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FULL-WAVE INTEGRAL-OPERATOR DESCRIPTION OF
PROPAGATION MODES EXCITED ON STRIPLINE STRUCTURES

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David John Infante

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of the requirements for

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Major professor

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**FULL-WAVE INTEGRAL-OPERATOR DESCRIPTION OF
PROPAGATION MODES EXCITED ON STRIPLINE STRUCTURES**

By

David John Infante

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1999

FULL-WAVE PROPAGATION

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ABSTRACT

FULL-WAVE INTEGRAL-OPERATOR DESCRIPTION OF PROPAGATION MODES EXCITED ON STRIPLINE STRUCTURES

By

David John Infante

Stripline waveguide structures are commonly used in microwave and millimeter-wave integrated circuit devices, and as field applicators for the electromagnetic characterization of materials. Despite widespread use as a waveguide device, analysis of the higher order modes excited on stripline structures has received surprisingly little attention. This dissertation presents investigations into several topics involving stripline waveguide devices.

First, a Hertz potential Green's function is developed representing a generalized stripline structure, excited by a three-dimensional volume current source. The associated Helmholtz equation is solved via a two-dimensional Fourier transform. Pole singularities and branch points associated with the resulting Sommerfeld integral Green's function representations are identified and examined.

The generalized Green's function is specialized to represent an isolated stripline structure. Electric field integral equations (EFIEs) are developed through enforcement of appropriate boundary conditions at the conducting center-strip surface. Current distributions associated with various propagation modes are determined using a Galerkin's Moment Method technique employing Chebychev basis functions. Because most stripline devices are operated in such way that only the fundamental TEM mode is excited, a TEM specialization of the stripline full-wave theory is developed, leading to an expression for the characteristic impedance of stripline.

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The analysis of isolated stripline is then extended to the case of a structure having a pair of coupled center strips. Two pairs of coupled EFIEs are developed through enforcement of boundary conditions at the surfaces of the conducting center strips. These coupled EFIEs are then solved numerically via a Galerkin's Moment Method. A TEM specialization of the coupled stripline full-wave theory is also developed, allowing for determination of the impedance parameters associated with this structure.

In order to reduce the complexity associated with solution of the coupled strip EFIEs, a full-wave perturbation theory for coupled strips is developed. Here it is assumed that, for loosely coupled center strips, the current distributions of the coupled system modes are approximately weighted replicas of the corresponding isolated currents. Numerical results using this method are compared to those obtained previously.

Finally, a technique for measuring the complex permittivity and permeability of sample materials placed in a stripline field applicator is presented. Limitations associated with frequency-domain characterization techniques currently in use can be overcome by exploiting the inherent time-domain response of the stripline applicator. This time-domain response is synthesized from frequency-domain measurements made at the applicator ports. Time windows are then applied to the resulting waveform to remove unwanted signal components. This technique requires fewer measurements than those currently in use, leading to increased efficiency and accuracy.

for my grandparents,

Rody and Angela Zivkovic

and

Charles and Mary Infante

I would not
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to her time and m
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passed over the
standing research
to express my
support and to Dr

In addition to
family has been ex
Matthew and Micha
through many diffi
Michael Hayvalla. C
a amount of thank
Sister Daniel, my
family

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In addition to those who guided me academically, the support of my friends and family has been equally important. Special thanks go to Daniel and Kathleen Glatz, Matthew and Michelle Adomaitis, and their families, for their friendship and support through many difficult years. I am also grateful to my fellow students and friends Michael Havrilla, Christopher Coleman, Ponniah Illavarasan, and John Ross. Finally, no amount of thanks are sufficient for the support and encouragement given to me by my father Daniel, my mother Marilyn, my sister Alicia, and everyone in my extended family.

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Stripline structure

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Chapter 1

INTRODUCTION

Stripline structures represent one of the most widely used types of waveguides. The stripline has been used in UHF, microwave, and millimeter-wave circuits for many years. More recently, stripline waveguides have been employed in the production of materials which exhibit specific reflection and transmission properties over wide bandwidths. Such materials may be used in microwave circuitry, in broadband transmission lines, and as coatings for reduced radar cross-section weaponry. Efficient development and testing of these materials involves measurements made using field applicators such as stripline, which are capable of operating over wide frequency ranges. The stripline structure supports TEM modes and has a zero cutoff frequency, which allows for low frequency measurements. Stripline also provides good field confinement at higher frequencies. The layered cross-section of stripline allows for relatively easy machining and placement of sample materials. Also, sample materials may be clamped into the applicator, allowing measurements to be made at high temperatures. Designing stripline applicators with maximum operational bandwidths requires an understanding of the behavior of such devices over a wide range of frequencies. Specifically, the frequencies at which higher order modes are excited must be determined.

Due to compatibility with contemporary integrated circuit fabrication techniques, stripline devices are also used extensively in micro-electronic circuits. As microprocessor clock speeds increase, the design of associated circuitry requires knowledge of the high frequency behavior of structures like stripline. In addition, the

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decreasing size of micro-electronic circuits has prompted the need for a mathematically rigorous understanding of coupling effects and leakage mechanisms which may lead to loss or distortion of signals.

Historically, analysis of the higher order modes associated with stripline devices has received surprisingly little attention. Although the associated geometry lends itself well to analysis, no rigorous full wave theory for single and coupled stripline devices currently exists. Early work in this area relied on conformal mapping techniques and focused on determining the characteristic impedance of the dominant (TEM) stripline mode [1]-[3]. Higher order modes were largely ignored by these methods. More recent efforts to determine the propagation-mode spectrum of stripline have involved the transverse resonance technique [4], [5]. These approaches have provided information about higher order mode cutoff frequencies, but not about the center-strip current distributions associated with the various higher-order modes, or about the excitation of improper (leaky-wave) modes on stripline structures.

This dissertation investigates the discrete higher order modes, both proper and leaky, of stripline transmission lines via a full-wave integral operator technique [6]. Full-wave integral operator formulations for both single and coupled stripline configurations will be presented. A Hertz potential Green's function is developed for a generalized stripline cross-sectional geometry, and is then specialized to represent single and coupled strip structures. The cross-sectional geometry of the devices under consideration prompts the use of two-dimensional Fourier transforms to solve the associated Helmholtz equations. Electric field integral equations (EFIEs) representing the single and coupled strip structures are developed based on the Hertz potential Green's

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The higher order modes of stripline are investigated through complex analysis in the axial Fourier-transform plane. Proper, discrete guided-wave modes of stripline are associated with pole singularities of this Green's function which occur on the top Riemann sheet, while improper (leaky-wave) modes are associated with singularities located on the bottom Riemann sheet. Finally, numerical solutions are implemented using moment method techniques.

This dissertation is presented in eight chapters. Chapter 2 consists of an overview of the analysis method which is to be employed. In Chapter 3, a Hertz potential Green's function is developed representing a generalized three-dimensional current source suspended between two infinitely wide, parallel, conducting ground planes [7]-[22]. This development is presented for the general case in which one of the ground planes is coated with a simple, non-magnetic dielectric layer (Figure 1.1). Analysis begins with the establishment of boundary conditions at the dielectric/free space interface, and at the surfaces of the conducting ground planes [23], [24]. The associated Helmholtz equation is solved via a two-dimensional Fourier transform, which is prompted by the uniform cross-sectional geometry of the generalized stripline structure. Pole singularities and branch points associated with the resulting Sommerfeld integral Green's function representations are identified and examined. From this, surface-wave and parallel-plate propagation modes are also identified.

The generalized Green's function is specialized in Chapter 4 to the case of an isolated, perfectly-conducting center strip suspended between two infinitely wide, parallel conducting ground planes (Figure 1.2). It is assumed that the center strip is immersed

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Figure

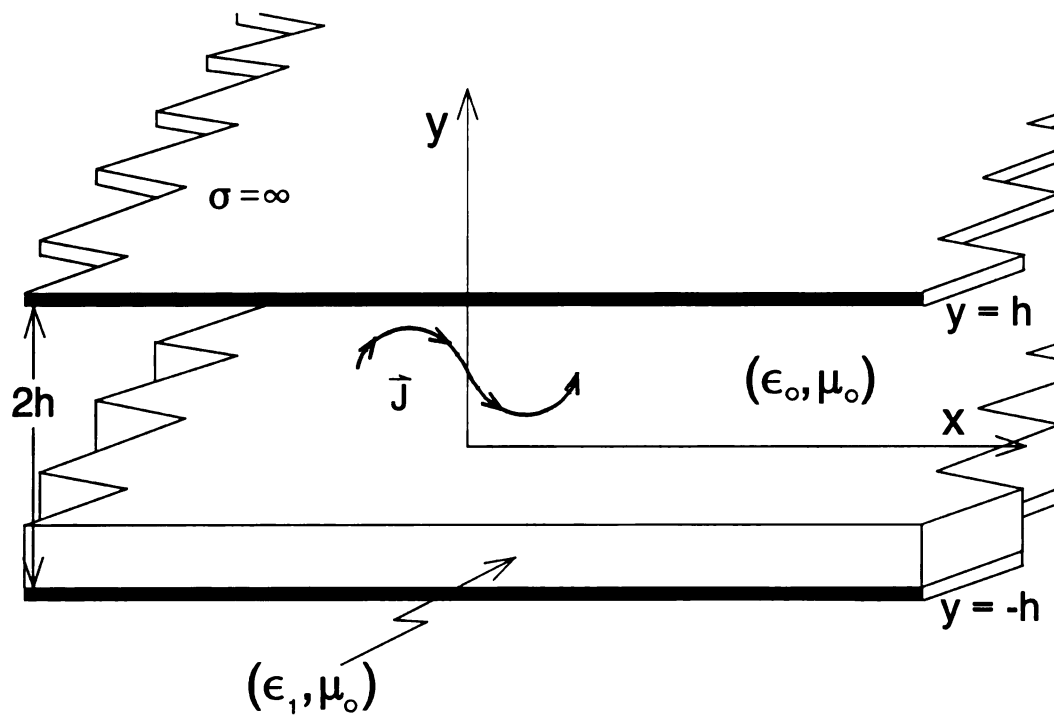


Figure 1.1 General stripline cross-section with volume source current.

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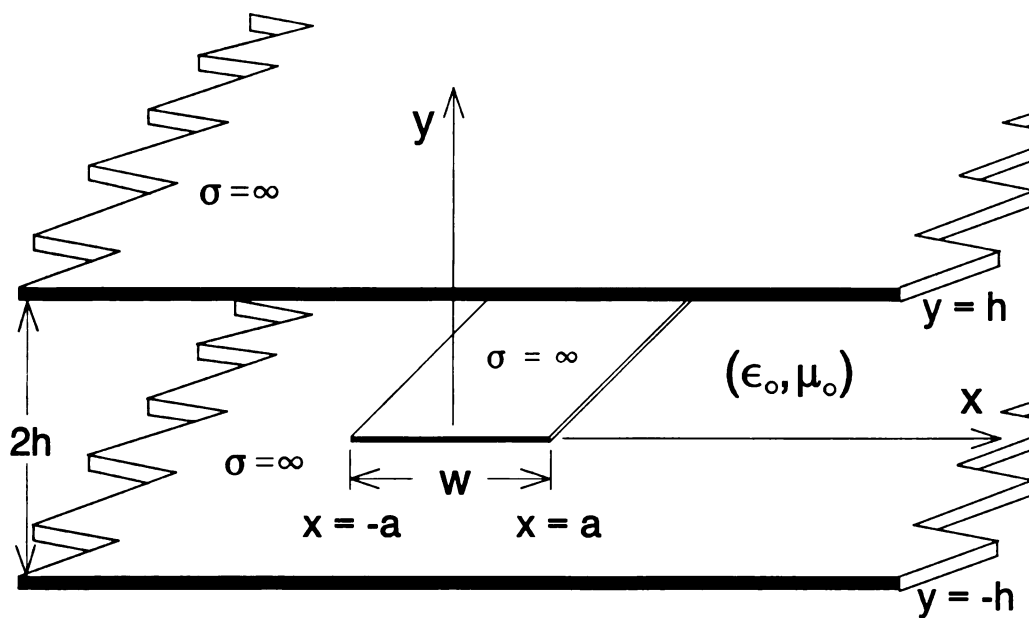


Figure 1.2 Isolated center strip cross-sectional configuration.

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in free space, and that no dielectric layer exists. EFIEs are developed through enforcement of appropriate boundary conditions on the tangential electric fields at the perfectly-conducting center-strip surface. Propagation modes, both proper and leaky, are then examined and identified [25]-[42]. Current distributions associated with various propagation modes are determined using a Galerkin's Moment Method technique employing entire domain (Chebychev) basis functions [43], [44].

In practice, most stripline transmission devices are operated in such a way that only the fundamental TEM mode is excited. This prompts the development of a TEM specialization of the isolated stripline full-wave theory. This TEM specialization allows for the determination of the stripline characteristic impedance. A pole series representation which is used in place of the real axis integration implicated in the Green's function greatly reduces the amount of computational time required for numerical results.

The analysis of the previous chapter is extended in Chapter 5 to the case of a stripline structure having two coupled center strips [47]. These center strips are assumed to be of arbitrary width and placement with respect to both the origin of coordinates and the ground planes (Figure 1.3). Enforcement of boundary conditions on the tangential electric fields at the surfaces of the conducting center strips leads to two pairs of coupled EFIEs. Current distributions are determined through a Galerkin's Moment Method solution, and the associated propagation modes (proper and leaky) are examined. An investigation of the symmetric and anti-symmetric propagation modes which are supported by this structure is also included. As with the analysis of the isolated strip structure, a TEM specialization of the full-wave theory for coupled stripline is developed. This allows for determination of the impedance parameters associated with the coupled

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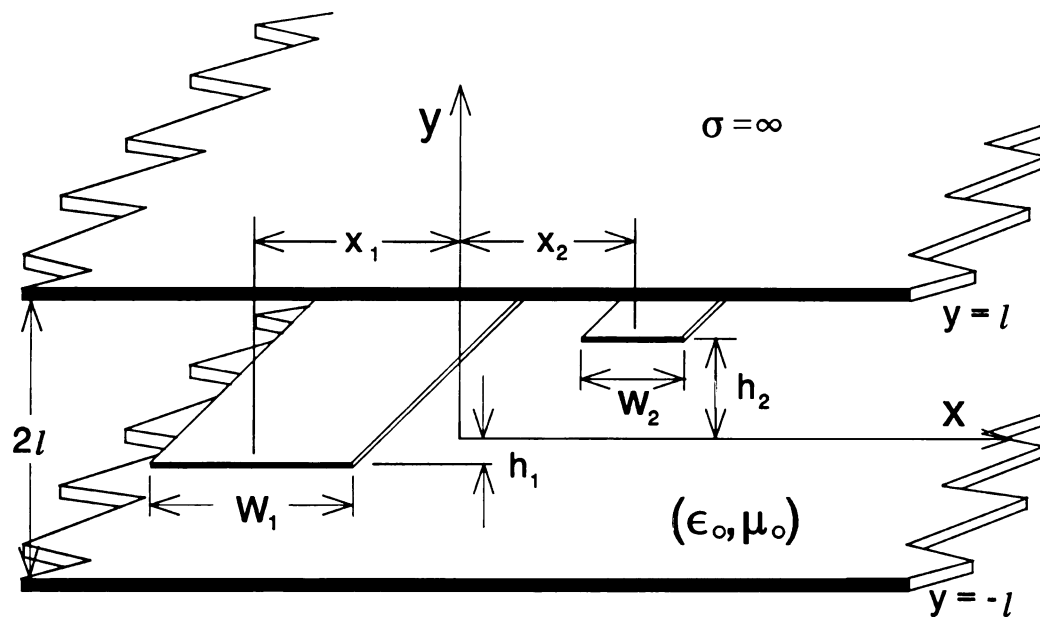


Figure 1.3 Coupled strip cross-sectional configuration.

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strip structure. Again, a pole series representation is used to reduce the amount of computational time required for numerical results.

Solution of the coupled strip EFIEs is formidable both analytically and numerically. To reduce complexity, a full-wave perturbation theory for coupled strips is developed in Chapter 6 [48]. Here, the eigenmode currents of an isolated center strip are used as a first approximation for the nearly degenerate eigenmode currents of a loosely coupled pair of center strips [49]. In addition to reducing the amount of time required for computation of propagation constants, this method typically provides insight into the nature of coupling effects. Results obtained using this method will be compared to those presented in Chapter 5.

In Chapter 7, a measurement technique for determining the constitutive parameters (complex permittivity and permeability) of sample materials placed in a stripline field applicator is presented. A frequency domain materials characterization scheme involving stripline field applicators presently exists and has been shown to yield good results [58]. However, this technique becomes cumbersome and of questionable accuracy when used to measure the properties of materials at high temperature. Several of the limitations associated with the frequency-domain technique can be overcome by exploiting the inherent time-domain response of the stripline applicator [59]. This time-domain response is synthesized from frequency-domain measurements made at the applicator ports. Time windows are then applied to the resulting waveform to remove unwanted signal components. This technique requires fewer measurements than those currently in use, leading to increased efficiency and accuracy when knowledge of high temperature material properties is required.

Finally,

Results of the

Finally, conclusions and ideas for future work are presented in Chapter 8.
Results of the previous chapters will be summarized and examined here.

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2.1 Establishment

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Chapter 2

OVERVIEW OF ANALYTICAL TECHNIQUE

The dominant (TEM) mode of stripline structures has been thoroughly investigated in the past, and is well understood. Little attention, however, has been given to the higher order modes of stripline. The earliest treatments of stripline [1] dealt with the determination of characteristic impedance. Other efforts in the past have determined the cutoff frequencies of higher order modes through such methods as conformal mapping [2],[3], finite-element analysis [4], and the transverse resonance technique [5]. These techniques have yielded only information about the proper discrete spectrum, and not the improper, higher order leaky modes supported by stripline. The method employed in this dissertation identifies the complete (proper) and leaky discrete propagation-mode spectral components for stripline structures. This is accomplished through a singularity expansion [6] of center-strip currents in the axial Fourier-transform plane. This chapter provides a general overview of the technique employed in this dissertation.

2.1 Establishment of Helmholtz Equation

Analysis begins by considering a generalized, three-dimensional impressed current source which is suspended in free space. An appropriately specialized system of Maxwell's equations is

$$\nabla \times \vec{E} = -j\omega\mu_o\vec{H} \quad (1)$$

$$\nabla \times \vec{H} = j\omega\epsilon_o\vec{E} + \vec{J}_i \quad (2)$$

Equation (4) indicates

where

where $\vec{H}$ is the electric
field

where

where $k_0^2 = \mu_0 \epsilon_0 \omega^2$
may be combined

and by vector identity
gradient of a scalar

leading to an expression

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o} \quad (3)$$

$$\nabla \cdot \vec{H} = 0 \quad (4)$$

Equation (4) indicates that $\vec{H}$ may be represented by the curl of a vector potential function

$$\vec{H} = j\omega\epsilon_o \nabla \times \vec{\Pi} \quad (5)$$

where $\vec{\Pi}$ is the electric Hertz potential [7]-[14]. Substitution of this expression into (1) yields

$$\nabla \times \vec{E} = -j\omega\mu_o(j\omega\epsilon_o \nabla \times \vec{\Pi}) = k_o^2(\nabla \times \vec{\Pi}) = \nabla \times (k_o^2 \vec{\Pi}) \quad (6)$$

where $k_o^2 = \mu_o\epsilon_o\omega^2$ is the free space wavenumber. Curl operators on both sides of (6) may be combined

$$\nabla \times (\vec{E} - k_o^2 \vec{\Pi}) = 0 \quad (7)$$

and by vector identity the terms of equation (7) in parenthesis may be represented as the gradient of a scalar

$$\vec{E} - k_o^2 \vec{\Pi} = -\nabla\Phi. \quad (8)$$

leading to an expression for the electric field

$$\vec{E} = k_o^2 \vec{\Pi} - \nabla\Phi \quad (9)$$

in terms of the

Substituting

in equations (1)

which results in

The standard of

Helmholtz equation

and the electric

2.2 Hertz Potential

The Hertz

three-dimensional

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non-magnetic ($\mu = \mu_0$)

in terms of the electric Hertz potential and the electric scalar potential $\nabla\Phi$.

Substitution of the Hertz potential representations for electric and magnetic fields in equations (5) and (9) respectively into equation (2) gives

$$\nabla \times (j\omega \epsilon_o \nabla \times \vec{\Pi}) = \vec{J}_i + j\omega \epsilon_o (k_o^2 \vec{\Pi} - \nabla \Phi) \quad (10)$$

which results in

$$\nabla(\nabla \cdot \vec{\Pi}) - \nabla^2 \vec{\Pi} = \frac{\vec{J}_i}{j\omega \epsilon_o} + k_o^2 \vec{\Pi} - \nabla \Phi. \quad (11)$$

The standard choice of the Lorentz condition $\nabla \cdot \vec{\Pi} = -\nabla \Phi$ leads to the inhomogeneous Helmholtz equation

$$\nabla^2 \vec{\Pi} + k_o^2 \vec{\Pi} = -\frac{\vec{J}_i}{j\omega \epsilon_o} \quad (12)$$

and the electric field is then recovered from

$$\vec{E} = (k_o^2 + \nabla \nabla \cdot) \vec{\Pi}. \quad (13)$$

2.2 Hertz Potential Green's Function Solution

The Hertz potential solution to (12) will be determined for the general case of a three-dimensional impressed current source suspended in a region of free space between two perfectly-conducting ground planes, one of which is coated with a linear, isotropic, non-magnetic ($\mu_r = 1$) dielectric layer. The resulting solution will later be specialized

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where $\bar{G}, \bar{F}, \bar{F}$) is

to the case in which no dielectric layer exists. The ground planes and dielectric layers are assumed to be infinitely wide and extend to infinity in the longitudinal direction. This uniformity of the structure along these directions prompts the use of a two-dimensional Fourier transform

$$\Psi(\vec{r}) = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} \tilde{\Psi}(\vec{\lambda}, y) e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad (14)$$

$$\tilde{\Psi}(\vec{\lambda}, y) = \int \int_{-\infty}^{\infty} \Psi(\vec{r}) e^{-j\vec{\lambda} \cdot \vec{r}} dx dz \quad (15)$$

in order to solve the Helmholtz equation (12). This two-dimensional transform reduces the second order partial differential equation (12) to an ordinary differential equation and leads to a solution in terms of unknown wave amplitudes. These wave amplitudes are associated with the various reflected and transmitted wave components present within the structure and are determined through application of boundary conditions on the remaining spatial variable (boundary conditions on the Hertz potential at the surfaces of the perfectly-conducting ground planes and at the dielectric interface are established through application of boundary conditions on individual electric and magnetic field components) [23], [24]. This results in space domain Hertz potential solutions to the Helmholtz equation (12) of the form

$$\bar{\Pi}(\vec{r}) = \int_V \tilde{G}(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' \quad (16)$$

where $\tilde{G}(\vec{r}|\vec{r}')$ is a Sommerfeld-integral representation of the resulting Green's function.

23 Electric Dyadic

The direct, free space Green's function for the Helmholtz potential Green's function is the Green's function employed in the direct, free space Green's function is used to obtain the relationship between the

and the electric dyadic

$$\bar{G}$$

Here $\bar{L} = -\nabla \nabla$ is known as the Laplacian dyadic. The Green's function is performed in the plane of the dyadic. The Green's function is expressed as

$$\bar{G}(\mathbf{r}) = \frac{1}{4\pi} \nabla \nabla \frac{1}{r}$$

Here $\bar{V} = \nabla \nabla - \nabla^2 \mathbf{I}$ is the vector Laplacian dyadic.

is the transformed dyadic. The Green's function is suspended in free space.

2.3 Electric Dyadic Green's Function

The direct, full-wave analysis of single and coupled stripline structures involves a Hertz potential Green's function, as described above. Due to the form of the testing operators employed in the perturbation method presented in Chapter 6, an electric dyadic Green's function is used in this chapter [48]-[51]. The electric field in the region of the stripline between the perfectly-conducting ground planes may be represented as

$$\vec{E}(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_V \vec{G}(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' \quad (17)$$

with the electric dyadic Green's function

$$\vec{G}(\vec{r}|\vec{r}') = P.V. (k_o^2 + \nabla \nabla \cdot) \vec{G}(\vec{r}|\vec{r}') + \vec{L} \delta(\vec{r} - \vec{r}'). \quad (18)$$

Here $\vec{L} = -\hat{y}\hat{y}$ is known as the depolarizing dyad, and $P.V.$ indicates that integration is performed in the principal value sense. The transform-domain electric dyadic Green's function is expressed

$$\vec{g}_\zeta(\vec{\rho}|\vec{\rho}') = P.V. (k_o^2 + \nabla \nabla \cdot) \vec{g}_\zeta(\vec{\rho}|\vec{\rho}') + \vec{L} \delta(\vec{\rho} - \vec{\rho}'). \quad (19)$$

Here $\tilde{\nabla} = \nabla_t + \hat{z}j\zeta$ with $\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$ where $t=x,z$, and

$$\vec{g}_\zeta(\vec{\rho}|\vec{\rho}') = (\hat{x}\hat{x} + \hat{z}\hat{z})g_\zeta^t(\vec{\rho}|\vec{\rho}') + \hat{y}\hat{y}g_\zeta^y(\vec{\rho}|\vec{\rho}') \quad (20)$$

is the transform-domain Hertz potential Green's function representing a current source suspended in free space between parallel conducting plates.

2.4 Development of

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2.4 Development of Electric Field Integral Equations

After the Hertz potential Green's function for the generalized stripline structure has been established, electric field integral equations (EFIEs) are developed by specializing the three-dimensional volume current to surface currents present within specific stripline geometries (single and coupled center strips). The general Green's function is also specialized to represent structures where no dielectric layer exists. The structures examined in this dissertation will consist of single and coupled strips immersed in a homogenous, lossless medium between infinite conducting ground planes. The examination of structures having center strips over a dielectric layer will be left for future work.

The perfectly-conducting center strips are modeled as infinitely thin surface current distributions. This is accomplished through enforcement of the boundary condition on tangential electric fields at the surface of the strip conductors, which requires that

$$\hat{t} \cdot (\vec{E}^i + \vec{E}^s) = 0 \quad (21)$$

where $\hat{t}$ is a unit vector tangent to the strip surface, $\vec{E}^i$ is the impressed field maintained by $\vec{J}^i$, and $\vec{E}^s$ is the scattered field maintained by surface current $\vec{K}$. Enforcement of this boundary condition leads to fundamental EFIEs of the form

$$\hat{t} \cdot (k_o^2 + \nabla \nabla \cdot) \int_C \vec{G}(\vec{r}|\vec{r}') \cdot \vec{K}(\vec{r}') ds' = -j\omega\epsilon_o \hat{t} \cdot \vec{E}^i(\vec{r}) \quad (22)$$

where C represents the integration path over the periphery of the strip conductors.

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Axially transforming the EFIEs term by term, and applying the differentiation and convolution theorems leads to

$$\hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_C \tilde{g}_\zeta(\bar{r}|\bar{r}') \cdot \vec{k}(x', \zeta) dx' = -j\omega\epsilon_o \hat{t} \cdot \vec{e}^i(\bar{r}, \zeta) \quad (23)$$

where $\tilde{\nabla} = \nabla_t + \hat{z}j\zeta$ with $\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$, and $\tilde{g}_\zeta(\bar{r}|\bar{r}')$ and $\vec{e}^i(\bar{r}, \zeta)$ are the transform-domain Green's function and impressed electric field, respectively, with ζ the axial transform variable. Discrete guided-wave modes of stripline are associated with the pole singularities of (23). Near these poles, the strip current can be represented as

$$\vec{k}(x, \zeta) \approx \frac{\vec{k}_p(x)}{(\zeta - \zeta_p)} \quad (24)$$

where $\vec{k}_p(x)$ is regular at $\zeta = \zeta_p$ [25]. Substitution of relationship (24) into the EFIE (23) yields

$$\hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_C \tilde{g}_\zeta(\bar{r}|\bar{r}') \cdot \frac{\vec{k}_p(x')}{(\zeta - \zeta_p)} dx' = -j\omega\epsilon_o \hat{t} \cdot \vec{e}^i(\bar{r}, \zeta). \quad (25)$$

The left hand side of (25) is singular at $\zeta = \zeta_p$, while the right-hand side is analytic everywhere. Resolving this paradox requires

$$\hat{t} \cdot (k^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_C \tilde{g}_\zeta(\bar{\rho}|\bar{\rho}') \cdot \vec{k}_p(x') dx' = 0 \quad (26)$$

which is the fundamental EFIE for discrete stripline modes, having non-trivial solutions that exist only for $\zeta = \zeta_p$.

2.3 Numerical Solution

A numerical solution

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2.5 Numerical Solution

A numerical solution is implemented through a Galerkin's Method of Moments (MoM) technique [43]. Chebychev polynomials are used as basis functions for both expansion and testing [44]. The use of Chebychev polynomials allows for closed form evaluation of spatial integrals. The EFIEs are thus converted to a homogeneous matrix equation of the form

$$[A_{mn}(\zeta)][a_n] = 0 \quad (27)$$

where $A_{mn}(\zeta)$ are spectral integrals and the a_n are the unknown amplitudes associated with current expansion functions. This matrix equation has non-trivial solutions for discrete-mode poles satisfying the determinantal characteristic equation

$$\det[A_{mn}(\zeta)] = 0. \quad (28)$$

For the case of a single conducting strip structure, the discrete stripline modes possess either even or odd symmetry about the origin of coordinates [26]. Therefore, in order to reduce the time required for computation, even or odd mode symmetry is pre-selected before solution of the matrix equation. When coupled-strip structures are examined, in general no such symmetry will exist [47]. The resulting discrete modes will therefore possess neither even nor odd symmetry about the origin.

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3.1 Representations

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Chapter 3

GREEN'S FUNCTION FOR A GENERALIZED VOLUME SOURCE IN A STRIPLINE STRUCTURE

In this chapter, a Green's function is established for a generalized three-dimensional volume source suspended in the space between two parallel, infinitely-wide, perfectly-conducting plates, one of which is coated with a non-magnetic dielectric material (Figure 3.1). This Green's function will serve as the building block for the EFIEs which will be determined for specific stripline geometries in later chapters. As outlined in the preceding chapter, a Helmholtz equation is established for Hertz potential representations of electric and magnetic fields. Boundary conditions on the Hertz potential at the surface of the perfect conductors, and the free space/dielectric interface are determined by examining boundary conditions on the electric and magnetic fields. The uniformity of the cross-sectional geometry prompts the use of Fourier transforms to solve the Helmholtz equation. This results in a solution in terms Sommerfeld integral Green's function representations [7]-[14]. Pole and branch point singularities associated with this Green's function will be identified and examined [15]-[22]. Parallel-plate and surface-wave modes will also be identified.

3.1 Representation of EM Field by Hertz Potential

For the purpose of this analysis, the electric field will be decomposed into a primary wave $\vec{E}^p$ maintained by the source $\vec{J}$ in unbounded space, and scattered waves $\vec{E}^r$, $\vec{E}^t$ consisting of reflected and transmitted waves excited at the dielectric and conducting

Figure

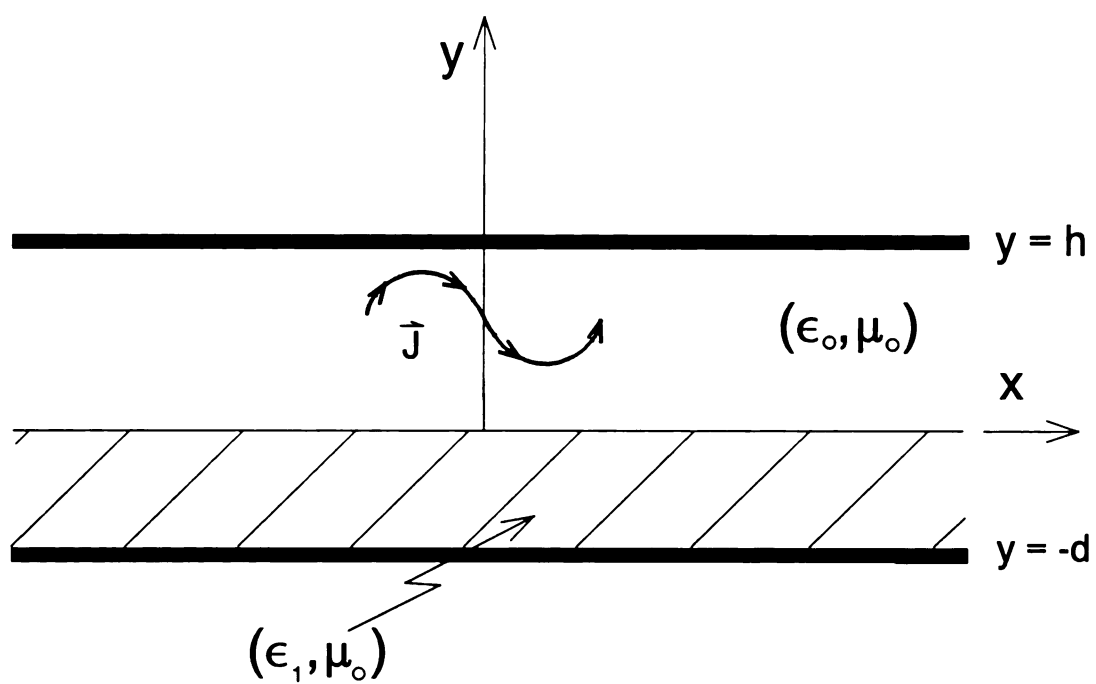


Figure 3.1 Generalized stripline cross-sectional geometry.

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interfaces. The electric and magnetic fields may be expressed in terms of the electric Hertz potential $\vec{\Pi}$ maintained by the general volume current $\vec{J}$. As stated in Chapter 2, these may be represented by

$$\vec{E} = k^2 \vec{\Pi} + \nabla(\nabla \cdot \vec{\Pi}) \quad (1)$$

$$\vec{H} = j\omega\epsilon \nabla \times \vec{\Pi} \quad (2)$$

leading to a Helmholtz equation for $\vec{\Pi}$ due to $\vec{J}$

$$\nabla^2 \vec{\Pi} + k^2 \vec{\Pi} = \frac{-\vec{J}}{j\omega\epsilon} . \quad (3)$$

Determination of $\vec{\Pi}$ from equation (3) requires the application of boundary conditions on the Hertz potential at the surfaces of the perfectly-conducting ground planes and at the dielectric/free space interface. These Hertz potential boundary conditions are determined from boundary conditions on the electric and magnetic fields. Boundary conditions at the dielectric/free space interface will be determined first [23], [24]. These will then be specialized to determine boundary conditions at the surface of the conducting ground planes. This procedure is detailed in Appendix A.

The inhomogeneous Helmholtz equation (3) may be decomposed into scalar components

$$\nabla^2 \Pi_\alpha + k_o^2 \Pi_\alpha = \frac{-J_\alpha}{j\omega\epsilon} \quad \dots \text{ for } \alpha = x, y, z \quad (4)$$

which will facilitate the application of boundary conditions. The individual scalar field

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Tangential electric field

Space interface

Here region (I)

5. (6), and (7)

$$E_{\text{ex}}(y) =$$

$$E_{\text{ex}}(y) =$$

$$H_{\text{ex}}(y) =$$

components may then be expressed in terms of Hertz potentials as

$$E_x = k^2 \Pi_x + \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}) \quad H_x = j\omega\epsilon \left(\frac{\partial \Pi_z}{\partial y} - \frac{\partial \Pi_y}{\partial z} \right) \quad (5)$$

$$E_y = k^2 \Pi_y + \frac{\partial}{\partial y}(\nabla \cdot \vec{\Pi}) \quad H_y = j\omega\epsilon \left(\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x} \right) \quad (6)$$

$$E_z = k^2 \Pi_z + \frac{\partial}{\partial z}(\nabla \cdot \vec{\Pi}) \quad H_z = j\omega\epsilon \left(\frac{\partial \Pi_y}{\partial x} - \frac{\partial \Pi_x}{\partial y} \right). \quad (7)$$

Tangential electric and magnetic fields ($\beta = x, z$) are continuous across the dielectric/free space interface located at $y=0$, therefore

$$E_{0\beta}(y=0^+) = E_{1\beta}(y=0^-) \quad (8)$$

$$H_{0\beta}(y=0^+) = H_{1\beta}(y=0^-). \quad (9)$$

Here region 0 is free space, and region 1 is the dielectric. Applying these conditions to (5), (6), and (7) results in the following relationships

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow k_o^2 \Pi_{0x} + \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}_0) = k_1^2 \Pi_{1x} + \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}_1) \quad (10)$$

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow k_o^2 \Pi_{0z} + \frac{\partial}{\partial z}(\nabla \cdot \vec{\Pi}_0) = k_1^2 \Pi_{1z} + \frac{\partial}{\partial z}(\nabla \cdot \vec{\Pi}_1) \quad (11)$$

$$H_{0x}(y=0^+) = H_{1x}(y=0^-) \rightarrow \epsilon_o \left(\frac{\partial \Pi_{0z}}{\partial y} - \frac{\partial \Pi_{0y}}{\partial z} \right) = \epsilon_1 \left(\frac{\partial \Pi_{1z}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial z} \right) \quad (12)$$

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$$H_{0z}(y=0^+) = H_{1z}(y=0^-) \rightarrow \epsilon_o \left(\frac{\partial \Pi_{0y}}{\partial x} - \frac{\partial \Pi_{0y}}{\partial y} \right) = \epsilon_1 \left(\frac{\partial \Pi_{1y}}{\partial x} - \frac{\partial \Pi_{1x}}{\partial y} \right). \quad (13)$$

For a general 3-D current source $\vec{J} = \hat{x}J_x + \hat{y}J_y + \hat{z}J_z$ suspended in the space above the dielectric region, (11)-(13) lead to the following boundary conditions on the Hertz potential at the dielectric/free space interface at $y=0$

$$\Pi_{0\alpha} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1\alpha} \quad \dots \text{for } \alpha = x, y, z \quad (14)$$

$$\frac{\partial \Pi_{0\beta}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1\beta}}{\partial y} \quad \dots \text{for } \beta = x, z \quad (15)$$

$$\left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) \left(\frac{\partial \Pi_{1x}}{\partial x} + \frac{\partial \Pi_{1z}}{\partial z} \right). \quad (16)$$

Now the boundary conditions on the Hertz potential at the surfaces of the perfectly-conducting ground planes, located at $y=h$ and $y=-d$, must be established. At the surface of a perfect conductor the tangential components ($\beta = x, z$) of electric field must vanish, thus

$$E_{0\beta}(y=h) = E_{1\beta}(y=-d) = 0. \quad (17)$$

The boundary conditions on the Hertz potential at the dielectric interface are specialized such that the material in region 1 becomes perfectly conducting ($\epsilon_1 \rightarrow -j\infty$), leading to

$$\Pi_{0x} = \Pi_{0z} = \frac{\partial \Pi_{0y}}{\partial y} = 0 \quad \dots \text{at } y = h \quad (18)$$

which apply at the surface

3.2 Application of F

Boundary conditions

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$$\Pi_{lx} = \Pi_{lz} = \frac{\partial \Pi_{ly}}{\partial y} = 0 \quad \dots \text{at } y = -d \quad (19)$$

which apply at the surfaces of the ground planes.

3.2 Application of Fourier Transform

Boundary conditions on the Hertz potential have been established, now $\vec{\Pi}$ may be determined from solution of the Helmholtz equation [24]. Several methods exist for the solution of second order, inhomogeneous partial differential equations. However, uniformity of the cross-sectional stripline geometry along the tangential (x- and z-) directions suggests the use of a two dimensional Fourier transform

$$\tilde{\pi}_\alpha(\vec{\lambda}, y) = \int_{-\infty}^{\infty} \int \Pi_\alpha(\vec{r}) e^{-j\vec{\lambda} \cdot \vec{r}} dx dz \quad (20)$$

$$\Pi_\alpha(\vec{r}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int \tilde{\pi}_\alpha(\vec{\lambda}, y) e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad (21)$$

to solve the Helmholtz equation (4). Here $\vec{\lambda} = \hat{x}\xi + \hat{z}\zeta$, $\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z$, $d^2\lambda = d\xi d\zeta$, and the symbol " $\sim$ " denotes transform domain quantities. Applying (20) and (21), the following Fourier transform pairs hold

$$\frac{\partial}{\partial x} \rightarrow j\xi, \quad \frac{\partial}{\partial z} \rightarrow j\zeta \quad (22)$$

$$\vec{E}(\vec{r}) \leftrightarrow \vec{e}(\vec{\lambda}, y), \quad \vec{H}(\vec{r}) \leftrightarrow \vec{h}(\vec{\lambda}, y) \quad (23)$$

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3.3 Decomposition

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$$\Pi_{\alpha}(\vec{r}) \leftrightarrow \tilde{\pi}_{\alpha}(\vec{\lambda}, y) . \quad (24)$$

Transform domain representations for electric and magnetic fields follow from equations (1) and (2)

$$\vec{e}(\vec{\lambda}, y) = (k^2 + \tilde{\nabla} \cdot \tilde{\nabla}) \tilde{\pi}(\vec{\lambda}, y) \quad (25)$$

$$\vec{h}(\vec{\lambda}, y) = j\omega\epsilon \tilde{\nabla} \times \tilde{\pi}(\vec{\lambda}, y) \quad (26)$$

where

$$\nabla \rightarrow \tilde{\nabla} = \hat{x}j\xi + \hat{y}\frac{\partial}{\partial y} + \hat{z}j\zeta = (\hat{x}j\xi + \hat{z}j\zeta) + \hat{y}\frac{\partial}{\partial y} \quad (27)$$

$$\nabla^2 \rightarrow \tilde{\nabla}^2 = \frac{\partial^2}{\partial y^2} - (\xi^2 + \zeta^2) . \quad (28)$$

3.3 Decomposition of Hertz Potential

Determination of $\tilde{\pi}(\vec{\lambda}, y)$ is facilitated by decomposing the Hertz potential in such a way that

$$\tilde{\pi}_{\alpha}(\vec{\lambda}, y) = \tilde{\pi}_{\alpha}^p(\vec{\lambda}, y) + \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y) . \quad (29)$$

Here, $\tilde{\pi}_{\alpha}^p(\vec{\lambda}, y)$ is the principal wave of potential that would be excited by $\tilde{J}_{\alpha}$ in an unbounded medium, in this case a region of free space, and $\tilde{\pi}_{\alpha}^s(\vec{\lambda}, y)$ represents scattered waves of potential reflected by the perfectly-conducting ground planes, and reflected and transmitted at the dielectric interface.

From the cross-sectional geometry of this device (Figure 3.2), it is expected that

Figure

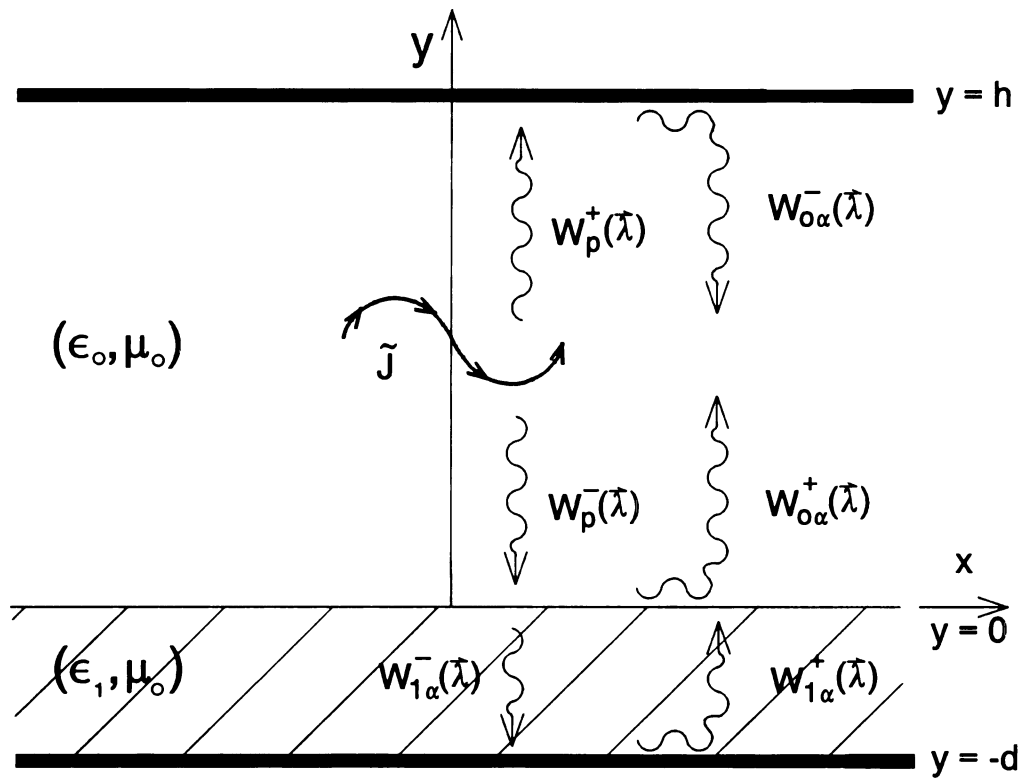


Figure 3.2 Principal and scattered Hertz potential wave components.

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3.4 Homogeneous

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for tangential and
to δ^2 and expansion

the principal potential component will consist of upward and downward traveling waves emanating from the source in the free space region. Likewise, the scattered potential will consist of waves reflected at the surface of the perfect conductors and dielectric, and also a transmitted wave in the dielectric region. The scattered potential is determined by the solution of the homogeneous Helmholtz equation

$$\nabla^2 \vec{\Pi}^s + k^2 \vec{\Pi}^s = 0 \quad (30)$$

in the free space and dielectric regions, while the principal potential will be determined by the solution of the inhomogeneous Helmholtz equation

$$\nabla^2 \vec{\Pi}^p + k_o^2 \vec{\Pi}^p = \frac{-\vec{J}}{j\omega\epsilon_o} \quad (31)$$

in an unbounded region of free space.

3.4 Homogeneous Solution for Scattered Waves of Hertz Potential

The homogeneous transform-domain Helmholtz equation for the scattered potential may be decomposed into scalar equations

$$\nabla^2 \tilde{\pi}_\alpha^s + k^2 \tilde{\pi}_\alpha^s = 0 \quad \dots \text{ for } \alpha = x, y, z \quad (32)$$

for tangential and normal components. Applying the two-dimensional Fourier transform to (32) and expanding the Laplacian operator gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} \tilde{\pi}_\alpha^s(\vec{\lambda}, y) e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda = 0. \quad (33)$$

The differential operator

is equal

which results in

$$\frac{\partial}{\partial x}$$

Comparing (3)
Fourier transform of

The inverse Fourier
theorem

where $p(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$

free space and dielectric

the free space refractive index

$$p(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$$

The differential operators and wavenumber quantity may be passed through the spectral integral

$$\int_{-\infty}^{\infty} \int \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y) e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda = 0 \quad (34)$$

which results in

$$\int_{-\infty}^{\infty} \int \left[\frac{\partial^2 \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y)}{\partial y^2} - (\xi^2 + \zeta^2 - k^2) \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y) \right] e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda = 0. \quad (35)$$

Comparing (35) to (21), it is apparent that equation (35) represents the inverse Fourier transform of the bracketed quantity

$$\left[\frac{\partial^2 \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y)}{\partial y^2} - (\xi^2 + \zeta^2 - k^2) \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y) \right]. \quad (36)$$

The inverse Fourier transform of (36) is zero, therefore by the Fourier transform theorem

$$\left(\frac{\partial^2}{\partial y^2} - p(\vec{\lambda})^2 \right) \tilde{\pi}_{\alpha}^s(\vec{\lambda}, y) = 0 \quad (37)$$

where $p(\vec{\lambda}) = \sqrt{\xi^2 + \zeta^2 - k^2}$ is referred to as the transform-domain wavenumber. The free space and dielectric regions possess different transform-domain wavenumbers. In the free space region, $p_0(\vec{\lambda}) = \sqrt{\xi^2 + \zeta^2 - k_o^2}$ and in the dielectric region, $p_1(\vec{\lambda}) = \sqrt{\xi^2 + \zeta^2 - k_1^2}$, therefore

Equations (35) and (36)

scalar Hertz potential

is represented by

$$\Phi_{\pm}^i(\vec{r})$$

$$\Phi_{\pm}^i(\vec{r})$$

where $W_{\pm}^i(\vec{r})$ and

respectively) in the

downward traveling

is zero (again see

covered through the

$$\Pi_{\pm}^i(\vec{r}) = \frac{1}{(2\pi)^{3/2}}$$

$$\Pi_{\pm}^i(\vec{r}) = \frac{1}{(2\pi)^{3/2}}$$

$$\left(\frac{\partial^2}{\partial y^2} - p_0(\vec{\lambda})^2 \right) \tilde{\pi}_\alpha^s(\vec{\lambda}, y) = 0 \quad \dots \text{for } h \leq y \leq 0 \quad (38)$$

$$\left(\frac{\partial^2}{\partial y^2} - p_1(\vec{\lambda})^2 \right) \tilde{\pi}_\alpha^s(\vec{\lambda}, y) = 0 \quad \dots \text{for } 0 \leq y \leq -d. \quad (39)$$

Equations (38) and (39) have solutions in terms of exponential functions. The transform-domain Hertz potentials for the scattered field in the free space and dielectric regions are thus represented by

$$\tilde{\pi}_{0\alpha}^s(\vec{\lambda}, y) = W_{0\alpha}^+(\vec{\lambda}) e^{-p_0 y} + W_{0\alpha}^-(\vec{\lambda}) e^{p_0 y} \quad \dots \text{for } h \leq y \leq 0 \quad (40)$$

$$\tilde{\pi}_{1\alpha}^s(\vec{\lambda}, y) = W_{1\alpha}^+(\vec{\lambda}) e^{-p_1 y} + W_{1\alpha}^-(\vec{\lambda}) e^{p_1 y} \quad \dots \text{for } 0 \leq y \leq -d \quad (41)$$

where $W_{0\alpha}^+(\vec{\lambda})$ and $W_{0\alpha}^-(\vec{\lambda})$ are upward and downward traveling wave amplitudes (respectively) in the free space region, and $W_{1\alpha}^+(\vec{\lambda})$ and $W_{1\alpha}^-(\vec{\lambda})$ are upward and downward traveling wave amplitudes (respectively) in the dielectric region, where $\alpha = x, y, z$ (again see Figure 3.2). The space-domain Hertz potentials may now be recovered through the inverse Fourier transform

$$\Pi_{0\alpha}^s(\vec{r}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left[W_{0\alpha}^+(\vec{\lambda}) e^{-p_0 y} + W_{0\alpha}^-(\vec{\lambda}) e^{p_0 y} \right] e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad \dots \text{for } h \leq y \leq 0 \quad (42)$$

$$\Pi_{1\alpha}^s(\vec{r}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left[W_{1\alpha}^+(\vec{\lambda}) e^{-p_1 y} + W_{1\alpha}^-(\vec{\lambda}) e^{p_1 y} \right] e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad \dots \text{for } 0 \leq y \leq -d. \quad (43)$$

3.5 Princip

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with $\vec{\lambda} = \vec{\lambda}_0 + \vec{\lambda}_1$

3.5 Principal Incident Wave of Hertz Potential in Free Space Region

The principal component of Hertz potential is determined from solution of the scalar inhomogeneous Helmholtz equation

$$\nabla^2 \pi_{0\alpha}^p + k_o^2 \pi_{0\alpha}^p = \frac{-J_\alpha}{j\omega\epsilon_o} \quad \dots \text{for } \alpha = x, y, z \quad (44)$$

in an unbounded region of free space. In such a region, the Green's function is defined by

$$\nabla^2 G^p(\vec{r}|\vec{r}') + k_o^2 G^p(\vec{r}|\vec{r}') = \delta(\vec{r} - \vec{r}') \quad (45)$$

where $G^p(\vec{r}|\vec{r}') = G^p(\vec{r} - \vec{r}')$. For simplicity the Green's function in equation (45) is evaluated for $\vec{r}' = 0$, therefore

$$(\nabla^2 + k_o^2)G^p(\vec{r}) = -\delta(\vec{r}) \quad (46)$$

and the final result can be shifted to any $\vec{r}' \neq 0$. The two-dimensional Fourier transform pair

$$G^p(\vec{r}) = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} \tilde{G}^p(\vec{\lambda}, y) e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad (47)$$

$$\tilde{G}^p(\vec{\lambda}, y) = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} G^p(\vec{\lambda}, y) e^{-j\vec{\lambda} \cdot \vec{r}} dx dz \quad (48)$$

with $\vec{\lambda} = \hat{x}\xi + \hat{z}\zeta$ may be applied given that the integral representation

ends. For the case

or

where $\delta(\vec{r}) = \delta(x)$

Application of (47)

$\vec{r} = \vec{r}'$

which may be expressed

$\vec{r} = \vec{r}'$

$$\frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} e^{j\vec{\lambda}(\vec{r}-\vec{r}')} d^2\lambda = \delta(x-x') \delta(z-z') \quad (49)$$

holds. For the case of $\vec{r}' = 0$, this becomes

$$\delta(x) \delta(z) = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} e^{j\vec{\lambda}(\vec{r})} d^2\lambda \quad (50)$$

or

$$\delta(\vec{r}) = \frac{\delta(y)}{(2\pi)^2} \int \int_{-\infty}^{\infty} e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad (51)$$

where $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$. As a result

$$(\nabla^2 + k_o^2) G^p(\vec{r}) = - \frac{\delta(y)}{(2\pi)^2} \int \int_{-\infty}^{\infty} e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda. \quad (52)$$

Application of (47) lead to

$$(\nabla^2 + k_o^2) \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} \tilde{G}^p(\vec{\lambda}, y) e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda = - \frac{\delta(y)}{(2\pi)^2} \int \int_{-\infty}^{\infty} e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda \quad (53)$$

which may be expressed

$$\int \int_{-\infty}^{\infty} \left\{ \left[\frac{\partial^2}{\partial y^2} - (\xi^2 + \zeta^2 - k_o^2) \right] \tilde{G}^p(\vec{\lambda}, y) + \delta(y) \right\} e^{j\vec{\lambda} \cdot \vec{r}} d^2\lambda = 0. \quad (54)$$

The Fourier

therefore

where $p_j(\lambda)$

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The Fourier transform theorem requires that the integrand of (54) be equal to zero, therefore

$$\left[\frac{\partial^2}{\partial y^2} - p_0^2(\vec{\lambda}) \right] \tilde{G}^p(\vec{\lambda}, y) = -\delta(y) \quad (55)$$

where $p_0(\vec{\lambda}) = \sqrt{\xi^2 + \zeta^2 - k_o^2}$.

The ordinary differential equation for $\tilde{G}^p(\vec{\lambda}, y)$ can be solved by exploiting the one dimensional transform pair

$$\bar{G}^p(\vec{\lambda}, \eta) = \int_{-\infty}^{\infty} \tilde{G}^p(\vec{\lambda}, y) e^{-j\eta y} dy \quad (56)$$

$$\tilde{G}^p(\vec{\lambda}, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}^p(\vec{\lambda}, \eta) e^{j\eta y} d\eta. \quad (57)$$

This leads to

$$\left[\frac{\partial^2}{\partial y^2} - p_0^2(\vec{\lambda}) \right] \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{G}^p(\vec{\lambda}, \eta) e^{j\eta y} d\eta = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\eta y} d\eta \quad (58)$$

or

$$\int_{-\infty}^{\infty} \left\{ \left[\eta^2 + p_0^2(\vec{\lambda}) \right] \bar{G}^p(\vec{\lambda}, \eta) - 1 \right\} e^{-j\eta y} d\eta = 0. \quad (59)$$

Again the Fourier transform theorem requires that the integrand of (59) be equal to zero

From this

and therefore

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is shown in

It is clear that
when $y < 0$

but

$$\left[\eta^2 + p_0^2(\bar{\lambda}) \right] \bar{G}^p(\bar{\lambda}, \eta) - 1 = 0. \quad (60)$$

From this, it is seen that

$$\bar{G}^p(\bar{\lambda}, \eta) = \frac{1}{\eta^2 + p_0^2(\bar{\lambda})} \quad (61)$$

and therefore

$$\tilde{G}^p(\bar{\lambda}, y) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\eta y}}{(\eta - jp_0)(\eta + jp_0)} d\eta. \quad (62)$$

The original inversion contour C which lies along the $\text{Re}\{\eta\}$ axis for $-\infty < \eta_r < \infty$ is shown in Figure 3.3. Because

$$e^{j\eta y} = e^{-\eta_i y} e^{j\eta_r y} \quad (63)$$

it is clear that C must be closed along C_{∞}^+ when $y > 0$, and C must be closed along C_{∞}^- when $y < 0$. For $y > 0$

$$\int_{-\infty}^{\infty} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta + \int_{C_{\infty}^+} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta = 2\pi j \frac{e^{j\eta y}}{(\eta + jp_0)} \Big|_{\eta = jp_0} \quad (64)$$

but

$$\int_{C_{\infty}^-} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta = 0 \quad (65)$$

C_{∞}^{+}

—

Figure 3.3 Cont.

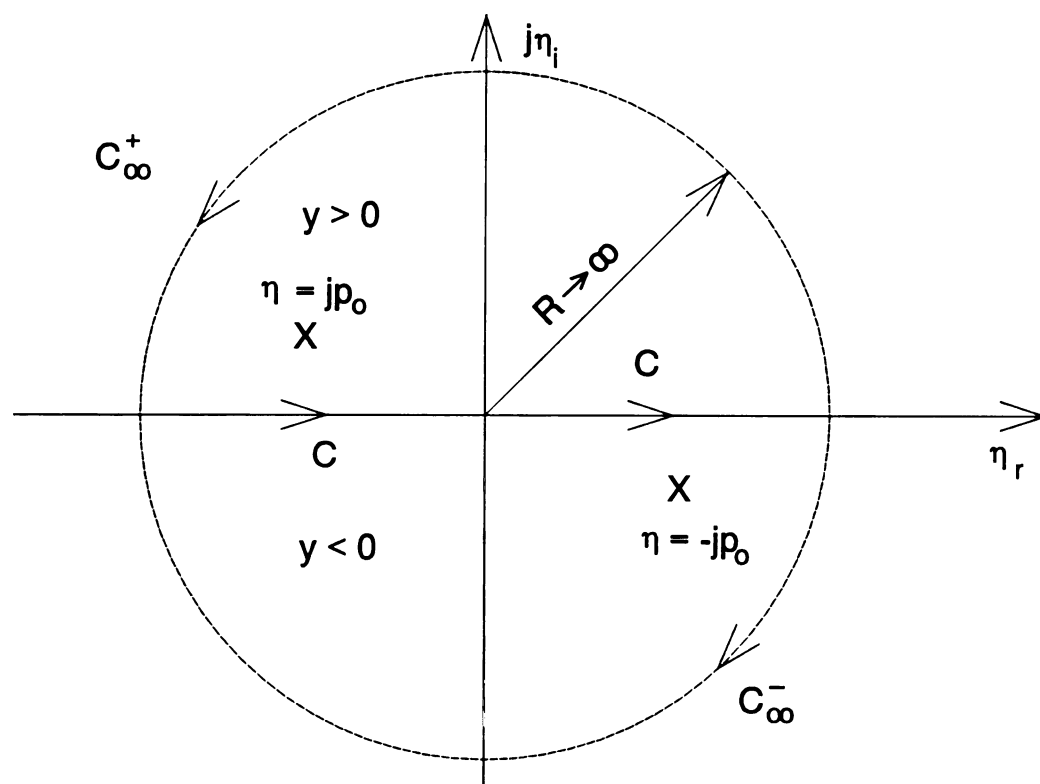


Figure 3.3 Complex η -plane inversion contour.

therefore

likewise, for $y < 0$

$\vdots$
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From this it can be

and therefore

therefore

$$\tilde{G}^p(\vec{\lambda}, y) = \frac{e^{-p_0 y}}{2p_0(\vec{\lambda})} \quad \dots \text{ for } y > 0. \quad (66)$$

Likewise, for $y < 0$

$$\int_{-\infty}^{\infty} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta + \int_{C_-} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta = -2\pi j \frac{e^{j\eta y}}{(\eta - jp_0)} \Big|_{\eta = -jp_0} \quad (67)$$

but

$$\int_{C_-} \frac{e^{j\eta y}}{\eta^2 + p_0^2} d\eta = 0 \quad (68)$$

therefore

$$\tilde{G}^p(\vec{\lambda}, y) = \frac{e^{p_0 y}}{2p_0(\vec{\lambda})} \quad \dots \text{ for } y < 0. \quad (69)$$

From this it can be seen that

$$\tilde{G}^p(\vec{\lambda}, y) = \frac{e^{-p_0 |y|}}{2p_0(\vec{\lambda})} \quad \dots \text{ for } -\infty < y < \infty \quad (70)$$

and therefore

$$G^p(\vec{r}|\vec{r}') = \iint_{-\infty}^{\infty} \frac{e^{j\vec{\lambda} \cdot \vec{r}} e^{-p_0(\vec{\lambda})|y|}}{2(2\pi)^2 p_0(\vec{\lambda})} d^2\lambda \quad (71)$$

for $\vec{r} = 0$. The re

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3.6 Determination

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$W_{1a}(\vec{r})$ and $W_{1a}(\vec{r})$

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for $\vec{r}' = 0$. The result of equation (71) can be extended to the case of $\vec{r}' \neq 0$, which yields the Green's function $G^P(\vec{r}|\vec{r}')$ for the three-dimensional scalar Helmholtz equation (44)

$$G^P(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \frac{e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} e^{-p_0(\vec{\lambda})|y - y'|}}{2(2\pi)^2 p_0(\vec{\lambda})} d^2\lambda. \quad (72)$$

The total potential maintained by J_α is represented by

$$\Pi_{0\alpha}^P(\vec{r}) = \int_V G^P(\vec{r}|\vec{r}') \cdot \frac{J_\alpha(\vec{r}')}{j\omega\epsilon_o} dv' \quad (73)$$

and the associated Hertz potential maintained by the current source embedded in the free space region is then

$$\Pi_{0\alpha}^P(\vec{r}) = \int_{-\infty}^{\infty} d^2\lambda \int_V \frac{J_\alpha(\vec{r}')}{j\omega\epsilon_o} \frac{e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} e^{-p_0(\vec{\lambda})|y - y'|}}{2(2\pi)^2 p_0(\vec{\lambda})} dv' \quad (74)$$

where volume and spectral integrations have been exchanged to facilitate matching of boundary conditions.

3.6 Determination of Unknown Wave Amplitudes

In order to determine the unknown wave amplitudes $W_{0\alpha}^+(\vec{\lambda})$, $W_{0\alpha}^-(\vec{\lambda})$, $W_{1\alpha}^+(\vec{\lambda})$, and $W_{1\alpha}^-(\vec{\lambda})$, boundary conditions must be matched at the surfaces of the perfect conductors and at the dielectric/free space interface. Boundary conditions on the tangential components of Hertz potential will be matched first, followed by the normal

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The total potential in the free space region ($h \leq y \leq 0$) consists of both principal and scattered components

$$\Pi_{0\alpha} = \Pi_{0\alpha}^p + \Pi_{0\alpha}^s \quad (75)$$

while the total potential in dielectric region ($0 \leq y \leq -d$) consists only of scattered components

$$\Pi_{1\alpha} = \Pi_{1\alpha}^s \quad (76)$$

Boundary conditions on the tangential ($\beta=x,z$) Hertz potential components were previously determined to be

$$\Pi_{0\beta} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1\beta} \quad , \quad \frac{\partial \Pi_{0\beta}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1\beta}}{\partial y} \quad \dots \text{at } y = 0 \quad (77)$$

at the dielectric/free space interface, and

$$\Pi_{0x} = \Pi_{0z} = 0 \quad \dots \text{at } y = h \quad (78)$$

$$\Pi_{1x} = \Pi_{1z} = 0 \quad \dots \text{at } y = -d \quad (79)$$

at the upper and lower conducting ground planes, respectively. The boundary conditions at the surface of the perfect conductors are applied first to relate the upward and downward traveling waves in each region, which greatly simplifies the solution.

At the surface of the lower conductor ($y=-d$), only the scattered potential exists, therefore

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plane $(y = h)$.

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$$\Pi_{1\beta}^s(x, -d, z) = 0 \quad (80)$$

which, when applied to equation (43), results in the relationship

$$W_{1\beta}^+(\vec{\lambda}) = -W_{1\beta}^-(\vec{\lambda})e^{-2p_1d}. \quad (81)$$

At the surface of the upper conducting ground plane both the principal and scattered potentials exist. According to (78), the total Hertz potential must vanish at the surface of the conductor, therefore

$$\Pi_{0\beta}^p(x, h, z) + \Pi_{0\beta}^s(x, h, z) = 0. \quad (82)$$

This condition, when applied to (42), results in

$$W_{0\beta}^-(\vec{\lambda}) = -e^{-p_0h} [V_h(\vec{\lambda}) + W_{0\beta}^+(\vec{\lambda})e^{-p_0h}] \quad (83)$$

where

$$V_{h\beta}(\vec{\lambda}) = \int_V \frac{J_{\beta}(\vec{r}')}{j\omega\epsilon_o} \frac{e^{-j\vec{\lambda} \cdot \vec{r}'} e^{-p_0(\vec{\lambda})(h-y)}}{2p_0(\lambda)} dv' \quad (84)$$

is the value of the principal potential evaluated along the surface of the upper ground plane ($y=h$).

The boundary conditions at the dielectric interface

$$\Pi_{0\beta}^p(x, 0, z) + \Pi_{0\beta}^s(x, 0, z) = \frac{\epsilon_1}{\epsilon_o} \Pi_{1\beta}^s(x, 0, z) \quad \dots \text{for } \beta = x, z \quad (85)$$

and

combined with

upward-traveling

wave amplitude

P_1

P_2

where

is the value

Equations (87

84 and 89)

potential comp

and

$$\frac{\partial \Pi_{0\beta}^p(x,0,z)}{\partial y} + \frac{\partial \Pi_{0\beta}^s(x,0,z)}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1\beta}^s(x,0,z)}{\partial y} \quad \dots \text{for } \beta = x, z \quad (86)$$

combined with the results above lead to a pair of coupled equations involving the upward-traveling wave amplitude in the free space region and the downward traveling wave amplitude in the dielectric region

$$V_{i\beta}(\vec{\lambda}) = V_{h\beta}(\vec{\lambda}) e^{-p_0 h} + W_{0\beta}^+(\vec{\lambda}) [1 + e^{-2p_0 h}] + \frac{\epsilon_1}{\epsilon_o} \frac{p_1(\vec{\lambda})}{p_0(\vec{\lambda})} W_{\beta}^- [1 + e^{-2p_1 d}] \quad (87)$$

$$V_{i\beta}(\vec{\lambda}) = V_{h\beta}(\vec{\lambda}) e^{-p_0 h} - W_{0\beta}^+(\vec{\lambda}) [1 - e^{-2p_0 h}] + \frac{\epsilon_1}{\epsilon_o} \frac{p_1(\vec{\lambda})}{p_0(\vec{\lambda})} W_{\beta}^- [1 - e^{-2p_1 d}] \quad (88)$$

where

$$V_{i\beta}(\vec{\lambda}) = \int_V \frac{J_{\beta}(\vec{r}')}{j\omega\epsilon_o} \frac{e^{-j\vec{\lambda} \cdot \vec{r}'} e^{-p_0(\vec{\lambda})y'}}{2p_0(\lambda)} dv' \quad (89)$$

is the value of the principal potential evaluated at the dielectric interface ($y=0$). Equations (87) and (88) can be solved simultaneously, and combined with expressions (84) and (89) above, yield the four unknown wave amplitudes for the tangential Hertz potential components

$$W_{0\beta}^+(\vec{\lambda}) = [V_i(\vec{\lambda}) - V_h(\vec{\lambda}) e^{-p_0 h}] R_t(\vec{\lambda}) \quad (90)$$

$$W_{0\beta}^-(\vec{\lambda}) = -V_h(\vec{\lambda}) e^{-p_0 h} + [V_h(\vec{\lambda}) e^{-p_0 h} - V_i(\vec{\lambda})] R_t(\vec{\lambda}) e^{-2p_0 h} \quad (91)$$

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$$W_{1\beta}^+(\vec{\lambda}) = \left[V_h(\vec{\lambda})e^{-p_0 h} - V_i(\vec{\lambda}) \right] T_t(\vec{\lambda})e^{-2p_1 d} \quad (92)$$

$$W_{1\beta}^-(\vec{\lambda}) = \left[V_i(\vec{\lambda}) - V_h(\vec{\lambda})e^{-p_0 h} \right] T_t(\vec{\lambda}) \quad (93)$$

which are expressed in terms of a reflection coefficient

$$R_t(\vec{\lambda}) = \frac{p_0(\vec{\lambda})(1 - e^{-2p_1 d}) - p_1(\vec{\lambda})(1 + e^{-2p_1 d})}{p_0(\vec{\lambda})(1 - e^{-2p_1 d})(1 + e^{-2p_0 h}) + p_1(\vec{\lambda})(1 + e^{-2p_1 d})(1 - e^{-2p_0 h})} \quad (94)$$

and a transmission coefficient

$$T_t(\vec{\lambda}) = \frac{2p_0(\vec{\lambda})}{\frac{\epsilon_1}{\epsilon_0} \left[p_0(\vec{\lambda})(1 - e^{-2p_1 d})(1 + e^{-2p_0 h}) + p_1(\vec{\lambda})(1 + e^{-2p_1 d})(1 - e^{-2p_0 h}) \right]} \quad (95)$$

It is noted that in the case that the conducting ground planes are moved an infinite distance away from the dielectric interface ($h, d \rightarrow \infty$) the expressions above specialize to the well known results obtained for a source suspended in space above a single dielectric interface (the downward traveling wave in free space and the upward traveling wave in the dielectric both vanish, and the reflection and transmission coefficients become those of a single dielectric interface).

The same process employed above is used to determine the four unknown wave amplitudes for the normal (y-directed) Hertz potential components. The boundary conditions on the normal components

$$\Pi_{0y} = \frac{\epsilon_1}{\epsilon_0} \Pi_{1y}, \quad \left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_0} \right) \left(\frac{\partial \Pi_{1x}}{\partial x} + \frac{\partial \Pi_{1z}}{\partial z} \right) \quad \dots \text{at } y=0 \quad (96)$$

are applied to the

boundary condition

to relate the up and

conditions at the

solved simultaneously

normal Hertz potential

$$W_{\alpha}^{(1)}(\vec{k}) = \int_{\alpha} V_{\alpha}^{(1)}(\vec{k})$$

$$W_{\alpha}^{(2)}(\vec{k}) = \int_{\alpha} V_{\alpha}^{(2)}(\vec{k})$$

$$W_{\alpha}^{(3)}(\vec{k}) = \int_{\alpha} V_{\alpha}^{(3)}(\vec{k})$$

$$\frac{\partial \Pi_{0y}}{\partial y} = 0 \quad \dots \text{at } y = h \quad (97)$$

$$\frac{\partial \Pi_{1y}}{\partial y} = 0 \quad \dots \text{at } y = -d \quad (98)$$

are applied to the space domain Hertz potential representations (42), (43), and (74). The boundary conditions at the surface of the perfect conductors are once again applied first to relate the upward and downward traveling waves in each region. Then the boundary conditions at the dielectric interface are applied, yielding a pair of equations which are solved simultaneously to determine the unknown scattered wave amplitudes for the normal Hertz potential components

$$\begin{aligned} W_{0y}^+(\vec{\lambda}) = & [V_{iy}(\vec{\lambda}) + V_{hy}(\vec{\lambda})e^{-p_0h}]R_n(\vec{\lambda}) + C_n(\vec{\lambda}) \left\{ j\xi [V_{ix}(\vec{\lambda}) - V_{hx}(\vec{\lambda})e^{-p_0h}] \right. \\ & \left. + j\zeta [V_{iz}(\vec{\lambda}) - V_{hz}(\vec{\lambda})e^{-p_0h}] \right\} (1 + e^{-2p_1d})(1 - e^{-2p_1d}) \end{aligned} \quad (99)$$

$$\begin{aligned} W_{0y}^-(\vec{\lambda}) = & [V_{iy}(\vec{\lambda}) + V_{hy}(\vec{\lambda})e^{-p_0h}]R_n(\vec{\lambda})e^{-2p_0h} + C_n(\vec{\lambda}) \left\{ j\xi [V_{ix}(\vec{\lambda}) - V_{hx}(\vec{\lambda})e^{-p_0h}] \right. \\ & \left. + j\zeta [V_{iz}(\vec{\lambda}) - V_{hz}(\vec{\lambda})e^{-p_0h}] \right\} (1 + e^{-2p_1d})(1 - e^{-2p_1d})e^{-2p_0h} + V_{hy}(\vec{\lambda})e^{-p_0h} \end{aligned} \quad (100)$$

$$\begin{aligned} W_{1y}^+(\vec{\lambda}) = & [V_{iy}(\vec{\lambda}) + V_{hy}(\vec{\lambda})e^{-p_0h}]T_n(\vec{\lambda})e^{-2p_1d} + \frac{\epsilon_1}{\epsilon_o} C_n(\vec{\lambda}) \left\{ j\xi [V_{ix}(\vec{\lambda}) - V_{hx}(\vec{\lambda})e^{-p_0h}] \right. \\ & \left. + j\zeta [V_{iz}(\vec{\lambda}) - V_{hz}(\vec{\lambda})e^{-p_0h}] \right\} (1 + e^{-2p_0h})(1 - e^{-2p_1d})e^{-2p_1d} \end{aligned} \quad (101)$$

$$\begin{aligned}
W_{1y}^-(\vec{\lambda}) = & \left[V_{iy}(\vec{\lambda}) + V_{hy}(\vec{\lambda}) e^{-p_0 h} \right] T_n(\vec{\lambda}) + \frac{\epsilon_1}{\epsilon_o} C_n(\vec{\lambda}) \left\{ j\xi \left[V_{ix}(\vec{\lambda}) - V_{hx}(\vec{\lambda}) e^{-p_0 h} \right] \right. \\
& \left. + j\zeta \left[V_{iz}(\vec{\lambda}) - V_{hz}(\vec{\lambda}) e^{-p_0 h} \right] \right\} (1 + e^{-2p_0 h}) (1 - e^{-2p_1 d}).
\end{aligned} \tag{102}$$

These wave amplitudes are expressed in terms of reflection and transmission coefficients

$$R_n(\vec{\lambda}) = \frac{p_0(\vec{\lambda}) \frac{\epsilon_1}{\epsilon_o} (1 + e^{-2p_1 d}) - p_1(\vec{\lambda}) (1 - e^{-2p_1 d})}{p_0(\vec{\lambda}) \frac{\epsilon_1}{\epsilon_o} (1 + e^{-2p_1 d}) (1 - e^{-2p_0 h}) + p_1(\vec{\lambda}) (1 - e^{-2p_1 d}) (1 + e^{-2p_0 h})} \tag{103}$$

and

$$T_n(\vec{\lambda}) = \frac{2p_0(\vec{\lambda})}{p_0(\vec{\lambda}) \frac{\epsilon_1}{\epsilon_o} (1 + e^{-2p_1 d}) (1 - e^{-2p_0 h}) + p_1(\vec{\lambda}) (1 - e^{-2p_1 d}) (1 + e^{-2p_0 h})} \tag{104}$$

as well as a coupling coefficient

$$C_n(\vec{\lambda}) = \frac{2 \left(\frac{\epsilon_1}{\epsilon_o} - 1 \right) p_0(\vec{\lambda})}{\left[X_0(\vec{\lambda}) + X_1(\vec{\lambda}) \right] \left[Y_0(\vec{\lambda}) \frac{\epsilon_1}{\epsilon_o} + Y_1(\vec{\lambda}) \right]} \tag{105}$$

where

$$X_0(\vec{\lambda}) = p_0(\vec{\lambda}) (1 - e^{-2p_1 d}) (1 + e^{-2p_0 h}) \tag{106}$$

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$$X_1(\vec{\lambda}) = p_1(\vec{\lambda})(1 + e^{-2p_1 d})(1 - e^{-2p_0 h}) \quad (107)$$

$$Y_0(\vec{\lambda}) = p_0(\vec{\lambda})(1 + e^{-2p_1 d})(1 - e^{-2p_0 h}) \quad (108)$$

$$Y_1(\vec{\lambda}) = p_1(\vec{\lambda})(1 - e^{-2p_1 d})(1 + e^{-2p_0 h}) \quad (109)$$

and the expressions for the principal potential evaluated along the dielectric/free space interface

$$V_{iy}(\vec{\lambda}) = \int_V \frac{J_y(\vec{r}')}{j\omega\epsilon_o} \frac{e^{-j\vec{\lambda} \cdot \vec{r}'} e^{-p_0(\vec{\lambda})y'}}{2p_0(\lambda)} dv' \quad (110)$$

and along the surface of the upper ground plane

$$V_{hy}(\vec{\lambda}) = \int_V \frac{J_y(\vec{r}')}{j\omega\epsilon_o} \frac{e^{-j\vec{\lambda} \cdot \vec{r}'} e^{-p_0(\vec{\lambda})(h-y)}}{2p_0(\lambda)} dv'. \quad (111)$$

Again these results show the correct behavior in the case where both conductors are moved an infinite distance away from the dielectric interface.

3.7 Hertz Potential Components and Associated Green's Functions

The normal and tangential components of Hertz potential in each region can now be determined by substituting the wave amplitudes into (42) and (43). The resulting potentials in each region can be expressed in terms of Green's functions. Thus, the total tangential potential in the free space region ($h \leq y \leq 0$) can be represented by

where

and

$$G_{\beta}^{\prime}(\vec{r})$$

with

$$A_{\beta}^{\prime}(\gamma)$$

for $\beta = x, z$. Likewise

$$\Pi_{\beta}^{\prime}(\vec{r}) = \int_V \frac{J}{J'} dV$$

where

$$G_{\beta}^{\prime}(\vec{r})$$

$$\Pi_{0\beta}(\vec{r}) = \int_V \frac{J_\beta(\vec{r}')}{j\omega\epsilon_o} \left[G^p(\vec{r}|\vec{r}') + G_{0\beta}^r(\vec{r}|\vec{r}') \right] d\mathbf{v}' \quad (112)$$

where

$$G^p(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \frac{e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} e^{-p_0(\vec{\lambda}) |y - y'|}}{2(2\pi)^2 p_0(\vec{\lambda})} d^2\lambda \quad (113)$$

and

$$G_{0\beta}^r(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \frac{\left[R_t(\vec{\lambda}) A_{0\beta}^r(y, y') - e^{-p_0(2h - y' - y)} \right]}{2(2\pi)^2 p_0(\vec{\lambda})} e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} d^2\lambda \quad (114)$$

with

$$A_{0\beta}^r(y, y') = \left[e^{-p_0(y' + y)} - e^{-p_0(2h + y' - y)} - e^{-p_0(2h - y' + y)} + e^{-p_0(4h - y' - y)} \right] \quad (115)$$

for $\beta = x, z$. Likewise, the normal component of potential in the free space region is

$$\Pi_{0y}(\vec{r}) = \int_V \left\{ \frac{J_y(\vec{r}')}{j\omega\epsilon_o} \left[G^p(\vec{r}|\vec{r}') + G_{0y}^r(\vec{r}|\vec{r}') \right] + \left[\frac{J_x(\vec{r}')}{j\omega\epsilon_o} \frac{\partial}{\partial x} + \frac{J_z(\vec{r}')}{j\omega\epsilon_o} \frac{\partial}{\partial z} \right] G_{0y}^c(\vec{r}|\vec{r}') \right\} d\mathbf{v}' \quad (116)$$

where

$$G_{0y}^r(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \frac{\left[R_n(\vec{\lambda}) A_{0y}^r(y, y') + e^{-p_0(2h - y' - y)} \right]}{2(2\pi)^2 p_0(\vec{\lambda})} e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} d^2\lambda \quad (117)$$

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$$A_{0y}^r(y, y') = \left[e^{-p_0(y' + y)} + e^{-p_0(2h + y' - y)} + e^{-p_0(2h - y' + y)} + e^{-p_0(4h - y' - y)} \right] \quad (118)$$

and

$$G_{0y}^c(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C_n(\vec{\lambda})(1 + e^{-2p_1 d})(1 - e^{-2p_1 d})A_{0y}^c(y, y')}{2(2\pi)^2 p_0(\vec{\lambda})} e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} d^2\lambda \quad (119)$$

with

$$A_{0y}^c(y, y') = \left[e^{-p_0(y' + y)} - e^{-p_0(2h - y' + y)} + e^{-p_0(2h + y' - y)} - e^{-p_0(4h - y' - y)} \right]. \quad (120)$$

The total tangential potential in the dielectric region ($0 \leq y \leq -d$) can be represented in similar fashion

$$\Pi_{1\beta}(\vec{r}) = \int_V \frac{J_{\beta}(\vec{r}')}{j\omega\epsilon_1} G_{1\beta}^t(\vec{r}|\vec{r}') dv' \quad \dots \text{ for } \beta = x, y \quad (121)$$

where

$$G_{1\beta}^t(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T_t(\vec{\lambda}) \frac{\left[(e^{-p_0(2h - y')} - e^{-p_0 y'}) (e^{-p_1(2d + y')} - e^{p_1 y'}) \right]}{2(2\pi)^2 p_0(\vec{\lambda})} e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} d^2\lambda. \quad (122)$$

Finally, the total normal potential in the dielectric region is

$$\Pi_{1y}(\vec{r}) = \int_V \left\{ \frac{J_y(\vec{r}')}{j\omega\epsilon_1} G_{1y}^t(\vec{r}|\vec{r}') + \frac{\epsilon_1}{\epsilon_0} \left[\frac{J_x(\vec{r}')}{j\omega\epsilon_1} \frac{\partial}{\partial x} + \frac{J_z(\vec{r}')}{j\omega\epsilon_1} \frac{\partial}{\partial z} \right] G_{1y}^c(\vec{r}|\vec{r}') \right\} dv' \quad (123)$$

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$$G_{1y}^t(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int T_n(\vec{\lambda}) \frac{(e^{-p_0 y'} + e^{-p_0(2h-y')})(e^{-p_1(2d+y)} + e^{p_1 y})}{2(2\pi)^2 p_0(\vec{\lambda})} e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} d^2 \lambda \quad (124)$$

and

$$G_{1y}^c(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int C_n(\vec{\lambda}) (1 + e^{-2p_0 h})(1 - e^{-2p_1 d}) \frac{(e^{-p_1(2d+y)} + e^{p_1 y})(e^{-p_0 y} - e^{-p_0(2h-y')})}{2(2\pi)^2 p_0(\vec{\lambda})} e^{j\vec{\lambda} \cdot (\vec{r} - \vec{r}')} d^2 \lambda. \quad (125)$$

3.8 Examination of Green's Function Singularities

The wavenumber parameters $p_0(\vec{\lambda}) = \sqrt{\lambda^2 - k_o^2}$ and $p_1(\vec{\lambda}) = \sqrt{\lambda^2 - k_1^2}$, introduce apparent branch points at $\lambda = \pm k_{\{o\}}$. These branch points are removable, because the associated Green's functions are all even in both $p_0(\vec{\lambda})$ and $p_1(\vec{\lambda})$. The inhomogeneously loaded stripline structure will support both parallel plate modes (which propagate in the total region between the parallel ground plates) and surface wave modes (which propagate only in the dielectric region, and are evanescent in the free space region). Poles associated with the parallel-plate background structure occur for particular spatial frequencies $\lambda_{p\{o\}}^2 = k_{\{o\}}^2 + p_{\{o\}}^2$ which lead to vanishing of the denominators in R_t , R_n , and C_n . When the operating frequency is sufficiently low, then the $\lambda_{p\{o\}}^2$ are imaginary, and both the parallel-plate and surface wave modes are evanescent (Figure 3.4). Propagating parallel plate modes occur when p_0 is imaginary ($\lambda_p < k_o$), and surface wave modes occur for p_0 real and p_1 imaginary ($k_o < \lambda_p < k_1$).

The relationship $\lambda_p^2 = \xi_p^2 + \zeta^2$ leads to the expression

evanescent
modes

Figure 3.4

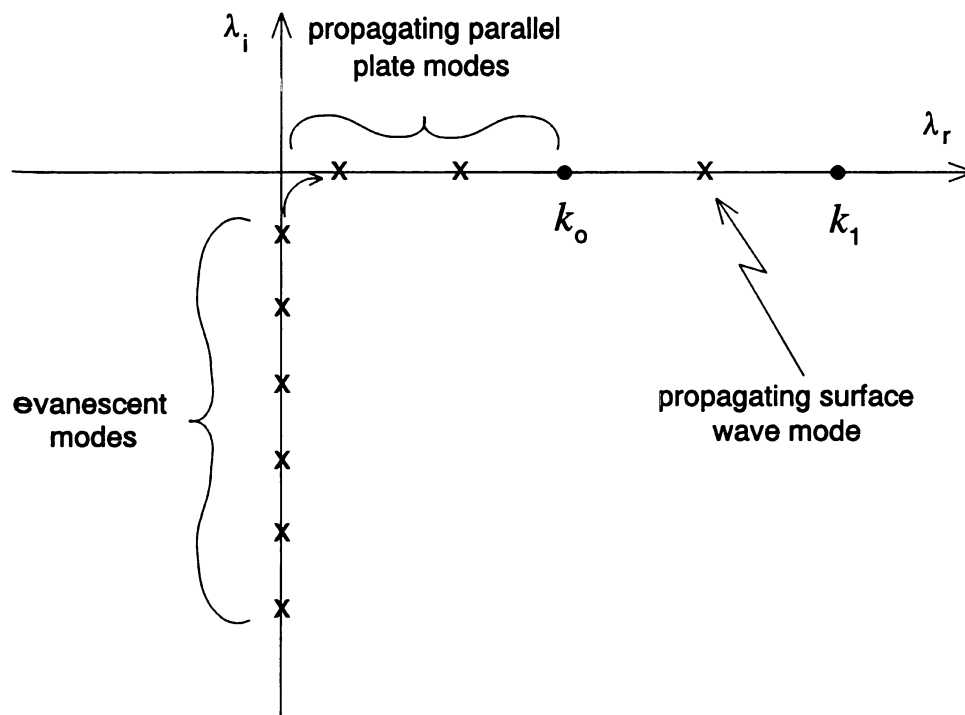


Figure 3.4 Poles associated with parallel-plate and surface-wave modes.

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$$\xi_p = -j\sqrt{\zeta - \lambda_p}\sqrt{\zeta + \lambda_p} \quad (126)$$

which implicates branch points at $\zeta = \pm \lambda_p$ in the complex ζ -plane. Branch cuts are chosen to maintain $\text{Im}\{\xi_p\} < 0$ on the top Riemann sheet. Leaky modes occur when the operating frequency is such that at least one λ_p lies on the real ζ -axis, and the real part of ζ_p is less than λ_p . In this case, roots are found only on the lower (improper) Riemann sheet [25], [27]-[42]. This will be discussed in greater detail in the chapters that follow.

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Chapter 4

FULL-WAVE ANALYSIS OF ISOLATED STRIPLINE STRUCTURES

In this chapter, the structure consisting of a single perfectly-conducting center strip suspended in free space between two parallel, infinitely-wide ground planes will be investigated. The complete propagation mode spectrum for this structure will be determined through a full-wave integral operator technique involving a singularity expansion of currents in the axial Fourier-transform plane [6]. The Hertz potentials established in the previous chapter will first be modified to represent a homogeneous stripline cross-section consisting of a three-dimensional current source suspended in free space between parallel conducting ground planes. Electric field integral equations (EFIEs) will be formulated by specializing the generalized three-dimensional current source to an infinitely-thin surface current through application of boundary conditions. The EFIEs will be solved using a Galerkin's Moment Method technique, and current distributions on the center strip will be determined for various propagation modes. The discrete higher-order modes, both proper and leaky, which are supported by this single strip structure will be investigated and identified using complex analysis.

4.1 Green's Function for Homogeneous Stripline Cross-Section

The stripline structures considered in this and the following chapters consist of perfectly-conducting center strips which are suspended in free space between parallel ground planes. Therefore, the Hertz potential functions established in the previous chapter for a generalized stripline cross-section are specialized for the case where no

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dielectric layer is present. This is accomplished by letting the permittivity of the dielectric region become that of free space ($\epsilon_1 \rightarrow \epsilon_o$, which implies $k_1 \rightarrow k_o$) and placing the ground planes equidistant from the origin at $y = \pm h$. For this case the transform-domain wavenumbers in each region are identical

$$p_1(\vec{\lambda}) = \sqrt{\xi^2 + \zeta^2 - k_1^2} = \sqrt{\xi^2 + \zeta^2 - k_o^2} = p_o(\vec{\lambda}) . \quad (1)$$

Substituting these conditions into the Hertz potentials for the generalized stripline cross-section results in reflection coefficients for tangential and normal potential components given by

$$R_t(\vec{\lambda}) = \frac{-e^{-2p_o h}}{(1 - e^{-4p_o h})}, \quad R_n(\vec{\lambda}) = \frac{e^{-2p_o h}}{(1 - e^{-4p_o h})} \quad (2)$$

transmission coefficients for normal and tangential components

$$T_t(\vec{\lambda}) = T_n(\vec{\lambda}) = \frac{1}{(1 - e^{-4p_o h})} \quad (3)$$

and normal component coupling coefficient

$$C_n(\vec{\lambda}) = 0. \quad (4)$$

The resulting space-domain Hertz potential functions represent a three-dimensional current source suspended in free space between two parallel ground planes (Figure 4.1). These can be represented in terms of Green's functions by

$$\Pi_\alpha(\vec{r}) = \int_V G_{o\alpha}(\vec{r}|\vec{r}') \cdot \frac{J_\alpha(\vec{r}')}{j\omega\epsilon_o} dv' \quad (5)$$

where the associated Green's function $G_{o\alpha}(\vec{r}|\vec{r}')$ consists of both principal and scattered

$$y = h \quad \triangleleft$$

$$y = -h \quad \blacksquare$$

Figure 4.1

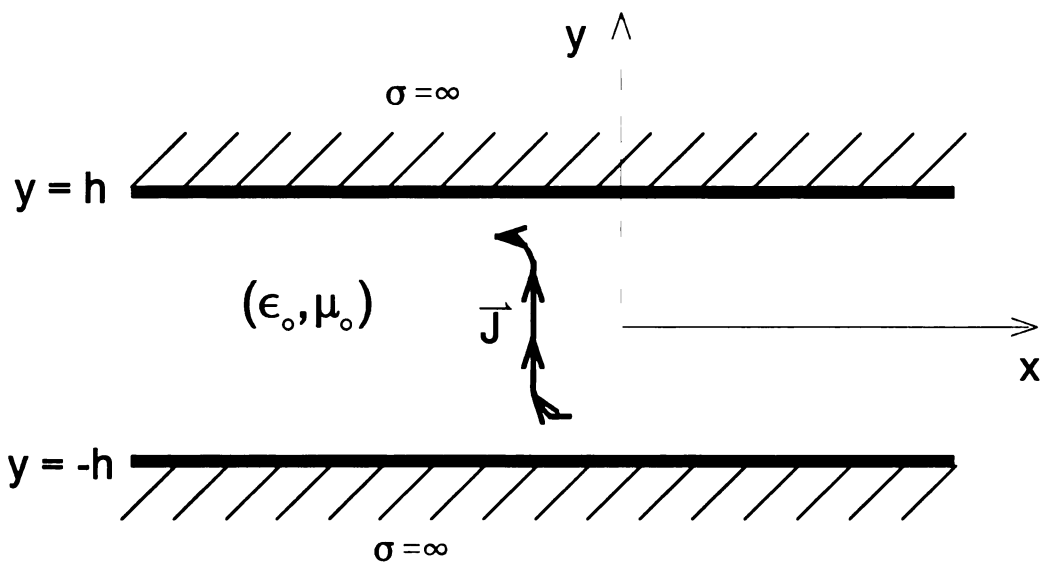


Figure 4.1 Generalized three-dimensional current source suspended between parallel conducting ground planes.

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for $\alpha = x, y, z$

$$G_{\alpha\beta}^i(\vec{r}, \vec{r}')$$

for the tangential

$$G_{\alpha\beta}^i(\vec{r}, \vec{r}')$$

for normal components

4.2 Integral

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$$G_{0\alpha}(\vec{r}|\vec{r}') = G_{0\alpha}^p(\vec{r}|\vec{r}') + G_{0\alpha}^s(\vec{r}|\vec{r}'). \quad (6)$$

Here the principal Green's function is

$$G_{0\alpha}^p(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-p_0|y-y'|} e^{j\vec{\lambda} \cdot (\vec{r}-\vec{r}')}}{2(2\pi)^2 p_0(\vec{\lambda})} d^2\lambda \quad (7)$$

for $\alpha=x,y,z$, and the scattered Green's functions are

$$G_{0\beta}^s(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[e^{-p_0(4h+y'-y)} + e^{-p_0(4h-y'+y)} - e^{-p_0(2h+y'+y)} - e^{-p_0(2h-y'-y)}]}{2(2\pi)^2 p_0(1-e^{-4p_0h})} e^{j\vec{\lambda} \cdot (\vec{r}-\vec{r}')} d^2\lambda \quad (8)$$

for the tangential components of Hertz potential $\beta=x,y$, and

$$G_{0y}^s(\vec{r}|\vec{r}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[e^{-p_0(4h+y'-y)} + e^{-p_0(4h-y'+y)} + e^{-p_0(2h+y'+y)} + e^{-p_0(2h-y'-y)}]}{2(2\pi)^2 p_0(1-e^{-4p_0h})} e^{j\vec{\lambda} \cdot (\vec{r}-\vec{r}')} d^2\lambda \quad (9)$$

for normal components.

4.2 Integral Equation Formulation

In order to represent a stripline structure, the generalized three-dimensional current source must be further specialized to the surface current on an infinitely-thin, perfectly-conducting strip, centered at the origin, which is suspended between parallel conducting ground planes (Figure 4.2). This is accomplished through enforcement of boundary conditions at the center-strip surface. The electric field may be represented in terms of a Green's function by

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Figure

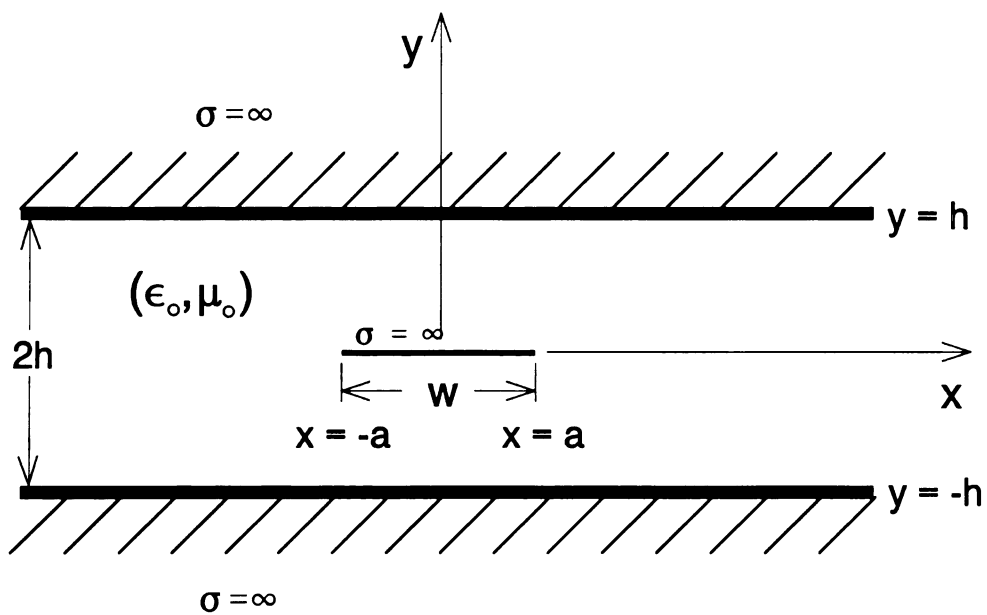


Figure 4.2 Stripline structure immersed in layered background environment.

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$$\vec{E}(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_V \vec{G}_0(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' \quad (10)$$

where the dyadic Green's function may be expressed as

$$\vec{G}_0(\vec{r}|\vec{r}') = (\hat{x}\hat{x} + \hat{z}\hat{z}) G_{0\alpha}(\vec{r}|\vec{r}') + \hat{y}\hat{y} G_{0\alpha}^s(\vec{r}|\vec{r}') . \quad (11)$$

The boundary conditions at the surface of the perfectly-conducting center strip require that the tangential components of electric field vanish there

$$\hat{t} \cdot \vec{E} = 0 \quad \dots \text{at } y=0, \text{ for } -a \leq x \leq a . \quad (12)$$

It is assumed that the current present on the conducting center strip consists of an induced current $\vec{K}$, and an impressed, excitatory current $\vec{J}^e$. In this case the total electric field is

$$\vec{E}(\vec{r}) = \vec{E}^i(\vec{r}) + \vec{E}^s(\vec{r}) \quad (13)$$

where

$$\vec{E}^i(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_V \vec{G}_0(\vec{r}|\vec{r}') \cdot \frac{\vec{J}^e(\vec{r}')}{j\omega\epsilon_o} dv' \quad (14)$$

is the impressed field maintained by $\vec{J}^e$, and

$$\vec{E}^s(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_S \vec{G}_0(\vec{r}|\vec{r}') \cdot \frac{\vec{K}(\vec{r}')}{j\omega\epsilon_o} ds' \quad (15)$$

is the scattered field maintained by the induced surface current $\vec{K}$. Enforcement of the boundary condition at the surface of the center-strip conductor results in

$$\hat{t} \cdot \vec{E}^s(\vec{r}) = -\hat{t} \cdot \vec{E}^i(\vec{r}) \quad (16)$$

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at $y=0$, for $-a \leq x \leq a$, and $\hat{t} = \hat{x}, \hat{z}$. This leads to the fundamental EFIE

$$\lim_{y \rightarrow 0} \left\{ \hat{t} \cdot (k_o^2 + \nabla \nabla \cdot) \int_S \tilde{G}_0(\vec{r} | \vec{r}') \cdot \vec{K}(\vec{r}') ds' \right\} = -j\omega \epsilon_o \lim_{y \rightarrow 0} \left\{ \hat{t} \cdot \vec{E}^i(\vec{r}) \right\}. \quad (17)$$

4.3 Axial Transform-Domain Integral Equation

The perfectly-conducting center strip is assumed to have zero thickness.

Therefore, the associated surface current has no normal ($\hat{y}$ -directed) variation, thus

$$\vec{K}(\vec{r}) = \vec{K}(x, z) \quad (18)$$

and

$$\tilde{G}_0(\vec{r} | \vec{r}') = \tilde{G}_0(\vec{\rho} | \vec{\rho}'; z - z') \quad (19)$$

where $\vec{\rho} = \hat{x}x + \hat{y}y$ is a two-dimensional position vector. The integral operator in (17) is convolutional with respect to the axial variable z ,

$$\int_S \tilde{G}_0(\vec{r} | \vec{r}') \cdot \vec{K}(\vec{r}') ds' = \int_{-a}^a \int_{-\infty}^{\infty} \tilde{G}_0(\vec{\rho} | \vec{\rho}'; z - z') \cdot \vec{K}(x', z') dz' dx' \quad (20)$$

which prompts application of the Fourier transform pair

$$f(\zeta) = \int_{-\infty}^{\infty} F(z) e^{-j\zeta z} dz \quad (21)$$

$$F(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) e^{j\zeta z} d\zeta. \quad (22)$$

Transforming the EFIE (17) term by term, and applying the differentiation and

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convolution theorems leads to the transform-domain EFIE

$$\lim_{y \rightarrow 0} \left\{ \hat{\mathbf{t}} \cdot (\mathbf{k}_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_{-a}^a \tilde{\mathbf{g}}_{\zeta}(\tilde{\rho} | \tilde{\rho}') \cdot \tilde{\mathbf{k}}(x', \zeta) dx' \right\} = -j\omega\epsilon_o \lim_{y \rightarrow 0} \left\{ \hat{\mathbf{t}} \cdot \tilde{\mathbf{e}}^i(\tilde{\rho}, \zeta) \right\} \quad (23)$$

where $\tilde{\nabla} = \nabla_t + \hat{\mathbf{z}}j\zeta$, $\nabla_t = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$, and $\tilde{\mathbf{k}}$ is the surface current residing on the periphery of the center strip. The transform-domain dyadic Green's function

$$\tilde{\mathbf{g}}_{\zeta}(\tilde{\rho} | \tilde{\rho}') = (\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{z}}\hat{\mathbf{z}})g_{\zeta}^{\beta}(\tilde{\rho} | \tilde{\rho}') + \hat{\mathbf{y}}\hat{\mathbf{y}}g_{\zeta}^{\gamma}(\tilde{\rho} | \tilde{\rho}') \quad (24)$$

consists of both tangential ($\beta=x,z$) and normal components, which in turn are composed of principal and scattered components

$$g_{\zeta}^{\beta}(\tilde{\rho} | \tilde{\rho}') = g_{\zeta}^{\beta}(\tilde{\rho} | \tilde{\rho}') + g_{\zeta}^{s\beta}(\tilde{\rho} | \tilde{\rho}') \quad (25)$$

$$g_{\zeta}^{\gamma}(\tilde{\rho} | \tilde{\rho}') = g_{\zeta}^{\gamma}(\tilde{\rho} | \tilde{\rho}') + g_{\zeta}^{s\gamma}(\tilde{\rho} | \tilde{\rho}') \quad (26)$$

where

$$g_{\zeta}^{\beta}(\tilde{\rho} | \tilde{\rho}') = \int_{-\infty}^{\infty} \frac{e^{-p_0|y-y'|} e^{j\xi(x-x')}}{4\pi p_0(\tilde{\lambda})} d\xi \quad (27)$$

and

$$g_{\zeta}^{s\beta}(\tilde{\rho} | \tilde{\rho}') = \int_{-\infty}^{\infty} \frac{[e^{-p_0(4h+y'-y)} + e^{-p_0(4h-y'+y)} - e^{-p_0(2h+y'+y)} - e^{-p_0(2h-y'-y)}]}{4\pi p_0(\tilde{\lambda})(1-e^{-4p_0h})} e^{j\xi(x-x')} d\xi \quad (28)$$

$$g_{\zeta}^{s\gamma}(\tilde{\rho} | \tilde{\rho}') = \int_{-\infty}^{\infty} \frac{[e^{-p_0(4h+y'-y)} + e^{-p_0(4h-y'+y)} + e^{-p_0(2h+y'+y)} + e^{-p_0(2h-y'-y)}]}{4\pi p_0(\tilde{\lambda})(1-e^{-4p_0h})} e^{j\xi(x-x')} d\xi. \quad (29)$$

Near simple pole singularities for propagation constant ζ_p , the transform-domain

current may

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Substitution

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

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current may be approximated by

$$\vec{k}(x, \zeta) \approx \frac{k_p(x)}{(\zeta - \zeta_p)} \quad (30)$$

where $k_p(x)$ is the eigenmode current associated with the p^{th} propagation mode [25].

Substitution of this approximation into (23) leads to

$$\lim_{\zeta \rightarrow \zeta_p} \left\{ \lim_{y \rightarrow 0^+} \left\{ \hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_{-a}^a \vec{g}_{\zeta}(\vec{\rho} | \vec{\rho}') \cdot \frac{\vec{k}_p(x')}{(\zeta - \zeta_p)} dx' \right\} \right\} = -j\omega\epsilon_o \lim_{\zeta \rightarrow \zeta_p} \left\{ \lim_{y \rightarrow 0^+} \left\{ \hat{t} \cdot \vec{e}^i(\vec{\rho}, \zeta) \right\} \right\}. \quad (31)$$

It can be seen that as $\zeta \rightarrow \zeta_p$, the left hand side of (31) becomes singular, while the right hand side remains regular. This apparent paradox is resolved by requiring that the left hand side of (31) become indeterminate as $\zeta \rightarrow \zeta_p$, thus

$$\hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_{-a}^a \vec{g}_{\zeta}(\vec{\rho} | \vec{\rho}') \cdot \vec{k}_p(x') dx' = 0. \quad (32)$$

Equation (32) is the fundamental EFIE for the discrete modes of stripline structures with isolated center strips.

4.4 Scalar x- and z-Component Integral Equations

Because the center strip has no y-directed current, the eigenmode current $\vec{k}_p(x')$ may be decomposed into tangential components

$$\vec{k}(x) = \hat{x}k_x(x) + \hat{z}k_z(x). \quad (33)$$

The transform-domain Green's function is

$$\vec{g}_{\zeta}(\vec{\rho} | \vec{\rho}') = (\hat{x}\hat{x} + \hat{z}\hat{z}) \left[g_{\zeta}^p(\vec{\rho} | \vec{\rho}') + g_{\zeta}^{sp}(\vec{\rho} | \vec{\rho}') \right] + \hat{y}\hat{y} \left[g_{\zeta}^p(\vec{\rho} | \vec{\rho}') + g_{\zeta}^{sy}(\vec{\rho} | \vec{\rho}') \right] \quad (34)$$

where $\beta = x$.

or

Now the scal.

$$\dot{x} = k_1^2$$

and

$$\dot{z} = k_2^2$$

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where $\beta = x, z$, and therefore

$$\vec{g}_\zeta(\vec{\rho} \mid \vec{\rho}') \cdot \vec{k}(x') = [\hat{x}k_x(x') + \hat{z}k_z(x')] [g_\zeta^p(\vec{\rho} \mid \vec{\rho}') + g_\zeta^{s\beta}(\vec{\rho} \mid \vec{\rho}')] \quad (35)$$

or

$$\vec{g}_\zeta(\vec{\rho} \mid \vec{\rho}') \cdot \vec{k}(x') = [\hat{x}k_x(x') + \hat{z}k_z(x')] g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}'). \quad (36)$$

Now the scalar x - and z -component equations are given by

$$\begin{aligned} \hat{x} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_{-a}^a g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') \cdot \vec{k}(x') dx' \\ = \int_{-a}^a \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') k_x(x') + j\zeta \frac{\partial}{\partial x} g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') k_z(x') \right] dx' \end{aligned} \quad (37)$$

and

$$\begin{aligned} \hat{z} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_{-a}^a g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') \cdot \vec{k}(x') dx' \\ = \int_{-a}^a \left[j\zeta \frac{\partial}{\partial x} g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') k_x(x') + (k_o^2 - \zeta^2) g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') k_z(x') \right] dx'. \end{aligned} \quad (38)$$

Assuming that the conducting center strip lies in the plane of the origin $y' = 0$, and the transform-domain Green's function takes the form

$$g_\zeta^\beta(\vec{\rho} \mid \vec{\rho}') = g_\zeta^\beta(x, y \mid x', 0) \quad (39)$$

a pair of coupled integral equations result

$$\int_{-a}^a [k_x(x') K_{xx}(x \mid x') + k_z(x') K_{xz}(x \mid x')] dx' = 0 \quad (40)$$

where

and

with

$W(\xi, y, \zeta)$

Substituting the

$\lim_{y \rightarrow c}$

$$\int_{-a}^a \left[k_x(x') K_{xx}(x|x') + k_z(x') K_{zz}(x|x') \right] dx' = 0 \quad (41)$$

where

$$K_{xx}(x|x') = \lim_{y \rightarrow 0} \left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_\zeta^\alpha(x, y|x', 0) \quad (42)$$

$$K_{xz}(x|x') = K_{zx}(x|x') = \lim_{y \rightarrow 0} j\zeta \frac{\partial}{\partial x} g_\zeta^\alpha(x, y|x', 0) \quad (43)$$

$$K_{zz}(x|x') = \lim_{y \rightarrow 0} (k_o^2 - \zeta^2) g_\zeta^\alpha(x, y|x', 0) \quad (44)$$

and

$$g_\zeta^\beta(x, y|x', 0) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, \zeta) d\xi \quad (45)$$

with

$$W(\xi, y, \zeta) = \frac{1}{p_0} \left\{ e^{-p_0|y|} + \frac{1}{(1 - e^{-4p_0h})} \left[-e^{-p_0(2h+y)} - e^{-p_0(2h-y)} + e^{-p_0(4h+y)} + e^{-p_0(4h-y)} \right] \right\}. \quad (46)$$

Substituting these expressions into (40) and (41) gives

$$\begin{aligned} \lim_{y \rightarrow 0} \left\{ \int_{-a}^a \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] k_x(x') dx' \right. \\ \left. + \int_{-a}^a \left[j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] k_z(x') dx' \right\} = 0 \end{aligned} \quad (47)$$

$$\lim_{y \rightarrow 0} \int_{-\infty}^{\infty} f(x) dx$$

4.5 Green's

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where

$$W(\xi, y, z)$$

For $y > 0$

$$W(\xi, y, z)$$

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$$\lim_{y \rightarrow 0} \left\{ \int_{-a}^a \left[j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] k_x(x') dx' + \int_{-a}^a \left[(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] k_z(x') dx' \right\} = 0. \quad (48)$$

4.5 Green's Function Singularities and Axial Transform-Plane Branch Cuts

The transform-domain Green's function for transverse components of Hertz potential is given by

$$g_{\zeta}^{\beta}(x, y | x', 0) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, \zeta) d\xi \quad (49)$$

where

$$W(\xi, y, \zeta) = \frac{1}{p_0} \left\{ e^{-p_0|y|} + \frac{1}{(1 - e^{-4p_0h})} \left[-e^{-p_0(2h+y)} - e^{-p_0(2h-y)} + e^{-p_0(4h+y)} + e^{-p_0(4h-y)} \right] \right\}. \quad (50)$$

For $y > 0$

$$\begin{aligned} W(\xi, y, \zeta) &= \frac{1}{p_0} \left\{ e^{-p_0y} + \frac{1}{(1 - e^{-4p_0h})} \left[-e^{-p_0(2h+y)} - e^{-p_0(2h-y)} + e^{-p_0(4h+y)} + e^{-p_0(4h-y)} \right] \right\} \\ &= \frac{e^{-p_0y} - e^{-p_0(2h-y)}}{p_0(1 + e^{-2p_0h})} = \frac{\sinh(p_0[h-y])}{p_0 \cosh(p_0h)}. \end{aligned} \quad (51)$$

It can be seen that (51) is even with respect to parameter p_0 . Likewise, for $y < 0$

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$$\begin{aligned}
W(\xi, y, \zeta) &= \frac{1}{p_0} \left\{ e^{p_0 y} + \frac{1}{(1 - e^{-4p_0 h})} \left[-e^{-p_0(2h+y)} - e^{-p_0(2h-y)} + e^{-p_0(4h+y)} + e^{-p_0(4h-y)} \right] \right\} \\
&= \frac{e^{-p_0 y} - e^{-p_0(2h+y)}}{p_0(1 + e^{-2p_0 h})} = \frac{\sinh(p_0[h+y])}{p_0 \cosh(p_0 h)}.
\end{aligned} \tag{52}$$

Equation (52) is also even with respect to parameter p_0 , and thus the branch point associated with (50) is removable.

Poles of $W(\xi, y, \zeta)$ occur for $\cosh(p_0 h) = 0$ or for

$$p_0 h = \pm j \frac{(2n-1)\pi}{2} \tag{53}$$

or

$$p_0 = \pm j \frac{(2n-1)\pi}{2h} \tag{54}$$

where n is a positive integer ($n=1,2,3,\dots$). As stated previously,

$$p_0^2 = \lambda^2 - k_o^2 \tag{55}$$

with $\lambda^2 = \xi^2 + \zeta^2$, therefore

$$-\left[\frac{(2n-1)\pi}{2h} \right]^2 = \lambda_p^2 - k_o^2. \tag{56}$$

From this the propagation eigenvalues λ_p at the poles of $W(\xi, y, \zeta)$ are found to be

$$\lambda_p = \sqrt{k_o^2 - \left[\frac{(2n-1)\pi}{2h} \right]^2}. \tag{57}$$

These poles are associated with waves which are guided by the parallel ground-plate

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background structure, having plate separation $2h$ [26]. When the frequency of operation is such that $k_o < (2n - 1)\pi/(2h)$ it is seen that λ_p is imaginary, and thus the associated parallel-plate modes are evanescent. When k_o^2 is sufficiently large, λ_p is real, and propagating modes exist which are guided by the background structure. The poles associated with the guided-wave modes of the background structure migrate from the imaginary axis to the real axis in the complex λ -plane as the operating frequency increases (Figure 4.3). While an infinite number of discrete poles exist in the complex λ -plane, only a finite number lie along the real- λ axis for a given operating frequency.

The location of poles in the complex ξ -plane is dependent upon the positions of λ_p and the value of ζ in the complex ζ -plane. The relationship between λ_p and the location of ξ_p is determined from

$$\xi_p^2 + \zeta^2 = \lambda_p^2. \quad (58)$$

From this it is seen that

$$\begin{aligned} \xi_p^2 &= \lambda_p^2 - \zeta^2 = -(\zeta^2 - \lambda_p^2) \\ &= -(\zeta - \lambda_p)(\zeta + \lambda_p) \end{aligned} \quad (59)$$

or

$$\xi_p = -j\sqrt{\zeta - \lambda_p}\sqrt{\zeta + \lambda_p}. \quad (60)$$

This expression implicates branch points at $\zeta = \pm\lambda_p$ in the complex ζ -plane. In order to insure solutions that lead to proper modes associated with fields which satisfy the radiation condition, branch cuts must be chosen to restrict migration of singularities in the complex ξ -plane (Figure 4.4). These singularities must be prevented from

Figure 4.

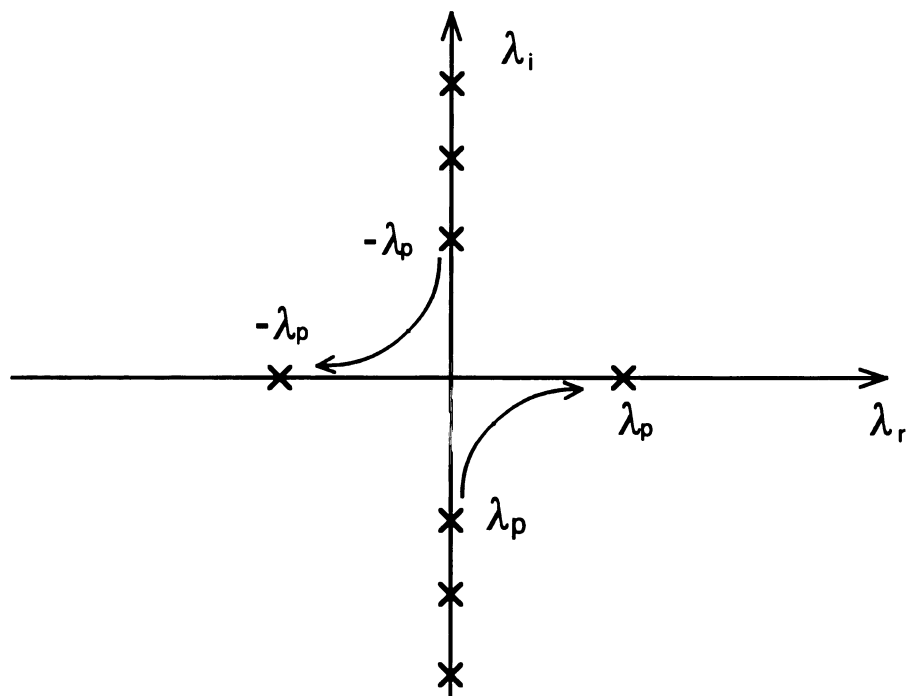


Figure 4.3 Location of parallel-plate guided-wave poles in complex λ -plane.

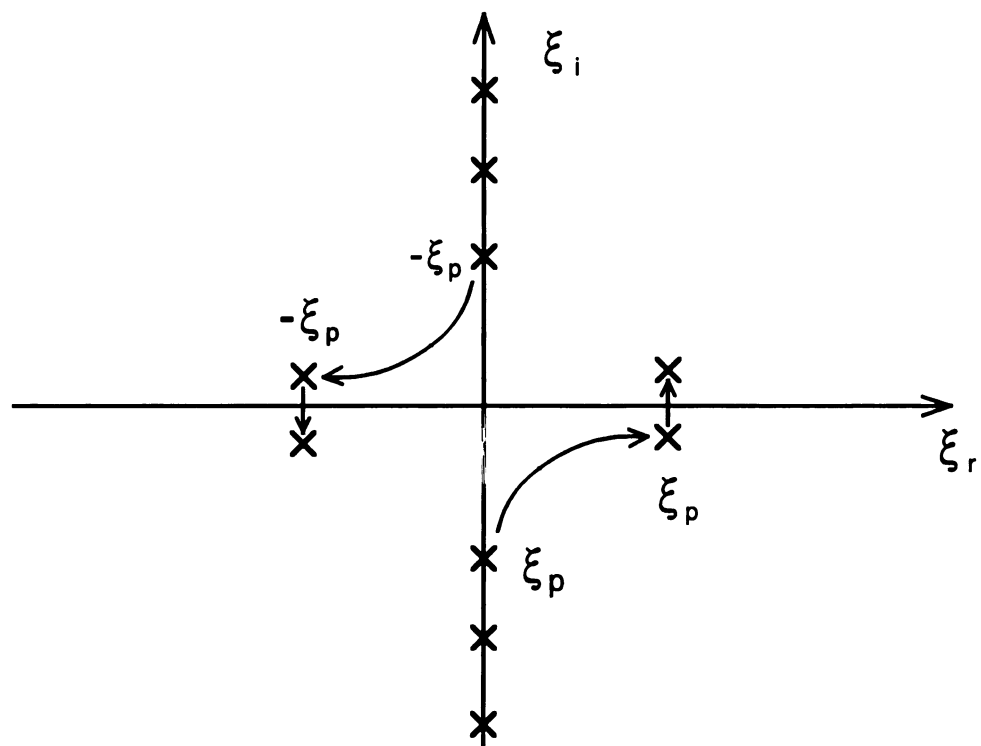


Figure 4.4 Location of parallel-plate guided-wave poles in complex ξ -plane.

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migrating across the $Re\{\xi\}$ axis to provide a strip of convergence, leading to fields which decay in directions away from the guiding structure. Thus $Im\{\xi_p\} < 0$, or

$$Re\left\{\sqrt{\zeta^2 - \lambda_p^2}\right\} = Re\left\{\sqrt{\zeta - \lambda_p}\sqrt{\zeta + \lambda_p}\right\} > 0 \quad (61)$$

is the appropriate criterion. This leads to the conventional Sommerfeld [8] branch cuts which emanate from the branch points at $\zeta = \pm \lambda_p$ (Figure 4.5).

4.6 Improper Discrete Stripline Modes

The discrete modes of the propagation spectrum identified above are proper guided-wave modes. In the absence of loss, the fields associated with the proper modes of stripline give rise to waves which propagate axially without attenuation, and are confined in directions transverse to the guiding structure. The poles associated with these proper modes lie on the top Riemann sheet of the multi-sheeted ζ -plane.

There also exist improper leaky-wave poles which lie on the lower Riemann sheets [25], [27]-[42]. These modes are associated with waves which attenuate axially, and grow in directions transverse to the guiding structure. The improper modes lose power to transversely propagating TM parallel plate modes. Leakage occurs when at least one ζ_p moves off the $Re\{\zeta\}$ axis, and becomes complex with $Re\{\zeta\} < \lambda_p$, and $Im\{\zeta\} < 0$. In this case, roots can no longer be found on the top sheet. If passage is made to the bottom sheet by violating (passing above and then down across) the ζ_p -related branch cut, then roots are again found which correspond to the leaky-wave modes. Passage to the bottom sheet causes the ξ -plane pole at ξ_p to migrate across the $Re\{\xi\}$ axis. The transform-domain Green's function for transverse components of Hertz

Figure

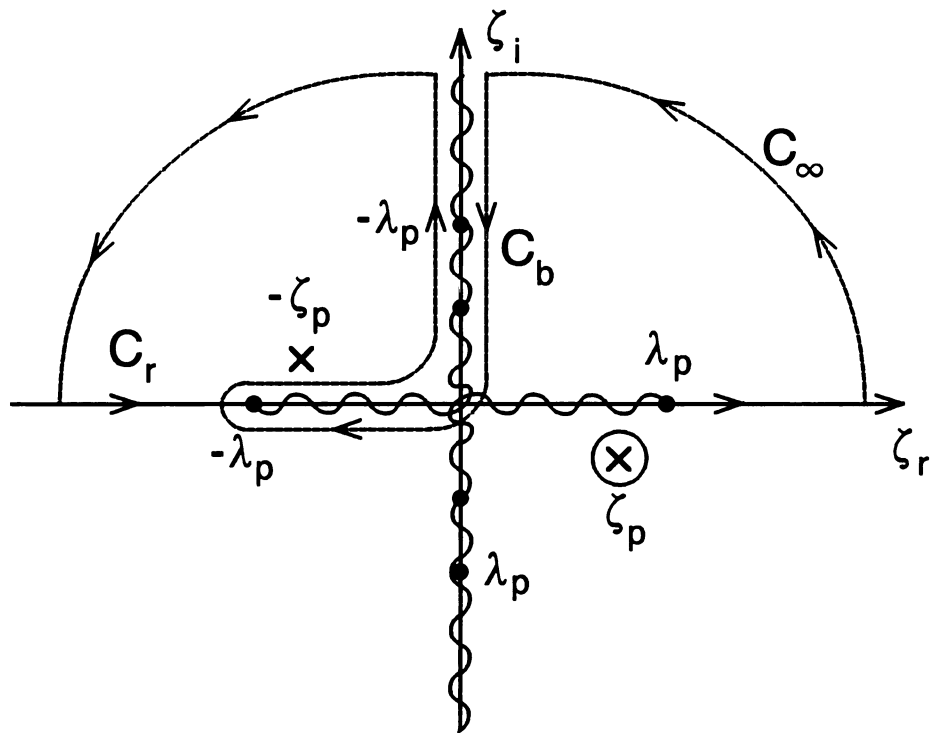


Figure 4.5 Branch-point singularities and inversion-contour deformation in the complex ζ -plane; branch cuts appropriate to restrict ξ -plane pole migration.

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potential (49) involves integration along the $Re\{\xi\}$ axis. To retain physical continuity of the mode after poles have migrated across the $Re\{\xi\}$ axis, the integration path must remain above or below the pole at ξ_p . The integration path is therefore deformed as indicated in Figure 4.6 when improper leaky-wave modes exist. This leads to a pole residue contribution to the real axis integration implicated in the transform-domain Green's function (49).

Determination of the pole residue contribution requires the evaluation of

$$\oint_{C_p} W(\xi, y, \zeta) \Big|_{\xi = \xi_p} d\xi . \quad (62)$$

The term $W(\xi, y, \zeta)$ may be expressed

$$W(\xi, y, \zeta) = \frac{N(\xi, y, \zeta)}{p_0(\tilde{\lambda}) D(\xi)} \quad (63)$$

where

$$N(\xi, y, \zeta) = \left\{ (1 - e^{-4p_0 h}) e^{-p_0 |y|} - e^{-p_0(2h+y)} - e^{-p_0(2h-y)} + e^{-p_0(4h+y)} + e^{-p_0(4h-y)} \right\} \quad (64)$$

and

$$D(\xi) = 1 - e^{-4p_0 h} . \quad (65)$$

At the pole

$$\xi_p^2 = -j\sqrt{\zeta^2 - \lambda_p^2} \quad (66)$$

and

Figure

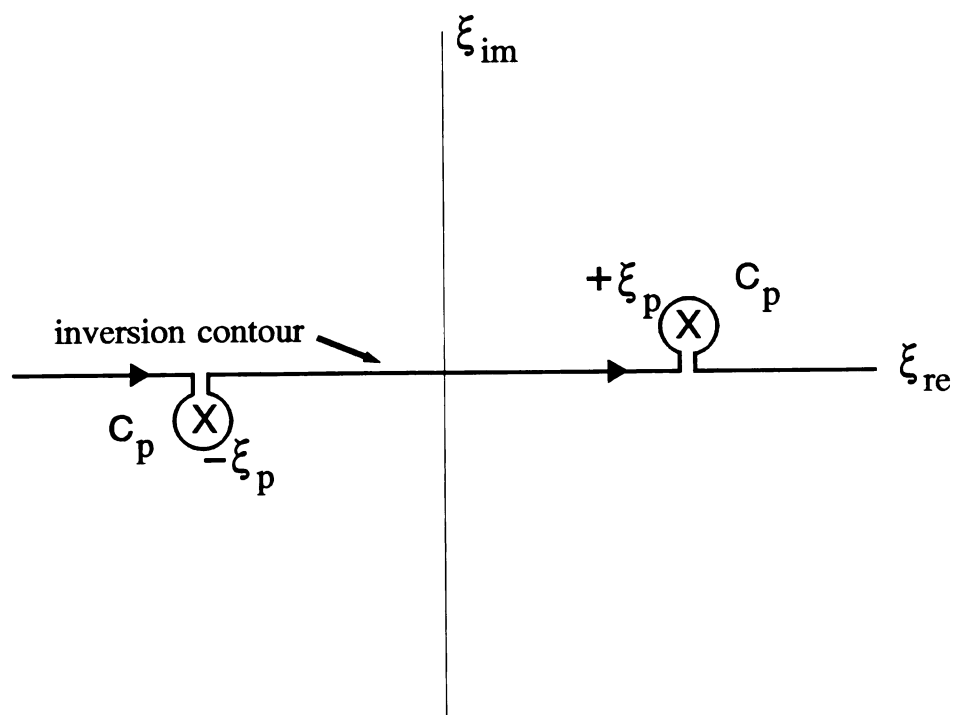


Figure 4.6 Inversion contour in complex ξ -plane.

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$$1 - e^{-4p_0h} = 0 \quad (67)$$

or

$$e^{-4p_0h} = 1 \quad (68)$$

and thus

$$N_p(\xi, y, \zeta) = \left\{ -e^{-p_0(2h+y)} - e^{-p_0(2h-y)} + e^{-p_0y} + e^{p_0y} \right\}. \quad (69)$$

Near ξ_p

$$\begin{aligned} D(\xi) &= 1 - e^{-4p_0h} \approx D(\xi_p) + \left. \frac{dD}{d\xi} \right|_{\xi=\xi_p} (\xi - \xi_p) \\ &= -4h \left(-e^{-4p_0h} \frac{dp_0}{d\xi} \right) \Big|_{\xi=\xi_p} (\xi - \xi_p) = 4h \left. \frac{dp_0}{d\xi} \right|_{\xi=\xi_p} (\xi - \xi_p) \end{aligned} \quad (70)$$

but

$$\frac{dp_0}{d\xi} = \frac{\xi}{p_0} \quad (71)$$

and therefore

$$D_p(\xi) \approx \frac{4h\xi_p}{p_0^p} (\xi - \xi_p) \quad (72)$$

where $p_0^p = \sqrt{\xi_p^2 + \zeta^2 - k_o^2}$. This leads to

$$W_p(\xi, y, \zeta) = \frac{N_p(\xi)}{p_0^p D_p(\xi)} = \frac{[-e^{-p_0^p(2h+y)} - e^{-p_0^p(2h-y)} + e^{-p_0^py} + e^{p_0^py}]}{4h\xi_p(\xi - \xi_p)}. \quad (73)$$

The integral

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The integral around C_p

$$\oint_{C_p} \frac{d\xi}{(\xi - \xi_p)} \quad (74)$$

is evaluated by letting

$$\xi = \xi_p + \epsilon e^{j\psi} \quad (75)$$

and therefore

$$d\xi = j\epsilon e^{j\psi} d\psi. \quad (76)$$

This gives

$$\oint_{C_p} \frac{d\xi}{(\xi - \xi_p)} = \int_0^{-2\pi} \frac{j\epsilon e^{j\psi}}{\epsilon e^{j\psi}} d\psi = -2\pi j \quad (77)$$

and then

$$\oint_{C_p} W_p(\xi, y, \zeta) d\xi = -j\pi \left[\frac{-e^{-p_0^p(2h+y)} - e^{-p_0^p(2h-y)} + e^{-p_0^p y} + e^{p_0^p y}}{2h\xi_p} \right]. \quad (78)$$

4.7 Method of Moments Numerical Solution

Equations (47) and (48) are solved using a Galerkin's Moment Method formulation [43]. The current functions are first expanded such that

$$k_\beta(x') = \sum_{n=1}^{N_\beta} a_\beta^n e_\beta^n(x') \quad (79)$$

where $e_\beta^n(x')$ is a basis set, the a_β^n represent unknown amplitude coefficients, and

where

Also let

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therefore

$$\int_{-a}^a k_{\beta}(x') e^{-j\xi x'} dx' = \sum_{n=1}^{N_{\beta}} a_{\beta}^n \int_{-a}^a e_{\beta}^n(x') e^{-j\xi x'} dx' = \sum_{n=1}^{N_{\beta}} a_{\beta}^n f_{\beta}^n(\xi) \quad (80)$$

where

$$f_{\beta}^n(\xi) = \int_{-a}^a e_{\beta}^n(x') e^{-j\xi x'} dx'. \quad (81)$$

Also let

$$g_{\beta}^m(\xi) = \int_{-a}^a t_{\beta}^m(x) e^{j\xi x} dx \quad (82)$$

where $t_{\beta}^m(x)$ is a testing function which will be applied later. Application of the partial derivatives in (40) and (41) gives

$$\frac{\partial^2}{\partial x^2} e^{j\xi(x-x')} = \frac{\partial}{\partial x} j\xi e^{j\xi(x-x')} = -\xi^2 e^{j\xi(x-x')} \quad (83)$$

therefore $\frac{\partial^2}{\partial x^2} \rightarrow -\xi^2$ and $\frac{\partial}{\partial x} \rightarrow j\xi$ in (47) and (48). These substitutions result in

$$\begin{aligned} \lim_{y \rightarrow 0} \left\{ \int_{-a}^a \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot k_x(x') dx' \right. \\ \left. - \int_{-a}^a \left[\int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot k_z(x') dx' \right\} = 0 \end{aligned} \quad (84)$$

$$\begin{aligned} \lim_{y \rightarrow 0} \left\{ - \int_{-a}^a \left[\int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot k_x(x') dx' \right. \\ \left. + \int_{-a}^a \left[\int_{-\infty}^{\infty} (k_o^2 - \zeta^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot k_z(x') dx' \right\} = 0. \end{aligned} \quad (85)$$

Expanding the current functions as specified in (79) gives

$$\lim_{y \rightarrow 0} \left\{ \int_{-a}^a \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot \sum_{n=1}^{N_x} a_x^n e_x^n(x') dx' \right. \\ \left. - \int_{-a}^a \left[\int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot \sum_{n=1}^{N_z} a_z^n e_z^n(x') dx' \right\} = 0 \quad (86)$$

$$\lim_{y \rightarrow 0} \left\{ - \int_{-a}^a \left[\int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot \sum_{n=1}^{N_x} a_x^n e_x^n(x') dx' \right. \\ \left. + \int_{-a}^a \left[(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, 0) d\xi \right] \cdot \sum_{n=1}^{N_z} a_z^n e_z^n(x') dx' \right\} = 0. \quad (87)$$

Now applying the testing operator

$$\int_{-a}^a t_\alpha^m(x) dx. \quad (88)$$

to both sides of (86) and (87), and exchanging the order of integration yields

$$\lim_{y \rightarrow 0} \left\{ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[(k_o^2 - \xi^2) W(\xi, y, 0) \sum_{n=1}^{N_x} a_x^n \int_{-a}^a e_x^n(x') e^{-j\xi x'} dx' \int_{-a}^a t_x^m(x) e^{j\xi x} dx \right] d\xi \right. \\ \left. - \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\zeta \xi W(\xi, y, 0) \sum_{n=1}^{N_z} a_z^n \int_{-a}^a e_z^n(x') e^{-j\xi x'} dx' \int_{-a}^a t_x^m(x) e^{j\xi x} dx \right] d\xi \right\} = 0 \quad (89)$$

$$\lim_{y \rightarrow 0} \left\{ - \frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\zeta \xi W(\xi, y, 0) \sum_{n=1}^{N_x} a_x^n \int_{-a}^a e_x^n(x') e^{-j\xi x'} dx' \int_{-a}^a t_x^m(x) e^{j\xi x} dx \right] d\xi \right. \\ \left. + \frac{1}{4\pi} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, 0) \sum_{n=1}^{N_z} a_z^n \int_{-a}^a e_z^n(x') e^{-j\xi x'} dx' \int_{-a}^a t_z^m(x) e^{j\xi x} dx \right] d\xi \right\} = 0 \quad (90)$$

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Letting

$$\lim_{y \rightarrow 0} \left\{ \frac{1}{4\pi} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, 0) \sum_{n=1}^{N_x} a_x^n f_x^n(\xi) g_x^m(\xi) d\xi \right. \\ \left. - \frac{1}{4\pi} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) \sum_{n=1}^{N_z} a_z^n f_z^n(\xi) g_x^m(\xi) d\xi \right\} = 0 \quad (91)$$

$$\lim_{y \rightarrow 0} \left\{ -\frac{1}{4\pi} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) \sum_{n=1}^{N_x} a_x^n f_x^n(\xi) g_z^m(\xi) d\xi \right. \\ \left. + \frac{1}{4\pi} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, 0) \sum_{n=1}^{N_z} a_z^n f_z^n(\xi) g_z^m(\xi) d\xi \right\} = 0. \quad (92)$$

Interchanging the order of summation and integration and multiplying both sides by 4π results in

$$\sum_{n=1}^{N_x} a_x^n \left\{ \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, 0) f_x^n(\xi) g_x^m(\xi) d\xi \right\} \\ - \sum_{n=1}^{N_z} a_z^n \left\{ \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) f_z^n(\xi) g_x^m(\xi) d\xi \right\} = 0 \quad (93)$$

$$- \sum_{n=1}^{N_x} a_x^n \left\{ \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) f_x^n(\xi) g_z^m(\xi) d\xi \right\} \\ + \sum_{n=1}^{N_z} a_z^n \left\{ \lim_{y \rightarrow 0} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, 0) f_z^n(\xi) g_z^m(\xi) d\xi \right\} = 0. \quad (94)$$

Letting

$$A_{xx}^{mn} = \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, 0) f_x^n(\xi) g_x^m(\xi) d\xi \quad (95)$$

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$$A_{xz}^{mn} = \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) f_z^n(\xi) g_x^m(\xi) d\xi \quad (96)$$

$$A_{zx}^{mn} = \lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) f_x^n(\xi) g_z^m(\xi) d\xi \quad (97)$$

$$A_{zz}^{mn} = \lim_{y \rightarrow 0} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, 0) f_z^n(\xi) g_z^m(\xi) d\xi \quad (98)$$

equations (93) and (94) become

$$\sum_{n=1}^{N_x} A_{xx}^{mn} a_x^n - \sum_{n=1}^{N_z} A_{xz}^{mn} a_z^n = 0 \quad m = 1, 2, \dots, N_x \quad (99)$$

$$-\sum_{n=1}^{N_x} A_{zx}^{mn} a_x^n + \sum_{n=1}^{N_z} A_{zz}^{mn} a_z^n = 0 \quad m = 1, 2, \dots, N_z. \quad (100)$$

For a Chebyshev Galerkin's solution

$$t_{\beta}^m(x) = T_m(x/a) [1 - (x/a)^2]^{\pm 1/2} \quad (101)$$

$$e_{\beta}^n(x) = T_n(x/a) [1 - (x/a)^2]^{\pm 1/2} \quad (102)$$

where the "+" sign is used for $\beta=x$, and the "-" sign is used for $\beta=z$ [44].

In the transform domain, the time-harmonic continuity equation is expressed

$$\vec{\nabla} \cdot \vec{k} = -j\omega \rho_s(x) \quad (103)$$

or

$$\frac{\partial k_x}{\partial x} + j\zeta k_z = -j\omega \rho_s(x). \quad (104)$$

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It can be seen from (104) that if $k_z(x)$ is an even function, and $k_x(x)$ is an odd function, then $\rho_s(x)$ will be an even function, and thus represent an even propagation mode. Likewise, if $k_z(x)$ is an odd function, and $k_x(x)$ is an even function, then $\rho_s(x)$ will be odd, and thus represent an odd mode. Therefore, for even modes $e_z^n(x)$ is even and $e_x^n(x)$ is odd, and

$$f_x^n(\xi) = \int_{-a}^a e_x^n(x') e^{-j\xi x'} dx' = -j2 \int_0^a e_x^n(x') \sin \xi x' dx' \quad (105)$$

$$f_z^n(\xi) = \int_{-a}^a e_z^n(x') e^{-j\xi x'} dx' = 2 \int_0^a e_z^n(x') \cos \xi x' dx'. \quad (106)$$

For odd modes $e_z^n(x)$ is odd and $e_x^n(x)$ is even, thus

$$f_x^n(\xi) = \int_{-a}^a e_x^n(x') e^{-j\xi x'} dx' = 2 \int_0^a e_x^n(x') \cos \xi x' dx' \quad (107)$$

$$f_z^n(\xi) = \int_{-a}^a e_z^n(x') e^{-j\xi x'} dx' = -j2 \int_0^a e_z^n(x') \sin \xi x' dx'. \quad (108)$$

Chebyshev polynomials of even order are even, and those of odd order are odd such that

$$T_n(-x) = (-1)^n T_n(x) \quad (109)$$

therefore, for even modes

$$e_x^n(x) = T_{2n+1}(x/a) \sqrt{1 - (x/a)^2} \quad (110)$$

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$$e_z^n(x) = \frac{T_{2n}(x/a)}{\sqrt{1 - (x/a)^2}}$$

and for odd modes

$$e_x^n(x) = T_{2n}(x/a)\sqrt{1 - (x/a)^2} \quad (112)$$

$$e_z^n(x) = \frac{T_{2n+1}(x/a)}{\sqrt{1 - (x/a)^2}}. \quad (113)$$

Likewise with the testing operator, for even modes $t_z^m(x)$ is even and $t_x^m(x)$ is odd, thus

$$g_x^m(\xi) = \int_{-a}^a t_x^m(x) e^{j\xi x} dx = j2 \int_0^a t_x^m(x) \sin \xi x dx \quad (114)$$

$$g_z^m(\xi) = \int_{-a}^a t_z^m(x) e^{j\xi x} dx = 2 \int_0^a t_z^m(x) \cos \xi x dx. \quad (115)$$

For odd modes $t_z^m(x)$ is odd and $t_x^m(x)$ is even, thus

$$g_x^m(\xi) = \int_{-a}^a t_x^m(x) e^{j\xi x} dx = 2 \int_0^a t_x^m(x) \cos \xi x dx \quad (116)$$

$$g_z^m(\xi) = \int_{-a}^a t_z^m(x) e^{j\xi x} dx = j2 \int_0^a t_z^m(x) \sin \xi x dx. \quad (117)$$

The β^{th} component equation is tested with the β^{th} expansion basis function, or $t_\alpha^m = e_\alpha^m$, and $g_\alpha^m(\xi) = \pm f_\alpha^m(\xi)$ with + (-) for even (odd) current components. It is seen from expressions (105) through (108), and (114) through (117) that four generic integral types are implicated [24]. The first of these is

$$I_z^{ne}(\xi) = \int_0^a \frac{T_{2n}(x/a)}{\sqrt{1-(x/a)^2}} \cos(\xi x) dx. \quad (118)$$

First a change of variables is made, letting $x/a = \tilde{x}$, and then $dx = a d\tilde{x}$. When $x = 0$, $\tilde{x} = 0$, and when $x = a$, $\tilde{x} = 1$ therefore

$$I_z^{ne}(\xi) = a \int_0^1 \frac{T_{2n}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x}. \quad (119)$$

The relationship

$$\int_0^1 T_{2n}(x) \frac{\cos(ax)}{\sqrt{1-x^2}} dx = (-1)^n \frac{\pi}{2} J_{2n}(a) \quad (120)$$

is found in [45], therefore (119) may be expressed in closed form

$$I_z^{ne}(\xi) = (-1)^n \frac{\pi a}{2} J_{2n}(\xi a). \quad (121)$$

The second generic integral type is given by

$$I_z^{no}(\xi) = \int_0^a \frac{T_{2n+1}(x/a)}{\sqrt{1-(x/a)^2}} \sin(\xi x) dx \quad (122)$$

which after application of the change of variable becomes

$$I_z^{no}(\xi) = a \int_0^1 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x}. \quad (123)$$

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Again a relationship found in [45] is applied

$$\int_0^1 T_{2n+1}(x) \frac{\sin(ax)}{\sqrt{1-x^2}} dx = (-1)^n \frac{\pi}{2} J_{2n+1}(a) \quad (124)$$

therefore (123) may be expressed in closed form

$$I_z^{no}(\xi) = (-1)^n \frac{\pi a}{2} J_{2n+1}(\xi a). \quad (125)$$

The third generic integral type is given by

$$I_x^{ne}(\xi) = \int_0^a T_{2n}(x/a) \sqrt{1-(x/a)^2} \cos(\xi x) dx \quad (126)$$

which becomes

$$I_x^{ne}(\xi) = \int_0^a T_{2n}(x/a) \frac{1-(x/a)^2}{\sqrt{1-(x/a)^2}} \cos(\xi x) dx. \quad (127)$$

After the change of variables applied previously, (127) becomes

$$I_x^{ne}(\xi) = a \int_0^1 T_{2n}(\tilde{x}) \frac{1-(\tilde{x})^2}{\sqrt{1-(\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} \quad (128)$$

which may be expressed

$$I_x^{ne}(\xi) = a \int_0^1 \frac{T_{2n}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} - a \int_0^1 (\tilde{x})^2 \frac{T_{2n}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x}. \quad (129)$$

The first term on the right side of (129) may be expressed in closed form through

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expression (120), yielding

$$I_x^{ne}(\xi) = (-1)^n \frac{\pi a}{2} J_{2n}(\xi a) - a \int_0^1 (\tilde{x})^2 \frac{T_{2n}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x}. \quad (130)$$

The second term in (130) must now be examined. In order to obtain a closed form expression a recursion formula for Chebshev polynomials [46]

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \quad (131)$$

is applied. From this the following relationships are obtained

$$xT_n(x) = \frac{1}{2} [T_{n+1}(x) + T_{n-1}(x)] \quad (132)$$

$$x^2 T_n(x) = \frac{1}{2} [xT_{n+1}(x) + xT_{n-1}(x)] \quad (133)$$

$$xT_{n+1}(x) = \frac{1}{2} [T_{n+2}(x) + T_n(x)] \quad (134)$$

$$xT_{n-1}(x) = \frac{1}{2} [T_n(x) + T_{n-2}(x)]. \quad (135)$$

Therefore

$$\begin{aligned} x^2 T_n(x) &= \frac{1}{2} \left\{ \frac{1}{2} [T_{n+2}(x) + T_n(x)] + \frac{1}{2} [T_n(x) + T_{n-2}(x)] \right\} \\ &= \frac{1}{4} \left\{ T_{n+2}(x) + 2T_n(x) + T_{n-2}(x) \right\}. \end{aligned} \quad (136)$$

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$$x^2 T_{2n}(x) = \frac{1}{4} \left\{ T_{2(n+1)}(x) + 2T_{2n}(x) + T_{2(n-1)}(x) \right\} \quad (137)$$

and therefore

$$\begin{aligned} a \int_0^1 (\tilde{x})^2 \frac{T_{2n}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} \\ = a \int_0^1 \frac{[T_{2(n+1)}(\tilde{x}) + 2T_{2n}(\tilde{x}) + T_{2(n-1)}(\tilde{x})]}{4\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x}. \end{aligned} \quad (138)$$

It is seen that when $n=0$ in (138), a Chebyshev polynomial of negative order appears in this expression. Chebyshev polynomials are defined by [45]

$$T_n(x) = \cos(n \arccos x) \quad \dots \text{for } -1 < x < 1. \quad (139)$$

From this it is seen that

$$T_n(x) = T_{-n}(x) = T_{|n|}(x) \quad \dots \text{for } -1 < x < 1 \quad (140)$$

and thus

$$T_{2(n-1)}(x) = T_{2|n-1|}(x) \quad \dots \text{for } -1 < x < 1.$$

Applying this to (138) gives

$$\begin{aligned} a \int_0^1 (\tilde{x})^2 \frac{T_{2n}(\tilde{x})}{\sqrt{1 - \tilde{x}^2}} \cos(\xi a \tilde{x}) d\tilde{x} = \frac{a}{4} \left\{ \int_0^1 \frac{T_{2(n+1)}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} \right. \\ \left. + 2 \int_0^1 \frac{T_{2n}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} + \int_0^1 \frac{T_{2|n-1|}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} \right\}. \end{aligned} \quad (142)$$

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Using relationship (120) with the terms in (142) gives

$$2 \int_0^1 \frac{T_{2n}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} = (-1)^n \pi J_{2n}(\xi a) \quad (143)$$

$$\int_0^1 \frac{T_{2(n+1)}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} = (-1)^{n+1} \frac{\pi}{2} J_{2(n+1)}(\xi a) \quad (144)$$

and

$$\int_0^1 \frac{T_{2|n-1|}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} = (-1)^{|n-1|} \frac{\pi}{2} J_{2|n-1|}(\xi a). \quad (145)$$

Equation (142) may therefore be expressed

$$\begin{aligned} a \int_0^1 (\tilde{x})^2 \frac{T_{2n}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \cos(\xi a \tilde{x}) d\tilde{x} &= \frac{a}{4} \left\{ (-1)^{n+1} \frac{\pi}{2} J_{2(n+1)}(\xi a) \right. \\ &\quad \left. + (-1)^n J_{2n}(\xi a) + (-1)^{|n-1|} \frac{\pi}{2} J_{2|n-1|}(\xi a) \right\}. \end{aligned} \quad (146)$$

Thus

$$\begin{aligned} I_x^{ne}(\xi) &= (-1)^n \frac{\pi a}{2} J_{2n}(\xi a) - \frac{\pi a}{8} (-1)^{n+1} \frac{\pi}{2} J_{2(n+1)}(\xi a) \\ &\quad - \frac{\pi a}{4} (-1)^n J_{2n}(\xi a) - \frac{\pi a}{8} (-1)^{|n-1|} \frac{\pi}{2} J_{2|n-1|}(\xi a). \end{aligned} \quad (147)$$

After algebraic manipulation, (147) becomes

$$I_x^{ne}(\xi) = (-1)^n \frac{\pi a}{4} \left[J_{2n}(\xi a) + \frac{1}{2} \left(J_{2(n+1)}(\xi a) + J_{2|n-1|}(\xi a) \right) \right]. \quad (148)$$

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The fourth and final generic integral type is given by

$$I_x^{no}(\xi) = \int_0^a T_{2n+1}(x/a) \sqrt{1 - (x/a)^2} \sin(\xi x) dx \quad (149)$$

which may be written

$$I_x^{no}(\xi) = \int_0^a T_{2n+1}(x/a) \frac{1 - (x/a)^2}{\sqrt{1 - (x/a)^2}} \sin(\xi x) dx. \quad (150)$$

After the change of variables (150) becomes

$$I_x^{no}(\xi) = a \int_0^1 T_{2n+1}(\tilde{x}) \frac{1 - (\tilde{x})^2}{\sqrt{1 - (\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} \quad (151)$$

or

$$I_x^{no}(\xi) = a \int_0^1 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} - a \int_0^1 (\tilde{x})^2 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x}. \quad (152)$$

By relationship (124), this becomes

$$I_x^{no}(\xi) = (-1)^n \frac{\pi a}{2} J_{2n+1}(\xi a) - a \int_0^1 (\tilde{x})^2 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x}. \quad (153)$$

Again a recursion formula for Chebyshev polynomials [46]

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \quad (154)$$

must be applied to obtain a closed form expression for the second term on the right side

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of (153). From (154) the following relationships are found

$$xT_n(x) = \frac{1}{2} [T_{n+1}(x) + T_{n-1}(x)] \quad (155)$$

$$xT_{2n+1}(x) = \frac{1}{2} [T_{2n+2}(x) + T_{2n}(x)] \quad (156)$$

$$x^2T_{2n+1}(x) = \frac{1}{2} [xT_{2n+2}(x) + xT_{2n}(x)] \quad (157)$$

$$xT_{2n+2}(x) = \frac{1}{2} [T_{2n+3}(x) + T_{2n+1}(x)] \quad (158)$$

$$xT_{2n}(x) = \frac{1}{2} [T_{2n+1}(x) + T_{2n-1}(x)]. \quad (159)$$

Combining (157), (158), and (159) yields

$$\begin{aligned} x^2T_{2n+1}(x) &= \frac{1}{2} \left\{ \frac{1}{2} [T_{2n+3}(x) + T_{2n+1}(x)] + \frac{1}{2} [T_{2n+1}(x) + T_{2n-1}(x)] \right\} \\ &= \frac{1}{4} \left\{ T_{2n+3}(x) + 2T_{2n+1}(x) + T_{2n-1}(x) \right\}. \end{aligned} \quad (160)$$

Therefore

$$\begin{aligned} a \int_0^1 (\tilde{x})^2 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1 - (\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} \\ = a \int_0^1 \frac{[T_{2n+3}(\tilde{x}) + 2T_{2n+1}(\tilde{x}) + T_{2n-1}(\tilde{x})]}{4\sqrt{1 - (\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} \end{aligned} \quad (161)$$

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$$\begin{aligned}
a \int_0^1 (\tilde{x})^2 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} &= \frac{a}{4} \left\{ \int_0^1 \frac{T_{2n+3}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} \right. \\
&\quad \left. + 2 \int_0^1 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} + \int_0^1 \frac{T_{2n-1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} \right\}.
\end{aligned} \tag{162}$$

Applying relationship (124) to the terms in equation (162) gives

$$\int_0^1 \frac{T_{2n+3}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} = (-1)^{n+1} \frac{\pi}{2} J_{2n+3}(\xi a) \tag{163}$$

$$2 \int_0^1 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} = (-1)^n \pi J_{2n+1}(\xi a) \tag{164}$$

$$\int_0^1 \frac{T_{2n-1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} = (-1)^{n-1} \frac{\pi}{2} J_{2n-1}(\xi a) \tag{165}$$

therefore

$$\begin{aligned}
a \int_0^1 (\tilde{x})^2 \frac{T_{2n+1}(\tilde{x})}{\sqrt{1-(\tilde{x})^2}} \sin(\xi a \tilde{x}) d\tilde{x} &= \frac{a}{4} \left\{ (-1)^{n+1} \frac{\pi}{2} J_{2n+3}(\xi a) \right. \\
&\quad \left. + (-1)^n \pi J_{2n+1}(\xi a) + (-1)^{n-1} \frac{\pi}{2} J_{2n-1}(\xi a) \right\}.
\end{aligned} \tag{166}$$

Thus

$$\begin{aligned}
I_x^{no}(\xi) &= (-1)^n \frac{\pi a}{2} J_{2n+1}(\xi a) - \frac{\pi a}{4} (-1)^n J_{2n+1}(\xi a) \\
&\quad - \frac{\pi a}{8} (-1)^{n+1} J_{2n+3}(\xi a) - \frac{\pi a}{8} (-1)^{n-1} J_{2n-1}(\xi a)
\end{aligned} \tag{167}$$

which after algebraic manipulation becomes

$$I_x^{no}(\xi) = (-1)^n \frac{\pi a}{4} \left[J_{2n+1}(\xi a) + \frac{1}{2} \left(J_{2n+3}(\xi a) + J_{2n-1}(\xi a) \right) \right]. \quad (168)$$

Summarizing the closed form expressions for the four generic integral types

$$I_z^{ne}(\xi) = (-1)^n \frac{\pi a}{2} J_{2n}(\xi a) \quad (169)$$

$$I_z^{no}(\xi) = (-1)^n \frac{\pi a}{2} J_{2n+1}(\xi a) \quad (170)$$

$$I_x^{ne}(\xi) = (-1)^n \frac{\pi a}{4} \left[J_{2n}(\xi a) + \frac{1}{2} \left(J_{2(n+1)}(\xi a) + J_{2|n-1|}(\xi a) \right) \right] \quad (171)$$

$$I_x^{no}(\xi) = (-1)^n \frac{\pi a}{4} \left[J_{2n+1}(\xi a) + \frac{1}{2} \left(J_{2n+3}(\xi a) + J_{2n-1}(\xi a) \right) \right]. \quad (172)$$

Applying the results above leads to

$$g_z^m(\xi) = \int_{-a}^a t_z^m(x) e^{j\xi x} dx = \begin{cases} 2 \int_0^a t_z^m(x) \cos \xi x dx = 2I_z^{me}(\xi) & \dots \text{for even } t_z^m \\ j 2 \int_0^a t_z^m(x) \sin \xi x dx = j 2I_z^{mo}(\xi) & \dots \text{for odd } t_z^m \end{cases} \quad (173)$$

$$g_x^m(\xi) = \int_{-a}^a t_x^m(x) e^{j\xi x} dx = \begin{cases} 2 \int_0^a t_x^m(x) \cos \xi x dx = 2I_x^{me}(\xi) & \dots \text{for even } t_x^m \\ j 2 \int_0^a t_x^m(x) \sin \xi x dx = j 2I_x^{mo}(\xi) & \dots \text{for odd } t_x^m \end{cases} \quad (174)$$

$$f_z^n(\xi) = \int_{-a}^a e_z^n(x') e^{-j\xi x'} dx' = \begin{cases} 2 \int_0^a e_z^n(x') \cos \xi x' dx' = 2I_z^{ne}(\xi) & \dots \text{for even } e_z^n \\ -j2 \int_0^a e_z^n(x') \sin \xi x' dx' = -j2I_z^{no}(\xi) & \dots \text{for odd } e_z^n \end{cases} \quad (175)$$

and

$$f_x^n(\xi) = \int_{-a}^a e_x^n(x') e^{-j\xi x'} dx' = \begin{cases} 2 \int_0^a e_x^n(x') \cos \xi x' dx' = 2I_x^{ne}(\xi) & \dots \text{for even } e_x^n \\ -j2 \int_0^a e_x^n(x') \sin \xi x' dx' = -j2I_x^{no}(\xi) & \dots \text{for odd } e_x^n. \end{cases} \quad (176)$$

Thus for even modes

$$A_{xx}^{mn} = \lim_{y \rightarrow 0} 8 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, 0) I_x^{no}(\xi) I_x^{mo}(\xi) d\xi \quad (177)$$

$$A_{xz}^{mn} = \lim_{y \rightarrow 0} 8j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) I_z^{ne}(\xi) I_x^{mo}(\xi) d\xi \quad (178)$$

$$A_{zx}^{mn} = \lim_{y \rightarrow 0} -8j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) I_x^{no}(\xi) I_z^{me}(\xi) d\xi \quad (179)$$

$$A_{zz}^{mn} = \lim_{y \rightarrow 0} 8 (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, 0) I_z^{ne}(\xi) I_z^{me}(\xi) d\xi \quad (180)$$

and for odd modes

$$A_{xx}^{mn} = \lim_{y \rightarrow 0} 8 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, 0) I_x^{ne}(\xi) I_x^{me}(\xi) d\xi \quad (181)$$

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$$A_{xz}^{mn} = \lim_{y \rightarrow 0} -8j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) I_z^{no}(\xi) I_x^{me}(\xi) d\xi \quad (182)$$

$$A_{zx}^{mn} = \lim_{y \rightarrow 0} 8j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, 0) I_x^{ne}(\xi) I_z^{mo}(\xi) d\xi \quad (183)$$

$$A_{zz}^{mn} = \lim_{y \rightarrow 0} 8 \left(k_o^2 - \zeta^2 \right) \int_{-\infty}^{\infty} W(\xi, y, 0) I_z^{no}(\xi) I_z^{mo}(\xi) d\xi . \quad (184)$$

4.8 Analysis of Numerical Results

The dispersion characteristics of various higher-order modes for several stripline geometries and the corresponding surface current distributions are presented [25], [26], [40], [41]. The mode dispersion characteristics of a stripline with dimensions $w=10.1$ cm, and $h=3.67$ cm are shown in Figure 4.7. Here the first two higher-order modes are shown with the $l=1$ TM parallel-plate guided wave pole λ_p . The first higher-order mode is the odd TE_1 mode, while the second higher-order mode is the even TE_2 mode. Both modes are proper non-leaky modes having real propagation constants $\zeta > \lambda_p$, and exhibit low frequency cut-off. The poles associated with these modes lie on the top Riemann sheet. The cross-sectional surface current distributions for the TE_1 and TE_2 are presented in Figure 4.8 and Figure 4.9, respectively. It is seen that for both modes the axially-directed currents possesses the expected edge singularities, while the transverse current amplitudes are large near cutoff, and decrease far from cutoff.

The dispersion characteristics of the first three higher-order modes of a stripline with dimensions $w=5.0$ cm, and $h=1.75$ cm are shown in Figure 4.10. The first two

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higher-order modes are again the non-leaky TE_1 and TE_2 modes. It is noted that the low-frequency cutoff of each of these modes is higher than the cutoff frequency of the corresponding modes of the previous stripline structure, which had larger dimensions. Additionally, the TE_2 dispersion curve differs to a greater extent from the $l=1$ TM parallel-plate guided wave pole λ_p than the TE_2 mode of the stripline with larger dimensions. The cross-sectional surface current distribution for the even TE_2 mode is shown in Figure 4.11. Again the axially-directed current displays the expected edge singularity, and the amplitude of the transverse current is large near cutoff.

The third higher-order odd TE_3 mode is an improper, non-spectral leaky mode. It is noted that this mode shows no sharp cutoff, but rather is characterized by an attenuation constant which becomes large as the phase constant decreases. The cross-sectional surface current distribution associated with this mode is shown in Figure 4.12.

The dispersion characteristics of the first three higher-order modes supported by a stripline structure with dimensions $w=1.07$ cm and $h=0.37$ cm are shown in Figure 4.13, while the first, second, and fourth higher-order modes of the same structure are shown in Figure 4.14. The $l=1$ and $l=2$ TM parallel-plate guided wave poles λ_p have been included in both figures. Here the non-leaky TE_1 and TE_2 modes display cutoff at much higher frequencies than the corresponding modes of the stripline structures with larger dimensions. Also the TE_2 is displaced further in frequency from the $l=1$ TM parallel-plate guided wave pole than for the structures with larger dimensions. Thus it is seen that as the dimensions of the device become smaller, the TE_2 mode begins to differ significantly from the $l=1$ TM parallel-plate mode. Cross-sectional surface current distributions for the TE_1 and TE_2 modes are shown in Figure 4.15 and Figure 4.16,

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The third higher-order mode associated with this structure is the leaky non-spectral TE_3 mode, while the fourth higher-order mode is the leaky, non-spectral TM_1 mode. The cross-sectional current distributions associated with these modes are shown in Figure 4.17 and Figure 4.18. It was observed that the fourth higher-order mode shows almost no transverse current at all frequencies, and thus is classified as being TM.

4.9 TEM-Mode Specialization

The expressions obtained in the previous sections can be greatly simplified when only the fundamental TEM mode of stripline operation is considered. The results of this specialization are valuable because most stripline devices are operated at frequencies such that only the TEM mode is excited. Also, a characteristic impedance for stripline devices may be defined for this mode.

For the TEM mode $\zeta = k_o$, and only axial currents exist on the center strip, therefore $k_x(x) = 0$. As a result

$$K_{zz}(x|x') = \lim_{y \rightarrow 0} (k_o^2 - k_z^2) g_{\zeta}^a(x, y|x', 0) = 0 \quad (185)$$

and equations (40) and (41) both reduce to

$$\int_{-a}^a k_z(x') K_{zz}(x|x') dx' = 0 \quad (186)$$

where

$$K_{zz}(x|x') = \lim_{y \rightarrow 0} j k_o \frac{\partial}{\partial x} g_{\zeta}^t(x, y|x', 0). \quad (187)$$

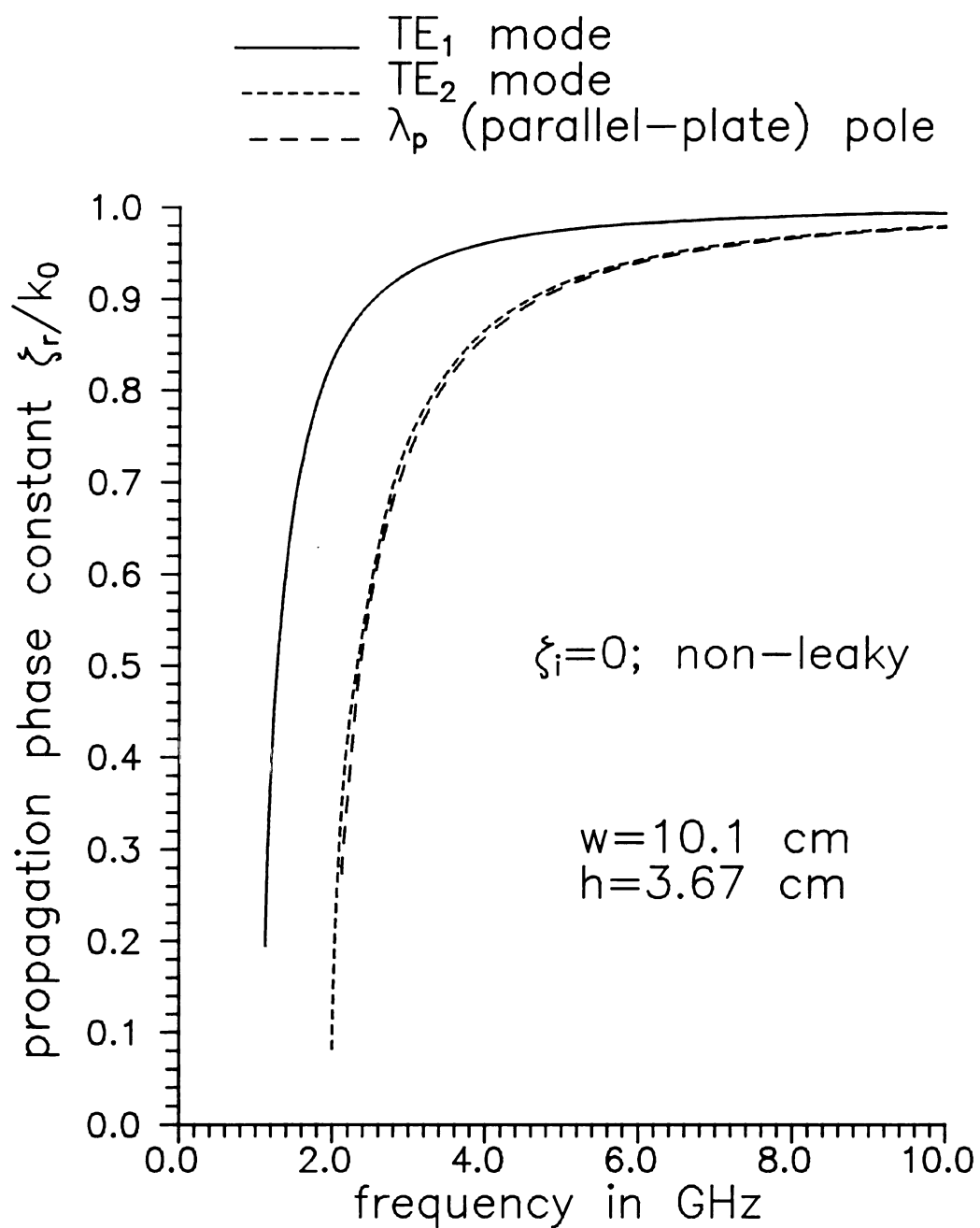


Figure 4.7 Dispersion characteristics of higher-order, non-leaky modes supported by stripline ($w=10.1$ cm, $h=3.67$ cm).

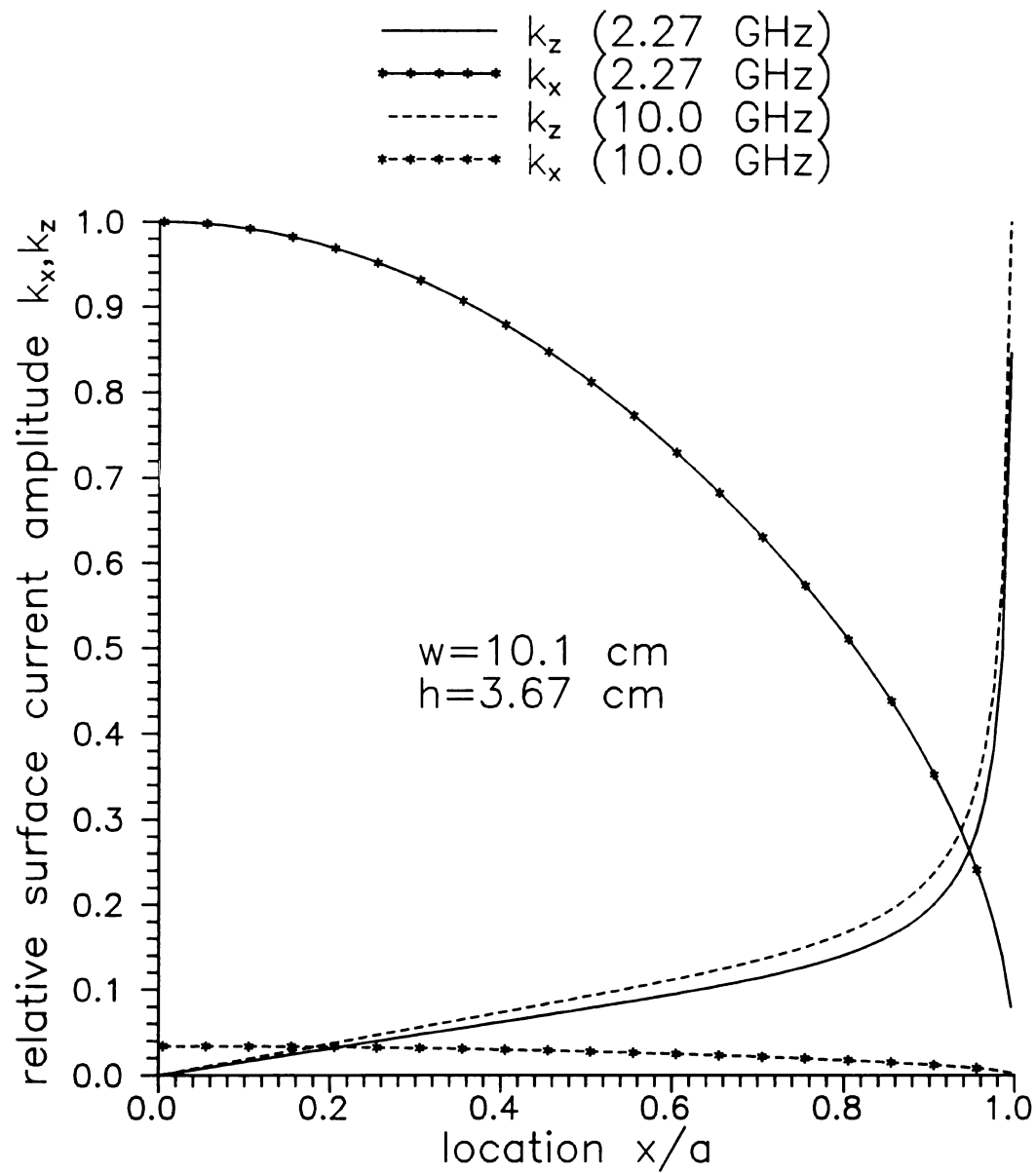


Figure 4.8 Cross-sectional current distribution for TE_1 mode of stripline, near and well above cutoff ($w=10.1$ cm, $h=3.67$ cm).

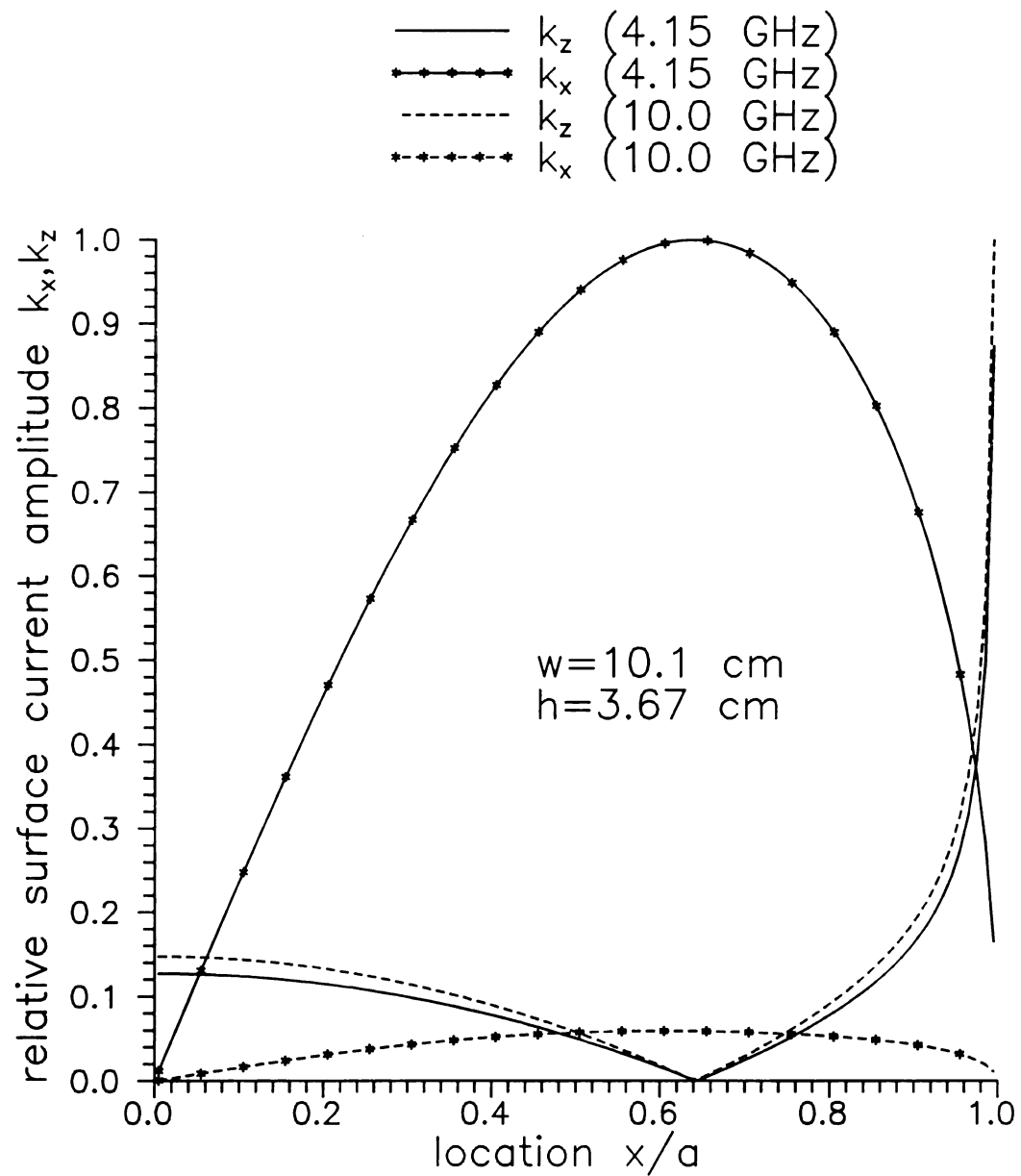


Figure 4.9 Cross-sectional current distribution for TE_2 mode of stripline, near and well above cutoff ($w=10.1$ cm, $h=3.67$ cm).

$w=5.0$ cm
 $h=1.75$ cm

— TE1 mode ($\xi_r, \xi_i=0.$)
 - - - TE2 mode ($\xi_r, \xi_i=0.$)
 ○ ○ ○ ○ ○ λ_p (parallel-plate) pole
 * * * * * TE3 mode (ξ_r)
 * * * * * TE3 mode ($-\xi_i$)

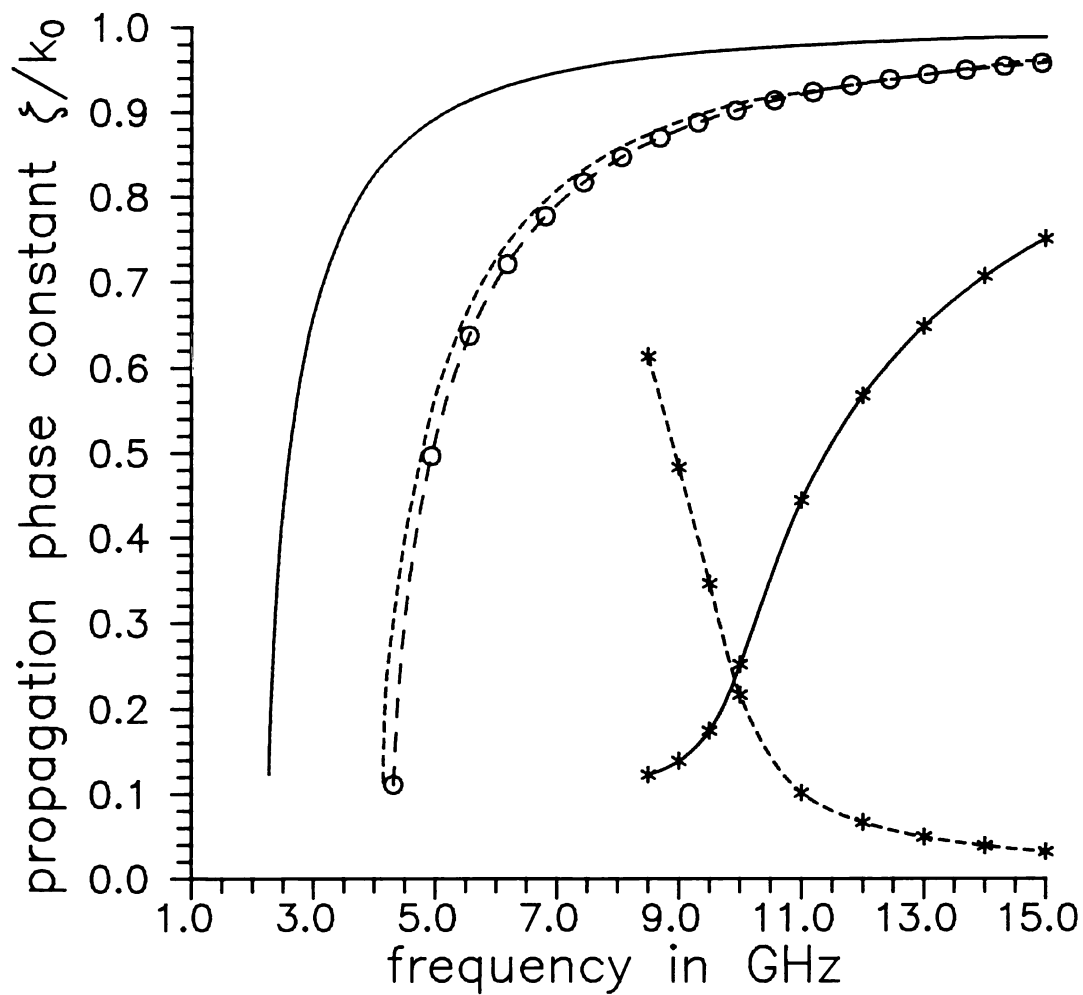


Figure 4.10 Dispersion characteristics of higher-order modes supported by stripline ($w=5.0$ cm, $h=1.75$ cm).

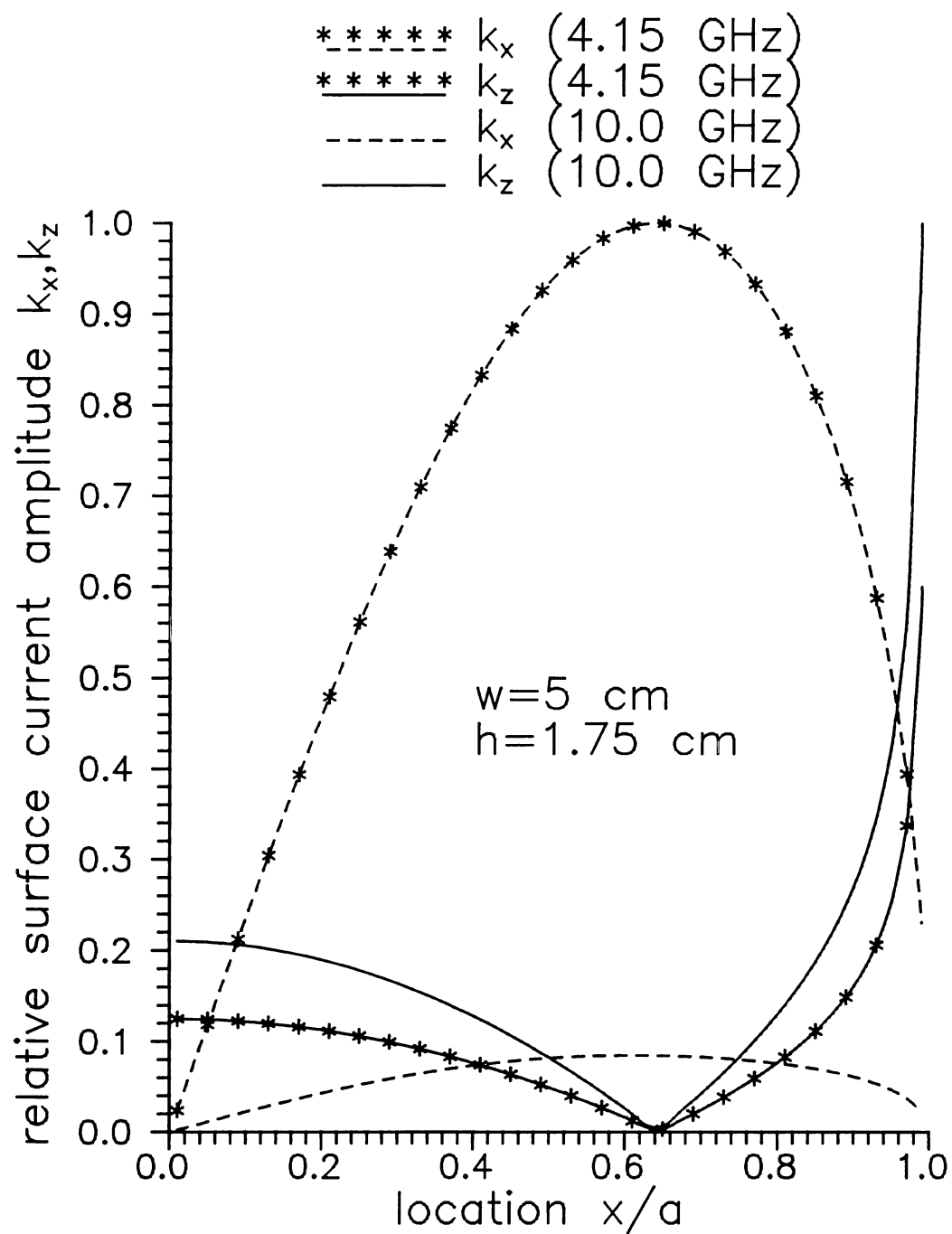


Figure 4.11 Cross-sectional current distribution for TE_2 mode of stripline, near and well above cutoff ($w=5.0$ cm, $h=1.75$ cm).

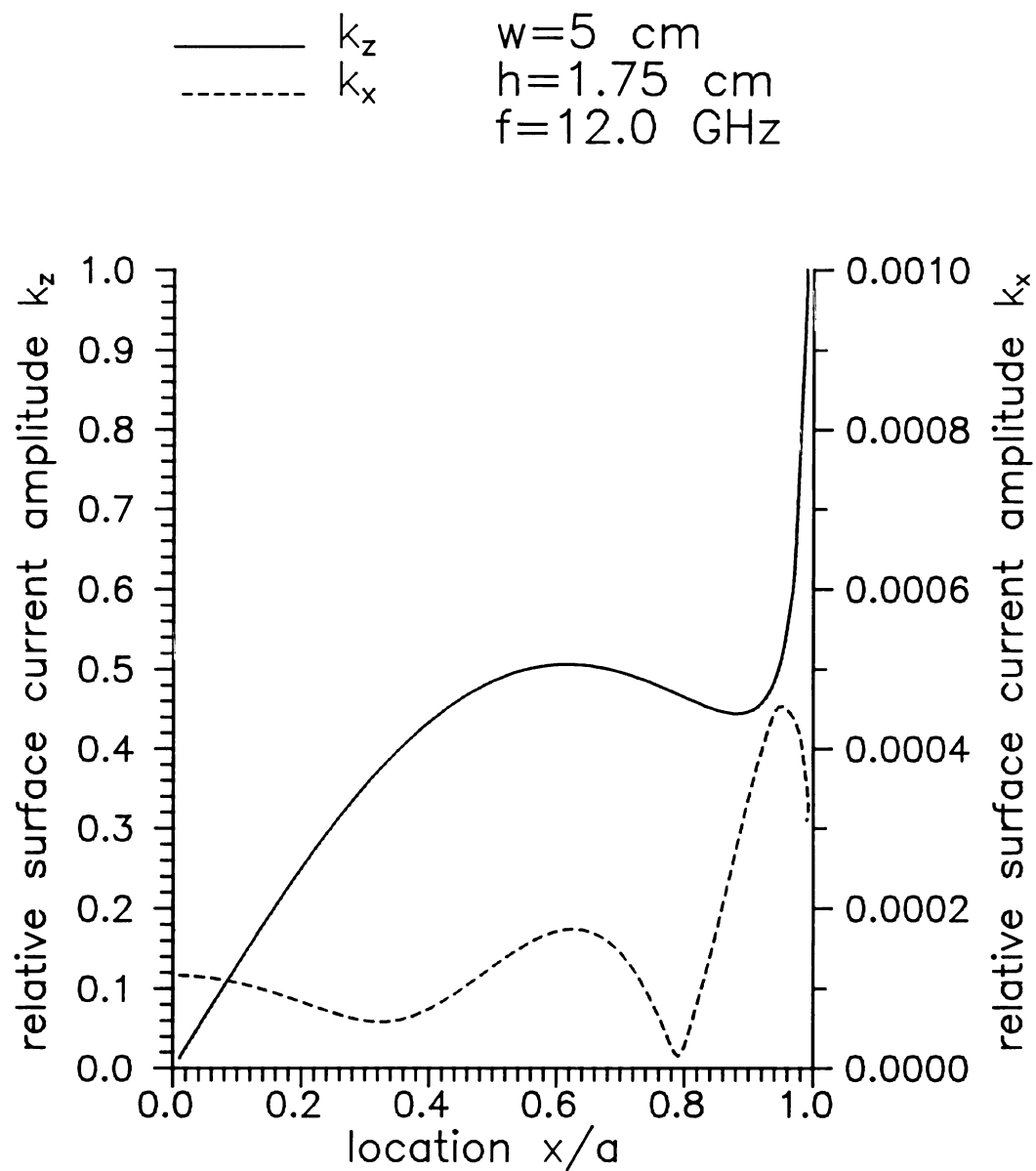


Figure 4.12 Cross-sectional current distribution for TE_3 (always leaky) stripline mode ($w=5.0 \text{ cm}$, $h=1.75 \text{ cm}$).

----- TE₁ stripline mode ($\zeta = \zeta_r + j0.$): ζ_r
 ----- TE₂ stripline mode ($\zeta = \zeta_r + j0.$): ζ_r
 * * * * * λ_{pole} of TM₁ parallel-plate mode
 ----- TE₃ stripline mode ($\zeta = \zeta_r + j\zeta_i$): ζ_r
 ▲ ▲ ▲ ▲ ▲ λ_{pole} of TM₂ parallel-plate mode
 x x x x x TE₃ stripline mode: ζ_i

$w = 1.07$ cm, $h = 0.37$ cm
 $Z_0 = 50.007$ ohms

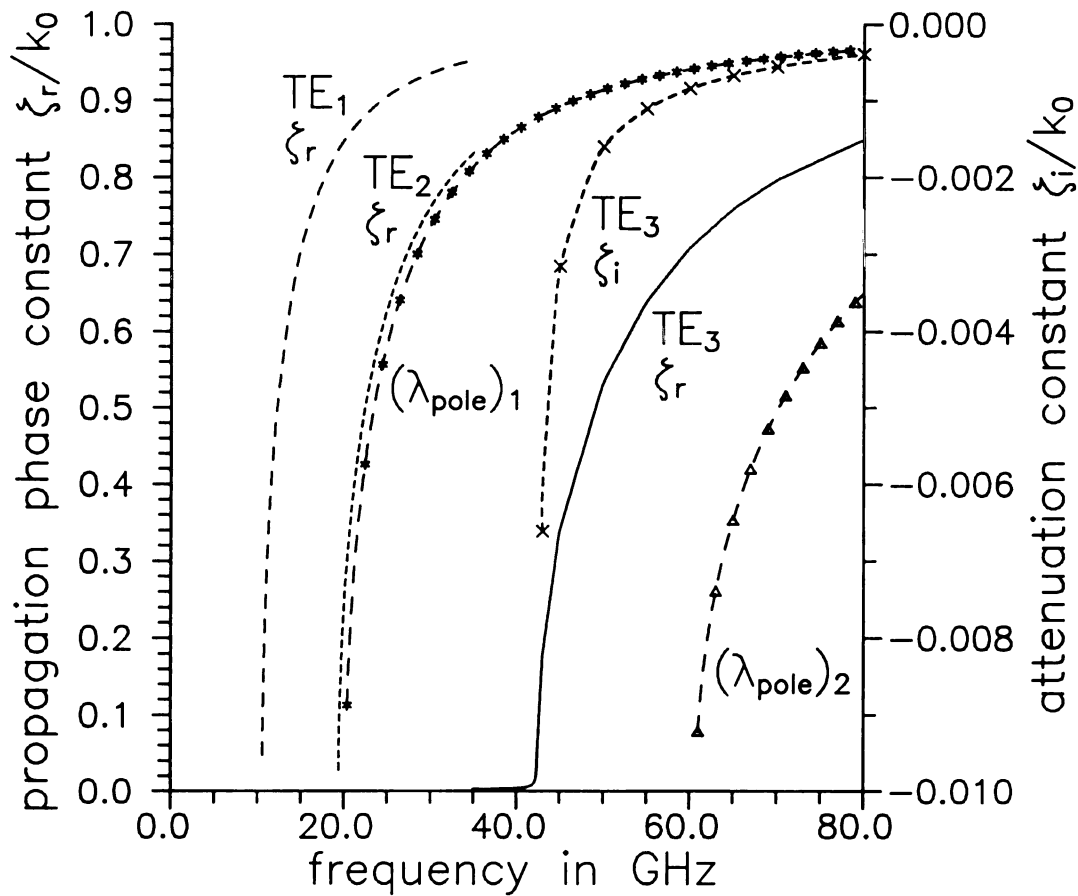


Figure 4.13 Dispersion characteristics of first three higher-order modes supported by stripline ($w = 1.07$ cm, $h = 0.37$ cm).

----- TE_1 stripline mode ($\xi = \xi_r + j0.$): ξ_r
 ----- TE_2 stripline mode ($\xi = \xi_r + j0.$): ξ_r
 * * * * * λ_{pole} of TM_1 parallel-plate mode
 ^ ^ ^ ^ ^ λ_{pole} of TM_2 parallel-plate mode
 ——— TM_1 stripline mode: ($\xi = \xi_r + j\xi_i$): ξ_r
 x x x x x TM_1 stripline mode: ξ_i

$w=1.07$ cm, $h=0.37$ cm
 $Z_0=50.007$ ohms

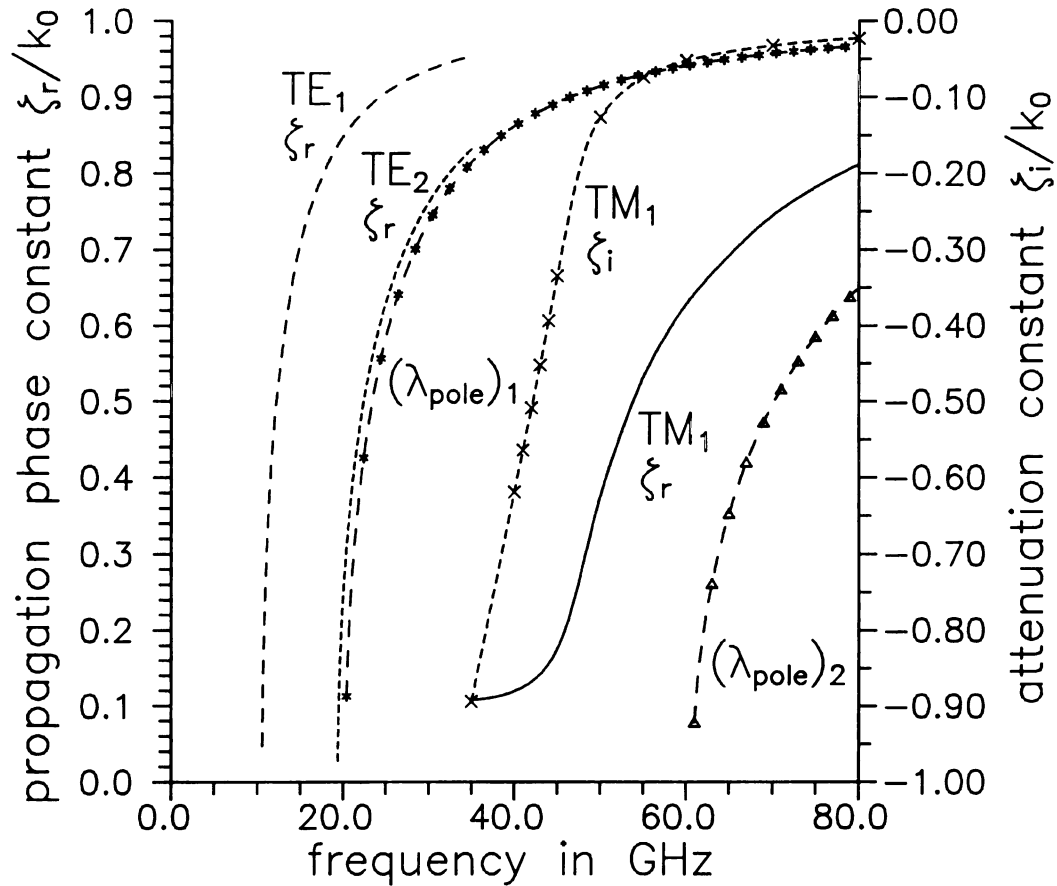


Figure 4.14 Dispersion characteristics of first (TE_1), second (TE_2), and fourth (TM_1) higher-order modes supported by stripline ($w=1.07$ cm, $h=0.37$ cm).

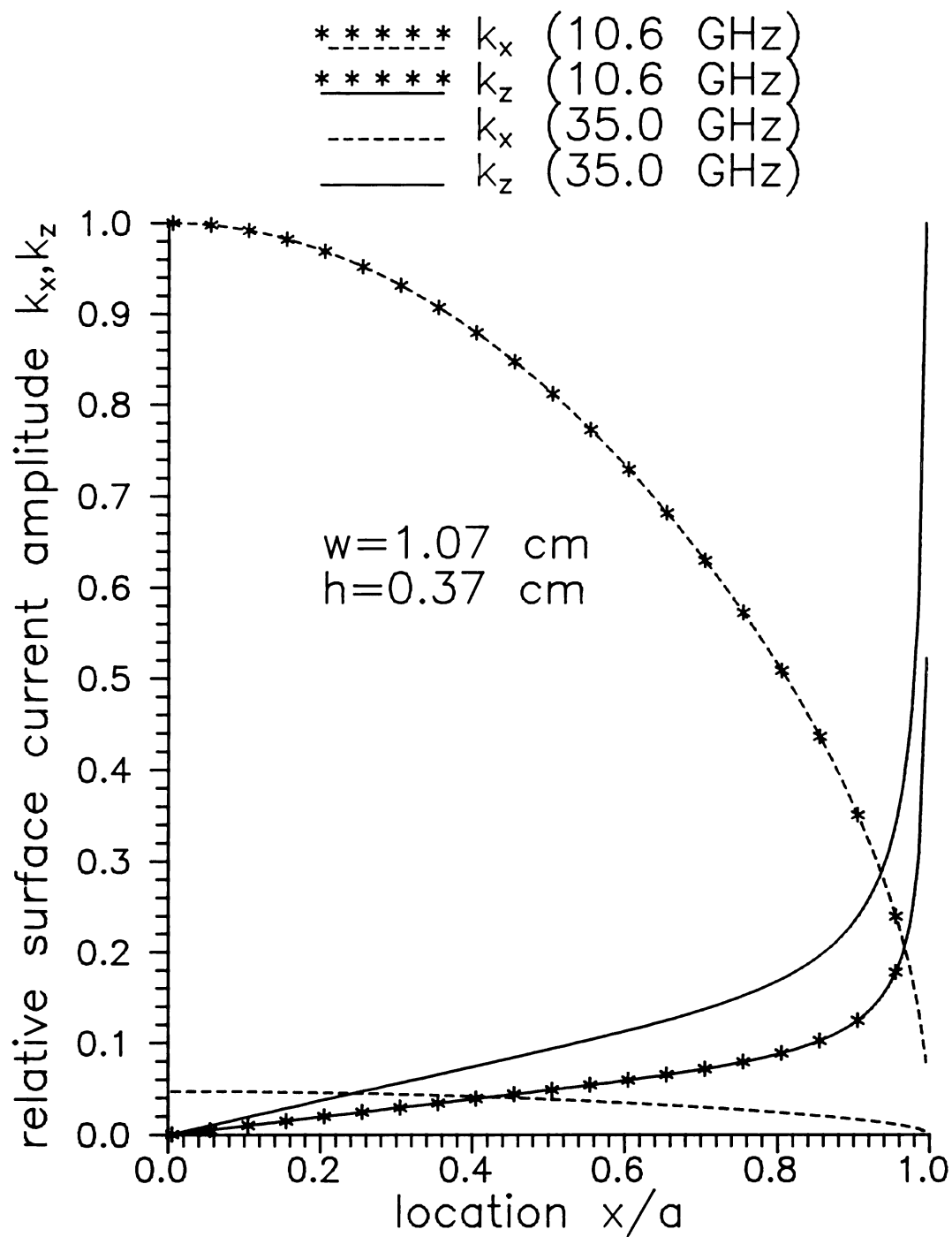


Figure 4.15 Cross-sectional current distribution for TE_1 mode of stripline, near and well above cutoff ($w=1.07$ cm, $h=0.37$ cm).

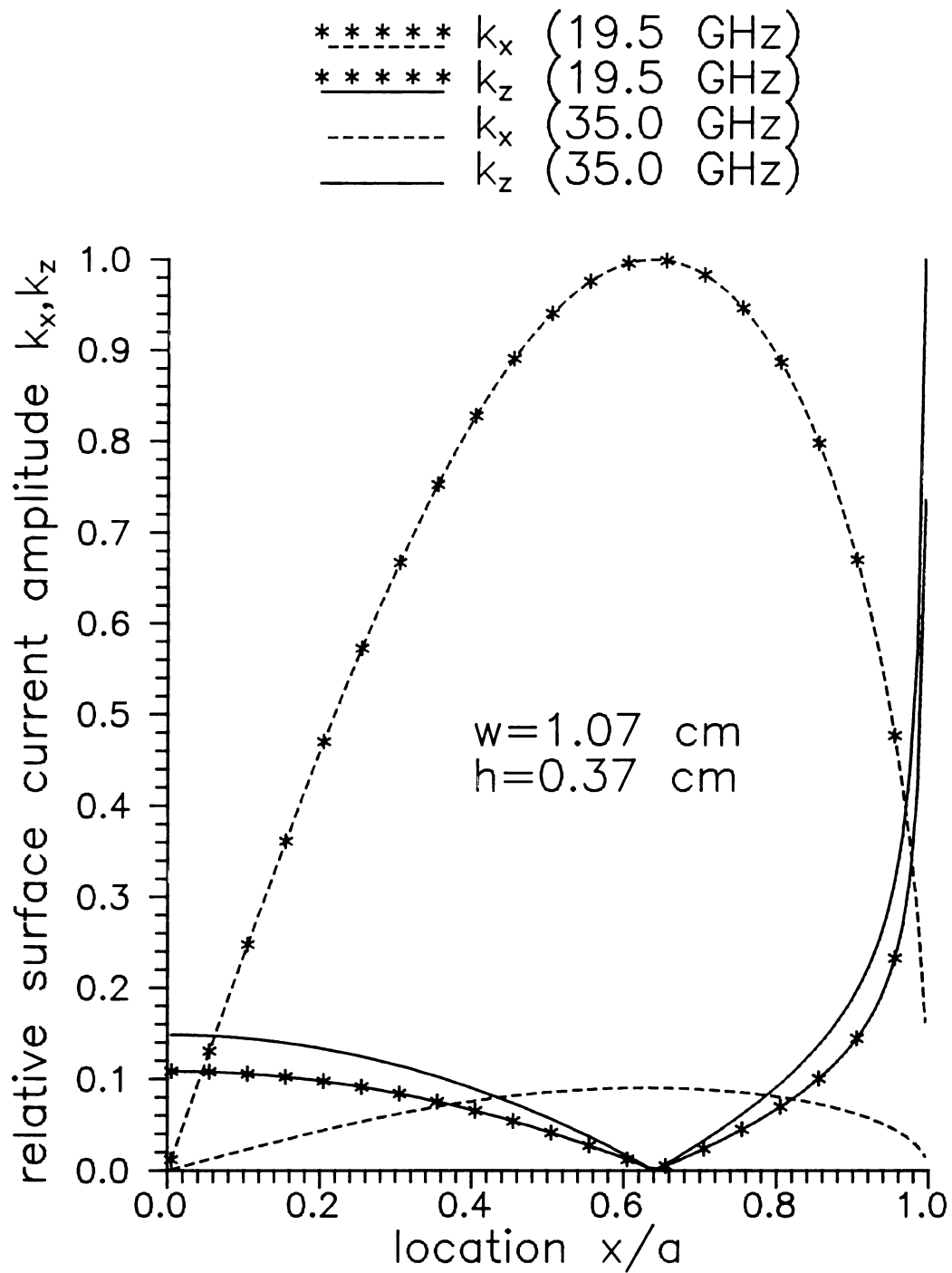


Figure 4.16 Cross-sectional current distribution for TE_2 mode of stripline, near and well above cutoff ($w=1.07 \text{ cm}$, $h=0.37 \text{ cm}$).

*****	k_x	{	43.0	GHz
*****	k_z			
-----	k_x	{	80.0	GHz
-----	k_z			

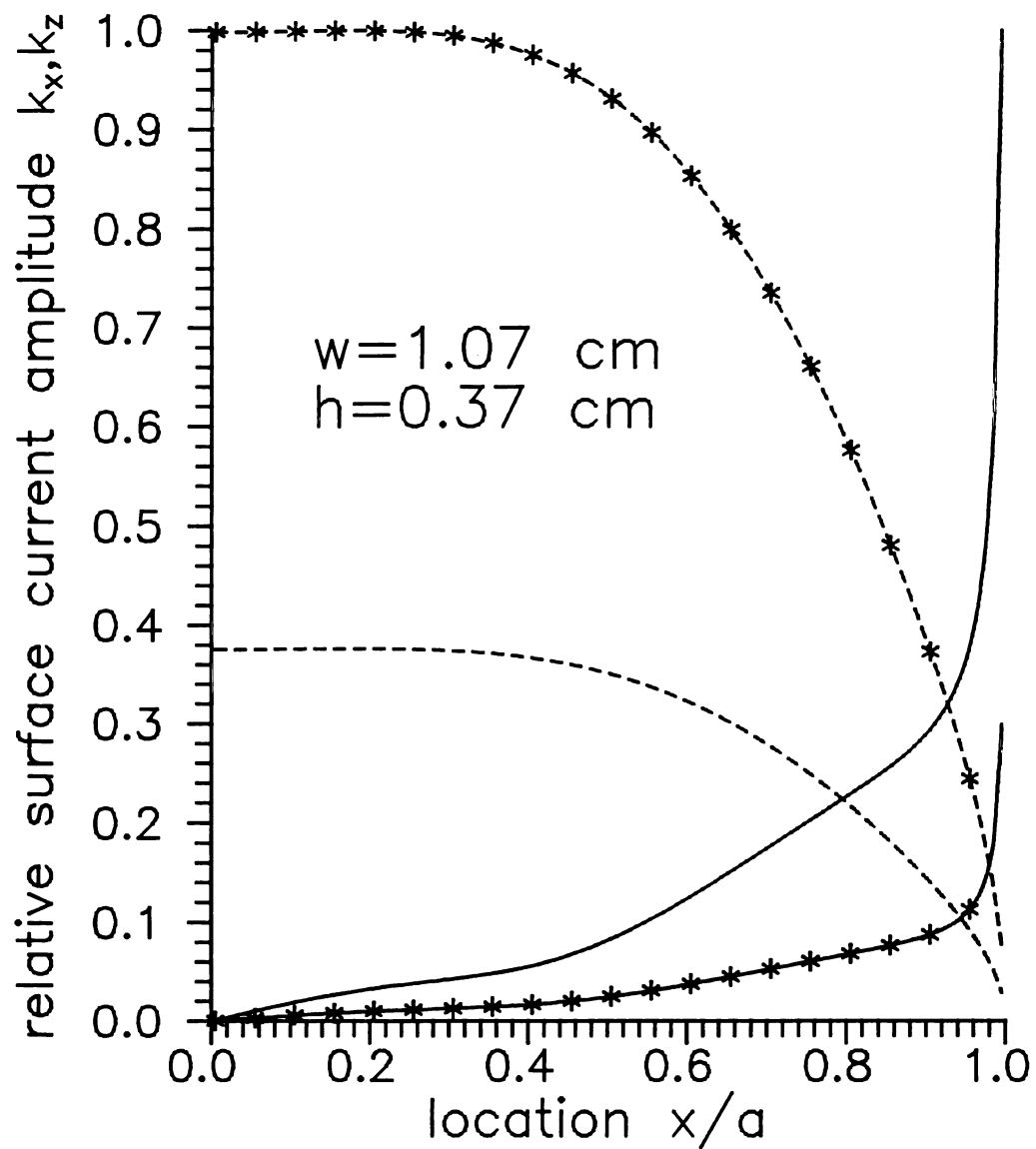


Figure 4.17 Cross-sectional current distribution for TE_3 (always leaky) mode of stripline, near and well above cutoff ($w=1.07$ cm, $h=0.37$ cm).

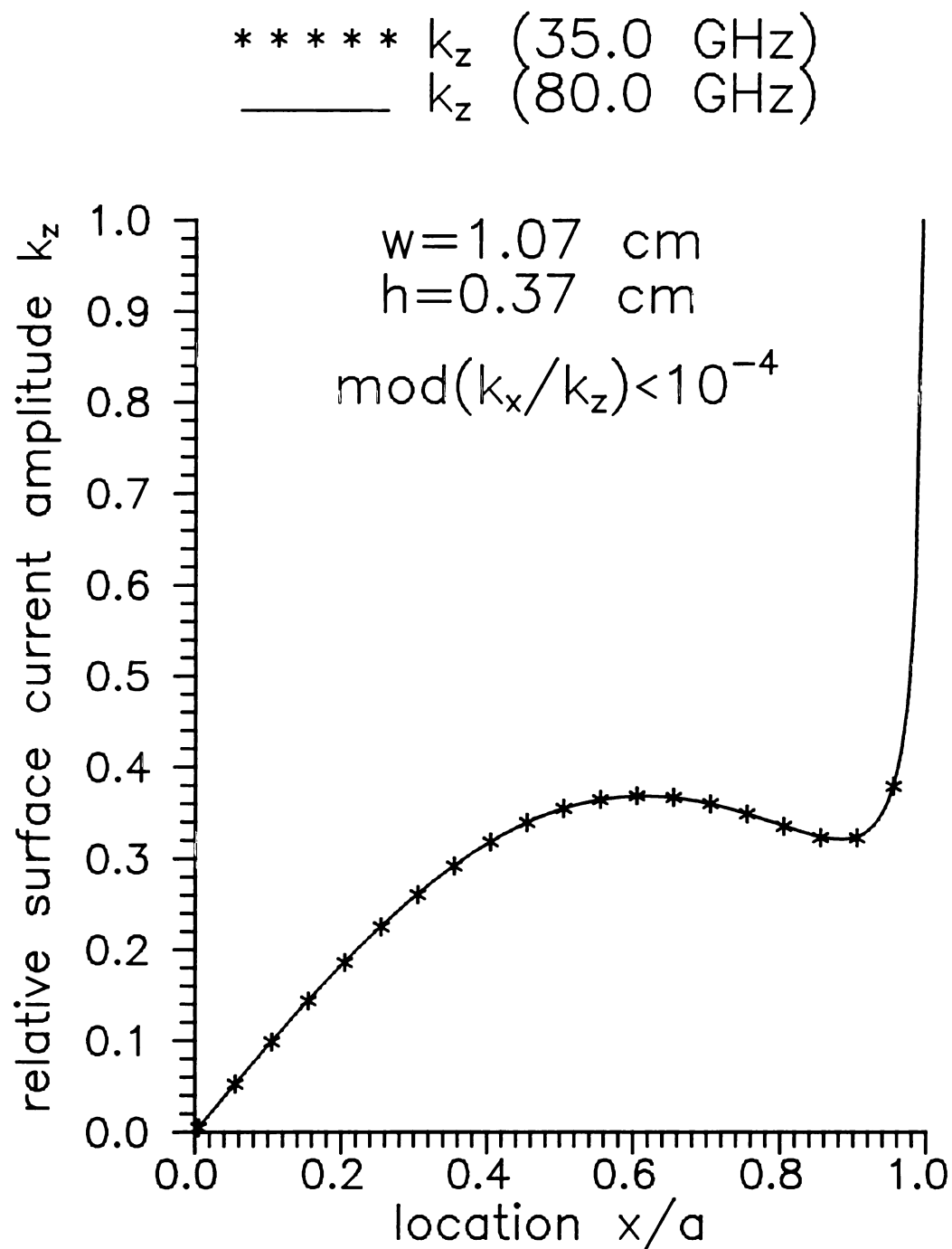


Figure 4.18 Cross-sectional current distribution for TM_1 (always leaky) mode of stripline, near and well above cutoff ($w=1.07 \text{ cm}$, $h=0.37 \text{ cm}$).

The derivative term $\partial/\partial x$ may be removed from the resulting integral expression

$$\lim_{y \rightarrow 0} \int_{-a}^a k_z(x') \frac{\partial}{\partial x} g_{\zeta}'(x, y | x', 0) dx' = 0 \quad (188)$$

so that

$$\frac{\partial}{\partial x} \left[\lim_{y \rightarrow 0} \int_{-a}^a k_z(x') g_{\zeta}'(x, y | x', 0) dx' \right] = 0 \quad (189)$$

and therefore

$$\lim_{y \rightarrow 0} \int_{-a}^a k_z(x') g_{\zeta}'(x, y | x', 0) dx' = C \quad (190)$$

for $|x| < a$. Here

$$g_{\zeta}'(x, y | x', 0) = g_{\zeta}^p(x, y | x', 0) + g_{\zeta}^{st}(x, y | x', 0). \quad (191)$$

For $\zeta = k_o$ the transform-domain wavenumber becomes

$$p_o(\bar{\lambda}) = \sqrt{\xi^2 + \zeta^2 - k_o^2} = \sqrt{\xi^2 + k_o^2 - k_o^2} = \sqrt{\xi^2} = |\xi| \quad (192)$$

therefore

$$\lim_{y \rightarrow 0} g_{\zeta}^p(x, y | x', 0) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi|\xi|} d\xi \quad (193)$$

and

$$\begin{aligned}
\lim_{y \rightarrow 0} g_{\zeta}^{sr}(x, y | x', 0) &= \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi|\xi|(1-e^{-4|\xi|h})} \left[-e^{-2|\xi|h} - e^{-2|\xi|h} + e^{-4|\xi|h} + e^{-4|\xi|h} \right] d\xi \\
&= \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{2\pi|\xi|(1-e^{-4|\xi|h})} \left[e^{-2|\xi|h} - 1 \right] d\xi.
\end{aligned} \tag{194}$$

The total transform-domain Green's function for the TEM mode is then given by

$$\lim_{y \rightarrow 0} g_{\zeta}'(x, y | x', 0) = \lim_{y \rightarrow 0} \left\{ g_{\zeta}^p(x, y | x', 0) + g_{\zeta}^{sr}(x, y | x', 0) \right\} = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi|\xi|} q(\xi) d\xi \tag{195}$$

where

$$\begin{aligned}
q(\xi) &= 1 - \frac{2e^{-2|\xi|h}}{1 - e^{-4|\xi|h}} (1 - e^{-2|\xi|h}) = 1 - \frac{2e^{-2|\xi|h}}{1 + e^{-2|\xi|h}} \\
&= \frac{1 - e^{-2|\xi|h}}{1 + e^{-2|\xi|h}} = \tanh(|\xi|h).
\end{aligned} \tag{196}$$

It is noted that for $\xi \ll 1$

$$q(\xi) \approx \frac{1 - (1 - 2|\xi|h)}{1 + (1 - 2|\xi|h)} \approx \frac{2|\xi|h}{2} = |\xi|h \tag{197}$$

and therefore

$$\lim_{\xi \rightarrow 0} \frac{q(\xi)}{|\xi|} = h. \tag{198}$$

Thus the integrand of (195) is not singular at $\xi = 0$.

The constant C in equation (190) is proportional to the potential difference which exists between the ground planes and the center conductor. Determination of this

potential difference requires an expression for the y-component of electric field which exists within the stripline device. As stated previously

$$\vec{E}(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_V \tilde{G}_0(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' \quad (199)$$

where

$$\vec{J}(\vec{r}') = \hat{x}J_x(\vec{r}') + \hat{z}J_z(\vec{r}') \quad (200)$$

but $J_x(\vec{r}') = 0$ for the TEM mode, therefore

$$\vec{E}(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_V \tilde{G}_0(\vec{r}|\vec{r}') \cdot \frac{[\hat{z}J_z(\vec{r}')] }{j\omega\epsilon_o} dv'. \quad (201)$$

The Green's function may be expressed

$$\tilde{G}_0(\vec{r}|\vec{r}') = (\hat{x}\hat{x} + \hat{z}\hat{z})G_0'(\vec{r}|\vec{r}') + \hat{y}\hat{y}G_0^y(\vec{r}|\vec{r}') \quad (202)$$

which then leads to

$$\vec{E}(\vec{r}) = \hat{z}k_o^2 \int_V G_0'(\vec{r}|\vec{r}') \frac{J_z(\vec{r}')}{j\omega\epsilon_o} dv' + \nabla \cdot \left\{ \nabla \cdot \hat{z} \int_V G_0'(\vec{r}|\vec{r}') \frac{J_z(\vec{r}')}{j\omega\epsilon_o} dv' \right\} \quad (203)$$

and the y-component of electric field is then given by

$$\begin{aligned} \hat{y} \cdot \vec{E}(\vec{r}) &= \hat{y} \cdot \hat{z}k_o^2 \int_V G_0'(\vec{r}|\vec{r}') \frac{J_z(\vec{r}')}{j\omega\epsilon_o} dv' + \hat{y} \cdot \nabla \left\{ \nabla \cdot \hat{z} \int_V G_0'(\vec{r}|\vec{r}') \frac{J_z(\vec{r}')}{j\omega\epsilon_o} dv' \right\} \\ &= \hat{y} \cdot \nabla \left\{ \frac{\partial}{\partial z} \int_V G_0'(\vec{r}|\vec{r}') \frac{J_z(\vec{r}')}{j\omega\epsilon_o} dv' \right\} \end{aligned} \quad (204)$$

or

$$E_y(\vec{r}) = \frac{\partial^2}{\partial y \partial z} \int_V G_0'(\vec{r}|\vec{r}') \cdot \frac{J_z(\vec{r}')}{j\omega\epsilon_o} dv'. \quad (205)$$

Because currents are confined to the surface of the center strip

$$J_z(\vec{r}) = K_z(x,z) \quad (206)$$

and

$$G_0'(\vec{r}|\vec{r}') = G_0'(\vec{\rho}|\vec{\rho}'; z - z') \quad (207)$$

where $\vec{\rho} = \hat{x}x + \hat{z}z$ is the transverse position vector. Thus the y-component of electric field may be expressed

$$E_y(\vec{r}) = \frac{\partial^2}{\partial y \partial z} \int_{-a}^a \int_{-\infty}^{\infty} G_0'(\vec{\rho}|\vec{\rho}'; z - z') \frac{K_z(x', z')}{j\omega\epsilon_o} dz' dx'. \quad (208)$$

From this it is apparent that the transform-domain electric field is given by

$$e_y(\vec{\rho}, \zeta) = j\zeta \frac{\partial}{\partial y} \int_{-a}^a \tilde{g}_\zeta'(\vec{\rho}|\vec{\rho}') \frac{k_z(x', \zeta)}{j\omega\epsilon_o} dx'. \quad (209)$$

For the TEM mode $\zeta = k_o$, and $p_0(\vec{\lambda}) = |\xi|$, therefore

$$e_y(\vec{\rho}, \zeta) = \frac{k_o}{\omega\epsilon_o} \frac{\partial}{\partial y} \int_{-a}^a \tilde{g}_{k_o}'(\vec{\rho}|\vec{\rho}') k_z(x', k_o) dx' \quad (210)$$

with the transform-domain Green's function given by

$$g_{k_o}^t(\vec{\rho}|\vec{\rho}') = g_{k_o}^p(\vec{\rho}|\vec{\rho}') + g_{k_o}^{st}(\vec{\rho}|\vec{\rho}') \quad (211)$$

where

$$g_{k_o}^p(\vec{\rho}|\vec{\rho}') = \int_{-\infty}^{\infty} \frac{e^{-j\xi(x-x')} e^{-|\xi||y-y'|}}{4\pi|\xi|} d\xi \quad (212)$$

and

$$g_{k_o}^{st}(\vec{\rho}|\vec{\rho}') = \int_{-\infty}^{\infty} \frac{e^{j|\xi|(x-x')}}{4\pi|\xi|(1 - e^{-4|\xi|h})} \left[-e^{-|\xi|(2h+y+y')} \right. \\ \left. - e^{-|\xi|(2h-y-y')} + e^{-|\xi|(4h+y-y')} + e^{-|\xi|(4h-y+y')} \right] d\xi. \quad (213)$$

With the y-component of electric field determined, the potential difference between the center conductor and the bottom ground plane is given by

$$V = - \int_{-h}^0 e_y(\vec{\rho}, \zeta) dy \\ = - \int_{-h}^0 \left[\frac{k_o}{\omega \epsilon_o} \frac{\partial}{\partial y} \int_{-a}^a \tilde{g}_{k_o}^t(\vec{\rho}|\vec{\rho}') k_z(x', k_o) dx' \right] dy \quad (214)$$

which leads to

$$V = - \frac{k_o}{\omega \epsilon_o} \int_{-a}^a \tilde{g}_{k_o}^t(x, y|x', y') k_z(x', k_o) dx' \Big|_{y=-h}^{y=0}. \quad (215)$$

Expression (215) is most conveniently evaluated at the center of the strip conductor located at $x = 0$. The strip conductor is assumed to lie equidistant from the ground planes, therefore $y' = 0$ and

$$V = \frac{k_o}{\omega \epsilon_o} \left[\int_{-a}^a \tilde{g}_{k_o}^t(0, -h|x', 0) k_z(x', k_o) dx' - \int_{-a}^a \tilde{g}_{k_o}^t(0, 0|x', 0) k_z(x', k_o) dx' \right] \quad (216)$$

but

$$\int_{-a}^a \tilde{g}_{k_o}^t(0, 0|x', 0) k_z(x', k_o) dx' = C \quad (217)$$

from condition (190), and $k_o/(\omega \epsilon_o) = \sqrt{\mu_o/\epsilon_o}$, therefore

$$V = \sqrt{\frac{\mu_o}{\epsilon_o}} \left[\int_{-a}^a \tilde{g}_{k_o}^t(0, -h|x', 0) k_z(x', k_o) dx' - C \right]. \quad (218)$$

For points along the lower ground plane at $y = -h$

$$g_{k_o}^t(0, -h|x', 0) = g_{k_o}^p(0, -h|x', 0) + g_{k_o}^{st}(0, -h|x', 0) \quad (219)$$

where

$$g_{k_o}^p(0, -h|x', 0) = \int_{-\infty}^{\infty} \frac{e^{j|\xi|x'} e^{-|\xi||-h|}}{4\pi|\xi|} d\xi \quad (220)$$

and

$$g_{k_o}^{st}(0, -h|x', 0) = \int_{-\infty}^{\infty} \frac{e^{-j|\xi|x'}}{4\pi|\xi|(1 - e^{-4|\xi|h})} \left[-e^{-|\xi|h} - e^{-|\xi|3h} + e^{-|\xi|3h} + e^{-|\xi|5h} \right] d\xi. \quad (221)$$

Combining terms leads to

$$g_{k_o}^{st}(0, -h|x', 0) = \int_{-\infty}^{\infty} \frac{e^{-j|\xi|x'}}{4\pi|\xi|(1 - e^{-4|\xi|h})} \left[-e^{-|\xi|h} (1 - e^{-4|\xi|h}) \right] d\xi \quad (222)$$

and thus

$$g'_{k_o}(0, -h|x', 0) = \int_{-\infty}^{\infty} \frac{e^{-j|\xi|x'} e^{-|\xi|h}}{4\pi|\xi|} d\xi - \int_{-\infty}^{\infty} \frac{e^{-j|\xi|x'} e^{-|\xi|h}}{4\pi|\xi|} d\xi = 0. \quad (223)$$

From this, the potential difference between the center conductor and the top ground plane is

$$V = \sqrt{\frac{\mu_o}{\epsilon_o}} (-C) = -\eta_o C \quad (224)$$

where η_o is the intrinsic impedance of the medium between the ground planes (in this case that medium is free space, and $\eta_o \approx 120\pi$). It is noted from expression (189) that the constant C may be either positive or negative. Thus for convenience

$$C \approx \frac{V}{120\pi} \quad (225)$$

is chosen. Normalizing the potential difference V to 1 volt, this becomes

$$C \approx \frac{1}{120\pi} \approx 2.653 \times 10^{-3} \Omega^{-1}. \quad (226)$$

4.10 Moment-Method Solution for TEM-Mode Specialization

From the geometry of the isolated stripline structure, it is apparent that the current distribution for the principle TEM mode is symmetric about the origin, therefore

$$k_z(-x) = k_z(x) \quad (227)$$

and thus

$$\int_{-a}^0 k_z(x') g_{k_o}^t(x, 0 | x', 0) dx' + \int_0^a k_z(x') g_{k_o}^t(x, 0 | x', 0) dx' = C. \quad (228)$$

Performing a change of variables by letting $x' = -u$ in the first integral term of (228) leads to

$$\int_{-a}^0 k_z(-u) g_{k_o}^t(x, 0 | -u, 0) du + \int_0^a k_z(x') g_{k_o}^t(x, 0 | x', 0) dx' = C \quad (229)$$

which may be expressed

$$\int_0^a k_z(x') [g_{k_o}^t(x | x') + g_{k_o}^t(x | -x')] dx' = C. \quad (230)$$

The current function is now expanded such that

$$k_z(x) = \sum_{n=1}^N a_n p_n(x) \quad (231)$$

using square pulse basis functions

$$p_n(x) = \begin{cases} 1 & \dots \text{ for } x_n - \overline{\Delta x} < x < x_n + \overline{\Delta x} \\ 0 & \dots \text{ elsewhere} \end{cases} \quad (232)$$

where

$$\Delta x = 2 \overline{\Delta x} = \frac{a}{N}, \quad x_n = \left(n - \frac{1}{2}\right) \Delta x. \quad (233)$$

Application of this expansion gives

$$\int_0^a \sum_{n=1}^N a_n p_n(x') [g_{k_o}^t(x | x') + g_{k_o}^t(x | -x')] dx' = C. \quad (234)$$

Interchanging the order of summation and integration leads to

$$\sum_{n=1}^N a_n \int_0^a p_n(x') \left[g_{k_0}'(x|x') + g_{k_0}'(x|-x') \right] dx' = C \quad (235)$$

or

$$\sum_{n=1}^N A_{mn} a_n = C \quad (236)$$

where

$$A_{mn} = \int_{-\infty}^{\infty} q(\xi) \frac{e^{j|\xi|x_m}}{4\pi|\xi|} \int_{x_n - \overline{\Delta x}}^{x_n + \overline{\Delta x}} \left[e^{-j|\xi|x'} + e^{j|\xi|x'} \right] dx' d\xi. \quad (237)$$

Evaluation of the inner integral in (237) gives

$$\begin{aligned} \int_{x_n - \overline{\Delta x}}^{x_n + \overline{\Delta x}} \left[e^{-j|\xi|x'} + e^{j|\xi|x'} \right] dx' &= \frac{e^{j\xi x'} - e^{-j\xi x'}}{j\xi} \Bigg|_{x_n - \overline{\Delta x}}^{x_n + \overline{\Delta x}} \\ &= 2 \left(e^{-j\xi x_n} + e^{j\xi x_n} \right) \frac{\sin \xi \overline{\Delta x}}{\xi} \end{aligned} \quad (238)$$

therefore

$$\begin{aligned} A_{mn} &= \int_{-\infty}^{\infty} q(\xi) \frac{\sin \xi \overline{\Delta x}}{\xi} \frac{e^{j\xi(x_m - x_n)} + e^{j\xi(x_m + x_n)}}{2\pi|\xi|} d\xi \\ &= \int_0^{\infty} \frac{q(\xi)}{\pi|\xi|} \frac{\sin \xi \overline{\Delta x}}{\xi} \left[\cos \xi(x_m - x_n) + \cos \xi(x_m + x_n) \right] d\xi. \end{aligned} \quad (239)$$

4.11 Pole Series Representation of Matrix Elements for TEM Modes

The numerical solution of matrix equation (236) may be greatly accelerated by replacing the real axis integration involved in (239) with an equivalent pole-series representation [26]. The matrix elements in (239) may be expressed in terms of complex exponentials as

$$A_{mn} = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{q(\xi)}{|\xi|} \frac{\sin \xi \overline{\Delta x}}{\xi} \left[e^{j\xi(x_m - x_n)} + e^{j\xi(x_m + x_n)} \right] d\xi \quad (240)$$

where

$$q(\xi) = \frac{1 - e^{-2|\xi|h}}{1 + e^{-2|\xi|h}}. \quad (241)$$

It was shown earlier that

$$\lim_{\xi \rightarrow 0} \frac{q(\xi)}{|\xi|} = h \quad (242)$$

therefore no pole exists at $\xi = 0$, however poles do occur for

$$1 + e^{-2|\xi|h} = e^{|\xi|h} + e^{-|\xi|h} = 0 \quad (243)$$

or

$$2 \cosh \xi h = 0. \quad (244)$$

At the roots of the matrix elements $\xi = j\xi_i$, therefore poles occur for

$$\cos \xi_i h = 0 \quad (245)$$

or

$$\xi_i = \frac{(2n-1)\pi}{2h} . \quad (246)$$

The value of ξ at the poles is thus given by

$$\xi_p = \pm \frac{j(2n-1)\pi}{2h} . \quad (247)$$

Equation (241) may be expressed

$$q(\xi) = \frac{N(\xi)}{D(\xi)} \quad (248)$$

where $N(\xi) = 2$ at $\xi = \xi_p$, and

$$\begin{aligned} D(\xi) &= 1 + e^{-2\xi h} \approx D(\xi_p) + \left. \frac{dD}{d\xi} \right|_{\xi=\xi_p} (\xi - \xi_p) \\ &\approx -2he^{-2\xi h} \Big|_{\xi=\xi_p} (\xi - \xi_p) \end{aligned} \quad (249)$$

or

$$D(\xi) \approx 2h(\xi - \xi_p) . \quad (250)$$

The matrix elements may be expressed

$$A_{mn} = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{q(\xi)}{\xi} \frac{\sin \xi \overline{\Delta x}}{\xi} e^{j\xi L} d\xi \quad (251)$$

where $L = (x_m - x_n)$ or $L = (x_m + x_n)$, and

$$\xi = \xi_r + j\xi_i \quad (252)$$

and thus

$$e^{j\xi L} = e^{-\xi_i L} e^{j\xi_r L} . \quad (253)$$

Here $L > 0$ requires that the integration contour be closed in the upper half of the complex ξ -plane, and $L < 0$ requires lower half-plane closure (Figure 4.19).

Cauchy's integral theorem states

$$\oint_C f(\xi) d\xi = \int_{C_R} f(\xi) d\xi + \int_{C_+^*} f(\xi) d\xi + \int_{C_p^*} f(\xi) d\xi = 0 \quad (254)$$

but it is apparent from (253) that

$$\int_{C_+^*} f(\xi) d\xi = 0 \quad (255)$$

therefore the real-axis integration may be expressed as the sum of contributions to the contour integration around the poles

$$\int_{C_R} f(\xi) d\xi = - \sum_p \int_{C_p^*} f(\xi) d\xi . \quad (256)$$

As $\xi \rightarrow \xi_p$

$$q(\xi) \rightarrow \frac{1}{h(\xi - \xi_p)} \quad (257)$$

and if

$$I(\xi) = \frac{1}{h} \frac{\sin(\xi \overline{\Delta x}) e^{j\xi L}}{\xi^2} \quad (258)$$

the matrix elements may be expressed

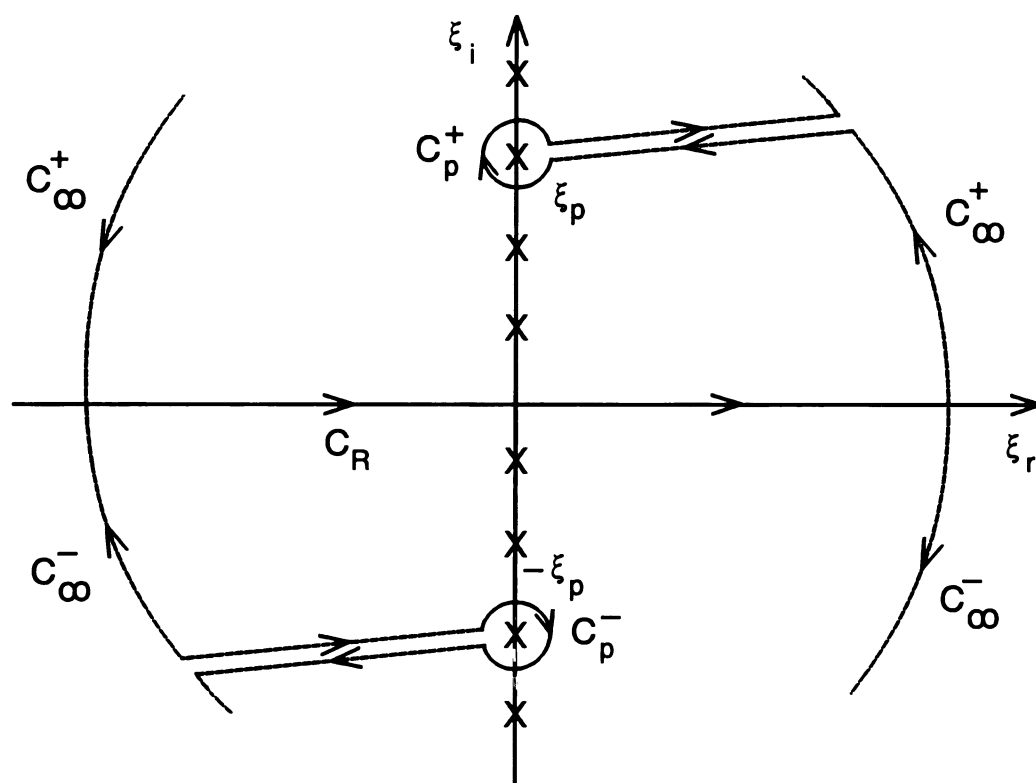


Figure 4.19 Poles and integration contours in the complex ξ -plane.

$$A_{mn} = - \sum_p \frac{1}{2\pi} \int_{C_p^+} \frac{I(\xi)}{(\xi \mp \xi_p)} d\xi . \quad (259)$$

Now at points along C_p^+

$$\xi = \pm \xi_p + \epsilon e^{j\psi} \quad (260)$$

and thus

$$d\xi = j\epsilon e^{j\psi} d\psi \quad (261)$$

therefore

$$\begin{aligned} \oint_{C_p^+} \frac{I(\xi)}{(\xi \mp \xi_p)} d\xi &= \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \frac{I(\xi)}{\epsilon e^{j\psi}} j\epsilon e^{j\psi} d\psi \\ &= \mp 2\pi j I(\pm \xi_p) \end{aligned} \quad (262)$$

which gives

$$A_{mn} = \pm j \sum_p I(\pm \xi_p) . \quad (263)$$

At the poles $\xi_p = j\xi_{pi}$, and

$$I(\pm \xi_p) = \frac{1}{h} \frac{\sin(\pm j\xi_{pi} \overline{\Delta x})}{(\pm j\xi_{pi})^2} e^{\mp \xi_{pi} L} = \pm j \frac{1}{h} \frac{\sinh(\xi_{pi} \overline{\Delta x})}{-\xi_{pi}^2} e^{\mp \xi_{pi} L} \quad (264)$$

leading to

$$A_{mn} = \frac{1}{h} \sum_p \frac{\sinh(\xi_{pi} \overline{\Delta x})}{\xi_{pi}^2} e^{\mp \xi_{pi} L} \quad \dots \text{ for } L < 0, L > 0 . \quad (265)$$

But

$$\begin{aligned} -\xi_{pi}L &= -\xi_{pi}|L| \quad \dots \text{ for } L > 0 \\ +\xi_{pi}L &= -\xi_{pi}|L| \quad \dots \text{ for } L < 0 \end{aligned} \quad (266)$$

thus

$$A_{mn} = \frac{1}{h} \sum_p \frac{\sinh(\xi_{pi} \overline{\Delta x})}{\xi_{pi}^2} e^{-\xi_{pi}|L|} \quad \dots \text{ for } L \neq 0. \quad (267)$$

Now the case when $L = 0$ must be examined more closely. When this condition occurs, (258) becomes

$$I(\xi) = \frac{1}{h} \frac{\sin(\xi \overline{\Delta x})}{\xi^2} = \frac{e^{j\xi \overline{\Delta x}} - e^{-j\xi \overline{\Delta x}}}{j2h\xi^2} \quad (268)$$

and

$$\begin{aligned} A_{mn} &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{q(\xi)}{|\xi|} \frac{\sin(\xi \overline{\Delta x})}{\xi} d\xi = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{q(\xi)}{|\xi|} \frac{e^{j\xi \overline{\Delta x}} - e^{-j\xi \overline{\Delta x}}}{2j\xi} d\xi \\ &= \int_0^{\infty} \frac{1}{2j} \frac{1}{2\pi} \frac{q(\xi)}{|\xi|} \left[\frac{e^{j\xi \overline{\Delta x}}}{\xi} - \frac{e^{-j\xi \overline{\Delta x}}}{\xi} \right] d\xi. \end{aligned} \quad (269)$$

Expanding the term $\sin(\xi \overline{\Delta x})$ into exponentials creates a pole at the origin (Figure 4.20), and Cauchy's integral theorem then states

$$\oint_C f(\xi) d\xi = \int_{C_R^+} f(\xi) d\xi + \int_{C_R^-} f(\xi) d\xi + \int_{C_\epsilon^+} f(\xi) d\xi + \int_{C_\epsilon^-} f(\xi) d\xi + \int_{C_p^+} f(\xi) d\xi = 0 \quad (270)$$

or

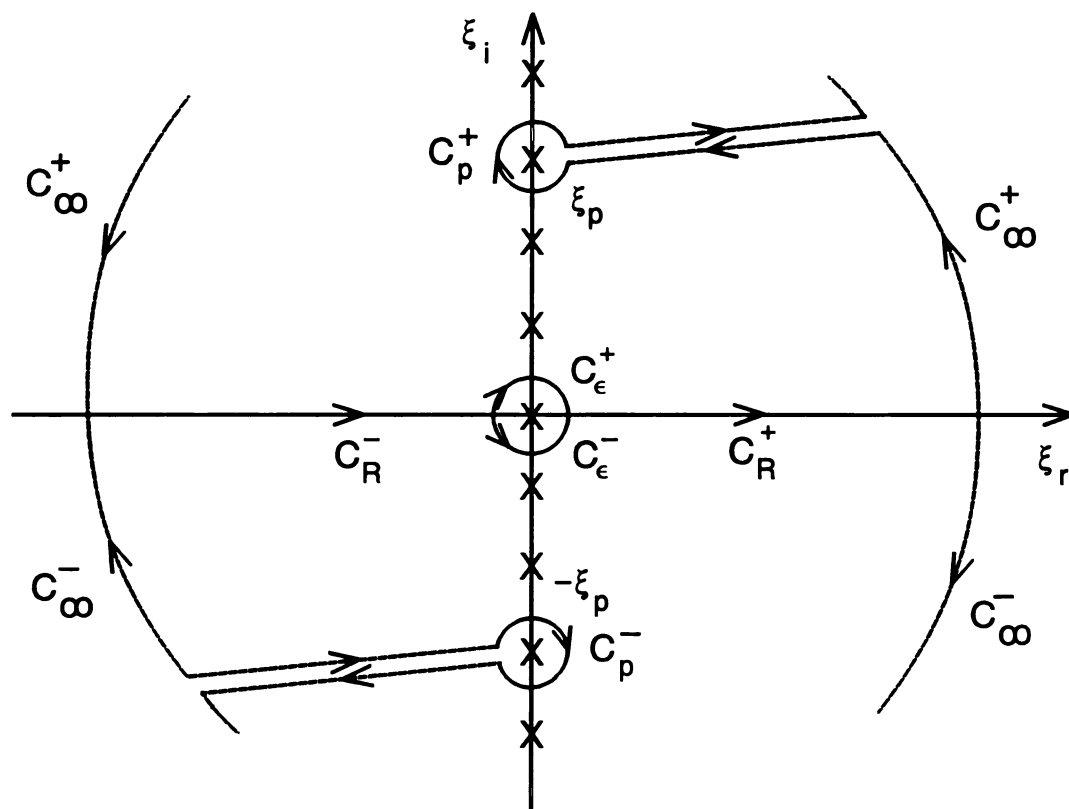


Figure 4.20 Location of additional pole at the origin of complex ξ -plane.

$$\int_{C_R^+} f(\xi) d\xi + \int_{C_R^-} f(\xi) d\xi = - \sum_p \int_{C_p^+} f(\xi) d\xi - \int_{C_\epsilon^+} f(\xi) d\xi. \quad (271)$$

As a result

$$A_{mn} = - \sum_p \int_{C_p^+} f(\xi) d\xi - \int_{C_\epsilon^+} f(\xi) d\xi. \quad (272)$$

Evaluation of the first term on the right side of (272) results in

$$\begin{aligned} - \sum_p \int_{C_p^+} f(\xi) d\xi &= - \sum_p \frac{1}{2\pi} \frac{1}{j2h} \left[\oint_{C_p^+} \frac{e^{j\xi \overline{\Delta x}}}{\xi^2 (\xi - \xi_p)} d\xi - \oint_{C_p^-} \frac{e^{-j\xi \overline{\Delta x}}}{\xi^2 (\xi + \xi_p)} d\xi \right] \\ &= - \sum_p \frac{1}{2\pi} \frac{1}{j2h} \left[-j2\pi \frac{e^{j\xi_p \overline{\Delta x}}}{\xi_p^2} - j2\pi \frac{e^{j\xi_p \overline{\Delta x}}}{(-\xi_p)^2} \right] \end{aligned} \quad (273)$$

which may be expressed

$$- \sum_p \int_{C_p^+} f(\xi) d\xi = \sum_p \frac{1}{h} \frac{e^{j\xi_p \overline{\Delta x}}}{\xi_p^2} = - \frac{1}{h} \sum_p \frac{e^{-\xi_{pi} \overline{\Delta x}}}{\xi_{pi}^2} \quad (274)$$

and thus

$$A_{mn} = - \frac{1}{h} \sum_p \frac{e^{-\xi_{pi} \overline{\Delta x}}}{\xi_{pi}^2} - \int_{C_\epsilon^+} f(\xi) d\xi. \quad (275)$$

Now the second term on the right side of (275) must be evaluated. As $\xi \rightarrow 0$, $q(\xi)/\xi \rightarrow h$, therefore

$$\begin{aligned}
\int_{C_\epsilon} f(\xi) d\xi &= \int_{C_\epsilon} \frac{1}{2j} \frac{h}{2\pi} \left[\frac{e^{j\xi\overline{\Delta x}}}{\xi} - \frac{e^{-j\xi\overline{\Delta x}}}{\xi} \right] d\xi \\
&= \int_{C_\epsilon} \frac{1}{2j} \frac{h}{2\pi} \frac{e^{j\xi\overline{\Delta x}}}{\xi} d\xi - \int_{C_\epsilon} \frac{1}{2j} \frac{h}{2\pi} \frac{e^{-j\xi\overline{\Delta x}}}{\xi} d\xi.
\end{aligned} \tag{276}$$

Letting $\xi = \epsilon e^{j\theta}$ leads to

$$\int_{C_\epsilon} f(\xi) d\xi = \int_{\theta=\pi}^0 \frac{1}{2j} \frac{h}{2\pi} \frac{e^{j\overline{\Delta x}(\epsilon e^{j\theta})}}{\epsilon e^{j\theta}} j\epsilon e^{j\theta} d\theta - \int_{\theta=-\pi}^0 \frac{1}{2j} \frac{h}{2\pi} \frac{e^{-j\overline{\Delta x}(\epsilon e^{j\theta})}}{\epsilon e^{j\theta}} j\epsilon e^{j\theta} d\theta. \tag{277}$$

Examining the behavior of this expression as $\epsilon \rightarrow 0$ gives

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} f(\xi) d\xi = \frac{1}{2j} \frac{h}{2\pi} j \int_{\pi}^0 d\theta - \frac{1}{2j} \frac{h}{2\pi} j \int_{-\pi}^0 d\theta = \frac{h}{4\pi} \int_{\pi}^0 d\theta - \frac{h}{4\pi} \int_{-\pi}^0 d\theta = \frac{h}{4\pi} (-\pi) - \frac{h}{4\pi} \pi \tag{278}$$

therefore

$$\int_{C_\epsilon} f(\xi) d\xi = -\frac{h}{4} - \frac{h}{4} = -\frac{h}{2}. \tag{279}$$

The pole-series expression for the matrix elements associated with TEM modes of stripline when $L = 0$ is thus

$$A_{mn} = \frac{h}{2} - \frac{1}{h} \sum_p \frac{e^{-\xi_{pi}\overline{\Delta x}}}{\xi_{pi}^2}. \tag{280}$$

Solution of (236), using pole-series representations for matrix elements (267) and (280), results in the cross-sectional current distribution associated with the TEM mode

of stripline shown in Figure 4.21. It is noted that this distribution displays the expected edge singularity behavior.

4.12 Characteristic Impedance of Stripline

A unique characteristic impedance can be determined for the TEM mode of stripline, given by

$$Z_o = \frac{V}{I} \quad (281)$$

where V is the potential difference between the center strip and the ground plates, and I is the total current flowing on the center strip. In the case of the TEM mode, only axially-directed currents flow on the center strip, therefore the total current is

$$I = \int_{-a}^a k_z(x) dx. \quad (282)$$

The surface current distribution associated with the TEM mode is even about the origin, thus

$$\int_{-a}^a k_z(x) dx = 2 \int_0^a k_z(x) dx. \quad (283)$$

Expanding the currents using the square pulse basis functions defined in (232) gives

$$I = 2 \sum_{n=1}^N a_n \int_0^a p_n(x) dx = 2 \Delta x \sum_{n=1}^N a_n. \quad (284)$$

If the potential difference between the center strip and the ground plates is normalized such that $V=1$ volt, then the characteristic impedance of the stripline is found from

$$Z_o = \frac{1}{I} \quad (285)$$

The characteristic impedance of stripline devices with center strip widths of $w=10.0\text{cm}$, $w=5.0\text{cm}$, and $w=2.5\text{cm}$ are presented in Figure 4.22. This figure shows the characteristic impedance for fixed center strip widths as the distance between the center strip and the ground planes (h) is varied. It is noted that for a particular characteristic impedance, the ratio w/h is the same for all three center strip widths. Examining the dimensions which produce a 50Ω characteristic impedance, it is seen that $(10.0\text{cm}/3.45\text{cm}) = (5.0\text{cm}/1.725\text{cm}) = (2.5\text{cm}/0.8625\text{cm}) \approx 2.9$.

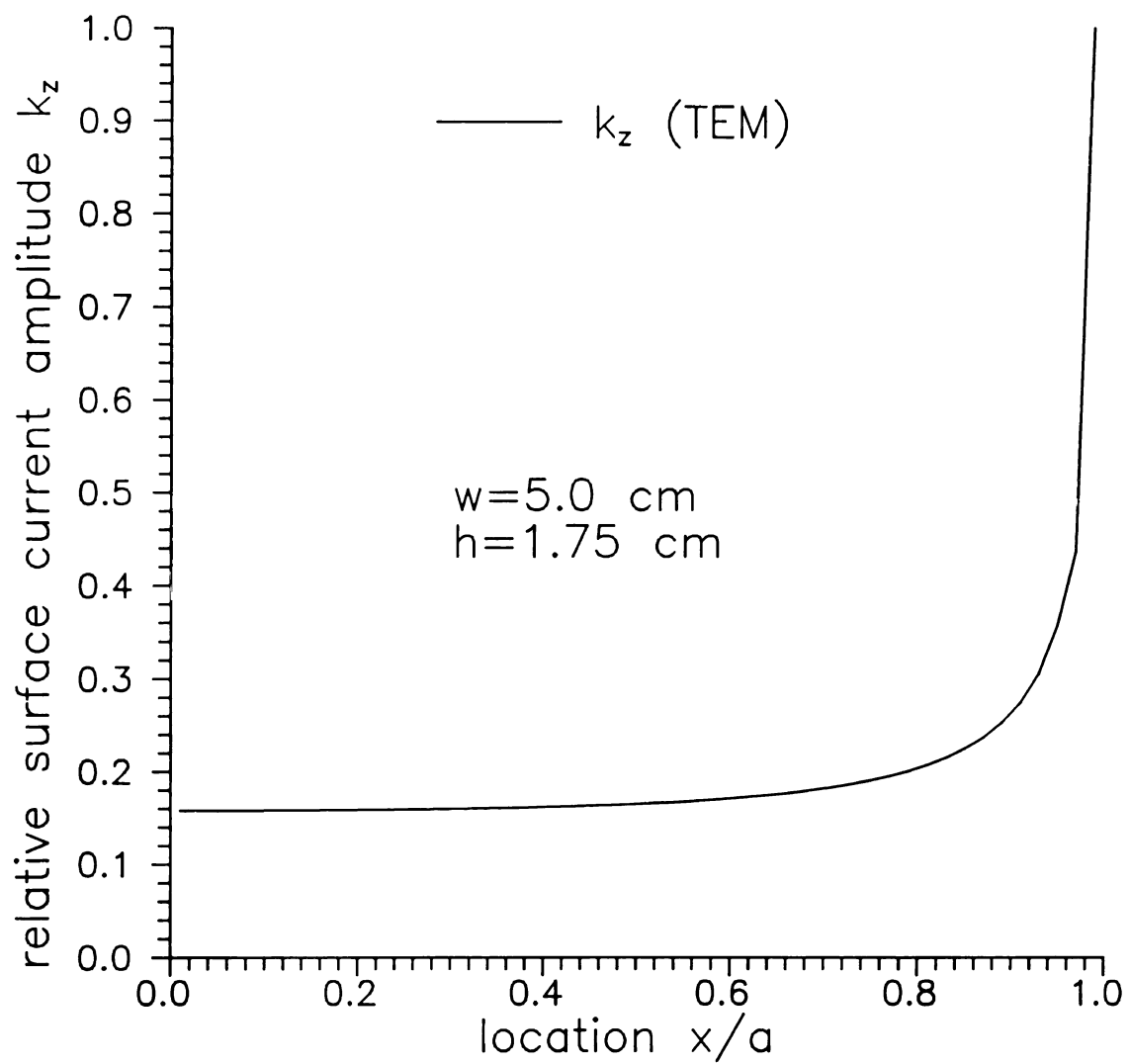


Figure 4.21 Cross-sectional current distribution for TEM mode of stripline.

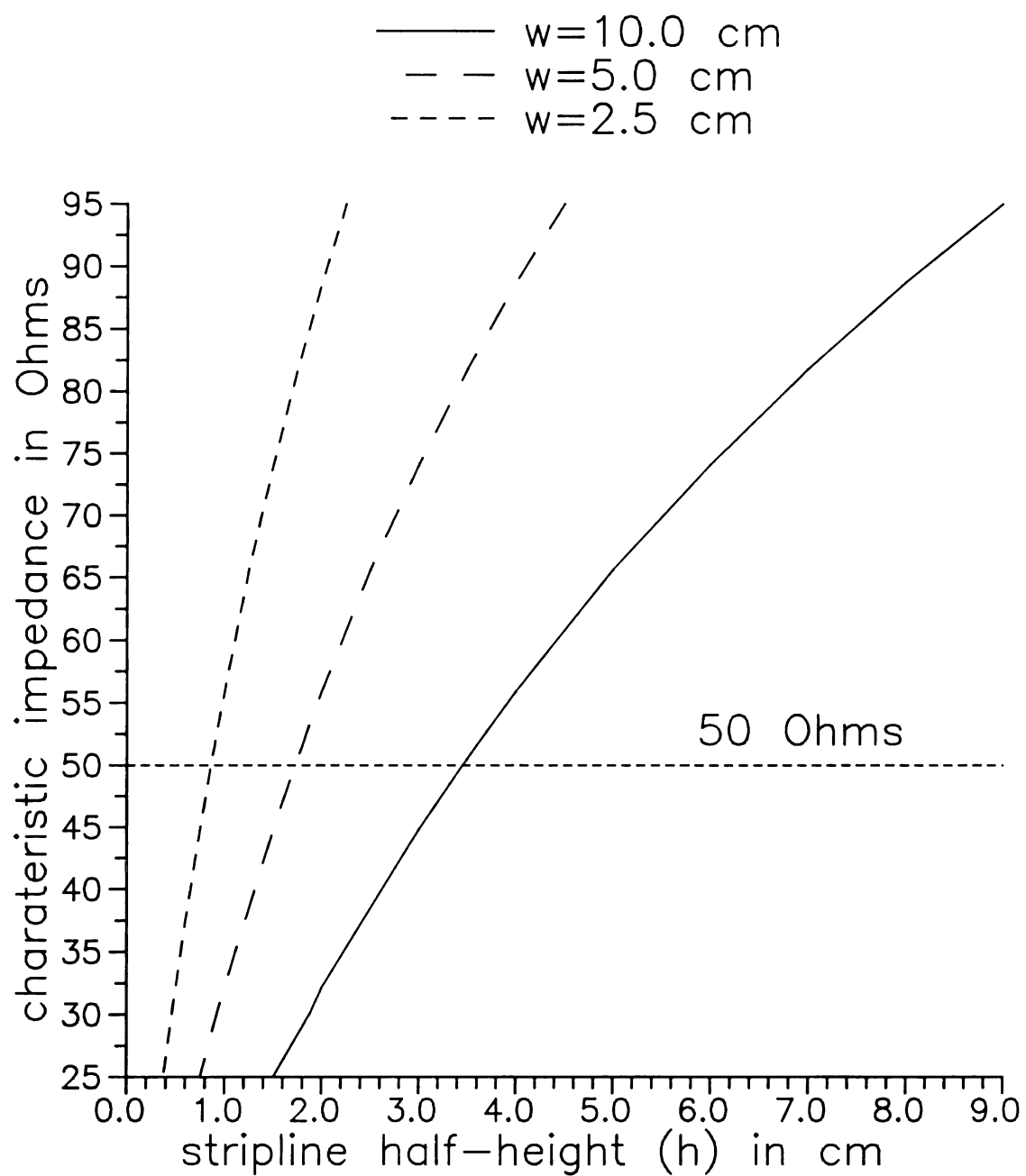


Figure 4.22 Relationship between stripline characteristic impedance and cross-sectional dimensions.

Chapter 5

FULL-WAVE ANALYSIS OF COUPLED STRIPLINE STRUCTURES

The analysis of the previous chapter is now extended to the case of a stripline structure having a pair of coupled center strips [47]. This structure is of considerable practical importance because many integrated circuit devices contain layered strip transmission lines that lie close to one another. Coupling between the center strips may lead to crosstalk and interference. Additionally, certain combinations of device geometry and operational frequency may lead to the excitation of leaky-wave modes. The propagation eigenmodes associated with a structure having widely separated center strips will approach those of the isolated center strip structure. However, closely spaced center strips will result in propagation eigenvalues which are different from those of the isolated strip structure. In this case, separate propagation constants will arise for symmetric and anti-symmetric current modes. In this chapter, the complete propagation-mode spectrum of a stripline structure with two conducting center strips will be investigated.

5.1 Coupled Strip Electric Field Integral Equations

Consider a stripline structure having perfectly-conducting center strips of widths w_1 and w_2 , centered at horizontal positions x_1 and x_2 from the origin, and located at heights h_1 and h_2 above or below the origin (Figure 5.1). In the previous chapter, an expression for the transform-domain electric field was presented

$$\vec{e}(\vec{\rho}, \zeta) = (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \int_C \vec{g}_\zeta(\vec{\rho} | \vec{\rho}') \cdot \frac{\vec{k}(\vec{\rho}', \zeta)}{j\omega\epsilon_o} dl' \quad (1)$$

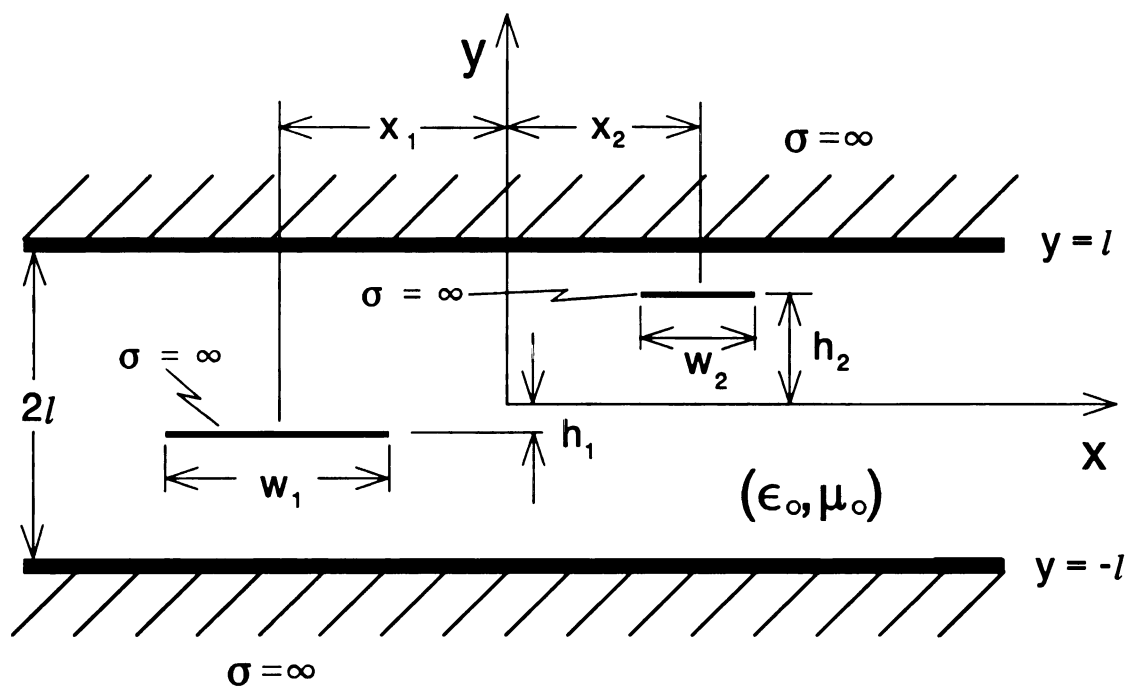


Figure 5.1 Cross-section of stripline structure with coupled center strips.

where $\tilde{\nabla} = \nabla_t + \hat{z}j\zeta$ with $\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. Extending this to a structure having a pair of center strips leads to

$$\begin{aligned} \bar{e}(x, h_1; \zeta) = \lim_{y \rightarrow h_1} \left(k_0^2 + \tilde{\nabla} \tilde{\nabla} \cdot \right) & \left\{ \int_{C_1} \tilde{g}_\zeta(x, y | x', h_1) \cdot \frac{\vec{k}_1(x', \zeta)}{j\omega\epsilon_o} dx' \right. \\ & \left. + \int_{C_2} \tilde{g}_\zeta(x, y | x', h_2) \cdot \frac{\vec{k}_2(x', \zeta)}{j\omega\epsilon_o} dx' \right\} \end{aligned} \quad (2)$$

$$\begin{aligned} \bar{e}(x, h_2; \zeta) = \lim_{y \rightarrow h_2} \left(k_0^2 + \tilde{\nabla} \tilde{\nabla} \cdot \right) & \left\{ \int_{C_1} \tilde{g}_\zeta(x, y | x', h_1) \cdot \frac{\vec{k}_1(x', \zeta)}{j\omega\epsilon_o} dx' \right. \\ & \left. + \int_{C_2} \tilde{g}_\zeta(x, y | x', h_2) \cdot \frac{\vec{k}_2(x', \zeta)}{j\omega\epsilon_o} dx' \right\} \end{aligned} \quad (3)$$

where C_1 and C_2 represent the peripheries of the center strips. The boundary condition at the surface of the perfectly-conducting center strips requires

$$\hat{t} \cdot \bar{e}^s = -\hat{t} \cdot \bar{e}^i \quad (4)$$

at $y=h_1, h_2$ and for $x_\alpha - w_\alpha/2 \leq x \leq x_\alpha + w_\alpha/2$, where $\alpha=1,2$. This leads to a pair of coupled EFIEs

$$\begin{aligned} \lim_{y \rightarrow h_1} \hat{t} \cdot \left(k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot \right) & \left\{ \int_{C_1} \tilde{g}_\zeta(x, y | x', h_1) \cdot \vec{k}_1(x', \zeta) dx' \right. \\ & \left. + \int_{C_2} \tilde{g}_\zeta(x, y | x', h_2) \cdot \vec{k}_2(x', \zeta) dx' \right\} = -j\omega\epsilon_o \hat{t} \cdot \bar{e}^i(x, h_1; \zeta) \end{aligned} \quad (5)$$

$$\begin{aligned} \lim_{y \rightarrow h_2} \hat{t} \cdot \left(k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot \right) & \left\{ \int_{C_1} \tilde{g}_\zeta(x, y | x', h_1) \cdot \vec{k}_1(x', \zeta) dx' \right. \\ & \left. + \int_{C_2} \tilde{g}_\zeta(x, y | x', h_2) \cdot \vec{k}_2(x', \zeta) dx' \right\} = -j\omega\epsilon_o \hat{t} \cdot \bar{e}^i(x, h_2; \zeta). \end{aligned} \quad (6)$$

As with the isolated center-strip structure, as $\zeta \rightarrow \zeta_p$, near a simple pole singularity, the current may be approximated as

$$\vec{k}_\alpha(x', \zeta) \approx \frac{k_{\alpha p}(x', \zeta)}{(\zeta - \zeta_p)} \quad (7)$$

where $\alpha=1,2$ and $k_{\alpha p}(x', \zeta)$ is analytic at ζ_p [25]. Substitution of this into (5) and (6) gives

$$\begin{aligned} \lim_{\zeta \rightarrow \zeta_p} \left\{ \lim_{y \rightarrow h_1} \hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \left[\int_{C_1} \tilde{g}_\zeta(x, y | x', h_1) \cdot \frac{\vec{k}_1(x', \zeta)}{(\zeta - \zeta_p)} dx' \right. \right. \\ \left. \left. + \int_{C_2} \tilde{g}_\zeta(x, y | x', h_2) \cdot \frac{\vec{k}_2(x', \zeta)}{(\zeta - \zeta_p)} dx' \right] \right\} = \lim_{\zeta \rightarrow \zeta_p} \left\{ -j\omega \epsilon_o \hat{t} \cdot \vec{e}^i(x, h_1; \zeta) \right\} \end{aligned} \quad (8)$$

$$\begin{aligned} \lim_{\zeta \rightarrow \zeta_p} \left\{ \lim_{y \rightarrow h_2} \hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \left[\int_{C_1} \tilde{g}_\zeta(x, y | x', h_1) \cdot \frac{\vec{k}_1(x', \zeta)}{(\zeta - \zeta_p)} dx' \right. \right. \\ \left. \left. + \int_{C_2} \tilde{g}_\zeta(x, y | x', h_2) \cdot \frac{\vec{k}_2(x', \zeta)}{(\zeta - \zeta_p)} dx' \right] \right\} = \lim_{\zeta \rightarrow \zeta_p} \left\{ -j\omega \epsilon_o \hat{t} \cdot \vec{e}^i(x, h_2; \zeta) \right\}. \end{aligned} \quad (9)$$

The left-hand sides of these expressions are singular at $\zeta = \zeta_p$, but the right-hand sides remain regular everywhere. This apparent paradox is resolved if the left-hand sides of (8) and (9) reduce to an indeterminate form at $\zeta = \zeta_p$. A pair of fundamental EFIEs results

$$\lim_{y \rightarrow h_1} \left\{ \hat{t} \cdot (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \left[\int_{x_1 - \frac{w_1}{2}}^{x_1 + \frac{w_1}{2}} \tilde{g}_\zeta(x, y | x', h_1) \cdot \vec{k}_{1p}(x') dx' + \int_{x_2 - \frac{w_2}{2}}^{x_2 + \frac{w_2}{2}} \tilde{g}_\zeta(x, y | x', h_2) \cdot \vec{k}_{2p}(x') dx' \right] \right\} = 0 \quad (10)$$

$$\lim_{y \rightarrow h_2} \left\{ \hat{t} \cdot \left(k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot \right) \left[\int_{x_1 - \frac{w_1}{2}}^{x_2 + \frac{w_2}{2}} \tilde{g}_\zeta(x, y | x', h_1) \cdot \vec{k}_{1p}(x') dx' + \int_{x_2 - \frac{w_2}{2}}^{x_2 + \frac{w_2}{2}} \tilde{g}_\zeta(x, y | x', h_2) \cdot \vec{k}_{2p}(x') dx' \right] \right\} = 0 \quad (11)$$

having non-trivial solutions only for $\zeta = \zeta_p$.

5.2 Scalar x - and z -Component Integral Equations

As with the analysis of isolated center-strip structures in the previous chapter, the coupled perfectly-conducting center strips considered here are assumed to be infinitely thin. As a result, the surface currents present on the strips have only x - and z -directed components, and are therefore represented by

$$\vec{k}_\beta(x') = \hat{x} k_{\beta x}(x') + \hat{z} k_{\beta z}(x') \quad (12)$$

where $\beta = 1, 2$. The transform-domain Green's function implicated in the coupled EFIEs (10) and (11) is associated with principal and scattered waves, having components which lie transverse and normal to the surface of the center strips

$$\begin{aligned} \tilde{g}_\zeta(x, y | x', y') = & (\hat{x}\hat{x} + \hat{z}\hat{z}) \left[g_\zeta^p(x, y | x', y') + g_\zeta^{st}(x, y | x', y') \right] \\ & + \hat{y}\hat{y} \left[g_\zeta^p(x, y | x', y') + g_\zeta^{sn}(x, y | x', y') \right]. \end{aligned} \quad (13)$$

When (13) is operated on by the transversely-directed current functions the result is

$$\begin{aligned} \tilde{g}_\zeta(x, y | x', y') \cdot \vec{k}_\alpha(x') &= \left[\hat{x} k_{\alpha x}(x') + \hat{z} k_{\alpha z}(x') \right] \left[g_\zeta^p(x, y | x', y') + g_\zeta^{st}(x, y | x', y') \right] \\ &= \left[\hat{x} k_{\alpha x}(x') + \hat{z} k_{\alpha z}(x') \right] g_\zeta^t(x, y | x', y') \end{aligned} \quad (14)$$

where

$$g_{\zeta}^t(x, y | x', y') = \int_{-\infty}^{\infty} \frac{e^{j\zeta(x-x')}}{4\pi} W(\xi, y, y') d\xi. \quad (15)$$

Here

$$W(\xi, y, y') = \frac{1}{p_0} \left\{ e^{-p_0|y-y'|} + \frac{1}{(1-e^{-4p_0l})} \left[e^{-p_0(4l+y-y')} + e^{-p_0(4l-y+y')} - e^{-p_0(2l+y+y')} - e^{-p_0(2l-y-y')} \right] \right\} \quad (16)$$

and $p_0 = \sqrt{\zeta^2 + \xi^2 - k_o^2}$ is the transform-domain wavenumber. Substituting (16) into the fundamental EFIEs (10) and (11) gives scalar x - and z -component equations for strip 1

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta}^t(x, y | x', h_1) k_{1x}(x') + j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_1) k_{1z}(x') \right] dx' \right\} \\ + \int_{a_2}^{b_2} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta}^t(x, h_1 | x', h_2) k_{2x}(x') + j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_1 | x', h_2) k_{2z}(x') \right] dx' = 0 \end{aligned} \quad (17)$$

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_1) k_{1x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, y | x', h_1) k_{1z}(x') \right] dx' \right\} \\ + \int_{a_2}^{b_2} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_1 | x', h_2) k_{2x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, h_1 | x', h_2) k_{2z}(x') \right] dx' = 0 \end{aligned} \quad (18)$$

and scalar x - and z -component equations for strip 2

$$\begin{aligned}
& \int_{a_1}^{b_1} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta}^t(x, h_2 | x', h_1) k_{1x}(x') + j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_2 | x', h_1) k_{1z}(x') \right] dx' \\
& + \lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta}^t(x, y | x', h_2) k_{2x}(x') + j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_2) k_{2z}(x') \right] dx' \right\} = 0
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \int_{a_1}^{b_1} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_2 | x', h_1) k_{1x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, h_2 | x', h_1) k_{1z}(x') \right] dx' \\
& + \lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_2) k_{2x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, y | x', h_2) k_{2z}(x') \right] dx' \right\} = 0
\end{aligned} \tag{20}$$

where

$$a_{\alpha} = x_{\alpha} - \frac{w_{\alpha}}{2}, \quad b_{\alpha} = x_{\alpha} + \frac{w_{\alpha}}{2} \tag{21}$$

for $\alpha=1,2$. Applying the Green's function expression (15) to the scalar x - and z -component EFIEs (17)-(20) gives

$$\begin{aligned}
& \lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1x}(x') \right. \right. \\
& \quad \left. \left. + j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1z}(x') \right] dx' \right\} \\
& \quad + \int_{a_2}^{b_2} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2x}(x') \right. \\
& \quad \left. + j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2z}(x') \right] dx' = 0
\end{aligned} \tag{22}$$

$$\begin{aligned}
& \lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1x}(x') \right. \right. \\
& \quad \left. \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1z}(x') \right] dx' \right\} \\
& \quad + \int_{a_2}^{b_2} \left[j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2x}(x') \right. \\
& \quad \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2z}(x') \right] dx' = 0
\end{aligned} \tag{23}$$

$$\begin{aligned}
& \int_{a_1}^{b_1} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1x}(x') \right. \\
& \quad \left. + j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1z}(x') \right] dx' \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2x}(x') \right. \right. \\
& \quad \left. \left. + j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2z}(x') \right] dx' \right\} = 0
\end{aligned} \tag{24}$$

$$\begin{aligned}
& \int_{a_1}^{b_1} \left[j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1x}(x') \right. \\
& \quad \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1z}(x') \right] dx' \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \left[j\zeta \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2x}(x') \right. \right. \\
& \quad \left. \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2z}(x') \right] dx' \right\} = 0.
\end{aligned} \tag{25}$$

Application of the partial derivatives present in the scalar component equations gives

$$\frac{\partial}{\partial x} e^{j\xi(x-x')} = j\xi e^{j\xi(x-x')} \quad (26)$$

and therefore

$$\frac{\partial^2}{\partial x^2} e^{j\xi(x-x')} = \frac{\partial}{\partial x} j\xi e^{j\xi(x-x')} = -\xi^2 e^{j\xi(x-x')} . \quad (27)$$

As a result, the scalar component equations (22)-(25) become

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1x}(x') \right. \right. \\ \left. \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1z}(x') \right] dx' \right\} \\ + \int_{a_2}^{b_2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2x}(x') \right. \\ \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2z}(x') \right] dx' = 0 \end{aligned} \quad (28)$$

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1x}(x') \right. \right. \\ \left. \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_1) d\xi k_{1z}(x') \right] dx' \right\} \\ + \int_{a_2}^{b_2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2x}(x') \right. \\ \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2z}(x') \right] dx' = 0 \end{aligned} \quad (29)$$

$$\begin{aligned}
& \int_{a_1}^{b_1} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1x}(x') \right. \\
& \quad \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1z}(x') \right] dx' \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2x}(x') \right. \right. \\
& \quad \left. \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2z}(x') \right] dx' \right\} = 0
\end{aligned} \tag{30}$$

$$\begin{aligned}
& \int_{a_1}^{b_1} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1x}(x') \right. \\
& \quad \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1z}(x') \right] dx' \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2x}(x') \right. \right. \\
& \quad \left. \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, h_2) d\xi k_{2z}(x') \right] dx' \right\} = 0.
\end{aligned} \tag{31}$$

5.3 Transformation to Local Coordinate Variables

In order to facilitate a numerical solution, a change of variables is performed. Coordinate variables of the scalar component EFIEs presented above are shifted to a system of local coordinates. This is done in such a way that the shifted coordinate systems originate at the centers of the strip conductors (Figure 5.2). First let

$$\bar{x}'_{\alpha} = x' - x_{\alpha} \tag{32}$$

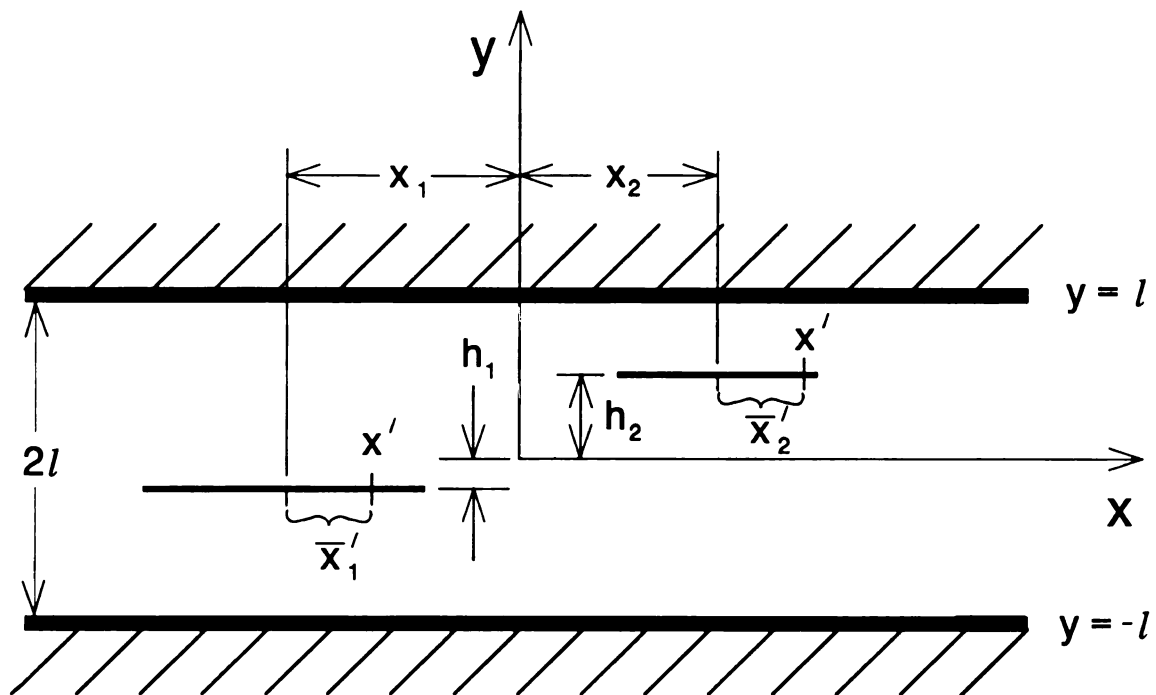


Figure 5.2 Local coordinate variables.

and then $dx' = d\bar{x}'_\alpha$. It is seen that when $x' = x_\alpha - w_\alpha/2$, $\bar{x}'_\alpha = -w_\alpha/2$, and when $x' = x_\alpha + w_\alpha/2$, $\bar{x}'_\alpha = w_\alpha/2$. Also let

$$\bar{x}_\alpha = x - x_\alpha \quad (33)$$

and then $dx = d\bar{x}_\alpha$. In this case, when $x = x_\alpha - w_\alpha/2$, $\bar{x}_\alpha = -w_\alpha/2$ and when $x = x_\alpha + w_\alpha/2$, $\bar{x}_\alpha = w_\alpha/2$. Applying this change of variables to the scalar component EFIEs (28)-(31) gives

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{-w_1/2}^{w_1/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi k_{1x}(\bar{x}'_1) \right. \right. \\ \left. \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi k_{1z}(\bar{x}'_1) \right] d\bar{x}'_1 \right\} \\ + \int_{-w_2/2}^{w_2/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_1 + x_1 - x_2 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2x}(\bar{x}'_2) \right. \\ \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 + x_1 - x_2 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2z}(\bar{x}'_2) \right] d\bar{x}'_2 = 0 \end{aligned} \quad (34)$$

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{-w_1/2}^{w_1/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi k_{1x}(\bar{x}'_1) \right. \right. \\ \left. \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi k_{1z}(\bar{x}'_1) \right] d\bar{x}'_1 \right\} \\ + \int_{-w_2/2}^{w_2/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 + x_1 - x_2 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2x}(\bar{x}'_2) \right. \\ \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 + x_1 - x_2 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) d\xi k_{2z}(\bar{x}'_2) \right] d\bar{x}'_2 = 0 \end{aligned} \quad (35)$$

$$\begin{aligned}
& \int_{-w_1/2}^{w_1/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_2 + x_2 - x_1 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1x}(\bar{x}'_1) \right. \\
& \quad \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 + x_2 - x_1 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1z}(\bar{x}'_1) \right] d\bar{x}'_1 \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{-w_2/2}^{w_2/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi k_{2x}(\bar{x}'_2) \right. \right. \\
& \quad \left. \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi k_{2z}(\bar{x}'_2) \right] d\bar{x}'_2 \right\} = 0
\end{aligned} \tag{36}$$

$$\begin{aligned}
& \int_{-w_1/2}^{w_1/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 + x_2 - x_1 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1x}(\bar{x}'_1) \right. \\
& \quad \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 + x_2 - x_1 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) d\xi k_{1z}(\bar{x}'_1) \right] d\bar{x}'_1 \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{-w_2/2}^{w_2/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi k_{2x}(\bar{x}'_2) \right. \right. \\
& \quad \left. \left. + (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi k_{2z}(\bar{x}'_2) \right] d\bar{x}'_2 \right\} = 0.
\end{aligned} \tag{37}$$

5.4 Moment-Method Solution

As in the previous chapter, the EFIEs (34)-(37) are solved using a Galerkin's Moment Method formulation [43]. The current functions are expanded such that

$$k_{\alpha\beta}(\bar{x}'_\alpha) = \sum_{n=1}^{N_\alpha} a_{\alpha\beta}^n e_{\alpha\beta}^n(\bar{x}'_\alpha) \tag{38}$$

where $e_{\alpha\beta}^n(\bar{x}'_\alpha)$ is the basis set, and the $a_{\alpha\beta}^n$ represent unknown amplitude coefficients for

$\alpha=1,2$, and $\beta=x,z$. As a result

$$\int_{-w_\alpha/2}^{w_\alpha/2} k_{\alpha\beta}(\bar{x}'_\alpha) e^{-j\xi\bar{x}'_\alpha} d\bar{x}'_\alpha = \sum_{n=1}^{N_\alpha} a_{\alpha\beta}^n \int_{-w_\alpha/2}^{w_\alpha/2} e_{\alpha\beta}^n(\bar{x}'_\alpha) e^{-j\xi\bar{x}'_\alpha} d\bar{x}'_\alpha = \sum_{n=1}^{N_\alpha} a_{\alpha\beta}^n f_{\alpha\beta}^n(\xi) \quad (39)$$

where

$$f_{\alpha\beta}^n(\xi) = \int_{-w_\alpha/2}^{w_\alpha/2} e_{\alpha\beta}^n(\bar{x}'_\alpha) e^{-j\xi\bar{x}'_\alpha} d\bar{x}'_\alpha. \quad (40)$$

In addition, let

$$g_{\alpha\beta}^m(\xi) = \int_{-w_\alpha/2}^{w_\alpha/2} t_{\alpha\beta}^m(\bar{x}_\alpha) e^{j\xi\bar{x}_\alpha} d\bar{x}_\alpha \quad (41)$$

where $t_{\alpha\beta}^m(\bar{x}_\alpha)$ is a testing function which is to be applied later.

Applying the expansion shown in (38) to the current functions in EFIEs (34)-(37)

results in

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{-w_1/2}^{w_1/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \sum_{n=1}^{N_{1x}} a_{1x}^n e_{1x}^n(\bar{x}'_1) \right. \right. \\ \left. \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \sum_{n=1}^{N_{1z}} a_{1z}^n e_{1z}^n(\bar{x}'_1) \right] d\bar{x}'_1 \right\} \\ + \int_{-w_2/2}^{w_2/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)} e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) d\xi \sum_{n=1}^{N_{2x}} a_{2x}^n e_{2x}^n(\bar{x}'_2) \right. \\ \left. - \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)} e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) d\xi \sum_{n=1}^{N_{2z}} a_{2z}^n e_{2z}^n(\bar{x}'_2) \right] d\bar{x}'_2 = 0 \end{aligned} \quad (42)$$

$$\begin{aligned}
\lim_{y \rightarrow h_1} \left\{ \int_{-w_1/2}^{w_1/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \sum_{n=1}^{N_{1x}} a_{1x}^n e_{1x}^n(\bar{x}'_1) \right. \right. \\
+ (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \sum_{n=1}^{N_{1z}} a_{1z}^n e_{1z}^n(\bar{x}'_1) \left. \right] d\bar{x}'_1 \Big\} \\
+ \int_{-w_2/2}^{w_2/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)} e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) d\xi \sum_{n=1}^{N_{2x}} a_{2x}^n e_{2x}^n(\bar{x}'_2) \right. \\
+ (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)} e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) d\xi \sum_{n=1}^{N_{2z}} a_{2z}^n e_{2z}^n(\bar{x}'_2) \left. \right] d\bar{x}'_2 = 0
\end{aligned} \tag{43}$$

$$\begin{aligned}
\int_{-w_1/2}^{w_1/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_1)} e^{j\xi(x_2 - x_1)}}{4\pi} W(\xi, h_2, h_1) d\xi \sum_{n=1}^{N_{1x}} a_{1x}^n e_{1x}^n(\bar{x}'_1) \right. \\
- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_1)} e^{j\xi(x_2 - x_1)}}{4\pi} W(\xi, h_2, h_1) d\xi \sum_{n=1}^{N_{1z}} a_{1z}^n e_{1z}^n(\bar{x}'_1) \left. \right] d\bar{x}'_1 \\
+ \lim_{y \rightarrow h_2} \left\{ \int_{-w_2/2}^{w_2/2} \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \sum_{n=1}^{N_{2x}} a_{2x}^n e_{2x}^n(\bar{x}'_2) \right. \right. \\
- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \sum_{n=1}^{N_{2z}} a_{2z}^n e_{2z}^n(\bar{x}'_2) \left. \right] d\bar{x}'_2 \Big\} = 0
\end{aligned} \tag{44}$$

$$\begin{aligned}
\int_{-w_1/2}^{w_1/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)} e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_2, h_1) d\xi \sum_{n=1}^{N_{1x}} a_{1x}^n e_{1x}^n(\bar{x}'_1) \right. \\
+ (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_1)} e^{j\xi(x_2 - x_1)}}{4\pi} W(\xi, h_2, h_1) d\xi \sum_{n=1}^{N_{1z}} a_{1z}^n e_{1z}^n(\bar{x}'_1) \left. \right] d\bar{x}'_1 \\
+ \lim_{y \rightarrow h_2} \left\{ \int_{-w_2/2}^{w_2/2} \left[- \int_{-\infty}^{\infty} \zeta \xi \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \sum_{n=1}^{N_{2x}} a_{2x}^n e_{2x}^n(\bar{x}'_2) \right. \right. \\
+ (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \sum_{n=1}^{N_{2z}} a_{2z}^n e_{2z}^n(\bar{x}'_2) \left. \right] d\bar{x}'_2 \Big\} = 0.
\end{aligned} \tag{45}$$

The testing operator

$$\cdot \int_{-w_\alpha/2}^{w_\alpha/2} t_{\alpha\beta}^m(\bar{x}_\alpha) d\bar{x}_\alpha \quad (46)$$

is now applied to equations (42)-(45), and the order of integration is interchanged, giving

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ \int_{-\infty}^{\infty} \left[\frac{(k_o^2 - \xi^2)}{4\pi} W(\xi, y, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n \int_{-w_1/2}^{w_1/2} e_{1x}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_1/2}^{w_1/2} t_{1x}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi \right. \\ \left. - \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} W(\xi, y, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n \int_{-w_1/2}^{w_1/2} e_{1z}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_1/2}^{w_1/2} t_{1x}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi \right\} \\ + \int_{-\infty}^{\infty} \left[(k_o^2 - \xi^2) \frac{e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n \int_{-w_1/2}^{w_1/2} e_{2x}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_1/2}^{w_1/2} t_{1x}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi \\ - \int_{-\infty}^{\infty} \left[\zeta\xi \frac{e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n \int_{-w_1/2}^{w_1/2} e_{2z}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_1/2}^{w_1/2} t_{1x}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi = 0 \end{aligned} \quad (47)$$

$$\begin{aligned} \lim_{y \rightarrow h_1} \left\{ - \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} W(\xi, y, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n \int_{-w_1/2}^{w_1/2} e_{1x}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_1/2}^{w_1/2} t_{1z}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi \right. \\ \left. + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[W(\xi, y, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n \int_{-w_1/2}^{w_1/2} e_{1z}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_1/2}^{w_1/2} t_{1z}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi \right\} \\ - \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n \int_{-w_2/2}^{w_2/2} e_{2x}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_1/2}^{w_1/2} t_{1z}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi \\ + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n \int_{-w_2/2}^{w_2/2} e_{2z}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_1/2}^{w_1/2} t_{1z}^m(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \right] d\xi = 0 \end{aligned} \quad (48)$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left[\frac{(k_o^2 - \xi^2)}{4\pi} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n \int_{-w_1/2}^{w_1/2} e_{1x}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_2/2}^{w_2/2} t_{2x}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \\
& - \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n \int_{-w_1/2}^{w_1/2} e_{1z}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_2/2}^{w_2/2} t_{2x}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \\
& + \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \left[\frac{(k_o^2 - \xi^2)}{4\pi} W(\xi, y, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n \int_{-w_2/2}^{w_2/2} e_{2x}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_2/2}^{w_2/2} t_{2x}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \right. \\
& \quad \left. - \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} W(\xi, y, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n \int_{-w_2/2}^{w_2/2} e_{2z}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_2/2}^{w_2/2} t_{2x}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \right\} = 0
\end{aligned} \tag{49}$$

$$\begin{aligned}
& - \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n \int_{-w_1/2}^{w_1/2} e_{1x}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_2/2}^{w_2/2} t_{2z}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \\
& + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n \int_{-w_1/2}^{w_1/2} e_{1z}^n(\bar{x}_1') e^{-j\xi\bar{x}_1'} d\bar{x}_1' \int_{-w_2/2}^{w_2/2} t_{2z}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \\
& - \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \left[\frac{\zeta\xi}{4\pi} W(\xi, y, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n \int_{-w_2/2}^{w_2/2} e_{2x}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_2/2}^{w_2/2} t_{2z}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \right. \\
& \quad \left. + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[W(\xi, y, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n \int_{-w_2/2}^{w_2/2} e_{2z}^n(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \int_{-w_2/2}^{w_2/2} t_{2z}^m(\bar{x}_2) e^{j\xi\bar{x}_2} d\bar{x}_2 \right] d\xi \right\} = 0.
\end{aligned} \tag{50}$$

The equations above may be expressed in terms of the definitions given in (40)

and (41). This results in a more compact representation, given by

$$\begin{aligned}
& \lim_{y \rightarrow h_1} \left\{ \int_{-\infty}^{\infty} \left[\frac{(k_o^2 - \xi^2)}{4\pi} W(\xi, y, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n f_{1x}^n(\xi) g_{1x}^m(\xi) \right] d\xi \right. \\
& \quad \left. - \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} W(\xi, y, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n f_{1z}^n(\xi) g_{1x}^m(\xi) \right] d\xi \right\} \\
& \quad + \int_{-\infty}^{\infty} \left[(k_o^2 - \xi^2) \frac{e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n f_{2x}^n(\xi) g_{1x}^m(\xi) \right] d\xi \\
& \quad - \int_{-\infty}^{\infty} \left[\zeta \xi \frac{e^{j\xi(x_1 - x_2)}}{4\pi} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n f_{2z}^n(\xi) g_{1x}^m(\xi) \right] d\xi = 0
\end{aligned} \tag{51}$$

$$\begin{aligned}
& \lim_{y \rightarrow h_1} \left\{ - \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} W(\xi, y, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n f_{1x}^n(\xi) g_{1z}^m(\xi) \right] d\xi \right. \\
& \quad \left. + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[W(\xi, y, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n f_{1z}^n(\xi) g_{1z}^m(\xi) \right] d\xi \right\} \\
& \quad - \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n f_{2x}^n(\xi) g_{1z}^m(\xi) \right] d\xi \\
& \quad + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n f_{2z}^n(\xi) g_{1z}^m(\xi) \right] d\xi = 0
\end{aligned} \tag{52}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left[\frac{(k_o^2 - \xi^2)}{4\pi} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n f_{1x}^n(\xi) g_{2x}^m(\xi) \right] d\xi \\
& \quad - \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n f_{1z}^n(\xi) g_{2x}^m(\xi) \right] d\xi \\
& \quad + \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \left[\frac{(k_o^2 - \xi^2)}{4\pi} W(\xi, y, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n f_{2x}^n(\xi) g_{2x}^m(\xi) \right] d\xi \right. \\
& \quad \left. - \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} W(\xi, y, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n f_{2z}^n(\xi) g_{2x}^m(\xi) \right] d\xi \right\} = 0
\end{aligned} \tag{53}$$

$$\begin{aligned}
& - \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1x}} a_{1x}^n f_{1x}^n(\xi) g_{2z}^m(\xi) \right] d\xi \\
& + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) \sum_{n=1}^{N_{1z}} a_{1z}^n f_{1z}^n(\xi) g_{2z}^m(\xi) \right] d\xi \\
& - \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \left[\frac{\zeta \xi}{4\pi} W(\xi, y, h_2) \sum_{n=1}^{N_{2x}} a_{2x}^n f_{2x}^n(\xi) g_{2z}^m(\xi) \right] d\xi \right. \\
& \quad \left. + \frac{(k_o^2 - \zeta^2)}{4\pi} \int_{-\infty}^{\infty} \left[W(\xi, y, h_2) \sum_{n=1}^{N_{2z}} a_{2z}^n f_{2z}^n(\xi) g_{2z}^m(\xi) \right] d\xi \right\} = 0.
\end{aligned} \tag{54}$$

Now the order of summation and integration is interchanged, resulting in

$$\begin{aligned}
& \sum_{n=1}^{N_{1x}} a_{1x}^n \left[\lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) f_{1x}^n(\xi) g_{1x}^m(\xi) d\xi \right] \\
& - \sum_{n=1}^{N_{1z}} a_{1z}^n \left[\lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) f_{1z}^n(\xi) g_{1x}^m(\xi) d\xi \right] \\
& + \sum_{n=1}^{N_{2x}} a_{2x}^n \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2x}^n(\xi) g_{1x}^m(\xi) d\xi \right] \\
& - \sum_{n=1}^{N_{2z}} a_{2z}^n \left[\int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2z}^n(\xi) g_{1x}^m(\xi) d\xi \right] = 0
\end{aligned} \tag{55}$$

$$\begin{aligned}
& - \sum_{n=1}^{N_{1x}} a_{1x}^n \left[\lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) f_{1x}^n(\xi) g_{1z}^m(\xi) d\xi \right] \\
& + \sum_{n=1}^{N_{1z}} a_{1z}^n \left[\lim_{y \rightarrow h_1} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_1) f_{1z}^n(\xi) g_{1z}^m(\xi) d\xi \right] \\
& - \sum_{n=1}^{N_{2x}} a_{2x}^n \left[\int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2x}^n(\xi) g_{1z}^m(\xi) d\xi \right] \\
& + \sum_{n=1}^{N_{2z}} a_{2z}^n \left[(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2z}^n(\xi) g_{1z}^m(\xi) d\xi \right] = 0
\end{aligned} \tag{56}$$

$$\begin{aligned}
& \sum_{n=1}^{N_{1x}} a_{1x}^n \left[\int_{-\infty}^{\infty} (k_o^2 - \xi^2) e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1x}^n(\xi) g_{2x}^m(\xi) d\xi \right] \\
& - \sum_{n=1}^{N_{1z}} a_{1z}^n \left[\int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1z}^n(\xi) g_{2x}^m(\xi) d\xi \right] \\
& + \sum_{n=1}^{N_{2x}} a_{1x}^n \left[\lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_2) f_{2x}^n(\xi) g_{2x}^m(\xi) d\xi \right] \\
& - \sum_{n=1}^{N_{2z}} a_{2z}^n \left[\lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) f_{2z}^n(\xi) g_{2z}^m(\xi) d\xi \right] = 0
\end{aligned} \tag{57}$$

$$\begin{aligned}
& - \sum_{n=1}^{N_{1x}} a_{1x}^n \left[\int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1x}^n(\xi) g_{2z}^m(\xi) d\xi \right] \\
& + \sum_{n=1}^{N_{1z}} \left[(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1z}^n(\xi) g_{2z}^m(\xi) d\xi \right] \\
& - \sum_{n=1}^{N_{2x}} a_{2x}^n \left[\lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) f_{2x}^n(\xi) g_{2z}^m(\xi) d\xi \right] \\
& + \sum_{n=1}^{N_{2z}} a_{2z}^n \left[\lim_{y \rightarrow h_2} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_2) f_{2z}^n(\xi) g_{2z}^m(\xi) d\xi \right] = 0.
\end{aligned} \tag{58}$$

Now, by making the following definitions

$$A_{xx}^{11mn} = \lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) f_{1x}^n(\xi) g_{1x}^m(\xi) d\xi \tag{59}$$

$$A_{xz}^{11mn} = \lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) f_{1z}^n(\xi) g_{1x}^m(\xi) d\xi \tag{60}$$

$$A_{xx}^{12mn} = \int_{-\infty}^{\infty} (k_o^2 - \xi^2) e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2x}^n(\xi) g_{1x}^m(\xi) d\xi \tag{61}$$

$$A_{xz}^{12mn} = \int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2z}^n(\xi) g_{1x}^m(\xi) d\xi \quad (62)$$

$$A_{zx}^{11mn} = \lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) f_{1x}^n(\xi) g_{1z}^m(\xi) d\xi \quad (63)$$

$$A_{zz}^{11mn} = \lim_{y \rightarrow h_1} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_1) f_{1z}^n(\xi) g_{1z}^m(\xi) d\xi \quad (64)$$

$$A_{zx}^{12mn} = \int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2x}^n(\xi) g_{1z}^m(\xi) d\xi \quad (65)$$

$$A_{zz}^{12mn} = (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} e^{j\xi(x_1 - x_2)} W(\xi, h_1, h_2) f_{2z}^n(\xi) g_{1z}^m(\xi) d\xi \quad (66)$$

$$A_{xx}^{21mn} = \int_{-\infty}^{\infty} (k_o^2 - \xi^2) e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1x}^n(\xi) g_{2x}^m(\xi) d\xi \quad (67)$$

$$A_{xz}^{21mn} = \int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1z}^n(\xi) g_{2x}^m(\xi) d\xi \quad (68)$$

$$A_{xz}^{21mn} = \lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_2) f_{2x}^n(\xi) g_{2x}^m(\xi) d\xi \quad (69)$$

$$A_{xz}^{22mn} = \lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) f_{2z}^n(\xi) g_{2z}^m(\xi) d\xi \quad (70)$$

$$A_{zx}^{21mn} = \int_{-\infty}^{\infty} \zeta \xi e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1x}^n(\xi) g_{2z}^m(\xi) d\xi \quad (71)$$

$$A_{zz}^{21mn} = (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} e^{j\xi(x_2 - x_1)} W(\xi, h_2, h_1) f_{1z}^n(\xi) g_{2z}^m(\xi) d\xi \quad (72)$$

$$A_{zx}^{22mn} = \lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) f_{2x}^n(\xi) g_{2z}^m(\xi) d\xi \quad (73)$$

$$A_{zz}^{22mn} = \lim_{y \rightarrow h_2} (k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_2) f_{2z}^n(\xi) g_{2z}^m(\xi) d\xi \quad (74)$$

equations (55)-(58) become

$$\sum_{n=1}^{N_{1x}} a_{1x}^n A_{xx}^{11mn} - \sum_{n=1}^{N_{1z}} a_{1z}^n A_{xz}^{11mn} + \sum_{n=1}^{N_{2x}} a_{2x}^n A_{xx}^{12mn} - \sum_{n=1}^{N_{2z}} a_{2z}^n A_{xz}^{12mn} = 0 \quad \dots \text{ with } m = 1, 2, \dots, N_x \quad (75)$$

$$- \sum_{n=1}^{N_{1x}} a_{1x}^n A_{zx}^{11mn} + \sum_{n=1}^{N_{1z}} a_{1z}^n A_{zz}^{11mn} - \sum_{n=1}^{N_{2x}} a_{2x}^n A_{zx}^{12mn} + \sum_{n=1}^{N_{2z}} a_{2z}^n A_{zz}^{12mn} = 0 \quad \dots \text{ with } m = 1, 2, \dots, N_z \quad (76)$$

$$\sum_{n=1}^{N_{1x}} a_{1x}^n A_{xx}^{21mn} - \sum_{n=1}^{N_{1z}} a_{1z}^n A_{xz}^{21mn} + \sum_{n=1}^{N_{2x}} a_{2x}^n A_{xx}^{22mn} - \sum_{n=1}^{N_{2z}} a_{2z}^n A_{xz}^{22mn} = 0 \quad \dots \text{ with } m = 1, 2, \dots, N_x \quad (77)$$

$$- \sum_{n=1}^{N_{1x}} a_{1x}^n A_{zx}^{21mn} + \sum_{n=1}^{N_{1z}} a_{1z}^n A_{zz}^{21mn} - \sum_{n=1}^{N_{2x}} a_{2x}^n A_{zx}^{22mn} + \sum_{n=1}^{N_{2z}} a_{2z}^n A_{zz}^{22mn} = 0 \quad \dots \text{ with } m = 1, 2, \dots, N_z. \quad (78)$$

For the Chebyshev Galerkin's solution, the basis set and testing function are

$$e_{\alpha\beta}^n(x) = T_n(x/a) [1 - (x/a)^2]^{\pm 1/2} \quad (79)$$

$$t_{\alpha\beta}^m(x) = T_m(x/a) \left[1 - (x/a)^2 \right]^{\pm 1/2} \quad (80)$$

where $\alpha=1,2$, $a=w_\alpha/2$, and the "+" sign is associated with $\beta=x$, and the "-" sign is associated with $\beta=z$ [44]. In the analysis of the single center-strip structure of the previous chapter, the time-harmonic, transform-domain continuity equation

$$\frac{\partial k_x}{\partial x} + j\zeta k_z = -j\omega \rho_s(x) \quad (81)$$

was examined to determine the relationship between basis functions $e_x(x)$ and $e_z(x)$, and modal structure. For the isolated stripline, it was observed from equation (81) that if $k_z(x)$ is an even function, and $k_x(x)$ is an odd function, then $\rho_s(x)$ is an even function, representing an even propagation mode [26]. Likewise, if $k_z(x)$ is an odd function, and $k_x(x)$ is an even function, then $\rho_s(x)$ is odd, and thus represents an odd mode. It was therefore concluded that, for even modes $e_z^n(x)$ is even and $e_x^n(x)$ is odd, and for odd modes $e_z^n(x)$ is odd and $e_x^n(x)$ is even.

The analysis above is not valid for coupled stripline. Due to interactions between the conducting strips, in general, the current distribution functions $k_{\alpha x}(x)$ and $k_{\alpha z}(x)$ will be neither even nor odd about the centers of the respective strip conductors. As a result, the relationships between modal index and basis function structure described above for the isolated stripline will not exist for the coupled structure. The expanded current distributions (38) associated with coupled stripline will therefore be sums of both even and odd basis functions.

In the previous chapter, it was noted that Chebyshev polynomials of even order are even, and those of odd order are odd such that

$$T_n(-x) = (-1)^n T_n(x). \quad (82)$$

As a result of relationship (82)

$$f_{\alpha\beta}^n(\xi) = \int_{-w_\alpha/2}^{w_\alpha/2} e_{\alpha\beta}^n(x') e^{-j\xi x'} dx' = \begin{cases} 2 \int_0^{w_\alpha/2} e_{\alpha\beta}^n(x') \cos \xi x' dx' & \dots \text{ for } e_{\alpha\beta}(x') \text{ even} \\ -2j \int_0^{w_\alpha/2} e_{\alpha\beta}^n(x') \sin \xi x' dx' & \dots \text{ for } e_{\alpha\beta}(x') \text{ odd} \end{cases} \quad (83)$$

where $\alpha=1,2$, $\beta=x,z$. Likewise, with the testing operator

$$g_{\alpha\beta}^m(\xi) = \int_{-w_\alpha/2}^{w_\alpha/2} t_{\alpha\beta}^m(x) e^{j\xi x} dx = \begin{cases} 2 \int_0^{w_\alpha/2} t_{\alpha\beta}^m(x) \cos \xi x dx & \dots \text{ for } t_{\alpha\beta}(x) \text{ even} \\ 2j \int_0^{w_\alpha/2} t_{\alpha\beta}^m(x) \sin \xi x dx & \dots \text{ for } t_{\alpha\beta}(x) \text{ odd} \end{cases} \quad (84)$$

for $\alpha=1,2$, $\beta=x,z$.

It is seen from expressions (83) and (84) that four generic integral types are implicated

$$I_{\alpha z}^{ne}(\xi) = \int_0^a \frac{T_{2n}(x/a)}{\sqrt{1-(x/a)^2}} \cos(\xi x) dx \quad (85)$$

$$I_{\alpha z}^{no}(\xi) = \int_0^a \frac{T_{2n+1}(x/a)}{\sqrt{1-(x/a)^2}} \sin(\xi x) dx \quad (86)$$

$$I_{\alpha x}^{ne}(\xi) = \int_0^a T_{2n}(x/a) \sqrt{1-(x/a)^2} \cos(\xi x) dx \quad (87)$$

$$I_{\alpha x}^{no}(\xi) = \int_0^a T_{2n+1}(x/a) \sqrt{1 - (x/a)^2} \sin(\xi x) dx \quad (88)$$

where $a = w_\alpha/2$ [24]. Following the methods of the previous chapter, closed form expressions can be obtained for the integrals (85)-(88). As a result, relationships (83) and (84) may be written as

$$f_{\alpha x}^n(\xi) = \begin{cases} 2I_{\alpha x}^{ne}(\xi) & \dots \text{ for } n \text{ even} \\ -2jI_{\alpha x}^{no}(\xi) & \dots \text{ for } n \text{ odd} \end{cases} \quad (89)$$

$$f_{\alpha z}^n(\xi) = \begin{cases} 2I_{\alpha z}^{ne}(\xi) & \dots \text{ for } n \text{ even} \\ -2jI_{\alpha z}^{no}(\xi) & \dots \text{ for } n \text{ odd} \end{cases} \quad (90)$$

$$g_{\alpha x}^m(\xi) = \begin{cases} 2I_{\alpha x}^{me}(\xi) & \dots \text{ for } m \text{ even} \\ 2jI_{\alpha x}^{mo}(\xi) & \dots \text{ for } m \text{ odd} \end{cases} \quad (91)$$

$$g_{\alpha z}^m(\xi) = \begin{cases} 2I_{\alpha z}^{me}(\xi) & \dots \text{ for } m \text{ even} \\ 2jI_{\alpha z}^{mo}(\xi) & \dots \text{ for } m \text{ odd} \end{cases} \quad (92)$$

where

$$I_{\alpha z}^{ne} = (-1)^n \frac{\pi w_\alpha}{4} J_{2n} \left(\xi \frac{w_\alpha}{2} \right) \quad (93)$$

$$I_{\alpha z}^{no} = (-1)^n \frac{\pi w_\alpha}{4} J_{2n+1} \left(\xi \frac{w_\alpha}{2} \right) \quad (94)$$

$$I_{\alpha x}^{ne} = (-1)^n \frac{\pi w_\alpha}{8} \left[J_{2n} \left(\xi \frac{w_\alpha}{2} \right) + \frac{1}{2} \left(J_{2(n+1)} \left(\xi \frac{w_\alpha}{2} \right) + J_{2|n-1|} \left(\xi \frac{w_\alpha}{2} \right) \right) \right] \quad (95)$$

$$I_{\alpha x}^{no} = (-1)^n \frac{\pi w_\alpha}{8} \left[J_{2n+1} \left(\xi \frac{w_\alpha}{2} \right) + \frac{1}{2} \left(J_{2n+3} \left(\xi \frac{w_\alpha}{2} \right) + J_{2n-1} \left(\xi \frac{w_\alpha}{2} \right) \right) \right]. \quad (96)$$

Using (89)-(92), the matrix elements (59)-(74) become

$$A_{xx}^{11mn} = \begin{cases} \lim_{y \rightarrow h_1} 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) I_{1x}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_1} -4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) I_{1x}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_1} 4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) I_{1x}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_1} 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) I_{1x}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (97)$$

$$A_{xz}^{11mn} = \begin{cases} \lim_{y \rightarrow h_1} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1z}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_1} -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1z}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_1} 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1z}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_1} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1z}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (98)$$

$$A_{xx}^{12mn} = \begin{cases} 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (99)$$

$$A_{xz}^{12mn} = \begin{cases} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (100)$$

$$A_{zx}^{11mn} = \begin{cases} \lim_{y \rightarrow h_1} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1x}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_1} -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1x}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_1} 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1x}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_1} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1x}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (101)$$

$$A_{zz}^{11mn} = \begin{cases} \lim_{y \rightarrow h_1} 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_1) I_{1z}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_1} -4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_1) I_{1z}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_1} 4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_1) I_{1z}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_1} 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_1) I_{1z}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (102)$$

$$A_{zx}^{12mn} = \begin{cases} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2x}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (103)$$

$$A_{zz}^{12mn} = \begin{cases} 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} I_{2z}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (104)$$

$$A_{xx}^{21mn} = \begin{cases} 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (105)$$

$$A_{xz}^{21mn} = \begin{cases} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (106)$$

$$A_{xx}^{22mn} = \begin{cases} \lim_{y \rightarrow h_2} 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_2) I_{2x}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_2} -4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_2) I_{2x}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_2} 4j \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_2) I_{2x}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_2} 4 \int_{-\infty}^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_2) I_{2x}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (107)$$

$$A_{xz}^{22mn} = \begin{cases} \lim_{y \rightarrow h_2} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2z}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_2} -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2z}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_2} 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2z}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_2} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2z}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (108)$$

$$A_{zx}^{21mn} = \begin{cases} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1x}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (109)$$

$$A_{zz}^{21mn} = \begin{cases} 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} I_{1z}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (110)$$

$$A_{zx}^{22mn} = \begin{cases} \lim_{y \rightarrow h_2} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2x}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{ for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_2} -4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2x}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{ for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_2} 4j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2x}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{ for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_2} 4 \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_2) I_{2x}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{ for } n \text{ odd, } m \text{ odd} \end{cases} \quad (111)$$

$$A_{zz}^{22mn} = \begin{cases} \lim_{y \rightarrow h_2} 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_2) I_{2z}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{ for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_2} -4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_2) I_{2z}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{ for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_2} 4j(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_2) I_{2z}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{ for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_2} 4(k_o^2 - \zeta^2) \int_{-\infty}^{\infty} W(\xi, y, h_2) I_{2z}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{ for } n \text{ odd, } m \text{ odd} \end{cases} \quad (112)$$

It is seen that $(k_o^2 - \xi^2)$ and $W(\xi, y, y')$ are both even functions of ξ , therefore matrix elements (97)-(112) become

$$A_{xx}^{11mn} = \begin{cases} \lim_{y \rightarrow h_1} 8 \int_0^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) I_{1x}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{ for } n \text{ even, } m \text{ even} \\ 0 & \dots \text{ for } n \text{ odd, } m \text{ even} \\ 0 & \dots \text{ for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_1} 8 \int_0^{\infty} (k_o^2 - \xi^2) W(\xi, y, h_1) I_{1x}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{ for } n \text{ odd, } m \text{ odd} \end{cases} \quad (113)$$

$$A_{xz}^{11mn} = \begin{cases} 0 & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_1} -8j \int_0^\infty \zeta \xi W(\xi, y, h_1) I_{1z}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_1} 8j \int_0^\infty \zeta \xi W(\xi, y, h_1) I_{1z}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (114)$$

$$A_{xx}^{12mn} = \begin{cases} 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2x}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2x}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ -8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2x}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2x}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (115)$$

$$A_{xz}^{12mn} = \begin{cases} 8j \int_0^\infty \zeta \xi W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2z}^{ne}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -8j \int_0^\infty \zeta \xi W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2z}^{no}(\xi) I_{1x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 8j \int_0^\infty \zeta \xi W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2z}^{ne}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8j \int_0^\infty \zeta \xi W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2z}^{no}(\xi) I_{1x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (116)$$

$$A_{zx}^{11mn} = \begin{cases} 0 & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_1} -8j \int_0^{\infty} \zeta \xi W(\xi, y, h_1) I_{1x}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_1} 8j \int_{-\infty}^{\infty} \zeta \xi W(\xi, y, h_1) I_{1x}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (117)$$

$$A_{zz}^{11mn} = \begin{cases} \lim_{y \rightarrow h_1} 8(k_o^2 - \zeta^2) \int_0^{\infty} W(\xi, y, h_1) I_{1z}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ even} \\ 0 & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_1} 8(k_o^2 - \zeta^2) \int_0^{\infty} W(\xi, y, h_1) I_{1z}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (118)$$

$$A_{zx}^{12mn} = \begin{cases} 8j \int_0^{\infty} \zeta \xi W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2x}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -8j \int_0^{\infty} \zeta \xi W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2x}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 8j \int_0^{\infty} \zeta \xi W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2x}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8j \int_0^{\infty} \zeta \xi W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2x}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (119)$$

$$A_{zz}^{12mn} = \begin{cases} 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2z}^{ne}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2z}^{no}(\xi) I_{1z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ -8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_1, h_2) \sin[\xi(x_1 - x_2)] I_{2z}^{ne}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_1, h_2) \cos[\xi(x_1 - x_2)] I_{2z}^{no}(\xi) I_{1z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (120)$$

$$A_{xx}^{21mn} = \begin{cases} 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1x}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1x}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ -8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1x}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1x}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (121)$$

$$A_{xz}^{21mn} = \begin{cases} 8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1z}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1z}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1z}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1z}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (122)$$

$$A_{xx}^{22mn} = \begin{cases} \lim_{y \rightarrow h_2} 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, y, h_2) I_{2x}^{ne}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ even} \\ 0 & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_2} 8 \int_0^\infty (k_o^2 - \xi^2) W(\xi, y, h_2) I_{2x}^{no}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (123)$$

$$A_{xz}^{22mn} = \begin{cases} 0 & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_2} -8j \int_0^\infty \zeta \xi W(\xi, y, h_2) I_{2z}^{no}(\xi) I_{2x}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_2} 8j \int_0^\infty \zeta \xi W(\xi, y, h_2) I_{2z}^{ne}(\xi) I_{2x}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (124)$$

$$A_{zx}^{21mn} = \begin{cases} 8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1x}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ -8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1x}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ 8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1x}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8j \int_0^\infty \zeta \xi W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1x}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (125)$$

$$A_{zz}^{21mn} = \begin{cases} 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1z}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1z}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ -8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_2, h_1) \sin[\xi(x_2 - x_1)] I_{1z}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, h_2, h_1) \cos[\xi(x_2 - x_1)] I_{1z}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (126)$$

$$A_{zx}^{22mn} = \begin{cases} 0 & \dots \text{for } n \text{ even, } m \text{ even} \\ \lim_{y \rightarrow h_2} -8j \int_0^\infty \zeta \xi W(\xi, y, h_2) I_{2x}^{no}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ even} \\ \lim_{y \rightarrow h_2} 8j \int_0^\infty \zeta \xi W(\xi, y, h_2) I_{2x}^{ne}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ odd} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (127)$$

$$A_{zz}^{22mn} = \begin{cases} \lim_{y \rightarrow h_2} 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, y, h_2) I_{2z}^{ne}(\xi) I_{2z}^{me}(\xi) d\xi & \dots \text{for } n \text{ even, } m \text{ even} \\ 0 & \dots \text{for } n \text{ odd, } m \text{ even} \\ 0 & \dots \text{for } n \text{ even, } m \text{ odd} \\ \lim_{y \rightarrow h_2} 8(k_o^2 - \zeta^2) \int_0^\infty W(\xi, y, h_2) I_{2z}^{no}(\xi) I_{2z}^{mo}(\xi) d\xi & \dots \text{for } n \text{ odd, } m \text{ odd} \end{cases} \quad (128)$$

5.5 Analysis of Green's Function Singularities

For simplicity, the singularities of the Green's function are first examined for the case of coplanar center strips which are located in the plane of the origin ($y' = 0$). The transform-domain Green's function associated with the transverse components of Hertz potential is

$$g_{\zeta}^{\beta}(x, y | x', 0) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, y') d\xi \quad (129)$$

where $\beta = x, z$. In the region above the strip conductors ($y > y' = 0$)

$$\begin{aligned} W(\xi, y, 0) &= \frac{1}{p_0} \left\{ e^{-p_0 y} + \frac{1}{(1 - e^{-4p_0 l})} \left[-e^{-p_0(2l+y)} - e^{-p_0(2l-y)} + e^{-p_0(4l+y)} + e^{-p_0(4l-y)} \right] \right\} \\ &= \frac{e^{-p_0 y} - e^{-p_0(2l-y)}}{p_0(1 + e^{-2p_0 l})} = \frac{\sinh(p_0[l-y])}{p_0 \cosh(p_0 l)}. \end{aligned} \quad (130)$$

It can be seen that (130) is even with respect to the transform-domain wavenumber parameter p_0 . In the region below the strips ($y < y' = 0$)

$$\begin{aligned} W(\xi, y, 0) &= \frac{1}{p_0} \left\{ e^{p_0 y} + \frac{1}{(1 - e^{-4p_0 l})} \left[-e^{-p_0(2l+y)} - e^{-p_0(2l-y)} + e^{-p_0(4l+y)} + e^{-p_0(4l-y)} \right] \right\} \\ &= \frac{e^{-p_0 y} - e^{-p_0(2l+y)}}{p_0(1 + e^{-2p_0 l})} = \frac{\sinh(p_0[l+y])}{p_0 \cosh(p_0 l)}. \end{aligned} \quad (131)$$

Equation (131) is also even with respect to parameter p_0 , and thus the branch point associated with (129) is removable.

For the more general case when the conducting strips are not coplanar

$$W(\xi, y, y') = \frac{1}{p_0} \left\{ e^{-p_0|y-y'|} + \frac{1}{(1 - e^{-4p_0 l})} \left[-e^{-p_0(2l+y+y')} - e^{-p_0(2l-y-y')} + e^{-p_0(4l+y-y')} + e^{-p_0(4l-y+y')} \right] \right\} \quad (132)$$

which may be expressed

$$W(\xi, y, y') = \frac{1}{p_0(1 - e^{-4p_0 l})} \left\{ (1 - e^{-4p_0 l}) e^{-p_0|y-y'|} - e^{-p_0(2l+y+y')} - e^{-p_0(2l-y-y')} + e^{-p_0(4l+y-y')} + e^{-p_0(4l-y+y')} \right\}. \quad (133)$$

It is seen that $W(\xi, y, \zeta)$ is singular at points where

$$1 - e^{-4p_0 l} = e^{2p_0 l} - e^{-2p_0 l} = 0 \quad (134)$$

or

$$2 \sinh(2p_0 l) = 0. \quad (135)$$

Thus poles of $W(\xi, y, \zeta)$ occur for

$$p_0 = \pm j \frac{n\pi}{2l} \quad (136)$$

where n is a positive integer ($n=1,2,3,\dots$). It is seen that the poles indicated in equation (136) differ from those associated with the isolated stripline (examined in the previous chapter), which occur for $\cosh(p_0 h) = 0$, or when

$$p_0 = \pm j \frac{(2n-1)\pi}{2h} \quad (137)$$

where again n is a positive integer ($n=1,2,3,\dots$). The poles associated with the isolated stripline given by (137) are actually a subset of the poles in (136). The difference is due to the fact that, in the analysis of the isolated stripline, it was assumed that the conducting strip was centered at the origin, and was equidistant from both ground planes. As a result, certain modes could not be excited due to the symmetry of the structure. The additional poles which appear in (136) include those which are introduced when both conducting strips are not assumed to lie in the plane of the origin.

Returning to the analysis of the coupled stripline structure, the parameter p_0 was previously defined as

$$p_0^2 = \lambda^2 - k_o^2 \quad (138)$$

with $\lambda^2 = \xi^2 + \zeta^2$, therefore at the poles

$$-\left(\frac{n\pi}{2l}\right)^2 = \lambda_p^2 - k_o^2. \quad (139)$$

From this the propagation eigenvalues λ_p at the poles of $W(\xi, y, \zeta)$ are seen to be

$$\lambda_p = \sqrt{k_o^2 - \left(\frac{n\pi}{2l}\right)^2}. \quad (140)$$

The poles implicated in the expressions above, and the associated propagation modes for coupled stripline exhibit the same behavior as those associated with the isolated stripline structure analyzed in Chapter 4. In this case the poles are associated with waves that are guided by the parallel ground-plate background structure, having plate separation $2l$. When the frequency of operation is such that $k_o < n\pi/(2l)$ it is seen

from expression (140) that λ_p is imaginary, and the associated parallel-plate modes are evanescent. When k_o^2 becomes sufficiently large, λ_p is real, and propagating modes exist which are guided by the background structure. During this transition, the poles associated with the guided-wave modes of the background structure migrate from the imaginary λ -axis to the real λ -axis in the manner discussed in the previous chapter.

Also as before, the relationship

$$\xi_p = -j\sqrt{\zeta - \lambda_p}\sqrt{\zeta + \lambda_p} \quad (141)$$

implicates branch points at $\zeta = \pm\lambda_p$ in the complex ζ -plane. Solutions which lead to proper modes (those associated with fields satisfying the radiation condition) are insured by choosing branch cuts which prevent singularities from migrating across the $Re\{\xi\}$ axis. This creates a strip of convergence, resulting in fields which fall off with distance in directions away from the guiding structure. Thus requiring that $Im\{\xi_p\} < 0$, or

$$Re\left\{\sqrt{\zeta^2 - \lambda_p^2}\right\} = Re\left\{\sqrt{\zeta - \lambda_p}\sqrt{\zeta + \lambda_p}\right\} > 0 \quad (142)$$

leads to Sommerfeld [20] branch cuts emanating from the branch points at $\zeta = \pm\lambda_p$.

In addition to the proper guided modes, the coupled stripline structure also supports improper leaky-wave modes which are associated with poles lying on the lower Riemann sheets [25], [27]-[42]. As with the isolated stripline, improper modes arise when at least one ζ_p moves off the $Re\{\zeta\}$ axis, and becomes complex with $Re\{\zeta\} < \lambda_p$, and $Im\{\zeta\} < 0$. In this case, roots can only be found if passage is made to the bottom sheet by violating the λ_p -related branch cut, which causes the ξ -plane pole at ξ_p to migrate across the $Re\{\xi\}$ axis, into the upper-half ξ -plane.

The integration path must be deformed so as to remain above or below the pole

at ξ_p when improper leaky-wave modes exist. This leads to a pole residue contribution to the real axis integration which occurs in the transform-domain Green's function. This residue contribution is determined through evaluation of

$$\oint_{C_p} W(\xi, y, \zeta) \Big|_{\xi=\xi_p} d\xi \quad (143)$$

where $W(\xi, y, \zeta)$ may be expressed

$$W(\xi, y, \zeta) = \frac{N(\xi, y, \zeta)}{p_0(\tilde{\lambda}) D(\xi)}. \quad (144)$$

The numerator in expression (144) is given by

$$N(\xi, y, \zeta) = \left\{ (1 - e^{-4p_0 l}) e^{-p_0 |y - y'|} - e^{-p_0 (2l + y + y')} \right. \\ \left. - e^{-p_0 (2l - y - y')} + e^{-p_0 (4l + y - y')} + e^{-p_0 (4l - y + y')} \right\} \quad (145)$$

and the denominator is

$$D(\xi) = 1 - e^{-4p_0 l}. \quad (146)$$

At the pole

$$\xi_p^2 = -j\sqrt{\zeta^2 - \lambda_p^2} \quad (147)$$

and

$$D(\xi) = 1 - e^{-4p_0 l} = 0 \quad (148)$$

or

$$e^{-4p_0 l} = 1 \quad (149)$$

and thus

$$\begin{aligned} N_p(\xi, y, \zeta) &= \left\{ \left(1 - e^{-4p_0 l} \right) e^{-p_0 |y - y'|} - e^{-p_0 (2l + y + y')} \right. \\ &\quad \left. - e^{-p_0 (2l - y - y')} + e^{-p_0 (4l + y - y')} + e^{-p_0 (4l - y + y')} \right\} \quad (150) \\ &= \left\{ -e^{-p_0 (2l + y + y')} - e^{-p_0 (2l - y - y')} + e^{-p_0 (y - y')} + e^{-p_0 (y' - y)} \right\}. \end{aligned}$$

The denominator term (146) is now expanded about the pole ξ_p , such that

$$\begin{aligned} D(\xi) &= 1 - e^{-4p_0 l} \approx D(\xi_p) + \left. \frac{dD}{d\xi} \right|_{\xi=\xi_p} (\xi - \xi_p) \\ &= -4l \left(-e^{-4p_0 l} \frac{dp_0}{d\xi} \right) \Big|_{\xi=\xi_p} (\xi - \xi_p) = 4l \left. \frac{dp_0}{d\xi} \right|_{\xi=\xi_p} (\xi - \xi_p). \end{aligned} \quad (151)$$

But

$$\frac{dp_0}{d\xi} = \frac{\xi}{p_0} \quad (152)$$

and therefore

$$D_p(\xi) \approx \frac{4l \xi_p}{p_{0p}} (\xi - \xi_p) \quad (153)$$

where $p_{0p} = \sqrt{\xi_p^2 + \zeta^2 - k_o^2}$ is the transform-domain wavenumber at the pole. This leads to

$$W_p(\xi, y, \zeta) = \frac{N_p(\xi)}{p_{0p} D_p(\xi)} = \frac{\left[-e^{-p_{0p}(y-y')} - e^{-p_{0p}(y'-y)} + e^{-p_{0p}(2l+y+y')} + e^{-p_{0p}(2l-y-y')} \right]}{4l\xi_p(\xi - \xi_p)}. \quad (154)$$

The line integral along the inversion contour C_p

$$\oint_{C_p} \frac{d\xi}{(\xi - \xi_p)} \quad (155)$$

is evaluated by letting

$$\xi = \xi_p + \epsilon e^{j\psi} \quad (156)$$

and therefore

$$d\xi = j\epsilon e^{j\psi} d\psi. \quad (157)$$

This gives

$$\oint_{C_p} \frac{d\xi}{(\xi - \xi_p)} = \int_0^{-2\pi} \frac{j\epsilon e^{j\psi}}{\epsilon e^{j\psi}} d\psi = -2\pi j \quad (158)$$

and then the pole residue contribution is

$$\oint_{C_p} W_p(\xi, y, \zeta) d\xi = -j\pi \frac{\left[e^{-p_{0p}(y-y')} + e^{-p_{0p}(y'-y)} - e^{-p_{0p}(2l+y+y')} - e^{-p_{0p}(2l-y-y')} \right]}{2l\xi_p}. \quad (159)$$

5.6 Analysis of Numerical Results

The dispersion characteristics of the first two higher-order modes excited on a coupled stripline structure having widely separated center strips are compared to those excited on an isolated stripline with similar dimensions in Figure 5.3. The coupled

stripline is configured such that both conducting center strips lie in the plane of the origin ($h_1=h_2=0$), the widths of both strips are identical ($w_1=w_2=5.0$ cm), the ground plane half-separation is $l=1.75$ cm, and the strips are widely separated, having a center-to-center distance of 15.0 cm. The isolated stripline is such that the center strip has a width $w=5.0$ cm and a ground plane half-separation of $h=1.75$ cm. The $l=1$ TM parallel-plate guided wave pole λ_p is also included with the dispersion curves for these structures.

For both the isolated and coupled stripline structures, the first higher-order mode excited is the TE_1 mode, while the second higher-order mode is the TE_2 mode. During the analysis of the isolated stripline, it was seen that the TE_1 mode was odd, while the TE_2 mode was even. It is noted that, in general due to the lack of symmetry of the coupled stripline, each of these modes is usually neither even nor odd. The poles associated with these modes lie on the top Riemann sheet, and thus both are proper non-leaky modes, having real propagation constants $\zeta > \lambda_p$. Both modes also exhibit a low frequency cut-off.

It is seen that the dispersion curves for both the TE_1 and TE_2 modes excited on a coupled stripline, with widely separated center strips, are very nearly the same as those associated with an isolated stripline (having similar dimensions). It is noted, however, that for both modes, the slight discrepancy between the curves for the two structures is greater at lower frequencies than at higher frequencies. This is due to the fact that as the frequency of operation decreases, the electrical distance between the center strips of the coupled stripline also decreases, leading to tighter coupling. This stronger coupling causes a slight change in the modal characteristics of the coupled stripline, as compared with the isolated stripline.

The cross-sectional surface current distributions associated with the TE_1 and TE_2 modes excited on the coupled stripline are presented in Figure 5.4 and Figure 5.5, respectively. The total surface currents which exist on the center strips consist of both axial and transverse components

$$\vec{k}_1(\vec{x}) = \hat{x}k_{1x}(\vec{x}) + \hat{z}k_{1z}(\vec{x}) \quad (160)$$

$$\vec{k}_2(\vec{x}) = \hat{x}k_{2x}(\vec{x}) + \hat{z}k_{2z}(\vec{x}) . \quad (161)$$

It is seen for both modes that the axially-directed currents show the expected edge singularities, while the transverse current amplitudes are large for frequencies near cutoff, and small for frequencies far from cutoff. While it is expected that coupling between the center strips will cause a perturbation of the current distributions, none is detected in this case. This is due to the relatively large center-to-center distance between the conducting strips.

Although the propagation modes excited on the coupled stripline are in general neither even nor odd, this structure supports modes which are either symmetric or anti-symmetric. For the case in which the center strips have equal widths, symmetric modes are denoted (in local coordinates) by

$$k_{1z}(\vec{x}) = k_{2z}(-\vec{x}) \quad (162)$$

and anti-symmetric modes are denoted by

$$k_{1z}(\vec{x}) = -k_{2z}(-\vec{x}) . \quad (163)$$

When the amount of coupling between the strips is small, the propagation constants for symmetric and anti-symmetric modes of the same order are nearly identical. This is the

case for the dispersion curves presented in Figure 5.3. Due to the wide separation between the center strips, the amount of coupling is small, the dispersion curves for the symmetric and anti-symmetric modes are nearly identical, and thus no distinction between the two is made.

As the amount of coupling increases, however, the degree to which the propagation constants for the symmetric and anti-symmetric modes differ also increases. This mode-splitting behavior can be observed in Figure 5.6. Here the propagation constants associated with the TE_1 symmetric and the TE_1 anti-symmetric modes at a single operating frequency are tracked for varying center-to-center strip separations on a coupled stripline for which $h_1=h_2=0$, $w_1=w_2=5.0$ cm, and $l=1.75$ cm. It is seen that a significant difference between the symmetric and anti-symmetric propagation constants occurs for center-to-center separations which are less than twice width of the center strips.

The axial current distributions associated with the TE_1 symmetric and the TE_1 anti-symmetric modes excited on the coupled stripline described above, are presented in Figure 5.7, Figure 5.8, Figure 5.9, and Figure 5.10. Here the center strips lie in the plane of the origin ($h_1=h_2=0$), with strip 1 lying to the left of the origin, and strip 2 lying to the right of the origin. It is seen that, as expected, the current distributions on the center strips perturb as the center-to-center distance between the strips is decreased. The currents associated with the symmetric mode repel. As the distance between the strips decreases, currents move away from the inner edges of the strips, and begin to accumulate on the outer edges of the strips. Conversely, the currents associated with the anti-symmetric mode attract. As the center-to-center distance decreases, currents begin

to accumulate on the inner edges of the strips.

A more interesting case is presented in Figure 5.11. Here the propagation constants associated with the TE_2 symmetric and TE_2 anti-symmetric modes are tracked for varying center-to-center strip separations on the same coupled stripline structure examined above, having $h_1=h_2=0$, $w_1=w_2=5.0$ cm, and $l=1.75$ cm (it should be noted that the TE_2 dispersion curve for the widely separated center strips shown in Figure 5.3 lies close to the $l=1$ TM parallel-plate guided wave pole λ_p). At a constant frequency, as the distance between the center strips is decreased, the propagation constants associated with the symmetric and anti-symmetric modes are again observed to diverge. However in this case the dispersion curve for the TE_2 symmetric mode is observed to approach, and very nearly cross the λ_p parallel-plate background pole.

Axial current distributions associated with the anti-symmetric mode are presented in Figure 5.12 and Figure 5.13. These figures show the expected behavior the TE_2 anti-symmetric mode as the distance between the strips decreases. An examination of the axial current distributions associated with the TE_2 symmetric mode (shown in Figure 5.14 and Figure 5.15) reveals an abrupt change at the center-to-center separation at which the TE_2 dispersion curve very nearly crosses the background pole. For strip separations greater than the critical value of approximately 7.0 cm, the expected behavior of the TE_2 symmetric mode is observed. However, for separations less than the critical value, the current distributions, which are no longer symmetric, appear to be non-physical. It may therefore be concluded that as the center strips are moved closer together, the TE_2 symmetric mode is excited until some critical center-to-center separation is exceeded, at which point a change in the propagation mode occurs. It is also concluded that the real

root found just above the background pole is associated with a spurious, non-physical mode. It is suspected that the TE₂ mode can be tracked below the λ_p parallel-plate background pole, although efforts to do this to date have not been successful. After the TE₂ propagation constant crosses the background pole, it is expected that roots associated with improper leaky wave modes may be found on the lower Riemann sheet. This investigation requires further attention and will be left for future work.

5.7 TEM Specialization for Coupled Stripline

It was noted previously that stripline devices are often operated at frequencies such that only the fundamental mode is excited. Here the behavior of the coupled stripline structure will be examined for the case when only the fundamental TEM mode is present. As with the isolated stripline, this specialization is also valuable because self and mutual impedance parameters for coupled stripline structures may be defined for the TEM mode.

In the analysis above, it was determined that the coupled strip structure is described by two pairs of coupled integral equations, one pair representing the x - and z -directed electric field components associated with strip 1, given by

$$\lim_{y \rightarrow h_1} \left\{ \int_{-a_1}^{a_1} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g'_\zeta(x, y | x', h_1) k_{1x}(x') + j\zeta \frac{\partial}{\partial x} g'_\zeta(x, y | x', h_1) k_{1z}(x') \right] dx' \right\} \\ + \int_{-a_2}^{a_2} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g'_\zeta(x, h_1 | x', h_2) k_{2x}(x') + j\zeta \frac{\partial}{\partial x} g'_\zeta(x, h_1 | x', h_2) k_{2z}(x') \right] dx' = 0 \quad (164)$$

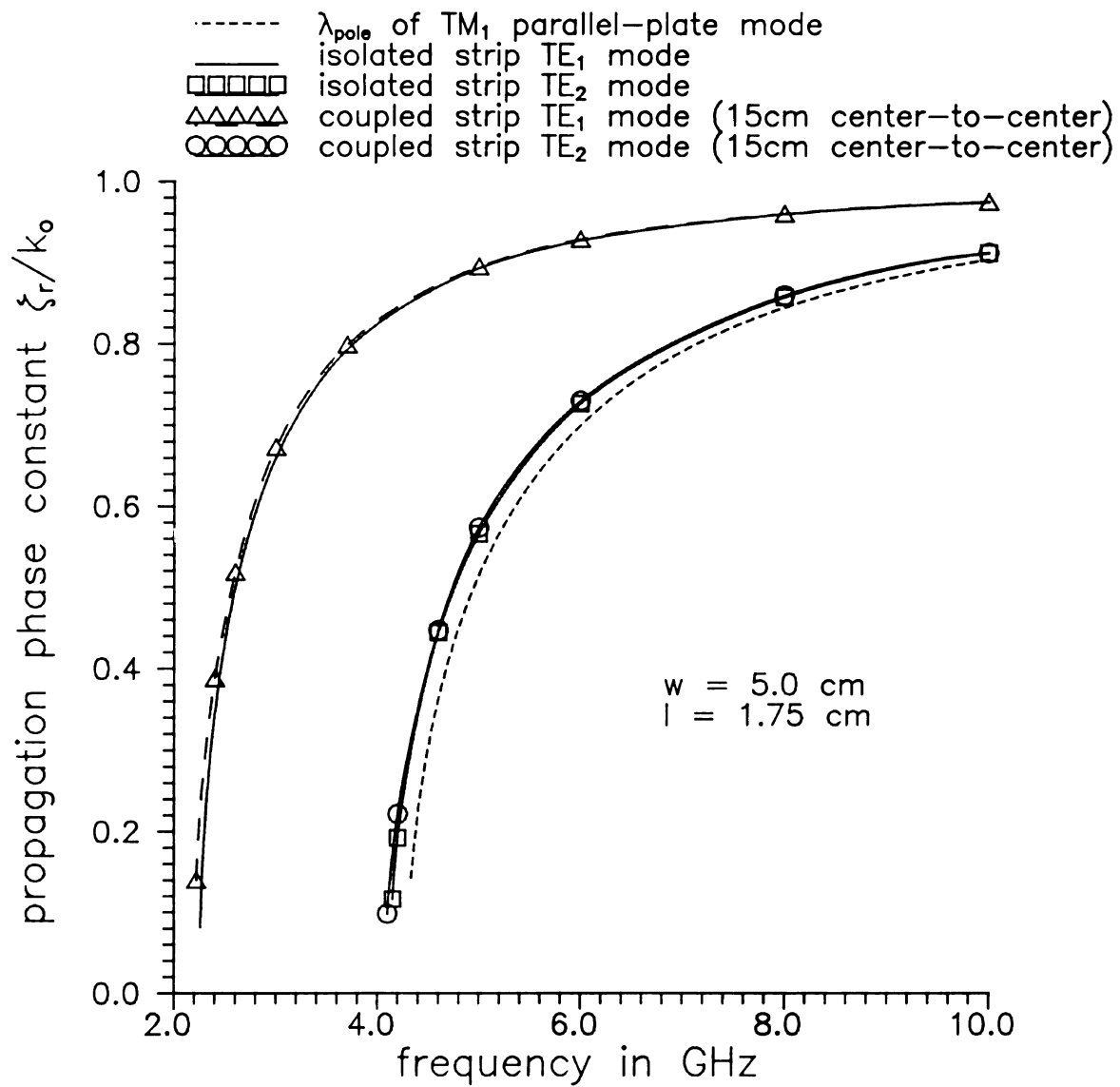


Figure 5.3. Comparison of dispersion characteristics of non-leaky, higher-order modes of isolated and widely-spaced coupled, coplanar stripline.

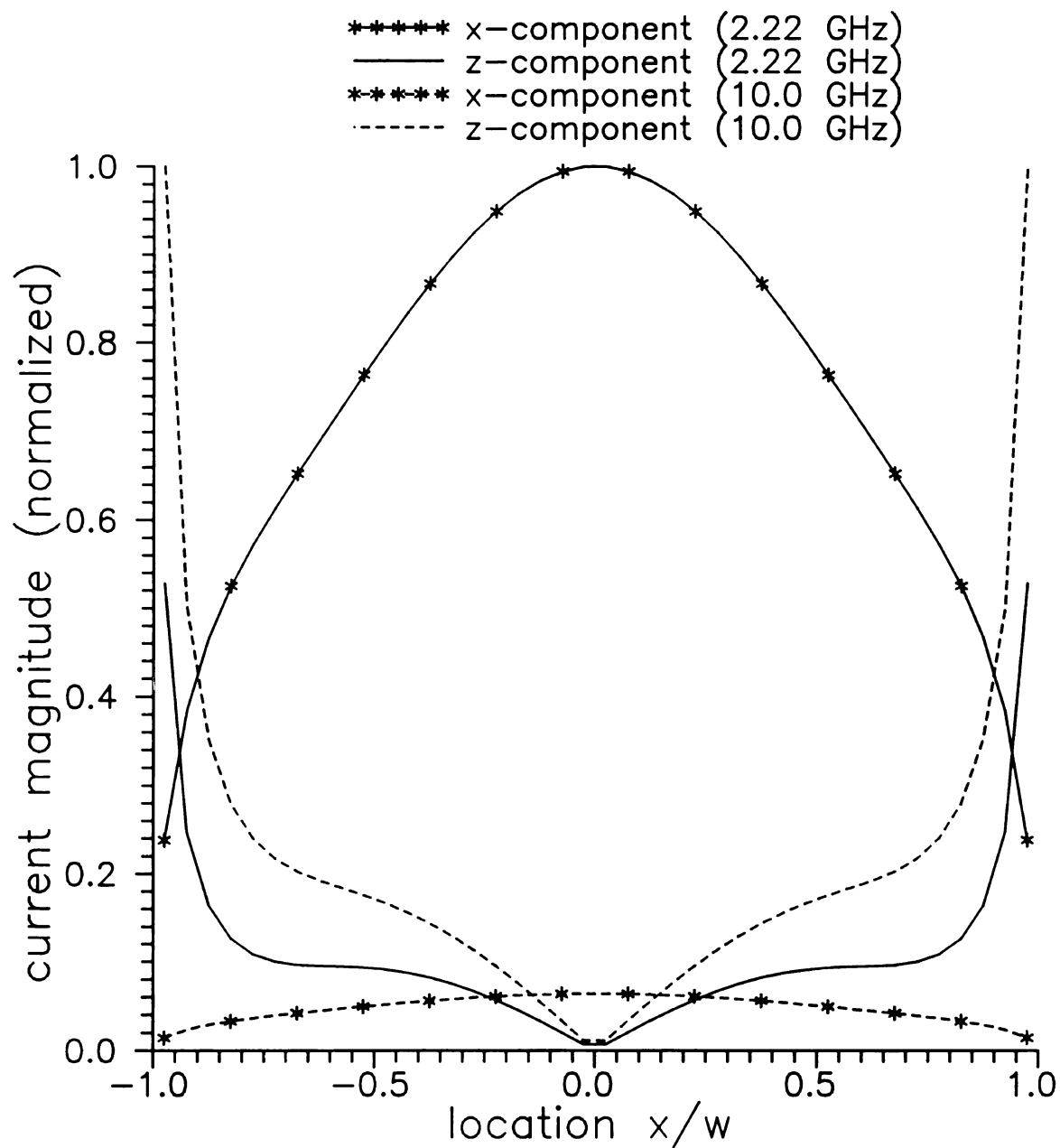


Figure 5.4. Cross-sectional current distribution for TE_1 mode, on center conductors of widely separated, coplanar strips (15cm center-to-center, 5cm width, 1.75cm ground plane half-spacing).

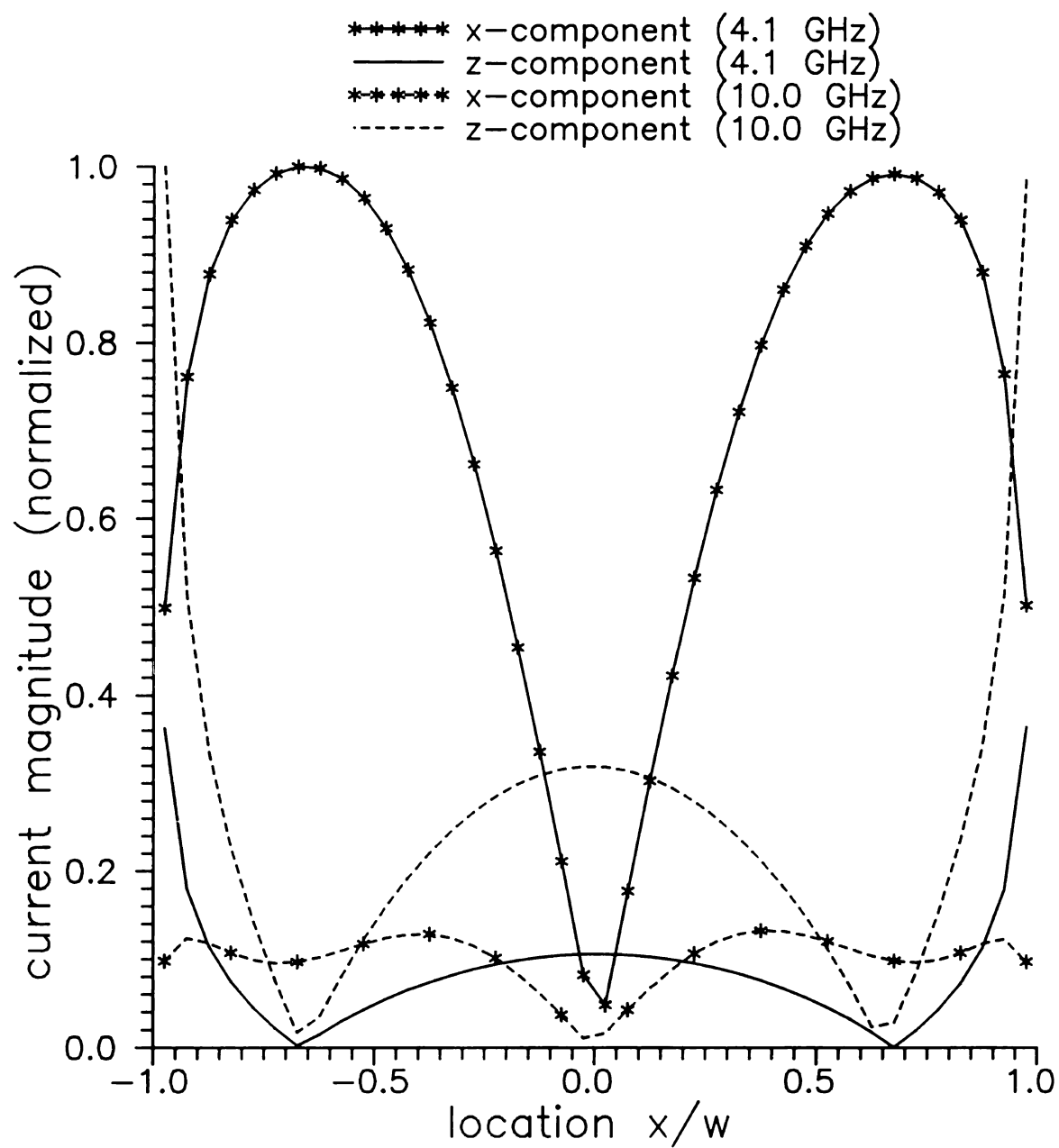


Figure 5.5. Cross-sectional current distribution for TE₂ mode, on center conductors of widely-separated, coplanar strips (15cm center-to-center, 5cm width, 1.75cm ground plane half-spacing).

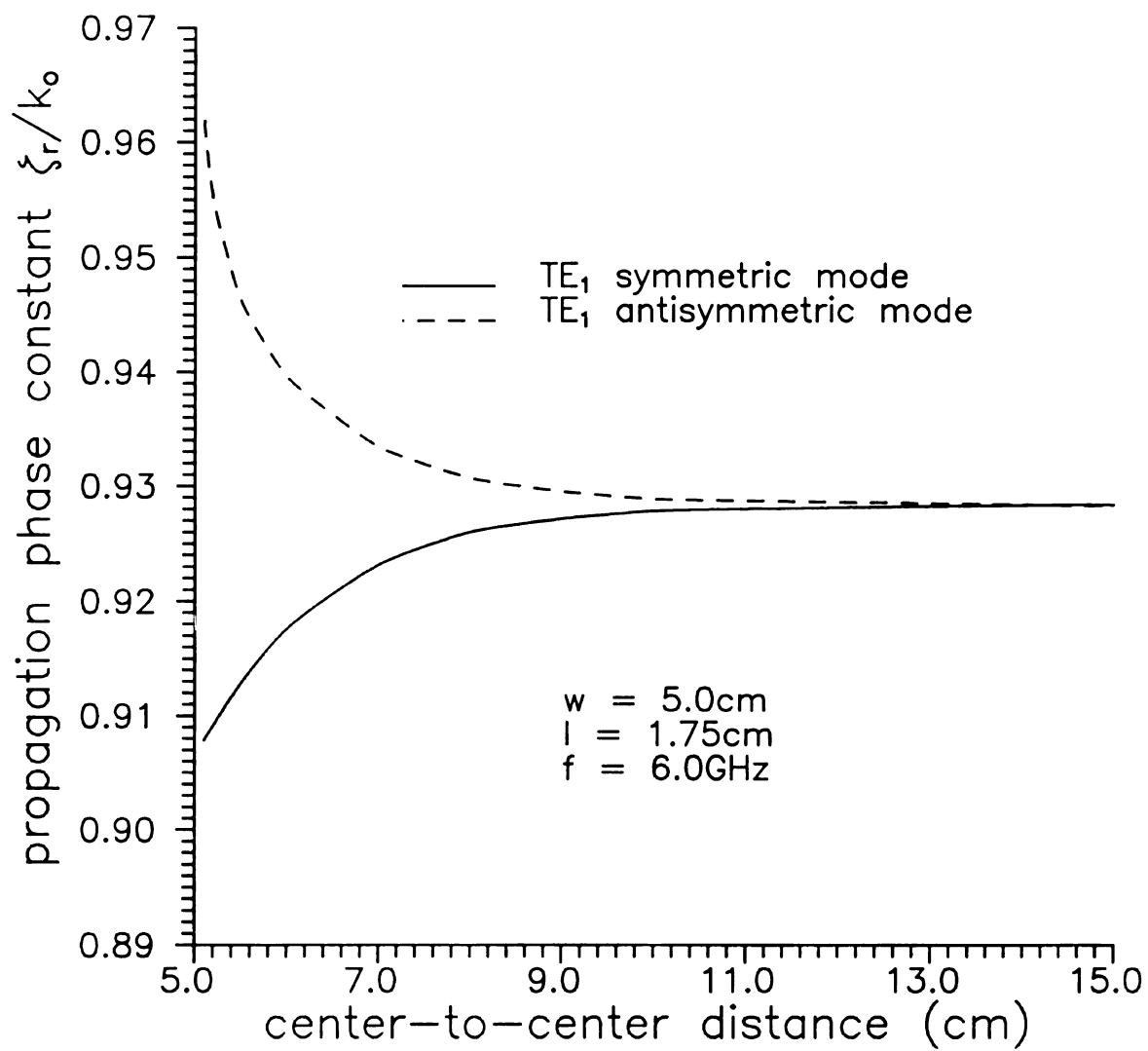


Figure 5.6. Dependence of propagation phase constant upon center-strip spacing for the TE₁ mode of coupled stripline.

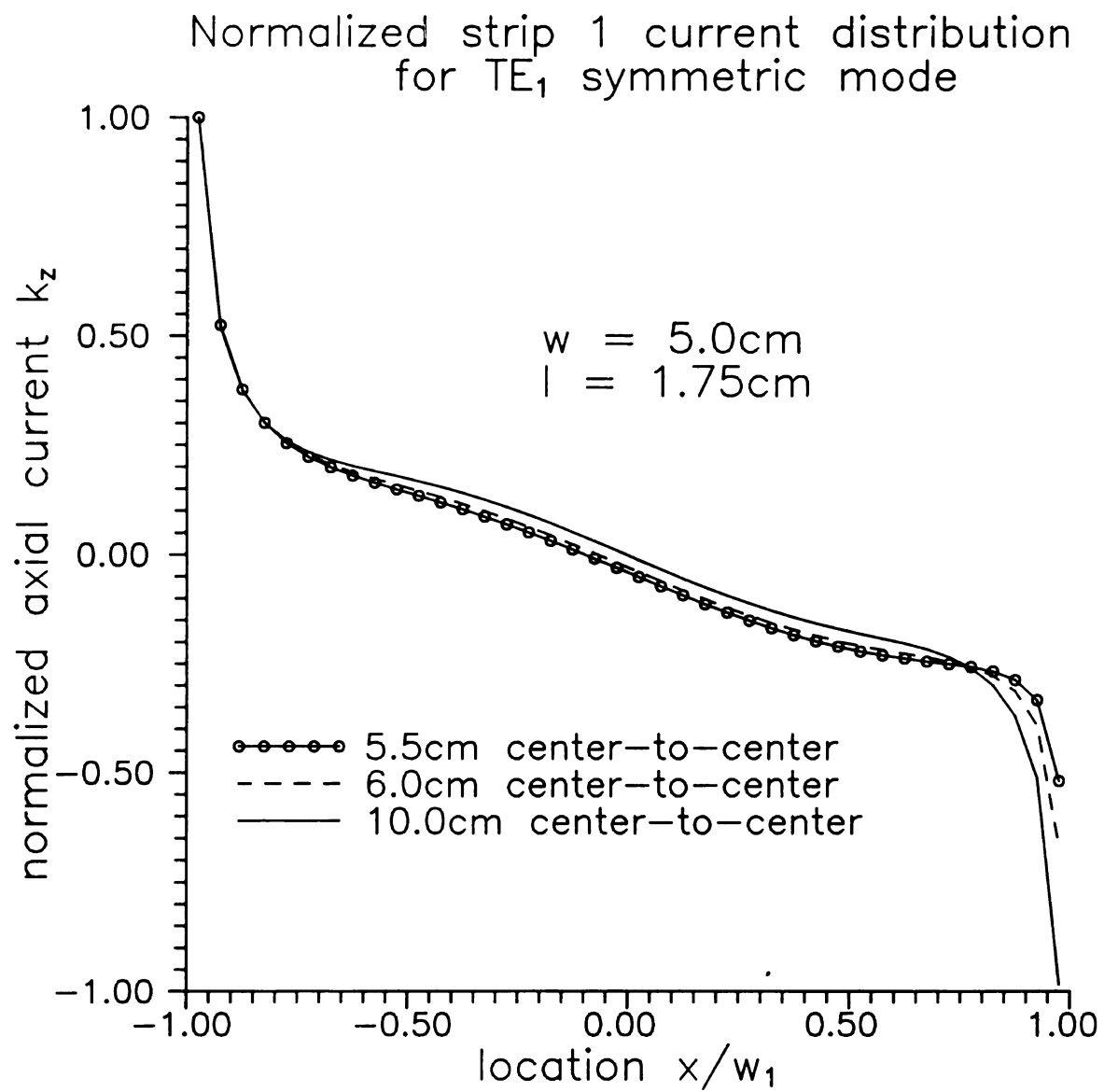


Figure 5.7. Strip 1 (left) normalized TE₁ symmetric current distribution.

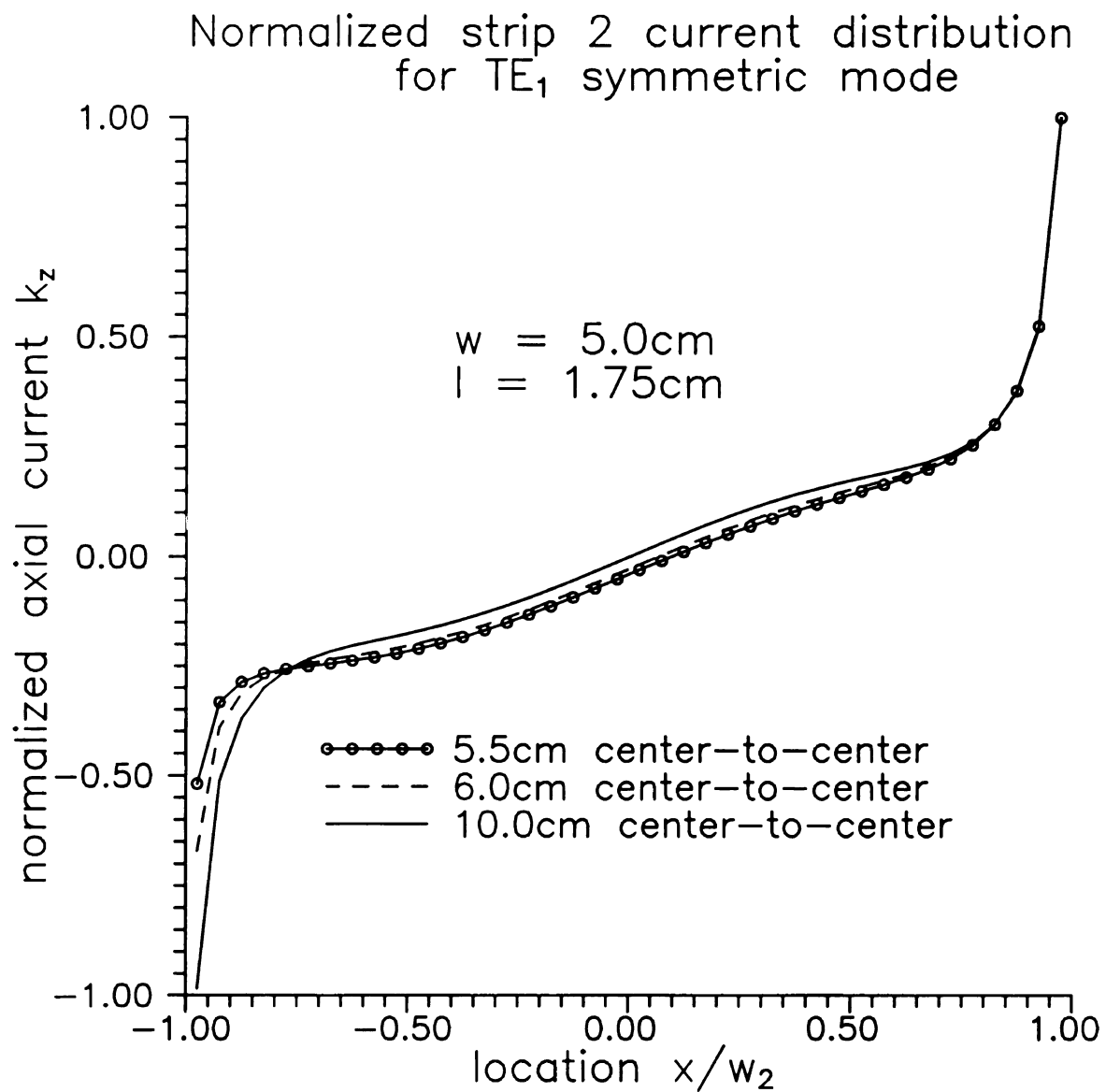


Figure 5.8. Strip 2 (right) normalized TE₁ symmetric current distribution.

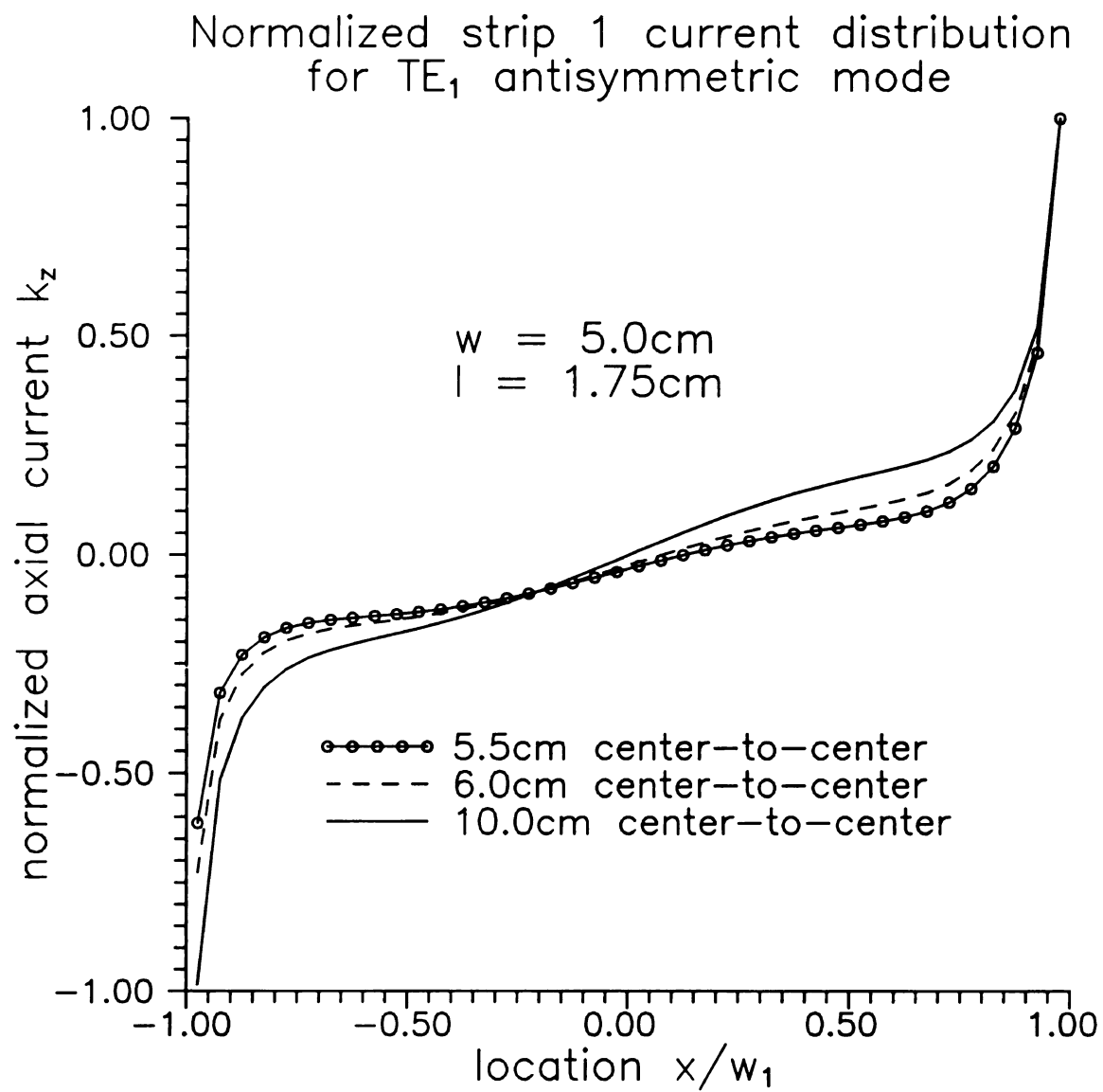


Figure 5.9 Strip 1 (left) normalized TE_1 anti-symmetric current distribution.

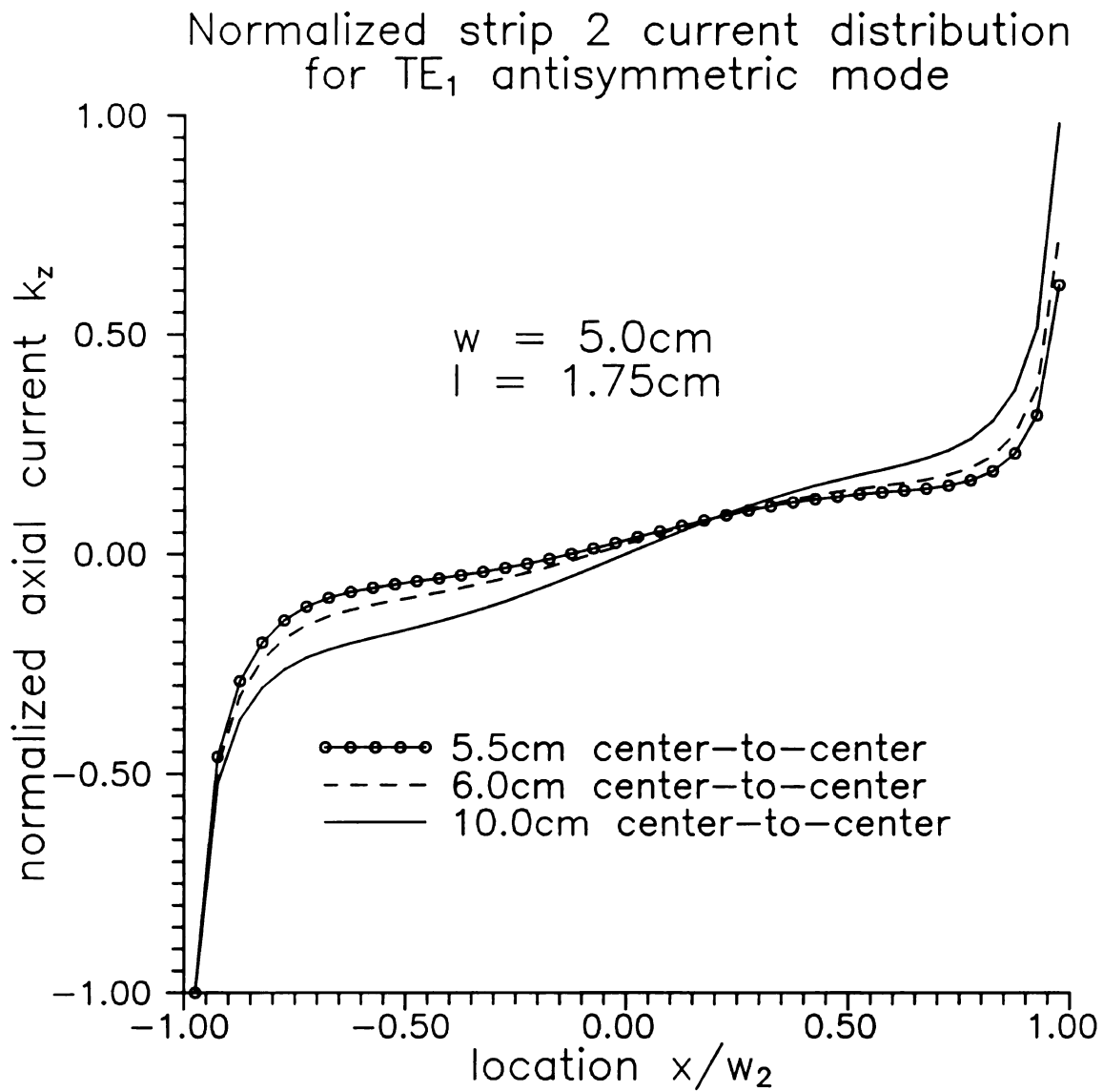


Figure 5.10 Strip 2 (right) normalized TE₁ anti-symmetric current distribution.

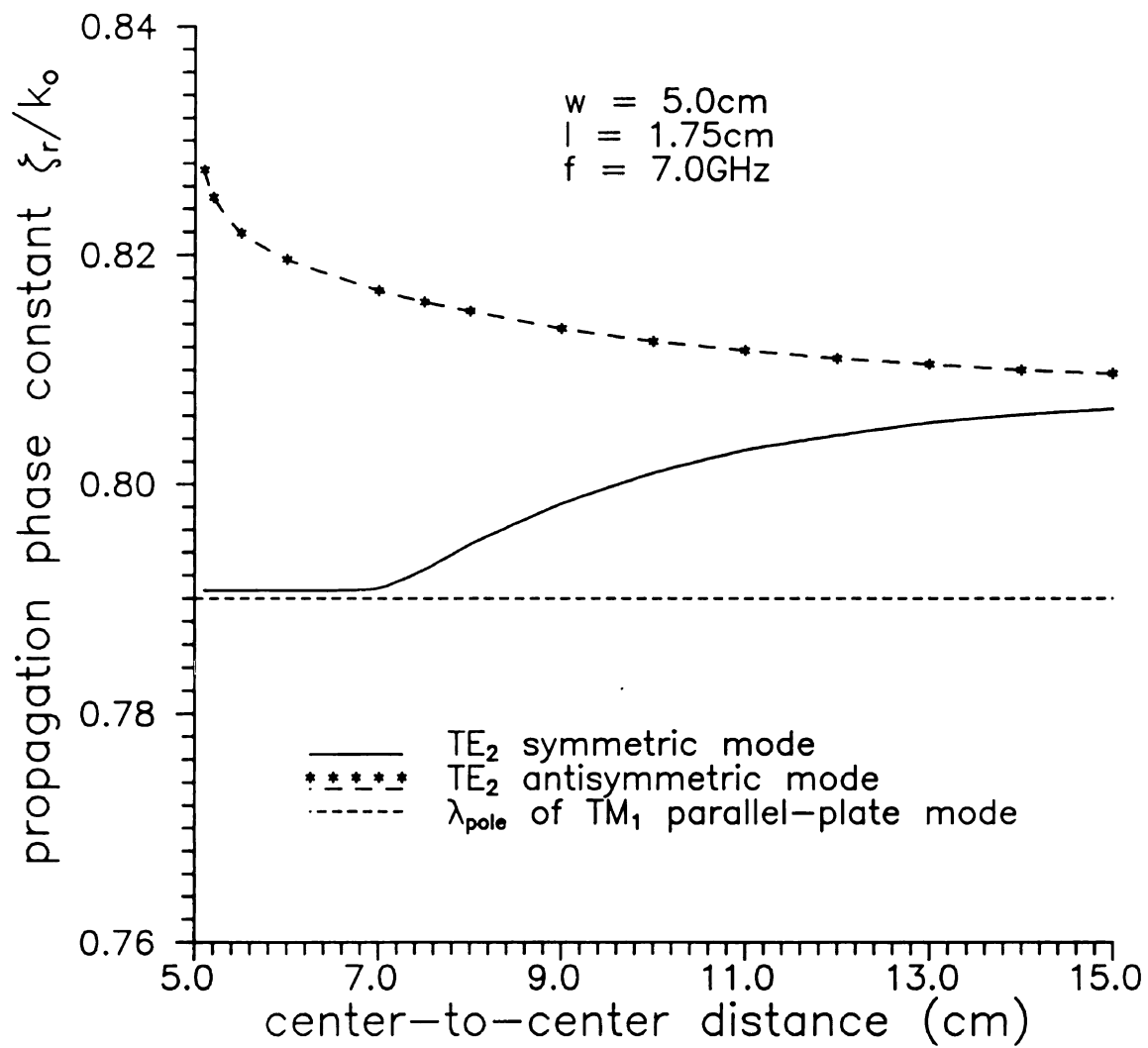


Figure 5.11 Dependence of propagation phase constant upon center-strip spacing for the TE_2 mode of coupled stripline.

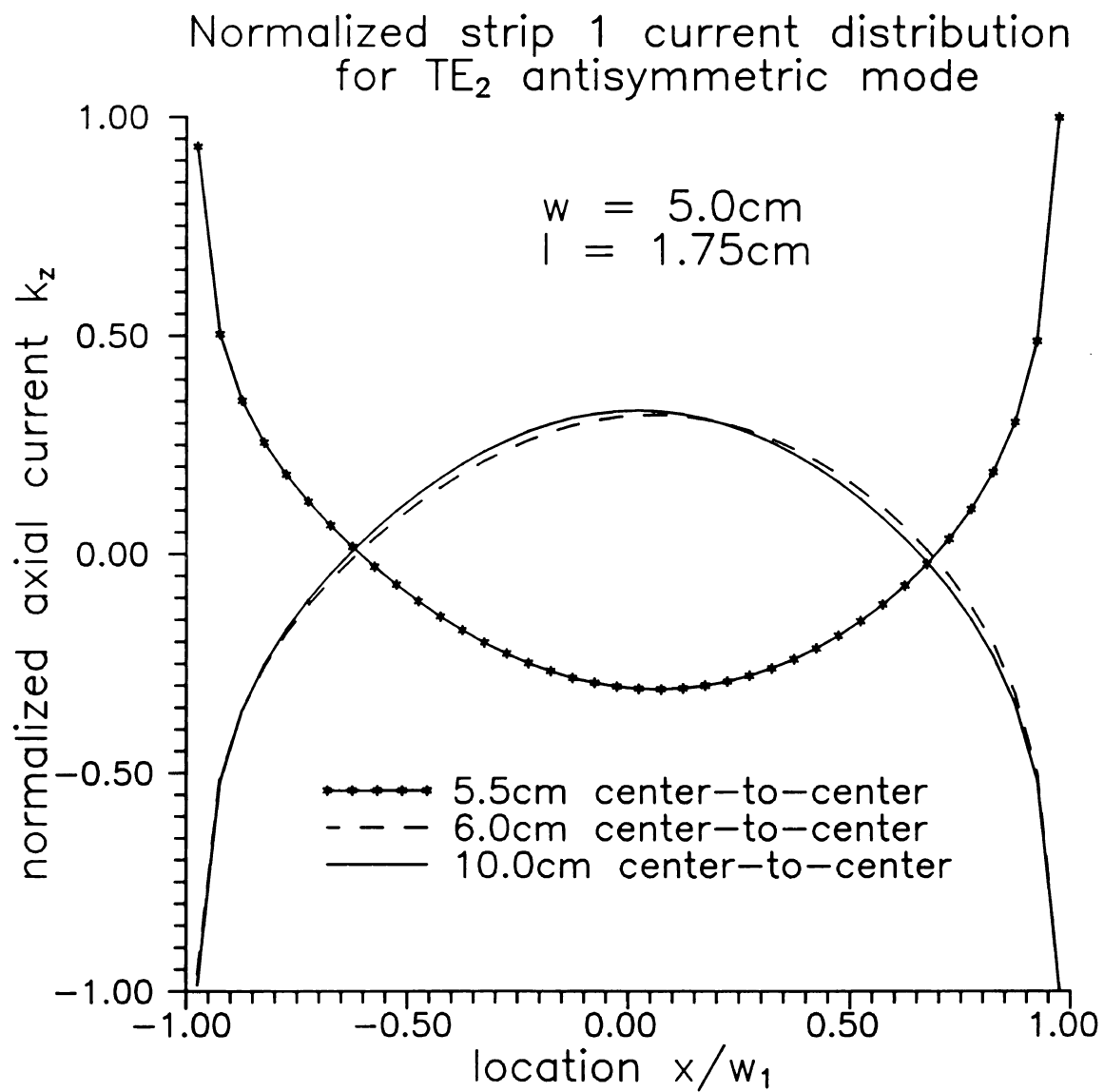


Figure 5.12 Strip 1 (left) normalized TE₂ anti-symmetric current distribution.

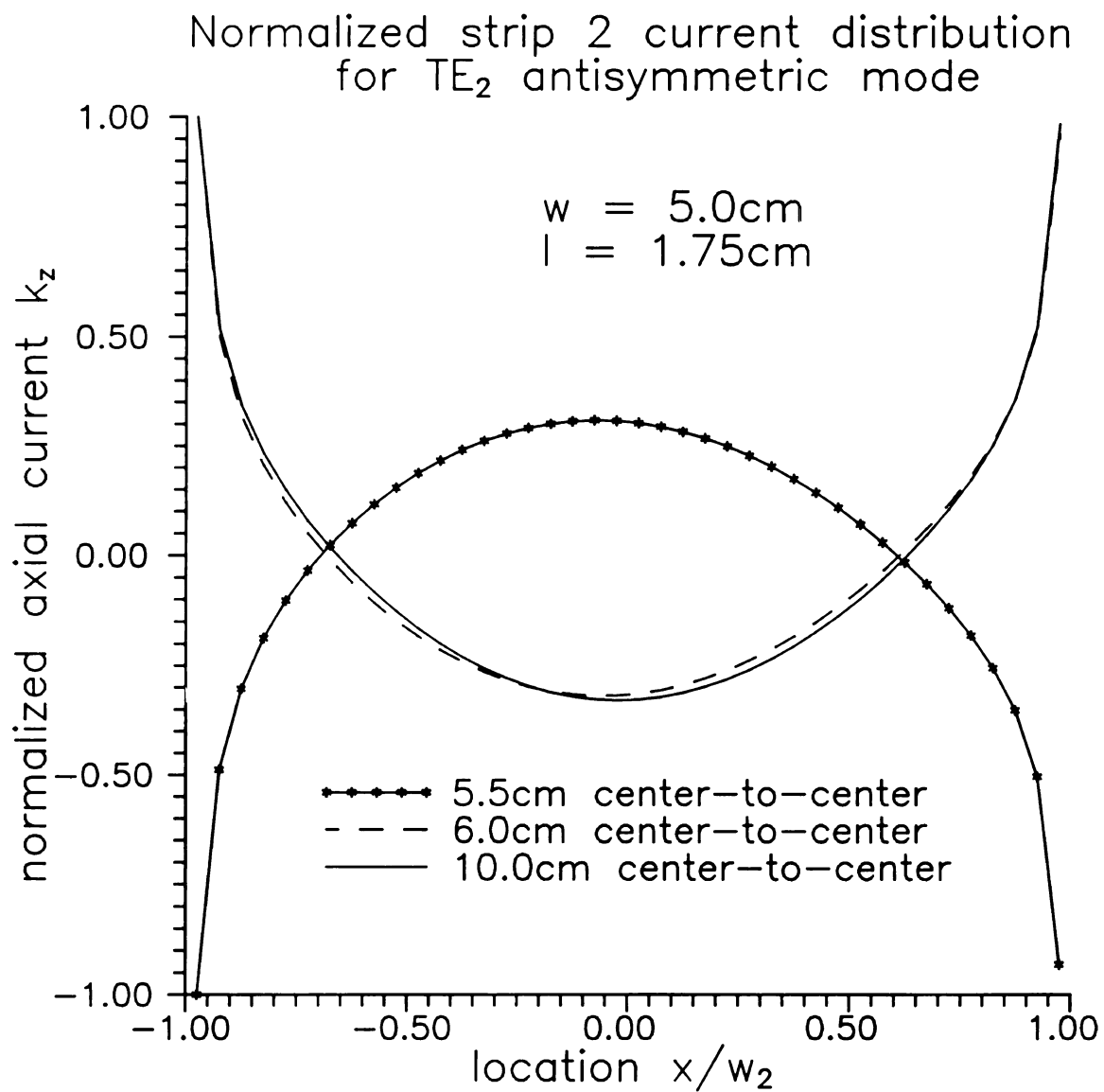


Figure 5.13 Strip 2 (right) normalized TE₂ anti-symmetric current distribution.

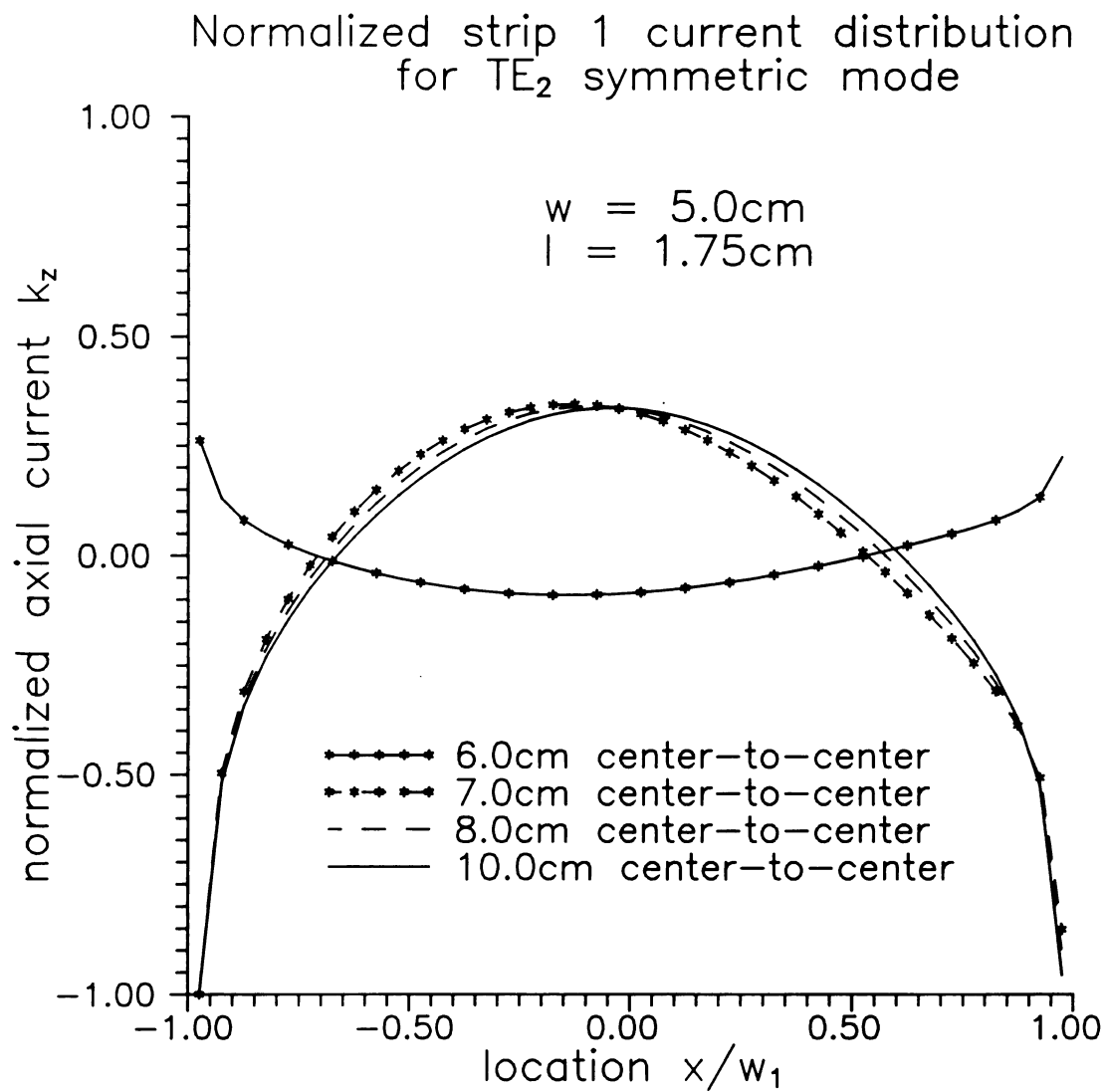


Figure 5.14 Strip 1 (left) normalized TE₂ symmetric current distribution.

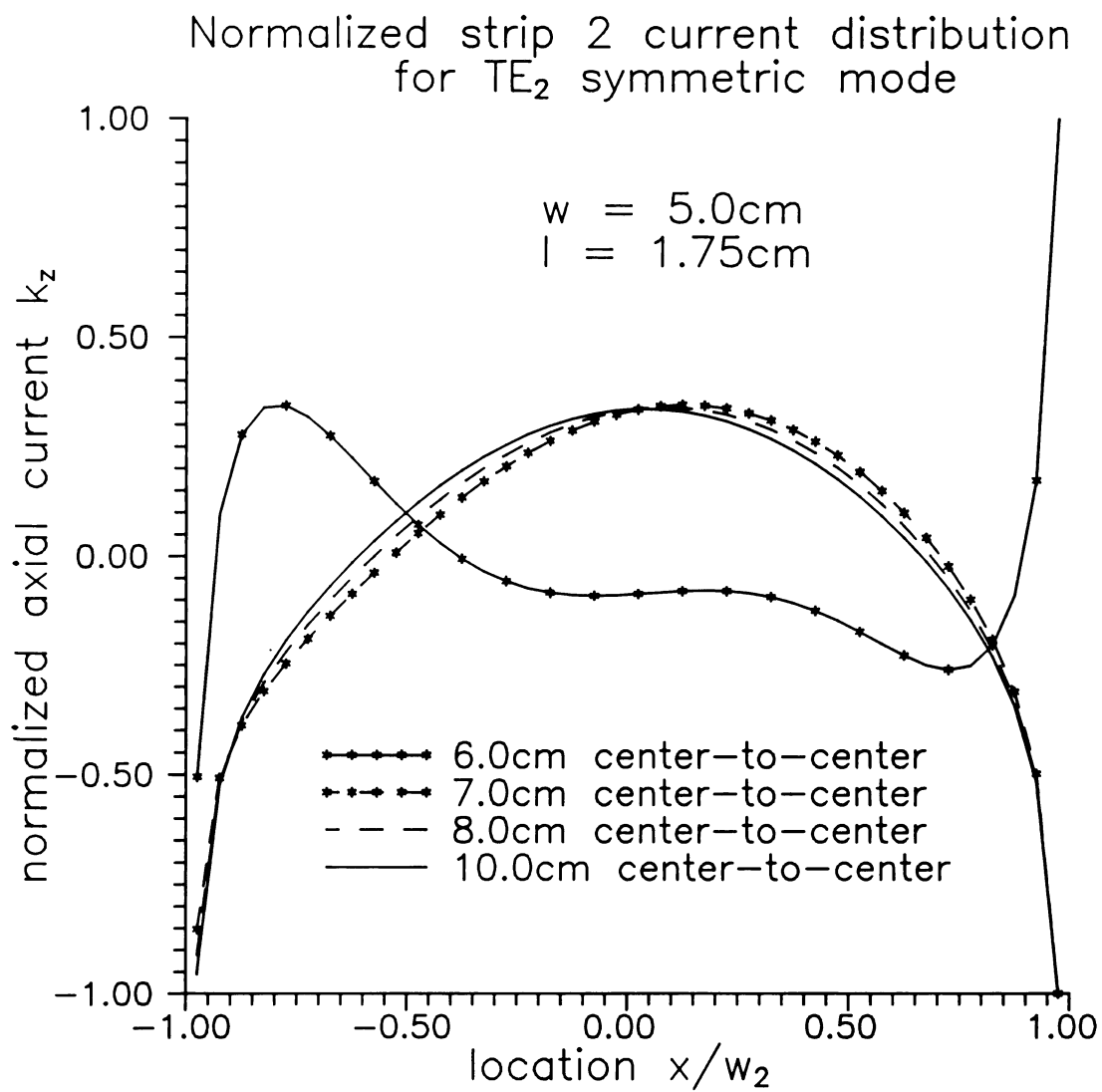


Figure 5.15 Strip 2 (right) normalized TE₂ symmetric current distribution.

$$\lim_{y \rightarrow h_1} \left\{ \int_{-a_1}^{a_1} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_1) k_{1x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, y | x', h_1) k_{1z}(x') \right] dx' \right\} \quad (165)$$

$$+ \int_{-a_2}^{a_2} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_1 | x', h_2) k_{2x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, h_1 | x', h_2) k_{2z}(x') \right] dx' = 0$$

and one pair representing the x- and z-directed electric field components associated with strip 2, given by

$$\int_{-a_1}^{a_1} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta}^t(x, h_2 | x', h_1) k_{1x}(x') + j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_2 | x', h_1) k_{1z}(x') \right] dx' \quad (166)$$

$$+ \lim_{y \rightarrow h_2} \left\{ \int_{-a_2}^{a_2} \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta}^t(x, y | x', h_2) k_{2x}(x') + j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_2) k_{2z}(x') \right] dx' \right\} = 0$$

$$\int_{-a_1}^{a_1} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, h_2 | x', h_1) k_{1x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, h_2 | x', h_1) k_{1z}(x') \right] dx' \quad (167)$$

$$+ \lim_{y \rightarrow h_2} \left\{ \int_{-a_2}^{a_2} \left[j\zeta \frac{\partial}{\partial x} g_{\zeta}^t(x, y | x', h_2) k_{2x}(x') + (k_o^2 - \zeta^2) g_{\zeta}^t(x, y | x', h_2) k_{2z}(x') \right] \right\} dx' = 0.$$

No transverse currents exist on the center strips when the TEM mode is excited, **therefore**

$$k_{nx}(x) \equiv 0 \quad (168)$$

for $n=1,2$, and also $\zeta = k_o$ in free space. As a result of the TEM assumption

$$(k_o^2 - \zeta^2) g_{\zeta}^b(x, y | x', y') k_{nz}(x) = 0 \quad (169)$$

for $n=1,2$, and $\beta=x,z$. As a result of (168) and (169), the equations representing the longitudinal (z-directed) component of electric field (equations (165) and (167)) for both strip 1 and strip 2 vanish, as expected. Equations (164) and (166), which represent the transverse (x-directed) component of electric field supported by currents on strips 1 and 2, then become

$$\lim_{y \rightarrow h_1} \left\{ \int_{-a_1}^{a_1} j k_o \frac{\partial}{\partial x} g_{k_o}^{\beta}(x, y | x', h_1) k_{1z}(x') dx' \right\} + \int_{-a_2}^{a_2} j k_o \frac{\partial}{\partial x} g_{k_o}^{\beta}(x, h_1 | x', h_2) k_{2z}(x') dx' = 0 \quad (170)$$

$$\int_{-a_1}^{a_1} j \zeta \frac{\partial}{\partial x} g_{k_o}^{\beta}(x, h_2 | x', h_1) k_{1z}(x') dx' + \lim_{y \rightarrow h_2} \left\{ \int_{-a_2}^{a_2} j k_o \frac{\partial}{\partial x} g_{k_o}^{\beta}(x, y | x', h_2) k_{2z}(x') dx' \right\} = 0. \quad (171)$$

The differential operator $\partial/\partial x$ acts on field points, while integration occurs over source points, therefore (170) and (171) may be expressed

$$j k_o \frac{\partial}{\partial x} \left\{ \lim_{y \rightarrow h_1} \int_{-a_1}^{a_1} g_{k_o}^{\beta}(x, y | x', h_1) k_{1z}(x') dx' + \int_{-a_2}^{a_2} g_{k_o}^{\beta}(x, h_1 | x', h_2) k_{2z}(x') dx' \right\} = 0 \quad (172)$$

$$j k_o \frac{\partial}{\partial x} \left\{ \int_{-a_1}^{a_1} g_{k_o}^{\beta}(x, h_2 | x', h_1) k_{1z}(x') dx' + \lim_{y \rightarrow h_2} \int_{-a_2}^{a_2} g_{k_o}^{\beta}(x, y | x', h_2) k_{2z}(x') dx' \right\} = 0. \quad (173)$$

Integrating both sides of (172)-(173) over x gives

$$\lim_{y \rightarrow h_1} \int_{-a_1}^{a_1} g_{k_o}^{\beta}(x, y | x', h_1) k_{1z}(x') dx' + \int_{-a_2}^{a_2} g_{k_o}^{\beta}(x, h_1 | x', h_2) k_{2z}(x') dx' = C_1 \quad (174)$$

$$\int_{-a_1}^{a_1} g_{k_o}^{\beta}(x, h_2 | x', h_1) k_{1z}(x') dx' + \lim_{y \rightarrow h_2} \int_{-a_2}^{a_2} g_{k_o}^{\beta}(x, y | x', h_2) k_{2z}(x') dx' = C_2 \quad (175)$$

where C_1 and C_2 are constants which are determined by the potential difference existing between the conducting strips and one of the ground planes.

The Green's function implicated in (174) and (175) is given by

$$g_{k_o}^{\beta}(x, y | x', y') = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi} W(\xi, y, y') d\xi. \quad (176)$$

As a result of the TEM approximation, the transform-domain wavenumber becomes

$$p_0 = \sqrt{\xi^2 + k_o^2 - k_o^2} = \sqrt{\xi^2} = |\xi| \quad (177)$$

therefore

$$W(\xi, y, y') = \frac{1}{|\xi|} \left\{ e^{-|\xi||y-y'|} + \frac{1}{(1 - e^{-4|\xi|l})} \left[-e^{-|\xi|(2l+y+y')} \right. \right. \\ \left. \left. - e^{-|\xi|(2l-y-y')} + e^{-|\xi|(4l+y-y')} + e^{-|\xi|(4l-y+y')} \right] \right\} \quad (178)$$

Or

$$W(\xi, y, y') = \frac{1}{|\xi|(1 - e^{-4|\xi|l})} \left\{ (1 - e^{-4|\xi|l}) e^{-|\xi||y-y'|} - e^{-|\xi|(2l+y+y')} \right. \\ \left. - e^{-|\xi|(2l-y-y')} + e^{-|\xi|(4l+y-y')} + e^{-|\xi|(4l-y+y')} \right\}. \quad (179)$$

It is seen that (179) may be expressed

$$W(\xi, y, y') = \frac{N(\xi, y, y')}{|\xi| D(\xi)} \quad (180)$$

where

$$N(\xi, y, y') = \left\{ \left(1 - e^{-4|\xi|l} \right) e^{-|\xi||y-y'|} - e^{-|\xi|(2l+y+y')} \right. \\ \left. - e^{-|\xi|(2l-y-y')} + e^{-|\xi|(4l+y-y')} + e^{-|\xi|(4l-y+y')} \right\} \quad (181)$$

and

$$D(\xi) = 1 - e^{-4|\xi|l}. \quad (182)$$

In order to determine if the integrand of (176) is singular when $\xi = 0$, the behavior of (180) is examined for small values of $|\xi|$. It is first noted that

$$\lim_{\xi \rightarrow 0} N(\xi) = \left\{ (1 - 1) \cdot 1 - 1 - 1 + 1 + 1 \right\} = 0. \quad (183)$$

For $\xi \ll 1$ the approximation

$$e^x \approx 1 + x + \frac{x^2}{2} \quad (184)$$

can be made, therefore

$$N(\xi) \approx \left\{ \left[1 - \left(1 - a + \frac{(-a)^2}{2} \right) \right] \left[1 - b + \frac{(-b)^2}{2} \right] - \left[1 - c + \frac{(-c)^2}{2} \right] \right. \\ \left. - \left[1 - d + \frac{(-d)^2}{2} \right] + \left[1 - e + \frac{(-e)^2}{2} \right] + \left[1 - f + \frac{(-f)^2}{2} \right] \right\} \quad (185)$$

where

$$a = 4|\xi|l, \quad b = |\xi||y-y'|, \quad c = |\xi|(2l+y+y') \quad (186)$$

and

$$d = |\xi|(2l-y-y'), \quad e = |\xi|(4l+y-y'), \quad f = |\xi|(4l-y+y'). \quad (187)$$

After a great deal of algebra, it is determined that

$$N(\xi) \approx -4|\xi|^2 l |y-y'| + 4|\xi|^2 l^2 - 4|\xi|^2 yy'. \quad (188)$$

It is also seen that

$$D(\xi) \approx 1 - \left(1 - a + \frac{(-a)^2}{2}\right) = 4|\xi|l - 8|\xi|^2 l^2 \quad (189)$$

for $\xi \ll 1$, therefore

$$\lim_{\xi \rightarrow 0} \frac{N(\xi)}{|\xi| D(\xi)} = \frac{-4|\xi|^2 l |y-y'|}{4|\xi|^2 l} = l - |y-y'| - \frac{yy'}{l}. \quad (190)$$

Thus the integrand of (176) is not singular for $\xi = 0$.

5.8 Moment-Method Solution for Coupled Stripline TEM Specialization

Once again, in order to facilitate a numerical solution, a coordinate shift is performed. The coordinate variables of the scalar component EFIEs are shifted to a system of local coordinates using the same technique applied in section 5.3 (refer again to Figure 5.2). This is done by letting

$$\bar{x}'_\alpha = x' - x_\alpha \quad (191)$$

and then $dx' = d\bar{x}'_\alpha$. It is seen that when $x' = x_\alpha - w_\alpha/2$, $\bar{x}'_\alpha = -w_\alpha/2$, and when

$x' = x_\alpha + w_\alpha/2$, $\bar{x}'_\alpha = w_\alpha/2$. The coordinate shift

$$\bar{x}_\alpha = x - x_\alpha \quad (192)$$

is also performed, and then $dx = d\bar{x}_\alpha$. In this case, when $x = x_\alpha - w_\alpha/2$, $\bar{x}_\alpha = -w_\alpha/2$, and when $x = x_\alpha + w_\alpha/2$, $\bar{x}_\alpha = w_\alpha/2$.

Applying this change of variables to the scalar component EFIEs (174) and (175) gives

$$\begin{aligned} \lim_{y \rightarrow h_1} \int_{a_1}^{b_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \right\} k_{1z}(\bar{x}'_1) d\bar{x}'_1 \\ + \int_{a_2}^{b_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} d\xi \right\} k_{2z}(\bar{x}'_2) d\bar{x}'_2 = C_1 \end{aligned} \quad (193)$$

$$\begin{aligned} \int_{a_1}^{b_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} d\xi \right\} k_{1z}(\bar{x}'_1) d\bar{x}'_1 \\ + \lim_{y \rightarrow h_2} \int_{a_2}^{b_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \right\} k_{2z}(\bar{x}'_2) d\bar{x}'_2 = C_2. \end{aligned} \quad (194)$$

The current functions in (193) and (194) are now expanded such that

$$k_{\alpha z}(\bar{x}) = \sum_{n=1}^N a_{\alpha n} p_{\alpha n}(\bar{x}) \quad (195)$$

using square pulse basis functions

$$p_{\alpha n}(\bar{x}) = \begin{cases} 1 & \dots \text{ for } \bar{x}_{\alpha n} - \tilde{x}_\alpha \leq \bar{x} \leq \bar{x}_{\alpha n} + \tilde{x}_\alpha \\ 0 & \dots \text{ elsewhere} \end{cases} \quad (196)$$

where

$$\overline{\Delta x}_\alpha = 2\bar{x}_\alpha = \frac{2a_\alpha}{N}, \quad \bar{x}_{\alpha n} = -a_\alpha + \left(n - \frac{1}{2}\right)\overline{\Delta x}_\alpha. \quad (197)$$

Here n is a positive integer ($n=1,2,3,\dots$), and $\alpha=1,2$. Application of this expansion gives

$$\begin{aligned} \lim_{y \rightarrow h_1} \int_{a_1}^{b_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \right\} \sum_{n=1}^N a_{1n} p_{1z}(\bar{x}'_1) d\bar{x}'_1 \\ + \int_{a_2}^{b_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} d\xi \right\} \sum_{n=1}^N a_{2n} p_{2z}(\bar{x}'_2) d\bar{x}'_2 = C_1 \end{aligned} \quad (198)$$

$$\begin{aligned} \int_{a_1}^{b_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} d\xi \right\} \sum_{n=1}^N a_{1n} p_{1n}(\bar{x}'_1) d\bar{x}'_1 \\ + \lim_{y \rightarrow h_2} \int_{a_2}^{b_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \right\} \sum_{n=1}^N a_{2n} p_{2n}(\bar{x}'_2) d\bar{x}'_2 = C_2. \end{aligned} \quad (199)$$

Interchanging the order of summation and integration leads to

$$\begin{aligned} \sum_{n=1}^N a_{1n} \int_{a_1}^{b_1} \lim_{y \rightarrow h_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \right\} p_{1z}(\bar{x}'_1) d\bar{x}'_1 \\ + \sum_{n=1}^N a_{2n} \int_{a_2}^{b_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_1 - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} d\xi \right\} p_{2z}(\bar{x}'_2) d\bar{x}'_2 = C_1 \end{aligned} \quad (200)$$

$$\begin{aligned}
& \sum_{n=1}^N a_{1n} \int_{a_1}^{b_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} d\xi \right\} p_{1z}(\bar{x}'_1) d\bar{x}'_1 \\
& + \sum_{n=1}^N a_{2n} \int_{a_2}^{b_2} \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_2 - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \right\} p_{2z}(\bar{x}'_2) d\bar{x}'_2 = C_2.
\end{aligned} \tag{201}$$

Now letting

$$A_{11}^{mn} = \int_{\bar{x}_{1n} - \bar{x}_1}^{\bar{x}_{1n} + \bar{x}_1} \lim_{y \rightarrow h_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{1m} - \bar{x}'_1)}}{4\pi} W(\xi, y, h_1) d\xi \right\} d\bar{x}'_1 \tag{202}$$

$$A_{12}^{mn} = \int_{\bar{x}_{2n} - \bar{x}_2}^{\bar{x}_{2n} + \bar{x}_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{1m} - \bar{x}'_2)}}{4\pi} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} d\xi \right\} d\bar{x}'_2 \tag{203}$$

$$A_{21}^{mn} = \int_{\bar{x}_{1n} - \bar{x}_1}^{\bar{x}_{1n} + \bar{x}_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{2m} - \bar{x}'_1)}}{4\pi} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} d\xi \right\} d\bar{x}'_1 \tag{204}$$

$$A_{22}^{mn} = \int_{\bar{x}_{2n} - \bar{x}_2}^{\bar{x}_{2n} + \bar{x}_2} \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{2m} - \bar{x}'_2)}}{4\pi} W(\xi, y, h_2) d\xi \right\} d\bar{x}'_2 \tag{205}$$

equations (200) and (201) can be expressed

$$\sum_{n=1}^N A_{11}^{mn} a_{1n} + \sum_{n=1}^N A_{12}^{mn} a_{2n} = C_1 \tag{206}$$

$$\sum_{n=1}^N A_{21}^{mn} a_{1n} + \sum_{n=1}^N A_{22}^{mn} a_{2n} = C_2. \tag{207}$$

It is noted that in contrast to the complete propagation mode MoM solution performed

previously, the matrix elements in (206) and (207) are determined from a forced solution,

where $\beta = k_o$ is known. The matrix elements (202)-(205) may be expressed

$$A_{11}^{mn} = \lim_{y \rightarrow h_1} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi \bar{x}_{1m}}}{4\pi} W(\xi, y, h_1) \int_{\bar{x}_{1n} - \bar{x}_1}^{\bar{x}_{1n} + \bar{x}_1} e^{-j\xi \bar{x}'_1} d\bar{x}'_1 d\xi \right\} \quad (208)$$

$$A_{12}^{mn} = \int_{-\infty}^{\infty} \frac{e^{j\xi \bar{x}_{1m}}}{4\pi} W(\xi, h_1, h_2) e^{j\xi(x_1 - x_2)} \int_{\bar{x}_{2n} - \bar{x}_2}^{\bar{x}_{2n} + \bar{x}_2} e^{-j\xi \bar{x}'_2} d\bar{x}'_2 d\xi \quad (209)$$

$$A_{21}^{mn} = \int_{-\infty}^{\infty} \frac{e^{j\xi \bar{x}_{2m}}}{4\pi} W(\xi, h_2, h_1) e^{j\xi(x_2 - x_1)} \int_{\bar{x}_{1n} - \bar{x}_1}^{\bar{x}_{1n} + \bar{x}_1} e^{-j\xi \bar{x}'_1} d\bar{x}'_1 d\xi \quad (210)$$

$$A_{22}^{mn} = \lim_{y \rightarrow h_2} \left\{ \int_{-\infty}^{\infty} \frac{e^{j\xi \bar{x}_{2m}}}{4\pi} W(\xi, y, h_2) \int_{\bar{x}_{2n} - \bar{x}_2}^{\bar{x}_{2n} + \bar{x}_2} e^{-j\xi \bar{x}'_2} d\bar{x}'_2 d\xi \right\} \quad (211)$$

and evaluation of the inner integral leads to

$$\begin{aligned} \int_{\bar{x}_{an} - \bar{x}_a}^{\bar{x}_{an} + \bar{x}_a} e^{-j\xi \bar{x}'_a} d\bar{x}'_a &= \frac{e^{-j\xi \bar{x}'_a}}{-j\xi} \Big|_{\bar{x}_{an} - \bar{x}_a}^{\bar{x}_{an} + \bar{x}_a} = \frac{e^{j\xi(\bar{x}_{an} + \bar{x}_a)} - e^{-j\xi(\bar{x}_{an} - \bar{x}_a)}}{-j\xi} \\ &= \frac{e^{-j\xi \bar{x}_{an}} e^{-j\xi \bar{x}_a} - e^{-j\xi \bar{x}_{an}} e^{j\xi \bar{x}_a}}{-j\xi} \\ &= \frac{e^{-j\xi \bar{x}_{an}} (e^{j\xi \bar{x}_a} - e^{-j\xi \bar{x}_a})}{j\xi} \\ &= \frac{2 \sin \xi \bar{x}_a}{\xi} e^{-j\xi \bar{x}_{an}}. \end{aligned} \quad (212)$$

As a result

$$A_{11}^{mn} = \lim_{y \rightarrow h_1} \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{1m} - x_{1n})}}{2\pi} W(\xi, y, h_1) \frac{\sin \xi \bar{x}_1}{\xi} d\xi \quad (213)$$

$$A_{12}^{mn} = \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{1m} + x_1 - x_{2n} - x_2)}}{2\pi} W(\xi, h_1, h_2) \frac{\sin \xi \bar{x}_2}{\xi} d\xi \quad (214)$$

$$A_{21}^{mn} = \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{2m} + x_2 - x_{1n} - x_1)}}{2\pi} W(\xi, h_2, h_1) \frac{\sin \xi \bar{x}_1}{\xi} d\xi \quad (215)$$

$$A_{22}^{mn} = \lim_{y \rightarrow h_2} \int_{-\infty}^{\infty} \frac{e^{j\xi(\bar{x}_{2m} - x_{2n})}}{2\pi} W(\xi, y, h_2) \frac{\sin \xi \bar{x}_2}{\xi} d\xi. \quad (216)$$

Now the constants C_1 and C_2 in equations (206) and (207) must be determined. These constants are proportional to the potential differences that exist between the conducting strips and the ground planes. In Chapter 4, the normal (y-directed) component of electric field in the region between the ground planes was found to be

$$E_y(\bar{r}) = \frac{\partial^2}{\partial y \partial z} \int_V G_0'(\bar{r}|\bar{r}') \cdot \frac{J_z(\bar{r}')}{j\omega\epsilon_o} dv'. \quad (217)$$

Because the center strips are assumed to be infinitely thin

$$J_z(\bar{r}') = K_z(x', z') \quad (218)$$

and

$$\tilde{G}_0'(\bar{r}|\bar{r}') = \tilde{G}_0'(\bar{\rho}|\bar{\rho}'; z - z') \quad (219)$$

where $\bar{\rho} = \hat{x}x + \hat{z}z$. Now let the cross-sectional surfaces of strips 1 and 2 be represented

by S_1 and S_2 , respectively, then

$$\begin{aligned} \int_V G_0'(\vec{r}|\vec{r}') \cdot \frac{K_z(\vec{r}')}{j\omega\epsilon_o} dv' &= \int_{S_1} G_0'(\vec{r}|\vec{r}') \cdot \frac{K_{1z}(\vec{r}')}{j\omega\epsilon_o} ds_1' \\ &+ \int_{S_2} G_0'(\vec{r}|\vec{r}') \cdot \frac{K_{2z}(\vec{r}')}{j\omega\epsilon_o} ds_2'. \end{aligned} \quad (220)$$

The normal component of electric field may therefore be expressed

$$\begin{aligned} E(\vec{r}) &= \frac{\partial^2}{\partial y \partial z} \left\{ \int_{a_1}^{b_1} \int_{-\infty}^{\infty} G'(\vec{r}|\vec{r}') \cdot \frac{K_{1z}(x', z')}{j\omega\epsilon_o} dz' dx' \right. \\ &\quad \left. + \int_{a_2}^{b_2} \int_{-\infty}^{\infty} G'(\vec{r}|\vec{r}') \cdot \frac{K_{2z}(x', z')}{j\omega\epsilon_o} dz' dx' \right\} \end{aligned} \quad (221)$$

and, after application of the convolution theorem, the transform-domain electric field is given by

$$e_y(\vec{\rho}, \zeta) = j\zeta \frac{\partial}{\partial y} \left\{ \int_{a_1}^{b_1} \tilde{g}_\zeta'(\vec{\rho}|\vec{\rho}') \cdot \frac{k_{1z}(x', \zeta)}{j\omega\epsilon_o} dx' + \int_{a_2}^{b_2} \tilde{g}_\zeta'(\vec{\rho}|\vec{\rho}') \cdot \frac{k_{2z}(x', \zeta)}{j\omega\epsilon_o} dx' \right\}. \quad (222)$$

For the TEM mode $\zeta = k_o$, therefore

$$e_y(\vec{\rho}, \zeta) = \frac{k_o}{\omega\epsilon_o} \frac{\partial}{\partial y} \left\{ \int_{a_1}^{b_1} \tilde{g}_{k_o}'(\vec{\rho}|\vec{\rho}') k_{1z}(x', \zeta) dx' + \int_{a_2}^{b_2} \tilde{g}_{k_o}'(\vec{\rho}|\vec{\rho}') k_{2z}(x', \zeta) dx' \right\}. \quad (223)$$

The transform-domain Green's function implicated in (223) is given by

$$g_{k_o}'(\vec{\rho}|\vec{\rho}') = g_{k_o}^p(\vec{\rho}|\vec{\rho}') + g_{k_o}^{st}(\vec{\rho}|\vec{\rho}') \quad (224)$$

where

$$g_{k_o}^p(\vec{\rho}|\vec{\rho}') = \int_{-\infty}^{\infty} \frac{e^{-j\xi(x-x')} e^{-|\xi||y-y'|}}{4\pi|\xi|} d\xi \quad (225)$$

and

$$g_{k_o}^{st}(\vec{\rho}|\vec{\rho}') = \int_{-\infty}^{\infty} \frac{e^{j|\xi|(x-x')}}{4\pi|\xi|(1 - e^{-4|\xi|l})} \left[-e^{-|\xi|(2l+y+y')} - e^{-|\xi|(2l-y-y')} \right. \\ \left. + e^{-|\xi|(4l+y-y')} + e^{-|\xi|(4l-y+y')} \right] d\xi. \quad (226)$$

With the normal component of electric field determined, the potential difference between the center strips and the ground planes can be found. Let V_l be the potential between a point x_l in the center of strip 1, and a point directly below on the nearest ground plane (Figure 5.16). Then

$$V_1 = - \int_{-l}^{h_1} e_y(\vec{\rho}, \zeta) \cdot d\vec{l} \Big|_{x=x_1} = - \int_{-l}^{h_1} \frac{k_o}{\omega \epsilon_o} \frac{\partial}{\partial y} \left\{ \int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_1, y|x', h_1) k_{1z}(x') dx' \right. \\ \left. + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_1, y|x', h_2) k_{2z}(x') dx' \right\} dy. \quad (227)$$

Application of the partial derivative to the outer integral gives

$$V_1 = - \frac{k_o}{\omega \epsilon_o} \left\{ \int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_1, y|x', h_1) k_{1z}(x') dx' \Big|_{-l}^{h_1} \right. \\ \left. + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_1, y|x', h_2) k_{2z}(x') dx' \Big|_{-l}^{h_1} \right\} \quad (228)$$

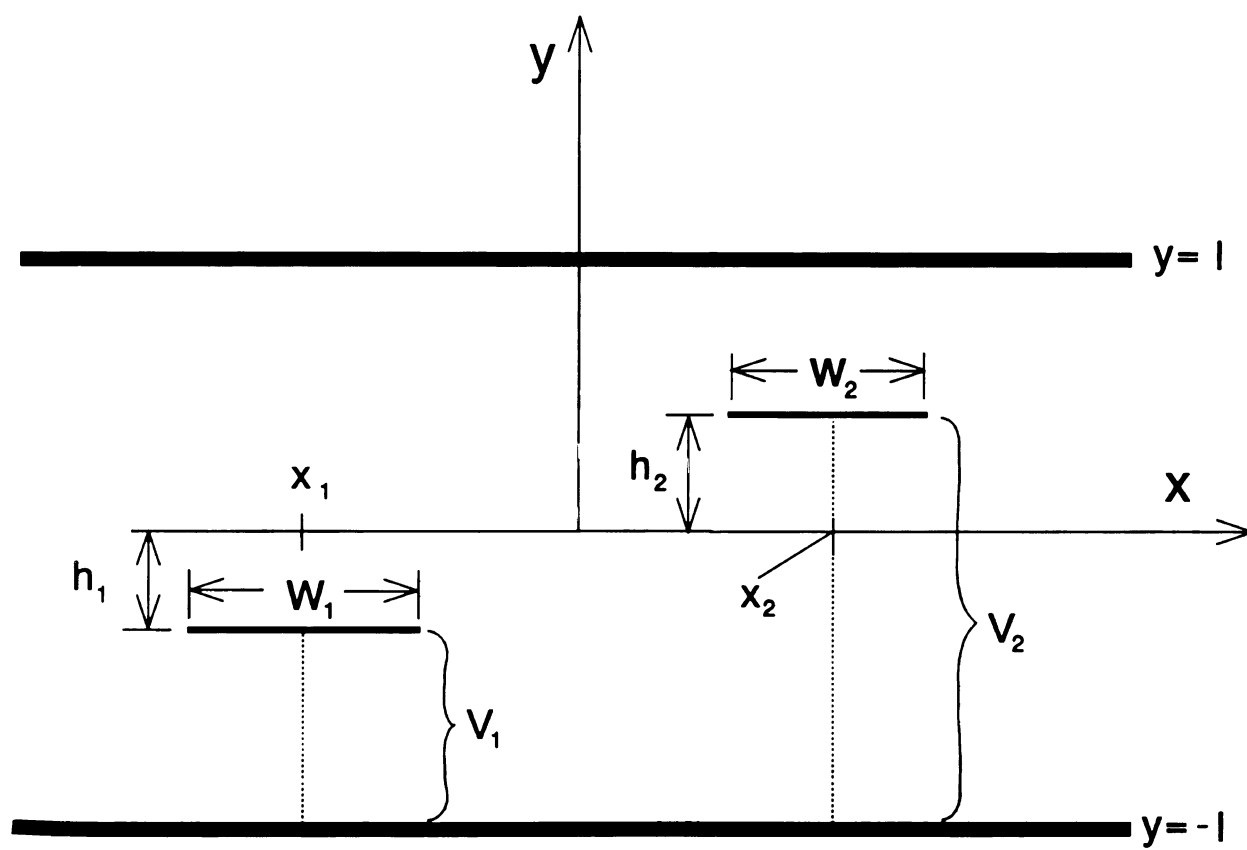


Figure 5.16. TEM mode center strip potentials relative to bottom ground plane.

or

$$V_1 = -\frac{k_o}{\omega\epsilon_o} \left\{ \int_{a_1}^{b_1} \left[\tilde{g}_{k_o}'(x_1, h_1 | x', h_1) k_{1z}(x') - \tilde{g}_{k_o}'(x_1, -l | x', h_1) k_{1z}(x') \right] dx' \right. \\ \left. + \int_{a_2}^{b_2} \left[\tilde{g}_{k_o}'(x_1, h_1 | x', h_2) k_{2z}(x') - \tilde{g}_{k_o}'(x_1, -l | x', h_2) k_{2z}(x') \right] dx' \right\} \quad (229)$$

leading to

$$V_1 = \frac{k_o}{\omega\epsilon_o} \left\{ \int_{a_1}^{b_1} \left[\tilde{g}_{k_o}'(x_1, -l | x', h_1) - \tilde{g}_{k_o}'(x_1, h_1 | x', h_1) \right] k_{1z}(x') dx' \right. \\ \left. + \int_{a_2}^{b_2} \left[\tilde{g}_{k_o}'(x_1, -l | x', h_2) - \tilde{g}_{k_o}'(x_1, h_1 | x', h_2) \right] k_{2z}(x') dx' \right\}. \quad (230)$$

But it can be seen that

$$\frac{k_o}{\omega\epsilon_o} = \frac{\omega\sqrt{\mu_o\epsilon_o}}{\omega\epsilon_o} = \frac{\sqrt{\mu_o}\sqrt{\epsilon_o}}{\sqrt{\epsilon_o}\sqrt{\epsilon_o}} = \sqrt{\frac{\mu_o}{\epsilon_o}} \quad (231)$$

therefore, the potential which exists between strip 1 and the ground plane located at $y=-l$

is

$$V_1 = \sqrt{\frac{\mu_o}{\epsilon_o}} \left\{ \int_{a_1}^{b_1} \left[\tilde{g}_{k_o}'(x_1, -l | x', h_1) - \tilde{g}_{k_o}'(x_1, h_1 | x', h_1) \right] k_{1z}(x') dx' \right. \\ \left. + \int_{a_2}^{b_2} \left[\tilde{g}_{k_o}'(x_1, -l | x', h_2) - \tilde{g}_{k_o}'(x_1, h_1 | x', h_2) \right] k_{2z}(x') dx' \right\}. \quad (232)$$

In a similar manner, let V_2 be the potential between a point x_2 in the center of

strip 2, and a point directly below on the nearest ground plane (again see Figure 5.16).

This leads to

$$V_2 = - \int_{-l}^{h_2} E_y(\vec{r}) \cdot d\vec{l} \Big|_{x=x_2} = - \int_{-l}^{h_2} \frac{k_o}{\omega \epsilon_o} \frac{\partial}{\partial y} \left\{ \int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_2, y | x', h_1) k_{1z}(x') dx' + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_2, y | x', h_2) k_{2z}(x') dx' \right\} dy \quad (233)$$

therefore, the potential which exists between strip 2 and the ground plane located at $y=-l$ is

$$V_2 = \sqrt{\frac{\mu_o}{\epsilon_o}} \left\{ \int_{a_1}^{b_1} [\tilde{g}_{k_o}^t(x_2, -l | x', h_1) - \tilde{g}_{k_o}^t(x_2, h_2 | x', h_1)] k_{1z}(x') dx' + \int_{a_2}^{b_2} [\tilde{g}_{k_o}^t(x_2, -l | x', h_2) - \tilde{g}_{k_o}^t(x_2, h_2 | x', h_2)] k_{2z}(x') dx' \right\}. \quad (234)$$

It was seen earlier that

$$\int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_1, h_1 | x', h_1) k_{1z}(x') dx' + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_1, h_1 | x', h_2) k_{2z}(x') dx' = C_1 \quad (235)$$

$$\int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_2, h_2 | x', h_1) k_{1z}(x') dx' + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_2, h_2 | x', h_2) k_{2z}(x') dx' = C_2 \quad (236)$$

therefore (232) and (234) become

$$V_1 = \sqrt{\frac{\mu_o}{\epsilon_o}} \left\{ \int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_1, -l | x', h_1) k_{1z}(x') dx' + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_1, -l | x', h_2) k_{2z}(x') dx' - C_1 \right\} \quad (237)$$

$$V_2 = \sqrt{\frac{\mu_o}{\epsilon_o}} \left\{ \int_{a_1}^{b_1} \tilde{g}_{k_o}^t(x_2, -l | x', h_1) k_{1z}(x') dx' + \int_{a_2}^{b_2} \tilde{g}_{k_o}^t(x_2, -l | x', h_2) k_{2z}(x') dx' - C_2 \right\}. \quad (238)$$

The principal and scattered components of the transform-domain Green's function implicated in (237) and (238) are given by

$$\tilde{g}_{k_o}^P(x_\alpha, -l | x', h_\alpha) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')} e^{-\xi|l - h_\alpha|}}{4\pi\xi} d\xi = \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')} e^{-\xi|l + h_\alpha|}}{4\pi\xi} d\xi \quad (239)$$

and

$$\tilde{g}_{k_o}^{st}(x_\alpha, -l | x', h_\alpha) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')}}{4\pi\xi(1 - e^{4\xi l})} \sum_{n=1}^4 s_n^t e^{-\xi\phi_n(-l|h_\alpha)} d\xi. \quad (240)$$

Here

$$\begin{aligned} \phi_1 &= 2l + (-l) + h_\alpha = l + h_\alpha & \phi_3 &= 4l + (-l) - h_\alpha = 3l - h_\alpha \\ \phi_2 &= 2l - (-l) - h_\alpha = 3l - h_\alpha & \phi_4 &= 4l - (-l) + h_\alpha = 5l + h_\alpha \end{aligned} \quad (241)$$

and as a result

$$\begin{aligned} \sum_{n=1}^4 s_n^t &= \left[-1e^{-\xi(l+h_\alpha)} - 1e^{-\xi(3l-h_\alpha)} + 1e^{-\xi(3l-h_\alpha)} + 1e^{-\xi(5l+h_\alpha)} \right] \\ &= \left[-1e^{-\xi(l+h_\alpha)} + 1e^{-\xi(5l+h_\alpha)} \right] = (e^{-\xi 5l} - e^{-\xi l}) e^{-\xi h_\alpha} \\ &= (e^{-\xi 4l} - 1) e^{-\xi l} e^{-\xi h_\alpha} \\ &= -e^{-\xi|l+h_\alpha|} (1 - e^{-\xi 4l}). \end{aligned} \quad (242)$$

From this it is seen that the scattered component of the transform-domain Green's

function is

$$\begin{aligned}
 \tilde{g}_{k_o}^{st}(x_\alpha, -l | x', h_\alpha) &= \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')}}{4\pi\xi(1 - e^{-4\xi l})} \left[-e^{-\xi|l+h_\alpha|} (1 - e^{-4\xi l}) \right] d\xi \\
 &= - \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')}}{4\pi\xi} e^{-\xi|l+h_\alpha|} d\xi
 \end{aligned} \tag{243}$$

and thus the total Green's function $\tilde{g}_{k_o}'(x_\alpha, -l | x', h_\alpha)$ vanishes

$$\begin{aligned}
 \tilde{g}_{k_o}'(x_\alpha, -l | x', h_\alpha) &= \tilde{g}_{k_o}^p(x_\alpha, -l | x', h_\alpha) + \tilde{g}_{k_o}^{st}(x_\alpha, -l | x', h_\alpha) \\
 &= \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')}}{4\pi\xi} e^{-\xi|l+h_\alpha|} d\xi - \int_{-\infty}^{\infty} \frac{e^{j\xi(x_\alpha - x')}}{4\pi\xi} e^{-\xi|l+h_\alpha|} d\xi \\
 &= 0.
 \end{aligned} \tag{244}$$

As a result, the potentials V_1 and V_2 can be expressed

$$V_1 = \sqrt{\frac{\mu_o}{\epsilon_o}} \left\{ -C_1 \right\} = -\eta_o C_1 \tag{245}$$

$$V_2 = \sqrt{\frac{\mu_o}{\epsilon_o}} \left\{ -C_2 \right\} = -\eta_o C_2 \tag{246}$$

where $\eta_o = \sqrt{\mu_o/\epsilon_o} \cong 120\pi$ in free space. As was noted in Chapter 4 the sign of the constants C_1 and C_2 in expressions (174) and (175) can be either positive or negative.

As a result, these constants are chosen for convenience to be

$$C_\alpha = \frac{V_\alpha}{120\pi} \tag{247}$$

for $\alpha = 1, 2$.

5.9 Pole Series Representation for Matrix Elements

As with the isolated strip structure, solution of the coupled matrix equations (206) and (207) can be greatly accelerated by replacing the real axis integration implicated in the matrix elements (213)-(216) with an equivalent pole-series representation (refer to section 4.11). For simplicity, the case of coplanar center strips which lie in the plane of the origin ($h_1 = h_2 = 0$) will be examined. If this specialization is made, (179) becomes

$$W(\xi) = \frac{1}{|\xi| (1 - e^{-4|\xi|l})} \left\{ (1 - e^{-4|\xi|l}) - e^{-2l|\xi|} - e^{-2l|\xi|} + e^{-4l|\xi|} + e^{-4l|\xi|} \right\} \quad (248)$$

which, simplifies to

$$W(\xi) = \frac{1 - e^{-2l|\xi|}}{|\xi| (1 + e^{-2l|\xi|})}. \quad (249)$$

From this it is seen that

$$W(\xi) \approx \frac{1 - (1 - 2l|\xi|)}{|\xi| [1 + (1 - 2l|\xi|)]} = \frac{2l|\xi|}{2|\xi| - 2l|\xi|^2} \approx l \quad (250)$$

for $\xi \ll 1$, and therefore poles occur for

$$1 + e^{-2|\xi_p|l} = 0 \quad (251)$$

or

$$e^{|\xi_p|l} + e^{-|\xi_p|l} = 2 \cosh(\xi_p l) = 0. \quad (252)$$

At the roots $\xi_p = j\xi_{pi}$, therefore

$$\cos(\xi_{pi}l) = 0 \quad (253)$$

and thus roots occur for

$$\xi_{pi}l = (2n - 1) \frac{\pi}{2} \quad \dots \text{ for } n = 1, 2, \dots \quad (254)$$

therefore

$$\xi_{pi} = \pm \frac{j(2n-1)\pi}{2l} \quad \dots \text{ for } n = 1, 2, \dots \quad (255)$$

Now

$$W(\xi) = \frac{N(\xi)}{|\xi| D(\xi)} = \frac{1 - e^{-2|\xi|l}}{|\xi| (1 + e^{-2|\xi|l})} \quad (256)$$

and from this it is seen that at $\xi = \xi_p$, $N(\xi) = 2$ and

$$\begin{aligned} D(\xi) &= 1 + e^{-2\xi l} \approx D(\xi_p) + \left. \frac{dD}{d\xi} \right|_{\xi = \xi_p} (\xi - \xi_p) \\ &\approx -2le^{-2\xi l} \Big|_{\xi = \xi_p} (\xi - \xi_p) \\ &\approx 2l(\xi - \xi_p) \end{aligned} \quad (257)$$

therefore

$$\lim_{\xi \rightarrow \xi_p} W(\xi) = \frac{2}{\xi 2l(\xi - \xi_p)} = \frac{1}{\xi l(\xi - \xi_p)} \quad (258)$$

As discussed in Chapter 4, Cauchy's integral theorem states

$$\oint_C f(\xi) d\xi = \int_{C_R} f(\xi) d\xi + \int_{C_{\infty}^+} f(\xi) d\xi + \int_{C_{\rho}^+} f(\xi) d\xi = 0. \quad (259)$$

The matrix elements (213)-(216) may be generically expressed as

$$A_{\alpha\beta}^{mn} = \int_{-\infty}^{\infty} \frac{1}{2\pi} W(\xi) \frac{\sin \xi \tilde{x}_{\beta}}{\xi} e^{j\xi L_{\alpha\beta}} d\xi \quad \dots \text{for } \alpha = 1, 2, \beta = 1, 2 \quad (260)$$

where

$$L_{11} = (\bar{x}_{1m} - x_{1n}) \quad (261)$$

$$L_{12} = (\bar{x}_{1m} + x_1 - x_{2n} - x_2) \quad (262)$$

$$L_{21} = (\bar{x}_{2m} + x_2 - x_{1n} - x_1) \quad (263)$$

and

$$L_{22} = (\bar{x}_{2m} - x_{2n}). \quad (264)$$

Because

$$\xi = \xi_r + j\xi_i \quad (265)$$

it is apparent that

$$e^{j\xi L_{\alpha\beta}} = e^{-\xi_i L_{\alpha\beta}} e^{j\xi_r L_{\alpha\beta}} \quad (266)$$

and therefore the case when $L_{\alpha\beta} > 0$ requires that the integration contour be closed in the upper half of the complex ξ -plane, and the case when $L_{\alpha\beta} < 0$ requires lower half-plane closure (refer to Figure 4.19). Thus it is seen that

$$\int_{C_{\pm}^{\pm}} f(\xi) d\xi = 0 \quad (267)$$

and the real-axis integration implicated in the matrix element expressions may be represented as the sum of contributions to the contour integration around the poles

$$\int_{C_R} f(\xi) d\xi = - \sum_p \int_{C_p^{\pm}} f(\xi) d\xi . \quad (268)$$

For $\xi \rightarrow \xi_p$ it was seen that

$$W(\xi) \rightarrow \frac{1}{\xi l(\xi - \xi_p)} \quad (269)$$

and thus the matrix elements may be written

$$A_{\alpha\beta}^{mn} = - \sum_p \frac{1}{2\pi} \int_{C_p^{\pm}} \frac{I(\xi)}{(\xi \mp \xi_p)} d\xi \quad (270)$$

if

$$I(\xi) = \frac{1}{l} \frac{\sin(\xi \tilde{x}_{\beta})}{\xi^2} e^{j\xi L_{\alpha\beta}} . \quad (271)$$

At points along $C_p^{\pm}$

$$\xi = \pm \xi_p + \epsilon e^{j\psi} \quad (272)$$

and

$$d\xi = j\epsilon e^{j\psi} d\psi \quad (273)$$

therefore

$$\begin{aligned}
\oint_{C_p^*} \frac{I(\xi)}{(\xi \mp \xi_p)} d\xi &= \lim_{\epsilon \rightarrow 0} \int_0^{\mp 2\pi} \frac{I(\xi)}{\epsilon e^{j\psi}} j\epsilon e^{j\psi} d\psi \\
&= \mp 2\pi j I(\pm \xi_p).
\end{aligned} \tag{274}$$

As a result, (270) becomes

$$A_{\alpha\beta}^{mn} = \pm j \sum_p I(\pm \xi_p). \tag{275}$$

At the poles $\xi_p = j\xi_{pi}$, leading to

$$I(\pm \xi_p) = \frac{1}{l} \frac{\sin(\pm j\xi_{pi}\tilde{x}_\beta)}{(\pm j\xi_{pi})^2} e^{\mp \xi_{pi}L_{\alpha\beta}} = \pm j \frac{1}{l} \frac{\sinh(\xi_{pi}\tilde{x}_\beta)}{-\xi_{pi}^2} e^{\mp \xi_{pi}L_{\alpha\beta}} \tag{276}$$

which in turn gives

$$A_{\alpha\beta}^{mn} = \frac{1}{l} \sum_p \frac{\sinh(\xi_{pi}\tilde{x}_\beta)}{\xi_{pi}^2} e^{\mp \xi_{pi}L_{\alpha\beta}} \quad \dots \text{ for } L_{\alpha\beta} < 0, \text{ or } L_{\alpha\beta} > 0. \tag{277}$$

It is noted that

$$\begin{aligned}
-\xi_{pi}L_{\alpha\beta} &= -\xi_{pi}|L_{\alpha\beta}| \quad \dots \text{ for } L_{\alpha\beta} > 0 \\
+\xi_{pi}L_{\alpha\beta} &= -\xi_{pi}|L_{\alpha\beta}| \quad \dots \text{ for } L_{\alpha\beta} < 0
\end{aligned} \tag{278}$$

thus

$$A_{\alpha\beta}^{mn} = \frac{1}{l} \sum_p \frac{\sinh(\xi_{pi}\tilde{x}_\beta)}{\xi_{pi}^2} e^{-\xi_{pi}|L_{\alpha\beta}|} \quad \dots \text{ for } L_{\alpha\beta} \neq 0. \tag{279}$$

It is observed that the generic representation of the matrix elements associated with the coupled strip structure in (279) closely resembles the expressions for the isolated stripline presented in Section 4.11. It was determined in this section that the case when

$L_{\alpha\beta}=0$ must be examined more closely. It can be seen from (261)-(264) that $L_{\alpha\beta}=0$ only occurs in the self terms A_{11}^{mn} and A_{22}^{mn} . This is due to the fact that the center strips cannot overlap, and therefore the distances L_{12} and L_{21} are never smaller than the center-to-center distance between the coupled strips (it should be noted that x_1 is defined to lie to the left of the origin, and is thus always negative, while x_2 lies to the right of the origin, and is thus always positive). For the case of $L_{\alpha\beta}=0$

$$I(\xi) = \frac{1}{l} \frac{\sin(\xi \tilde{x}_\beta)}{\xi^2} = \frac{e^{j\xi \tilde{x}_\beta} - e^{-j\xi \tilde{x}_\beta}}{j2l\xi^2} \quad (280)$$

and

$$\begin{aligned} A_{\alpha\beta}^{mn} &= \int_{-\infty}^{\infty} \frac{1}{2\pi} W(\xi) \frac{\sin(\xi \tilde{x}_\beta)}{\xi} d\xi = \int_{-\infty}^{\infty} \frac{1}{2\pi} W(\xi) \frac{e^{j\xi \tilde{x}_\beta} - e^{-j\xi \tilde{x}_\beta}}{2j\xi} d\xi \\ &= \int_0^{\infty} \frac{1}{2j} \frac{1}{2\pi} W(\xi) \left[\frac{e^{j\xi \tilde{x}_\beta}}{\xi} - \frac{e^{-j\xi \tilde{x}_\beta}}{\xi} \right] d\xi. \end{aligned} \quad (281)$$

As noted previously, expanding the term $\sin(\xi \tilde{x}_\beta)$ into exponentials creates a pole at the origin (refer to Figure 4.20), and Cauchy's integral theorem then states

$$\oint_C f(\xi) d\xi = \int_{C_R^+} f(\xi) d\xi + \int_{C_R^-} f(\xi) d\xi + \int_{C_\epsilon^+} f(\xi) d\xi + \int_{C_\epsilon^-} f(\xi) d\xi + \int_{C_p^+} f(\xi) d\xi = 0 \quad (282)$$

resulting in

$$A_{\alpha\beta}^{mn} = - \sum_p \int_{C_p^+} f(\xi) d\xi - \int_{C_\epsilon^+} f(\xi) d\xi. \quad (283)$$

Following the development presented in Section 4.11, it is determined that

$$-\sum_p \int_{C_p^*} f(\xi) d\xi = -\frac{1}{l} \sum_p \frac{e^{-\xi_{pi} \bar{\xi}_\beta}}{\xi_{pi}^2} \quad (284)$$

and

$$\int_{C_\epsilon} f(\xi) d\xi = -\frac{l}{2}. \quad (285)$$

Thus the pole-series expressions for the matrix elements in the case when $L_{\alpha\beta}=0$ are given by

$$A_{\alpha\beta}^{mn} = \frac{l}{2} - \frac{1}{l} \sum_p \frac{e^{-\xi_{pi} \bar{\xi}_\beta}}{\xi_{pi}^2}. \quad (286)$$

It is again noted that this special $L_{\alpha\beta}=0$ case only occurs in the self terms A_{11}^{mn} and A_{22}^{mn} .

The coupling terms A_{12}^{mn} and A_{21}^{mn} are *always* given by expression (277).

The cross-sectional surface current distributions associated with the TEM mode of coupled stripline can now be determined through solution of (206) and (207). In Figure 5.17 a comparison is made between the current distribution associated with widely-separated coupled center strips, and that obtained for the isolated stripline using the method developed in Chapter 4. It is seen here that these two methods produce identical results. Figure 5.18 and Figure 5.19 present current distributions associated with a coupled strip structure having identical ($w_1=w_2$) coplanar center strips, lying in the plane of the origin, for the case of $V_1 = 1V$, $V_2 = 1V$. It is apparent from these figures that large center-strip separations result in the expected current distributions, which are identical to those obtained for the isolated stripline. As the center-to-center strip separation decreases, however, current begins to accumulate on the outer edges of the center strips (the edge farthest from the opposing strip). This continues until the

center strips come into contact, at which point the current distribution resembles that of a structure having a single center strip of width $w=2w_1=2w_2$.

5.10 Impedance Parameters of Coupled Stripline

The TEM specialization for coupled stripline presented in this chapter allows for the determination of the total center-strip surface currents I_1 and I_2 for specific strip potentials V_1 and V_2 . These total center-strip currents are determined from

$$I_\alpha = \int_{-w_\alpha/2}^{w_\alpha/2} k_{\alpha z}(\bar{x}) d\bar{x} = \sum_{n=1}^{N_\alpha} a_{\alpha n} \int_{-w_\alpha/2}^{w_\alpha/2} p_{\alpha n}(\bar{x}) d\bar{x} \quad (287)$$

which becomes

$$I_\alpha = \overline{\Delta x}_\alpha \sum_{n=1}^{N_\alpha} a_{\alpha n} \quad (288)$$

for $\alpha=1,2$, where $\overline{\Delta x}_\alpha$ is defined in (197). The current and potential quantities are related through the conventional circuit admittance descriptions

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 \\ I_2 &= Y_{21}V_1 + Y_{22}V_2. \end{aligned} \quad (289)$$

Here the various admittance parameters are determined by solving the system (206)-(207) first for $V_1 = 1V$, $V_2 = 0V$ and then relating the resulting currents and potentials through the expressions in (289), which gives

$$Y_{11} = I_1 \Big|_{\substack{V_1=1, \\ V_2=0}}, \quad Y_{21} = I_2 \Big|_{\substack{V_1=1, \\ V_2=0}} \quad (290)$$

and then solving the system (206)-(207) again for $V_1 = 0V$, $V_2 = 1V$, leading to

$$Y_{12} = I_1 \bigg|_{\substack{V_1=0, \\ V_2=1}}, \quad Y_{22} = I_2 \bigg|_{\substack{V_1=0, \\ V_2=1}}. \quad (291)$$

Once the admittance parameters have been computed, impedance parameters for the coupled stripline are found by inverting the admittance matrix, which gives

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1} = \frac{1}{Y_{11}Y_{22} - Y_{12}Y_{21}} \begin{bmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{bmatrix}. \quad (292)$$

The behavior of impedance parameters associated with a coplanar, coupled stripline structure is presented in Figure 5.20 and Figure 5.21. For this case, the center strips lie in the plane of the origin, and have identical dimensions ($w_1 = w_2$), resulting in $Z_{11} = Z_{22}$, while $Z_{12} = Z_{21}$ due to reciprocity. In Figure 5.20 the variation of Z_{11} , the self impedance of strip 1, is plotted versus center-strip spacing, for various center-strip widths. It is seen that, for large center-strip separations, the self impedance of a single strip is identical to the characteristic impedance obtained for the isolated stripline using the method outlined in Chapter 4 (refer to Figure 4.21). As the distance between the center strips decreases, the self impedance is seen to decrease until the strips come into contact.

Variation of the mutual impedance parameter Z_{12} is plotted versus center-strip spacing, for various center-strip widths, in Figure 5.21. Here it is seen that the mutual impedance that exists between two coupled strips is vanishingly small for large center-strip separations, but becomes significant as the center-to-center strip separation decreases.

Normalized current distributions associated with the impedance plots discussed above are also presented. In Figure 5.22, the normalized strip 1 (left) surface current distribution is presented for the case of $V_1 = 1V$, $V_2 = 0V$. Here the expected distribution, identical to that obtained for the isolated stripline, is displayed when the center strips are widely separated. As the distance between the center strips decreases, it is seen that current begins to accumulate on the side of the center strip which is nearest to strip 2 (right).

From Figure 5.23, it is apparent that, for the case of $V_1 = 1V$, $V_2 = 0V$, a current is induced on strip 2 which accumulates on the side of the center strip nearest to strip 1. Although the absolute magnitude of this induced current differs with center-strip spacing (as evidenced by the change in impedance parameters), the normalized distribution is seen to differ very little as the distance between the center strips changes. A slight shift of the distribution toward the edge of the center strip is evident in the case of a small center-to-center strip separation, which is consistent with tighter coupling.

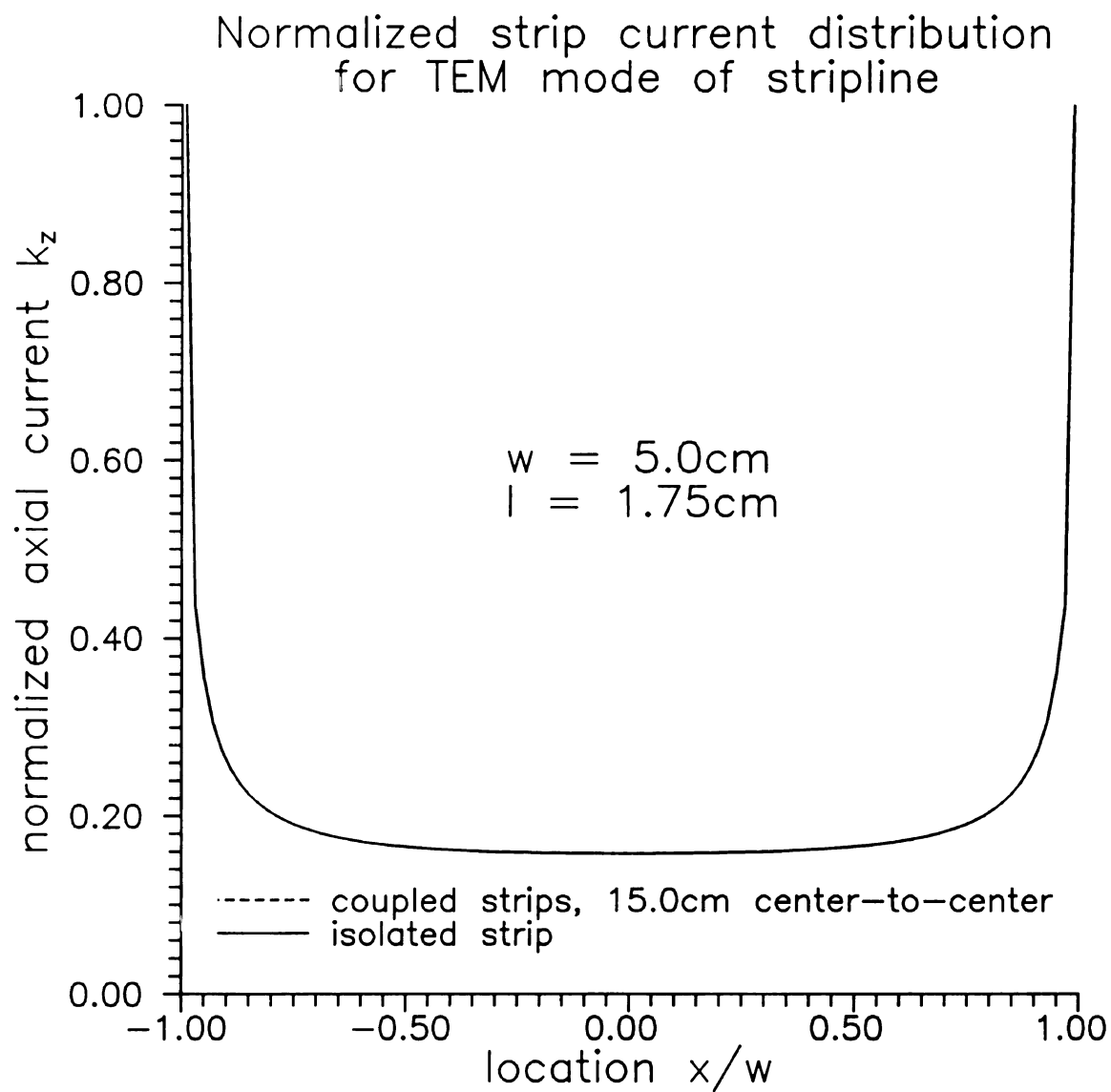


Figure 5.17 Comparison of surface current distributions associated with TEM mode, for isolated stripline, and widely-separated coupled center strips.

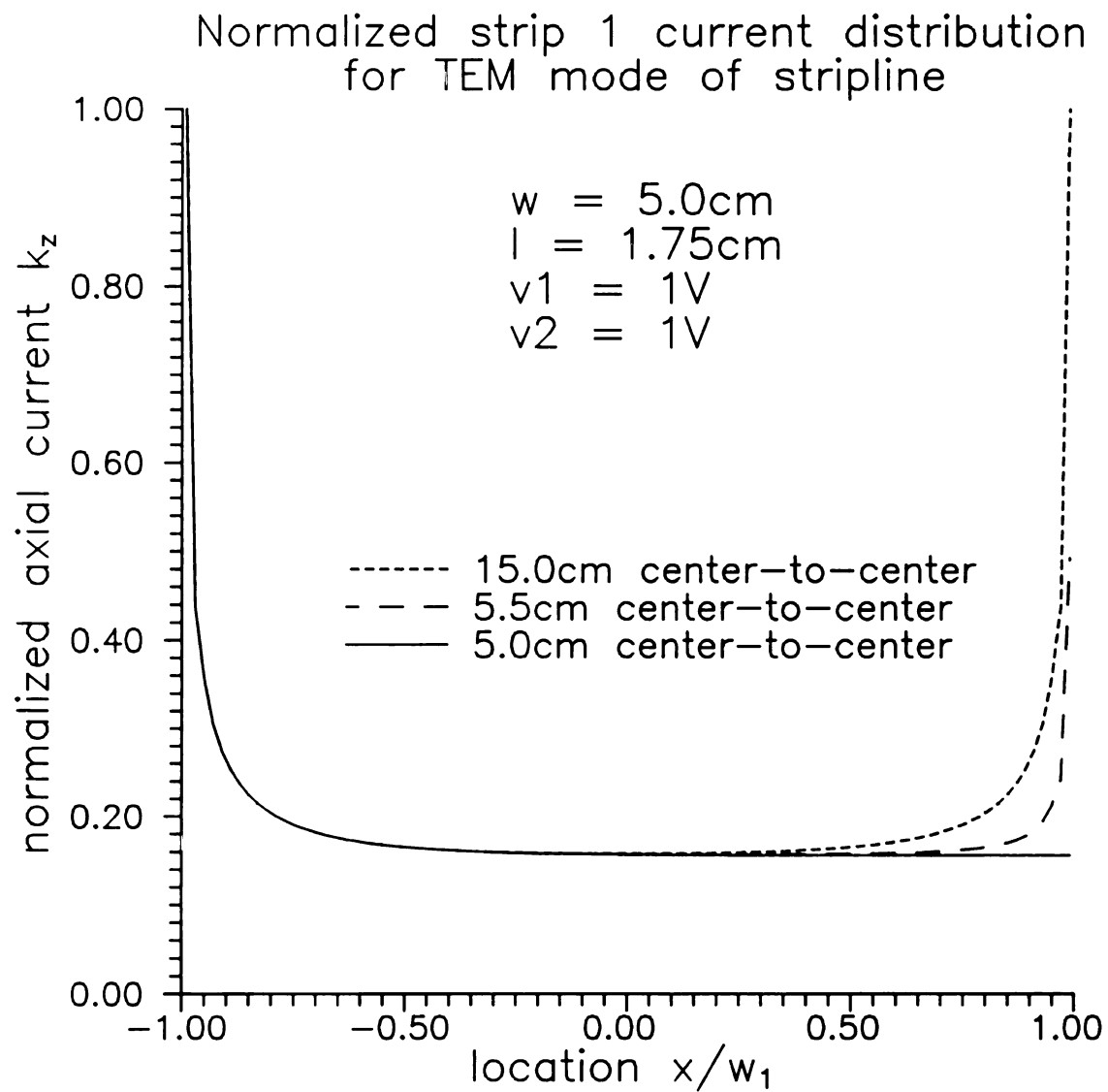


Figure 5.18 Strip 1 (left) normalized TEM mode current distribution.

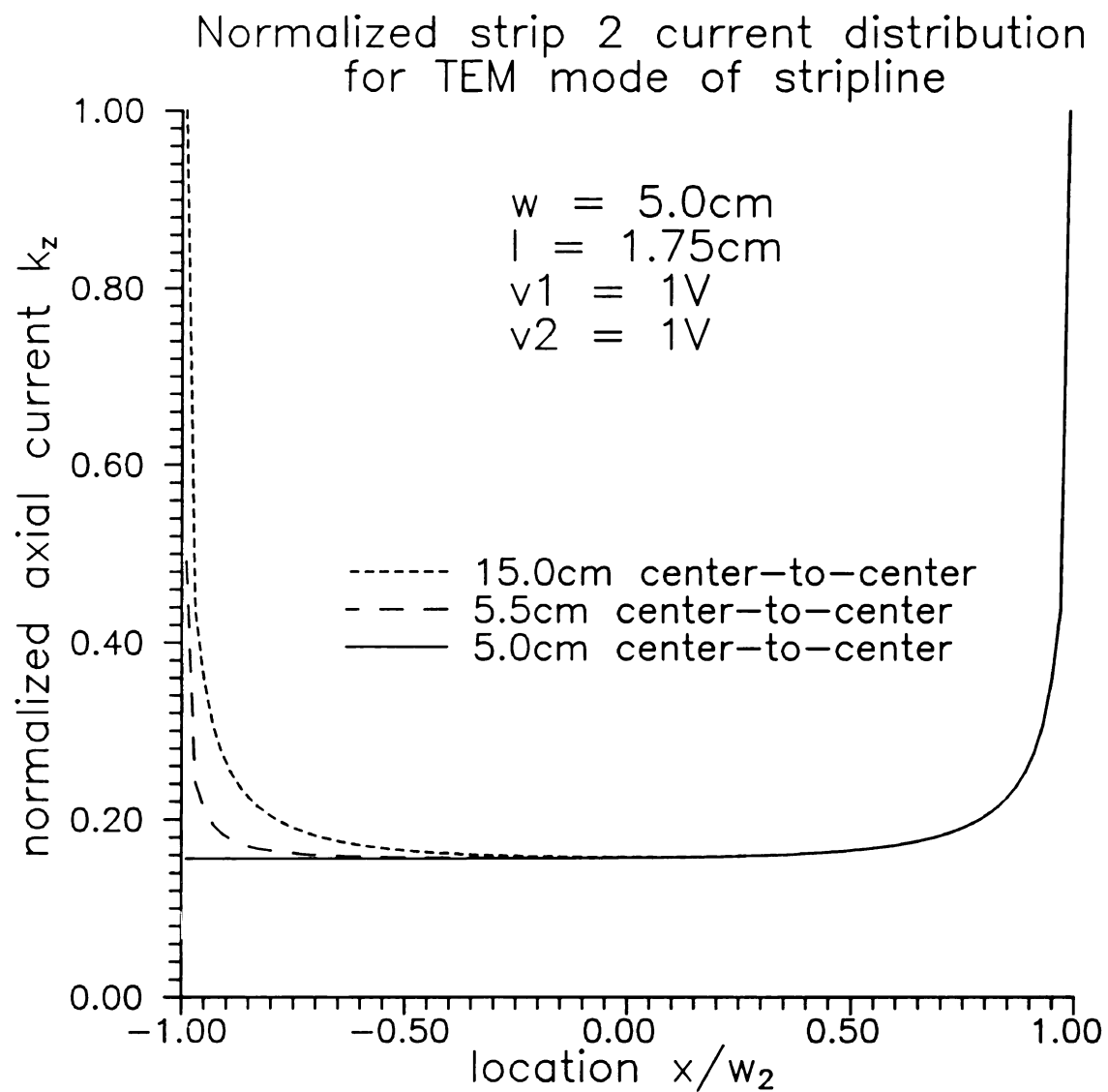


Figure 5.19 Strip 2 (right) normalized TEM mode current distribution.

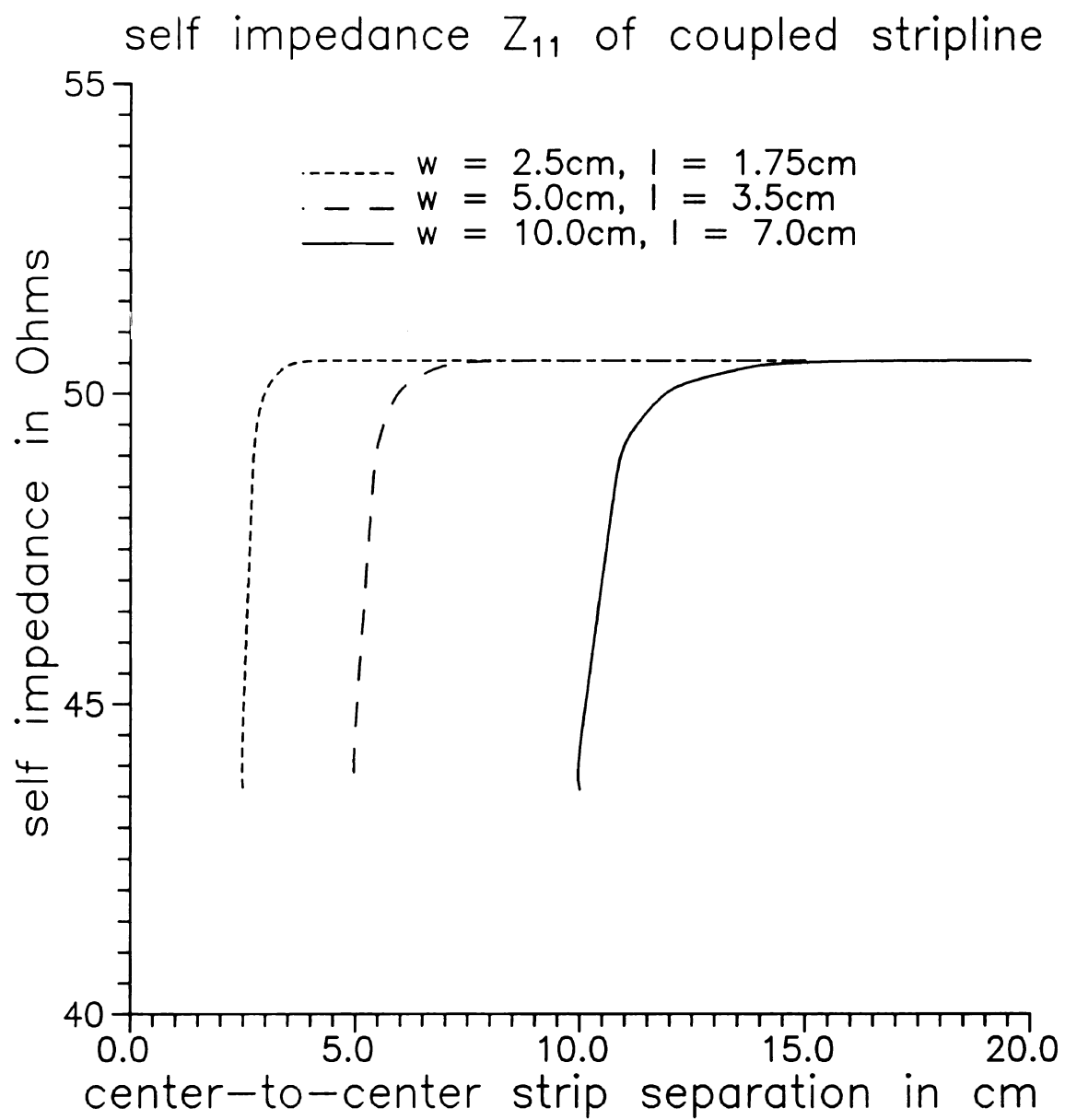


Figure 5.20 Relationship between coupled stripline self impedance Z_{11} and center-strip spacing.

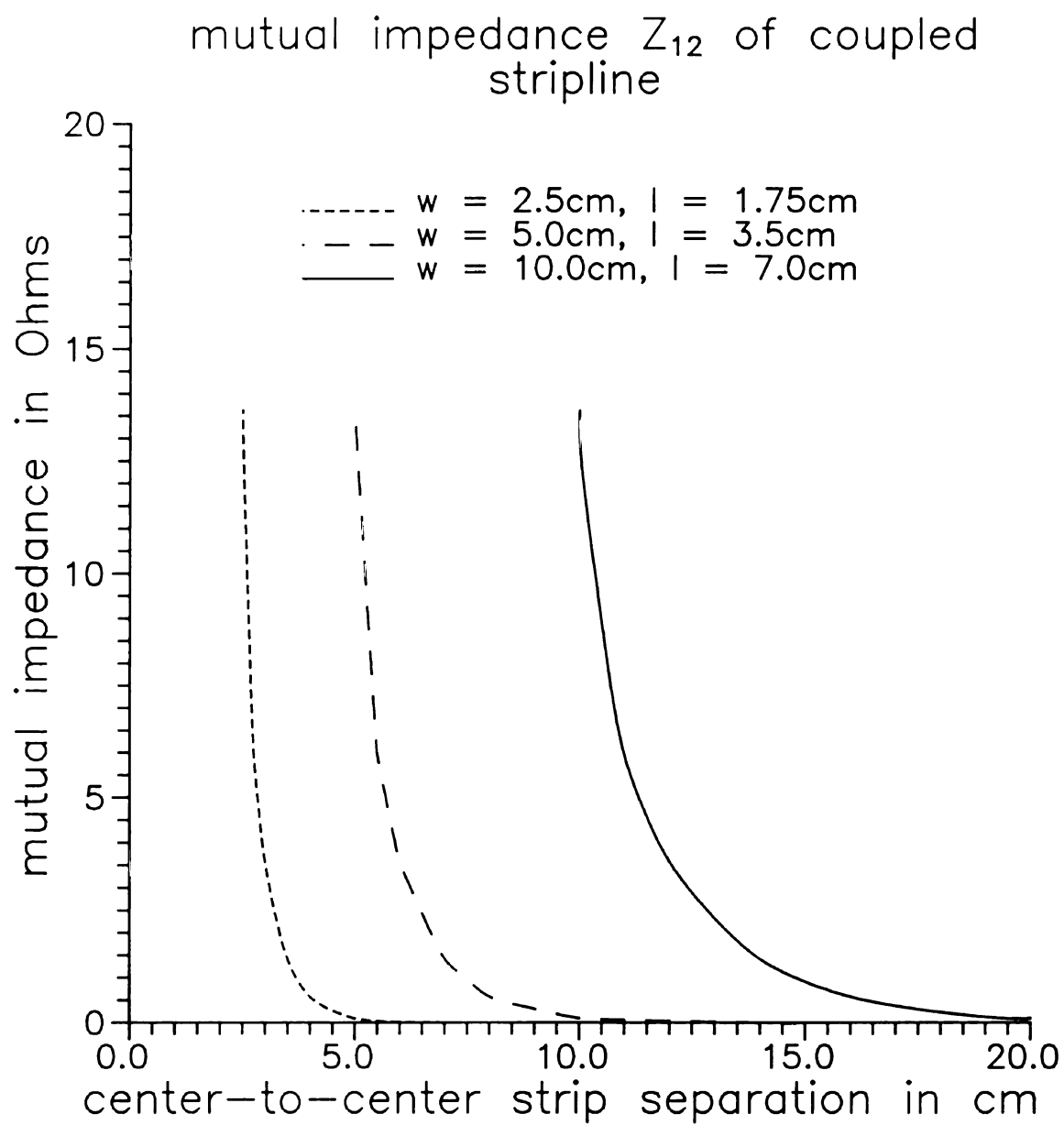


Figure 5.21 Relationship between coupled stripline mutual impedance Z_{12} and center-strip spacing.

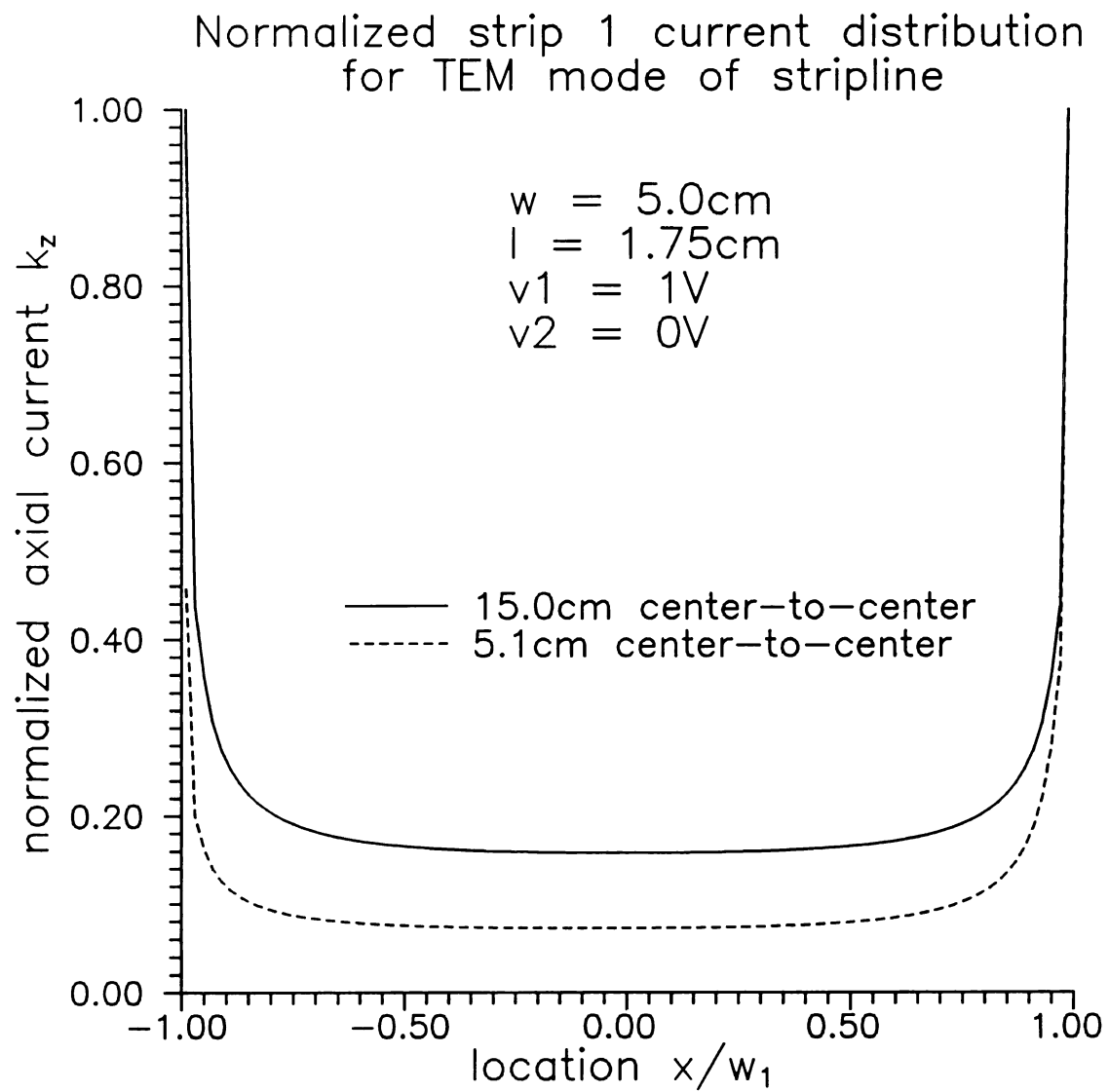


Figure 5.22 Strip 1 (left) normalized TEM mode current distribution.

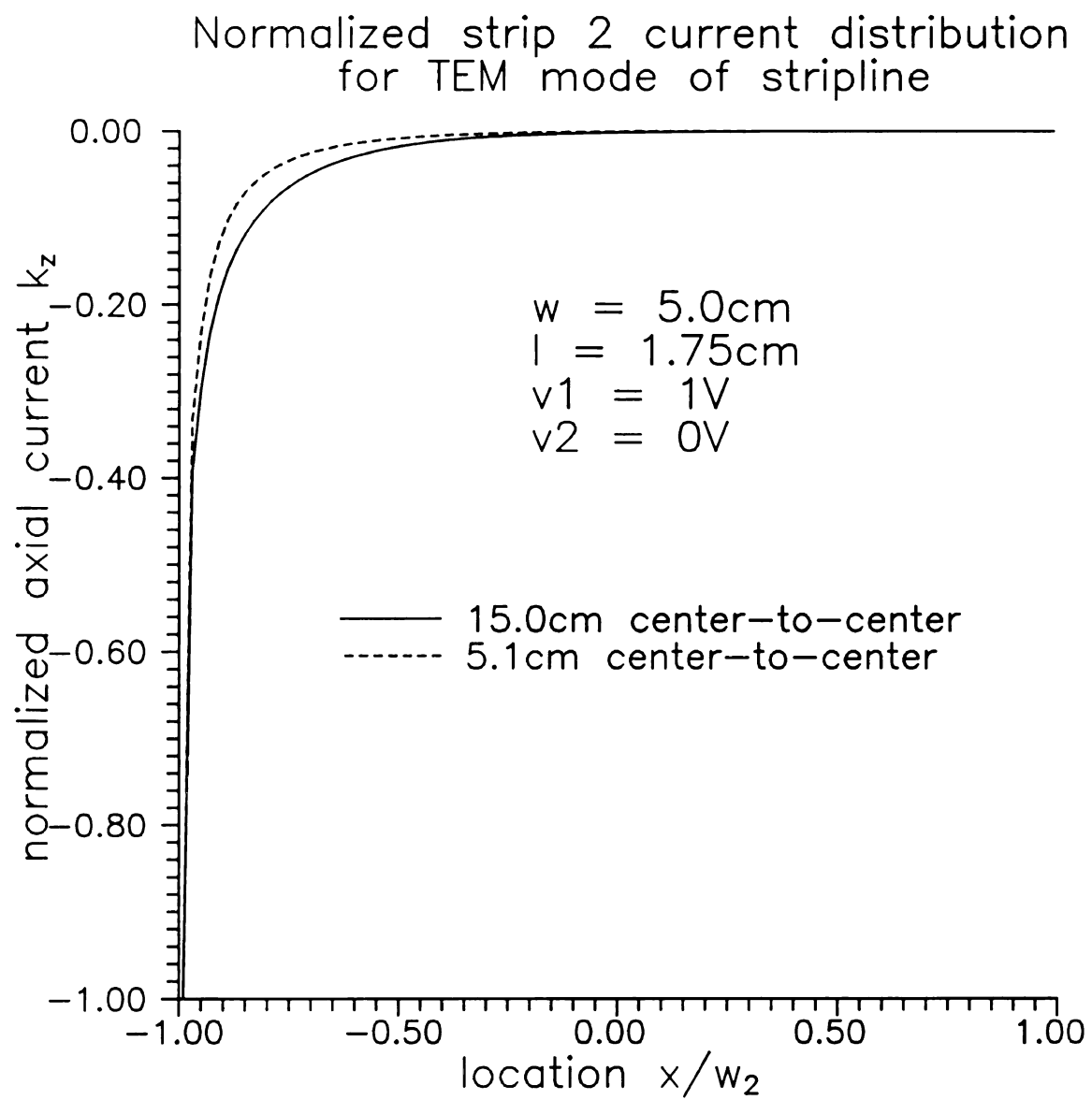


Figure 5.23 Strip 2 (right) normalized TEM mode current distribution.

Chapter 6

FULL-WAVE PERTURBATION METHOD FOR COUPLED STRIPLINE

In the previous chapter the propagation modes excited on a coupled stripline were determined through the development and numerical solution of a pair of coupled EFIEs. Although this method yielded the complete propagation-mode spectrum of the coupled stripline structure, the solution of the coupled EFIEs is a formidable task both analytically and numerically. The complexity of this task can be greatly reduced through implementation of a full-wave perturbation method for the coupled stripline [48]. This method involves the use of the eigenmode currents of an isolated strip as a first approximation for the nearly degenerate eigenmode currents of a loosely coupled pair of center strips [49]. In addition to reducing the amount of time required for computation of current distributions, this method provides insight into the nature of coupling effects. The results of this method also provide a convenient first approximation with which to begin searching for the eigenmodes of the coupled system using the full-wave EFIE method of Chapter 5. Results obtained using the perturbation method will be compared to those obtained using the full-wave EFIE method which were presented in the previous chapter.

6.1 Electric Dyadic Green's Function

In the previous analysis of coupled stripline structures, electric field integral equations were developed based on a Hertz potential Green's function. Due to the form of the testing operators which will be employed in the perturbation method, an electric dyadic Green's function must be used [50], [51]. It was determined previously that the

electric field in the free space region of the stripline which exists between the perfectly-conducting ground planes may be represented by

$$\vec{E}(\vec{r}) = (k_o^2 + \nabla \nabla \cdot) \int_V \vec{G}(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' . \quad (1)$$

The Green's function that appears in this expression contains components associated with both the principal and scattered Hertz potentials. The component of the Green's function associated with the scattered potential is bounded, and remains finite everywhere. However the Green's function component associated with the principal Hertz potential contains a source-point singularity. As a result the term

$$\nabla \nabla \cdot \int_V \vec{G}(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' \quad (2)$$

must be examined more closely. Application of Leibnitz's rule to equation (2) yields

$$\nabla \nabla \cdot \int_V \vec{G}(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' = P.V. \int_V \nabla \nabla \cdot [\vec{G}^P(\vec{r}|\vec{r}') \cdot \vec{J}(\vec{r}')] dv' + \hat{y} J_y(\vec{r}) . \quad (3)$$

Here $P.V.$ indicates that the integration is performed in the principle value sense. The electric field between the conducting plates may now be expressed as

$$\vec{E}(\vec{r}) = \int_V \vec{G}^e(\vec{r}|\vec{r}') \cdot \frac{\vec{J}(\vec{r}')}{j\omega\epsilon_o} dv' \quad (4)$$

where the electric field dyadic Green's function is given by

$$\vec{G}^e(\vec{r}|\vec{r}') = P.V. (k_o^2 + \nabla \nabla \cdot) \vec{G}(\vec{r}|\vec{r}') + \vec{L} \delta(\vec{r} - \vec{r}') . \quad (5)$$

Here the term $\vec{L} = -\hat{y}\hat{y}$ is known as the depolarizing dyad. The transform-domain

electric dyadic Green's function is expressed

$$\tilde{g}_\zeta^e(\vec{\rho}|\vec{\rho}') = P.V.(k_o^2 + \nabla\nabla\cdot)\tilde{g}_\zeta(\vec{\rho}|\vec{\rho}') + \tilde{L}\delta(\vec{\rho} - \vec{\rho}') \quad (6)$$

where $\tilde{\nabla} = \nabla_t + \hat{z}j\zeta$, and $\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$.

In the previous chapter, transform-domain electric field expressions for coupled strips were established

$$\vec{e}(x, h_1; \zeta) = \lim_{y \rightarrow h_1} \left\{ \int_{C_1} \tilde{g}^e(x, y | x', h_1) \cdot \frac{\vec{k}_1(x', \zeta)}{j\omega\epsilon_o} dx' + \int_{C_2} \tilde{g}^e(x, y | x', h_2) \cdot \frac{\vec{k}_2(x', \zeta)}{j\omega\epsilon_o} dx' \right\} \quad (7)$$

$$\vec{e}(x, h_2; \zeta) = \lim_{y \rightarrow h_2} \left\{ \int_{C_1} \tilde{g}^e(x, y | x', h_1) \cdot \frac{\vec{k}_1(x', \zeta)}{j\omega\epsilon_o} dx' + \int_{C_2} \tilde{g}^e(x, y | x', h_2) \cdot \frac{\vec{k}_2(x', \zeta)}{j\omega\epsilon_o} dx' \right\} \quad (8)$$

where C_1 and C_2 represent the cross-sections of the conducting center strips. Application of the boundary condition at the surface of the perfectly-conducting center strips

$$\hat{t} \cdot \vec{e}^s = -\hat{t} \cdot \vec{e}^i \quad (9)$$

and the approximate form of the current functions near the simple pole singularities

$$\vec{k}(x', \zeta) \approx \frac{k_p(x')}{(\zeta - \zeta_p)} \quad (10)$$

leads to a pair of fundamental EFIEs having non-trivial solutions only for $\zeta = \zeta_p$

$$\lim_{y \rightarrow h_1} \left\{ \hat{t} \cdot \left[\int_{a_1}^{b_1} \tilde{g}^e(x, y | x', h_1) \cdot \vec{k}_{1p}(x') dx' + \int_{a_2}^{b_2} \tilde{g}^e(x, y | x', h_2) \cdot \vec{k}_{2p}(x') dx' \right] \right\} = 0 \quad (11)$$

$$\lim_{y \rightarrow h_2} \left\{ \hat{t} \cdot \left[\int_{a_1}^{b_1} \tilde{g}^e(x, y | x', h_1) \cdot \vec{k}_{1p}(x') dx' + \int_{a_2}^{b_2} \tilde{g}^e(x, y | x', h_2) \cdot \vec{k}_{2p}(x') dx' \right] \right\} = 0 \quad (12)$$

where $a_\alpha = x_\alpha - w_\alpha/2$ and $b_\alpha = x_\alpha + w_\alpha/2$.

Now the transverse operator $\hat{t} \cdot (...)$ in EFIEs (11) and (12) is replaced with the testing operator

$$\int_{C_m} dl \vec{k}_{mp}^{\mp(0)} . \quad (13)$$

This leads to a pair of modified integral equations

$$\lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \vec{k}_{1p}^{\mp(0)}(x) \cdot \left[\int_{a_1}^{b_1} \tilde{g}^e(x, y | x', h_1) \cdot \vec{k}_{1p}^{\pm}(x') dx' + \int_{a_2}^{b_2} \tilde{g}^e(x, y | x', h_2) \cdot \vec{k}_{2p}^{\pm}(x') dx' \right] dx \right\} = 0 \quad (14)$$

$$\lim_{y \rightarrow h_2} \left\{ \int_{a_2}^{b_2} \vec{k}_{2p}^{\mp(0)}(x) \cdot \left[\int_{a_1}^{b_1} \tilde{g}^e(x, y | x', h_1) \cdot \vec{k}_{1p}^{\pm}(x') dx' + \int_{a_2}^{b_2} \tilde{g}^e(x, y | x', h_2) \cdot \vec{k}_{2p}^{\pm}(x') dx' \right] dx \right\} = 0 . \quad (15)$$

In expressions (14) and (15), the term $\vec{k}_{mp}^{\mp(0)}$ represents the current associated with the p^{th} propagation mode which exists along the m^{th} isolated center strip which has propagation constant $\pm \zeta_p^{(0)}$. This current function satisfies the EFIE for a single center-strip structure

$$\hat{t} \cdot \int_{a_\alpha}^{b_\alpha} \tilde{g}^e(x, y | x', y'; \pm \zeta_{mp}^{(0)}) \cdot \vec{k}_{mp}^{\mp(0)} dx' = 0. \quad (16)$$

The transform-domain electric dyadic Green's function has the reciprocity property

$$\tilde{g}^e(\vec{\rho} | \vec{\rho}'; \zeta) = \tilde{g}^e(\vec{\rho}' | \vec{\rho}; -\zeta). \quad (17)$$

Applying this property to the modified integral equations (14) and (15) results in

$$\lim_{y \rightarrow h_1} \left\{ \int_{a_1}^{b_1} \left[\vec{k}_{1p}^\pm(x') \int_{a_1}^{b_1} \tilde{g}^e(x', h_1 | x, y; \pm \zeta) \cdot \vec{k}_{1p}^{\mp(0)}(x) dx \right] dx' \right. \\ \left. + \int_{a_2}^{b_2} \left[\vec{k}_{2p}^\pm(x') \int_{a_1}^{b_1} \tilde{g}^e(x', h_2 | x, y; \pm \zeta) \cdot \vec{k}_{1p}^{\mp(0)}(x) dx \right] dx' \right\} = 0 \quad (18)$$

$$\lim_{y \rightarrow h_2} \left\{ \int_{a_1}^{b_1} \left[\vec{k}_{1p}^\pm(x') \int_{a_2}^{b_2} \tilde{g}^e(x', h_1 | x, y; \pm \zeta) \cdot \vec{k}_{2p}^{\mp(0)}(x) dx \right] dx' \right. \\ \left. + \int_{a_2}^{b_2} \left[\vec{k}_{2p}^\pm(x') \int_{a_2}^{b_2} \tilde{g}^e(x', h_2 | x, y; \pm \zeta) \cdot \vec{k}_{2p}^{\mp(0)}(x) dx \right] dx' \right\} = 0. \quad (19)$$

These expressions simplify for the case of center strips which lie in the plane of the origin ($h_1 = h_2 = 0$), becoming

$$\lim_{y \rightarrow 0} \left\{ \left[\int_{a_1}^{b_1} dx' \vec{k}_{1p}^\pm(x') + \int_{a_2}^{b_2} dx' \vec{k}_{2p}^\pm(x') \right] \cdot \int_{a_1}^{b_1} \tilde{g}^e(x', 0 | x, y; \pm \zeta) \cdot \vec{k}_{1p}^{\mp(0)}(x) dx \right\} = 0 \quad (20)$$

$$\lim_{y \rightarrow 0} \left\{ \left[\int_{a_1}^{b_1} dx' \vec{k}_{1p}^\pm(x') + \int_{a_2}^{b_2} dx' \vec{k}_{2p}^\pm(x') \right] \cdot \int_{a_2}^{b_2} \tilde{g}^e(x', 0 | x, y; \pm \zeta) \cdot \vec{k}_{2p}^{\mp(0)}(x) dx \right\} = 0. \quad (21)$$

6.2 Perturbation Approximation

If the geometry of the stripline, and frequency of operation are such that the center strips are only loosely coupled, then the system propagation mode eigenvalues are nearly the same as those of a stripline structure with only a single center strip (i.e the coupling effect is only a perturbation of the isolated system response). This suggests that a Taylor's series expansion of the Green's function be performed about the propagation constant of the single strip structure $\zeta = \mp \zeta_{np}^{(0)}$. This results in the expressions

$$\lim_{y \rightarrow 0} \left\{ \sum_{n=1}^2 \int_{a_n}^{b_n} dx' \bar{k}_{np}^{\pm}(x') \cdot \int_{a_1}^{b_1} \left[\tilde{g}^e(x', 0 | x, y; \pm \zeta_{np}^{(0)}) + \frac{\partial \tilde{g}^e(x', 0 | x, y; -\zeta)}{\partial \zeta} \Big|_{\mp \zeta_{np}^{(0)}} (\zeta \pm \zeta_{np}^{(0)} + \dots) \right] \cdot \bar{k}_{1p}^{\mp(0)}(x) dx \right\} = 0 \quad (22)$$

$$\lim_{y \rightarrow 0} \left\{ \sum_{n=1}^2 \int_{a_n}^{b_n} dx' \bar{k}_{np}^{\pm}(x') \cdot \int_{a_2}^{b_2} \left[\tilde{g}^e(x', 0 | x, y; \pm \zeta_{np}^{(0)}) + \frac{\partial \tilde{g}^e(x', 0 | x, y; -\zeta)}{\partial \zeta} \Big|_{\mp \zeta_{np}^{(0)}} (\zeta \pm \zeta_{np}^{(0)} + \dots) \right] \cdot \bar{k}_{2p}^{\mp(0)}(x) dx \right\} = 0. \quad (23)$$

Due to relationship (16), the leading terms of these expressions vanish when $m=n$.

Neglecting the second-order and higher terms, and retaining only the leading non-zero terms leaves

$$\lim_{y \rightarrow 0} \left\{ \int_{a_1}^{b_1} dx' \bar{k}_{1p}^{\pm(0)}(x') \cdot \int_{a_1}^{b_1} \frac{\partial \tilde{g}^e(x', 0 | x, y; -\zeta)}{\partial \zeta} \Big|_{\mp \zeta_{1p}^{(0)}} (\zeta \pm \zeta_{1p}^{(0)}) \cdot \bar{k}_{1p}^{\mp}(x) dx + \int_{a_2}^{b_2} dx' \bar{k}_{2p}^{\pm(0)}(x') \cdot \int_{a_1}^{b_1} \tilde{g}^e(x', 0 | x, y; \pm \zeta_{2p}^{(0)}) \cdot \bar{k}_{1p}^{\mp}(x) dx \right\} = 0 \quad (24)$$

$$\lim_{y \rightarrow 0} \left\{ \int_{a_1}^{b_1} dx' \vec{k}_{1p}^{\pm(0)}(x') \cdot \int_{a_2}^{b_2} \tilde{g}^e(x', 0 | x, y; \pm \zeta_{1p}^{(0)}) \cdot \vec{k}_{2p}^{\mp}(x) dx \right. \\ \left. + \int_{a_2}^{b_2} dx' \vec{k}_{2p}^{\pm(0)}(x') \cdot \int_{a_1}^{b_1} \frac{\partial \tilde{g}^e(x', 0 | x, y; -\zeta)}{\partial \zeta} \Big|_{\mp \zeta_{2p}^{(0)}} (\zeta \pm \zeta_{2p}^{(0)}) \cdot \vec{k}_{2p}^{\mp}(x) dx \right\} = 0. \quad (25)$$

Because the center strips are considered to be loosely coupled, the current distributions associated with the coupled system propagation modes are assumed to be nearly the same as the current distributions of a single center-strip system modified by multiplicative amplitude coefficients

$$\vec{k}_{np}^{\pm}(x) \approx a_n \vec{k}_{np}^{\pm(0)}. \quad (26)$$

Here the a_n are unknown amplitude coefficients. This relationship represents the basic perturbation approximation. Substituting (26) into expressions (24) and (25), and again applying the reciprocity property of the dyadic Green's function yields

$$\lim_{y \rightarrow 0} \left\{ a_1^{\pm} (\zeta \pm \zeta_{1p}^{(0)}) \int_{a_1}^{b_1} \left[\vec{k}_{1p}^{\mp(0)}(x) \cdot \int_{a_1}^{b_1} \frac{\partial \tilde{g}^e(x, y | x', h_1; \zeta)}{\partial \zeta} \Big|_{\mp \zeta_{1p}^{(0)}} \cdot \vec{k}_{1p}^{\mp(0)}(x') dx' \right] dx \right. \\ \left. + a_2^{\pm} \int_{a_2}^{b_2} \left[\vec{k}_{2p}^{\mp(0)}(x) \cdot \int_{a_1}^{b_1} \tilde{g}^e(x, y | x', h_1; \mp \zeta_{2p}^{(0)}) \cdot \vec{k}_{1p}^{\pm(0)}(x') dx' \right] dx \right\} = 0 \quad (27)$$

$$\lim_{y \rightarrow h_2} \left\{ a_1^{\pm} \int_{a_1}^{b_1} \left[\vec{k}_{1p}^{\mp(0)}(x) \cdot \int_{a_2}^{b_2} \tilde{g}^e(x, y | x', h_2; \mp \zeta_{1p}^{(0)}) \cdot \vec{k}_{2p}^{\mp}(x') dx' \right] dx \right. \\ \left. + a_2^{\pm} (\zeta \pm \zeta_{2p}^{(0)}) \int_{a_2}^{b_2} \left[\vec{k}_{2p}^{\mp(0)}(x) \cdot \int_{a_2}^{b_2} \frac{\partial \tilde{g}^e(x, y | x', h_2; \zeta)}{\partial \zeta} \Big|_{\mp \zeta_{2p}^{(0)}} \cdot \vec{k}_{2p}^{\mp}(x') dx' \right] dx \right\} = 0. \quad (28)$$

Now let

$$C_{11} = \lim_{y \rightarrow 0} \left\{ \int_{a_1}^{b_1} \left[\vec{k}_{1p}^{\mp(0)}(x) \cdot \int_{a_1}^{b_1} \frac{\partial \tilde{g}^e(x, y | x', 0; \zeta)}{\partial \zeta} \Big|_{\mp \zeta_{1p}^{(0)}} \cdot \vec{k}_{1p}^{\pm(0)}(x') dx' \right] dx \right\} \quad (29)$$

$$C_{21} = \lim_{y \rightarrow 0} \left\{ \int_{a_2}^{b_2} \left[\vec{k}_{2p}^{\mp(0)}(x) \cdot \int_{a_1}^{b_1} \tilde{g}^e(x, y | x', 0; \mp \zeta_{2p}^{(0)}) \cdot \vec{k}_{1p}^{\pm(0)}(x') dx' \right] dx \right\} \quad (30)$$

$$C_{12} = \lim_{y \rightarrow 0} \left\{ \int_{a_1}^{b_1} \left[\vec{k}_{1p}^{\mp(0)}(x) \cdot \int_{a_2}^{b_2} \tilde{g}^e(x, y | x', 0; \mp \zeta_{1p}^{(0)}) \cdot \vec{k}_{2p}^{\pm(0)}(x') dx' \right] dx \right\} \quad (31)$$

$$C_{22} = \lim_{y \rightarrow 0} \left\{ \int_{a_2}^{b_2} \left[\vec{k}_{2p}^{\mp(0)}(x) \cdot \int_{a_2}^{b_2} \frac{\partial \tilde{g}^e(x, y | x', 0; \zeta)}{\partial \zeta} \Big|_{\mp \zeta_{2p}^{(0)}} \cdot \vec{k}_{2p}^{\pm(0)}(x') dx' \right] dx \right\}. \quad (32)$$

Assuming that ζ_1 and ζ_2 are the isolated propagation eigenvalues of the coupled center strips, and applying relationships (29)-(32), the following matrix results

$$\begin{bmatrix} C_{11}(\zeta - \zeta_1) & C_{12} \\ C_{21} & C_{22}(\zeta - \zeta_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0. \quad (33)$$

It is also assumed that $a_1 \neq 0$ and $a_2 \neq 0$, therefore

$$\det \begin{bmatrix} C_{11}(\zeta - \zeta_1) & C_{12} \\ C_{21} & C_{22}(\zeta - \zeta_2) \end{bmatrix} = 0. \quad (34)$$

Equation (34) can be solved for ζ to determine the approximate perturbed propagation constant in terms of the isolated propagation constants, thus

$$C_{11}(\zeta - \zeta_1)C_{22}(\zeta - \zeta_2) - C_{21}C_{12} = 0 \quad (35)$$

which may be expressed

$$\zeta^2 - \zeta(\zeta_1 + \zeta_2) + \zeta_1\zeta_2 + \frac{C_{21}C_{12}}{C_{11}C_{22}} = 0. \quad (36)$$

Now letting

$$a = 1, \quad b = \zeta_1 + \zeta_2, \quad c = \zeta_1\zeta_2 + \frac{C_{21}C_{12}}{C_{11}C_{22}} \quad (37)$$

the solution to (36) is seen to be

$$\zeta = \frac{(\zeta_1 + \zeta_2) \pm \sqrt{(\zeta_1 - \zeta_2)^2 - 4\left(\zeta_1\zeta_2 + \frac{C_{21}C_{12}}{C_{11}C_{22}}\right)}}{2}. \quad (38)$$

However

$$\begin{aligned} (\zeta_1 + \zeta_2)^2 - 4\left(\zeta_1\zeta_2 + \frac{C_{21}C_{12}}{C_{11}C_{22}}\right) &= \zeta_1^2 + 2\zeta_1\zeta_2 + \zeta_2^2 - 4\zeta_1\zeta_2 + 4\frac{C_{21}C_{12}}{C_{11}C_{22}} \\ &= (\zeta_1 - \zeta_2)^2 + 4\frac{C_{21}C_{12}}{C_{11}C_{22}} \end{aligned} \quad (39)$$

therefore

$$\zeta = \frac{(\zeta_1 + \zeta_2)}{2} \pm \sqrt{\left[\frac{(\zeta_1 - \zeta_2)}{2}\right]^2 + \frac{C_{21}C_{12}}{C_{11}C_{22}}}. \quad (40)$$

Equation (40) simplifies further if both center strips have identical cross-sectional dimensions ($w_1 = w_2 = w$). In this case $C_{21} = C_{12}$, $C_{11} = C_{22}$, and $\zeta_1 = \zeta_2 = \zeta_o$, thus

$$\zeta = \frac{2\zeta_o}{2} \pm \sqrt{\frac{(C_{12})^2}{(C_{11})^2}} = \zeta_o \pm \left(\frac{C_{12}}{C_{11}} \right) \quad (41)$$

where ζ_o is the propagation constant of an isolated single center-strip system. Here the coefficient C_{11} will be referred to as the normalization term, and the coefficient C_{12} will be referred to as the coupling term.

6.3 Determination of Normalization and Coupling Terms

It can be seen that the perturbation propagation constant expression (41) requires determination of the normalization and coupling coefficients. It is assumed that currents are confined to the surfaces of the infinitely thin center strips. The coupled stripline structure supports modes which are either symmetric, or anti-symmetric, therefore

$$\vec{k}_{\alpha p}^{\pm}(x) = \hat{x}k_{\alpha px}(x) \pm \hat{z}k_{\alpha pz}(x). \quad (42)$$

For simplicity, the center strips are also assumed to be identical ($w_1=w_2=w$), therefore

$\vec{k}_{1p}(x) = \vec{k}_{2p}(x) = \vec{k}_p(x)$. As a result

$$\begin{aligned} \vec{g}^e(\vec{\rho}|\vec{\rho}'; \mp \zeta_p^{(0)}) \cdot \vec{k}^{\pm(0)}(x') &= \left[P.V. (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \vec{g}_{\zeta}(\vec{\rho}|\vec{\rho}') - \hat{y}\hat{y} \delta(\vec{\rho} - \vec{\rho}') \right] \cdot \left(\hat{x}k_{px}^{(0)}(x') \pm \hat{z}k_{pz}^{(0)}(x') \right) \\ &= \left\{ P.V. (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \left[(\hat{x}\hat{x} + \hat{z}\hat{z}) g_{\zeta}^x(\vec{\rho}|\vec{\rho}') + \hat{y}\hat{y} g_{\zeta}^y(\vec{\rho}|\vec{\rho}') \right. \right. \\ &\quad \left. \left. - \hat{y}\hat{y} \delta(\vec{\rho} - \vec{\rho}') \right] \right\} \cdot \left(\hat{x}k_{px}^{(0)}(x') \pm \hat{z}k_{pz}^{(0)}(x') \right) \end{aligned} \quad (43)$$

which leads to

$$\vec{g}^e(\vec{\rho}|\vec{\rho}'; \mp \zeta_p^{(0)}) \cdot \vec{k}^{\pm(0)}(x') = P.V. (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \left[\hat{x}g_{\zeta}^x(\vec{\rho}|\vec{\rho}')k_{px}^{(0)}(x') \pm \hat{z}g_{\zeta}^x(\vec{\rho}|\vec{\rho}')k_{pz}^{(0)}(x') \right]. \quad (44)$$

Substituting (44) into expression (29) for the coupling term gives

$$C_{12} = \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left[\hat{x} k_{px}^{(0)}(x) + \hat{z} k_{pz}^{(0)}(x) \right] \cdot \int_{a_2}^{b_2} \left\{ P.V. (k_o^2 + \tilde{\nabla} \tilde{\nabla} \cdot) \left[\hat{x} \tilde{g}_\zeta^\alpha(x, y | x', 0) \bar{k}_{px}^{(0)}(x') \right. \right. \\ \left. \left. + \hat{z} g_\zeta^\alpha(x, y | x', 0) k_{pz}^{(0)}(x') \right] \right\} dx' \Big] dx. \quad (45)$$

Here

$$\tilde{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} j \zeta \quad (46)$$

represents the transform-domain differential operator, and $\zeta_{np}^{(0)} = -\zeta_o$, $-\zeta_{np}^{(0)} = \zeta_o$, therefore

$$\tilde{\nabla} \cdot \hat{x} g_\zeta^\alpha(x, y | x', 0) k_{px}^{(0)}(x') = \frac{\partial}{\partial x} g_\zeta^\alpha(x, y | x', 0) k_{px}^{(0)}(x') \quad (47)$$

$$\tilde{\nabla} \cdot \hat{z} g_\zeta^\alpha(x, y | x', 0) k_{pz}^{(0)}(x') = j \zeta g_\zeta^\alpha(x, y | x', 0) k_{pz}^{(0)}(x') \quad (48)$$

and thus

$$\tilde{\nabla} \tilde{\nabla} \cdot \hat{x} g_\zeta^\alpha(x, y | x', 0) k_{px}^{(0)}(x') = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} j \zeta \right) \frac{\partial}{\partial x} g_\zeta^\alpha(x, y | x', 0) k_{px}^{(0)}(x') \quad (49)$$

$$\tilde{\nabla} \tilde{\nabla} \cdot \hat{z} g_\zeta^\alpha(x, y | x', 0) k_{pz}^{(0)}(x') = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} j \zeta \right) j \zeta g_\zeta^\alpha(x, y | x', 0) k_{pz}^{(0)}(x'). \quad (50)$$

Applying the relationships above to expression (45) gives

$$C_{12} = \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left[\left(\hat{x} k_{px}^{(0)}(x) - \hat{z} k_{pz}^{(0)}(x) \right) \cdot \int_{a_2}^{b_2} P.V. \left\{ \hat{x} k_o^2 \tilde{g}_{\zeta_o}^\alpha(x, y | x', 0) k_{px}^{(0)}(x') + \hat{z} k_o^2 g_{\zeta_o}^\alpha(x, y | x', 0) k_{pz}^{(0)}(x') \right. \right. \\ \left. \left. + \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} j \zeta_o \right) \left[\frac{\partial}{\partial x} g_{\zeta_o}^\alpha(x, y | x', 0) k_{px}^{(0)}(x') + j \zeta_o g_{\zeta_o}^\alpha(x, y | x', 0) k_{pz}^{(0)}(x') \right] \right\} dx' \right] dx \quad (51)$$

or

$$\begin{aligned}
C_{12} = \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ k_{px}^{(0)}(x) \int_{a_2}^{b_2} P.V. \left[k_o^2 g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{px}^{(0)}(x') + \frac{\partial^2}{\partial x^2} g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{px}^{(0)}(x') \right. \right. \\
\left. \left. + j \zeta_o \frac{\partial}{\partial x} g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{pz}^{(0)}(x') \right] dx' + k_{pz}^{(0)}(x) \int_{a_2}^{b_2} P.V. \left[-k_o^2 g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{pz}^{(0)}(x') \right. \right. \\
\left. \left. - j \zeta_o \frac{\partial}{\partial x} g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{px}^{(0)}(x') + \zeta_o^2 g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{pz}^{(0)}(x') \right] dx' \right\} dx. \quad (52)
\end{aligned}$$

Note that in equations (51) and (52) a direction for the axial current has been chosen in order to simplify the expressions. Because the center strips are identical, the cross terms integrate to the same values

$$\begin{aligned}
& \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} P.V. \left[j \zeta_o \frac{\partial}{\partial x} g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{pz}^{(0)}(x') \right] dx' \right\} k_{px}^{(0)}(x) dx \\
& = \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} P.V. \left[j \zeta_o \frac{\partial}{\partial x} g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{px}^{(0)}(x') \right] dx' \right\} k_{pz}^{(0)}(x) dx \quad (53)
\end{aligned}$$

and vanish from expression (52), therefore

$$\begin{aligned}
C_{12} = \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} P.V. \left[\left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{px}^{(0)}(x') \right] dx' \right\} k_{px}^{(0)}(x) dx \\
- \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} P.V. \left[(k_o^2 - \zeta_o^2) g_{\zeta_o}^{\alpha}(x, y | x', 0) k_{pz}^{(0)}(x') \right] dx' \right\} k_{pz}^{(0)}(x) dx. \quad (54)
\end{aligned}$$

For the sake of simplicity, only the case of coplanar center strips which lie in the plane of the origin ($h_1=h_2=0$) will be examined. Thus, for $y'=0$

$$\lim_{y \rightarrow 0} g_{\zeta_o}^p(x, y | x', 0) = \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi p_0(\vec{\lambda})} d\xi \quad (55)$$

and

$$\begin{aligned} \lim_{y \rightarrow 0} g_{\zeta_o}^{s\beta}(x, y | x', 0) &= \int_{-\infty}^{\infty} \frac{[e^{-p_0 4l} + e^{-p_0 4l} - e^{-p_0 2l} - e^{-p_0 2l}]}{4\pi p_0(\vec{\lambda})(1 - e^{-4p_0 l})} e^{j\xi(x-x')} d\xi \\ &= \int_{-\infty}^{\infty} \frac{2e^{-2p_0 l}(e^{-2p_0 l} - 1)}{4\pi p_0(\vec{\lambda})(1 - e^{-4p_0 l})} e^{j\xi(x-x')} d\xi \end{aligned} \quad (56)$$

therefore the total transform-domain Green's function is given by

$$\begin{aligned} \lim_{y \rightarrow 0} g_{\zeta_o}^{\beta}(x, y | x', 0) &= \int_{-\infty}^{\infty} \frac{[(1 - e^{-4p_0 l}) + 2e^{-2p_0 l}(e^{-2p_0 l} - 1)]}{4\pi p_0(\vec{\lambda})(1 - e^{-4p_0 l})} e^{j\xi(x-x')} d\xi \\ &= \int_{-\infty}^{\infty} \frac{e^{j\xi(x-x')}}{4\pi p_0(\vec{\lambda})} \left[\frac{1 - e^{-2p_0 l}}{1 + e^{-2p_0 l}} \right] d\xi . \end{aligned} \quad (57)$$

It is noted that $\frac{\partial^2}{\partial x^2} \rightarrow -\xi^2$, and therefore the coupling term may be expressed

$$\begin{aligned} C_{12} &= \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ (k_o^2 - \xi^2) \frac{(1 - e^{-2p_0 l})}{p_0(\vec{\lambda})(1 + e^{-2p_0 l})} \int_{-a_1}^{a_1} k_{px}^{(0)}(x) e^{j\xi x} dx \int_{-a_2}^{a_2} k_{px}^{(0)}(x') e^{-j\xi x'} dx' \right. \\ &\quad \left. - (k_o^2 - \zeta_o^2) \frac{(1 - e^{-2p_0 l})}{p_0(\vec{\lambda})(1 + e^{-2p_0 l})} \int_{-a_1}^{a_1} k_{pz}^{(0)}(x) e^{j\xi x} dx \int_{-a_2}^{a_2} k_{pz}^{(0)}(x') e^{-j\xi x'} dx' \right\} d\xi . \end{aligned} \quad (58)$$

Now the normalization term C_{11} must be determined. It is seen that

$$\begin{aligned}
\left. \frac{\partial \tilde{g}^e(\vec{\rho} | \vec{\rho}'; \zeta_p)}{\partial \zeta} \right|_{\vec{\tau} \zeta_o} \cdot \vec{k}^{\pm(0)}(x') &= \frac{\partial}{\partial \zeta} \left[P.V. (k_o^2 + \vec{\nabla} \vec{\nabla} \cdot) \tilde{g}_{\zeta_o}(\vec{\rho} | \vec{\rho}') \right. \\
&\quad \left. - \hat{y} \hat{y} \delta(\vec{\rho} - \vec{\rho}') \right] \cdot \left(\hat{x} k_{px}^{(0)}(x') \pm \hat{z} k_{pz}^{(0)}(x') \right) \\
&= \frac{\partial}{\partial \zeta} \left\{ P.V. (k_o^2 + \vec{\nabla} \vec{\nabla} \cdot) \left[\hat{x} g_{\zeta_o}^a(\vec{\rho} | \vec{\rho}') k_{px}^{(0)}(x') \pm \hat{z} g_{\zeta_o}^a(\vec{\rho} | \vec{\rho}') k_{pz}^{(0)}(x') \right] \right\} \\
&\quad (59)
\end{aligned}$$

therefore

$$\begin{aligned}
C_{11} &= \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left[(\hat{x} k_{px}^{(0)}(x) \mp \hat{z} k_{pz}^{(0)}(x)) \cdot \int_{a_1}^{b_1} \frac{\partial}{\partial \zeta} \left\{ P.V. (k_o^2 + \vec{\nabla} \vec{\nabla} \cdot) \left[\hat{x} g_{\zeta_o}^a(x, y | x', 0) k_{px}^{(0)}(x') \right. \right. \right. \\
&\quad \left. \left. \left. \pm \hat{z} g_{\zeta_o}^a(x, y | x', 0) k_{pz}^{(0)}(x') \right] \right\} dx' \right] dx \\
&= \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} \frac{\partial}{\partial \zeta} \left[P.V. \left\{ \left(\frac{\partial^2}{\partial x^2} + k_o^2 \right) g_{\zeta_o}^a(x, y | x', 0) k_{px}^{(0)}(x') \right\} dx' \right] k_{px}^{(0)}(x) \right\} dx \\
&\quad - \lim_{y \rightarrow 0} \int_{a_1}^{b_1} \left\{ \int_{a_2}^{b_2} \frac{\partial}{\partial \zeta} \left[P.V. \left\{ (k_o^2 - \zeta_o^2) g_{\zeta_o}^a(x, y | x', 0) k_{pz}^{(0)}(x') \right\} dx' \right] k_{pz}^{(0)}(x) \right\} dx \\
&\quad (60)
\end{aligned}$$

where again a direction for the axial current has been chosen in order to simplify the expression. Substituting (57) into expression (60) and noting that $\frac{\partial^2}{\partial^2 x} \rightarrow -\xi^2$, gives

$$\begin{aligned}
C_{11} &= \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \zeta} \left[(k_o^2 - \xi^2) \frac{(1 - e^{-2p_o d})}{p_o(\vec{\lambda})(1 + e^{-2p_o d})} \int_{a_1}^{b_1} k_{px}^{(0)}(x) e^{j\xi x} dx \int_{a_1}^{b_1} k_{px}^{(0)}(x') e^{-j\xi x'} dx' \right] \right. \\
&\quad \left. - \frac{\partial}{\partial \zeta} \left[(k_o^2 - \zeta_o^2) \frac{(1 - e^{-2p_o d})}{p_o(\vec{\lambda})(1 + e^{-2p_o d})} \int_{a_1}^{b_1} k_{pz}^{(0)}(x) e^{j\xi x} dx \int_{a_1}^{b_1} k_{pz}^{(0)}(x') e^{-j\xi x'} dx' \right] \right\} d\xi. \\
&\quad (61)
\end{aligned}$$

After a great deal of effort it can be seen that

$$\frac{\partial}{\partial \zeta} \left[\frac{(1 - e^{-2p_o l})}{p_o(\vec{\lambda})(1 + e^{-2p_o l})} \right] = \frac{\zeta [4lp_o(\vec{\lambda})e^{-2p_o l} - (1 - e^{-4p_o l})]}{p_o(\vec{\lambda})[p_o(\vec{\lambda})(1 + e^{-2p_o l})]^2} \quad (62)$$

and after an even greater amount of effort, it is determined that

$$\begin{aligned} \frac{\partial}{\partial \zeta} \left[(k_o^2 - \zeta^2) \frac{(1 - e^{-2p_o l})}{p_o(\vec{\lambda})(1 + e^{-2p_o l})} \right] \\ = \frac{(k_o^2 - \zeta^2) \zeta [4p_o(\vec{\lambda})le^{-2p_o l} - (1 - e^{-4p_o l})] - 2\zeta p_o(\vec{\lambda})^2(1 - e^{-4p_o l})}{p_o(\vec{\lambda})[p_o(\vec{\lambda})(1 + e^{-2p_o l})]^2} . \end{aligned} \quad (63)$$

As a result, the normalization term may be expressed

$$\begin{aligned} C_{11} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ M_{xx}^{11}(\zeta_o, \xi) \int_{a_1}^{b_1} k_{px}^{(0)}(x) e^{j\xi x} dx \int_{a_1}^{b_1} k_{px}^{(0)}(x') e^{-j\xi x'} dx' \right. \\ \left. + M_{zz}^{11}(\zeta_o, \xi) \int_{a_1}^{b_1} k_{pz}^{(0)}(x) e^{j\xi x} dx \int_{a_1}^{b_1} k_{pz}^{(0)}(x') e^{-j\xi x'} dx' \right\} d\xi \end{aligned} \quad (64)$$

where

$$\begin{aligned} M_{xx}^{11}(\zeta, \xi) &= \frac{\partial}{\partial \zeta} \left[(k_o^2 - \xi^2) \frac{(1 - e^{-2p_o l})}{p_o(\vec{\lambda})(1 + e^{-2p_o l})} \right] \\ &= \frac{\zeta [4lp_o(\vec{\lambda})e^{-2p_o l} - (1 - e^{-4p_o l})]}{p_o(\vec{\lambda})[p_o(\vec{\lambda})(1 + e^{-2p_o l})]^2} (k_o^2 - \xi^2) \end{aligned} \quad (65)$$

and

$$\begin{aligned}
M_{zz}^{11}(\zeta, \xi) &= \frac{\partial}{\partial \zeta} \left[(k_o^2 - \zeta^2) \frac{(1 - e^{-2p_o l})}{p_o(\vec{\lambda})(1 + e^{-2p_o l})} \right] \\
&= \frac{(k_o^2 - \zeta^2) \zeta \left[4p_o(\vec{\lambda}) l e^{-2p_o l} - (1 - e^{-4p_o l}) \right] - 2 \zeta p_o(\vec{\lambda})^2 (1 - e^{-4p_o l})}{p_o(\vec{\lambda}) \left[p_o(\vec{\lambda})(1 + e^{-2p_o l}) \right]^2}.
\end{aligned} \tag{66}$$

The coupling term may likewise be written

$$\begin{aligned}
C_{12} &= \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{j\xi(x_1 - x_2)} \left\{ M_{xx}^{12}(\zeta_o, \xi) \int_{a_1}^{b_1} k_{px}^{(0)}(x) e^{j\xi x} dx \int_{a_2}^{b_2} k_{px}^{(0)}(x') e^{-j\xi x'} dx' \right. \\
&\quad \left. + M_{zz}^{12}(\zeta_o, \xi) \int_{a_1}^{b_1} k_{pz}^{(0)}(x) e^{j\xi x} dx \int_{a_2}^{b_2} k_{pz}^{(0)}(x') e^{-j\xi x'} dx' \right\} d\xi
\end{aligned} \tag{67}$$

where

$$M_{xx}^{12}(\zeta, \xi) = (k_o^2 - \xi^2) \frac{(1 - e^{-2p_o l})}{p_o(\vec{\lambda})(1 + e^{-2p_o l})} \tag{68}$$

and

$$M_{zz}^{12}(\zeta, \xi) = (k_o^2 - \zeta^2) \frac{(1 - e^{-2p_o l})}{p_o(\vec{\lambda})(1 + e^{-2p_o l})}. \tag{69}$$

6.4 Transformation to Local Coordinates

Once again the numerical solution is facilitated by a shift to local coordinates.

As was done previously, this is accomplished through the change of variables

$$\bar{x}'_{\alpha} = x' - x_{\alpha} \tag{70}$$

and then $dx' = d\bar{x}'_\alpha$. This leads to a change in the associated limits of integration such that when $x' = x_\alpha - w_\alpha/2$, $\bar{x}'_\alpha = -w_\alpha/2$, and when $x' = x_\alpha + w_\alpha/2$, $\bar{x}'_\alpha = w_\alpha/2$. Also let

$$\bar{x}_\alpha = x - x_\alpha \quad (71)$$

and then $dx = d\bar{x}_\alpha$. In this case, when $x = x_\alpha - w_\alpha/2$, $\bar{x}_\alpha = -w_\alpha/2$ and when $x = x_\alpha + w_\alpha/2$, $\bar{x}_\alpha = w_\alpha/2$. This results in

$$\begin{aligned} C_{11} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ M_{xx}^{11}(\zeta_o, \xi) \int_{-w_1/2}^{w_1/2} k_{px}^{(0)}(\bar{x}_1) e^{j\xi(\bar{x}_1 + x_1)} d\bar{x}_1 \int_{-w_1/2}^{w_1/2} k_{px}^{(0)}(\bar{x}'_1) e^{-j\xi(\bar{x}'_1 + x_1)} d\bar{x}'_1 \right. \\ \left. + M_{zz}^{11}(\zeta_o, \xi) \int_{-w_1/2}^{w_1/2} k_{pz}^{(0)}(\bar{x}_1) e^{j\xi(\bar{x}_1 + x_1)} d\bar{x}_1 \int_{-w_1/2}^{w_1/2} k_{pz}^{(0)}(\bar{x}'_1) e^{-j\xi(\bar{x}'_1 + x_1)} d\bar{x}'_1 \right\} d\xi \end{aligned} \quad (72)$$

$$\begin{aligned} C_{12} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ M_{xx}^{12}(\zeta_o, \xi) \int_{-w_1/2}^{w_1/2} k_{px}^{(0)}(\bar{x}_1) e^{j\xi(\bar{x}_1 + x_1)} d\bar{x}_1 \int_{-w_2/2}^{w_2/2} k_{px}^{(0)}(\bar{x}'_2) e^{-j\xi(\bar{x}'_2 + x_2)} d\bar{x}'_2 \right. \\ \left. + M_{zz}^{12}(\zeta_o, \xi) \int_{-w_1/2}^{w_1/2} k_{pz}^{(0)}(\bar{x}_1) e^{j\xi(\bar{x}_1 + x_1)} d\bar{x}_1 \int_{-w_2/2}^{w_2/2} k_{pz}^{(0)}(\bar{x}'_2) e^{-j\xi(\bar{x}'_2 + x_2)} d\bar{x}'_2 \right\} d\xi \end{aligned} \quad (73)$$

but the center strips ($w_1 = w_2 = w$), and therefore

$$\begin{aligned} C_{11} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ M_{xx}^{11}(\zeta_o, \xi) \int_{-w/2}^{w/2} k_{px}^{(0)}(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \int_{-w/2}^{w/2} k_{px}^{(0)}(\bar{x}'_1) e^{-j\xi\bar{x}'_1} d\bar{x}'_1 \right. \\ \left. + M_{zz}^{11}(\zeta_o, \xi) \int_{-w/2}^{w/2} k_{pz}^{(0)}(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \int_{-w/2}^{w/2} k_{pz}^{(0)}(\bar{x}'_1) e^{-j\xi\bar{x}'_1} d\bar{x}'_1 \right\} d\xi \end{aligned} \quad (74)$$

$$C_{12} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{j\xi(x_1 - x_2)} \left\{ M_{xx}^{12}(\zeta_o, \xi) \int_{-w/2}^{w/2} k_{px}^{(0)}(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \int_{-w/2}^{w/2} k_{px}^{(0)}(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \right. \\ \left. + M_{zz}^{12}(\zeta_o, \xi) \int_{-w/2}^{w/2} k_{pz}^{(0)}(\bar{x}_1) e^{j\xi\bar{x}_1} d\bar{x}_1 \int_{-w/2}^{w/2} k_{pz}^{(0)}(\bar{x}_2') e^{-j\xi\bar{x}_2'} d\bar{x}_2' \right\} d\xi. \quad (75)$$

6.5 Method of Moments Solution

A numerical solution is performed by expanding the current functions such that

$$k_{\alpha\beta}(\bar{x}'_{\alpha}) = \sum_{n=1}^{N_{\alpha}} a_{\alpha\beta}^n e_{\alpha\beta}^n(\bar{x}'_{\alpha}) \quad (76)$$

where $e_{\alpha\beta}^n(\bar{x}'_{\alpha})$ is the basis set, and the $a_{\alpha\beta}^n$ represent unknown amplitude coefficients for $\alpha=1,2$, and $\beta=x,z$ [44]. As a result

$$\int_{-w_{\alpha}/2}^{w_{\alpha}/2} k_{\alpha\beta}(\bar{x}'_{\alpha}) e^{-j\xi\bar{x}'_{\alpha}} d\bar{x}'_{\alpha} = \sum_{n=1}^{N_{\alpha}} a_{\alpha\beta}^n \int_{-w_{\alpha}/2}^{w_{\alpha}/2} e_{\alpha\beta}^n(\bar{x}'_{\alpha}) e^{-j\xi\bar{x}'_{\alpha}} d\bar{x}'_{\alpha} = \sum_{n=1}^{N_{\alpha}} a_{\alpha\beta}^n f_{\alpha\beta}^n(\xi) \quad (77)$$

where

$$f_{\alpha\beta}^n(\xi) = \int_{-w_{\alpha}/2}^{w_{\alpha}/2} e_{\alpha\beta}^n(\bar{x}'_{\alpha}) e^{-j\xi\bar{x}'_{\alpha}} d\bar{x}'_{\alpha}. \quad (78)$$

In addition, let

$$g_{\alpha\beta}^m(\xi) = \int_{-w_{\alpha}/2}^{w_{\alpha}/2} t_{\alpha\beta}^m(\bar{x}_{\alpha}) e^{j\xi\bar{x}_{\alpha}} d\bar{x}_{\alpha} \quad (79)$$

where $t_{\alpha\beta}^m(\bar{x}_{\alpha})$ is a testing function which is to be applied later. Applying the relationships above to (74) and (75) results in

$$C_{11} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ M_{xx}^{11}(\zeta_o, \xi) \sum_{n=1}^N a_{1x}^n f_{1x}^n(\xi) \sum_{m=1}^N a_{1x}^m g_{1x}^m(\xi) \right. \\ \left. + M_{zz}^{11}(\zeta_o, \xi) \sum_{n=1}^N a_{1z}^n f_{1z}^n(\xi) \sum_{m=1}^N a_{1z}^m g_{1z}^m(\xi) \right\} d\xi \quad (80)$$

and

$$C_{12} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{j\xi(x_1 - x_2)} \left\{ M_{xx}^{12}(\zeta_o, \xi) \sum_{n=1}^N a_{2x}^n f_{2x}^n(\xi) \sum_{m=1}^N a_{1x}^m g_{1x}^m(\xi) \right. \\ \left. + M_{zz}^{12}(\zeta_o, \xi) \sum_{n=1}^N a_{2z}^n f_{2z}^n(\xi) \sum_{m=1}^N a_{1z}^m g_{1z}^m(\xi) \right\} d\xi. \quad (81)$$

Now the order of integration and summation are interchanged, to give

$$C_{11} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \left\{ \sum_{n=1}^{N_1} a_{1x}^n \sum_{m=1}^{N_1} a_{1x}^m \int_{-\infty}^{\infty} M_{xx}^{11}(\zeta_o, \xi) f_{1x}^n(\xi) g_{1x}^m(\xi) d\xi \right. \\ \left. + \sum_{n=1}^{N_1} a_{1z}^n \sum_{m=1}^{N_1} a_{1z}^m \int_{-\infty}^{\infty} M_{zz}^{11}(\zeta_o, \xi) f_{1z}^n(\xi) g_{1z}^m(\xi) d\xi \right\} \quad (82)$$

$$C_{12} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \left\{ \sum_{m=1}^{N_1} a_{1x}^m \sum_{n=1}^{N_2} a_{2x}^n \int_{-\infty}^{\infty} M_{xx}^{12}(\zeta_o, \xi) g_{1x}^m(\xi) f_{2x}^n(\xi) e^{j\xi(x_1 - x_2)} d\xi \right. \\ \left. + \sum_{m=1}^{N_1} a_{1z}^m \sum_{n=1}^{N_2} a_{2z}^n \int_{-\infty}^{\infty} M_{zz}^{12}(\zeta_o, \xi) g_{1z}^m(\xi) f_{2z}^n(\xi) e^{j\xi(x_1 - x_2)} d\xi \right\}. \quad (83)$$

For a Chebychev Galerkin's solution

$$e_{\alpha\beta}^n(x) = T_n(x/a) [1 - (x/a)^2]^{\pm 1/2} \quad (84)$$

and

$$t_{\alpha\beta}^m(x) = T_m(x/a) \left[1 - (x/a)^2 \right]^{\pm 1/2} \quad (85)$$

where $\alpha=1,2$, $a = w/2$, and the "+" sign is associated with $\beta=x$, and the "-" sign is associated with $\beta=z$ [45]. Chebyshev polynomials of even order are even, and those of odd order are odd such that

$$T_n(-x) = (-1)^n T_n(x). \quad (86)$$

As a result

$$f_{\alpha\beta}^n(\xi) = \int_{-w/2}^{w/2} e_{\alpha\beta}^n(x') e^{-j\xi x'} dx' = \begin{cases} 2 \int_0^{w/2} e_{\alpha\beta}^n(x') \cos \xi x' dx' & \dots \text{for } e_{\alpha\beta}(x') \text{ even} \\ -2j \int_0^{w/2} e_{\alpha\beta}^n(x') \sin \xi x' dx' & \dots \text{for } e_{\alpha\beta}(x') \text{ odd} \end{cases} \quad (87)$$

where $\alpha=1,2$, $\beta=x,z$. Likewise, with the testing operator

$$g_{\alpha\beta}^m(\xi) = \int_{-w/2}^{w/2} t_{\alpha\beta}^m(x) e^{j\xi x} dx = \begin{cases} 2 \int_0^{w/2} t_{\alpha\beta}^m(x) \cos \xi x dx & \dots \text{for } t_{\alpha\beta}(x) \text{ even} \\ 2j \int_0^{w/2} t_{\alpha\beta}^m(x) \sin \xi x dx & \dots \text{for } t_{\alpha\beta}(x) \text{ odd} \end{cases} \quad (88)$$

for $\alpha=1,2$, $\beta=x,z$.

It is seen from expressions (87) and (88) that four generic integral types are implicated. In Chapter 4 these generic integral forms were examined, and it was determined that

$$f_{\alpha x}^n(\xi) = \begin{cases} 2I_{\alpha x}^{ne}(\xi) & \dots \text{for } n \text{ even} \\ -2jI_{\alpha x}^{no}(\xi) & \dots \text{for } n \text{ odd} \end{cases} \quad (89)$$

$$f_{\alpha z}^n(\xi) = \begin{cases} 2I_{\alpha z}^{ne}(\xi) & \dots \text{ for } n \text{ even} \\ -2jI_{\alpha z}^{no}(\xi) & \dots \text{ for } n \text{ odd} \end{cases} \quad (90)$$

$$g_{\alpha x}^m(\xi) = \begin{cases} 2I_{\alpha x}^{me}(\xi) & \dots \text{ for } m \text{ even} \\ 2jI_{\alpha x}^{mo}(\xi) & \dots \text{ for } m \text{ odd} \end{cases} \quad (91)$$

and

$$g_{\alpha z}^m(\xi) = \begin{cases} 2I_{\alpha z}^{me}(\xi) & \dots \text{ for } m \text{ even} \\ 2jI_{\alpha z}^{mo}(\xi) & \dots \text{ for } m \text{ odd} \end{cases} \quad (92)$$

where

$$I_{\alpha z}^{ne} = (-1)^n \frac{\pi w}{4} J_{2n}\left(\xi \frac{w}{2}\right) \quad (93)$$

$$I_{\alpha z}^{no} = (-1)^n \frac{\pi w}{4} J_{2n+1}\left(\xi \frac{w}{2}\right) \quad (94)$$

$$I_{\alpha x}^{ne} = (-1)^n \frac{\pi w}{8} \left[J_{2n}\left(\xi \frac{w}{2}\right) + \frac{1}{2} \left(J_{2(n+1)}\left(\xi \frac{w}{2}\right) + J_{2|n-1|}\left(\xi \frac{w}{2}\right) \right) \right] \quad (95)$$

and

$$I_{\alpha x}^{no} = (-1)^n \frac{\pi w}{8} \left[J_{2n+1}\left(\xi \frac{w}{2}\right) + \frac{1}{2} \left(J_{2n+3}\left(\xi \frac{w}{2}\right) + J_{2n-1}\left(\xi \frac{w}{2}\right) \right) \right]. \quad (96)$$

Now let

$$A_{xx}^{11} = \int_{-\infty}^{\infty} M_{xx}^{11}(\zeta_o, \xi) f_{1x}^n(\xi) g_{1x}^m(\xi) d\xi \quad (97)$$

$$A_{zz}^{11} = \int_{-\infty}^{\infty} M_{zz}^{11}(\zeta_o, \xi) f_{1z}^n(\xi) g_{1z}^m(\xi) d\xi \quad (98)$$

$$A_{xx}^{12} = \int_{-\infty}^{\infty} M_{xx}^{12}(\zeta_o, \xi) g_{1x}^m(\xi) f_{2x}^n(\xi) e^{j\xi(x_1 - x_2)} d\xi \quad (99)$$

$$A_{zz}^{12} = \int_{-\infty}^{\infty} M_{zz}^{12}(\zeta_o, \xi) g_{1z}^m(\xi) f_{2z}^n(\xi) e^{j\xi(x_1 - x_2)} d\xi . \quad (100)$$

This leads to

$$C_{11} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \left\{ \sum_{n=1}^N a_{1x}^n \sum_{m=1}^N a_{1x}^m A_{xx}^{11} + \sum_{n=1}^N a_{1z}^n \sum_{m=1}^N a_{1z}^m A_{zz}^{11} \right\} \quad (101)$$

$$C_{12} = \lim_{y \rightarrow 0} \frac{1}{4\pi} \left\{ \sum_{n=1}^N a_{2x}^n \sum_{m=1}^N a_{1x}^m A_{xx}^{12} + \sum_{n=1}^N a_{2z}^n \sum_{m=1}^N a_{1z}^m A_{zz}^{12} \right\} \quad (102)$$

where the $a_{\alpha\beta}^{n,m}$ terms are determined during the solution of ζ_o .

It is seen that for even modes

$$A_{xx}^{11} = \int_{-\infty}^{\infty} M_{xx}^{11}(\zeta_o, \xi) (-2j) I_x^{no}(\xi) (2j) I_x^{mo}(\xi) d\xi = 4 \int_{-\infty}^{\infty} M_{xx}^{11}(\zeta_o, \xi) I_x^{no}(\xi) I_x^{mo}(\xi) d\xi \quad (103)$$

$$A_{zz}^{11} = \int_{-\infty}^{\infty} M_{zz}^{11}(\zeta_o, \xi) 2I_z^{ne}(\xi) 2I_z^{me}(\xi) d\xi = 4 \int_{-\infty}^{\infty} M_{zz}^{11}(\zeta_o, \xi) I_z^{ne}(\xi) I_z^{me}(\xi) d\xi \quad (104)$$

$$\begin{aligned}
A_{xx}^{12} &= \int_{-\infty}^{\infty} M_{xx}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} (2j) I_x^{mo}(\xi) (-2j) I_x^{no}(\xi) d\xi \\
&= 4 \int_{-\infty}^{\infty} M_{xx}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} I_x^{mo}(\xi) I_x^{no}(\xi) d\xi
\end{aligned} \tag{105}$$

and

$$\begin{aligned}
A_{zz}^{12} &= \int_{-\infty}^{\infty} M_{zz}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} 2I_z^{me}(\xi) 2I_z^*(\xi) d\xi \\
&= 4 \int_{-\infty}^{\infty} M_{zz}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} I_z^{me}(\xi) I_z^{ne}(\xi) d\xi .
\end{aligned} \tag{106}$$

Likewise, for odd modes

$$A_{xx}^{11} = \int_{-\infty}^{\infty} M_{xx}^{11}(\zeta_o, \xi) 2I_x^{ne}(\xi) 2I_x^{me}(\xi) d\xi = 4 \int_{-\infty}^{\infty} M_{xx}^{11}(\zeta_o, \xi) I_x^{ne}(\xi) I_x^{me}(\xi) d\xi \tag{107}$$

$$A_{zz}^{11} = \int_{-\infty}^{\infty} M_{zz}^{11}(\zeta_o, \xi) (-2j) I_z^{no}(\xi) (2j) I_z^{mo}(\xi) d\xi = 4 \int_{-\infty}^{\infty} M_{zz}^{11}(\zeta_o, \xi) I_z^{no}(\xi) I_z^{mo}(\xi) d\xi \tag{108}$$

$$\begin{aligned}
A_{xx}^{12} &= \int_{-\infty}^{\infty} M_{xx}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} 2I_x^{me}(\xi) 2I_x^{ne}(\xi) d\xi \\
&= 4 \int_{-\infty}^{\infty} M_{xx}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} I_x^{me}(\xi) I_x^{ne}(\xi) d\xi
\end{aligned} \tag{109}$$

$$\begin{aligned}
A_{zz}^{12} &= \int_{-\infty}^{\infty} M_{zz}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} (2j) I_z^{mo}(\xi) (-2j) 2I_z^{no}(\xi) d\xi \\
&= 4 \int_{-\infty}^{\infty} M_{zz}^{12}(\zeta_o, \xi) e^{j\xi(x_1 - x_2)} I_z^{mo}(\xi) I_z^{no}(\xi) d\xi .
\end{aligned} \tag{110}$$

Now it is noted

$$\begin{aligned}
 M_{xx}^{12}(\zeta_o, \xi) &= (k_o^2 - \xi^2) \frac{(1 - e^{-2p_0 l})}{p_0(\bar{\lambda})(1 + e^{-2p_0 l})} = (k_o^2 - \xi^2) \frac{(e^{p_0 l} - e^{-p_0 l})}{p_0(\bar{\lambda})(e^{p_0 l} + e^{-p_0 l})} \\
 &= (k_o^2 - \xi^2) \frac{\sinh(p_0 l)}{p_0(\bar{\lambda}) \cosh(p_0 l)}
 \end{aligned} \tag{111}$$

and likewise

$$M_{zz}^{12}(\zeta_o, \xi) = (k_o^2 - \zeta_o^2) \frac{(1 - e^{-2p_0 l})}{p_0(\bar{\lambda})(1 + e^{-2p_0 l})} = (k_o^2 - \zeta_o^2) \frac{\sinh(p_0 l)}{p_0(\bar{\lambda}) \cosh(p_0 l)} \tag{112}$$

$$\begin{aligned}
 M_{xx}^{11}(\zeta, \xi) &= \frac{\zeta \left[4l p_0(\bar{\lambda}) e^{-2p_0 l} - (1 - e^{-4p_0 l}) \right]}{p_0(\bar{\lambda}) \left[p_0(\bar{\lambda})(1 + e^{-2p_0 l}) \right]^2} (k_o^2 - \xi^2) \\
 &= \frac{4l \zeta_o (k_o^2 - \xi^2)}{\left[2p_0(\bar{\lambda}) \cosh(p_0 l) \right]^2} - \frac{2\zeta_o \sinh(p_0 l) (k_o^2 - \xi^2)}{p_0(\bar{\lambda}) \left[2p_0(\bar{\lambda}) \cosh(p_0 l) \right]^2}
 \end{aligned} \tag{113}$$

$$\begin{aligned}
 M_{zz}^{11}(\zeta_o, \xi) &= \frac{(k_o^2 - \zeta_o^2) \zeta_o \left[4p_0(\bar{\lambda}) l e^{-2p_0 l} - (1 - e^{-4p_0 l}) \right] - 2\zeta_o p_0(\bar{\lambda})^2 (1 - e^{-4p_0 l})}{p_0(\bar{\lambda}) \left[p_0(\bar{\lambda})(1 + e^{-2p_0 l}) \right]^2} \\
 &= \frac{4\zeta_o (k_o^2 - \zeta_o^2) l}{\left[2p_0(\bar{\lambda}) \cosh(p_0 l) \right]^2} - \frac{2\zeta_o (k_o^2 - \zeta_o^2) \sinh(p_0 l)}{p_0(\bar{\lambda}) \left[2p_0(\bar{\lambda}) \cosh(p_0 l) \right]^2} - \frac{4\zeta_o p_0(\bar{\lambda}) \sinh(p_0 l)}{\left[2p_0(\bar{\lambda}) \cosh(p_0 l) \right]^2}.
 \end{aligned} \tag{114}$$

Through careful examination of the symmetry of expressions (111)-(114), it is determined that, for even modes

$$A_{xx}^{11} = \lim_{y \rightarrow 0} 8 \int_0^\infty M_{xx}^{11}(\zeta_o, \xi) I_x^{no}(\xi) I_x^{mo}(\xi) d\xi \tag{115}$$

$$A_{zz}^{11} = \lim_{y \rightarrow 0} 8 \int_0^{\infty} M_{zz}^{11}(\zeta_o, \xi) I_z^{ne}(\xi) I_z^{me}(\xi) d\xi \quad (116)$$

$$A_{xx}^{12} = \lim_{y \rightarrow 0} 8 \int_0^{\infty} M_{xx}^{12}(\zeta_o, \xi) \cos[\xi(x_1 - x_2)] I_x^{mo}(\xi) I_x^{no}(\xi) d\xi \quad (117)$$

$$A_{zz}^{12} = \lim_{y \rightarrow 0} 8 \int_0^{\infty} M_{zz}^{12}(\zeta_o, \xi) \cos[\xi(x_1 - x_2)] I_z^{me}(\xi) I_z^{ne}(\xi) d\xi \quad (118)$$

and for odd modes

$$A_{xx}^{11} = \lim_{y \rightarrow 0} 8 \int_0^{\infty} M_{xx}^{11}(\zeta_o, \xi) I_x^{ne}(\xi) I_x^{me}(\xi) d\xi \quad (119)$$

$$A_{zz}^{11} = \lim_{y \rightarrow 0} 8 \int_0^{\infty} M_{zz}^{11}(\zeta_o, \xi) I_z^{no}(\xi) I_z^{mo}(\xi) d\xi \quad (120)$$

$$A_{xx}^{12} = 8 \int_0^{\infty} M_{xx}^{12}(\zeta_o, \xi) \cos[\xi(x_1 - x_2)] I_x^{me}(\xi) I_x^{ne}(\xi) d\xi \quad (121)$$

$$A_{zz}^{12} = \lim_{y \rightarrow 0} 8 \int_0^{\infty} M_{zz}^{12}(\zeta_o, \xi) \cos[\xi(x_1 - x_2)] I_z^{mo}(\xi) I_z^{no}(\xi) d\xi . \quad (122)$$

6.6 Analysis of Numerical Results

Propagation mode constants found using the perturbation method outlined in this chapter and the full-wave technique developed previously are presented in Figure 6.1.

Here the structure analyzed is a coplanar coupled stripline having $w_1 = w_2 = 5.0\text{cm}$,

$l=1.75\text{cm}$, and operated at 6.0 GHz (the same structure examined in Chapter 5). The mode splitting behavior of symmetric and antisymmetric modes is evident as the center-to-center distance between the center strips is varied. It is seen that the propagation constants found using the perturbation method differ slightly from those acquired using the full-wave technique (which are accepted as being more accurate), particularly as the distance between the center strips becomes small. It must be noted, however, that the amount of time and computational resources needed to generate the perturbation method results were several orders of magnitude less than those required for the full-wave technique.

The perturbation method may also be employed to avoid an additional shortcoming of the full-wave technique, namely that the associated root search requires an initial guess for the propagation constant. Due to the complex nature of the propagation mode spectra of coupled stripline structures, such initial values are sometimes difficult to determine. This difficulty may be overcome by using the perturbation method to provide initial guess values for the root search associated with the full-wave technique.

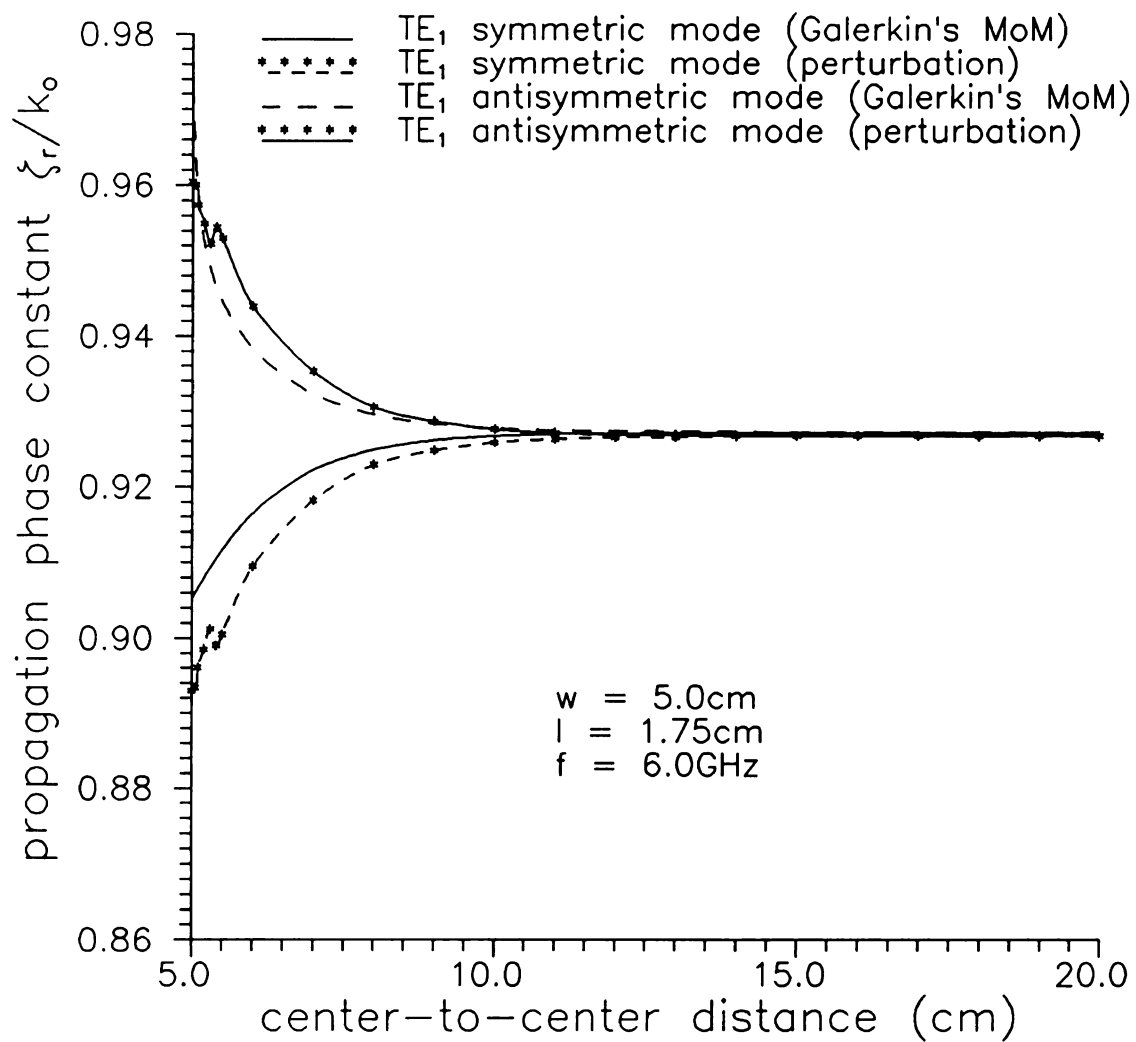


Figure 6.1 Comparison of propagation constants for symmetric and anti-symmetric modes obtained using full-wave, and perturbation methods.

Chapter 7

TIME DOMAIN TECHNIQUE FOR MEASUREMENT OF MATERIAL PROPERTIES USING STRIPLINE FIELD APPLICATORS

Stripline devices are widely used in measuring the electromagnetic properties of materials [52]-[58]. The stripline geometry lends itself well to analytic schemes by which information about sample materials placed in a field applicator can be extracted from measurements made at the applicator ports. Such measurements are facilitated by the fact that, if properly designed, stripline devices functioning in the principal TEM mode can be operated over wide frequency ranges. Additionally, effective transitions from the stripline structure to coaxial cable can be manufactured with relative ease. As a result, several materials characterization techniques employing stripline field applicators exist, and have been shown to yield good results. However, when used to measure the properties of materials which are heated to a high temperature, current characterization methods become cumbersome, and of questionable accuracy. Such measurements are necessary for the proper design of radar absorbing materials which are placed in areas subjected to high temperature, such as on the leading edges of aircraft wings, or near engine intakes, and exhaust ports.

Many of the difficulties associated with using stripline devices to make high temperature material measurements can be overcome by taking advantage of the inherent time-domain response of the stripline applicator [59]. The use of time-windowing techniques can reduce the number of measurements required for materials characterization, as well as produce insight into the workings of the stripline itself. This

chapter describes a time-domain technique used with stripline field applicators for electromagnetic materials characterization.

7.1 Advantages of Stripline Field Applicators

Materials characterization is typically performed by interrogating sample materials either using antennas in a free-field scattering apparatus, or by using some type of waveguide applicator. Often sample materials are machined and placed into sections of coaxial line, or within rectangular waveguides which are transitioned to coaxial connectors. The applicator connectors are then mated to coaxial cables which are attached to a measurement device such as a sampling oscilloscope, or automatic network analyzer (ANA).

Stripline field applicators (Figure 7.1, and Figure 7.2) are well suited for making materials measurements for several reasons. In order to obtain accurate results, sample materials must be machined so that good electrical contact with the walls of the waveguide applicator is maintained throughout the measurement process. When coaxial or closed rectangular waveguides are employed, electrical contact often cannot be maintained when the applicator and sample are heated, because of differing thermal expansion coefficients. Due to the planar cross-section of the stripline structure, sample materials can be firmly clamped into the applicator to prevent air gaps from forming between the sample and conducting surfaces during high temperature measurements. Sample materials are machined into simple shapes which interlock above and below the center conductor (Figure 7.3). Once samples are placed within the applicator they can be repositioned with relative ease.

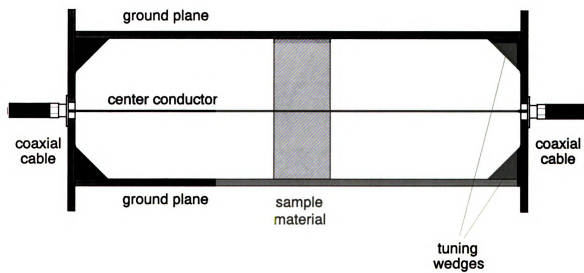


Figure 7.1 Side view of stripline field applicator.

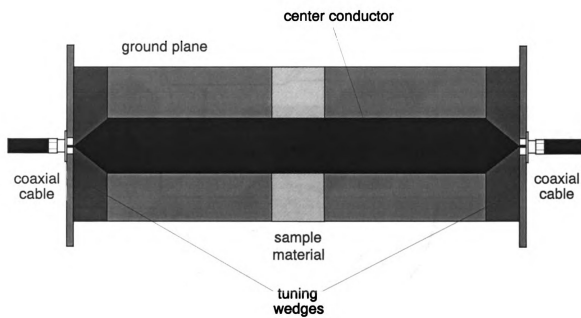


Figure 7.2. Top view of stripline field applicator (top ground plate removed).

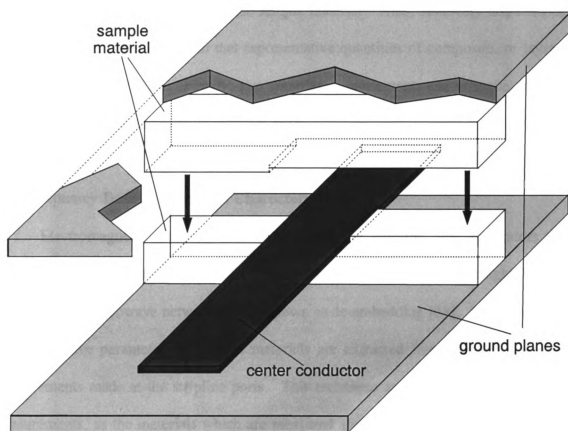


Figure 7.3. Placement of sample material in stripline field applicator.

Stripline applicators provide other advantages over different types of waveguides used for materials characterization. The stripline structure supports TEM modes with zero frequency cutoff, and if properly designed, can provide broadband frequency coverage (spectra of interest typically lie within the range from 100 MHz to 18 GHz). In addition, the stripline produces unidirectional field polarizations which are useful for identifying anisotropic behaviors of the sample material. Also, relatively large sample volumes may be interrogated, so that representative quantities of composite, or layered materials can be examined. Finally, two complex measurements can be made, because the applicator has two accessible terminal ports.

7.2 Frequency Domain Materials Characterization

Electromagnetic materials characterization using stripline field applicators was originally accomplished at Michigan State University through a straight-forward application of microwave network theory known as de-embedding [58]. In this process, the constitutive parameters of sample materials are extracted from frequency domain measurements made at the stripline ports. This technique accommodates two complex measurements, as the materials which are measured are likely to have properties which may be both dielectric and magnetic.

Transmission lines are normally characterized by measuring the response of a system to some combination of standard terminations such as short circuits, matched loads, or open circuits. Due to the structure of the stripline, scattering parameters can only be measured at the applicator ports. Additionally, the only practical type of standard termination available for characterization is a short circuit, which normally

consists of a pair of notched metallic blocks that interlock above and below the center conductor (again see Figure 7.3). Thus, characterization of the stripline applicator is accomplished using measurements of short-circuit terminations, and the empty applicator.

As seen in Figure 7.4, the stripline applicator may be modeled as three cascaded two-port microwave networks. Here, the area from the front coaxial connector port to the front of the sample material is referred to as transition region *a*, the area occupied by sample materials is referred to as the sample region *s*, and the area extending from the back of the sample material to the rear coaxial connector port is referred to as transition region *b*.

Using the de-embedding method, the forward scattering parameter associated with a sample material placed within the stripline applicator is determined to be

$$S_{11}^s = \frac{P - S_{11}^b Q^2}{1 - (S_{11}^b Q)^2} \quad (1)$$

and the through scattering parameter of the sample material is given by

$$S_{21}^s = \frac{Q(1 - S_{11}^b P)}{1 - (S_{11}^b Q)^2} \quad (2)$$

where

$$P = \frac{\Gamma_1^m - S_{11}^a}{(\Gamma_1^m - S_{11}^a)S_{22}^a + (S_{21}^a)^2} \quad (3)$$

and

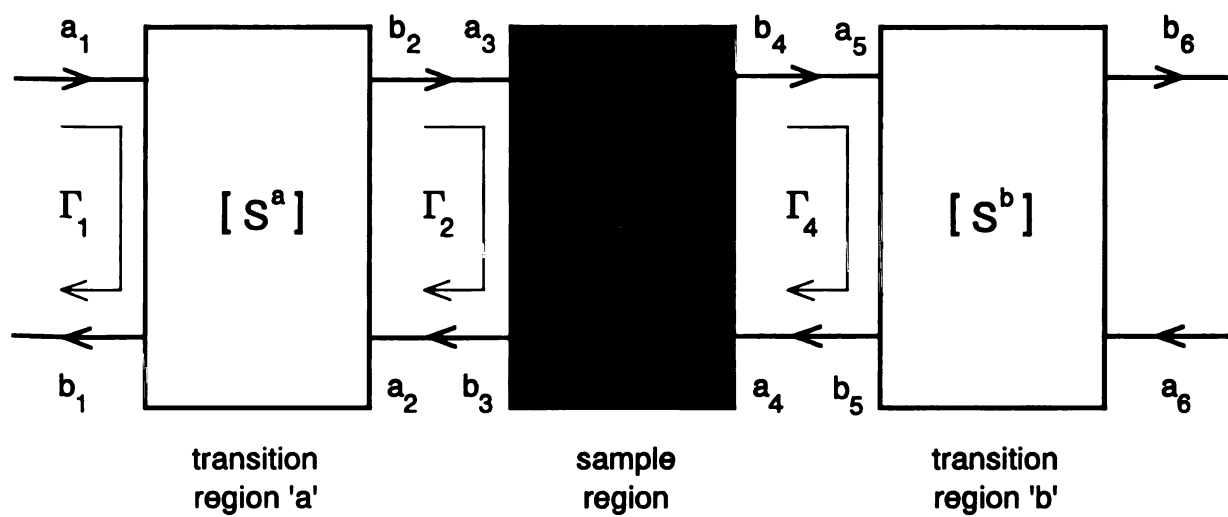


Figure 7.4. Cascaded two-port network representation of stripline applicator.

$$Q = S_{21}^m \frac{(1 - \Gamma_2 S_{22}^a)}{S_{21}^a S_{21}^b}. \quad (4)$$

Here the quantities Γ_1^m and S_{21}^m are measured by the ANA at the stripline applicator ports ($S_{11}^m = \Gamma_1^m$ when port 6 is match terminated).

It can be seen from (1)-(4) above that the expressions for the sample region scattering parameters require determination of unknown quantities associated with transition regions a , and b , these being S_{11}^a , $S_{12}^a S_{21}^a$, S_{22}^a , S_{11}^b , and $S_{21}^a S_{21}^b$. Three short-circuit measurements located at the sample material terminal plane ($l_1 = 0$), a known distance in front of the sample plane ($l_2 = d$), and a known distance behind the sample plane ($l_3 = -d$), provide calibration reflection coefficients

$$\Gamma_2^{ci} = -1e^{-2jk_0 l_i} \quad (5)$$

which are used to determine the necessary characteristics of transition region 'a', given by

$$S_{11}^a = \Gamma_1^{c1} - \frac{S_{12}^a S_{21}^a \Gamma_2^{c1}}{1 - S_{22}^a \Gamma_2^{c1}} \quad (6)$$

$$S_{12}^a S_{21}^a = (\Gamma_1^{c2} - \Gamma_1^{c1}) \frac{(1 - S_{22}^a \Gamma_2^{c1})(1 - S_{22}^a \Gamma_2^{c2})}{\Gamma_2^{c2} - \Gamma_2^{c1}} \quad (7)$$

and

$$S_{22}^a = \frac{K^c - 1}{K^c \Gamma_2^{c3} - \Gamma_2^{c2}} \quad (8)$$

where

$$K^c = \frac{(\Gamma_1^{c3} - \Gamma_1^{c1}) (\Gamma_2^{c2} - \Gamma_2^{c1})}{(\Gamma_1^{c2} - \Gamma_1^{c1}) (\Gamma_2^{c3} - \Gamma_2^{c1})}. \quad (9)$$

The remaining transition region characteristics required for de-embedding S_{11}^b , and $S_{21}^a S_{21}^b$, are determined from measurements of the empty stripline applicator. An air-filled sample region is reciprocal and symmetric, thus

$$S_{11}^s = S_{22}^s = 0 \quad (10)$$

and

$$S_{12}^s = S_{21}^s = e^{-jk_o l_s} \quad (11)$$

where l_s is the length of the sample region. Utilizing condition (11), it is determined that S_{11}^b is related to the reflection off of the front of the empty sample region, defined as Γ_2^e , modified by a phase shift factor, or

$$S_{11}^b = \Gamma_4 = \Gamma_2^e e^{2jk_o l_s}. \quad (12)$$

The through measurement of the empty applicator is given by

$$S_{21}^e = \frac{S_{21}^a S_{21}^b}{1 - S_{22}^a \Gamma_2^e} e^{-jk_o l_s} \quad (13)$$

and from equation (13), $S_{21}^a S_{21}^b$ can be found as

$$S_{21}^a S_{21}^b = S_{21}^e (1 - S_{22}^a \Gamma_2^e) e^{jk_o l_s}. \quad (14)$$

Once the sample region S-parameters have been obtained, the material constitutive

parameters are determined using the Nicholson-Ross-Weir (NRW) technique [60], [61].

According to this technique, the sample region S-parameters are represented by

$$S_{11}^s = \frac{\Gamma [1 - (e^{-j\beta l_s})^2]}{1 - (\Gamma e^{-j\beta l_s})^2} \quad (15)$$

and

$$S_{21}^s = \frac{e^{-j\beta l_s} (1 - \Gamma^2)}{1 - (\Gamma e^{-j\beta l_s})^2} \quad (16)$$

where l_s is the length of the sample, β is the propagation constant for the fundamental TEM stripline mode, and Γ is the interfacial reflection coefficient at the front air/sample interface, given by

$$\Gamma = \frac{Z_s - Z_o}{Z_s + Z_o} \quad (17)$$

where Z_s is the impedance of the sample region when a sample material is present, and Z_o is the impedance of the empty stripline applicator. The expressions for the sample S-parameters (15) and (16) can be solved simultaneously to yield

$$e^{-j\beta l_s} = \frac{(S_{11}^s + S_{21}^s) - \Gamma}{1 - \Gamma (S_{11}^s + S_{21}^s)} \quad (18)$$

and

$$\Gamma = K \pm \sqrt{K^2 - 1} \quad (19)$$

where

$$K = \frac{1 + (S_{11}^s + S_{21}^s)(S_{11}^s - S_{21}^s)}{2S_{11}^s} . \quad (20)$$

Equating measured values of $\beta_m(\omega)$ and $\Gamma_m(\omega)$ to the corresponding theoretical values leads to a pair of equations which can be solved for the complex permittivity ϵ and the complex permeability μ

$$\beta(\epsilon, \mu, \omega) - \beta_m(\omega) = 0 \quad (21)$$

$$\Gamma(\epsilon, \mu, \omega) - \Gamma_m(\omega) = 0 . \quad (22)$$

It is assumed that measurements are made while the stripline applicator is operated such that only the fundamental TEM mode is excited. The propagation constant associated with this mode is given by

$$\beta = k = \frac{\omega}{c} \sqrt{\epsilon_r \mu_r} \quad (23)$$

where ϵ_r and μ_r are the relative permittivity and permeability of the sample region (respectively), and the TEM impedance of the stripline applicator is

$$Z = \eta f_g = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} f_g \quad (24)$$

where f_g is a purely geometrical factor that is dependent on the stripline cross-section.

Applying expressions (23) and (24) to (18) and (19) results in

$$\Gamma = \frac{\sqrt{\mu_r/\epsilon_r} - 1}{\sqrt{\mu_r/\epsilon_r} + 1} \quad (25)$$

which provides

$$\frac{\mu_r}{\epsilon_r} = \left(\frac{1+\Gamma}{1-\Gamma} \right)^2 = X \quad (26)$$

and

$$\ln(T) = -j\beta l_s \quad (27)$$

which provides

$$\epsilon_r \mu_r = - \left[\frac{c}{\omega l_s} \ln(T) \right]^2 = Y \quad (28)$$

where

$$T = \frac{(S_{11}^s + S_{21}^s) - \Gamma}{1 - \Gamma(S_{11}^s + S_{21}^s)} . \quad (29)$$

Consequently, the relative permittivity and permeability of the sample are determined to be

$$\epsilon_r = \sqrt{\frac{Y}{X}} , \quad \mu_r = \sqrt{XY} \quad (30)$$

or

$$\epsilon_r = \frac{\beta}{k_o} \left(\frac{1-\Gamma}{1+\Gamma} \right) \quad (31)$$

and

$$\mu_r = \frac{\beta}{k_o} \left(\frac{1+\Gamma}{1-\Gamma} \right) . \quad (32)$$

Thus it is seen that the de-embedding procedure requires seven frequency domain measurements, these being

1. S_{11} of the applicator with a sample material present
2. S_{21} of the applicator with a sample material present
3. S_{11} of the empty applicator
4. S_{21} of the empty applicator
- 5-7. S_{11} of the applicator with a short located at three different positions.

7.3 Disadvantages of Frequency Domain Technique

The frequency-domain de-embedding process has been used extensively, and has been shown to provide good results for a variety of sample materials. In spite of this, the frequency-domain technique has several disadvantages. The first is that the short-circuit termination standards must be placed with extreme accuracy, in order to obtain good results. The distances from the sample terminal plane in expression (5) must be precisely known, or errors will result in the calculated sample permittivity and permeability. When high temperature measurements are made, the positions of the standard terminations may be extremely difficult to determine, due to expansion of the stripline applicator.

The de-embedding procedure is made more difficult by the fact that the short-circuit terminations must be placed within one half wavelength of the sample terminal plane ($l = \pm \lambda/2$) in order to provide unique calibration standards. In addition, if any of the short-circuit terminations are located exactly one-half wavelength apart, the calibration constant of expression (9) becomes unity, making (8) indeterminate. Thus

proper positioning of the required three short-circuit terminations may become nearly impossible at high frequency.

An additional shortcoming of the de-embedding technique involves the number of required measurements. The repeated process of positioning the short-circuit terminations and sample materials can be time consuming. This impacts the accuracy of the process because the calibration of the ANA may drift in the time required for a complete set of measurements. High temperature measurements made using the de-embedding method may be particularly time consuming. After the sample-containing region of the stripline applicator is positioned inside a furnace, a certain period of time is required to heat the device to the proper temperature for measurement. Several hours may then be required for the applicator and sample to cool sufficiently for the next adjustment to be made. The placement of samples and movement of shorts requires a minimum of five such heating/cooling cycles.

In addition to limiting the accuracy of measurements, the multiple heating/cooling cycles required for high temperature measurements limit the life of the applicator and sample materials. The repeated heating and cooling eventually deforms the applicator and sample materials. These deformations result in the formation of air gaps between the sample and conductive surfaces of the applicator. Deformation of the applicator also affects the proper positioning of the three short-circuit terminations required for de-embedding.

7.4 Time Domain Materials Characterization

An alternate method of measuring the constitutive parameters of sample materials

that employs time windowing techniques can overcome some of the shortcomings of the frequency domain de-embedding technique. By taking advantage of the inherent time-domain response of the stripline applicator, the number of measurements required for materials characterization can be reduced.

Time-domain materials characterization is typically performed by interrogating sample materials with short duration EM pulses, and then monitoring the response with a sampling oscilloscope. The technique described in this chapter involves measurements of the stripline applicator which are first made in the frequency domain using an ANA. The Inverse Fast Fourier Transform (IFFT) of these measurements is then calculated. What results is a synthesized, time-domain response representing what would occur if a pulse of very short duration, containing all the frequencies present in the measured spectrum were launched into the applicator. This time-domain response consists of reflections from or transmissions through (depending on the measurement) the various structures inside the stripline applicator.

Unlike the frequency-domain de-embedding technique, the time-windowing technique requires only four measurements and, more importantly, only three applicator adjustments [59]. The required frequency-domain measurements are

1. S_{11} of the applicator with a sample material present
2. S_{21} of the applicator with a sample material present
3. S_{21} of the empty applicator
4. S_{11} of the applicator with a short located at the sample terminal plane.

To implement this technique, frequency domain measurements are transformed

to the time-domain through use of the IFFT. The resulting time-domain responses are windowed to remove unwanted reflections or transmissions. The processed time-domain signals are then transformed back to frequency-domain through the Fast Fourier Transform (FFT). From this, two pairs of equations in two unknowns are established, which can be solved to determine the scattering parameters associated with the unknown sample. The complex constitutive parameters (permittivity ϵ , and permeability μ) of the sample material are then calculated from these scattering parameters through the Nicholson-Ross-Weir method.

7.5 Materials Characterization Signal Processing Scheme

The time-windowing materials measurement scheme begins with making the required frequency-domain measurements at the stripline applicator ports. An ANA is calibrated to the ends of the attached coaxial cables, which are in turn connected to the terminals of the stripline field applicator. These frequency domain measurements are then processed to obtain the scattering parameters associated with the unknown sample material.

The first measurement examined is the response at the stripline applicator terminals of a short-circuit termination (S_{11}^{scf}) placed within the applicator (Figure 7.5). In order to reduce the magnitude of distortions due to the inverse Fourier transform process, a weighting function is applied to the raw frequency domain measurement. This is accomplished by multiplying the raw S_{11}^{scf} data by the weighting function so that the ends of the resulting frequency-domain response are made to approach zero. Many types of weighting functions are available. Here a Gaussian Modulated Cosine (GMC) taper

[62] weighting function is applied (Figure 7.6), and the weighted frequency-domain response is shown in Figure 7.7. This GMC weighting function has a spectrum

$$W_{GMC}(f) = \tau \left[e^{-\pi(f-f_c)\tau^2} + e^{-\pi(f+f_c)\tau^2} \right] \quad (33)$$

where f_c is the cosine frequency, and τ is a user-adjusted shape factor. These parameters must be defined so as to provide the proper shape for the weighted waveform. More will be discussed about this process in a later section.

The IFFT of the weighted S_{11}^{scf} is now calculated. What results is the synthesized time-domain reflection response of the stripline applicator (with the short-circuit termination in place) to a short duration EM pulse. Several reflections are usually visible in the resulting time-domain waveform (Figure 7.8). Applying the general rule of thumb that the speed of light in free space is approximately one foot per nanosecond, these reflections can easily be correlated with structures within the stripline applicator. A small reflection from the coax-to-stripline transition at the front of the stripline applicator is visible first. A much larger primary reflection from the short-circuit termination is seen several nanoseconds later. Finally, several secondary reflections from the front of the applicator and the short circuit are usually visible after the primary reflection from the short circuit.

In order to remove reflections from the front of the applicator, as well as any secondary reflections, a rectangular time window is applied to the synthesized time-domain response (Figure 7.9). The signal that remains consists only of that part of the synthesized pulse which travels through transition region a , reflects off of the short circuit, and then travels back through transition region a to the front stripline port.

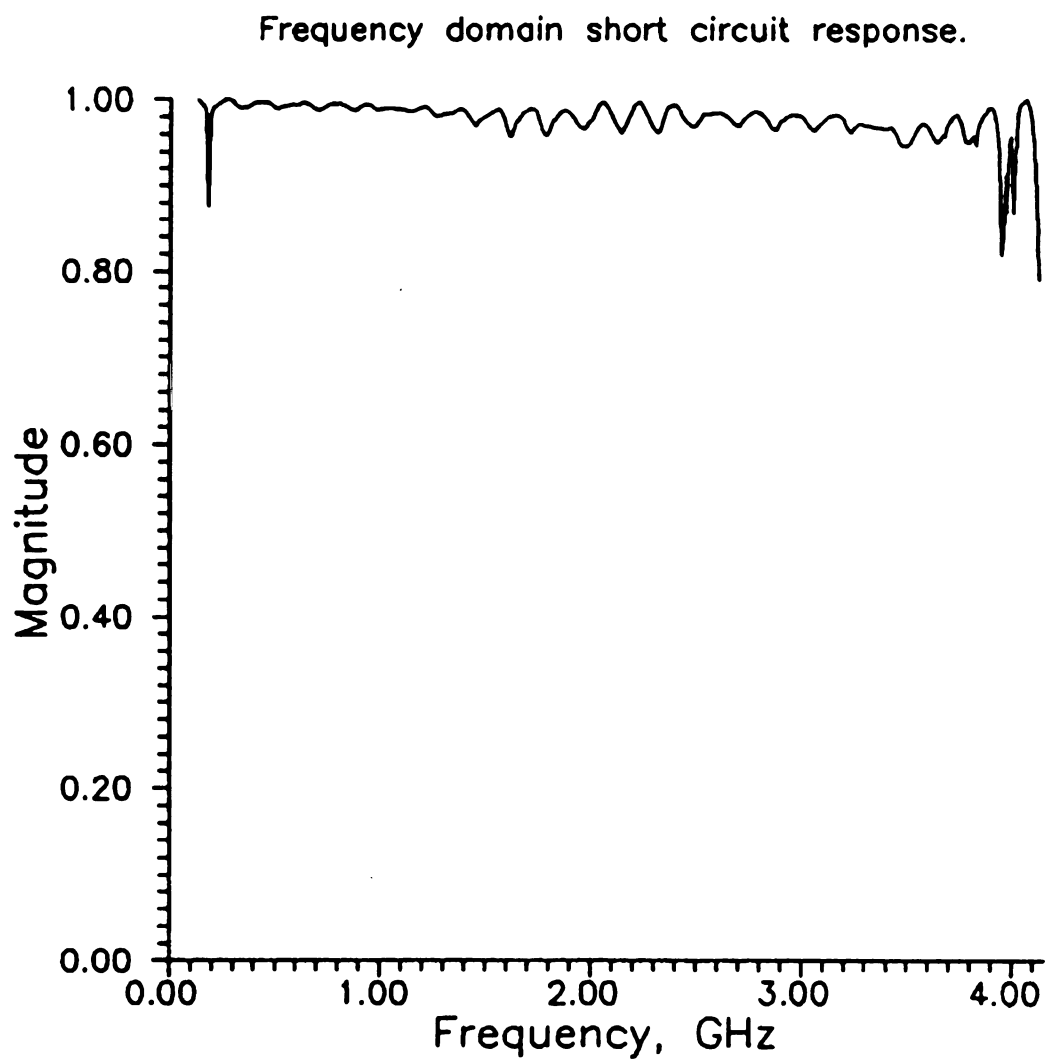


Figure 7.5. Frequency-domain response of short-circuit termination, measured at front applicator port.

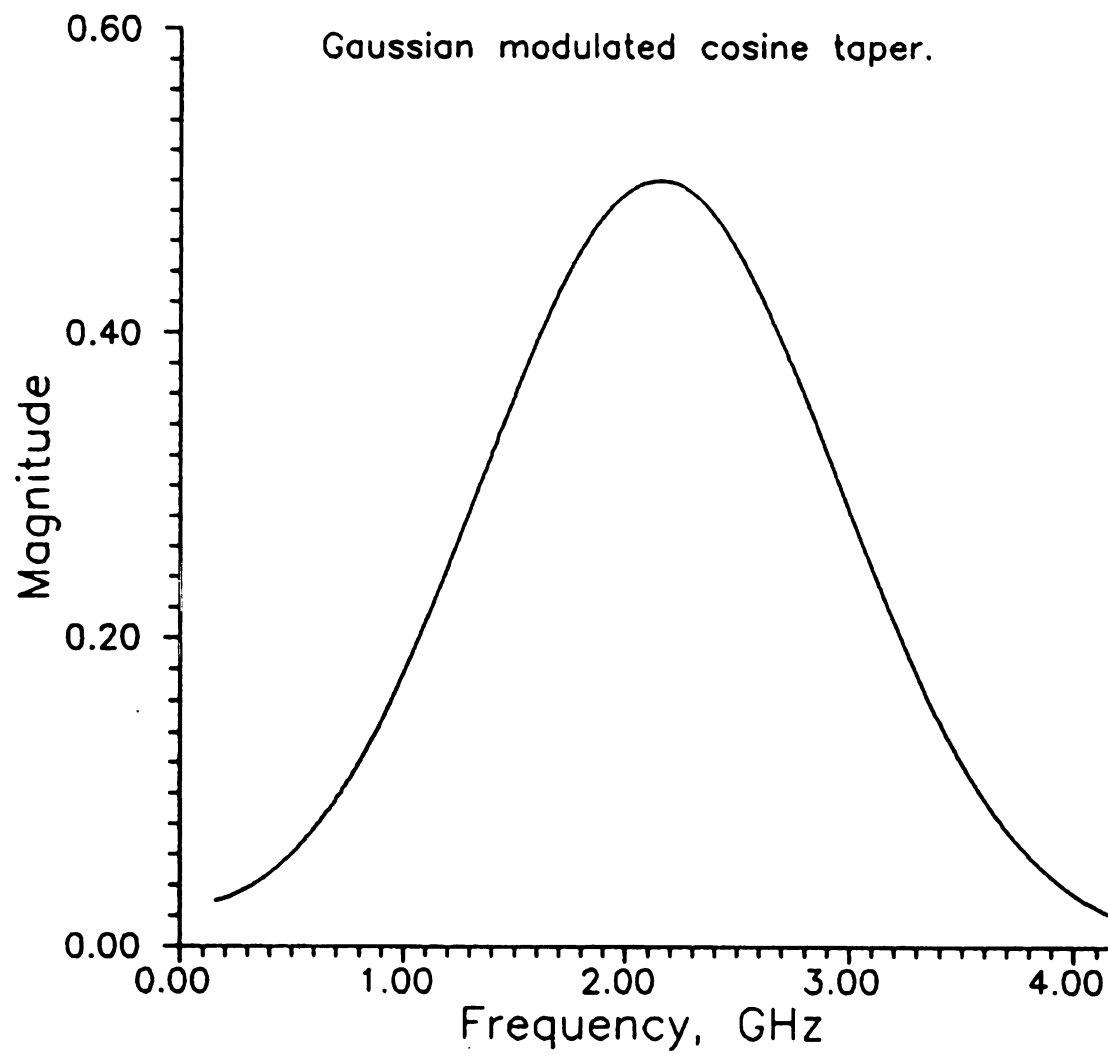


Figure 7.6. Gaussian Modulated Cosine weighting function.

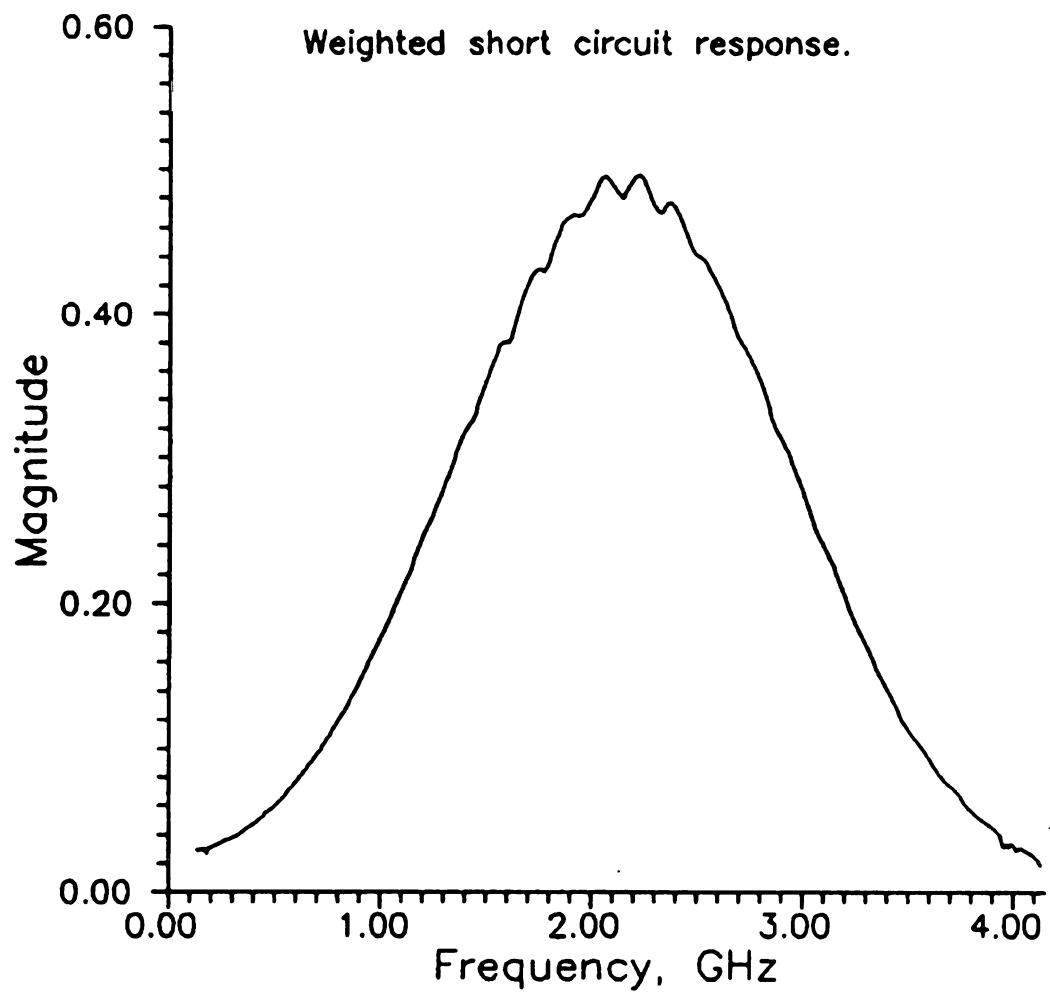


Figure 7.7 Weighted frequency-domain response of short-circuit termination.

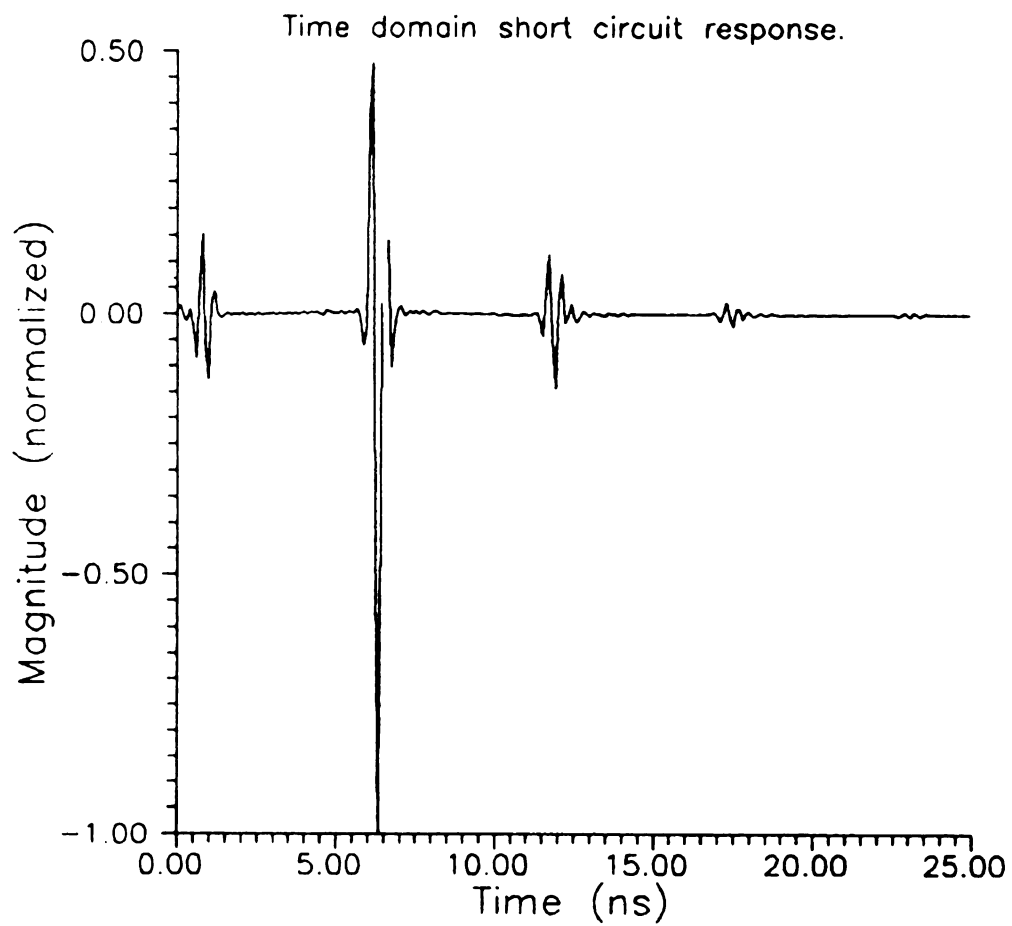


Figure 7.8. Time-domain response of short-circuit termination, measured at front applicator port.

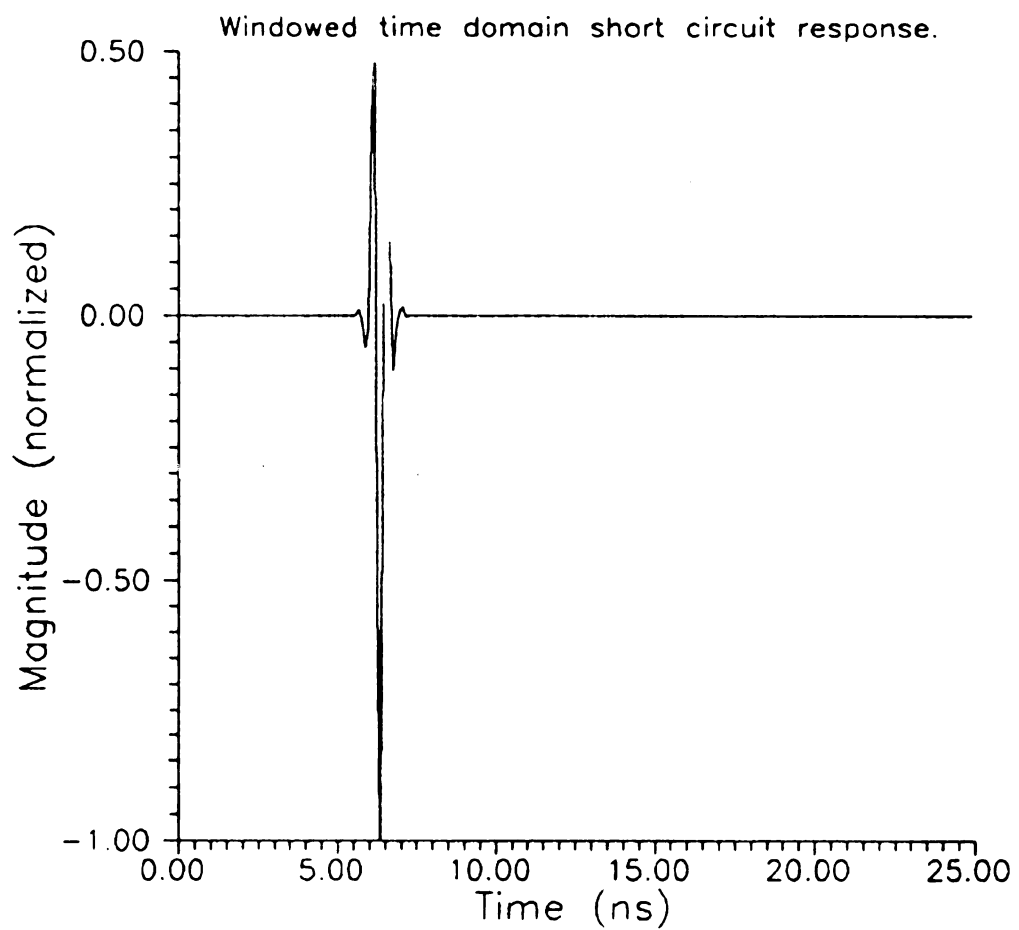


Figure 7.9. Windowed time-domain response of short-circuit termination, measured at front applicator port.

The FFT of this windowed time-domain response is then calculated, resulting in a processed frequency-domain response. As a final step, the weighting function is removed by dividing this processed data by the original GMC. The remaining signal consists of the forward scattering parameter associated with transition region a (S_{11}^a), multiplied by the reflection coefficient off of the short-circuit termination, multiplied by the backward scattering parameter through transition region a (S_{21}^a), or

$$R_1 = S_{12}^a S_{21}^a S_{11}^{short} = -S_{12}^a S_{21}^a \quad (34)$$

because the ideal response of the short-circuit termination is such that

$$S_{11}^{short} = -1. \quad (35)$$

The amplitude of R_1 is scaled by factor of -1, and the remaining data is designated as the reflection calibration

$$C_R = S_{12}^a S_{21}^a \quad (36)$$

which will be used to remove the effects of transition region a from the reflection sample measurement.

The terminal sample S_{11}^{smr} measurement is now examined. Again, in order to reduce the distortions caused by the Fourier transform, the GMC taper weighting function is applied to this data. The IFFT this calibrated sample S_{11}^{smr} is then calculated. Once again, a rectangular time window is used to remove reflections from the transition regions. The FFT of this windowed time-domain data is calculated, and the weighting function is removed from the resulting frequency-domain response. The remaining data consists of the frequency response of signals which travel forward through transition

region a (S_{11}^a), reflect off of the sample material (S_{11}^s), and then travel back through transition region a (S_{21}^a), or

$$R_2 = S_{12}^a S_{21}^a S_{11}^s. \quad (37)$$

When this reflection measurement is divided by the reflection calibration (36), what remains is the desired sample region S-parameter S_{11}^s

$$\frac{R_2}{C_R} = \frac{S_{12}^a S_{21}^a S_{11}^s}{S_{12}^a S_{21}^a} = S_{11}^s. \quad (38)$$

Now the through scattering parameter of the sample S_{21}^s must be determined. A weighting function is applied to the terminal frequency domain measurement of the empty stripline applicator S_{21}^{te} , and the IFFT of this weighted data is calculated. A rectangular time window is applied to the resulting signal to remove any secondary transmissions that might occur. Then the FFT of this data is calculated, and the weighting function is removed. The resulting data consists of the frequency response of signals which travel forward through transition region a (S_{21}^a), experience a phase shift through the empty sample region ($e^{-jk_o l_s}$), and then travel forward through transition region b (S_{21}^b), or

$$T_1 = S_{21}^a S_{21}^b e^{-jk_o l_s}. \quad (39)$$

This processed transmission measurement is divided by the phase factor $e^{-jk_o l_s}$ to remove the effects of the empty sample region. The remaining data is designated the transmission calibration, given by

$$C_T = \frac{S_{21}^a S_{21}^b e^{-jk_o l_s}}{e^{-jk_o l_s}} = S_{21}^a S_{21}^b \quad (40)$$

and will be used to remove the effects of transition region a , and transition region b from the terminal sample transmission measurement.

Now the final terminal frequency-domain measurement S_{21}^{mt} is examined. As with the previous waveforms, the GMC taper is applied to this data, and the IFFT of the weighted sample S_{21}^{mt} measurement is calculated. A rectangular time window is applied to remove any secondary transmissions which may be present in the resulting time-domain signal. The FFT of the windowed time-domain waveform is calculated, and the weighting function is removed from the resulting frequency-domain data. What remains is the frequency-domain response of signals which travel forward through transition region a (S_{21}^a), propagate through the sample material (S_{21}^m), and then travel forward through transition region b (S_{21}^b), given by

$$T_2 = S_{21}^a S_{21}^b S_{21}^s. \quad (41)$$

Now the expression (41) is divided by the transmission calibration factor (40), giving

$$\frac{T_2}{C_T} = \frac{S_{21}^a S_{21}^b S_{21}^s}{S_{21}^a S_{21}^b} = S_{21}^s \quad (42)$$

which yields the desired sample-region S-parameter S_{21}^s .

The constitutive parameters of the sample material may now be determined from scattering parameters (38) and (42), using the Nicholson-Ross-Weir method as discussed in the previous section (see Figure 7.10, Figure 7.11). Comparisons of the complex

permittivity and permeability of plexiglas and teflon samples obtained using both the de-embedding and time-windowing techniques are shown in Figure 7.12, Figure 7.13, Figure 7.14, and Figure 7.15.

In practice, the procedure outlined above can be simplified by combining several of the steps. The raw sample terminal S_{11}^{sm} measurement can be immediately divided by the reflection calibration factor, and then processed. Also the raw sample through measurement S_{21}^{mt} can also be immediately divided by the transmission calibration factor and then processed. In addition to shortening the time required to process the data, this tends to increase the accuracy of the measurement process, because in each case a Fourier transform is eliminated (anytime data is transformed information is inevitably lost).

7.6 Application of Fourier Transforms and Weighting Functions

The materials characterization procedure outlined in the sections above involves the use of the FFT. The Fourier transform, when used as a signal processing tool, must always be applied with caution, for data can be easily lost or distorted if transform/inverse transform procedures are performed improperly. Of particular significance is the fact that the Fourier transform assumes that the signal being processed is infinite in extent. However, the actual data collected using any real measurement scheme only covers a finite frequency spectrum. The measured data begins abruptly at some start frequency, and stops abruptly at some end frequency. In the time-windowing materials characterization method discussed in this chapter, the finite extent of the measured frequency data results in distortions in the calculated time-domain response.

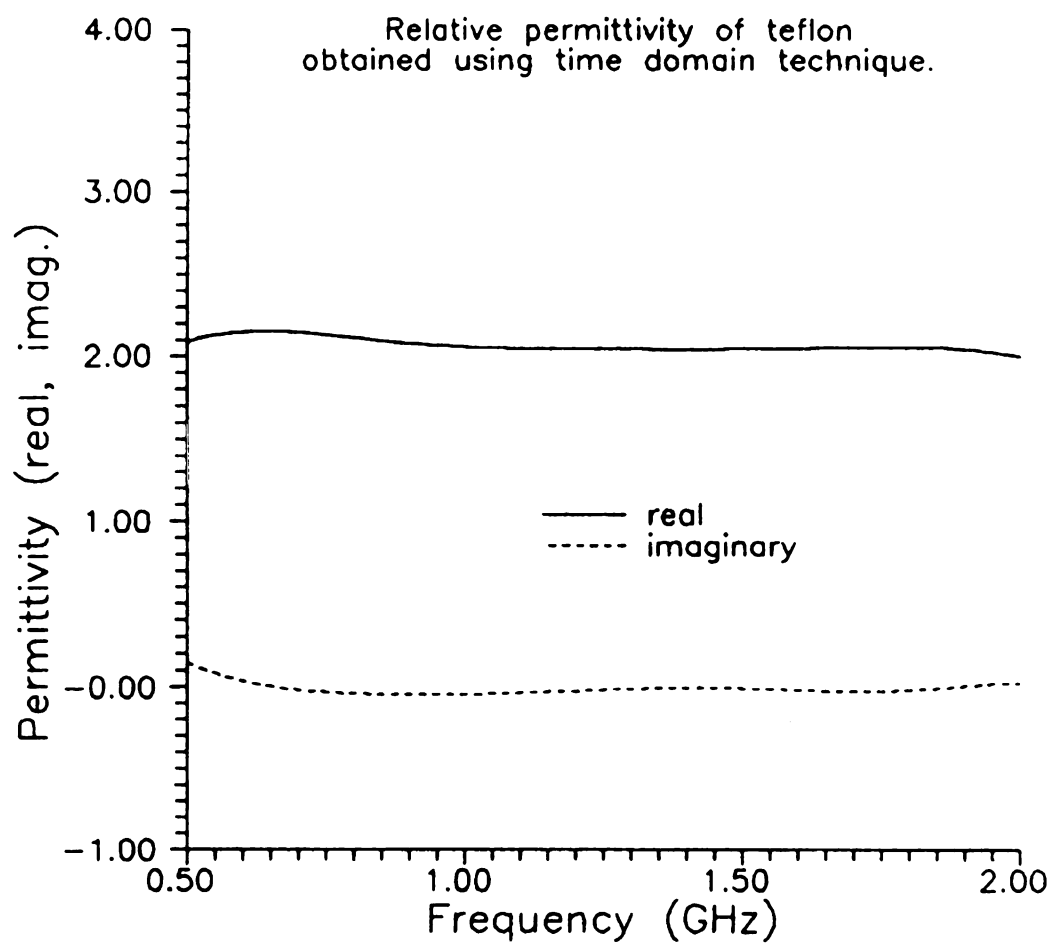


Figure 7.10. Relative permittivity of teflon sample obtained using stripline time-domain technique.

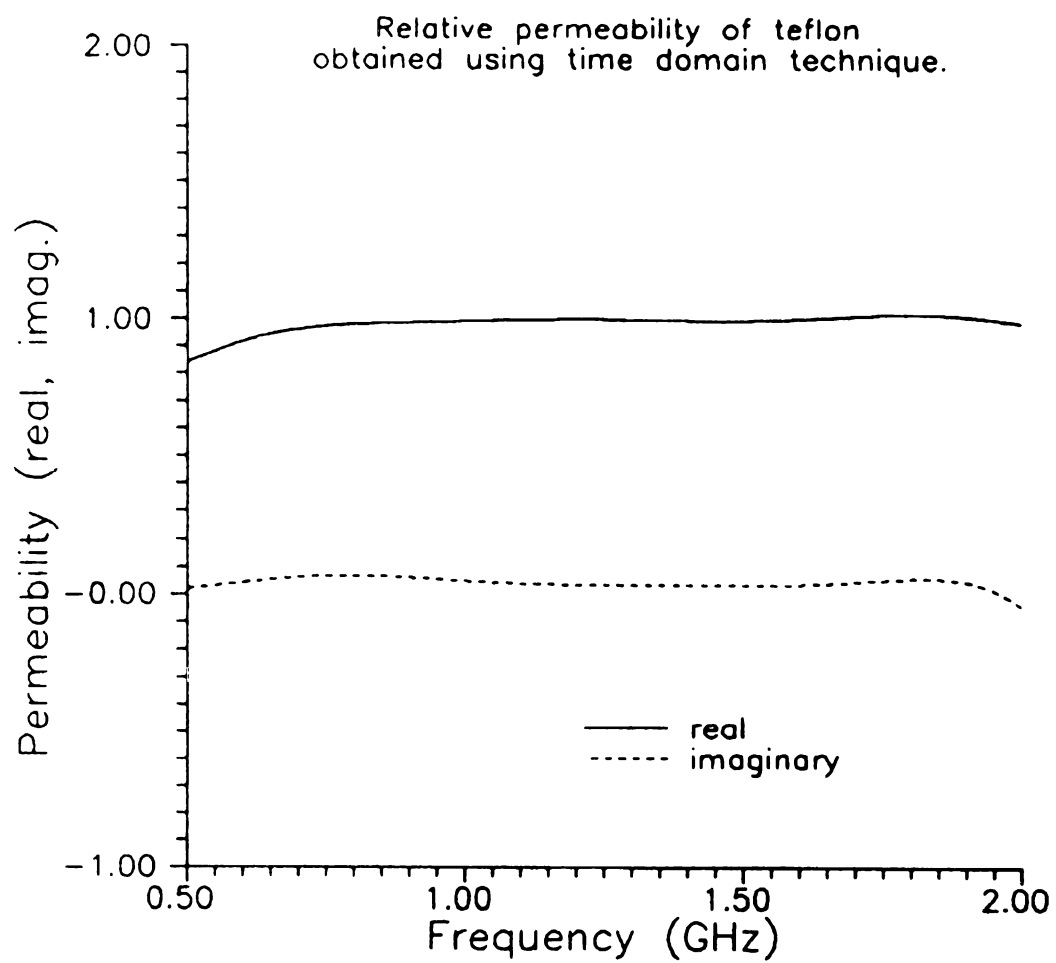


Figure 7.11. Relative permeability of teflon sample obtained using stripline time-domain technique.

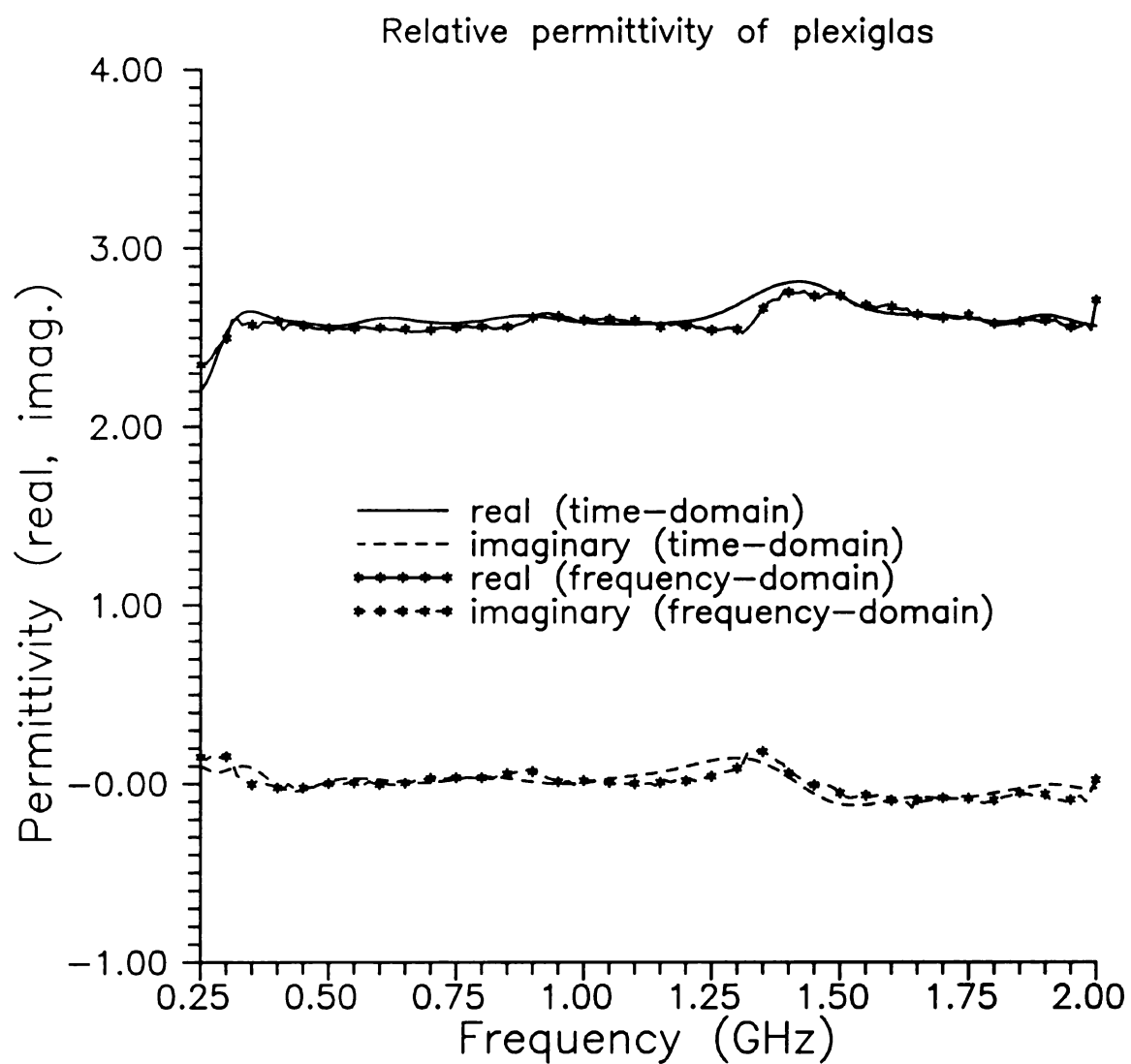


Figure 7.12 Comparison of relative permittivity of plexiglas obtained using time- and frequency-domain methods.

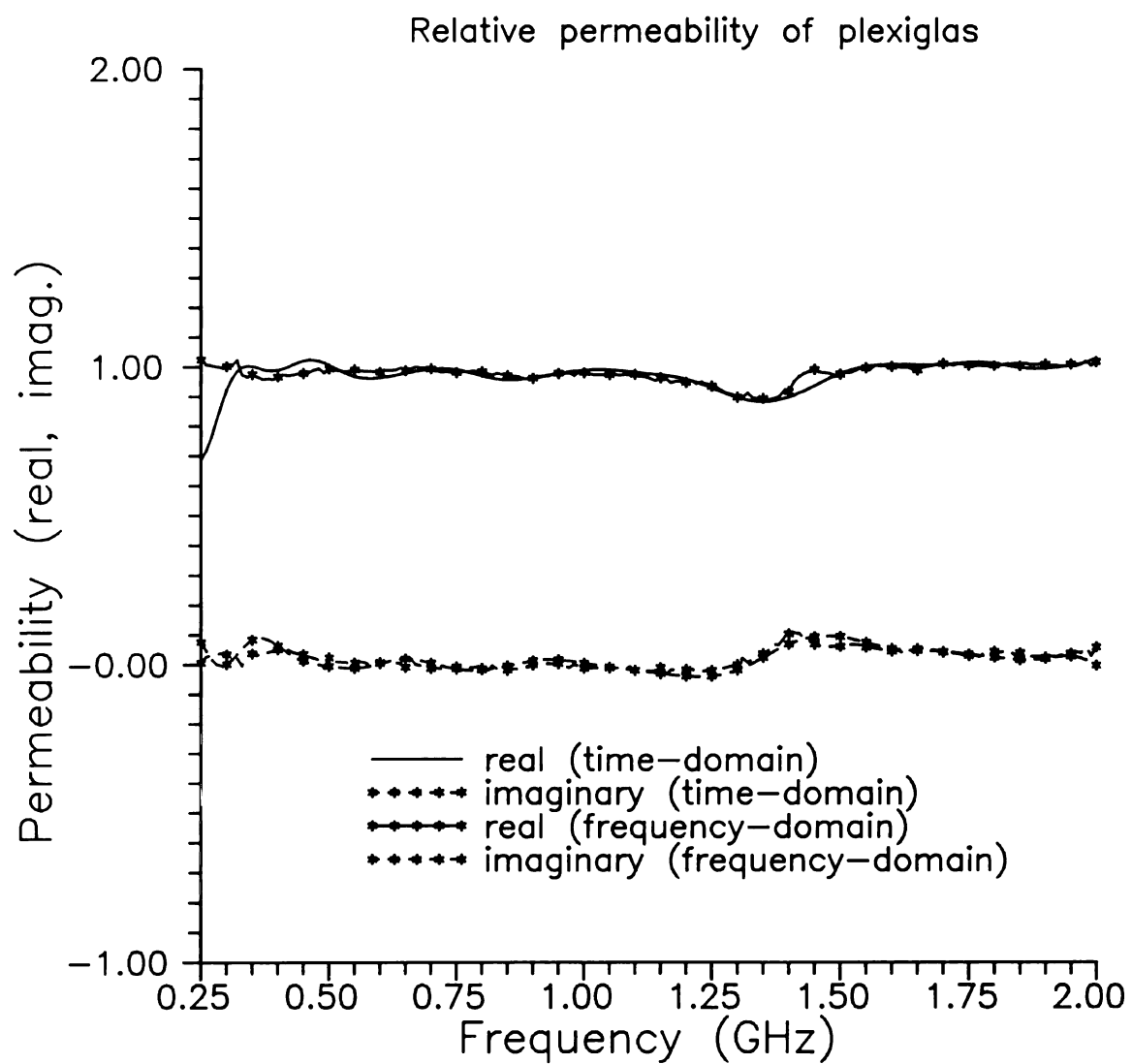


Figure 7.13 Comparison of relative permeability of plexiglas obtained using time- and frequency-domain techniques.

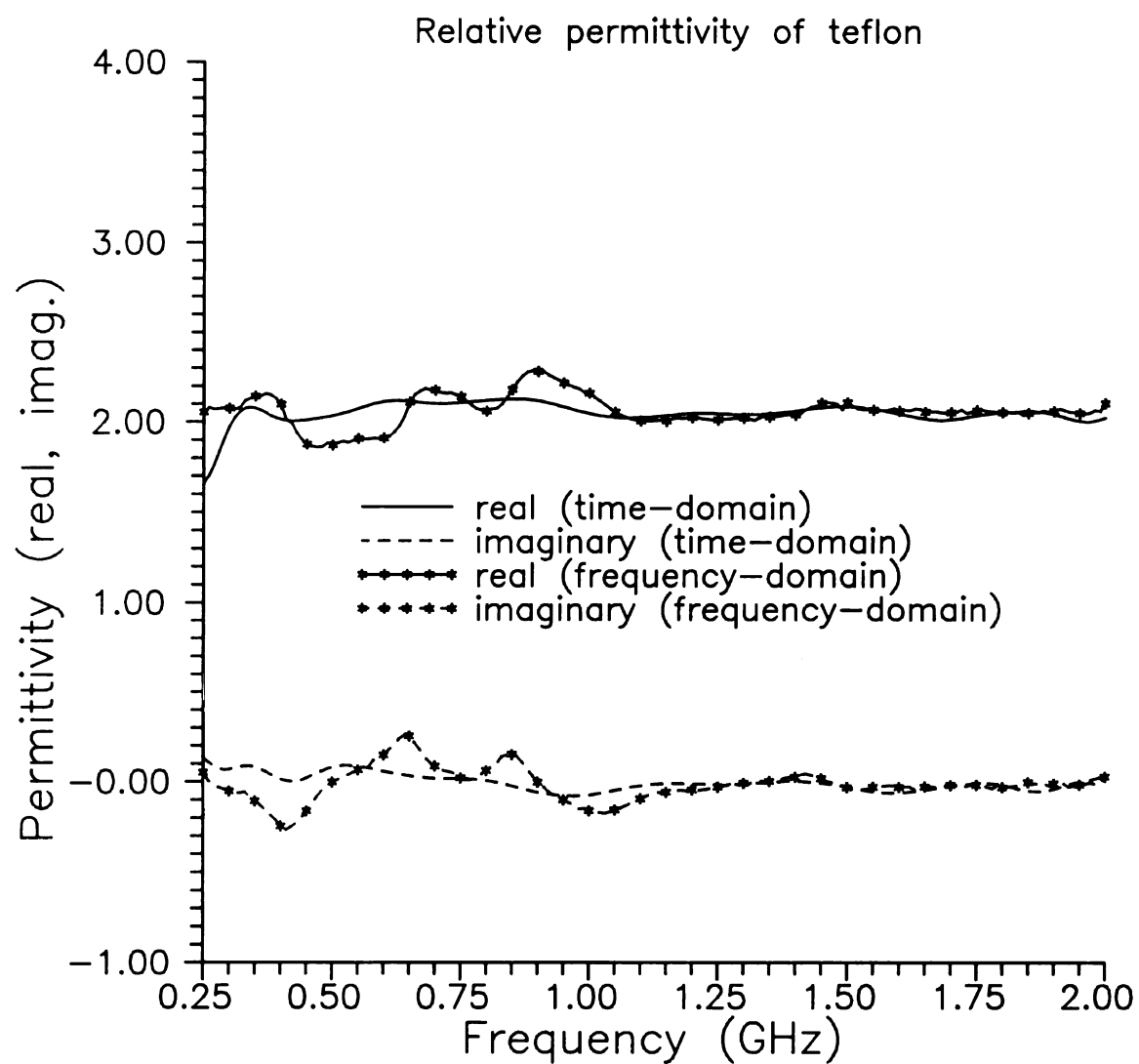


Figure 7.14 Comparison of relative permittivity of teflon obtained using time- and frequency-domain methods.

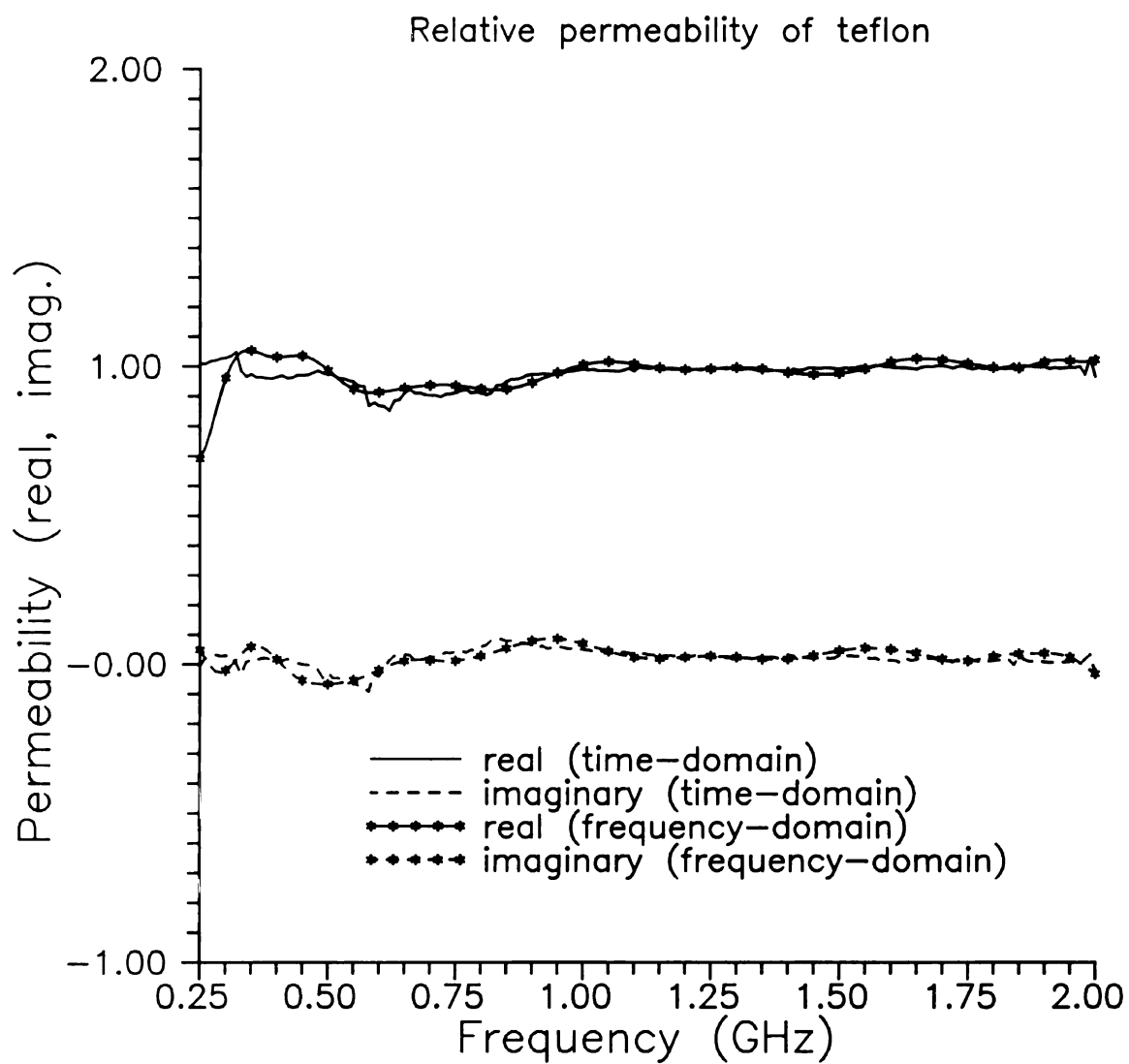


Figure 7.15 Comparison of relative permeability of teflon obtained using time- and frequency-domain methods.

These distortions are analogous to the Gibb's phenomenon associated with discontinuous time waveforms.

The effects of such distortions can be greatly reduced if the edges of the measured frequency-domain data are "rolled off", or made to smoothly approach zero in some systematic fashion which may be easily undone at a later time. This is accomplished by multiplying raw frequency measurements by a "weighting function" prior to calculating the IFFT. If the weighting function is carefully selected and applied, it may be removed after data processing with minimal effect.

Many types of weighting functions exist. The amount and type of "roll off" due to the weighting function must be carefully adjusted depending on the type of data being operated upon, and the desired result. The unweighted raw frequency domain response of the stripline applicator with a short-circuit termination present is shown in Figure 7.5. The time-domain response of this data is shown in Figure 7.16. Reflections from the short-circuit standard, and the coax-to-stripline transitions are apparent in this plot. However, due to the distortions caused by the abrupt data terminations, it is impossible to resolve where the reflections from one structure end, and those from another begin. This would make the proper application of a time-window extremely difficult.

The same raw short-circuit data with a moderate weighting function applied is shown in Figure 7.17. The time domain response of this moderately weighted data is shown in Figure 7.18. The same reflections are evident in this plot, however the distortions due to the abrupt data terminations have been reduced. In spite of this, it is still somewhat difficult to resolve distinct structures in the stripline.

An ideal weighting function has been applied to the raw short-circuit data in

Figure 7.19. In the resulting time-domain response, shown in Figure 7.20, reflections from the short, and coaxial connectors are distinct, making the task of windowing unambiguous. Great care must be taken to avoid using a weighting function which rolls off too much of the data, as in Figure 7.21. In this case data at the edges of the frequency spectrum has been lost. It must be remembered that use of the weighting function will always cause some loss of data at the edges of the frequency spectrum, however this can be minimized with careful selection of the weighting function.

Another aspect of the Fourier transform is that if the width of the spectrum in the frequency domain is increased, resolution in the resulting time-domain response is increased. This is evident in Figure 7.22, and Figure 7.23. The short-circuit time-domain response of a stripline applicator measured from 150 MHz to 2.15 GHz is plotted in Figure 7.22. In this plot it is difficult to determine where the primary reflection from the short-circuit termination ends and the secondary reflection from the front coaxial connection begins. This difficulty is resolved by widening the frequency measurement spectrum. Figure 7.23 shows the response of the same applicator as measured from 150 Mhz to 4.15 GHz. In this plot all reflections are easily resolvable.

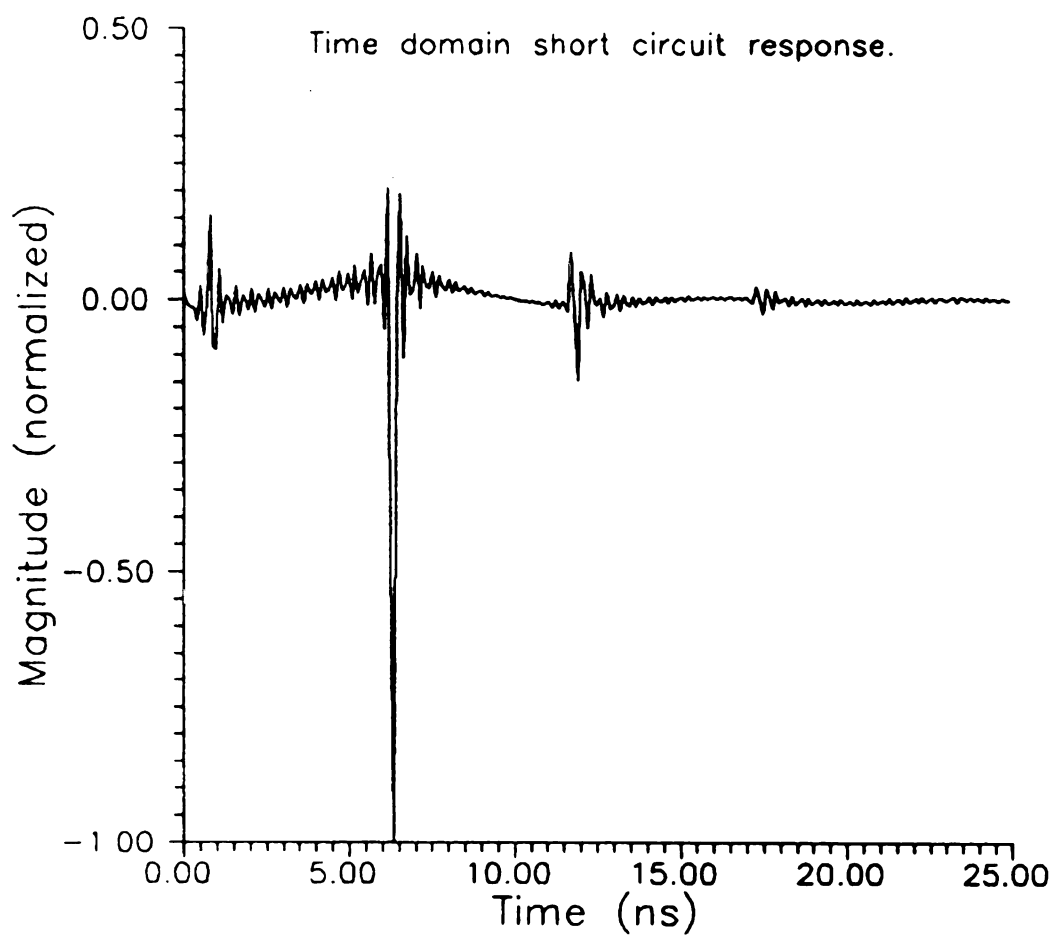


Figure 7.16. Time-domain response of unweighted frequency-domain stripline short-circuit response.

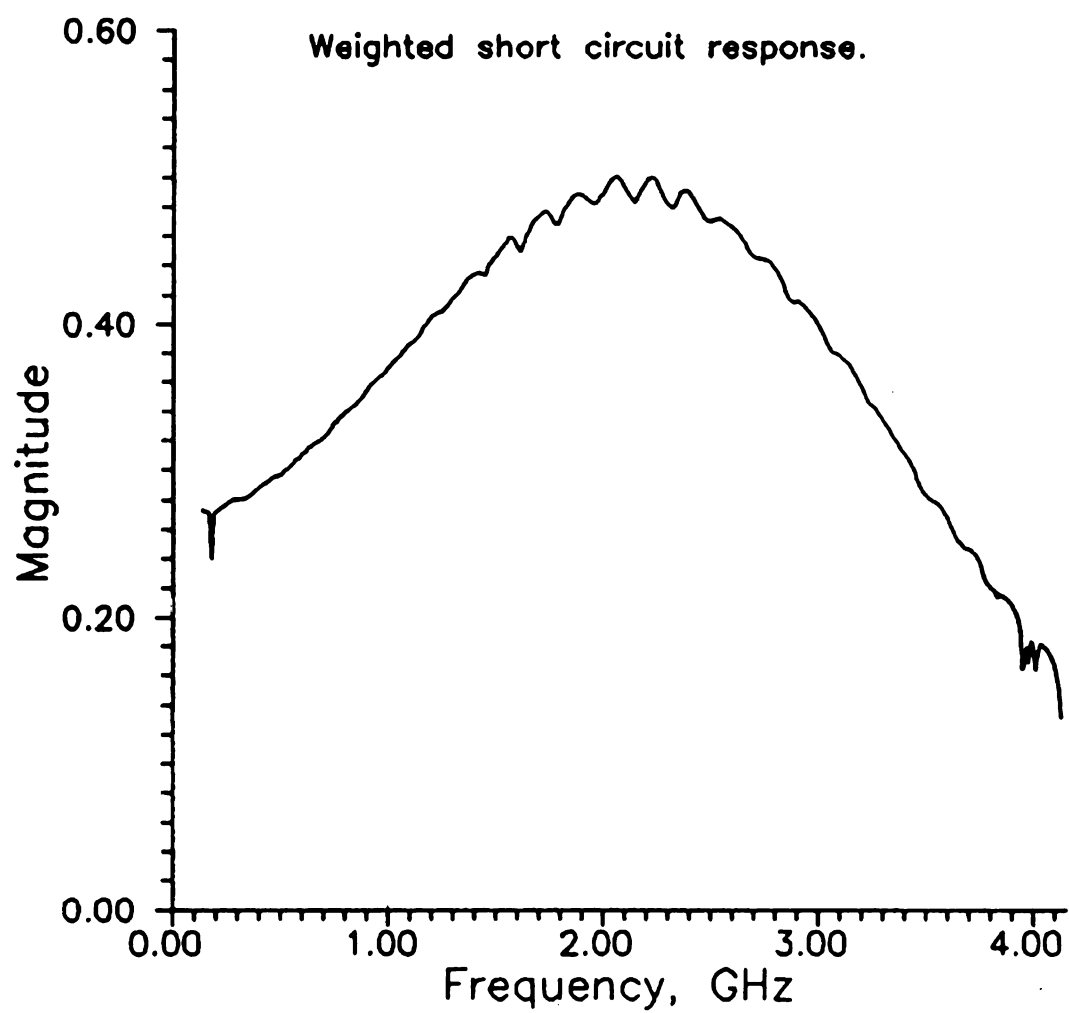


Figure 7.17. Moderately weighted frequency-domain short-circuit response.

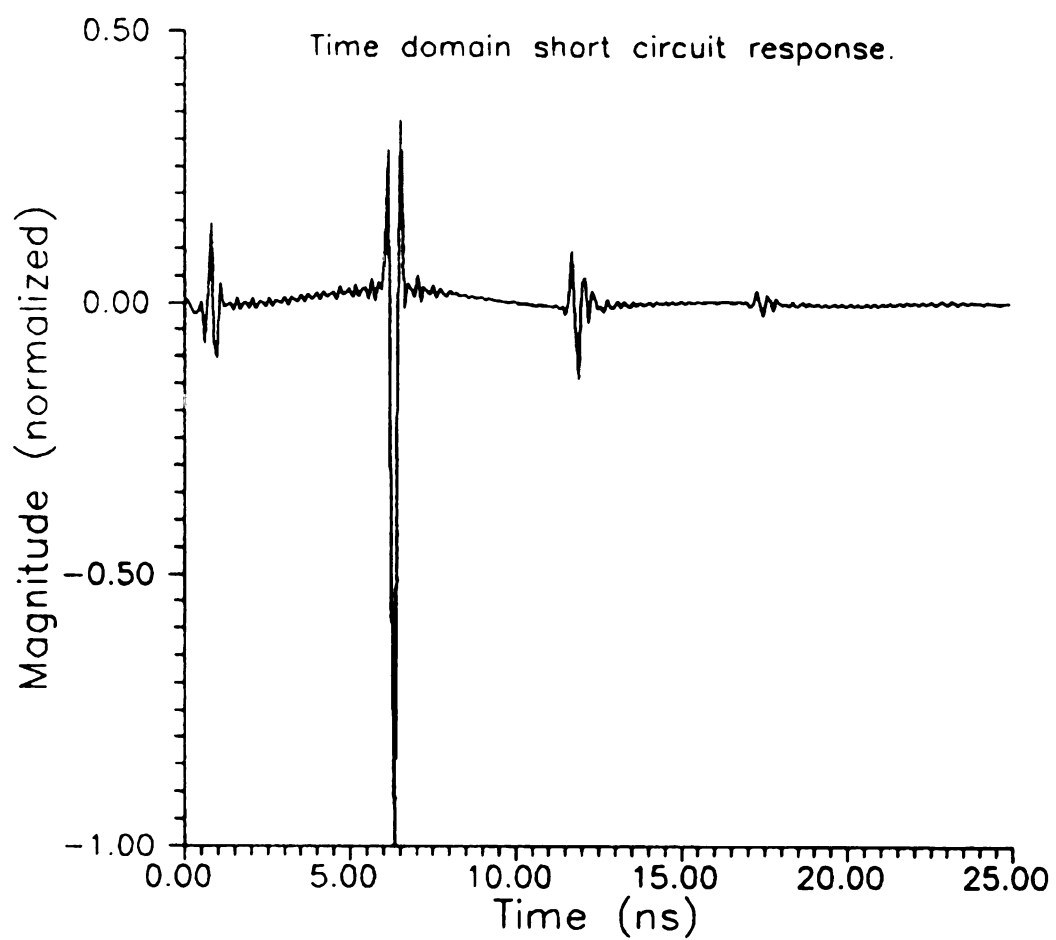


Figure 7.18. Time-domain response of moderately weighted frequency-domain short-circuit data.

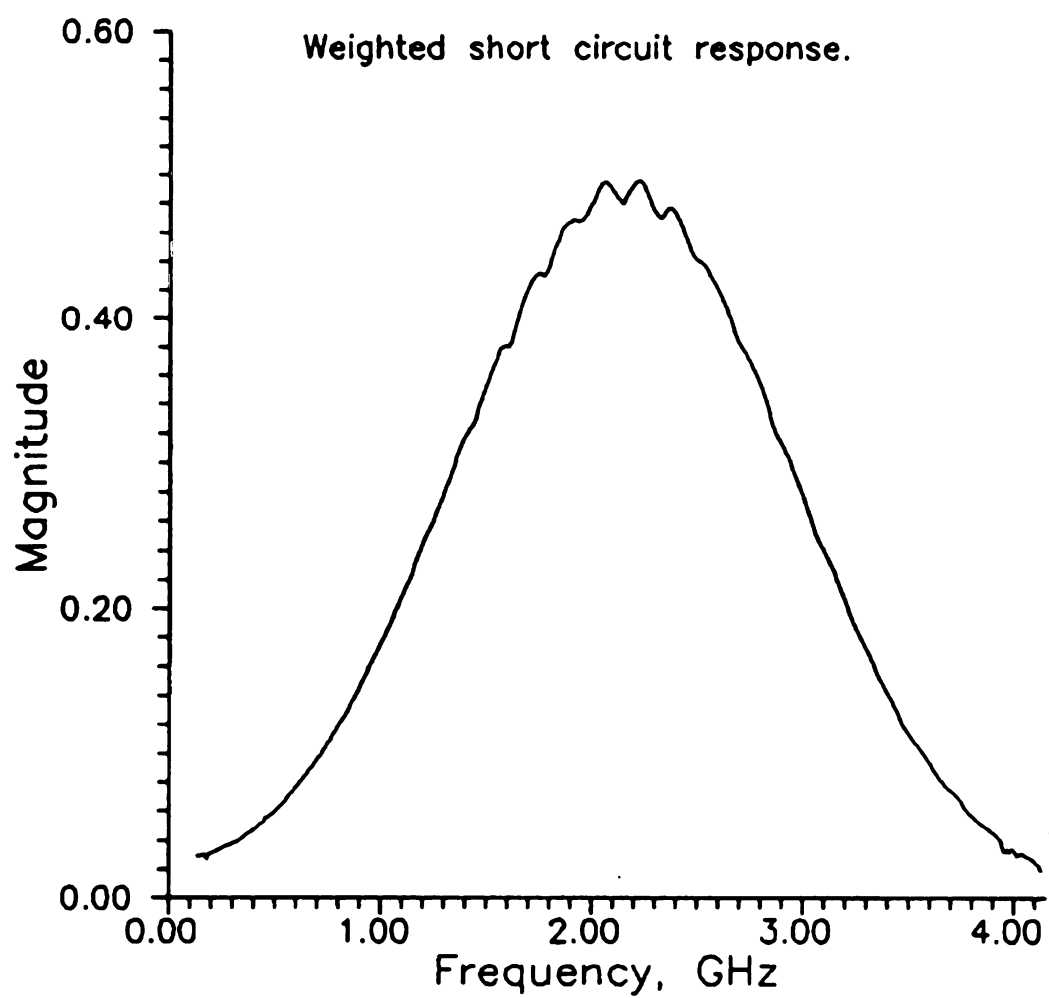


Figure 7.19. Correctly weighted frequency-domain short-circuit response.

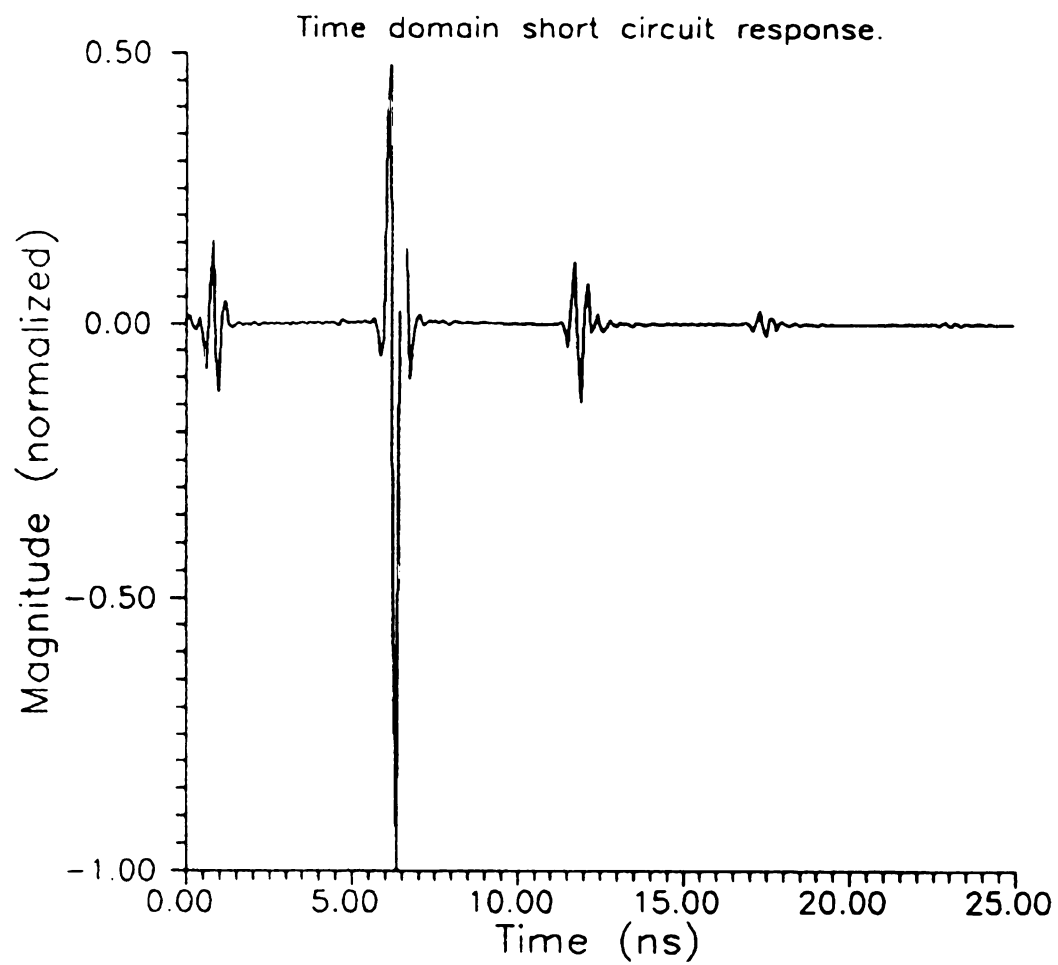


Figure 7.20. Time-domain response of correctly weighted frequency-domain short-circuit data.

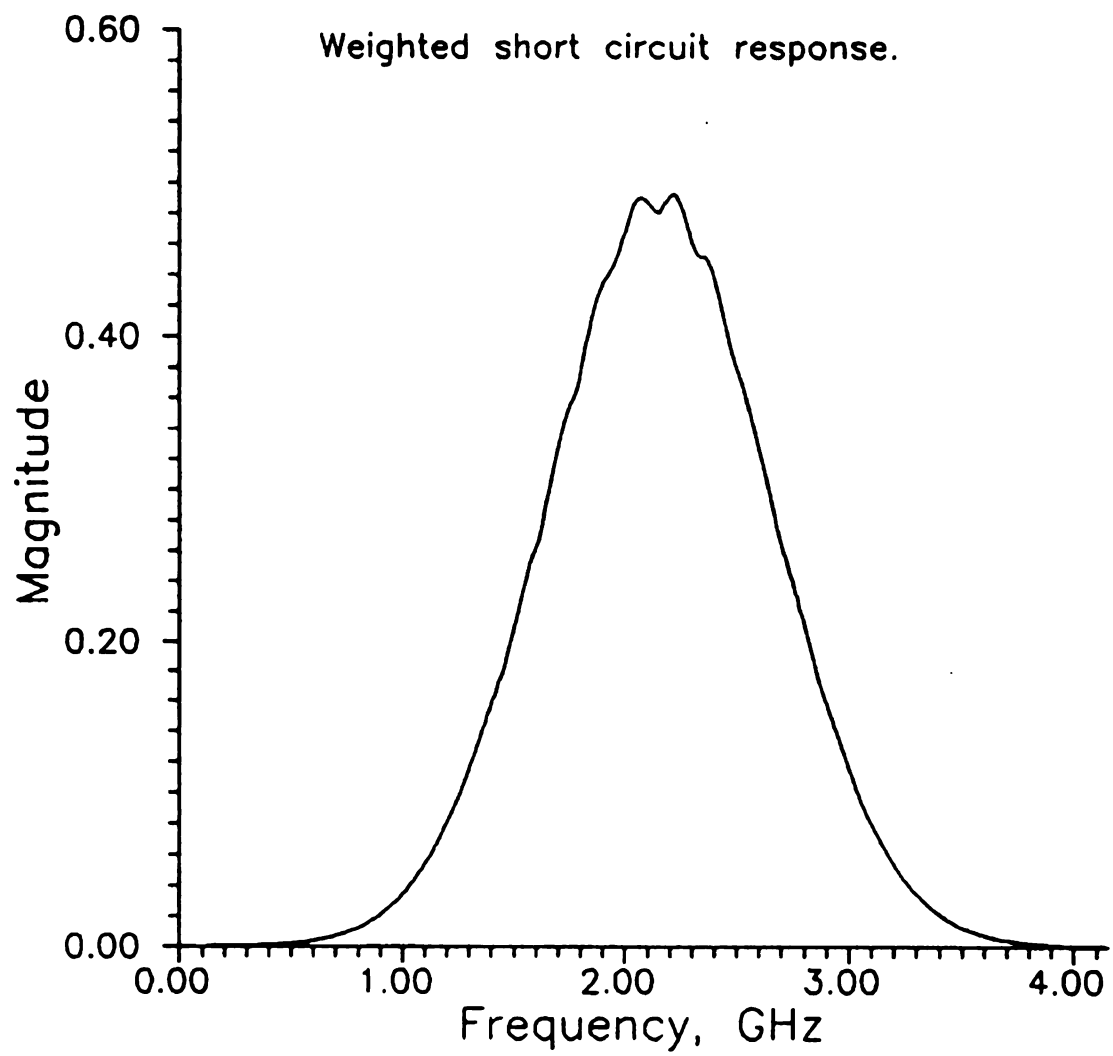


Figure 7.21. Incorrectly weighted frequency-domain short-circuit response.

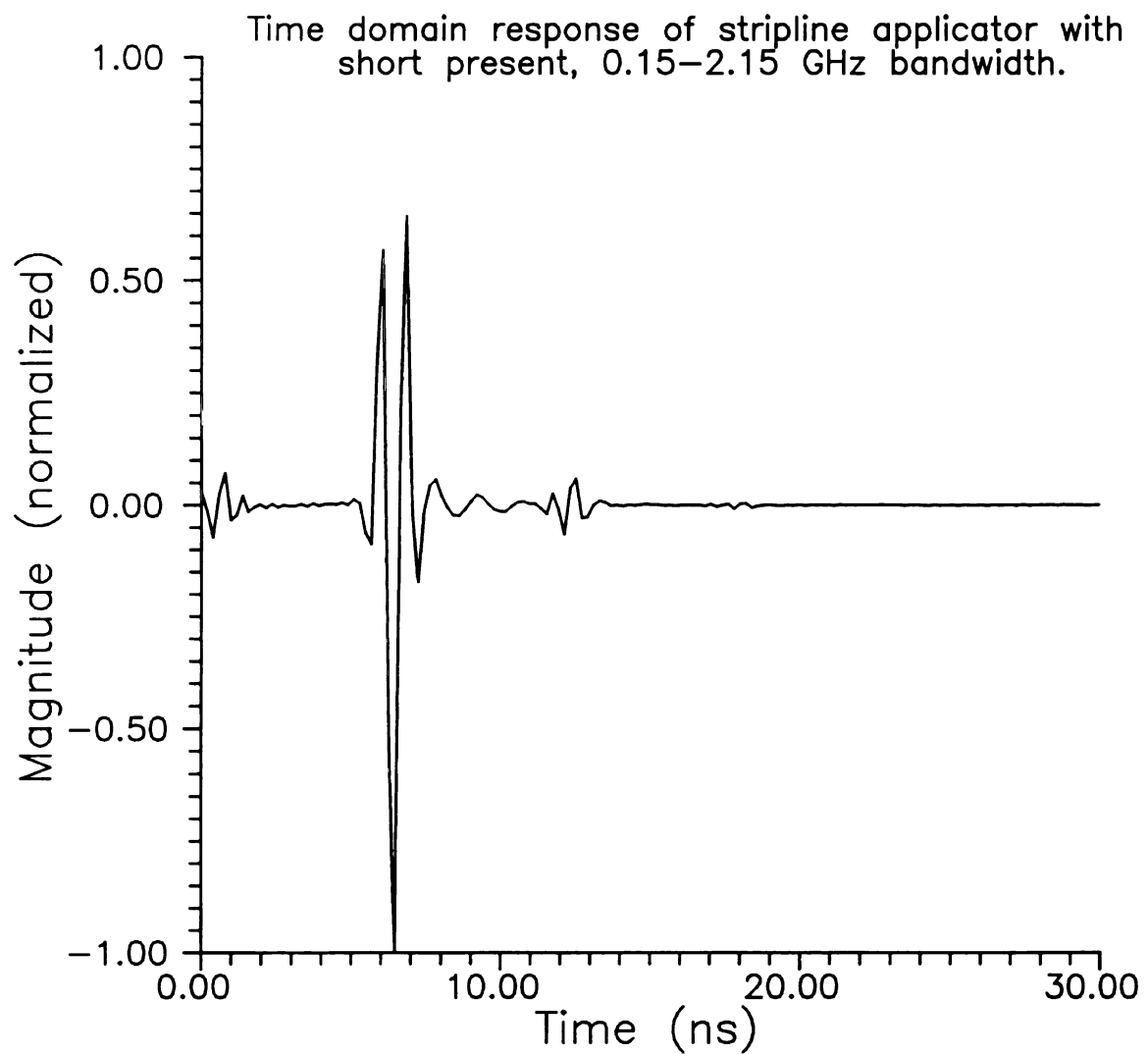


Figure 7.22. Time-domain response of stripline applicator, synthesized using 0.15-2.15 GHz frequency-domain data.

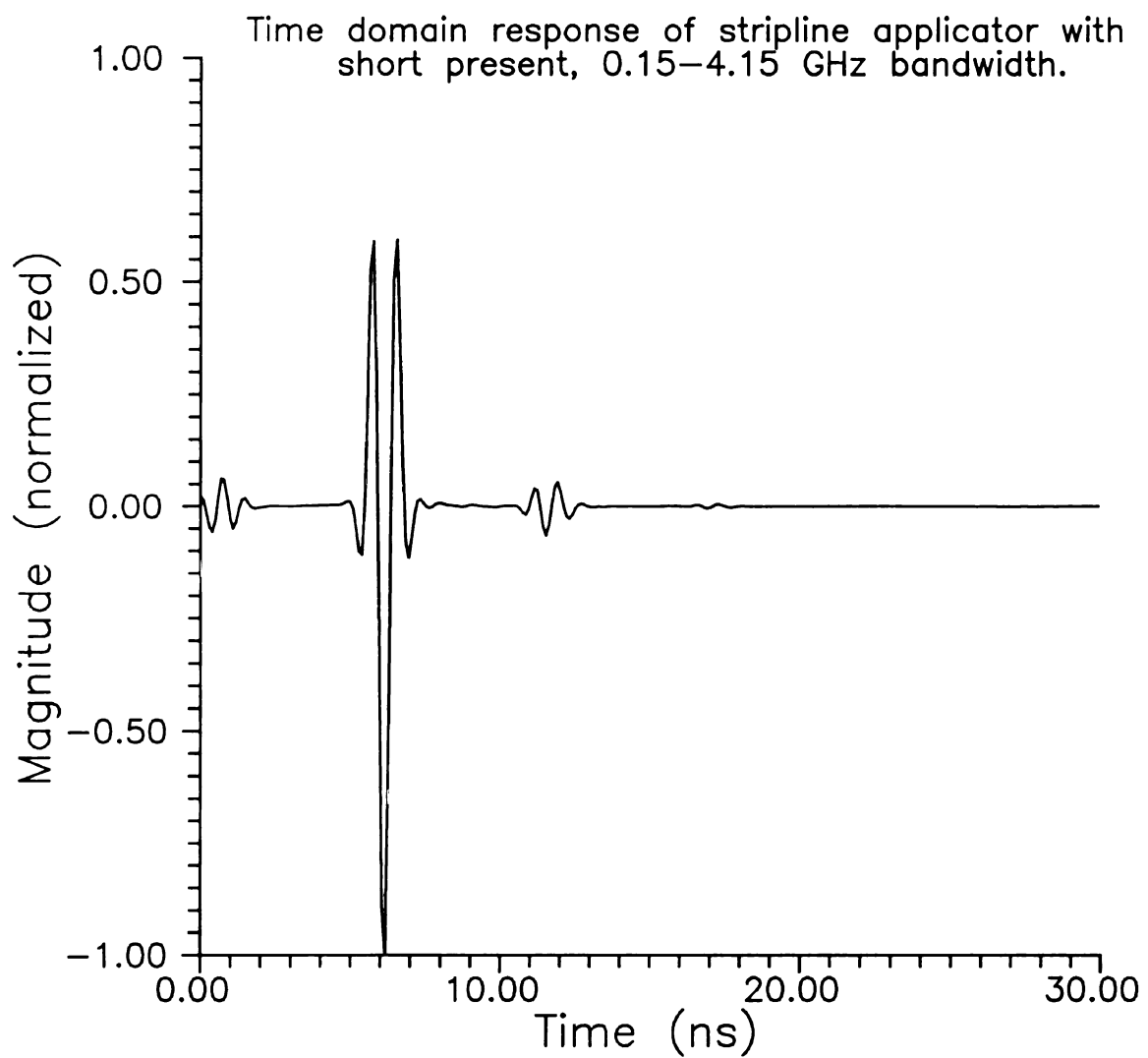


Figure 7.23. Time-domain response of stripline applicator, synthesized using 0.15–4.15 GHz frequency-domain data.

Chapter 8

CONCLUSION

This dissertation has presented investigations into several topics involving stripline transmission lines. Chapter 2 provided an overview of the analysis techniques which were employed. In Chapter 3, a Hertz potential Green's function was developed for a three-dimensional current source suspended within a generalized stripline structure. In this case the current source was considered to be suspended between parallel, infinitely-wide, perfectly-conducting ground planes. A layer composed of a simple, non-magnetic dielectric material was assumed to coat one of the ground planes. Boundary conditions at the dielectric/free space interface and at the surfaces of the perfectly-conducting ground planes were first established. The use of a two-dimensional Fourier transform to solve the Helmholtz equation was prompted by the uniform cross-sectional geometry of the stripline structure. The result was a Hertz potential solution in terms of Sommerfeld integral Green's function representations. Pole singularities and branch points associated with the Green's function were examined and identified. It was seen that both surface-wave and parallel-plate propagation modes are supported by this structure.

In Chapter 4, the generalized Green's function developed in the previous chapter was specialized to represent a stripline structure having a single, planar conducting center strip. For this case it was assumed that the center strip was immersed in free space, and that no dielectric layer was present between the conducting ground planes. A pair of EFIEs was established by enforcing the proper boundary conditions on the tangential

electric fields at the surface of the perfectly-conducting center strip. Current distributions associated with various higher order modes were determined through application of a Galerkin's moment method technique. Here entire domain (Chebychev) basis functions were employed, leading to closed form expressions for matrix elements. It was seen that proper, discrete guided-wave modes excited on the stripline structure are associated with pole singularities of the specialized Green's function which occur on the top Riemann sheet. Improper (leaky-wave) modes are associated with singularities located on the bottom Riemann sheet. It was also noted that current distributions associated with the discrete modes supported by the isolated stripline possessed either even or odd symmetry about the longitudinal axis of the center strip.

Most stripline devices are operated in such a way that only the fundamental TEM mode is excited. This prompted the development of a TEM specialization of the isolated stripline full-wave theory. The amount of computational time required for the numerical solution was reduced for this technique by using a pole series representation in place of the real axis integration implicated in the Green's function. The TEM specialization allowed for determination of the isolated stripline characteristic impedance.

The analysis of the isolated stripline was extended in Chapter 5 to the case of a stripline structure having two coupled center strips. These center strips were assumed to have arbitrary width and placement within the stripline cross-section. In this case, two pairs of EFIEs were established through enforcement of the appropriate boundary conditions at the surfaces of the conducting center strips. Once again the center-strip current distributions were determined through application of a Galerkin's Moment Method solution. It was seen that while the current distributions associated with modes

excited on the coupled stripline were in general neither even nor odd (as was the case with the isolated center strip), both symmetric and anti-symmetric modes are supported by this structure. When the center strips are widely separated, the propagation constants associated with the symmetric and anti-symmetric modes are nearly identical. However, when the center-to-center distance between the strips is decreased, the propagation constants become significantly different. Several of the discrete, higher-order propagation modes associated with the coupled stripline structure, both proper and leaky, were examined and identified. As with the isolated stripline, a TEM specialization of the full-wave theory describing this structure was performed. A pole series representation was developed to replace the real axis integration implicated in the matrix element Green's functions. From this the associated impedance parameters were determined.

Solution of the coupled strip EFIEs was seen to be formidable both analytically and numerically. In order to avoid the complexity of the integral equation solution, a full-wave perturbation theory for coupled stripline was developed in Chapter 6. For this method, the eigenmode currents of an isolated center strip were used as a first approximation for nearly the degenerate eigenmode currents of a loosely coupled system. The results obtained using this method were compared to those presented in the previous chapter and seen to agree well for widely separated center strips. This technique was found to provide a good initial propagation mode for use with the full-wave technique.

In Chapter 7, a technique for measuring the constitutive parameters (complex permittivity and permeability) of sample materials placed in a stripline field applicator was presented. For this method, the time domain response of the stripline is synthesized

by calculating the IFFT of frequency domain measurements made at the applicator terminals. Rectangular time-windows are applied to remove unwanted components of the resulting time-waveform. The scattering parameters associated with the unknown sample materials are then determined from the FFT of the processed time-domain data using the Nicholson-Ross-Weir technique. This time-windowing technique requires fewer measurements for materials characterization than frequency-domain techniques currently available. This reduction leads to increased accuracy when high temperature measurements are made.

During the course of the research involved in this dissertation, much was learned about the behavior of stripline field applicators. In spite of this, much work in this area remains. The conducting center strips of all structures considered in this dissertation were modeled as being infinitely thin. In practice, stripline field applicators having very thin center conductors are often cumbersome, and difficult to use. Proper time-gating for materials characterization is facilitated by using a very long applicator. This allows transients excited at the coax-to-stripline transitions to settle before reaching the sample region, thus insuring single mode propagation required for the Nicholson-Ross-Weir technique. For long applicators, proper support of the center conductor is a difficulty. Thin center conductors tend to bow and warp (particularly at high temperature), causing changes in the characteristic impedance of the applicator. This also causes a change in the structure of the applicator between when sample materials are present and when the applicator is empty. As a result it is desirable to use a thicker center conductor when making materials measurements. However, any departure from a thin strip structure causes changes in the stripline characteristic impedance, as well as the cutoff frequencies

of the higher order modes. Thus a full-wave theory for stripline devices having thick center strips is needed. The Hertz potential Green's function specialized for the homogeneous stripline cross-section which was developed at the beginning of Chapter 4 can be used to represent an extended volume current source. However, a change in how the EFIEs are established is required, as additional boundary conditions must be satisfied. Analysis of stripline having a finite thickness center conductor may also prove interesting because such a structure should support a continuous radiation spectrum, as well as discrete bound modes.

Finally, the generalized Hertz potential Green's function which was established in Chapter 3 can be specialized to represent a thin center strip which lies on top of a dielectric material, between two parallel conducting ground planes, forming a covered microstrip. The theory for an isolated center strip can then be extended to describe a multi-strip structure. In addition to a full wave theory for the coupled structure, a perturbation method can also be developed. Both single and coupled covered microstrip are of importance due to widespread use in integrated circuits. An understanding of the propagation mode spectra of these structures would aid in the design of high frequency integrated circuits.

APPENDIX

Appendix A

HERTZ POTENTIAL BOUNDARY CONDITIONS AT DIELECTRIC AND CONDUCTING INTERFACES

A.1 Boundary Conditions on Hertz Potential at a Dielectric Interface

Consider an interface between free space and a differing (non-magnetic) medium. It is assumed that free space occupies the half-space $y > 0$ (region 0), the dielectric material occupies the half-space $y < 0$ (region 1), and electric and magnetic fields are supported by a generalized current source which is suspended in the free space region above the material interface at $y = 0$ (Figure A.1). It was seen in Chapter 3 that the associated inhomogeneous Helmholtz equation may be decomposed into scalar components

$$\nabla^2 \Pi_\alpha + k_o^2 \Pi_\alpha = \frac{-J_\alpha}{j\omega\epsilon} \quad \dots \text{for } \alpha = x, y, z. \quad (\text{A.1})$$

This facilitates the application of boundary conditions, allowing the individual scalar field components to be expressed in terms of Hertz potentials as

$$E_x = k^2 \Pi_x + \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}) \quad H_x = j\omega\epsilon \left(\frac{\partial \Pi_z}{\partial y} - \frac{\partial \Pi_y}{\partial z} \right) \quad (\text{A.2})$$

$$E_y = k^2 \Pi_y + \frac{\partial}{\partial y}(\nabla \cdot \vec{\Pi}) \quad H_y = j\omega\epsilon \left(\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x} \right) \quad (\text{A.3})$$

$$E_z = k^2 \Pi_z + \frac{\partial}{\partial z}(\nabla \cdot \vec{\Pi}) \quad H_z = j\omega\epsilon \left(\frac{\partial \Pi_y}{\partial x} - \frac{\partial \Pi_x}{\partial y} \right). \quad (\text{A.4})$$

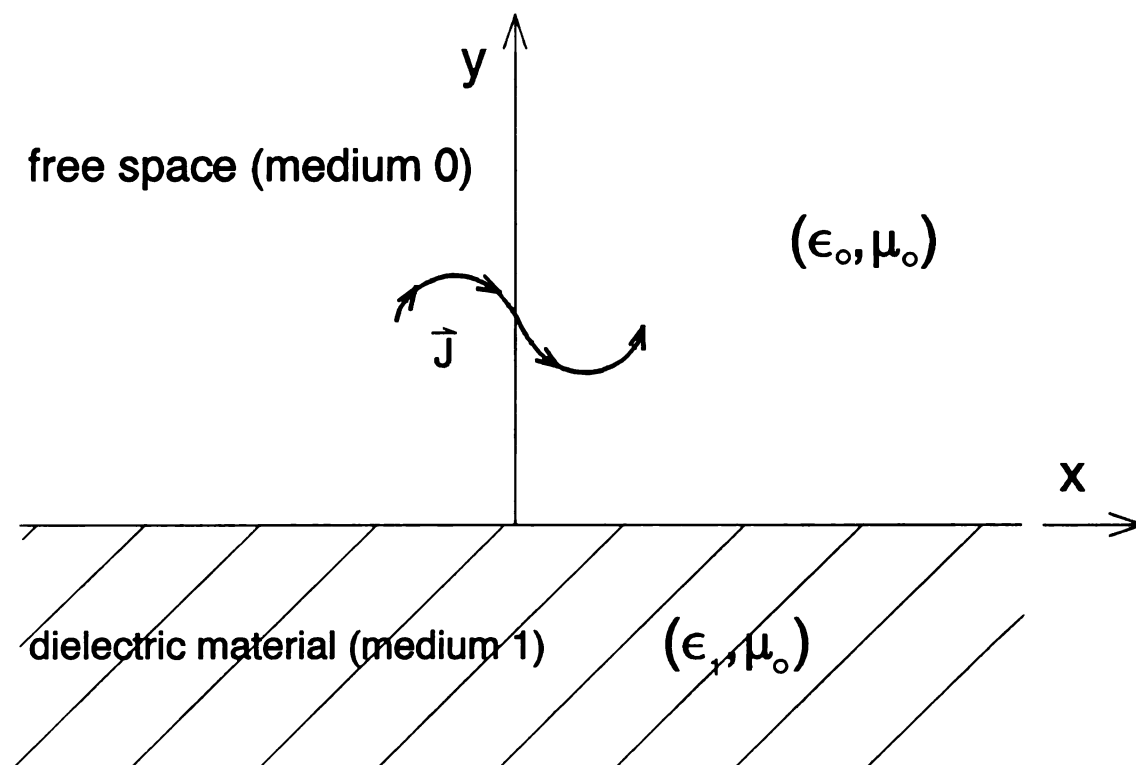


Figure A.1 Interface between free space and non-magnetic, dielectric material.

Tangential electric and magnetic fields ($\beta=x,z$) are continuous across the interface at $y = 0$, or

$$E_{0\beta}(y=0^+) = E_{1\beta}(y=0^-) \quad (\text{A.5})$$

$$H_{0\beta}(y=0^+) = H_{1\beta}(y=0^-). \quad (\text{A.6})$$

These boundary conditions lead to relationships between the various components of Hertz potential in each region

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow k_o^2 \Pi_{0x} + \frac{\partial}{\partial x} (\nabla \cdot \vec{\Pi}_0) = k_1^2 \Pi_{1x} + \frac{\partial}{\partial x} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.7})$$

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow k_o^2 \Pi_{0z} + \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_0) = k_1^2 \Pi_{1z} + \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.8})$$

$$H_{0x}(y=0^+) = H_{1x}(y=0^-) \rightarrow \epsilon_o \left(\frac{\partial \Pi_{0z}}{\partial y} - \frac{\partial \Pi_{0y}}{\partial z} \right) = \epsilon_1 \left(\frac{\partial \Pi_{1z}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial z} \right) \quad (\text{A.9})$$

$$H_{0z}(y=0^+) = H_{1z}(y=0^-) \rightarrow \epsilon_o \left(\frac{\partial \Pi_{0y}}{\partial x} - \frac{\partial \Pi_{0x}}{\partial y} \right) = \epsilon_1 \left(\frac{\partial \Pi_{1y}}{\partial x} - \frac{\partial \Pi_{1x}}{\partial y} \right). \quad (\text{A.10})$$

The boundary conditions on the Hertz potential at the dielectric interface due to the general, three-dimensional current source will be constructed through the linear superposition of the boundary conditions on Hertz potentials associated with simpler, one-dimensional current sources. Care must be taken to maintain consistency in the implementation of these boundary conditions. For simplicity, a vertical current source is first considered

$$\vec{J} = \hat{y} J_y. \quad (\text{A.11})$$

The principal outgoing wave for this source in the free space region is given by

$$\vec{\Pi}_0^P = \hat{y} \Pi_{0y}^P \quad (\text{A.12})$$

and the associated magnetic field is

$$\vec{H}_0^P = \hat{x} H_{0x}^P + \hat{z} H_{0z}^P. \quad (\text{A.13})$$

As a result, it is conjectured that the total fields are determined by $\vec{\Pi} = \hat{y} \Pi_y$ with $\vec{H} = \hat{x} H_x + \hat{z} H_z$. This conjecture is applied to the boundary conditions at the interface to obtain

$$H_{0x}(y=0^+) = H_{1x}(y=0^-) \rightarrow \epsilon_o \frac{\partial \Pi_{0y}}{\partial z} = \epsilon_1 \frac{\partial \Pi_{1y}}{\partial z} \quad (\text{A.14})$$

$$H_{0z}(y=0^+) = H_{1z}(y=0^-) \rightarrow \epsilon_o \frac{\partial \Pi_{0y}}{\partial x} = \epsilon_1 \frac{\partial \Pi_{1y}}{\partial x} \quad (\text{A.15})$$

which are valid for all (x,z) . Thus the condition on the Hertz potential which satisfies the boundary conditions on the tangential magnetic field is

$$\Pi_{0y} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1y} \quad (\text{A.16})$$

for all (x,z) at $y = 0$. Now from the boundary conditions on the electric field comes

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow \frac{\partial}{\partial x} \left(\frac{\partial \Pi_{0y}}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial \Pi_{1y}}{\partial y} \right) \quad (\text{A.17})$$

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow \frac{\partial}{\partial z} \left(\frac{\partial \Pi_{0y}}{\partial y} \right) = \frac{\partial}{\partial z} \left(\frac{\partial \Pi_{1y}}{\partial y} \right) \quad (\text{A.18})$$

for all (x,z) , and thus the condition on the Hertz potential which satisfies the boundary

conditions on the tangential electric field is

$$\frac{\partial \Pi_{0y}}{\partial y} = \frac{\partial \Pi_{1y}}{\partial y} \quad (\text{A.19})$$

for all (x,z) at $y = 0$. It is thus seen that the conjecture was correct and the EM field supported by the vertical current source can be obtained from $\vec{\Pi} = \hat{y}\Pi_y$.

Next consider the case of a horizontally oriented current source

$$\vec{J} = \hat{x}J_x. \quad (\text{A.20})$$

The principal outgoing wave associated with this source in the free space region is given by

$$\vec{\Pi}_0^p = \hat{x}\Pi_{0x}^p. \quad (\text{A.21})$$

Because of the previous result for the vertical current source, it is initially tempting to conjecture that the total fields supported by the horizontal current source are determined by $\vec{\Pi} = \hat{x}\Pi_x$. Applying this conjecture, it is apparent that the boundary condition

$$H_{0x}(y=0^+) = H_{1x}(y=0^-) \quad (\text{A.22})$$

for all (x,z) is satisfied because $H_x = 0$ for this case. The remaining boundary condition on the tangential magnetic field leads to the relationship

$$H_{0z}(y=0^+) = H_{1z}(y=0^-) \rightarrow \epsilon_0 \frac{\partial \Pi_{0x}}{\partial y} = \epsilon_1 \frac{\partial \Pi_{1x}}{\partial y} \quad (\text{A.23})$$

for all (x,z) , and the boundary condition on the z -directed electric field gives

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow \frac{\partial}{\partial z} \left(\frac{\partial \Pi_{0x}}{\partial x} \right) = \frac{\partial}{\partial z} \left(\frac{\partial \Pi_{1x}}{\partial x} \right) \quad (\text{A.24})$$

for all (x,z) . It is seen that the conditions on the Hertz potential which satisfy boundary conditions (A.23) and (A.24) are

$$\frac{\partial \Pi_{0x}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1x}}{\partial y}, \quad \Pi_{0x} = \Pi_{1x} \quad (\text{A.25})$$

for all (x,z) . Application of the final boundary condition on the x -directed electric field yields

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow k_o^2 \Pi_{0x} + \frac{\partial^2 \Pi_{0x}}{\partial x^2} = k_1^2 \Pi_{1x} + \frac{\partial^2 \Pi_{1x}}{\partial x^2} \quad (\text{A.26})$$

which leads to the condition on the Hertz potential

$$\epsilon_o \Pi_{0x} = \epsilon_1 \Pi_{1x} \quad (\text{A.27})$$

for all (x,z) at $y = 0$. It is apparent that (A.27) can only be consistent with (A.25) if $\epsilon_o = \epsilon_1$, which is obviously incorrect for this case. It is thus seen that the initial conjecture was incorrect and $\vec{\Pi} \neq \hat{x} \Pi_x$ because the resulting EM field does not satisfy the boundary conditions at $y = 0$.

The determination of the total fields supported by the horizontal current source therefore requires a new conjecture, namely

$$\vec{\Pi} = \hat{x} \Pi_x + \hat{y} \Pi_y. \quad (\text{A.28})$$

Application of this conjecture to the boundary condition on the z -directed electric field gives

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_0) = \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.29})$$

which requires

$$\nabla \cdot \vec{\Pi}_0 = \nabla \cdot \vec{\Pi}_1 \quad (\text{A.30})$$

or

$$\frac{\partial \Pi_{0x}}{\partial x} - \frac{\partial \Pi_{1x}}{\partial x} = - \left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) \quad (\text{A.31})$$

for all (x,z) at $y = 0$. The remaining boundary condition on the tangential electric field leads to

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow k_o^2 \Pi_{0x} + \frac{\partial}{\partial x} (\nabla \cdot \vec{\Pi}_0) = k_1^2 \Pi_{1x} + \frac{\partial}{\partial x} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.32})$$

but, because of (A.30)

$$k_o^2 \Pi_{0x} = k_1^2 \Pi_{1x} \quad (\text{A.33})$$

and therefore the resulting condition on the Hertz potential is

$$\Pi_{0x} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1x} \quad (\text{A.34})$$

for all (x,z) at $y = 0$. The boundary condition on the x -directed magnetic field gives

$$H_{0x}(y=0^+) = H_{1x}(y=0^-) \rightarrow \epsilon_o \frac{\partial \Pi_{0y}}{\partial z} = \epsilon_1 \frac{\partial \Pi_{1y}}{\partial z} \quad (\text{A.35})$$

which leads to the condition on the Hertz potential

$$\Pi_{0y} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1y} \quad (\text{A.36})$$

for all (x,z) at $y = 0$. Finally, the boundary condition on the z -directed magnetic field

yields the relationship

$$H_{0z}(y=0^+) = H_{1z}(y=0^-) \rightarrow \epsilon_o \frac{\partial \Pi_{0x}}{\partial x} - \epsilon_1 \frac{\partial \Pi_{1x}}{\partial y} = \epsilon_o \frac{\partial \Pi_{0y}}{\partial x} - \epsilon_1 \frac{\partial \Pi_{1y}}{\partial x} = 0 \quad (\text{A.37})$$

due to (A.36), therefore

$$\frac{\partial \Pi_{0x}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1x}}{\partial y} \quad (\text{A.38})$$

and thus

$$\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} = - \left(\frac{\partial \Pi_{0x}}{\partial x} - \frac{\partial \Pi_{1x}}{\partial x} \right) = - \left(\frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1x}}{\partial x} - \frac{\partial \Pi_{1x}}{\partial x} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) \frac{\partial \Pi_{1x}}{\partial x} \quad (\text{A.39})$$

for all (x,z) at $y = 0$. It is therefore apparent that the new conjecture was correct, and the EM fields supported by the horizontal current source (A.20) can be obtained from $\vec{\Pi} = \hat{x}\Pi_x + \hat{y}\Pi_y$.

Now consider a second horizontal current source, oriented such that $\vec{J} = \hat{z}J_z$ with the associated Hertz potential being $\vec{\Pi} = \hat{y}\Pi_y + \hat{y}\Pi_y$. Application of this to the boundary condition on the z -directed electric field gives

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}_0) = \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}_1) \quad (\text{A.40})$$

or

$$\nabla \cdot \vec{\Pi}_0 = \nabla \cdot \vec{\Pi}_1 \quad (\text{A.41})$$

which leads to the condition on the Hertz potential

$$\frac{\partial \Pi_{0z}}{\partial z} - \frac{\partial \Pi_{1z}}{\partial z} = - \left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) \quad (\text{A.42})$$

for all (x,z) at $y = 0$. The remaining boundary condition on the tangential electric field leads to

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow k_o^2 \Pi_{0z} + \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_0) = k_1^2 \Pi_{1z} + \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.43})$$

and, because of relationship (A.41), the resulting condition on the Hertz potential is

$$\Pi_{0z} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1z} \quad (\text{A.44})$$

for all (x,z) at $y = 0$. The boundary condition on the z -directed magnetic field gives

$$H_{0z}(y=0^+) = H_{1z}(y=0^-) \rightarrow \epsilon_o \frac{\partial \Pi_{0y}}{\partial x} = \epsilon_1 \frac{\partial \Pi_{1y}}{\partial x} \quad (\text{A.45})$$

which leads to the condition on the Hertz potential

$$\Pi_{0y} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1y} \quad (\text{A.46})$$

for all (x,z) at $y = 0$. The final boundary condition on the tangential magnetic field results in the relationship

$$H_{0x}(y=0^+) = H_{1x}(y=0^-) \rightarrow \epsilon_o \frac{\partial \Pi_{0z}}{\partial y} - \epsilon_1 \frac{\partial \Pi_{1z}}{\partial y} = \epsilon_o \frac{\partial \Pi_{0y}}{\partial z} - \epsilon_1 \frac{\partial \Pi_{1y}}{\partial z} = 0 \quad (\text{A.47})$$

due to (A.45), therefore

$$\frac{\partial \Pi_{0z}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1z}}{\partial y} \quad (\text{A.48})$$

and then

$$\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} = - \left(\frac{\partial \Pi_{0z}}{\partial z} - \frac{\partial \Pi_{1z}}{\partial z} \right) = - \left(\frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1z}}{\partial z} - \frac{\partial \Pi_{1z}}{\partial z} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) \frac{\partial \Pi_{1z}}{\partial z} \quad (\text{A.49})$$

for all (x,z) at $y = 0$.

Now based upon the cases above, the Hertz potential boundary conditions for a general three-dimensional current source $\vec{J} = \hat{x}J_x + \hat{y}J_y + \hat{z}J_z$ can be constructed through linear superposition. The boundary conditions on the scalar components of Hertz potential (Dirichlet boundary conditions) are thus

$$\Pi_{0\alpha} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1\alpha} \quad \dots \text{for } \alpha = x, y, z \quad (\text{A.50})$$

for all (x,z) at $y = 0$. Boundary conditions on the normal derivatives of Hertz potential (Neumann boundary conditions) are determined from conditions on tangential electric fields

$$E_{0x}(y=0^+) = E_{1x}(y=0^-) \rightarrow \frac{\partial}{\partial x} (\nabla \cdot \vec{\Pi}_0) = \frac{\partial}{\partial x} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.51})$$

$$E_{0z}(y=0^+) = E_{1z}(y=0^-) \rightarrow \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_0) = \frac{\partial}{\partial z} (\nabla \cdot \vec{\Pi}_1) \quad (\text{A.52})$$

which require

$$\nabla \cdot \vec{\Pi}_0 = \nabla \cdot \vec{\Pi}_1 \quad (\text{A.53})$$

for all (x,z) at $y = 0$, and conditions on tangential magnetic fields

$$H_{0x}(y=0^-) = H_{1x}(y=0^+) \rightarrow \frac{\partial \Pi_{0z}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1z}}{\partial y} \quad (\text{A.54})$$

$$H_{0z}(y=0^-) = H_{1z}(y=0^+) \rightarrow \frac{\partial \Pi_{0x}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1x}}{\partial y} \quad (\text{A.55})$$

(since $\epsilon_o \Pi_{0y} = \epsilon_1 \Pi_{1y}$) for all (x,z) at $y = 0$. As a result of (A.53)

$$\nabla \cdot \vec{\Pi}_0 = \nabla \cdot \vec{\Pi}_1 \rightarrow \frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} = - \left(\frac{\partial \Pi_{0x}}{\partial x} - \frac{\partial \Pi_{1x}}{\partial x} \right) - \left(\frac{\partial \Pi_{0z}}{\partial z} - \frac{\partial \Pi_{1z}}{\partial z} \right). \quad (\text{A.56})$$

Combining (A.56) with (A.54) and (A.55) gives

$$\left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) \left(\frac{\partial \Pi_{1x}}{\partial x} + \frac{\partial \Pi_{1z}}{\partial z} \right). \quad (\text{A.57})$$

Summarizing these results, the boundary conditions on the Hertz potential at the $y = 0$ interface between free space and a non-magnetic dielectric material are

$$\Pi_{0\alpha} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1\alpha} \quad \dots \text{for } \alpha = x, y, z \quad (\text{A.58})$$

$$\frac{\partial \Pi_{0\beta}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1\beta}}{\partial y} \quad \dots \text{for } \beta = x, z \quad (\text{A.59})$$

and

$$\left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) \left(\frac{\partial \Pi_{1x}}{\partial x} + \frac{\partial \Pi_{1z}}{\partial z} \right). \quad (\text{A.60})$$

A.2 Boundary Conditions on Hertz Potential at a Conducting Interface

At the surface of a perfect conductor, the tangential components of electric field vanish

$$E_{0\beta}(y=0) = 0 \quad (\text{A.61})$$

Here the boundary conditions on the Hertz potential at the surface of a perfect conductor are determined by examining the Hertz potential boundary conditions at a dielectric interface (A.58)-(A.60) in the limit that medium 1 becomes perfectly conducting ($\epsilon_1 \rightarrow -j\infty$).

For this analysis, it is assumed that free space occupies the half-space $y > 0$ (region 0), and a conducting material occupies the half-space $y < 0$ (region 1) (see again Figure A.1). Suppose a plane wave originates in the free space region and is incident on the conducting boundary at $y = 0$. Two incident wave polarizations must be considered. For the case of a tangentially polarized incident wave, the total potential in the free space region consists of the superposition of forward (incident) and backward traveling (reflected) waves, given by

$$\Pi_{0x} = A_0(e^{-jk_0 y} + R_t e^{jk_0 y}) \quad (\text{A.62})$$

where R_t is a reflection coefficient. The total potential in the conductive material (medium 1) consists solely of a forward traveling wave given by

$$\Pi_{1x} = A_0 T_t e^{-jk_1 y} \quad (\text{A.63})$$

where T_t is a transmission coefficient. Applying the boundary conditions on the Hertz potential at the $y = 0$ interface

$$\epsilon_0 \Pi_{0\alpha} = \epsilon_1 \Pi_{1\alpha} \quad (\text{A.64})$$

and

$$\epsilon_0 \frac{\partial \Pi_{0x}}{\partial y} = \epsilon_1 \frac{\partial \Pi_{1x}}{\partial y} \quad (\text{A.65})$$

to the wave expressions (A.62) and (A.63) leads to

$$\epsilon_o A_0 T_t = \epsilon_o A_0 (1 + R_t) \quad (\text{A.66})$$

and

$$-jk_1 \epsilon_1 A_0 T_t = jk_o \epsilon_o A_0 (-1 + R_t) \quad (\text{A.67})$$

respectively. It is seen that (A.66) may be expressed

$$T_t = \frac{\epsilon_o}{\epsilon_1} (1 + R_t) \quad (\text{A.68})$$

and from (A.67) comes

$$k_1 \epsilon_1 T_t = k_o \epsilon_o (1 - R_t). \quad (\text{A.69})$$

Substituting (A.68) into (A.69) yields

$$k_o \epsilon_o (1 - R_t) = k_1 \epsilon_1 \left[\frac{\epsilon_o}{\epsilon_1} (1 + R_t) \right] \quad (\text{A.70})$$

which simplifies to

$$k_o (1 - R_t) = k_1 (1 + R_t) \quad (\text{A.71})$$

or

$$R_t = \frac{k_o - k_1}{k_o + k_1}. \quad (\text{A.72})$$

Thus the reflection coefficient R_t takes the expected form. Now substituting (A.72) into (A.68) gives

$$T_t = \frac{\epsilon_o}{\epsilon_1} \left[1 + \frac{k_o - k_1}{k_o + k_1} \right] = \frac{\epsilon_o}{\epsilon_1} \left[\frac{2k_o}{k_o + k_1} \right] \quad (\text{A.73})$$

and so the reflection and transmission coefficients associated with a tangentially polarized plane wave incident from a free space region onto a differing conductive material at $y = 0$ are given by

$$R_t = \frac{k_o - k_1}{k_o + k_1}, \quad T_t = \frac{\epsilon_o}{\epsilon_1} \left[\frac{2k_o}{k_o + k_1} \right]. \quad (\text{A.74})$$

The coefficients in (A.74) are now examined in the limit that medium 1 becomes perfectly conducting ($\epsilon_1 \rightarrow -j\infty$). It is assumed that $\omega_o = \omega_1$, and $\mu_o = \mu_1$, therefore

$$R_t = \frac{\epsilon_o - \epsilon_1}{\epsilon_o + \epsilon_1} \quad (\text{A.75})$$

and thus

$$\lim_{\epsilon_1 \rightarrow -j\infty} R_t = -1. \quad (\text{A.76})$$

Likewise

$$T_t = \frac{\epsilon_o}{\epsilon_1} \left[\frac{2\epsilon_o}{\epsilon_o + \epsilon_1} \right] \quad (\text{A.77})$$

therefore

$$\lim_{\epsilon_1 \rightarrow -j\infty} T_t = 0. \quad (\text{A.78})$$

Now the conditions on the x -directed scalar components of Hertz potential at the surface of a perfect conductor are determined from

$$\Pi_{0x} = A_0(1 + R_t) = A_0(1 - 1) = 0 \quad (\text{A.79})$$

and

$$\Pi_{1x} = A_0 T_t = A_0(0) = 0 \quad (\text{A.80})$$

as $\epsilon_1 \rightarrow -j\infty$, for all (x,z) at $y = 0$. By a similar development

$$\Pi_{0z} = \Pi_{1z} = 0 \quad (\text{A.81})$$

for all (x,z) at $y = 0$. Thus it is seen that for a tangentially polarized incident wave, the tangential components of the scalar Hertz potential vanish at the surface of a perfect conductor.

Now consider the case of a normally polarized plane wave which is incident from the free space region onto medium 1. As before, the total potential in the free space region consists of the superposition of forward (incident) and backward traveling (reflected) waves

$$\Pi_{0y} = A_0(e^{-jk_o y} + R_n e^{jk_o y}) \quad (\text{A.82})$$

where R_n is a reflection coefficient. The total potential in the conductive material (medium 1) consists a forward traveling wave given by

$$\Pi_{1y} = A_0 T_n e^{-jk_1 y} \quad (\text{A.83})$$

where T_n is a transmission coefficient. From condition (A.58) on the Hertz potential

$$\epsilon_o \Pi_{0y} = \epsilon_1 \Pi_{1y} \quad (\text{A.84})$$

for all (x,z) at $y = 0$, comes the relationship

$$\epsilon_o A_0 (1 + R_n) = \epsilon_1 A_0 T_n \quad (\text{A.85})$$

which may be expressed

$$T_n = \frac{\epsilon_o}{\epsilon_1} (1 + R_n). \quad (\text{A.86})$$

It was shown above that $\Pi_{0x} = \Pi_{0z} = 0$ at $y = 0$, therefore

$$\frac{\partial \Pi_{0x}}{\partial x} = \frac{\partial \Pi_{0z}}{\partial z} = 0. \quad (\text{A.87})$$

As a result of (A.87), boundary condition (A.60) yields

$$\left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) (0) \quad (\text{A.88})$$

and thus

$$\frac{\partial \Pi_{0y}}{\partial y} = \frac{\partial \Pi_{1y}}{\partial y}. \quad (\text{A.89})$$

Applying condition (A.89) to (A.82) and (A.83) results in

$$-jk_1 A_0 T_n = jk_o A_0 (-1 + R_n) \quad (\text{A.90})$$

or

$$k_1 T_n = k_o (1 - R_n). \quad (\text{A.91})$$

Now substituting (A.90) into (A.91) gives

$$k_o(1 - R_n) = k_1 \frac{\epsilon_o}{\epsilon_1} (1 + R_n) \quad (\text{A.92})$$

which yields the reflection coefficient associated with a normally polarized plane wave incident upon the interface at $y = 0$

$$R_n = \frac{\epsilon_1 k_o - \epsilon_o k_1}{\epsilon_1 k_o + \epsilon_o k_1}. \quad (\text{A.93})$$

Substituting (A.93) into expression (A.86) gives the transmission coefficient associated with the normally polarized wave

$$\begin{aligned} T_n &= \frac{\epsilon_o}{\epsilon_1} (1 + R_n) = \frac{\epsilon_o}{\epsilon_1} \left[1 + \frac{\epsilon_1 k_o - \epsilon_o k_1}{\epsilon_1 k_o + \epsilon_o k_1} \right] \\ &= \frac{\epsilon_o}{\epsilon_1} \left[\frac{2\epsilon_1 k_1}{\epsilon_1 k_o + \epsilon_o k_1} \right] = \frac{2\epsilon_o k_1}{\epsilon_1 k_o + \epsilon_o k_1}. \end{aligned} \quad (\text{A.94})$$

The reflection coefficient (A.93) may be written

$$R_n = \frac{\epsilon_1 k_o - \epsilon_o k_1}{\epsilon_1 k_o + \epsilon_o k_1} = \frac{k_o - k_1 \epsilon_o / \epsilon_1}{k_o + k_1 \epsilon_o / \epsilon_1} \quad (\text{A.95})$$

and from this it is seen that

$$\lim_{\epsilon_1 \rightarrow -j\infty} R_n = 1. \quad (\text{A.96})$$

Likewise, from examination of (A.94) it is clear that

$$\lim_{\epsilon_1 \rightarrow -j\infty} T_n = 0. \quad (\text{A.97})$$

Now

$$\Pi_{1y} = A_0 T_n e^{-jk_1 y} = 0 \quad (\text{A.98})$$

for $\epsilon_1 \rightarrow -j\infty$, as expected, and

$$\Pi_{0y} = A_0 (e^{-jk_o y} + R_n e^{jk_o y}) = 2A_0 \quad (\text{A.99})$$

again for $\epsilon_1 \rightarrow -j\infty$. It is therefore seen that $\Pi_{0y} \neq 0$ at the surface of the perfect conductor (at $y = 0$). And finally

$$\frac{\partial \Pi_{0y}}{\partial y} = A_0 [-jk_o e^{-jk_o y} + jk_o R_n e^{jk_o y}] = -jk_o A_0 (1 - 1) = 0 \quad (\text{A.100})$$

for all (x, z) at $y = 0$. Thus the boundary conditions on the Hertz potential at the surface of the perfectly-conducting material are

$$\Pi_{0x} = \Pi_{0z} = \frac{\partial \Pi_{0y}}{\partial y} = 0 \quad (\text{A.101})$$

for all (x, z) at $y = 0$.

A.3 Application of Hertz Potential Boundary Conditions to Stripline Cross-Section

The boundary conditions established on the Hertz potential in the preceding sections are now applied to the generalized stripline structure shown in Figure A.2. This results in the conditions

$$\Pi_{0x} = \Pi_{0z} = \frac{\partial \Pi_{0y}}{\partial y} = 0 \quad (\text{A.102})$$

at $y = h$ the free space/perfect conductor boundary,

$$\Pi_{1x} = \Pi_{1z} = \frac{\partial \Pi_{1y}}{\partial y} = 0 \quad (\text{A.103})$$

at $y = -d$ the dielectric/perfect conductor boundary, and

$$\Pi_{0\alpha} = \frac{\epsilon_1}{\epsilon_o} \Pi_{1\alpha} \quad \dots \text{for } \alpha = x, y, z \quad (\text{A.104})$$

$$\frac{\partial \Pi_{0\beta}}{\partial y} = \frac{\epsilon_1}{\epsilon_o} \frac{\partial \Pi_{1\beta}}{\partial y} \quad \dots \text{for } \beta = x, z \quad (\text{A.105})$$

$$\left(\frac{\partial \Pi_{0y}}{\partial y} - \frac{\partial \Pi_{1y}}{\partial y} \right) = \left(1 - \frac{\epsilon_1}{\epsilon_o} \right) \left(\frac{\partial \Pi_{1x}}{\partial x} + \frac{\partial \Pi_{1z}}{\partial z} \right) \quad (\text{A.106})$$

at $y = 0$ the free space/dielectric boundary.

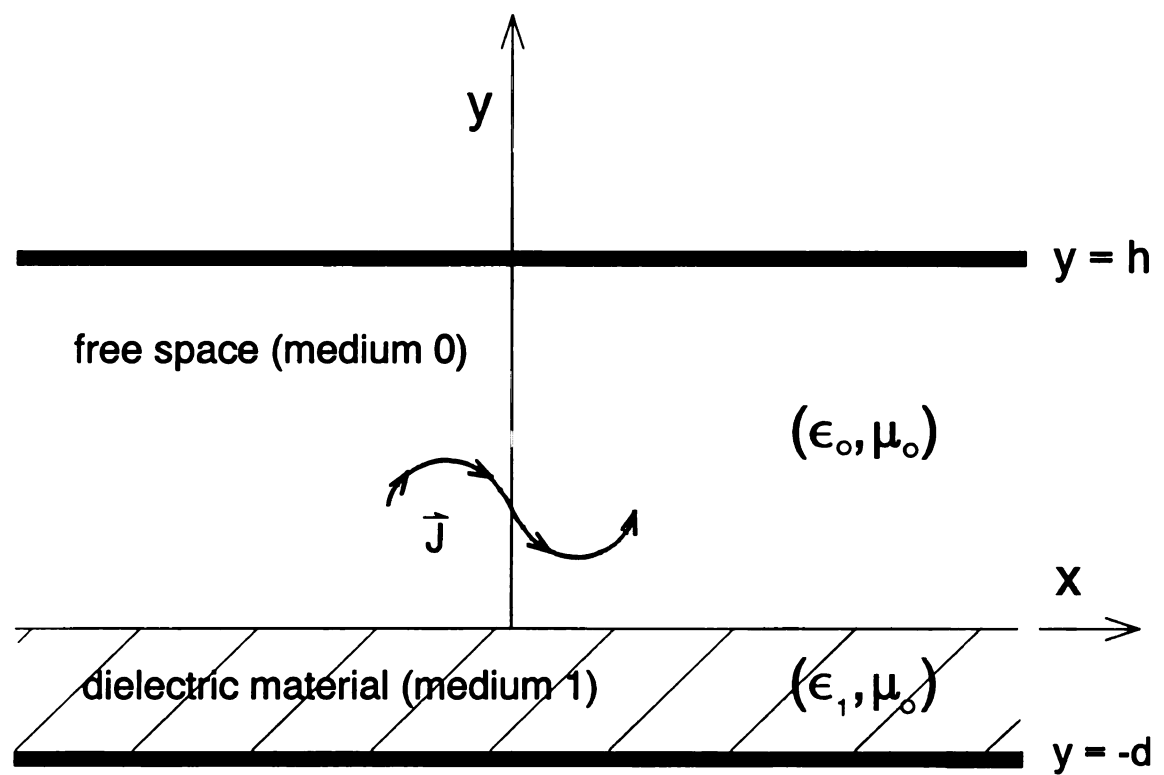


Figure A.2 Generalized stripline cross-sectional structure.

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