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THE APPLICATION OF THE MODULAR MODELING METHOD TO FINITE ELEMENTS ANALYSIS

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Clark Radcliffe
Major professor

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THE APPLICATION OF MODULAR MODELING METHODS IN FINITE ELEMENT ANALYSIS

By

Xin Li

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ABSTRACT

The Application of Modular Modeling Methods in Finite Element Analysis

By

Xin Li

Structural modeling analysis is important to engineering design and development. An important issue is the interconnectivity problem for multi degree of freedom subsystems. The traditional finite element method requires re-meshing subsystems for mesh compatibility at interconnections. As the number of subsystems increase, the computational burden of re-meshing increases rapidly, sometimes exponentially.

The Modular Modeling method is applied here to develop a new finite element analysis approach. Modular Modeling through the use of input / output ports on the boundaries of interconnected parts is introduced for this purpose. Using input / output ports and the Modular Modeling method to connect parts in system design does not require expensive remodeling and re-meshing. This approach uses the original FEA model and combines parts with the input / output ports among their boundaries. As the number of available alternate parts in a structural system increases, the modular FEA method minimizes the time and cost for the system integration.

A two-dimensional example of FEA elastic strain-stress problems is used to illustrate this technique. This example shows it is not necessary to re-mesh components to assemble a structural system. Computation error is evaluated to show that this modular finite element method produces accurate system models for engineering design.

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INTRODUCTION

During the design of complex systems, reliable predictions of detailed stress states for coupled components are often critical to correctly predict the performance of those systems. The finite element analysis (FEA) yields accurate, detailed stress predictions for individual system components with the resolution of mesh distribution. When more precision is needed, a higher precision mesh needs to be generated. In engineering practice, system components can be so complex that the finite element method is mandatory for the design phase. It is computationally intensive and often requires extensive re-modeling effort.

Coupling system components into assemblies is a time intensive task frequently requiring reformulation for finite element analysis models. For example, the Chrysler large car models used for mechanical geometry studies contain representations of over 5500 interconnected physical subsystems [Computers in Engineering: "Chrysler designs paperless cars", 1998]. To reduce the length and cost of the design process for a complex system, efficient modeling techniques that guarantee reliable predictions of detailed stress states are needed. These techniques must be not only computationally efficient, but also dramatically reduce or eliminate the necessity of re-modeling. According to [Computers in Engineering: "Chrysler designs paperless cars", 1998], Chrysler engineers solved design issues by developing a computer model instead of physical prototypes, reducing the cycle time from 39 to 31 months and saving the company more than $75 million.

Finite element analysis of system of components typically requires compatible node geometry at each component connection. When constructing the two-dimensional FEA models, it is a mandatory requirement to keep mesh nodally compatible [Zienkiewicz and Taylor, 1989]. At present, the methods available to ensure FEA nodal compatibility include the global /local analysis method, the coupling analysis method, and the interface element method, [M. Aminpour, 1995; V. S. Kothnur, 1999; J. B. Ransom, 1993]. These methods have drawbacks that prevent them from being widely accepted as standard design tools, [Z. Zhao, 1998]. The common limitation of these methods is that they can only be applied to mesh discretizations with a one-to-one nodal correspondence across boundaries between structural subsystems. Methods that do not require nodal compatibility on the component boundaries have much more modeling flexibility, and eliminate the need for re-meshing.

A new finite element analysis approach is introduced in this paper which employs the fixed input / output structure "modular modeling method" [B. Byam and C. Radcliffe, 1999]. In Modular Modeling, the mathematical model that describes each component remains the same independent of the system model in which it is used. Fixed input / output structure means the input and output ports are standardized, therefore the internal equations of mathematical subsystem models of engineering systems have the same modularity as the engineering system. Fixed input / output structure modular modeling is a power-based, physically intuitive, top-down methodology systematic modeling methodology that eliminates equation reformulation from large model development across multiple energy domain [B. Byam and C. Radcliffe, 1999]

This paper demonstrates the modular modeling method in FEA. The work presented here developed an analytical framework with the Modular Modeling analysis method, a solution strategy, and a demonstration of this solution on a FEA example. Finally, the accuracy and convergence of a Modular FEA Model will be evaluated by comparison with various conventional FEA solutions to a 2-D plate problem (Fig, 1.1).

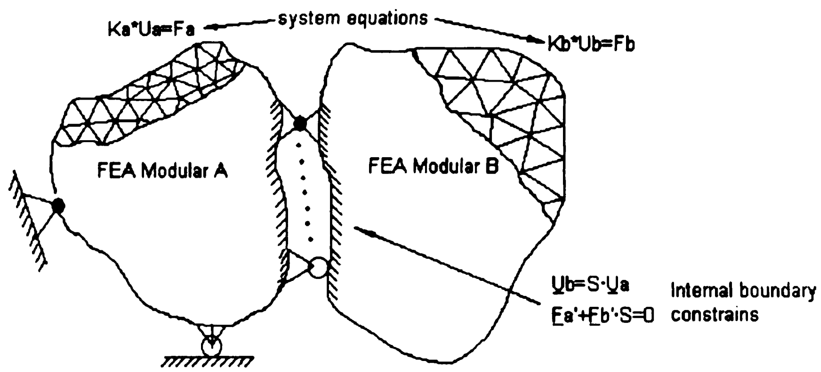


Figure 1.1. The modular modeling using input / output Ports (joint)

MULTI-COMPONENT MODULAR MODELING FOR FINITE ELEMENT ANALYSIS

2.1. FEA Modular Modeling and the related DOF analysis

Modular modeling is a new modeling method designed to eliminate the model reformulation and enhance model performance verification. It is defined by a standard input-output structure, in which connection model physical interaction between components via power transfer. Power Transfer between components requires the product of two signed variables such as force times velocity, pressure times flow rate, and voltage times current. The two physical variables form a component input-output pair. For each type of physical interaction there is a standard input-output definition for modular modeling components. [B. Byam and C. Radcliffe, 1999]

Modular modeling can be applied to FEA model components. In the discussion below, a basic 2-D elastic stress-strain FEA problem will demonstrate the basic approach for connecting different FEA component models together, even when their nodes are geometrically incompatible. The following figure shows the basic concept for modular modeling in FEA. In this approach, a single, multi-node input/output port is defined to connect two FEA models. The power, force and displacement transfers through this connection port have standard definitions [B. Byam and C. Radcliffe, 1999].

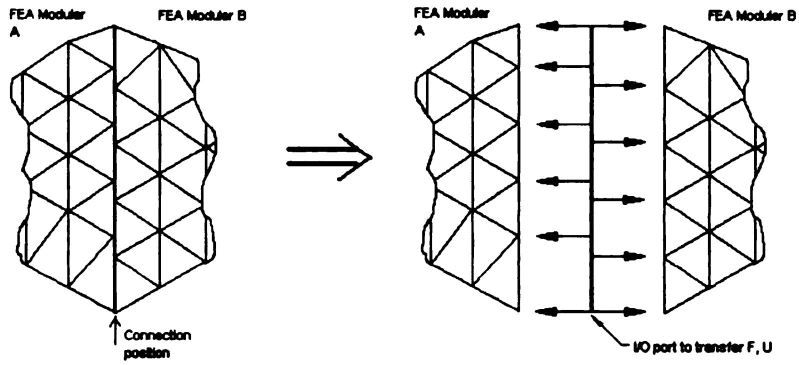


Figure 2.1. The modular modeling using input / output Ports (connectors)

The analysis of system degrees of freedom (DOF) is the foundation of all other analysis. We can not begin any static or dynamic analysis before we decide the system DOF. In FEA modular modeling, the connection of two independent FEA components produces a modular model with a unique solution if it is the zero DOF. In another words, modular connectors provide enough constraints for whole system to generate a non-singular system equation. For example, if we combine two independent FEA components, each with 6, unconstrained DOF, we need to find at least 2x6=12 constraints for displacement to generate a system model with stable, unique solutions for different force inputs. These constraints should provide both external boundary conditions and internal boundary conditions. In Figure 1.1, a constraint set would fix one component to the ground (external constraint) and the other component is fixed on the first one (internal constraint). This combination of constraints will remove all rigid body degrees of freedom and generate system equations with a unique solution.

Supplemental boundary constraint equations are critical to control the FEA modular model DOF. In the first step of the proposed analysis, we combine system equations and their degrees of freedom. For example, in 2-D stress-strain analysis, the three degree of freedom unconstrained components have a singular stiffness matrix, whose rank is N–3 where N is the dimension of the stiffness matrix. When we combine two three degree of freedom components together, the combined stiffness matrix has a rank of N+M–6, where N and M are the dimensions of the two component stiffness matrices. External boundary conditions combined with connection constraints must increase the combined system rank to N+M to generate a model with unique solutions to external force inputs. This idea can be easily understood if we consider this with a physical model. In two-dimensional space, a 3-DOF subject can be held fixed if 3 boundary conditions are provided. When we connect two 3-DOF subjects together, an additional 3 boundary conditions prevent relative motion between the two components. When the 3 boundary conditions on one component are combined with 3 internal constraints, the resulting stiffness matrix is non-singular and yields unique solutions to input forcing.

2.2. Visual Component Boundary Port — FEA Input/Output Connector

The modular modeling method generates non-singular equations by using the original component system equations and the modular input/output equation. Actually, this is one of the advantages of the modular modeling method. With this method, a great deal of effort can be saved because no remodeling and re-verification is needed [B. Byam and C. Radcliffe, 1999]. When applying this idea into FEA, the benefit is clear: an FEA model can be used directly in system re-mesh and re-verification of component models are not necessary.

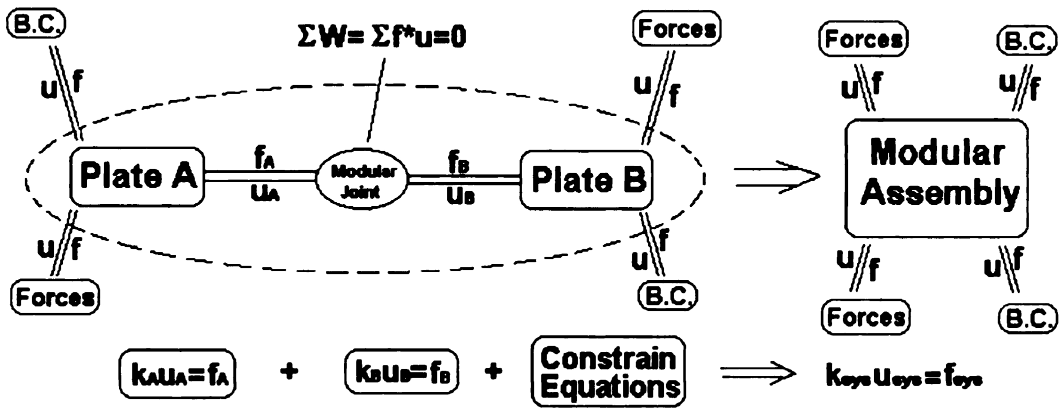


Figure 2.2. The modular modeling method for the FEA application

Definition of an I/O port on each component connection will provide proper constraint equations to make the combined system equation non-singular. The component system equations will be applied directly to generate the combined system equations. The Modular Modeling Method (Fig. 2.2) allows the development of standard integration and computer automation methods. Development of I/O port constraint equations begins with the basic properties of FEA model. Force and displacement transferred across a component connection port must satisfy basic energy and work conservation requirements [Yang, T.Y. 1986,]. The total work transferred through the I/O port must sum to zero.

$$\delta W_A + \delta W_B = O \quad \Rightarrow \quad \delta W_A = -\delta W_B \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2.1)$$

in which :

$$\delta W_A = \Sigma f_{aN} \cdot u_{aN} = f_{2a1-1} \cdot u_{2a1-1} + f_{2a1} \cdot u_{2a1} + f_{2a2-1} \cdot u_{2a2-1} + f_{2a2} \cdot u_{2a2} + \cdots + f_{aN} \cdot u_{aN}$$

$$\delta W_B = \Sigma f_{BM} \cdot u_{BM} = f_{2b1-1} \cdot u_{2b1-1} + f_{2b1} \cdot u_{2b1} + f_{2b2-1} \cdot u_{2b2-1} + f_{2b2} \cdot u_{2b2} + \cdots + f_{Bm} \cdot u_{Bm}$$

In the upper equations, the indices a1, a2,... aN, and b1,b2,... bN refer to the nodal number along the connection forming the I/O port.

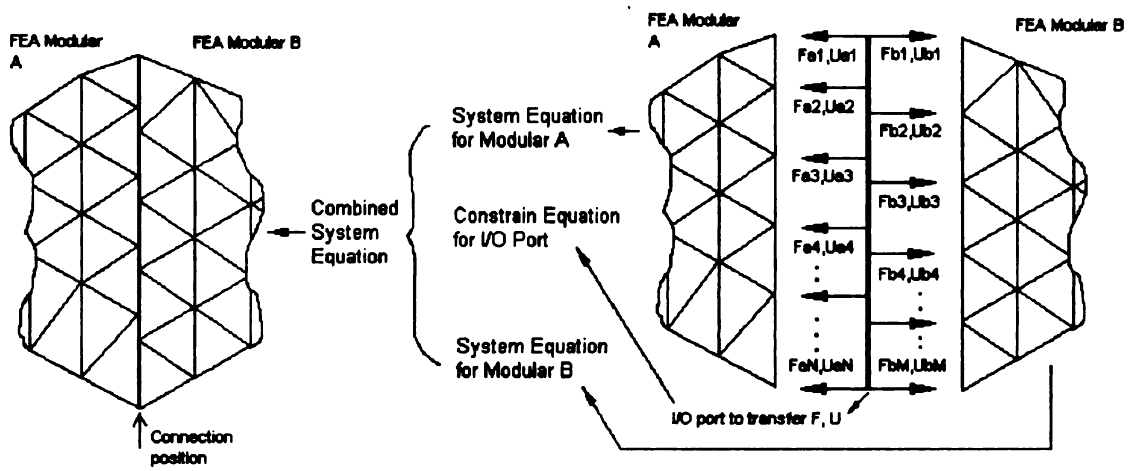


Figure 2.3. The modular modeling input / output Ports along the FEA boundary

Theoretical solution is provided here for a two-dimensional, straight-line boundary and three degree of freedom elastic stress-strain analyses. Future work will be needed for three-dimensional, curve-surface boundary, different type of elements and six degree of freedom finite element analysis.

2.3. FEA Modular Modeling Procedures and Formulations

New formulations for the modular modeling need to be developed for the two-dimensional FEA models. The procedures and formulations for one-dimensional FEA modular modeling have been demonstrated [B. Byam and C. Radcliffe, 1999]. The two dimensional problems are different and need updated procedures and formulations. Two-dimensional FEA modular modeling does not require mesh nodal compatibility along the connection boundary. To make FEA modular modeling, we must develop new procedures and formulations for making the FEA modular model work under the situation of mesh nodal non-compatibility along the connection boundary. Methods to connect modular FEA models without compatible nodal point geometry are given below.

In this paper, four steps develop a two-dimensional FEA modular modeling formulation, which is demonstrated in section §2.3.1, §2.3.2, §2.3.3 and §2.3.4. The combined system equation of FEA models is provided by step one. Step two applies the boundary displacement constraint equations to transform the system equations. Step three puts the system equations into standard linear algebraic equation form by moving the unknown force into the unknown variable vector. In step four, the boundary force constraint equation will supplement the system equation, and make the system stiffness matrix square. The system model equations are now complete, non-singular and have a unique solution given a set of independent inputs.

2.3.1. Sub-system Equation Combination

The first step is to get the combined system equation. Consider 2 FEA components: A and B. A has N nodes and B has M nodes and their stress-strain equation is following. To avoid confusion, The displacement and force on subject A and B are represented by u_A, u_B and f_A, f_B. They have same definitions and units.

$$\mathbf{K}_A \cdot \mathbf{U}_A = \mathbf{F}_A \qquad \textit{or}$$

$$\begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \cdots & \mathbf{k}_{1N} \\ \mathbf{k}_{21} & \mathbf{k}_{22} & \cdots & \mathbf{k}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{k}_{N1} & \mathbf{k}_{N2} & \cdots & \mathbf{k}_{NN} \end{bmatrix}_A \cdot \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{bmatrix}_A = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_N \end{bmatrix}_A \quad \text{..............................(2.2)}$$

$$\mathbf{K}_B \cdot \mathbf{U}_B = \mathbf{F}_B \qquad \textit{or}$$

$$\begin{bmatrix} \mathbf{k}_{11} & \mathbf{k}_{12} & \cdots & \mathbf{k}_{1M} \\ \mathbf{k}_{21} & \mathbf{k}_{22} & \cdots & \mathbf{k}_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{k}_{M1} & \mathbf{k}_{M2} & \cdots & \mathbf{k}_{MM} \end{bmatrix}_B \cdot \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_M \end{bmatrix}_B = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_M \end{bmatrix}_B \quad \text{..............................(2.3)}$$

In both equations, K_A and K_B, are singular. For two dimentional problems the subsystem models allow "free body" motion with two orthogonal deflection and one rotation. So the ranks of K_A and K_B are N−3 and M−3. For every individual equation could be solved if boundary condition such as following provided

$$\mathbf{u}^{\bullet}_A = \begin{pmatrix} \mathbf{u}_a \\ \mathbf{u}_b \\ \mathbf{u}_c \end{pmatrix}_A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \quad \text{and} \quad \mathbf{u}^{\bullet}_B = \begin{pmatrix} \mathbf{v}_a \\ \mathbf{v}_b \\ \mathbf{v}_c \end{pmatrix}_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{..........................(2.4)}$$

Any 3 U value equal to zero could eliminate the rank of K by 3, and make it non-singular. Now we combine both components together without any re-mesh work. From this equation, we can see that it contains all of the old elements and nodes.

$$\mathbf{K}_{sys} \cdot \mathbf{u}_{sys} = \mathbf{f}_{sys} \quad \cdots\cdots (2.5)$$

in which

$$\mathbf{K}_{sys} = \begin{pmatrix} \mathbf{K}_A & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{K}_B \end{pmatrix}; \quad (\mathbf{u}_{sys}) = \begin{pmatrix} \mathbf{u}_A \\ \cdots \\ \mathbf{u}_B \end{pmatrix}; \quad (\mathbf{f}_{sys}) = \begin{pmatrix} \mathbf{f}_A \\ \cdots \\ \mathbf{f}_B \end{pmatrix}$$

Obviously, the rank of $\mathbf{K}_{sys}$ could be available from following equation:

$$\mathrm{Rank}(K_{sys}) = \mathrm{Rank}(K_A) + \mathrm{Rank}(K_B) = (N-3)+(M-3) = N+M-6 \ldots\ldots (2.6)$$

From equation 2.6 we know that the combined system needs at least 6 internal and external boundary condition to become solvable.

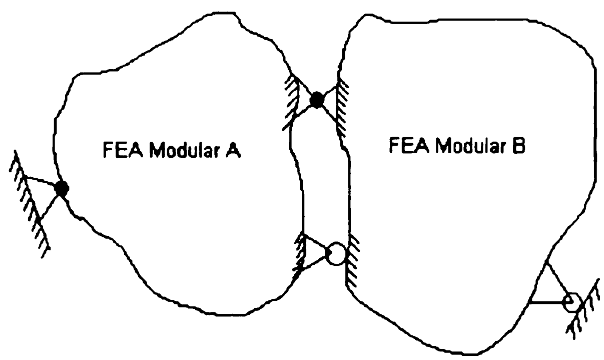


Figure 2.4. At least 6 constraint are needed for 2-D FEA modular modeling

2.3.2. Applying Boundary Displacements Constraints

The boundary displacement constraint needs to be defined at first, like most regular FEA procedures. This procedure will make the system equation (2.5) a full-rank linear equation. Therefore, this is the crucial step to make the system solvable. According to the previous analysis, the internal (boundary between 2 components) displacement constraint equation must be more than rank of 3. Assume the following equation:

$$(\mathbf{u}_A)_{n\times n} = \mathbf{S}_{m\times n} \cdot (\mathbf{u}_B)_{m\times m} \quad \cdots\cdots\cdots\cdots \quad (2.7)$$

is the boundary displacements constraint equation, we need to make sure that:

$$Rank(\mathbf{S}_{m\times n}) \geq 3 \quad \cdots\cdots\cdots\cdots \quad (2.8)$$

The simulation result with the FEA model is a linear approximation for real situation [Zienkiewicz, O. C. and Taylor, R. L., 1989]. Suppose we are using linear elements, the boundary displacements constraint analysis should be based on every nodal value on both components boundaries linear approximation. The finite element method provides a displacement value within every element only according to the nodal value in itself. Therefore the boundary constraint equation could be found by the local boundary elements interpolation. Figure 2.4 demonstrates the mean of the local boundary elements nodal displacement relationship.

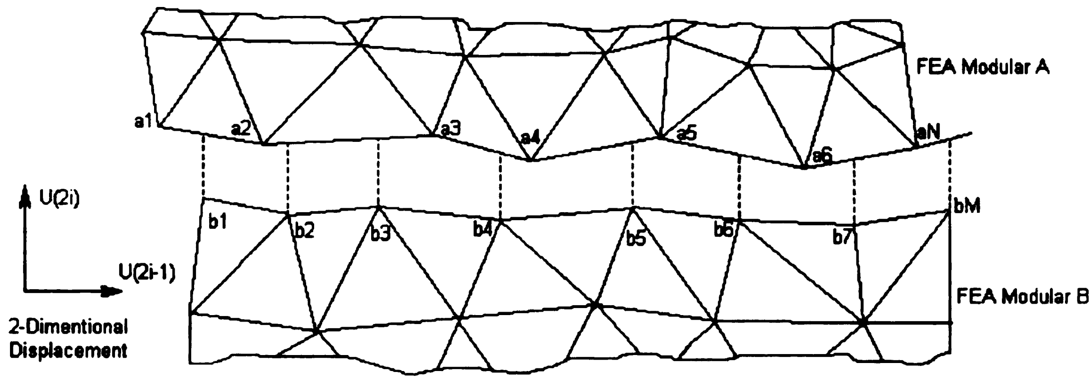


Figure 2.5. Nodal displacements along the boundary elements

The boundary displacement in Figure 2.4 at node "i" was defined as "u2i-1" and "u2i". If the boundary nodes on modular A and B is a1, a2,...aN and b1, b2,...bM, then the boundary displacement constraint equation could be written as.

$$(\mathbf{u}_{b1} \quad \mathbf{u}_{b2} \quad \cdots \quad \mathbf{u}_{bM})_A^T = \mathbf{S}\cdot(\mathbf{u}_{a1} \quad \mathbf{u}_{a2} \quad \cdots \quad \mathbf{u}_{aM})_B^T \cdots\cdots\cdots\cdots\cdots\cdots (2.9)$$

specially, the 2 - D equation situation :

$$(u_{2b1-1} \quad u_{2b1} \quad u_{2b2-1} \quad u_{2b2} \quad \cdots \quad u_{2bM-1} \quad u_{2bM})_A^T =$$
$$\mathbf{S}_{2M\times 2N}\cdot(u_{2a1-1} \quad u_{2a1} \quad u_{2a2-1} \quad u_{2a2} \quad \cdots \quad u_{2aN-1} \quad u_{2aN})$$

In most cases the rank of equation (2.9) will be more than 3, otherwise at least one component has only one node on the boundary. So, if we can develop boundary displacement constraint relationships, equation (2.9) will have enough constraints as required. For both subjects (Fig. 2.5), the nodal displacement value along the boundary should satisfy the boundary interpolation [Langhe, K.De, 1995].

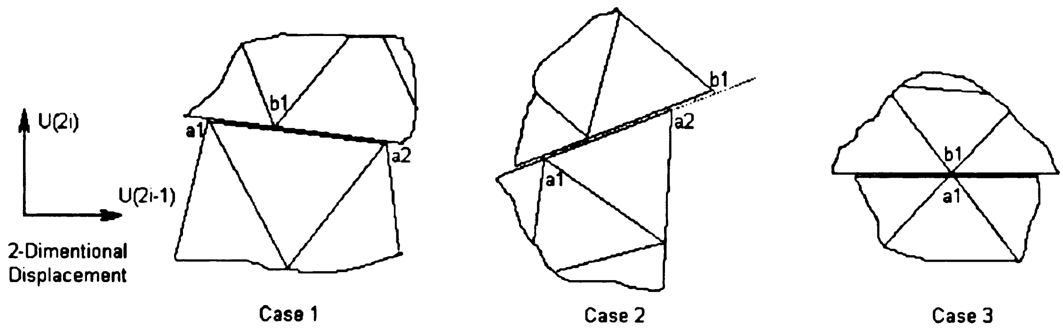


Figure 2.6. Nodal compatible situation along the boundary elements

Figure 2.5 shows the position of node "b1" has only 3 possibilities relative to the corresponding elements. Case 1 is located on the edge of corresponding boundary element. Case 2 is located on the straight extended line of the edge of corresponding boundary element. Case 3 is coincided with one node of the corresponding boundary element.

Independent of which component has more boundary nodes than another, both sides of the port can be used to develop boundary displacement constraint equations. In other words, equation 2.9 has 2 equivalent forms and they are equivalent constraints. Here we present results for nodes on module B along the boundary.

CASE 1. From the definition of finite element, any internal point value can be represented by the linear transformation of nodal values [Segerlind, L. J., 1984]. For the boundary nodes in another component, we also use similar linear interpolation approach to figure out their values [Langhe, K.De., 1995 and Kothnur, V.S. , Yu Xie, 1999]. In this paper, the discussion was focused in the basic theoretical solution for two-dimensional, straight-line boundary and three degree of freedom elastic stress-strain problem. So we can apply the similar approach from reference [Langhe, K.De., 1995].

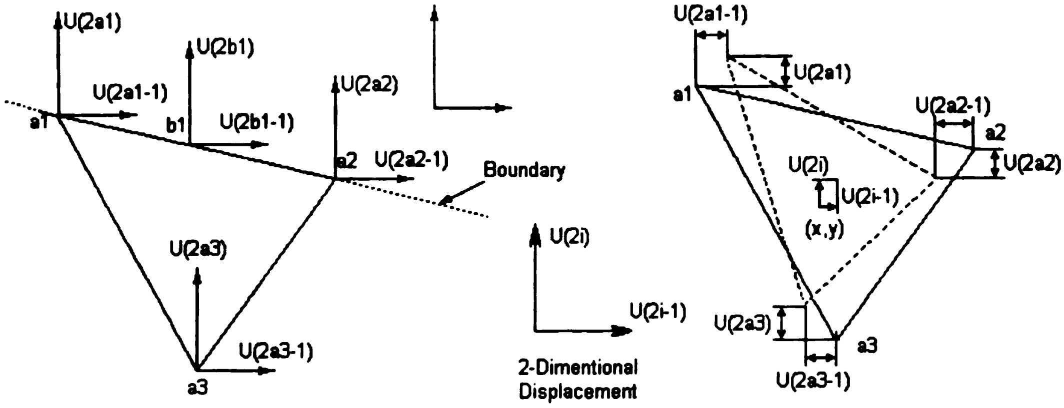


Figure 2.7. Nodal displacements in the boundary elements

As it shows in figure 2.6, we can get the shape function of an arbitrary point in the element which has the coordinates of (x, y) [Yang, T.Y. 1986]:

$$
\begin{aligned}
u_{2i}(x,y) &= u_{2a1} + c_1 x + c_2 y \\
u_{2i-1}(x,y) &= u_{2a1-1} + c_3 x + c_4 y
\end{aligned}
\quad \cdots\cdots\cdots\cdots\cdots \quad (2.10)
$$

The c1, c2, c3, c4 is determined by the shape of element and can be derived from the following equation:

$$
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{a_j b_k - a_k b_j} \begin{bmatrix} b_j - b_k & 0 & b_k & 0 & -b_j & 0 \\ a_k - a_j & 0 & -a_k & 0 & a_j & 0 \\ 0 & b_j - b_k & 0 & b_k & 0 & -b_j \\ 0 & a_k - a_j & 0 & -a_k & 0 & a_j \end{bmatrix} \cdot \begin{pmatrix} u_{2a1-1} \\ u_{2a1} \\ u_{2a2-1} \\ u_{2a2} \\ u_{2a3-1} \\ u_{2a3} \end{pmatrix} \cdots\cdots \quad (2.11)
$$

From equation (2.10) and (2.11), we get a general interpolation form for any point located in the boundary element. Because the values of c1, c2, c3, c4 are determined linearly only by properties of the corresponding boundary element, we could conclude that the u2b1-1 and u2b1 in (2.10) are linear functions of u2a1, u2a1-1, u2a2, and u2a2-1, u2a3, u2a3-1 correspondingly.

The transformation matrix S1, S2 in the upper equation is decided by the corresponding boundary element and the x, y coordinate value of node "b1". After we figure out all of the equations (2.12) for all boundary nodes on component B, we combine them and get the equation (2.9).

$$u_{2b1-1} = \mathbf{S}_1 \cdot \left(u_{2a1-1} \quad u_{2a2-1} \quad u_{2a3-1}\right)^T$$
$$u_{2b1} = \mathbf{S}_2 \cdot \left(u_{2a1} \quad u_{2a2} \quad u_{2a3}\right)^T \quad \cdots\cdots\cdots\cdots \quad (2.12)$$

In this paper, we are discussing the straight-line boundary problem. So all boundary node "b1", "b2", ... "bM" is locate on the edge of boundary elements. In this situation, the equation (2.12) could be simplified and decided only by nodal value on boundary edge [Segerlind, L. J., 1984].

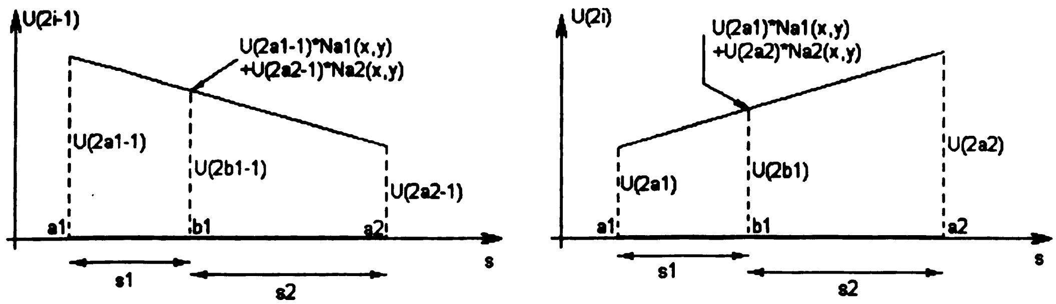


Figure 2.8. Linear transformation on the boundary elements (case 1: interpolation)

Figure 2.7 shows the linear interpolation of node "b1". Suppose the distance from node "b1" and node "a1" is s1, and from node "b1" and node "a1" is s2, which could be derived from the their coordinate values, we can get the linear interpolation equation from the shape function [Langhe, K.De., 1995]:

$$u_{2b1-1} = N_{a1}(s) \cdot u_{2a1-1} + N_{a2}(s) \cdot u_{2a2-1} = \frac{s_2}{s_1+s_2} \cdot u_{2a1-1} + \frac{s_1}{s_1+s_2} \cdot u_{2a2-1}$$

$$u_{2b1} = N_{a1}(s) \cdot u_{2a1} + N_{a2}(s) \cdot u_{2a2} = \frac{s_2}{s_1+s_2} \cdot u_{2a1} + \frac{s_1}{s_1+s_2} \cdot u_{2a2} \quad \cdots\cdots\cdots \quad (2.13)$$

The distance from node “b1” and node “a1”, which is s1, and from node “b1” and node “a1”, which is s2, could be derived from following coordinate computation.

$$s_1 = \sqrt{(x_{b1} - x_{a1})^2 + (y_{b1} - y_{a1})^2}$$
$$s_2 = \sqrt{(x_{b1} - x_{a2})^2 + (y_{b1} - y_{a2})^2} \quad \cdots\cdots\cdots\cdots \quad (2.14)$$

It is necessary to mention that the s1 and s2 values are always positive in case 1.

CASE 2. In this case, node “b1” is located on the straight extended line of the edge of corresponded boundary element. This happens because node “b1” is the terminal point on the boundary, and it does not contact with the corresponded component. In the following figure (2.8), we can see that equation (2.13) is still valid. The only thing that needs to be modified is that we should to re-define s1 and s2 according to the situation of “b1” location. If the “b1” location comes with the sequence of “a1, a2, b1”, the s1 and s2 should be:

$$s_1 = \sqrt{(x_{b1} - x_{a1})^2 + (y_{b1} - y_{a1})^2}$$
$$s_2 = -\sqrt{(x_{b1} - x_{a2})^2 + (y_{b1} - y_{a2})^2} \quad \cdots\cdots\cdots\cdots \quad (2.15)$$

If the “b1” location comes with the sequence of “b1, a1, a2”, the s1 and s2 should be:

$$s_1 = -\sqrt{(x_{b1} - x_{a1})^2 + (y_{b1} - y_{a1})^2}$$
$$s_2 = \sqrt{(x_{b1} - x_{a2})^2 + (y_{b1} - y_{a2})^2} \quad \cdots\cdots\cdots\cdots \quad (2.16)$$

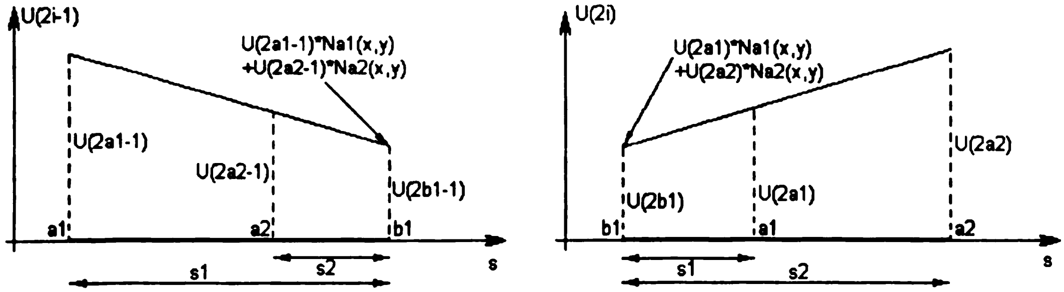


Figure 2.9. Linear transformation on the boundary elements (case 2: extrapolation)

CASE 3. In this case, node "b1" is located on the exact same nodal point of corresponded element. Clearly, this is a "nodal compatibility" situation. It is easy to conclude that if "b1" is located on same nodal point of "a1"

$$u_{2b1-1} = u_{2a1-1} = 1 \cdot u_{2a1-1} + 0 \cdot u_{2a2-1} = \frac{s_2}{s_1+s_2} \cdot u_{2a1-1} + \frac{s_1}{s_1+s_2} \cdot u_{2a2-1}$$

$$u_{2b1} = u_{2a1} = 1 \cdot u_{2a1} + 0 \cdot u_{2a2} = \frac{s_2}{s_1+s_2} \cdot u_{2a1} + \frac{s_1}{s_1+s_2} \cdot u_{2a2}$$

$$s_1 = 0, \qquad \cdots\cdots \qquad (2.17)$$

If "b1" is located on same nodal point of "a2"

$$u_{2b1-1} = u_{2a2-1} = 0 \cdot u_{2a1-1} + 1 \cdot u_{2a2-1} = \frac{s_2}{s_1+s_2} \cdot u_{2a1} + \frac{s_1}{s_1+s_2} \cdot u_{2a2}$$

$$u_{2b1} = u_{2a2} = 0 \cdot u_{2a1} + 1 \cdot u_{2a2} = \frac{s_2}{s_1+s_2} \cdot u_{2a1} + \frac{s_1}{s_1+s_2} \cdot u_{2a2}$$

$$s_2 = 0, \qquad \cdots\cdots \qquad (2.18)$$

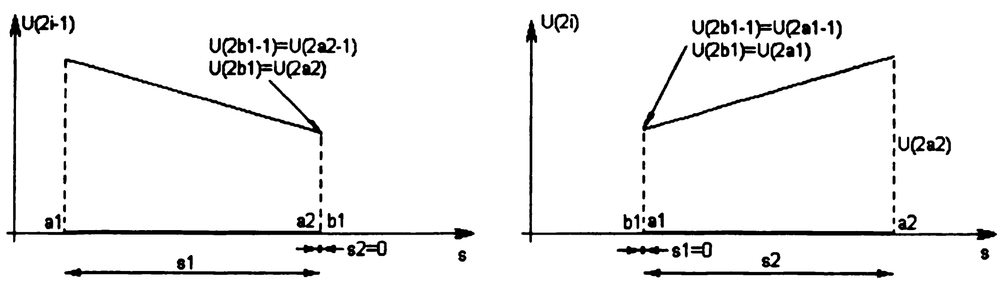


Figure 2.10. Linear transformation on the boundary elements (case 3: equality)

Therefore, equation (2.13) is valid too for this case. When the condition of "s1=0 or s1=0" was represented in (2.13), the output gets the simplified result as equation (2.17) or (2.18). Overall, we could conclude that in the 2-dimentional elastic problem, the displacement constraint equation along the straight-line boundary is equation (2.13). Based on different nodal location, the interpolation parameter s1 and s2 should be computed according to equations (2.14), (2.15), (2.16), (2.17) or (2.18).

From these equations, we could get all interpolation functions for "b1, b2,...bM" in figure 2.4. In equation (2.19), sb11 and sb12 represent the s1 and s2 value for "b1", and Lb1 represent s1+s2 value for "b1",

$$
\begin{array}{ll}
u_{2b1-1} = \dfrac{s_2^{b1}}{L_{b1}} \cdot u_{2a1-1} + \dfrac{s_1^{b1}}{L_{b1}} \cdot u_{2a2-1}, & u_{2b1} = \dfrac{s_2^{b1}}{L_{b1}} \cdot u_{2a1} + \dfrac{s_1^{b1}}{L_{b1}} \cdot u_{2a2} \\
u_{2b2-1} = \dfrac{s_2^{b2}}{L_{b2}} \cdot u_{2a2-1} + \dfrac{s_1^{b2}}{L_{b2}} \cdot u_{2a3-1}, & u_{2b2} = \dfrac{s_2^{b2}}{L_{b2}} \cdot u_{2a2} + \dfrac{s_1^{b2}}{L_{b2}} \cdot u_{2a3} \\
\vdots \quad\quad \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \quad\quad \vdots \\
u_{2bM-1} = \dfrac{s_2^{bM}}{L_{bM}} \cdot u_{2a(N-1)-1} + \dfrac{s_1^{bM}}{L_{bM}} \cdot u_{2aN-1}, & u_{2bM} = \dfrac{s_2^{bM}}{L_{bM}} \cdot u_{2a(N-1)} + \dfrac{s_1^{bM}}{L_{bM}} \cdot u_{2aN}
\end{array}
\quad \cdots\cdots \quad (2.19)
$$

If we define displacements by node as following:

$$
\mathbf{u}_{b1} = \begin{pmatrix} u_{2b1-1} \\ u_{2b1} \end{pmatrix}, \quad \mathbf{u}_{b2} = \begin{pmatrix} u_{2b2-1} \\ u_{2b2} \end{pmatrix}, \quad \cdots\cdots \mathbf{u}_{bM} = \begin{pmatrix} u_{2bM-1} \\ u_{2bM} \end{pmatrix};
$$

$$
\mathbf{u}_{a1} = \begin{pmatrix} U_{2a1-1} \\ U_{2a1} \end{pmatrix}, \quad \mathbf{u}_{a2} = \begin{pmatrix} u_{2a2-1} \\ u_{2a2} \end{pmatrix}, \quad \cdots\cdots \mathbf{u}_{aN} = \begin{pmatrix} u_{2aN-1} \\ u_{2aN} \end{pmatrix}
$$

Equation group (2.19) could be gotten if all boundary nodal coordinate values are available. Also, the equations of (2.19) could be re-written as the boundary displacement constraint equation (2.20).

$$\begin{pmatrix} \mathbf{u}_{b1} \\ \mathbf{u}_{b2} \\ \vdots \\ \mathbf{u}_{bM} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{b1*a1} & \mathbf{C}_{b1*a2} & \mathbf{0} & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{b2*a2} & \mathbf{C}_{b2*a3} & \vdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & . & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \vdots & \mathbf{C}_{bM*a(N-1)} & \mathbf{C}_{bM*aN} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u}_{a1} \\ \mathbf{u}_{a2} \\ \mathbf{u}_{a3} \\ \vdots \\ \mathbf{u}_{a(N-1)} \\ \mathbf{u}_{aN} \end{pmatrix} \quad \cdots \quad (2.20)$$

In upper constraint equation, the constraint elements was defined as:

$$\mathbf{C}_{b1*a1} = \begin{pmatrix} \frac{s_2^{b1}}{L_{b1}} & 0 \\ 0 & \frac{s_2^{b1}}{L_{b1}} \end{pmatrix}; \quad \mathbf{C}_{b1*a2} = \begin{pmatrix} \frac{s_1^{b1}}{L_{b1}} & 0 \\ 0 & \frac{s_1^{b1}}{L_{b1}} \end{pmatrix}; \quad \cdots\cdots \mathbf{C}_{bX*aY} = \begin{pmatrix} \frac{s_i^{bX}}{L_{bX}} & 0 \\ 0 & \frac{s_i^{bX}}{L_{bX}} \end{pmatrix};$$

$$i = 1 \quad or \quad 2; \quad decided \quad by \quad aY \quad position$$

Or the equation (2.20) could be write as:

$$\begin{pmatrix} u_{2b1-1} \\ u_{2b1} \\ u_{2b2-1} \\ u_{2b2} \\ \vdots \\ u_{2bM-1} \\ u_{2bM} \end{pmatrix} = \begin{pmatrix} \frac{s_2^{b1}}{L_{b1}} & 0 & \frac{s_1^{b1}}{L_{b1}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{s_2^{b1}}{L_{b1}} & 0 & \frac{s_1^{b1}}{L_{b1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{s_2^{b2}}{L_{b2}} & 0 & \frac{s_1^{b2}}{L_{b2}} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{s_2^{b2}}{L_{b2}} & 0 & \frac{s_1^{b2}}{L_{b2}} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{s_2^{bM}}{L_{bM}} & 0 & \frac{s_1^{bM}}{L_{bM}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{s_2^{bM}}{L_{bM}} & 0 & \frac{s_1^{bM}}{L_{bM}} \end{pmatrix} \cdot \begin{pmatrix} u_{2a1-1} \\ u_{2a1} \\ u_{2a2-1} \\ u_{2a2} \\ u_{2a3-1} \\ u_{2a3} \\ \vdots \\ u_{2a(N-1)-1} \\ u_{2a(N-1)} \\ u_{2aN-1} \\ u_{2aN} \end{pmatrix}$$

2.3.3. Applying Boundary Forces Constraints

Boundary force constraints analysis is required to solve the system equation. From the description of §2.3.1, we know that after boundary displacement constraint equations convert the system equation to full rank, the boundary force constraint equations and initial conditions are also required to solve the system. In fact, the boundary force constraints equation is very important to enable the system equation to get the final solution. The corresponding mathematical description is, if we need to solve $\mathbf{K}'_{m\times n}\cdot\mathbf{u}_{1\times m}=\mathbf{f}_{1\times m};\quad n>m$. After applying the system boundary displacement constraints equation, the equality of the rank of stiffness matrix K' and f ' can only guarantee the solution exist, but it can not guarantee that the solution is unique. It is absolutely necessary to have the force constraint equations for the system equation.

The physical sense of this statement is: both the boundary displacement and force constraint equations are required to ensure that the total power (for dynamic model) or work (for static model) flow through the boundary is conserved. Modular modeling ports provide the power constraint. Power is conserved across modular ports because modular ports are power (or work) transfer mechanisms. The power at the connected modular modeling ports always sum to zero. For the elastic stress-strain problem, we need not make dynamic analysis work. Therefore, the work at the connected modular modeling ports always sum to zero.

The work along the boundary defined as: $\delta\mathbf{W}=\sum_{i=1}^{aN}\mathbf{W}_i=\sum_{i=1}^{aN}(\mathrm{f}_i\cdot\mathrm{u}_i)=\mathbf{f}^{\mathrm{T}}\cdot\mathbf{u}$

Because work across the connection of component A and component B will be

$$\delta\mathbf{W}_A+\delta\mathbf{W}_B=0\quad\Rightarrow\quad\delta\mathbf{W}_A=-\delta\mathbf{W}_B\cdots\cdots\cdots\cdots(2.21)$$

$$\therefore\ (\mathbf{f}^{\mathrm{T}}\cdot\mathbf{u})_{\mathrm{A}}+(\mathbf{f}^{\mathrm{T}}\cdot\mathbf{u})_{\mathrm{B}}=\mathbf{f}_A^T\cdot\mathbf{u}_A+\mathbf{f}_B^A\cdot\mathbf{u}_B=0$$

from Equation (2.20), we get

$$(\mathbf{f}_A^T \cdot \mathbf{u}_A) + (\mathbf{f}_B^T \cdot \mathbf{C} \cdot \mathbf{u}_A) = \mathbf{0} \Rightarrow (\mathbf{f}_A^T + \mathbf{f}_B^T \cdot \mathbf{C}) \cdot \mathbf{u}_A = 0$$

Because the power conservation is valid for any displacement under elastic range, we can say that the equation (2.22) is valid for any of those displacement. Therefore, we could conclude

$$\mathbf{f}_A^T + \mathbf{f}_B^T \cdot \mathbf{C} = \mathbf{0} \quad \text{and} \quad \mathbf{f}_A + \mathbf{C} \cdot \mathbf{f}_B = \mathbf{0}$$

Especially, for 2-dimentional case:

$$\delta \mathbf{W}_A = \Sigma \mathbf{f}_{AM} \cdot \mathbf{u}_{AN} = \begin{pmatrix} f_{2a1} \\ f_{2a1-1} \\ f_{2a2-1} \\ f_{2a2} \\ \vdots \\ f_{2aN-1} \\ f_{2aN} \end{pmatrix}^T \cdot \begin{pmatrix} u_{2a1-1} \\ u_{2a1} \\ u_{2a2-1} \\ u_{2a2} \\ \vdots \\ u_{2aN-1} \\ u_{2aN} \end{pmatrix}; \quad \delta \mathbf{W}_B = \Sigma \mathbf{f}_{BM} \cdot \mathbf{u}_{BM} = \begin{pmatrix} f_{2b1} \\ f_{2b1-1} \\ f_{2b2-1} \\ f_{2b2} \\ \vdots \\ f_{2bM-1} \\ f_{2bM} \end{pmatrix}^T \cdot \begin{pmatrix} u_{2b1-1} \\ u_{2b1} \\ u_{2b2-1} \\ u_{2b2} \\ \vdots \\ u_{2bM-1} \\ u_{2bM} \end{pmatrix}$$

The equation (2.1) should be the beginning step of the constraint equation for nodal forces. From equation (2.1), we get

$$\delta W_A + \delta W_B = \Sigma f_{AN} \cdot u_{AN} + \Sigma f_{BM} \cdot u_{BM} = \begin{pmatrix} f_{2a1} \\ f_{2a1-1} \\ f_{2a2-1} \\ f_{2a2} \\ \vdots \\ f_{2aN-1} \\ f_{2aN} \end{pmatrix}^T \cdot \begin{pmatrix} u_{2a1-1} \\ u_{2a1} \\ u_{2a2-1} \\ u_{2a2} \\ \vdots \\ u_{2aN-1} \\ u_{2aN} \end{pmatrix} + \begin{pmatrix} f_{2b1} \\ f_{2b1-1} \\ f_{2b2-1} \\ f_{2b2} \\ \vdots \\ f_{2bM-1} \\ f_{2bM} \end{pmatrix}^T \cdot \begin{pmatrix} u_{2b1-1} \\ u_{2b1} \\ u_{2b2-1} \\ u_{2b2} \\ \vdots \\ u_{2bM-1} \\ u_{2bM} \end{pmatrix} = 0$$

When we place equation (2.20) into the above equation,

$$
\delta W_A + \delta W_B = \begin{pmatrix} f_{2a1} \\ f_{2a1-1} \\ f_{2a2-1} \\ f_{2a2} \\ \vdots \\ f_{2aN-1} \\ f_{2aN} \end{pmatrix}^T \cdot \begin{pmatrix} u_{2a1-1} \\ u_{2a1} \\ u_{2a2-1} \\ u_{2a2} \\ \vdots \\ u_{2aN-1} \\ u_{2aN} \end{pmatrix} + \begin{pmatrix} f_{2b1} \\ f_{2b1-1} \\ f_{2b2-1} \\ f_{2b2} \\ \vdots \\ f_{2bM-1} \\ f_{2bM} \end{pmatrix}^T \cdot \mathbf{S} \cdot \begin{pmatrix} u_{2a1-1} \\ u_{2a1} \\ u_{2a2-1} \\ u_{2a2} \\ \vdots \\ u_{2aN-1} \\ u_{2aN} \end{pmatrix} = 0 \quad \cdots\cdots\cdots \quad (2.22)
$$

$$
in \quad which \quad \mathbf{S} = \begin{pmatrix}
\frac{s_2^{b1}}{L_{b1}} & 0 & \frac{s_1^{b1}}{L_{b1}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \frac{s_2^{b1}}{L_{b1}} & 0 & \frac{s_1^{b1}}{L_{b1}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{s_2^{b2}}{L_{b2}} & 0 & \frac{s_1^{b2}}{L_{b2}} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{s_2^{b2}}{L_{b2}} & 0 & \frac{s_1^{b2}}{L_{b2}} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{s_2^{bM}}{L_{bM}} & 0 & \frac{s_1^{bM}}{L_{bM}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{s_2^{bM}}{L_{bM}} & 0 & \frac{s_1^{bM}}{L_{bM}}
\end{pmatrix}
$$

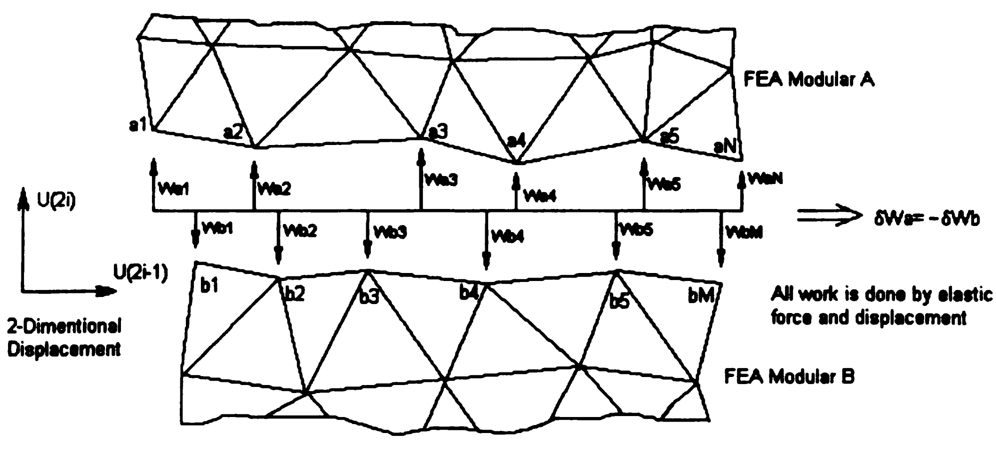


Figure 2.11. The power conservation along the Boundary

Figure 2.10 shows the sum of work transferred along the boundary equal ZERO under the situation of the elastic range.

2.3.4. Combine the Supplemental Boundary Constraint

From the boundary constraint equations (2.20) and (2.23), we get the total 2M constraint equations (if M<N) or 2N constraint equations (if M>N). From figure 2.12, we can see that when the system DOF is 0, there must be M>3 and N>3, and the system equation is mathematically non-singular.

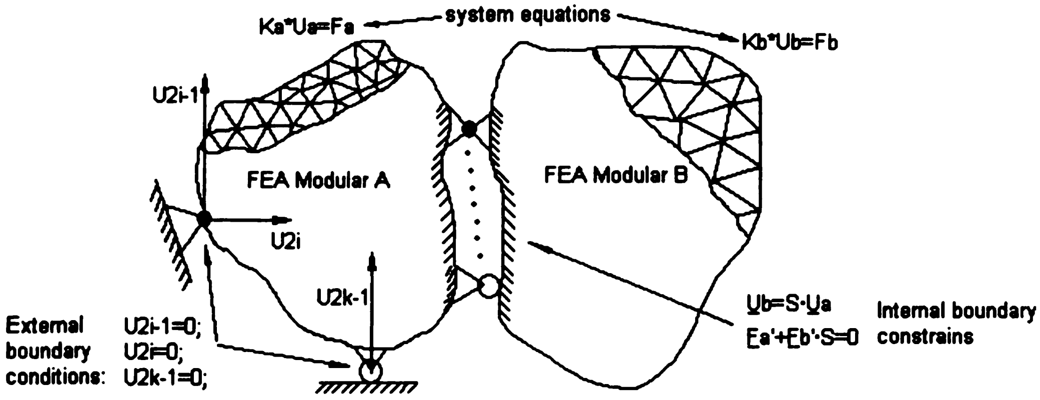


Figure 2.12. The "solvable system" and related equations.

For any solvable system like figure 2.12, the system equation, boundary condition and boundary constraint equation should be combined through the following procedures.

Step 1. Use the FEA models to get the combined system equation (2.5), the boundary displacement constraint equation (2.20), and the boundary force constraint equation (2.23). All of these constraint equations could come out from nodal coordinate values and I/O port shape. If KA is pxp, KB is qxq matrix, then

$$\begin{pmatrix} \mathbf{K}_A & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{K}_B \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u}_A \\ \cdots \\ \mathbf{u}_B \end{pmatrix} = \mathbf{K}_{AB} \cdot \begin{pmatrix} \mathbf{u}_A \\ \cdots \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_A \\ \cdots \\ \mathbf{f}_B \end{pmatrix} \cdots\cdots\cdots\cdots (2.23)$$

KAB is a (p+q) ×(p+q) matrix.

Step 2. Apply the external boundary conditions and boundary displacement constraint equation (2.20) to transform system equation. This step will replace

$$
\begin{aligned}
&[\mathbf{u}_{2i-1} \quad \mathbf{u}_{2i} \quad \mathbf{u}_{2k-1}] = [0 \quad 0 \quad 0]; \qquad \text{and} \\
&\left[\mathrm{U}_{2b1-1} \quad \mathrm{U}_{2b1} \quad \mathrm{U}_{2b2-1} \quad \mathrm{U}_{2b2} \quad \cdots \quad \mathrm{U}_{2bM-1} \quad \mathrm{U}_{2bM}\right]_{B^T}^T \\
&= \mathbf{S} \cdot \left[\mathrm{U}_{2a1-1} \quad \mathrm{U}_{2a1} \quad \mathrm{U}_{2a2-1} \quad \mathrm{U}_{2a2} \quad \cdots \quad \mathrm{U}_{2aN-1} \quad \mathrm{U}_{2aN}\right]_A^T
\end{aligned}
$$

the displacement of some boundary nodes. For example in figure 2.11 the external boundary conditions and boundary displacement constraint equation is

Using upper equations, we can get rid of u2i-1; u2i; u2k-1; and u2b1-1; u2b1; u2b2-1; u2b2; ... U2bM-1 and U2bM-1 from UA and UB, and get following equation (assume N>M):

$$
\begin{pmatrix} \mathbf{u}_A \\ \cdots \\ \mathbf{u}_B \end{pmatrix} = \mathbf{T}_U \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{K}_A & \vdots & \mathbf{0} \\ \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{K}_B \end{pmatrix} \cdot \mathbf{T}_U \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \end{pmatrix} = \mathbf{K'}_{AB} \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_A \\ \cdots \\ \mathbf{f}_B \end{pmatrix} \quad \cdots \quad (2.24)
$$

In which TU is a (p+q-3-M) ×(p+q) transformation matrix, and K'AB is a (p+q-3-M) ×(p+q) matrix that is transformed from KAB.

Step 3. Move the unknown forces (external and internal boundary forces) to the left side, and leave the right side of the equation as a totally known force vector. For example, if fj, fk, fl, fm in [FA FB]T is unknown, we can simply replace it by "0". This will make the system equation become a standard linear algebra equation, in which TF is a (2M+3) ×(p+q) matrix and K"AB is a (p+q+M) ×(p+q) matrix.

$$
\mathbf{K'}_{AB} \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_A \\ \cdots \\ \mathbf{f}_B \end{pmatrix} \Rightarrow (\mathbf{K'}_{AB} \ \vdots \ \mathbf{T}_F) \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \\ \cdots \\ \mathbf{f'}_{A+B} \end{pmatrix} = \mathbf{K''}_{AB} \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \\ \cdots \\ \mathbf{f'}_{A+B} \end{pmatrix} = \begin{pmatrix} \mathbf{f'}_A \\ \cdots \\ \mathbf{f'}_B \end{pmatrix} \quad \cdots \quad (2.25)
$$

Step 4. Apply the boundary force constraint equation. Because all boundary forces in equation(2.23) is unknown forces that have been moved to left side, we can directly apply the boundary force constraint equation into the system equation.

$$\begin{pmatrix} \mathbf{f}_{2a1} \\ \mathbf{f}_{2a2} \\ \vdots \\ \mathbf{f}_{2aN} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{2b1} \\ \mathbf{f}_{2b2} \\ \vdots \\ \mathbf{f}_{2bM} \end{pmatrix} \cdot \mathbf{S}_F = \underline{\mathbf{0}} \quad \Rightarrow \quad \begin{pmatrix} \mathbf{K'}_{AB} & \vdots & \mathbf{T}_F \\ \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{S'}_F \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \\ \cdots \\ \mathbf{f'}_{A+B} \end{pmatrix} = \mathbf{K''}_{AB} \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \\ \cdots \\ \mathbf{f'}_{A+B} \end{pmatrix} = \begin{pmatrix} \mathbf{f'}_A \\ \cdots \\ \mathbf{f'}_A \\ \cdots \\ \mathbf{o} \end{pmatrix} \quad \cdots \quad (2.26)$$

Because the boundary force could provide M constraint, the system equation matrix becomes a square matrix with full rank, which S'F is a (2M+3) ×(M) matrix, and K'"AB is a (p+q+M) ×(p+q+M) matrix. Therefore, the final integrated system is

$$\begin{pmatrix} \mathbf{K'}_{AB} & \vdots & \mathbf{T}_F \\ \cdots & \cdot & \cdots \\ \mathbf{0} & \vdots & \mathbf{S'}_F \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u'}_A \\ \cdots \\ \mathbf{u'}_B \\ \cdots \\ \mathbf{f'}_{A+B} \end{pmatrix} = \begin{pmatrix} \mathbf{f'}_A \\ \cdots \\ \mathbf{f'}_B \\ \cdots \\ \mathbf{0} \end{pmatrix} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2.27)$$

By solving equation (2.27), the system internal forces f'A+B and displacement distribution u'A and u'B are available. In following chapter, I am going to demonstrate this approach with a example. Through this example, the system modeling and integrating details could be shown clearly.

2.4. Result and Discussion

Modular modeling system FEA equations can be developed from the subsystem equations, external boundary conditions, and internal constraint equations. If enough constraint equations are available and the DOF is zero, the modular modeling FEA equations can be solvable. The general form of the combined system equation (2.27) has been developed here for the two-dimensional elastic problem.

Modular modeling FEA equations could be realized automatically with a proper algorithm. The reason is that all of parameters used in combined systems come from individual FEA model and connection description, such as nodal coordinates and boundary conditions. This point is valuable for the modular modeling method because this means that the system could be integrated by the user's selection of parts and connection type.

Solution of the modular modeling formulation can be shown to be equivalent to solving the associated Lagrangian formulation [Appendix I, Radcliffe and Diaz, 2000]. Although the modular modeling formulation is derived based on only work constraint at the boundary, its solution is identical to the solution of the Lagrangian formulation which analysis minimum potential energy for the system. It is gratifying that the solution for the modular modeling method coincides with the traditional energy-based Lagrangian approach.

TWO-DIMENSIONAL EXAMPLE FOR ELASTIC STRAIN-STRESS PROBLEM

3.1. FEA static stress-strain analysis by modular modeling

To demonstrate the last chapter work about integrating the modular modeling FEA systems, a simple example for a 2-D elastic plane force-displacement problem will be solved by the approach and procedures in the chapter 2. After the combined system has been solved, the result was compared with same object but derived from FEM using two different mesh densities.

In figure 3.1, there is a combined plane will be a subject of load. The two triangular components have FEA models: plane A has a FEA stiffness matrix K_A, plane B has a FEA stiffness matrix K_B,

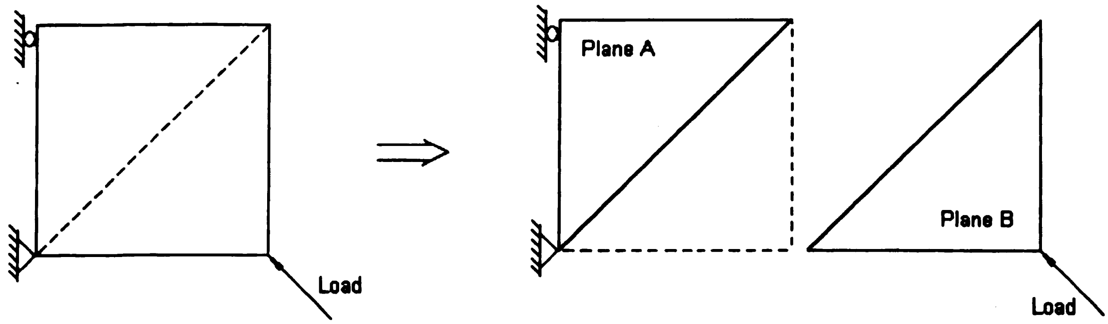
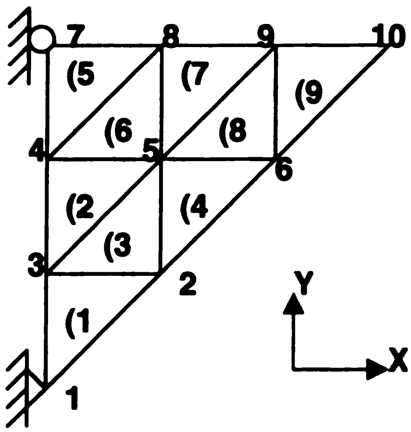


Figure 3.1. Two-Dimensional Example for Elastic Strain-Stress Problem

From figure 3.1, the subject has 3 external boundary conditions so it is 0 DOF, using the approach in last chapter and original equation (3.1) and (3.2), applying the external and internal boundary constraint (port) equations. The system equations are singular.

3.1.1 2-D Elastic Plane A:

We need to get the individual plane meshing and system equation. From the following procedures, it could be solved like any traditional examples [L. Segerlind, 1984]. The related system properties are: Young's modulus: E = 15×106 (N/cm2); Poisson's ratio: μ = 1/4; Plane thickness: t = 0.1cm.



Finite element summary:

element	i	j	k
(1)	1	3	2
(2)	3	4	5
(3)	2	3	5
(4)	2	5	6
(5)	4	7	8
(6)	5	4	8
(7)	5	8	9
(8)	6	5	9
(9)	6	9	10

Nodes' coordinator values:

Node	1	2	3	4	5	6	7	8	9	10
X	0	2.0	0	0	2.0	4.0	0	2.0	4.0	6.0
Y	0	2.0	2.0	4.0	4.0	4.0	6.0	6.0	6.0	6.0

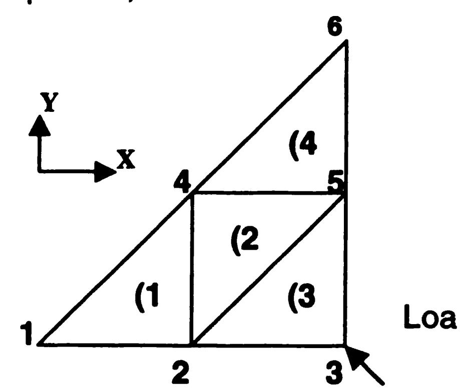
Figure 3.2 Plane A meshing details and nodal distribution

By the traditional method [L. Segerlind, 1984], we get all element stiffness matrixes in plane A: [K(1)], [K(2)], [K(3)], [K(4)], [K(5)], [K(6)] and combine them together as following. See the Appendix II for the detail of computer software.

$$
K_A = \begin{vmatrix}
3 & 0 & 0 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 2 & 0 & -2 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 22 & 5 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 5 & 22 & -5 & -6 & 0 & 0 & -5 & -16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -2 & -16 & -5 & 22 & 5 & -3 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -8 & -5 & -6 & 5 & 22 & -2 & -8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -2 & 22 & 5 & -16 & -5 & 0 & 0 & -3 & -3 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -8 & 5 & 22 & -5 & -6 & 0 & 0 & -2 & -8 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -6 & -5 & 0 & 5 & -16 & -5 & 44 & 10 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 5 & 0 & 0 \\
0 & 0 & -5 & -16 & 5 & 0 & -5 & -6 & 10 & 44 & -5 & -6 & 0 & 0 & -5 & -16 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -16 & -5 & 22 & 5 & 0 & 0 & 0 & 0 & -6 & -5 & 0 & 3 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -5 & -6 & 5 & 22 & 0 & 0 & 0 & 0 & -5 & -16 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 & 11 & 5 & -8 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -8 & 0 & 0 & 0 & 0 & 5 & 11 & -2 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -8 & -2 & 22 & 5 & -8 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & -3 & 5 & 22 & -2 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -8 & -2 & 22 & 5 & -8 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & -3 & 5 & 22 & -2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -8 & -2 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & 0 & 3
\end{vmatrix} \times 10^5
$$

3.1.2. 2-D Elastic Plane B:

The meshing and system equation for plane B could be done by similar way. It has same static properties as plane A: Young's modulus: E = 15×106 (N/cm2); Poisson's ratio: $\mu = 1/4$; Plane thickness: t = 0.1cm.



Finite element summary:

element	i	j	k
(1)	1	4	2
(2)	2	4	5
(3)	2	5	3
(4)	4	6	5

Nodes' coordinator values:

Node	1	2	3	4	5	6
X	0	3.0	6.0	3.0	6.0	6.0
Y	0	0	0	3.0	3.0	6.0

Figure 3.3. Plane B meshing details and nodal distribution

By the traditional method [L. Segerlind, 1984], we could get all element stiffness matrixes in plain A: [K(1)], [K(2)], [K(3)], [K(4)], and combine them together as following. See the Appendix III for the detail of computer software.

$$K_B = \begin{vmatrix} 8 & 0 & -8 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ -8 & -3 & 22 & 5 & -8 & -2 & -6 & -5 & 0 & 5 & 0 & 0 \\ -2 & -3 & 5 & 22 & -3 & -3 & -5 & -16 & 5 & 0 & 0 & 0 \\ 0 & 0 & -8 & -3 & 11 & 5 & 0 & 0 & -3 & -2 & 0 & 0 \\ 0 & 0 & -2 & -3 & 5 & 11 & 0 & 0 & -3 & -8 & 0 & 0 \\ 0 & 3 & -6 & -5 & 0 & 0 & 22 & 5 & -16 & -5 & 0 & 2 \\ 2 & 0 & -5 & -16 & 0 & 0 & 5 & 22 & -5 & -6 & 3 & 0 \\ 0 & 0 & 0 & 5 & -3 & -3 & -16 & -5 & 22 & 5 & -3 & -2 \\ 0 & 0 & 5 & 0 & -2 & -8 & -5 & -6 & 5 & 22 & -3 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & -8 & 0 & 8 \end{vmatrix} \times 10^5$$

3.1.3. Using Modular Modeling To Integrate 2-D Elastic Components

We are using this example to demonstrate the details of the modular modeling FEA approach. We need to follow the steps one by one according to the description in previous chapter. The final solvable system equation is too large to be shown here. For detailed computing procedure and result, please review Appendix V.

3.2. Error Estimation and Analysis

To validate the result from §3.1, we need to compare it with a traditional model. Both higher and lower meshing resolution models are needed for the comparison to verify the modular modeling result. For a linear problem, higher resolution meshes yield a more accurate result. The modular model mesh has more resolution than then low resolution reference model but less resolution than the higher resolution model. We expect the modular model to be more accurate than the lower resolution mesh model and less accurate than the higher resolution mesh model. Comparing these models will provide useful information about the detail in the model and component connections at the boundary.

3.2.1. Standard Reference Models for Accuracy Comparison

Two standard reference FEA models (Fig. 3.4) will be used to evaluate the accuracy of the 13 element modular modeling example (Fig. 3.1). The higher resolution standard reference model has 72 elements while the lower resolution standard reference model has only eight elements. We use the same coordinate systems, load and FEA equations to ensure the accuracy of the formulations are directly comparable.

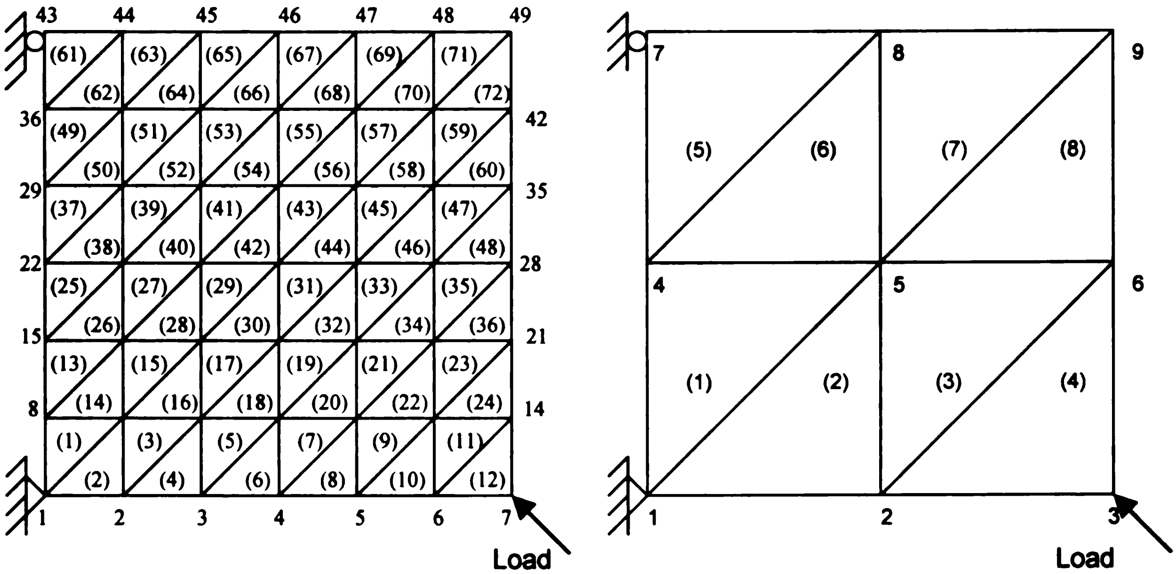


Figure 3.4. Re-meshed reference model with higher or lower meshing resolution

3.2.2. Modular Modeling Method Accuracy Comparison

Displacement contour lines were used for comparing the overall system accuracy. Displacement means the total displacement at every node. If a node has "a" displacement on X direction and "b" displacement on "Y" direction, its total displacement is $\left(\sqrt{a^2+b^2}\right)$. Because we are focusing on the linear case and the load is same, the linear static elastic problem should have the same and repeatable result. Therefore this comparing is reliable.

Predicted displacement for the modular model were compared to the displacements predicted by the higher resolution standard reference model. EXCEL was used to compute and plot total model displacements (Fig. 3.5). The 3-D displacement difference surface (Fig. 3.6) shows the largest model differences are less than 10%. In particular, note that the modular model results are almost universally smaller displacement than the higher resolution reference model. This indicates that the modular model is stiffer than the higher resolution reference model.

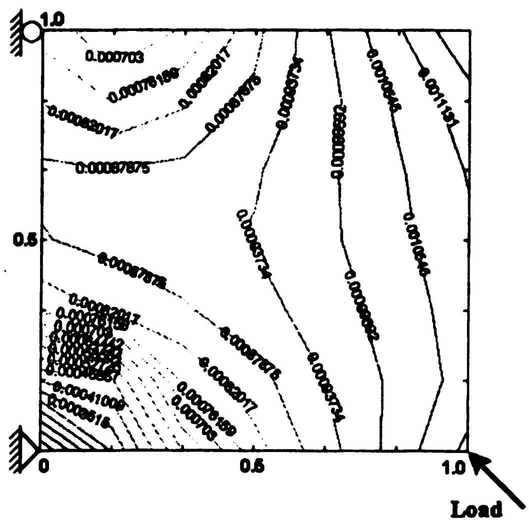


Figure 3.5(a). Displacement contour lines for reference model (7x7).

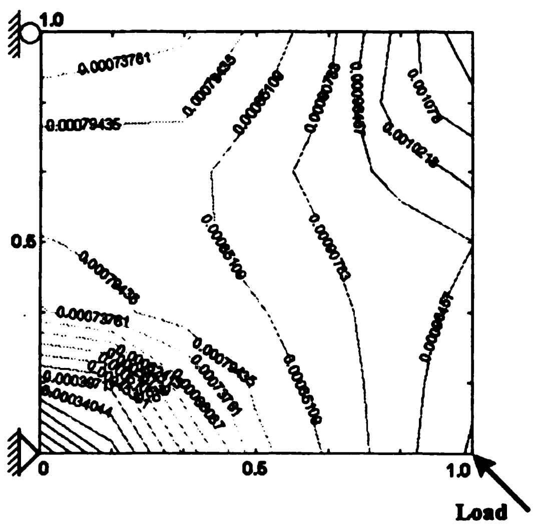


Figure 3.5(b). Displacement contour lines for the Joint Modular Model

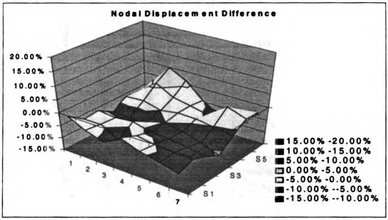


Figure 3.6. Displacement difference between the Joint Modular Model and reference model (7x7)

Predicted displacement for the modular model were next compared to the displacements predicted by the lower resolution standard reference model. EXCEL was again used to compute and plot total model displacements (Fig. 3.7). The 3-D displacement difference surface (Fig. 3.8) shows that in this case, the largest model differences approach 25%. In particular, note that the modular model results are almost universally larger displacement than the lower resolution reference model. In contrast to the comparison with the higher resolution reference model, this results indicates that the modular model is less stiff than the lower resolution reference model.

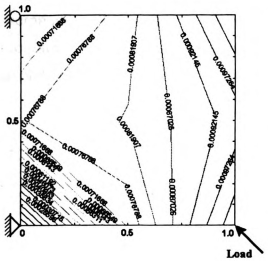


Figure 3.7(a). Displacement contour lines for reference model (3x3).

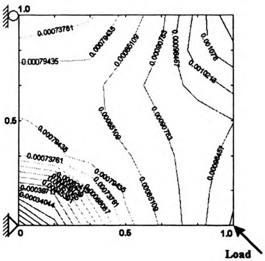


Figure 3.7(b). Displacement contour lines for the Joint Modular Model

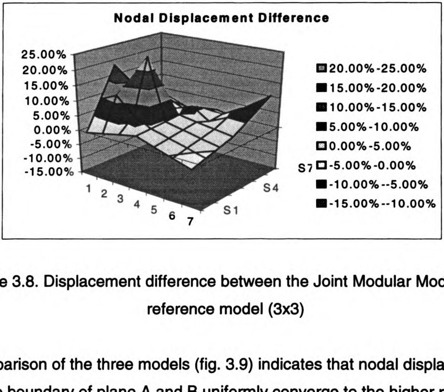


Figure 3.8. Displacement difference between the Joint Modular Model and reference model (3x3)

Comparison of the three models (fig. 3.9) indicates that nodal displacements along the boundary of plane A and B uniformly converge to the higher resolution standard model result as the number of elements in the model increase. Because the number of elements in the modular model lies between the lower and higher resolution reference models, its displacements are bracketed by the two reference model displacements. As expected, the model stiffness uniformly decreases as the number of elements is increased, indicating the modular model accuracy is comparable to traditional FEM models with the same mesh resolution.

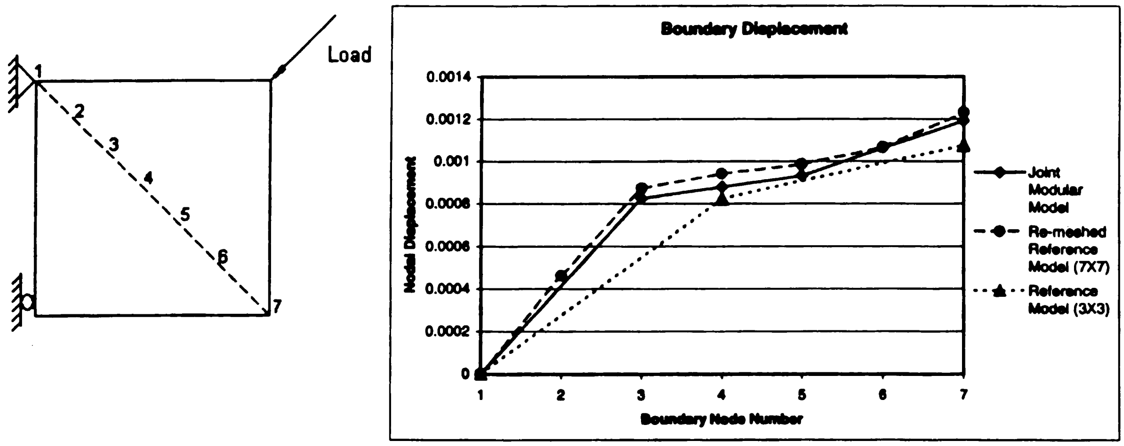


Figure 3.9. The difference of nodal displacements along the boundary

CONCLUSIONS

4.1 Contributions

Application of the modular modeling method in FEA has been discussed; its advantages and efficiency have been shown in examples and numerical results. Most importantly, the 2-dimensional boundary constraint equation that is used to connect FEA modules has been developed and verified in examples. Modular modeling method is shown to use the original FEA data, removing the need for re-meshing and verification, so it saves cost and time.

From a general description and example, we believe that the equation (2.5), (2.20) and (2.23) is the proper way to get the combined system equation and boundary constraint equations. For actual engineering problem, just like the example, we can sequentially use the equations (2.5), (2.24), (2.25), (2.26) and (2.27) to get the solution. For 2-Dimentional elastic stress-strain problem, if the system have 0 DOF, we can conclude that the developed system equation of FEA modular model is non-singular and could be solved with the boundary constraint equations, external boundary conditions and loads.

The modular model has good computational accuracy and quality that is close to the re-meshed higher resolution model and is much better than the relatively low-resolution model. The modular modeling method was shown to provide accuracy comparable to traditional FEA models with comparable mesh resolution. The ability to achieve this accuracy without remeshing is a significant improvement to FEA modeling technology.

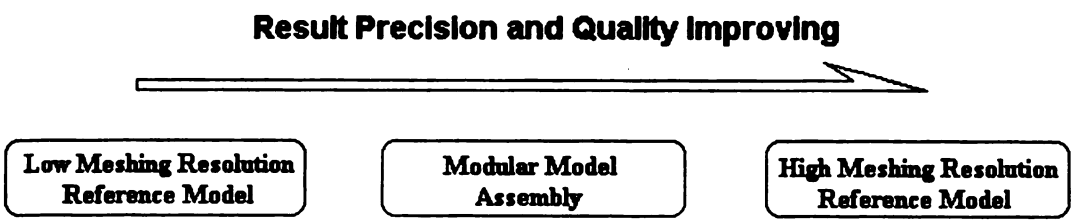


Figure 4.1. Comparing the Modular Model and other models

4.2 Future work

The equation developed in this paper can be only applied to a simple 2-D problem, the accuracy of this approach will be decided by the equality of boundary equation to make this approach practically useful in industrial applications, we must develop the following topics:

- Convergence of the stiffness matrix
- All kinds of meshes
- The 3 dimensional flat boundary
- The 3 dimensional arbitrary boundaries
- The Boundary Constraint Equation development automation,
- Apply modular modeling method into the nonlinear FEA problem.

There are 2 ways to develop the boundary constraint equation. From equations (2.20) and (2.23), we can see that component B was treated as the subject of interpolation from component A. It is easy to understand, the contrary way could also provide similar constraint equations. Those equations should also be enough to make the system solvable. But the problem is, "which one is better?" I believe that this is a more complex and practical problem and needs research through great number of actual case, cause the meshing and constraint have so many kind of types that it is very hard to simply identify them.

Theoretically, the boundary interface input / output equation between components will be only decided by their shape function. Therefore, the automatically formulation should be practically realized by proper programming. This feature should be one of the critical block in the future modular modeling FEA software.

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APPENDIX

I. Lagrangian FEM Modular Modeling notes

II. The Mathematica® program for plant A. system stiffness matrix computation.
Filename: *Plane_a.nb*

III. The Mathematica® program for plant B. system stiffness matrix computation.
Filename: *Plane_b.nb*

IV. The Mathematica® program for re-meshed plant. Work as the reference for error analysis.
Filename: Plane_all.nb

V. The Mathematica® program for combined plant. System integration.
Filename: *Plane_a+b_connect_6.nb*

VI. The MS Excel data table for the result of FEA computation and error analysis.

TO: Mr. Xin Li
From: Clark Radcliffe
Subject: Lagrangian FEM Modular Modeling
Date: April 7, 1999

Per discussion with A. Diaz 4/6/00 at your MS defense, I have outlined below the relationship between the Modular Modeling Method and the Lagrangian approach to the joining of two bodies with constraints.

The Modular FEM Modeling Problem (Xin Li MS thesis,4/6/00) is formulated as

$$\begin{bmatrix} \mathbf{K}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_B \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_A \\ \mathbf{u}_B \end{bmatrix} = \begin{bmatrix} \mathbf{f}_A \\ \mathbf{f}_B \end{bmatrix} \tag{1a}$$

subject to the constraints

$$\hat{\mathbf{u}}_B = \hat{\mathbf{S}} \cdot \hat{\mathbf{u}}_A \text{ and} \tag{1b}$$

$$\hat{\mathbf{W}}_A + \hat{\mathbf{W}}_B = \mathbf{0} \tag{1c}$$

where $\mathbf{u}_A = \begin{bmatrix} \bar{\mathbf{u}}_A \\ \hat{\mathbf{u}}_A \end{bmatrix}$, $\mathbf{u}_B = \begin{bmatrix} \bar{\mathbf{u}}_B \\ \hat{\mathbf{u}}_B \end{bmatrix}$, $\mathrm{f}_A = \begin{bmatrix} \bar{\mathrm{f}}_A \\ \hat{\mathrm{f}}_A \end{bmatrix}$, $\mathbf{f}_B = \begin{bmatrix} \bar{\mathbf{f}}_B \\ \hat{\mathbf{f}}_B \end{bmatrix}$

and the work done on the plates A and B by the connection are $\hat{\mathbf{W}}_A = \hat{\mathbf{f}}_A \cdot \hat{\mathbf{u}}_A$ and $\hat{\mathbf{W}}_B = \hat{\mathbf{f}}_B \cdot \hat{\mathbf{u}}_B$ respectively. Substitution of the connection forces and displacements into (1c) yields the relationship between internal joint forces

$$\hat{\mathbf{f}}_A = -\hat{\mathbf{S}}^T \cdot \hat{\mathbf{f}}_B \tag*{(1c)'}$$

All variables denoted "bar" variables ($\bar{\mathbf{x}}$) refer to "external" quantities not involved in a joint while all "hat" variables ($\hat{\mathbf{x}}$) refer to "internal" joint variables. Partitioning the equations and rewriting them

$$\begin{bmatrix} \bar{\mathrm{K}}_A & \hat{\bar{\mathrm{K}}}_A & 0 & 0 \\ \hat{\bar{\mathrm{K}}}_A^T & \hat{\mathrm{K}}_A & 0 & 0 \\ 0 & 0 & \bar{\mathrm{K}}_B & \hat{\bar{\mathrm{K}}}_B \\ 0 & 0 & \hat{\bar{\mathrm{K}}}_B^T & \hat{\mathrm{K}}_B \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathrm{u}}_A \\ \hat{\mathrm{u}}_A \\ \bar{\mathrm{u}}_B \\ \hat{\mathrm{u}}_B \end{bmatrix} = \begin{bmatrix} \bar{\mathrm{f}}_A \\ \hat{\mathrm{f}}_A \\ \bar{\mathrm{f}}_B \\ \hat{\mathrm{f}}_B \end{bmatrix} \tag{2a}$$

subject to constraints
$$\hat{\mathbf{u}}_B = \hat{\mathbf{S}}\hat{\mathbf{u}}_A \tag{2b}$$
and
$$\hat{\mathbf{f}}_A = -\hat{\mathbf{S}}^T\hat{\mathbf{f}}_B \tag{2c}$$

Question: Is the solution of this problem equivalent to the solution of the Lagrangian formulation?

The Lagrangian Problem (A. Diaz, 4/6/00)
Given the potential energy function

$$\Pi = \frac{1}{2}\mathbf{u}_A^T\mathbf{K}_A\mathbf{u}_A - \overline{\mathbf{u}}_A^T\overline{\mathbf{f}}_A + \frac{1}{2}\mathbf{u}_B^T\mathbf{K}_B\mathbf{u}_B - \overline{\mathbf{u}}_B^T\overline{\mathbf{f}}_B \tag{3a}$$

subject to displacement constraints

$$\hat{\mathbf{u}}_B = \hat{\mathbf{S}} \cdot \hat{\mathbf{u}}_A \tag{3b}$$

The Lagrangian is

$$\begin{aligned} \mathbf{L} &= \Pi + (\hat{\mathbf{u}}_B - \hat{\mathbf{S}} \cdot \hat{\mathbf{u}}_A)^T \cdot \lambda \\ &= \frac{1}{2}\mathbf{u}_A^T\mathbf{K}_A\mathbf{u}_A - \overline{\mathbf{u}}_A^T\overline{\mathbf{f}}_A + \frac{1}{2}\mathbf{u}_B^T\mathbf{K}_B\mathbf{u}_B - \overline{\mathbf{u}}_B^T\overline{\mathbf{f}}_B + (\hat{\mathbf{u}}_B - \hat{\mathbf{S}} \cdot \hat{\mathbf{u}}_A)^T \cdot \lambda \end{aligned} \tag{4}$$

The solution of (3a) and (3b) is a stationary point of (4) that minimizes the potential energy (3a).

Taking the derivative of $\mathcal{L}$ with respect to each of the independent variables $\mathbf{u}_A, \mathbf{u}_B, \lambda$.

$$\frac{\partial L}{\partial \overline{\mathbf{u}}_A} = 0 = \overline{\mathbf{K}}_A\overline{\mathbf{u}}_A + \hat{\overline{\mathbf{K}}}_A\hat{\mathbf{u}}_A - \overline{\mathbf{f}}_A \tag{4a}$$

$$\frac{\partial L}{\partial \hat{\mathbf{u}}_A} = 0 = \hat{\overline{\mathbf{K}}}_A^T\overline{\mathbf{u}}_A + \hat{\mathbf{K}}_A\hat{\mathbf{u}}_A - \hat{\mathbf{S}}^T\lambda \tag{4b}$$

$$\frac{\partial L}{\partial \overline{\mathbf{u}}_B} = 0 = \overline{\mathbf{K}}_B\overline{\mathbf{u}}_B + \hat{\overline{\mathbf{K}}}_B\hat{\mathbf{u}}_B - \overline{\mathbf{f}}_B \tag{4c}$$

$$\frac{\partial L}{\partial \hat{\mathbf{u}}_B} = 0 = \hat{\overline{\mathbf{K}}}_B^T\overline{\mathbf{u}}_B + \hat{\mathbf{K}}_B\hat{\mathbf{u}}_B - \lambda \tag{4d}$$

$$\frac{\partial L}{\partial \lambda} = 0 = \hat{\mathbf{u}}_B - \hat{\mathbf{S}}\hat{\mathbf{u}}_A \tag{4e}$$

Putting (4a)-(4e) into matrix form yields,

$$\begin{bmatrix} \overline{\mathbf{K}}_A & \hat{\overline{\mathbf{K}}}_A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hat{\overline{\mathbf{K}}}_A^T & \hat{\mathbf{K}}_A & \mathbf{0} & \mathbf{0} & -\mathbf{S}^T \\ \mathbf{0} & \mathbf{0} & \overline{\mathbf{K}}_B & \hat{\overline{\mathbf{K}}}_B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\overline{\mathbf{K}}}_B^T & \hat{\mathbf{K}}_B & \mathbf{I} \\ \mathbf{0} & -\hat{\mathbf{S}} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \overline{\mathbf{u}}_A \\ \hat{\mathbf{u}}_A \\ \overline{\mathbf{u}}_B \\ \hat{\mathbf{u}}_B \\ \lambda \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{f}}_A \\ \mathbf{0} \\ \overline{\mathbf{f}}_B \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{5}$$

Note that (5) is a symmetric formulation.

It can be shown that solutions to (4a-e) rewritten as (5) are also solutions to (2a-c). First note that the Lagrange formulation equation (4e) is equivalent to Modular Modeling constraint (2b).

$$0 = \hat{\mathbf{u}}_B - \hat{\mathbf{S}}\hat{\mathbf{u}}_A \Leftrightarrow \hat{\mathbf{u}}_B = \hat{\mathbf{S}}\hat{\mathbf{u}}_A \tag{6}$$

Let the Lagrange multiplier, $\lambda = -\hat{\mathbf{f}}_B$, the internal boundary force on plate B. Then, the Lagrange formulation equation (4d) is equivalent to the fourth equation in the Modular Modeling matrix equation (2a).

$$\mathbf{0} = \hat{\overline{\mathbf{K}}}_B^T \overline{\mathbf{u}}_B + \hat{\mathbf{K}}_B \hat{\mathbf{u}}_B + \lambda \Leftrightarrow \hat{\overline{\mathbf{K}}}_B^T \overline{\mathbf{u}}_B + \hat{\mathbf{K}}_B \hat{\mathbf{u}}_B = \hat{\mathbf{f}}_B \text{ for } \lambda = -\hat{\mathbf{f}}_B \tag{7}$$

The Lagrange formulation equation (4b) is equivalent to the second equation in the Modular Modeling matrix equation (2a). Use the force constraint (2c) from the Modular Modeling formulation to redefine the internal force $\hat{\mathbf{f}}_A$ appearing in the second equation in the matrix equation (2a) to write

$$\mathbf{0} = \hat{\overline{\mathbf{K}}}_A^T \overline{\mathbf{u}}_A + \hat{\mathbf{K}}_A \hat{\mathbf{u}}_A - \hat{\mathbf{S}}^T \lambda \Leftrightarrow \hat{\overline{\mathbf{K}}}_A^T \overline{\mathbf{u}}_A + \hat{\mathbf{K}}_A \hat{\mathbf{u}}_A = \hat{\mathbf{f}}_A \text{ for } \hat{\mathbf{f}}_B = -\hat{\mathbf{S}}^T \hat{\mathbf{f}}_B = \hat{\mathbf{S}}^T \lambda \tag{8}$$

The Lagrange formulation equations (4a,c) are identical to the first and third equations in the Modular Modeling matrix equation (2a).

$$\mathbf{0} = \overline{\mathbf{K}}_A \overline{\mathbf{u}}_A + \hat{\overline{\mathbf{K}}}_A \hat{\mathbf{u}}_A - \overline{\mathbf{f}}_A \Leftrightarrow \overline{\mathbf{K}}_A \overline{\mathbf{u}}_A + \hat{\overline{\mathbf{K}}}_A \hat{\mathbf{u}}_A = \overline{\mathbf{f}}_A \tag{9}$$

$$\mathbf{0} = \overline{\mathbf{K}}_B \overline{\mathbf{u}}_B + \hat{\overline{\mathbf{K}}}_B \hat{\mathbf{u}}_B - \overline{\mathbf{f}}_B \Leftrightarrow \overline{\mathbf{K}}_B \overline{\mathbf{u}}_B + \hat{\overline{\mathbf{K}}}_B \hat{\mathbf{u}}_B = \overline{\mathbf{f}}_B \tag{10}$$

The conclusion is that the two formulations have identical solutions with the recognition that the Lagrange multiplier $\lambda = -\hat{\mathbf{f}}_B$, the joint internal force.

General Form of the Displacement Constraint

A more general form of the displacement constraint is the linear relationship

$$\mathbf{S}_A \mathbf{u}_A = \mathbf{S}_B \mathbf{u}_B \tag{11}$$

with this relationship, the Lagrangian formulation becomes

$$\Pi = \frac{1}{2}\mathbf{u}_A^T \mathbf{K}_A \mathbf{u}_A - \mathbf{u}_A^T \mathbf{f}_A + \frac{1}{2}\mathbf{u}_B^T \mathbf{K}_B \mathbf{u}_B - \mathbf{u}_B^T \mathbf{f}_B \tag{12a}$$

subject to displacement constraints

$$\mathbf{S}_A \mathbf{u}_A = \mathbf{S}_B \mathbf{u}_B \tag{12b}$$

Note that in this case, I have not partitioned the displacements $\mathbf{u}_A$ and $\mathbf{u}_B$. The body forces $\mathbf{f}_A$ and $\mathbf{f}_B$ represent all external forces applied to the two bodies and are simply a superset of the previous external forces $\overline{\mathbf{f}}_A$ and $\overline{\mathbf{f}}_B$ that also allow external tractions to be applied at the position of joint connections. The Lagrangian is

$$\begin{aligned} \mathbf{L} &= \Pi + (\mathbf{S}_B \mathbf{u}_B - \mathbf{S}_A \cdot \mathbf{u}_A)^T \cdot \lambda \\ &= \frac{1}{2}\mathbf{u}_A^T \mathbf{K}_A \mathbf{u}_A - \mathbf{u}_A^T \mathbf{f}_A + \frac{1}{2}\mathbf{u}_B^T \mathbf{K}_B \mathbf{u}_B - \mathbf{u}_B^T \mathbf{f}_B + (\mathbf{S}_B \mathbf{u}_B - \mathbf{S}_A \cdot \mathbf{u}_A)^T \cdot \lambda \end{aligned} \tag{13}$$

whose stationary value occurs at the solution to

$$\frac{\partial L}{\partial \mathbf{u}_A} = \mathbf{0} = \mathbf{K}_A\mathbf{u}_A - \bar{\mathbf{f}}_A - \mathbf{S}_A^T\lambda \tag{14a}$$

$$\frac{\partial L}{\partial \mathbf{u}_B} = \mathbf{0} = \mathbf{K}_B\mathbf{u}_B - \bar{\mathbf{f}}_B - \mathbf{S}_B^T\lambda \tag{14b}$$

$$\frac{\partial L}{\partial \lambda} = \mathbf{0} = \mathbf{S}_B\mathbf{u}_B - \mathbf{S}_A\mathbf{u}_A \tag{14c}$$

Putting (14a)-(14c) into matrix form yields,

$$\begin{bmatrix} \mathbf{K}_A & 0 & -\mathbf{S}_A^T \\ 0 & \mathbf{K}_A & \mathbf{S}_B^T \\ -\mathbf{S}_A & \mathbf{S}_A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_A \\ \mathbf{u}_B \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_A \\ \mathbf{f}_B \\ 0 \end{bmatrix} \tag{15}$$

Note that (15) is a symmetric formulation with $\mathbf{u}_A$ and $\mathbf{u}_B$ defined as in (1c). With this definition, the joint constraint matrices in (15) can be used to define the original Modular Modeling formulation by defining,

$$\mathbf{S}_A = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}}_A \end{bmatrix} \tag{16a}$$

and

$$\mathbf{S}_B = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \tag{16b}$$

APPENDIX II

$$EE = 15000000;\ u = \frac{1}{4};\ t = 0.1;$$

```
"System properties: Young's modulus(N/cm²); Poisson's ratio; Plane thickness(cm); ";

NI = {1, 3, 2, 2, 4, 5, 5, 6, 6}; NJ = {3, 4, 3, 5, 7, 4, 8, 5, 9};
NK = {2, 5, 5, 6, 8, 8, 9, 9, 10};
"mesh for plane A: nodes distribution, pick up the i;j;k; of #n element by: ";
"in=NI[[n]]; jn=NJ[[n]]; kn=NK[[n]]";

X = {0, 2, 0, 0, 2, 4, 0, 2, 4, 6}; Y = {6, 4, 4, 2, 2, 2, 0, 0, 0, 0};
"X,Y value for plane A nodes, pick up the Xn & Ynj of #n node by: ";
"Xn=X[[n]]; Yn=Y[[n]] ";

Array[Bi, 9]; Array[Bj, 9]; Array[Bk, 9]; Array[Ci, 9]; Array[Cj, 9]; Array[Ck, 9];
Do[ Bi[n] = (Y[[(NJ[[n]])]]) - (Y[[(NK[[n]])]]), {n, 9} ];
Do[ Bj[n] = (Y[[(NK[[n]])]]) - (Y[[(NI[[n]])]]), {n, 9} ];
Do[ Bk[n] = (Y[[(NI[[n]])]]) - (Y[[(NJ[[n]])]]), {n, 9} ];
Do[ Ci[n] = (X[[(NK[[n]])]]) - (X[[(NJ[[n]])]]), {n, 9} ];
Do[ Cj[n] = (X[[(NI[[n]])]]) - (X[[(NK[[n]])]]), {n, 9} ];
Do[ Ck[n] = (X[[(NJ[[n]])]]) - (X[[(NI[[n]])]]), {n, 9} ];
"Define Bi,Bj,Bk,Ci,Cj,Ck for all elements, compute every value by: ";
"Bi[n]=Yj[n]-Yk[n];   Bj[n]=Yk[n]-Yi[n];   Bk[n]=Yi[n]-Yj[n]";
"Ci[n]=Xk[n]-Xj[n];   Cj[n]=Xi[n]-Xk[n];   Ck[n]=Xj[n]-Xi[n]";

 "t = Table[{X[[(NK[[n]])]],Y[[(NK[[n]])]]},
     {n,9}];ListPlot[t,PlotJoined -> True, PlotLabel->" e "]";

Do[ Print["Bi(", i, ")=", (Bi[i]), "        Bj(", i, ")=",
   (Bj[i]), "        Bk(", i, ")=", (Bk[i]), "        Ci(", i, ")=", (Ci[i]),
   "        Cj(", i, ")=", (Cj[i]), "        Ck(", i, ")=", (Ck[i])], {i, 9} ];
```

$$\text{Do}\left[A = \frac{1}{2}\times \text{Det}\left[\begin{pmatrix} 1 & X[[(NI[[n]])]] & Y[[(NI[[n]])]] \\ 1 & X[[(NJ[[n]])]] & Y[[(NJ[[n]])]] \\ 1 & X[[(NK[[n]])]] & Y[[(NK[[n]])]] \end{pmatrix}\right];\ \text{"Print[A]"},\ \{n = 9\}\right];$$

$$\text{Array}[BB,\ 9];\ \text{Do}\left[\ BB[n] = \frac{1}{2\,A}\begin{pmatrix} Bi[n] & 0 & Bj[n] & 0 & Bk[n] & 0 \\ 0 & Ci[n] & 0 & Cj[n] & 0 & Ck[n] \\ Ci[n] & Bi[n] & Cj[n] & Bj[n] & Ck[n] & Bk[n] \end{pmatrix};\right.$$

$$\left.\text{"Print[BB[n]]"},\ \{n,\ 9\}\ \right];\ DD = \frac{EE}{1-u^2}\begin{pmatrix} 1 & u & 0 \\ u & 1 & 0 \\ 0 & 0 & \frac{1-u}{2} \end{pmatrix};$$

```
Array[Kelement, 9];
Do[KK = (Transpose[(BB[n])].DD);
 KK2 = KK.BB[n]; K = t * A * KK2; Kelement[n] = Rationalize[K];
       "Print[Kelement[n]]", {n, 9}]
```

$$
\text{Kall} = \left(\begin{array}{cccccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right);
$$

```
Array[ijk, 6];
"2i-1=ijk[1],2i=ijk[2],2j-1=ijk[3],2j-1=ijk[4],2k-1=ijk[5],2k=ijk[6],";
Do[Print["element: ", e]; ijk[1] = (2×NI[[e]]) - 1;
  ijk[2] = (2×NI[[e]]); ijk[3] = (2×NJ[[e]]) - 1; ijk[4] = (2×NJ[[e]]);
      ijk[5] = (2×NK[[e]]) - 1; ijk[6] = (2×NK[[e]]); Ktemp = Kelement[e];
     Do [Do[Kall[[ijk[i], ijk[j]]] = Kall[[ijk[i], ijk[j]]] + Ktemp[[i, j]], {i, 6}],
   {j, 6}], {e, 9}];
"Fill the element K into the integral K(Kall), based on every corrdinated nodes";
```

$$\frac{\mathbf{Kall}}{\mathbf{100000}}$$

$$\begin{pmatrix}
3 & 0 & 0 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 2 & 0 & -2 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 22 & 5 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 5 & 22 & -5 & -6 & 0 & 0 & -5 & -16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -2 & -16 & -5 & 22 & 5 & -3 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -8 & -5 & -6 & 5 & 22 & -2 & -8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -2 & 22 & 5 & -16 & -5 & 0 & 0 & -3 & -3 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -8 & 5 & 22 & -5 & -6 & 0 & 0 & -2 & -8 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -6 & -5 & 0 & 5 & -16 & -5 & 44 & 10 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 5 & 0 & 0 \\
0 & 0 & -5 & -16 & 5 & 0 & -5 & -6 & 10 & 44 & -5 & -6 & 0 & 0 & -5 & -16 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -16 & -5 & 22 & 5 & 0 & 0 & 0 & 0 & -6 & -5 & 0 & 3 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -5 & -6 & 5 & 22 & 0 & 0 & 0 & 0 & -5 & -16 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 & 11 & 5 & -8 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -8 & 0 & 0 & 0 & 0 & 5 & 11 & -2 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -8 & -2 & 22 & 5 & -8 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & -3 & 5 & 22 & -2 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -8 & -2 & 22 & 5 & -8 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & -3 & 5 & 22 & -2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -8 & -2 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -3 & -3 & 0 & 3
\end{pmatrix}$$

APPENDIX III

```
EE = 15000000; u = 1/4; t = 0.1;
"System properties: Young's modulus(N/cm^2); Poisson's ratio; Plane thickness(cm); ";

NI = {1, 2, 2, 4}; NJ = {4, 4, 5, 6}; NK = {2, 5, 3, 5};
"mesh for plane A: nodes distribution, pick up the i;j;k; of #n element by: ";
"in=NI[[n]]; jn=NJ[[n]]; kn=NK[[n]]";

X = {0, 3, 6, 3, 6, 6}; Y = {0, 0, 0, -3, -3, -6};
"X,Y value for plane A nodes, pick up the Xn & Ynj of #n node by: ";
"Xn=X[[n]]; Yn=Y[[n]] ";

Array[Bi, 4]; Array[Bj, 4]; Array[Bk, 4]; Array[Ci, 4]; Array[Cj, 4]; Array[Ck, 4];
Do[ Bi[n] = (Y[[(NJ[[n]])]]) - (Y[[(NK[[n]])]]), {n, 4} ];
Do[ Bj[n] = (Y[[(NK[[n]])]]) - (Y[[(NI[[n]])]]), {n, 4} ];
Do[ Bk[n] = (Y[[(NI[[n]])]]) - (Y[[(NJ[[n]])]]), {n, 4} ];
Do[ Ci[n] = (X[[(NK[[n]])]]) - (X[[(NJ[[n]])]]), {n, 4} ];
Do[ Cj[n] = (X[[(NI[[n]])]]) - (X[[(NK[[n]])]]), {n, 4} ];
Do[ Ck[n] = (X[[(NJ[[n]])]]) - (X[[(NI[[n]])]]), {n, 4} ];
"Define Bi,Bj,Bk,Ci,Cj,Ck for all elements, compute every value by: ";
"Bi[n]=Yj[n]-Yk[n];   Bj[n]=Yk[n]-Yi[n];   Bk[n]=Yi[n]-Yj[n]";
"Ci[n]=Xk[n]-Xj[n];   Cj[n]=Xi[n]-Xk[n];   Ck[n]=Xj[n]-Xi[n]";

 "t = Table[{X[[(NJ[[n]])]],Y[[(NJ[[n]])]]},
    {n,4}];ListPlot[t,PlotJoined -> True, PlotLabel->" e "]";

 Do[ Print["Bi(", i, ")=", (Bi[i]), "        Bj(", i, ")=", (Bj[i]),
  "       Bk(", i, ")=", (Bk[i]), "        Ci(", i, ")=", (Ci[i]),
  "       Cj(", i, ")=", (Cj[i]), "        Ck(", i, ")=", (Ck[i])], {i, 4} ]
```

$$\texttt{Do}\left[\texttt{A} = \frac{1}{2}\times\texttt{Det}\left[\begin{pmatrix} 1 & \texttt{X[[(NI[[n]])]]} & \texttt{Y[[(NI[[n]])]]} \\ 1 & \texttt{X[[(NJ[[n]])]]} & \texttt{Y[[(NJ[[n]])]]} \\ 1 & \texttt{X[[(NK[[n]])]]} & \texttt{Y[[(NK[[n]])]]} \end{pmatrix}\right]; \texttt{"Print[A]", \{n = 4\}}\right];$$

$$\texttt{Array[BB, 9]; Do}\left[\texttt{BB[n]} = \frac{1}{2\,\texttt{A}}\begin{pmatrix} \texttt{Bi[n]} & 0 & \texttt{Bj[n]} & 0 & \texttt{Bk[n]} & 0 \\ 0 & \texttt{Ci[n]} & 0 & \texttt{Cj[n]} & 0 & \texttt{Ck[n]} \\ \texttt{Ci[n]} & \texttt{Bi[n]} & \texttt{Cj[n]} & \texttt{Bj[n]} & \texttt{Ck[n]} & \texttt{Bk[n]} \end{pmatrix};\right.$$

$$\left.\texttt{"Print[BB[n]]", \{n, 4\}}\right]; \texttt{DD} = \frac{\texttt{EE}}{1-\texttt{u}^2}\begin{pmatrix} 1 & \texttt{u} & 0 \\ \texttt{u} & 1 & 0 \\ 0 & 0 & \frac{1-\texttt{u}}{2} \end{pmatrix}$$

```
Array[Kelement, 4];
Do[KK = (Transpose[(BB[n])].DD);
 KK2 = KK.BB[n]; K = t * A * KK2; Kelement[n] = Rationalize[K];
      "Print[Kelement[n]]", {n, 4}]
```

$$
Kall = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix};
$$

```
Array[ijk, 6];
"2i-1=ijk[1],2i=ijk[2],2j-1=ijk[3],2j-1=ijk[4],2k-1=ijk[5],2k=ijk[6],";
Do[Print["element: ", e]; ijk[1] = (2×NI[[e]]) - 1;
  ijk[2] = (2×NI[[e]]); ijk[3] = (2×NJ[[e]]) - 1; ijk[4] = (2×NJ[[e]]);
      ijk[5] = (2×NK[[e]]) - 1; ijk[6] = (2×NK[[e]]); Ktemp = Kelement[e];
     Do [Do[Kall[[ijk[i], ijk[j]]] = Kall[[ijk[i], ijk[j]]] + Ktemp[[i, j]], {i, 6}],
   {j, 6}], {e, 4}];
"Fill the element K into the integral K(Kall), based on every corrdinated nodes";

Kall / 100000
```

$$
\begin{pmatrix}
8 & 0 & -8 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 3 & -3 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
-8 & -3 & 22 & 5 & -8 & -2 & -6 & -5 & 0 & 5 & 0 & 0 \\
-2 & -3 & 5 & 22 & -3 & -3 & -5 & -16 & 5 & 0 & 0 & 0 \\
0 & 0 & -8 & -3 & 11 & 5 & 0 & 0 & -3 & -2 & 0 & 0 \\
0 & 0 & -2 & -3 & 5 & 11 & 0 & 0 & -3 & -8 & 0 & 0 \\
0 & 3 & -6 & -5 & 0 & 0 & 22 & 5 & -16 & -5 & 0 & 2 \\
2 & 0 & -5 & -16 & 0 & 0 & 5 & 22 & -5 & -6 & 3 & 0 \\
0 & 0 & 0 & 5 & -3 & -3 & -16 & -5 & 22 & 5 & -3 & -2 \\
0 & 0 & 5 & 0 & -2 & -8 & -5 & -6 & 5 & 22 & -3 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & -8 & 0 & 8
\end{pmatrix}
$$

APPENDIX IV

```
EE = 15000000; u = 1/4; t = 0.1;
"System properties: Young's modulus(N/cm²); Poisson's ratio; Plane thickness(cm); ";

NI = {1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 15, 15, 16,
   16, 17, 17, 18, 18, 19, 19, 20, 20, 22, 22, 23, 23, 24, 24, 25, 25, 26, 26, 27, 27, 29, 29,
   30, 30, 31, 31, 32, 32, 33, 33, 34, 34, 36, 36, 37, 37, 38, 38, 39, 39, 40, 40, 41, 41};
NJ = {8, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 14, 15, 16, 16, 17, 17, 18, 18,
   19, 19, 20, 20, 21, 22, 23, 23, 24, 24, 25, 25, 26, 26, 27, 27, 28, 29,
   30, 30, 31, 31, 32, 32, 33, 33, 34, 34, 35, 36, 37, 37, 38, 38, 39, 39,
   40, 40, 41, 41, 42, 43, 44, 44, 45, 45, 46, 46, 47, 47, 48, 48, 49};
NK = {9, 2, 10, 3, 11, 4, 12, 5, 13, 6, 14, 7, 16, 9, 17, 10, 18, 11, 19, 12,
   20, 13, 21, 14, 23, 16, 24, 17, 25, 18, 26, 19, 27, 20, 28, 21, 30,
   23, 31, 24, 32, 25, 33, 26, 34, 27, 35, 28, 37, 30, 38, 31, 39, 32, 40,
   33, 41, 34, 42, 35, 44, 37, 45, 38, 46, 39, 47, 40, 48, 41, 49, 42};
"mesh for plane A: nodes distribution, pick up the i;j;k; of #n element by: ";
"in=NI[[n]]; jn=NJ[[n]]; kn=NK[[n]]";

X = {0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, 0, 1,
   2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6};
Y = {6, 6, 6, 6, 6, 6, 6, 5, 5, 5, 5, 5, 5, 5, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3,
   3, 3, 3, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0};
"X,Y value for plane A nodes, pick up the Xn & Ynj of #n node by: ";
"Xn=X[[n]]; Yn=Y[[n]] ";

Array[Bi, 72]; Array[Bj, 72]; Array[Bk, 72];
Array[Ci, 72]; Array[Cj, 72]; Array[Ck, 72];
Do[ Bi[n] = (Y[[(NJ[[n]])]]) - (Y[[(NK[[n]])]]), {n, 72} ];
Do[ Bj[n] = (Y[[(NK[[n]])]]) - (Y[[(NI[[n]])]]), {n, 72} ];
Do[ Bk[n] = (Y[[(NI[[n]])]]) - (Y[[(NJ[[n]])]]), {n, 72} ];
Do[ Ci[n] = (X[[(NK[[n]])]]) - (X[[(NJ[[n]])]]), {n, 72} ];
Do[ Cj[n] = (X[[(NI[[n]])]]) - (X[[(NK[[n]])]]), {n, 72} ];
Do[ Ck[n] = (X[[(NJ[[n]])]]) - (X[[(NI[[n]])]]), {n, 72} ];
"Define Bi,Bj,Bk,Ci,Cj,Ck for all elements, compute every value by: ";
"Bi[n]=Yj[n]-Yk[n];  Bj[n]=Yk[n]-Yi[n];  Bk[n]=Yi[n]-Yj[n]";
"Ci[n]=Xk[n]-Xj[n];  Cj[n]=Xi[n]-Xk[n];  Ck[n]=Xj[n]-Xi[n]";

Do[ Print["Bi(", i, ")=", (Bi[i]), "        Bj(", i, ")=",
  (Bj[i]), "        Bk(", i, ")=", (Bk[i]), "        Ci(", i, ")=", (Ci[i]),
  "        Cj(", i, ")=", (Cj[i]), "        Ck(", i, ")=", (Ck[i])], {i, 72} ]
```

$$\text{Do}\left[A = \frac{1}{2}\times\text{Det}\left[\begin{pmatrix} 1 & \text{X[[(NI[[n]])]]} & \text{Y[[(NI[[n]])]]} \\ 1 & \text{X[[(NJ[[n]])]]} & \text{Y[[(NJ[[n]])]]} \\ 1 & \text{X[[(NK[[n]])]]} & \text{Y[[(NK[[n]])]]} \end{pmatrix}\right];\ \text{"Print[A]"},\ \{n, 72\}\right]$$

$$\texttt{Array[BB, 72]; Do}\left[\texttt{BB[n]} = \frac{1}{2\,A}\begin{pmatrix} \texttt{Bi[n]} & 0 & \texttt{Bj[n]} & 0 & \texttt{Bk[n]} & 0 \\ 0 & \texttt{Ci[n]} & 0 & \texttt{Cj[n]} & 0 & \texttt{Ck[n]} \\ \texttt{Ci[n]} & \texttt{Bi[n]} & \texttt{Cj[n]} & \texttt{Bj[n]} & \texttt{Ck[n]} & \texttt{Bk[n]} \end{pmatrix},\ \{\texttt{n, 72}\}\right];$$

$$\texttt{DD} = \frac{\texttt{EE}}{1-\texttt{u}^2}\begin{pmatrix} 1 & u & 0 \\ u & 1 & 0 \\ 0 & 0 & \frac{1-u}{2} \end{pmatrix}$$

$$\begin{pmatrix} 16000000 & 4000000 & 0 \\ 4000000 & 16000000 & 0 \\ 0 & 0 & 6000000 \end{pmatrix}$$

```
Array[Kelement, 72];
Do[KK = (Transpose[(BB[n])].DD);
 KK2 = KK.BB[n]; K = t * A * KK2; Kelement[n] = Rationalize[K];
        "Print[Kelement[n]]", {n, 72}]
```

$$\texttt{Kall} = \begin{pmatrix} 0 & \vdots & 0 \\ .. & \vdots & .. \\ 0 & \vdots & 0 \end{pmatrix}_{98\times98};$$

```
Array[ijk, 6];
"2i-1=ijk[1],2i=ijk[2],2j-1=ijk[3],2j-1=ijk[4],2k-1=ijk[5],2k=ijk[6],";
Do[Print["element: ", e]; ijk[1] = (2×NI[[e]]) - 1;
  ijk[2] = (2×NI[[e]]); ijk[3] = (2×NJ[[e]]) - 1; ijk[4] = (2×NJ[[e]]);
      ijk[5] = (2×NK[[e]]) - 1; ijk[6] = (2×NK[[e]]); Ktemp = Kelement[e];
     Do [Do[Kall[[ijk[i], ijk[j]]] = Kall[[ijk[i], ijk[j]]] + Ktemp[[i, j]], {i, 6}],
   {j, 6}], {e, 72}];
"Fill the element K into the integral K(Kall), based on every corrdinated nodes";

"Do[Print[Kall[[i]]],{i,98}]";
    "use to list all Kall element";
```

$$\texttt{U} = \begin{pmatrix} U1 \\ U2 \\ U3 \\ \vdots \\ \vdots \\ U12 \\ U13 \\ U14 \\ U15 \\ \vdots \\ \vdots \\ U84 \\ U85 \\ U86 \\ \vdots \\ \vdots \\ U98 \end{pmatrix};\ \texttt{F} = \begin{pmatrix} f1 \\ f2 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -100 \\ -100 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ f85 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix};$$

```
Solve[{Kall.U == F, U1 == 0, U2 == 0, U85 == 0},
 {f1, f2, f85, U1, U2, U3, U4, U5, U6, U7, U8, U9, U10, U11, U12, U13, U14, U15, U16,
  U17, U18, U19, U20, U21, U22, U23, U24, U25, U26, U27, U28, U29, U30, U31, U32, U33,
  U34, U35, U36, U37, U38, U39, U40, U41, U42, U43, U44, U45, U46, U47, U48, U49, U50,
  U51, U52, U53, U54, U55, U56, U57, U58, U59, U60, U61, U62, U63, U64, U65, U66,
  U67, U68, U69, U70, U71, U72, U73, U74, U75, U76, U77, U78, U79, U80, U81, U82,
  U83, U84, U85, U86, U87, U88, U89, U90, U91, U92, U93, U94, U95, U96, U97, U98}]
```

$\{\{f1 \to 0,\ f2 \to 100,\ f85 \to 100,\ U1 \to 0,\ U2 \to 0,$

$U3 \to -\frac{24075920469190150494123730208034627941724498147485521256707108872468174964936 51}{34754280563764201964667714108193024093549345336569725904035771913447301943108 3040000},$

$U4 \to -\frac{1240118079660558731107203417149144524903115498671995043630743888585903681914279}{6516427605705787868375196395286192017540502250606823607006707233771369114332807000},$

$U5 \to -\frac{2059543348633698800959755489598990805529647954073087896933575375917428236382627}{57923800939607003274446190180321706822582242227616209840059619855745503238513840000},$

$U6 \to -\frac{1481837928189712029859843475336704801854606190320752555888029446692548885359 11839}{52131420845646302947001571162289536140324018004854588856053657870170952914662456000},$

$U7 \to -\frac{2213717000576217295794204802513165796625502067526731468343245273774573969782651 3}{34754280563764201964667714108193024093549345336569725904035771913447301943108 3040000},$

$U8 \to -\frac{1022794547568881169902438630364855134411101948110483020096642720950587875331467}{2896190046980350163722309509016085341129112111380810492002980992787275161925692000},$

$U9 \to -\frac{8159494195900848084706599775574783566907639926889935255106568031324591080886601}{86885701409410504911669285270482560233873363341424314760089429783618254857770760000},$

$U10 \to -\frac{3626936133420891916259467703259369739392842964358328380235199402938555582338309}{8688570140941050491166928527048256023387336334142431476008942978361825485777076000},$

$U11 \to -\frac{311261400596633478201160767300198386486351955936267201508021301127272941289760 1}{23169520375842801309778476072128682729032896891046483936023847942298201295405536000},$

$U12 \to -\frac{2123383606361286489971817918389014384523235983528035869708374677817852767676479}{4344285070470525245583464263524128011693668167071215738004471489180912742888538000},$

$U13 \to -\frac{336921023644136703529726737095690613179023540008810576598144439921509947136238 03}{17377140281882100982333857054096512046774672668284862952017885956723650971554152 0000},$

$U14 \to -\frac{17774722960773464381322239379519487628 33}{298265120989306232853340630969426565071 2000},$

$U15 \to -\frac{10279347345505920387782176010704113180633680566364116940940140541349400676203011}{1042628416912926058940031423245790722806480360097091777121073157403419058293249 12000},$

$U16 \to -\frac{33743948374049089314060247323220758465272862625047116106898080268663218604372589}{34754280563764201964667714108193024093549345336569725904035771913447301943108 3040000},$

$U17 \to -\frac{117947659526741963745845406028882056214757888385323187667415408875588265304610003}{10426284169129260589400314232457907228064803600970917771210731574034190582932491 20000},$

$U18 \to -\frac{213339190069810914120041910443423915327120387149070243633644408859972757170772107}{10426284169129260589400314232457907228064803600970917771210731574034190582932491 20000},$

$U19 \to -\frac{15053847884544485964477221380063016033688175936937182069317454636531416384727727}{13032855211411575736750392790572384035081004501213647214013414467542738228665614 0000},$

$U20 \to -\frac{302539156432708362842088963187000869264877392172223109936881021805737136971417647}{10426284169129260589400314232457907228064803600970917771210731574034190582932491 20000},$

$U21 \to -\frac{130177900967636184385373402405152572901289037448588859903504928231878516958476929}{10426284169129260589400314232457907228064803600970917771210731574034190582932491 20000},$

$U22 \to -\frac{41302365069422922908959207825707820490596016876953136039477074853308481029922423}{11584760187921400654889238036064341364516448445523241968011923971149100647702768 0000},$

$U23 \to -\frac{72852482455170235647385151398271334582444842343627923547161990691485780222679101}{52131420845646302947001571162289536140324018004854588856053657870170952914662456 0000},$

$U24 \to -\frac{11164325855744279166436188539559201070554814753817030072565585032757399046996393}{26734061972126309203590549313994633918114881028130558387719824548805616879314080000},$

$$U25 \to -\frac{65349066667637854553296350845489868753113142797103604772484147305759552987 15143}{417051366765170423576012569298316289122592144038836710848429262961367623317299 64800},$$

$$U26 \to -\frac{5548178996168444605114110790645733223831855954297049707161397558657736349584 5883}{1158476018792140065488923803606434136451644844552324196801192397114910064770276 80000},$$

$$U27 \to -\frac{33324724802330733327854202993157205136506598132585505387573044053857675393 97179}{2005054647909473190269291198549597543858616077109791879078986841160421265948 5560000},$$

$$U28 \to -\frac{451431750357121510883861500865835968840823270473151706865 8567}{841610387819992252090104841537480113349364670240546273753 0720000},$$

$$U29 \to -\frac{2496862648840747844467259601357865909858310209913095644597484957554680630 48327}{16291069014264469670937990988215480043851255626517059017516768084428422785832 01750},$$

$$U30 \to -\frac{29331382326278761244717801491485661579522416819081876822557832756103433442 920039}{17377140281882100982333857054096512046774672668284862952017885956723650971554 1520000},$$

$$U31 \to -\frac{1712420497606071306341314139004593295151517034869056440055834439196998667621 92333}{10426284169129260589400314232457907228064803600970917771210731574034190582932 49120000},$$

$$U32 \to -\frac{1193605371605328454053387886783468362477424548981497898805324564642820085922 71787}{52131420845646302947001571162289536140324018004854588856053657870170952914662 4560000},$$

$$U33 \to -\frac{66069995094000897496203262869345358843891578897262644045797658011426367250 61671}{40101092958189463805385823970991950877172321542195837581579736823208425318971 120000},$$

$$U34 \to -\frac{1172186910110865051565635286706917344294104319880091806933618499250102362472 2519}{40101092958189463805385823970991950877172321542195837581579736823208425318971 120000},$$

$$U35 \to -\frac{17225876320008073282208547332348756548787368138480274994303895522305606097933 7759}{10426284169129260589400314232457907228064803600970917771210731574034190582932 49120000},$$

$$U36 \to -\frac{61255812647354562237994479654706518751284055803039417542784485565367218972801 889}{17377140281882100982333857054096512046774672668284862952017885956723650971554 1520000},$$

$$U37 \to -\frac{33872977863794263889563792085237096490342130306752753203254333344090847745976 71}{2005054647909473190269291198549597543858616077109791879078986841160421265948 5560000},$$

$$U38 \to -\frac{78954039598365824976000092805774277820972948035137954090613829088291566524173 31}{19307933646535667758148730060107235607527414075872069946686539951915167746171 280000},$$

$$U39 \to -\frac{36104812635842126964911830173314115967865492388956046941073884832955188359327 749}{20852568338258521178800628464915814456129607201941835542421463148068381165864 9824000},$$

$$U40 \to -\frac{42073711821613044515068331872439857171101732964944632098713208739230168140600 71}{91458633062537373591230826600507958140919329833078226063252031351177110376600 80000},$$

$$U41 \to -\frac{89986670214447333434445684987128919247461387734983151063053774042967763159563 569}{52131420845646302947001571162289536140324018004854588856053657870170952914662 4560000},$$

$$U42 \to -\frac{86303717143006777665096798036634252486294163071228952332937565728607499364890 589}{17377140281882100982333857054096512046774672668284862952017885956723650971554 1520000},$$

$$U43 \to -\frac{62275937974143485749038558923514762237208873854653749344861184980024167233199 03}{34754280563764201964667714108193024093549345336569725904035771913447301943108 304000},$$

$$U44 \to -\frac{25924735093365357472005661275391681280608760492804995720616183795161787779498 789}{11584760187921400654889238036064341364516448445523241968011923971149100647702 7680000},$$

$$U45 \to -\frac{65084661062741094076799989718636365807312853834884255632851204851128360734151 401}{34754280563764201964667714108193024093549345336569725904035771913447301943108 3040000},$$

$$U46 \to -\frac{26184989766330698257027631894840575743512850307614361281092923791599434603346 6861}{10426284169129260589400314232457907228064803600970917771210731574034190582932 49120000},$$

$$U47 \to -\frac{16356096827774845307641326560853193641363115602755890710806956817582979944427 523}{86885701409410504911669285270482560233873363341424314760089429783618254857770 760000},$$

$$U48 \to -\frac{30724374758462489930859823753024413991992272343941714404446734203817178607162 9021}{10426284169129260589400314232457907228064803600970917771210731574034190582932 49120000},$$

$$U49 \to -\frac{65180635574278047374675685565629304138723620186076554274934705704064262923842 723}{34754280563764201964667714108193024093549345336569725904035771913447301943108 3040000},$$

$$U50 \to -\frac{12024277287670368383031306962068535194947066227794589824293438877797682099332 8987}{34754280563764201964667714108193024093549345336569725904035771913447301943108 3040000},$$

$$U51 \to -\frac{10864960485416698843397382362362667294003058836891391030235754248629037634775 559}{57923800939607003274446190180321706822582242227616209840059619855745503238513 840000},$$

$$U52 \to -\frac{10622986420959809679966565871589323923111425272507632309340362775797318291117619}{26734061972126309203590549313994633918114881028130558387719824548805616879314080000},$$

$$U53 \to -\frac{1041978925199901181530481831079023294134015295146426865941141664485488354486 21}{5560684890202272314346834257310883854967895253851156144645723506151568310897328 64},$$

$$U54 \to -\frac{15362071263291148381398363484857824010889897643080654753423867312171808685618 1607}{34754280563764201964667714108193024093549345336569725904035771913447301943108304 0000},$$

$$U55 \to -\frac{50237502982093806312388021037288591463377469851970945501493340103793494956278 7}{27151781690440782784896651647025800073085426044195098362527946807380704643053362 50},$$

$$U56 \to -\frac{54569418839793723608839296273310983778475856721124419056235282228659359181907 949}{11584760187921400654889238036064341364516448445523241968011923971149100647702768 0000},$$

$$U57 \to -\frac{32802769230862019067622768352095441995373570867715003541871424440731405519091}{18101187793627188523264434431350533382056950696130065575018631204920469762035575 0},$$

$$U58 \to -\frac{23334421602208627137667188614407646234756650913811311735704306697045571346251 719}{86885701409410504911669285270482560233873363341424314760089429783618254857770760 000},$$

$$U59 \to -\frac{18716600938515252262871626904862321850617468800116568607386261552470258794062 1}{99015044341208552605890923385165310807832892696779845880443794625205988441904000 0},$$

$$U60 \to -\frac{69736674015738563518992457174669853821091945124770471978262095166173834807040 137}{26065710422823151473500785581144768070162009002427294428026828935085476457331228 0000},$$

$$U61 \to -\frac{33676509916384171380182634558744486189368057605028803901880398283207229145207 081}{17377140281882100982333857054096512046774672668284862952017885956723650971554152 0000},$$

$$U62 \to -\frac{37046579853175535164148810035975079951074112735021955369583709210764625232559 9}{12531591549434207439183069990934984649116350481936199244243667757252632912178475 00},$$

$$U63 \to -\frac{17588469663004193690498178234956186903896580179099036852452749164129830463484 87}{89113539907087697345301831046648779727049603427101861292399415162685389597713600 00},$$

$$U64 \to -\frac{98421685167765788383010082309600566324447432840256258815935483253176204587763 43}{28961900469803501637223095090160853411291121113808104920029809927872751619256920 000},$$

$$U65 \to -\frac{17365748488109925029153220874059270795600089500752498143336820685147702331812 201}{86885701409410504911669285270482560233873363341424314760089429783618254857770760 000},$$

$$U66 \to -\frac{33662301624615491112327411998360574033372080324044804413403558448832488241449 209}{86885701409410504911669285270482560233873363341424314760089429783618254857770760 000},$$

$$U67 \to -\frac{46467083572886051267948552081849748426107789512628866650426795979260694316287 93}{23169520375842801309778476072128682729032896891046483936023847942298201295405536 000},$$

$$U68 \to -\frac{37258995748189859724797941642052847058073323863856330971957776054848256671469 209}{86885701409410504911669285270482560233873363341424314760089429783618254857770760 000},$$

$$U69 \to -\frac{34623129341501359717008923557308795897371679545611760630370476975129315867640 763}{17377140281882100982333857054096512046774672668284862952017885956723650971554152 0000},$$

$$U70 \to -\frac{19813311467682074619613044826639884691334519981927871095177613899728324617962 977}{43442850704705252455834642635241280116936681670712157380044714891809127428885380 000},$$

$$U71 \to -\frac{56949190075991910940873746433683149152717845397290828762062521142780700662395 7}{38615867293071335516297460120214471215054828151744139893373079903830335492342560 00},$$

$$U72 \to -\frac{21505592277973830952031830044283220947056365271440607304402493869186865692082 941}{69508561127528403929335428216386048187098690673139451808071543826894603886216608 000},$$

$$U73 \to -\frac{19319182827010288570102868537440440735752278910585267827931016430176483074792 267}{11584760187921400654889238036064341364516448445523241968011923971149100647702768 0000},$$

$$U74 \to -\frac{57897718357627718592769238134274778080604825169951266972605382449380409049959 107}{20852568338258521178800628464915814456129607201941835542421463148068381165864982 4000},$$

$$U75 \to -\frac{42669117199202115629531441523689970346339984629389593987314919635397645008942 1}{22864658265634343397807706650126989535229832458269556515813007837794277594150200 00},$$

$$U76 \to -\frac{61633912587799874770528044217820582467076154761957619890131678483232456394992 711}{20852568338258521178800628464915814456129607201941835542421463148068381165864982 4000},$$

$$U77 \to -\frac{70393685154110708973348588959486050341962532865588987347521882063141661925604 5083}{34754280563764201964667714108193024093549345336569725904035771913447301943108304 0000},$$

$$U78 \to -\frac{23363563748425462001166282985384432574609487107112025803332032179276923505785 437}{69508561127528403929335428216386048187098690673139451808071543826894603886216608 000},$$

$$U79 \to -\frac{123178078648732834208476715695980443515384155674532167706063092506963419270 02029}{579238009396070032744461901803217068225822422276162098400596198557455032385138 40000},$$

$$U80 \to -\frac{265379672263363802569329926182793524331466689325111643391743461198513198501 52669}{695085611275284039293354282163860481870986906731394518080715438268946038862166 08000},$$

$$U81 \to -\frac{120746789515786911786578163981978144763398211020798512940026909535096184188325}{556068489020227231434683425731088385496789525385115614464572350615156831089732864},$$

$$U82 \to -\frac{977181112062493959826954313750552624133913058527675493302353158572831399793 6339}{231695203758428013097784760721286827290328968910464839360238479422982012954055 36000},$$

$$U83 \to -\frac{630100558696481472457685094713614851507276695130785664917110189305478703954 2873}{289619004698035016372230950901608534112911211138081049200298099278727516192569 20000},$$

$$U84 \to -\frac{762904414743221335785898096792034908599766709485547776983386950482238862103661}{169533075920800985193501044430209873627069977251559638556272058114377082649308 8000}, U85 \to 0,$$

$$U86 \to -\frac{632590365209458839006295105791518736886482228566184492936607625665748085428 09717}{173771402818821009823338570540965120467746726682848629520178859567236509715541 520000},$$

$$U87 \to -\frac{4141303147335917247565264958354339161974 81647}{3696747318729169550998017741225752060941858080000},$$

$$U88 \to -\frac{115324969446734333740586260427769135459435716796973715029756571721001092016 69597}{401010929581894638053858239709919508771723215421958375815797368232084253189711 20000},$$

$$U89 \to -\frac{311782206193166695317361275277033503698002993836345516817572990120014207621 15561}{173771402818821009823338570540965120467746726682848629520178859567236509715541 520000},$$

$$U90 \to -\frac{155904276176392353915865267715108669026569681354812131179811791541002846036 440981}{521314208456463029470015711622895361403240180048545888560536578701709529146624 560000},$$

$$U91 \to -\frac{760359135581425065331097956542717316732458973297558567397265176218687833837 61973}{347542805637642019646677141081930240935493453365697259040357719134473019431083 040000},$$

$$U92 \to -\frac{651096093684435846564120868433202244358704259722989995590619536669539271281 8783}{193079336465356677581487300601072356075274140758720699466865399519151677461712 80000},$$

$$U93 \to -\frac{689365875368744962866616611187747685484627793673963258137828845782911173968 0707}{289619004698035016372230950901608534112911211138081049200298099278727516192569 20000},$$

$$U94 \to -\frac{662376962822067571289542836562002630320097959905977324158627274847652763536 72537}{173771402818821009823338570540965120467746726682848629520178859567236509715541 520000},$$

$$U95 \to -\frac{413857284088650504348038792669732326250329859144444776998109692535591123855 871}{169533075920800985193501044430209873627069977251559638556272058114377082649308 8000},$$

$$U96 \to -\frac{730156426025698882589968791535443750791768580011222286495458960401260328293 29627}{173771402818821009823338570540965120467746726682848629520178859567236509715541 520000},$$

$$U97 \to -\frac{471679342659904223176590588335165082120348355397129520764273360447373362733 8467}{193079336465356677581487300601072356075274140758720699466865399519151677461712 80000},$$

$$U98 \to -\frac{260252991266536944617225101733284051999811880655796737976725725617750876569 37209}{579238009396070032744461901803217068225822422276162098400596198557455032385138 40000}\}\}$$

```
Ulist =
 {U1, U2, U3, U4, U5, U6, U7, U8, U9, U10, U11, U12, U13, U14, U15, U16, U17, U18, U19, U20,
   U21, U22, U23, U24, U25, U26, U27, U28, U29, U30, U31, U32, U33, U34, U35, U36, U37, U38,
   U39, U40, U41, U42, U43, U44, U45, U46, U47, U48, U49, U50, U51, U52, U53, U54, U55,
   U56, U57, U58, U59, U60, U61, U62, U63, U64, U65, U66, U67, U68, U69, U70,
   U71, U72, U73, U74, U75, U76, U77, U78, U79, U80, U81, U82, U83, U84, U85,
   U86, U87, U88, U89, U90, U91, U92, U93, U94, U95, U96, U97, U98} /. %
"Give U(n) value to list of Ulist. Ulist[[1,n]]=Un";

"double check if Ulist is correct."; Ulist[[1, 3]]
```

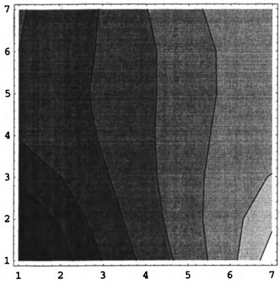
$$TD = Table\left[\sqrt{(Ulist[[1, (2\times(7\times(j-1)+i))]])^2 + (Ulist[[1, (2\times(7\times(j-1)+i)-1)]])^2},\ \{j, 7\}, \{i, 7\}\right];$$

```
  "TD[[i,j]] is the displacement at position {i,j}";
```

```
N[%]
```

$$\begin{pmatrix} 0 & 0.000397902 & 0.000636563 & 0.000793737 & 0.000913013 & 0.00101909 & 0.00112839 \\ 0.00055708 & 0.000663627 & 0.000770334 & 0.000857346 & 0.000936392 & 0.00101069 & 0.00107818 \\ 0.000776718 & 0.000833543 & 0.000872305 & 0.000914931 & 0.000965958 & 0.00102364 & 0.0010854 \\ 0.00087507 & 0.00089985 & 0.000915151 & 0.000942044 & 0.000984314 & 0.00103857 & 0.00110172 \\ 0.000890793 & 0.000878236 & 0.000893197 & 0.000931388 & 0.000985405 & 0.00104844 & 0.00111826 \\ 0.000816166 & 0.000759257 & 0.000811204 & 0.000892521 & 0.00097927 & 0.00106492 & 0.00114791 \\ 0.000740533 & 0.000549315 & 0.000709462 & 0.000864815 & 0.000992764 & 0.00110837 & 0.00123026 \end{pmatrix}$$

```
ListContourPlot[TD]
```



- ContourGraphics -

APPENDIX V

$$KA = \begin{pmatrix}
3 & 0 & 0 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 2 & 0 & -2 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 22 & 5 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 5 & 22 & -5 & -6 & 0 & 0 & -5 & -16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -2 & -16 & -5 & 22 & 5 & -3 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -8 & -5 & -6 & 5 & 22 & -2 & -8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -2 & 22 & 5 & -16 & -5 & 0 & 0 & -3 & -3 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -8 & 5 & 22 & -5 & -6 & 0 & 0 & -2 & -8 & 5 & 0 & 0 & 0 \\
0 & 0 & -6 & -5 & 0 & 5 & -16 & -5 & 44 & 10 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 5 \\
0 & 0 & -5 & -16 & 5 & 0 & -5 & -6 & 10 & 44 & -5 & -6 & 0 & 0 & -5 & -16 & 5 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -16 & -5 & 22 & 5 & 0 & 0 & 0 & 0 & -6 & -5 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -5 & -6 & 5 & 22 & 0 & 0 & 0 & 0 & -5 & -16 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 & 11 & 5 & -8 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -8 & 0 & 0 & 0 & 0 & 5 & 11 & -2 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -8 & -2 & 22 & 5 & -8 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & -3 & 5 & 22 & -2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -8 & -2 & 22 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & -3 & 5 & 22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -8 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -3 & -3
\end{pmatrix}$$

$$KB = \begin{pmatrix}
8 & 0 & -8 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 3 & -3 & -3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
-8 & -3 & 22 & 5 & -8 & -2 & -6 & -5 & 0 & 5 & 0 & 0 \\
-2 & -3 & 5 & 22 & -3 & -3 & -5 & -16 & 5 & 0 & 0 & 0 \\
0 & 0 & -8 & -3 & 11 & 5 & 0 & 0 & -3 & -2 & 0 & 0 \\
0 & 0 & -2 & -3 & 5 & 11 & 0 & 0 & -3 & -8 & 0 & 0 \\
0 & 3 & -6 & -5 & 0 & 0 & 22 & 5 & -16 & -5 & 0 & 2 \\
2 & 0 & -5 & -16 & 0 & 0 & 5 & 22 & -5 & -6 & 3 & 0 \\
0 & 0 & 0 & 5 & -3 & -3 & -16 & -5 & 22 & 5 & -3 & -2 \\
0 & 0 & 5 & 0 & -2 & -8 & -5 & -6 & 5 & 22 & -3 & -8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & -8 & 0 & 8
\end{pmatrix} * 100000;$$

$$Kcombined = \begin{pmatrix}
KA & : & O \\
.. & : & .. \\
O & : & KB
\end{pmatrix}$$

$$
\mathrm{UI} = \begin{pmatrix} U1 \\ U2 \\ U3 \\ U4 \\ U5 \\ U6 \\ U7 \\ U8 \\ U9 \\ U10 \\ U11 \\ U12 \\ U13 \\ U14 \\ U15 \\ U16 \\ U17 \\ U18 \\ U19 \\ U20 \\ V1 \\ V2 \\ V3 \\ V4 \\ V5 \\ V6 \\ V7 \\ V8 \\ V9 \\ V10 \\ V11 \\ V12 \end{pmatrix};\ F = \begin{pmatrix} f1 \\ f2 \\ f3 \\ f4 \\ f50 \\ f60 \\ f70 \\ f80 \\ f90 \\ f100 \\ f11 \\ f12 \\ f13 \\ f140 \\ f150 \\ f160 \\ f170 \\ f180 \\ f19 \\ f20 \\ g1 \\ g2 \\ g30 \\ g40 \\ g50 \\ g60 \\ g7 \\ g8 \\ g90 \\ g100 \\ g11 \\ g12 \end{pmatrix};\ \mathrm{UII} = \begin{pmatrix} U1 \\ U2 \\ U3 \\ U4 \\ U5 \\ U6 \\ U7 \\ U8 \\ U9 \\ U10 \\ U11 \\ U12 \\ U13 \\ U14 \\ U15 \\ U16 \\ U17 \\ U18 \\ U19 \\ U20 \\ V3 \\ V4 \\ V5 \\ V6 \\ V9 \\ V10 \end{pmatrix};\ \mathrm{Uall} = \begin{pmatrix} U3 \\ U4 \\ U5 \\ U6 \\ U7 \\ U8 \\ U9 \\ U10 \\ U11 \\ U12 \\ U14 \\ U15 \\ U16 \\ U17 \\ U18 \\ U19 \\ U20 \\ V3 \\ V4 \\ V5 \\ V6 \\ V9 \\ V10 \end{pmatrix};
$$

$$
\mathrm{TU1} = \left(\begin{array}{cccccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right);
$$

$$
TU2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix};
$$

TU = TU1.TU2; Test Transformation matrix by : TU.Uall "

KII = Kcombined.TU;

$$UF = \begin{pmatrix} U3 \\ U4 \\ U5 \\ U6 \\ U7 \\ U8 \\ U9 \\ U10 \\ U11 \\ U12 \\ U14 \\ U15 \\ U16 \\ U17 \\ U18 \\ U19 \\ U20 \\ V3 \\ V4 \\ V5 \\ V6 \\ V9 \\ V10 \\ f1 \\ f2 \\ f3 \\ f4 \\ f11 \\ f12 \\ f13 \\ f19 \\ f20 \\ g1 \\ g2 \\ g7 \\ g8 \\ g11 \\ g12 \end{pmatrix}; \; Zero = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ -100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \; KIII =$$

$$
\left(\begin{array}{ccccccccccccccccccc}
0 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & -2 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
22 & 5 & -16 & -5 & 0 & 0 & -6 & -5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 22 & -5 & -6 & 0 & 0 & -5 & -16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-16 & -5 & 22 & 5 & -3 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & -6 & 5 & 22 & -2 & -8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & -2 & 22 & 5 & -16 & -5 & 0 & 0 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & -8 & 5 & 22 & -5 & -6 & 0 & 0 & -8 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -5 & 0 & 5 & -16 & -5 & 44 & 10 & -16 & -5 & 0 & -6 & -5 & 0 & 5 & 0 & 0 & 0 & 0 \\
-5 & -16 & 5 & 0 & -5 & -6 & 10 & 44 & -5 & -6 & 0 & -5 & -16 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & -16 & -5 & 22 & 5 & 0 & 0 & 0 & -6 & -5 & 0 & 3 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & -5 & -6 & 5 & 22 & 0 & 0 & 0 & -5 & -16 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 & 5 & -8 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & -8 & 0 & 0 & 0 & 0 & 11 & -2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & 0 & -2 & 22 & 5 & -8 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & 0 & -3 & 5 & 22 & -2 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & -5 & 0 & -8 & -2 & 22 & 5 & -8 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & -5 & -16 & 0 & -3 & -3 & 5 & 22 & -2 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -8 & -2 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 & -3 & 0 & 3 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -2 \\
\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3 \\
-3 & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22 & 5 \\
-\frac{5}{2} & -8 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -3 \\
11 & \frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 11 & \frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -6 & -5 \\
\frac{5}{2} & 11 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & 11 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -5 & -16 \\
-8 & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 0 & 5 \\
-\frac{5}{2} & -3 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} & -3 & 0 & 0 & 0 & 0 & 0 & -3 & -8 & 5 & 0 \\
0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.
$$

KUF = KIII +

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

```
Det[KUF]
```

2184935404930738421760000000000000000000000000000000000000000000000000000.
000000000000000000000000000000000000000000000000000000000

```
Solve[{KUF.UF == Zero}, {U3, U4, U5, U6, U7, U8, U9, U10, U11,
  U12, U14, U15, U16, U17, U18, U19, U20, V3, V4, V5, V6, V9, V10,
  f1, f2, f3, f4, f11, f12, f13, f19, f20, g1, g2, g7, g8, g11, g12}]
```

$$\{\{U3 \to -\frac{-2041150230705}{13317750352961700},\ U4 \to -\frac{25769698430570}{31723750352961700},\ U5 \to -\frac{8662215172565}{56517750352961700},$$
$$U6 \to -\frac{79359420331675}{110981252941347500},\ U7 \to \frac{-12067138811082}{66588751764808500},\ U8 \to -\frac{91853071812403}{110981252941347500},$$
$$U9 \to \frac{-105415086089231}{543943758824042500},\ U10 \to \frac{269461324093314}{332943758824042500},\ U11 \to$$
$$\frac{-247924400314217}{867943758824042500},\ U12 \to \frac{866230479658018}{975943758824042500},\ U14 \to \frac{666645560737}{5549062647067375 0},$$
$$U15 \to \frac{-19912355731059}{110981252941347500},\ U16 \to \frac{397587017928351}{563943758824042500},\ U17 \to$$
$$\frac{-26416321906499}{110981252941347500},\ U18 \to \frac{564259691405924}{656943758824042500},\ U19 \to \frac{-282201884005113}{987943758824042500},$$
$$U20 \to \frac{500819713669427}{432943758824042500},\ V3 \to -\frac{-678632499652849}{10654200282369360000},\ V4 \to$$
$$-\frac{1641723224448110}{21308400564738720 00},\ V5 \to -\frac{66686394550089}{343943758824042500},\ V6 \to \frac{2697536739171}{266355007059234 0},$$
$$V9 \to -\frac{-10056869673993300}{54354200282369360000},\ V10 \to \frac{83087993445792600}{87867200282369360000},\ f1 \to$$
$$\frac{84346766723995}{88785002353078},\ f2 \to \frac{8621943225249440}{78646502353078},\ f3 \to -\frac{183863918674800}{44392501176539},\ f4 \to$$
$$-\frac{93998899125853}{4345401176539},\ f11 \to \frac{694453503415314}{54654654176539},\ f12 \to -\frac{703562101587}{142634576539},\ f13 \to$$
$$-\frac{469089364343894000}{4756754745176539},\ f19 \to \frac{7609309641345}{456456456539},\ f20 \to \frac{1721670875290200}{88957644176539},$$
$$g1 \to -\frac{31790674735717}{33463465653078},\ g2 \to -\frac{9733417591862140}{88785002353078},\ g7 \to \frac{55046800022816}{6534524876539},$$
$$g8 \to \frac{87351542020393}{6576579176539},\ g11 \to -\frac{9463422511822380}{567678345353078},\ g12 \to -\frac{19307074104135}{997584296102}\}\}$$

```
N[%]

{{U3→ -0.00015326539217269, U4→ -0.00015326539217269, U5→
  -0.00015326539217269, U6→ -0.00071475324324991, U7→ -0.0001812188769314,
  U8→ -0.00024856457649176903,U9→ -0.0001937977674698, U10→
  -0.000285645697424133, U11→ -0.000285645697424133, U12→
  -0.000285645697424133, U14→ -0.00036403594316897, U15→ -0.00017942089500091,
  U16→ -0.00029906009398439, U17→ -0.000238025082672836,
  U18→ -0.00038117719721275, U19→ -0.000285645697424133,
  U20→ -0.000285645697424133, V3→ -0.00006369624013694,V4→
  -0.00035315173762, V5→ -0.0001938874971248, V6→ -0.0005959370274947,
  V9→ -0.00018502470134319, V10→ -0.00047104487235471, f1→
  0.95001142635067, f2→ 109.62907398655632, f3→ -4.141778764472281, f4→
  -21.631811496107005,f11→ 12.70620981649933, f12→ -4.932619555920048, f13→
  98.61541943476531, f19→ 16.670395461247498, f20→ 19.353827219991384, g1→
  -0.95001142635067, g2→ -109.62907398655632, g7→ 8.423994255565799, g8→
  13.2822155220159, g11→ -16.670395461247498, g12→ -19.353827219991384}}

U1 = 0; U2 = 0; U13 = 0; V1 = 0; V2 = 0; V7 = (U3 + U11) / 2;
V8 = (U4 + U12) / 2; V11 = U19; V12 = U20;
"Boundary Conditions and Displacement Constrain Equation";

Ulist = {U1, U2, U3, U4, U5, U6, U7, U8, U9,
   U10, U11, U12, U13, U14, U15, U16, U17, U18, U19, U20};
Vlist = {V1, V2, V3, V4, V5, V6, V7, V8, V9, V10, V11, V12};
"Give displacement value to list of Ulist and Vlist";

    "Compute the general displacement at nodal position}";
UD = Table[(√((Ulist[[1, ((2×n) - 1)]])^2 + (Ulist[[1, (2×n)]])^2)), {n, 1, 10}] ;
N[%]

{0.000000000000000000, 0.000826648034066665,
  0.000731311222584203, 0.000847252275786886,
  0.000832209342095123, 0.000932414037579913, 0.000684510360745850,
  0.000727484384202640, 0.000891287339200160, 0.001191523504541550}

VD = Table[(√((Vlist[[1, ((2×n) - 1)]])^2 + (Vlist[[1, (2×n)]])^2)), {n, 1, 6}] ;
N[%]

{0.000000000000000000, 0.000773086716148328, 0.001031152270031470,
   0.000879531000000000, 0.000963540372081521,0.001191523504541550}
```

APPENDIX VI

The Result of FEA computation and Displacement Difference Analysis

Reference Model Result: (From re-meshed 7x7 modal, 72 Elements)

0	0.0003	0.00059	0.00079	0.00091	0.00102	0.001128
0.00042	0.00046	0.00077	0.00086	0.00094	0.00101	0.001078
0.00078	0.00083	0.00087	0.00091	0.00097	0.00102	0.001085
0.00088	0.0009	0.00092	0.00094	0.00098	0.00104	0.001102
0.00089	0.00088	0.00089	0.00093	0.00099	0.00105	0.001118
0.00082	0.00076	0.00081	0.00089	0.00098	0.00106	0.001148
0.00074	0.00065	0.00071	0.00086	0.00099	0.00111	0.00123

Modular Model Result (interpolation value in yellow area).

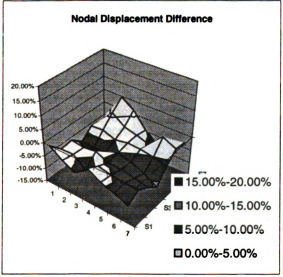
0	0.00026	0.00052	0.00077	0.00086	0.00095	0.001031
0.00037	0.00041	0.00067	0.00081	0.00088	0.00094	0.001009
0.00073	0.00078	0.00083	0.00084	0.00089	0.00094	0.000986
0.00079	0.00081	0.00083	0.00088	0.00091	0.00094	0.000964
0.00085	0.00084	0.00083	0.00088	0.00093	0.00099	0.00104
0.00077	0.00077	0.00078	0.00085	0.00091	0.00106	0.001116
0.00068	0.00071	0.00073	0.00081	0.00089	0.00104	0.001192

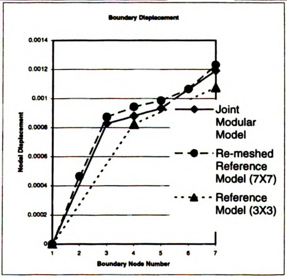
Boundary Displacement:

1	0
4	0
7	0
1	0
3	0
4	0
5	0
7	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0

Displacement difference: (result-reference)/MAX(reference) (%)

0.00%	-3.27%	-5.79%	-1.68%	-4.38%	-6.01%	-7.91%
-4.18%	-4.09%	-8.07%	-3.97%	-4.97%	-5.59%	-5.66%
-3.69%	-4.44%	-3.71%	-5.76%	-6.06%	-6.90%	-8.07%
-6.97%	-7.36%	-6.97%	-5.08%	-6.24%	-8.38%	-11.23%
-3.54%	-3.13%	-4.96%	-3.99%	-4.31%	-5.08%	-6.40%
-4.09%	1.11%	-2.55%	-3.79%	-5.48%	-0.24%	-2.63%
-4.55%	4.61%	1.47%	-4.51%	-8.25%	-5.44%	-3.15%





The Result of FEA computation and Displacement Difference Analysis (2)

Reference Model Result: (From 3x3 (8 elements) model).

0	0.00024	0.00049	0.00073	0.00084	0.00095	0.00106
0.00026	0.00027	0.00059	0.00076	0.00085	0.00093	0.00102
0.00051	0.0006	0.00055	0.00079	0.00086	0.00092	0.00098
0.00077	0.00078	0.0008	0.00082	0.00086	0.0009	0.00094
0.00073	0.00076	0.00079	0.00081	0.00091	0.00093	0.00099
0.0007	0.00073	0.00077	0.0008	0.00088	0.00099	0.00103
0.00066	0.00071	0.00075	0.00079	0.00089	0.00098	0.00108

Modular Model Result (interpolation value in yellow area).

0	0.00026	0.00052	0.00077	0.00086	0.00095	0.00103
0.00037	0.00041	0.00067	0.00081	0.00088	0.00094	0.00101
0.00073	0.00078	0.00083	0.00084	0.00089	0.00094	0.00099
0.00079	0.00081	0.00083	0.00088	0.00091	0.00094	0.00096
0.00085	0.00084	0.00083	0.00088	0.00093	0.00099	0.00104
0.00077	0.00077	0.00078	0.00085	0.00091	0.00106	0.00112
0.00068	0.00071	0.00073	0.00081	0.00089	0.00104	0.00119

Displacement difference: (result-reference)/MAX(reference) (%)

0.00%	1.16%	2.31%	3.47%	1.52%	-0.43%	-2.38%
9.66%	12.13%	6.73%	3.95%	2.31%	0.68%	-0.95%
19.32%	15.24%	24.25%	4.42%	3.11%	1.79%	0.48%
2.11%	2.17%	2.22%	4.90%	3.90%	2.90%	1.91%
10.22%	7.13%	4.03%	5.97%	2.19%	5.00%	4.66%
6.16%	3.59%	1.03%	3.62%	2.80%	6.18%	7.42%
2.09%	0.06%	-1.97%	1.27%	0.27%	5.22%	10.17%