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ABSTRACT

SUPER-REPLICATION OF EUROPEAN EXOTIC OPTIONS

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We study the continuous time problem of hedging a European style Asian call option in the presence of transaction costs. Under the assumption that the price process of the relevant stock both fluctuates and does not fluctuate with positive probability, we find a portfolio that super-replicates the option. **Most important,** we prove that the portfolio that we found is optimal in the sense that it requires the smallest initial investment among all the super-replicating portfolios.

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INTRODUCTION

In their fundamental paper, Black and Scholes (1973) discovered how to price options in continuous–time financial markets where the stock price follows a geometric Brownian motion and the market is free of transaction costs. By "option," we mean here a contract between a buyer and a seller whose value at some future date, the "exercise time," will be equal to a given function of the underlying stock. The value of the option when it will be exercised will be transferred from the seller to the buyer. For the right to receive that transfer of wealth in the future, the buyer pays the seller a certain amount of money which is the option price. The main idea in Black and Scholes (1973) is that the option price should be the exact difference between the value of the option at the exercise time and the "capital gain" achieved from some "replicating portfolio." This replicating portfolio is based on the underlying stock and money market account. By using the replicating portfolio, the seller is able to "hedge" his or her liability; namely, the seller will not lose any money from the option contract.

The main problem in the Black and Scholes theory is that the replicating portfolio demands continuous trading. This makes the theory not practical in the presence of transaction costs that are proportional to the monetary value of the trades. The replicating portfolio will create an infinite amount of trading and hence an infinite amount of transaction costs. The transaction costs are called "two–sided" when they are being charged in both buying and selling of shares. They are called "one–sided" when they are being charged only in buying shares (or only in selling shares). We only consider the two–sided transaction costs case in this thesis.

It was discovered in Bensaid, Lense, Pages, and Scheinkman (1992), in the context of a discrete time model, that if the requirement of exact replication portfolio is relaxed, it is sometimes possible to lower the option price. That is why we will only require here that the hedging portfolio will dominate almost surely ("**super-replicate**") the value of the option at payoff time. This is, of course, enough protection from the seller's point of view so that is why we will not deal with exact replication in this thesis.

Davis and Clark (1994) has formally conjectured the "conventional wisdom" concerning hedging of options in the presence of transaction costs. More precisely, they conjectured that the only possible way to hedge a European style call option is by a trivial hedging portfolio: buy one share and hold it till expiration day.

Soner, Shreve, and Cvitanic (1995) have proved the conjecture in a setup where the stock price is modeled by a geometric Brownian motion. In their proof they have used some ideas from convex function theory. We like their proof and we believe that their methods can be applied to other problems as well. An example would be a problem where the super-replication requirement is relaxed.

Levental and Skorohod (1997) (hereforth referred to as LS) deal with both generalized American and European style call options. They require only that the stock price will be modeled by a non-degenerate, continuous, positive semimartingale rather than by a geometric Brownian motion. LS only use only fundamental properties of stochastic integrals of continuous semimartingales. It is very exciting from the mathematical point of view, but their approach has its limitations. Their method works well only in super-replication context and it will be less useful when this requirement is relaxed.

Finally Civtanic and Karatzas (1996) state a general result about the minimal price that is needed to super-replicate European style options. They found essentially that this price is the supremum of the expected discounted value of the claim with respect to all equivalent probability measures under which all portfolios are supermartingales.

In our model we use the same model as LS with one difference. We use a stronger assumption on the fluctuations of the stock price process than the one used by LS. The option that we use is "European style Asian call option" which has a payoff similar to the classical call option with one difference: The stock price at payoff time is being replaced by averaging the stock price throughout the option's lifetime. Our achievement is that we find a portfolio that super-replicate the option and, **most important,** we prove that among all possible hedging portfolios it is the one that requires the smallest initial investment.

Chapter 1

The model, basic definitions and main results

We consider a financial market in which one stock is traded in the time interval $0 \le t \le 1$. The price of this stock is represented by a stochastic process $Z = \{Z(t)\colon 0 \le t \le 1\}$, which is defined on a complete probability space $(\Omega, \boldsymbol{F}, P)$. Assume that Z is a continuous semimartingale with respect to a filtration $\{F_t\colon 0 \le t \le 1\}$ that is right continuous, and such that F_t contains all P null sets, $0 \le t \le 1$, and F_0 is the trivial σ-algebra. Since Z represents a price of stock, we will assume that Z is strictly positive process. For simplicity we will assume that $Z(0) = 1$.

We will assume that the interest rate equals 0 in our model. Since one can always work with discounted price, rather than the actual ones, this entails no loss of generality. We want to assume that Z both fluctuates and does not fluctuates with positive probability. We will give a precise definition later in this chapter.

Definition 1.1 A portfolio is an adapted stochastic process $N = \{N(t)\colon 0 \le t \le 1\}$, which has almost surely (a.s.) left limit and right continuous sample paths, and satisfies

(1.2) $$P\left(\int_0^1 |dN|(t) < \infty\right) = 1.$$

We denote the class of all portfolios by FV.

For $N \in FV$, We define two processes N^+ and N^-, which are associated with N:

$$N^+(t) = \frac{N(0) + N(t) + \int_0^t |dN|(s)}{2},$$

$$N^-(t) = \frac{N(0) - N(t) + \int_0^t |dN|(s)}{2}.$$

The process N^+ and N^- are nondecreasing a.s. and satisfy:

(1.3) $\quad N = N^+ - N^-,$

$|dN| = dN^+ + dN^-,$

$N^+(0) = N(0)$, and

$N^-(0) = 0.$

The process $N^+(t)$ (respectively $N^-(t)$) represents the accumulated number of shares that the owner of the N portfolio has bought (sold) up to time t, and $N(t)$ represents the number of shares in the account at time t.

Let $0 < \lambda < 1$, $0 < \mu < 1$. In what follows λ, respectively μ, represents the fractional transaction costs when one is buying, respectively selling, shares.

Remark: It is assumed that no transaction costs are being paid due to holding of $N(0)$ shares at time $t = 0$.

The accumulated capital gain generated by a portfolio N is a stochastic process $\{S_M(t): 0 < t \leq 1\}$ defined by,

(1.4) $$S_N(t) = \int_0^t N(s)dZ(s) - \lambda \int_0^t Z(s)dN^+(s) - \mu \int_0^t Z(s)dN^-(s).$$

The financial interpretation of (1.4) is the following: $N(t)\, dZ(t)$ represents instant gain (lost) of the portfolio due to the change of the share price $dZ(t)$, while $\lambda\, Z(t)\, dN^{+}(t)$ ($\mu\, Z(t)\, dN^{-}(t)$) represents the transaction cost paid at time t due to buying (selling) of $dN^{+}(t)$ (respectively, $dN^{-}(t)$) shares.

We describe now the European type Asian call option that we deal with in this paper. This option is a contract between two persons: a seller and buyer. The option can be exercised only at time $t = 1$. At that time the seller has the obligation to pay the buyer

$$g\left(\int_0^1 Z(t)dt\right) \text{ where } g(x) = ((1-\mu)\,x - q)^{+},$$

where q is a given positive number. The function g is called the payoff function of the option. From now on, when we mention option, we mean the option that we have just defined.

Remark. We find it easier to work with the option above. However some readers will think that the payoff function $g\left(\int_0^1 Z(t)dt\right)$ with $g(x) = (x - q)^{+}$ is a more natural choice. Those readers are directed to the Corollary 1.9 at the end of this chapter.

The question that we are asking here is, what price should the seller charge the potential buyer at $t = 0$, for the right to own the option? The idea is that the seller will charge the minimal amount of money that will allow him or her to hedge their liability. This means that the seller will create a portfolio skillfully. This portfolio's capital gain at $t = 1$ plus the money received from the buyer at $t = 0$ will be at least as large as the payment that the seller has to transfer to the buyer at $t = 1$. In that way there is a certainty that seller will not lose any money.

More precisely we define for each $N \in FV$.

(1.5) $$x_N = \inf\{x \in \boldsymbol{R} : x + S_N(1) \geq g(\int_0^1 Z(t)dt)\}$$

If the set is empty, x_N will be taken to be ∞.

We define the selling price of the option to be

(1.6). $$b_E = \inf\{x_N : N \in FV\}.$$

Observe that $b_E \leq 1$. To show this we take $N(t) = 1 - t$ (so $N^+(t) = 1$ and $N^-(t) = t$) and we get

$$g(\int_0^1 Z(t)dt)$$

$$= \{(1-\mu) \int_0^1 Z(t)dt - q\}^+$$

$$\leq (1-\mu) \int_0^1 Z(t)dt$$

$$= \int_0^1 Z(t)dt - \mu \int_0^1 Z(t)dt$$

$$= \int_0^1 (1-t)dZ(t) + 1 - \mu \int_0^1 Z(t)dN^-(t)$$

$$= 1 + S_N(1).$$

So we conclude that $b_E \leq 1$.

In this paper we will assume that the stock price $Z(t)$, in addition to being a positive semimartingale, also satisfies some extra assumptions. To define them precisely, we need some notation.

For every $0 < d \leq 1$ and $\delta, \gamma > 0$ and stopping time $T < d$, we define the following stopping time.

$$(1.7)\ \ T^d_{\delta,\gamma} = \begin{cases} \inf\{T \le t \le d : Z(t) = e^{-\delta} Z(T) \text{ or } Z(t) = e^{\gamma} Z(T)\}, \\ d \text{ if no such } t \text{ exists.} \end{cases}$$

We denote $T^d_\delta = T^d_{\delta,\delta}$, $T_{\delta,\gamma} = T^1_{\delta,\gamma}$ and $T_\delta = T^1_\delta$.

The following basic assumption on Z will hold throughout this paper.

Assumption 1.8 For every $\delta > 0$, $d > 0$ and stopping time $0 \le T \le 1$, the following holds

(i) On the event $\{T < d\}$ we have a.s.

$$\begin{cases} P\{T^d_\delta < d, Z(T^d_\delta) = Z(T)e^{-\delta} / F_T) > 0, \\ P(T^d_\delta < d, Z(T^d_\delta) = Z(T)e^{\delta} / F_T) > 0. \end{cases}$$

(ii) $P(T_\delta = 1 / F_T) > 0$.

It will be convenient for us to modify Assumption 1.8. We will state here an equivalent form of Assumption 1.8. In the proof of our main theorem we will use this equivalent form. We state it here as Lemma A.25. The proof of this lemma will appear in the Appendix.

Lemma A.25 If Assumption 1.8 is satisfied, then for every $0 < \varepsilon < 1$, there exists $\delta°(\varepsilon) > 0$ that satisfies: $e^{2\delta°} \le (1 - 2\varepsilon) / (1 - 3\varepsilon)$, $1 - e^{-3\delta°} \le \mu$, $e^{3\delta°} - 1 \le \lambda$ so that for every stopping times $0 \le T \le \varepsilon$, and $1 - \varepsilon \le \tau < 1$, we have

(i) $P(T_{\delta°} = 1 / F_T) > 0$ a.s,

(ii) $P(\tau_{\beta,\delta°} < 1, Z(\tau_{\beta,\delta°}) = e^{-\beta} Z(\tau) / F_\tau) > 0$ a.s, where $\beta = -[\log(1 - \mu) + \delta°]$, and

(iii) $P(1 - \varepsilon \le T_{\delta°} < 1, Z(T_{\delta°}) = e^{-\delta°} Z(T) / F_T) > 0$ a.s.

Finally, we state the main result of this thesis.

Theorem 2.1 If Assumption 1.8 is satisfied then $b_E = 1$.

We already saw that $b_E \leq 1$. We need then to show that $b_E \geq 1$. The idea behind the proof of that is simple but the details are complicated. First we create a discrete version of the problem. We are doing it in Lemma **A**.1. Then we need to know how to handle that discrete version. This is done with the help of Lemma **A**.6. The actual proof the theorem will appear in Chapter 2. The proofs of all the lemmas that we use during the proof of the theorem in Chapter 2 will appear in the Appendix.

Corollary 1.9 Let the payoff function be: $(\int_0^1 Z(t)dt - q)^+$. Under Assumption 1.8 we have

$$b_E = \frac{1}{1-\mu}.$$

Proof. We have $(\int_0^1 Z(t)dt - q)^+ = \frac{1}{1-\mu}\ \{(1-\mu)\int_0^1 Z(t)dt - (1-\mu)\,q\}^+$.

Now observe that for every random variable $H \geq 0$ and constant $a \geq 0$ we have $b_E(aH) = a\, b_E(H)$, where $b_E(H)$, the selling price of a European option with payoff H, is defined by (1.5) with H replacing $g(\int_0^1 Z(t)dt)$ and by (1.6). By taking

$H = \{(1-\mu)\int_0^1 Z(t)dt - (1-\mu)\,q\}^+$, we see that it is enough to prove that $b_E(H) = 1$ but this follows form **Theorem 2.1** when we use $(1-\mu)\,q$ instead of q.

Chapter 2

Proof of the theorem about the option

In this chapter we will prove that $b_E \geq 1$. This will be achieved by proving that $x_N \geq 1 - \varepsilon$ for any $N \in FV$ where $0 < \varepsilon < 1$ is arbitrary. This implies that $x_N \geq 1$ and since N is arbitrary we conclude that $b_E \geq 1$. In Chapter 1 we have already proved that $b_E \leq 1$. Putting the two together gives $b_E = 1$.

We will quote here four lemmas (**A**.1, **A**.3, **A**.5 and **A**.6) that are essential to our proof. The proof of these lemmas will be given in the Appendix. We hope that this will make our proof easier to read. The idea behind the proof is simple. First we create a discrete version of the problem in Lemma **A**.1, then we handle that discrete version in Lemma **A**.6 and finally we convert the result on the discrete version back into the setup of our original continuous–time problem in Lemmas **A**.3 and **A**.5.

In order to state Lemma **A**.1 we need a new definition. Let $N \in FV$, and let $T \leq \tau$ be stopping times. We define:

$$S_N(T,\tau) = \int_{(T,\tau]} N(s)\,dZ(s) - \lambda \int_{(T,\tau]} Z(s)\,dN^+(s) - \mu \int_{(T,\tau]} Z(s)\,dN^-(s).$$

$S_N(T,\tau)$ is the capital gain generated by the portfolio N between the T and τ.

Lemma A.1 Let $N \in FV$ and $0 \leq T \leq \tau \leq 1$ be two stopping time so that $e^{-\delta} Z(T) \leq Z(t) \leq e^{\delta} Z(T)$, for all $t \in [T, \tau]$ a.s, where $\delta > 0$ satisfies $1 - e^{-2\delta} \leq \mu$ and $e^{2\delta} - 1 \leq \lambda$, then

$$S_N(T,\tau) \leq N(T)\,(Z(\tau) - Z(T)).$$

Furthermore,

(i) If $Z(\tau) = e^{-\delta} Z(\mathrm{T})$ then

$$S_M(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^+ - \mu_\delta\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^-$$

, where $\mu_\delta = e^{2\delta}\,(\mu + e^{-2\delta} - 1) \geq 0$.

(ii) If $Z(\tau) = e^{\delta} Z(\mathrm{T})$ then

$$S_M(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda_\delta\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^+ - \mu\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^-$$

, where $\lambda_\delta = e^{-2\delta}\,(\lambda + 1 - e^{2\delta}) \geq 0$.

Lemma **A**.3 is an asymmetric extension of Lemma **A**.1.

Lemma A.3 Let $N \in FV$ and $0 \leq \mathrm{T} \leq \tau \leq 1$ be two stopping time so that $e^{-\beta} Z(\mathrm{T}) \leq Z(t) \leq e^{\gamma} Z(\mathrm{T})$, $\mathrm{T} \leq t \leq \tau$ a.s, and $Z(\tau) = e^{-\beta} Z(\mathrm{T})$, where $\beta, \gamma > 0$ satisfy that $1 - e^{-(\beta+\gamma)} \leq \mu$, then

$$S_M(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})).$$

Lemma **A**.5 shows that after each stopping time there is a positive probability that our capital gain will be non-positive regardless of the trading strategy that we are using.

Lemma A.5 Let $N \in FV$ and $0 \leq \mathrm{T} \leq 1$ be a stopping time. Then

$$P(S_M(\mathrm{T},1) \leq 0 \,/\, F_{\mathrm{T}}) > 0.$$

The last lemma that we quote is Lemma **A**.6. To state it we need the following notations. Let $0 < \varepsilon < 1$ and let $\delta > 0$ so that $1 - e^{-2\delta} \leq \mu$, $e^{2\delta} - 1 \leq \lambda$ as in Lemma **A**.1.

Let $b > 1 / (1 - \mu)$ so that $c_2 \equiv 1 - \dfrac{3\varepsilon}{4} - \dfrac{q}{(1-\mu)b} > 1 - \varepsilon$, where q is the strike price of the option. The quantities μ_δ and λ_δ are as in Lemma **A**.1.

Lemma A.6 There exists an integer $M = M(\varepsilon,\delta) \geq 1$ and a sequence of measurable functions $Z_k : \boldsymbol{R}^k \to \boldsymbol{R}^+$, $k \geq 0$, so that $Z_0 = 1$, and for every sequence of numbers N_k, $k \geq 0$, we have

$$\frac{Z_k(N_0,\ldots,N_{k-1})}{Z_{k-1}(N_0,\ldots,N_{k-2})} = e^{\pm\delta}, \quad k \geq 1 \text{ such that if } \inf_{0\leq n\leq M} S_n > \varepsilon - 1,$$

then $\exists\, 0 \leq n \leq M$ for which

(i). $\qquad z_n = b$, and

(ii). $\qquad c_2\,(b - 1) - \mu\, b\,(N_n - c_2)^- \geq S_n$,

where $S_n = \sum_{k=0}^{n-1} N_k(z_{k+1} - z_k)$

$$- \{z_{k+1} = e^{\delta} z_k\}[\lambda_\delta\,(N_{k+1} - N_k)^+ + \mu\,(N_{k+1} - N_k)^-]\, z_{k+1}$$

$$- \{z_{k+1} = e^{-\delta} z_k\}[\lambda\,(N_{k+1} - N_k)^+ + \mu_\delta\,(N_{k+1} - N_k)^-]\, z_{k+1},$$

and $z_k = Z_k(N_0,\ldots,N_{k-1})$, $k \geq 0$.

Finally we are able to prove our main theorem.

Theorem 2.1 If Assumption 1.8 is satisfied then $b_E = 1$.

Proof.

Let $N \in FV$ and $0 < \varepsilon < 1$. To apply Lemma A.6 we select some constants.

Let $\delta > 0$ satisfies $1 - e^{-2\delta} \leq \mu$ and $e^{2\delta} - 1 \leq \lambda$. Choose $b > 1 / (1 - \mu)$ so that

$$c_2 \equiv 1 - \frac{3\varepsilon}{4} - \frac{q}{(1-\mu)b}$$

satisfies $c_2 > 1 - \varepsilon$. We denote $d = \varepsilon / 4$ for notational simplicity.

Next we define a sequence of stopping times

$$\tau_0 = 0, \quad \tau_{k+1} = (\tau_k)^d_\delta, \quad k \geq 0.$$

By Lemma A.1, we have, for every $n \geq 0$

$S_M(0,\tau_n) \leq \check{S}_n$ a.s. on $\{\tau_n < d\}$, where

$$\breve{S}_n = \sum_{k=0}^{n-1} N(\tau_k)(Z(\tau_{k+1}) - Z(\tau_k))$$

$$- \{Z(\tau_{k+1}) = e^{\delta} Z(\tau_k)\} [\lambda_\delta (N(\tau_{k+1}) - N(\tau_k))^+ + \mu (N(\tau_{k+1}) - N(\tau_k))^-] Z(\tau_{k+1})$$

$$- \{Z(\tau_{k+1}) = e^{-\delta} Z(\tau_k)\} [\lambda (N(\tau_{k+1}) - N(\tau_k))^+ + \mu_\delta (N(\tau_{k+1}) - N(\tau_k))^-] Z(\tau_{k+1}).$$

By Lemma **A**.6, there is integer $M(\varepsilon,\delta) \geq 1$, and there are measurable functions Z_k : $\boldsymbol{R}^k \to \boldsymbol{R}^+$, $k \geq 0$, so that if $\inf_{0 \leq n \leq M} S_n > \varepsilon - 1$ then $\exists\ 0 \leq n \leq M$ for which

(2.2). $\qquad z_n = b$, and

$$c_2 (b - 1) - \mu\, b\, (N_n - c_2)^- \geq S_n,$$

where $S_n = \sum_{k=0}^{n-1} N(\tau_k)(z_{k+1} - z_k)$

$$- \{z_{k+1} = e^{\delta} z_k\} [\lambda_\delta (N(\tau_{k+1}) - N(\tau_k))^+ + \mu (N(\tau_{k+1}) - N(\tau_k))^-] z_{k+1}$$

$$- \{z_{k+1} = e^{-\delta} z_k\} [\lambda (N(\tau_{k+1}) - N(\tau_k))^+ + \mu_\delta (N(\tau_{k+1}) - N(\tau_k))^-] z_{k+1},$$

$z_k = Z_k(N(\tau_0),\ldots,N(\tau_{k-1}))$, $k \geq 0$, and $z_{k+1} = e^{\pm\delta} z_k$, $k \geq 0$.

We will show that

(2.3). $\qquad P(Z(\tau_k) = z_k,\ 1 \leq k \leq M,\ \tau_M < d) > 0.$

In principle, (2.3) follows because the price process Z fluctuates according to Assumption 1.8 (i). We will prove it formally by induction. To start the induction, we assume that $P(A_k) > 0$, for some k between 1 and $M - 1$, where $A_k = \{Z(\tau_i) = z_i,\ 1 \leq i \leq k,\ \tau_k < d\} \in F_{\tau_k}$.

Since

$$P(A_k) = P(A_k, z_{k+1} = e^{-\delta} z_k) + P(A_k, z_{k+1} = e^{\delta} z_k),$$

we will assume without loss of generality, that $P(A_k, z_{k+1} = e^{\delta} z_k) > 0$.

Since $z_{k+1} \in F_{\tau_k}$, it follows that $(A_k, z_{k+1} = e^{\delta} z_k) \in F_{\tau_k}$. From Assumption 1.8 (i), we get

$$P(A_k, z_{k+1} = e^{\delta} z_k, Z(\tau_{k+1}) = e^{\delta} Z(\tau_k), \tau_{k+1} < d) > 0.$$

But

$$A_{k+1} \supseteq \{A_k, z_{k+1} = e^{\delta} z_k, Z(\tau_{k+1}) = e^{\delta} Z(\tau_k), \tau_{k+1} < d\},$$

so $P(A_{k+1}) > 0$. By induction we get (2.3).

From (2.2) and (2.3) we conclude that $\exists\, 0 \le n \le M$ so that either

$P(\check{S}_n \le \varepsilon - 1) > 0$ or

(2.4). $$P(Z(\tau_n) = b, c_2(b - 1) - \mu b (N(\tau_n) - c_2)^- \ge \check{S}_n) > 0.$$

We claim that $x_N \ge 1 - \varepsilon$. The proof will be divided to 3 cases.

Remark. Throughout the proof all the inequalities will be understood to hold **with positive probability** and we will not repeat it.

Case 1: $P(\check{S}_n \le \varepsilon - 1) > 0$.

By using Lemma **A**.5, we have $S_M(0,1) \le S_N(0,\tau_n)$. We have already seen that $S_N(0,\tau_n) \le \check{S}_n$. So we conclude that $S_M(0,1) \le \varepsilon - 1$. It follows from the positivity of the payoff function g that $x_N \ge 1 - \varepsilon$ under the assumption of Case 1.

Next we assume that (2.4) holds and we split it into two cases based on the value of $N(\tau_n)$.

Case 2: $P(Z(\tau_n) = b,\ c_2(b - 1) - \mu b (N(\tau_n) - c_2)^- \ge \check{S}_n, N(\tau_n) \le 0) > 0$.

We denote $T = \tau_n$, $N^* = N(\tau_n)$ and $z = Z(1)$. We need to estimate $S_M(0,1)$.

We claim that

(2.5). $S_N(0,1) \le c_2\,(1 - \mu)\,b - c_2.$

We choose $\delta°$, $\beta > 0$ so that

$$e^{2\delta°} \le (1-\varepsilon/2)/(1 - 3\varepsilon/4),\ 1 - e^{-3\delta°} \le \mu,\ e^{3\delta°} - 1 \le \lambda \text{ and } e^{-(\beta+\delta°)} = 1-\mu.$$

By using our assumption on the process Z in the form of Lemma **A**.25 (i) with $\varepsilon/4$ instead of ε, we have $P(T_{\delta°} = 1 / F_T) > 0$. Now we use Lemma **A**.1 and (2.4) with $(N^* - c_2)^- = c_2 - N^*$ as follows from $N^* \le 0$ and $c_2 > 1-\varepsilon$.

We get

$$S_N(0,T) \le \check{S}_n \le c_2\,(b-1) - \mu\,(c_2 - N^*)\,b$$
$$= c_2\,(b-1) - \mu\,c_2\,b + \mu\,N^*\,b.$$

We apply again Lemma **A**.1 and we get

$$S_N(T,1) \le N^*\,(z - b) \le N^*\,(e^{-\delta°} - 1)\,b < -\mu\,N^*\,b,$$

because $T_{\delta°} = 1$, $N^* \le 0$, $z \ge e^{-\delta°}\,b$, and $\mu \ge 1 - e^{-3\delta°} > 1 - e^{-\delta°}$.

So we have

$$S_N(0,1) = S_N(0,T) + S_N(T,1)$$
$$\le c_2\,(b-1) - \mu\,c_2\,b$$
$$= c_2\,(1-\mu)\,b - c_2.$$

Thus we get (2.5).

Next we will estimate the payoff $g(Z_0^1)$ where $Z_0^1 = \int_0^1 Z(t)\,dt$.

We calculate

$$g(Z_0^1) \ge (1 - \mu) \int_{\varepsilon/4}^1 Z(t)\,dt - q$$

(2.6). $\geq (1-\mu)(1-\varepsilon/2)\, e^{-\delta^\circ} b - q,$

by $e^{-\delta^\circ} b \leq Z(t) \leq e^{\delta^\circ} b$, $T \leq t \leq T_{\delta^\circ}$ a.s, $T \leq \varepsilon/4$ and $T_{\delta^\circ} = 1$.

Finally we claim that $g(Z_0^1) - S_M(0,1) \geq 1-\varepsilon$.

To see this, we use (2.5) and (2.6)

$$g(Z_0^1) - S_M(0,1)$$

$$\geq (1-\mu)(1-\varepsilon/2)\, e^{-\delta^\circ} b - q - (c_2 (1-\mu) b - c_2)$$

$$= (1-\mu)(1-\frac{\varepsilon}{2} - e^{\delta^\circ}(1-\frac{3\varepsilon}{4}))e^{-\delta^\circ} b + \frac{q}{(1-\mu)b}(1-\mu)b - q + c_2$$

$$(\text{use } c_2 = 1 - \frac{3\varepsilon}{4} - \frac{q}{(1-\mu)b})$$

$$\geq (1-\mu)(1-\frac{\varepsilon}{2} - e^{2\delta^\circ}(1-\frac{3\varepsilon}{4}))e^{-\delta^\circ} b + q - q + c_2$$

$$\geq (1-\mu)(1-\varepsilon/2 - \frac{(1-\varepsilon/2)}{(1-\frac{3\varepsilon}{4})}(1-\frac{3\varepsilon}{4}))e^{-\delta^\circ} b + c_2$$

$$(\text{use } e^{2\delta^\circ} \leq (1-\varepsilon/2)/(1-3\varepsilon/4))$$

$$= (1-\mu)(1-\varepsilon/2 - (1-\varepsilon/2))\, e^{-\delta^\circ} b + c_2$$

$$= c_2 > 1-\varepsilon.$$

So we have $g(Z_0^1) - S_M(0,1) \geq 1-\varepsilon$.

It follows now that $\boldsymbol{x_N \geq 1-\varepsilon}$ under the assumption of Case 2.

Case 3: $P(Z(\tau_n) = b,\ c_2(b-1) - \mu b\,(N(\tau_n) - c_2)^- \geq \check{S}_n,\ N(\tau_n) > 0) > 0$.

Again we denote $T = \tau_n$ and $N^* = N(\tau_n)$. We introduce now some quantities to simplify the calculations. We choose δ°, $\beta > 0$ as in Case 2 and the following:

$$c_9 = (1-\mu)\, c_2\, e^{\delta^\circ} b - c_2,$$

$c_3 = c_2\,(b-1) \;-\mu\,(N^* \;- c_2)^-\, b$, and

$c_4 = N^*\,(e^{-\delta^\circ} \;-1)\,b \;-\mu_{\delta^\circ}\,N^*\,e^{-\delta^\circ}\,b.$

By Lemma **A**.1 and the assumption of Case 3, we get

(2.7). $$c_3 \geq S_M(0,\mathrm{T}).$$

The first calculation is to compare c_9 with c_3 and c_4.

We claim that under the assumption $N^* > 0$, we have

(2.8). $$c_9 \geq c_3 + c_4.$$

Proof of (2.8). We split the proof into two cases according to the relation between N^* and c_2.

First we assume that $N^* \geq c_2 \;> 0$.

Here $(N^* \;- c_2)^- \;= 0$. So we have

$$c_3 + c_4 = c_2\,(b-1) \;+ N^*\,(e^{-\delta^\circ} \;-1)\,b \;-\mu_{\delta^\circ}\,N^*\,e^{-\delta^\circ}\,b$$

$$\leq c_2\,(b-1) \;+\; c_2\,(e^{-\delta^\circ} \;-1)\,b \;-\mu_{\delta^\circ}\,c_2\,e^{-\delta^\circ}\,b$$

(use $N^* \geq c_2$ and $e^{-\delta^\circ} \;-1 < 0$)

$$= c_2\,(e^{-\delta^\circ} b \;-1) \;-\mu_{\delta^\circ}\,c_2\,e^{-\delta^\circ}\,b$$

$$= (1 \;-\mu_{\delta^\circ})\,c_2\,e^{-\delta^\circ} b - c_2$$

$$= e^{2\delta^\circ}\,(1 \;-\mu)\,c_2\,e^{-\delta^\circ}\,b - c_2$$

(use $1 \;-\mu_{\delta^\circ} = 1 - e^{2\delta^\circ}\,(\mu \;+ e^{-2\delta^\circ} \;-1) = 1 - e^{2\delta^\circ}\mu - 1 + e^{2\delta^\circ} \;= e^{2\delta^\circ}\,(1 \;-\mu)$)

$$= (1-\mu)\,c_2\,e^{\delta^\circ} b - c_2$$

$$= c_9.$$

To finish the proof of (2.8) we assume that $0 < N^* < c_2$.

Here $(N^* - c_2)^- = c_2 - N^*$. So we have

$$c_3 + c_4 = c_2\,(b-1) - \mu\,(c_2 - N^*)\,b + N^*\,(e^{-\delta^\circ} - 1)\,b - \mu_{\delta^\circ} N^*\,e^{-\delta^\circ}\,b$$

$$= (c_2 - N^*)\,b + N^*\,b + N^*\,(e^{-\delta^\circ} - 1)\,b - \mu\,(c_2 - N^*)\,b - \mu_{\delta^\circ} N^*\,e^{-\delta^\circ}\,b - c_2$$

$$= (1-\mu)\,(c_2 - N^*)\,b + N^*\,e^{-\delta^\circ}\,b - \mu_{\delta^\circ} N^*\,e^{-\delta^\circ}\,b - c_2$$

$$= (1-\mu)\,(c_2 - N^*)\,b + (1-\mu_{\delta^\circ})\,N^*\,e^{-\delta^\circ}\,b - c_2$$

$$= (1-\mu)\,(c_2 - N^*)\,b + e^{2\delta^\circ}(1-\mu)\,N^*\,e^{-\delta^\circ}\,b - c_2$$

(use $1 - \mu_{\delta^\circ} = e^{2\delta^\circ}\,(1-\mu)$)

$$\le (1-\mu)\,(c_2 - N^*)\,e^{\delta^\circ}\,b + (1-\mu)\,N^*\,e^{\delta^\circ}\,b - c_2$$

(use $1 - \mu > 0$ and $c_2 > N^*$)

$$= (1-\mu)\,(c_2 - N^* + N^*)\,e^{\delta^\circ}\,b - c_2$$

$$= (1-\mu)\,c_2\,e^{\delta^\circ}\,b - c_2$$

$$= c_9.$$

After establishing (2.8) we will estimate $S_M(0,1)$.

We start by defining a stopping time $\tau = T_{\delta^\circ}$. Then we use Lemma **A**.25 (iii) with $\varepsilon / 4$ instead of ε, and we get that

$$P(1 - \varepsilon/4 \le \tau < 1,\ Z(\tau) = e^{-\delta^\circ} b \,/\, F_T) > 0.$$

Next we denote $\tilde{N} = N(\tau)$ and define a stopping time L

$$L = \begin{cases} \tau & \text{if } \tilde{N} \le 0, \\ \tau_{\beta,\delta^\circ} & \text{if } \tilde{N} > 0. \end{cases}$$

We claim that on the event $\{N^* > 0\}$

(2.9). $$P(S_M(T,L) \le c_4 \,/\, F_T) > 0.$$

Proof of (2.9).

First we assume $\tilde{N} \le 0$, so $(\tilde{N} - N^*)^+ = 0$, $(\tilde{N} - N^*)^- = N^* - \tilde{N}$ and $L = \tau$.

To see (2.9), we use Lemma **A**.1 (i) with $\delta°$

$$S_M(T,L) = S_M(T,\tau) \le N^* (e^{-\delta°} - 1)\, b - \mu_{\delta°} (N^* - \tilde{N})\, e^{-\delta°} b$$

$$= c_4 + \mu_{\delta°} \tilde{N} e^{-\delta°} b \le c_4,$$

by $\tilde{N} \le 0$.

Now we assume $\tilde{N} > 0$ (namely $L = \tau_{\beta,\delta°}$). By using our assumption in the form of Lemma **A**.25 (ii) (with $\varepsilon / 4$ instead of ε) we get $P(L < 1, Z(L) = e^{-\beta} Z(\tau) / F_\tau) > 0$.

Using Lemma **A**.3 with β, $\delta°$, $e^{-(\beta+\delta°)} = 1 - \mu$, and $Z(L) = e^{-\beta} Z(\tau)$, we get

(2.10). $$S_M(\tau,L) \le \tilde{N} (e^{-\beta} - 1)\, e^{-\delta°} b.$$

Using Lemma **A**.1 (i) with $\delta°$ and (2.10) we have

(2.11). $$S_M(T,L) = S_M(T,\tau) + S_M(\tau,L)$$

$$\le N^* (e^{-\delta°} - 1)\, b - \lambda (\tilde{N} - N^*)^+ e^{-\delta°} b - \mu_{\delta°} (\tilde{N} - N^*)^- e^{-\delta°} b + \tilde{N} (e^{-\beta} - 1)\, e^{-\delta°} b.$$

We need to show that the RHS of (2.11) is less than c_4.

We do it first under the assumption $\tilde{N} < N^*$.

Here $(\tilde{N} - N^*)^+ = 0$, $(\tilde{N} - N^*)^- = N^* - \tilde{N}$, so (2.11) gives

$$S_M(T,L) \le N^* (e^{-\delta°} - 1)\, b - \mu_{\delta°} (N^* - \tilde{N})\, e^{-\delta°} b + \tilde{N} (e^{-\beta} - 1)\, e^{-\delta°} b$$

$$= c_4 + \mu_{\delta°} \tilde{N} e^{-\delta°} b + \tilde{N} (e^{-\beta} - 1)\, e^{-\delta°} b$$

$$= c_4 + (\mu_{\delta°} + e^{-\beta} - 1)\, \tilde{N} e^{-\delta°} b$$

$$= c_4 + (e^{-\beta} - (1 - \mu_{\delta°}))\, \tilde{N} e^{-\delta°} b$$

$$\le c_4,$$

by $1 - \mu_{\delta°} = e^{2\delta°} (1 - \mu) = e^{-\beta+\delta°} > e^{-\beta}$ and $\tilde{N} > 0$.

To finish the proof of (2.9), we work with the assumption $\tilde{N} > N^*$ which gives

$(\tilde{N} - N^*)^+ = \tilde{N} - N^*$, $(\tilde{N} - N^*)^- = 0$. So (2.11) gives

$$S_M(\mathrm{T,L}) \le N^* (e^{-\delta^\circ} - 1)\, b - \lambda\, (\tilde{N} - N^*)\, e^{-\delta^\circ}\, b + \tilde{N}\, (e^{-\beta} - 1)\, e^{-\delta^\circ}\, b$$

$$\le N^* (e^{-\delta^\circ} - 1)\, b + N^* (e^{-\beta} - 1)\, e^{-\delta^\circ}\, b$$

(use $\tilde{N} > N^* > 0$, $e^{-\beta} - 1 < 0$)

$$= N^* (e^{-\delta^\circ} - 1)\, b - (1 - e^{-\beta})\, N^*\, e^{-\delta^\circ}\, b$$

$$\le c_4,$$

by $1 - e^{-\beta} > \mu_{\delta^\circ}$ and $N^* > 0$.

We have established (2.9).

We will use (2.7) (2.8) and (2.9) to estimate $S_M(0,1)$. Under the assumption of Case 3 we have

(2.12). $$S_M(0,\mathrm{L}) = S_M(0,\mathrm{T}) + S_M(\mathrm{T,L})$$

$$\le c_3 + c_4 \le c_9.$$

By using Lemma A.5 and (2.12) we finally have

(2.13). $$S_M(0,1) = S_M(0,\mathrm{L}) + S_M(\mathrm{L},1)$$

$$\le c_9.$$

Next we will estimate the payoff function.

We observe that

(2.14). $$g(Z_0^1) = (1 - \mu) \int_{\mathrm{T}}^{\tau} Z(s)\, ds - q$$

$$\ge (1 - \mu)\, (1 - \varepsilon / 2)\, e^{-\delta^\circ}\, b - q,$$

by $e^{-\delta^\circ} Z(\mathrm{T}) \le Z(t) \le e^{\delta^\circ} Z(\mathrm{T})$, $\mathrm{T} \le t \le \tau$, $\mathrm{T} \le \varepsilon / 4$ and $1 - \varepsilon / 4 \le \tau$.

Finally we claim that $g(Z_0^1) - S_M(0,1) \ge 1 - \varepsilon$.

To see this, we use (2.13) and (2.14)

$$g(Z_0^1) - S_N(0,1)$$

$$\geq (1-\mu)\,(1-\varepsilon/2)\,e^{-\delta^\circ} b - (1-\mu)\,c_2\,e^{\delta^\circ} b - q + c_2$$

$$= (1-\mu)e^{-\delta^\circ} b(1 - \frac{\varepsilon}{2} - (1 - \frac{3\varepsilon}{4})e^{2\delta^\circ}) + \frac{q}{(1-\mu)b}(1-\mu)e^{\delta^\circ} b - q + c_2$$

(use $c_2 = 1 - 3\varepsilon/4 - q/[(1-\mu)\,b]$)

$$\geq (1-\mu)e^{-\delta} b(1-\varepsilon/2-(1-3\varepsilon/4)\frac{1-\varepsilon/2}{1-3\varepsilon/4}) + qe^{\delta^\circ} - q + c_2$$

(use $e^{2\delta^\circ} \leq (1-\varepsilon/2)/(1-3\varepsilon/4)$)

$$\geq (1-\mu)\,e^{-\delta^\circ}\,b\,(1-\varepsilon/2-(1-\varepsilon/2)) + c_2$$

$$= c_2 > 1 - \varepsilon.$$

So we get that $g(Z_0^1) - S_N(0,1) \geq 1 - \varepsilon$. It follows now that $\boldsymbol{x_N \geq 1 - \varepsilon}$ under the assumption of Case 3.

By combing the 3 cases we see that $\boldsymbol{x_N \geq 1 - \varepsilon}$. As we explained in the first paragraph of this chapter this leads to $b_E = 1$.

APPENDIX

A.1 The tools of the proof

In this appendix, we will prove four lemmas that will be useful for us.

The first lemma is **A**.1. It is a simple result of the integration by parts formula. It allows us to create a discrete–time version of the problem by looking at the hedging portfolio at the times where the price process Z is going up or down by a factor of e^{δ}, where $\delta > 0$ is related to the order λ and μ.

The second lemma is **A**.3. This lemma is an asymmetric extension of Lemma **A**.1.

The third lemma is **A**.5. We call this lemma the "closing lemma," since this lemma helps us to finish the proof after finding a stopping time in which our goal is achieved.

The fourth lemma is a main lemma. This lemma (Lemma **A**.6) shows how to deal with that discrete time version of our problem.

Recall that

$$S_N(\mathrm{T},\tau) = \int_{(\mathrm{T},\tau]} N(s)\,dZ(s) - \lambda \int_{(\mathrm{T},\tau]} Z(s)\,dN^+(s) - \mu \int_{(\mathrm{T},\tau]} Z(s)\,dN^-(s),$$

where $N \in FV$, and $\mathrm{T} \le \tau$ are stopping times. $S_N(\mathrm{T},\tau)$ is the capital gain generated by the portfolio N between the T and τ.

Lemma A.1 Let $N \in FV$ and $0 \le \mathrm{T} \le \tau \le 1$ be two stopping time so that $e^{-\delta} Z(\mathrm{T}) \le Z(t) \le e^{\delta} Z(\mathrm{T})$, $\mathrm{T} \le t \le \tau$ a.s, where $\delta > 0$ satisfies $1 - e^{-2\delta} \le \mu$ and $e^{2\delta} - 1 \le \lambda$, then

$$S_N(\mathrm{T},\tau) \le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})).$$

Furthermore,

(i) If $Z(\tau) = e^{-\delta} Z(\mathrm{T})$ then

$$S_M(\mathrm{T},\tau) \le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^+ - \mu_\delta\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^-$$

, where $\mu_\delta = e^{2\delta}\,(\mu + e^{-2\delta} - 1) \ge 0$.

(ii) If $Z(\tau) = e^{\delta} Z(\mathrm{T})$ then

$$S_M(\mathrm{T},\tau) \le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda_\delta\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^+ - \mu\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^-$$

, where $\lambda_\delta = e^{-2\delta}\,(\lambda + 1 - e^{2\delta}) \ge 0$.

Proof.

We will use the following notations:

$Z_* = \min\{Z(t)\colon \mathrm{T} \le t \le \tau\}$,

$Z^* = \max\{Z(t)\colon \mathrm{T} \le t \le \tau\}$,

$h_1 = N^+(\tau) - N^+(\mathrm{T})$,

$h_2 = N^-(\tau) - N^-(\mathrm{T})$,

$h_3 = h_1 - h_2 = N(\tau) - N(\mathrm{T})$.

By integration by part and the definitions of Z_*, Z^* we have:

$$S_M(\mathrm{T},\tau) = N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + \int_{(\mathrm{T},\tau]} (Z(\tau) - Z(s))\, dN^+(s)$$

$$- \int_{(\mathrm{T},\tau]} (Z(\tau) - Z(s))\, dN^-(s) - \lambda \int_{(\mathrm{T},\tau]} Z(s)\, dN^+(s) - \mu \int_{(\mathrm{T},\tau]} Z(s)\, dN^-(s)$$

$$= N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + \int_{(\mathrm{T},\tau]} (Z(\tau) - (1+\lambda)\, Z(s))\, dN^+(s)$$

$$- \int_{(\mathrm{T},\tau]} (Z(\tau) - (1 - \mu)\, Z(s))\, dN^-(s)$$

$$\le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + \int_{(\mathrm{T},\tau]} (Z(\tau) - (1 + \lambda)\, Z_*)\, dN^+(s)$$

$$- \int_{(T,\tau]} (Z(\tau) - Z^*(1-\mu))\, dN^-(s)$$

$$= N(T)\,(Z(\tau) - Z(T)) + (Z(\tau) \ - (1+\lambda)\,Z_*)\,h_1 - (Z(\tau) - (1-\mu)\,Z^*)\,h_2$$

$$= N(T)\,(Z(\tau) - Z(T)) + Z(\tau)\,h_3 \ - (1+\lambda)\,Z_*\,h_1 + (1-\mu)\,Z^*\,h_2$$

$$= N(T)\,(Z(\tau) - Z(T)) + Z(\tau)\,h_3 - (1+\lambda)\,Z_*\,h_1 + (1-\mu)\,Z^*\,(h_3 - h_1)$$

$$= N(T)\,(Z(\tau) - Z(T)) + (Z(\tau) + (1-\mu)\,Z^*)\,h_3 - ((1+\lambda)\,Z_* - (1-\mu)\,Z^*)\,h_1.$$

Since $e^{-\delta} Z(T) \ \le \ Z_* \le \ Z(T) \le Z^* \le e^{\delta} Z(T)$, we have

$$Z^* / Z_* \le e^{\delta} / e^{-\delta} \ = e^{2\delta} \le (1+\lambda) \wedge 1/(1-\mu).$$

This implies that $(1+\lambda)\,Z_* - \ (1-\mu)\,Z^* \ge 0$, so the last term is maximized when $h_1 = h_3^+$

(and then necessarily $h_2 = h_3^-$).

We conclude that

(A.2). $\quad S_M(T,\tau) \le N(T)\,(Z(\tau) - Z(T)) + (Z(\tau) \ - (1+\lambda)\,Z_*)\,h_3^+ \ - (Z(\tau) - (1-\mu)\,Z^*)\,h_3^-.$

The RHS of (**A**.2) is an increasing in Z^* (decreasing in Z_*), so we can use

$e^{\delta} Z(T)$ and $e^{-\delta} Z(T)$ instead of Z^* and Z_* respectively for estimation.

Since $Z(\tau) \ - (1+\lambda)\,Z_* \le 0$ and $Z(\tau) \ - (1-\mu)\,Z^* \ge 0$, we get immediately that

$$S_M(T,\tau) \ \le N(T)\,(Z(\tau) \ - Z(T)).$$

Proof of (i). We use (**A**.2) and the assumptions $Z_* = Z(\tau) = e^{-\delta} Z(T)$ and $Z^* \le e^{\delta} Z(T)$,

and we get

$$S_M(T,\tau) \le N(T)\,(Z(\tau) \ - Z(T)) - \lambda\, Z(\tau)\,h_3^+ \ - \ (Z(\tau) - (1-\mu)\,e^{\delta} Z(T))\,h_3^-$$

$$= N(T)\,(Z(\tau) \ - Z(T)) - \lambda\, Z(\tau)\,h_3^+ \ - \ (Z(\tau) - (1-\mu)\,e^{2\delta} Z(\tau))\,h_3^-$$

$$= N(T)\,(Z(\tau) \ - Z(T)) \ - \lambda\, Z(\tau)\,h_3^+ \ - (1 - e^{2\delta}\,(1-\mu))\,Z(\tau)\,h_3^-$$

$$= N(T)\,(Z(\tau) \ - Z(T)) - \lambda\, Z(\tau)\,(N(\tau) \ - N(T))^+ \ - \mu_\delta\, Z(\tau)\,(N(\tau) - N(T))^- .$$

Proof of (ii). We use (A.2) and the assumptions $Z^* = Z(\tau) = e^{\delta} Z(\mathrm{T})$ and $Z_* \geq e^{-\delta} Z(\mathrm{T})$, and we get

$$S_N(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + (Z(\tau) - (1+\lambda)\,e^{-\delta} Z(\mathrm{T}))\,h_3^+ - \mu\, Z(\tau) h_3^-$$

$$= N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + (Z(\tau) - e^{-2\delta}(1+\lambda)\,Z(\tau))\,h_3^+ - \mu\, Z(\tau) h_3^-$$

$$= N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + (1 - e^{-2\delta}(1+\lambda))\,Z(\tau)\,h_3^+ - \mu\, Z(\tau) h_3^-$$

$$= N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda_\delta\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^+ - \mu\, Z(\tau)\,(N(\tau) - N(\mathrm{T}))^- .$$

■

Lemma A.3 Let $N \in FV$ and $0 \leq \mathrm{T} \leq \tau \leq 1$ be two stopping time so that $e^{-\beta} Z(\mathrm{T}) \leq Z(t) \leq e^{\gamma} Z(\mathrm{T})$, $\mathrm{T} \leq t \leq \tau$ a.s, and $Z(\tau) = e^{-\beta} Z(\mathrm{T})$, where $\beta, \gamma > 0$ satisfy that $1 - e^{-(\beta+\gamma)} \leq \mu$, then

$$S_N(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})).$$

Proof.

We will use the notation from Lemma A.1.

By integration by part, the definitions of Z_*, Z^* we have:

$$S_N(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + Z(\tau)\,h_3 - (1+\lambda)\,Z_*\,h_1 + (1-\mu)\,Z^*\,h_2$$

$$= N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + (Z(\tau) + Z^*(1-\mu))\,h_3 - (Z_*(1+\lambda) - Z^*(1-\mu))\,h_1.$$

Since $e^{-\beta} Z(\mathrm{T}) \leq Z_* \leq Z(\mathrm{T}) \leq Z^* \leq e^{\gamma} Z(\mathrm{T})$, we have

$$Z^* / Z_* \leq e^{\gamma} / e^{-\beta} = e^{\beta+\gamma} \leq 1 / (1-\mu).$$

This implies that $(1+\lambda)\,Z_* - (1-\mu)\,Z^* \geq 0$, so the last term is maximized when $h_1 = h_3^+$ (and then necessarily $h_2 = h_3^-$).

We conclude that

$$\text{(A.4).}\quad S_N(\mathrm{T},\tau) \leq N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) + (Z(\tau) - (1+\lambda)\,Z_*)\,h_3^+ - (Z(\tau) - (1-\mu)\,Z^*)\,h_3^- .$$

The RHS of (A.4) is an increasing in Z^*, so we can use $e^\gamma Z(\mathrm{T})$ instead of Z^* for estimation.

We use (A.4) and the assumptions $Z_* = Z(\tau) = e^{-\beta} Z(\mathrm{T})$ and $Z^* \le e^\gamma Z(\mathrm{T})$, and we get

$$S_N(\mathrm{T},\tau) \le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda\, Z(\tau)\, h_3^+ - (Z(\tau) - (1-\mu)\, e^\gamma Z(\mathrm{T}))\, h_3^-$$

$$= N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) - \lambda\, Z(\tau)\, h_3^+ - (e^{-\beta} - e^\gamma (1-\mu))\, Z(\mathrm{T})\, h_3^-$$

$$\le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})),$$

by $e^{-\beta} - e^\gamma(1-\mu) \ge 0$. ■

Next we state and prove the "closing lemma."

Lemma A.5 Let $N \in FV$ and $0 \le \mathrm{T} \le 1$ be a stopping time. Then

$$P(S_M(\mathrm{T},1) \le 0 \,/\, F_\mathrm{T}) > 0.$$

Proof. Let $\alpha > 0$ satisfies $1 - e^{-3\alpha} \le \mu$ and $e^{3\alpha} - 1 \le \lambda$, let $0 \le \mathrm{T} \le 1$ be a stopping time and let $\tau = \mathrm{T}_\alpha$.

Case 0: $N(\mathrm{T}) = 0$. We may use Lemma A.1 with $N(\mathrm{T}) = 0$ and we get

$$S_M(\mathrm{T},\tau) \le N(\mathrm{T})\,(Z(\tau) - Z(\mathrm{T})) = 0.$$

The result now follows because $\mathrm{P}(\tau = 1 \,/\, F_\mathrm{T}) > 0$ via Assumption 1.8 (ii).

Case 1: $N(\mathrm{T}) > 0$.

We can assume that $\tau < 1$, $Z(\tau) = e^{-\alpha} Z(\mathrm{T})$ and $\tau_\alpha = 1$ with positive probability, by Assumption 1.8. We denote that $p = Z(\mathrm{T})$, $N^* = N(\mathrm{T})$, $\tilde{N} = N(\tau)$ and $z = Z(1)$ for notational simplicity.

We calculate using Lemma A.1

$$S_M(\mathrm{T},1) \le N^*(e^{-\alpha} - 1)\,p - (\tilde{N} - N^*)^-\, \mu_\alpha\, e^{-\alpha} p - (\tilde{N} - N^*)^+\, \lambda\, e^{-\alpha} p + \tilde{N}(z - e^{-\alpha} p).$$

Now we split case 1 into 3 sub cases:

Case 1 (i): $\tilde{N} \geq N^* > 0$.

Here $(\tilde{N} - N^*)^- = 0$, and $(\tilde{N} - N^*)^+ = \tilde{N} - N^*$. So

$$S_M(\mathrm{T},1) \leq N^* (e^{-\alpha} - 1)\,p - (\tilde{N} - N^*)\,\lambda\, e^{-\alpha} p + \tilde{N}\,(z - e^{-\alpha} p)$$

$$\leq N^* (e^{-\alpha} - 1)\,p - (\tilde{N} - N^*)\,\lambda\, e^{-\alpha} p + \tilde{N}\,(1 - e^{-\alpha})\,p$$

(use $z \leq p$ and $\tilde{N} > 0$)

$$= (\tilde{N} - N^*)\,(1 - e^{-\alpha})\,p - (\tilde{N} - N^*)\,\lambda\, e^{-\alpha} p$$

$$= (\tilde{N} - N^*)\,(1 - e^{-\alpha} - \lambda\, e^{-\alpha})\,p \;\leq\; 0,$$

by $\tilde{N} \geq N^*$ and $\lambda\, e^{-\alpha} \geq (e^{3\alpha} - 1)\, e^{-\alpha} = e^{2\alpha} - e^{-\alpha} > 1 - e^{-\alpha}$.

Case 1 (ii): $N^* > \tilde{N} \geq 0$.

Here $(\tilde{N} - N^*)^+ = 0$, and $(\tilde{N} - N^*)^- = N^* - \tilde{N}$. So

$$S_M(\mathrm{T},1) \leq N^* (e^{-\alpha} - 1)\,p - (N^* - \tilde{N})\,\mu_\alpha\, e^{-\alpha} p + \tilde{N}\,(z - e^{-\alpha} p)$$

$$\leq N^* (e^{-\alpha} - 1)\,p - (N^* - \tilde{N})\,\mu_\alpha\, e^{-\alpha} p + \tilde{N}\,(1 - e^{-\alpha})\,p$$

(use $z \leq p$ and $N^* > \tilde{N} \geq 0$)

$$= (N^* - \tilde{N})\,(e^{-\alpha} - 1)\,p - (N^* - \tilde{N})\,\mu_\alpha\, e^{-\alpha} p$$

$$= (N^* - \tilde{N})\,(e^{-\alpha} - 1 - \mu_\alpha\, e^{-\alpha})\,p < 0,$$

by $e^{-\alpha} - 1 < 0$, $N^* - \tilde{N} > 0$ and $\mu_\alpha > 0$.

Case 1 (iii): $N^* > 0 > \tilde{N}$.

Here $(\tilde{N} - N^*)^+ = 0$, $(\tilde{N} - N^*)^- = N^* - \tilde{N}$. So

$$S_M(\mathrm{T},1) \leq N^* (e^{-\alpha} - 1)\,p - (N^* - \tilde{N})\,\mu_\alpha\, e^{-\alpha} p + \tilde{N}\,(z - e^{-\alpha} p)$$

$$\leq N^* (e^{-\alpha} - 1)\,p - (N^* - \tilde{N})\,\mu_\alpha\, e^{-\alpha} p + \tilde{N}\,(e^{-\alpha} - 1)\, e^{-\alpha} p$$

(use $\tilde{N} < 0$ and $z \geq e^{-2\alpha} p$)

$$\leq N^{*}(e^{-\alpha} - 1 - \mu_\alpha e^{-\alpha})\,p + \tilde{N}(e^{-\alpha} - 1 + \mu_\alpha)\,e^{-\alpha} p < 0,$$

by $N^{*} > 0 > \tilde{N}$, $e^{-\alpha} - 1 < 0$ and

$$\mu_\alpha = e^{2\alpha}(\mu + e^{-2\alpha} - 1) \geq e^{2\alpha}(1 - e^{-3\alpha} + e^{-2\alpha} - 1) = 1 - e^{-\alpha} > 0.$$

Case 2: $N(\mathrm{T}) > 0$.

We can assume that $\tau < 1$, $Z(\tau) = e^{\alpha} Z(\mathrm{T})$ and $\tau_\alpha = 1$ with positive probability, by Assumption 1.8. We denote that $p = Z(\mathrm{T})$, $N^{*} = N(\mathrm{T})$, $\tilde{N} = N(\tau)$ and $z = Z(1)$ for notational simplicity.

We calculate using Lemma **A**.1

$$S_N(\mathrm{T},1) \leq N^{*}(e^{\alpha} - 1)\,p - (\tilde{N} - N^{*})^{-}\,\mu\, e^{\alpha} p - (\tilde{N} - N^{*})^{+}\,\lambda_\alpha e^{\alpha} p + \tilde{N}(z - e^{\alpha} p).$$

Now we split Case 2 into 3 sub cases:

Case 2 (i): $\tilde{N} > 0 > N^{*}$.

Here $(\tilde{N} - N^{*})^{-} = 0$, and $(\tilde{N} - N^{*})^{+} = \tilde{N} - N^{*}$. So

$$S_N(\mathrm{T},1) \leq N^{*}(e^{\alpha} - 1)\,p - (\tilde{N} - N^{*})\,\lambda_\alpha e^{\alpha} p + \tilde{N}(z - e^{\alpha} p)$$

$$\leq N^{*}(e^{\alpha} - 1)\,p - (\tilde{N} - N^{*})\,\lambda_\alpha e^{\alpha} p + \tilde{N}(e^{\alpha} - 1)\,e^{\alpha} p$$

(use $\tilde{N} > 0$ and $z \leq e^{2\alpha} p$)

$$= N^{*}(e^{\alpha} - 1 + \lambda_\alpha e^{\alpha})\,p + \tilde{N}(e^{\alpha} - 1 - \lambda_\alpha)\,e^{\alpha} p < 0,$$

by $\tilde{N} > 0 > N^{*}$, $e^{\alpha} - 1 > 0$ and

$$\lambda_\alpha = e^{-2\alpha}(\lambda + 1 - e^{2\alpha}) \geq e^{-2\alpha}(e^{3\alpha} - 1 + 1 - e^{2\alpha}) = e^{\alpha} - 1 > 0.$$

Case 2 (ii): $0 \geq \tilde{N} > N^{*}$.

Here $(\tilde{N} - N^{*})^{-} = 0$, and $(\tilde{N} - N^{*})^{+} = \tilde{N} - N^{*}$. So

$$S_N(\mathrm{T},1) \leq N^{*}(e^{\alpha} - 1)\,p - (\tilde{N} - N^{*})\,\lambda_\alpha e^{\alpha} p + \tilde{N}(z - e^{\alpha} p)$$

$\leq N^* (e^\alpha - 1) p - (\tilde{N} - N^*) \lambda_\alpha e^\alpha p + \tilde{N} (1 - e^\alpha) p$

(use $\tilde{N} \leq 0$ and $z \geq p$)

$= (\tilde{N} - N^*) (1 - e^\alpha) p - (\tilde{N} - N^*) \lambda_\alpha e^\alpha p$

$= (\tilde{N} - N^*) (1 - e^\alpha - \lambda_\alpha e^\alpha) p < 0,$

by $\tilde{N} > N^*$, $1 - e^\alpha < 0$ and $\lambda_\alpha > 0$.

Case 2 (iii): $0 > N^* \geq \tilde{N}$.

Here $(\tilde{N} - N^*)^+ = 0$, and $(\tilde{N} - N^*)^- = N^* - \tilde{N}$. So

$S_M(\mathrm{T},1) \leq N^* (e^\alpha - 1) p - (N^* - \tilde{N}) \mu e^\alpha p + \tilde{N} (z - e^\alpha p)$

$\leq N^* (e^\alpha - 1) p - (N^* - \tilde{N}) \mu e^\alpha p + \tilde{N} (1 - e^\alpha) p$

(use $\tilde{N} < 0$ and $z \geq p$)

$= (N^* - \tilde{N}) (e^\alpha - 1) p - (N^* - \tilde{N}) \mu e^\alpha p$

$= (N^* - \tilde{N}) (e^\alpha - 1 - \mu e^\alpha) p \leq 0,$

by $N^* \geq \tilde{N}$, and $\mu e^\alpha \geq (1 - e^{-3\alpha}) e^\alpha = e^\alpha - e^{-2\alpha} > e^\alpha - 1$.

We conclude that $P(S_M(\mathrm{T},1) \leq 0 / F_\mathrm{T}) > 0$. ■

We will start with the setup of Lemma **A**.6.

Let $0 < \varepsilon < 1$ and let $\delta > 0$ so that $1 - e^{-2\delta} \leq \mu$, $e^{2\delta} - 1 \leq \lambda$ as in Lemma **A**.1.

Let $b > 1 / (1 - \mu)$ so that $c_2 \equiv 1 - 3\varepsilon / 4 - q / [(1 - \mu) b] > 1 - \varepsilon$ where q is the strike price of the option.

Finally recall the notation of Lemma **A**.1:

$\mu_\delta = e^{2\delta} (\mu + e^{-2\delta} - 1)$, and $\lambda_\delta = e^{-2\delta} (\lambda + 1 - e^{2\delta})$.

Lemma A.6 There exists an integer $M = M(\varepsilon,\delta) \geq 1$ and a sequence of measurable functions $Z_k : \boldsymbol{R}^k \to \boldsymbol{R}^+$, $k \geq 0$, so that $Z_0 = 1$, and for every sequence of numbers N_k, $k \geq 0$, we have

$$\frac{Z_k(N_0,\ldots,N_{k-1})}{Z_{k-1}(N_0,\ldots,N_{k-2})} = e^{\pm\delta}, \quad k \geq 1 \text{ such that if } \inf_{0\leq n\leq M} S_n > \varepsilon - 1,$$

then $\exists\ 0 \leq n \leq M$ for which

(i). $\quad z_n = b$, and

(ii). $\quad c_2\,(b - 1) - \mu\, b\,(N_n - c_2)^- \geq S_n$,

where $S_n = \sum_{k=0}^{n-1} N_k (z_{k+1} - z_k)$

$$- \{z_{k+1} = e^{\delta} z_k\}[\lambda_\delta\,(N_{k+1} - N_k)^+ + \mu\,(N_{k+1} - N_k)^-]\, z_{k+1}$$

$$- \{z_{k+1} = e^{-\delta} z_k\}[\lambda\,(N_{k+1} - N_k)^+ + \mu_\delta\,(N_{k+1} - N_k)^-]\, z_{k+1},$$

and $z_k = Z_k(N_0,\ldots,N_{k-1})$, $k \geq 0$.

Proof. Let $a > 0$ so that $c_1 \equiv 1 - \varepsilon + a < c_2$.

Let f: $\boldsymbol{R}^+ \to (0,1)$ be a strictly increasing function so that

$$c_1 = f(0) < f(\infty) = c_2.$$

We define $z_k = Z_k(N_0,\ldots,N_{k-1})$, $k \geq 0$, as follows:

$$z_0 = 1,$$

$$z_{k+1} = e^{\delta} z_k \text{ if } N_k < f(z_k),\ k \geq 0,$$

$$= e^{-\delta} z_k \text{ if } N_k \geq f(z_k),\ k \geq 0.$$

Here we explain our basic ideas. Our starting point is LS (1997) paper. In addition to that paper we are using an important new idea. We define a dominant portfolio. This new portfolio helps us in checking the inequality of Lemma A.6. When the stock price is going up the number of shares in this new portfolio ($N^1{}_k$ in what follows) is more than the number of shares in the original one, N_k. Furthermore, when $z_n = b$ the sequence $N^1{}_k$ is increasing for $k \le n-1$, while at the time n there is a reduction in the number of shares of the new portfolio as we choose $N^1{}_n = N_n$.

We need some sequences ($k \ge 0$) for simplicity.

Let $L_k = \{N_k < f(z_k)\}$.

Let $l_k = \{N_k \ge f(z_k)\}$.

Let $H_k = (N_{k+1} - N_k)^+$.

Let $h_k = (N_{k+1} - N_k)^-$.

Let $\alpha_k = N_k\,(z_{k+1} - z_k) - L_k\,[\lambda_\delta\, H_k + \mu\, h_k]\, z_{k+1} - l_k\,[\lambda\, H_k + \mu_\delta\, h_k]\, z_{k+1}$.

With this notation we can now rewrite the sequence $\{S_n\}$ as:

$S_0 = 0$.

$$S_n = \sum_{k=0}^{n-1} \alpha_k, n \ge 1.$$

Let $q_k = \max\{1 \le i \le k: L_{i-1} = l_i\}$, (= 0 if the set is empty).

Let $N^1{}_k = \max\{N_i: q_k \le i \le k\}$ if $L_k = 1$,

$= \min\{N_i: q_k \le i \le k\}$ if $l_k = 1$.

Next, we define the following sequence:

$S^2{}_0 = 0$.

$$S_n^2 = \sum_{k=0}^{n-1} N_k^1 (z_{k+1} - z_k), n \ge 1.$$

Claim 1 $S^2_n \geq S_n, n \geq 0$.

Proof of Claim 1. If $L_k = 1$ (namely $z_{k+1} > z_k$) then $N^1_k = \max\{N_i: q_k \leq i \leq k\} \geq N_k$ and if $l_k = 1$ (namely $z_{k+1} < z_k$) then $N^1_k = \min\{N_i: q_k \leq i \leq k\} \leq N_k$.

We can conclude that $N^1_k (z_{k+1} - z_k) \geq N_k (z_{k+1} - z_k), k \geq 0$, and $S^2_n \geq S_n, n \geq 0$.

Claim 2 The sequence $\{S^2_n\}$ satisfies the following:

(A.7) (i) If $m > k$ and $z_m < z_k$, then

$$S^2_m - S^2_k \leq f(z_m)(z_m - z_k) < c_1 (z_m - z_k).$$

(ii) If $m > k$ and $z_m > z_k$, then

$$S^2_m - S^2_k \leq f(z_m)(z_m - z_k) < c_2 (z_m - z_k).$$

(iii) If $m \geq k$ and $z_m = z_k$, then $S^2_m - S^2_k \leq 0$.

(iv) Assume (w.l.o.g.) $\exists\, 0 < r, s \in \boldsymbol{N}$ so that $a = e^{-r\delta}, b = e^{s\delta}$. If $a \leq z_k \leq b, 0 \leq k \leq n$, then

$$S^2_n \leq c_2 (b - a) - (n - (s + r))\, \theta / 2,$$

where

$$\theta = \inf_{-r \leq k < s} \{(f(e^{(k+1)\delta}) - f(e^{k\delta}))(e^{(k+1)\delta} - e^{k\delta})\}.$$

Proof of Claim 2.

We first observe that $N^1_k \leq f(z_k)$ if $L_k = 1$.

To see this, we calculate:

If $L_k = 1$ and $q_k \leq i \leq k$ then $f(z_i) \leq f(z_k)$, and

$$N^1_k = \max\{N_i: q_k \leq i \leq k\}$$

$$\leq \max\{f(z_i): q_k \leq i \leq k\}$$

$$= f(z_k).$$

We also observe that $N^1{}_k \geq f(z_k)$ if $l_k = 1$.

Similarly, we calculate:

If $l_k = 1$ and $q_k \leq i \leq k$ then $f(z_i) \geq f(z_k)$, and

$$\begin{aligned} N^1{}_k &= \min\{N_i\colon q_k \leq i \leq k\} \\ &\geq \min\{f(z_i)\colon q_k \leq i \leq k\} \\ &= f(z_k). \end{aligned}$$

So for every $k \geq 0$ we have

$$\begin{aligned} S^2{}_{k+1} - S^2{}_k &= N^1{}_k\,(z_{k+1} - z_k) \\ &\leq f(z_k)\,(z_{k+1} - z_k) \end{aligned}$$

(**A**.8). $$\leq f(z_m)\,(z_{k+1} - z_k) < c_1\,(z_{k+1} - z_k), \text{ if } z_{k+1} < z_k \text{ and } z_m \leq z_k,$$

$$\text{or} \leq f(z_m)\,(z_{k+1} - z_k) < c_2\,(z_{k+1} - z_k), \text{ if } z_{k+1} > z_k \text{ and } z_m \geq z_k.$$

We also observe that

If $z_k = z_{m+1}$ and $z_{k+1} = z_m$, $k \neq m$, then

(**A**.9) $$\begin{aligned} &(S^2{}_{k+1} - S^2{}_k) + (S^2{}_{m+1} - S^2{}_m) \\ &\leq -[f(z_{k+1}) - f(z_k)]\,(z_{k+1} - z_k) < 0. \end{aligned}$$

To see (**A**.9), we calculate

$$\begin{aligned} &(S^2{}_{k+1} - S^2{}_k) + (S^2{}_{m+1} - S^2{}_m) \\ &\leq f(z_k)\,(z_{k+1} - z_k) + f(z_m)\,(z_{m+1} - z_m) \\ &= f(z_k)\,(z_{k+1} - z_k) + f(z_{k+1})\,(z_k - z_{k+1}) \\ &= -[f(z_{k+1}) - f(z_k)]\,(z_{k+1} - z_k). \end{aligned}$$

Next we define, for every integer v and $0 \le k < m$,

$$u(v,k,m) = \sum_{n=k}^{m-1} \{(z_n, z_{n+1}) = (e^{v\delta}, e^{(v+1)\delta})\},$$
$$d(v,k,m) = \sum_{n=k}^{m-1} \{(z_n, z_{n+1}) = (e^{(v+1)\delta}, e^{v\delta})\},$$

where we identify sets with their indication functions. In word, $u(v, k, m)$ and $d(v, k, m)$ are the number of changes $e^{v\delta} \uparrow e^{(v+1)\delta}$, $e^{(v+1)\delta} \downarrow e^{v\delta}$, respectively, of the sequence $(z_k, \ldots, z_m)$.

Now we verify (**A**.7) (i). We define for each v that satisfies $z_k > e^{v\delta} \ge z_m$:

$$n(v) = \min\{n \ge k: (z_n, z_{n+1}) = (e^{(v+1)\delta}, e^{v\delta})\}.$$

We get

$$\begin{aligned}
S_m^2 - S_k^2 &= \sum_{n=k}^{m-1} (S_{n+1}^2 - S_n^2) \\
&< \sum_{z_k > e^{v\delta} \ge z_m} S_{n(v)+1}^2 - S_{n(v)}^2 \\
&= \sum_{z_k > e^{v\delta} \ge z_m} N_{n(v)}^1 (z_{n(v)+1} - z_{n(v)}) \\
&\le \sum_{z_k > e^{v\delta} \ge z_m} f(z_m)(z_{n(v)+1} - z_{n(v)}) \\
&= f(z_m)\,(z_m - z_k) < c_1\,(z_m - z_k).
\end{aligned}$$

The first inequality follows from (**A**.9) and the fact that $z_k > e^{v\delta} \ge z_m$ implies $d(v, k, m) = u(v, k, m) + 1$, while that $e^{v\delta} \ge z_k$ or $z_m > e^{v\delta}$ implies $d(v, k, m) = u(v, k, m)$. The second inequality follows from (**A**.8).

The proof of (**A**.7) (ii) is similar to the proof of (**A**.7) (i) and will be omitted.

We prove (**A**.7) (iii). Since each v is either $e^{v\delta} \ge z_k = z_m$ or $z_k = z_m > e^{v\delta}$, this implies $d(v, k, m) = u(v, k, m)$, and $S^2_m - S^2_k \le 0$.

Next we prove (**A**.7) (iv). We have

$$n = \sum_{t=-r}^{s-1} d(t,0,n) + u(t,0,n)$$
$$\leq \sum_{t=-r}^{s-1} 2\,[d(t,0,n) \wedge u(t,0,n)] + 1.$$

So

$$\text{(A.10)} \qquad \sum_{t=-r}^{s-1} d(t,0,n) \wedge u(t,0,n) \geq \frac{n-(r+s)}{2}.$$

Next we define

$$A_t = \{0 \leq k \leq n-1 : (z_k, z_{k+1}) = (e^{t\delta}, e^{(t+1)\delta}) \text{ or } (e^{(t+1)\delta}, e^{t\delta})\}, \; -r \leq t \leq s-1.$$

By using (**A**.8) and (**A**.9) we get

$$\sum_{k \in A_t} (S_{k+1}^2 - S_k^2) \leq c_2 (e^{(t+1)\delta} - e^{t\delta}) - [d(t,0,n) \wedge u(t,0,n)]\theta.$$

Using (**A**.10) we now have

$$S_n^2 = \sum_{t=-r}^{s-1} \sum_{k \in A_t} (S_{k+1}^2 - S_k^2)$$
$$\leq c_2 (e^{s\delta} - e^{-r\delta}) - (n-(s+r))\,\theta / 2$$
$$= c_2 (b - a) - (n-(s+r))\,\theta / 2.$$

This is the end of the proof of Claim 2.

We will use now (**A**.7) to prove Lemma **A**.6.

We define: $M = [(s+r) + 2\,(c_2\,(b - a) + 1 - \varepsilon) / \theta\,] + 1$.

First we need to divide Lemma **A**.6 into three cases.

Case 1: If $a = e^{-r\delta} < z_k < e^{s\delta} = b$, $k \geq 0$, then by **A**.7 (iv), we have

$$S_n \leq S^2_n \leq c_2 (b - a) - (n - (s + r))\,\theta / 2$$

$$< \varepsilon - 1,$$

whenever $n > (s + r) + 2\,(c_2\,(b - a) + 1 - \varepsilon) / \theta$.

So $\inf_{0 \leq n \leq M} S_n \leq \varepsilon - 1$ in this case.

Case 2: If there is $0 \leq n \leq M$ such that $z_n = a$, then by **A**.7 (i) and $a < \varepsilon / 4$, we have

$$\begin{aligned} S_n \leq S^2_n &\leq c_1 (a - 1) \\ &= (1 - \varepsilon + a)(a - 1) \\ &= a - a\,\varepsilon + a^2 - 1 + \varepsilon - a \\ &= \varepsilon - 1 - a\,(\varepsilon - a) \\ &< \varepsilon - 1. \end{aligned}$$

So $\inf_{0 \leq n \leq M} S_n \leq \varepsilon - 1$ in this case.

Case 3: If there is $0 \leq n \leq M - 1$ such that $z_{n+1} = b$, then

we claim that $\boldsymbol{S_{n+1} \leq c_2\,(b - 1) - \mu\, b\,(N_{n+1} - c_2)^-}$.

We will call this inequality the "*main inequality*."

We need to go through some very long and painful steps to achieve the main inequality.

First we need some notations.

Let $H^1{}_k = (N^1{}_{k+1} - N^1{}_k)^+$, $0 \leq k \leq n-1$.

Let $h^1{}_k = (N^1{}_{k+1} - N^1{}_k)^-$, $0 \leq k \leq n-1$.

Let $\beta_k = N^1{}_k\,(z_{k+1} - z_k) - [L_k\,\lambda_\delta\,H^1{}_k + l_k\,\mu_\delta\,h^1{}_k]\,z_{k+1}$, $0 \leq k \leq n-1$.

Next we compare β_k with α_k. This is a basic element of the proof.

Claim 3 $\beta_k \geq \alpha_k$, $0 \leq k \leq n-1$.

Proof of Claim 3. We divide the proof into the two cases.

Case (i): $L_k = 1$.

Here $z_{k+1} = e^{\delta} z_k$ and $N_k \leq N^1{}_k < f(z_k)$.

We observe that $N^1{}_{k+1} \geq N^1{}_k$, and $N^1{}_{k+1} = N_{k+1} \vee N^1{}_k$, $0 \leq k \leq n-1$.

To see these, we calculate that

If $N_{k+1} \geq f(z_{k+1})$ then $q_{k+1} = k+1$,

$$N^1{}_{k+1} = N_{k+1} \geq f(z_{k+1}) > f(z_k) > N^1{}_k, \text{ and}$$

$$N^1{}_{k+1} = N_{k+1} \vee N^1{}_k,$$

If $N_{k+1} < f(z_{k+1})$ then $q_{k+1} = q_k$ and

$$N^1{}_{k+1} = \max\{N_i\colon q_k \leq i \leq k+1\} = N_{k+1} \vee N^1{}_k \geq N^1{}_k.$$

We also observe that $H^1{}_k \leq H_k$, $0 \leq k \leq n-1$.

To see this, we calculate

$$\begin{aligned} H^1{}_k &= (N^1{}_{k+1} - N^1{}_k)^+ \\ &= \{(N_{k+1} \vee N^1{}_k) - N^1{}_k\}^+ \\ &= (N_{k+1} - N^1{}_k)^+ \\ &\leq (N_{k+1} - N_k)^+ = H_k, \end{aligned}$$

by $N_k \leq N^1{}_k$.

So we can easily get the claim:

$$\begin{aligned} \beta_k &= N^1{}_k (z_{k+1} - z_k) - \lambda_\delta H^1{}_k z_{k+1} \\ &\geq N_k (z_{k+1} - z_k) - \lambda_\delta H_k z_{k+1} - \mu\, h_k z_{k+1} = \alpha_k,\ 0 \leq k \leq n-1, \end{aligned}$$

by $L_k = 1$, $z_{k+1} > z_k$, $N^1{}_k \geq N_k$ and $H^1{}_k \leq H_k$.

The second case is very similar to the first one.

Case (ii): $l_k = 1$.

Here $z_{k+1} = e^{-\delta} z_k$, $N_k \geq N^1{}_k \geq f(z_k)$.

We can easily have that $N^1{}_{k+1} \leq N^1{}_k$, and $N^1{}_{k+1} = N^1{}_k \wedge N_{k+1}$, $0 \leq k \leq n-1$ by a similar calculation.

We observe that $h^1{}_k \leq h_k$, $0 \leq k \leq n-1$.

To see this, we calculate

$$\begin{aligned} h^1{}_k &= (N^1{}_{k+1} - N^1{}_k)^- \\ &= \{(N^1{}_k \wedge N_{k+1}) - N^1{}_k\}^- \\ &= (N_{k+1} - N^1{}_k)^- \\ &\leq (N_{k+1} - N_k)^- = h_k, \end{aligned}$$

by $N_k \geq N^1{}_k$.

So we easily have the claim:

$$\begin{aligned} \beta_k &= N^1{}_k (z_{k+1} - z_k) - \mu_\delta h^1{}_k z_{k+1} \\ &\geq N_k (z_{k+1} - z_k) - \mu_\delta h_k z_{k+1} - \lambda H_k z_{k+1} = \alpha_k, \ 0 \leq k \leq n-1, \end{aligned}$$

by $l_k = 1$, $z_{k+1} < z_k$, $N^1{}_k \leq N_k$ and $h^1{}_k \leq h_k$.

Thus we conclude that $\beta_k \geq \alpha_k$, $0 \leq k \leq n-1$.

This is the end of the proof of Claim 3.

The next claim is of the fundamental importance.

Claim 4 $S^1{}_{n+1} \geq S_{n+1}$.

Here $n = \inf\{k: z_{k+1} = b\}$ and $S^1_{n+1} = \gamma_n + \sum_{k=0}^{n-1} \beta_k$

, where $\gamma_n = N^1_n (z_{n+1} - z_n) - [\lambda_\delta (N_{n+1} - N^1_n)^+ + \mu (N_{n+1} - N^1_n)^-] z_{n+1}$.

We have: $L_n = 1$, $z_{n+1} > z_n$ and $N_n \leq N^1_n < f(z_n)$.

Proof of Claim 4. We need to split the proof into three cases.

The first case is a trivial one.

Case (I): $N_{n+1} \geq N^1_n$.

Here $(N_{n+1} - N^1_n)^- = 0$.

We observe $\gamma_n \geq \alpha_n$:

$$\begin{aligned}\gamma_n &= N^1_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N^1_n)^+ z_{n+1} \\ &\geq N_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N_n)^+ z_{n+1} - \mu (N_{n+1} - N^1_n)^- = \alpha_n,\end{aligned}$$

by $L_n = 1$, $N^1_n \geq N_n$ and $z_{n+1} > z_n$.

We will use this to prove our claim:

$$S^1_{n+1} = \sum_{k=0}^{n-1} \beta_k + \gamma_n \geq \sum_{k=0}^{n-1} \alpha_k + \alpha_n = \sum_{k=0}^{n} \alpha_k = S_{n+1}.$$

We get that $\boldsymbol{S^1_{n+1} \geq S_{n+1}}$ **if** $N_{n+1} \geq N^1_n$.

Before we discuss the other 2 cases, we need one more notation.

Let $p_n = \max\{q_n \leq k \leq n \colon N_k = N^1_n\}$.

In words: p_n is the last time when the number of shares in the portfolio $\{N_k \colon q_n \leq k \leq n\}$ is maximized. In particular: $N^1_n = N_{p_n}$.

Case (II): $N_{n+1} < N^1_n$ and $p_n = n$.

Here $N^1_n = N_n$, $(N_{n+1} - N^1_n)^+ = (N_{n+1} - N_n)^+ = 0$, and $L_n = 1$.

It is similar to the previous case and it is also easy.

We observe that

$$\gamma_n = N'_n (z_{n+1} - z_n) - \mu (N_{n+1} - N'_n)^- z_{n+1}$$

$$= N_n (z_{n+1} - z_n) - \mu (N_{n+1} - N_n)^- z_{n+1} = \alpha_n.$$

So we have

$$S^1_{n+1} = \sum_{k=0}^{n-1} \beta_k + \alpha_n \geq \sum_{k=0}^{n-1} \alpha_k + \alpha_n = \sum_{k=0}^{n} \alpha_k = S_{n+1}.$$

We get that $\boldsymbol{S'_{n+1}} \geq \boldsymbol{S_{n+1}}$ in this case.

Next we go to the hardest case. It has a long proof.

Case (III): $N_{n+1} < N'_n$ and $p_n < n$.

Here $(N_{n+1} - N'_n)^+ = 0$, $\{k: p_n \leq k \leq n-1\} \neq \phi$, and when $q_n \leq k \leq n$ we have:

$$L_k = 1, q_k = q_n.$$

First we observe the property N'_k related with p_n.

By definition of p_n, when $p_n \leq k \leq n$ we have:

$$N'_k = N_{p_n} \text{ and } h'_k = H'_k = 0.$$

So we have

(A.11). $$\beta_k = N_{p_n} (z_{k+1} - z_k), \quad p_n \leq k \leq n-1.$$

Next, we need some notations. Let

$$\tilde{N}_k = \max\{N_i: k \leq i \leq n\}, \quad p_n \leq k \leq n,$$

$$h^*_k = (\tilde{N}_{k+1} - \tilde{N}_k)^-, \qquad p_n \leq k \leq n-1,$$

$$a_k = \tilde{N}_k (z_{k+1} - z_k) - \mu\, h^*_k z_{k+1}, \quad p_n \leq k \leq n-1.$$

We first observe that when $p_n \leq k \leq n-1$ we have

$$\tilde{N}_k = N_k \vee \tilde{N}_{k+1} \ \geq \ \tilde{N}_{k+1},$$

$$h^*_k = \ \tilde{N}_k \ - \ \tilde{N}_{k+1} \geq 0, \text{ and}$$

(A.12). $$a_k = \tilde{N}_k\,(z_{k+1} - z_k) - \mu\,(\tilde{N}_k - \tilde{N}_{k+1})\ z_{k+1}.$$

Next we observe that:

$$N^1_k \ \geq \ \tilde{N}_k, \qquad p_n \leq \ k \ \leq \ n.$$

To see this, we calculate

$$\begin{aligned} N^1_k &= \max\{N_i\colon q_n \ \leq i \leq k\} \\ &\geq \ N_{p_n} = N^1_n \\ &= \max\{N_i\colon q_n \ \leq i \leq n\} \\ &\geq \max\{N_i\colon k \ \leq i \leq n\} \\ &= \tilde{N}_k, \end{aligned}$$

by $q_k \ = q_n \ \leq \ p_n \ \leq \ k \ \leq \ n.$

Next we observe that

$$h^*_k \ \leq \ h_k, \qquad p_n \leq \ k \leq n-1.$$

Indeed,

$$\begin{aligned} h^*_k &= (\tilde{N}_{k+1} \ - \ \tilde{N}_k)^- \\ &= \{\tilde{N}_{k+1} - (N_k \ \vee \ \tilde{N}_{k+1})\}^- \\ &= (\tilde{N}_{k+1} - N_k)^- \\ &\leq (N_{k+1} - N_k)^- \ = \ h_k, \end{aligned}$$

by $\tilde{N}_{k+1} \geq N_{k+1}$.

We also observe that

$$a_k \ \geq \ \alpha_k, \qquad p_n \leq \ k \leq n-1.$$

Indeed, $a_k = \tilde{N}_k (z_{k+1} - z_k) - \mu\, h^*_k z_{k+1}$

$$\geq N_k (z_{k+1} - z_k) - \mu\, h_k z_{k+1} - \lambda_\delta H_k z_{k+1} = \alpha_k,$$

by $L_k = 1$, $\tilde{N}_k \geq N_k$, $z_{k+1} > z_k$ and $h^*_k \leq h_k$.

Next we define a temporary summation that is useful for us.

Let $\breve{S}_{n+1} = \sum_{k=0}^{p_n-1} \beta_k + \sum_{k=p_n}^{n-1} a_k + \alpha_n$.

First, we observe that $\breve{S}_{n+1} \geq S_{n+1}$.

To see this, we calculate

$$\breve{S}_{n+1} = \sum_{k=0}^{p_n-1} \beta_k + \sum_{k=p_n}^{n-1} a_k + \alpha_n \geq \sum_{k=0}^{n-1} \alpha_k + \sum_{k=p_n}^{n-1} \alpha_k + \alpha_n = \sum_{k=0}^{n} \alpha_k = S_{n+1}.$$

To finish Case (III), all we need is to show that: $S^1_{n+1} \geq \breve{S}_{n+1}$.

We start with some basic simplifications. First we observe

$$\gamma_n = N^1_n (z_{n+1} - z_n) - \mu\, (N_{n+1} - N^1_n)^- z_{n+1}$$

$$= N^1_n (z_{n+1} - z_n) - \mu\, (N^1_n - N_{n+1})\, z_{n+1},$$

because in our case $N_{n+1} < N^1_n$.

We also observe

$$\alpha_n = N_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N_n)^+ z_{n+1} - \mu\, (N_{n+1} - N_n)^- z_{n+1}$$

$$\leq N_n (z_{n+1} - z_n) - \mu\, (N_{n+1} - N_n)^- z_{n+1},$$

because $L_n = 1$.

So we have

$$\gamma_n - \alpha_n \geq (N^1_n - N_n)(z_{n+1} - z_n) - \mu\, [(N^1_n - N_{n+1}) - (N_{n+1} - N_n)^-]\, z_{n+1}$$

$$\geq (N^1_n - N_n)(z_{n+1} - z_n) - \mu\, (N^1_n - N_n)\, z_{n+1}$$

(**A**.13) $\qquad = (\widetilde{N}_{p_n} - \tilde{N}_n)(z_{n+1} - z_n) - \mu(\widetilde{N}_{p_n} - \tilde{N}_n) z_{n+1},$

where the second inequality follows from (**A**.26), and the equality follows from $\tilde{N}_n = N_n$ and from $N^1_{p_n} \leq N^1{}_n = N_{p_n} \leq \widetilde{N}_{p_n} \leq N^1_{p_n}$, so $\widetilde{N}_{p_n} = N_{p_n} = N^1{}_n$.

Next we observe

$$\beta_k - a_k = \widetilde{N}_{p_n}(z_{k+1} - z_k) - (\tilde{N}_k(z_{k+1} - z_k) - \mu(\tilde{N}_k - \tilde{N}_{k+1}) z_{k+1})$$

(**A**.14) $\qquad = (\widetilde{N}_{p_n} - \tilde{N}_k)(z_{k+1} - z_k) + \mu(\tilde{N}_k - \tilde{N}_{k+1}) z_{k+1}, \qquad p_n \leq k \leq n-1,$

by (**A**.11), (**A**.12) and $\widetilde{N}_{p_n} = N_{p_n}$.

In particular $\beta_{p_n} - a_{p_n} = \mu(\widetilde{N}_{p_n} - \widetilde{N}_{p_n+1}) z_{p_n+1}$.

Now we use (**A**.13), (**A**.14) and $\widetilde{N}_{p_n} - \widetilde{N}_k = \sum_{i=p_n}^{k-1}(\widetilde{N}_i - \widetilde{N}_{i+1}), \qquad p_n + 1 \leq k \leq n,$

and we prove that $S^1{}_{n+1} \geq \check{S}_{n+1}$. Indeed,

$$S^1_{n+1} - \breve{S}_{n+1} = (\sum_{k=0}^{n-1}\beta_k + \gamma_n) - (\sum_{k=0}^{p_n-1}\beta_k + \sum_{k=p_n}^{n-1} a_k + \alpha_n)$$

$$= \sum_{k=p_n}^{n-1}(\beta_k - a_k) + (\gamma_n - \alpha_n)$$

$$\geq \sum_{k=p_n+1}^{n-1}(\widetilde{N}_{p_n} - \widetilde{N}_k)(z_{k+1} - z_k) + \mu\sum_{k=p_n}^{n-1}(\widetilde{N}_k - \widetilde{N}_{k+1})z_{k+1}$$

$$+ (\widetilde{N}_{p_n} - \widetilde{N}_n)(z_{n+1} - z_n) - \mu(\widetilde{N}_{p_n} - \widetilde{N}_n)z_{n+1}$$

$$= \sum_{k=p_n+1}^{n}(\widetilde{N}_{p_n} - \widetilde{N}_k)(z_{k+1} - z_k) + \mu\sum_{k=p_n}^{n-1}(\widetilde{N}_k - \widetilde{N}_{k+1})z_{k+1} - \mu\sum_{k=p_n}^{n-1}(\widetilde{N}_k - \widetilde{N}_{k+1})z_{n+1}$$

$$= \sum_{k=p_n+1}^{n}(\widetilde{N}_{p_n} - \widetilde{N}_k)(z_{k+1} - z_k) - \mu\sum_{k=p_n}^{n-1}(\widetilde{N}_k - \widetilde{N}_{k+1})(z_{n+1} - z_{k+1})$$

$$= \sum_{k=p_n+1}^{n}\sum_{i=p_n}^{k-1}(\widetilde{N}_i - \widetilde{N}_{i+1})(z_{k+1} - z_k) - \mu\sum_{k=p_n}^{n-1}(\widetilde{N}_k - \widetilde{N}_{k+1})(z_{n+1} - z_{k+1})$$

$$= \sum_{i=p_n}^{n-1}\sum_{k=i+1}^{n}(z_{k+1} - z_k)(\widetilde{N}_i - \widetilde{N}_{i+1}) - \mu\sum_{k=p_n}^{n-1}(\widetilde{N}_k - \widetilde{N}_{k+1})(z_{n+1} - z_{k+1})$$

(use index change)

$$= \sum_{i=p_n}^{n-1}(z_{n+1} - z_{i+1})(\widetilde{N}_i - \widetilde{N}_{i+1}) - \mu \sum_{k=p_n}^{n-1}(z_{n+1} - z_{k+1})(\widetilde{N}_k - \widetilde{N}_{k+1})$$

$$= \sum_{k=p_n}^{n-1}(z_{n+1} - z_{k+1})(\widetilde{N}_k - \widetilde{N}_{k+1}) - \mu \sum_{k=p_n}^{n-1}(z_{n+1} - z_{k+1})(\widetilde{N}_k - \widetilde{N}_{k+1})$$

$$= (1-\mu)\sum_{k=p_n}^{n-1}(z_{n+1} - z_{k+1})(\widetilde{N}_k - \widetilde{N}_{k+1}) \geq 0$$

, as follows from $z_{n+1} > z_{k+1}$, $\tilde{N}_k \geq \tilde{N}_{k+1}$, $p_n \leq k \leq n-1$.

We get that $\boldsymbol{S^1_{n+1} \geq \check{S}_{n+1} \geq S_{n+1}}$ **if** $N_{n+1} < N^1{}_n$ and $p_n < n$.

So we conclude that $\boldsymbol{S^1_{n+1} \geq S_{n+1}}$ **in all the 3 cases.**

We call $\{N^1{}_k\}$ the "dominant portfolio" of $\{N_k\}$.

This is the end of the proof of Claim 4.

Finally, we can prove our main inequality.

We will divide this proof into two cases according to the value of z_{q_n}.

Case (I): $z_{q_n} \leq 1$.

First we need to divide $S^1{}_{n+1}$ into two sub summations: $S^1{}_{n+1} = W_o + (S^1{}_{n+1} - W_o)$.

There exists o so that $z_o = 1$, $o \geq q_n$ and we define $W_o = \sum_{k=0}^{o-1} \beta_k$.

We get

(A.15) $$W_o \leq \sum_{k=0}^{o-1} N^1_k (z_{k+1} - z_k) = S^2_o \leq 0,$$

by $\beta_k = N^1{}_k (z_{k+1} - z_k) - [L_k \lambda_\delta H^1{}_k + l_k \mu_\delta h^1{}_k] z_{k+1}$,

and (A.7) (iii) with $z_m = z_k = 1$, $m = o$, and $k = 0$.

As for the other summation (the more interesting one)

$$S^1{}_{n+1} - W_o$$

$$= \sum_{k=o}^{n-1} \beta_k + \gamma_n$$

$$= \sum_{k=o}^{n-1} N_k^1 (z_{k+1} - z_k) - \lambda_\delta H_k^1 z_{k+1}$$

$$+ N^1{}_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N^1{}_n)^+ z_{n+1} - \mu (N_{n+1} - N^1{}_n)^- z_{n+1}$$

$$\leq \sum_{k=o}^{n} N_n^1 (z_{k+1} - z_k) - \mu (N_{n+1} - N_n^1)^- z_{n+1}$$

(use $N^1{}_n \geq N^1{}_k$ and $z_{k+1} > z_k$ when $q_n \leq o \leq k \leq n-1$)

$$= N^1{}_n (z_{n+1} - z_o) - \mu (N_{n+1} - N^1{}_n)^- z_{n+1}$$

(**A**.16) $$= N^1{}_n (b-1) - \mu (N_{n+1} - N^1{}_n)^- b,$$

by $z_{n+1} = b$, $z_o = 1$.

From (**A**.15) and (**A**.16) we get

(**A**.17). $$S^1{}_{n+1} \leq N^1{}_n (b-1) - \mu (N_{n+1} - N^1{}_n)^- b.$$

We call this the "**basic inequality**."

We will use the basic inequality to prove our main inequality.

Recall that $b > 1/(1-\mu) > 1$, namely: $b - 1 - \mu b > 0$.

Since $c_2 = f(\infty) > f(z_n) > N^1{}_n$, we only need to consider three sub cases.

Sub case (I.i): $N_{n+1} \geq c_2 > N^1{}_n$.

Here $(N_{n+1} - N^1{}_n)^- = 0$ and $(N_{n+1} - c_2)^- = 0$.

Using (**A**.17) with $(N_{n+1} - N^1{}_n)^- = 0$, we have

$$S^1{}_{n+1} \leq N^1{}_n (b-1)$$

$$\leq c_2 (b-1) - \mu N^1{}_n (N_{n+1} - c_2)^-,$$

by $b > 1$, $c_2 > N^1_n$ and $(N_{n+1} - c_2)^- = 0$.

Sub case (I.ii): $c_2 > N_{n+1} \geq N^1_n$.

Here $(N_{n+1} - N^1_n)^- = 0$ and $(N_{n+1} - c_2)^- = c_2 - N_{n+1}$.

Using (**A**.17) with $(N_{n+1} - N^1_n)^- = 0$, we get

$$S^1_{n+1} \leq N^1_n (b-1)$$

$$\leq N^1_n (b-1) + (c_2 - N^1_n)(b - 1 - \mu b)$$

(use $c_2 > N^1_n$ and $b - 1 - \mu b > 0$)

$$= c_2 (b-1) - \mu b (c_2 - N^1_n)$$

$$\leq c_2 (b-1) - \mu b (c_2 - N_{n+1})$$

(use $N_{n+1} \geq N^1_n$)

$$= c_2 (b-1) - \mu b (N_{n+1} - c_2)^-.$$

Sub case (I.iii): $c_2 > N^1_n > N_{n+1}$.

Here $(N_{n+1} - N^1_n)^- = N^1_n - N_{n+1}$, and $(N_{n+1} - c_2)^- = c_2 - N_{n+1}$.

Using (**A**.17) with $(N_{n+1} - N^1_n)^- = N^1_n - N_{n+1}$, we also get

$$S^1_{n+1} \leq N^1_n (b-1) - \mu (N^1_n - N_{n+1}) b$$

$$\leq N^1_n (b-1) - \mu (N^1_n - N_{n+1}) b + (c_2 - N^1_n)(b - 1 - \mu b)$$

(use $c_2 > N^1_n$ and $b - 1 - \mu b > 0$)

$$= c_2 (b-1) - \mu b (c_2 - N_{n+1})$$

$$= c_2 (b-1) - \mu b (N_{n+1} - c_2)^-.$$

So $S^1_{n+1} \leq c_2 (b-1) - \mu b (N_{n+1} - c_2)^-$ if $z_{q_n} \leq 1$.

We conclude that $\boldsymbol{S_{n+1} \leq S^1_{n+1} \leq c_2 (b-1) - \mu b (N_{n+1} - c_2)^-}$ **if** $z_{q_n} \leq 1$.

Case (II): $z_{q_n} > 1$.

We need another long and painful steps for this result.

First we observe simple facts.

Since $z_{n-1} = e^{-2\delta} b$, $z_n = e^{-\delta} b$ and $z_{n+1} = b$, $q_n \le n-1$.

By definition of q_n and $N^1{}_k$, $L_k = 1$, $z_{k+1} > z_k$ and $N^1{}_{k+1} \ge N^1{}_k$ when $q_n \le k \le n-1$.

Next we need new notations.

We denote $N_* = f(z_{q_n})$ for notational simplicity.

Let $N^0{}_k = N^1{}_k \vee N_*$, $\quad q_n \le k \le n$.

We first observe that

$$N^0{}_k = N^1{}_k \vee N_* \le f(z_k) \vee f(z_{q_n}) = f(z_k), \qquad q_n \le k \le n,$$

by $z_k \ge z_{q_n}$.

We also observe that

$$N^0{}_{k+1} = N^1{}_{k+1} \vee N_* \ge N^1{}_k \vee N_* = N^0{}_k, \quad q_n \le k \le n-1,$$

by $N^1{}_{k+1} \ge N^1{}_k$.

Thus $(N^0{}_{k+1} - N^0{}_k)^+ = N^0{}_{k+1} - N^0{}_k \ge 0, \qquad q_n \le k \le n-1.$

Let $H^\circ{}_k = N^0{}_{k+1} - N^0{}_k, \qquad q_n \le k \le n-1.$

Let $\beta^0_{q_n-1} = N^1_{q_n-1}\,(z_{q_n} - z_{q_n-1}) - \mu_\delta\,(N^1_{q_n-1} - N_*)\, z_{q_n}$,

Let $\beta^\circ{}_k = N^0{}_k\,(z_{k+1} - z_k) - \lambda_\delta\, H^0 z_{k+1}$, $\; q_n \le k \le n-1$.

We compare $\beta^\circ{}_k$ with β_k, it is a basic element of this step.

Claim 5 $\beta^\circ{}_k \ge \beta_k$, $q_n - 1 \le k \le n-1$.

Proof of Claim 5. We first consider $q_n - 1$ case:

$$\beta^{0}_{q_n-1} = N^{1}_{q_n-1}(z_{q_n} - z_{q_n-1}) - \mu_\delta (N^{1}_{q_n-1} - N_*)\, z_{q_n}$$

$$\geq N^{1}_{q_n-1}(z_{q_n} - z_{q_n-1}) - \mu_\delta (N^{1}_{q_n-1} - N^{1}_{q_n})\, z_{q_n} = \beta_{q_n-1},$$

by $N^{1}_{q_n} < N_* < N^{1}_{q_n-1}$.

For general case, we need some algebra:

$$H^{\circ}_k = (N^{1}_{k+1} \vee N_*) - (N^{1}_k \vee N_*)$$

$$\leq N^{1}_{k+1} - N^{1}_k = H^{1}_k, \quad q_n \leq k \leq n-1,$$

by (A.27) with $N^{1}_{k+1} \geq N^{1}_k$.

So we get

$$\beta^{\circ}_k = N^{\circ}_k (z_{k+1} - z_k) - \lambda_\delta H^{\circ}_k z_{k+1}$$

$$\geq N^{1}_k (z_{k+1} - z_k) - \lambda_\delta H^{1}_k z_{k+1} = \beta_k, \quad q_n \leq k \leq n-1,$$

by $L_k = 1$, $N^{\circ}_k \geq N^{1}_k$, $z_{k+1} > z_k$ and $H^{1}_k \geq H^{\circ}_k$.

This is the end of the proof of Claim 5.

Next we define new summation.

Let $S^{0}_{n+1} = \sum_{k=0}^{q_n-2} \beta_k + \sum_{k=q_n-1}^{n-1} \beta^{0}_k + \gamma^{0}_n$,
where $\gamma^{\circ}_n = N^{\circ}_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N^{\circ}_n)^+ z_{n+1} - \mu\, (N_{n+1} - N^{\circ}_n)^- z_{n+1}$.

Claim 6 $S^{\circ}_{n+1} \geq S^{1}_{n+1}$.

Proof of Claim 6. We divide the claim proof into two cases.

Case (i): $N^{1}_n \geq N_*$ or $N_{n+1} \geq N_*$.

We first observe that $\gamma^{\circ}_n = \gamma_n$ if $N^{1}_n \geq N_*$.

Here $N^{\circ}_n = N^{1}_n \vee N_* = N^{1}_n$. So we have

$$\gamma^{\circ}_n = N^{\circ}_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N^{\circ}_n)^+ z_{n+1} - \mu\, (N_{n+1} - N^{\circ}_n)^- z_{n+1}$$

$= N^1{}_n\,(z_{n+1} - z_n) - \lambda_\delta\,(N_{n+1} - N^1{}_n)^+ z_{n+1} - \mu\,(N_{n+1} - N^1{}_n)^- z_{n+1} = \gamma_n.$

We also observe that $\gamma^\circ{}_n \geq \gamma_n$ if $N_{n+1} \geq N_* > N^1{}_n$.

Here $N^0{}_n = N^1{}_n \vee N_* = N_*$, and $(N_{n+1} - N^0{}_n)^- = 0$. So we also have

$$\gamma^\circ{}_n = N^0{}_n\,(z_{n+1} - z_n) - \lambda_\delta\,(N_{n+1} - N^0{}_n)^+ z_{n+1}$$

$$\geq N^1{}_n\,(z_{n+1} - z_n) - \lambda_\delta\,(N_{n+1} - N^1{}_n)^+ z_{n+1} \geq \gamma_n,$$

by $N^0{}_n \geq N^1{}_n$ and $z_{n+1} > z_n$.

We get that $\gamma^\circ{}_n \geq \gamma_n$, if $N^1{}_n \geq N_*$ or $N_{n+1} \geq N_*$.

We will use this fact to prove Claim 6:

$$S^0_{n+1} = \sum_{k=0}^{q_n-2} \beta_k + \sum_{k=q_n-1}^{n-1} \beta^0_k + \gamma^0_n \geq \sum_{k=0}^{n-1} \beta_k + \gamma_n = S^1_{n+1}.$$

So we get $\boldsymbol{S^0{}_{n+1}} \geq \boldsymbol{S^1{}_{n+1}}$ if $N^1{}_n \geq N_*$ or $N_{n+1} \geq N_*$.

Case (ii): $(N^1{}_n \vee N_{n+1}) < N_*$.

Here $N^0{}_k = N^1{}_k \vee N_* = N_*, \quad q_n \leq k \leq n.$

It is a hard part of the proof of Claim 6.

Since $q_n \leq n - 1$, $n \geq q_n + 1$.

We denote that $q = q_n$ and $p = z_{q_n}$ for notational simplicity in this case.

By the definition of q, $z_{q-1} = z_{q+1} = e^\delta p$, $z_{q+2} = e^{2\delta} p$.

If $n = q + 1$ then let $N^1{}_{q+2} = N^1{}_{q+1} \vee N_{q+2}$.

By the definition of q and $(N^1{}_n \vee N_{n+1}) < N_*$, $N^1{}_{q-1} > N_* = f(z_q) > N^1{}_{q+2} \geq N^1{}_{q+1} \geq N^1{}_q$.

First we need two constants.

Let $V_n = N^1{}_{q-1}\,(z_q - z_{q-1}) + N_*\,(z_{q+2} - z_q) - \mu_\delta\,(N^1{}_{q-1} - N_*)\,z_q - \mu\,(N_* - N^1{}_{q+2})\,z_{q+2}$

(**A**.18) $\quad = N^1_{q-1}(1-e^{\delta})\,p + N_*(e^{2\delta}-1)\,p - \mu_\delta(N^1_{q-1}-N_*)\,p - \mu\,(N_*-N^1_{q+2})\,e^{2\delta}p.$

Let $U_n = N^1_{q-1}(z_q - z_{q-1}) + N^1_q(z_{q+1}-z_q) + N^1_{q+1}(z_{q+2}-z_{q+1})$

$$- \mu_\delta(N^1_{q-1}-N^1_q)\,z_q - \lambda_\delta(N^1_{q+1}-N^1_q)\,z_{q+1} - \lambda_\delta(N^1_{q+2}-N^1_{q+1})\,z_{q+2}.$$

We start with some basic simplifications. By $N^1_{q+1} \geq N^1_q$ and $z_{q+1} > z_q$ we get

$$U_n \leq N^1_{q-1}(z_q - z_{q-1}) + N^1_{q+1}(z_{q+2}-z_q) - \mu_\delta(N^1_{q-1}-N^1_q)\,z_q$$

(**A**.19) $\quad = N^1_{q-1}(1-e^{\delta})\,p + N^1_{q+1}(e^{2\delta}-1)\,p - \mu_\delta(N^1_{q-1}-N^1_q)\,p.$

Here we explain the meaning of these constants.

If $n = q+1$ then $V_n = \beta^\circ_{n-2} + \beta^\circ_{n-1} + \gamma^\circ_n + \mu\,(N_{n+1}-N^1_n)^-\,z_{n+1}$, and $U_n = \beta_{n-2} + \beta_{n-1} + \gamma_n + \mu\,(N_{n+1}-N^1_n)^-\,z_{n+1}$.

If $n \geq q+2$ then $V_n = \beta^\circ_{q-1} + \beta^\circ_q + \beta^\circ_{q+1} - \mu\,(N^1_{q+2}-N^\circ_{q+1})^-\,z_{q+2}$, and $U_n = \beta_{q-1} + \beta_q + \beta_{q+1}$.

We claim $V_n \geq U_n$.

To see this, we use (**A**.18) and (**A**.19)

$$V_n - U_n \geq N^1_{q-1}(1-e^{\delta})\,p + N_*(e^{2\delta}-1)\,p - \mu_\delta(N^1_{q-1}-N_*)\,p - \mu\,(N_*-N^1_{q+2})\,e^{2\delta}p$$

$$- \;[N^1_{q-1}(1-e^{\delta})\,p + N^1_{q+1}(e^{2\delta}-1)\,p - \mu_\delta(N^1_{q-1}-N^1_q)\,p]$$

$$= (N_*-N^1_{q+1})(e^{2\delta}-1)\,p + \mu_\delta(N_*-N^1_q)\,p - \mu\,(N_*-N^1_{q+2})\,e^{2\delta}p$$

$$\geq (N_*-N^1_{q+2})(e^{2\delta}-1)\,p + \mu_\delta(N_*-N^1_{q+2})\,p - \mu\,(N_*-N^1_{q+2})\,e^{2\delta}p$$

(use $N_* > N^1_{q+2} \geq N^1_{q+1} \geq N^1_q$)

$$= (N_*-N^1_{q+2})\,p\,(e^{2\delta}-1+\mu_\delta-\mu\,e^{2\delta}) = 0,$$

by $(1-\mu_\delta) = 1 - e^{2\delta}(\mu + e^{-2\delta} - 1) = 1 - e^{2\delta}(\mu-1) - 1 = e^{2\delta}(1-\mu)$.

We will use this to prove that $\boldsymbol{S}^\circ_{n+1} \geq \boldsymbol{S}^1_{n+1}$.

We split this case into two sub cases.

Sub case (ii.a): $n = q + 1$.

Here $N_{n+1} = N_{q+2}$, $N^1{}_n = N^1{}_{q+1}$.

We observe

$$\begin{aligned} S^0_{n+1} &= \sum_{k=0}^{q-2} \beta_k + V_n - \mu(N_{n+1} - N^1_n)^- z_{n+1} \\ &\geq \sum_{k=0}^{q-2} \beta_k + U_n - \mu(N_{n+1} - N^1_n)^- z_{n+1} \\ &= S^1_{n+1}. \end{aligned}$$

We get that $S^\circ{}_{n+1} \geq S^1{}_{n+1}$ in this sub case.

Sub case (ii.b): $n \geq q + 2$.

First we define the following temporary summation

$$\breve{S}_{n+1} = \sum_{k=0}^{q-2} \beta_k + V_n + \sum_{k=q+2}^{n-1} \beta_k + \gamma_n.$$

Here we explain the meaning of $\breve{S}_{n+1}$.

$\breve{S}_{n+1}$ is a capital gain from portfolio $\{\tilde{N}_k\}$, where $\tilde{N}_k = N^1{}_k$, $k \geq 0$ except $\tilde{N}_q = \tilde{N}_{q+1} = N_*$.

When $q \leq k \leq n - 1$, $L_k = 1$, and the dominant portfolio of $\{\tilde{N}_k\}$ is $\{N^\circ{}_k\}$ as follows from $\max\{\tilde{N}_i\colon q \leq i \leq k\} = N_* \vee N^1{}_k = N^\circ{}_k$. By the same argument of Case (I) we conclude $S^\circ{}_{n+1} \geq \breve{S}_{n+1}$.

We also observe that $\breve{S}_{n+1} \geq S^1{}_{n+1}$.

To see this, we calculate:

$$\breve{S}_{n+1} = \sum_{k=0}^{q-2} \beta_k + V_n + \sum_{k=q+2}^{n-1} \beta_k + \gamma_n$$
$$\geq \sum_{k=0}^{q-2} \beta_k + U_n + \sum_{k=q+2}^{n-1} \beta_k + \gamma_n$$
$$= \sum_{k=0}^{n-1} \beta_k + \gamma_n = S^1_{n+1}.$$

So we get $S^o{}_{n+1} \geq \breve{S}_{n+1} \geq S^1{}_{n+1}$ in this sub case.

We conclude that $\boldsymbol{S^o{}_{n+1} \geq S^1{}_{n+1} \geq S_{n+1}}$ in all cases.

This is the end of the proof of Claim 6.

We use q_n instead of q after this point.

Finally, we can prove the main inequality in this case.

Recall that we claim that $\boldsymbol{S_{n+1} \leq S^o{}_{n+1} \leq c_2\,(b-1) - \mu\, b\,(N_{n+1} - c_2)^-}$.

First we need to divide $S^o{}_{n+1}$ into two sub summations: $S^o{}_{n+1} = W_{q_n} + (S^o{}_{n+1} - W_{q_n})$.

We define $W_{q_n} = \sum_{k=0}^{q_n-2} \beta_k + \beta^0_{q_n-1}$.

We get

$$\text{(A.20)}\quad W_{q_n} \leq \sum_{k=0}^{q_n-1} N^1_k (z_{k+1} - z_k) = S^2_{q_n} \leq f(z_{q_n})(z_{q_n} - 1),$$

by $\beta^0_{q_n-1} = N^1_{q_n-1}\,(z_{q_n} - z_{q_n-1}) - \mu_\delta\,(N^1_{q_n-1} - N_*)\, z_{q_n}$ and (**A**.7) (ii) with $m = q_n$, $k = 0$,

$z_m = z_{q_n}$ and $z_k = 1$.

As for the other summation (the more interesting one)

$$S^0_{n+1} - W_{q_n} = \sum_{k=q_n}^{n-1} \beta^0_k + \gamma^0_n$$

$$= \sum_{k=q_n}^{n-1} N^0_k (z_{k+1} - z_k) - \lambda_\delta H^0_k z_{k+1}$$

$$+ N^\circ_n (z_{n+1} - z_n) - \lambda_\delta (N_{n+1} - N^\circ_n)^+ z_{n+1} - \mu (N_{n+1} - N^\circ_n)^- z_{n+1}$$

$$\leq \sum_{k=q_n}^{n} N^0_n (z_{k+1} - z_k) - \mu (N_{n+1} - N^0_n)^- z_{n+1}$$

(use $N^\circ_n \geq N^\circ_k$ and $z_{k+1} > z_k$, when $q_n \leq k \leq n$)

$$= N^\circ_n (z_{n+1} - z_{q_n}) - \mu (N_{n+1} - N^\circ_n)^- z_{n+1}$$

(**A**.21) $$= N^\circ_n (b - z_{q_n}) - \mu (N_{n+1} - N^\circ_n)^- b,$$

by $z_{n+1} = b$.

From (**A**.20) and (**A**.21) we get

$$S^\circ_{n+1} = (S^\circ_{n+1} - W_{q_n}) + W_{q_n}$$

$$\leq N^\circ_n (b - z_{q_n}) - \mu (N_{n+1} - N^\circ_n)^- b + f(z_{q_n}) (z_{q_n} - 1)$$

(**A**.22). $$\leq N^\circ_n (b - 1) - \mu (N_{n+1} - N^\circ_n)^- b,$$

by $N^\circ_n \geq N_* = f(z_{q_n})$ and $b > z_{q_n} > 1$.

We will use this basic inequality to prove our main inequality.

Recall that $b > 1 / (1 - \mu) > 1$, namely: $b - 1 - \mu b > 0$.

Since $c_2 = f(\infty) > f(z_n) > N^\circ_n$, we only consider three sub cases.

Sub case (II.i): $N_{n+1} \geq c_2 > N^\circ_n$.

Using (**A**.22) with $(N_{n+1} - N^\circ_n)^- = 0$, we get

$$S^\circ_{n+1} \leq N^\circ_n (b - 1)$$

$$\leq c_2 (b - 1) - \mu b (N_{n+1} - c_2)^-,$$

by $b > 1$, $c_2 > N^{\circ}_n$ and $(N_{n+1} - c_2)^- = 0$.

Sub case (II.ii) $c_2 > N_{n+1} \geq N^{\circ}_n$.

Here $(N_{n+1} - N^{\circ}_n)^- = 0$, and $(N_{n+1} - c_2)^- = c_2 - N_{n+1}$.

Using (**A**.22) with $(N_{n+1} - N^{\circ}_n)^- = 0$, we have

$$S^{\circ}_{n+1} \leq N^{\circ}_n (b-1)$$

$$\leq N^{\circ}_n (b-1) + (c_2 - N^{\circ}_n)(b - 1 - \mu b)$$

(use $c_2 > N^{\circ}_n$ and $b - 1 - \mu b > 0$)

$$= c_2 (b-1) - \mu b (c_2 - N^{\circ}_n)$$

$$\leq c_2 (b-1) - \mu b (c_2 - N_{n+1})$$

(use $N_{n+1} \geq N^{\circ}_n$)

$$= c_2 (b-1) - \mu b (N_{n+1} - c_2)^-.$$

Sub case (II.iii) $c_2 > N^{\circ}_n > N_{n+1}$.

Here $(N_{n+1} - N^{\circ}_n)^- = N^{\circ}_n - N_{n+1}$, $(N_{n+1} - c_2)^- = c_2 - N_{n+1}$.

Using (**A**.22) with $(N_{n+1} - N^{\circ}_n)^- = N^{\circ}_n - N_{n+1}$, we also have

$$S^{\circ}_{n+1} \leq N^{\circ}_n (b-1) - \mu (N^{\circ}_n - N_{n+1})\, b$$

$$\leq N^{\circ}_n (b-1) - \mu (N^{\circ}_n - N_{n+1})\, b + (c_2 - N^{\circ}_n)(b - 1 - \mu b)$$

(use $c_2 > N^{\circ}_n$ and $b - 1 - \mu b > 0$)

$$= c_2 (b-1) - \mu b (c_2 - N_{n+1})$$

$$= c_2 (b-1) - \mu b (N_{n+1} - c_2)^-.$$

So we have $S^{\circ}_{n+1} \leq c_2 (b-1) - \mu b (N_{n+1} - c_2)^-$ in all three case.

We get that $S_{n+1} \leq S^{\circ}_{n+1} \leq c_2 (b-1) - \mu b (N_{n+1} - c_2)^-$ if $z_{q_n} > 1$.

We conclude that $\boldsymbol{S_{n+1} \leq c_2 (b-1) - \mu b (N_{n+1} - c_2)^-}$ if $z_{n+1} = b$.

A.2 Equivalent forms of Assumption 1.8

Finally we modify Assumption 1.8 to a more convenient form and we need three lemmas for that purpose. The form of Assumption 1.8 that is achieved in the third lemma is the one that we use in the proof of the main theorem.

Lemma A.23 If Assumption 1.8 (i) is satisfied, then for every l, $m \geq 1$, $\delta > 0$, $0 < d \leq 1$ and stopping time T, we have, on the event $\{T < d\}$ a.s.

(i) $P(T^d_{l\delta, m\delta} < d, Z(T^d_{l\delta, m\delta}) = e^{-l\delta} Z(T) / F_T) > 0$ and

(ii) $P(T^d_{l\delta, m\delta} < d, Z(T^d_{l\delta, m\delta}) = e^{m\delta} Z(T) / F_T) > 0$.

Proof of Lemma A.23 (i).

First we define the followings.

Let $T_0 = T$, $T_k = (T_{k-1})^d_\delta$, $k \geq 1$.

Let $A_k = \{T_k < d, Z(T_i) = e^{-\delta} Z(T_{i-1}), 1 \leq i \leq k\}$, $1 \leq k \leq l$.

Then $A_{k+1} = \{A_k, T_{k+1} < d, Z(T_{k+1}) = e^{-\delta} Z(T_k)\}$, $1 \leq k \leq l-1$.

We claim that $P(A_l / F_T) > 0$ a.s.

We will prove it formally by induction.

By Assumption 1.8 (i), $P(A_1 / F_T) > 0$ a.s.

We assume that $P(A_k / F_T) > 0$ a.s. for some k between $1, l-1$.

$$P(A_{k+1} / F_T) = \int_{A_k} P(T_{k+1} < d, Z(T_{k+1}) = e^{-\delta} Z(T_k) / F_{T_k})(x) P(dx / F_T) > 0 \text{ a.s,}$$

by Assumption 1.8 (i) with T_k ($< d$) and $P(A_k / F_T) > 0$ a.s.

So $P(A_l / F_T) > 0$ a.s, by induction.

Let $B = \{T^d_{l\delta, m\delta} < d, Z(T^d_{l\delta, m\delta}) = e^{-l\delta} Z(T)\}$. We have

$B \supset A_l$, and $P(B / F_T) \geq P(A_l / F_T) > 0$ a.s. So we have proved Lemma **A**.23 (i).

The proof of Lemma **A**.23 (ii) is similar to the proof of Lemma **A**.23 (i) and will be omitted.

Lemma A.24 If Assumption 1.8 is satisfied, then for every δ, δ_2, $\delta_3 > 0$, $0 < d \leq 1$ and stopping time T, we have, on the event $\{T < d\}$ a.s.

(i) $P(T^d_{\delta_2,\delta_3} < d, Z(T^d_{\delta_2,\delta_3}) = e^{-\delta_2} Z(T) / F_T) > 0$,

(ii) $P(T^d_{\delta_2,\delta_3} < d, Z(T^d_{\delta_2,\delta_3}) = e^{\delta_3} Z(T) / F_T) > 0$, and

(iii) $P(T_\delta = 1 / F_T) > 0$.

Proof of Lemma **A**.24 (i). Assume (w.l.o.g.) that $\exists\, l, m \geq 1$ so that $(l-1)\,\delta < \delta_2 \leq l\,\delta$ and $\delta_3 = m\,\delta$.

By Lemma **A**.23 (i), $P(T^d_{l\delta,m\delta} < d, Z(T^d_{l\delta,m\delta}) = e^{-l\delta} Z(T) / F_T) > 0$ a.s.

Since $\{T^d_{\delta_2,\delta_3} < d, Z(T^d_{\delta_2,\delta_3}) = e^{-\delta_2} Z(T)\} \supset \{T^d_{l\delta,m\delta} < d, Z(T^d_{l\delta,m\delta}) = e^{-l\delta} Z(T)\}$,

$P(T^d_{\delta_2,\delta_3} < d, Z(T^d_{\delta_2,\delta_3}) = e^{-\delta_2} Z(T) / F_T) > 0$ a.s.

We prove Lemma **A**.24 (i).

The proof of Lemma **A**.24 (ii) is similar to the proof of Lemma **A**.24 (i) and will be omitted.

The proof of Lemma **A**.24 (iii) is trivial by Assumption 1.8 (ii).

Lemma A.25 If Assumption 1.8 is satisfied, then for every $0 < \varepsilon < 1$, there exists $\delta°(\varepsilon) > 0$ that satisfies: $e^{2\delta°} \leq (1-2\varepsilon)/(1-3\varepsilon)$, $1 - e^{-3\delta°} \leq \mu$, $e^{3\delta°} - 1 \leq \lambda$ so that for every stopping times $0 \leq T \leq \varepsilon$, and $1 - \varepsilon \leq \tau < 1$, we have

(i) $P(T_{\delta°} = 1 / F_T) > 0$ a.s,

(ii) $P(\tau_{\beta,\delta^\circ} < 1, Z(\tau_{\beta,\delta^\circ}) = e^{-\beta} Z(\tau) / F_\tau) > 0$ a.s, where $\beta = -[\log(1 - \mu) + \delta^\circ]$, and

(iii) $P(1 - \varepsilon \le T_{\delta^\circ} < 1, Z(T_{\delta^\circ}) = e^{-\delta^\circ} Z(T) / F_T) > 0$ a.s.

Proof.

By Lemma A.24 (iii) with $\delta = \delta^\circ$, the proof of (i) is trivial.

By using Lemma A.24 (i) with $d = 1$, $\delta_2 = \beta$ and $\delta_3 = \delta^\circ$, the proof of (ii) is trivial.

We need some calculation for (iii).

We define stopping time $L = (1 - \varepsilon) \wedge T_{\delta^\circ}$.

By $P(T_{\delta^\circ} = 1 / F_T) > 0$ a.s, $P(L = 1 - \varepsilon / F_T) > 0$ a.s.

$e^{-(\beta+\delta^\circ)} = 1 - \mu \le e^{-3\delta^\circ}$ implies that $e^{-\beta} \le e^{-2\delta^\circ}$.

If $L = 1 - \varepsilon$ then $T_{\delta^\circ} \ge 1 - \varepsilon$, $Z(L) \le e^{\delta^\circ} Z(T)$ and

$$e^{-\beta} Z(L) \le e^{-\beta} e^{\delta^\circ} Z(T) \le e^{-2\delta^\circ} e^{\delta^\circ} Z(T) \le e^{-\delta^\circ} Z(T). \text{ So}$$

$$P(1 - \varepsilon \le T_{\delta^\circ} < 1 : Z(T_{\delta^\circ}) = e^{-\delta^\circ} Z(T) / F_T)$$

$$\ge P(L_{\beta,\delta^\circ} < 1, Z(L_{\beta,\delta^\circ}) = e^{-\beta} Z(L) / L = 1 - \varepsilon)\, P(L = 1 - \varepsilon / F_T) > 0 \text{ a.s,}$$

by $P(L = 1 - \varepsilon / F_T) > 0$ a.s, and (ii) with $\tau = L$ on $\{L = 1 - \varepsilon\}$.

We have proved part (iii).

A.3 Two simple inequalities

In this part of appendix we prove two inequalities that were used in A.1.

First Inequality:

Let a, b, $c \in \boldsymbol{R}$. We have

(A.26). $$(a - c) - (c - b)^- \leq a - b.$$

Proof. If $c \geq b$ then $(c - b)^- = 0$.

$(a - c) - (c - b)^- = a - c \leq a - b$,

by $c \geq b$.

If $c < b$ then $(c - b)^- = b - c$.

$(a - c) - (c - b)^- = a - c - (b - c) = a - b$.

Second Inequality:

Let $a \geq b$, $c \in \boldsymbol{R}$. We have

(A.27). $$a \vee c - b \vee c \leq a - b.$$

Proof. If $c \geq a \geq b$ then $a \vee c = c$, $b \vee c = c$.

$a \vee c - b \vee c = c - c = 0 \leq a - b > 0$.

If $a > c \geq b$ then $a \vee c = a$, $b \vee c = c$.

$a \vee c - b \vee c = a - c \leq a - b$,

by $c \geq b$.

If $a > b > c$ then $a \vee c = a$, $b \vee c = b$.

$a \vee c - b \vee c = a - b$.

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