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Buckling and Post-Buckling Analysis of Neo-Hookean Plates and its Correlation to a Direct Energetic Stability Analysis

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Sangwoo Kim

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Ph.D. degree in Mechanics

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BUCKLING AND POST-BUCKLING ANALYSIS OF NEO-HOOKEAN PLATES AND ITS CORRELATION TO A DIRECT ENERGETIC STABILITY ANALYSIS

By

Sangwoo Kim

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Materials Science and Mechanics

1999

ABSTRACT

BUCKLING AND POST-BUCKLING ANALYSIS OF NEO-HOOKEAN PLATES AND ITS CORRELATION TO A DIRECT ENERGETIC STABILITY ANALYSIS

By

Sangwoo Kim

The elastic stability of buckling and post-buckling deformations for incompressible neo-Hookean rectangular plate subjected to a uni-axial thrust is investigated. The buckling deformation is described by the small deformation superposed on finite homogeneous deformations. Throughout the investigation the thickness of the plate is not limited. The resulting nonlinear boundary value problem is analyzed by using the perturbation expansion method in which an associated linear problem is solved at each order.

Buckling onset is determined from the first order expansion and can occur in either flexural or barreling mode shapes with any integer number of half wavelength in the direction of thrust. The solutions from the higher order expansion correspond to post-buckling deformations. The higher order problems inherit information from problems of the previous order, both by the expansion procedure and by the application of mathematical solvability conditions. The stability criterion for post-buckling deformations is based on energy competition between the buckled deformation and unbuckled homogeneous deformation in the vicinity of buckling onset. It is formally established that the energetic favoribility correlates with the load following character of the buckled solutions (progressive buckling vs. snap buckling).

Based on the expressions obtained by these procedures, it is found that the flexural buckled deformation is energetically favored over the unbuckled homogeneous deformation when mode number is small, otherwise the homogeneous deformation is favored. The barreling buckled deformation is always energetically favored over the homogeneous deformation. This contrasts with previous results of Sawyers and Rivlin (1982), who obtain essentially opposite results for the elastic stability of homogeneous deformation. The approach to evaluate the elastic stability by using the perturbation method considered in this research gives more insights to understand the buckling phenomena and is systematically applicable to higher order analysis.

Besides the main topic of stability evaluation, several approximate schemes for the critical buckling load in neo-Hookean three-ply sandwich type plate were developed in view of practical application. The schemes are based mostly on the Rayleigh quotients approach and trial solutions. These schemes can be expanded to general multi-ply composite plates and so reduce the effort to determine the critical buckling load.

To my parents
for their endless love, support
and teaching me the value of challenge and perseverance

ACKNOWLEDGEMENTS

Thank God for giving me the opportunity to write this acknowledgment at last. He always guided and strengthened me through His Words and taught me the values of humility and integrity.

During the course of my graduate school experience, I have learned a lot about the knowledge of applied mechanics as well as the life itself from many persons. The first person I should thank was my respectful advisor Prof. Thomas Pence who accepted me, taught me the detailed works of scientific research and advised me with valuable tips of life. Your practical remarks such as "when you stuck, step back and consider a simple example" and "follow the principles" will exist firmly in my memory. Also you showed me the importance of honesty and integrity and I apologize to have you hard times due to my insincerity. I appreciate your continuous encouragement to me when I am suffering.

My thanks also goes to my dear committee members - Prof. Dashin Liu, who made me to think about the theoretical and experimental works, Prof. Ronald Averill, who taught me the applied methods of mechanics and the fun of scientific jokes and Prof. Baisheng Yan who granted me the confidence for my works and made me to continue. They deserve my gratitude for their valuable supports and contributions to my work.

Very little of this work could have been accomplished without the help of my family for their continuous support and encouragement. I am thankful to my father and mother, Chang-Il Kim and Sung-Sil Baik for their enormous pray and priceless support. I hope I will have a chance to recompense their sufferings. Also I thank to my parents-in-law Hong-Soon Shin and Wol-Hwa Kim for their timeless pray and rigid assurance to my work.

Special recognition should be bestowed upon my close family members. I cannot find how to thank to my only blood brother Prof. Junwoo Kim who rescued me from the temptation to give up all my works by numerous emails and phone calls. His existence have given me the strength to overcome every difficulties. And I owe everything to my precious wife Kyoungsook Shin, who always showed the confidence in me and made me comfortable in home. In addition, I deeply thank you for your patience. My first son David made me to consider what is my problems and twin sons Samuel (Gom) and Daniel (Kan) gave me not only sufferings but also the pleasures and reasons to live. The contributions you all gave to my life are indescribable.

I also give my thanks to the staffs in Dept. of Materials Science and Mechanics for creating warm and supportive atmosphere. Many friends and elders in Korean community also deserve my thanks for their supports and advice to me and my family during our stay in this town. Finally I want to give my appreciation to the people who maintain the nature centers in Lansing area. In these places, I have relieved, thought, made a decision and recovered my confidence. It is so fortunate for me to have these places in a close distance.

Now I am standing top of one small hill in the mountains. Soon I will climb up another hill and may suffer the hard times. However I believe my experiences will help much to overcome these obstacles. Oh God, bless me and my path, and strengthen me just as you have given to me.

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CHAPTER 1

INTRODUCTION

1.1 Overview

Mechanical instabilities that lead to sudden structural rearrangement, have been a considerable factor in structural design. Two significant examples of mechanical instability are concerned with internal rupture (cavitation), in which a hole forms and grows in the interior of a solid body under the tensile loading, and buckling, in which the structure reconfigure its shape under the compressive loading. Both could eventually lead to failure. These phenomena involve large deformations so that they cannot be easily explained by the classical linear theory of elasticity, since the theory of elasticity on the material subjected to large deformations is inherently nonlinear. Analysis on this subject can predict a critical load at which the material fails its structural task and modes of instability which enables us to prevent the possible failure.

Above examples can be described in mathematical terminology as a bifurcation from a simple configuration known as the trivial solution. The concern of this research is bifurcation due to buckling in rectangular slabs. The post-buckling behavior considers the character of the buckled equilibrium paths in the vicinity of buckling initiation (rigorously, initial post-buckling). The post-buckling analysis provides not only the stability of equilibrium solutions but also the possibility of snap-buckling to be expected in the case of imperfect loading.

Hyperelasticity is the theory of nonlinear (finite) elasticity for hyperelastic materials whose elastic potential energy can be described by a strain energy function. The mechanical behavior of rubberlike materials which bear large deformations such as synthetic elastomers, polymers, and biological tissues as well as natural rubbers, can be analyzed by hyperelasticity. The stability problem of such materials mentioned above has been focused by many researchers since some theoretical materials have been suggested. These ideal materials have a specific strain energy functions whose characteristics are similar to rubberlike materials (Beatty, 1987). The neo-Hookean material is the simplest model of an incompressible, isotropic, hyperelastic materials.

The void nucleation and growth in a hyperelastic material can be understood as the bifurcation from a critical load at onset of void formation (Ball, 1982). The method of energy competition for minimum between the deformations with void (bifurcated) and without void (trivial) can be used to determine the stable - physically obtained - configurations. The void or bifurcated solution grows smoothly for an isotropic material with increasing tensile load on the exterior of material after the critical load of void formation is attained. This phenomenon may change for a composite material depending on its initial geometry as sudden cavity formation (snap-buckling) before the critical load may occur.

Numerous investigations on the buckling instability for rectangular plates or cylinder by using the linear theory of elasticity have been developed after von Karman theory was formulated (Matkowsky and Putnick, 1974). This theory considers the higher order terms added to small deformations. The governing equations are also nonlinear so that the proper linearization such as perturbation method can be used. However this approach is restricted to small strain conditions and therefore the material to be considered is applied only to thin plate.

For the buckling problem of hyperelastic rectangular plates, the buckling deformation can be described by using the technique of small deformation superposed on finite homogeneous deformations (Biot, 1963). Two buckling solution for the linearized equilibrium equations are possible with an arbitrary integer number of half-wave lengths. The flexural mode has symmetric shape and the barreling mode, which always occurs at higher load than the flexural mode, has antisymmetric shape with respect to loading direction. The buckling load of the flexure (barreling) mode is monotonically increasing (decreasing) with the geometrical aspect ratio and mode number. Hence the critical, or minimum, buckling load is always obtained in mode-1 flexural deformation. Then at each potential point of buckling initiation, the question is whether the buckling actually occurs as the load is increased or whether the unbuckled state persists. The answer to this question is related to the initial post-buckling analysis and stability evaluation. Here the buckled and unbuckled states compete for stability. Due to the concepts of elastic stability in a static setting which is independent on time, the state which has smaller energy holds stability.

Some characteristics will change for the multi-layered hyperelastic composite plate. The investigations on two- and three-ply sandwich plates reveal that the monotonic ordering of the buckling loads may change its behavior and this change depends on the initial geometry and mechanical properties of plies. The critical buckling which has minimum buckling load may be either mode-1 flexural or wrinkling deformation. It has also been shown that there exist a new family of buckling solutions in addition to the original family which is a continuation of the buckling solutions of noncomposite plate.

The analysis on prediction of critical buckling load for a N-ply plate eventually requires the nontrivial solutions of a $4N\times4N$ matrix equation originating from 4 boundary conditions and 4(N-1) interface conditions. The entries of this matrix involve the load parameter in a nonlinear way, giving rise to a nonlinear eigenvalue problem. The dimension of this matrix can be reduced to $2N\times2N$ for a symmetrically stacked plate. When the stacking number of plies goes higher, the complicated expression on matrix elements and errors caused by standard numerical procedures can be expected so that it is desirable to consider the approximate analysis for the buckling prediction. To establish the proper approximate schemes constitutes the second subject of this research.

1.2 Literature Review

For the structual problem, mechanical instabilities make the structure to reconfigure itself such that it often cannot achieve its assigned structual tasks. Furthermore, these phenomena usually appear below the critical limit of material property. The governing equations of equilibrium state for these cases will be nonlinear so that they give more than one solution. Of all these solutions, one solution gives the minimum total stored energy and typically the structure follows this solution in a physical sense. This energetically favorable solution leads to the stable configuration. Other solutions, which will be energetically unfavorable, leads to the unstable configurations. The buckled and unbuckled configurations compete for energy minimizer and typically the buckled form which is undesirable from the engineering point of view, appears to be stable. Eventually these phenomena carry out large deformations so that the classical linear theory of elasticity no longer applies.

The major contributions on elastic stability theory within the framework on finite elasticity and on von Karman theory which is derived from the infinitesimal or linear elasticity will be reviewed at first. Furthermore the literatures on bifurcation theory, which include both the cavitation and buckling problems, will be examined. The research works on buckling analysis of hyperelastic rectangular plate will be also reviewed in detail. Finally, the literature on the critical buckling load of multi-ply composite plate will be reviewed. This review is purposefully broad so as to emphasize how disparate phenomena in solid mechanics (structual buckling, cavitation rupture) and fluid mechanics (transition to turbulence) can be treated in a similar mathematical framework. Readers who are not interested in this level of detail can proceed directly to Section 1.3.

1.2.1 Elastic Stability

The theory of buckling and postbuckling behaviors of elastic structures was enunciated by Koiter (1981) for the case of small finite deformations from the configuration of equilibrium. The general theory on these can be found in an explanatory article by Budiansky (1974) where he presents the virtual work and energy approaches and showed that they are equivalent. In an energy approach, the post-buckling analysis is performed by variation of the potential energy functional and perturbation expansions of a load parameter and displacements within the class of general elasticity.

The theory of elastic stability was studied firstly by Euler more than 200 years ago for the lateral buckling of compressed slender bars and he used the idea of the method of adjacent equilibrium. From this emerged, the energy theory of stability for thin bodies which have large deflections but small strains was emerged. This type of problem leads to von Karman plate theory which eventually gives a nonlinear eigenvalue problem. In the most cases, the closed form solutions do not exist. For a thin elastic simply supported rectangular plate subjected to a compressive thrust, Bauer and Reiss (1965) obtained the approximate buckled solutions by using perturbation method, energy methods and series expansions. These buckled solutions are bifurcated from each eigenvalue of the linearized problem. Here the lowest eigenvalue is equal to the buckling load. Matkowsky and Putnick (1974) also studied possible equilibrium states after buckling onset as a multiplicity of eigenvalue. They evaluated the stability of each buckled solution by the amplitude decay of initial deviation on power series expansion. They showed that four of the nine possible equilibrium solutions are stable while the other five are unstable. For the buckling problem of a rectangular three-layered sandwich plate with soft core, the work of He and Cheng (1992) is based on Reissner's equation of sandwich plate and they found the similar results to previous authors. In addition, the other kind of equilibrium states which did not appear in noncomposite plate was shown. Above works were based on modified linear theory of elasticity so that the materials to be considered were limited to thin plates despite their large deflections.

The concept of elastic stability within the framework of finite elasticity theory was developed by Pearson (1955). He formulated the energy criterion for stability under the dead loading and pressure loading conditions by means of approximation in the nonlinearity of the stress-strain law. He then focused on the relationship between the existence of adjacent equilibrium and the energy criterion. It is found that both approaches are equivalent for special situations such as at points where an originally stable structure first becomes unstable. A review of various stability criteria may be found in the survey articles by Beatty (1965, 1987). The energy criterion of stability is eventually equivalent to the positiveness of second variation of energy function. The stability evaluation of an equilibrium state varies on each investigation for the specific material type and loading condition. Hill (1957) also studied the criterion for stability of an elastic solid in a state of finite strain under dead loading and showed that the stability criterion is closely related to the unique solution of associated boundary value problem in a series of adjacent equilibrium. Holden (1964) derived an inequality condition for stability based on energy criterion and obtained an estimate of the critical load of a circular column. Beatty (1965) considered static and dynamic implications of the stability theory generated from the above papers and modified the criterion. Then Beatty (1971) estimates the critical load for incompressible ideal models such as neo-Hookean and Mooney-Rivlin materials. In summary, the energy criterion for stability (which is equivalent to the existence of adjacent equilibrium) requires a minimum potential energy at that state.

The problem on void formation and growth in solids and sudden void formation in vulcanized rubber has been focused as failure mechanism to many applied scientists. The work of Ball (1982) which gives a theoretical basis to the most developments thereafter, considered various problems of uniform radial traction or displacement at the boundary of an elastic solid sphere. He investigated the bifurcation problems for the equations of nonlinear elasticity as a branch of radially symmetric solutions with hole bifurcate from a path of homogeneous radial expansions with no hole. It is also shown that such bifurcated solution is the only stable solution for sufficiently large loads by minimization of the total energy integral. According to the linear theory, a material without hole remains in the same shape despite the amount of external loads. The changes in bifurcation behavior when the material has inhomogeneity was examined by Horgan and Pence (1989) for a composite sphere composed of two different neo-Hookean materials. Unlike the homogeneous sphere, the bifurcation diagram for composite sphere may fold back in a plot of hole radius vs. external load. This gives a discontinuity in path for stable equilibrium configurations which turn out to be a snap-buckling by using the energy competition for minimizer to a stable configuration. This concept is also used for non-radially symmetric solutions by James and Spector (1991) for a large class of nonlinear elastic materials. They found that the radial deformation of spherical void is energetically unfavorable to the formation of long, thin filamentary void. An extensive bibliography on the various application of void formation problems can be found by Polignone and Horgan (1993) in which they studied the combined effects of material anisotropy and inhomogeneity. Horgan and Polignone (1995) reviewed and summarized the investigations on radially symmetric cavitation in nonlinearly elastic solids which focused on results established using the bifurcation analysis.

The bifurcation approach has been considered as a useful tool in the wide fields of nonlinear mechanics (Keller and Antman, 1967), especially when the qualitatively different behaviors emerge from the original state. In nonlinear hydrodynamics, the fluid flow changes from the laminar to the turbulent state at a critical value of certain parameter such as Reynolds number. These two states compete for stability. Kirchgassner (1975) reviewed the works on nonlinear hydrodynamic stability, especially for the Taylor and the Benard models. The Couette-Taylor problem (Tagg, 1994) deals with the viscous flow between two coaxial infinitely long cylinders rotating in the same direction. If the angular velocity surpasses a critical value, the basic Couette flow changes to a new state as Taylor vortices. The Benard problem deals with a viscous fluid in a horizontal layer which is heated from below. If the temperature difference between lower and upper planes passes a critical value, the convective motion is observed from the purely conductive state in which the fluid remains at rest.

The method of incremental deformation superposed on finite deformations to examine the buckled shape was introduced by Biot (1963). He then applied this theory to incompressible thick rubberlike slabs in a uniaxial compression. He determined the flexural - bending type - bifurcations. Wu and Widera (1969) formulated the general nonlinear theory of a rectangular Mooney-Rivlin type solid subjected to a biaxial loading by considering small deformations superposed on finite homogeneous deformation. The bending (flexure) and bulging (barreling) type solutions were obtained. For compressible Blatz-Ko material and incompressible neo-Hookean material in a biaxial loading, Burgess and Levinson (1972) also found two kinds of buckling instabilities. Most of applications are restricted to a plane strain condition so that all deformations can be described in a two dimensional setting. Rivlin's cube problem as a fundamental application of stability was studied by Rivlin (1974) for the purely homogeneous deformation of a unit cube of incompressible neo-Hookean material subjected to three pairs of equal and opposite forces. From the variation of the energy functional, he showed that there exist seven possible equilibrium states in a tensile loading condition. Also, based on the criterion that an equilibrium state is stable if the second variation of the energy functional is positive, he found that one trivial and three nontrivial solutions are unstable and the other three nontrivial solutions are stable. For incompressible Mooney-Rivlin material in the equi-triaxial loading, Ball and Schaeffer (1983) investigated the bifurcation and stability of equilibrium solutions from the view of absolute minimizer of potential energy. Sawyers

(1976) studied the case of neo-Hookean cube where two pairs of loads on the cube are the same, but different from the third. MacSithigh and Chen (1992) developed the stability conditions based on energy minimization for general incompressible material in equi-biaxial loading and applied this condition to Mooney-Rivlin material. For an infinitely long neo-Hookean cylinder subjected to radial loading, Haughton (1992) found the analytic nontrivial solutions and their stability based on the criterion that the second variation of energy functional must be positive for its stability. The analytic solutions have the form of modified Bessel functions in a polar coordinate system and he concluded that all the nontrivial solutions are stable.

The brief summary on incremental deformation equations were presented in Ogden (1992). Furthermore he examined the stability of the underlying deformation and the connection between stability and uniqueness of solution. Ogden (1995) also discussed the stability of the finite deformation near the point of bifurcation from the view point of dynamics.

The perturbation expansion method has been well used to analyze the cases concerned with the nonlinear boundary value problem. Elgindi et. al. (1992) considered the case of long elastic cylindrical tube submerged in a liquid by means of perturbation method. The obtained solutions in the neighborhood of the critical buckling pressure, showed that the deformed shape changes drastically from the unbuckled circular tube. Parker and Mote (1996) developed a perturbation method for self-adjoint eigenvalue problems with perturbed boundary conditions. The finite order of expressions for the eigenvalue perturbations are derived for distinct unperturbed and degenerate eigensolutions. The eigensolutions are simpler than traditional eigenfuction expansion and are convenient for applications to further analysis. An asymptotic expansion technique using power series, is applied to a nonlinear asymptotic membrane theory for thin hyperelastic plates by Erbay (1997).

For the rectangular neo-Hookean plate subjected to uniaxial compression, the buckled shapes and their critical loads depending on slenderness of initial geometry have been the main issues. Levinson (1968) considered the small disturbance of homogeneous deformation by using the first variation of energy functional with displacement potential function. Nowinski (1969) and Sawyers and Rivlin (1974) analyzed the linearized equilibrium equation of small deformation superposed on the finite homogeneous deformation. They revealed that there exist two kinds of buckling - flexural and barreling deformations - depending on the load parameter. The buckling load of flexural mode is monotonically increasing and that of barreling mode is monotonically decreasing when the mode number is increased. Finally both modes meet at the infinite mode or plate-like geometry as shown in the figures from Chapter 4 so that the barreling occurs at higher loading than flexure.

The subsequent paper by Sawyers and Rivlin (1982) investigated the stability of homogeneous deformation at the buckling onset for a neo-Hookean rectangular plate on the basis of an energy criterion that is related to the sign of second variation of the difference in potential energies between buckled and homogeneous deformation. The flexural buckling deformation, was found to be stable only if the aspect ratio l_2/l_1, is less than about 0.2 so that after this range, the buckling can occur. And the barreling buckling deformation was found to be always stable. These results were obtained from the linearized energy equation with linearized solutions and remainders. Meijers (1987) studied the post-buckling behaviors of surface waves when a neo-Hookean half space is compressed in perfect and imperfect geometries by using approach of previous paper. An asymptotic expansion of the potential energy as a linear combination of two buckling modes and remainder terms was used to show the reduction of critical loads in a imperfect geometry. Lazopoulos (1996) has utilized this type of methodology to determine the features of the post-bifurcation displacement solutions and given associated numerical examples for incompressible materials including Mooney-Rivlin type which is more general than the neo-Hookean type used here. His second order displacement solution were compared with the experimental work by Beatty and Dadras (1976) where the barreling type displacement exists. In this respect his work is similar to our work as well as Sawyers and Rivlin (1982) in a respect of post-buckled solution space which will be developed in Chapter 4. We, however, correlate the stability behavior directly to the energy difference between the homogeneous (unbuckled) solution and the potential buckled solution. In addition, we show how the third order displacement solution can be eliminated from the immediate post-bifurcation analysis. Bajenitchev (1996) developed a numerical procedure for incompressible material in nonlinear elasticity based on perturbation methods and finite element approximations. The application to a plane deformation of Mooney-Rivlin type rectangular body are compared with analytic results for the behavior of force-displacement dependence. Beatty and Pan (1998) investigated the elastic stability of thick plate as hyperelastic Bell material and incompressible neo-Hookean model on the basis of Euler dead load criterion. They found the similar type buckling solutions - symmetric and asymmetric mode. Also the critical load for extremely thin plate was deduced to compare with classical Euler buckling load.

1.2.2 Buckling Instability of Composite Plates

Buckling instability of sandwich composite plate based on the results of a noncomposite plate was studied by Pence and Song (1991). They investigated a symmetric three-ply plate consisting of neo-Hookean materials within the framework of incremental deformation. Unlike the noncomposite case, the buckling onset prediction curve shows non-monotonicity depending on some initial geometrical settings as shown in Figure 3. Therefore the critical buckling load - the smallest load for buckling onset - is not always mode-1 flexure. Based on this algorithm to predict the buckling onset, Song and Pence (1992) implemented the optimal design scheme for a neo-Hookean sandwich plate. For the three-ply plate, they compared the buckling load of one configuration (which has the stiffer central layer) and its conjugate configuration (which has the stiffer outer layers). It is found that there exists a transition point which changes the configurations having lower critical load.

Further study on symmetric sandwich plate by Qiu, Kim and Pence (1994) showed that there exist another family of onset buckling solutions above the original family as shown in the figures of Chapter 6. The wrinkling load of the original family converges to that of noncomposite case regardless of the initial geometry and material properties. The wrinkling load of new family converges to a value that depends only on the shear modulus ratio. Also, their investigation for the asymmetric 2-ply sandwich plate shows that there are three onset buckling curves. The lower two solutions of these are the counterpart of the original family of symmetric case. However, due to the nature of asymmetry, each solution does not represent the flexure or barreling characteristics seen in the symmetric case. For the onset buckling prediction of general N-ply neo-Hookean sandwich plate, investigation predicts that 4N×4N nonlinear determinant equation should be solved. The dimension of the matrix can be reduced to 2N×2N for a symmetrically stacked composite plate. If the number of plies in a plate is larger, the mathematical analysis encountered will be difficult to handle even by the numerical computation. This motivates the consideration of approximate methods.

The Rayleigh quotient approach is widely used for an approximation technique to determine the natural frequency in vibration problems and the critical buckling load for structual problems. With reasonable test functions, the Rayleigh quotient based on displacement and stress as the independent field yields an upper bounds to the exact value (Sagan, 1961). For the layered elastic composites, where the material properties are discontinuous across an interface, Nemat-Nasser and Minagawa (1975) proposed a new quotient which is obtained from combining the displacement and stress Rayleigh quotients. Lang and Nemat-Nasser (1977) applied these quotients to the problems of vibration and buckling. Horgan et. al. (1978) discussed the bound for various quotients and showed the closer upper and lower bounds of the new quotient.

1.3 Thesis Organization

The main purpose of this thesis is to investigate the post-buckling instabilities of thick rectangular plate near buckling initiation. This is the content of Chapter 2 through 5. A somewhat related, and more applied issue, is the determination of convenient approximate schemes for the buckling loads themselves in more complicated geometries, such as a multi-layered composite plate. We give some development on this issue in Chapter 6, but detailed further inquiry into this issue is not developed here. In all of these efforts, we will concentrate our attention to an incompressible, isotropic, neo-Hookean type hyperelastic material in which the strain energy function is known to have the simplest form of all ideal models and closest behavior to the rubber materials.

In Chapter 2, the basis hyperelasticity theories necessary for the buckling analysis of neo-Hookean single layer plate will be presented with the associated boundary conditions. The buckling phenomena are explained by an incremental deformation superposed onto the finite homogeneous deformations (the trivial or unbuckled deformation). The equilibrium equations and boundary conditions for buckling deformations with the condition of incompressibility generate a fully nonlinear second order boundary value problem. Also the stability criterion based on the concept of minimum energy where the stable deformation achieves the lowest energy under that of competitive deformations, will be constructed for stability evaluation. The potential energies are derived for the buckled state and the homogeneous deformation which compete for energy minimum. Then we will seek to determine which deformation is energetically stable and whether this stability evaluation is directly related to the behavior of the load parameter.

In Chapter 3, the formal perturbation expansion method in which the buckling load parameter is expanded from the buckling onset will be introduced and applied to the previously developed nonlinear problem. The nonlinear problem then produce the set of linear boundary value problems. The solutions of each linear problem will construct a full buckled deformation. Though this methodology will give an approximate result because of the limitation of expansion, it is anticipated that the necessary post-buckling behaviors in the vicinity of buckling onset will be obtained. Hence, the second term in the expansion of load parameter is of interest since the sign of this term plays an important role in buckling development phenomena such as load shortening and snap-buckling in an imperfect loading condition. Furthermore it will show that the load parameter can be obtained from the relation of the first order solution and the nonhomogeneous terms in the higher order problems since the differential operators between the first and higher order problems are the same.

Chapter 4 will devote to find the nontrivial solutions of each linear boundary value problem. The separation of variables will generate the partial differential equations to the fourth order ordinary differential equation. Then by using the series expansion method, the solutions of each order will be developed. The first order results show the behaviors of buckling initiation. To evaluate the stability after bifurcation occurs, the higher order solutions are necessary. However, it will show that the symmetric nature of the load parameter reduces the efforts to find the third and higher order solutions.

The analytical procedure to find the stability parameter will be presented in Chapter 5. The numerical results and their explanation will be followed. Because the approach involves a highly complex calculation, an asymptotic analysis in limit cases will be determined and their results for stability will be presented for comparison. This concludes the major topic area of this thesis.

In Chapter 6, we will explore a related application, that of determining buckling initiation in more complicated geometries by approximate methods. At first, the buckling onset analysis of a single ply plate will be extended to multi-ply sandwich type plate. After developing the energy criterion of stability, we will then apply this to determine the critical buckling load for multi-layered composite plate composed of neo-Hookean materials with two different moduli. The formulations on single ply plate will modify to adapt the composite plate. Since the onset of buckling corresponds to the homogeneous equation of the first order in perturbation expansion, we will not need to consider the complex higher order equations. Beside the formal approach to find the buckling onset load (which will encounter much complex mathematical analysis), we will construct various approximate schemes based on rather simple buckling solutions of the noncomposite plate. These schemes will satisfy some parts of complete conditions which will be discussed later. The result on the prediction of the buckling onset load for each scheme can be compared with the exact results of the three-ply plate. Then we will implement the schemes which will give the closest results to the general ply plate.

Finally, conclusions of this research and recommendations for future works derived from this research are given in Chapter 7. In addition, this thesis includes two Appendices. Appendix A discusses the stability evaluation by perturbation expansion methods for relatively simpler example problems so as to better outline the structure of the procedures for comparison to the main topic of thesis. Appendix B contains a collection of detailed formulations used in Section 5.2 which apply to a stability parameter equation.

The procedures developed in this research for the elastic stability of post-buckled deformation may have the importance to determine the possible buckled shapes and may contribute to design the structures. Also the results will be a basis of the analysis of imperfect loading. The procedure can be expandable to higher order analysis to get more accurate anticipation for structual stability.

CHAPTER 2

PRELIMINARY WORKS ON NEO-HOOKEAN PLATE

2.1 Introduction

The theory of elastic materials subjected to large deformations has been evolved through the investigations on rubberlike materials and founded a basis of finite elasticity (Beatty, 1987). A hyperelastic material for which there exists an elastic potential energy function has been also focused in the study of finite elasticity which is known as hyperelasticity. The neo-Hookean material is the widely used theoretical model of incompressible isotropic hyperelastic material and the simplest model of rubberlike elastic behavior. Throughout the thesis on nonlinear elastic stability, our attention is restricted to neo-Hookean materials. In this Chapter, the necessary equations used to describe finite deformations of neo-Hookean rectangular plate are formulated and buckling behaviors of thickness-independent plate are presented in the context of finite elasticity. For the minimum energy principle of elastic stability analysis, the differences in energy between unbuckled and buckled deformations are also formulated. It is shown that the buckling deformations of neo-Hookean plate are characterized by solving a nonlinear boundary value problem.

2.2 Problem Descriptions

We shall consider an rectangular plate of incompressible, isotropic, homogeneous hyperelastic material which occupies a dimension of $2l_1 \times 2l_2 \times 2l_3$ before any external loads are applied. The three dimensional rectangular Cartesian coordinate system $\mathbf{X}=\mathbf{X}(X_1,X_2,X_3)$ is located in the center of the plate as its origin and its axes are parallel to the edges of the plate. Then the equal and opposite thrusts are applied to both ends on $X_1 = \pm l_1$. The geometry of considered neo-Hookean plate is described in Figure 2.1

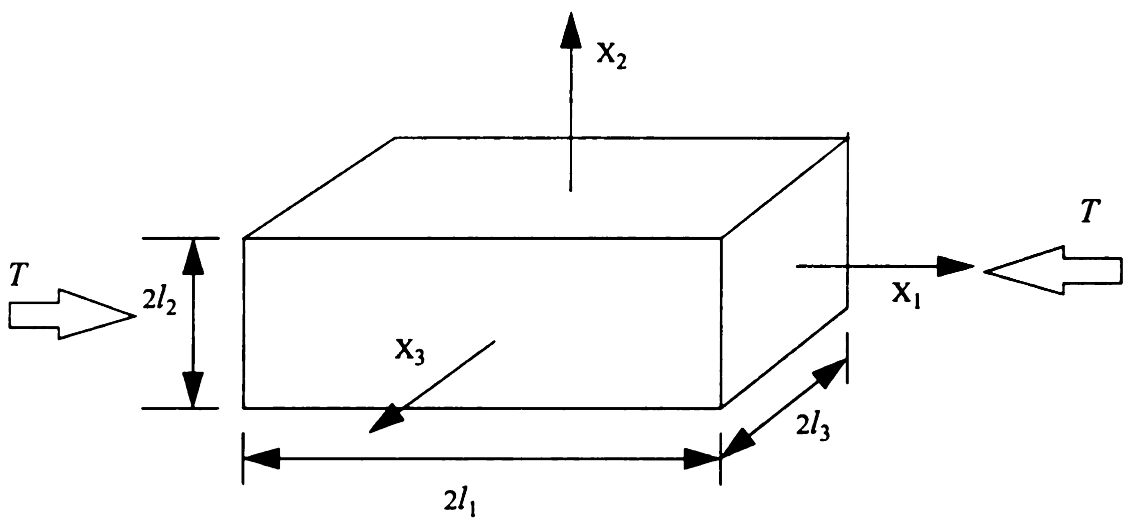


Figure 2.1 Description of the neo-Hookean rectangular plate under consideration. The thrusts T are applied to the ends of surfaces at $X_1=\pm l_1$ and the plate has a dimension of $2l_1 \times 2l_2 \times 2l_3$.

The current configuration is defined by undeformed or reference coordinates so that the deformation tensor is given by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{2.1}$$

where $\mathbf{X}(X_1,X_2,X_3)$ and $\mathbf{x}(x_1,x_2,x_3)$ are undeformed and deformed coordinates, respectively. The deformation gradient tensor and Green's deformation tensor are given as

$$\mathbf{F} = (\partial\mathbf{x}/\partial\mathbf{X}), \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T. \tag{2.2}$$

The condition of material incompressibility requires that the volume does not change after deformation so that

$$\det \mathbf{F} = 1. \tag{2.3}$$

The Cauchy stress tensor for the incompressible isotropic hyperelastic material is then given by

$$\tau = -p\mathbf{I} + 2\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right)\mathbf{B} - 2\left(\frac{\partial W}{\partial I_2}\right)\mathbf{B}^2, \tag{2.4}$$

where p is the hydrostatic pressure due to the incompressibility constraint, I_1 and I_2 are the first and second invariants of $\mathbf{B}$ and $W = W(I_1, I_2)$ is the strain energy density function of hyperelastic material. The symmetric Cauchy stress tensor τ described in a current configuration is transformed to the Piola-Kirchoff stress tensor $\mathbf{S}$ in a reference configuration which is given by

$$\mathbf{S} = \mathbf{F}^{-1}\tau, \tag{2.5}$$

after applying the incompressibility condition (2.3). The equilibrium equations in a reference frame are then expressed as

$$\mathrm{Div}\, \mathbf{S}^T = 0. \tag{2.6}$$

The plate under consideration is subjected to a compressive load on each surfaces $X_1 = \pm l_1$. The boundary conditions for a frictionless thrust with an overall stretch ratio of ρ are

$$S_{12} = S_{13} = 0, \qquad \text{on} \qquad X_1 = \pm l_1, \tag{2.7}$$

$$x_1 = \pm\rho l_1 \qquad \text{on} \qquad X_1 = \pm l_1, \tag{2.8}$$

where the case of compression requires $0<\rho<1$. The traction free surfaces $X_2 = \pm l_2$ require

$$S_{21} = S_{22} = S_{23} = 0 \qquad \text{on} \qquad X_2 = \pm l_2. \tag{2.9}$$

The surfaces $X_3 = \pm l_3$, corresponding to a frictionless clamp, give boundary conditions

$$S_{31} = S_{32} = 0 \qquad \text{on} \qquad X_3 = \pm l_3, \tag{2.10}$$

$$x_3 = \pm l_3 \qquad \text{on} \qquad X_3 = \pm l_3. \tag{2.11}$$

This type of plate problem was considered by Sawyers and Rivlin (1974,1982) and extended to multi-layered plates by Pence and Song (1991), Song and Pence (1993) and Qiu, Kim and Pence (1994).

From now on, we will focus to neo-Hookean type material for which the strain energy density function is given by

$$W = \frac{\mu(I_1 - 3)}{2}, \tag{2.12}$$

where μ is the shear modulus. Then the Cauchy stress tensor (2.4) is reduced to

$$\tau = -p\mathbf{I} + \mu\mathbf{B}. \tag{2.13}$$

The boundary value problem given by (2.6) to (2.11) with incompressibility condition (2.3) has only one homogeneous deformation solution to within a rigid body motion. This deformation is expressed as

$$x_1 = \rho X_1, \qquad x_2 = \rho^{-1} X_2, \qquad x_3 = X_3, \tag{2.14}$$

where the principal stretches are $\lambda^I = \rho$, $\lambda^{II} = \rho^{-1}$ and $\lambda^{III} = 1$. With the Piola-Kirchoff stress tensor (2.5) and the condition $(2.9)_2$, the hydrostatic pressure becomes

$$p = \mu\rho^{-2}. \tag{2.15}$$

Let A_S be the original area of the surface normal to the X_1 direction. Then the total (compressive) thrust T applied to the faces $X_1 = \pm l_1$ for homogeneous deformation (2.14), is given by

$$T = -S_{11}A_S = -4\mu l_2 l_3(\rho - \rho^{-3}). \tag{2.16}$$

Thus T is monotonically decreasing with respect to ρ and vanishes when $\rho=1$ as shown in Figure 2.2(a), by scaled thrust $T_s=T/(4\mu l_2 l_3)$ vs. ρ. Introducing a new stretch ratio as

$$\lambda = \lambda^{II}/\lambda^{I} = \rho^{-2}, \tag{2.17}$$

the thrust becomes monotonically increasing along the increasing λ as shown in Figure 2.2(b), by scaled thrust T_s vs. λ, so that λ can play a role as a *load parameter*. However, the simplicity in mathematical formulation urges us to use ρ so that, at the stage of physical interpretation, the value ρ will be converted to λ according to (2.17). Compressive loading, which is of concern in this study, corresponds to $0<\rho<1$ and $\lambda > 1$.

2.3 Bifurcation from Homogeneous Deformation

Motivated by boundary condition (2.11), we restrict attention to states of plane-strain buckling taking place in the (X_1, X_2)-plane. The buckling can be described as the bifurcation from the solution of homogeneous deformation (2.14) so that the incremental deformations of buckling are superposed on finite homogeneous deformations. The fully finite deformation is then expressed as

$$\begin{aligned} x_1 &= \rho X_1 + v_1(X_1, X_2), \\ x_2 &= \rho^{-1} X_2 + v_2(X_1, X_2), \\ x_3 &= X_3, \end{aligned} \tag{2.18}$$

and the pressure field is accordingly

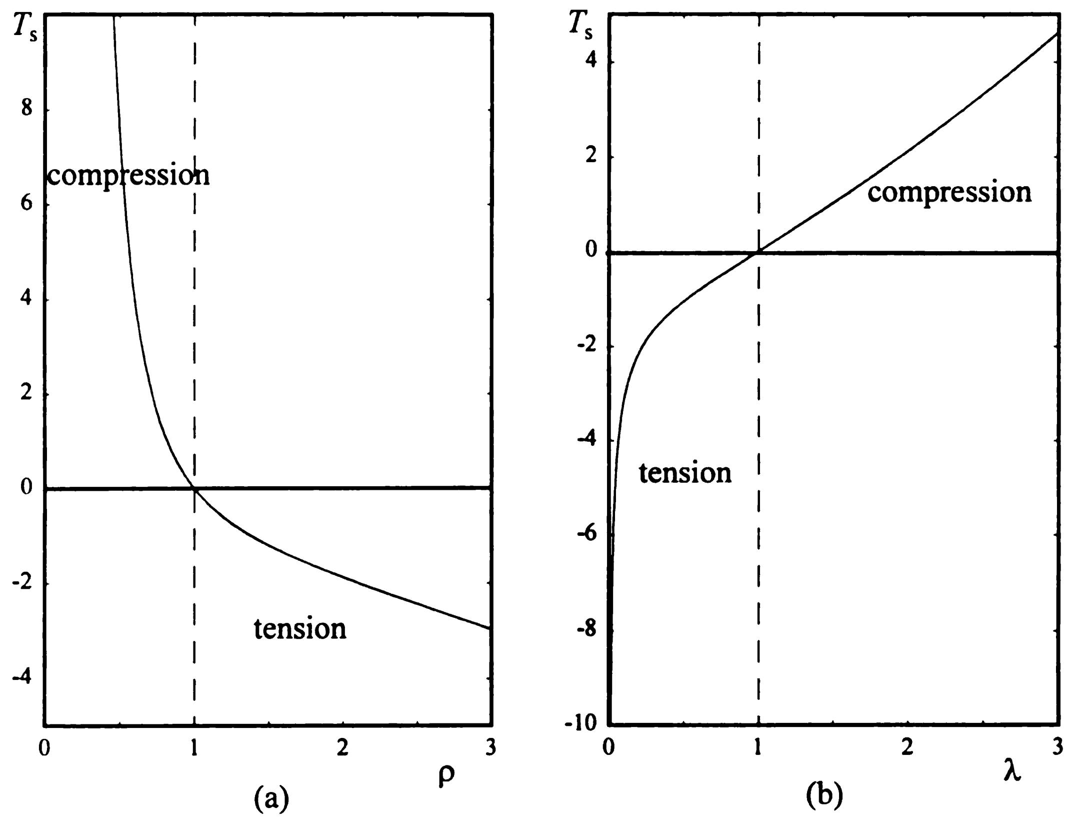


Figure 2.2 The relation between the scaled thrust T_s and load parameters ρ and λ. The thrust is the compressive load so for this study, the load parameter is restricted to $0<\rho<1$ or $\lambda>1$.

$$p = \mu\rho^{-2} + q(X_1, X_2, X_3). \tag{2.19}$$

The functions $v_1(X_1,X_2)$, $v_2(X_1,X_2)$, $q(X_1,X_2,X_3)$, as well as the values of ρ at which bifurcation can occur (nontrivial v_1 and v_2), are the unknowns in the mathematical problem. Substituting from (2.18) and (2.19) into the condition of incompressibility (2.3) gives

$$v_{1,1}v_{2,2} - v_{1,2}v_{2,1} + \rho v_{2,2} + \rho^{-1}v_{1,1} = 0. \tag{2.20}$$

The Piola-Kirchoff stress tensor (2.5) after applying the incompressibility condition (2.20) becomes

$$
\begin{aligned}
S_{11} &= \mu[(v_{1,1}+\rho)-\rho^{-2}(v_{2,2}+\rho^{-1})]-q(v_{2,2}+\rho^{-1}),\\
S_{12} &= \mu(v_{2,1}+\rho^{-2}v_{1,2})+qv_{1,2},\\
S_{21} &= \mu(v_{1,2}+\rho^{-2}v_{2,1})+qv_{2,1},\\
S_{22} &= \mu(v_{2,2}-\rho^{-2}v_{1,1})-q(v_{1,1}+\rho),\\
S_{33} &= \mu(1-\rho^{-2})-q,\\
S_{13} &= S_{23} = S_{31} = S_{32} = 0.
\end{aligned}
\tag{2.21}
$$

In equations (2.20) and (2.21), the commas in the subscript denotes differentiation with respect to the associated coordinate X_i, i=1,2,3.

The equilibrium equations $(2.6)_{1,2}$ for the buckled deformation now become

$$
\begin{aligned}
\mu(v_{1,11}+v_{1,22})-q_{,1}(v_{2,2}+\rho^{-1})+q_{,2}v_{2,1} &= 0,\\
\mu(v_{2,11}+v_{2,22})+q_{,1}v_{1,2}-q_{,2}(v_{1,1}+\rho) &= 0,
\end{aligned}
\tag{2.22}
$$

and the third equation $(2.6)_3$ is simply $-q_{,3} = 0$ so that q is confined to be a function of X_1 and X_2 only. Then the boundary value problem reduces to a two dimensional problem in which the plate is described in the domain Π ($-l_1<X_1<l_1$, $-l_2<X_2<l_2$) surrounded by the boundaries Γ_1 ($X_1 = \pm l_1$) and Γ_2 ($X_2 = \pm l_2$). The boundary conditions on v_1, v_2 and q associated with $(2.7)_1$, (2.8) and $(2.9)_{1,2}$ with the Piola-Kirchoff stress tensor (2.21) become

$$\mu(v_{2,1}+\rho^{-2}v_{1,2})+qv_{1,2} = 0, \qquad \text{on} \qquad \Gamma_1, \tag{2.23}$$

$$v_1 = 0, \qquad \text{on} \qquad \Gamma_1, \tag{2.24}$$

$$\mu(v_{1,2}+\rho^{-2}v_{2,1})+qv_{2,1} = 0, \qquad \text{on} \qquad \Gamma_2, \tag{2.25}$$

$$\mu(v_{2,2} - \rho^{-2} v_{1,1}) - q(v_{1,1} + \rho) = 0, \qquad \text{on} \qquad \Gamma_2. \tag{2.26}$$

The other boundary conditions $(2.7)_2$, $(2.9)_3$, (2.10) and (2.11) are automatically satisfied. Condition (2.24) implies $v_{1,2} = 0$, and this reduces the condition (2.23) to

$$v_{2,1} = 0, \qquad \text{on} \qquad \Gamma_1. \tag{2.27}$$

Note that an arbitrary constant can always be added to v_2 without effecting the solution of (2.22); this corresponds to the rigid body motion in X_2 mentioned earlier. Thus the complete nonlinear boundary value problem for the buckling of neo-Hookean plate is summarized as: Find $v_1(X_1,X_2)$, $v_2(X_1,X_2)$, $q(X_1,X_2)$ such that the following equations are satisfied.

$$\left.\begin{aligned} \mu(v_{1,11} + v_{1,22}) - q_{,1}(v_{2,2} + \rho^{-1}) + q_{,2} v_{2,1} &= 0, \\ \mu(v_{2,11} + v_{2,22}) + q_{,1} v_{1,2} - q_{,2}(v_{1,1} + \rho) &= 0, \\ v_{1,1} v_{2,2} - v_{1,2} v_{2,1} + \rho v_{2,2} + \rho^{-1} v_{1,1} &= 0, \end{aligned}\right\} \quad \text{in} \quad \Pi,$$
$$v_1 = 0, \qquad v_{2,1} = 0, \qquad \text{on} \quad \Gamma_1, \tag{2.28}$$
$$\left.\begin{aligned} \mu(v_{1,2} + \rho^{-2} v_{2,1}) + q v_{2,1} &= 0, \\ \mu(v_{2,2} - \rho^{-2} v_{1,1}) - q(v_{1,1} + \rho) &= 0, \end{aligned}\right\} \quad \text{on} \quad \Gamma_2.$$

The trivial solution $\mathbf{v} = (v_1, v_2, q) = \mathbf{0}$ in which the system has no deformation away from the homogeneous deformation (2.14), obviously satisfies the boundary value problem (2.28). However the concern here is in configurations that buckle away from this homogeneous deformation. Hence we seek nontrivial solutions $\mathbf{v}$; these will only occur for particular values of ρ and so will give particular values of thrust T according to (2.16). At the instant of bifurcation, these will correspond to distinct values ρ_{0i} and T_{0i} where the subscript i indexes the potential multiplicity of bifurcation points.

2.4 Energy Minimization of the Deformed Configuration

The potential energy of the deformation in a neo-hookean plate is formulated as the strain energy by the strain energy density function (2.12) and work done by the external load. The energy competition between the buckled and unbuckled state gives the physical preference of elastic stability after buckling occurs. Obviously the plate will follow the state which has smaller energy. The difference between the potential energies of the buckled state (2.18) and homogeneous deformed or unbuckled state (2.14), denoted by subscripts b and h respectively, is given by

$$\begin{aligned} \Delta I &= \iiint_V (W_b - W_h)\,dX_1 dX_2 dX_3 + (\overline{W}_b - \overline{W}_h) \\ &= 2l_3 \iint_\Pi \frac{\mu}{2}(2\rho v_{1,1} + 2\rho^{-1} v_{2,2} + v_{1,1}^2 + v_{2,2}^2 + v_{1,2}^2 + v_{2,1}^2)\,dX_1 dX_2. \end{aligned} \tag{2.29}$$

where W is the strain energy density function of neo-Hookean material, $\overline{}$ is the work associated with the external loading in each state and V is the domain of the undeformed configuration. Here the work difference vanishes since, on each external boundary, either the difference in traction vanishes or the difference in displacement vanishes. The plate is subjected to an incompressibility condition (2.20) so that the energy formulation for this problem is reconstructed by the Lagrange multiplier method as

$$\Delta E = \Delta I - 2l_3 \iint_\Pi \xi(v_{1,1}v_{2,2} - v_{1,2}v_{2,1} + \rho v_{2,2} + \rho^{-1} v_{1,1})\,dX_1 dX_2 \tag{2.30}$$

with Lagrange multiplier $\xi(X_1, X_2)$. It is then found that the multiplier ξ becomes the difference in hydrostatic pressures between the buckled and homogeneous deformations, that is q in (2.19). The equilibrium states are obtained from the first variation of the energy functional (2.30) which subsequently yields the nonlinear boundary value problem (2.28).

If $\Delta E<0$, that is if the potential energy of the buckled state is less than that of the homogeneous deformed state, then the deformation favors the buckled state. In other words, the buckled state is energetically stable. The opposite statement is also clear. At the instant of buckling initiation, the buckled state is not yet distinguished from the trivial solution so both states have the same potential energy, that is, $\Delta E=0$. However the sign of ΔE shows which deformation is stable out of the possible postbuckling solution paths.

2.5 Summary

In this Chapter, we described the neo-Hookean plate under consideration and formulated the nonlinear boundary value problem for the buckled deformation in view of incremental deformation on the finite homogeneous deformation. With the boundary conditions expressed in (2.28), the buckling equation reduced to two dimensional problem. The energy difference between buckled and unbuckled homogenous states are formulated in (2.30) for the elastic stability based on energy minimization.

CHAPTER 3

BIFURCATION ANALYSIS BY PERTURBATION EXPANSION METHODS

3.1 Introduction

Solutions to the nonlinear boundary value problem (2.28) involve nontrivial solutions v_1, v_2, q, at specific value of load parameter ρ which characterize the behavior of buckled deformations. The elastic stability of buckled deformations can be determined through the consideration into the energy difference equations (2.30) based on energy minimization scheme. However the direct analytical solutions for the nonlinear problem may not be obtained in a formal linear type process in view of the nonlinear natures. Introducing perturbation expansion methods in which the solutions are expanded with respect to the small parameter ε, makes the nonlinear problem to cast into an infinite set of iteratively coupled linear problems (see Bauer and Reiss, 1965; Matkowsky and Putnick, 1974; Budiansky, 1974). Upon truncation, this method will give approximate values to a degree of accuracy that is quantified in terms of the expansion parameter ε. In particular, behaviors near the bifurcation initiation can be captured with a relative accuracy by utilizing the perturbation expansion method. A discussion of this method for nonlinear boundary value problems that are simpler than (2.28) can be found in Appendix A.

The linearized problems for the buckling of neo-Hookean plate will be formulated via perturbation expansion methods in the subsequent section. Then analysis on the characteristics of resulting equations are followed. The investigations on the load parameters and energy difference equations based on the perturbation expansion methods will be treated finally.

3.2 Perturbation Expansion Methods

The buckling phenomena is well explainable as the bifurcation from the homogeneous deformation. Mathematically, nontrivial solution to the problem (2.28) corresponding to buckled deformation is bifurcated from the trivial solutions which is homogeneous deformation in (2.14). Obviously if we have the nontrivial solutions (v_1,v_2,q), then we can describe the buckled deformation of neo-Hookean plate. However the boundary value problem for the buckled deformation contains nonlinear nature so that it may not obtain the solution by linear type analysis. The perturbation expansion methods have been adopted for nonlinear analysis in various areas of applied mathematics by deriving groups of linear equations. Hence the formal linear analysis can be utilized but it shows the approximate results at moderate degree of accuracy based on the limitation that only the first few terms may be considered. The incremental terms in the fully finite deformation $(2.18)_{1,2}$ and the pressure field (2.19) are expanded with respect to relatively small parameter ε such as

$$\begin{aligned} v_1(X_1,X_2) &= \sum_{k=1}^{\infty} \varepsilon^k u_1^{(k)}(X_1,X_2), \\ v_2(X_1,X_2) &= \sum_{k=1}^{\infty} \varepsilon^k u_2^{(k)}(X_1,X_2), \\ q(X_1,X_2) &= \sum_{k=1}^{\infty} \varepsilon^k p^{(k)}(X_1,X_2). \end{aligned} \tag{3.1}$$

In vector notations with $\mathbf{v}=(v_1,v_2,q)$ and $\mathbf{u}^{(k)}=(u_1^{(k)},u_2^{(k)},p^{(k)})$, the deviations $\mathbf{v}$ away from the homogeneous deformation state are expressed as

$$\mathbf{v} = \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + \varepsilon^3\mathbf{u}^{(3)} + \ldots . \tag{3.2}$$

Here ε is a measure of the amount of deformation away from the homogeneous solution which is defined as

$$\varepsilon = \langle \mathbf{v}, \mathbf{u}^{(1)} \rangle / \langle \mathbf{u}^{(1)}, \mathbf{u}^{(1)} \rangle, \tag{3.3}$$

in accordance with the orthogonality condition for the vector functions of each order $\mathbf{u}^{(j)}$ and $\mathbf{u}^{(1)}$

$$\langle \mathbf{u}^{(1)}, \mathbf{u}^{(j)} \rangle = 0, \qquad j \neq 1. \tag{3.4}$$

The brackets $\langle \; , \; \rangle$ denote the bilinear inner product on pairs of vector functions in a domain Π so that

$$\langle \Psi, \Phi \rangle = \frac{1}{4 l_1 l_2} \iint_{\Pi} (\Psi^T \Phi) dX_1 dX_2. \tag{3.5}$$

In order to acknowledge the evolution of the postbuckling path with the thrust, the overall stretch ratio ρ is also expanded from the stretch ratio on buckling onset ρ_0 as

$$\rho = \rho_0 + \sum_{k=1}^{\infty} \varepsilon^k \rho_k. \tag{3.6}$$

The stretch ratios ρ or λ represent the load parameters as shown in Figure 2.2. and their expansions are related by the expansion of equation (2.17) such that

$$\begin{gathered} \lambda_0 = \rho_0^{-2}, \qquad \lambda_1 = -2\rho_0^{-3}\rho_1, \qquad \lambda_2 = -\rho_0^{-3}(2\rho_2 - 3\rho_0^{-1}\rho_1^2), \\ \lambda_3 = -2\rho_0^{-3}(\rho_3 - 3\rho_0^{-1}\rho_1\rho_2 + 2\rho_0^{-2}\rho_1^3), \qquad \ldots \end{gathered} \tag{3.7}$$

where $\lambda=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\varepsilon^3\lambda_3+O(\varepsilon^4)$. For the simpler analysis, the expansions of ρ will be used but for the interpretation of the thrust load, λ will be used after converting by the relations (3.7). If the odd terms in (3.6) vanish, $\rho_k=0$, k=1,3,5,..., then the deformation bifurcates symmetrically with respect to the trivial solution path. The symmetric buckling which is common to the perfectly loading plate and the analysis much easier than the case of asymmetric buckling.

3.3 Analysis on Neo-Hookean Plate

Entering the boundary value problem (2.28) with the expansions (3.1) and (3.6) and collecting together common orders of ε, give rise to the following general linearized boundary value problem at each order ε^k:

$$
\left.
\begin{aligned}
\mu(u^{(k)}_{1,11} + u^{(k)}_{1,22}) - \rho_0^{-1} p^{(k)}_{,1} &= f_1^{(k)}(X_1, X_2),\\
\mu(u^{(k)}_{2,11} + u^{(k)}_{2,22}) - \rho_0 p^{(k)}_{,2} &= f_2^{(k)}(X_1, X_2),\\
\rho_0^{-1} u^{(k)}_{1,1} + \rho_0 u^{(k)}_{2,2} &= f_3^{(k)}(X_1, X_2),
\end{aligned}
\right\} \quad \text{in } \Pi, \tag{3.8}
$$

$$
\begin{aligned}
u_1^{(k)} = 0, \qquad \mu(u^{(k)}_{2,1} + \rho_0^{-2} u^{(k)}_{1,2}) &= 0, && \text{on } \Gamma_1,\\
\mu(u^{(k)}_{1,2} + \rho_0^{-2} u^{(k)}_{2,1}) &= g_1^{(k)}(X_1, \pm l_2), && \\
\mu(u^{(k)}_{2,2} - \rho_0^{-2} u^{(k)}_{1,1}) - \rho_0 p^{(k)} &= g_2^{(k)}(X_1, \pm l_2), && \text{on } \Gamma_2,
\end{aligned} \tag{3.9}
$$

here k=1,2,.... The second in Γ_1 boundary conditions can be reduced to $u^{(k)}_{2,1} = 0$ because the first condition gives $u^{(k)}_{1,2} = 0$. Primarily the above equation is a nonhomogeneous second order partial differential equation except for order ε (k=1). The expressions on the right hand side in (3.8) and (3.9), $\mathbf{f}^{(k)}$ and $\mathbf{g}^{(k)}$, depend upon the lower order solutions $\mathbf{u}^{(j)} = [u_1^{(j)}, u_2^{(j)}, p^{(j)}]$ and $\rho_j, j<k$ for each order k. The first few of $\mathbf{f}^{(k)}$ and $\mathbf{g}^{(k)}$ are given as:

Order ε (k = 1):

$$
\begin{aligned}
f_1^{(1)} = 0, \qquad f_2^{(1)} = 0, \qquad f_3^{(1)} = 0,\\
g_1^{(1)} = 0, \qquad g_2^{(1)} = 0.
\end{aligned} \tag{3.10}
$$

Order ε^2 (k = 2):

$$
\begin{aligned}
f_1^{(2)} &= p^{(1)}_{,1} u^{(1)}_{2,2} - p^{(1)}_{,2} u^{(1)}_{2,1} - \rho_0^{-2}\rho_1 p^{(1)}_{,1},\\
f_2^{(2)} &= p^{(1)}_{,2} u^{(1)}_{1,1} - p^{(1)}_{,1} u^{(1)}_{1,2} + \rho_1 p^{(1)}_{,2},\\
f_3^{(2)} &= u^{(1)}_{1,2} u^{(1)}_{2,1} - u^{(1)}_{1,1} u^{(1)}_{2,2} - \rho_1 u^{(1)}_{2,2} + \rho_0^{-2}\rho_1 u^{(1)}_{1,1},\\
g_1^{(2)} &= 2\mu\rho_0^{-3}\rho_1 u^{(1)}_{2,1} - p^{(1)} u^{(1)}_{2,1},\\
g_2^{(2)} &= \rho_1 p^{(1)} + p^{(1)} u^{(1)}_{1,1} - 2\mu\rho_0^{-3}\rho_1 u^{(1)}_{1,1}.
\end{aligned} \tag{3.11}
$$

Order ε^3 ($k = 3$):

$$
\begin{aligned}
f_1^{(3)} &= \{p_{,1}^{(2)}u_{2,2}^{(1)} + p_{,1}^{(1)}u_{2,2}^{(2)}\} - \{p_{,2}^{(2)}u_{2,1}^{(1)} + p_{,2}^{(1)}u_{2,1}^{(2)}\} \\
&\quad -\rho_0^{-2}\{\rho_1 p_{,1}^{(2)} + (\rho_2 - \rho_0^{-1}\rho_1^2)p_{,1}^{(1)}\}, \\
f_2^{(3)} &= \{p_{,2}^{(2)}u_{1,1}^{(1)} + p_{,2}^{(1)}u_{1,1}^{(2)}\} - \{p_{,1}^{(2)}u_{1,2}^{(1)} + p_{,1}^{(1)}u_{1,2}^{(2)}\} + \{\rho_2 p_{,2}^{(1)} + \rho_1 p_{,2}^{(2)}\}, \\
f_3^{(3)} &= \{u_{1,2}^{(2)}u_{2,1}^{(1)} + u_{1,2}^{(1)}u_{2,1}^{(2)}\} - \{u_{1,1}^{(2)}u_{2,2}^{(1)} + u_{1,1}^{(1)}u_{2,2}^{(2)}\} - \{\rho_2 u_{2,2}^{(1)} + \rho_1 u_{2,2}^{(2)}\} \\
&\qquad + \rho_0^{-2}\{\rho_1 u_{1,1}^{(2)} + (\rho_2 - \rho_0^{-1}\rho_1^2)u_{1,1}^{(1)}\}, \\
g_1^{(3)} &= \mu\rho_0^{-3}\{2\rho_1 u_{2,1}^{(2)} + (2\rho_2 - 3\rho_0^{-1}\rho_1^2)u_{2,1}^{(1)}\} - \{p^{(2)}u_{2,1}^{(1)} + p^{(1)}u_{2,1}^{(2)}\}, \\
g_2^{(3)} &= \{\rho_2 p^{(1)} + \rho_1 p^{(2)}\} + \{p^{(2)}u_{1,1}^{(1)} + p^{(1)}u_{1,1}^{(2)}\} \\
&\quad -\mu\rho_0^{-3}\{2\rho_1 u_{1,1}^{(2)} + (2\rho_2 - 3\rho_0^{-1}\rho_1^2)u_{1,1}^{(1)}\}.
\end{aligned}
\tag{3.12}
$$

Order ε^4 ($k = 4$):

$$
\begin{aligned}
f_1^{(4)} &= \{p_{,1}^{(3)}u_{2,2}^{(1)} + p_{,1}^{(2)}u_{2,2}^{(2)} + p_{,1}^{(1)}u_{2,2}^{(3)}\} - \{p_{,2}^{(3)}u_{2,1}^{(1)} + p_{,2}^{(2)}u_{2,1}^{(2)} + p_{,2}^{(1)}u_{2,1}^{(3)}\} \\
&\quad -\rho_0^{-2}\{\rho_1 p_{,1}^{(3)} + (\rho_2 - \rho_0^{-1}\rho_1^2)p_{,1}^{(2)} + (\rho_3 - 2\rho_0^{-1}\rho_1\rho_2 + \rho_0^{-2}\rho_1^3)p_{,1}^{(1)}\}, \\
f_2^{(4)} &= \{p_{,2}^{(3)}u_{1,1}^{(1)} + p_{,2}^{(2)}u_{1,1}^{(2)} + p_{,2}^{(1)}u_{1,1}^{(3)}\} - \{p_{,1}^{(3)}u_{1,2}^{(1)} + p_{,1}^{(2)}u_{1,2}^{(2)} + p_{,1}^{(1)}u_{1,2}^{(3)}\} \\
&\qquad + \{\rho_3 p_{,2}^{(1)} + \rho_2 p_{,2}^{(2)} + \rho_1 p_{,2}^{(3)}\}, \\
f_3^{(4)} &= \{u_{1,2}^{(3)}u_{2,1}^{(1)} + u_{1,2}^{(2)}u_{2,1}^{(2)} + u_{1,2}^{(1)}u_{2,1}^{(3)}\} - \{u_{1,1}^{(3)}u_{2,2}^{(1)} + u_{1,1}^{(2)}u_{2,2}^{(2)} + u_{1,1}^{(1)}u_{2,2}^{(3)}\} \\
&\quad -\{\rho_3 u_{2,2}^{(1)} + \rho_2 u_{2,2}^{(2)} + \rho_1 u_{2,2}^{(3)}\} \\
&\qquad + \rho_0^{-2}\{\rho_1 u_{1,1}^{(3)} + (\rho_2 - \rho_0^{-1}\rho_1^2)u_{1,1}^{(2)} + (\rho_3 - 2\rho_0^{-1}\rho_1\rho_2 + \rho_0^{-2}\rho_1^3)u_{1,1}^{(1)}\}, \\
g_1^{(4)} &= \mu\rho_0^{-3}\{2\rho_1 u_{2,1}^{(3)} + (2\rho_2 - 3\rho_0^{-1}\rho_1^2)u_{2,1}^{(2)} + 2(\rho_3 - 3\rho_0^{-1}\rho_1\rho_2 + 2\rho_0^{-2}\rho_1^3)u_{2,1}^{(1)}\} \\
&\quad -\{p^{(3)}u_{2,1}^{(1)} + p^{(2)}u_{2,1}^{(2)} + p^{(1)}u_{2,1}^{(3)}\}, \\
g_2^{(4)} &= \{\rho_3 p^{(1)} + \rho_2 p^{(2)} + \rho_1 p^{(3)}\} + \{p^{(3)}u_{1,1}^{(1)} + p^{(2)}u_{1,1}^{(2)} + p^{(1)}u_{1,1}^{(3)}\} \\
&\quad -\mu\rho_0^{-3}\{2\rho_1 u_{1,1}^{(3)} + (2\rho_2 - 3\rho_0^{-1}\rho_1^2)u_{1,1}^{(2)} + 2(\rho_3 - 3\rho_0^{-1}\rho_1\rho_2 + 2\rho_0^{-2}\rho_1^3)u_{1,1}^{(1)}\}.
\end{aligned}
\tag{3.13}
$$

The linear operators in the left hand sides of (3.8) and (3.9) are the same for all orders of ε.

In the operator form, the boundary value problem (3.8) and (3.9) can be stated as

$$
F\mathbf{u}^{(k)} \equiv \begin{bmatrix} \mu\nabla & 0 & -\rho_0^{-1}\dfrac{\partial}{\partial X_1} \\ 0 & \mu\nabla & -\rho_0\dfrac{\partial}{\partial X_2} \\ \rho_0^{-1}\dfrac{\partial}{\partial X_1} & \rho_0\dfrac{\partial}{\partial X_2} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix}^{(k)} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}^{(k)} \equiv \mathbf{f}^{(k)}, \qquad \text{in } \Pi \tag{3.14}
$$

where $\nabla = \partial^2/\partial X_1^2 + \partial^2/\partial X_2^2$ is Laplacian and

$$
\begin{aligned}
G_1\mathbf{u}^{(k)} &= \begin{bmatrix} 1 & 0 & 0 \\ \mu\rho_0^{-2}\dfrac{\partial}{\partial X_2} & \mu\dfrac{\partial}{\partial X_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix}^{(k)} = \mathbf{0}, \quad \text{on } \Gamma_1, \\
G_2\mathbf{u}^{(k)} &= \begin{bmatrix} \mu\dfrac{\partial}{\partial X_2} & \mu\rho_0^{-2}\dfrac{\partial}{\partial X_1} & 0 \\ -\mu\rho_0^{-2}\dfrac{\partial}{\partial X_1} & \mu\dfrac{\partial}{\partial X_2} & -\rho_0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix}^{(k)} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}^{(k)} \equiv \mathbf{g}^{(k)}, \quad \text{on } \Gamma_2.
\end{aligned} \tag{3.15}
$$

Note for k=1, $\mathbf{f}^{(1)}=\mathbf{0}$ and $\mathbf{g}^{(1)}=\mathbf{0}$ so that the boundary value problem for order ε is homogeneous and is given by

$$
\begin{aligned}
F\,\mathbf{u}^{(1)} &= \mathbf{0}, && \text{in } \Pi, \\
G_1\mathbf{u}^{(1)} &= \mathbf{0}, && \text{on } \Gamma_1, \\
G_2\mathbf{u}^{(1)} &= \mathbf{0}, && \text{on } \Gamma_2.
\end{aligned} \tag{3.16}
$$

The solution for (3.16) determines the initiation of buckling and has been studied by Sawyers and Rivlin (1974, 1982) who gave the nontrivial solutions for special values of ρ_0. These solutions will be obtained and reviewed in sections 4.3.1 and 4.4 later. The cases of k=2,3,... extend the solution into the postbuckling region bifurcated from the trivial solution.

3.3.1 Linear Differential Operator

Let $\mathbf{H}_h(\Pi) \equiv \{u_1, u_2, p \in \Pi \times \Pi \times \Pi \rightarrow \Re^3 : G_1\mathbf{u}=0 \text{ on } \Gamma_1, G_2\mathbf{u} = 0 \text{ on } \Gamma_2\}$ and let $\mathbf{u} \in \mathbf{H}_h(\Pi)$ be continuously differentiable. The subscript h as used here is to indicate homogeneous boundary conditions. Notice also that $\mathbf{H}_h(\Pi)$ is dependent on ρ_0 because the boundary operators G_1 and G_2 depend on ρ_0. In this Section 3.3, we disregard the order superscript k in the equations for simplicity. Let F be the second order differential operator defined in (3.14) and consider

$$\langle F\mathbf{u}, \bar{\mathbf{u}}\rangle = \frac{1}{4l_1l_2}\iint_{\Pi}[(F\mathbf{u})^T\bar{\mathbf{u}}]dX_1dX_2. \tag{3.17}$$

Then integration by parts twice upon (3.17) with respect to the variables either X_1 or X_2 yields

$$\begin{aligned}\langle F\mathbf{u}, \bar{\mathbf{u}}\rangle = \frac{1}{4l_1l_2}[&\int_{\Gamma_2}[\{\mu(u_{1,1} - \rho_0^{-2}u_{2,2}) - \rho_0^{-1}p\}\bar{u}_1 + \mu(u_{2,1} + \rho_0^{-2}u_{1,2})\bar{u}_2]\big|_{\Gamma_1}dX_2\\
&+\int_{\Gamma_1}[\mu(u_{1,2} + \rho_0^{-2}u_{2,1})\bar{u}_1 + \{\mu(u_{2,2} - \rho_0^{-2}u_{1,1}) - \rho_0 p\}\bar{u}_2]\big|_{\Gamma_2}dX_1\\
&-\int_{\Gamma_2}[\{\mu(\bar{u}_{1,1} - \rho_0^{-2}\bar{u}_{2,2}) - \rho_0^{-1}\bar{p}\}u_1 + \mu(\bar{u}_{2,1} + \rho_0^{-2}\bar{u}_{1,2})u_2]\big|_{\Gamma_1}dX_2\\
&-\int_{\Gamma_1}[\mu(\bar{u}_{1,2} + \rho_0^{-2}\bar{u}_{2,1})u_1 + \{\mu(\bar{u}_{2,2} - \rho_0^{-2}\bar{u}_{1,1}) - \rho_0\bar{p}\}u_2]\big|_{\Gamma_2}dX_1\\
&+\iint_{\Pi}[u_1\{\mu(\bar{u}_{1,11} + \bar{u}_{1,22}) - \rho_0^{-1}\bar{p}_{,1}\} + u_2\{\mu(\bar{u}_{2,11} + \bar{u}_{2,22}) - \rho_0\bar{p}_{,2}\}\\
&+p(\rho_0^{-1}\bar{u}_{1,1} + \rho_0\bar{u}_{2,2})]dX_1dX_2].\end{aligned} \tag{3.18}$$

Applying the boundary condition requirements on $\mathbf{u}$ inherent in $\mathbf{H}_h(\Pi)$ to (3.18) shows that the second of the two terms in the first integral vanishes, as does the entire second integral, and the first of the two terms in the third integral. Hence

$$\langle F\mathbf{u}, \bar{\mathbf{u}}\rangle = \frac{1}{4l_1l_2}[\int_{\Gamma_2}[\{\mu(u_{1,1}-\rho_0^{-2}u_{2,2})-\rho_0^{-1}p\}\bar{u}_1-\mu(\bar{u}_{2,1}+\rho_0^{-2}\bar{u}_{1,2})u_2]|_{\Gamma_1}dX_2$$
$$-\int_{\Gamma_1}[\mu(\bar{u}_{1,2}+\rho_0^{-2}\bar{u}_{2,1})u_1+\{\mu(\bar{u}_{2,2}-\rho_0^{-2}\bar{u}_{1,1})-\rho_0\bar{p}\}u_2]|_{\Gamma_2}dX_1 \tag{3.19}$$
$$+\iint_{\Pi}[u_1\{\mu(\bar{u}_{1,11}+\bar{u}_{1,22})-\rho_0^{-1}\bar{p}_{,1}\}+u_2\{\mu(\bar{u}_{2,11}+\bar{u}_{2,22})-\rho_0\bar{p}_{,2}\}$$
$$+p(\rho_0^{-1}\bar{u}_{1,1}+\rho_0\bar{u}_{2,2})]dX_1dX_2].$$

This defines the adjoint operator to the triple consisting of the field operator F and the boundary operators G_1 and G_2. Let the associated adjoint operators be F^c and $G_1{}^c$, $G_2{}^c$. Then the integral over Γ_1 shows that $G_1{}^c=G_1$, the integral over Γ_2 shows that $G_2{}^c=G_2$ and the integral over Π shows that $F^c=F$. In other words, the linear differential operator F restricted to $\mathbf{H}_h(\Pi)$ is self-adjoint.

3.3.2 Different Formulations

The nonhomogeneous boundary value problems (3.14) and (3.15) can be formulated in different ways (see Reddy, 1986). The *weak (variational) formulation* is stated as follows: find $\mathbf{u}=(u_1,u_2,p) \in \mathbf{H}(\Pi)$ such that

$$\mathbf{B}(\mathbf{w},\mathbf{u}) = l(\mathbf{w}), \qquad \text{for all } \mathbf{w} \in \{(w_1, w_2, r)|w_1=0 \text{ on } \Gamma_1\}, \tag{3.20}$$

where $\mathbf{B}(\mathbf{w},\mathbf{u})$ is the bilinear form and $l(\mathbf{w})$ is the linear form given by

$$\mathbf{B}(\mathbf{w},\mathbf{u}) = \mu\iint_{\Pi}[(u_{1,1}w_{1,1}+u_{1,2}w_{1,2}+u_{2,1}w_{2,1}+u_{2,2}w_{2,2})$$
$$+\rho_0^{-2}(u_{1,2}w_{2,1}+u_{2,1}w_{1,2}-u_{2,2}w_{1,1}-u_{1,1}w_{2,1})]dX_1dX_2 \tag{3.21}$$
$$-\iint_{\Pi}[p(\rho_0^{-1}w_{1,1}+\rho_0w_{2,2})+r(\rho_0^{-1}u_{1,1}+\rho_0u_{2,2})]dX_1dX_2.$$

$$l(\mathbf{w}) = -\iint_{\Pi}(w_1f_1+w_2f_2+w_3f_3)dX_1dX_2+\int_{\Gamma_1}(g_1w_1+g_2w_2)|_{\Gamma_2}dX_1. \tag{3.22}$$

The solution space $\mathbf{H}(\Pi)$ indicates the boundary conditions that should be satisfied

$$\mathbf{H}(\Pi) = \{(u_1, u_2, p)|G_1\mathbf{u} = 0 \quad \text{on } \Gamma_1, \quad G_2\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_2\}. \tag{3.23}$$

Note that **B(w,u)** is symmetric, i.e. **B(w,u)** = **B(u,w)**. For sufficiently smooth functions, the weak formulation (3.20) is equivalent to the direct formulation (3.14) and (3.15).

The energy functional on **H**(Π) corresponding to symmetric **B(w,u)** becomes (Reddy, 1986)

$$\begin{aligned} E(\mathbf{u}) &= \frac{1}{2}\mathbf{B}(\mathbf{u}, \mathbf{u}) - l(\mathbf{u}) \\ &= \frac{\mu}{2}\iint_{\Pi}[u_{1,1}^2 + u_{1,2}^2 + u_{2,1}^2 + u_{2,2}^2 + 2\rho_0^{-2}(u_{1,2}u_{2,1} - u_{1,1}u_{2,2})]dX_1dX_2 \\ &-\iint_{\Pi}[p(\rho_0^{-1}u_{1,1} + \rho_0 u_{2,2}) + (u_1f_1 + u_2f_2 + u_3f_3)]dX_1dX_2 \\ &-\int_{\Gamma_1}(g_1u_1 + g_2u_2)|_{\Gamma_2}dX_1. \end{aligned} \tag{3.24}$$

The *energy (functional) formulation* is to find $\mathbf{u} \in \mathbf{H}(\Pi)$ which minimizes $E(\mathbf{u})$. If **B(u,u)** is positive for $\mathbf{u} \in \mathbf{H}(\Pi)$, then the weak and energy formulations are equivalent. In the problem under study here, it is not clear under what circumstances **B(u,u)** is positive. The first variation of $E(\mathbf{u})$ gives (3.14) as its Euler equation, when the following boundary conditions are specified:

$$\text{on } \Gamma_1 \quad \begin{aligned} &\mu(u_{1,1} - \rho_0^{-2}u_{2,2}) - \rho_0^{-1}p = 0 \quad &\text{or} \quad u_1 = 0 \\ &\mu(u_{2,1} + \rho_0^{-2}u_{1,2}) = 0 \quad &\text{or} \quad u_2 = 0 \end{aligned} \tag{3.25}$$

$$\text{on } \Gamma_2 \quad \begin{aligned} &\mu(u_{1,2} + \rho_0^{-2}u_{2,1}) = g_1 \quad &\text{or} \quad u_1 = 0 \\ &\mu(u_{2,2} - \rho_0^{-2}u_{1,1}) - \rho_0 p = g_2 \quad &\text{or} \quad u_2 = 0 \end{aligned} \tag{3.26}$$

In the condition (3.25) and (3.26), the right sides correspond to essential boundary conditions and the left sides to natural boundary conditions. Comparison of (3.25) and (3.26) with (3.9), shows that the boundary condition $u_1 = 0$ on Γ_1 of the direct formulation contributes the only essential boundary condition.

3.4 Load Parameters

The load parameters ρ represent buckling behavior on and after the buckling initiation. The homogeneous problem (3.16) for the case of k=1 will only have nontrivial solutions for certain special values ρ_0 which define buckling initiation modes. At these special values ρ_0, the differential operator trio $\{F,G_1,G_2\}$ is singular. The same differential operator trio as the homogeneous problem appears in (3.14) for the case of k=2,3,..., and the special values ρ_0 are used here. These cases of $k \geq 2$ will be a problem for solving nonhomogeneous boundary value problem (3.14) and (3.15) for a singular operator trio $\{F,G_1,G_2\}$. For most right hand sides, solutions will not exist. But for certain special right hand side of equation, the solution can exist - eventually this is explained by the *Fredholm Alternative Theorem* for solvability of the nonhomogeneous equation as developed next for this particular problem.

At a fixed value of ρ_0, let $\mathbf{w}=(w_1,w_2,r)$ be a nontrivial solution of the homogeneous equation (3.16) and let $\mathbf{u}=(u_1,u_2,p)$ be a solution to the nonhomogeneous equations (3.14) and (3.15) for given $\mathbf{f}$ and $\mathbf{g}$. Then consider the expression

$$\langle F\mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{u}, F\mathbf{w}\rangle = \frac{1}{4l_1l_2}\iint_{\Pi}\{(F\mathbf{u})^T\mathbf{w} - \mathbf{u}^T(F\mathbf{w})\}dX_1dX. \tag{3.27}$$

Since $F\mathbf{w} = \mathbf{0}$ and $F\mathbf{u} = \mathbf{f}$ in Π, the left side of the equation (3.27) is equivalent to $\langle \mathbf{f}, \mathbf{w}\rangle$. After integration by parts twice and applying the boundary conditions $G_1\mathbf{u} = \mathbf{0}$ and $G_1\mathbf{w} = \mathbf{0}$ on Γ_1 and $G_2\mathbf{u} = \mathbf{g}$ and $G_2\mathbf{w} = \mathbf{0}$ on Γ_2 to the right hand side of equation (3.27), it becomes

$$\langle \mathbf{f}, \mathbf{w}\rangle = \frac{1}{4l_1l_2}\int_{-l_1}^{l_1}(w_1g_1 + w_2g_2)\Big|_{-l_2}^{l_2}dX_1. \tag{3.28}$$

Thus if the nonhomogeneous problem (3.14) and (3.15) is to have solutions, then it is necessary that **f**, g_1 and g_2 obey the solvability condition (3.28). In particular, since $\mathbf{u}^{(1)}$ is a nontrivial solution of the homogeneous equation (3.16), any nonhomogeneous solution $(u_1^{(k)}, u_2^{(k)}, p^{(k)})$, k=2,3,... to (3.14) and (3.15) for given $\mathbf{f}^{(k)}$ and $\mathbf{g}^{(k)}$ must satisfy

$$\iint_\Pi \mathbf{f}^{(k)T}\mathbf{u}^{(1)} d\mathbf{X} - \int_{\Gamma_1} (u_1^{(1)} g_1^{(k)} + u_2^{(1)} g_2^{(k)})\big|_{\Gamma_2} dX_1 = 0. \tag{3.29}$$

In order for the nonhomogeneous problem of order ε^2 (k=2) to have a solution $\mathbf{u}^{(2)}$, the condition (3.29) must be satisfied with corresponding terms $\mathbf{f}^{(2)}$ and $\mathbf{g}^{(2)}$ in (3.11). Substituting $\mathbf{f}^{(2)}$ and $\mathbf{g}^{(2)}$ into condition (3.29) gives

$$R_1 + \rho_1 R_2 = 0, \tag{3.30}$$

where R_1 and R_2 are constants defined as

$$\begin{aligned} R_1 = {} & \iint_\Pi [u_1^{(1)}(p^{(1)}u_{2,2}^{(1)})_{,1} - u_2^{(1)}(p^{(1)}u_{1,2}^{(1)})_{,1}] d\mathbf{X} \\ & + \iint_\Pi 2p^{(1)}(u_{1,2}^{(1)}u_{2,1}^{(1)} - u_{1,1}^{(1)}u_{2,2}^{(1)}) d\mathbf{X}, \end{aligned} \tag{3.31}$$

$$\begin{aligned} R_2 = {} & \iint_\Pi [-\rho_0^{-2}(u_1^{(1)}p_{,1}^{(1)} - u_{1,1}^{(1)}p^{(1)}) + (u_2^{(1)}p_{,2}^{(1)} - p^{(1)}u_{2,2}^{(1)})] d\mathbf{X} \\ & - 2\mu\rho_0^{-3}\int_{\Gamma_1} (u_1^{(1)}u_{2,1}^{(1)} - u_{1,1}^{(1)}u_2^{(1)})\big|_{\Gamma_2} dX_1 - \int_{\Gamma_1} p^{(1)}u_2^{(1)}\big|_{\Gamma_2} dX_1. \end{aligned} \tag{3.32}$$

If R_2 is not zero, then ρ_1 can be expressed as

$$\rho_1 = -R_1/R_2. \tag{3.33}$$

Solutions $\mathbf{u}^{(2)}$ to the linear equations of order two, will exist only if the equation (3.33) is satisfied. Similarly for the nonhomogeneous problem of order ε^3 (k=3) with $\mathbf{f}^{(3)}$ and $\mathbf{g}^{(3)}$ in (3.12), the solvability condition (3.29) for the existence of $\mathbf{u}^{(3)}$, gives

$$R_3 + R_4\rho_1 + R_5\rho_1^2 + R_2\rho_2 = 0, \tag{3.34}$$

where R_3, R_4 and R_5 are

$$
\begin{aligned}
R_3 = \iint_\Pi [& u_1^{(1)}(p^{(2)}u_{2,2}^{(1)} + p^{(1)}u_{2,2}^{(2)})_{,1} - u_2^{(1)}(p^{(2)}u_{1,2}^{(1)} + p^{(1)}u_{1,2}^{(2)})_{,1} \\
& + 2p^{(1)}(u_{1,2}^{(1)}u_{2,1}^{(2)} - u_{2,2}^{(1)}u_{1,1}^{(2)}) \\
& + u_{2,1}^{(1)}(p^{(1)}u_{1,2}^{(2)} + p^{(2)}u_{1,2}^{(1)}) - u_{1,1}^{(1)}(p^{(1)}u_{2,2}^{(2)} + p^{(2)}u_{2,2}^{(1)})]d\mathbf{X},
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
R_4 = & \iint_\Pi [-\rho_0^{-2}(u_1^{(1)}p_{,1}^{(2)} - u_{1,1}^{(2)}p^{(1)}) + (u_2^{(1)}p_{,2}^{(2)} - p^{(1)}u_{2,2}^{(2)})]d\mathbf{X} \\
& - 2\mu\rho_0^{-3}\int_{\Gamma_1}(u_1^{(1)}u_{2,1}^{(2)} - u_{1,1}^{(2)}u_2^{(1)})\big|_{\Gamma_2}dX_1 - \int_{\Gamma_1}p^{(2)}u_2^{(1)}\big|_{\Gamma_2}dX_1,
\end{aligned}
\tag{3.36}
$$

$$
R_5 = \iint_\Pi [\rho_0^{-3}(u_1^{(1)}p_{,1}^{(1)} - p^{(1)}u_{1,1}^{(1)})]d\mathbf{X} + 3\mu\rho_0^{-4}\int_{\Gamma_1}(u_1^{(1)}u_{2,1}^{(1)} - u_2^{(1)}u_{1,1}^{(1)})\big|_{\Gamma_2}dX_1, \tag{3.37}
$$

so that

$$
\rho_2 = -\{R_3 + (\rho_1 R_4 + \rho_1^2 R_5)\}/R_2, \tag{3.38}
$$

if R_2 is not zero. The higher order parameter ρ_i, i=3,4,5,..., can be obtained in a similar way. Note that solving for a specific ρ_N requires full determination of $\mathbf{u}^{(i)}$'s, i=1,..., N. In particular, the conditions (3.33) and (3.38) must be satisfied for the existence of solutions $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$. The freedom to choose the coefficients ρ_1, ρ_2,... in (3.6) are used to meet the solvability condition (3.28). Each term ρ_i in the expansion will give a key to the postbuckling behavior of the system. Budiansky (1974) has discussed the mathematical structure of general post-buckling problems through variational analysis and shows that equation of the form (3.33) and (3.38) can be expected in the general case.

3.5 Auxiliary Conditions Associated with Incompressibility Constraint

A set of extra conditions from $(3.8)_3$ and $(3.9)_1$ are derived for simpler calculations of necessary formulations. First double integration on $(3.8)_3$ gives

$$\rho_0^{-1}\iint_{\Pi} u_{1,1}^{(k)}\, d\mathbf{X} + \rho_0 \iint_{\Pi} u_{2,2}^{(k)}\, d\mathbf{X} = \iint_{\Pi} f_3^{(k)}\, d\mathbf{X}, \tag{3.39}$$

and on $(3.9)_1$ gives

$$\iint_{\Pi} u_{1,1}^{(k)}\, d\mathbf{X} = \int u_1^{(k)}\Big|_{\Gamma_1} dX_2 = 0, \tag{3.40}$$

for k=1,2,3,..., so that

$$\iint_{\Pi} u_{2,2}^{(k)}\, d\mathbf{X} = \rho_0^{-1}\iint_{\Pi} f_3^{(k)}\, d\mathbf{X}. \tag{3.41}$$

Using the expressions for $f_3^{(k)}$ given in (3.10) to (3.13) yields

$$\begin{aligned}
\iint_{\Pi} u_{2,2}^{(1)}\, dX_1 dX_2 &= 0, \\
\iint_{\Pi} u_{2,2}^{(2)}\, dX_1 dX_2 &= \rho_0^{-1}\Phi_1, \\
\iint_{\Pi} u_{2,2}^{(3)}\, dX_1 dX_2 &= -\rho_1\rho_0^{-2}\Phi_1 + \rho_0^{-1}\Phi_2, \\
\iint_{\Pi} u_{2,2}^{(4)}\, dX_1 dX_2 &= \rho_0^{-2}(\rho_1^2\rho_0^{-1} - \rho_2)\Phi_1 - \rho_1\rho_0^{-2}\Phi_2 + \rho_0^{-1}\Phi_3,
\end{aligned} \tag{3.42}$$

where

$$\Phi_1 = \iint_{\Pi} (u_{1,2}^{(1)} u_{2,1}^{(1)} - u_{1,1}^{(1)} u_{2,2}^{(1)})\, dX_1 dX_2, \tag{3.43}$$

$$\Phi_2 = \iint_{\Pi} [(u_{1,2}^{(1)} u_{2,1}^{(2)} + u_{1,2}^{(2)} u_{2,1}^{(1)}) - (u_{1,1}^{(1)} u_{2,2}^{(2)} + u_{1,1}^{(2)} u_{2,2}^{(1)})]\, dX_1 dX_2, \tag{3.44}$$

$$\begin{aligned}
\Phi_3 = \iint_{\Pi} [&(u_{1,2}^{(1)} u_{2,1}^{(3)} + u_{1,2}^{(2)} u_{2,1}^{(2)} + u_{1,2}^{(3)} u_{2,1}^{(1)}) \\
&-(u_{1,1}^{(1)} u_{2,2}^{(3)} + u_{1,1}^{(2)} u_{2,2}^{(2)} + u_{1,1}^{(3)} u_{2,2}^{(1)})]\, dX_1 dX_2.
\end{aligned} \tag{3.45}$$

Here the results of each step in (3.42) was used for the calculations of subsequent step.

3.6 Relation between Thrust and Load Parameter

After bifurcation away from the homogeneous deformation solution (2.14) and (2.15), the associated values of the thrust T is obtained by substituting the expansion (3.6) into the expression (2.16) as follows:

$$\begin{aligned} T &= -4\mu l_2 l_3[(\rho_0 - \rho_0^{-3}) + (\rho_1 + 3\rho_0^{-4}\rho_1)\varepsilon + (\rho_2 + 3\rho_0^{-4}\rho_2 - 6\rho_0^{-5}\rho_1^2)\varepsilon^2 \\ &\quad + (\rho_3 + 3\rho_0^{-4}\rho_3 - 12\rho_0^{-5}\rho_1\rho_2 + 10\rho_0^{-6}\rho_1^3)\varepsilon^3] + O(\varepsilon^4) \\ &\equiv T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + O(\varepsilon^4). \end{aligned} \tag{3.46}$$

The first term T_0 represents the thrust at buckling onset and the other terms in (3.46) will show the change in thrust load after buckling. If the terms except T_0 have a positive value, then the thrust must increase to get a larger buckling deformation after buckling initiates - *progressive buckling*. For the opposite case, the thrust must decrease when the buckling deformation grows so that there is a possibility of *snap buckling* in which the buckling mode jumps to another mode. If the bifurcation growth is symmetric with respect to the trivial solution, then ρ_1=ρ_3=...=0 so that the thrust (3.46) becomes

$$\begin{aligned} T &= -4 l_2 l_3 \mu\left[(\rho_0 - \rho_0^{-3}) + \rho_2\left(1 + \frac{3}{\rho_0^4}\right)\varepsilon^2\right] + O(\varepsilon^4) \\ &= T_0 + \varepsilon^2 T_2 + O(\varepsilon^4). \end{aligned} \tag{3.47}$$

For this symmetric case, if ρ_2 <0, then T is an increasing function of ε so that the buckling is progressive. Otherwise, if ρ_2 >0, then T is an decreasing function of ε so that the snap buckling is possible.

3.7 Energy Formulation

The energy equation (2.30) may be expanded accordingly. Substituting the perturbation expansion (3.1) and (3.6) into the energy functional (2.30) and applying the incompressibility $(3.8)_3$, the energy functional for each order becomes

$$\Delta E = 2l_3(\varepsilon E_1 + \varepsilon^2 E_2 + \varepsilon^3 E_3 + \varepsilon^4 E_4) + O(\varepsilon^5). \tag{3.48}$$

where E_1, E_2, E_3 and E_4 are given as follows:

$$E_1 = \iint_\Pi \mu(\rho_0^{-1} u_{2,2}^{(1)} + \rho_0 u_{1,1}^{(1)})dX_1 dX_2, \tag{3.49}$$

$$E_2 = \iint_\Pi \mu[\rho_1(u_{1,1}^{(1)} - \rho_0^{-2} u_{2,2}^{(1)}) + (\rho_0 u_{1,1}^{(2)} + \rho_0^{-1} u_{2,2}^{(2)})]dX_1 dX_2 + \frac{\mu}{2}\Xi_{11}, \tag{3.50}$$

$$\begin{aligned} E_3 = \iint_\Pi \mu[& \{\rho_2 u_{1,1}^{(1)} + \rho_0^{-3}(\rho_1^2 - \rho_0\rho_2)u_{2,2}^{(1)}\} \\ & + \rho_1(u_{1,1}^{(2)} - \rho_0^{-2}u_{2,2}^{(2)}) + (\rho_0 u_{1,1}^{(3)} + \rho_0^{-1}u_{2,2}^{(3)})]dX_1 dX_2 + \mu\Xi_{12}, \end{aligned} \tag{3.51}$$

$$\begin{aligned} E_4 = \iint_\Pi \mu[& \{\rho_3 u_{1,1}^{(1)} + \rho_0^{-4}(-\rho_1^3 + 2\rho_0\rho_1\rho_2 - \rho_0^2\rho_3)u_{2,2}^{(1)}\} \\ & + \{\rho_2 u_{1,1}^{(2)} + \rho_0^{-3}(\rho_1^2 - \rho_0\rho_2)u_{2,2}^{(2)}\} + \rho_1(u_{1,1}^{(3)} - \rho_0^{-2}u_{2,2}^{(3)}) \\ & + (\rho_0 u_{1,1}^{(4)} + \rho_0^{-1}u_{2,2}^{(4)})]dX_1 dX_2 + \frac{\mu}{2}\Xi_{22} + \mu\Xi_{13}. \end{aligned} \tag{3.52}$$

Here we have introduced the notation:

$$\Xi_{ij} = \iint_\Pi (u_{1,1}^{(i)}u_{1,1}^{(j)} + u_{1,2}^{(i)}u_{1,2}^{(j)} + u_{2,1}^{(i)}u_{2,1}^{(j)} + u_{2,2}^{(i)}u_{2,2}^{(j)})dX_1 dX_2. \tag{3.53}$$

Now direct application of the extra conditions (3.40) to (3.42) into the equations (3.49) to (3.52) gives the following simplifications

$$E_1 = 0, \tag{3.54}$$

$$E_2 = \mu\rho_0^{-2}\Phi_1 + \frac{\mu}{2}\Xi_{11}, \tag{3.55}$$

$$E_3 = \mu((-2\rho_1\rho_0^{-3})\Phi_1 + \rho_0^{-2}\Phi_2) + \mu\Xi_{12}, \tag{3.56}$$

$$E_4 = \mu\{\rho_0^{-3}(3\rho_1^2\rho_0^{-1} - 2\rho_2)\Phi_1 - 2\rho_1^2\rho_0^{-4}\Phi_2 + \rho_0^{-2}\Phi_3\} + \frac{\mu}{2}\Xi_{22} + \mu\Xi_{13}. \tag{3.57}$$

Notice that each of the terms E_k, k=2,3,... make use of the solution expressions up to the (k-1)th order, $\mathbf{u}^{(k-1)}$.

If the parameter ε is small enough, then the evaluation of ΔE depends upon the first non-zero term in expansion (3.48). Hence if the first non-zero term appears in E_k, then the solution sets up to the (k-1)th order are required. If the odd terms in the expansion of load parameter vanish (ρ_1=ρ_3=...=0), then the whole formulation becomes much simpler. Specifically, if ρ_1=0, then various terms within ρ_2 in (3.38), E_3 in (3.56) and E_4 in (3.57) will vanish. If this is the case of the neo-Hookean plate, then the buckling grows symmetrically with respect to the trivial solution path.

3.8 Summary

The nonlinear boundary value problem for buckling deformations of neo-Hookean plate is analyzed by reducing to a set of linear type problems according to the perturbation expansion method. The corresponding sets of linear differential equations and energy equations are formulated. The load parameters in expansion are determined from a solvability condition that arises because the differential operators in each set are the same. The leading order analysis for buckling onset renders this differential operator singular, hence the need for a solvability condition in the higher order analysis. The conditions for the existence of solutions are generated by considering the first order homogeneous problem and the higher order nonhomogeneous problems.

CHAPTER 4

BUCKLING DEFORMATIONS FOR NEO-HOOKEAN PLATE

4.1 Introduction

The nonlinear boundary value problem for buckling deformations and energy equations of neo-Hookean plate was reduced to the set of linear type problems through the perturbation expansion methods in Chapter 3. The solvability evaluation near the bifurcation initiation depends on the deformation solutions of each order in the expanded set of linear problems so that we will focus to determine these solutions in this Chapter. Since the differential operators in the linear equations of each order are the same, the solvability of equations on each order (and the possibility of symmetric bifurcation) are firstly checked. Then the solutions for generalized problems represented to all linear problems of each order are investigated and the solutions of specific order are followed. The solvability conditions of each order will give the relations between the load parameter and deformations. In particular, symmetric bifurcation is verified. With the symmetric behaviors of the load parameter, the formulations developed previously may be reduced to much simpler forms which is beneficial to further calculations. We will examine the symmetric behavior and the reduced equations thereafter.

4.2 General Solution for the Governing Linear Differential Operator

We will investigate in this section the solutions of the generalized nonhomogeneous boundary value problem (3.8) and (3.9) for general order k under the assumption that the associated solvability condition (3.28) has been met. Applying the method of separation of variables to two dimensional boundary value problem (3.14) and

(3.15) and matching the functions of X_1, indicates that the basis for generalized solutions are four trigonometric functions, $\cos(\Omega_{1;n}X_1)$, $-\sin(\Omega_{2;n}X_1)$, $\sin(\Omega_{1;n}X_1)$ and $\cos(\Omega_{2;n}X_1)$. The basis functions which are orthogonal to each other, form a complete set on the domain $-l_1 < X_1 < l_1$ and modes are

$$\Omega_{1;n} = \frac{2n-1}{2l_1}\pi, \qquad \Omega_{2;n} = \frac{n}{l_1}\pi, \qquad n = 1, 2, \ldots \tag{4.1}$$

Hence the solutions can be expressed as the infinite series in the following forms:

$$\begin{aligned} u_1 &\sim \sum_{n=1}^{\infty} [A_{1;n}(X_2)\cos(\Omega_{1;n}X_1) - A_{2;n}(X_2)\sin(\Omega_{2;n}X_1)] \\ u_2 &\sim \sum_{n=1}^{\infty} [B_{1;n}(X_2)\sin(\Omega_{1;n}X_1) + B_{2;n}(X_2)\cos(\Omega_{2;n}X_1)] + \frac{1}{2}B_0(X_2), \\ p &\sim \sum_{n=1}^{\infty} [C_{1;n}(X_2)\sin(\Omega_{1;n}X_1) + C_{2;n}(X_2)\cos(\Omega_{2;n}X_1)] + \frac{1}{2}C_0(X_2). \end{aligned} \tag{4.2}$$

An expression of the form (4.2) will apply to each order k, consequently in (4.2) the order superscripts (k) are ignored for the generalized view. In the expressions u_1, u_2, p, the boundary conditions $(3.9)_1$ on Γ_1 has been applied so that the zeroth term in u_1 does not appear. The *coefficient functions*, $A_{i;n}$, $B_{i;n}$, $C_{i;n}$, which are in fact functions of X_2, are obtained as follows:

$$\begin{aligned} A_{i;n}(X_2) &= \frac{1}{l_1}\int_{-l_1}^{l_1} u_1 \begin{Bmatrix} \cos(\Omega_{1;n}X_1) \\ -\sin(\Omega_{2;n}X_1) \end{Bmatrix} dX_1, \\ B_{i;n}(X_2) &= \frac{1}{l_1}\int_{-l_1}^{l_1} u_2 \begin{Bmatrix} \sin(\Omega_{1;n}X_1) \\ \cos(\Omega_{2;n}X_1) \end{Bmatrix} dX_1, \\ C_{i;n}(X_2) &= \frac{1}{l_1}\int_{-l_1}^{l_1} p \begin{Bmatrix} \sin(\Omega_{1;n}X_1) \\ \cos(\Omega_{2;n}X_1) \end{Bmatrix} dX_1. \end{aligned} \tag{4.3}$$

and

$$B_0(X_2) = \frac{1}{l_1}\int_{-l_1}^{l_1} u_2 dX_1, \qquad C_0(X_2) = \frac{1}{l_1}\int_{-l_1}^{l_1} p dX_1. \tag{4.4}$$

Here the upper terms correspond to i=1 and lower terms to i=2 and orthogonality of different kinds of trigonometric functions has been considered.

The next procedure is to determine the coefficient functions of X_2 by eliminating the functions of X_1 in boundary value problems which is done by the following ways. First, multiply the differential equation (3.8) and boundary conditions (3.9) by the basis functions and then integrate with respect to X_1 over $-l_1$ to l_1. This gives

$$\begin{aligned}
\frac{1}{l_1}\int_{-l_1}^{l_1} \{\mu(u_{1,11}+u_{1,22}) - \rho_0^{-1}p_{,1}\}\begin{Bmatrix} \cos(\Omega_{1;n}X_1) \\ -\sin(\Omega_{2;n}X_1) \end{Bmatrix} dX_1 &= I_i(X_2), \\
\frac{1}{l_1}\int_{-l_1}^{l_1} \{\mu(u_{2,11}+u_{2,22}) - \rho_0 p_{,2}\}\begin{Bmatrix} \sin(\Omega_{1;n}X_1) \\ \cos(\Omega_{2;n}X_1) \end{Bmatrix} dX_1 &= J_i(X_2), \\
\frac{1}{l_1}\int_{-l_1}^{l_1} \{\rho_0^{-1}u_{1,1}+\rho_0 u_{2,2}\}\begin{Bmatrix} \sin(\Omega_{1;n}X_1) \\ \cos(\Omega_{2;n}X_1) \end{Bmatrix} dX_1 &= K_i(X_2),
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\frac{1}{l_1}\int_{-l_1}^{l_1} \{\mu(u_{1,2}+\rho_0^{-2}u_{2,1})\}\begin{Bmatrix} \cos(\Omega_{1;n}X_1) \\ -\sin(\Omega_{2;n}X_1) \end{Bmatrix} dX_1 &= G_i(\pm l_2), \\
\frac{1}{l_1}\int_{-l_1}^{l_1} \{\mu(u_{2,2}-\rho_0^{-2}u_{1,1}) - \rho_0 p\}\begin{Bmatrix} \sin(\Omega_{1;n}X_1) \\ \cos(\Omega_{2;n}X_1) \end{Bmatrix} dX_1 &= H_i(\pm l_2),
\end{aligned} \tag{4.6}$$

where the upper (lower) terms are for i=1 (i=2). The right hand side notations are

$$
\begin{aligned}
I_i(X_2) &\equiv \frac{1}{l_1}\int_{-l_1}^{l_1} f_1(X_1, X_2)\begin{pmatrix}\cos(\Omega_{1;n}X_1)\\ -\sin(\Omega_{2;n}X_1)\end{pmatrix}dX_1,\\
J_i(X_2) &\equiv \frac{1}{l_1}\int_{-l_1}^{l_1} f_2(X_1, X_2)\begin{pmatrix}\sin(\Omega_{1;n}X_1)\\ \cos(\Omega_{2;n}X_1)\end{pmatrix}dX_1,\\
K_i(X_2) &\equiv \frac{1}{l_1}\int_{-l_1}^{l_1} f_3(X_1, X_2)\begin{pmatrix}\sin(\Omega_{1;n}X_1)\\ \cos(\Omega_{2;n}X_1)\end{pmatrix}dX_1,
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
G_i(\pm l_2) &\equiv \frac{1}{l_1}\int_{-l_1}^{l_1} g_1(X_1, \pm l_2)\begin{pmatrix}\cos(\Omega_{1;n}X_1)\\ -\sin(\Omega_{2;n}X_1)\end{pmatrix}dX_1,\\
H_i(\pm l_2) &\equiv \frac{1}{l_1}\int_{-l_1}^{l_1} g_2(X_1, \pm l_2)\begin{pmatrix}\sin(\Omega_{1;n}X_1)\\ \cos(\Omega_{2;n}X_1)\end{pmatrix}dX_1.
\end{aligned}
\tag{4.8}
$$

Then apply integration by parts with respect to X_1 twice to terms with a double-differentiated variable and once to terms with a single-differentiated variable. After applying the Γ_1-boundary conditions $(3.9)_1$ and substituting l_1 and $-l_1$ to the trigonometric functions in the boundary terms, the partial differential equation (3.8) becomes an ordinary differential equation with coefficient functions $A_{i;n}$, $B_{i;n}$, $C_{i;n}$ of X_2 defined in (4.3). In conclusion, the differential equations (3.8) yield:

$$
\begin{aligned}
\mu A_i'' - \mu\Omega_i^2 A_i - \rho_0^{-1}\Omega_i C_i &= I_i(X_2),\\
\mu B_i'' - \mu\Omega_i^2 B_i - \rho_0 C'_i &= J_i(X_2),\\
-\rho_0^{-1}\Omega_i A_i + \rho_0 B_i' &= K_i(X_2),
\end{aligned}
\tag{4.9}
$$

where i=1 or 2 and prime denotes the differentiation with respect to X_2. Note that the resulting equations (4.9) are consistent for either i=1 or 2. The mode numbers n in the subscripts are suppressed to have a simpler formulation. That is, Ω_i and $I_i(X_2)$, $J_i(X_2)$, $K_i(X_2)$ are in fact dependent on n, and so give the dependence of $A_i(X_2)$, $B_i(X_2)$ and $C_i(X_2)$ on n as is required by (4.2) and (4.3). The boundary conditions on Γ_2 in $(3.9)_2$ become

$$\begin{aligned} \mu A_i' + \mu\rho_0^{-2}\Omega_i B_i &= G_i(\pm l_2), \\ \mu B_i' + \mu\rho_0^{-2}\Omega_i A_i - \rho_0 C_i &= H_i(\pm l_2). \end{aligned} \tag{4.10}$$

Furthermore the three equations in (4.9) can be combined into one ordinary differential equation with respect to one coefficient function B_i as

$$B_i^{iv} - \Omega_i^2(1+\rho_0^{-4})B_i'' + \Omega_i^4\rho_0^{-4}B_i = \frac{\rho_0^{-2}}{\mu}\Omega_i I_i' - \frac{\rho_0^{-4}}{\mu}\Omega_i^2 J_i + \rho_0^{-1}(K_i'' - \Omega_i^2 K_i)'. \tag{4.11}$$

The other coefficient functions A_i and C_i are then related to B_i by

$$\begin{aligned} A_i &= -\frac{\rho_0}{\Omega_i}K_i + \frac{\rho_0^2}{\Omega_i}B_i', \\ C_i &= -\frac{\rho_0}{\Omega_i}I_i - \mu\left(\frac{\rho_0}{\Omega_i}\right)^2 K_i'' + \mu\rho_0^3 K_i + \mu\frac{\rho_0^3}{\Omega_i^2}(B_i'' - \Omega_i^2 B_i)'. \end{aligned} \tag{4.12}$$

The boundary conditions (4.10) on Γ_2 are also expressed as

$$\begin{aligned} B_i'' + \Omega_i^2\rho_0^{-4}B_i &= \rho_0^{-1}K_i' + \frac{\rho_0^{-2}}{\mu}\Omega_i G_i, \\ B_i''' - \Omega_i^2(1+2\rho_0^{-4})B_i' &= \frac{\rho_0^{-2}}{\mu}\Omega_i I_i + \rho_0^{-1}K_i'' - \Omega_i^2\rho_0^{-1}(1+\rho_0^{-4})K_i - \frac{\rho_0^{-4}}{\mu}\Omega_i^2 H_i. \end{aligned} \tag{4.13}$$

For the zeroth terms B_0 and C_0, the ordinary differential equation and its boundary conditions become

$$\mu B_0' - \rho_0 C_0 = \frac{1}{l_1}\int_{-l_1}^{l_1} g_2(X_1, \pm l_2)\,dX_1, \qquad \rho_0 B_0' = \frac{1}{l_1}\int_{-l_1}^{l_1} f_3\,dX_1, \tag{4.14}$$

$$\mu B_0' - \rho_0 C_0 = \frac{1}{l_1}\int_{-l_1}^{l_1} g_2(X_1, \pm l_2)\,dX_1, \qquad \text{on } \Gamma_2. \tag{4.15}$$

Note that $B_0(X_2)$ is determined only to within a constant, thus $B_0(X_2)$ reflects the previously mentioned possibility of arbitrary rigid body motion in the X_2 direction.

4.3 Nonhomogeneous Ordinary Differential Equation

The equation (4.11) and the boundary conditions (4.13) form a nonhomogeneous boundary value problem with respect to only X_2. Denoting the right hand sides of (4.11) and (4.13) as the notations $\tilde{f}, \tilde{g}_1, \tilde{g}_2$ respectively gives

$$B^{iv} - \Omega^2(1+\rho_0^{-4})B'' + \Omega^4\rho_0^{-4}B = \tilde{f}(X_2), \tag{4.16}$$

$$\left.\begin{aligned} B'' + \Omega^2\rho_0^{-4}B &= \tilde{g}_1(\pm l_2), \\ B''' - \Omega^2(1+2\rho_0^{-4})B' &= \tilde{g}_2(\pm l_2), \end{aligned}\right\} \quad \text{on } \Gamma_2, \tag{4.17}$$

where the subscripts i in B and Ω are also ignored for a generalized discussion. Next we consider the possibility of nontrivial solutions to the problem (4.16) and (4.17) for the case of zero right hand sides, the homogeneous problem. Then we consider the solution to the problem (4.16) and (4.17) for the case of nonzero right hand sides, the nonhomogeneous problem.

4.3.1 Nontrivial Solutions to Homogeneous Problem

The homogeneous problem consists of (4.16) and (4.17) with zero right hand sides, $\tilde{f} = \tilde{g}_1 = \tilde{g}_2 = 0$. Clearly the trivial solution $B_h = 0$ is one solution to this problem. Any nontrivial solution B_h, which is our concern for the general problem (4.16), can be written as

$$B_h = \tilde{L}_1\cosh(\Omega X_2) + \tilde{L}_2\sinh(\Omega X_2) + \tilde{M}_1\cosh(\Omega\rho_0^{-2}X_2) + \tilde{M}_2\sinh(\Omega\rho_0^{-2}X_2), \tag{4.18}$$

where the coefficients $\tilde{L}_1$, $\tilde{L}_2$, $\tilde{M}_1$, $\tilde{M}_2$ are constants and can be determined by applying the boundary conditions (4.17). Substituting the form (4.18) into these boundary conditions on Γ_2 ($X_2 = \pm l_2$) gives four algebraic equations which can be written in matrix form as

$$\mathbf{C}_{4\times4}\mathbf{I}_{4\times1} = \mathbf{0}. \tag{4.19}$$

where

$$\mathbf{C} = \begin{bmatrix} (1+\rho_0^{-4})\tilde{C}_1 & (1+\rho_0^{-4})\tilde{S}_1 & 2\rho_0^{-4}\tilde{C}_2 & 2\rho_0^{-4}\tilde{S}_2 \\ (1+\rho_0^{-4})\tilde{C}_1 & -(1+\rho_0^{-4})\tilde{S}_1 & 2\rho_0^{-4}\tilde{C}_2 & -2\rho_0^{-4}\tilde{S}_2 \\ 2\rho_0^{-4}\tilde{S}_1 & 2\rho_0^{-4}\tilde{C}_1 & \rho_0^{-2}(1+\rho_0^{-4})\tilde{S}_2 & \rho_0^{-2}(1+\rho_0^{-4})\tilde{C}_2 \\ -2\rho_0^{-4}\tilde{S}_1 & 2\rho_0^{-4}\tilde{C}_1 & -\rho_0^{-2}(1+\rho_0^{-4})\tilde{S}_2 & \rho_0^{-2}(1+\rho_0^{-4})\tilde{C}_2 \end{bmatrix} \tag{4.20}$$

$$\mathbf{I} = \begin{bmatrix} \tilde{L}_1 & \tilde{L}_2 & \tilde{M}_1 & \tilde{M}_2 \end{bmatrix}^T. \tag{4.21}$$

The new symbols used above are defined as

$$\begin{aligned} \tilde{C}_1 &= \cosh\eta, & \tilde{S}_1 &= \sinh\eta, \\ \tilde{C}_2 &= \cosh(\rho_0^{-2}\eta), & \tilde{S}_2 &= \sinh(\rho_0^{-2}\eta). \end{aligned} \tag{4.22}$$

The scale parameter

$$\eta = \Omega l_2, \tag{4.23}$$

is eventually determined by the geometry of the plate considered (l_2/l_1, see Figure 2.1) and the mode value n in (4.1) of the nontrivial solutions. Considering the coefficient matrix $\mathbf{C}$ in (4.20) with respect to symmetry reveals that there exist two different kinds of solution. The first kind is a symmetric solution with respect to X_2 so that $B_h(-X_2)=B_h(X_2)$ which is known as a *flexural solution.* The other kind is an antisymmetric solution with respect to X_2 so that $B_h(-X_2)=-B_h(X_2)$ which is known as a *barreling solution.* These types of solution can be also obtained by applying fundamental operations of matrix algebra to the coefficient matrix in (4.20) and decoupling into two separate independent pairs.

For the symmetric solution, $B_h(-X_2) = B_h(X_2)$, set $\tilde{L}_2 = \tilde{M}_2 = 0$. Then the decoupled matrix equation reduces to

$$\begin{bmatrix} (1+\rho_0^{-4})\tilde{C}_1 & 2\rho_0^{-4}\tilde{C}_2 \\ 2\rho_0^{-4}\tilde{S}_1 & (1+\rho_0^{-4})\rho_0^{-2}\tilde{S}_2 \end{bmatrix} \begin{bmatrix} \tilde{L}_1 \\ \tilde{M}_1 \end{bmatrix} = \mathbf{0}. \tag{4.24}$$

To have the nontrivial solutions, $\tilde{L}_1$ and $\tilde{M}_1$, the determinant of coefficient matrix in (4.24) should vanish so that the following solvability condition must hold

$$(1+\rho_0^{-4})^2 \tanh(\rho_0^{-2}\eta) - 4\rho_0^{-6}\tanh\eta = 0. \tag{4.25}$$

Then the symmetric solution becomes

$$B_h = M[\cosh(\Omega X_2) - s\cosh(\Omega\rho_0^{-2}X_2)], \tag{4.26}$$

where M is an arbitrary constant for general solutions and the *aspect ratio* s denotes

$$s = \frac{(1+\rho_0^{-4})\cosh\eta}{2\rho_0^{-4}\cosh(\rho_0^{-2}\eta)}. \tag{4.27}$$

For the antisymmetric solution, $B_h(-X_2) = -B_h(X_2)$, set $\tilde{L}_1 = \tilde{M}_1 = 0$. Then the decoupled matrix equation from the coefficient matrix (4.20) reduces to

$$\begin{bmatrix} (1+\rho_0^{-4})\tilde{S}_1 & 2\rho_0^{-4}\tilde{S}_2 \\ 2\rho_0^{-4}\tilde{C}_1 & (1+\rho_0^{-4})\rho_0^{-2}\tilde{C}_2 \end{bmatrix} \begin{bmatrix} \tilde{L}_2 \\ \tilde{M}_2 \end{bmatrix} = \mathbf{0}. \tag{4.28}$$

To have nontrivial solutions, $\tilde{L}_2$ and $\tilde{M}_2$, the following solvability condition from vanishing determinant of coefficient matrix in (4.28) must hold

$$(1+\rho_0^{-4})^2 \tanh\eta - 4\rho_0^{-6}\tanh(\rho_0^{-2}\eta) = 0. \tag{4.29}$$

Then the asymmetric solution becomes

$$B_h = M[\sinh(\Omega X_2) - s\sinh(\Omega\rho_0^{-2}X_2)], \tag{4.30}$$

where the aspect ratio s is

$$s = \frac{(1+\rho_0^{-4})\sinh\eta}{2\rho_0^{-4}\sinh(\rho_0^{-2}\eta)}. \tag{4.31}$$

The symmetric and antisymmetric solutions B_h, in (4.26) and (4.30) represent the homogeneous solutions to the problem (4.16) and (4.17) for the case of $\tilde{f} = \tilde{g}_1 = \tilde{g}_2 = 0$. For a given η in (4.23), such solutions only exist for special values ρ_0 that satisfy either (4.25) or (4.29). These values, which have been obtained previously by Sawyers and Rivlin (1974) for the buckling of neo-Hookean plate, will be discussed in section 4.4 later. The two aspect ratio in (4.27) and (4.31) can be rewritten as

$$s = \frac{(1+\rho_0^{-4})}{2\rho_0^{-4}}\frac{\sinh 2\eta}{\sinh(1+\rho_0^{-2})\eta + \nu\sinh(1-\rho_0^{-2})\eta}, \tag{4.32}$$

where ν=1 for flexural deformations and ν=-1 for barreling deformations. The two solvability conditions (4.25) and (4.29) can then be rewritten into one expression as

$$\frac{\sinh(1+\rho_0^{-2})\eta}{\sinh(1-\rho_0^{-2})\eta} = \nu\frac{(1+\rho_0^{-4})^2 + 4\rho_0^{-6}}{(1+\rho_0^{-4})^2 - 4\rho_0^{-6}}. \tag{4.33}$$

The expression for s^2 by using the combined solvability (4.33) becomes

$$s^2 = \frac{\sinh 2\eta}{\rho_0^{-2}\sinh 2\rho_0^{-2}\eta}, \tag{4.34}$$

which is independent of the type of deformation. However s^2 depends on the value of ρ_0 so that at a fixed value of η, the value of (4.34) for flexural deformations is different from that for barreling deformations.

4.3.2 Solutions to Nonhomogeneous Problem

In order to determine the general solutions to the nonhomogeneous problem (4.16) and (4.17), the method of variation of parameters can be used. The nonhomogeneous solution B is found from the homogeneous solutions (4.18) by allowing the constant coefficients $\tilde{L}_1, \tilde{L}_2, \tilde{M}_1, \tilde{M}_2$ to become functions of X_2: $L_1(X_2)$, $L_2(X_2)$, $M_1(X_2)$, $M_2(X_2)$, thus giving

$$\begin{aligned} B = {} & L_1(X_2)\cosh(\Omega X_2) + L_2(X_2)\sinh(\Omega X_2) \\ & + M_1(X_2)\cosh(\Omega\rho_0^{-2}X_2) + M_2(X_2)\sinh(\Omega\rho_0^{-2}X_2). \end{aligned} \tag{4.35}$$

These coefficient functions are subject to the following requirements

$$\begin{aligned} L_1'C_1 + L_2'S_1 + M_1'C_2 + M_2'S_2 &= 0, \\ L_2'\Omega C_1 + L_1'\Omega S_1 + M_2'\Omega\rho_0^{-2}C_2 + M_1'\Omega\rho_0^{-2}S_2 &= 0, \\ L_1'\Omega^2 C_1 + L_2'\Omega^2 S_1 + M_1'\Omega^2\rho_0^{-4}C_2 + M_2'\Omega^2\rho_0^{-4}S_2 &= 0, \end{aligned} \tag{4.36}$$

with the similar notations as in (4.22):

$$\begin{aligned} C_1 &= \cosh(\Omega X_2), & S_1 &= \sinh(\Omega X_2), \\ C_2 &= \cosh(\rho_0^{-2}\Omega X_2), & S_2 &= \sinh(\rho_0^{-2}\Omega X_2). \end{aligned} \tag{4.37}$$

Substituting the form (4.35) into the nonhomogeneous equation (4.16) gives the fourth equation to complement the three equations (4.36). Together this gives a 4×4 matrix equation with respect to the first derivative of the functions $L_1(X_2)$, $L_2(X_2)$, $M_1(X_2)$, $M_2(X_2)$:

$$\begin{bmatrix} C_1 & S_1 & C_2 & S_2 \\ \Omega S_1 & \Omega C_1 & \Omega\rho_0^{-2}S_2 & \Omega\rho_0^{-2}C_2 \\ \Omega^2 C_1 & \Omega^2 S_1 & \Omega^2\rho_0^{-4}C_2 & \Omega^2\rho_0^{-4}S_2 \\ \Omega^3 S_1 & \Omega^3 C_1 & \Omega^3\rho_0^{-6}S_2 & \Omega^3\rho_0^{-6}C_2 \end{bmatrix} \begin{bmatrix} L_1' \\ L_2' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{f} \end{bmatrix} \tag{4.38}$$

Note that the 4×4 coefficient matrix is not singular so that L_1', L_2', M_1', M_2' can be determined directly from pre-multiplying the right hand side of (4.38) by the inverse of the 4×4 coefficient matrix function. Then the coefficient functions L_1, L_2, M_1, M_2 after integrating with respect to X_2 are

$$\begin{bmatrix} L_1 \\ L_2 \\ M_1 \\ M_2 \end{bmatrix} = \frac{1}{(1-\rho_0^{-4})\Omega^3} \begin{bmatrix} -\int \tilde{f} S_1 dX_2 + \overline{L_1} \\ \int \tilde{f} C_1 dX_2 + \overline{L_2} \\ \rho_0^2 \int \tilde{f} S_2 dX_2 + \overline{M_1} \\ -\rho_0^2 \int \tilde{f} C_2 dX_2 + \overline{M_2} \end{bmatrix} \tag{4.39}$$

Here the integrals are *indefinite* and $\overline{L_1}, \overline{L_2}, \overline{M_1}, \overline{M_2}$ are integration constants. The full form of solution B is

$$\begin{aligned} B(X_2) = \frac{1}{(1-\rho_0^{-4})\Omega^3} [&\{ C_1 (-\int \tilde{f} S_1 dX_2 + \overline{L_1}) + S_1 (\int \tilde{f} C_1 dX_2 + \overline{L_2}) \} \\ &+ \rho_0^2 \{ C_2 (\int \tilde{f} S_2 dX_2 + \overline{M_1}) + S_2 (-\int \tilde{f} C_2 dX_2 + \overline{M_2}) \}]. \end{aligned} \tag{4.40}$$

The function $\tilde{f}$ is specified in the right hand side of (4.11). The complete solution B contains four as yet undetermined constant coefficients $\overline{L_1}, \overline{L_2}, \overline{M_1}, \overline{M_2}$ that are available to satisfy the four boundary conditions (4.17). Substituting the solution (4.40) into these four boundary conditions (4.17), the four equations for these constant coefficients are expressed in matrix form as:

$$\mathbf{C}_{4\times 4} \mathbf{J}_{4\times 1} = \mathbf{G}_{4\times 1}, \tag{4.41}$$

where

$$\mathbf{J} = \begin{bmatrix} \overline{L_1} & \overline{L_2} & \overline{M_1} & \overline{M_2} \end{bmatrix}^T, \tag{4.42}$$

$$\mathbf{G} = \begin{bmatrix} \dfrac{\tilde{g}_1^+}{\Omega^2} \\ \dfrac{\tilde{g}_1^-}{\Omega^2} \\ -\dfrac{\tilde{g}_2^+}{\Omega^3} \\ -\dfrac{\tilde{g}_2^-}{\Omega^3} \end{bmatrix} - \begin{bmatrix} (1+\rho_0^{-4})\{L_1^+\tilde{C}_1 + L_2^+\tilde{S}_1\} + 2\rho_0^{-4}\{M_1^+\tilde{C}_2 + M_2^+\tilde{S}_2\} \\ (1+\rho_0^{-4})\{L_1^-\tilde{C}_1 - L_2^-\tilde{S}_1\} + 2\rho_0^{-4}\{M_1^-\tilde{C}_2 - M_2^-\tilde{S}_2\} \\ 2\rho_0^{-4}\{L_1^+\tilde{S}_1 + L_2^+\tilde{C}_1\} + \rho_0^{-2}(1+\rho_0^{-4})\{M_1^+\tilde{S}_2 + M_2^+\tilde{C}_2\} \\ 2\rho_0^{-4}\{-L_1^-\tilde{S}_1 + L_2^-\tilde{C}_1\} + \rho_0^{-2}(1+\rho_0^{-4})\{-M_1^-\tilde{S}_2 + M_2^-\tilde{C}_2\} \end{bmatrix} \tag{4.43}$$

and the coefficient matrix **C** is the same as that appearing in the homogeneous problem (4.20). The functions with superscripts + or - denote that the functions are evaluated at $X_2=l_2$ or $-l_2$ respectively. Note that the matrix **C** is singular if and only if ρ_0 satisfies either (4.25) or (4.29). If ρ_0 does not satisfy either of these equations, then (4.41) yields a unique vector expression for **J**. On the other hand, if **C** is singular, then (4.41) gives solutions if and only if **G** is orthogonal to the null space of $\mathbf{C}^T$ which generates the solvability condition as we will show in the following section. For this singular case, if the solvability condition is not satisfied, then there is no solution to the problem (4.16) and (4.17). If the solvability condition is satisfied, then there exist infinite solutions which is shown in (4.40). The constant vector **J** can be obtained by using the psudoinverse matrix of the singular matrix **C** or by connecting the solvability condition to (4.41).

4.3.3 Solvability Condition

The existence of a solution $\mathbf{J} = \left[\overline{L_1}\ \overline{L_2}\ \overline{M_1}\ \overline{M_2}\right]^T$ for the linear algebraic equation (4.41) when ρ_0 satisfies either (4.25) or (4.29) requires satisfaction of an orthogonality condition with the right hand side vector **G** and the null space of $\mathbf{C}^T$. If ρ_0 obeys (4.25) which corresponds to the symmetric case, then the null space of $\mathbf{C}^T$ is given by

$$NS_S = \Omega^3(1-\rho_0^{-4})\left[-\frac{\tilde{S}_1}{1+\rho_0^{-4}}\ -\frac{\tilde{S}_1}{1+\rho_0^{-4}}\ \frac{\tilde{C}_1}{2\rho_0^{-4}}\ -\frac{\tilde{C}_1}{2\rho_0^{-4}}\right]^T. \tag{4.44}$$

If ρ_0 obeys (4.29), which corresponds to the antisymmetric case, then the null space of $\mathbf{C}^T$ is given by

$$NS_A = \Omega^3(1-\rho_0^{-4})\left[-\frac{\tilde{C}_1}{1+\rho_0^{-4}}\ \frac{\tilde{C}_1}{1+\rho_0^{-4}}\ \frac{\tilde{S}_1}{2\rho_0^{-4}}\ \frac{\tilde{S}_1}{2\rho_0^{-4}}\right]^T. \tag{4.45}$$

Together the orthogonality of the null space and the matrix $\mathbf{G}$ can be written

$$\mathbf{G}^T \cdot NS_{S\text{ or A}} = 0, \tag{4.46}$$

and so yields the following solvability conditions with the relations (4.25) and (4.29). For the case associated with (4.25) and hence a symmetric nontrivial homogeneous solution,

$$\int_{\Gamma_2} \tilde{f}(C_1 - sC_2)dX_2 = -(1-\rho_0^{-4})\left[\frac{C_1}{2\rho_0^{-4}}\tilde{g}_2 + \frac{\Omega S_1}{(1+\rho_0^{-4})}\tilde{g}_1\right]\Bigg|_{-l_2}^{l_2}. \tag{4.47}$$

For the case associated with (4.29) and hence an antisymmetric nontrivial homogeneous solution,

$$\int_{\Gamma_2} \tilde{f}(S_1 - sS_2)dX_2 = -(1-\rho_0^{-4})\left[\frac{S_1}{2\rho_0^{-4}}\tilde{g}_2 + \frac{\Omega C_1}{(1+\rho_0^{-4})}\tilde{g}_1\right]\Bigg|_{-l_2}^{l_2}. \tag{4.48}$$

Thus, if there exist nontrivial solutions B_h to homogeneous versions of (4.16) and (4.17), then there exist solutions to (4.16) and (4.17) with nonzero $\tilde{f}, \tilde{g}_1, \tilde{g}_2$ if and only if these functions satisfy either the condition (4.47) or (4.48), as appropriate

The solvability conditions (4.47) and (4.48) can be also obtained directly from the ordinary differential equation problem (4.16) and (4.17) via similar procedure developed in Section 3.4 to get (3.28) for the partial differential equation problem. For the problem (4.16) and (4.17), exchanging **u** to B, **w** to B_h, and **f** to $\tilde{f}$ in the formulation (3.27) gives

$$\langle \tilde{F}B, B_h \rangle - \langle B, \tilde{F}B_h \rangle = \frac{1}{4l_1 l_2}\int_{\Gamma_2} [(\tilde{F}B)B_h - B(\tilde{F}B_h)]dX_2. \tag{4.49}$$

Here $\tilde{F}$ is the differential operator used in the left hand side of (4.16) so that $\tilde{F}B = \tilde{f}$ and $\tilde{F}B_h = 0$. Applying the integration by parts and boundary conditions in (4.17) yields

$$\langle \tilde{f}, B_h \rangle = \frac{1}{4l_1 l_2}(\tilde{g}_2 B_h - \tilde{g}_1 B_h')\Big|_{-l_2}^{l_2}. \tag{4.50}$$

Then substituting B_h in (4.26) and (4.30) into (4.50) gives the same conditions as (4.47) and (4.48) after rigorous calculations with the relations of (4.25) and (4.29). Hence the condition (3.28) represents solvability at the partial differential equation level, the condition (4.50) represents solvability at the ordinary differential equation level and conditions (4.47) and (4.48) represent solvability at the linear algebra level. The three conditions (3.28), (4.50) and either (4.47) or (4.48) for solvability are perfectly matched to each other.

4.4 Buckling Onset (the First Order Solution k=1)

The first order solution $\mathbf{u}^{(1)}=(u_1^{(1)}, u_2^{(1)}, p^{(1)})$ of homogeneous boundary value problem (3.16) represents the status of buckling initiation and was investigated by Sawyers and Rivlin (1974). Since all the right hand side terms $\mathbf{f}^{(1)}$ and $\mathbf{g}^{(1)}$ vanish, the terms I_i, J_i, K_i in (4.7) and G_i, H_i in (4.8) as well as the right hand sides of the equations (4.11) and (4.13) vanish. Therefore the boundary value problem for the case of k=1 becomes

$$B_{i;n}^{(1)iv} - \Omega_{i;n}^2(1+\rho_0^{-4})B_{i;n}^{(1)''} + \Omega_{i;n}^4\rho_0^{-4}B_{i;n}^{(1)} = 0, \tag{4.51}$$

$$\left.\begin{aligned} B_{i;n}^{(1)''} + \Omega_{i;n}^2\rho_0^{-4}B_{i;n}^{(1)} &= 0, \\ B_{i;n}^{(1)'''} - \Omega_{i;n}^2(1+2\rho_0^{-4})B_{i;n}^{(1)'} &= 0, \end{aligned}\right. \quad \text{on} \quad \Gamma_2. \tag{4.52}$$

In view of the relations (4.12), the other coefficients in (4.3) become

$$A_{i;n}^{(1)} = \frac{\rho_0^2}{\Omega_{i;n}}B_{i;n}^{(1)'}, \qquad C_{i;n}^{(1)} = \mu\frac{\rho_0^3}{\Omega_{i;n}^2}(B_{i;n}^{(1)''} - \Omega_{i;n}^2 B_{i;n}^{(1)})'. \tag{4.53}$$

The first order solution of (4.51) and (4.52) can be derived from the solution of homogeneous problem in (4.26) for the symmetric case, and the solutions of (4.30) for the antisymmetric case by substituting $B_{i;n}^{(1)}$ into B_h and adding subscript i to Ω and η.

4.4.1 Buckling Initiation by Flexure and by Barreling

Two different kinds of solution in view of symmetry represents two shapes of deformation: flexural and barreling deformation. For the flexural deformation, the lateral deflection $u_2^{(1)}$ is symmetric with respect to X_2, so that $B_{i;n}^{(1)}(X_2)$ is an even function. It follows essentially from (4.25) - (4.27) as

$$B_{i;n}^{(1)}(X_2) = M[\cosh(\Omega_{i;n}X_2) - s_{i;n}\cosh(\Omega_{i;n}\rho_0^{-2}X_2)], \tag{4.54}$$

under the solvability condition of

$$(1+\rho_0^{-4})^2\tanh(\eta_{i;n}\rho_0^{-2}) - 4\rho_0^{-6}\tanh\eta_{i;n} = 0. \tag{4.55}$$

Here $\eta_{i;n}$ which is given from (4.1) and (4.23) as $\eta_{i;n} = \Omega_{i;n}l_2$. The constant M denotes the amount of buckling from homogeneous deformation so that M will be determined according to the normalization convention. The value of M will be determined later in this section. For the barreling case, the lateral deflection is antisymmetric so that $B_{i;n}^{(1)}(X_2)$ is odd function and expressed from (4.29) - (4.31) as

$$B_{i;n}^{(1)}(X_2) = M[\sinh(\Omega_{i;n}X_2) - s_{i;n}\sinh(\Omega_{i;n}\rho_0^{-2}X_2)], \tag{4.56}$$

under the solvability condition of

$$(1+\rho_0^{-4})^2\tanh\eta_{i;n} - 4\rho_0^{-6}\tanh(\eta_{i;n}\rho_0^{-2}) = 0. \tag{4.57}$$

Nontrivial solutions will only exist, at fixed mode number n and initial geometries l_1 and l_2 (which will be shown in $\Omega_{i;n}$ and $\eta_{i;n}$), for particular values of ρ_0. The conditions (4.55) and (4.57) show the relations between ρ_0 and $\eta_{i;n}$ which represent the load parameter curve. The aspect ratio s for both types of deformation is

$$s_{i;n} = \frac{(1+\rho_0^{-4})}{2\rho_0^{-4}}\frac{\sinh 2\eta_{i;n}}{\sinh(1+\rho_0^{-2})\eta_{i;n} + \nu\sinh(1-\rho_0^{-2})\eta_{i;n}}, \tag{4.58}$$

where ν=1 (ν=-1) for flexural (barreling) deformation.

The value of the constant coefficient M in either (4.54) or (4.56) is determined from normalization for which we define as

$$\frac{1}{4l_1l_2}\iint_{\Pi}\left[\left(\frac{u_1^{(1)}}{l_1}\right)^2 + \left(\frac{u_2^{(1)}}{l_2}\right)^2 + \left(\frac{p^{(1)}}{\mu}\right)^2\right]dX_1dX_2 = 1. \tag{4.59}$$

This is in contrast to the normalization used by Sawyers and Rivlin (1982) who instead require that $u_2^{(1)}(l_1, 0) = \pm 1$ for flexure and $u_2^{(1)}(l_1, l_2) = \pm 1$ for barreling deformation. Their results are simple and procedures are relatively easy by introducing above special rules of normalization. But in this work, we will follow the definition of normalization, $\|\mathbf{u}^{(1)}\| = 1$, as shown in (4.59). Applying the solutions (4.2) for k=1 to (4.59) with the relations (4.53) and integrating over Γ_1 give

$$\frac{1}{4l_2}\int_{\Gamma_2}\left[\left(\frac{\rho_0^2}{\Omega_{i;n}l_1}B_{i;n}^{(1)\prime}\right)^2+\left(\frac{B_{i;n}^{(1)}}{l_2}\right)^2+\left\{\frac{\rho_0^3}{\Omega_{i;n}^2}(B_{i;n}^{(1)\prime\prime}-\Omega_{i;n}^2B_{i;n}^{(1)})'\right\}^2\right]dX_2 = 1. \tag{4.60}$$

Then substituting the solution expression for $B_{i;n}{}^{(1)}$ in the separate cases of flexural deformation (4.54) and barreling deformation (4.56) and the aspect ratio s in (4.58) into (4.60) and applying the solvability conditions (4.55) and (4.57) give the following equation,

$$\begin{aligned}\frac{M^2}{4l_2^2}\Bigg[&\left\{\nu(1-\sigma^2\rho_0^4)+(1+\sigma^2\rho_0^4)\frac{\sinh(2\eta_{i;n})}{2\eta_{i;n}}\right\}-\eta_{i;n}^2s^2\rho_0^2(1-\rho_0^{-4})^2\left(\nu-\frac{\sinh(2\rho_0^{-2}\eta_{i;n})}{2\rho_0^{-2}\eta_{i;n}}\right)\\&+s^2\left\{\nu(1-\sigma^2)+(1+\sigma^2)\frac{\sinh(2\rho_0^{-2}\eta_{i;n})}{2\rho_0^{-2}\eta_{i;n}}\right\}\\&-2s\left\{\nu(1-\sigma^2\rho_0^2)\frac{\sinh\eta_{i;n}(1-\rho_0^{-2})}{\eta_{i;n}(1-\rho_0^{-2})}+(1+\sigma^2\rho_0^2)\frac{\sinh\eta_{i;n}(1+\rho_0^{-2})}{\eta_{i;n}(1+\rho_0^{-2})}\right\}\Bigg] = 1.\end{aligned} \tag{4.61}$$

where $\sigma=l_2/l_1$ and $\nu=1$ for flexural deformation, and $\nu=-1$ for barreling deformation. When the geometry l_1, l_2 and mode number i, n of given plate are supplied, the value of M for flexural or barreling deformation is determined by (4.61).

4.4.2 Load Parameters Associated with Buckling Initiation

The load parameter values ρ_0 at which the plate may initiate buckling, is dependent upon the initial geometry through mode number, $\eta_{i;n}$. Their relations for the case of flexure and barreling deformations are given in (4.55) and (4.57) and are shown in Figure 4.1. Sawyers and Rivlin (1974, 1982) first reached these first order solutions in terms of $\lambda=\rho_0^{-2}$ and η by using the variational approach. It can be shown for each fixed value of i and for each n (which then specifies a value for $\eta_{i;n}>0$) that there exists exactly one solution ρ_0^F which satisfies (4.55). Also for each fixed value of i and for each n, there

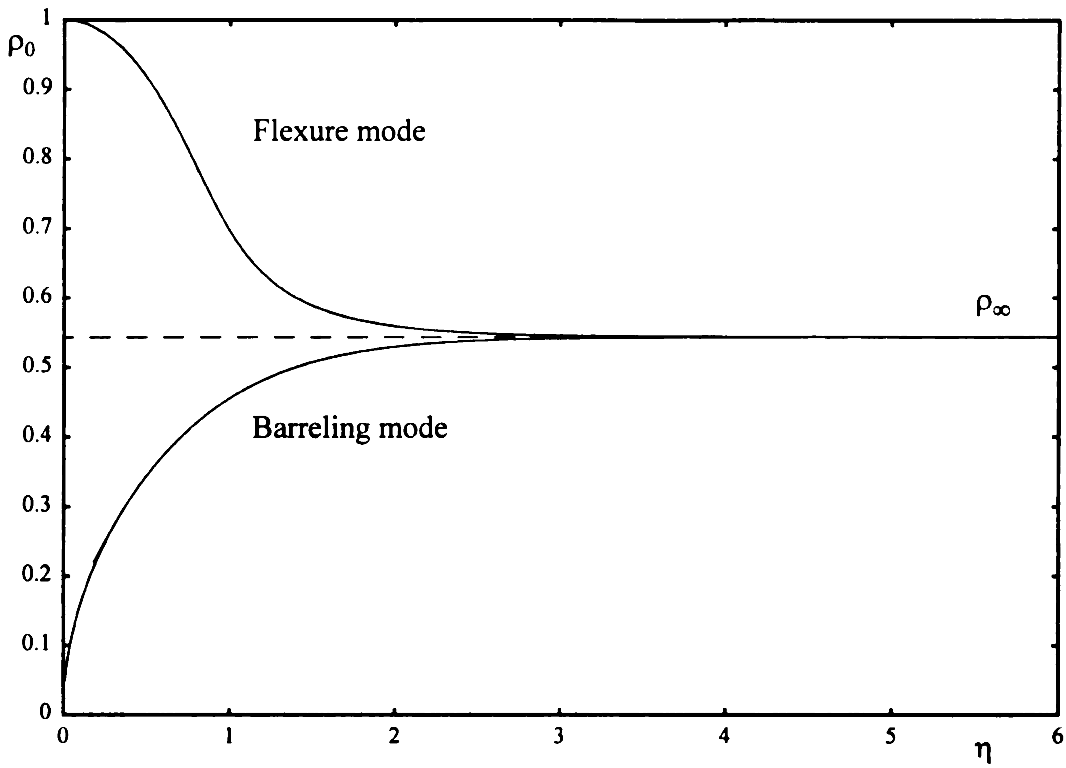


Figure 4.1 The load parameters for flexure and barreling modes. When η goes to infinity, both modes converge to ρ_∞=0.543689 as shown in (4.65).

exists exactly one solution ρ_0^B which satisfies (4.57). Therefore there exist only two possible solutions for each fixed $\Omega_{i;n}$. The indices i=1 or 2 and n=1,2,3,... determine special values for each solution

$$\rho_0 = \rho_0^F(i;n),\ \rho_0^B(i;n), \qquad i = 1,2, \qquad n = 1,2,3,\dots . \tag{4.62}$$

If ρ_0 is not equal to one of the two special values, then no solution, other than the trivial solution $\mathbf{u}^{(1)}$=0, exists for the homogeneous problem of k=1. That is buckling can not initiate at loads other than those given by (4.62). On the other hand, if ρ_0 is equal to one of the special values given by (4.62), then the solution of the first order problem consists of the single function from each of the infinite series in (4.2) that corresponds to the particular i and n which satisfy (4.62).

Note that the two curves in Figure 4.1 are each monotonic and approach the common asymptote ρ_∞ as $\eta \to \infty$ in which they share the same load parameter. When $\eta \to \infty$, both $\tanh\eta_\infty$ and $\tanh\rho_0^{-2}\eta_\infty$ have the same value since ρ_0^{-2} is finite and hence the conditions (4.55) and (4.57) lead to

$$(1+\rho_\infty^{-4})^2 - 4\rho_\infty^{-6} = 0. \tag{4.63}$$

The solution except for trivial case of $\rho_\infty=1$ in (4.63) is the real root of

$$\rho_\infty^{-6} - 3\rho_\infty^{-4} - \rho_\infty^{-2} - 1 = 0, \tag{4.64}$$

and becomes by using Cardano's solution for cubic equation (see Qiu, et al., 1993)

$$\begin{gathered}\rho_\infty = \left[1 + \sqrt[3]{R + \sqrt{Q^3+R^2}} + \sqrt[3]{R - \sqrt{Q^3+R^2}}\right]^{-1/2} = 0.543689\ldots \\ Q = -(4/3), \qquad R = 2.\end{gathered} \tag{4.65}$$

The deformed shape of the plate at $\eta \to \infty$ involves an infinite number of wrinkles. In view of the loading mechanism, when the thrust T is increasing, the load parameter ρ_0 is decreasing starting from $\rho_0=1$ according to (2.16) so that the buckling initiation modes occur sequentially such as

$$T_1^F < T_2^F < \ldots < T_\infty^F = T_\infty^B < \ldots < T_2^B < T_1^B. \tag{4.66}$$

For the zeroth terms, the coefficient functions in (4.14) and (4.15), ignoring the rigid body motion, become

$$B_0^{(1)}(X_2) = 0, \qquad C_0^{(1)}(X_2) = 0. \tag{4.67}$$

Hence the solutions of the homogeneous equation of order ε (3.8) and (3.9) for $k=1$ are

$$u_1^{(1)} = \begin{cases} \dfrac{\rho_0^2}{\Omega_{1;n}} B_{1;n}^{(1)\prime} \cos(\Omega_{1;n} X_1), \\ -\dfrac{\rho_0^2}{\Omega_{2;n}} B_{2;n}^{(1)\prime} \sin(\Omega_{2;n} X_1), \end{cases} \tag{4.68}$$

$$u_2^{(1)} = \begin{cases} B_{1;n}^{(1)} \sin(\Omega_{1;n} X_1), \\ B_{2;n}^{(1)} \cos(\Omega_{2;n} X_1), \end{cases} \tag{4.69}$$

$$p^{(1)} = \begin{cases} \mu \dfrac{\rho_0^3}{\Omega_{1;n}^2} (B_{1;n}^{(1)\prime\prime} - \Omega_{1;n}^2 B_{1;n}^{(1)})' \sin(\Omega_{1;n} X_1), \\ \mu \dfrac{\rho_0^3}{\Omega_{2;n}^2} (B_{2;n}^{(1)\prime\prime} - \Omega_{2;n}^2 B_{2;n}^{(1)})' \cos(\Omega_{2;n} X_1), \end{cases} \tag{4.70}$$

where $B_{i;n}^{(1)}(X_2)$ are given by (4.54) for flexure and (4.56) for barreling. The particular value of ρ_0^F or ρ_0^B is obtained as the unique root of (4.55) or (4.57) for the given value of $\eta_{i;n}$.

4.4.3 Asymptotic Expressions for Load Parameters

Equations (4.55) and (4.57) do not show the explicit form for ρ_0 and this will be a difficulty to fully analytic study. Hence it is convenient to analyze the expressions (4.55) and (4.57) in the separate limits of $\eta \to 0$ and $\eta \to \infty$. This gives four separate cases corresponding to: (i) flexural deformations at low mode number ((4.55) as $\eta \to 0$); (ii) barreling deformation at low mode number ((4.57) as $\eta \to 0$); (iii) flexural deformation corresponds to wrinkling ((4.55) as $\eta \to \infty$); (iv) barreling deformation corresponds to wrinkling ((4.57) as $\eta \to \infty$). Later in Section 5.4.2, the low mode number flexure expansion (i) is used in an asymptotic stability analysis. The other cases (ii)-(iv) are given here for completeness.

(i) ρ_0 for flexural deformation when η goes to zero

When η goes to zero for the flexural deformation in (4.55), the parameter ρ_0 goes to one as shown in Figure 4.1, so that we assume the series polynomial expansion of ρ_0 as

$$\rho_0 = 1 + k_1\eta + k_2\eta^2 + k_3\eta^3 + \dots \tag{4.71}$$

The hyperbolic tangent can be expanded in a series form when η has small value as

$$\tanh\eta = \eta - \frac{1}{3}\eta^3 + \frac{2}{15}\eta^5 - \dots \tag{4.72}$$

Introducing (4.71) and (4.72) to the condition (4.55) gives the polynomial equation of η in which each coefficient function vanishes simultaneously. The lowest order becomes η^3 and its coefficient shows

$$k_1 = 0. \tag{4.73}$$

Then the coefficient functions after substituting (4.73) become

$$\begin{aligned}&\frac{16}{3}k_2(1+3k_2)\eta^5 + 16k_3\left(\frac{1}{3}+2k_2\right)\eta^6\\ &+8\left(-\frac{64}{15}k_2 - \frac{152}{3}k_2^2 - 112k_2^3 + 16k_3^2 + \frac{16}{3}k_4 + 32k_2k_4\right)\eta^7\\ &+16\left(\frac{4}{15}k_3 - \frac{19}{3}k_2k_3 - 21k_2^2k_3 + 2k_3k_4 + \frac{1}{3}k_5 + 2k_2k_5\right)\eta^8 + O(\eta^9) = 0.\end{aligned} \tag{4.74}$$

Because of small η, each term in (4.74) vanishes simultaneously so that the first term gives

$$k_2 = -\frac{1}{3}, \tag{4.75}$$

excluding the trivial case, k_2=0. Subsequently the other coefficients show that

$$k_3 = 0, \qquad k_4 = -\frac{1}{90}, \qquad k_5 = 0, \qquad k_6 = -\frac{19}{1890}, \qquad k_7 = 0. \tag{4.76}$$

Therefore the asymptotic equation for ρ_0 when η goes to zero gives

$$\rho_0 \sim 1 - \frac{1}{3}\eta^2 - \frac{1}{90}\eta^4 - \frac{19}{1890}\eta^6 + O(\eta^8), \tag{4.77}$$

which is shown in Figure 4.2 up to the fourth order accompanied with the exact values which is computed by using (4.55). As we can see in (4.77), ρ_0 is an even function of η. The asymptotic equation (4.77) were also obtained by Sawyers and Rivlin (1982) in terms of λ as $\lambda = \rho_0^{-2} = 1 + \frac{2}{3}\eta^2 + \frac{16}{45}\eta^4 + O(\eta^6)$.

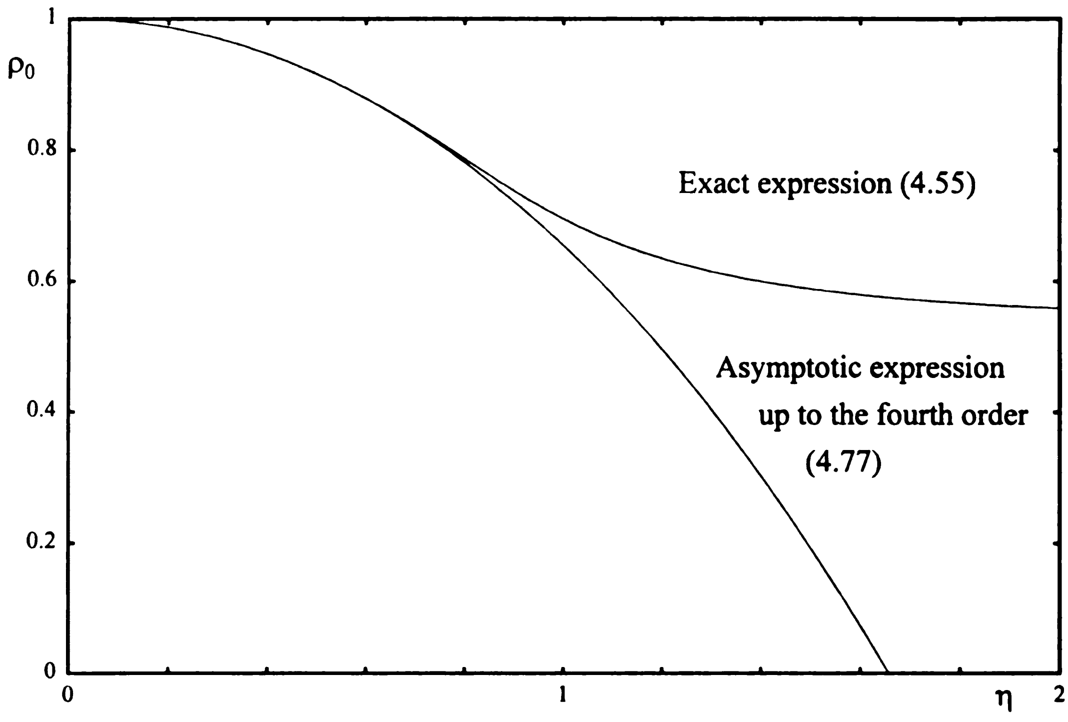


Figure 4.2 Asymptotic equations of load parameter ρ_0 for the flexural deformation when η has small value by using (4.77) up to the fourth order. For η<1, asymptotic expression is quite close to exact expression.

(ii) ρ_0 for barreling deformation when η goes to zero

For the case of barreling deformation shown in (4.57), the parameter ρ_0 goes to zero when η goes to zero. However $\eta\rho_0^{-2}$ in hyperbolic tangent does not go to zero. A consistent analysis of possible forms for the expansion of $\rho_0(\eta)$ near η=0 shows that the correct expansion form is

$$\rho_0 = k_1\eta^{1/2} + k_2\eta + k_3\eta^{3/2} + k_4\eta^2 + k_5\eta^{5/2} + O(\eta^3). \tag{4.78}$$

Then the expansion

$$\eta\rho_0^{-2} = k_1^{-2} - 2k_2k_1^{-3}\eta^{1/2} + (3k_2^2k_1^{-4} - 2k_3k_1^{-3})\eta + \ldots \tag{4.79}$$

so that

$$\lim_{\eta\to 0}(\eta\rho_0^{-2}) = k_1^{-2}. \tag{4.80}$$

Substituting (4.78) into the condition (4.57) and the Taylor series expansion of $\tanh(\eta\rho_0^{-2})$ with respect to the value of (4.80) give an algebraic equation of η in which each coefficient function vanishes. Then the first term becomes

$$k_1^{-8}(1 - 4k_1^2\tanh k_1^{-2})\eta^{-3} = 0 \tag{4.81}$$

which shows the numerical value of k_1=0.500169... The following terms show that k_2, k_3 and k_4 equal zero and the fifth term shows

$$k_5 = \frac{k_1(1 - 6k_1^4)}{24(-\tanh^2 k_1^{-2} + 3k_1^2\tanh k_1^{-2})}. \tag{4.82}$$

Using the value of k_1, the equation (4.82) shows k_5=-0.05234..., numerically. Hence the asymptotic equation of ρ_0 for barreling deformation when η goes to zero becomes

$$\rho_0 \sim 0.500169\eta^{1/2} - 0.05234\eta^{5/2} + O(\eta^3) \tag{4.83}$$

which is shown in Figure 4.3 with exact equation.

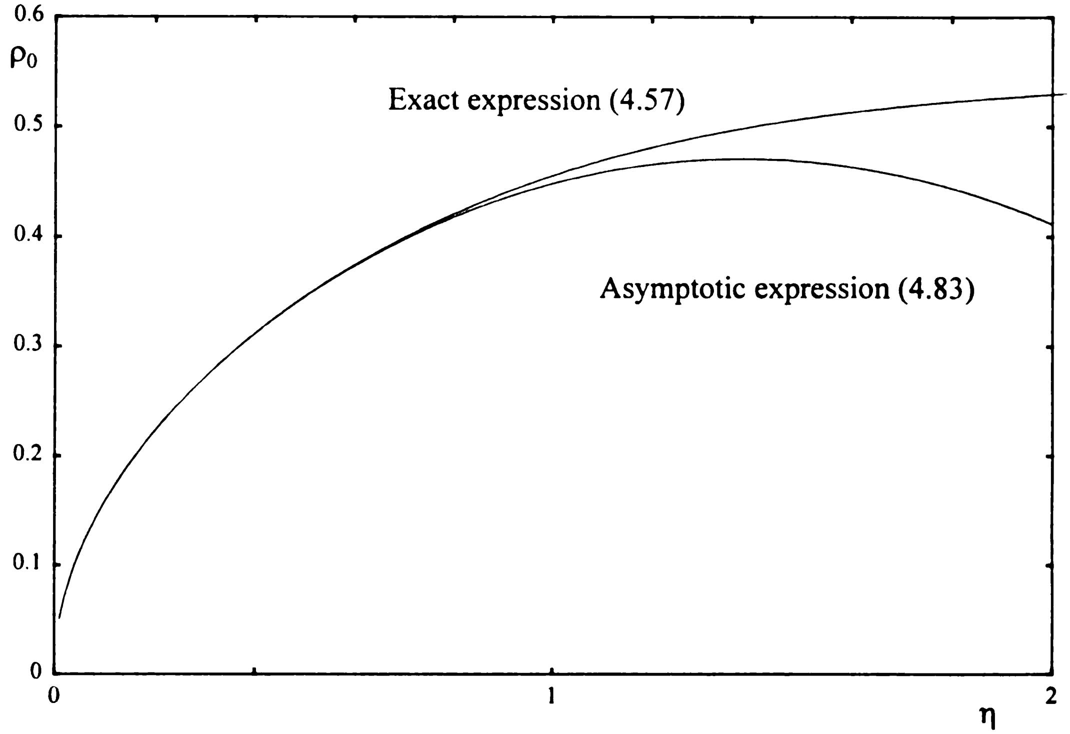


Figure 4.3 Two-term asymptotic equation of load parameter ρ_0 for barreling deformation when η has small value by using (4.83). Again when $\eta<1$, the asymptote is quite close to exact expression.

(iii) ρ_0 for flexural deformation when η goes to infinity

When η goes to infinity, the parameter ρ_0 goes to a finite value ρ_w so that ρ_0 can be written as

$$\rho_0 = \rho_w\{1+\delta(\eta)\}, \tag{4.84}$$

where δ is remainder term that vanishes as η goes to infinity. The hyperbolic tangents are expanded by using infinite series of exponential equation as

$$\tanh\eta = \frac{1-e^{-2\eta}}{1+e^{-2\eta}} = 1-2e^{-2\eta}+2e^{-4\eta}-2e^{-6\eta}+\ldots$$
$$\tanh(\eta\rho_0^{-2}) = 1-2e^{-2\eta\rho_0^{-2}}+2e^{-4\eta\rho_0^{-2}}-2e^{-6\eta\rho_0^{-2}}+\ldots \tag{4.85}$$

where

$$\eta\rho_0^{-2} = \eta\rho_w^{-2}(1-2\delta+3\delta^2-4\delta^3+\ldots). \tag{4.86}$$

The possible form of $\delta(\eta)$ can be obtained from considering (4.85) with the conditions (4.55) or (4.57) as

$$\delta = k_1 e^{-2\eta}+k_2 e^{-4\eta}+\ldots \tag{4.87}$$

Substituting (4.84) and (4.85) into (4.55) for flexural deformation and equating each coefficient function of each order of e to zero give algebraic equations which contain the unknowns k_i. The first term which is independent on η shows

$$(1+\rho_w^{-4})^2-4\rho_w^{-6} = 0 \tag{4.88}$$

and the real solution to (4.88) give a limit value of ρ_w=0.5437... excluding the case of unloading, ρ_0=1, by using Cardano's rule as shown in (4.65). The next leading terms show that

$$k_1 = \frac{\rho_w^2}{1-3\rho_w^2+\rho_w^4},$$
$$k_2 = \frac{(9-21\rho_w^2+5\rho_w^4)k_1^2-2\rho_w^2(1+6k_1)}{2(1-3\rho_w^2+\rho_w^4)}, \tag{4.89}$$

with the numerical values of ρ_w, the parameter ρ_0 when η goes to infinity becomes

$$\rho_0 \sim \rho_w(1+1.47395e^{-2\eta}+2.98066e^{-4\eta}+\ldots), \tag{4.90}$$

which is shown in Figure 4.4 with exact solution.

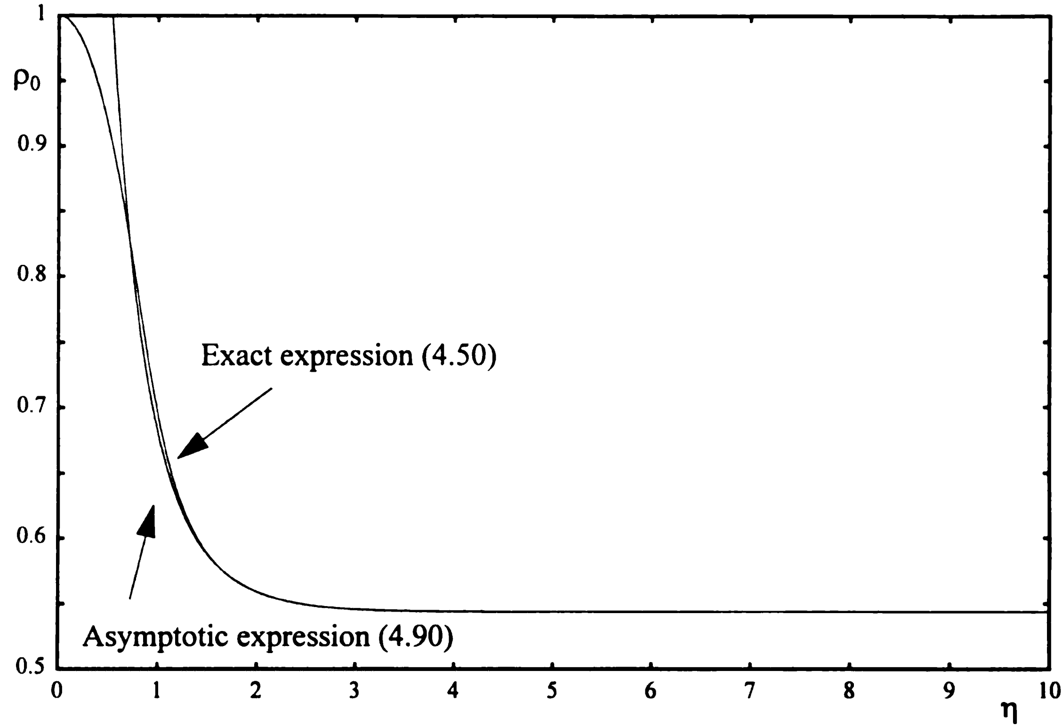


Figure 4.4 Three-term asymptotic equations of load parameter ρ_0 for flexural deformation when η goes to infinity. When $\eta>1.5$, asymptotes is quite close to exact expression.

(iv) ρ_0 for barreling deformation when η goes to infinity

For barreling deformation, the similar approaches can be used in the case of flexural deformation when η goes to infinity. The expansions (4.84) and (4.87) are also possible candidates for barreling. Then the results shows that

$$\begin{aligned} k_1 &= -\frac{(1+\rho_w^4)^2}{4(1-3\rho_w^2+\rho_w^4)}, \\ k_2 &= \frac{(18-42\rho_w^2+10\rho_w^4)k_1^2+(1+\rho_w^4)(1+\rho_w^4-8k_1)}{4(1-3\rho_w^2+\rho_w^4)}, \end{aligned} \tag{4.91}$$

which determines the asymptotic equation numerically as

$$\rho_0 \sim \rho_w(1-1.47391e^{-2\eta}+34.9454e^{-4\eta}+\ldots). \tag{4.92}$$

The asymptotic equation (4.92) with exact equation are shown in Figure 4.5. and it

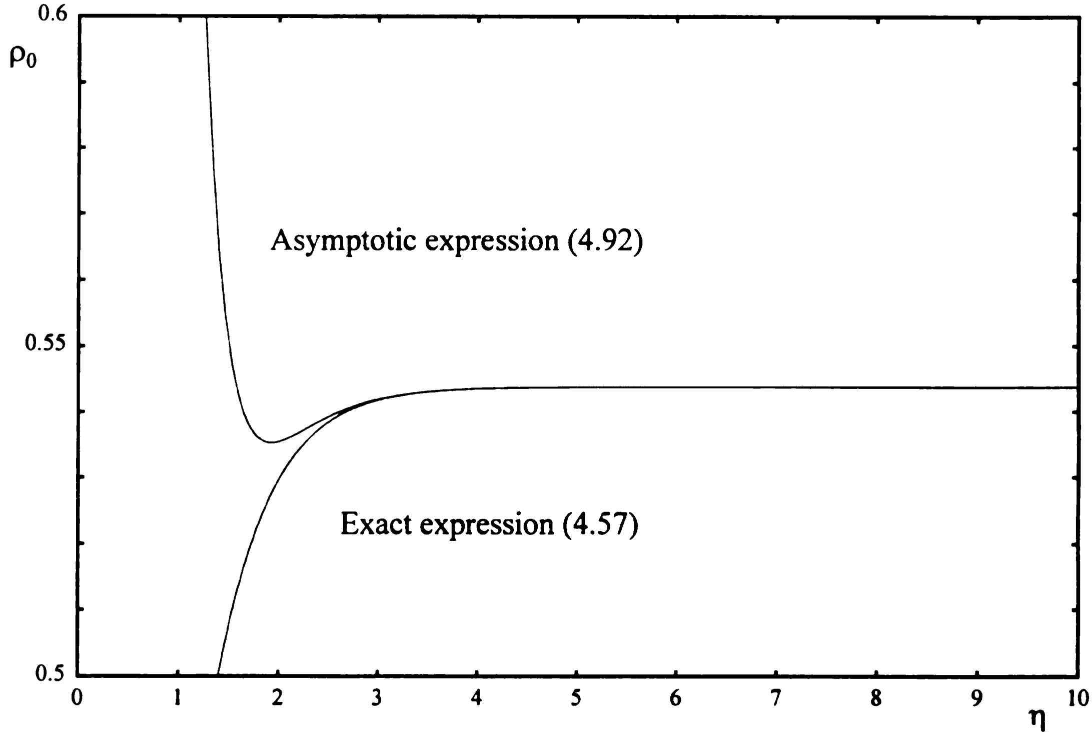


Figure 4.5 Three-term asymptotic equation of load parameter ρ_0 for barreling deformation when η goes to infinity. When $\eta>2.5$, asymptotes is quite close to exact expression. Otherwise, two expressions show totally different values.

is shown that the asymptotic equations are well matched when η has larger values in both Figure 4.4 and Figure 4.5.

4.5 Post-buckling Deformation (Second Order Solution)

The higher order solutions $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$,... are the expansion in ε as given by (3.1) to account for the postbuckled deformation. This is determined by the nonhomogeneous nature of the equations which effect $\mathbf{f}$ and $\mathbf{g}$ in (3.8) and (3.9). To determine the second order solution bifurcated from one mode of the first order solution, we now choose either flexure or barreling and also fix i=1 or 2 and mode number n=1, 2, 3... in the first order buckling solution $\mathbf{u}^{(1)}$ in (4.68) to (4.70). It will be convenient to rename i to j and n to m as we wish to use j and m in what follows as the expansion indices for the chosen first order mode. Thus for fixed j=1 or 2 and m=1,2,3,... as determined from the bifurcation mode under consideration, we seek the coefficient functions $B_{i;n}{}^{(2)}$, $A_{i;n}{}^{(2)}$, $C_{i;n}{}^{(2)}$, i=1,2, n=1,2,3,..., in the second order case.

For the zeroth terms, the general equations (4.14) and (4.15) with the right hand sides of the second order equation (3.11) become

$$\mu B_0^{(2)''} - \rho_0 C_0^{(2)} = -\mu\frac{\rho_0^5}{\Omega_{j;m}^2}(\beta_{j;m}' B_{j;m}^{(1)'})' , \qquad \rho_0 B_0^{(2)'} = \rho_0^2 (B_{j;m}^{(1)} B_{j;m}^{(1)'})',$$
$$\text{on } \Gamma_2 \qquad \mu B_0^{(2)'} - \rho_0 C_0^{(2)} = -\mu\frac{\rho_0^5}{\Omega_{j;m}^2}\beta_{j;m}' B_{j;m}^{(1)'}, \tag{4.93}$$

where

$$\beta_{j;m} = B_{j;m}^{(1)''} - \Omega_{j;m}^2 B_{j;m}^{(1)}. \tag{4.94}$$

To within the rigid body motion given by a constant in $B_0^{(2)}$, the solution for (4.93) is

$$B_0^{(2)} = \rho_0 B_{j;m}^{(1)} B_{j;m}^{(1)'}, \qquad C_0^{(2)} = \mu\left[(B_{j;m}^{(1)} B_{j;m}^{(1)'})' + \frac{\rho_0^4}{\Omega_{j;m}^2}\beta_{j;m}' B_{j;m}^{(1)'}\right]. \tag{4.95}$$

For the other coefficient functions, we will utilize the generalized solution for nonhomogeneous equation, (4.43) with the solvability conditions (4.47) and (4.48). Substituting $\mathbf{f}^{(2)}$, $\mathbf{g}^{(2)}$ in (3.11) into the equations (4.7) and (4.8) shows the following results for the right hand sides of the equation (4.9) and (4.10):

(A) If the mode of $\mathbf{u}^{(1)}$ is j=1 and m, then only the expressions for i=1, n=m and for i=2, n=2m-1 of $\mathbf{u}^{(2)}$ are non-vanishing:

$$\begin{aligned} I_{1;m}^{(2)} &= -\rho_1\rho_0^{-2}\Omega_{1;m}C_{1;m}^{(1)}, \qquad J_{1;m}^{(2)} = \rho_1 C_{1;m}^{(1)\prime}, \\ K_{1;m}^{(2)} &= -\rho_1(B_{1;m}^{(1)\prime} + \rho_0^{-2}\Omega_{1;m}A_{1;m}^{(1)}), \\ G_{1;m}^{(2)} &= 2\mu\rho_1\rho_0^{-3}\Omega_{1;m}B_{1;m}^{(1)}, \qquad H_{1;m}^{(2)} = \rho_1(C_{1;m}^{(1)} + 2\mu\rho_0^{-3}\Omega_{1;m}A_{1;m}^{(1)}), \end{aligned} \tag{4.96}$$

and

$$\begin{aligned} I_{2;2m-1}^{(2)} &= \frac{1}{2}\Omega_{1;m}(C_{1;m}^{(1)\prime}B_{1;m}^{(1)} - C_{1;m}^{(1)}B_{1;m}^{(1)\prime}), \\ J_{2;2m-1}^{(2)} &= \frac{1}{2}\Omega_{1;m}(C_{1;m}^{(1)\prime}A_{1;m}^{(1)} - C_{1;m}^{(1)}A_{1;m}^{(1)\prime}), \\ K_{2;2m-1}^{(2)} &= \frac{1}{2}\Omega_{1;m}(A_{1;m}^{(1)\prime}B_{1;m}^{(1)} - A_{1;m}^{(1)}B_{1;m}^{(1)\prime}), \\ G_{2;2m-1}^{(2)} &= \frac{1}{2}\Omega_{1;m}B_{1;m}^{(1)}C_{1;m}^{(1)}, \qquad H_{2;2m-1}^{(2)} = \frac{1}{2}\Omega_{1;m}A_{1;m}^{(1)}B_{1;m}^{(1)}. \end{aligned} \tag{4.97}$$

(B) If the mode of $\mathbf{u}^{(1)}$ is j=2 and m, then only the expressions for i=2, n=m and for i=2, n=2m of $\mathbf{u}^{(2)}$ are non-vanishing:

$$\begin{aligned} I_{2;m}^{(2)} &= -\rho_1\rho_0^{-2}\Omega_{2;m}C_{2;m}^{(1)}, \qquad J_{2;m}^{(2)} = \rho_1 C_{2;m}^{(1)\prime}, \\ K_{2;m}^{(2)} &= -\rho_1(B_{2;m}^{(1)\prime} + \rho_0^{-2}\Omega_{2;m}A_{2;m}^{(1)}), \\ G_{2;m}^{(2)} &= 2\mu\rho_1\rho_0^{-3}\Omega_{2;m}B_{2;m}^{(1)}, \qquad H_{2;m}^{(2)} = \rho_1(C_{2;m}^{(1)} + 2\mu\rho_0^{-3}\Omega_{2;m}A_{2;m}^{(1)}), \end{aligned} \tag{4.98}$$

and

$$\begin{aligned} I_{2;2m}^{(2)} &= -\frac{1}{2}\Omega_{2;m}(C_{2;m}^{(1)\prime}B_{2;m}^{(1)} - C_{2;m}^{(1)}B_{2;m}^{(1)\prime}), \\ J_{2;2m}^{(2)} &= -\frac{1}{2}\Omega_{2;m}(C_{2;m}^{(1)\prime}A_{2;m}^{(1)} - C_{2;m}^{(1)}A_{2;m}^{(1)\prime}), \\ K_{2;2m}^{(2)} &= -\frac{1}{2}\Omega_{2;m}(A_{2;m}^{(1)\prime}B_{2;m}^{(1)} - A_{2;m}^{(1)}B_{2;m}^{(1)\prime}), \\ G_{2;2m}^{(2)} &= -\frac{1}{2}\Omega_{2;m}B_{2;m}^{(1)}C_{2;m}^{(1)}, \qquad H_{2;2m}^{(2)} = -\frac{1}{2}\Omega_{2;m}A_{2;m}^{(1)}B_{2;m}^{(1)}. \end{aligned} \tag{4.99}$$

For the modes which are not mentioned above, the right hand sides of the equation (4.9) and (4.10) vanishes so that these modes have the same solutions as those for the homogeneous case or $\mathbf{u}^{(1)}$. However, according to the orthogonality rule (3.4), the solution of these modes must be eliminated. For the cases of n=m in (4.96) and (4.98), the differential operator are the same as those of homogeneous equation so that in most cases, the solution does not exist. But for the special arrangement of the nonhomogeneous terms which satisfy the solvability condition, the solutions exist. Substituting (4.96) and (4.98) into the right hand side of (4.11) and (4.13) with the relations (4.53) for $A^{(1)}{}_{j;m}$ and $C^{(1)}{}_{j;m}$ gives

$$\tilde{f} = -4\rho_1\rho_0^{-1}(B_{j;m}^{(1)\prime\prime} - \Omega_{j;m}^2 B_{j;m}^{(1)})^{\prime\prime}, \tag{4.100}$$

and

$$\begin{aligned} \tilde{g}_1 &= -2\rho_1\rho_0^{-1}(B_{j;m}^{(1)\prime\prime} - \Omega_{j;m}^2\rho_0^{-4}B_{j;m}^{(1)}), \\ \tilde{g}_2 &= -4\rho_1\rho_0^{-1}(B_{j;m}^{(1)\prime\prime} - \Omega_{j;m}^2 B_{j;m}^{(1)})^{\prime}. \end{aligned} \tag{4.101}$$

Substituting from (4.100) and (4.101) into the solvability conditions (4.50) with B_h=$B^{(1)}{}_{j;m}$ and integration by parts to the left hand side gives

$$\begin{aligned} &2\rho_1\rho_0^{-1}\Big[\{(B_{j;m}^{(1)\prime\prime} + \Omega_{j;m}^2\rho_0^{-4}B_{j;m}^{(1)})B_{j;m}^{(1)\prime}\}\Big|_{-l_2}^{l_2} \\ &-2\int_{-l_2}^{l_2}\{(B_{j;m}^{(1)\prime\prime})^2 + \Omega_{j;m}^2(B_{j;m}^{(1)})^2\}dX_2\Big] = 0. \end{aligned} \tag{4.102}$$

Considering the first order Γ_2-boundary condition $(4.52)_1$, the condition (4.102) becomes ρ_1=0 since the integral is positive definite. Hence the solution for the case of n=m exists only if ρ_1=0. If the first order load parameter ρ_1=0, the right hand sides, (4.96) and (4.98) vanish so the modes n=m have the same solutions as the homogeneous solutions. Due to the orthogonality between the first order and higher order solutions, the solutions of modes n=m must be eliminated. Therefore if the mode of j=1, m is chosen for $\mathbf{u}^{(1)}$, then only the mode associated with i=2 and n=2m-1 in the second order solution expansion $\mathbf{u}^{(2)}$

is governed by an equation that has nonhomogeneous terms. That is, only $B_{2;2m-1}^{(2)}$ has nonhomogeneous solutions so that $B_{1;n}^{(2)} = 0$ for all n and $B_{2;n}^{(2)} = 0$ unless n=2m-1. On the other hand, if j=2, m of $\mathbf{u}^{(1)}$ is chosen, then only the mode i=2 and n=2m, that is, $B_{2;2m}^{(2)}$ has nonhomogeneous solutions. Thus for fixed mode variables m=1,2,3,... and j=1,2, there is exactly one nonzero $B_{j;m}^{(1)}$ which corresponds to only $B_{i;n}^{(2)}$ governed by a nonhomogeneous equation. The index i for this nonzero second order $B_{i;n}^{(2)}$ is always i=2 and n is given by n=2m-1 if j=1 and n=2m if j=2. Note that the mode number n of the second order is different from the mode number m of the first order so that the differential operators of the second order are different from those of the first order. Hence there exists a unique nontrivial solution to the second order equation (4.16). After combining the two cases in which the nontrivial solutions are possible to exist, the right hand sides become

$$I_{2;n}^{(2)} = \frac{1}{2}\mu\rho_0^3\begin{cases}\dfrac{1}{\Omega_{1;m}}\gamma_{1;m}\delta_{n;2m-1}, \\ -\dfrac{1}{\Omega_{2;m}}\gamma_{2;m}\delta_{n;2m},\end{cases} \qquad J_{2;n}^{(2)} = \frac{1}{2}\mu\rho_0^5\begin{cases}\dfrac{1}{\Omega_{1;m}^2}\kappa_{1;m}\delta_{n;2m-1}, \\ -\dfrac{1}{\Omega_{2;m}^2}\kappa_{2;m}\delta_{n;2m},\end{cases} \tag{4.103}$$

$$K_{2;n}^{(2)} = \frac{1}{2}\rho_0^2\begin{cases}\alpha_{1;m}\delta_{n;2m-1}, \\ -\alpha_{2;m}\delta_{n;2m},\end{cases}$$

and

$$G_{2;n}^{(2)} = \frac{1}{2}\mu\rho_0^3\begin{cases}\dfrac{1}{\Omega_{1;m}}\beta'_{1;m}B_{1;m}^{(1)}\delta_{n;2m-1}, \\ -\dfrac{1}{\Omega_{2;m}}\beta'_{2;m}B_{2;m}^{(1)}\delta_{n;2m},\end{cases} \qquad H_{2;n}^{(2)} = \frac{1}{2}\mu\rho_0^5\begin{cases}\dfrac{1}{\Omega_{1;m}^2}\beta'_{1;m}B_{1;m}^{(1)\prime}\delta_{n;2m-1}, \\ -\dfrac{1}{\Omega_{2;m}^2}\beta'_{2;m}B_{2;m}^{(1)\prime}\delta_{n;2m},\end{cases} \tag{4.104}$$

where the upper terms are for j=1 and the lower terms are for j=2 and $A_{j;m}$ and $C_{j;m}$ have been converted to $B_{j;m}$. The notations used above are

$$\begin{aligned} \alpha_{j;m} &= B_{j;m}^{(1)\prime\prime} B_{j;m}^{(1)} - B_{j;m}^{(1)\prime} B_{j;m}^{(1)\prime}, \qquad \gamma_{j;m} = \beta_{j;m}^{\prime\prime} B_{j;m}^{(1)} - \beta_{j;m}^{\prime} B_{j;m}^{(1)\prime}, \\ \kappa_{j;m} &= \beta_{j;m}^{\prime\prime} B_{j;m}^{(1)\prime} - \beta_{j;m}^{\prime} B_{j;m}^{(1)\prime\prime}, \end{aligned} \tag{4.105}$$

and $\delta_{p,q}$ is the Kronecker delta ($\delta_{p;q} = 1$ if $p = q$ and $\delta_{p,q} = 0$ if $p \neq q$).

For the equations (4.103) and (4.104), the mode numbers i and n in the left hand side of (4.9) and (4.10) are changed accordingly. And from the definitions of mode number in (4.1), $\Omega_{i;n}$ can be written in terms of $\Omega_{j;m}$ as

$$\Omega_{2;2m-1} = 2\Omega_{1;m}, \qquad \Omega_{2;2m} = 2\Omega_{2;m}. \tag{4.106}$$

so that the formulations for the second order can be considered as the generalized formulations developed in Section 4.2 with $2\Omega_{j;m}$ in place of $\Omega_{i;n}$. Applying the relations (4.106) and the right hand side terms in (4.103) and (4.104) into the boundary value problem (4.11) and (4.13) for both cases of j=1 and 2, have the following format

$$\begin{aligned} &B^{(2)iv} - 4\Omega_{j;m}^2(1+\rho_0^{-4})B^{(2)\prime\prime} + 16\Omega_{j;m}^4\rho_0^{-4}B^{(2)} \\ &= \omega\rho_0\left(\gamma_{j;m}{}' - 2\kappa_{j;m} + \frac{1}{2}\alpha_{j;m}^{\prime\prime\prime} - 2\Omega_{j;m}^2\alpha_{j;m}^{\prime}\right) \equiv \tilde{f}^{(2)}, \end{aligned} \tag{4.107}$$

on Γ_2,

$$\begin{aligned} &B^{(2)\prime\prime} + 4\Omega_{j;m}^2\rho_0^{-4}B^{(2)} = \omega\rho_0\left(\frac{1}{2}\alpha_{j;m}^{\prime} + \beta_{j;m}^{\prime}B_{j;m}^{(1)}\right) \equiv \tilde{g}_1^{(2)}, \\ &B^{(2)\prime\prime\prime} - 4\Omega_{j;m}^2(1+2\rho_0^{-4})B^{(2)\prime} \\ &\quad = \omega\rho_0\left[\gamma_{j;m} + \frac{1}{2}\alpha_{j;m}^{\prime\prime} - 2(1+\rho_0^{-4})\Omega_{j;m}^2\alpha_{j;m} - 2\beta_{j;m}^{\prime}B_{j;m}^{(1)\prime}\right] \equiv \tilde{g}_2^{(2)}, \end{aligned} \tag{4.108}$$

where

$$B^{(2)} = \begin{cases} B_{2;2m-1}^{(2)}, \\ B_{2;2m}^{(2)}, \end{cases} \qquad \omega = \begin{cases} 1, & \text{for } j = 1, \\ -1, & \text{for } j = 2. \end{cases} \tag{4.109}$$

According to the relations (4.12), the other coefficients become

$$
\begin{aligned}
A^{(2)} &= \frac{\rho_0^2}{2\Omega_{j;m}}\left(B^{(2)\prime} - \frac{\omega\rho_0}{2}\alpha_{j;m}\right), \\
C^{(2)} &= \frac{\mu\rho_0^3}{4\Omega_{j;m}^2}\left[B^{(2)\prime\prime\prime} - 4\Omega_{j;m}^2 B^{(2)\prime} - \frac{\omega\rho_0}{2}(\alpha_{j;m}{}'' - 4\Omega_{j;m}^2\alpha_{j;m} + 2\gamma_{j;m})\right],
\end{aligned}
\tag{4.110}
$$

where

$$
A^{(2)} = \begin{cases} A_{2;2m-1}^{(2)}, \\ A_{2;2m}^{(2)}, \end{cases} \qquad C^{(2)} = \begin{cases} C_{2;2m-1}^{(2)}, & j = 1, \\ C_{2;2m}^{(2)}, & j = 2. \end{cases} \tag{4.111}
$$

Applying the first order solutions (4.54) and (4.56) with zero right hand side into (4.94) and (4.105), then the right hand sides of (4.107) and (4.108) becomes

$$
\tilde{f}^{(2)} = \frac{3}{2}\omega\rho_0\Omega_{j;m}^2(\rho_0^{-4} - 1)\alpha'_{j;m}, \tag{4.112}
$$

and

$$
\begin{aligned}
\tilde{g}_1^{(2)} &= \frac{1}{2}\omega\rho_0\Omega_{j;m}^2(7\rho_0^{-4} + 1)B_{j;m}^{(1)}B_{j;m}^{(1)\prime} \\
\tilde{g}_2^{(2)} &= -2\omega\rho_0\Omega_{j;m}^2(2\rho_0^{-4} - 1)(B_{j;m}^{(1)\prime})^2
\end{aligned}
\tag{4.113}
$$

Substituting $B_{j;m}^{(1)}$ for flexure in (4.54) or for barreling in (4.56) reduces the equations (4.112) and (4.113) to

$$
\begin{aligned}
\tilde{f}^{(2)} = {}& \frac{3}{4}\omega\nu M^2 s_{j;m}\Omega_{j;m}^5\rho_0(1 - \rho_0^{-4})^2[\nu(1 - \rho_0^{-2})\sinh\{\Omega_{j;m}(1 + \rho_0^{-2})X_2\} \\
& + (1 + \rho_0^{-2})\sinh\{\Omega_{j;m}(1 - \rho_0^{-2})X_2\}],
\end{aligned}
\tag{4.114}
$$

$$
\begin{aligned}
\tilde{g}_1^{(2)} = {}& \frac{1}{4}\omega M^2\rho_0(7\rho_0^{-4} + 1)\Omega_{j;m}^3[\sinh(2\Omega_{j;m}X_2) + s_{j;m}^2\rho_0^{-2}\sinh(2\Omega_{j;m}\rho_0^{-2}X_2) \\
& - s_{j;m}\{(1 + \rho_0^{-2})\sinh\{\Omega_{j;m}(1 + \rho_0^{-2})X_2\} + \nu(1 - \rho_0^{-2})\sinh\{\Omega_{j;m}(1 - \rho_0^{-2})X_2\}\}],
\end{aligned}
\tag{4.115}
$$

$$\tilde{g}_2^{(2)} = -\omega M^2 \rho_0 (2\rho_0^{-4} - 1)\Omega_{j;m}^4 [\cosh(2\Omega_{j;m} X_2) + s_{j;m}^2 \rho_0^{-4} \cosh(2\Omega_{j;m}\rho_0^{-2} X_2) \\ - \nu(1 + s^2\rho_0^{-4}) - 2s_{j;m}\rho_0^{-2}\{\cosh\{\Omega_{j;m}(1+\rho_0^{-2})X_2\} - \nu\cosh\{\Omega_{j;m}(1-\rho_0^{-2})X_2\}\}], \tag{4.116}$$

where ν=1 for flexural mode and ν=-1 for barreling mode. The general solution to the differential equation (4.107) with the boundary conditions (4.108) has the expression

$$B^{(2)} = N_1 \sinh(2\rho_0^{-2}\Omega_{j;m} X_2) + N_2 \sinh(2\Omega_{j;m} X_2) \\ + N_3 \sinh\{(\rho_0^{-2}+1)\Omega_{j;m} X_2\} + N_4 \sinh\{(\rho_0^{-2}-1)\Omega_{j;m} X_2\}. \tag{4.117}$$

Here the first two terms are from the homogeneous part and the rest are from nonhomogeneous part with constants N_i's. Introducing (4.117) to the differential equation (4.112) and matching the coefficients give N_3 and N_4 as

$$N_3 = \frac{3}{4}\omega M^2 s_{j;m}\Omega_{j;m}\rho_0 \frac{(\rho_0^{-4}-1)(\rho_0^{-2}+1)}{(3\rho_0^{-2}+1)(\rho_0^{-2}+3)}, \\ N_4 = \frac{3}{4}\omega\nu M^2 s_{j;m}\Omega_{j;m}\rho_0 \frac{(\rho_0^{-4}-1)(\rho_0^{-2}-1)}{(3\rho_0^{-2}-1)(\rho_0^{-2}-3)}, \tag{4.118}$$

and to the boundary conditions (4.113) give the values of other coefficients as

$$N_1 = -\frac{\omega M^2 \Omega_{j;m}\rho_0^3}{32\Delta}[4\rho_0^{-4} D_1 \cosh(2\eta) - (1+\rho_0^{-4}) D_2 \sinh(2\eta)], \\ N_2 = -\frac{\omega M^2 \Omega_{j;m}\rho_0}{16\Delta}[\rho_0^{-2} D_2 \sinh(2\rho_0^{-2}\eta) - (1+\rho_0^{-4}) D_1 \cosh(2\rho_0^{-2}\eta)], \tag{4.119}$$

where

$$= 4\rho_0^{-6}\sinh(2\rho_0^{-2}\eta)\cosh(2\eta) - (1+\rho_0^{-4})^2\cosh(2\rho_0^{-2}\eta)\sinh(2\eta), \tag{4.120}$$

and

$$D_1 = d_1(\rho_0^{-2}) + \nu d_1(-\rho_0^2) - (7\rho_0^{-4}+1)[\sinh(2\eta) + s_{j;m}^2\rho_0^{-2}\sinh(2\rho_0^{-2}\eta)], \tag{4.121}$$

$$D_2 = d_2(\rho_0^{-2}) + \nu d_2(-\rho_0^{-2})$$
$$-4(2\rho_0^{-4} - 1)[(\cosh 2\eta - \nu) + s_{j;m}^2 \rho_0^{-4}(\cosh 2\rho_0^{-2}\eta - \nu)], \tag{4.122}$$

$$d_1(\xi) = -s_{j;m}(1+\xi)\Big[3\frac{(1-\xi^2)(1+2\xi+5\xi^2)}{(1+3\xi)(3+\xi)} - (1+7\xi^2)\Big]\sinh(1+\xi)\eta, \tag{4.123}$$

$$d_2(\xi) = -s_{j;m}\Big[3\frac{(1-\xi^2)(1+\xi)^2(3-2\xi+7\xi^2)}{(1+3\xi)(3+\xi)} + 8\xi(1-2\xi^2)\Big]\cosh(1+\xi)\eta. \tag{4.124}$$

The denominator Δ represents the combination of the conditions (4.55) for flexure and (4.57) for barreling such as

$$\Delta = -2\tilde{C}_1\tilde{C}_2[\tilde{S}_1\tilde{S}_2\{(1+\rho_0^{-4})^2\tanh\rho_0^{-2}\eta - 4\rho_0^{-6}\tanh\eta\}$$
$$+\tilde{C}_1\tilde{C}_2\{(1+\rho_0^{-4})^2\tanh\eta - 4\rho_0^{-6}\tanh\rho_0^{-2}\eta\}]. \tag{4.125}$$

The solution (4.117) can be also obtained by modifying a generalized solution (4.43) with replacing $\Omega_{i;n}$ by $2\Omega_{j;m}$. The particular solution (4.40) contributes to the last two terms including N_3 and N_4 in (4.117). Using the variational analysis, Sawyers and Rivlin (1982) also found the same type of second order solution (4.117) which only differs in the notations. Finally, the corresponding second order nonhomogeneous solutions $\mathbf{u}^{(2)}=(u_1^{(2)},u_2^{(2)},p^{(2)})$ become from (4.2) with (4.95) for the zeroth term and (4.110) and (4.111) for the higher order as

$$u_1^{(2)} = -A^{(2)}\sin(2\Omega_{j;m}X_1),$$
$$u_2^{(2)} = \frac{1}{2}\rho_0 B_{j;m}^{(1)}B_{j;m}^{(1)\prime} + B^{(2)}\cos(2\Omega_{j;m}X_1), \tag{4.126}$$
$$p^{(2)} = \frac{\mu}{2}\Big[(B_{j;m}^{(1)}B_{j;m}^{(1)\prime})' + \frac{\rho_0^4}{\Omega_{j;m}^2}\beta_{j;m}{}'B_{j;m}^{(1)\prime}\Big] + C^{(2)}\cos(2\Omega_{j;m}X_1),$$

where $B_{1;n}^{(2)}$ has only homogeneous solution and $B_{2;n}^{(2)}$ has nonhomogeneous solutions $B_{2;2m-1}^{(2)}$ or $B_{2;2m}^{(2)}$ for j=1 or 2, respectively. The nature of coefficients $B^{(2)}$ depends on the choice of the first order solution - flexure or barreling mode and $j=1$ or $j=2$ and mode number m.

4.6 Summary of Full Buckled Deformations

The fully finite buckled deformations $v_1(X_1,X_2)$, $v_2(X_1,X_2)$ with pressure $p(X_1,X_2)$ after perturbation expansion are defined in (3.1) and solutions of the first and second orders are determined in the previous two sections. Now we will summarize the results as a reminder of complex procedure. The buckled deformation with respect to X_2 axis is given from $(3.1)_2$ as

$$v_2(X_1,X_2) = \varepsilon u_2^{(1)}(X_1,X_2) + \varepsilon^2 u_2^{(2)}(X_1,X_2) + O(\varepsilon^3). \tag{4.127}$$

More specific result is from (4.1), (4.69) and $(4.126)_2$ as

$$v_2(X_1,X_2) = \begin{cases} \varepsilon B_{1;m}^{(1)}(X_2)\sin\left(\dfrac{2m-1}{2l_1}\pi X_1\right) \\ \quad + \varepsilon^2\left\{\dfrac{\rho_0}{2}B_{1;m}^{(1)}(X_2)B_{1;m}^{(1)}{}'(X_2) + B_{1;m}^{(2)}(X_2)\cos\left(\dfrac{2m-1}{l_1}\pi X_1\right)\right\} + O(\varepsilon^3), \\ \varepsilon B_{2;m}^{(1)}(X_2)\cos\left(\dfrac{m}{l_1}\pi X_1\right) \\ \quad + \varepsilon^2\left\{\dfrac{\rho_0}{2}B_{2;m}^{(1)}(X_2)B_{2;m}^{(1)}{}'(X_2) + B_{2;m}^{(2)}(X_2)\cos\left(\dfrac{2m}{l_1}\pi X_1\right)\right\} + O(\varepsilon^3), \end{cases} \tag{4.128}$$

where the upper solution is for $j=1$ and the lower solution is for $j=2$. The coefficients $B_{j;m}^{(1)}$ are given in (4.54) for flexural deformation and (4.56) for barreling deformation and the coefficients $B_{j;m}^{(2)}$ are given in (4.117). The subscripts $j=1,2$ and $m=1,2,3,...$ are indices of the chosen onset bifurcation modes. Then deformation $v_1(X_1,X_2)$ along the X_1 axis and pressure $p(X_1,X_2)$ follow v_2 solution (4.128) in an appropriate order of ε from (4.2), (4.53) and (4.110). In conclusion, there exist four different types of solution which depend on the deformation types such as flexure or barreling and chosen value of j=1 or 2. In each type, there exist infinite modes of solution.

4.7 Reduced Formulations Due to Symmetric Load Parameter

The first order solutions $\mathbf{u}^{(1)}$ obtained in (4.68) - (4.70) may affect the formulations on the load parameter and the energy equations so that their equations will be reduced to simpler forms. These simplified forms will make less efforts to investigate the stability near buckling initiation.

4.7.1 Load Parameters

With the results on the first order solution in hand, the postbuckling formulations given in solvability conditions can be much reduced. In particular, we now show that ρ_1 as given by (3.33) must vanish. Applying integration by parts once with respect to X_1 to the first two terms of R_1 in (3.31) yields

$$\begin{aligned} R_1 &= \int_{\Gamma_2} p^{(1)}\left(u_1^{(1)} u_{2,2}^{(1)} - u_2^{(1)} u_{1,2}^{(1)}\right)\Big|_{\Gamma_1} dX_2 \\ &\quad + \iint_{\Pi} 3p^{(1)}\left(u_{1,2}^{(1)} u_{2,1}^{(1)} - u_{1,1}^{(1)} u_{2,2}^{(1)}\right) dX_1 dX_2. \end{aligned} \tag{4.129}$$

Then applying the Γ_1-boundary conditions in $(3.9)_1$ to the first term, the numerator R_1 in (3.33) becomes

$$R_1 = 3\iint_{\Pi} p^{(1)}\left(u_{1,2}^{(1)} u_{2,1}^{(1)} - u_{1,1}^{(1)} u_{2,2}^{(1)}\right) dX_1 dX_2. \tag{4.130}$$

Substituting from either option for the first order solution (4.68) into (4.130) and performing the associated integration gives

$$R_1 = 0. \tag{4.131}$$

Rearranging R_2 in (3.32) yields

$$\begin{aligned} R_2 = & -\iint_{\Pi}(\rho_0^{-2}u_1^{(1)}p_{,1}^{(1)} - u_2^{(1)}p_{,2}^{(1)})d\mathbf{X} + \iint_{\Pi}p^{(1)}(\rho_0^{-2}u_{1,1}^{(1)} - u_{2,2}^{(1)})d\mathbf{X} \\ & - 2\mu\rho_0^{-3}\int_{\Gamma_1}(u_1^{(1)}u_{2,1}^{(1)} - u_{1,1}^{(1)}u_2^{(1)})\Big|_{\Gamma_2} dX_1 - \int_{\Gamma_1}(p^{(1)}u_2^{(1)})|_{\Gamma_2} dX_1. \end{aligned} \tag{4.132}$$

Integrating by parts on the first integral with respect to X_1 to the first term and X_2 to the second term produces the same terms as the second integral in (4.132). And in the third integral, integration by part with respect to X_1 on the first term yields the same term as the second. Then collecting all the remaining terms gives

$$\begin{aligned} R_2 = & -\rho_0^{-2}\int_{\Gamma_2}u_1^{(1)}p^{(1)}|_{\Gamma_1}dX_2 - 2\mu\rho_0^{-3}u_1^{(1)}u_2^{(1)}|_{\Gamma_1,\Gamma_2} \\ & + 4\mu\rho_0^{-3}\int_{\Gamma_1}u_{1,1}^{(1)}u_2^{(1)}|_{\Gamma_2}dX_1 + 2\iint_{\Pi}p^{(1)}(\rho_0^{-2}u_{1,1}^{(1)} - u_{2,2}^{(1)})d\mathbf{X}. \end{aligned} \tag{4.133}$$

Entering the Γ_1-boundary conditions in $(3.9)_1$ to the first two integrals in (4.133) shows

$$R_2 = 2[2\mu\rho_0^{-3}\int_{\Gamma_1}u_{1,1}^{(1)}u_2^{(1)}|_{\Gamma_2}dX_1 + \iint_{\Pi}p^{(1)}(\rho_0^{-2}u_{1,1}^{(1)} - u_{2,2}^{(1)})d\mathbf{X}]. \tag{4.134}$$

Substituting the first order solutions (4.68) - (4.70) into (4.134) and integration with respect to X_1 now yields

$$R_2 = -4\mu l_1\frac{\rho_0^3}{\Omega_i^2}B_i^{(1)\prime}(\Omega_i^2\rho_0^{-4}B_i^{(1)} + B_i^{(1)\prime\prime})|_{\Gamma_2} - \int_{\Gamma_2}\{(B_i^{(1)\prime\prime})^2 + \Omega_{i2}(B_i^{(1)\prime})^2\}dX_2. \tag{4.135}$$

The Γ_2 boundary term in (4.135) vanishes by virtue of the homogeneous form of the Γ_2-boundary condition $(4.13)_1$. Then

$$R_2 = 4\mu l_1\frac{\rho_0^3}{\Omega_i^2}\int_{\Gamma_2}\{(B_i^{(1)\prime\prime})^2 + \Omega_i^2(B_i^{(1)\prime})^2\}dX_2. \tag{4.136}$$

It is significant to note that

$$R_2 > 0, \tag{4.137}$$

for nonconstant $B_i^{(1)}$. The results on R_1 in (4.131) and R_2 in (4.137) in conjunction with (3.33) show that

$$\rho_1 = 0. \tag{4.138}$$

This result simplifies the expression of ρ_2 in (3.38) to

$$\rho_2 = -\frac{R_3}{R_2}. \tag{4.139}$$

The numerator R_3 can be reduced via integration by parts with respect to X_1 on the first two terms in (3.35) such as

$$\begin{aligned} R_3 = & \int_{\Gamma_2} [u_1^{(1)}(p^{(2)}u_{2,2}^{(1)} + p^{(1)}u_{2,2}^{(2)}) - u_2^{(1)}(p^{(2)}u_{1,2}^{(1)} + p^{(1)}u_{1,2}^{(2)})]\big|_{\Gamma_1} dX_2 \\ & -\iint_{\Pi} [p^{(1)}(u_{1,1}^{(1)}u_{2,2}^{(2)} - u_{2,1}^{(1)}u_{1,2}^{(2)}) + p^{(2)}(u_{1,1}^{(1)}u_{2,2}^{(1)} - u_{2,1}^{(1)}u_{1,2}^{(1)})]d\mathbf{X} \\ & + \iint_{\Pi} 2p^{(1)}(u_{1,2}^{(1)}u_{2,1}^{(2)} - u_{2,2}^{(1)}u_{1,1}^{(2)})d\mathbf{X} \\ & + \iint_{\Pi} [p^{(1)}(u_{2,1}^{(1)}u_{1,2}^{(2)} - u_{1,1}^{(1)}u_{2,2}^{(2)}) + p^{(2)}(u_{2,1}^{(1)}u_{1,2}^{(1)} - u_{1,1}^{(1)}u_{2,2}^{(1)})]d\mathbf{X}. \end{aligned} \tag{4.140}$$

After introducing the Γ_1-boundary conditions $(3.9)_1$ for k=1 and 2, the first row in (4.140) vanishes and rearranging all the remaining terms yields

$$\begin{aligned} R_3 = & 2\iint_{\Pi} [p^{(1)}\{(u_{1,2}^{(1)}u_{2,1}^{(2)} + u_{1,2}^{(2)}u_{2,1}^{(1)}) - (u_{1,1}^{(1)}u_{2,2}^{(2)} + u_{1,1}^{(2)}u_{2,2}^{(1)})\} \\ & + p^{(2)}(u_{1,2}^{(1)}u_{2,1}^{(1)} - u_{1,1}^{(1)}u_{2,2}^{(1)})]dX_1 dX_2. \end{aligned} \tag{4.141}$$

4.7.2 Energy Equations

With ρ_1=0 in the condition (4.138), the energy terms in (3.55)-(3.57) can also be reduced. However, it is convenient to simplify first the terms in (3.43) - (3.45) and (3.53).

After integration by parts with respect to the variables of $u_2^{(1)}$ on (3.43), the variables of $u_i^{(2)}$, i=1,2 on (3.44) and the variables of $u_2^{(2)}$ and $u_i^{(3)}$ on (3.45), then Φ_k, k=1,2,3, become

$$\Phi_1 = \int_{\Gamma_2} u_{1,2}^{(1)} u_2^{(1)}\Big|_{\Gamma_1} dX_2 - \int_{\Gamma_1} u_{1,1}^{(1)} u_2^{(1)}\Big|_{\Gamma_2} dX_1, \tag{4.142}$$

$$\Phi_2 = \int_{\Gamma_2} (u_{1,2}^{(1)} u_2^{(2)} - u_{2,2}^{(1)} u_1^{(2)})\Big|_{\Gamma_2} dX_2 + \int_{\Gamma_1} (u_{2,1}^{(1)} u_1^{(2)} - u_{1,1}^{(1)} u_2^{(2)})\Big|_{\Gamma_2} dX_1, \tag{4.143}$$

$$\begin{aligned}\Phi_3 = &\int_{\Gamma_2} (u_{1,2}^{(2)} u_2^{(2)} - u_{2,2}^{(1)} u_1^{(3)} + u_{2,1}^{(1)} u_2^{(3)})\Big|_{\Gamma_1} dX_2 \\ &-\int_{\Gamma_1} (u_{1,1}^{(2)} u_1^{(2)} - u_{2,1}^{(1)} u_1^{(3)} + u_{1,1}^{(1)} u_2^{(3)})\Big|_{\Gamma_2} dX_1.\end{aligned} \tag{4.144}$$

Substituting Γ_1 - boundary condition in $(3.9)_1$, all the first integrals in (4.142) to (4.144) vanish so that

$$\begin{aligned}\Phi_1 &= -\int_{\Gamma_1} u_{1,1}^{(1)} u_2^{(1)}\Big|_{\Gamma_2} dX_1, \\ \Phi_2 &= \int_{\Gamma_1} (u_{2,1}^{(1)} u_1^{(2)} - u_{1,1}^{(1)} u_2^{(2)})\Big|_{\Gamma_2} dX_1, \\ \Phi_3 &= -\int_{\Gamma_1} (u_{1,1}^{(2)} u_1^{(2)} - u_{2,1}^{(1)} u_1^{(3)} + u_{1,1}^{(1)} u_2^{(3)})\Big|_{\Gamma_2} dX_1.\end{aligned} \tag{4.145}$$

Similarly, after integration by part with respect to the variables for $u_i^{(j)}$, the definition $\Xi_{i,j}$,'s in (3.53) are reduced to

$$\begin{aligned}\Xi_{ij} = &\int_{\Gamma_2} (u_{1,1}^{(i)} u_1^{(j)} + u_{2,1}^{(i)} u_2^{(j)})\Big|_{\Gamma_1} dX_2 + \int_{\Gamma_1} (u_{1,2}^{(i)} u_1^{(j)} + u_{2,2}^{(i)} u_2^{(j)})\Big|_{\Gamma_2} dX_1 \\ &-\iint_{\Pi} [(u_{1,11}^{(i)} + u_{1,22}^{(i)}) u_1^{(j)} + (u_{2,11}^{(i)} + u_{2,22}^{(i)}) u_2^{(j)}] d\mathbf{X},\end{aligned} \tag{4.146}$$

where i=1,2 and j=1,2,3,.... For i=1, the coefficients of $u_1^{(j)}$ and $u_2^{(j)}$ in the third integral are substituted by the governing equations in $(3.8)_{1,2}$ for k=1 and due to the Γ_1-boundary condition in $(3.9)_1$, the first integral vanishes. Hence

$$\Xi_{1j} = \int_{\Gamma_1} (u_{1,2}^{(1)} u_1^{(j)} + u_{2,2}^{(1)} u_2^{(j)})\Big|_{\Gamma_2} dX_1 - \frac{1}{\mu}\iint_{\Pi} (\rho_0^{-1} p_{,1}^{(1)} u_1^{(j)} + \rho_0 p_{,2}^{(1)} u_2^{(j)}) d\mathbf{X}. \tag{4.147}$$

Integration by parts with respect to the variables of $p^{(1)}$ in the second integral in (4.147) and application of the boundary condition $(3.9)_1$ yield

$$\begin{aligned}\Xi_{1j} &= \int_{\Gamma_1}\left(u^{(1)}_{1,2}u^{(j)}_{1} + u^{(1)}_{2,2}u^{(j)}_{2} - \frac{\rho_0}{\mu}p^{(1)}u^{(j)}_{2}\right)\Bigg|_{\Gamma_2} dX_1 \\ &\quad + \frac{1}{\mu}\iint_{\Pi}(\rho_0^{-1}u^{(j)}_{1,1} + \rho_0 u^{(j)}_{2,2})p^{(1)}d\mathbf{X}.\end{aligned} \tag{4.148}$$

The parentheses inside of the second integral in (4.148) is the left hand sides of incompressibility conditions in $(3.8)_3$ so that substituting the right hand sides of the condition to (4.148) becomes for each j

$$\Xi_{11} = \int_{\Gamma_1}\left(u^{(1)}_{1,2}u^{(1)}_{1} + u^{(1)}_{2,2}u^{(1)}_{2} - \frac{\rho_0}{\mu}p^{(1)}u^{(1)}_{2}\right)\Bigg|_{\Gamma_2} dX_1, \tag{4.149}$$

$$\begin{aligned}\Xi_{12} &= \int_{\Gamma_1}\left(u^{(1)}_{1,2}u^{(2)}_{1} + u^{(1)}_{2,2}u^{(2)}_{2} - \frac{\rho_0}{\mu}p^{(1)}u^{(2)}_{2}\right)\Bigg|_{\Gamma_2} dX_1 \\ &\quad + \frac{1}{\mu}\iint_{\Pi}p^{(1)}(u^{(1)}_{1,2}u^{(1)}_{2,1} - u^{(1)}_{1,1}u^{(1)}_{2,2})d\mathbf{X},\end{aligned} \tag{4.150}$$

$$\begin{aligned}\Xi_{13} &= \int_{\Gamma_1}\left(u^{(1)}_{1,2}u^{(3)}_{1} + u^{(1)}_{2,2}u^{(3)}_{2} - \frac{\rho_0}{\mu}p^{(1)}u^{(3)}_{2}\right)\Bigg|_{\Gamma_2} dX_1 \\ &\quad + \frac{1}{\mu}\iint_{\Pi}p^{(1)}\{\rho_2(\rho_0^{-2}u^{(1)}_{1,1} - u^{(1)}_{2,2}) + (u^{(1)}_{1,2}u^{(2)}_{2,1} - u^{(1)}_{1,1}u^{(2)}_{2,2}) \\ &\quad + (u^{(2)}_{1,2}u^{(1)}_{2,1} - u^{(2)}_{1,1}u^{(1)}_{2,2})\}d\mathbf{X}.\end{aligned} \tag{4.151}$$

When $i \neq 1$, in the case of Ξ_{22}, the formulation is slightly different from the previous i=1 case. Since the coefficients of $u_1^{(j)}$ and $u_2^{(j)}$ in the third integral in (4.146) cannot be substituted by the simple terms, the formulation has more terms than i=1 case as

$$\Xi_{22} = \int_{\Gamma_1} \left(u_{1,2}^{(2)} u_1^{(2)} + u_{2,2}^{(2)} u_2^{(2)} - \frac{\rho_0}{\mu} p^{(2)} u_2^{(2)} \right) \Bigg|_{\Gamma_2} dX_1$$

$$+ \frac{1}{\mu} \iint_{\Pi} (u_{1,2}^{(1)} u_{2,1}^{(1)} - u_{1,1}^{(1)} u_{2,2}^{(1)}) p^{(2)} d\mathbf{X} \tag{4.152}$$

$$- \frac{1}{\mu} \iint_{\Pi} \{ u_1^{(2)} (p_{,1}^{(1)} u_{2,2}^{(1)} - p_{,2}^{(1)} u_{2,1}^{(1)}) + u_2^{(2)} (p_{,2}^{(1)} u_{1,1}^{(1)} - p_{,1}^{(1)} u_{1,2}^{(1)}) \} d\mathbf{X}.$$

Then, from the abbreviated equations (4.145) and (4.149) to (4.152), the terms in the expansion of energy difference (3.55) to (3.57) have reduced formulation. Substituting (4.145) and (4.149) into (3.55) yields

$$E_2 = \frac{\mu}{2} \int_{\Gamma_1} \left[\left\{ (u_{2,2}^{(1)} - \rho_0^{-2} u_{1,1}^{(1)}) - \frac{\rho_0}{\mu} p^{(1)} \right\} u_2^{(1)} + (u_{1,2}^{(1)} u_1^{(1)} - \rho_0^{-2} u_{1,1}^{(1)} u_2^{(1)}) \right] \Bigg|_{\Gamma_2} dX_1. \tag{4.153}$$

The coefficient of $u_2^{(1)}$ in the first term in (4.153) vanish due to the Γ_2-boundary condition $(3.9)_2$, and after integration by part with respect to X_1 to the second term and applying the Γ_1-boundary condition $(3.9)_1$, E_2 becomes

$$E_2 = \frac{\mu}{2} \int_{\Gamma_1} (u_{1,2}^{(1)} + \rho_0^{-2} u_{2,1}^{(1)}) u_1^{(1)} \big|_{\Gamma_2} dX_1. \tag{4.154}$$

Substituting again the Γ_2-boundary condition in $(3.9)_2$ to the coefficient of $u_1^{(1)}$ in (4.154) yields

$$E_2 = 0. \tag{4.155}$$

Similarly E_3 in (3.56) with (4.138), (4.143) and (4.150) becomes

$$
\begin{aligned}
E_3 &= \mu(\rho_0^{-2}\Phi_2 + \Xi_{12}) \\
&= \int_{\Gamma_1}\left[\left\{(u_{2,2}^{(1)} - \rho_0^{-2}u_{1,1}^{(1)}) - \frac{\rho_0}{\mu}p^{(1)}\right\}u_2^{(2)} + (u_{1,2}^{(1)} + \rho_0^{-2}u_{2,1}^{(1)})u_1^{(2)}\right]\Bigg|_{\Gamma_2} dX_1 \\
&\quad + \frac{1}{\mu}\iint_{\Pi} p^{(1)}(u_{1,2}^{(1)}u_{2,1}^{(1)} - u_{1,1}^{(1)}u_{2,2}^{(1)})dX_1 dX_2.
\end{aligned}
\tag{4.156}
$$

Applying the Γ_2-boundary conditions in $(3.9)_2$ to the first integral, E_3 becomes

$$
E_3 = \frac{1}{\mu}\iint_{\Pi} p^{(1)}(u_{1,2}^{(1)}u_{2,1}^{(1)} - u_{1,1}^{(1)}u_{2,2}^{(1)})dX_1 dX_2. \tag{4.157}
$$

The integrand of E_3 in (4.157) is the same as that of R_1 in (4.130) so that the procedure leading to (4.131) also gives

$$
E_3 = 0. \tag{4.158}
$$

Based on the results (3.54), (4.155) and (4.158), the first nonzero term in the energy difference is, at minimum, the fourth order term E_4 so that

$$
\Delta E = 2l_3 E_4 \varepsilon^4 + O(\varepsilon^5). \tag{4.159}
$$

With (4.138), the formulation of E_4 in (3.57) becomes

$$
E_4 = \mu\left(-2\rho_0^{-3}\rho_2\Phi_1 + \rho_0^{-2}\Phi_3 + \frac{1}{2}\Xi_{22} + \Xi_{13}\right). \tag{4.160}
$$

Here Φ_1 requires use of $\mathbf{u}^{(1)}$, Ξ_{22} requires use of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, and Φ_3 and Ξ_{13} requires use of $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, and $\mathbf{u}^{(3)}$.

We now establish, however, that E_4 can in fact be determined without first having obtained $\mathbf{u}^{(3)}$. Using $(4.145)_3$ and (4.151) we may write

$$
\begin{aligned}
\rho_0^{-2}\Phi_3 + \Xi_{13} = {} & \int_{\Gamma_1}\Big[-\rho_0^{-2}(u_{1,1}^{(2)}u_2^{(2)} - u_{2,1}^{(1)}u_1^{(3)} + u_{1,1}^{(1)}u_2^{(3)}) \\
& + u_{1,2}^{(1)}u_1^{(3)} + u_{2,2}^{(1)}u_2^{(3)} - \frac{\rho_0}{\mu}p^{(1)}u_2^{(3)}\Big]\Big|_{\Gamma_2} dX_1 \\
& + \frac{1}{\mu}\iint_{\Pi} p^{(1)}[\rho_2(\rho_0^{-2}u_{1,1}^{(1)} - u_{2,2}^{(1)}) + (u_{1,2}^{(1)}u_{2,1}^{(2)} + u_{1,2}^{(2)}u_{2,1}^{(1)}) \\
& -(u_{1,1}^{(1)}u_{2,2}^{(2)} + u_{1,1}^{(2)}u_{2,2}^{(1)})]d\mathbf{X}.
\end{aligned}
\tag{4.161}
$$

Note that the third order solution $\mathbf{u}^{(3)}$ appears only in the first boundary integral in (4.161) and it may be rearranged into the form

$$
\int_{\Gamma_1}\Big[u_2^{(3)}\Big\{(u_{2,2}^{(1)} - \rho_0^{-2}u_{1,1}^{(1)}) - \frac{\rho_0}{\mu}p^{(1)}\Big\} + u_1^{(3)}(u_{1,2}^{(1)} + \rho_0^{-2}u_{2,1}^{(1)}) - \rho_0^{-2}u_{1,1}^{(2)}u_2^{(2)}\Big]\Big|_{\Gamma_2} dX_1. \tag{4.162}
$$

Note, however, that the multipliers of $u_1^{(3)}$ and $u_2^{(3)}$ in (4.162) vanish by virtue of the Γ_2-boundary conditions in $(3.9)_2$ for k=1. Hence the form of (4.161) is reduced to

$$
\begin{aligned}
\rho_0^{-2}\Phi_3 + \Xi_{13} = {} & -\rho_0^{-2}\int_{\Gamma_1} u_{1,1}^{(2)}u_2^{(2)}\big|_{\Gamma_2} dX_1 \\
& + \frac{1}{\mu}\iint_{\Pi} p^{(1)}\{\rho_2(\rho_0^{-2}u_{1,1}^{(1)} - u_{2,2}^{(1)}) + (u_{1,2}^{(1)}u_{2,1}^{(2)} + u_{1,2}^{92}u_{2,1}^{(1)}) \\
& -(u_{1,1}^{(1)}u_{2,2}^{2} + u_{1,1}^{(2)}u_{2,2}^{(1)})\}d\mathbf{X}.
\end{aligned}
\tag{4.163}
$$

It is shown in (4.163) that the third order solution no longer appears. This means that determining of $\mathbf{u}^{(3)}$ is not necessary to evaluate the immediate postbuckling stability competition between the buckled state and the homogeneous solution provided $E_4 \neq 0$. Substituting (4.145), (4.152) and (4.163) into (4.160) gives

$$
E_4 = R_6 + R_7\rho_2, \tag{4.164}
$$

where

$$
\begin{aligned}
R_6 = {} & \frac{1}{2}\mu\int_{\Gamma_1}\left(-2\rho_0^{-2}u_{1,1}^{(2)}u_2^{(2)} + u_{1,2}^{(2)}u_1^{(2)} + u_{2,2}^{(2)}u_2^{(2)} + \frac{-\rho_0}{\mu}p^{(2)}u_2^{(2)}\right)\bigg|_{\Gamma_2} dX_1 \\
& + \iint_{\Pi} p^{(1)}\{(u_{1,2}^{(1)}u_{2,1}^{(2)} + u_{1,2}^{(2)}u_{2,1}^{(1)}) - (u_{1,1}^{(1)}u_{2,2}^{(2)} + u_{1,1}^{(2)}u_{2,2}^{(1)})\}\,d\mathbf{X} \\
& + \frac{1}{2}\iint_{\Pi} p^{(2)}(u_{1,2}^{(1)}u_{2,1}^{(1)} - u_{2,2}^{(1)}u_{1,1}^{(1)})\,d\mathbf{X} \\
& - \frac{1}{2}\iint_{\Pi}\{u_1^{(2)}(p_{,1}^{(1)}u_{2,2}^{(1)} - p_{,2}^{(1)}u_{2,1}^{(1)}) + u_2^{(2)}(p_{,2}^{(1)}u_{1,1}^{(1)} - p_{,1}^{(1)}u_{1,2}^{(1)})\}\,d\mathbf{X},
\end{aligned}
\tag{4.165}
$$

$$
R_7 = 2\mu\rho_0^{-3}\int_{\Gamma_1} u_{1,1}^{(1)}u_2^{(1)}\big|_{\Gamma_2} dX_1 + \iint_{\Pi} p^{(1)}(\rho_0^{-2}u_{1,1}^{(1)} - u_{2,2}^{(1)})\,d\mathbf{X}. \tag{4.166}
$$

Rearranging the first integral in (4.165) yields

$$
\frac{1}{2}\mu\int_{\Gamma_1}\left[u_2^{(2)}\left\{(u_{2,2}^{(2)} - \rho_0^{-2}u_{1,1}^{(2)}) - \frac{\rho_0}{\mu}p^{(2)}\right\} + (u_{1,2}^{(2)}u_1^{(2)} - \rho_0^{-2}u_{1,1}^{(2)}u_2^{(2)})\right]\Bigg|_{\Gamma_2} dX_1. \tag{4.167}
$$

Substituting the Γ_2-boundary conditions in $(3.9)_2$ for k=2 into the first term and integration by part with respect to X_1 to the second of second term in (4.167) give

$$
\frac{1}{2}\int_{\Gamma_1} u_2^{(2)}p^{(1)}u_{1,1}^{(1)}\big|_{\Gamma_2} dX_1 + \frac{\mu}{2}\int_{\Gamma_1}(u_{1,2}^{(2)} + \rho_0^{-2}u_{2,1}^{(2)})u_1^{(2)}\big|_{\Gamma_2} dX_1. \tag{4.168}
$$

Again substituting the Γ_2 boundary condition in $(3.9)_2$ for k=2 into the second integral yields

$$
\frac{1}{2}\int_{\Gamma_1} p^{(1)}(u_2^{(2)}u_{1,1}^{(1)} - u_1^{(2)}u_{2,1}^{(1)})\big|_{\Gamma_2} dX_1. \tag{4.169}
$$

Hence R_6 in (4.165) becomes

$$
\begin{aligned}
R_6 = & \frac{1}{2}\int_{\Gamma_1} p^{(1)}(u_2^{(2)}u_{1,1}^{(1)} - u_1^{(2)}u_{2,1}^{(1)})\big|_{\Gamma_2} dX_1 \\
& + \iint_{\Pi} p^{(1)}[(u_{1,2}^{(1)}u_{2,1}^{(2)} + u_{1,2}^{(2)}u_{2,1}^{(1)}) - (u_{1,1}^{(1)}u_{2,2}^{(2)} + u_{1,1}^{(2)}u_{2,2}^{(1)})]d\mathbf{X} \\
& + \frac{1}{2}\iint_{\Pi} p^{(2)}(u_{1,2}^{(1)}u_{2,1}^{(1)} - u_{2,2}^{(1)}u_{1,1}^{(1)})d\mathbf{X} \\
& -\frac{1}{2}\iint_{\Pi}[u_1^{(2)}(p_{,1}^{(1)}u_{2,2}^{(1)} - p_{,2}^{(1)}u_{2,1}^{(1)}) + u_2^{(2)}(p_{,2}^{(1)}u_{1,1}^{(1)} - p_{,1}^{(1)}u_{1,2}^{(1)})]d\mathbf{X}.
\end{aligned}
\tag{4.170}
$$

4.7.3 Energy and Load Parameter

In equation (3.47), the relation between the thrust T and the load parameter ρ_2 is revealed for the case of symmetric bifurcation. Now we will investigate the relation between the energy difference ΔE and the postbuckling behavior. The first non-zero term of ΔE appears in E_4 and the postbuckling behavior depends on the sign of ρ_2. If ρ_2 is negative, then the progressive buckling occurs in view of the relation between T and ρ_2. Right after the buckling initiates, there exist an extension of trivial solution path and the buckled paths. The actual deformation will follow the energy minimizer between these solution paths - stable deformation.

Integration by parts to the last integral for R_6 in (4.170) with respect to the variables of $p^{(1)}$ gives

$$
\begin{aligned}
& -\frac{1}{2}\int_{\Gamma_2} p^{(1)}(u_1^{(2)}u_{2,2}^{(1)} - u_2^{(2)}u_{1,2}^{(1)})\big|_{\Gamma_1} dX_2 - \frac{1}{2}\int_{\Gamma_1} p^{(1)}(u_2^{(2)}u_{1,1}^{(1)} - u_1^{(2)}u_{2,1}^{(1)})\big|_{\Gamma_2} dX_1 \\
& -\frac{1}{2}\iint_{\Pi} p^{(1)}(u_{1,2}^{(1)}u_{2,1}^{(2)} + u_{2,1}^{(1)}u_{1,2}^{(2)} - u_{1,1}^{(1)}u_{2,2}^{(2)} - u_{2,2}^{(1)}u_{1,1}^{(2)})d\mathbf{X}.
\end{aligned}
\tag{4.171}
$$

The first integral in (4.171) vanish after applying the boundary condition on Γ_1 in $(3.9)_1$. Then R_6 in (4.170) becomes

$$
\begin{aligned}
R_6 &= \frac{1}{2}\iint_{\Pi} p^{(1)}[(u_{1,2}^{\{1\}}u_{2,1}^{\{2\}} + u_{1,2}^{\{2\}}u_{2,1}^{\{2\}}) - (u_{1,1}^{\{1\}}u_{2,2}^{\{2\}} + u_{1,1}^{\{2\}}u_{2,2}^{\{1\}})]d\mathbf{X} \\
&+ \iint_{\Pi} p^{(2)}(u_{1,2}^{\{1\}}u_{2,1}^{\{1\}} - u_{1,1}^{\{1\}}u_{2,2}^{\{1\}})d\mathbf{X}.
\end{aligned}
\tag{4.172}
$$

Comparing R_6 in (4.172) with the result on R_3 in (4.141) and using the relation for R_3 in (4.139), establish the following relation.

$$
R_6 = \frac{1}{4}R_3 = -\frac{1}{4}R_2\rho_2. \tag{4.173}
$$

Note also that R_2 in (4.134) and R_7 in (4.166) are related by

$$
R_7 = \frac{1}{2}R_2. \tag{4.174}
$$

In conclusion, the first non-zero term in the energy difference E_4 in (4.164) with (4.173) and (4.174) becomes

$$
E_4 = \frac{1}{4}R_2\rho_2. \tag{4.175}
$$

The relation (4.175) shows that E_4 is simply related to ρ_2. Since in (4.136), R_2 is always positive for the nontrivial solutions so that the sign of E_4 depends on the sign of ρ_2. If $\rho_2<0$, then $E_4<0$ so that $\Delta E<0$. In other words, the solutions corresponding to buckling have lower energy than the trivial solution at the same load - the buckled path is stable. Therefore when the progressive buckling occurs, the buckled path is stable in the vicinity of the buckling onset. The other case is also clear. When the snap buckling is possible, then the trivial solution path is stable in the vicinity of buckling onset. In view of the above statement, if the values of ρ_2 are known, then the postbuckled behavior and the stability of each path can be obtained.

The parameter ρ_2 is given by (4.139) involves R_2 and R_3 so that the energy (4.175) becomes

$$E_4 = -\frac{1}{4} R_3. \tag{4.176}$$

The parameter R_3 shown in (4.141) involves the first and second order solutions $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ determined in Section 4.6.

4.8 Summary

The solutions of the first and second order in the expanded linear type boundary value problems are obtained by using the separation of variables and infinite series method. Physically, the higher order solutions are explained as the bifurcation from one of infinite modes of the first order solution or buckling initiation. With the first order solution, it is shown that the first load parameter ρ_1 vanish so that the buckling behaviors become symmetric with respect to the trivial or homogeneous deformation. According to the symmetric nature, the formulations on the higher order load parameters and energy formulations are much reduced. By substituting the first order solutions and ρ_1=0 into the energy formulations, it is revealed that the first nonzero terms come from the fourth order energy equation for which only the first and second order solutions are necessary to solve.

CHAPTER 5

STABILITY EVALUATION FOR A NEO-HOOKEAN PLATE

5.1 Introduction

The energy difference (3.48) between the buckled and homogeneous deformations determines the energy minimal, or stable, path after bifurcation occurs. The equations (3.54), (4.155) and (4.158) show that the first nonzero term in the energy difference is the fourth order, E_4, so that the energy difference (3.48) becomes

$$\Delta E = 2l_3E_4\varepsilon^4 + O(\varepsilon^5). \qquad (5.1)$$

Also as we have seen in (4.175), E_4 is linearly related to the second order load parameter ρ_2 as E_4=$R_2\rho_2/4$. Specifically, the sign of E_4 is the same as the sign of ρ_2 since R_2>0 in (4.137) so that if ρ_2<0, then E_4<0 which becomes ΔE<0. Then from (4.139), Sign(ρ_2)=Sign(-R_3) which shows Sign(E_4)=Sign(-R_3). These correlations are what one would expect for a ΔE that is $O(\varepsilon^4)$. That they have emerged here after a great deal of tedious reduction is therefore comforting. The analogous correlations do not appear to be present in the work of Sawyers and Rivlin (1982). A comparison between our methodology and the Sawyers and Rivlin methodology is the subject of Appendix A in the context of some simple problems. In particular, this shows that the two methodologies can give different stability predictions.

The complete equation for E_4 is shown in (4.176) as E_4=-R_3/4 so that the energy difference (5.1) becomes

$$\Delta E = -\frac{1}{2}l_3R_3\varepsilon^4 + O(\varepsilon^5). \qquad (5.2)$$

Hence it is necessary to evaluate the sign of R_3 to determine the stability of buckled deformations and from now on, we refer to R_3 as the *stability parameter*. If $R_3>0$, then the configurations on the bifurcated path have less energy than those on the homogeneous solution path. Hence $R_3>0$ gives that the nonhomogeneous deformation is more stable. Conversely if $R_3<0$, then the homogeneous deformation is more stable. In this Chapter, the stability of the bifurcated nonhomogeneous deformations and homogeneous deformations near the buckling initiation will be investigated by using the reduced formulations and the buckling solutions.

Our comparison is to Sawyers and Rivlin (1982) who analyzed the stability of those type of homogeneous deformations in a neo-Hookean rectangular plate by comparing the energy of homogeneous deformation with that of the bifurcated deformation. This was done in the vicinity of the bifurcation points. Their analysis, however, is apparently not a direct energy comparison between the homogeneous and nonhomogeneous bifurcated deformations, either flexure or barreling, *at the same level of loading condition* as analyzed in this work. On the basis of their analysis, the following conclusions were obtained: (1) From Figure 3 of Sawyers and Rivlin (1982), the homogeneous deformation is more stable than flexural buckled deformation when $\eta<0.32$ and the flexural buckled deformation is more stable than homogeneous deformation when $\eta>0.32$. This is based on their equation (8.4). (2) From Figure 4 of Sawyers and Rivlin (1982), the homogeneous deformation is more stable than barreling buckled deformation at all values of η. This is based on their equation (8.7). Note from (4.55) and Figure 4.1 that the value $\eta=0.32$ on the flexure branch corresponds to $\rho=0.966$. Also $\eta\rightarrow\infty$ corresponds to $\rho_\infty\rightarrow0.5437$. Thus the analysis of Sawyers and Rivlin predicts the followings: For increasing compression ratio, and hence ρ decreasing from $\rho=1$, and near bifurcation onset, (1) the homogeneous deformation is more stable than flexural buckled deformation for $\rho:1\rightarrow 0.966$, (2) the flexural buckled deformation is more stable than homogeneous deformation for $\rho:0.966\rightarrow 0.5437$, and (3) the homogeneous deformation is more stable than barreling buckled deformation for $\rho:0.5437\rightarrow 0$.

5.2 Formal Determination of Stability Parameter

We first examine ρ_2, which is related with stability parameter R_3. The denominator R_2 of ρ_2 in (4.139) is always positive for the nontrivial solution in view of (4.137). Hence the formulation (4.139) shows Sign(ρ_2)=Sign($-R_3$). Direct substitution of the first order solutions (4.68) - (4.70) and the second order solutions (4.126) into the simplified equation of R_3 in (4.141) give upon collecting values with X_1 and X_2:

$$
\begin{aligned}
R_3 = 2\Omega\iint_{\Pi}\Bigg[& -C^{(1)}(2A^{(1)\prime}B^{(2)} + A^{(2)\prime}B^{(1)})\begin{pmatrix}\bar{C}_1\bar{S}_1\bar{S}_3\\ -\bar{C}_1\bar{S}_1\bar{S}_3\end{pmatrix} \\
& + C^{(1)}(A^{(1)}B^{(2)\prime} + 2A^{(2)}B^{(1)\prime})\begin{pmatrix}\bar{S}_1^2\bar{C}_3\\ \bar{C}_1^2\bar{C}_3\end{pmatrix} + \frac{1}{2}C^{(1)}A^{(1)}B_0^{(2)\prime}\begin{pmatrix}\bar{S}_1^2\\ \bar{C}_1^2\end{pmatrix} \\
& + C^{(2)}\left\{A^{(1)\prime}B^{(1)}\begin{pmatrix}\bar{C}_3\bar{C}_1^2\\ \bar{C}_3\bar{S}_1^2\end{pmatrix} + A^{(1)}B^{(1)\prime}\begin{pmatrix}\bar{C}_3\bar{S}_1^2\\ \bar{C}_3\bar{C}_1^2\end{pmatrix}\right\} \\
& + \frac{1}{2}C_0^{(2)}\left\{A^{(1)\prime}B^{(1)}\begin{pmatrix}\bar{C}_1^2\\ \bar{S}_1^2\end{pmatrix} + A^{(1)}B^{(1)\prime}\begin{pmatrix}\bar{S}_1^2\\ \bar{C}_1^2\end{pmatrix}\right\}\Bigg]d\mathbf{X}.
\end{aligned}
\tag{5.3}
$$

Here the following trigonometric notations have been used,

$$
\begin{aligned}
\bar{C}_1 &= \cos\Omega X_1, & \bar{S}_1 &= \sin\Omega X_1,\\
\bar{C}_3 &= \cos 2\Omega X_1, & \bar{S}_3 &= \sin 2\Omega X_1,
\end{aligned}
\tag{5.4}
$$

and upper (lower) terms are for j=1 (j=2). Note that the subscripts j and m of the coefficients $A^{(k)}$, $B^{(k)}$, $C^{(k)}$ and Ω which denote the chosen mode of the first order solution, are ignored for simpler expressions. Integration (5.3) with respect to X_1 gives

$$
\begin{aligned}
R_3 &= \Omega l_1 \int_{-l_2}^{l_2} [-\omega C^{(1)}(A^{(2)\prime}B^{(1)} + A^{(1)}B^{(2)\prime} + 2A^{(1)\prime}B^{(2)} + 2A^{(2)}B^{(1)\prime}) \\
&\quad -\omega C^{(2)}(A^{(1)}B^{(1)\prime} - A^{(1)\prime}B^{(1)}) + C^{(1)}A^{(1)}B_0^{(2)\prime} + C_0^{(2)}(A^{(1)}B^{(1)})'\,]dX_2,
\end{aligned}
\tag{5.5}
$$

where ω=1 or -1 for j=1 or 2. According to the relations $A^{(k)}$ and $C^{(k)}$ to $B^{(k)}$ shown in (4.53) and (4.110), the equation (5.5) becomes the function of only $B^{(k)}$ as

$$
R_3 = \mu\rho_0^2 l_1 \int_{-l_2}^{l_2} \frac{\rho_0^4}{8\Omega^2}(2\omega\rho_0^{-1}Q_A + Q_B)dX_2, \tag{5.6}
$$

where

$$
\begin{aligned}
Q_A &= -2\beta'\{B^{(2)\prime\prime}B^{(1)} + 4(B^{(1)\prime}B^{(2)})'\} + \alpha(B^{(2)\prime\prime} - 4\Omega^2 B^{(2)})', \\
Q_B &= -\alpha\{\alpha'' - 4\Omega^2\alpha + 2(\beta'' B^{(1)} - \beta' B^{(1)\prime})\} + 2\beta'(\alpha' B^{(1)} + 2\alpha B^{(1)\prime}) \\
&\quad + 8(B^{(1)}B^{(1)\prime})'\{2\beta' B^{(1)\prime} + \Omega^2\rho_0^{-4}(B^{(1)}B^{(1)\prime})'\}.
\end{aligned}
\tag{5.7}
$$

Here β and α are defined in (4.94) and (4.105). Note from the new notations that Q_A is function of $B^{(1)}$ and $B^{(2)}$ and that Q_B is function of only $B^{(1)}$. Note also that the second order solution $B^{(2)}$ is more complex than $B^{(1)}$. One approach to evaluating (5.6) is directly substituting $B^{(1)}$ in (4.54) for flexure or (4.56) for barreling and $B^{(2)}$ in (4.117) into (5.6) and integration over $-l_2<X_2<l_2$. However this direct approach may be modified by reducing the order of differentiation for $B^{(2)}$ in the integrand Q_A. The third and first order differentiations of $B^{(2)}$ reduce their order by one as shown in the followings.

$$
\begin{aligned}
\alpha\bar{B}''' &= (\alpha\bar{B}'')' - \alpha'\bar{B}'', \\
B'''B'\bar{B}' &= (B'B''\bar{B}' - B''^2\bar{B})' + 2B''B'''\bar{B} - B'B''\bar{B}'', \\
BB'\bar{B}' &= (BB'\bar{B}' - B'^2\bar{B})' + 2B'B''\bar{B} - BB'\bar{B}''.
\end{aligned}
\tag{5.8}
$$

Here the simpler symbols B=$B^{(1)}$ and $\bar{B}$ =$B^{(2)}$ are used. Hence Q_A becomes

$$\begin{aligned} Q_A &= \{\alpha\bar{B}'' - 4(2B'B'' + \Omega^2 BB')\bar{B}' + 8(B''^2 + 2\Omega^2 B'^2)\bar{B}\}' \\ &\quad - 8\{\beta' B'' + 2(B''B''' + 2\Omega^2 B'B'')\}\bar{B} - (\alpha' + 2\beta' B - 8B'B'' - 4\Omega^2 BB')\bar{B}''. \end{aligned} \tag{5.9}$$

Q_B can be also rewritten as

$$\begin{aligned} Q_B &= 2(\alpha\beta' B)' - \alpha\{\alpha'' + 4(\beta'' B - \beta' B' - \Omega^2\alpha)\} \\ &\quad + 8(BB')'\{2\beta' B' + \Omega^2\rho_0^{-4}(BB')'\}. \end{aligned} \tag{5.10}$$

The symbols β, α and solutions B and $\bar{B}$ are substituted into (5.9) and (5.10), then the equation (5.6) after linearization of the products of hyperbolic equations becomes

$$\begin{aligned} R_3 &= \mu l_1 M^4 \Omega^3 \Big[\textstyle\sum_{k=1}^{12} Y[1,k]\sinh(Z_k\Omega X_2)\Big|_{-l_2}^{l_2} \\ &\quad + \Omega\int_{-l_2}^{l_2}[Y[2,0] + \textstyle\sum_{k=1}^{12} Y[2,k]\cosh(Z_k\Omega X_2)]dX_2\Big]. \end{aligned} \tag{5.11}$$

The coefficients $Y[1,k]$ and $Y[2,k]$ are functions of the load parameter ρ_0, the aspect ratio $s(\rho_0,\eta)$, the switching constant $\nu = \pm 1$ along the deformation types and the new dimensionless definitions, $\tilde{N}_i$, i=1,2,3 associated with the coefficient functions N_i for $\bar{B}$. The full definitions of Y's with variables $\lambda_0=\rho_0^{-2}$ introduced in (3.7) are shown in Appendix B. In general, these Y's involve products and quotients of hyperbolic functions. The 12 different kinds of arguments Z_k in the hyperbolic functions after considering $\eta=\Omega l_2$ are

$$\begin{aligned} Z_1 &= (1-\lambda_0), & Z_2 &= (1+\lambda_0), \\ Z_3 &= (1-3\lambda_0), & Z_4 &= (1+3\lambda_0), \\ Z_5 &= 2, & Z_6 &= 2\lambda_0, \\ Z_7 &= 2(1-\lambda_0), & Z_8 &= 2(1+\lambda_0), \\ Z_9 &= (3-\lambda_0), & Z_{10} &= (3+\lambda_0), \\ Z_{11} &= 4, & Z_{12} &= 4\lambda_0. \end{aligned} \tag{5.12}$$

Since Y's do not include the variable X_2, the simple calculus leads the equation (5.11) into

$$R_3 = 2\mu l_1 M^4 \Omega^3 \left\{ \eta Y[2,0] + \sum_{k=1}^{12} \left(Y[1,k] + \frac{Y[2,k]}{Z_k} \right) \sinh(Z_k \eta) \right\}. \quad (5.13)$$

Note that the terms of k=1 and 2, k=3 and 4, k=7 and 8, k=9 and 10 are antisymmetric for flexural deformation of ν=1 and symmetric for barreling deformation of ν=-1 with respect to λ_0. Applying the coefficient notations $\tilde{N}_i$ from (4.118) and (4.119), $s=s(\rho_0,\eta)$ from (4.58), ν=1 for flexure and ν=-1 for barreling deformations and Δ from (4.120), into Y's in (5.13) gives the full expression for the stability parameter R_3. Note that s and Δ also contain the hyperbolic functions which are shown in denominator of fully evaluated stability parameter R_3.

5.3 Numerical Determination of Stability Parameter

Numerical calculation with the parameter $\eta = \Omega l_2$ is handled by inserting specific values of η>0 and corresponding values of load parameter ρ_0 according to the relation (4.55) or (4.57) into s in (4.58) and Δ in (4.120), and then substituting obtained values into the stability parameter R_3 in (5.13). For the purpose of numerical setting, we introduce the dimensionless stability parameter $R_S = R_3/(\mu l_1 M^4 \Omega^3)$. Then

$$R_S = 2\left\{ \eta Y[2,0] + \sum_{k=1}^{12} \left(Y[1,k] + \frac{Y[2,k]}{Z_k} \right) \sinh(Z_k \eta) \right\}. \quad (5.14)$$

The Figure 5.1 and Figure 5.4 show the relations between R_S and η for flexure and barreling modes, respectively. For the flexural deformation, Figure 5.1 and its detailed Figure 5.2 and Figure 5.3 show that R_S is positive when $0<\eta<\eta_c=0.6443...$, negative when $\eta_c<\eta<1.305$, positive when $1.305<\eta<1.6283$ and negative when $\eta>1.6283$. According to the relation (5.2), the signs of R and R_S are different from the sign of ΔE. Therefore when $\eta<\eta_c$ and $1.305<\eta<1.6283$, the buckled state has less energy than the unbuckled state so that unbuckled state (the homogeneous deformation) is unstable. When $\eta_c<\eta<1.305$ and $\eta>1.6283$, the unbuckled state is energetically favored and hence stable. Note that when η goes to zero, the undeformed geometry looks like a rod subjected to thrust at its ends which is similar to the conventional elastica problem. At $\eta=1.6283$ which corresponds to $\lambda_0 = \rho_0^{-2} = 3$, there exists discontinuity. This comes from the fact that N_4 becomes infinity at this value according to (4.118). For the case of barreling, Figure 5.4 shows that R_S is

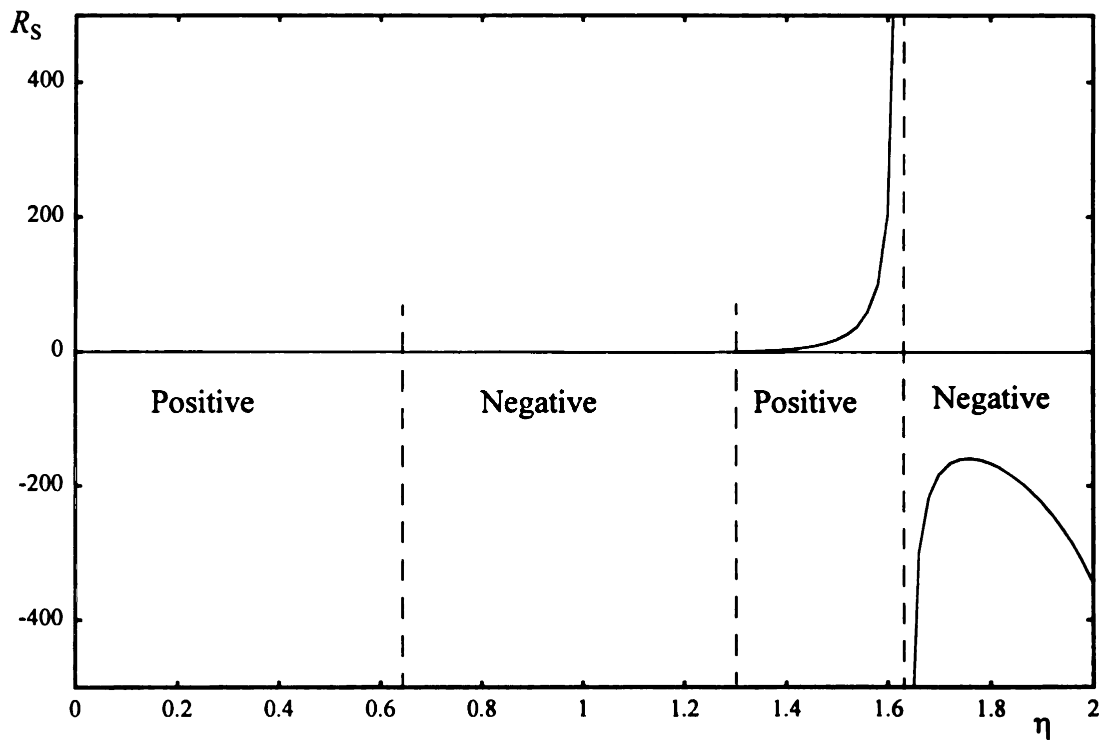


Figure 5.1 Dimensionless stability parameter $R_S=R_3/(\mu l_1 M^4\Omega^3)$ for the flexural deformation in (5.14) with $\nu=1$. At $\eta=1.6283$, the curve R_S has a discontinuity. Right before this, it is positive and after this, R_S is negative. There are also sign changes in $\eta=0.6443$ and $\eta=1.305$. Their details are shown in Figure 5.2 and Figure 5.3.

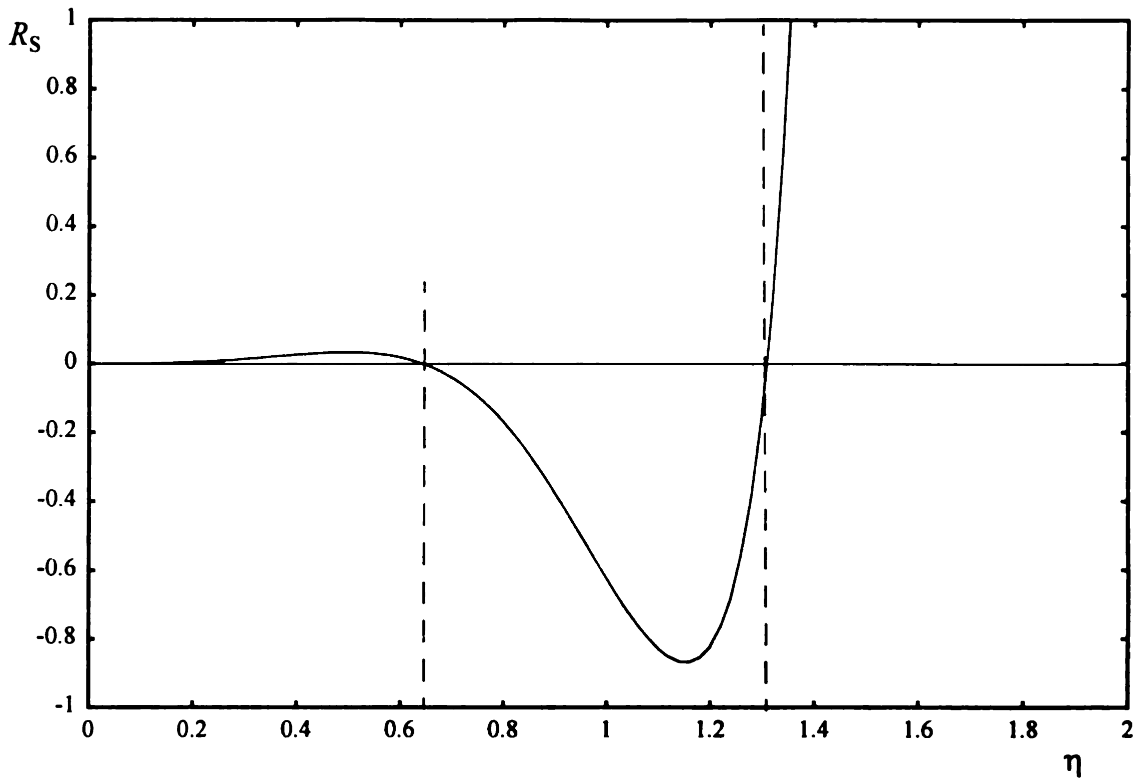


Figure 5.2 The detailed curve of dimensionless stability parameter R_S for the flexural deformation in (5.14). At $\eta=\eta_c=0.6443$, the sign of R_S changes from positive to negative and at $\eta=1.305$ the sign changes from negative to positive. More detail near $\eta=\eta_c$ is shown in Figure 5.3.

positive in a whole range of η so that the buckled state is always stable. Hence all buckled barreling deformations has larger energy than homogeneous deformation.

Compared with results by Sawyers and Rivlin (1982), they conclude in Figure 3 and 4 of their paper that the homogeneous state at which bifurcation occurs is stable when $\eta<\eta_c=0.32$ and otherwise the homogeneous state is unstable for flexural deformation. For barreling deformation, Sawyers and Rivlin found that the homogeneous state at which bifurcation occurs is always stable. In contrast our results show that there are different regions of sign R_S for flexural deformation and that the first transition occurs at $\eta=0.6443$. We also find that the sign of R_S are different when η goes to infinity for flexural and

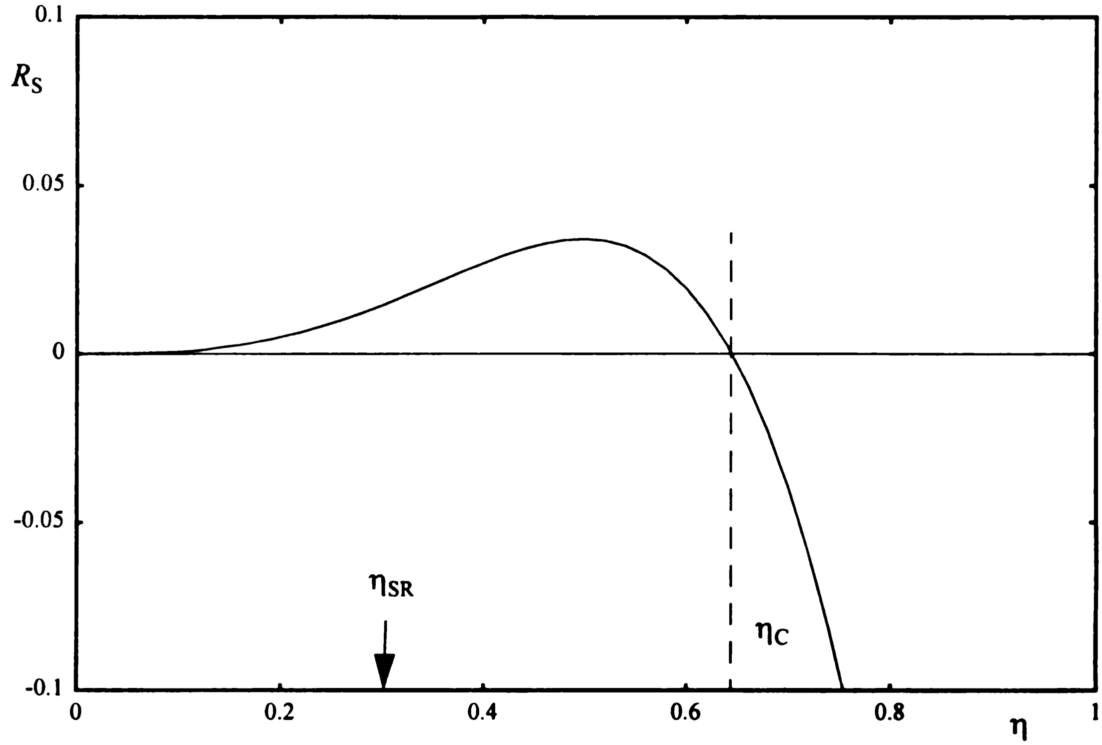


Figure 5.3 The detailed curve of dimensionless stability parameter R_S for the flexural deformation in (5.14). At $\eta=\eta_c=0.6443$, the sign of R_S changes from positive to negative. Hence when $\eta<\eta_c$, the homogeneous state has less energy than bifurcated state. Sawyers and Rivlin (1982) find that the stability of flexural deformation changes at $\eta_{SR}=0.32$.

barreling deformations are different. Recall that when η goes to infinity, ρ goes to ρ_∞ which corresponds to the wrinkling mode.

5.4 Asymptotic Study for Stability Parameter in Flexural Buckling at Low Mode

We now consider an asymptotic study of R_S as η goes to zero on the flexure branch. However, first we consider an analogy to well known Euler buckling - the extreme case of plate. Then the asymptotic analysis of R_3 near $\eta=0$ will be developed.

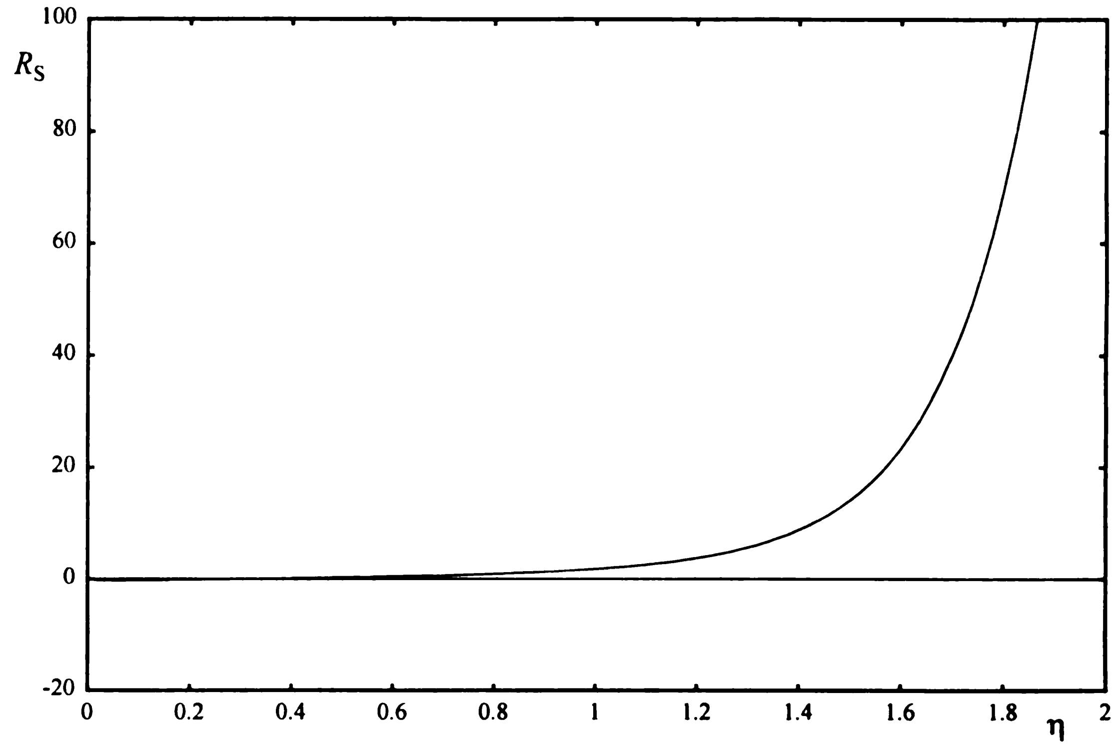


Figure 5.4 Dimensionless stability parameter $R_S=R_3/(\mu l_1 M^4 \Omega^3)$ for the barreling deformation in (5.14) with ν=-1. It is shown that the stability parameter has positive sign on all range of η which means the bifurcated state involves less energy than the homogeneous state (the homogeneous state is unstable).

5.4.1 Critical Load in Classical Euler Buckling

When η goes to zero, the geometrical shape of the considered plate approaches a thin column compressed at its ends. This resembles the classical Euler column explained in Timoshenko and Gere (1963). The critical buckling load, the lowest thrust load out of infinite buckling mode and important factor for structual stability is given by the material property and shape of the column.

The critical buckling load in this study is presented from the lowest load parameter ρ which occurs at the first mode in flexure deformation as shown in Figure 2.2. Recall that the thrust equation (2.16), $T = -4\mu l_2 l_3(\rho - \rho^{-3})$. Here l_1, l_2, l_3 are the half length of the considered plate. When η approaches to zero which means l_2 becomes very thin, the load parameter ρ becomes the equation (4.77). According to (4.1), the first mode of flexure deformation occurs at

$$\eta = \frac{l_2}{2l_1}\pi. \tag{5.15}$$

Then the thrust equation (2.16) becomes

$$T = \mu l_2 l_3 \left\{ \frac{4}{3}\pi^2\left(\frac{l_2}{l_1}\right)^2 + \frac{8}{45}\pi^4\left(\frac{l_2}{l_1}\right)^4 + \frac{43}{1512}\pi^6\left(\frac{l_2}{l_1}\right)^6 + O\left(\frac{l_2}{l_1}\right)^8 \right\}. \tag{5.16}$$

Now we consider the equivalent material properties of incompressible plate after linearization. For incompressible neo-Hookean material in the conditions of plane strain described in boundary conditions (2.7) to (2.11), the stress-strain relation is described as (2.13). With (2.15), the stress in X_1 direction becomes

$$\tau_1 = \mu(\rho^2 - \rho^{-2}). \tag{5.17}$$

Let the principal stretch $\rho = 1 + \varepsilon_1$ where ε_1 is the strain in X_1 direction, then (5.17) becomes

$$\tau_1 = 4\mu\varepsilon_1 + O(\varepsilon_1^2). \tag{5.18}$$

Ignoring the higher order term in (5.18) for linearization, the relation between the Young's modulus $\mathbf{E}$ and the shear modulus μ is obtained as

$$\mathbf{E} = 4\mu. \tag{5.19}$$

Also simple calculation reveals the Poisson's ratio equals one and the linearized shear modulus becomes $\mathbf{G} = \mu$.

Now for geometry, we introduce the new geometric variables $L=2l_1$, $t=2l_2$, $w=2l_3$ for convenience. The second moments of inertia of the rectangular cross section with respect to mid-point gives $\mathbf{I} = \frac{wt^3}{12}$. Substituting (5.19) into (5.16) with new variables L, t, w, the thrust becomes

$$T = \frac{\mathbf{EI}}{L^2}\pi^2\left\{1 + \frac{2}{15}\pi^2\left(\frac{t}{L}\right)^2 + O\left(\frac{t}{L}\right)^4\right\}. \tag{5.20}$$

For Euler column, the ratio of height and length is negligible, $\frac{t}{L} \ll 1$. Therefore the thrust equation (5.20) becomes

$$T_{Euler} = \frac{\mathbf{EI}}{L^2}\pi^2. \tag{5.21}$$

The critical thrust (5.21) is the same critical buckling thrust as the Euler column with the same boundary conditions such that one end is built-in and the other end is free to move laterally but is guided in a manner that the tangent to the column remains vertical shown in Figure 5.5 (Timoshenko and Gere, 1963). The result gives the verification for this study in an extreme case of thin plate.

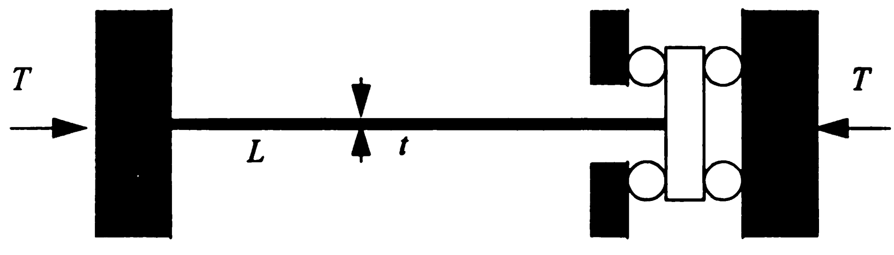


Figure 5.5 The Euler column subjected to the boundary conditions considered here. Here the height-length ratio t/L has negligible value. The critical thrust for this column is equated as in (5.21) which is the same critical thrust of plate studied here after linearization.

5.4.2 Buckled Deformations with Asymptotic Equations

The load parameters ρ_0 of two types of buckled deformations when η has an extremely small or large values were found in Chapter 4. The asymptotic equations of buckled deformation then can be obtained by substituting associated asymptotic load parameters into the solution terms of each order. Here we consider the case of flexural deformation when η is small. The corresponding asymptotic relation between ρ_0 and η in this case is given in equation (4.55). To develop the asymptotic analysis of this case further, we introduce the dimensionless variable ζ as the replacement of X_2 by

$$\zeta = \frac{X_2}{l_2}. \tag{5.22}$$

Then the boundary value problem (4.51) and (4.52) for the first order solution $B(l_2\zeta)=B^{(1)}(X_2)$ are rewritten as

$$\frac{1}{\eta^4}\frac{d^4B}{d\zeta^4} - \frac{(1+\rho_0^{-4})}{\eta^2}\frac{d^2B}{d\zeta^2} + \rho_0^{-4}B = 0, \tag{5.23}$$

$$\left.\begin{aligned} \frac{1}{\eta^2}\frac{d^2B}{d\zeta^2} + \rho_0^{-4}B &= 0, \\ \frac{1}{\eta^3}\frac{d^3B}{d\zeta^3} - \frac{(1+2\rho_0^{-4})}{\eta}\frac{dB}{d\zeta} &= 0, \end{aligned}\right\} \quad \text{on } \zeta = \pm 1. \tag{5.24}$$

The solution $B(\eta,\zeta)$ for problem (5.23) and (5.24) with the expansion for $\rho_0(\eta)$ which has the expression in (4.77) can be obtained by proposing a series type solution. Here we note that $\rho_0^{-4}(\eta) = 1 + \frac{4}{3}\eta^2 + \frac{52}{45}\eta^4 + \frac{808}{945}\eta^6 + O(\eta^8)$. The previously obtained analytic solution (4.54) is an even function with respect to both ζ and η. This motivates the consideration of a small η expansion for $B(\eta,\zeta)$ in the following form

$$B(\eta,\zeta) = C\{1 + (a_{10} + a_{11}\zeta^2)\eta^2 + (a_{20} + a_{21}\zeta^2 + a_{22}\zeta^4)\eta^4 \\ + (a_{30} + a_{31}\zeta^2 + a_{32}\zeta^4 + a_{33}\zeta^6)\eta^6 + O(\eta^8)\}, \tag{5.25}$$

where C is constant. Here the other terms like $\zeta^i\eta^j$, $i>j$ vanish in the process. The undetermined coefficients a_{10}, a_{11}, a_{20},..., a_{ij} are calculated by substituting the proposed solution (5.25) into (5.23) and (5.24) with (4.77) and equating the coefficients of various products of ζ and η to zero. In fact, we immediately find that a_{i0}=0, for i=1,2,3,... by considering the $O(\eta^{2i})$ term in (5.23). From $(5.24)_1$ after substituting (5.25), the $O(1)$ term gives

$$a_{11} = -\frac{1}{2}. \tag{5.26}$$

Substituting (5.26) into the expanded equations of (5.23), the $O(1)$ terms give

$$a_{22} = -\frac{1}{8}. \tag{5.27}$$

The same result is separately obtained by considering the $O(\eta)$ term in $(5.24)_2$. The other coefficients can be obtained similarly. Namely the $O(\zeta^2\eta^2)$ term in (5.23) gives $a_{33} = -\frac{1}{144}$ and the $O(\eta^2)$ term in $(5.24)_1$ gives $a_{21} = \frac{1}{3}$. Then the $O(\eta^2)$ term in (5.23) after substituting known coefficients give $a_{32} = -\frac{1}{18}$. The same result for a_{32} is obtained simultaneously considering the $O(\eta^3)$ term in $(5.24)_2$. Finally the $O(\eta^4)$ term in $(5.24)_1$ gives $a_{31} = \frac{4}{45}$. In summary, the first order solution $B(\eta,\zeta)$ becomes

$$B(\eta,\zeta) = C\left\{1-\frac{\zeta^2}{2}\eta^2 + \left(\frac{\zeta^2}{3}-\frac{\zeta^4}{8}\right)\eta^4 + \left(\frac{4\zeta^2}{45}-\frac{\zeta^4}{18}-\frac{\zeta^6}{144}\right)\eta^6 + O(\eta^8)\right\}. \tag{5.28}$$

The constant coefficient C will be obtained from the normalization process explained in Section 4.4.1. The reduced normalization equation (4.60) can be rewritten by using new variable (5.22) as

$$\frac{1}{4l_2^2}\int_{-1}^{1}\left[\frac{\sigma^2\rho_0^4}{\eta^2}\left(\frac{\mathrm{d}B}{\mathrm{d}\zeta}\right)^2 + B^2 + \rho_0^6\left\{\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{\eta^2}\frac{\mathrm{d}^2B}{\mathrm{d}\zeta^2} - B\right)\right\}^2\right]d\zeta = 1, \tag{5.29}$$

where $\sigma = l_2/l_1$. Substituting the solution (5.28) into (5.29) with ρ_0 in (4.77) gives

$$C = \sqrt{2}l_2\{1 + O(\eta^2)\}. \tag{5.30}$$

The $O(\eta^2)$ correction to (5.30) will put terms of order η^{2k}, k=2,3,4,... in (5.28), but we have previously shown that a_{i0}=0, for all i. Hence we conclude that $C = \sqrt{2}l_2$ so that

$$B(\eta,\zeta) = \sqrt{2}l_2\left\{1 - \frac{\zeta^2}{2}\eta^2 + \left(\frac{\zeta^2}{3} - \frac{\zeta^4}{8}\right)\eta^4 + \left(\frac{4\zeta^2}{45} - \frac{\zeta^4}{18} - \frac{\zeta^6}{144}\right)\eta^6 + O(\eta^8)\right\}. \tag{5.31}$$

The complete solution (5.28) with (5.30) may be compared with the direct small η expansion of analytic solution (4.54). Here one finds that

$$B(\eta,\zeta) = M\left\{\frac{2}{3}\eta^2 + \left(\frac{16}{45} - \frac{\zeta^2}{3}\right)\eta^4 + \left(-\frac{16}{945} - \frac{2\zeta^2}{45} - \frac{\zeta^4}{12}\right)\eta^6 + O(\eta^8)\right\}, \tag{5.32}$$

and the coefficient M in (4.61) for small η expanded flexural deformation is found as

$$M = \sqrt{2}l_2\left\{\frac{3}{2}\eta^{-2} - \frac{4}{5} + \frac{244}{525}\eta^2 + O(\eta^4)\right\}. \tag{5.33}$$

Note that M is required to be positive by virtue of (4.61). Therefore this expansion is valid only for $\eta \ll 1.37$. Thus both solutions (5.31) and (5.32) with (5.33) are the same in the view of small η. The same expression for solution (5.31) can be also found in (7.7) of

Sawyers and Rivlin (1982) except for the constant coefficient $C = \sqrt{2}l_2$ which is caused by adopting the different normalization condition.

For the second order solution $\bar{B}(l_2\zeta) = B^{(2)}(X_2)$, the boundary value problem (4.107) and (4.108) is rewritten here in terms of $B(\eta,\zeta)$ and $\bar{B}(\eta,\zeta)$ as:

$$\frac{1}{\eta^4}\frac{d^4\bar{B}}{d\zeta^4} - 4\frac{(1+\rho_0^{-4})}{\eta^2}\frac{d^2\bar{B}}{d\zeta^2} + 16\rho_0^{-4}\bar{B} = \frac{3}{2}\frac{\omega}{l_2}\rho_0\frac{(\rho_0^{-4}-1)}{\eta^2}\frac{d}{d\zeta}\left(B\frac{d^2B}{d\zeta^2} - \frac{dB}{d\zeta}\frac{dB}{d\zeta}\right). \tag{5.34}$$

And on $\zeta = \pm 1$,

$$\begin{aligned} \frac{1}{\eta^2}\frac{d^2\bar{B}}{d\zeta^2} + 4\rho_0^{-4}\bar{B} &= \frac{1}{2}\frac{\omega}{l_2}\rho_0(7\rho_0^{-4}+1)B\frac{dB}{d\zeta}, \\ \frac{1}{\eta^3}\frac{d^3\bar{B}}{d\zeta^3} - 4\frac{(1+2\rho_0^{-4})}{\eta}\frac{d\bar{B}}{d\zeta} &= -2\frac{\omega}{l_2}\rho_0\frac{(2\rho_0^{-4}-1)}{\eta}\left(\frac{dB}{d\zeta}\right)^2. \end{aligned} \tag{5.35}$$

Then substituting small η expansion of the first order solution (5.31) into the right hand sides of (5.34) and (5.35) gives

$$\frac{1}{\eta^4}\frac{d^4\bar{B}}{d\zeta^4} - 4\frac{(1+\rho_0^{-4})}{\eta^2}\frac{d^2\bar{B}}{d\zeta^2} + 16\rho_0^{-4}\bar{B} = -2\omega l_2\zeta\left\{8\eta^4 + \left(\frac{64}{15}+\frac{8}{3}\zeta^2\right)\eta^6\right\} + O(\eta^8), \tag{5.36}$$

and on $\zeta = \pm 1$,

$$\begin{aligned} \frac{1}{\eta^2}\frac{d^2\bar{B}}{d\zeta^2} + 4\rho_0^{-4}\bar{B} &= -2\omega l_2\left(4\eta^2 + \frac{2}{3}\eta^4 + \frac{26}{15}\eta^6\right) + O(\eta^8), \\ \frac{1}{\eta^3}\frac{d^3\bar{B}}{d\zeta^3} - 4\frac{(1+2\rho_0^{-4})}{\eta}\frac{d\bar{B}}{d\zeta} &= -2\omega l_2\left(2\eta^3 + 4\eta^5 + \frac{5}{3}\eta^7\right) + O(\eta^9). \end{aligned} \tag{5.37}$$

After considering the solution (4.117), the expression for $\bar{B}(\eta,\zeta)$ in (5.36) and (5.37) must be an even function in η and an odd function in ζ. This motivates

$$\begin{aligned} \bar{B}(\eta,\zeta) = -2\omega l_2\zeta\{&b_{11}\eta^2 + (b_{21}+b_{22}\zeta^2)\eta^4 + (b_{31}+b_{32}\zeta^2+b_{33}\zeta^4)\eta^6 \\ &+ (b_{41}+b_{42}\zeta^2+b_{43}\zeta^4+b_{44}\zeta^6)\eta^8 + O(\eta^{10})\}. \end{aligned} \tag{5.38}$$

Substitution of (5.38) into (5.36) and (5.37) and equating the coefficients of left and right hand sides in terms of the product ζ and η, give the undetermined coefficients b_{ij}'s. The details to obtain b_{ij}'s are as follows: $O(\eta^2)$ term in $(5.37)_1$ and $O(\eta)$ term in $(5.37)_2$ give b_{11} and b_{22}, $O(\zeta\eta^2)$ term in (5.36) gives b_{33}, $O(\zeta^2\eta^4)$ term in (5.36) gives b_{44}, $O(\eta^4)$ term in $(5.37)_1$ and $O(\eta^3)$ term in $(5.37)_2$ give b_{21} and b_{32}, $O(\zeta\eta^4)$ term in (5.36) gives b_{43}, and $O(\eta^6)$ term in $(5.37)_1$ and $O(\eta^5)$ term in $(5.37)_2$ give b_{31} and b_{42}. In summary, the asymptotic second order solution $\bar{B}(\eta,\zeta)$ for flexural deformation when η is small, becomes

$$\begin{aligned}\bar{B}(\eta,\zeta) &= -2\omega l_2\zeta\left\{\frac{1}{4}\eta^2-\left(\frac{7}{6}-\frac{1}{2}\zeta^2\right)\eta^4+\left(\frac{23}{72}-\frac{2}{9}\zeta^2+\frac{1}{6}\zeta^4\right)\eta^6\right.\\&\quad\left.+\left(b_{41}-\frac{23}{60}\zeta^2+\frac{2}{9}\zeta^4+\frac{1}{45}\zeta^6\right)\eta^8+O(\eta^{10})\right\}.\end{aligned} \tag{5.39}$$

The coefficient b_{41} is undetermined yet but it is not necessary for future calculation. As we expected, since the $\bar{B}$ matches the second order solution of Sawyers and Rivlin (1982), the asymptotic second order solution (5.39) is the same as the series equation (7.14) of theirs except for the sign which is due to the difference of definition of ω and the normalized coefficient. The asymptotic solution (5.39) can be compared with the small η expansion of analytic solution (4.117) for verification of its accuracy. Now we find from (4.118) and (4.119) that

$$\begin{aligned}N_3 &= \omega C^2\frac{\Omega}{\eta^2}\left\{\frac{9}{32}-\frac{69}{160}\eta^2+\frac{6537}{22400}\eta^4+O(\eta^6)\right\},\\N_4 &= \omega C^2\Omega\left\{-\frac{3}{8}+\frac{1}{2}\eta^2-\frac{107}{240}\eta^4+O(\eta^6)\right\},\end{aligned} \tag{5.40}$$

and

$$
\begin{aligned}
N_1 &= \omega C^2 \frac{\Omega}{\eta^2}\left\{-\frac{21}{64}+\frac{107}{640}\eta^2+\frac{103}{40}\eta^4+O(\eta^6)\right\}, \\
N_2 &= \omega C^2 \frac{\Omega}{\eta^2}\left\{\frac{3}{64}+\frac{169}{640}\eta^2-\frac{447}{280}\eta^4+O(\eta^6)\right\},
\end{aligned}
\tag{5.41}
$$

where $C = \sqrt{2}l_2$. Here small η expansions of coefficient M in (5.33) and the notation Δ in (4.120) as

$$
\Delta = \frac{32}{3}\eta^3+\frac{512}{15}\eta^5+\frac{64256}{945}\eta^7+O(\eta^9),
\tag{5.42}
$$

have been used. Then the second order solution $\bar{B}(\eta,\zeta)$ in (4.117) becomes

$$
\begin{aligned}
\bar{B}(\eta,\zeta) &= 2z_1\zeta\eta+\frac{2}{3}(z_2\zeta+2z_1\zeta^3)\eta^3+\frac{4}{45}(4z_2\zeta+15z_3\zeta^3+3z_1\zeta^5)\eta^5 \\
&\quad +\frac{4}{315}(15z_2\zeta+7z_4\zeta^3+35z_3\zeta^5+2z_1\zeta^7)\eta^7+O(\eta^9),
\end{aligned}
\tag{5.43}
$$

where

$$
\begin{aligned}
z_1 &= N_1+N_2+N_3 = \omega C^2\frac{\Omega}{\eta^2}\left\{\frac{28457}{22400}\eta^4+O(\eta^6)\right\}, \\
z_2 &= N_3+N_4+2N_1 = -\omega C^2\frac{\Omega}{\eta^2}\left\{\frac{3}{8}+\frac{151}{320}\eta^2-\frac{133097}{22400}\eta^4+O(\eta^6)\right\}, \\
z_3 &= N_3+2N_1 = -\omega C^2\frac{\Omega}{\eta^2}\left\{\frac{3}{8}+\frac{31}{320}\eta^2-\frac{121897}{22400}\eta^4+O(\eta^6)\right\}, \\
z_4 &= 13N_3+36N_1 = -\omega C^2\frac{\Omega}{\eta^2}\left\{\frac{261}{32}-\frac{33}{80}\eta^2-\frac{2161461}{22400}\eta^4+O(\eta^6)\right\}.
\end{aligned}
\tag{5.44}
$$

Substituting (5.40) and (5.41) with $\Omega = \eta l_2$ and $C = \sqrt{2}l_2$ into (5.43) becomes

$$\bar{B}(\eta,\zeta) = -2\omega l_2\zeta\left\{\frac{1}{4}\eta^2-\left(\frac{70321}{33600}-\frac{1}{2}\zeta^2\right)\eta^4+\left(c_u-\frac{14107}{16800}\zeta^2+\frac{1}{6}\zeta^4\right)\eta^6+O(\eta^8)\right\}, \quad (5.45)$$

where c_u is yet undetermined constant since it includes the terms of $O(\eta^6)$ in (5.40) and (5.41) which are too difficult to obtain directly. It is shown that of the five common terms in (5.39) and (5.45) for which the numerical prefactors have been obtained, that 3 of the 5 terms match and 2 terms do not match. The terms that do not match, $O(\eta^4)$ and $O(\zeta^2\eta^6)$, involve terms of $O(\eta^4)$ or higher in (5.40) and (5.41).

5.4.3 Stability Parameters with Asymptotic Equations

In this section, we will investigate the behavior of stability parameter R_3 which is directly related the energy difference ΔE for the flexural deformation when η is small. This analysis will give the verification for the analytic results of stability parameter when η has relatively small value. The asymptotic load parameter ρ_0 and solutions of the first and second orders B and $\bar{B}$ were obtained in previous sections. When η is small, l_2 is much shorter than l_1 in which the geometry considered may resemble to the thin plate with thrusts at both ends. This type of buckling problem for thin plate known as elastica, has been widely studied on the context of linear elasticity theory.

By using the new variable ζ in (5.22), the terms Q_A and Q_B in (5.7) is rewritten as

$$Q_A = \frac{\Omega^5}{\eta^3}\left[-2\frac{d\tilde{\beta}}{d\zeta}\left\{B\frac{d^2\bar{B}}{d\zeta^2}+4\frac{d}{d\zeta}\left(\frac{dB}{d\zeta}\bar{B}\right)\right\}+\tilde{\alpha}\frac{d}{d\zeta}\left(\frac{d^2\bar{B}}{d\zeta^2}-4\eta^2\bar{B}\right)\right],$$

$$Q_B = \frac{\Omega^6}{\eta^2}\left[-\tilde{\alpha}\left\{\frac{d^2\tilde{\alpha}}{d\zeta^2}-4\eta^2\tilde{\alpha}+2\left(\frac{d^2\tilde{\beta}}{d\zeta^2}B-\frac{d\tilde{\beta}}{d\zeta}\frac{dB}{d\zeta}\right)\right\}+2\frac{d\tilde{\beta}}{d\zeta}\left(\frac{d\tilde{\alpha}}{d\zeta}B+2\tilde{\alpha}\frac{dB}{d\zeta}\right)\right] \quad (5.46)$$

$$+8\frac{\Omega^6}{\eta^4}\frac{d}{d\zeta}\left(B\frac{dB}{d\zeta}\right)\left\{2\frac{d\tilde{\beta}}{d\zeta}\frac{dB}{d\zeta}+\rho_0^{-4}\frac{d}{d\zeta}\left(B\frac{dB}{d\zeta}\right)\right\},$$

where

$$\tilde{\beta} = \frac{\beta}{\Omega^2} = \frac{1}{\eta^2}\left(\frac{d^2B}{d\zeta^2} - \eta^2 B\right), \qquad \tilde{\alpha} = \frac{\alpha}{\Omega^2} = \frac{1}{\eta^2}\left\{B\frac{d^2B}{d\zeta^2} - \left(\frac{dB}{d\zeta}\right)^2\right\}. \tag{5.47}$$

Substituting the asymptotic load parameter ρ_0 in (4.77), solutions B in (5.28) into (5.47) gives

$$\begin{aligned}
\tilde{\beta} &= -\sqrt{2}l_2\left\{2 - \left(\frac{2}{3} - \zeta^2\right)\eta^2 - \left(\frac{8}{45} - \zeta^2 - \frac{1}{12}\zeta^4\right)\eta^4 + O(\eta^6)\right\}, \\
\tilde{\alpha} &= -2l_2^2\left\{1 - \left(\frac{2}{3} - 2\zeta^2\right)\eta^2 - \left(\frac{8}{45} - \frac{1}{3}\zeta^4\right)\eta^4 + O(\eta^6)\right\},
\end{aligned} \tag{5.48}$$

and then substituting $\bar{B}$ in (5.39) as well as ρ_0 and B into (5.46) becomes

$$\begin{aligned}
Q_A(\eta) &= 4\omega\Omega^6 l_2^4\left\{2 + 2(1 + 2\zeta^2)\eta^2 - \left(\frac{277}{45} + \frac{64}{3}\zeta^2 - 42\zeta^4\right)\eta^4 + O(\eta^6)\right\}, \\
Q_B(\eta) &= 16\Omega^6 l_2^4\{1 - (1 + 8\zeta^2)\eta^2 + O(\eta^4)\}.
\end{aligned} \tag{5.49}$$

Here $\eta = \Omega l_2$ are used. Then the stability parameter R_3 in (5.6) in terms of new variable ζ becomes

$$R_3 = \frac{\mu\rho_0^6 l_1 l_2}{8\Omega^2}\int_{-1}^{1}(2\omega\rho_0^{-1}Q_A + Q_B)d\zeta. \tag{5.50}$$

Substituting ρ_0 in (4.77), Q_A, Q_B in (5.49) and $\omega^2=1$ into the integrand of (5.50) gives

$$2\omega\rho_0^{-1}Q_A + Q_B = 16\Omega^6 l_2^4\left\{2 + \left(\frac{1}{3} - 6\zeta^2\right)\eta^2 + O(\eta^4)\right\}. \tag{5.51}$$

Also the stability parameter (5.51) shows that the small η behavior of $R_3(\eta)$ is $O(\eta^4)$ since $\eta = \Omega l_2$. We note that since $\rho_0=O(1)$, the $O(\eta^4)$ behavior of $R_3(\eta)$ is completely determined from the $O(\eta^6)=O(\Omega^6)$ behavior in $Q_A(\eta)$ and $Q_B(\eta)$ as given by the leading order terms in (5.48) and (5.49). The $O(\eta^6)$ behavior in $Q_A(\eta)$ and $Q_B(\eta)$ can be obtained from the investigation of leading orders in each term of (5.46) which is shown as

$$\begin{aligned}
Q_A(\eta) &= C^4\frac{\Omega^5\omega}{\eta^3 l_2}[\{O(\eta^6)\}+\{O(\eta^6)\}+\{3\eta^4+O(\eta^6)\}+\{-\eta^4+O(\eta^6)\}] \\
&= 2C^4\Omega^5\eta\frac{\omega}{l_2}+O(\Omega^5\eta^3), \\
Q_B(\eta) &= C^4\frac{\Omega^6}{\eta^2}[\{-4\eta^2+O(\eta^4)\}+\{4\eta^2+O(\eta^4)\}+\{-4\eta^2+O(\eta^4)\}+\{O(\eta^4)\} \\
&\quad +\{O(\eta^4)\}+\{O(\eta^4)\}]+C^4\frac{\Omega^6}{\eta^4}[\{O(\eta^6)\}+\{8\eta^4+O(\eta^6)\}] \\
&= 4C^4\Omega^6+O(\Omega^6\eta^2).
\end{aligned} \tag{5.52}$$

Now integrating (5.50) with respect to ζ gives

$$R_3(\eta) = 8\mu l_1 l_2\eta^4\left\{1-\frac{17}{6}\eta^2+O(\eta^4)\right\}. \tag{5.53}$$

After substituting (5.53), the energy difference ΔE in (5.2) for the flexural deformation for small η gives

$$\Delta E = -4\mu l_1 l_2 l_3\eta^4\left\{1-\frac{17}{6}\eta^2+O(\eta^4)\right\}\varepsilon^4+O(\varepsilon^6). \tag{5.54}$$

This gives a critical η values at which ΔE changes signs near $\eta=\sqrt{\frac{6}{17}}=0.594$. When $\eta<0.594$, $\Delta E<0$ so that the buckled deformation has less energy than the homogeneous deformation. When $\eta>0.594$, the homogeneous deformation has less energy.

These stability conclusions are similar to those obtained by the numerical procedures in Section 5.3, except that the asymptotic analysis predicts a critical $\eta=0.594$, while the numerical analysis gave a critical $\eta=0.6443$. Note that the curve generated by

(5.53) can not be directly compared with the numerical curve of Figure 5.3 because of the presence of the normalizing M in (5.13). Performing a similar normalization on (5.53) using asymptotic coefficient M in (5.33) makes dimensionless stability parameter R_S as

$$R_S(\eta) = \frac{R_3(\eta)}{\mu l_1 M^4 \Omega^3} = \frac{16}{81}\eta^9\left\{2 - \frac{7}{5}\eta^2 + O(\eta^4)\right\}, \tag{5.55}$$

where $\eta = \Omega l_2$ was used. The transition value here is η=1.195 but this value is not important since the expansion for M in (5.33) is valid only for $\eta \ll 1.37$. The comparison of R_S in numerical results (5.14) and asymptotic results (5.55) is shown in Figure 5.6.

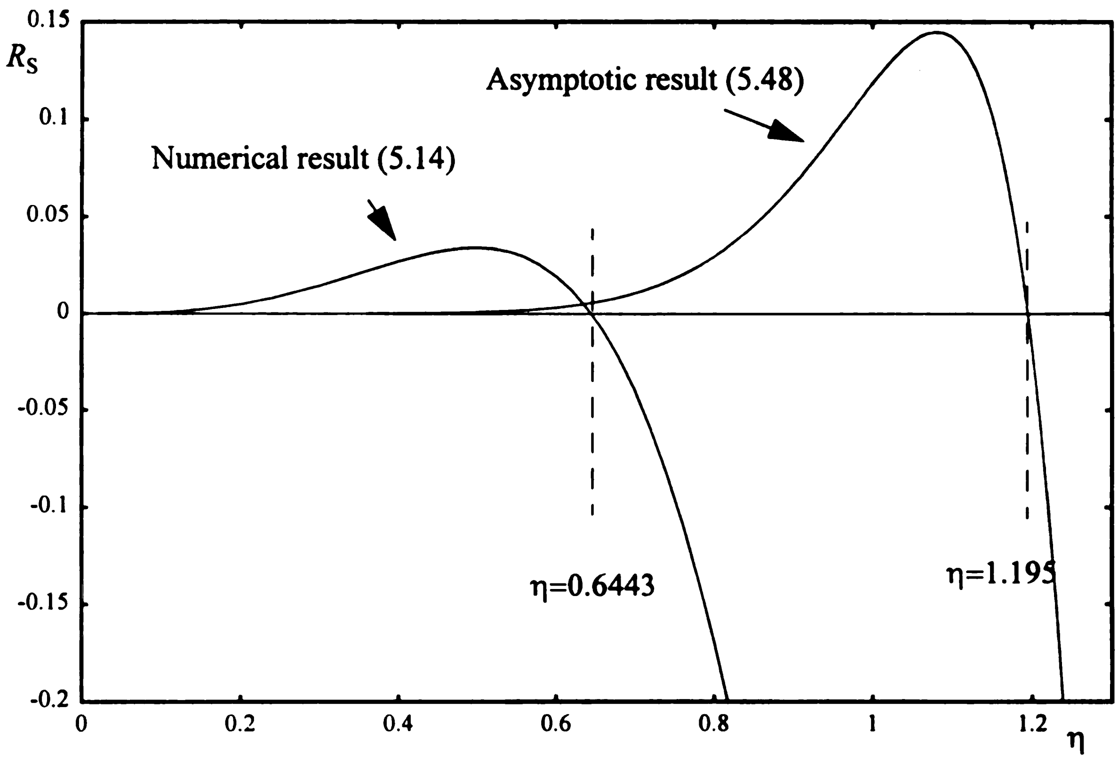


Figure 5.6 Comparison of dimensionless stability parameters R_S of numerical result (5.14) and asymptotic result (5.55) for flexural deformation. The asymptotic result is valid only for very small η.

When η is near zero, the signs and slopes are close so that the results are consistent with the numerical procedures.

5.5 Discussion

The sign of ρ_2 is opposite to the sign of R_3 according to (4.173) and the sign of E_4 is the same as that of ρ_2 according to (4.175). E_4 is the leading term in the energy difference ΔE. Positive (negative) values of ρ_2 denotes that the load must decrease (increase) after the buckling onset value ρ_0 in order to follow the bifurcated branch of buckling solutions. In other words, by converting ρ_2 to λ_2 with $\lambda_2 = -2\rho_2\rho_0^{-3}$, the load must decrease if $\lambda_2<0$.

For flexural deformation the numerical results show if $\eta<0.6443$ then $R_S>0$ which means the bifurcated path involves less energy than the trivial solution so that the homogeneous deformation near the bifurcation initiation is unstable. Otherwise when $\eta>0.6443$, the homogeneous deformation near the bifurcation initiation is stable. The numerical results also suggest additional stability transitions near $\eta=1.305$ and $\eta=1.6283$ (Figure 5.1). However, these results are highly sensitive to the numerical evaluation procedure and so are rather suspect since they involve $R_S\to\infty$. Further it is not obvious how to treat these by asymptotic or perturbation procedures. These $R_S\to\infty$ transitions that are only detected numerically will be dismissed from further discussion. In contrast, the stability for small η on the flexural branch is consistent with the asymptotic analysis near $\eta=0$ (although the value of η for stability transition found by the asymptotic procedure is different).

The numerical results also show for barreling deformation that $R_S>0$ for all η so that the homogeneous deformation is unstable compared to the barreling deformation.

These trends are in fact opposite to those found by Sawyers and Rivlin (1974) in the view of stability evaluation. They also find that η=0.32 gives the transition in stability on the flexural branch.

5.6 Summary

The stability of post-buckled deformation near buckling onset was evaluated by using energy minimization scheme. Extensive use of symbolic algebra procedures enabled certain simplifications, but the problem still remained very complex. Accordingly, a combination of asymptotic and numerical procedures were employed to attempt to determine stability transitions. The physical buckling behaviors are as follows. As the thrust load increases from the original zero value, a family of infinitesimal flexural deformation competes with the homogeneous deformation until the thrust reaches the value associated with ρ_0=0.5437 (T_S=5.6786 in (2.16)). This thrust is known as the wrinkling load. Then as the thrust exceeds the wrinkling load, the infinitesimal flexural deformation family ceases to exist and is replaced by an infinitesimal barreling deformation, which again competes with the homogeneous deformation family.

The energy analysis shows that the infinitesimal flexural deformation family is energetically favored over the homogeneous deformation family at small loads (implying small mode number), but that the homogeneous deformation family is energetically favored at large loads (again dismissing $R_S \rightarrow \infty$ transitions). Thus there is a transition load value, and a corresponding transition mode value η=0.6443 (See Equation (5.14)) for this exchange in stability. In contrast, the infinitesimal barreling deformation family is always found to be energetically favored over the homogeneous deformation family.

Asymptotic analysis, valid only for small η, was employed to attempt to verify the behaviors of stability pattern for the flexural deformation. The results agree with those of the numerical approach. A leading order value for transition of stability gives $\eta=1.195$ (See Equation (5.55)). This precise value ($\eta=1.195$) is not of importance since the asymptotic analysis is only valid for η near zero.

Clearly there remain significant questions with respect to this work, especially with regard to precise numerical transition value. It must also be admitted, since the stability interpretation is dependent on the (+/-) sign of very complicated expressions (Equation (5.13)), that additional efforts are necessitated for confirming those results. This, however, should not obscure the fundamental basis provided by this work. Notably the consistent perturbation analysis provides strict order expansions both of the energy competition between homogeneous and bifurcated solutions (Equation (4.159)) and of the relation to the bifurcated path near buckling onset (Equations (4.2), (4.69), (4.126), (4.127)). Here the essential and consistent coupling between these is provided by (4.141) and (4.176).

CHAPTER 6

APPROXIMATE SCHEMES FOR BUCKLING LOAD OF MULTI-LAYERED COMPOSITE PLATES

6.1 Introduction

The previous Chapters were concerned with the buckling and post-buckling behavior for a noncomposite single ply plate and the stability of the various competing solutions with respect to each other. However in this Chapter, we will investigate somewhat practical topic - the critical buckling load on a composite plate. In a structure, the critical buckling load plays an important role since it gives the lowest load to resist against the compressed load. We had the critical buckling load on a single ply plate by solving rather simple equations in (4.55) and (4.57) and showed the result in Figure 4.1 as the relation between the load parameter ρ_0 and the mode number η. The curves of buckling onset which give the load at the buckling onset for specified geometry and mode number, are monotonic with mode number (increasing for the flexural deformation and decreasing for barreling deformation). Hence the critical (lowest) buckling load is always mode-1 flexural deformation. But for a multi-layered plate such as the three-dimensional geometry of Figure 6.1, this behavior may be seriously altered. Pence and Song (1991) and Qiu et. al. (1994) showed that in symmetric three-ply plate composed of two different types of neo-Hookean material, there exist another family of buckling paths and they are not always monotonic. This means the mode of the lowest critical buckling load is not always mode-1 flexural deformation.

As the number of layers in a composite plate increases, the direct algebraic analysis of the bifurcation conditions becomes increasingly complicated since it involves

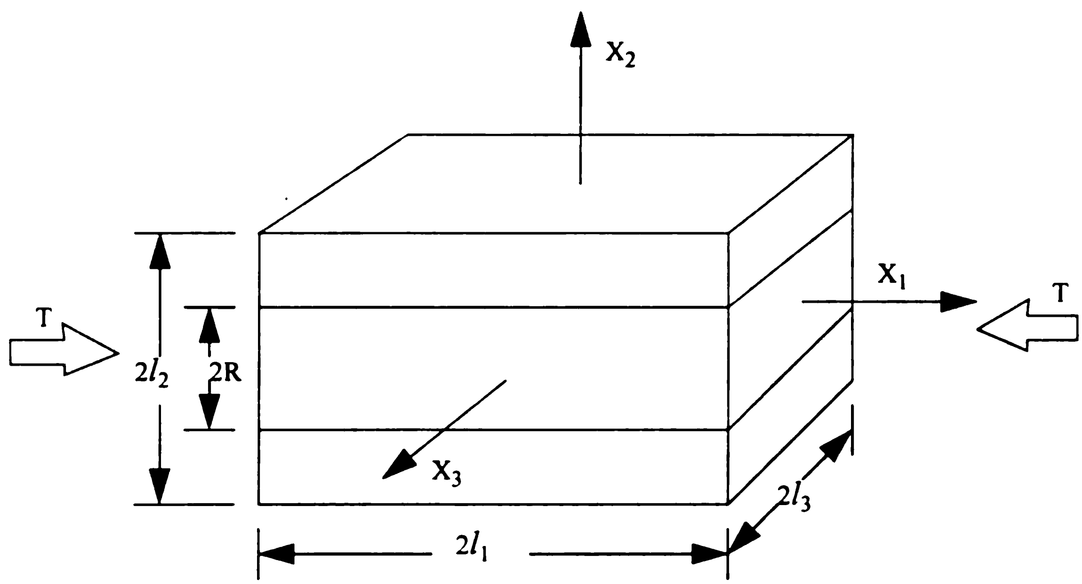


Figure 6.1 Geometry of the symmetric three-ply composite layer. The buckled configurations involve deformations in the (X_1,X_2)-plane.

seeking roots to a determinant equation for which the matrix dimension grows with the numbers of plies. Thus it is useful to seek alternate methods for determining buckling onset load in the manner of simpler approaches and closer to the exact values. The purposes of this final chapter is to present some developed observations on these issues.

6.2 Buckling Load of Multi-Layered Plates

The prediction for buckling onset load may change if the material is composed of multi-layered composite plate. In this Chapter, we will consider the general N-ply plate stacked along the X_2 direction. The undeformed configuration of whole plate occupies the region $2l_1 \times 2l_2 \times 2l_3$ and all plies are neo-Hookean materials. The shear modulus of each ply is either μ^{I} or μ^{II} and alternate by ply. Perfect bonding is assumed across the ply interfaces. Again our attention is restricted to plane strain deformation where buckling takes place in the (X_1,X_2)-plane as shown in (2.18). Then the mathematical formulation of composite plate problem is similar to those of single ply plate studied in Chapter 2. The differences are (i) the shear modulus μ of single ply plate alternates between μ^{I} and μ^{II}, and (ii) the assumption of perfect bonding yields the following interface conditions on the traction and displacement.

$$\begin{aligned} x_i(X_2^+) &= x_i(X_2^-) \\ S_{2i}(X_2^+) &= S_{2i}(X_2^-) \end{aligned} \qquad \text{on} \qquad \text{interfaces} \qquad i = 1, 2, 3. \tag{6.1}$$

Here the conditions for i=3 are automatically satisfied in a plane strain setting.

Let $A^{(j)}$ be the sum of original areas normal to X_1 direction of plies whose shear modulus is μ^j (j=I,II) so that $A^{\mathrm{I}} + A^{\mathrm{II}} = 4l_2l_3$. Then the total thrust on $X_1 = \pm l_1$ for homogeneous deformation can be modified from (2.16) to

$$T = -(\rho - \rho^{-3})(\mu^{\mathrm{I}}A^{\mathrm{I}} + \mu^{\mathrm{II}}A^{\mathrm{II}}). \tag{6.2}$$

The buckling onset can be analyzed by the incremental deformation superposed onto homogeneous deformation such as (2.18). The complete boundary value problem of a composite plate then consists of that of noncomposite plate (2.28) in which μ changes to μ^j and the interface continuity conditions

$$\begin{aligned} [\mu^j(v_{1,2} + \rho^{-2}v_{2,1}) + qv_{2,1}]_{X_2^+} &= [\mu^{\hat{j}}(v_{1,2} + \rho^{-2}v_{2,1}) + qv_{2,1}]_{X_2^-} \\ [\mu^j(v_{2,2} - \rho^{-2}v_{1,1}) - q(v_{1,1} + \rho)]_{X_2^+} &= [\mu^{\hat{j}}(v_{2,2} - \rho^{-2}v_{1,1}) - q(v_{1,1} + \rho)]_{X_2^-} \\ [v_1]_{X_2^+} &= [v_1]_{X_2^-} \\ [v_2]_{X_2^+} &= [v_2]_{X_2^-} \end{aligned} \tag{6.3}$$

on interface. Here μ^j is the shear modulus of top ply and $\mu^{\hat{j}}$ is of bottom ply on that interface. Since the buckling onset occurs at the first order (k=1) in perturbation expansion of deformation, we will consider only the homogeneous boundary value problem modified from (3.16) with appropriate interface conditions (6.1) such that,

$$
\begin{aligned}
F^j\mathbf{u} &= \mathbf{0} \qquad \text{in} \qquad \Pi, \\
G_1^j\mathbf{u} &= \mathbf{0} \qquad \text{on} \qquad \Gamma_1, \\
G_2^j\mathbf{u} &= \mathbf{0} \qquad \text{on} \qquad \Gamma_2,
\end{aligned}
\tag{6.4}
$$

and

$$
\left.
\begin{aligned}
[G^j{}_2\mathbf{u}]_{X_2^+} &= [G^{\hat{j}}{}_2\mathbf{u}]_{X_2^-} \\
[G_3\mathbf{u}]_{X_2^+} &= [G_3\mathbf{u}]_{X_2^-}
\end{aligned}
\right. \qquad \text{on interfaces}
\tag{6.5}
$$

where superscript j=I,II, denotes the differential operator of ply j in a composite plate. For simplicity in the expression of the first order equation, the superscript $^{(1)}$ will be suppressed here and after. Here G_3 is the constant matrix

$$
G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\tag{6.6}
$$

The difference in potential energy E in (2.30) for noncomposite plate can be used for composite plate as the sum of the energy of each ply. For buckling onset, the energy equation have the value up to the second order so that

$$
E_2 = 2l_3 \iint_\Pi \frac{\mu^{(j)}}{2} \{ 2\rho_0^{-2}(u_{1,2}u_{2,1} - u_{1,1}u_{2,2}) + u_{1,1}^2 + u_{2,2}^2 + u_{1,2}^2 + u_{2,1}^2 \} dX_1 dX_2 \tag{6.7}
$$

where the integration on X_2 is sum of the integrations of each ply.

Following the similar analysis to noncomposite case, this composite plate problem is reduced to one homogeneous ordinary differential equation, 4 boundary conditions and 4(N-1) interface conditions. Introducing the proper general solutions which is similar to (4.18) with discrete constants $L_1^{(m)}, L_2^{(m)}, M_1^{(m)}, M_2^{(m)}$, m=1,2,...,N of differential equation $(6.4)_1$ to the boundary and interface conditions $(6.4)_{2,3}$ and (6.5) form a homogeneous

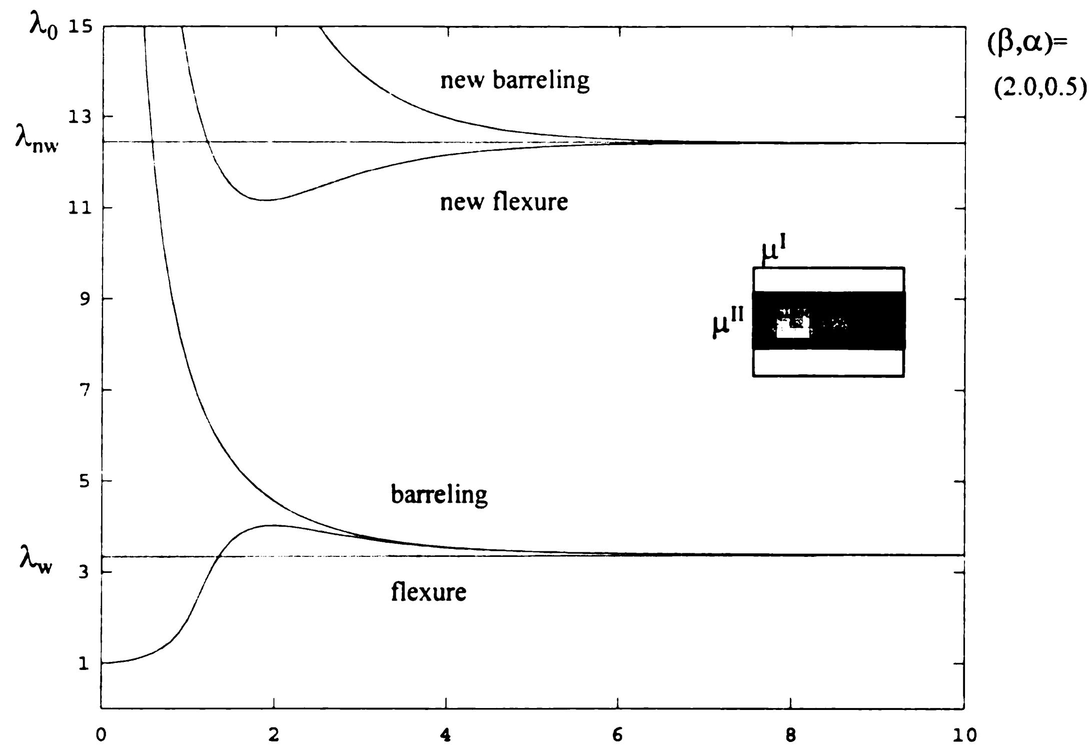
4N×4N matrix equation. For the buckling onset load (nontrivial solution), the determinant of this matrix must vanish.

In particular case of symmetric plate, the dimension of matrix can be reduced to two 2N×2N - one for symmetric mode and the other for antisymmetric mode as explained in Chapter 4. For the simple example of symmetric plate, the three-ply composite plate (N=3) was considered by Pence and Song (1991, 1993). Here the central ply (material II) has the thickness $2R$ ($<2l_2$) and the shear modulus μ^{II} so that the shear moduli of outer plies (material I) are μ^{I}. This problem then simplifies to 12 homogeneous linear equations for the 12 constants $L_1^{(m)}, L_2^{(m)}, M_1^{(m)}, M_2^{(m)}$, m=1,2,3. The vanishing of the determinant of 12×12 coefficient matrix gives the loads for buckling onset. Due to the aspect of symmetry, this problem can be considered by two 6×6 matrix equation - symmetric (flexure) deformation and antisymmetric (barreling) deformation along the X_2 direction with four dimensionless parameters as

$$\lambda_0 = \rho_0^{-2}, \qquad \eta = \Omega l_2, \qquad \beta = \mu^{II}/\mu^{I}, \qquad \alpha = R/l_2. \tag{6.8}$$

When β=1 or α=0 or α=1, this problem reduces to the noncomposite case which is analyzed previously by Sawyers and Rivlin (1974,1982).

The numerical computation for three-ply plate under various parameter sets shows in Figure 6.2 that (i) the buckling onset load for composite plate does not guarantee its monotonicity, i.e., the critical load is either mode-1 flexural deformation or wrinkling deformation in which the mode number is infinity (Pence and Song, 1991), (ii) there exists additional solutions for each original solution of flexure and barreling - we categorize these into a new family and the original family of solutions, respectively, and (iii) the wrinkling load of original family converges to that of noncomposite plate which is



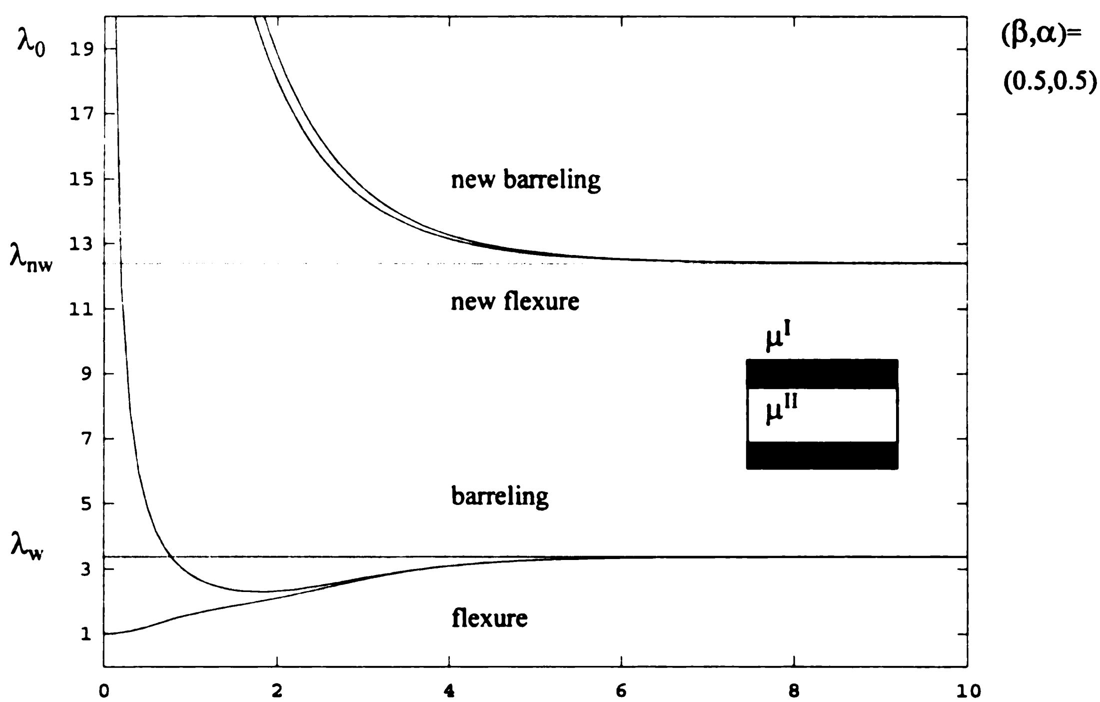


Figure 6.2 The buckling onset prediction curve for a symmetric tri-layer with different values of $\beta=\mu^{II}/\mu^{I}$. The shear moduli of shaded plies are twice as large as those in the unshaded plies. In both cases, the volume fraction of central ply, α, is 1/2. Nonmonotonic behaviors, new curves at large λ values, and asymptotes of all curves are shown (Qiu, et al, 1993)

constant for the material parameters and the wrinkling load of the new family converges to a higher value than that of original family depending on the stiffness ratio β (Qiu, et al., 1993).

An asymmetric two-ply plate (N=2) is discussed in Qiu, et al. (1993). Due to the lack of symmetric property in shape, the buckling deformations become a mixed mode of flexure and barreling. Solving the resulting 8×8 determinant shows that there are three solutions. The lower two curves are original family and the higher one is new family. The wrinkling load of original family have the same values of noncomposite plate and the wrinkling load of new family depends on the shear modulus ratio.

6.3 Approximate Schemes to Determine Buckling Load

To construct buckling onset prediction curves of buckling load vs. mode parameter at fixed values of appropriate volume fraction and stiffness ratio, will be a key to determine the critical buckling load and eventually require complicated numerical procedures. For the general N-ply sandwich plate, this problem is reduced to solving a $4N\times4N$ determinant equation. An exact analytic solution satisfies the *complete conditions*: the nonlinear constitutive equation (CE) in (2.13) for noncomposite plate, equilibrium equation (EE) in (2.6), boundary conditions of a free surface condition (FSC) in (2.9) and conditions of interface displacement continuity (IDC) in $(6.1)_1$ and traction continuity (ITC) in $(6.1)_2$. The simultaneous satisfaction of all these conditions gives much difficulty when the plate consists of large number of ply stacking. This difficulty is stems from the fact that standard numerical procedures to find the roots of the necessary determinant are subject to various numerical errors and numerical instabilities. The possible approximation schemes may involve procedures (specifically trial functions) that do not satisfy certain conditions mentioned above. Satisfaction of all these conditions gives an exact solution and thus an exact prediction of the buckling load. Therefore the goal of the research described in this Chapter is to construct useful approximation schemes which by sacrificing some of the conditions, give a simpler mathematical formulation. The effect on accuracy of these sacrifices will then be examined.

The simplest approximate scheme is based on equivalent modulus where the composite structure is treated as a homogeneous media with volume averaged stiffness modulus. For example as described in Figure 6.3, the three-ply plate which the central ply

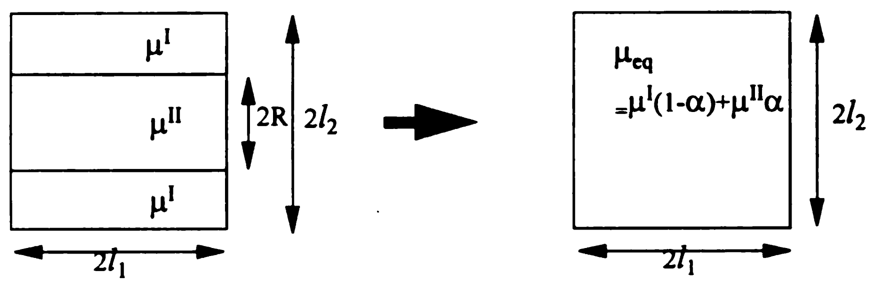


Figure 6.3 In the equivalent modulus scheme, the composite plate is treated as a single layer of volume averaged stiffness, μ_{eq}.

has μ^{II} and top and bottom plies have μ^I considered previously, is treated as a single ply which has the equivalent stiffness modulus $\mu_{eq} = \mu^I(1-\alpha) + \mu^{II}\alpha$ where α is the volume fraction explained in (6.8). Note that this scheme does not satisfy constitutive equation pointwise, but does so in a volume averaged sense. However the buckling onset prediction curves of single ply plate does not show the dependency on system parameters so that any combinations of plies have the same buckling onset prediction curves as the noncomposite plate. Therefore the critical buckling load for equivalent modulus scheme is always mode-1 flexure.

6.3.1 Variational scheme

Most approximation methods used in structual analysis are based on variational mechanics in which the approximate solutions satisfy the weak (variational) form or minimizes the energy functional (Reddy, 1986). The buckling load in a structual problem or the natural frequency in a vibration problem can be determined approximately by so called Rayleigh quotient obtained from the variational method.

The critical buckling load for the composite plate considered here can be constructed from the boundary value problem of composite plate (6.4) and (6.5). The inner product

$$\langle \mathbf{u}, F^j \mathbf{u} \rangle = 0 \tag{6.9}$$

followed by the integration by parts once and applying boundary conditions $(6.4)_{2,3}$ and continuity conditions (6.5) gives the critical buckling load as

$$\lambda_0 = \frac{\iint_{\Pi} \frac{\mu^{(j)}}{2}(u_{1,1}^2 + u_{2,2}^2 + u_{1,2}^2 + u_{2,1}^2)dX_1 dX_2}{\iint_{\Pi} \mu^{(j)}(u_{1,1}u_{2,2} - u_{1,2}u_{2,1})dX_1 dX_2} \tag{6.10}$$

where $\lambda_0 = \rho_0^{-2}$ is the load parameter and the integration on X_2 is sum of the integrations of each ply. This quotient can be also obtained by energy formulation based on the fact that deformed configurations with less strain energy than the unbuckled homogeneous configuration only become available once the buckling load is attained. The difference in energy from the homogeneous state to buckled state of a composite plate in (6.7) must be negative when the buckling takes place. At the buckling initiation, E_2 becomes zero. Equating E_2=0 gives the same quotient for the buckling load $\lambda_0 = \rho_0^{-2}$ as (6.10). If we have the exact solutions u_1 and u_2 of buckling onset, then the quotient (6.10) will give the exact buckling onset load. However the procedure to determine the exact buckling solutions u_1 and u_2 of the general ply composite plate is not that easy. So the approximate (trial) solutions which will satisfy part of required complete conditions must be considered. These will give a closer prediction to the exact buckling load.

6.3.2 Trial solutions

Recall that the exact solutions satisfy the requirements of complete conditions: CE, EE, FSC, ITC, IDC. A simple approximation for the composite plate is to use exact solutions of noncomposite plate (4.68) with (4.26) for flexure and (4.56) for barreling. One approach is that the composite material can be considered as the combination of corresponding single plies (combined single ply solution). For example as shown in Figure 6.4, the geometry of mode-m flexural buckling of composite plate is similar to mode-m flexure in each ply, while mode-m barreling of composite material is similar to mode-m barreling in the central layer with mode-m flexure in the outer layers. Note that in this approach, the length of X_2 as well as η in each single ply are scaled to those of single ply and the X_2 coordinates in each ply are transformed to the origin. This approximation does not satisfy the interface displacement $(6.5)_2$, although the displacements are close.

For a symmetric three-ply plate with (β,α)=(0.1,0.5), the buckling onset curves in Figure 6.5, are generated by the quotient with this combined single ply solutions. The exact curves and the curves by equivalent modulus scheme are also shown. For flexure, the variational scheme with combined single ply solutions gives better results than the

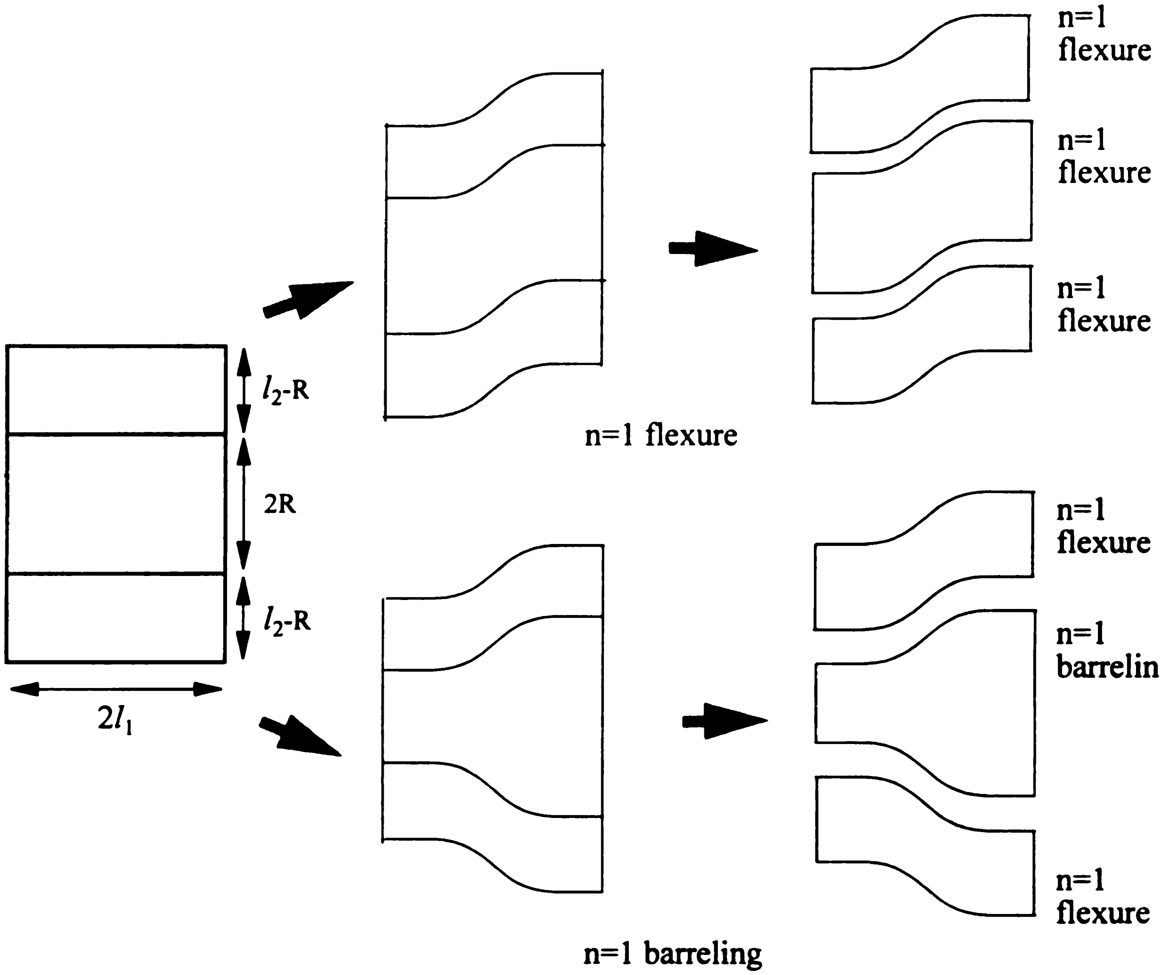


Figure 6.4 In the direct energy scheme, the overall deformation is approximated by the single layer deformations as shown. The deformation of the central layer distinguishes overall flexure from overall barreling.

equivalent scheme. This is because the single deformed shapes are well matched to the three-ply plate for flexure. However they are not well matched for the barreling case.

The other approach is that the single ply solutions can be applied directly to all the plies (direct single ply solution) since the final displacement of deformed shape of composite plate is similar to that of single ply plate. For example of three-ply case, the mode-m flexural buckling of composite plate may use the solutions of single ply mode-m

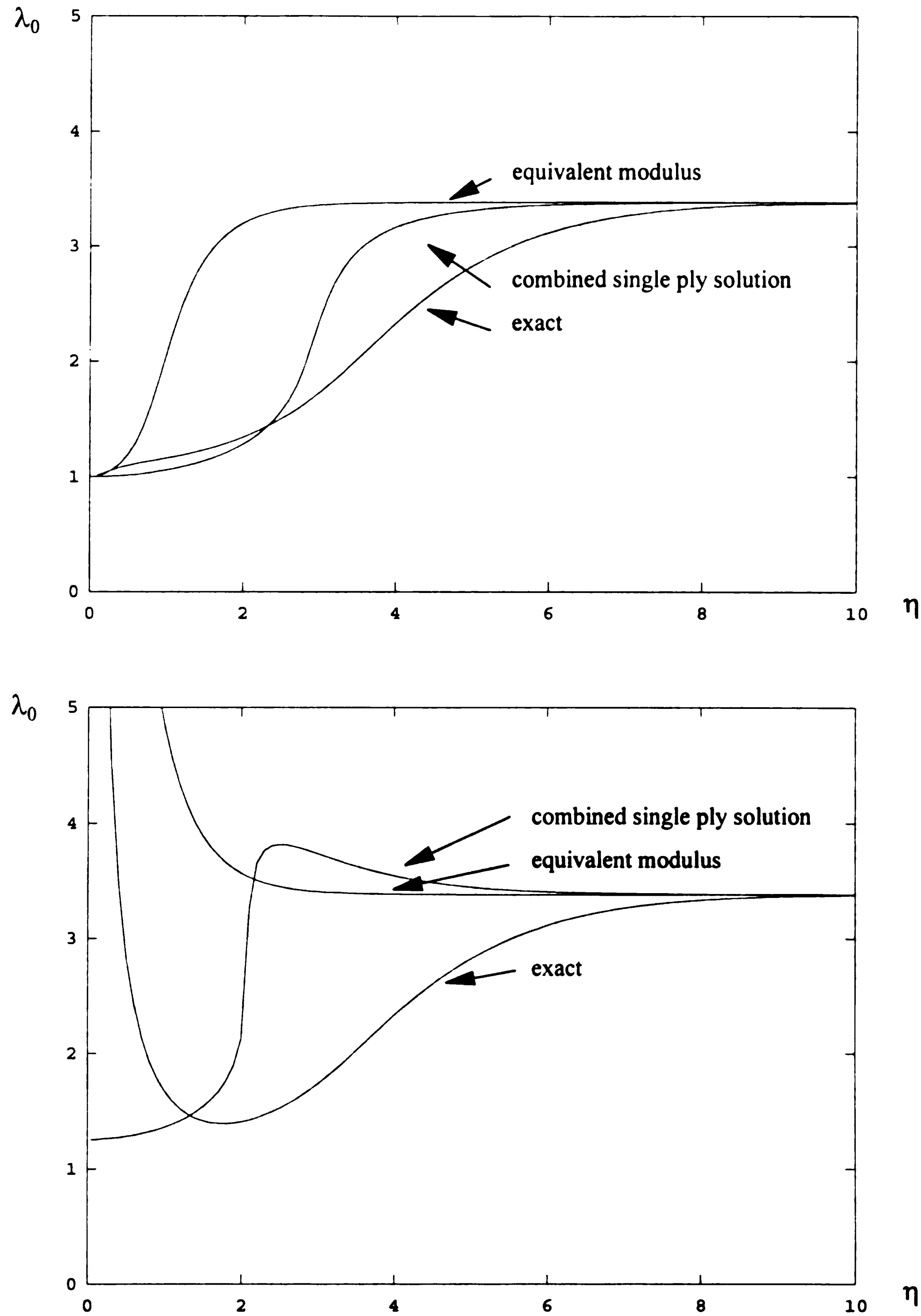


Figure 6.5 The onset prediction curves as given by the equivalence scheme, exact scheme and the variational scheme with combined single ply solution for flexure (top) and barreling (bottom) for tri-layer with $(\beta,\alpha)=(0.1,0.5)$.

flexure. This approximate solution satisfies the boundary and interface conditions, but not the constitutive equation.

For a three-ply plate with fixed pairs of $(\beta,\alpha)=(0.5,0.5)$ shown in Figure 6.6, the buckling onset curves are close to the exact solutions for both flexural and barreling modes. They form an upper bound as expected from the fact that the Rayleigh quotient gives an upper bound.

6.4 Discussion

For an analysis for determining buckling onset load of a composite plate, we examined some approximation schemes in which the approximate trial solutions satisfy some parts of the complete conditions. Since general multi-ply plate is combination of even or odd stacking, the approximate schemes developed previously can be applied to a different ply stacking (like two-ply plate) in order to determine whether these schemes can predict the buckling onset curves well enough.

Also we can consider other trial solutions based on combined single ply solution in the variational scheme so as to satisfy the interface boundary conditions (IDC) in $(6.1)_1$. One alternative is obtained by multiplying a suitable ***mollifier*** function of X_2 to a single layer solution (4.26) for flexure or (4.56) for barreling. The example of ***mollifier*** function is a simple polynomial forms with undetermined coefficients and play a role as a single ply solution in each separated ply in a composite plate according to its shape. The disadvantage of using ***mollifier*** function is the sacrifice of satisfaction on field conditions (CE) and (EE) but the satisfaction of (IDC) may improve the accuracy of whole approximation. Satisfaction with complete conditions of various schemes studied here and suggested scheme is summarized in TABLE 6-1.

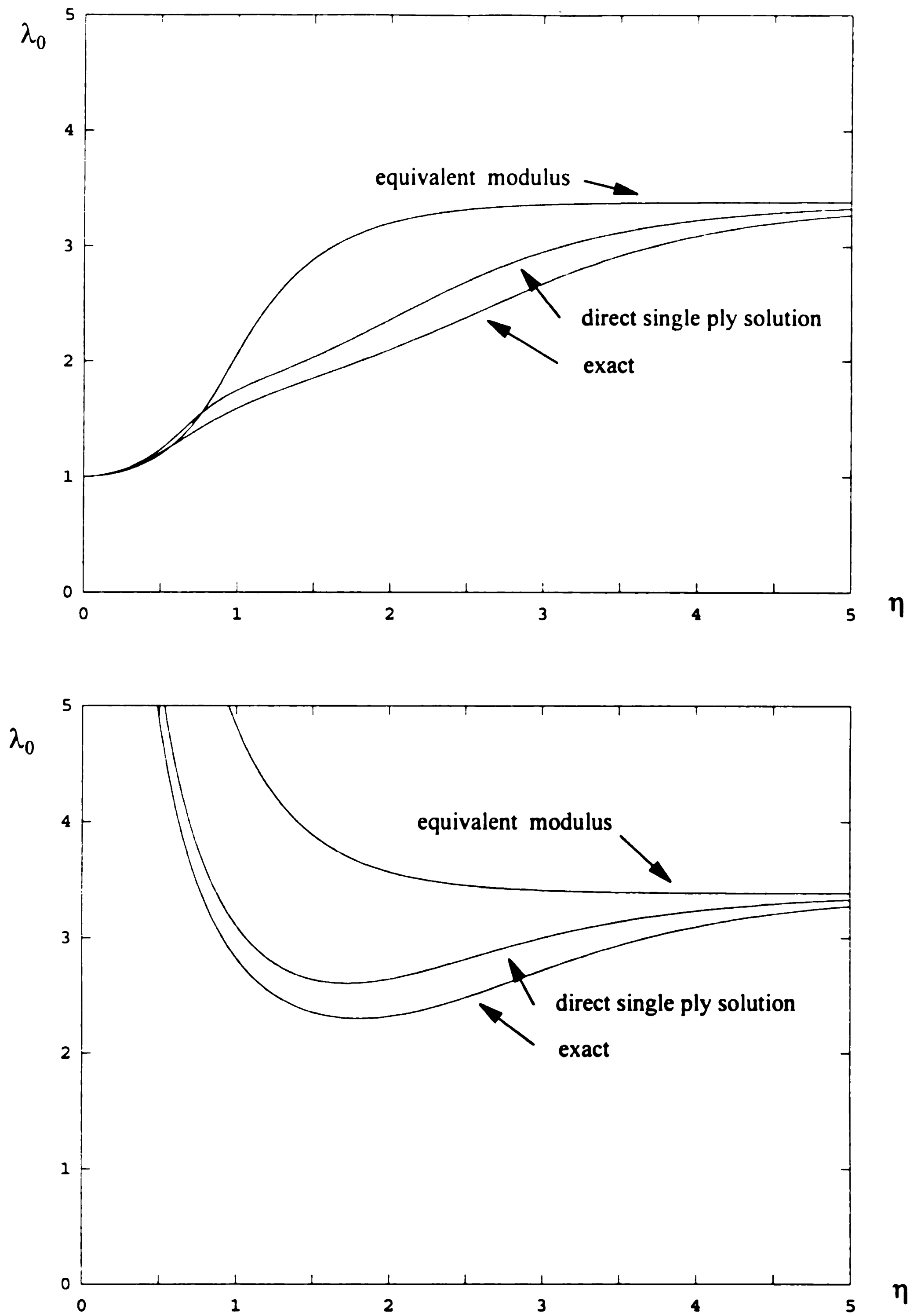


Figure 6.6 Comparison of the flexure (top) and barreling (bottom) onset prediction curves for symmetric tri-layer with $(\beta,\alpha)=(0.5,0.5)$ as generated by the exact scheme, the equivalent modulus scheme, and the variational scheme with direct single ply solutions. The upper bound property of variational scheme is evident.

Table 6-1 Summary of complete conditions and satisfaction for various schemes

	CE[1]	EE[1]	FSC[1]	ITC[1]	IDC[1]
Exact solution	yes	yes	yes	yes	yes
Equivalent modulus scheme	no[2]	yes	yes	yes	yes
Variational scheme (combined single ply solution)	yes	yes	yes	yes	no[3]
Variational scheme (direct single ply solution)	no[2]	yes	yes	yes	yes
Variational scheme - suggested (mollified single ply solution)	no[5]	no[5]	possible[4]	possible[4]	yes

1. CE: Constitutive Equation (2.13); EE: Equilibrium Equation (2.6); FSC: Free Surface Condition (2.9); ITC: Interface Traction Condition $(6.1)_2$; IDC: Interface Displacement Condition $(6.1)_1$.

2. satisfied only in a volume averaged sense, not pointwise.

3. however expect the result to be close since the mode shapes should be well approximated by the single layer theory.

4. these interface conditions can be ensured for mollifier functions of sufficiently many terms.

5. greater sacrifice of CE and EE will yield more improvement in FSC and ITC.

6.5 Summary

Three approximate schemes to determine the critical buckling load for three-ply neo-Hookean plate were investigated. Each scheme satisfies with parts of the perfect conditions and gives reliable values of critical load. The schemes developed here can estimate the critical buckling load of general multi-ply plate in a simpler manner.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions of the Thesis

The elastic stability analysis for the post-buckled and homogeneous deformations of single-ply neo-Hookean plate and the approximate schemes for buckling load of multi-ply neo-Hookean plate have been investigated. The elastic stability near buckling onset gives the prediction for physically existed deformation and can be evaluated by comparing the energy of all possible deformations. The buckling behavior are understood by the bifurcation theory in mathematical terminology.

Under these basis, two dimensional nonlinear boundary value problem for single ply neo-Hookean rectangular plate was generated in the context of finite elasticity. The perturbation expansion method was then applied to analyze the nonlinear problem into the set of linear equations by the order of ε. Each set of linear equations gives rise to a nonhomogeneous boundary value problem (except for the first order) and the solutions are related to the solutions and parameters of previous order. The first order equation is homogeneous and gives the thrust on buckling onset which is the critical buckling load. Also the load parameter that is barometer for thrust, are determined by Fredholm Alternative Theorem as a solvability conditions. The process for obtaining the solutions and load parameter for each order was developed in a systematic way and the more accurate results can be obtained from investigation of higher order equations. The deformations and their behavior at buckling initiation was at first obtained from the first order problem and shows the same results as other works.

For the stability evaluation of post-buckled deformation, the solutions and load parameters of second and third orders were obtained to analyze the energy difference which has the fourth order as the first appearing term (see Equation (4.159)). Since formulations is quite complex, a numerical analysis was applied at final evaluation of stability parameter (R_S in (5.14)). The numerical analysis shows that for flexural deformation, the buckled deformation has less energy than the homogeneous (unbuckled) deformation when $\eta<\eta_c=0.6443$ and $1.305<\eta<1.6283$ and the opposite is true when $\eta_c<\eta<1.305$ and $\eta>1.6283$. For barreling deformation, the buckled deformation has always less energy than the homogeneous deformation (see Section 5.3).

The complexity of formulation and non-explicity of load parameter lead to an asymptotic analysis for post-buckled deformation. The analysis on extreme shape gives the same critical load as Euler buckling. Also for extreme case of flexural deformation, the buckled deformation has less energy than the homogeneous deformation when $\eta<1.195$ and the opposite is true when $\eta>1.195$ (see Equation (5.55) and Figure 5.6). This asymptotic analysis is valid when η is extremely small. Hence the analytic results of energy comparison for the flexural deformation are true for limiting case of small η.

In this study, the perturbation expansion approach was used for evaluating the stability of homogeneous deformation. As a comparative work, Sawyers and Rivlin (1974) applied the variational methods to determine the stability of homogeneous deformation near the critical state. Their results are that the homogeneous deformation of flexural deformation is stable (has less energy) when $\eta<0.32$ and unstable elsewhere and the homogeneous deformation of barreling deformation is always stable. Hence the results of their works and this study have the opposite pattern. Also the transitional points η

obtained by Sawyers and Rivlin are different from those found here. The perturbation method used in this study is more direct and reasonable compared to the variational methods but the procedure is equally complex. One possible source of the difference in results stems from the different predictions that may be expected in general, as discussed in Appendix A in the context of some simpler problems. A second possible source of difference may be due to the difference in normalizations as explained in (4.59).

As an extended work from the stability of homogeneous deformation, the analytical way of the determination of critical load for composite plate was also studied (see Section 6.2). The critical load which can be found from the first order equation, plays a major role in the design of load-bearing structures. However for multiple stack composite plates, the determination of critical load requires the solution of a nonlinear eigenvalue problem for a 4Nx4N matrix. As stack grows, the equation will be so complex that we need simpler albeit approximate methods. In a beginning stage, three-ply symmetric plate was analyzed for the critical load in three different schemes. Compared with exact critical load in previous study by Qiu, et al. (1993), the results are quite close (see Figure 6.5 and Figure 6.6). These schemes can be expanded to more general ply plates.

7.2 Recommendations for the Future Work

Direct energy comparison for the stability of homogeneous deformation was performed and the results showed the differences compared with those of previous works. Hence as a verification of results, other limiting cases of infinite η for flexure, near zero η for barreling and infinite η for barreling should be investigated.

Though the computations on the values of higher order will give more accurate behaviors of post-buckling for hyperelastic plate, this will also be confined in local behavior because of the limitation of perturbation method. However the local stability criterion such as the stability near critical buckling load will be ensured.

APPENDICES

APPENDIX A

STABILITY EVALUATION BY PERTURBATION EXPANSION METHODS

A.1. Introduction

The analysis for the stability of buckled deformations in the vicinity of buckling initiation for a neo-Hookean plate requires complex mathematical calculations, so that the clarity of evaluations for each step is confused. In this Appendix, more simplified examples than considered problem in the context will be investigated in order to obtain the thorough acknowledgments for the concept of stability.

The well-known elastica problem will be considered at first as the simplest model. A modified elastica problem will be considered next to investigate the relation between the post-buckling behavior which is dependent upon a second order term in load parameter and the stability. Finally a higher order problem, which in certain ways resembles the neo-Hookean plate problem, will be examined. For the methods on stability evaluation near the buckling initiation, the perturbation analysis (PA) which has been used in the context and the analysis method adopted by Sawyers and Rivlin (SR) in (1982), will be used for the comparison. In particular, it is shown that these methods can give different stability predictions.

These examples will consider an energy equation $E(u,\lambda)$ where λ and u are load parameter and buckled deformation in the buckling problem or eigenvalue and solution in the mathematical bifurcation problem, respectively. Then the first variation of the energy equation gives a governing equation in a domain Ω and boundary conditions on a boundary Γ,

$$F(\lambda)u = 0 \qquad \text{in } \Omega, \qquad G(\lambda)u = 0 \qquad \text{on } \Gamma, \tag{A.1}$$

where F and G are differential operators. The statement in (A.1) constitutes a nonlinear boundary value problem.

With respect to (A.1), it is assumed that there is an obvious trivial solution u_{triv} for all values of λ. Thus $u = u_{triv}(\lambda)$ which is the family of trivial solutions. We now seek additional solutions (competitors) that bifurcate from this trivial solution. These additional solutions would also depend on λ, say $u = u_{bif}(\lambda)$, so that a continuous parametric dependence on λ also defines a family or branch of those additional solutions. Unlike the trivial solutions, the family $u_{bif}(\lambda)$ may exist for only a restricted range of λ. Now the solution family $u_{bif}(\lambda)$ is said to bifurcate from the trivial family $u_{triv}(\lambda)$ at the value λ_0 if $u_{triv}(\lambda_0) = u_{bif}(\lambda_0)$.

The stability evaluation is well explainable under the concept of energy minimization. If, at a given load parameter λ, the energy of the one equilibrium solution is less than that of another competing equilibrium solution, then the original solution is energetically preferable to that of the competitor (it is more stable). The energy difference between the trivial solution and the buckled solution at certain load level λ.

$$\Delta E(\lambda) = E(u_{bif}(\lambda), \lambda) - E(u_{triv}(\lambda), \lambda) \tag{A.2}$$

will be considered in the following analysis. According to the energy minimization scheme, if ΔE>0, then the state corresponding to the trivial solution is stable.

A.2. The Perturbation Expansion Method

One of the well-established approaches to solve the nonlinear boundary value problem is by using the perturbation method. This approach utilizes an expanded solution which is perturbed from the trivial solution $u_0 = u_{triv}$ with a small parameter ε such as

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots \qquad \text{(A.3)}$$

The load parameter λ is also expanded accordingly,

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \varepsilon^3\lambda_3 + \dots \qquad \text{(A.4)}$$

Here λ_0 is the critical load for bifurcation initiation so that the bifurcated solution $u = u_{bif}$ in (A.3) is branched from the bifurcation initiation $u_0 = u_{triv}(\lambda)$ at $\lambda = \lambda_0$. Substituting (A.3) and (A.4) into (A.1) and (A.2), and collecting along the same order of ε, give the set of linearized boundary value problem for each order of ε. By solving each set of equations, we can construct a complete expansion on u and λ. Budiansky (1974) also used the similar procedures to this study for post-buckling analysis.

For the purpose of stability evaluation, the energy difference ΔE compares the energy between the bifurcated and trivial path at a load level λ as shown in (A.2). The general solution (A.3) in this approach is an expansion in the vicinity of buckling initiation so that the results will be limited to the local analysis. Again by using the expansions (A.3) and (A.4), the energy difference (A.2) becomes along the order of ε as

$$\Delta E = \varepsilon^2 E_2 + \varepsilon^3 E_3 + \varepsilon^4 E_4 + \dots \qquad \text{(A.5)}$$

Here E_1 is vanished automatically if we substitute the equilibrium solution. For i-th set of the order ε, the solution u_i can be solved by the conventional way used in the analysis of linear differential equation and the eigenvalue λ_{i-1} can be obtained through the *Fredholm*

Alternate Theorem (FAT). The solutions and eigenvalues obtained are substituted into the energy equation then E_{i+1} can be determined. Since ε is small, the first nonzero term on energy equation (A.5) becomes the leading term on energy difference. If ΔE becomes positive then $E(u_{bif}) > E(u_{triv})$ so that the trivial solution is energetically preferable at the same load level λ. For the opposite case, the bifurcated solution is preferable. The schematic diagram of the procedure is shown in Figure A.1. During the process, it is sometimes hard to find all the solutions u_i. Instead of direct application, we introduce a certain orthogonal condition to make some terms in energy equation vanish. This condition can be obtained by using the integration by parts to the linearized equation. Detailed calculation will be explained later for a specific examples.

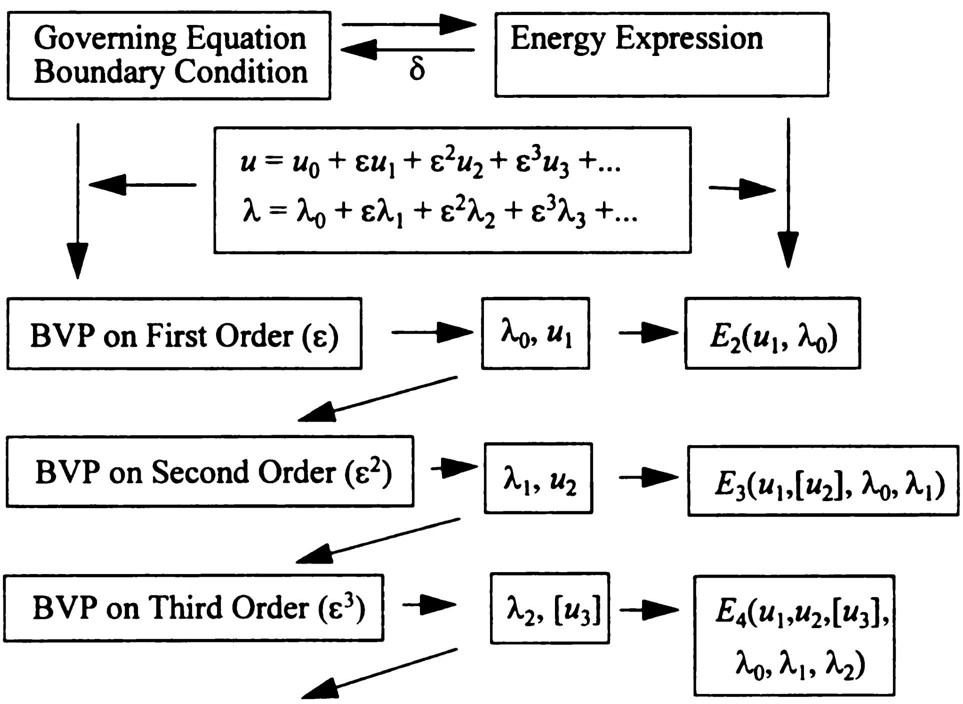


Figure A.1 The procedure for a perturbation method. Here the variables with a bracket are not necessary if we introduce a certain orthogonal conditions.

A.3. The Analysis Scheme by Sawyers and Rivlin

The approach to evaluate the stability of fundamental state which is the buckling initiation for the buckling problem, makes reference to the work of Koiter (1981). The energy of the admissible displacement u from the fundamental state characterize the stability of the fundamental state as the second variation to the energy is positive definite. Later Sawyers and Rivlin (1982) applied this approach to evaluate the stability of buckling initiation for the neo-Hookean rectangular plate.

If the second variation $P_2(u)$ for the potential energy difference $P(u)$ which is eventually the same as ΔE in (A.2) under the dead loading condition, is determined as positive definite then the fundamental state is regarded as stable. The neutral equilibrium solution u which becomes buckling deformations, can be obtained by a zero minimum of the second variation $P_2(u)$. Here $P_2(u)$ represents a linear version in the whole energy analysis so that $P_2(u)$ vanishes with the neutral equilibrium solution. Hence for the further condition on stability, the bifurcated path u is decomposed into a linear version of the neutral equilibrium $\hat{u}$ and an additional term v. Substituting this new solution into the energy equation $P(u)$ leads to a new energy equation $P(v)$. Again by solving the equilibrium solution for v and substituting the solution into $P(v)$, one can evaluate whether $P(v)$ as well as $P(u)$ is positive definite. This approach is also based on energy minimization scheme but the objects for competition is different from the perturbation expansion scheme.

A.4. Example 1: Elastica Problem

An elastica problem in which a long slender beam is compressed axially, has been a model for the buckling analysis (Thomson and Hunt, 1969). The vertical deformation or the buckled shape u is described mathematically as the bifurcation from the unbuckled deformation at a critical load parameter λ_0. The potential energy for a slender beam with a normalized length $0<x<1$, is given as

$$E(u, \lambda) = \int_0^1 \left(\frac{1}{2}u'^2 + \lambda \cos u\right) dx. \tag{A.6}$$

The primes denotes the differentiation with respect to x. The governing equation and boundary condition of (A.6) through the first variation lead to

$$u'' + \lambda \sin u = 0, \qquad u'(0) = u'(1) = 0, \tag{A.7}$$

where λ is an eigenvalue which represents the load parameter applied to the beam axially and the boundary conditions are characterized as a natural condition. Clearly one solution to the problem (A.7), valid for all possible λ, is $u(\lambda) = 0$. This is therefore the trivial solution to this problem, $u_{triv} = 0$.

The linearized version for the nonlinear boundary value problem (A.7) is stated as

$$u'' + \lambda u = 0, \qquad u'(0) = u'(1) = 0. \tag{A.8}$$

For a variable v, the inner product $\langle u'' + \lambda u, v\rangle = 0$ gives the adjoint problem to the linearized problem (A.8) through the integration by parts. The adjoint problem with respect to v has the same differential operator as that in (A.8) so that the linear differential operator in (A.8) is self-adjoint. The bracket used in inner product is defined as

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx. \tag{A.9}$$

The solution to the linearized problem (A.8), u_{lin} becomes

$$u_{lin} = A\cos m\pi x, \qquad \lambda = (m\pi)^2, \qquad m = 1, 2, 3, \ldots \tag{A.10}$$

where A is an arbitrary constant. The linear solution reveals that there is an infinite sequence of bifurcation load (one for each m).

A.4.1 Perturbation Analysis (PA)

We now seek additional solutions $u = u_{bif}$, bifurcated from $u_0 = u_{triv} = 0$ in (A.3) such as

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots \tag{A.11}$$

Here the perturbation parameter ε is defined as $\varepsilon = \langle u, u_1 \rangle$ with an orthogonality condition

$$\langle u_1, u_i \rangle = 0, \qquad i \neq 1. \tag{A.12}$$

The bracket denotes the inner product defined in (A.9). Substituting the expanded solution (A.11) and load parameter (A.4) into the boundary value problem (A.7) leads to the set of governing equations and boundary conditions which form the separate boundary value problems that originate from the various order of ε. Each set of governing equation consists of a linear differential operator with i-th order deformation u_i in the left hand side. The right hand side of i-th equation involves all of the previous solutions including u_{i-1} and λ_{i-1}. The boundary conditions of each equation becomes

$$u_i'(0) = u_i'(1) = 0, \qquad i = 1, 2, \dots \tag{A.13}$$

For the coefficient of ε, the first order equation becomes

$$u_1'' + \lambda_0 u_1 = 0. \tag{A.14}$$

The solutions of (A.14) with the normalized coefficient are

$$u_1 = \sqrt{2}\cos m\pi x, \qquad \lambda_0 = (m\pi)^2, \qquad m = 1, 2, 3, \dots \tag{A.15}$$

The first order equation (A.14) and the solutions (A.15) are in fact the same as those found in linearized problem (A.8) and (A.10) except for the subscript. Hence the first order equation represents the linearized problem. There is an infinite sequence of bifurcation loads according to the eigenvalue parameters λ_0 in (A.15) but, from now on, we will consider the lowest value, that is the case of m=1. The second order equation is shown as

$$u_2'' + \lambda_0 u_2 = -\lambda_1 u_1. \tag{A.16}$$

Since the differential operator in the left hand side of (A.16) is same as that in (A.14), the solutions u_2 will exist only if the right hand side of (A.16) is orthogonal to the solutions of (A.14), namely $\langle -\lambda_1 u_1, u_1 \rangle = 0$ according to the *Fredholm Alternate Theorem* (FAT). The existence condition and orthogonality (A.12) give

$$\lambda_1 = 0. \tag{A.17}$$

Introducing (A.17) into (A.16), the solutions with the orthogonality (A.12) become

$$u_2 = 0. \tag{A.18}$$

In a similar way, the third order BVP is expressed as

$$u_3'' + \lambda_0 u_3 = \frac{1}{6}\lambda_0 u_1^3 - \lambda_1 u_2 - \lambda_2 u_1. \tag{A.19}$$

Applying the FAT again to the right hand side of (A.19) and the first order solutions in (A.14) gives

$$\lambda_2 = \frac{1}{4}\lambda_0. \tag{A.20}$$

With the eigenvalues (A.17) and (A.20) into (A.19), the solution for the third order becomes

$$u_3 = -\frac{\sqrt{2}}{96}\cos 3m\pi x. \tag{A.21}$$

For the fourth order problem, the similar approaches give the boundary value problem as

$$u_4'' + \lambda_0 u_4 = \frac{1}{2}\lambda_0 u_1^2 u_2 - \lambda_1\left(u_3 - \frac{1}{6}u_1^3\right) - \lambda_2 u_2 - \lambda_3 u_1, \tag{A.22}$$

and the results are

$$\lambda_3 = 0 \qquad \text{and} \qquad u_4 = 0. \tag{A.23}$$

The higher order solutions can be obtained in a similar way. Then the buckled solution and load parameter up to the fourth order are summarized as follows.

$$u = u_{bif} = (\sqrt{2}\cos\pi x)\varepsilon - \left(\frac{\sqrt{2}}{96}\cos 3\pi x\right)\varepsilon^3 + O(\varepsilon^5), \tag{A.24}$$

$$\lambda = \lambda_0 + \frac{1}{4}\lambda_0\varepsilon^2 + O(\varepsilon^4), \qquad \lambda_0 = \pi^2. \tag{A.25}$$

The equation (A.25) represents the post-buckled solution path in λ-ε curve.

To evaluate the stability of the equilibrium solution near the buckling initiation, the energy minimization between the bifurcated and trivial solutions at the load level λ, is used. The energy difference (A.2) becomes

$$\Delta E = \int_0^1 \left[\frac{1}{2}u'^2 + \lambda(\cos u - 1)\right]dx. \tag{A.26}$$

If $\Delta E > 0$, then the trivial solution is stable. Substitution the bifurcated solution u_{bif} in (A.24) and eigenvalues (A.25) into (A.26) give the even-ordered series of energy difference

$$\Delta E = \varepsilon^2 E_2 + \varepsilon^4 E_4 + \ldots \tag{A.27}$$

The odd-order terms vanish automatically. The second order term, E_2 in (A.27) is

$$E_2 = \frac{1}{2}\int_0^1 (u_1'^2 - \lambda_0 u_1^2)dx. \tag{A.28}$$

With the solutions u_1 in (A.15), E_2 vanishes. The fourth order term, E_4 is

$$E_4 = \int_0^1 \left[u_1'u_3' - \lambda_0 u_1 u_3 - \frac{1}{2}\lambda_2 u_1^2 + \frac{1}{24}\lambda_0 u_1^4\right]dx. \tag{A.29}$$

Substituting all the solutions and eigenvalues in (A.24) and (A.25) into (A.29) gives

$$E_4 = -\frac{1}{2}\lambda_2 + \frac{1}{16}\lambda_0 = -\frac{1}{16}\lambda_0. \tag{A.30}$$

In fact, it is not necessary to know u_3 explicitly to evaluate (A.29). To see this, multiplying the first order problem (A.14) by u_k, k=1,2,3,... and integrating over the domain $0<x<1$, gives

$$\int_0^1 (u_1'' + \lambda_0 u_1)u_k dx = 0. \tag{A.31}$$

Then applying the integration by parts to the first term in (A.31) and the boundary conditions (A.13) for i=1, gives

$$\int_0^1 (u_1'u_k' - \lambda_0 u_1 u_k)dx = 0, \qquad k = 1, 2, 3, \ldots \tag{A.32}$$

Hence in (A.28), E_2 and in (A.29), the first two terms in E_4 vanish according to (A.32). For these terms, we do not need the solution u_3.

Regardless of how E_4 is obtained, the energy difference (A.27) now becomes

$$\Delta E = -\frac{1}{16}\lambda_0\varepsilon^4 + O(\varepsilon^6), \qquad \lambda_0 = \pi^2. \tag{A.33}$$

The dominant term E_4 in (A.33) is always negative giving $E(u_{bif}) < E(u_{triv})$ so that the bifurcated solutions are always stable in the vicinity of bifurcation initiation.

In this problem, $\lambda_1=0$ in (A.17) so that the post-buckled path is symmetric and $\lambda_2>0$ in (A.21) so that the load is increased when the bifurcation grows. Hence, considering the local behavior near the first bifurcation, $m=1$, there exists only one solution before bifurcation initiation, which is trivial and stable. After bifurcation initiation, there exist three local solutions of which one solution is trivial and unstable, and two other solutions follow the bifurcated path and are stable.

A.4.2 The Sawyers and Rivlin (SR) Scheme

Now consider the elastica problem with the approach used in the paper by Sawyers and Rivlin (1982). Starting from the potential energy equation for the admissible displacement u defined in (A.6). The energy difference which is in (A.26) is now rewritten as

$$P(u) = E(u, \lambda) - E(u_{triv}, \lambda) = \int_0^1 \left\{ \frac{1}{2}u'^2 + \lambda(\cos u - 1) \right\} dx, \qquad \text{(A.34)}$$

where $u = u_{bif}$, the bifurcated branch of solutions that we seek to construct. With the power series expansion of cos u, equation (A.34) becomes

$$P(u) = \int_0^1 \left\{ \frac{1}{2}u'^2 + \lambda\left(-\frac{1}{2}u^2 + \frac{1}{24}u^4 - \dots\right) \right\} dx. \qquad \text{(A.35)}$$

The second variational term in the expansion (A.35) is

$$P_2 = \frac{1}{2}\int_0^1 (u'^2 - \lambda u^2) dx. \qquad \text{(A.36)}$$

The necessary condition for stability of trivial solution is non-negative $P_2 \geq 0$. To find the stationary P_2, set δP_2=0

$$\delta P_2 = \int_0^1 (u'\delta u' - \lambda u \delta u)dx = 0. \tag{A.37}$$

After integration by parts, equation (A.37) yields

$$u'' + \lambda u = 0, \qquad u'(0) = u'(1) = 0. \tag{A.38}$$

The solutions of (A.38) are

$$u = A\cos m\pi x, \qquad \lambda = (m\pi)^2, \qquad m = 1, 2, 3, \dots , \tag{A.39}$$

which retrieves the bifurcation initiation values λ_0 and u_1 previously given in (A.15) and also the linearized solution u_{lin} in (A.10). Again we will stick to the lowest bifurcation value m=1. Here u is only a linearized version of the solution branch u_{bif} near the point of bifurcation initiation. By substituting (A.39) into (A.36), the result is

$$P_2 = 0. \tag{A.40}$$

Hence the state for which P_2 has a stationery value, is regarded as a state of neutral equilibrium.

Now we decompose $u = u_{bif}$ into the linearized solution (A.39) and the remainder term v as

$$u = a\hat{u} + v \qquad \hat{u} = \cos\pi x, \tag{A.41}$$

where $\hat{u} = A\cos\pi x$ comes from the linearized u in (A.39) and a is the coefficient given by $a = \langle\hat{u}, u\rangle / \langle\hat{u}, \hat{u}\rangle$. The solution components $\hat{u}$ and v have the orthogonality condition

$$\langle \hat{u}, v \rangle = 0. \tag{A.42}$$

Substituting the bifurcation solution u_{bif} as given in (A.41) into the energy difference (A.35) gives

$$P(u) = \int_0^1 \left\{ \frac{1}{2}a^2(\hat{u}' - \lambda\hat{u}^2) + a(\hat{u}'v' - \lambda\hat{u}v) + \frac{1}{2}(v'^2 - \lambda v^2) + \frac{1}{24}\lambda(a\hat{u} + v)^4 + \dots \right\} dx. \tag{A.43}$$

It is to be noted that, unlike the PA method, the SR scheme always use $\lambda = \lambda_0$ associated with bifurcation initiation. Now consider the multiplication v to the equation (A.38) with changed variable $\hat{u}$ and integration over the domain $0<x<1$ as

$$\int_0^1 (\hat{u}'' + \lambda\hat{u}) v dx = 0. \tag{A.44}$$

Using integration by parts and boundary conditions in (A.38) will give

$$\int_0^1 (\hat{u}'v' - \lambda\hat{u}v) dx = 0. \tag{A.45}$$

Introducing the condition (A.45) and the equation (A.38), the energy equation (A.43) becomes

$$P(u) = \int_0^1 \left\{ \frac{1}{2}(v'^2 - \lambda v^2) + \frac{1}{24}\lambda(a\hat{u} + v)^4 \right\} dx + O(a^5). \tag{A.46}$$

Now we determine the new solution v for which *P(u)* has a stationary value. According to (A.46), v has the value $O(a^2)$. Then let $v = a^2\bar{u}$ with the orthogonality

$$\langle \hat{u}, \bar{u} \rangle = 0. \tag{A.47}$$

Neglecting terms of order higher than a^4, equation (A.46) becomes

$$P(u) = \frac{1}{2}a^4\int_0^1\left\{(\bar{u}'^2 - \lambda\bar{u}^2) + \frac{1}{12}\lambda\hat{u}^4\right\}dx + O(a^5). \tag{A.48}$$

To find $\bar{u}$ for which *P(u)* has stationary value, the first variation with respect to $\bar{u}$ is applied so that

$$\delta P(u) = \frac{1}{2}a^4\left\{2\bar{u}'\delta\bar{u}\Big|_0^1 - 2\int_0^1(\bar{u}'' + \lambda\bar{u})\delta\bar{u}dx\right\} = 0. \tag{A.49}$$

Substituting for $\hat{u}$ from (A.39) and using $\lambda=\pi^2$ and boundary conditions $(A.38)_2$, equation (A.49) is rewritten as

$$-\int_0^1(\bar{u}'' + \lambda\bar{u})\delta\bar{u}dx + \bar{u}'\delta\bar{u}\Big|_0^1 = 0. \tag{A.50}$$

This yields

$$\bar{u}'' + \lambda\bar{u} = 0, \qquad \bar{u}'(0) = \bar{u}'(1) = 0. \tag{A.51}$$

The solution of (A.51) with orthogonality (A.47) gives

$$\bar{u} = 0. \tag{A.52}$$

Substituting the additional solution (A.52) into the energy difference (A.48) results

$$P(u) = \frac{1}{64}\lambda A a^4 + O(a^5). \tag{A.53}$$

Since *P*(*u*) in (A.53) is always positive for small value of *a*, the trivial solution in the vicinity of bifurcation initiation is regarded as stable. This result is different from that of perturbation analysis analyzed in Section A.4.1. This difference arises from the corresponding load level λ. The λ used in PA scheme is the load level on the bifurcated mode, however λ used in SR scheme is that on the bifurcation initiation. In fact, note from

(A.30) that omitting the λ_2 term from the PA scheme would give a conclusion similar to the SR scheme.

A.5. Example 2: Modified Elastica Problem

To determine the relationship between the second term in expanded load parameter, λ_2 and the stability of its solution path, the coefficient of the fourth term in an expansion of the elastica problem is replaced by an arbitrary constant. The sign of this constant coefficient represents the shape of post-buckled deformation in the vicinity of buckling initiation. Now we modify the elastica problem so as to include a coefficient α in the following energy functional.

$$E = \int_0^1 \left[\frac{1}{2}u' + \lambda\left\{ 1 - \frac{1}{2}u^2 - \frac{\alpha}{4}u^4 + O(u^6) \right\} \right] dx. \tag{A.54}$$

Note that if α=-1/6, then this problem is consistent with the original elastica problem. The boundary value problem for (A.54) becomes

$$u'' + \lambda\{u + \alpha u^3 + O(u^5)\} = 0, \qquad u'(0) = u'(1) = 0. \tag{A.55}$$

The trivial solution for the problem (A.55) is obviously u_{triv}=0 for any load level λ, and the linearized problem becomes

$$u'' + \lambda u = 0, \qquad u'(0) = u'(1) = 0. \tag{A.56}$$

The linearized problem (A.56) is the same as that in previous example (A.8) so that the linear operator is self-adjoint and the linearized solution is in (A.10).

A.5.1 PA Scheme

By substituting the expanded bifurcation solution $u{=}u_{bif}$ in (A.11) and the eigenvalues λ in (A.4) to the nonlinear problem (A.55), the boundary value problems for each order becomes

$$O(\varepsilon): \qquad u_1'' + \lambda_0 u_1 = 0, \tag{A.57}$$

$$O(\varepsilon^2): \qquad u_2'' + \lambda_0 u_2 = -\lambda_1 u_1, \tag{A.58}$$

$$O(\varepsilon^3): \qquad u_3'' + \lambda_0 u_3 = -\lambda_0 \alpha u_1^3 - \lambda_1 u_2 - \lambda_2 u_1, \tag{A.59}$$

and so on. The boundary conditions are the same as those shown in (A.13). With the orthogonality (A.12) and the FAT, the solutions for each order become

$$u = (\sqrt{2}\cos\pi x)\varepsilon + \left(\frac{\sqrt{2}}{16}\alpha\cos 3\pi x\right)\varepsilon^3 + O(\varepsilon^5), \tag{A.60}$$

$$\lambda = \lambda_0 - \frac{3}{2}\alpha\lambda_0\varepsilon^2 + O(\varepsilon^4) \qquad \lambda_0 = \pi^2. \tag{A.61}$$

Here we omitted the detailed process because this and previous examples are the same except for the parameter α. With the trivial solution u_{triv}=0, the energy difference (A.2) between the bifurcated and trivial solutions at a load level λ based on the energy (A.54) is stated as

$$\Delta E = \int_0^1 \left[\frac{1}{2}u'^2 - \lambda\left(\frac{1}{2}u^2 + \frac{\alpha}{4}u^4 + O(u^4)\right)\right]dx. \tag{A.62}$$

Substituting the perturbation expansions (A.4) and (A.11) into (A.62) becomes

$$\Delta E = \int_0^1 \left[\frac{1}{2}(u_1'^2 - \lambda_0 u_1^2)\varepsilon^2 + \left(u_1'u_3' - \lambda_0 u_1 u_3 - \frac{1}{2}\lambda_2 u_1^2 - \frac{\alpha}{4}\lambda_0 u_1^4\right)\varepsilon^4 + O(\varepsilon^6)\right]dx. \tag{A.63}$$

By applying the orthogonality (A.12) and the results obtained in (A.32) to the energy equation (A.63), finally we have

$$\Delta E = \frac{3}{8}\alpha\lambda_0\varepsilon^4 + O(\varepsilon^6) = E_4\varepsilon^4 + O(\varepsilon^6). \tag{A.64}$$

If $\alpha>0$, then from (A.61) and (A.64), $\lambda_2<0$ and $E_4>0$. Since E_4 is the dominant term in energy difference, the trivial solution has smaller energy than the bifurcated solution, that is, energetically stable. If $\alpha<0$, there exist three solutions before bifurcation initiation and $\lambda_2>0$ and $E_4<0$ so that the trivial solution is unstable.

A.5.2 SR Scheme

We now analyze this problem using the SR method used in Section A.4.2. The potential energy equation is expressed in (A.54) as

$$P(u) = \int_0^1\left[\frac{1}{2}u' + \lambda\left\{1 - \frac{1}{2}u^2 - \frac{\alpha}{4}u^4 + O(u^6)\right\}\right]dx. \tag{A.65}$$

The second variational term P_2 in (A.65) is exactly same as (A.36) in previous example so that the resulting equations and their equilibrium solutions are in (A.37) to (A.39).

$$u = A\cos\pi x \qquad \lambda = \pi^2. \tag{A.66}$$

By using the same decomposed new solution in (A.41), the new energy equations with the similar orthogonal condition (A.45) are

$$P(u) = \frac{1}{2}a^4\int_0^1\left[(\bar{u}'^2 - \lambda\bar{u}^2) - \frac{\alpha}{2}\lambda\hat{u}^4\right]dx. \tag{A.67}$$

The only difference between (A.67) and (A.48) in the previous example is the last term which contains α. But this term has only $\hat{u}$ so that the procedures to find $\bar{u}$ are the same as (A.49) to (A.52). Then the result for energy equation is

$$P(u) = -\frac{3\alpha}{32}\lambda A a^4 + O(a^5). \tag{A.68}$$

If $\alpha>0$, *P(u)* becomes negative so that the trivial solution is unstable. This result is opposite to the PA method.

A.6. Example 3: A Higher Order Problem Represented Neo-Hookean Buckling

The buckling and post-buckling problem for a neo-Hookean plate considered in the context by using the perturbation analysis, have a slightly different form from the previous two examples. Namely, the neo-Hookean plate involves $u_2 \neq 0$ (see (4.128)) and $\lambda_1=0$ ($\rho_1=0$, (4.138))so that the expansion for the deformation u and eigenvalue λ become

$$\begin{aligned} u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \ldots, \\ \lambda &= \lambda_0 + \varepsilon^2\lambda_2 + \varepsilon^4\lambda_4 + \ldots. \end{aligned} \tag{A.69}$$

In order to understand this type of expansion, it is useful to examine a simpler model than the neo-Hookean plate buckling problem. Such a simple model may be provided by considering the following energy expression

$$E(u, \lambda) = \int_0^1 \left[\frac{1}{2}u'^2 - \frac{1}{2}\lambda u^2 + \frac{1}{6}u'(u^2 + u'^2)\right]dx, \tag{A.70}$$

for a function u obeying $u(0) = u(1) = 0$. The governing Euler equation for the energy equation (A.70) are given by the first variation as

$$u'' + \lambda u + u'u'' = 0. \tag{A.71}$$

The trivial solution for the problem (A.71) is obviously u_{triv}=0. The linearized problem to (A.71) is stated as

$$u'' + \lambda u = 0, \qquad u(0) = u(1) = 0. \tag{A.72}$$

The boundary condition in (A.72) is different from previous two examples. However this does not affect the self-adjointness for the linear operator. The linear solution is obtained by solving (A.72) directly

$$u = A\sin\pi x, \qquad \lambda = \pi^2. \tag{A.73}$$

Here we considered the lowest value of bifurcation (m=1).

A.6.1 PA Scheme

The bifurcated solution u=u_{bif} is an expansion from the trivial solution u_{triv}=0. Substituting the expansions in (A.69) into the problem (A.71), the resulting boundary value problems for each order become

$$O(\varepsilon): \qquad u_1'' + \lambda_0 u_1 = 0, \tag{A.74}$$

$$O(\varepsilon^2): \qquad u_2'' + \lambda_0 u_2 = -\lambda_1 u_1 - u_1' u_1'', \tag{A.75}$$

$$O(\varepsilon^3): \qquad u_3'' + \lambda_0 u_3 = -\lambda_1 u_2 - \lambda_2 u_1 - u_1' u_2'' - u_2' u_1'', \tag{A.76}$$

and so on. The boundary conditions are

$$u_i(0) = u_i(1) = 0 \qquad i = 1, 2, \ldots \tag{A.77}$$

Since the trivial solution for this problem is u_{triv}=0, the energy difference between the bifurcated and trivial solutions is the same as (A.70). By using the expansions (A.69), this energy difference ΔE can be expanded accordingly.

$$\Delta E = E_2\varepsilon^2 + E_3\varepsilon^3 + E_4\varepsilon^4 + O(\varepsilon^5), \tag{A.78}$$

where the components are

$$E_2 = \int_0^1 \frac{1}{2}(u_1'^2 - \lambda_0 u_1^2)dx, \tag{A.79}$$

$$E_3 = \int_0^1 \left[u_1'u_2' - \lambda_0 u_1 u_2 + \frac{1}{2}\left(-\lambda_1 u_1^2 + \frac{1}{3}u_1'u_1^2 + \frac{1}{3}(u_1')^3\right)\right]dx, \tag{A.80}$$

$$\begin{aligned} E_4 = & \int_0^1 \frac{1}{2}\{2(u_1'u_3' - \lambda_0 u_1 u_3) + (u_2')^2 - \lambda_0 u_2^2 - 2\lambda_1 u_1 u_2 - \lambda_2 u_1^2\}dx \\ & + \frac{1}{2}\int_0^1 \left\{\frac{1}{3}(2u_1 u_1' u_2 + u_1^2 u_2') + u_1^{2\prime} u_2'\right\}dx, \end{aligned} \tag{A.81}$$

and so on. Now we consider the first order equation. By solving directly the solution becomes for the lowest value of bifurcation (m=1) as

$$u_1 = \sqrt{2}\sin\pi x, \qquad \lambda_0 = \pi^2. \tag{A.82}$$

In a similar way in (A.32), consider the following integration

$$\int_0^1 (u_1'' + \lambda_0 u_1)u_k dx = 0 \qquad k = 1, 2, 3, \ldots \tag{A.83}$$

Integration by part once and applying the boundary condition in (A.77) lead to the equation (A.32). This equation can apply to E_2 in (A.79) and the first two terms in (A.80) and (A.81) so that it is not necessary to solve u_3 in (A.81).

The solution in (A.82) is substituted into the second order solution and the results become

$$u_2 = -\frac{\sqrt{\lambda_0}}{3}\sin 2\pi x, \qquad \lambda_1 = 0. \tag{A.84}$$

Then applying the inner product with the right hand side of (A.76) and u_1 and the FAT gives the next order eigenvalue as

$$\lambda_2 = -\frac{1}{3}\lambda_0^2. \tag{A.85}$$

With the previous solutions (A.82), (A.84) and (A.85), the third order solution for the equation (A.76) becomes

$$u_3 = \frac{\lambda_0}{4\sqrt{2}}\sin 3\pi x. \tag{A.86}$$

However the third order solution (A.86) is not necessary to evaluate E_4 since there are no u_3 terms in (A.81) after considering the equation (A.83)[1]. Substituting all the solutions and eigenvalues obtained previously into the energy equation, then E_2 in (A.79) and E_3 in (A.80) vanish and E_4 in (A.81) gives

$$E_4 = \frac{1}{12}\lambda_0^2, \tag{A.87}$$

so that the energy difference becomes

$$\Delta E = \frac{1}{12}\lambda_0^2\varepsilon^4 + O(\varepsilon^5). \tag{A.88}$$

In this case, λ_2 is always negative and then ΔE is always positive so that the trivial solution near the bifurcation initiation is always stable.

1. This parallels the results in Section 4.7.2 where the $\mathbf{u}^{(3)}$ terms are eliminated in the energy expression (4.164) by using the boundary conditions $(3.9)_2$ for k=1.

A.6.2 SR Scheme

The energy equation for the neo-Hookean plate is expressed in (A.70). The second variation in energy P_2 to (A.70) is given by

$$P_2 = \int_0^1 \left(\frac{1}{2}(u')^2 - \frac{1}{2}\lambda u^2\right) dx. \tag{A.89}$$

After first variation to P_2, the linearized governing equation becomes

$$u'' + \lambda u = 0, \qquad u(0) = u(1) = 0. \tag{A.90}$$

The solution to the boundary value problem (A.90) is

$$u = A \sin \pi x \qquad \lambda = \pi^2. \tag{A.91}$$

Again we stick to the first mode m=1. Substituting the solution (A.91) into (A.89) gives

$$P_2 = 0. \tag{A.92}$$

Then the decomposition of $u = u_{bif}$ into the linearized solution $u = u_{lin}$ (A.91) and remainder term as

$$u = a\hat{u} + v, \tag{A.93}$$

where a is a small coefficient and the following orthogonality holds

$$\langle \hat{u}, v \rangle = 0. \tag{A.94}$$

After applying the new solution (A.93) to the energy equation (A.70) and set $v = a^2\bar{u}$, the energy equation after ignoring the order higher than 4 becomes

$$P(u) = a^4 \int_0^1 \left[\left\{\frac{1}{3}\hat{u}'(\hat{u}\bar{u} + \hat{u}'\bar{u}') + \frac{1}{6}\bar{u}'(\hat{u}^2 + (\hat{u}')^2) + \frac{1}{2}((\bar{u}')^2 - \lambda \bar{u}^2)\right\}\right] dx. \tag{A.95}$$

The first variation for the equation (A.95) gives the governing equation about $\bar{u}$ as

$$\bar{u}'' + \lambda\bar{u} = -\hat{u}'\hat{u}'' \qquad \bar{u}(0) = \bar{u}(1) = 0. \tag{A.96}$$

The solution for (A.96) with orthogonality condition (A.94) becomes

$$\bar{u} = \frac{A^2}{6}\pi\sin(2\pi x). \tag{A.97}$$

Substitution the solution (A.97) and linearized solution (A.91) to the energy equation (A.95) gives

$$P = -\frac{\lambda^2 A^4}{48}a^4. \tag{A.98}$$

The equation (A.98) reveals that P is negative so that the trivial solution is regarded as unstable.

A.7. Discussions

In this Appendix, we examined the stability in the vicinity of buckling initiation for more simplified problems than the buckling problem of neo-Hookean plate by using the perturbation method and the method used in the paper by Sawyers and Rivlin (1982). The whole procedure and their corresponding equations are summarized in Table 1. and 2.

The perturbation approach compares the energy between the bifurcation and the trivial solutions on the same load level of λ and on the first mode (m=1). The sign of an energy difference determines the lower energy level of two competing solutions which shows more energetically preferable solution. Also the results tell the relation between the post-buckling behavior (λ_2) and the energy difference (E_4). From the results on λ_2 and E_4, (A.20) and (A.30) for the elastica problem, (A.61) and (A.64) for the modified elastica problem and (A.85) and (A.87) for the simplified neo-Hookean buckling problem, it can be shown that

$$E_4 = -\frac{1}{4}\lambda_2. \tag{A.99}$$

Therefore the signs of λ_2 and E_4 are different so that, if λ_2 is negative then the trivial solution is energetically preferable. Otherwise the bifurcated solution is preferable.

In the SR method, the eigenvalue is not expanded so that the energy difference used in this method just compares the energy on the bifurcation initiation. This will give a confusion that on the eigenvalue (load parameter) at the bifurcation initiation. It also can generate opposite conclusions as to the stability of the bifurcated branch of solutions.

Table A.1 Summary of the procedures and their corresponding equations for the perturbation expansion.

Step	Elastica	Modified Elastica	Simplified neo-Hookean	Neo-Hookean plate (context)
1. Nonlinear BVP, Energy equation	(A.6), (A.26)	(A.55),(A.62)	(A.71)	(2.28), (2.30)
2. Applying the expansion	(A.11),(A.4)	(A.11),(A.4)	(A.11),(A.4)	(3.1)
3-1. 1st order BVP	(A.14)	(A.57)	(A.74)	(3.8), (3.9), (3.10)
4-1. λ_0	(A.15)	(A.61)	(A.82)	(4.62)
5-1. u_1	(A.15)	(A.60)	(A.82)	(4.68)-(4.70)
6-1. E_2	(A.28), E_2=0	(A.64), E_2=0	(A.88), E_2=0	(4.155),E_2=0
3-2. 2nd order BVP	(A.16)	(A.58)	(A.75)	(3.8), (3.9), (3.11)
4-2. λ_1	(A.17), λ_1=0	(A.61), λ_1=0	(A.84), λ_1=0	(4.138), ρ_1=0
5-2. u_2	(A.18), u_2=0	(A.60), u_2=0	(A.84)	(4.126)
6-2. E_3	E_3=0	E_3=0	(A.88),E_3=0	(4.158),E_3=0
3-3. 3rd order BVP	(A.19)	(A.59)	(A.76)	(3.8),(3.9), (3.12)
4-3. λ_2	(A.20), λ_2=1/4 λ_0	(A.61), λ_2=-3/2 $\alpha\lambda_0$	(A.85), λ_2=-1/3 λ_0^2	
5-3. u_3	(A.21)	(A.60)	(A.86)	N/A
6-3. E_4	(A.30), E_4=-1/16 $\lambda_0 <$ 0	(A.64), E_4=3/8 $\alpha\lambda_0$	(A.87), E_4=1/12 $\lambda_0^2 >$ 0	

Table A.2 Summary of the procedures and their corresponding equations for the Sawyers-Rivlin method.

Step	**Elastica**	**Modified Elastica**	**Simplified neo-Hookean**	**Neo-Hookean plate (1984)**
1. Energy equation, $P(u)$	(A.35)	(A.65)	(A.70)	(2.12)
2. Second variation, P_2	(A.36)	(A.36)	(A.89)	(3.1)
3. Solving for neutral equilibrium	(A.37) - (A.39)	(A.37) - (A.39), (A.66)	(A.90) - (A.91)	(3.8) (3.14), (3.16)
4. Define new solution	(A.41)	(A.41)	(A.93)	(4.1)
5. Energy with new solution	(A.48)	(A.67)	(A.95)	(4.7), (4.10)
6. Solving for new solution	(A.49) - (A.52)	(A.49) - (A.52)	(A.97)	(4.13) (5.5)
7. Evaluate energy with new solution (4th order)	(A.53), P=1/64 $\lambda A > 0$	(A.68), P=-3/32 $\alpha\lambda A$	(A.98), P=-1/48 $\lambda^2 A^4$	(6.1) Fig.3, Fig.4

APPENDIX B

COEFFICIENTS FOR STABILITY PARAMETER

The coefficient functions $Y[i,j]$ used in equations (5.11) and (5.12) for stability parameter R_3 are defined as follows:

$$Y[1,1] = \frac{1}{32\lambda_0^3}[y_{1,1}(\lambda_0) + \bar{N}_1 y_{1,2}(\lambda_0) - \bar{N}_2 y_{1,3}(\lambda_0) + \bar{N}_3 y_{1,4}(\lambda_0) - \nu\bar{N}_4 y_{1,5}(\lambda_0)]$$

$$Y[1,2] = \frac{\nu}{32\lambda_0^3}[y_{1,1}(-\lambda_0) - \bar{N}_1 y_{1,2}(-\lambda_0) - \bar{N}_2 y_{1,3}(-\lambda_0) - \nu\bar{N}_4 y_{1,4}(-\lambda_0) + \bar{N}_3 y_{1,5}(-\lambda_0)]$$

$$Y[1,3] = \frac{\nu s}{32\lambda_0^3}[y_{1,6}(\lambda_0) + \bar{N}_1 y_{1,7}(\lambda_0) - \nu\bar{N}_4 y_{1,8}(\lambda_0)],$$

$$Y[1,4] = -\frac{s}{32\lambda_0^3}[y_{1,6}(-\lambda_0) - \bar{N}_1 y_{1,7}(-\lambda_0) + \bar{N}_3 y_{1,8}(-\lambda_0)],$$

$$Y[1,5] = \frac{\nu s}{16\lambda_0^3}[-2s\lambda_0^2(1-\lambda_0^2)(1+8\bar{N}_2) - \bar{N}_3 y_{1,9}(\lambda_0) + \nu\bar{N}_4 y_{1,9}(-\lambda_0)],$$

$$Y[1,6] = -\frac{\nu}{16\lambda_0^3}[s^2\lambda_0(1-\lambda_0^2) + 16\bar{N}_1(1-\lambda_0^2)(1+2s^2\lambda_0^2) + \bar{N}_3 y_{1,10}(\lambda_0) + \nu\bar{N}_4 y_{1,10}(-\lambda_0)],$$

$$Y[1,7] = -\frac{(1+\lambda_0)}{32\lambda_0^3}[y_{1,11}(\lambda_0) - 2\{\nu\bar{N}_4 y_{1,12}(\lambda_0) - 24\bar{N}_1 + \bar{N}_2 y_{1,13}(\lambda_0)\}],$$

$$Y[1,8] = -\frac{(1-\lambda_0)}{32\lambda_0^3}[y_{1,11}(-\lambda_0) - 2\{-\bar{N}_3 y_{1,12}(-\lambda_0) + 24\bar{N}_1 + \bar{N}_2 y_{1,13}(-\lambda_0)\}],$$

$$Y[1,9] = -\frac{\nu(1+\lambda_0)}{4\lambda_0^3}\{3\nu\bar{N}_4 - \bar{N}_2 s(1-\lambda_0)\},$$

$$Y[1,10] = \frac{(1-\lambda_0)}{4\lambda_0^3}\{3\bar{N}_3 + \bar{N}_2 s(1+\lambda_0)\},$$

$$Y[1,11] = 0,$$

$$Y[1,12] = \frac{\bar{N}_1 s^2}{2\lambda_0}(1-\lambda_0^2).$$

$$Y[2,0] = -\frac{3s}{16\lambda_0^3}(1-\lambda_0^2)^2\{-\bar{N}_3(1-\lambda_0)+\nu\bar{N}_4(1+\lambda_0)\}$$
$$+\frac{(1+\lambda_0^2)}{32\lambda_0^3}\{16(1+s^4\lambda_0^4)+s^2(3\lambda_0^4+58\lambda_0^2+3)\},$$

$$Y[2,1] = -\frac{1}{16\lambda_0^3}[y_{2,1}(\lambda_0)+\bar{N}_1 y_{2,2}(\lambda_0)+\bar{N}_2 y_{2,3}(\lambda_0)+\bar{N}_3 y_{2,4}(\lambda_0)],$$

$$Y[2,2] = -\frac{\nu}{16\lambda_0^3}[y_{2,1}(-\lambda_0)-\bar{N}_1 y_{2,2}(-\lambda_0)+\bar{N}_2 y_{2,3}(-\lambda_0)-\nu\bar{N}_4 y_{2,4}(-\lambda_0)],$$

$$Y[2,3] = \frac{\nu s}{8\lambda_0^2}[y_{2,5}(\lambda_0)+\bar{N}_1 y_{2,6}(\lambda_0)-\nu\bar{N}_4 y_{2,7}(\lambda_0)],$$

$$Y[2,4] = -\frac{s}{8\lambda_0^2}[y_{2,5}(-\lambda_0)-\bar{N}_1 y_{2,6}(-\lambda_0)+\bar{N}_3 y_{2,7}(-\lambda_0)],$$

$$Y[2,5] = \frac{\nu s}{32\lambda_0^3}[s(1-\lambda_0^2)(3+64\lambda_0^2-3\lambda_0^4)-6\{\bar{N}_3 y_{2,8}(\lambda_0)-\nu\bar{N}_4 y_{2,8}(-\lambda_0)\}],$$

$$Y[2,6] = \frac{\nu s}{32\lambda_0^3}[s(1-\lambda_0^2)(3+32\lambda_0^2-3\lambda_0^4+32s^2\lambda_0^4)+6\{\bar{N}_3 y_{2,8}(\lambda_0)-\nu\bar{N}_4 y_{2,8}(-\lambda_0)\}],$$

$$Y[2,7] = \frac{1}{64\lambda_0^3}[y_{2,9}(\lambda_0)+192\bar{N}_1(1-\lambda_0^2)+\bar{N}_2 y_{2,10}(\lambda_0)+\nu\bar{N}_4 y_{2,11}(\lambda_0)],$$

$$Y[2,8] = \frac{1}{64\lambda_0^3}[y_{2,9}(-\lambda_0)-192\bar{N}_1(1-\lambda_0^2)+\bar{N}_2 y_{2,10}(-\lambda_0)-\bar{N}_4 y_{2,11}(-\lambda_0)],$$

$$Y[2,9] = -\frac{\nu}{4\lambda_0^3}[y_{2,12}(\lambda_0)+\nu\bar{N}_4 y_{2,13}(\lambda_0)+\bar{N}_2 y_{2,14}(\lambda_0)],$$

$$Y[2,10] = -\frac{1}{4\lambda_0^3}[y_{2,12}(-\lambda_0)-\bar{N}_3 y_{2,13}(-\lambda_0)+\bar{N}_2 y_{2,14}(-\lambda_0)],$$

$$Y[2,11] = \frac{1}{2\lambda_0},$$

$$Y[2,12] = -\frac{s^4}{2}\lambda_0(1-2\lambda_0^2).$$

Here the 27 notations $y_{i,j}(\zeta)$ used in coefficient functions $Y[i,j]$ are defined as;

$$y_{1,1}(\zeta) = -s\zeta(1-\zeta^2)\{4+s^2(1-2\zeta+5\zeta^2)\},$$

$$y_{1,2}(\zeta) = 8s\zeta(1+\zeta)(10+\zeta+\zeta^2),$$

$$y_{1,3}(\zeta) = 24s(1+\zeta)(1+3\zeta),$$

$$y_{1,4}(\zeta) = 8\{s^2\zeta(1+5\zeta+2\zeta^2+4\zeta^3)-3(3+\zeta)\},$$

$$y_{1,5}(\zeta) = -8(1+\zeta)\{(3-\zeta)+s^2\zeta^2(7-5\zeta)\},$$

$$y_{1,6}(\zeta) = s^2(1-\zeta)(1+\zeta)^3,$$

$$y_{1,7}(\zeta) = -8(1+\zeta)(1-\zeta)(6+\zeta),$$

$$y_{1,8}(\zeta) = 8s(1+\zeta)(1+2\zeta),$$

$$y_{1,9}(\zeta) = -(1-\zeta)(3+23\zeta-3\zeta^2+\zeta^3),$$

$$y_{1,10}(\zeta) = s(1-\zeta)(5-31\zeta-13\zeta^2-\zeta^3),$$

$$y_{1,11}(\zeta) = -s^2\zeta(1-\zeta)(1+\zeta)^2,$$

$$y_{1,12}(\zeta) = -s(3+13\zeta-7\zeta^2-\zeta^3),$$

$$y_{1,13}(\zeta) = 8s^2\zeta(1+\zeta+\zeta^2),$$

$$y_{2,1}(\zeta) = s(1+\zeta)\{(7+5\zeta+17\zeta^2+3\zeta^3)+s^2\zeta^2(15-19\zeta+41\zeta^2-5\zeta^3)\},$$

$$y_{2,2}(\zeta) = 12s\zeta(1-\zeta)(1+\zeta)(2+3\zeta-\zeta^2),$$

$$y_{2,3}(\zeta) = -12s(1-\zeta)(1+\zeta)(1+3\zeta),$$

$$y_{2,4}(\zeta) = -6(1-\zeta)\{2(3+\zeta)-s^2\zeta(1+3\zeta)(1+\zeta^2)\},$$

$$y_{2,5}(\zeta) = 2s^2\zeta(1-\zeta)(1-3\zeta-5\zeta^2+3\zeta^3),$$

$$y_{2,6}(\zeta) = -6(1+\zeta)^2(2+3\zeta-\zeta^2),$$

$$y_{2,7}(\zeta) = -3s(1+\zeta)(1-3\zeta)(1+\zeta^2),$$

$$y_{2,8}(\zeta) = (1-\zeta)(1+6\zeta-16\zeta^2-6\zeta^3-\zeta^4),$$

$$y_{2,9}(\zeta) = s^2(3 + 34\zeta - 19\zeta^2 - 4\zeta^3 + 173\zeta^4 - 62\zeta^5 + 3\zeta^6),$$

$$y_{2,10}(\zeta) = -96s^2\zeta(1 - \zeta^2)(1 + \zeta^2),$$

$$y_{2,11}(\zeta) = 12s(1 - \zeta^2)^2(1 + \zeta),$$

$$y_{2,12}(\zeta) = 2s\zeta(1 - \zeta)(1 + 2\zeta - \zeta^2),$$

$$y_{2,13}(\zeta) = -3(1 + \zeta)(3 - \zeta),$$

$$y_{2,14}(\zeta) = 3s(1 + \zeta)(1 - \zeta)(1-3\zeta).$$

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