcal{C}_{int}\mathcal{B}_{int}\mathcal{D}_{intint}^*\right)(\underline{S}_o - \underline{S}_e)^T + (\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}\underline{X}_{int}(t_0,t)\mathcal{C}_{int}^*(\underline{S}_o - \underline{S}_e)^T \tag{2.21a}$$

$$\underline{X}_{int}(t_0,t) = \int_{t_0}^{t} \Phi(t,\sigma)\mathcal{B}_{int}\mathcal{B}_{int}^*\Phi^*(t,\sigma)d\sigma. \tag{2.21b}$$

The * represents the complex conjugate transpose of a matrix, ε is arbitrarily small positive constant, $\underline{X}_{int}(t_0,t)$ is the system's "internal controllability grammian". The internal input is defined by the construction of (2.19).

$$U_{int}(\sigma) = -\underline{G}^*(\sigma)\underline{Y}^{-1}(t)(\underline{S}_o - \underline{S}_e)\cdot \left(\mathcal{C}_{int}\Phi(t,t_0)X_0 + \int_{t_0}^{t}\left(\mathcal{C}_{int}\Phi(t,\sigma)\mathcal{B}_{ext} + \mathcal{D}_{intext}\delta(t-\sigma)\right)U_{ext}(\sigma)d\sigma\right) \tag{2.22}$$

The internal input, U_{int}, exists if the matrix $\underline{Y}(t)$ is nonsingular. The matrix $\underline{Y}(t)$ is nonsingular if it is positive definite. This derivation is similar to derivation of output controllability in Skelton, 1988, which defines a positive definite matrix made up from the system matrices and state transition matrix similar to (2.21a). The derivation here is concerned with the controllability the internal inputs have over the internal outputs or its internal output controllability. If a modular system model (2.5a-2.5b) has a positive definite matrix $\underline{Y}(t)$, it is internally output controllable and the internal input U_{int} exists.

The condition of existence of U_{int} is less restrictive using the output controllability approach of modular modeling then previous methods. Hogan found that assembling linear modular component models with simple "nonenergic" connections required either the invertability of a matrix involving only the *D* matrix or that the *D* matrix be zero. Modular modeling clearly is less restrictive because the invertability of $\underline{Y}(t)$ is dependent on the *A*, *B*, *C*, and *D* matrices where all can be nonzero.

The internal port input pairs, U_{int}, are ordered the same way as the internal port output pairs, so the $m \times 2m$ equations selector matrices $\underline{S}_o$ and $\underline{S}_e$ can be used to define the $m \times 1$ constrained input U^c. The constrained input U^c is the result of the modular connector power constraint (2.2b) constraining internal port input pairs to be equal and opposite.

$$U_{int} = (\underline{S}_o - \underline{S}_e)^T U^c \tag{2.23}$$

Isolate the constrained input U^c by pre-multiplying (2.23) by $(\underline{S}_o - \underline{S}_e)$.

$$(\underline{S}_o - \underline{S}_e) U_{int} = 2\underline{I}_c U^c \tag{2.24}$$

The matrix $\underline{I}_c$ is an $m \times m$ identity matrix. Substitute the internal input (2.22) into (2.24) and solve for the constrained internal input, U^c.

$$\begin{aligned} U^c(\sigma) = -\frac{1}{2}(\underline{S}_o - \underline{S}_e)\underline{G}^*(\sigma)\underline{Y}^{-1}(t)(\underline{S}_o - \underline{S}_e)\cdot \\ \left(\mathcal{C}_{int}\Phi(t,t_0)X_0 + \int_{t_0}^{t} \left(\mathcal{C}_{int}\Phi(t,\sigma)\mathcal{B}_{ext} + \mathcal{D}_{intext}\delta(t-\sigma) \right) U_{ext}(\sigma) d\sigma \right) \end{aligned} \tag{2.25}$$

The fully connected modular system equations are realized by substituting (2.23) into (2.15a-2.15c).

$$\dot{X} = \mathcal{A}X + \mathcal{B}_{ext}U_{ext} + \mathcal{B}_{int}(\underline{S}_o - \underline{S}_e)^T U^c \tag{2.26a}$$

$$Y_{ext} = \mathcal{C}_{ext}X + \mathcal{D}_{extext}U_{ext} + \mathcal{D}_{extint}(\underline{S}_o - \underline{S}_e)^T U^c \tag{2.26b}$$

$$(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}X + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}(\underline{S}_o - \underline{S}_e)^T U^c = \underline{0} \tag{2.26c}$$

Solving the three equations (2.26a-2.26c) for the three unknowns, X, U^c, and Y_{ext} requires the modular system (2.5a-2.5b) to be internally output controllable where U^c is given by (2.25). The connected modular system (2.26a-2.26c) has maintained its modularity using "nonenergic" connectors with less restrictive conditions for solution then previous methods designed to maintain modularity (Hogan, 1987).

2.2.2 Linear Algebraic Equations

Modular modeling elements with user-defined linear algebraic equations have the same form as (2.4) but the matrices $\underline{A}$, $\underline{B}$, and $\underline{C}$, are zero.

$$\underset{\sim}{y} = \underline{D}\underset{\sim}{u} \tag{2.27}$$

Following the same procedure as in (2.5a-2.5b), consider a modular model in unconnected form as a concatenation of n independently formulated user-defined modular modeling elements in the algebraic form (2.27) with p total input-output power ports.

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_n \end{bmatrix}_{p\times 1} = \begin{bmatrix} \underline{D}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{D}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{D}_n \end{bmatrix}_{p\times p} \begin{bmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \vdots \\ \underset{\sim}{u}_n \end{bmatrix}_{p\times 1} \tag{2.28}$$

$$Y = \mathcal{D}U$$

The same input-output topology, reordering, and output constraint analysis applies to the algebraic output constrained modular system equations. These equations are written in terms of the external inputs, U_{ext}, the external outputs, Y_{ext}, and the internal inputs, U_{int}.

$$(\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}U_{int} = \underline{0}_{m\times 1} \tag{2.29a}$$

$$Y_{ext} = \mathcal{D}_{extext}U_{ext} + \mathcal{D}_{extint}U_{int} \tag{2.29b}$$

Solving (2.29a) for internal inputs, U_{int}, in terms of the U_{ext}, the external outputs using the same techniques as the linear algebraic differential case (2.22).

$$U_{int}(\sigma) = -\mathcal{D}_{intint}{}^{*}(\underline{S}_o - \underline{S}_e)^T\left((\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}\mathcal{D}_{intint}{}^{*}(\underline{S}_o - \underline{S}_e)^T\right)^{-1}(\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext}(\sigma) \tag{2.30}$$

The internal input, U_{int}, exists if the matrix $(\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}\mathcal{D}_{intint}{}^{*}(\underline{S}_o - \underline{S}_e)^T$ is nonsingular. This matrix is nonsingular if it is positive definite or the modular system (2.28) is internally output controllable. The constrained internal input, U^c, for the algebraic modular system is found following the same analysis in (2.23)-(2.24).

$$U^c(\sigma) = -\frac{1}{2}(\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}{}^{*}(\underline{S}_o - \underline{S}_e)^T\left((\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}\mathcal{D}_{intint}{}^{*}(\underline{S}_o - \underline{S}_e)^T\right)^{-1}(\underline{S}_o - \underline{S}_e)\cdot \mathcal{D}_{intext}U_{ext}(\sigma) \tag{2.31}$$

The fully input and output constrained algebraic modular system equations are written in terms of the two unknowns constrained internal input, U^c, and the external outputs, Y_{ext}, where U^c is given by (2.31).

$$(\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}(\underline{S}_o - \underline{S}_e)^T U^c = \underline{0} \tag{2.32a}$$

$$Y_{ext} = \mathcal{D}_{extext}U_{ext} + \mathcal{D}_{extint}(\underline{S}_o - \underline{S}_e)^T U^c \tag{2.32b}$$

2.2.3 Linear Algebraic Boundary Value Problem Equations

The linear algebraic equations that result from a boundary value problem (BVP) has the form of the generalized responses, $\tilde{x}$, pre-multiplied by a stiffness matrix, $\underline{K}$, equal to the generalized excitations, $\tilde{w}$ (Segerlind, 1984).

$$\underline{K}\tilde{x} = \tilde{w} \tag{2.33}$$

Without boundary conditions the stiffness matrix, $\underline{K}$, is singular. These equations occur in structural, solid mechanics, heat transfer, and irrotational flow models typically developed by Finite Element Analysis (FEA). Consider a modular model in unconnected form as a concatenation of n of these modeling elements without boundary conditions applied.

$$\begin{bmatrix} \underline{K}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{K}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{K}_n \end{bmatrix}_{s\times s} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix}_{s\times 1} = \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \tilde{w}_n \end{bmatrix}_{s\times 1} \tag{2.34}$$

$$\mathcal{K}X = W$$

The generalized responses, X, and the generalized excitations, W, are defined by element dependent geometry's which differ for each of the n modeling elements. In order to maintain modularity, the n modeling elements in (34) will be connected through inputs, U, and outputs, Y, that are linear interpolations of the generalized responses, X, and the generalized excitations, W.

$$\begin{bmatrix} \underline{K}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{K}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{K}_n \end{bmatrix}_{s\times s} \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_n \end{bmatrix}_{s\times 1} = \begin{bmatrix} \underset{\sim}{w}_1 \\ \underset{\sim}{w}_2 \\ \vdots \\ \underset{\sim}{w}_n \end{bmatrix}_{s\times 1} = \begin{bmatrix} \underline{C}_1^T & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{C}_2^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{C}_n^T \end{bmatrix}_{s\times p} \begin{bmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \vdots \\ \underset{\sim}{u}_n \end{bmatrix}_{p\times 1} \tag{2.35a}$$

$$\mathcal{K}X = W = \mathcal{C}^T U$$

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_n \end{bmatrix}_{p\times 1} = \begin{bmatrix} \underline{C}_1 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{C}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \underline{C}_n \end{bmatrix}_{p\times s} \begin{bmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \\ \vdots \\ \underset{\sim}{x}_n \end{bmatrix}_{s\times 1} \tag{2.35b}$$

$$Y = \mathcal{C}X$$

Notice the input-output matrix $\mathcal{C}$ is the same in both (2.35a) and (2.35b). This is because the co-located inputs and outputs are defined as ports. BVP modeling elements in this form (2.35a-2.35b) can be connected together maintaining modularity.

The fixed input-output functional form (2.3) of a modular model is seen in (2.35a-2.35b). The modular BVP model (2.35a-2.35b) has an equal number of inputs and outputs because all modular elements power ports always has one input and one output. Each modular BVP element of (2.35a-2.35b) is completely independent and uncoupled from the other modular elements. Given the element's input-output topology the modular connector constraints (2.2a-2.2b) provide the coupling between the modular element equations.

The same input-output topology, reordering scheme, and input-output constraint analysis applied to the algebraic differential and algebraic equations applies to the modular BVP system equations (2.35a-2.35b). These fully connected equations are written in terms of the external inputs, U_{ext}, the external outputs, Y_{ext}, and the constrained internal inputs, U^c, using the equation selector matrices, $\underline{S}_o$ and $\underline{S}_e$.

$$\mathcal{K}X = \mathcal{C}_{ext}{}^T U_{ext} + \mathcal{C}_{int}{}^T (\underline{S}_o - \underline{S}_e)^T U^c \tag{2.36a}$$

$$Y_{ext} = \mathcal{C}_{ext} X \tag{2.36b}$$

$$(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int} X = \underline{0}_{m\times 1} \tag{2.36c}$$

The input constraint is applied immediately to the BVP equation (2.36a) where $U_{int} = (\underline{S}_o - \underline{S}_e)^T U^c$. This realization is different then the state equations (2.26a-2.26c) and (2.32a-2.32b) because the stiffness matrix $\mathcal{K}$ is singular prior to application of boundary conditions.

Solving (2.36a-2.36c) for the generalized responses, X, the constrained inputs, U^c, and the external outputs, Y_{ext}, in terms of the external inputs, U_{ext} requires finding a set of generalized responses that satisfy the output constraints (2.36c). Any basis of the nullspace of $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}$ will define a set of output constrained generalized responses that satisfy (2.36c). The nullspace of $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}$ has dimension $s-m$ because it has rank of m (Leon, 1986). It has rank of m because it represents m connectors at m different physical locations resulting in m independent rows seen in the structure of $(\underline{S}_o - \underline{S}_e)$. Define a $s \times s-m$ matrix, $\mathcal{V}$, that is a basis of the nullspace of $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}$ to transform the output constrained generalized responses, X^c, to the generalized responses, X.

$$X = \mathcal{V}X^c \tag{2.37}$$

This procedure is similar to the procedure Meirovitch uses to eliminate rigid body modes from differential equations (Meirovitch, 1967). In Meirovitch a basis of the nullspace of the rigid body modes eigenvectors defines the transformed responses. Substitute (2.37) into (2.36a) and pre-multiply by $\mathcal{V}^T$ with the knowledge that $\mathcal{V}^T \mathcal{C}_{int}{}^T (\underline{S}_o - \underline{S}_e)^T = \underline{0}$ because of the orthogonality of nullspaces (Leon, 1986).

$$\mathcal{V}^T \mathcal{K} \mathcal{V} X^c = \mathcal{V}^T \mathcal{C}_{ext}{}^T U_{ext} \tag{2.38}$$

The matrix $\mathcal{V}^T \mathcal{K} \mathcal{V}$ has dimension $s-m \times s-m$. The matrix $\mathcal{V}$ combines rows and columns of the system stiffness matrix $\mathcal{K}$ connecting the otherwise independent element stiffness matrices $\underline{K}_i$. This pre and post multiplication has the same effect as applying the direct stiffness method. The direct stiffness method is used to build up stiffness matrices in BVP (Segerlind, 1984). Solving for X^c requires the matrix $\mathcal{V}^T \mathcal{K} \mathcal{V}$ to be nonsingular. This requires the elimination of all rigid body modes, which can be done by applying more output constraints or a further coordinate transformation. The solution of (2.38) X^c can be substituted in (2.37) and then used in (2.36b) to find the external outputs Y_{ext}. Solving for the constrained internal inputs U^c requires pre-multiplying (2.36a) by $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}$.

$$(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}\mathcal{C}_{int}^T(\underline{S}_o - \underline{S}_e)^T U^c = (\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}\mathcal{K}X - (\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}\mathcal{C}_{ext}^T U_{ext} \quad (2.39)$$

The matrix $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}\mathcal{C}_{int}^T(\underline{S}_o - \underline{S}_e)^T$ is nonsingular because $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}$ has rank of m. This solves the modular linear BVP (2.36a-2.36c).

The linear algebraic differential, linear algebraic, and linear BVP constrained modular system equations (2.26a-2.26c), (2.32a-2.32b), and (2.36a-2.36c) respectively are written in terms of independently formulated modular modeling elements. The reordering and selector matrices, $\underline{T}$ and $(\underline{S}_o - \underline{S}_e)$, are comprised of ones and zeros and defined by the size and input-output topology of the modular system model. The modular modeling element matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{K}$ can be directly inserted into the constrained modular system equations and solved like any other system of equations. The advantage of modular modeling is that each multi DOF modeling element is independent and can be connected together with simple connectors and assembled with a simple physically intuitive top-down direct insertion method.

2.3 Guaranteed Computational Sparseness

Linear modular models have a guaranteed diagonal modular structure because of their standardized input-output formulation. This guaranteed modular structure defines an advantageous guaranteed sparseness for computation at the cost of additional constraint equations not found in an arbitrary input-output structure or non-modular formulated models. These extra equations add computations to modular models but enable an advantageous modular construction for computation.

A number of computations (*NOC*) analysis shows the computational advantages of the modular construction. A *NOC* analysis is a count of the number of multiplications and additions in a linear system.. The modular system *NOC* is determined by analyzing the constrained modular system equations (2.26a-2.26c). Each of the n modular elements has states, internal port inputs, external port inputs, and external port outputs. Let a modular element i have s_i states, $2m_i$ internal ports, and q_i external ports. The contribution from (2.26a) to the modular system *NOC* comes from the state matrix times the states plus the internal input matrix times the constrained internal inputs plus the external input matrix times the external inputs (Redish, 1961). Because of modular construction the state matrix times the states and the external input matrix times the external inputs can be computed on an element basis which is the modular advantage. The internal input matrix times the constrained internal inputs must be computed on a system basis where m is the number of connectors in the modular system.

$$(2.26a)\ NOC = \underbrace{\sum_{i=1}^{n} s_i^2 + \sum_{i=1}^{n} s_i q_i}_{\text{Modular Advantage NOC Per Element}} + \underbrace{m\sum_{i=1}^{n} s_i}_{\text{Constraints NOC Per System}} \tag{2.40a}$$

Similarly, the contributions from equations (2.26b-2.26c) add to the system *NOC*.

$$(2.26b)\ NOC = \underbrace{\sum_{i=1}^{n} s_i q_i + \sum_{i=1}^{n} q_i^2}_{\text{Modular Advantage NOC Per Element}} + \underbrace{m\sum_{i=1}^{n} q_i}_{\text{Constraints NOC Per System}} \tag{2.40b}$$

$$(2.26c)\ NOC = \underbrace{m\sum_{i=1}^{n} s_i + m^2 + m\sum_{i=1}^{n} q_i}_{\text{Constraints NOC Per System}} \tag{2.40c}$$

Summing the *NOC* contributions (2.40a-2.40c) together and then simplifying results in an expression for the *NOC* for a modular system model.

$$NOC_{Modular\ System} = \underbrace{\sum_{i=1}^{n} (s_i + q_i)^2}_{\text{Modular Advantage NOC Per Element}} \underbrace{+ m^2 + 2m\sum_{i=1}^{n} (s_i + q_i)}_{\text{Constraints NOC Per System}} \tag{2.41}$$

The *NOC* for a modular system (2.41) is guaranteed by the construction of modular models.

The guaranteed computational sparseness of a modular system results in comparable computational efficiency when there are a large number of interconnected multi DOF subsystems. A non-modular system model of the form (2.4) has a *NOC* of

$(s+p)^2$ where s is the number of states and p is the number of input-output power ports. In this case, there are no guarantees about the system's sparseness. So, s is the number of states for the entire system and p is the number of power ports for the entire system. In other words, the *NOC* is determined by summing the number of all subsystem states and ports and squaring the sum. Notice in modular advantage portion of (2.41) that the number of multi-DOF elements' states and ports are first squared and then summed. Modular modeling is not, by any means, the minimal realization of a system. Other non-modular methods can realize models for the same complex multi-port multi DOF systems with fewer states and fewer ports with some effort. Modular models have a guaranteed sparseness, which results in a comparable and in some cases better computational efficiency.

For example, consider a system model with $n = 10$ multi DOF elements with $s_i = 5$ states each. A modular model requires at least $m = 9$ connectors to connect the 10 elements and, say, 2 external ports for a total of $\sum_{i=1}^{10} p_i = 20$ ports. Each element will have $p_i = 2$ ports. Two elements will have 1 external port and 1 internal port leaving 8 elements with 2 internal ports. The modular model *NOC* has $s_i = 5$, $q_1 = q_{10} = 1$, and $q_{2-9} = 2$.

$$NOC_{Modular} = \sum_{i=1}^{10}(5+q_i)^2 + 9^2 + 2(9)\sum_{i=1}^{10}(5+q_i) = 1289 \qquad (2.42)$$

A non-modular model of the same system may require only, say 75% of the states in the modular model $s = 0.75 * 50 = 37.5$ and only the external ports $p = 2$. The non-modular model *NOC*, with 75% the modular model states and a tenth of the modular model ports is larger.

$$NOC_{Non-Modular} = (37.5+2)^2 = 1560.25 \qquad (2.43)$$

Consider the same system but with $n = 2$ elements, $m = 1$ connector, and external ports $q_1 = q_2 = 1$ for the modular model. The *NOC* for modular and non-modular models become closer in magnitude with the non-modular being less.

$$NOC_{Modular} = \sum_{i=1}^{2}(5+1)^2 + 1^2 + 2(1)\sum_{i=1}^{2}(5+1) = 97 \tag{2.44a}$$

$$NOC_{Non-Modular} = (7.5+2)^2 = 90.25 \tag{2.44b}$$

The above *NOC* examples show cases where the computational efficiency for modular models is better and comparable then non-modular methods. Guaranteed computational sparseness makes modular modeling one of the modeling methods of choice for large systems.

2.4 Examples

An electric automotive vehicle power train example model is used to illustrate modular modeling of linear algebraic differential modular models and linear algebraic modular models. A structural three bar and connecting shear pin fixed to a point example model is used to illustrate modular modeling of linear algebraic BVP modular model. The electric automotive vehicle power train example model is a combination of three simple dynamic models found in Phillips and Harbor, 1996 and Minor, 1996. The three bar and connecting shear pin example model is a combination of four simple BVP models from Segerlind, 1984.

The electric automotive vehicle power train example model contains three simple modular modeling elements, an electric motor model, a clutch model, and a transmission model. The three modular modeling elements are all defined with standardized modular modeling causality (Table 1.1). The electric motor modular modeling element has one electrical power port with current input i voltage output e causality and one rotational mechanical power port with torque input τ angular velocity output ω causality (Fig. 2.3a).

$$\dot{x}_m = [0]x_m + \begin{bmatrix} \frac{K_t}{J} & \frac{1}{J} \end{bmatrix} \begin{bmatrix} i \\ \tau \end{bmatrix} \tag{2.45a}$$

$$\begin{bmatrix} e \\ \omega \end{bmatrix} = \begin{bmatrix} K_b \\ 1 \end{bmatrix} x_m + \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau \end{bmatrix} \tag{2.45b}$$

The motor model has a state, x_m, a lumped motor-shaft rotational inertia, J, a motor coil resistance, R, a motor back emf constant, K_b, and a motor torque constant, K_t. The transmission modular modeling element has two rotational mechanical power ports with torque τ_u and τ_d input angular velocity ω_u and ω_d output causality (Fig. 2.3b).

$$\dot{x}_{tr} = -\frac{c_{tr}}{J_{tr}} x_{tr} + \begin{bmatrix} \frac{GR}{J_{tr}} & \frac{1}{J_{tr}} \end{bmatrix} \begin{bmatrix} \tau_u \\ \tau_d \end{bmatrix} \tag{2.46a}$$

$$\begin{bmatrix} \omega_u \\ \omega_d \end{bmatrix} = \begin{bmatrix} GR \\ 1 \end{bmatrix} x_{tr} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_u \\ \tau_d \end{bmatrix} \tag{2.46b}$$

The transmission model a state, x_{tr}, a lumped transmission rotational inertia, J_{tr}, an equivalent transmission damping, c_{tr}, and a transmission gear ratio, GR. The clutch modular modeling element has two rotational mechanical power ports with torque τ_m and τ_t input angular velocity ω_m and ω_t output causality (Fig. 2.3c).

$$\begin{bmatrix} \dot{x}_{c1} \\ \dot{x}_{c2} \\ \dot{x}_{c3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{K_{cl}}{J_{cl}} \\ 0 & 0 & \frac{K_{cl}}{J_{cl}} \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{c1} \\ x_{c2} \\ x_{c3} \end{bmatrix} + \begin{bmatrix} \frac{1}{J_{cl}} & 0 \\ 0 & \frac{1}{J_{cl}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_m \\ \tau_t \end{bmatrix} \tag{2.47a}$$

$$\begin{bmatrix} \omega_m \\ \omega_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{c1} \\ x_{c2} \\ x_{c3} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_m \\ \tau_t \end{bmatrix} \tag{2.47b}$$

The clutch model has three states, $[x_{c1}, x_{c2}, x_{c3}]^T$, a lumped clutch rotational inertia, J_{cl}, and an equivalent torsional stiffness, K_{cl}.

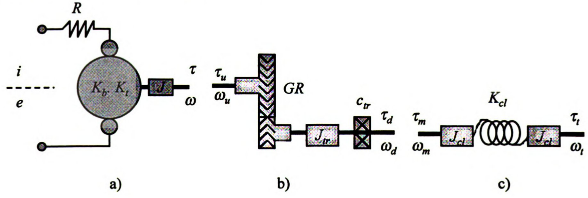

Figure 2.3: Electric Automotive Vehicle Power Train Modular Modeling Elements a) Electric Motor b) Transmission c) Clutch

These three modular modeling elements are assembled in two different configurations to illustrate the flexibility of modular modeling. The first configuration considers a power train with the modular motor modeling element and modular transmission modeling element (Fig. 2.4).

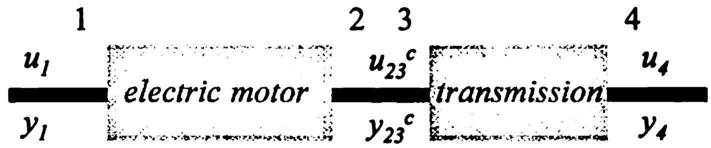

Figure 2.4: Electric Automotive Vehicle Power Train Modular Model With An Electric Motor And A Transmission Modular Modeling Elements

The unconnected two element modular system is built up from directly inserting the motor and transmission modular element matrices into (2.5a-2.5b).

$$\dot{X} = \begin{bmatrix}\dot{x}_m \\ \hline \dot{x}_t\end{bmatrix} = \mathcal{A}X + \mathcal{B}U = \left[\begin{array}{c|c}0 & 0\\ \hline 0 & -c_{tr}/J_{tr}\end{array}\right]\begin{bmatrix}x_m\\ \hline x_t\end{bmatrix} + \left[\begin{array}{cc|cc}K_t/J & -1/J & 0 & 0\\ \hline 0 & 0 & GR/J_{tr} & 1/J_{tr}\end{array}\right]\begin{bmatrix} i\\ \tau \\ \hline \tau_u \\ \tau_d\end{bmatrix} \tag{2.48a}$$

$$Y = \begin{bmatrix} e \\ \omega \\ \hline \omega_u \\ \omega_d\end{bmatrix} = \mathcal{C}X + \mathcal{D}U = \left[\begin{array}{c|c} K_b & 0 \\ 1 & 0 \\ \hline 0 & GR \\ 0 & 1\end{array}\right]\begin{bmatrix}x_m \\ \hline x_t\end{bmatrix} + \left[\begin{array}{cc|cc} R & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\begin{bmatrix} i \\ \tau \\ \hline \tau_u \\ \tau_d\end{bmatrix} \tag{2.48b}$$

The input-output topology of the two element power train modular model defines the size and structure of the modular modeling reordering and equation selector matrix $\underline{T}$ and $(\underline{S}_o - \underline{S}_e)$ respectively. This modular model has four power ports and one connector. This results in a 4×4 reordering matrix $\underline{T}$ and a 1×2 equation selector matrix $(\underline{S}_o - \underline{S}_e)$.

$$\underline{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}, \quad (\underline{S}_o - \underline{S}_e) = \begin{bmatrix}1 & -1\end{bmatrix} \tag{2.49}$$

Ports 1 and 4 are the external power ports so $\underline{T}$ reorders the input $U = [i, \tau, \tau_u, \tau_d]^T$ and output $Y = [e, \omega, \omega_u, \omega_d]^T$ so i, τ_d, e, and ω_d appear first in the vectors. Ports 2 and 3 are connected internal power ports reordered to appear in connected pairs after the external ports. Apply the reordering matrix $\underline{T}$ to the modular model (2.48a-2.48b) to partition the inputs and outputs internally and externally.

$$\begin{bmatrix} U_{ext} \\ \hline U_{int} \end{bmatrix} = \underline{T}^T U = \begin{bmatrix} i \\ \tau_d \\ \hline \tau \\ \tau_u \end{bmatrix} \tag{2.50a}$$

$$\begin{bmatrix} Y_{ext} \\ \hline Y_{int} \end{bmatrix} = \underline{T}^T Y = \begin{bmatrix} e \\ \omega_d \\ \hline \omega \\ \omega_u \end{bmatrix} \tag{2.50b}$$

$$\mathcal{B}\underline{T}^T = [\mathcal{B}_{ext}\ \mathcal{B}_{int}] = \left[\begin{array}{cc|cc} K_t/J & 0 & -1/J & 0 \\ 0 & 1/J_{tr} & 0 & GR/J_{tr} \end{array}\right] \tag{2.50c}$$

$$\underline{T}\mathcal{C} = \begin{bmatrix} \mathcal{C}_{ext} \\ \mathcal{C}_{int} \end{bmatrix} = \left[\begin{array}{cc} K_b & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & GR \end{array}\right] \tag{2.50d}$$

$$\underline{T}\mathcal{D}\underline{T}^T = \begin{bmatrix} \mathcal{D}_{extext} & \mathcal{D}_{extint} \\ \mathcal{D}_{intext} & \mathcal{D}_{intint} \end{bmatrix} = \left[\begin{array}{cc|cc} R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{2.50e}$$

The two element electric automotive vehicle power train modular model (2.48a-2.48b) is now in the form to apply modular modeling constraints (2.2a-2.2b).

Apply the internal input and output constraints to the two element electric power train modular model (Fig. 2.4) and rewrite in the fully connected modular system form (2.26a-2.26c).

$$\dot{X} = \mathcal{A}X + \mathcal{B}_{ext}U_{ext} + \mathcal{B}_{int}(\underline{S}_o - \underline{S}_e)^T U^c$$

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -c_{tr}/J_{tr} \end{bmatrix}\begin{bmatrix} x_m \\ x_t \end{bmatrix} + \begin{bmatrix} K_t/J & 0 \\ 0 & 1/J \end{bmatrix}\begin{bmatrix} i \\ \tau_d \end{bmatrix} + \begin{bmatrix} -1/J & 0 \\ 0 & GR/J_{tr} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} u_{23}^c \tag{2.51a}$$

$$Y_{ext} = \mathcal{C}_{ext} X + \mathcal{D}_{extext} U_{ext} + \mathcal{D}_{extint} \left(\underline{S}_o - \underline{S}_e\right)^T U^c$$

$$\begin{bmatrix} e \\ \omega_d \end{bmatrix} = \begin{bmatrix} K_b & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ x_t \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau_d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_{23}{}^c \tag{2.51b}$$

$$\left(\underline{S}_o - \underline{S}_e\right) \mathcal{C}_{int} X + \left(\underline{S}_o - \underline{S}_e\right) \mathcal{D}_{intext} U_{ext} + \left(\underline{S}_o - \underline{S}_e\right) \mathcal{D}_{intint} \left(\underline{S}_o - \underline{S}_e\right)^T U^c = \underline{0}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & GR \end{bmatrix} \begin{bmatrix} x_m \\ x_t \end{bmatrix} + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau_d \end{bmatrix} + \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_{23}{}^c = 0 \tag{2.51c}$$

The internal input $u_{23}{}^c$ exists if (2.51a-2.51c) is internally output controllable or the matrix (2.21) is positive definite. Since $\mathcal{D}_{intint}$ is zero, in this example, showing $\mathcal{C}_{int}\mathcal{B}_{int}$ has rank equal to the number of internal outputs shows positive definiteness of (2.21). This is typical practice to show output controllability (Skelton, 1988).

$$\mathcal{C}_{int}\mathcal{B}_{int} = \begin{bmatrix} 1 & 0 \\ 0 & GR \end{bmatrix} \begin{bmatrix} -1/J & 0 \\ 0 & GR/J_{tr} \end{bmatrix} = \begin{bmatrix} -1/J & 0 \\ 0 & GR^2/J_{tr} \end{bmatrix} \tag{2.52}$$

Clearly $\mathcal{C}_{int}\mathcal{B}_{int}$ has a rank of 2, which is the number of internal outputs. The internal input $u_{23}{}^c$ exists and the modular two element electric vehicle model (2.51a-2.51c) can be solved like any other state system model.

The second configuration considers a power train with the modular motor modeling element, a modular compliant clutch modeling element, and modular transmission modeling element (Fig. 2.5).

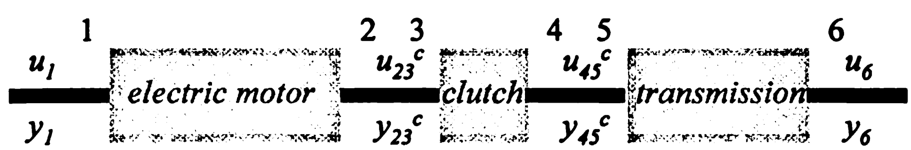

Figure 2.5: Electric Automotive Vehicle Power Train Modular Model With An Electric Motor, A Compliant Clutch, And A Transmission

The unconnected three element modular system is built up from directly inserting the motor, clutch, and transmission modular element matrices into (2.5a-2.5b).

$$\begin{bmatrix} \dot{x}_m \\ \hdashline \dot{x}_{cl_1} \\ \dot{x}_{cl_2} \\ \dot{x}_{cl_3} \\ \hdashline \dot{x}_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{K_{cl}}{J_{cl}} & 0 \\ 0 & 0 & 0 & \frac{K_{cl}}{J_{cl}} & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{C_{tr}}{J_{tr}} \end{bmatrix} \begin{bmatrix} x_m \\ \hdashline x_{cl_1} \\ x_{cl_2} \\ x_{cl_3} \\ \hdashline x_t \end{bmatrix} + \begin{bmatrix} \frac{K_t}{J} & -\frac{1}{J} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{J_{cl}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J_{cl}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{GR}{J_{tr}} & \frac{1}{J_{tr}} \end{bmatrix} \begin{bmatrix} i \\ \tau \\ \hdashline \tau_m \\ \tau_t \\ \hdashline \tau_u \\ \tau_d \end{bmatrix} \quad (2.53a)$$

$$\begin{bmatrix} e \\ \omega \\ \hdashline \omega_m \\ \omega_t \\ \hdashline \omega_u \\ \omega_d \end{bmatrix} = \begin{bmatrix} K_b & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & GR \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ \hdashline x_{cl_1} \\ x_{cl_2} \\ x_{cl_3} \\ \hdashline x_t \end{bmatrix} + \begin{bmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau \\ \hdashline \tau_m \\ \tau_t \\ \hdashline \tau_u \\ \tau_d \end{bmatrix} \quad (2.53b)$$

Apply reordering, internal input and output constraints, and rewrite in the fully connected modular system form (2.26a-2.26c).

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_{cl_1} \\ \dot{x}_{cl_2} \\ \dot{x}_{cl_3} \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{K_{cl}}{J_{cl}} & 0 \\ 0 & 0 & 0 & \frac{K_{cl}}{J_{cl}} & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{C_{tr}}{J_{tr}} \end{bmatrix} \begin{bmatrix} x_m \\ x_{cl_1} \\ x_{cl_2} \\ x_{cl_3} \\ x_t \end{bmatrix} + \begin{bmatrix} \frac{K_t}{J} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{J} \end{bmatrix} \begin{bmatrix} i \\ \tau_d \end{bmatrix} + \begin{bmatrix} -\frac{1}{J} & 0 & 0 & 0 \\ 0 & \frac{1}{J_{cl}} & 0 & 0 \\ 0 & 0 & \frac{1}{J_{cl}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{GR}{J_{tr}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_{23}^c \\ u_{45}^c \end{bmatrix} \quad (2.54a)$$

$$\begin{bmatrix} e \\ \omega_d \end{bmatrix} = \begin{bmatrix} K_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_m \\ x_{cl_1} \\ x_{cl_2} \\ x_{cl_3} \\ x_t \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau_d \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_{23}^c \\ u_{45}^c \end{bmatrix} \quad (2.54b)$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & GR \end{bmatrix} \begin{bmatrix} x_m \\ x_{cl_1} \\ x_{cl_2} \\ x_{cl_3} \\ x_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau_d \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_{23}^c \\ u_{45}^c \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.54c)$$

The internal input $[u_{23}^c, u_{45}^c]^T$ exists if (2.54a-2.54c) is internally output controllable or the matrix (2.21) is positive definite. Since $\mathcal{D}_{intint}$ is zero, in this example, showing $\mathcal{C}_{int}\mathcal{B}_{int}$ has rank equal to the number of internal outputs shows positive definiteness of (2.21).

$$\mathcal{C}_{int}\mathcal{B}_{int} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & GR \end{bmatrix} \begin{bmatrix} -\frac{1}{J} & 0 & 0 & 0 \\ 0 & \frac{1}{J_{cl}} & 0 & 0 \\ 0 & 0 & \frac{1}{J_{cl}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{GR}{J_{tr}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{J} & 0 & 0 & 0 \\ 0 & \frac{1}{J_{cl}} & 0 & 0 \\ 0 & 0 & \frac{1}{J_{cl}} & 0 \\ 0 & 0 & 0 & \frac{GR^2}{J_{tr}} \end{bmatrix}$$

(2.55)

Clearly $\mathcal{C}_{int}\mathcal{B}_{int}$ has a rank of 4, which is the number of internal outputs. The internal input $\left[u_{23}{}^{c}, u_{45}{}^{c}\right]^T$ exists and the modular 3 element electric vehicle model (2.54a-2.54c) can be solved like any other state system model.

The linear algebraic modular model example equations are defined by taking the Laplace transform of the linear algebraic differential model equations. The two element electric automotive vehicle power train modular model equations (2.48a-2.48b) after a Laplace transformation have the form of the unconnected linear algebraic modular model (2.28).

$$Y = \mathcal{D}U = \begin{bmatrix} e \\ \omega \\ \hdashline \omega_u \\ \omega_d \end{bmatrix} = \left[\begin{array}{cc:cc} \frac{K_t K_b/J + R}{s} & \frac{K_b/J}{s} & 0 & 0 \\ \frac{K_t/J}{s} & \frac{-1/J}{s} & 0 & 0 \\ \hdashline 0 & 0 & \frac{GR^2/J_{tr}}{s + c_{tr}/J_{tr}} & \frac{GR/J_{tr}}{s + c_{tr}/J_{tr}} \\ 0 & 0 & \frac{GR^2/J_{tr}}{s + c_{tr}/J_{tr}} & \frac{1/J_{tr}}{s + c_{tr}/J_{tr}} \end{array}\right] \begin{bmatrix} i \\ \tau \\ \tau_u \\ \tau_d \end{bmatrix} \tag{2.56}$$

The two element linear algebraic electric vehicle power train modular model (2.56) has the same input-output topology (Fig. 4) as the linear algebraic differential system (2.48a-2.48b). So, (2.56) can be reordered and constrained with the $\underline{T}$ and $(\underline{S}_o - \underline{S}_e)$ in (2.49) to obtain the fully connected linear algebraic modular model form (2.32a-2.32b).

$$(\underline{S}_o - \underline{S}_e)\mathcal{D}_{intext}U_{ext} + (\underline{S}_o - \underline{S}_e)\mathcal{D}_{intint}(\underline{S}_o - \underline{S}_e)^T U^c = \underline{0}$$

$$[1 \;\; -1]\begin{bmatrix} \frac{K_t/J}{s} & 0 \\ 0 & \frac{GR/J_{tr}}{s + c_{tr}/J_{tr}} \end{bmatrix}\begin{bmatrix} i \\ \tau_d \end{bmatrix} + [1 \;\; -1]\begin{bmatrix} \frac{-1/J}{s} & 0 \\ 0 & \frac{GR^2/J_{tr}}{s + c_{tr}/J_{tr}} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} u_{23}{}^{c} = 0 \tag{2.57a}$$

$$Y_{ext} = \mathcal{D}_{extext} U_{ext} + \mathcal{D}_{extint} (\underline{S}_o - \underline{S}_e)^T U^c$$

$$\begin{bmatrix} e \\ \omega_d \end{bmatrix} = \begin{bmatrix} \dfrac{K_t K_b / J + R}{s} & 0 \\ 0 & \dfrac{1/J_{tr}}{s + c_{tr}/J_{tr}} \end{bmatrix} \begin{bmatrix} i \\ \tau_{ext} \end{bmatrix} + \begin{bmatrix} \dfrac{K_b/J}{s} & 0 \\ 0 & \dfrac{GR/J_{tr}}{s + c_{tr}/J_{tr}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_{23}{}^c \quad (2.57b)$$

The internal input $u_{23}{}^c$ exists if (2.57a-2.57b) is internally output controllable or the matrix $(\underline{S}_o - \underline{S}_e) \mathcal{D}_{intint} \mathcal{D}_{intint}{}^* (\underline{S}_o - \underline{S}_e)^T$ is positive definite.

$$(\underline{S}_o - \underline{S}_e) \mathcal{D}_{intint} \mathcal{D}_{intint}{}^* (\underline{S}_o - \underline{S}_e)^T = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{-1/J}{s} & 0 \\ 0 & \dfrac{GR^2/J_{tr}}{s + c_{tr}/J_{tr}} \end{bmatrix} \begin{bmatrix} \dfrac{-1/J}{s} & 0 \\ 0 & \dfrac{GR^2/J_{tr}}{s + c_{tr}/J_{tr}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1/J^2}{s^2} + \frac{GR^4/J_{tr}{}^2}{(s + c_{tr}/J_{tr})^2} > 0$$

(2.58)

The internal input $u_{23}{}^c$ exists and the modular linear algebraic two element electric vehicle power train model (2.57a-2.57b) can be solved like any other linear algebraic system model. A similar analysis can be done for the modular three element electric vehicle power train model.

The linear algebraic BVP modular model example is a structural BVP model with three bars and connecting shear pin fixed to a point. This structural modular BVP model example contains three simple modular modeling elements, an axial force bar model, a bending pin model, and a fixed point model. The three modular modeling elements are all defined with standardized modular modeling causality (Table 1.1). Structural BVP models do not have traditional power ports because the output variable is a displacement instead of velocity. They are representations of physical points with two variables that require a causal definition, which fits the application of modular modeling and the measurement perspective standardized causality (Byam and Radcliffe, 1999).

The axial force bar modular modeling element is a two node Finite Element Model (FEA) from Segerlind, 1984. This model has two mechanical ports with force input F axial displacement output δ causality (Fig. 2.6a), which is written in linear algebraic BVP modular modeling form (2.35a-2.35b).

$$\underline{K}\,\underset{\sim}{x} = \underline{C}^T \underset{\sim}{u}$$

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{2.59a}$$

$$\underset{\sim}{y} = \underline{C}\,\underset{\sim}{x}$$

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} \tag{2.59b}$$

The modular bar BVP (2.59a-2.59b) has two states $[x_a, x_b]^T$, a cross-sectional area A, a modulus of elasticity E, and a length L.

The bending pin modular modeling element is a three node FEA from Segerlind, 1984. This model has three mechanical ports with force input F transverse displacement output δ causality and three mechanical rotational ports with torque input τ angular displacement output θ causality (Fig. 2.6b), which is written in linear algebraic BVP modular modeling form (2.35a-2.35b).

$$\underline{K}\,\underset{\sim}{x} = \underline{C}^T \underset{\sim}{u}$$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} x_c \\ x_d \\ x_e \\ x_f \\ x_g \\ x_h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} F_3 \\ \tau_3 \\ F_4 \\ \tau_4 \\ F_5 \\ \tau_5 \end{bmatrix}$$

(2.60a)

$$\underset{\sim}{y} = \underline{C}\,\underset{\sim}{x}$$

$$\begin{bmatrix} \delta_3 \\ \theta_3 \\ \delta_4 \\ \theta_4 \\ \delta_5 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_c \\ x_d \\ x_e \\ x_f \\ x_g \\ x_h \end{bmatrix} \tag{2.60b}$$

The modular pin BVP (2.60a-2.60b) has 6 states $\left[x_c, x_d, x_e, x_f, x_g, x_h\right]^T$, a modulus of elasticity E, a length L, and a 2nd moment of area I.

The fixed point modular modeling element is unique to modular modeling because of the standardized measurement perspective causality. The one port fixed point modular modeling element outputs a zero displacement regardless of the input and applies to any energy domain (Fig. 2.6c). The fixed point modular modeling element is written in linear algebraic BVP modular modeling form (2.35a-2.35b).

$$\underline{K}\,\underset{\sim}{x} = \underline{C}^T \underset{\sim}{u} \qquad [1]x_i = [0]F_6 \tag{2.61a}$$

$$\underset{\sim}{y} = \underline{C}^T \underset{\sim}{x} \qquad \delta_6 = [0]x_i \tag{2.61b}$$

The modular fixed point modular modeling element has one state x_i

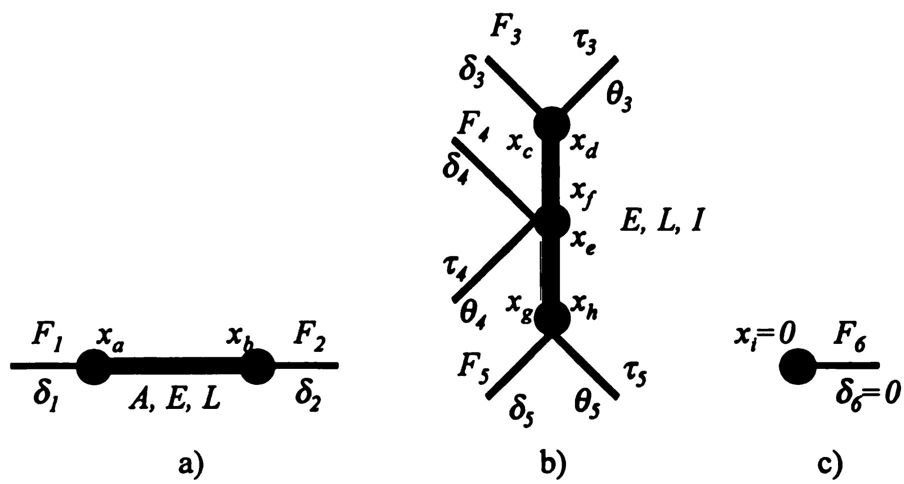

Figure 2.6: Three Structural Linear Algebraic BVP Modular Modeling Elements: a) Axial Force Bar b) Bending Pin c) Fixed Point

Two example linear algebraic BVP modular models will use the three modular modeling elements (Fig. 2.6) in two configurations. First, three axial force bars and a fixed point will be connected together in a bracket-pin-clevis configuration with no pin bending deflection (Fig. 2.7). Second, the three axial force bars will be connected with the bending pin (Fig. 2.8).

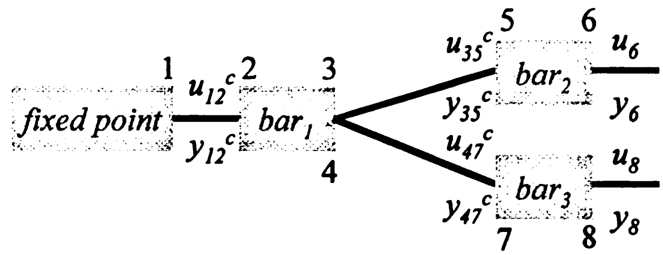

Figure 2.7: Three Axial Force Bar And A Fixed Point Modular Modeling Elements In A Bracket-Pin-Clevis Configuration With No Pin Deflection

The input-output topology of the three bar no pin modular model shows ports 6 and 8 are external ports and three connectors. The internal ports are connected in pairs 1 to 2, 3 to 5, and 4 to 7. This simple topology analysis defines the reordering matrix $\underline{T}$ and the equation selector matrix $(\underline{S}_o - \underline{S}_e)$.

$$\underline{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (\underline{S}_o - \underline{S}_e) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad (2.62)$$

The first two rows of $\underline{T}$ identify the external ports and the remaining six rows order the internal ports in connected pairs.

Build up the unconnected modular model from the three axial force bar and fixed point modular modeling elements and write in linear algebraic BVP modular form (2.35a-2.35b).

$$\mathcal{K}X = \mathcal{C}^T U$$

$$\begin{bmatrix} \underline{K}_{fix\ pt} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K}_{bar_1} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K}_{bar_2} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{K}_{bar_3} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_{fix\ pt} \\ \underset{\sim}{x}_{bar_1} \\ \underset{\sim}{x}_{bar_2} \\ \underset{\sim}{x}_{bar_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} \quad (2.63a)$$

$$Y = \mathcal{C}X$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_{fix\ pt} \\ \underset{\sim}{x}_{bar_1} \\ \underset{\sim}{x}_{bar_2} \\ \underset{\sim}{x}_{bar_3} \end{bmatrix} \quad (2.63b)$$

The $\underline{K}$ matrices and $\underset{\sim}{x}$ vectors in (2.63a-2.63b) are given in the modular modeling element equations (2.59) and (2.61). Note that the $\underline{C}$ matrix for bar_1 has an extra row to accommodate the input output ports 3 and 4 on the right hand side of the bar. These two ports occur at the same geometric point so they have the same output and the inputs sum in the modular elements internal junction structure (Byam and Radcliffe, 1999). Apply reordering and input output constraints and rewrite (2.63a-2.63b) in constrained BVP modular form (2.36a-2.36c)

$$\mathcal{K}X = \mathcal{C}_{ext}^T U_{ext} + \mathcal{C}_{int}^T (\underline{S}_o - \underline{S}_e)^T U^c$$

$$\begin{bmatrix} \underline{K}_{fix\ pt} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K}_{bar_1} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K}_{bar_2} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{K}_{bar_3} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_{fix\ pt} \\ \underset{\sim}{x}_{bar_1} \\ \underset{\sim}{x}_{bar_2} \\ \underset{\sim}{x}_{bar_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_6 \\ u_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{12}^c \\ u_{35}^c \\ u_{47}^c \end{bmatrix} \quad (2.64a)$$

$$Y_{ext} = \mathcal{C}_{ext} X$$

$$\begin{bmatrix} y_6 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{fix\ pt} \\ \underset{\sim}{x}_{bar_1} \\ \underset{\sim}{x}_{bar_2} \\ \underset{\sim}{x}_{bar_3} \end{bmatrix} \tag{2.64b}$$

$$(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int} X = \underline{0}_{m/2\times 1}$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{fix\ pt} \\ \underset{\sim}{x}_{bar_1} \\ \underset{\sim}{x}_{bar_2} \\ \underset{\sim}{x}_{bar_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.64c}$$

Find the basis for the nullspace of $(\underline{S}_o - \underline{S}_e)\mathcal{C}_{int}$.

$$\mathcal{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.65}$$

At this point the linear algebraic BVP modular system (2.63a-2.63b) can be solved using (2.37), (2.38), and (2.39). Linear algebraic BVP modular modeling elements were directly inserted into the modular model with given input output topology. The input output topology determines the reordering and equation selector matrices. The modular system model is rewritten in constrained form and the basis of the nullspace of the output constraint is found with a computer software package. The model is solved like any other linear algebraic BVP model.

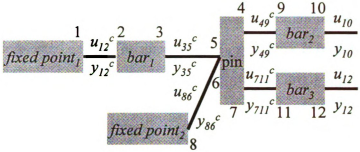

Figure 2.8: Three Axial Force Bar, Two Fixed Point, And A Bending Pin Modular Modeling Elements In A Bracket-Pin-Clevis Configuration

The three bar bending pin modular BVP model (Fig. 2.8) is solved similarly to the previous model. There are two fixed point modular modeling elements in the modular model because the bending pin modular modeling element requires a rotational constraint to eliminate the rotational rigid body mode. The modular modeling element *fixed point*$_2$ eliminates the rotation.

The matrices in this example model are quite large and will be present separately. The input-output topology of the three bar bending pin modular model shows ports 10 and 12 are external ports and five connectors. The internal ports are connected in pairs 1 to 2, 3 to 5, 4 to 9, 8 to 6, and 7 to 11. This simple topology analysis defines the reordering matrix $\underline{T}$ and the equation selector matrix $(\underline{S}_o - \underline{S}_e)$.

$$\underline{T} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ \hline 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ \hline 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ \hline 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ \hline 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ \hline 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}, (\underline{S}_o - \underline{S}_e) = \begin{bmatrix} 1&-1&0&0&0&0&0&0&0&0 \\ 0&0&1&-1&0&0&0&0&0&0 \\ 0&0&0&0&1&-1&0&0&0&0 \\ 0&0&0&0&0&0&1&-1&0&0 \\ 0&0&0&0&0&0&0&0&1&-1 \end{bmatrix} \quad (2.66)$$

The first two rows of $\underline{T}$ identify the external ports and the remaining ten rows order the internal ports in connected pairs. The unconnected $\mathcal{K}$ matrix is built up from the modular modeling elements $\underline{K}$ matrices in (2.59a-2.59b), (2.60a-2.60b), and (2.61a-2.61b).

$$\mathcal{K} = \left[\begin{array}{c|c|c|c|c|c}
\underline{K}_{fix\ pt_1} & \underline{0} & & \cdots & & \underline{0} \\
\hline
\underline{0} & \underline{K}_{bar_1} & & & & \\
\hline
& & \underline{K}_{fix\ pt_1} & \ddots & & \vdots \\
\hline
\vdots & & \ddots & \underline{K}_{pin} & & \\
\hline
& & & & \underline{K}_{bar_2} & \underline{0} \\
\hline
\underline{0} & & \cdots & & \underline{0} & \underline{K}_{bar_3}
\end{array}\right] \tag{2.67}$$

The $\mathcal{C}$ matrix and the $\mathcal{V}$ matrix complete the model.

$$\mathcal{C} = \left[\begin{array}{c|cc|c|cccccc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \tag{2.68}$$

$$
\mathcal{V} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{2.69}
$$

Note that in the $\mathcal{C}$ matrix the $\underline{C}$ matrix for the bending pin is missing the 2nd and 6th rows because those inputs are zero. Using (2.37), (2.38), and (2.39) this example model can be solved like any other linear algebraic BVP model.

Chapter 3

ASSEMBLY AND SOLUTION OF NONLINEAR MODULAR MODELS USING FIXED INPUT-OUTPUT STRUCTURE

3.1 Nonlinear Modular Modeling Analysis

Nonlinear modular modeling analysis is a systematic direct-insertion realization of nonlinear modular models of the form (2.3). Using a known input-output topology constrained by the output and power constraints of the connector (2.2) the constrained modular model is realized. Nonlinear modular modeling elements are independently user-formulated power-based multi-port multi-degree of freedom nonlinear modeling equations with a standardized input-output causality and sign convention. Connectors are two-port output and conservative power constraints between modular modeling element ports. Modular connector constraints are known. Modular elements have a known form but their equations are user-defined.

The possible user-defined nonlinear modular element equations are nonlinear algebraic differential equations and algebraic equations. The nonlinear algebraic differential equations can be represented in a form convenient for the application of the modular modeling fixed input-output functional form. A state-space form is the best fit because inputs and outputs are explicit. Any algebraic differential equation model can be written in state-space form. Algebraic equation models are typically expressed in an explicit input-output form.

3.1.1 Nonlinear Algebraic Differential Equations

Modular modeling elements with user-defined nonlinear algebraic differential equations have a traditional state-space form where the inputs and outputs are explicit.

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, \underset{\sim}{u})$$
$$\underset{\sim}{y} = g(\underset{\sim}{x}, \underset{\sim}{u}) \tag{3.1}$$

The fixed input-output functional form (2.1) of modular elements is seen in (3.1), where $\underset{\sim}{u}$ is a vector of port inputs and $\underset{\sim}{y}$ is a vector of port outputs. $\underset{\sim}{x}$ is a vector of states, $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are vectors of functions of the x-variables and u-variables.

Consider a modular model in unconnected form as a concatenation of n independently formulated user-defined nonlinear modular elements in state space form. Let this model have s total states and p total input-output power ports.

$$\begin{bmatrix} \dot{\underset{\sim}{x}}_1 \\ \dot{\underset{\sim}{x}}_2 \\ \vdots \\ \dot{\underset{\sim}{x}}_n \end{bmatrix}_{s\times 1} = \begin{bmatrix} f_1(\underset{\sim}{x}_1, \underset{\sim}{u}_1) \\ f_2(\underset{\sim}{x}_2, \underset{\sim}{u}_2) \\ \vdots \\ f_n(\underset{\sim}{x}_n, \underset{\sim}{u}_n) \end{bmatrix}_{s\times 1} \tag{3.2a}$$
$$\dot{X} = \mathbf{f}(X, U)$$

$$\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_n \end{bmatrix}_{p\times 1} = \begin{bmatrix} g_1(\underset{\sim}{x}_1, \underset{\sim}{u}_1) \\ g_2(\underset{\sim}{x}_2, \underset{\sim}{u}_2) \\ \vdots \\ g_n(\underset{\sim}{x}_n, \underset{\sim}{u}_n) \end{bmatrix}_{p\times 1} \tag{3.2b}$$
$$Y = \mathbf{g}(X, U)$$

For example, the modular model vector X is the concatenation of the n modular element state vectors with a total size $s \times 1$ and U is the concatenation of the n modular element input vectors with a total size $p \times 1$. The fixed input-output functional form (2.3) of a modular model is seen in (3.2). The modular model (3.2) has an equal number of inputs and outputs because all modular elements power ports always has one input and one output. Each modular element of (3.2) is completely independent and uncoupled from the other modular elements. Given the element's input-output topology the modular connector constraints (2.2a-2.2b) provide the coupling between the modular element equations.

The key concept of modular model analysis is isolating the internal element input-output power ports from the external element input-output power ports in a known input-output topology. External input-output power ports have known inputs. Internal input-output power ports are ports joined to other element ports through connectors (Fig. 2.2). The input-output topology of (3.2) has a total of *p* input-output power ports from the *n* modular elements. Let *m* be the number of connectors, hence there are $2m$ internal element input-output power ports, which leaves $q = p - 2m$ external element input-output power ports. The standardized form of modular elements (1) and modular connectors (2.2) makes isolating external and internal element ports in the unconnected modular model (3.2) a simple reordering of the systems' concatenated input and output vectors U and Y.

A transformation matrix reorders the vectors of (3.2). The vectors are reordered so all external ports' variables appear first in the vectors followed by all the internal ports' variables. The internal ports' variables are ordered such that connected port pairs' appear together. For example, if port *i* and port *j* are connected u_i should be followed immediately by u_j, similarly for the outputs y_i and y_j. The reordered input and output vectors are the input and output vectors of (3.2) pre-multiplied by the transformation matrix $\underline{T}$.

$$\underline{T}U = \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix} \tag{3.3a}$$

$$\underline{T}Y = \begin{bmatrix} Y_{ext} \\ Y_{int} \end{bmatrix} \tag{3.3b}$$

The transformation matrix $\underline{T}$ does not add, remove, or combine variables of the original vectors; it only changes the order in which variables appear. This makes $\underline{T}$ a linear and nonsingular reordered $p \times p$ identity matrix.

$$\underline{T}\underline{T}^T = \underline{I} \tag{3.4}$$

The external input U_{ext} is the $q \times 1$ vector of external port inputs. The internal input U_{int} is the $2m \times 1$ vector of internal port inputs. The external output Y_{ext} is the $q \times 1$ vector of external port outputs. The internal output Y_{int} is the $2m \times 1$ vector of internal port outputs. The mechanism for reorganizing the input-output topology of (3.2) to isolate the external and internal ports is the matrix $(\underline{T})_{p \times p}$.

The reordered modular model equations are written in terms of the external and internal inputs and outputs where $U = \underline{T}^T [U_{ext}, U_{int}]^T$.

$$\dot{X} = \mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.5a}$$

$$\underline{T}Y = \begin{bmatrix} Y_{ext} \\ Y_{int} \end{bmatrix} = \underline{T}\mathbf{g}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.5b}$$

Using the transformation matrix $\underline{T}$ the function vector $\mathbf{g}$ in (3.5) is partitioned to further isolate internal and external equations. The function vector $\mathbf{g}$ is partitioned by $\underline{T}$ into external and internal outputs.

$$\underline{T}\mathbf{g} = \begin{bmatrix} \mathbf{g}_{ext} \\ \mathbf{g}_{int} \end{bmatrix} \tag{3.6}$$

The external partition of the $\mathbf{g}$ vector $\mathbf{g}_{ext}$ is a $q \times 1$ vector. The internal partition of the $\mathbf{g}$ vector $\mathbf{g}_{int}$ is a $2m \times 1$ vector. The reordered modular model equations are rewritten in terms of the external and internal inputs and outputs and the partitioned output function vector.

$$\dot{X} = \mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.7a}$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.7b}$$

$$Y_{int} = \mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.7c}$$

The internal outputs Y_{int} of (3.7c) are ordered in connected internal port pairs. At each connected pair, the two output values are constrained by (2.2a) to have equal values. In order to apply the outputs of (3.7c) to the constraint (2.2a) the connected output pairs need to be selected from the matrix equation (3.7c). Define two $m \times 2m$ equation selector matrices $\underline{S}_o$ and $\underline{S}_e$ to select the odd and even equations of (3.7c) respectively.

$$\underline{S}_o = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & \vdots & 1 & \vdots & & \vdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix}_{m\times 2m} \qquad \underline{S}_e = \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ \vdots & 0 & \vdots & 1 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 \end{bmatrix}_{m\times 2m} \tag{3.8}$$

Pre-multiplying (3.7c) by $\underline{S}_o$ selects the odd (1^{st}, 3^{rd}, 5^{th}, … , $m-1^{st}$) internal port output equations. Pre-multiplying (3.7c) by $\underline{S}_e$ selects the even (2^{nd}, 4^{th}, 6^{th}, … , m^{th}) internal port output equations. Substitute selected equations in the output constraint (2.2a).

$$\underline{S}_o Y_{int} = Y^c \tag{3.9a}$$

$$\underline{S}_e Y_{int} = Y^c \tag{3.9b}$$

The constrained internal ports' output Y^c is a $m \times 1$ vector of the m constrained inputs at the $2m$ connected internal ports (Fig. 2.2) of the system (3.2a-3.2b). Rewrite the output constraint (3.9a-3.9b) in terms of a difference to eliminate the constrained internal port's output.

$$(\underline{S}_o - \underline{S}_e) Y_{int} = \underline{0}_{m\times 1} \tag{3.10}$$

Substituting the internal output equation (3.7c) into the internal output constraint (3.10) results in the output constraints of the modular model.

$$(\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) = \underline{0} \tag{3.11}$$

The three output constrained modular system equations are written in terms of the states, X, the external inputs, U_{ext}, the external outputs, Y_{ext}, and the internal inputs, U_{int}.

$$\dot{X} = \mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.12a}$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) \tag{3.12b}$$

$$(\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ U_{int} \end{bmatrix}) = \underline{0} \tag{3.12c}$$

The internal port input pairs, U_{int}, are ordered the same way as the internal port output pairs, so the $m \times 2m$ equations selector matrices $\underline{S}_o$ and $\underline{S}_e$ can be used to define the $m \times 1$ constrained input U^c. The constrained input U^c is the result of the modular connector power constraint (2.2b) constraining internal port input pairs to be equal and opposite. The constrained internal port's input U^c is a $m \times 1$ vector of the m constrained inputs at the $2m$ connected internal ports (Fig. 2.2) of the system (3.2a-3.2b).

$$U_{int} = (\underline{S}_o - \underline{S}_e)^T U^c \tag{3.13}$$

Substitute the internal input (3.13) into (3.12a-3.12c).

$$\dot{X} = \mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) \tag{3.14a}$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) \tag{3.14b}$$

$$(\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) = \underline{0} \tag{3.14c}$$

The three output constrained modular model equations (3.14a-3.14c) have three unknowns, the states, X, the external outputs, Y_{ext}, and the constrained internal inputs, U^c. The objective is to find the constrained internal input, U^c, in terms of the states, X, and external inputs, U_{ext}, that satisfies (3.14c).

The nonlinear modular system constraint equation (3.14c) is implicit in terms of the constrained internal input, U^c. Rewrite (3.14c) in terms of an error subtracting the right-hand side from the left-hand side.

$$\mathbf{e} = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) \tag{3.15}$$

The internal output constraint error, $\mathbf{e}$, is a $m \times 1$ vector of internal output constraint errors. These errors are a measure of the amount the constraint is violated at each modular connector. The internal output constraint error, $\mathbf{e}$, is driven to zero by the constrained internal inputs, U^c. There are many approaches for solving (3.15). One option is using an iterative approach like Newton-Raphson (Burden and Faires, 1985). Another option is a physical approach defining compliant functions to hold the error at zero. Modular modeling chooses a control logic approach.

A control logic approach to driving an error to a zero set point is a control stabilization problem (Khalil, 1996). Define a new $m \times 1$ constraint state vector, X^c, as the integral of the internal output constraint error, $\bullet$. Let the constrained internal inputs, U^c, be the results of multiplying constraint state vector X^c by a $m \times m$ matrix of constant gains, $\underline{K}_I$, (Fig. 3.1).

$$\dot{X}^c = \bullet \tag{3.16a}$$

$$U^c = \underline{K}_I X^c \tag{3.16b}$$

The internal constraint stabilizer (3.16a-3.16b) defines the constrained internal inputs that zero the internal output constraint error, $\bullet$. Rewrite the internal constraint stabilizer by substituting the internal constraint stabilizer output (3.16b) into the internal output constraint error (3.15) then substitute into internal constraint stabilizer state (3.16a) eliminating the unknown, $\bullet$.

$$\dot{X}^c = (\underline{S}_o - \underline{S}_e) g_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.17}$$

The internal constraint stabilizer (3.17) may be solved for the constrained states, X^c, in terms of the states, X, and the external inputs, U_{ext}. The constrained internal inputs, U^c, that satisfy the nonlinear modular system constraint equation (3.14c) are found from (3.16b).

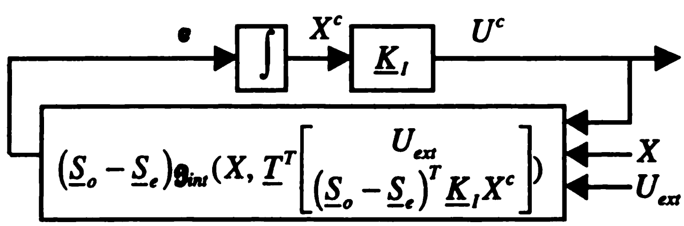

Figure 3.1: Nonlinear Modular Modeling Internal Constraint Stabilizer Block Diagram

The internal output constraint error depends on the states, X, which depends on initial conditions. In this environment where the error is dependent on joining outputs of models presetting the initial conditions would not be unusual because knowledge of the modular elements initial conditions exists. Presetting the initial conditions requires the knowledge of all the states in of all the modular elements in the modular system model (3.2a-3.2b).

The internal constraint stabilizer (3.16a-3.16b) is defined as dynamic function with integral control logic. Using the integral control logic does three things. First, integral control provides a robust stabilization if there are no perturbations in the system's parameters (Khalil, 1996). Since the internal constraint stabilizer is regulating the outputs of models, there are no parameter perturbations in system parameters unless it is part of the model. Second, it avoids having to define a physical compliant function with "fast" eigenvalues to quickly reduce the error resulting in stiff systems (Aiken, 1985). Third, no iteration is required so the constraint equations are combined and solved with the system equations. Combining the internal constraint stabilizer (3.17) with the constrained modular system equations (3.14a-3.14b) results in the fully connected modular system equations, which can be solved for the unknown states, X, the unknown constrained states, X^c, and the unknown external outputs, Y_{ext}, in terms of the known external inputs, U_{ext}.

$$\dot{X} = \mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.18a}$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.18b}$$

$$\dot{X}^c = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.18c}$$

The stability analysis of the feedback system in the nonlinear algebraic differential modular system equations (3.18a-3.18c) is done with the passivity approach. A passive system does not generate its own energy. The system absorbed energy must be greater than or equal to the system stored energy. If the absorbed energy is greater than the stored energy the difference must be the dissipated energy by the conservation of energy and the system is strictly passive. A passive system is globally uniformly stable and a strictly passive system is globally uniformly asymptotically stable (Khalil, 1996 & Krstic etal, 1995). A feedback system is strictly passive if both systems are strictly passive and passive if at least one of the systems is passive (Khalil, 1996 & Krstic etal, 1995).

Stability proofs for feedback systems from Khalil and Krstic etal can be easily applied to the nonlinear algebraic differential modular system (3.18a-3.18b) nontraditional feedback connection with some simple relationships. These proofs use the product of input and output and an instantaneous power inequality to show passivity (Khalil, 1996 & Krstic etal, 1995). The nonlinear algebraic differential modular system has a product of input and output with two inputs, the external inputs, U_{ext}, and constrained internal inputs, U^c, and two outputs, the external outputs, Y_{ext}, and internal output constraint error, $\bullet$.

$$Y^T U = \left(T^T \begin{bmatrix} Y_{ext} \\ \bullet \end{bmatrix} \right)^T T^T \begin{bmatrix} U_{ext} \\ U^c \end{bmatrix} = \begin{bmatrix} Y_{ext}{}^T & \bullet^T \end{bmatrix} T T^T \begin{bmatrix} U_{ext} \\ U^c \end{bmatrix} = Y_{ext}{}^T U_{ext} + \bullet^T U^c \qquad (3.19)$$

The integrator feedback system's input is the constraint error and outputs the constrained internal input to the nonlinear algebraic differential modular system with a sign change due to the standardized sign convention of modular modeling (Byam & Radcliffe, 1999)

$$Y^T U = -U^{c^T} \bullet \qquad (3.20)$$

The input and output products for the algebraic differential modular system (3.19) and the integrator feedback system (3.20) are in a form applicable to the passivity stability proofs of Khalil or Krstic etal. If the algebraic differential modular system is passive and the integrator feedback system is passive the feedback system (3.18a-3.18b) is globally uniformly stable.

The assembly and solution of nonlinear modular models using the control logic approach of modular modeling is less restrictive and avoids the stiff system problem of previous methods. Hogan found that assembling nonlinear modular component models with simple "nonenergic" connections required the output to be totally state dependent, which is restrictive. Modular modeling clearly is less restrictive because the outputs can be dependent on the inputs. Further Hogan stated that "energic" connections must be used to maintain modularity resulting in stiff systems. Modular modeling uses simple "nonenergic" connections and a control logic approach to avoid stiff systems.

3.1.2 Nonlinear Algebraic Equations

Modular modeling elements with user-defined nonlinear algebraic equations have the same form as (3.1) but $\dot{\underset{\sim}{x}} = \underline{0}$.

$$\begin{aligned} \underset{\sim}{f}(\underset{\sim}{x}, \underset{\sim}{u}) &= \underline{0} \\ \underset{\sim}{y} &= \underset{\sim}{g}(\underset{\sim}{x}, \underset{\sim}{u}) \end{aligned} \tag{3.21}$$

Following the same procedure as in (3.2a-3.2b), consider a modular model in unconnected form as a concatenation of n independently formulated user-defined modular modeling elements in the algebraic form (3.21) with p total input-output power ports.

$$\begin{bmatrix} \underset{\sim}{f_1}(\underset{\sim}{x_1}, \underset{\sim}{u_1}) \\ \underset{\sim}{f_2}(\underset{\sim}{x_2}, \underset{\sim}{u_2}) \\ \vdots \\ \underset{\sim}{f_n}(\underset{\sim}{x_n}, \underset{\sim}{u_n}) \end{bmatrix}_{s\times 1} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix}_{s\times 1} \tag{3.22a}$$

$$\mathbf{f}(X, U) = \underline{0}$$

$$\begin{bmatrix} \underset{\sim}{y_1} \\ \underset{\sim}{y_2} \\ \vdots \\ \underset{\sim}{y_n} \end{bmatrix}_{p\times 1} = \begin{bmatrix} \underset{\sim}{g_1}(\underset{\sim}{x_1}, \underset{\sim}{u_1}) \\ \underset{\sim}{g_2}(\underset{\sim}{x_2}, \underset{\sim}{u_2}) \\ \vdots \\ \underset{\sim}{g_n}(\underset{\sim}{x_n}, \underset{\sim}{u_n}) \end{bmatrix}_{p\times 1} \tag{3.22b}$$

$$Y = \mathbf{g}(X, U)$$

The same input-output topology, reordering, and input-output constraint analysis applies to the algebraic output constrained modular system equations. These equations are written in terms of the states, X, the external inputs, U_{ext}, the external outputs, Y_{ext}, and the constrained internal inputs, U^c.

$$\mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) = \underline{0} \tag{3.23a}$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) \tag{3.23b}$$

$$(\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T U^c \end{bmatrix}) = \underline{0} \tag{3.23c}$$

Solve (3.23c) for constrained internal inputs, U^c, in terms of external inputs U_{ext} using the same techniques as the nonlinear algebraic differential case (3.17 & Fig. 3.1).

$$\dot{X}^c = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(\underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.24}$$

The internal constraint stabilizer is defined in the same way as in the nonlinear algebraic differential equation case. The fully connected nonlinear algebraic modular system equations can be solved for the unknown constrained states, X^c, the unknown states, X, and the unknown external outputs, Y_{ext}, in terms of the known external inputs, U_{ext}.

$$\mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) = \underline{0} \tag{3.25a}$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.25b}$$

$$\dot{X}^c = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \tag{3.25c}$$

The stability analysis of the feedback system of the nonlinear algebraic modular system equations (3.25a-3.25c) is similar to the stability analysis done in the nonlinear algebraic differential case (3.18a-3.18c, 3.19, 3.20). If the algebraic modular system is passive and the integrator feedback system is passive the feedback system (3.25a-3.25c) is globally uniformly stable.

The nonlinear algebraic differential and algebraic constrained modular system equations (3.18a-3.18c) and (3.25a-3.25c) respectively are written in terms of independently formulated nonlinear modular modeling elements. The reordering and constraint matrices, $\underline{T}$ and $(\underline{S}_o - \underline{S}_e)$, are comprised of ones and zeros and defined by the size and input-output topology of the modular system model. The modular modeling element functions, $\mathbf{f}$ and $\mathbf{g}$, can be directly inserted into the constrained modular system equations and solved like any other nonlinear system of equations. The advantage of modular modeling is that each multi DOF modeling element is independent and can be connected together with simple connectors and assembled with a simple physically intuitive top-down direct insertion method.

3.2 Some Modular Computational Advantages

Nonlinear modular models have a guaranteed modular structure because of their standardized input-output formulation. Modular models never condense the model topology relative to the input-output structure. This uncondensed guaranteed modular structure has an advantage of ease of computation at a cost of additional computations. The additional computations come from the constraint equations not found in an arbitrary input-output structure or non-modular formulated models.

The ease of computation advantage arises from the modular construction. Modular construction enables evaluations to be performed for each uncondensed element independently. Independent evaluation is set up to avoid complex multi-step solutions and separate convergence problems. Modular modeling does not completely eliminate multi-step solutions and convergence problems but it does ease the computation by maintaining model element modularity.

Another computational advantage of modular modeling is parallel computation. Parallel computation is a perfect fit for modular modeling because the elements can be evaluated independently. Modular element evaluation can be run on separate networked processors. Evaluation of modular models on networked processors will be a subject of a future paper. Each multi-port multi-DOF element in a modular model can be distributed to separate processors. The constraint equations (3.18c) or (3.23b) must reside on a single processor.

The constraint equations add computations to modular models and another computational advantage. These extra dynamic equations have stability defined by the constraint control gains, $\underline{K}_I$. Manipulating the constraint control gains in the constraint equations, modularity allows for convergence control and adaptation. Convergence control and adaptation will be a subject of a future paper.

3.3 Examples

An electric automotive vehicle power train example model is used to illustrate modular modeling of nonlinear algebraic differential modular models. A structural three bar configuration fixed to a point example model is used to illustrate modular modeling of nonlinear algebraic modular models. The electric automotive vehicle power train example model is a combination of three simple dynamic models found in Phillips and Harbor, 1996 and Minor, 1996. The three bar structural example model is a combination of a simple nonlinear model from Ross, 1990.

The electric automotive vehicle power train example model contains three simple modular modeling elements, an electric motor model, a fluid clutch model, and a transmission model. The three modular modeling elements are all defined with standardized modular modeling causality (Table 1.1). The electric motor modular modeling element has one electrical power port with current input i voltage output e causality and one rotational mechanical power port with torque input τ angular velocity output ω causality (Fig. 3.2a).

$$\dot{x}_m = [0]x_m + \begin{bmatrix} \frac{K_t}{J} & \frac{1}{J} \end{bmatrix} \begin{bmatrix} i \\ \tau \end{bmatrix} \tag{3.26a}$$

$$\begin{bmatrix} e \\ \omega \end{bmatrix} = \begin{bmatrix} K_b \\ 1 \end{bmatrix} x_m + \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \tau \end{bmatrix} \tag{3.26b}$$

The motor model has a state, x_m, a lumped motor-shaft rotational inertia, J, a motor coil resistance, R, a motor back emf constant, K_b, and a motor torque constant, K_t. The transmission modular modeling element has two rotational mechanical power ports with torque τ_u and τ_d input angular velocity ω_u and ω_d output causality (Fig. 3.2b).

$$\dot{x}_{tr} = -\frac{c_{tr}}{J_{tr}} x_{tr} + \begin{bmatrix} \frac{GR}{J_{tr}} & \frac{1}{J_{tr}} \end{bmatrix} \begin{bmatrix} \tau_u \\ \tau_d \end{bmatrix} \tag{3.27a}$$

$$\begin{bmatrix} \omega_u \\ \omega_d \end{bmatrix} = \begin{bmatrix} GR \\ 1 \end{bmatrix} x_{tr} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_u \\ \tau_d \end{bmatrix} \tag{3.27b}$$

The transmission model a state, x_{tr}, a lumped transmission rotational inertia, J_{tr}, an equivalent transmission damping, c_{tr}, and a transmission gear ratio, GR. The fluid clutch modular modeling element has two rotational mechanical power ports with torque τ_m and τ_t input angular velocity ω_m and ω_t output causality (Fig. 3.2c).

$$\begin{aligned} \dot{x}_{c1} &= -\frac{R_{cl}}{J_{cl}}\left(x_{c1} - x_{c2}\right)^2 + \frac{1}{J_{cl}}\tau_m \\ \dot{x}_{c2} &= \frac{R_{cl}}{J_{cl}}\left(x_{c1} - x_{c2}\right)^2 + \frac{1}{J_{cl}}\tau_t \end{aligned} \tag{3.28a}$$

$$\begin{bmatrix} \omega_m \\ \omega_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} \tag{3.28b}$$

The clutch model has three states, $[x_{c1}, x_{c2}]^T$, a lumped clutch rotational inertia, J_{cl}, and an equivalent resistance, R_{cl}.

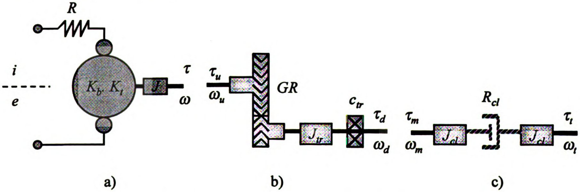

Figure 3.2: Electric Automotive Vehicle Power Train Modular Modeling Elements a) Electric Motor b) Transmission c) Fluid Clutch

These three modular modeling elements are assembled in a motor-clutch-transmission configuration to illustrate the flexibility of modular modeling. The modular motor modeling element is connected to the modular fluid clutch modeling element, which is connected to the modular transmission modeling element (Fig. 3.3).

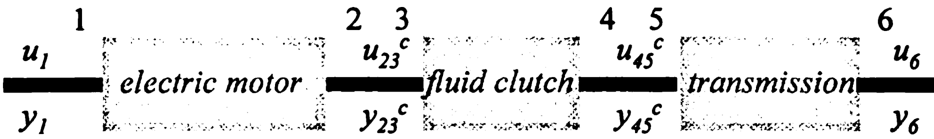

Figure 3.3: Electric Automotive Vehicle Power Train Modular Model With An Electric Motor, A Fluid Clutch, And A Transmission Modular Modeling Elements

The unconnected three element modular system is built up from directly inserting the motor, clutch, and transmission modular element matrices into (3.2a-3.2b).

$$\dot{X} = \mathbf{f}(X,U) = \begin{bmatrix} \dot{x}_m \\ \hline \dot{x}_{c1} \\ \dot{x}_{c2} \\ \hline \dot{x}_{tr} \end{bmatrix} = \begin{bmatrix} \frac{K_t}{J} i - \frac{1}{J}\tau \\ \hline -\frac{R_{cl}}{J_{cl}}(x_{c1}-x_{c2})^2 + \frac{1}{J_{cl}}\tau_m \\ \frac{R_{cl}}{J_{cl}}(x_{c1}-x_{c2})^2 + \frac{1}{J_{cl}}\tau_t \\ \hline -\frac{c_{tr}}{J_{tr}}x_{tr} + \frac{GR}{J_{tr}}\tau_u + \frac{1}{J_{tr}}\tau_d \end{bmatrix} \tag{3.29a}$$

$$Y = \mathbf{g}(X,U) = \begin{bmatrix} e \\ \omega \\ \hline \omega_m \\ \omega_t \\ \hline \omega_u \\ \omega_d \end{bmatrix} = \begin{bmatrix} K_b x_m + Ri \\ x_m \\ \hline x_{c1} \\ x_{c2} \\ \hline GRx_{tr} \\ x_{tr} \end{bmatrix} \tag{3.29b}$$

The input-output topology of the three element modular power train model shows ports 1 and 6 are external ports and two connectors (Fig. 3.3). The internal ports are connected in pairs 2 to 3 and 4 to 5. This simple topology analysis defines the reordering matrix $\underline{T}$ and the equation selector matrix $(\underline{S}_o - \underline{S}_e)$.

$$\underline{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (\underline{S}_o - \underline{S}_e) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.30}$$

Ports 1 and 6 are the external power ports so $\underline{T}$ reorders the input $U = [i, \tau, \tau_m, \tau_t, \tau_u, \tau_d]^T$ and output $Y = [e, \omega, \omega_m, \omega_t, \omega_u, \omega_d]^T$ so i, τ_d, e, and ω_d appear first in the vectors. Ports 2 and 3 and ports 4 and 5 are connected internal power ports reordered to appear in connected pairs after the external ports. Apply the reordering matrix $\underline{T}$ to the modular model (3.29a-3.29b) to partition the inputs and outputs internally and externally.

$$\begin{bmatrix} U_{ext} \\ \hline U_{int} \end{bmatrix} = \underline{T}U = \begin{bmatrix} i \\ \tau_d \\ \hline \tau \\ \tau_m \\ \tau_t \\ \tau_u \end{bmatrix} \tag{3.31a}$$

$$\begin{bmatrix} Y_{ext} \\ \hline Y_{int} \end{bmatrix} = \underline{T}Y = \begin{bmatrix} e \\ \omega_d \\ \hline \omega \\ \omega_m \\ \omega_t \\ \omega_u \end{bmatrix} = \begin{bmatrix} \mathfrak{g}_{ext}(X,U) \\ \hline \mathfrak{g}_{int}(X,U) \end{bmatrix} = \begin{bmatrix} K_b x_m + Ri \\ x_t \\ \hline x_m \\ x_{c1} \\ x_{c2} \\ GRx_t \end{bmatrix} \tag{3.31b}$$

The two element electric automotive vehicle power train modular model (3.29a-3.29b) is now in the form to apply modular modeling constraints (2.2a-2.2b).

The constraints are applied to the internal inputs and outputs. The internal inputs are constrained to be equal and opposite (3.13).

$$U_{int} = \begin{bmatrix} \tau \\ \tau_m \\ \tau_t \\ \tau_u \end{bmatrix} = (\underline{S}_o - \underline{S}_e)^T U^c = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_{23}^c \\ u_{45}^c \end{bmatrix} \tag{3.32}$$

The internal outputs are constrained to be equal using integral control logic (3.16a-3.16b)

$$\dot{X}^c = \begin{bmatrix} \dot{x}_{23}^c \\ \dot{x}_{45}^c \end{bmatrix} = \bullet = (\underline{S}_o - \underline{S}_e)\mathfrak{g}_{int}(X,U) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_m \\ x_{c1} \\ x_{c2} \\ GRx_t \end{bmatrix} = \begin{bmatrix} x_m - x_{c1} \\ x_{c2} - GRx_t \end{bmatrix} \tag{3.33a}$$

$$U^c = \begin{bmatrix} u_{23}^c \\ u_{45}^c \end{bmatrix} = \underline{K}_I X^c = \begin{bmatrix} k_{I1} & 0 \\ 0 & k_{I2} \end{bmatrix} \begin{bmatrix} x_{23}^c \\ x_{45}^c \end{bmatrix} \tag{3.33b}$$

Apply the internal input and output constraints to the three element electric power train modular model (Fig. 3.3) and rewrite in the fully connected modular system form (3.18a-3.18c).

$$\dot{X} = \mathbf{f}\left(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}\right) = \begin{bmatrix} \dot{x}_m \\ \dot{x}_{c1} \\ \dot{x}_{c2} \\ \dot{x}_{tr} \end{bmatrix} = \begin{bmatrix} \frac{K_t}{J} i - \frac{1}{J} k_{I1} x_{23}{}^c \\ -\frac{R_{cl}}{J_{cl}}(x_{c1} - x_{c2})^2 + \frac{1}{J_{cl}}(-k_{I1} x_{23}{}^c) \\ \frac{R_{cl}}{J_{cl}}(x_{c1} - x_{c2})^2 + \frac{1}{J_{cl}} k_{I2} x_{45}{}^c \\ -\frac{c_{tr}}{J_{tr}} x_{tr} + \frac{GR}{J_{tr}} \tau_u + \frac{1}{J_{tr}}(-k_{I2} x_{45}{}^c) \end{bmatrix} \tag{3.34a}$$

$$Y_{ext} = \mathbf{g}_{ext}\left(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}\right) = \begin{bmatrix} e \\ \omega_d \end{bmatrix} = \begin{bmatrix} K_b x_m + Ri \\ x_{tr} \end{bmatrix} \tag{3.34b}$$

$$\dot{X}^c = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}\left(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}\right) = \begin{bmatrix} \dot{x}_{23}{}^c \\ \dot{x}_{45}{}^c \end{bmatrix} = \begin{bmatrix} x_m - x_{c1} \\ x_{c2} - GRx_t \end{bmatrix} \tag{3.34c}$$

The fully connected 3 element electric vehicle power train modular model (3.34a-3.34c) can be solved like any other nonlinear algebraic differential model. The three modular modeling elements in this modular model are all strictly passive because they all have positive damping. Any model with positive damping is strictly passive (Khalil,
→ that do not deliver power
1996 & Krstic etal, 1995). The integrator of the output constraint (36a-36b), as are all integrators, is passive (Khalil, 1996). The modular model is then passive and globally uniformly stable (Khalil, 1996).

The nonlinear algebraic modular model example is a structural model with three bars fixed to a point. This structural modular model example contains two simple modular modeling elements, an axial force bar model and a fixed point model. The two modular modeling elements are all defined with standardized modular modeling causality (Table 1.1). These structural models do not have traditional power ports because the output variable is a displacement instead of velocity. They are representations of physical points with two variables that require a causal definition, which fits the application of modular modeling and the measurement perspective standardized causality (Byam and Radcliffe, 1999).

The axial force bar modular modeling element is a four node Finite Element Model (FEA) from Ross, 1990. This model has four mechanical ports with force input F displacement output δ causality (Fig. 3.4a), which is written in nonlinear algebraic modular modeling form (3.22a-3.22b).

$$\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) = \underline{0} = \frac{AE}{L}\begin{bmatrix} x_a - x_c - F_a \\ (x_c - x_a)(x_b - x_d) - F_b \\ -x_a + x_c - F_c \\ (x_c - x_a)(-x_b + x_d) - F_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.35a}$$

$$\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}) = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} \tag{3.35b}$$

The modular bar (3.35a-3.35b) has four states $[x_a, x_b, x_c, x_d]^T$, a cross-sectional area A, a modulus of elasticity E, and a length L.

The fixed point modular modeling element is unique to modular modeling because of the standardized measurement perspective causality. The one port fixed point modular modeling element outputs a zero displacement regardless of the input and applies to any energy domain (Fig. 3.4b). The fixed point modular modeling element is written in nonlinear algebraic modular modeling form (3.22a-3.22b).

$$\underset{\sim}{f}(\underset{\sim}{x},\underset{\sim}{u}) = \underline{0} = x_e = 0 \tag{3.36a}$$

$$\underset{\sim}{y} = \underset{\sim}{g}(\underset{\sim}{x},\underset{\sim}{u}) = \delta_5 = 0 \tag{3.36b}$$

The modular fixed point modular modeling element has one state x_e.

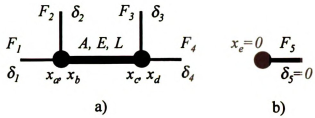

Figure 3.4: Three Structural Nonlinear Algebraic Modular Modeling Elements: a) Axial Force Bar b) Fixed Point

The example nonlinear algebraic modular model will use the two modular modeling elements (Fig. 3.4) in a bracket-pin-clevis configuration. Three axial force bars and a fixed point will be connected together in configuration with no pin bending deflection (Fig. 3.5).

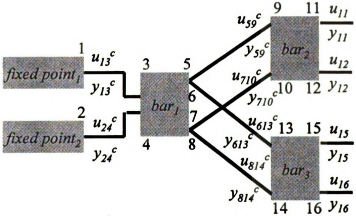

Figure 3.5: Three Axial Force Bar And A Fixed Point Modular Modeling Elements In A Bracket-Pin-Clevis Configuration With No Pin Deflection

The unconnected five element modular system is built up from directly inserting the bar and fixed point modular elements into (3.22a-3.22b) where the $\tilde{f}$, $\tilde{g}$, $\tilde{x}$, $\tilde{y}$, and $\tilde{u}$ of each element are found in the element equations (3.35a-3.35b) and (3.36a-3.36b).

$$\mathbf{f}(X,U)=\underline{0}=\begin{bmatrix} \underset{\sim}{f}_{fix\,pt_1}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{f}_{fix\,pt_2}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{f}_{bar_1}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{f}_{bar_2}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{f}_{bar_3}(\underset{\sim}{x},\underset{\sim}{u}) \end{bmatrix}=\begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} \tag{3.37a}$$

$$Y=\mathbf{g}(X,U)=\begin{bmatrix} \underset{\sim}{y}_{fix\,pt_1} \\ \underset{\sim}{y}_{fix\,pt_2} \\ \underset{\sim}{y}_{bar_1} \\ \underset{\sim}{y}_{bar_2} \\ \underset{\sim}{y}_{bar_3} \end{bmatrix}=\begin{bmatrix} \underset{\sim}{g}_{fix\,pt_1}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{g}_{fix\,pt_2}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{g}_{bar_1}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{g}_{bar_2}(\underset{\sim}{x},\underset{\sim}{u}) \\ \underset{\sim}{g}_{bar_3}(\underset{\sim}{x},\underset{\sim}{u}) \end{bmatrix} \tag{3.37b}$$

The input-output topology of the three bar no pin modular model shows ports 11, 12, 15, and 16 are external ports and six connectors. The internal ports are connected in pairs 1 to 3, 2 to 4, 5 to 9, 6 to 13, 7 to 10, and 8 to 14. This simple topology analysis defines the reordering matrix $\underline{T}$ and the equation selector matrix $(\underline{S}_o - \underline{S}_e)$.

$$\underline{T}=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
\hdashline
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hdashline
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
\hdashline
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
\hdashline
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
\hdashline
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
\hdashline
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{bmatrix} \tag{3.38a}$$

$$(\underline{S}_o - \underline{S}_e) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.38b}$$

Ports 11, 12, 15, and 16 are the external power ports so $\underline{T}$ reorders the input $U = [F_1, F_2, F_3, \cdots, F_{16}]^T$ and output $Y = [\delta_1, \delta_2, \delta_3, \cdots, \delta_{16}]^T$ so, F_{11}, F_{12}, F_{15}, F_{16}, δ_{11}, δ_{12}, δ_{15}, and δ_{16} appear first in the vectors. Ports 1 and 3, 2 and 4, 5 and 9, 6 and 13, 7 and 10, and 8 and 14 are connected internal power ports reordered to appear in connected pairs after the external ports. Apply the reordering matrix $\underline{T}$ to the modular model (3.37a-3.37b) to partition the inputs and outputs internally and externally.

$$\begin{bmatrix} U_{ext} \\ \hline U_{int} \end{bmatrix} = \underline{T}U = \begin{bmatrix} F_{11} \\ F_{12} \\ F_{15} \\ F_{16} \\ \hline F_1 \\ F_3 \\ \hdashline F_2 \\ F_4 \\ \hdashline F_5 \\ F_9 \\ \hdashline F_6 \\ F_{13} \\ \hdashline F_7 \\ F_{10} \\ \hdashline F_8 \\ F_{14} \end{bmatrix} \tag{3.39a}$$

$$\begin{bmatrix} Y_{ext} \\ \hline Y_{int} \end{bmatrix} = \underline{T}Y = \begin{bmatrix} \delta_{11} \\ \delta_{12} \\ \delta_{15} \\ \delta_{16} \\ \hline \delta_{1} \\ \delta_{3} \\ \hdashline \delta_{2} \\ \delta_{4} \\ \hdashline \delta_{5} \\ \delta_{9} \\ \hdashline \delta_{6} \\ \delta_{13} \\ \hdashline \delta_{7} \\ \delta_{10} \\ \hdashline \delta_{8} \\ \delta_{14} \end{bmatrix} = \begin{bmatrix} \mathfrak{g}_{ext}(X,U) \\ \hline \mathfrak{g}_{int}(X,U) \end{bmatrix} = \begin{bmatrix} x_{b2_c} \\ x_{b2_d} \\ x_{b3_c} \\ x_{b3_d} \\ \hline x_{fp_1} \\ x_{b1_a} \\ \hdashline x_{fp_2} \\ x_{b1_b} \\ \hdashline x_{b1_c} \\ x_{b2_a} \\ \hdashline x_{b1_c} \\ x_{b3_a} \\ \hdashline x_{b1_d} \\ x_{b2_b} \\ \hdashline x_{b1_d} \\ x_{b3_b} \end{bmatrix} \tag{3.39b}$$

The five element structural three bar modular model (3.37a-3.37b) is now in the form to apply modular modeling constraints (2.2a-2.2b).

The constraints are applied to the internal inputs and outputs. The internal inputs are constrained to be equal and opposite (3.13).

$$U_{int} = \begin{bmatrix} F_1 \\ F_3 \\ \hdashline F_2 \\ F_4 \\ \hdashline F_5 \\ F_9 \\ \hdashline F_6 \\ F_{13} \\ \hdashline F_7 \\ F_{10} \\ \hdashline F_8 \\ F_{14} \end{bmatrix} = \left(\underline{S}_o - \underline{S}_e\right)^T U^c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{13}{}^c \\ u_{24}{}^c \\ u_{59}{}^c \\ u_{613}{}^c \\ u_{710}{}^c \\ u_{814}{}^c \end{bmatrix} \tag{3.40}$$

The internal outputs are constrained to be equal using integral control logic (3.24)

$$\dot{X}^c = \begin{bmatrix} \dot{x}_{13}^c \\ \dot{x}_{24}^c \\ \dot{x}_{59}^c \\ \dot{x}_{613}^c \\ \dot{x}_{710}^c \\ \dot{x}_{814}^c \end{bmatrix} = \mathbf{e} = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}(X,U) = \begin{bmatrix} x_{fp_1} - x_{b1_a} \\ x_{fp_2} - x_{b1_b} \\ x_{b1_c} - x_{b2_a} \\ x_{b1_c} - x_{b3_a} \\ x_{b1_d} - x_{b2_b} \\ x_{b1_d} - x_{b3_b} \end{bmatrix} \quad (3.41a)$$

$$U^c = \begin{bmatrix} u_{13}^c \\ u_{24}^c \\ u_{59}^c \\ u_{613}^c \\ u_{710}^c \\ u_{814}^c \end{bmatrix} = \underline{K}_I X^c = \begin{bmatrix} k_{I1} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{I2} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{I3} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{I4} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{I5} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{I6} \end{bmatrix} \begin{bmatrix} x_{13}^c \\ x_{24}^c \\ x_{59}^c \\ x_{613}^c \\ x_{710}^c \\ x_{814}^c \end{bmatrix} \quad (3.41b)$$

Apply the internal input and output constraints to the five element electric power train modular model (Fig. 3.5) and rewrite in the fully connected modular system form (3.25a-2.25c).

$$\dot{X} = \mathbf{f}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) = \underline{0} = \begin{bmatrix} f_{fix\,pt_1}(\underset{\sim}{x}, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \\ f_{fix\,pt_2}(\underset{\sim}{x}, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \\ f_{bar_1}(\underset{\sim}{x}, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \\ f_{bar_2}(\underset{\sim}{x}, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \\ f_{bar_3}(\underset{\sim}{x}, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \end{bmatrix} \quad (3.42a)$$

$$Y_{ext} = \mathbf{g}_{ext}(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}) = \begin{bmatrix} \delta_{11} \\ \delta_{12} \\ \delta_{15} \\ \delta_{16} \end{bmatrix} = \begin{bmatrix} x_{b2_c} \\ x_{b2_d} \\ x_{b3_c} \\ x_{b3_d} \end{bmatrix} \quad (3.42b)$$

$$\dot{X}^c = (\underline{S}_o - \underline{S}_e)\mathbf{g}_{int}\left(X, \underline{T}^T \begin{bmatrix} U_{ext} \\ (\underline{S}_o - \underline{S}_e)^T \underline{K}_I X^c \end{bmatrix}\right) = \begin{bmatrix} \dot{x}_{13}{}^c \\ \dot{x}_{24}{}^c \\ \dot{x}_{59}{}^c \\ \dot{x}_{613}{}^c \\ \dot{x}_{710}{}^c \\ \dot{x}_{814}{}^c \end{bmatrix} = \begin{bmatrix} x_{fp_1} - x_{b1_a} \\ x_{fp_2} - x_{b1_b} \\ x_{b1_c} - x_{b2_a} \\ x_{b1_c} - x_{b3_a} \\ x_{b1_d} - x_{b2_b} \\ x_{b1_d} - x_{b3_b} \end{bmatrix} \qquad (3.42c)$$

The fully connected five element three bar structural modular model (3.42a-3.42c) can be solved like any other nonlinear algebraic differential model. The two modular modeling elements in this modular model are all strictly passive because they all memoryless functions that do not generate energy (Khalil, 1996). The integrator of the output constraint (3.41a-3.41b), as are all integrators, is passive (Khalil, 1996). The modular model is then passive and globally uniformly stable (Khalil, 1996).

Chapter 4

CONCLUSIONS

4.1 Contributions

We have successfully developed mechanisms for modular modeling of engineering systems with fixed input-output structure. Modular modeling elements with a power-based fixed input-output structure have fixed power port input-output causality, fixed sign convention, and a fixed internal equation formulation. Modular modeling connectors join modular element ports with fixed power constraints. The standardized internal formulation of modular modeling elements enables top-down modeling and enhances model verification via modularity. Because a modular modeling element's formulation does not change, modular modeling yields more easily verified models.

Unverified models are of no use in today's industrial environment. A model with n power ports has 2^n possible formulations. The performance of all 2^n formulations simply cannot be simultaneously verified. Reducing the number of verifications from 2^n different verifications to a single verification is an enhancement in modeling technology. Modular modeling elements with a known physically validated performance are an asset to modelers of engineering systems.

Model complexity increases with the modular modeling method because of the connector requirement. Modular connectors have a fixed definition, which is compatible with the modular modeling element's fixed functional definition. The compatible fixed functional definitions enable a systematic assembly method for modular models with a fixed mathematical formulation and a fixed computational sparseness.

The measurement perspective fixed port causality used in modular modeling has the ability to avoid using energic junctions to maintain modularity upon assembly. Hogan concluded that use of energic junctions guaranteed modularity at the cost of "stiff" system equations with widely spread eigenvalues. Modular modeling can assemble modular modeling elements with fixed measurement perspective causality and maintain modularity with the simple power constraint of the modular modeling connector.

Modular modeling reduces the verification task of large model design, development, and refinement by standardizing the functional form of all multi degree of freedom modeling elements. The fixed power-based measurement perspective port causality results in a standardized multi degree of freedom modular modeling element with a single mathematical model formulation. This formulation has the flexibility to support any number of physically disconnected ports without reformulation. Assembling incompatible modular modeling elements requires a causally compatible 2-port modular connector that facilitates a standardized constrained power transfer between modular modeling element ports. The separate modular modeling elements and connectors with explicitly different functions enable subsystem level modeling with no reformulation. Fixed formulation enhances model verification. Modular modeling has the flexibility to model any engineering system across multiple energy domains with the benefits of a fixed input-output structure.

The solution to linear modular models is a systematic direct insertion solution. Modular modeling elements with nonlinear equations are the subject of a future paper. Given n modular modeling elements and an input-output topology of their interconnections a systematic reordering and constraint analysis realizes a built up modular model that can be solved like any other linear system of equations. Modular modeling elements maintain their modularity in the assembled modular model. This method's maintenance of element modularity while assembling with simple connectors with a less restrictive condition on the modular elements is previously unavailable in other methods. The reordering and constraint processes add complexity to the solution process but the modular model has a fixed computational sparseness. This formulation results in cases where the number of computations for modular models is less or comparable to non-modular models.

The solution to nonlinear modular models is a systematic direct insertion solution. Given n modular modeling elements and an input-output topology of their interconnections a systematic reordering and constraint analysis realizes a built up modular model that can be solved like any other nonlinear system of equations. Modular modeling elements maintain their modularity in the assembled modular model. This method's maintenance of element modularity while assembling with simple connectors with a less restrictive condition on the modular elements and no stiff equations is previously unavailable in other methods. The reordering and constraint processes add complexity to the solution process but the modular model has some computational advantages. This formulation results in cases where the parts of the computation can be done on separate processors.

4.2 Future Work

Future work in this new modeling technology has several exciting possibilities. Application of adaptive control strategies to the control logic solution of nonlinear modular models. Using modular modeling with fixed input-output structure to connect complex Finite Element Analysis (FEA) models with incompatible meshes. Modular modeling using Statistical Energy Analysis (SEA) techniques. Utilization of modular modeling technology in internet-based or web-based modeling.

Adaptive control strategies such, gain scheduling, could be applied to the assembly and solution of nonlinear modular models. Each modular modeling element may have a unique response requiring a unique feedback gain to reduce the assembly error.

Connecting complex FEA models typically requires remeshing for connection. Modular modeling with fixed input-output structure is formulated to avoid this problem. The $\underline{C}$ matrix in the boundary value problem formulation provides for interpolation between mesh node points and the inputs and outputs. Interpolated outputs of different FEA models from number of mesh node points could be connected through a modular modeling connector.

Statistical Energy Analysis (SEA) models, which are an equal sharing of high frequency vibration and sound energy between modes, are applicable to modular modeling with fixed input-output structure (Lyon, 1975). SEA models have elements and connectors the store and distribute energy. SEA models have grown to be quite large and complex with numerous elements and interconnections. Modular SEA models would be an advancement in SEA modeling technology.

Internet-based or web-based modeling is an emerging modeling technology well suited for modular modeling with fixed input-output structure. A key design specification of web-based modeling in security of proprietary model information. A fixed input-output structure enables such security by restricting the type of input and output. Reverse engineering is not a trivial exercise when restricted to input and output information only.

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