This is to certify that the

dissertation entitled

Exponentiated Gradient Portfolios
in Continuous Trading

presented by

Alexander K. White

has been accepted towards fulfillment
of the requirements for

Ph.D degree in Statistics

Major professor

Date December 7, 1999

EXPONENTIATED GRADIENT PORTFOLIOS IN CONTINUOUS TRADING

By

Alexander K. White

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Statistics and Probability

1999

Abstract

Alexander K. White

In a non-probabilistic setting, with discrete time trading, Helmbold et. al. (1998) introduce the discrete exponentiated gradient (DEG) portfolio. They prove that under specified conditions it achieves nearly the same wealth as the best constant rebalanced portfolio (bcrp) determined retrospectively from the actual market outcomes. For continuous time trading and a stochastic model, we prove that the DEG portfolio converges to the solution of a stochastic differential equation. Under specified conditions this continuous EG portfolio achieves an exponential growth greater than the bcrp, recovering a portion of the additional exponential growth from the best limit of piecewise constant rebalanced portfolios. These results do not require any prior knowledge of market parameters.

Para Alejandra mi amor, mi alma y mi vida.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor Dr. LePage whose patience and optimism were essential to the completion of this work. I also owe sincere thanks to other members of my committee: Dr. Erickson with whom I shared a great teaching experience, Dr. Levental who taught me probability and introduced me to mathematical finance and stochastic integration and Dr. Frazier the best mathematics professor I have ever known. I would like thank the department as a whole which has given me such a great opportunity and much support through the years. Especially I want to mention Dr. Salehi who has always treated me as more than just a student, Cathy Sparks who makes everything work and Dr. Stapleton who brought me to MSU. Many thanks as well to Dr. Melfi for all his help especially with TeX and my statistics professor Dr. Koul.

I would be remiss if I did not mention the many fellow students who enlightened my stay at MSU. Srikanth Rajagopolan, Alexei Stepanov, Rudolf Blažek and James Demopolos taught me more than I can say about mathematics and life. Finally I give thanks to my family: my wife Alejandra and daughter Isabel for all their patience and love, my mother Ursula for instilling in me the love of learning and finally my father Roger who told me to take Statistics because then I could always get a job. I only wish my father were here to see the finale.

Contents

List of Figures

Introduction

A fundamental problem in finance is to choose an investment strategy which maximizes wealth. Strategies that require the investor to peer into the future are seemingly unattainable. Cover (1991) proposes to *target* one such strategy namely the retrospectively best constant rebalanced portfolio (bcrp). A constant rebalanced portfolio is an investment strategy which maintains a fixed proportion of total wealth in each asset. The retrospectively best of these constant rebalanced portfolios is the one which, for the actual market fluctuations experienced, would have earned the most money. This target varies with time, outperforming the best asset and the value line (geometric mean) index. In discrete time trading without transaction costs and without any probabilistic assumptions Cover constructs a "universal portfolio" depending only upon the past asset prices which, in the worst case, grows nearly as fast as this target, losing at most order $log(n)$ in the exponent, where n is the number of trading periods. Under specified regularity conditions, Jamshidian (1992) extends Cover's results to continuous time.

In the same discrete time context, Helmbold, Schapire, Singer and Warmuth (1998) present a simpler, more market responsive, algorithm which likewise at least

achieves almost the same wealth as the best constant rebalanced portfolio, losing at most order $\sqrt{n}$ in the exponent. This algorithm, here termed the discrete exponentiated gradient (DEG) portfolio, employs a multiplicative update derived using the framework introduced by Kivinen and Warmuth (1994) for a problem of linear prediction. A key feature is that the DEG portfolio depends only upon the current asset price relatives and the portfolio in the previous trading period (by price relative we mean the inter-period ratio of the price). The universal portfolio, by contrast, depends upon the entire past history and is highly computationally intensive. For the continuous time stochastic model usually employed in option pricing we prove that the DEG portfolio does indeed converge to the solution of a stochastic differential equation. This solution, which we call the EG portfolio, under specified conditions achieves at least nearly the same wealth as the bcrp for continuous time. Prior knowledge of market parameters is not required. Exploiting the time local nature of the EG portfolio we then examine the class of better time varying targets which are limits of piecewise constant rebalanced portfolios. Our formulas identify market conditions in which the latter earn substantially more than what the best constant portfolio would earn and the EG captures a portion of the exponent of this additional return. We examine market conditions which resoundingly illustrate this point.

Chapter 1 develops the continuous time model for the market and the possible portfolios as well as the target strategies whose wealth we would like to approximately achieve. The DEG portfolio is defined and the key results from Helmbold et. al are presented in section 2 of Chapter 1 . All of our main results are presented without proof in Chapter 2. In section 1 of Chapter 3 we examine the universal portfolio and compare its behavior to the EG portfolio. In section 2 of Chapter 3 the behavior of targets and their corresponding wealths is investigated. Examples where the EG portfolio outperforms the bcrp and the universal portfolio are given in Chapter 4. Chapter 5 contains proofs of the main results. Finally, Chapter 6 summarizes these results and discusses possible extensions.

Chapter 1

1.1 Setting

Consider a financial market in which one bond, with price process β, and $d \geq 1$ stocks with price processes $S = (S_1, S_2, ..., S_m)^*$ are traded continuously in the time interval $0 \leq t < \infty$, where * denotes matrix transpose. Unless otherwise specified, all processes will be defined for $0 \leq t < \infty$. The underlying source of uncertainty in the market is an m-dimensional standard Brownian Motion $B = (B_1, B_2, ..., B_m)^*$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We assume this space is rich enough to accommodate a random variable ξ independent of B. The term "adapted" will refer to the filtration

$$\{\mathcal{F}_t : 0 \leq t < \infty\} = \sigma\left(\{\{\xi, B(s)\} : 0 \leq s \leq t\} \cup \mathcal{N}\right)$$

where $\mathcal{N} = \{A \in \mathcal{F} : P(A) = 0\}$.

The price processes of the assets evolve according to the equations

$$d\beta = \beta r dt, \qquad \beta(0) = 1 \tag{1.1}$$

and for $1 \leq i \leq m$

$$dS_i = S_i \left[\mu_i dt + \sum_{j=1}^{m} \sigma_{ij} dB_j \right], \quad S_i(0) = 1, \tag{1.2}$$

where the real valued interest rate r, the $\mathbb{R}^m$-valued drift μ and the $m \times m$ matrix valued volatility σ are all adapted processes. Let $\Sigma = \sigma^* \sigma$ be the covariance process. In order that (1.1) and (1.2) have well defined solutions, we require that almost surely for each $T < \infty$ we have

$$\int_0^T \left[|r| + \sum_{i=1}^{m} (|\mu_i| + \Sigma_{ii}) \right] dt < \infty. \tag{1.3}$$

At each moment in time a trader is allowed to shift resources between the various assets. An adapted process which defines the proportion of wealth in each stock is called a portfolio and is formally defined by

Definition 1.1. *An adapted portfolio is an adapted $\mathbb{R}^m$-valued process p which satisfies the integrability constraint*

$$\int_0^T \left(p^* \Sigma p + |(\mu - r\mathbf{1_m})^* p| \right) dt < \infty \tag{1.4}$$

a.s. for each $T > 0$ where $\mathbf{1_m} = (1, ..., 1)^* \in \mathbb{R}^m$. Let $\mathcal{P}(A)$ be the collection of all w

di

ai

D

is

adapted portfolios taking values in $A \subseteq \mathbb{R}^M$.

Since for each portfolio p_i represents the proportion of wealth in S_i, we define $p_0 = 1 - p^*\mathbf{1_m}$ to be the proportion of wealth in the bond. Accordingly, when convenient, we refer to the bond as the "0 th" stock. The wealth process, W, generated by a portfolio p evolves as

$$dW = W\left(p_0 r dt + \sum_{i=1}^{m} p_i \frac{dS_i}{S_i}\right). \tag{1.5}$$

By Ito's formula

$$\begin{aligned} d\log(W) &= \frac{dW}{W} - \frac{1}{2}\frac{(dW)^2}{W^2} \\ &= \left[p_o r + p^*\mu - \frac{1}{2}p^*\Sigma p\right] dt + p^*\sigma dB \\ &= \left[r + \frac{1}{2}\sum_{i=1}^{m} p_i \Sigma_{ii} - \frac{1}{2}p^*\Sigma p\right] dt + p^* dZ \end{aligned}$$

where $Z = (Z_1, ..., Z_m)$ with $Z_i = \log(\beta^{-1}S_i)$, $i = 1, ..., m$ represents the vector of discounted stocks. The assumptions (1.3) and (1.4) insure that $\beta(t) > 0$ for $t \geq 0$ and that the semi-martingale, $LW^{(p)}$ in (1.6) below is a well defined process.

Definition 1.2. *The semi-martingale $LW^{(p)}$ given by*

$$LW^{(p)}(t) = \int_0^t r ds + \int_0^t p^* dZ + \frac{1}{2}\int_0^t \sum_{i=1}^{m} p_i \Sigma_{ii} ds - \frac{1}{2}\int_0^t p^*\Sigma p ds \tag{1.6}$$

is the log wealth generated by the adapted portfolio p.

We sometimes use the notation $LW_{[s,t)} = LW(t) - LW(s)$. We wish to investigate the behavior of special portfolios, in particular their ability to generate wealth in comparison to ideal (and impossible) investment strategies that may use future information. We shall refer to these ideal strategies (or their corresponding wealths) as *targets*.

Definition 1.3. *A $\mathbb{R}^m$ valued process, u, which for all $t > 0$*

$$LW^{(u)}(t) = \int_0^t r ds + \int_0^t u^* dZ + \frac{1}{2}\int_0^t \sum_{i=1}^m u_i \Sigma_{ii} ds - \frac{1}{2}\int_0^t u^* \Sigma u ds \qquad (1.7)$$

is well defined and finite a.s. is called a target. The process $LW^{(u)}$ is called the target log wealth. Let $\mathcal{T}(A)$ be the set of all targets taking values in $A \subseteq \mathbb{R}^m$. Any constant maximizing $\{LW^{(u)}\}$ is called a best constant rebalanced portfolio (bcrp).

Not every $\mathbb{R}^m$ valued process is a target. It is well known that Brownian motion is not of bounded variation (and hence neither is Z) and the stochastic integral in (1.7) need not be defined for non-anticipating u. An example is given in Chapter 3. For targets of bounded variation, however, the stochastic integral on the rhs of (1.7) can be defined by "integration by parts" (see Propostion 3.7). Also in Chapter 3, the class of constant targets $\mathcal{T}^0(A) = \{u \in \mathcal{T} : u_t \equiv a, \text{ for some } a \in A\}$ and piecewise constant targets are examined. Let $D_m = \{x \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i \leq 1\}$ be the m dimensional simplex. We show for constant targets that $\operatorname{argsup}_{u \in D_m} \{LW^{(u)}\}$ always exists but may not be unique. It is important to realize that these targets are continuously trading but are maintaining fixed proportions a_i, $1 \leq i \leq m$ in the stocks S_i, $1 \leq i \leq m$.

Please note that since it is determined from the actual observed price process the

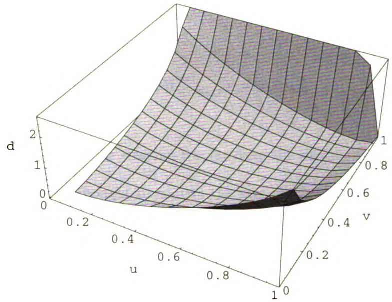

Figure 1.1: The relative entropy function: $d(u\|v)$, $(u, v) \in (0, 1)^2$.

bcrp is not an adapted portfolio.

In the next chapter we shall restrict our targets and (adapted) portfolios to the simplex D_m which prohibits borrowing and short-selling. By including the reciprocal of each stock in the model as well as a "margin" component we can, however, allow an investor to sell short and buy on margin in a limited sense. See Cover (1991).

1.2 Discrete Exponentiated Gradient

The discrete exponentiated gradient portfolio developed by Helmbold, et. al. is a modification of on-line learning strategies first used in regression. Given an initial value $q(k) \in \mathbb{R}^m$ which represents the value of the portfolio to be used at time t_k, c

r

v

T

w

fi

m

L

a

D

th

N

maximize:

$$F(p) = \eta LW^{(p)}_{[t_k, t_{k+1})} - d(p, (\pi_k)) \tag{1.8}$$

where d is a penalty term for straying too far from the initial value and $\eta > 0$ is a constant, which can be thought to control the rate of response of the algorithm to changing market fluctuations. They consider various choices for d but focus on the relative entropy defined for $u, v \in (0,1)^m$ by $d(u,v) = \sum_{i=0}^m u_i \log\left(\frac{u_i}{v_i}\right)$. This corresponds to the Kullback-Leibler distance between two $m+1$-dimensional probability vectors. For $m = 1$ the graph of the relative entropy is presented in Figure (1.1). This graph reveals that d is relatively flat except at the boundaries. This observation will prove useful for bounding terms involving d. Replacing $LW^{(p)}_{t_k, t_{k+1}}$ in (1.8) by a first order approximation Helmbold et. al. find a simple closed form solution to a modified version of (1.8).

For a bounded stopping time S and a positive constant $\tau > 0$ let

$$Part(S, \tau, \Delta) = \{t_k, 0 \leq k \leq N\}$$

be a partition with non-random increments such that $S = t_0 \leq t_1 \leq \cdots \leq t_N = \tau$ and mesh size $\Delta = sup\{t_{k+1} - t_k : 0 \leq k \leq N-1\}$.

Definition 1.4. *Let $\xi \in D_m$ be $\mathcal{F}_S$ measurable and $\eta > 0$. The price relatives for the stocks are denoted $X_i(k) = \frac{S_i(t_k)}{S_i(t_{k-1})}$, $k = 1, ..., n$ and $i = 1, ...m$ and for the bond $X_0(k) = \frac{\beta(t_k)}{\beta(t_{k-})}$. The expression DEG($\eta$, ξ), i.e. discrete exponentiated gradient, will*

denote the portfolio with

$$
\begin{aligned}
q(t_0) &= \xi, \\
q_i(t) &= q_i(t_{k-1}) \exp\left\{\frac{\eta X_i(k)}{q(t_{k-1})^* X(k) + q_0(t_{k-1}) X_0(k)}\right\} V_k^{-1}, \quad t_k \le t < t_{k+1}
\end{aligned}
\tag{1.9}
$$

where

$$
\begin{aligned}
V_k = &\sum_{i=1}^{m} q_i(t_k) \exp\left\{\frac{\eta X_i(k)}{q(t_{k-1})^* X(k) + q_0(t_{k-1}) X_0(k)}\right\} \\
&+ q_0(t_k) \exp\left\{\frac{\eta X_0(k)}{q(t_{k-1})^* X(k) + q_0(t_{k-1}) X_0(k)}\right\}
\end{aligned}
$$

is a normalizer which ensures that $q \in D_m$.

Although suppressed in the notation, the DEG portfolio depends on the choice of η and the trading times (i.e. the partition).

To see how the DEG portfolio works notice that $q(t_{k-1})^* X(k) + q_0(t_{k-1}) X_0(k)$ is the wealth growth obtained over the interval $[t_{k-1}, t_k)$ by managing the assets according to $q(t_{k-1})$ at t_{k-1} and holding them until t_k. The portfolio is then updated at time t_k according to the ratio of the price relative $X_i(t_k)$ to this wealth growth. If holding the *ith* stock would have made a lot money relative to what we just made previously using $q(t_{k-1})$ we increase the amount of money invested in the stock. The greediness parameter η determines how sensitive our algorithm is to shifts in the price relatives.

A key result from Helmbold et.al. is restated below. Let $\tilde{X} = (X_o, X^*)^*$ and

$\tilde{q} = (q_o, q^*)^*$. By (1.3) the price relatives are positive. If we only trade at the time points t_k the log wealth generated by the DEG is $\sum_k \log\left(\tilde{q}^*(t_k)\tilde{X}(t_k)\right)$.

Theorem 1.1. *(Helmbold, Schapire, Singer and Warmuth, 1998) Let $u \in D_{m+1}$ be constant and $\frac{\min_{k,i} X_i(t_k)}{max_{k,j} X_j(t_k)} = c > 0$. For $\eta > 0$,*

$$\sum_k \log\left(\tilde{q}^*(t_k)\tilde{X}(t_k)\right) - \sum_k \log\left(u^*\tilde{X}(t_k)\right) \geq -\frac{d(u\|q(t_0))}{\eta} - \frac{\eta n}{8c^2}. \tag{1.10}$$

Furthermore, if $q(t_0) = (m+1)^{-1}\mathbf{1_{m+1}}$ and we set $\eta = 2c\sqrt{2\log(m+1)/n}$ then we have

$$\sum_k \log\left(\tilde{q}^*(t_k)\tilde{X}(t_k)\right) - \sum_k \log\left(u^*\tilde{X}(t_k)\right) \geq -\frac{\sqrt{2n\log(m+1)}}{2c} \tag{1.11}$$

The left hand sides of (1.10) and (1.11) represent the difference in log wealth between the DEG and an arbitrary it constant target u through n trading periods. Therefore they have found a lower bound on the performance versus the non-adapted *best constant rebalanced portfolio* determined retrospectively from the observed price processes. We note for future reference that their bound, due to its dependence on the square root of the number of trading periods, must be modified if we are to extend this result, through rapid trading, to continuous trading. It is important to mention that in their work Helmbold, et. al. do not assume any probablistic model and prove that (1.10) and (1.11) hold generally for any sequence of positive price relatives satisfying the boundedness constraint. In a market with exponential growth and $\Delta = 1$, then $n = \tau - S$ and by (1.11) the ratio of wealth generated by the DEG to the wealth generated by the bcrp is of the order $e^{\sqrt{(\tau - S)}}$. Hence the DEG is capturing at least the first order exponential growth of the bcrp. From (1.11) we can see that a good choice of η is of the order $\eta = (\tau - S)^{-\frac{1}{2}}$. In Theorem 1.1 it is assumed that the ratio of the maximum price relative to the minimum price relative is bounded. In their paper they are able to remove this assumption for a modified version of the DEG and an expression similar to (1.11) but weakening the bound to order $(\tau - S)^{\frac{3}{4}}$.

These lower bounds for the DEG vs. the bcrp are weaker than Cover obtains for the universal portfolio vs. bcrp. We shall see, however, that the DEG, perhaps because it responds more readily to market fluctuations, can under specified conditions outperform the universal portfolio and even the bcrp, whereas a result from Jamshidian proves that the universal portfolio exhibits the same exponential growth rate as the bcrp. Helmbold et. al. performed experiments with the few specific examples of actual data from the New York Stock Exchange accumulated over a 22 year periodwhich first appeared in Cover (1991). In these experiments the DEG portfolio outperformed the buy and hold strategy for the best stock and Cover's universal portfolio and achieved only slightly less wealth than that achieved by the bcrp. See Figure (1.2) which is reproduced from their paper.

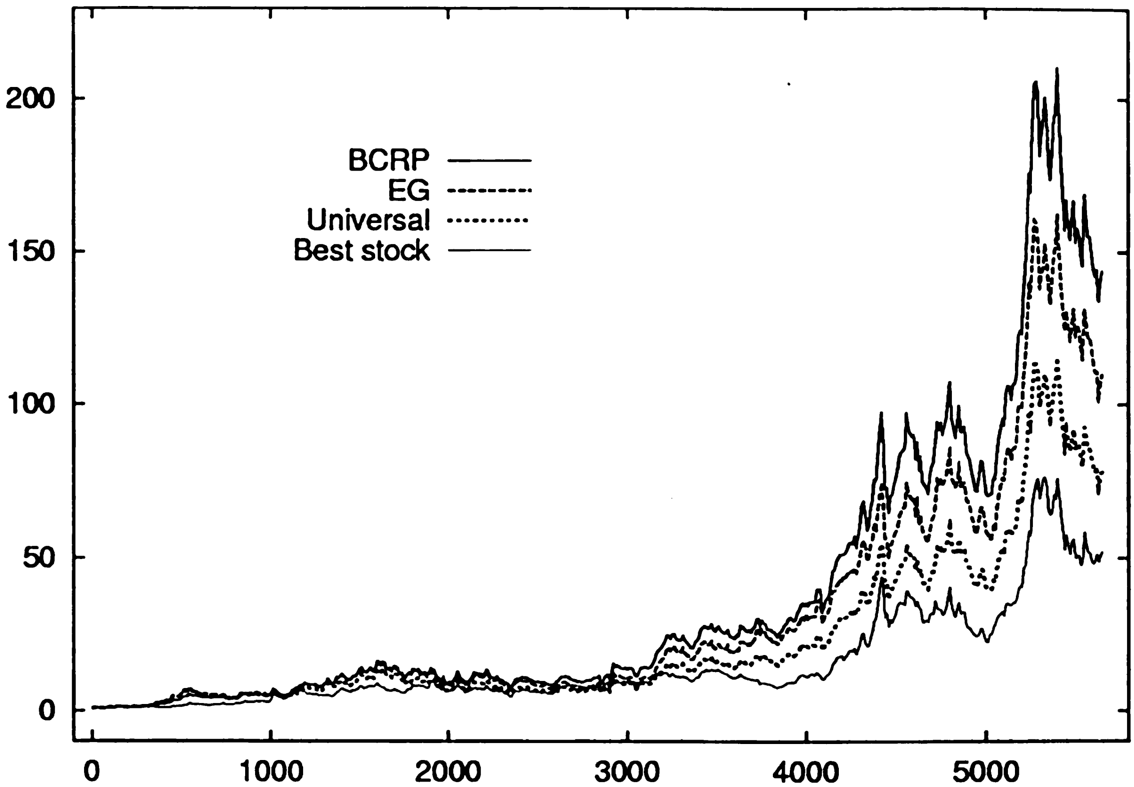

Figure 1.2: Comparison of wealths achieved by the best constant rebalanced portfolio, the DEG portfolio, and the universal portfolio. The market consists of Commercial Metals and Kin Ark. The wealth achieved by the EG portfolio is close to the wealth of the bcrp and exceeds that achieved by the universal portfolio.

c

li

le

si

m

de

Pa

le

T

a

Chapter 2

Exponentiated Gradient

For clarity of exposition no proofs of the results are presented in this chapter. Proofs of all results may be found in Chapter five.

It is not a-priori clear that the DEG portfolio has a continuous limit or that the limit will exhibit the same good properties as the discrete portfolio. In particular one looming difficulty, which we must overcome, is the presence of $\sqrt{n}$ in the right hand side of (1.10) which makes it appear that the bound blows up. In this chapter we manage to extend the DEG portfolio to continuous time trading and present results describing the wealth achieved in comparison with constant targets and limits of piecewise constant targets. We now establish the continuous version of the DEG by letting the mesh size $\Delta \to 0$.

Before we can find the continuous limit of the DEG we need the following:

Theorem 2.1. *Suppose μ, r and Σ are $\mathcal{F}_t$ adapted processes which satisfy (1.3), S is a bounded stopping time and $\xi \in \mathbb{R}^m$ is $\mathcal{F}_S$ measurable. Let η be an adapted $\mathbb{R}^+$*

valued process such that for each $t > 0$

$$\int_0^t \left[|\eta r| + \sum_{i=1}^m \left(|\eta \mu_i| + \eta^2 \Sigma_{ii} \right) \right] ds < \infty \tag{2.1}$$

almost surely. There exists a unique strong solution $\{\alpha(S,t), t \geq S\} \in \mathbb{R}^m$ to the following stochastic differential equation

$$d\alpha_i = \eta dZ_i + \eta \left[\frac{1}{2} \Sigma_{ii} - \Sigma_i f(\alpha) \right] dt \tag{2.2}$$

with initial $\alpha(S,0) = \xi$ and where $f : \mathbb{R}^m \to D_m$ is given by

$$f_i(a) = e^{a_i} \left(1 + \sum_{j=1}^m e^{a_j} \right)^{-1}, \quad i = 1, ..., m. \tag{2.3}$$

Using the process α we define an adapted portfolio which continuously updates.

Definition 2.1. *Let $\alpha(S,t)$ be the solution to the stochastic differential equation (2.2) with initial condition $\alpha(S,S) = \xi \in \mathcal{F}_S$ and $\eta > 0$ as in Theorem 2.1. The D_m valued process $\pi(t)$, $t > S$, defined by*

$$\pi_i = e^{\alpha_i} \left(1 + \sum_{j=1}^m e^{\alpha_j} \right)^{-1}, \quad i = 1, ..., m$$

is called an exponentiated gradient with learning parameter process η with starting time S (or EG(η,S)) portfolio.

The corresponding amount invested in the bond is $\pi_0 = \left(1 + \sum_{j=1}^m e^{\alpha_j}\right)^{-1}$. Since $\log \frac{\pi_i}{\pi_0} = \alpha_i$ we see that the EG portfolio places more money in the *ith* stock where the discounted stock value, Z_i increases.

Theorem 2.2. *Suppose that τ is a positive constant, S, T are bounded stopping times with $0 < T - S \leq \tau$ and $\eta > 0$ is constant. Under the conditions of Theorem 2.1 the DEG(η) portfolio q converges uniformly in probability to the EG(η) portfolio π and the $LW^{(q)}$ converges uniformly in probability to $LW^{(\pi)}$ on the interval $[S, T]$.*

Although in Theorem 2.2 we see that the π is the limit of q, we cannot use Theorem 1.1 to evaluate the performance of π since the bound in (1.10) increases with the number of trades n which tends to ∞ as the mesh size is decreased. The following continuous time counterpart of Theorem 1.1 resolves this question. In effect trading rapidly (i.e. over small increments) involves price relatives close to one and produces an identity, rather than lower bound, in the limit when we keep track of all terms. Stochastic calculus facilitates the bookkeeping.

Theorem 2.3. *Let $\eta > 0$ be constant. For any bounded stopping times S and T with $0 < T - S \leq \tau$ and any non short-selling constant target $u \in \mathcal{T}^0(D_m)$ the EG(η,S) portfolio π satisfies*

$$\begin{aligned} LW^{(\pi)}_{[S,T)} - LW^{(u)}_{[S,T)} &= \frac{1}{2}\int_S^T (\pi - u)^* \Sigma (\pi - u)\, dt + \frac{\eta}{2}\int_S^T \pi^* \Sigma \pi dt \\ &\quad -\frac{\eta}{2}\int_S^T \sum_{i=1}^m \pi_i \Sigma_{ii} dt + \frac{d(u \| \pi(T)) - d(u \| \pi(S))}{\eta} \end{aligned} \tag{2.4}$$

Comparing (2.4) to (1.10) we see some similarities and many differences. First of all (2.4) is an identity which retains and gives meaning to terms discarded in (1.10).

The two negative terms of (2.4) are $-\frac{\eta}{2}\int_S^T \sum_{i=1}^m \pi_i \Sigma_{ii} dt$ and $\frac{-d(u\|\pi(S))}{\eta}$, only the second of which appears in (1.10). Using the fact that $\pi \in D_m$ we can see that the term $-\frac{\eta}{2}\int_S^T \sum_{i=1}^m \pi_i \Sigma_{ii} dt$ is the counterpart of $-\frac{\eta m}{8c^2}$ in (1.10) . The intuition here is that both terms measure the total variation in stock prices over time. The positive terms $\frac{1}{2}\int_S^T (\pi - u)^* \Sigma (\pi - u)\, dt$ and $\frac{d(u\|\pi(T))}{\eta}$ of (2.4) are large if u is distant from π. This may seem paradoxical but, as we shall learn, the EG can do well against non-constant targets where the bcrp does poorly. By noting that $d(u\|\frac{1}{(m+1)}\mathbf{1_m}) \leq \log(m+1)$ we get

Corollary 2.4. *Let* $S = 0$, $M > 0$ *and* $T_M = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii}\, ds = M\right\}$. *Choose* $\eta = \sqrt{\frac{2\log(m+1)}{mM}}$ *then the EG(η,0) portfolio* π *with* $\pi(0) = \frac{1}{(m+1)}\mathbf{1_m}$ *satisfies*

$$LW_{[S,t)}^{(\pi)} - LW_{[S,t)}^{(u)} \geq \frac{1}{2}\int_S^T (\pi - u)^* \Sigma (\pi - u)\, dt - \sqrt{2\log(m+1)mM}$$

for each constant target u. *And*

$$\sup_{0\leq t\leq T_M}\left\{\frac{\eta}{2}\int_0^t \sum_{i=1}^m \pi_i \Sigma_{ii}\, dt + \frac{d(u\|\pi(0))}{\eta}\right\} \leq \sqrt{2\log(m+1)mM}. \qquad (2.5)$$

The stopping T_M is a measurable function of the paths of the stock price process (see Lemma 5.2). Hence for a specified level of variation M, and the above choice of η, the contribution of the negative terms to (2.4) up to stopping time T_M is no greater than square root of the variation.

In a reasonable market the wealth of the bcrp exhibits exponential growth on the order of the variation (see Proposition 3.3 and the following discussion, for details).

As in the discrete case the EG portfolio is at least capturing the first order exponential growth (if there is any) of the bcrp. This idea is made more precise in the following corollary.

Corollary 2.5. *Let* $S = 0$, $M > 0$ *and* $T_M = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii}^2 ds = M\right\}$. *Choose* $\eta = \sqrt{\frac{2\log(m+1)}{mM}}$ *then the EG(η,0) portfolio* π *with* $\pi(0) = \frac{1}{(m+1)}\mathbf{1_m}$ *satisfies*

$$\frac{LW_{[S,T_M)}^{(\pi)} - LW_{[S,T_M)}^{(u^\dagger)}}{LW_{[S,T_M)}^{(u^\dagger)}} \geq O\left(M^{-\frac{1}{2}}\right) \tag{2.6}$$

whenever $LW_{[S,t)}^{(u^\dagger)} = O(\max_i \int_0^t \Sigma_{ii}^2\, ds)$.

In Theorem 2.3 and 2.5 we employ a constant learning parameter and compare with a constant target. In the following generalizations of Theorem 2.3 we allow both to vary over time. First we define a very general class of targets.

Definition 2.2. *A target* $u \in \mathcal{T}$ *is called piecewise constant if there exist a finite sequence* $u_k \in \mathbb{R}^m$, $k = 1, \ldots, N$ *and a (possibly adapted) partition* $S = t_0 \leq t_1 \leq \cdots \leq t_N = \tau$ *such that*

$$u_t = \sum_{k=0}^{N-1} u_k \{t_k \leq t \leq t_{k+1}\}.$$

Let $\mathcal{T}^{pc}(A)$ *be the set of all piecewise constant targets taking values in* $A \subseteq \mathbb{R}^m$ *and let*

$$\mathcal{T}^{pcl}(A) = \left\{u \in \mathcal{T}(A) : \exists u_n \in \mathcal{T}^{pc}(A) \ni u_n \to u, LW^{(u_n)} \to_n LW^{(u)}\right\}$$

where the convergence is uniform in probability.

To compare performance against piecewise constant targets we generalize Corollary 2.5. Suppose $u \in \mathcal{T}^{pc}$ is a piecewise target with one jump at stopping time $T > S$. For $t > T$, using Theorem 2.3 on the two subintervals $[S,T)$ and $[T,t)$ we have almost surely

$$\begin{aligned} LW_{[S,t)}^{(\pi)} - LW_{[S,t)}^{(u)} &= \frac{1}{2}\int_S^t (\pi - u)^* \Sigma (\pi - u)\, ds + \frac{\eta}{2}\int_S^t \pi^* \Sigma \pi ds \\ &\quad - \frac{\eta}{2}\int_S^T \sum_{i=1}^m \pi_i \Sigma_{ii} ds + \frac{d(u(t)\,\|\pi(t)) - d(u(T-)\,\|\pi(T))}{\eta} \\ &\quad + \frac{d(u(T-)\,\|\pi(T)) - d(u(S)\,\|\pi(S))}{\eta} \end{aligned}$$

Comparing with (2.4) we see an additional term involving the difference in the relative entropy at the jump point. Hence using the inequality $d(u\|v) \le \max_{0 \le i \le m} \{-\log(v_i)\}$ we can generalize Corollary (2.5) to the case where we allow the target to have a finite number of jumps.

Corollary 2.6. *Let $u \in \mathcal{T}^{pc}$ be a piecewise target with n jumps. Let $S = 0$, $M > 0$ and $T_M = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii} ds = M\right\}$. Choose $\eta = \sqrt{\frac{2}{mM}}$ then the EG(η,0) portfolio π with $\pi(0) = \frac{1}{(m+1)}\mathbf{1_m}$ satisfies*

$$\begin{aligned} LW_{[0,t)}^{(\pi)} - LW_{[0,t)}^{(u)} &\ge \frac{1}{2}\int_0^T (\pi - u)^* \Sigma (\pi - u)\, dt \\ &\quad - \sqrt{\frac{mM}{2}}\left(1 + nK + \log(m+1)\right) \end{aligned}$$

where $K = \max_i \sup_{0 < s \le T_M} \{-\log(\pi_i(s))\}$.

From Corollary (2.6) we see that the EG can track the larger piecewise constant targets as long as the market does not force π towards the boundary of D_m.

Theorem 2.7. *Let $u \in \mathcal{T}^{pcl}(D_m)$. Suppose that η_t is a positive real valued process satisfying (2.1). If η_t is right continuous and bounded almost surely then for the EG portfolio π we have*

$$\begin{aligned} LW^{(\pi)}_{[S,T)} - LW^{(u)}_{[S,T)} &= \frac{1}{2}\int_S^T (\pi - u)^* \Sigma (\pi - u)\, dt + \frac{1}{2}\int_S^T \eta\pi^* \Sigma \pi dt \\ &\quad -\frac{1}{2}\int_S^T \sum_{i=1}^m \eta\pi_i \Sigma_{ii} dt + \int_S^T d\Psi_t \end{aligned} \tag{2.7}$$

where $\Psi_t = \frac{d(u_t \| \pi_t)}{\eta_t}$.

The expression $\int_S^T d\Psi_t$ is understood to be the limit in probability of

$$\sum_k \left[\frac{d\left(u_{t_{k+1}} \| \pi_{t_{k+1}}\right)}{\eta_{t_{k+1}}} - \frac{d\left(u_{t_k} \| \pi_{t_k}\right)}{\eta_{t_k}} \right] \tag{2.8}$$

which must exist since all other terms in (2.7) do. The proof is given in Chapter 5. We now the consider the form of (2.8) in the case where η is constant and u is smooth.

Lemma 2.8. *Let u be a continuous process of bounded variation taking values in the interior of D_m i.e. for all $t \geq 0$ $u_t \in A = \{u \in D_m : \sum_{i=1}^m u_i < 1, u_i > 0\}$. Then $u \in \mathcal{T}(D_m)$, i.e. up to time t the process u generates finite wealth. And for $\eta > 0$*

constant and the EG(η) portfolio we have

$$\int_S^T d\Psi_t = \eta^{-1}\left(d\left(u(T)\|\pi(T)\right) - d\left(u(S)\|\pi(S)\right) - \sum_{i=1}^m \int_S^T \left[\log\left(\frac{u_i}{u_0}\right) - \alpha_i\right] du\right).$$

The final result of the chapter gives conditions under which the EG portfolio can perform as well as smooth targets. As we have seen above the key restriction is that the portfolio must stay away from the boundary.

Theorem 2.9. *Let u be as in Theorem 2.8 with $\int_0^\infty d|u| = K$. Let $S = 0$, $M > 0$, $c > 0$ and*

$$T_{(M,c)} = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii}^2 ds = M or \max_i \left|\log(\frac{\pi_i}{\pi_0})\right| = c\right\}$$

Choose $\eta = 2\sqrt{\frac{\log(m+1)}{mM}}$ then the EG(η,0) portfolio π with $\pi(0) = \frac{1}{(m+1)}\mathbf{1_m}$ satisfies

$$\begin{aligned} LW^{(\pi)}_{[0,T_{(M,c)})} - LW^{(u)}_{[0,T_{(M,c)})} &\geq \frac{1}{2}\int_0^T (\pi - u)^* \Sigma (\pi - u)\, dt + \frac{\eta}{2}\int_0^T \pi^*\Sigma\pi dt \\ &\quad -2\sqrt{\log(m+1)mM} - cK\sqrt{\frac{mM}{4\log(m+1)}}. \end{aligned} \tag{2.9}$$

Chapter 3

3.1 Cover's Universal Portfolio

Cover's universal portfolio, introduced in 1991, uses an averaging method to pick the portfolio. The portfolio vector used at time t is the past performance weighted average of all constant portfolios. Cover and Ordentlich (1996) introduce the notion of side information and generalize Cover's algorithm by using the Dirichlet($\frac{1}{2}, ..., \frac{1}{2}$) and the Dirichlet(1,...,1) priors over the set of all portfolio vectors, i.e.

$$\hat{p}_i(t) = \frac{\int_{D_m} p_i W^{(p)}(t)\, d\lambda}{\int_{D_m} W^{(p)}(t)\, d\lambda}, \quad i = 1, ..., m \tag{3.1}$$

where λ is one of the Dirichlet priors mentioned above. In discrete time trading, Cover and Ordentlich prove that under no assumptions on the price relative vector (except non negativity) the Dirichlet($\frac{1}{2}, ..., \frac{1}{2}$) weighted universal portfolio, in the worst possible case, grows nearly as fast as the bcrp losing at most $\frac{m}{2}\log(n)$ in the exponent where n is the number of trading days and m is the number of stocks. Using a recursion scheme they can compute the portfolio on-line with storage requirements growing like n^{m-1}.

Employing the model (1.1) and (1.2), Jamshidian (1992) extends Cover's original portfolio to continuous time. He assumes that the following limits exist

$$\Sigma^{\infty} = \lim_{t\to\infty} \frac{\mathrm{E}\left[\int_0^t \Sigma ds\right]}{t}, \qquad \nu^{\infty} = \lim_{t\to\infty} \frac{\mathrm{E}\left[\log(S_t)\right]}{t} \tag{3.2}$$

and sets $\mu^{\infty} = \nu^{\infty} + \frac{1}{2}\Sigma_{ii}^{\infty}$. Under these conditions there exists an asymptotically optimal constant portfolio p^{∞} which is determined by

$$p^{\infty} = \mathrm{argmax}_{p\in D_m} \left\{ p^{*}\mu^{\infty} - \frac{1}{2} p^{*}\Sigma^{\infty} p \right\}. \tag{3.3}$$

Stock i is said to be asymptotically active if $p_i^{\infty} > 0$ and a market is asymptotically active if all assets are. Jamshidian's main result is

Theorem 3.1. *(Jamshidian) If the market is asymptotically active then*

$$LW^{(\hat{p})}(t) - LW^{\dagger}(t) \sim \log(m!) + \frac{1}{2}\log(|\Sigma^{\infty}|) - \frac{m}{2}\log(t) + C$$

where $LW^{(\hat{p})}(t)$ is the wealth generated by (3.1), $LW^{\dagger}$ is wealth generated by the bcrp, $|\Sigma^{\infty}|$ is the determinant of the asymptotic covariance, $C > 0$ is a constant independent of t and Σ^{∞} and the notation $X(t) \sim Y(t)$ means $X(t)/Y(t)$ converges to 1 in probability.

Jamshidian proved a similar result in the case where the market is asymptotically k-inactive ($p_i^{\infty} = 0$ for k of the m stocks) in which case the bound on the rhs is of the order $\frac{m+k}{2}\log(t)$. The bound achieved in Theorem 3.1 is superior to the one we obtain in Theorem 2.3 for the EG portfolio. Just as in discrete time, in comparison with the wealth generated by the bcrp, Cover's algorithm loses at most on the order $\frac{m}{2}\log(t)$ in the exponent while the exponentiated gradient method loses at most on the order of $\log(m)\sqrt{t}$. However, as seen in Corollary 2.6 and Theorem 2.9, the exponentiated portfolio can perform well versus better non constant targets where Cover's algorithm tracks the bcrp. Furthermore the EG portfolio is far simpler computationally than the universal portfolio[1].

The following proposition gives conditions under which the EG portfolio will outperform the universal portfolio. From (2.4) we see that EG portfolio will exhibit especially good growth whenever π differs from the bcrp sufficiently so that the term $\frac{1}{2}\int\left(\pi-u^{\dagger}\right)^{*}\Sigma\left(\pi-u^{\dagger}\right)dt$ is large, e.g. on the order of $\max_i\int\Sigma_{ii}dt$. One expects this to occur in cases where the drift coefficients oscillate. See the example presented in Chapter 4.

Proposition 3.1. *If $u_i^{\dagger}(t)>0$ for all $i=1,\dots,m$ then*

$$\begin{aligned}LW_{[S,t)}^{(\pi)}-LW_{[S,t)}^{(\hat{p})} &= \frac{1}{2}\int_S^t\left(\pi-u^{\dagger}\right)^{*}\Sigma\left(\pi-u^{\dagger}\right)dt+\frac{\eta}{2}\int_S^t\pi^{*}\Sigma\pi dt\\ &-\frac{\eta}{2}\int_S^t\sum_{i=1}^m\pi_i\Sigma_{ii}dt+\frac{d\left(u^{\dagger}\|\pi\left(T\right)\right)-d\left(u^{\dagger}\|\pi\left(S\right)\right)}{\eta}\\ &-\log(m!)+\frac{1}{2}\log\left(\left|\int_0^t\Sigma ds\right|\right)-C_t\end{aligned}$$

where $C_t=\log\left(\int_{A_t}\exp\left\{-\frac{\|x\|^2}{2}\right\}dx\right)\le\frac{m}{2}\log(7)$ *with* $A_t=\Sigma^{\frac{1}{2}}(D_m-u^{\dagger})$.

[1] Although it is adapted, given the computational requirements for discrete time especially for large m, a continuous implementation of (3.1) may not be feasible.

For η and T_M are as in Corollary (2.4) and $0 < c < 1$ suppose that

$$\frac{1}{2}\int_0^t \left(\pi - u^\dagger\right)^* \Sigma \left(\pi - u^\dagger\right) ds - \sqrt{\max_i \int_0^t \Sigma_{ii}^2 ds} = O(t^c)$$

as $t \to \infty$ and the market is active for all time $t > S$. Then for M sufficiently large

$$LW^{(\pi)}_{(S,T_M)} - LW^{(\hat{p})}_{(S,T_M)} \geq O(T_M^c).$$

Proof. In the proof of (3.1) Jamshidian compares the log wealth obtained by (3.1) and $u^\dagger$ as

$$LW^{(\hat{p})}(t) - LW^{(u^\dagger)}(t) = \log(m!) - \frac{1}{2}\log(\left|\int_0^t \Sigma ds\right|) + C_t. \tag{3.4}$$

The first result follows by combining the above with Theorem 2.3. The second result is a direct consequence of Corollary 2.4. □

The relation (3.4) demonstrates that under rather general market conditions the universal portfolio obtains the same asymptotic exponential growth rate of the bcrp but never better, even in cases where the bcrp does not perform so well. In fact in an active market where the variation tends to infinity, the wealth generated by bcrp is asymptotically infinitely greater than that of the universal portfolio. In Chapter 4 we present an example of a market with oscillatory drifts where the EG portfolio decidedly outperforms both the bcrp and the universal portfolio.

3.2 Targets

In chapter 2 Theorems (2.3) and (2.7) relate the wealth acheived by the EG portfolio to that of the bcrp and limits of piecewise bcrp respectively. In this chapter we examine the wealth generated by such targets. We exhibit examples of reasonable markets where the wealth achieved is quite large.

A simple example of rebalancing

To appreciate the power of constant rebalancing consider a simple two asset model presented in Helmbold, Schapire, Singer and Warmuth (1998). There is one risky asset or Stock whose value is halved on "down" days and doubled on "up" days. And one risk-free asset with zero growth rate. Suppose the relative returns of the stock are a random permutation of the sequence $\frac{1}{2}, 2, \frac{1}{2}, 2, ..., \frac{1}{2}, 2$ and the relative returns of the bond are $1, 1, ..., 1$. If given a dollar, an investor buys the stock and holds on to it the wealth for entire period can at best double. However, if the investor constantly rebalances to one-half of the total wealth each day in each asset, then on the "down" days the relative wealth decreases by a factor of $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$. On the "up" days the relative wealth grows by $\frac{1}{2} \times 1 + \frac{1}{2} \times 2 = \frac{3}{2}$. Thus pairing an "up" day and a "down" day the investor's wealth grows by a factor of $\frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$. Hence without any prior knowledge of the ordering of the $\frac{1}{2}$'s and 2's (or even which asset is the bond) the wealth after $2n$ days grows by a factor of $\left(\frac{9}{8}\right)^n$. Therefore, even in a flat or oscillatory market, constant rebalancing can be used to achieve exponential growth.

The BCRP

Let $\theta_i(t) = Z_i(t) + \frac{1}{2}\int_0^t \Sigma_{ii} ds$ for $i = 1, ..., m$ and $t \geq 0$. By (1.6) the log wealth growth obtained by constant portfolio vector $u \in \mathbb{R}^m$ over the interval $[s, \tau)$ is quadratic in u:

$$LW^{(u)}_{[s,\tau)} = \int_s^\tau r dt + u^*\theta - \frac{1}{2}u^*\left(\int_s^\tau \Sigma dt\right)u \tag{3.5}$$

and is maximized by any u satisfying

$$u = \left(\int_s^\tau \Sigma dt\right)^{-1}\theta$$

where $\left(\int_s^\tau \Sigma dt\right)^{-1}$ is a generalized inverse of $\int_s^\tau \Sigma dt$. The log wealth growth produced by such a u is

$$LW^{(u^\dagger)}(\tau) = \int_s^\tau r dt + \frac{1}{2}\theta^*\left(\int_s^\tau \Sigma dt\right)^{-1}\theta.$$

If the covariance matrix $\int_s^\tau \Sigma dt$ is non-singular we denote the (unrestricted) retrospective best constant target by $u^\dagger$. We now restrict to the simplex D_m. The bcrp is for the interval $[s, \tau]$

$$u^\ddagger = \mathrm{argsup}_{u\in D_m}\left\{u^* Z + \frac{1}{2}\sum_{i=1}^m u_i \int_s^\tau \Sigma_{ii} dt - \frac{1}{2}u^*\left(\int_s^\tau \Sigma dt\right)u\right\} \tag{3.6}$$

The wealth achieved by $u^\ddagger$ clearly exceeds the arithmetic mean, the geometric mean and the maximum of the assets in the market. For the following definition we use the interpretation that the bond is stock 0 and consider bcrp over the interval $[0, t)$.

Definition 3.1. *Let $u^{\ddagger}(t)$ be the bcrp.*

- *A stock i is active at time t if $u_i^{\ddagger}(t) > 0$.*
- *A stock is asymptotically active if* $\liminf_t u_i^{\ddagger}(t) > 0$.
- *A market is (asymptotically) active if all stocks are (asymptotically) active.*
- *A market is (asymptotically) k-inactive if exactly k stocks are (asymptotically) inactive.*

The following proposition from Jamshidian describes $u^{\ddagger}$ in terms of market activity.

Proposition 3.2. *(Jamshidian,1992) Assume $\int_0^t \Sigma ds$ is invertible. At time $t \geq 0$*

- *A market is active if and only if $u_i^{\dagger}(t) > 0$ for each i. In which case $u^{\ddagger} = u^{\dagger}$.*
- *Define the $m - k$ vector $u^{(k)}$ to be the solution of*

$$\sum_{l=k+1}^{m} u_l^{(k)} \left[\int_0^t \Sigma_{jl} ds \right] = \theta_j, \quad k+1 \leq j \leq m.$$

The market is k-inactive if and if $u_l^{(k)} > 0$ for $k+1 \leq j \leq m$ and then

$$u^{\ddagger} = (0, 0, ..., u^{(k)*})^*.$$

In particular in the $m = 1$ case $u^{\ddagger} = 0 \vee u^{\dagger} \wedge 1$.

Using Proposition 3.2 we can find a lower bound for the wealth generated by the bcrp.

Proposition 3.3. *Suppose the market is k-inactive for $k < m$. Let*

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^*$$

be the eigenvector process of the covariance $\int_0^t \Sigma ds$ with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m$. The log wealth generated by $u^\dagger$ is bounded below by

$$LW^{(u^\dagger)}(t) \geq \int_0^t rds + \frac{1}{2}\lambda_m u^{\dagger *} u^\dagger.$$

Proof. Suppose $\lambda_m = 0$ then the result trivially holds since $u^\dagger$ beats the bond. Assume $\lambda_m > 0$. Let $\Lambda = \left(\int_0^t \Sigma_{ij} ds\right)_{i,j=k+1}^m$ be the covariance submatrix corresponding to stocks $k+1, ..., m$ and define the $m-k$ vector $u = (u_{k+1}^\dagger, ..., u_m^\dagger)^*$. Then by Proposition 3.2 the log wealth is the quadratic form

$$\begin{aligned} LW^{(u^\dagger)}(t) &= \int_0^t rdt + \frac{1}{2} u^* \Lambda u \\ &\geq \int_0^t rdt + \lambda_m \frac{1}{2} u^* u = \int_0^t rdt + \lambda_m \frac{1}{2} u^{\dagger *} u^\dagger \end{aligned}$$

where the inequality follows from the basic result about quadratic forms and the last equality follows from the fact that the market is k-inactive. □

From Proposition 3.3 we see that if the interest rate r is non-negative, and $\lim_t \lambda_m/\lambda_1 > \epsilon$ for some $\epsilon > 0$ then $LW^{(u^\dagger)}(t)$ will grow exponentially on the order of λ_1.

Targets in Two Asset Market

In this section we further examine the nature of targets restricting ourselves to the $m = 1$ case. To avoid issues of degeneracy, we assume throughout that $\sigma > 0$ for almost all t and that $\int_0^\infty \sigma^2 dt = \infty$ almost surely. Define the stopping time $T(s) := \inf\left\{\tau > 0 : \int_0^\tau \sigma^2 dt > s\right\}$. By our assumptions on σ, T is 1-1 and onto with $s(T) = \int_0^T \sigma^2 dt$, hence $\tilde{B}_s = \int_0^{T(s)} \sigma dB$, $0 \le s < \infty$ is a Brownian motion (see Karatzas and Shreve pg 174).

Proposition 3.4. *Let $\int_0^\infty \sigma^2 dt = \infty$ almost surely. The stock is asymptotically active if and only if*

$$\liminf_{t\to\infty} \frac{\int_0^t (\mu - r)\, ds}{\int_0^t \sigma^2 ds} > 0$$

almost surely.

Proof. In this case the bcrp is

$$u_t^\ddagger = 0 \vee \left(\int_0^t \sigma^2 dt\right)^{-1} \left(\int_0^t (\mu - r)\, ds + \int_0^t \sigma dB\right) \wedge 1.$$

By the law of the iterated logarithm

$$\lim_{t\to 0} \left(\int_0^t \sigma^2 dt\right)^{-1} \int_0^t \sigma dB = \lim_{s\to\infty} s^{-1}\tilde{B}_s \to 0.$$

So $\liminf u^\ddagger > 0$ if and only if $\liminf_t \frac{\int_0^t (\mu-r)ds}{\int_0^t \sigma^2 ds} > 0$. $\square$

The next proposition shows that if $\mu < r$ the bcrp wealth is essentially equal to the bond.

Proposition 3.5. *If almost surely $\mu < r$, $\int_0^\infty \sigma^2 dt = \infty$ and $\int_0^t r ds = \infty$ then $u^\dagger \to 0$ and*

$$\frac{LW^{(u^\dagger)}(t)}{\int_0^t r ds} \to 1$$

almost surely as $t \to \infty$.

Proof. By proposition 3.4 $u^\dagger \to 0$. Since $u^\dagger$ must be in $[0,1]$ the corresponding log wealth is bounded above and below by

$$\int_0^t r ds \leq LW^{(u^\dagger)}(t) \leq \int_0^t r ds + \left(2\int_0^t \sigma^2 ds\right)^{-1} \left(\int_0^t \sigma dB\right)^2.$$

But for $s = \int_0^{T(t)} \sigma^2 d\tau$, $\left(\int_0^t \sigma^2 ds\right)^{-1} \left(\int_0^t \sigma dB\right)^2$ is equal in distribution to $(s)^{-1} \tilde{B}_s^2$ which follows a χ^2 distribution. Hence $\left(\int_0^t \sigma^2 ds\right)^{-1} \left(\int_0^t \sigma dB\right)^2$ is finite almost surely and its distribution does not depend on t. Therefore

$$\frac{\left(2\int_0^t \sigma^2 ds\right)^{-1} \left(\int_0^t \sigma dB\right)^2}{\int_0^t r ds} \to 0$$

almost surely as $t \to \infty$. □

The following proposition shows that piecewise constant targets can generate very large wealth. In the proof we exhibit a sequence of piecewise constant targets whose wealth tends to infinity. The argument takes advantage of the unbounded variation

of Brownian motion.

Proposition 3.6. *Fix $t > 0$, if $\mu \geq r \geq 0$ then* $\sup\{LW^u(t) : u \in \mathcal{T}^{pc}(D_m)\} = \infty$.

Proof. Let $N > 0$ and $\Delta = \frac{\int_0^t \sigma^2 ds}{N}$ and consider the random partition

$$\{t_k = T(k\Delta) : k = 0, 1, ..., N\}$$

where T is the time change given above. The bcrp over the interval $[t_{k-1}, t_k]$ is

$$u_k^\ddagger = 0 \vee \left(\int_{t_{k-1}}^{t_k} \sigma^2 dt\right)^{-1} \left(\int_{t_{k-1}}^{t_k} (\mu - r)\, dt + \int_{t_{k-1}}^{t_k} \sigma dB\right) \wedge 1$$

and hence on the event $u^\ddagger = 1$ the log wealth generated is

$$\begin{aligned} LW^{(u_k^\ddagger)}_{[t_{k-1}, t_k)} &= \int_{t_{k-1}}^{t_k} \mu dt - \frac{1}{2}\int_{t_{k-1}}^{t_k} \sigma^2 dt + \int_{t_{k-1}}^{t_k} \sigma dB \\ &\geq -\frac{1}{2}\int_{t_{k-1}}^{t_k} \sigma^2 dt + \int_{t_{k-1}}^{t_k} \sigma dB \\ &= -\frac{1}{2}\Delta + \Delta\tilde{B}_k \end{aligned}$$

where $\Delta\tilde{B}_k = \tilde{B}_{k\Delta} - \tilde{B}_{(k-1)\Delta}$ are Brownian motion differences and the inequality follows from the assumptions on μ. On the event $u^\ddagger < 1$ the log wealth is

$$LW^{(u_k^\ddagger)}_{[t_{k-1}, t_k)} = \int_{t_{k-1}}^{t_k} r dt + \frac{1}{2}\left(u^\ddagger\right)^2 \int_{t_{k-1}}^{t_k} \sigma^2 dt \geq 0.$$

Since μ is non-negative $\Delta\tilde{B}_k > \Delta$ implies that $u_k^\ddagger > 1$. So if we knit together the bcrp's, define a piecewise constant portfolio by $u = u_k^\ddagger$ on $[t_{k-1}, t_k)$, $k = 1, ..., N$ we obtain a log wealth of at least

$$\begin{aligned} LW^{(u)}(t) &\geq \sum_{k=1}^{N}\left[\Delta\tilde{B}_k - \frac{1}{2}\Delta\right]\left\{\Delta\tilde{B}_k > \Delta\right\} \\ &\geq \sum_{k=1}^{N}\Delta\tilde{B}_k\left\{\Delta\tilde{B}_k > \Delta\right\} - t/2. \end{aligned}$$

Finally $\Delta\tilde{B}_k\left\{\Delta\tilde{B}_k > \Delta\right\}$ are *iid* with expected value:

$$\begin{aligned} \mathrm{E}\left[\Delta\tilde{B}_k\left\{\Delta\tilde{B}_k > \Delta\right\}\right] &= \int_{\Delta}^{\infty}\frac{1}{\sqrt{\Delta}}x\phi\left(\frac{x}{\sqrt{\Delta}}\right)dx \\ &= \int_{\Delta}^{\infty}\sqrt{\Delta}d\left(-\phi\left(\frac{x}{\sqrt{\Delta}}\right)\right) \\ &= \sqrt{\Delta}\phi\left(\sqrt{\Delta}\right) \\ &\geq \frac{1}{\sqrt{2\pi}}\sqrt{\Delta} - \frac{1}{2\sqrt{2\pi}}\Delta^{\frac{3}{2}} \end{aligned}$$

where the inequality follows from Taylor's formula. Hence by the law of large numbers

$$\liminf N^{-1}\frac{1}{\sqrt{\Delta}}\sum\Delta\tilde{B}_k\left\{\frac{\Delta\tilde{B}_k}{\Delta} > 1\right\} \geq \frac{1}{\sqrt{2\pi}}.$$

But $N^{-1}\frac{1}{\sqrt{\Delta}} = \frac{\sqrt{\Delta}}{T} \to 0$ almost surely hence $\sum\Delta\tilde{B}_k\left\{\frac{\Delta\tilde{B}_k}{\Delta} > 1\right\} \to \infty$ as $\Delta \to 0$. Therefore $\lim_{\Delta\to 0} LW^{(u)}(t) = \infty$ almost surely. □

From the previous proposition we see that there exist limits of piecewise constant targets which produce infinite wealth in finite time and hence are not targets. The problem is that if we view the bcrp as a function of t it is not continuous at 0. For example if we assume $\mu = r$ and $\sigma = 1$ then the bcrp is $u_t^\dagger = 0 \vee t^{-1}B_t \wedge 1$ and by the law of the iterated logarithm we get

$$\limsup_{t\to 0} u_t^\ddagger = 1 \qquad \text{and} \qquad \liminf_{t\to 0} u_t^\ddagger = 0.$$

In the following proposition we prove that smooth processes are targets (i.e. do not produce infinite wealth in finite time).

Proposition 3.7. *Any process with the paths of bounded variation is a target.*

Proof. We must show that u generates at most finite wealth at each time $0 \le t < \infty$. Fix $t > 0$, let $\{t_{k:n} : 0 \le k \le n\}$ be a sequence of partitions of $[0, t]$ with mesh size converging to 0. Define the piecewise constant target $u_n = u(t_{k:n})$ on $[t_{k-1:n}, t_{k:n})$ for $0 \le k \le n$. From (1.7) the log wealth

$$\begin{aligned} LW^{\cdot(u_n)} &= \int_0^t r ds + \sum_{k=1}^{n} \left[u(t_{k:n}) \int_{t_{(k-1):n}}^{t_{k:n}} (\mu - r)\, dt - \frac{1}{2} u^2(t_{k:n}) \int_{t_{(k-1):n}}^{t_{k:n}} \sigma^2 dt \right] \\ &\quad + \sum_{k=1}^{n} \left[u(t_{k:n}) \int_{t_{(k-1):n}}^{t_{k:n}} \sigma dB \right] \end{aligned}$$

By (1.3) the first term on the right hand side is finite. From (1.3) we have that the time integrals $\int_0^t (\mu - r)\, dt$ and $\int_0^t \sigma^2 dt$ are continuous and have bounded variation and we have assumed that u is bounded. Hence the second term above converges almost surely to the difference of finite integrals $\int_0^t u\,(\mu - r)\, dt - \frac{1}{2}\int_0^t u^2\sigma^2 dt$ as $n \to 0$. By summation by parts we obtain

$$\begin{aligned}\sum_{k=1}^{n}\left[u(t_{k:n})\int_{t_{(k-1):n}}^{t_{k:n}}\sigma dB\right] &= \sum_{k=1}^{n-1}\int_0^{t_{k:n}}\sigma dB\left(u(t_{k:n})-u(t_{k-1:n})\right)+u(t-)\int_0^t\sigma dB\\ &\quad -u(t-)\int_0^{t_{1:n}}\sigma dB\\ &\to \int_0^t\int_0^s\sigma dB du+u(t-)\int_0^t\sigma dB\end{aligned}$$

where the convergence follows since u is of bounded variation and the stochastic integral is continuous. □

Chapter 4

Example

We consider markets where with probability arbitrarily close to 1 the EG portfolio outperforms the bcrp and Cover's universal portfolio by order of t in the exponent, and where each earns order t in the exponent. The idea of the example is simple. Take a drift value so that the best piecewise constant target for one half of the interval is very near one and for the other half is very near zero. So the best constant for the entire interval is near $\frac{1}{2}$. In this oscillatory market we expect π to track the piecewise constant target and so the $\int(\pi - \frac{1}{2})^2\sigma^2 dt$ term in (2.4) is large. On the other hand the universal portfolio will perform approximately as well as the bcrp, which is earning less in this market.

Definition 4.1. *For $m = 1$, we call the quadruplet (r, μ, σ^2, T) Model 1 if $r = 0$, $\sigma > 0$ is deterministic constant and*

$$\mu_t = \begin{cases} \sigma^2 & \textit{for } 0 \leq t < \frac{T}{2} \\ 0 & \textit{for } \frac{T}{2} \leq t \leq T \end{cases}$$

Proposition 4.1. *Let $0 < \epsilon < \frac{1}{8}$ and π be the EG portfolio with learning parameter $\eta = T^{-\frac{1}{2}}$ and initial value $\pi(0) = \frac{1}{2}$ and $u^\dagger$ be the bcrp. Then*

$$LW^{(\pi)}_{[0,T)} - LW^{(u^\dagger)}_{[0,T)} \geq T\sigma^2 \frac{1 - 8(\epsilon + \epsilon^2)}{16} - KT^{\frac{1}{2}}$$

where $K = \left(\frac{5\sigma^2}{32} + \frac{3\log(2)}{2}\right)$ and

$$LW^{(\pi)}_{[0,T)} - LW^{(\hat{p})}_{[0,T)} \geq T\sigma^2 \frac{1 - 8(\epsilon + \epsilon^2)}{16} - KT^{\frac{1}{2}} + \frac{1}{2}\log(\frac{T\sigma^2}{7})$$

on an event A with $P(A) > 1 - 3\Phi\left(-c\epsilon\sqrt{T}\right)$ with $c = \sigma \wedge \frac{1}{8}$ and where Φ is the cumulative distribution function for the standard normal distribution.

Proof. The piecewise constant target which takes the best constant value over each of the interval $[0, T]$ is given by

$$u = \begin{cases} 0 \vee \left[1 + \frac{2B_{T/2}}{\sigma T}\right] \wedge 1 & \text{for } 0 \leq t < \frac{T}{2} \\ 0 \vee \left[\frac{2\left(B_T - B_{T/2}\right)}{\sigma T}\right] \wedge 1 & \text{for } \frac{T}{2} \leq t \leq T \end{cases}$$

On the other hand, the best constant target over the entire interval $[0, T]$ is

$$u^\dagger = 0 \vee \left[\frac{1}{2} + \frac{B_T}{\sigma T}\right] \wedge 1.$$

So for large T we see that u is nearly one on the first half of the interval and zero on the second half while $u^\dagger$ is essentially $\frac{1}{2}$.

From Theorem 2.3 with $\eta = T^{-\frac{1}{2}}$ and $\pi(0) = \frac{1}{2}$ we get

$$\begin{aligned} LW^{(\pi)}_{[0,T/2)} - LW^{(u)}_{[0,T/2)} &= \frac{1}{2}\int_0^{T/2} (\pi - u)^2 \sigma^2 dt - T^{-\frac{1}{2}}\frac{1}{2}\int_0^t \pi(1-\pi)\sigma^2 \\ &\quad + T^{\frac{1}{2}} \left(d(u(o)\|\pi(T/2)) - d(u(0)\|1/2)\right) \\ &\geq \frac{1}{2}\int_0^{T/2} (\pi - u)^2 \sigma^2 dt - KT^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

where $K = (\frac{\sigma^2}{16} + log(2))$. Applying the general expression for log wealth (1.6) to our special model we get

$$\begin{aligned} LW^{(u)}_{[0,T/2)} - LW^{(\pi)}_{[0,T/2)} &= \frac{1}{2}u^2(0)\int_0^{T/2} \sigma^2 dt + \int_0^{T/2} \left(\frac{\pi^2}{2} - \pi\right)\sigma^2 dt \\ &\quad - \int_0^{T/2} \pi\sigma dB \\ &= \frac{1}{2}\int_0^{T/2} (\pi - u)^2\sigma^2 dt - \int_0^{T/2} \pi\left(1 - u\right)\sigma^2 dt \\ &\quad - \int_0^{T/2} \pi\sigma dB \end{aligned}$$

Hence on the event $A' = \left\{B_{T/2} > -T\epsilon\sigma \text{ and } \int_0^{T/2} \pi\sigma dB < \frac{T\epsilon\sigma^2}{4}\right\}$

$$LW^{(u)}_{[0,T/2)} - LW^{(\pi)}_{[0,T/2)} \geq \frac{1}{2}\int_0^{T/2} (\pi - u)^2\sigma^2 dt - \frac{T}{2}\epsilon\sigma^2.$$

Hence by (4.1) on A' we have

$$\int_0^{T/2} (\pi - u)^2\sigma^2 dt < \frac{T}{2}\epsilon\sigma^2 + KT^{\frac{1}{2}}.$$

Now we compare with $u^{\dagger}$. Let $A'' = \{B_T \leq \sqrt{\epsilon}\sigma T\}$ then on the event $A = A' \cap A''$

we have

$$
\begin{aligned}
\frac{1}{2}\int_0^{T/2}(\pi-u^{\dagger})^2\sigma^2 dt &\geq \frac{1}{2}\int_0^{T/2}(1-\frac{1}{2})^2\sigma^2 dt-\frac{1}{2}\int_0^{T/2}(\frac{1}{2}-u^{\dagger})^2\sigma^2 dt \\
&\quad -\frac{1}{2}\int_0^{T/2}(u-1)^2\sigma^2 dt-\frac{1}{2}\int_0^{T/2}(\pi-u)^2\sigma^2 dt \\
&\geq \frac{1-8\left(\epsilon+\epsilon^2\right)}{16}T\sigma^2-\frac{K}{2}T^{\frac{1}{2}}. \qquad (4.2)
\end{aligned}
$$

Using Theorem 2.3 once more but with the best constant target on the interval $[0, T]$

$$
\begin{aligned}
LW_{[0,T)}^{(\pi)}-LW_{[0,T)}^{(u^{\dagger})} &= \frac{1}{2}\int_0^{T}\left(\pi-u^{\dagger}\right)^2\sigma^2 dt-T^{-\frac{1}{2}}\frac{1}{2}\int_0^{T}\pi(1-\pi)\sigma^2 dt \\
&\quad +T^{\frac{1}{2}}\left(d(u(o)\|\pi(T))-d(u(0)\|1/2)\right) \\
&\geq T\sigma^2\frac{1-8\left(\epsilon+\epsilon^2\right)}{16}-K_2T^{\frac{1}{2}} \qquad (4.3)
\end{aligned}
$$

on $A'\cap A''$ and where $K_2=\left(\frac{5\sigma^2}{32}+\frac{3\log(2)}{2}\right)$. Since the market is active by Proposition (3.1) we have

$$
\begin{aligned}
LW_{[0,T)}^{(\pi)}-LW_{[0,T)}^{(\hat{p})} &= LW_{[0,T)}^{(\pi)}-LW_{[0,T)}^{(\hat{p})}+\frac{1}{2}\log(T\sigma^2)-C_T \\
&\geq \frac{1-8\left(\epsilon+\epsilon^2\right)}{16}T\sigma^2-K_2T^{\frac{1}{2}}+\frac{1}{2}\log(\frac{T\sigma^2}{7}).
\end{aligned}
$$

It remains to show the bound on the probability of A. Clearly

$$
\begin{aligned}
P\left(B_{T/2}>-\epsilon\sigma T\right) &= \Phi\left(\epsilon\sigma\sqrt{2T}\right) \\
P\left(B_T<\sqrt{\epsilon}\sigma T\right) &= \Phi\left(\sigma\sqrt{\epsilon T}\right)
\end{aligned}
$$

and by a time change argument

$$
\begin{aligned}
P\left(\int_0^{T/2} \pi\sigma dB < \frac{\epsilon\sigma^2}{4}T\right) &\geq P\left(\int_0^{T/2} \pi\sigma dB < \frac{\epsilon}{2}\int_0^{T/2} \pi^2\sigma^2 dt\right) \\
&= \Phi\left(\epsilon\sqrt{\frac{T}{8}}\right).
\end{aligned}
$$

□

Hence for T sufficiently large with probability close to 1, the EG portfolio is outperforming the bcrp and the universal portfolio by on the order of T in the exponent.

Chapter 5

Proofs of Ch. 2 Results

We now present the proofs of the results contained in Chapter 2. For the reader's convenience the theorems are restated and previously displayed expressions retain their original numbering.

Theorem (2.1). *Suppose μ, r and Σ are $\mathcal{F}_t$ adapted processes which satisfy (1.3), S is a bounded stopping time and $\xi \in \mathbb{R}^m$ is $\mathcal{F}_S$ measurable. Let η be an adapted $\mathbb{R}^+$ valued process such that for each $t > 0$*

$$\int_0^t \left[|\eta r| + \sum_{i=1}^{m} \left(|\eta \mu_i| + \eta^2 \Sigma_{ii} \right) \right] ds < \infty \tag{2.1}$$

almost surely. There exists a unique strong solution $\{\alpha(S,t), t \geq S\} \in \mathbb{R}^m$ to the following stochastic differential equation

$$d\alpha_i = \eta dZ_i + \eta \left[\frac{1}{2} \Sigma_{ii} - \Sigma_i f(\alpha) \right] dt \tag{2.2}$$

with initial $\alpha(S,0)=\xi$ *and where* $f:\mathbb{R}^m \to D_m$ *is given by*

$$f_i(a) = e^{a_i}\left(1+\sum_{j=1}^{m} e^{a_j}\right)^{-1}, \quad i=1,...,m. \tag{2.3}$$

Proof. By our assumptions (2.1) the processes $\int_0^t \eta dZ$ and $\int_0^t \eta\Sigma_{ij}ds$ are semi-martingales. From Theorem V.3.7 on page 197 of Protter (1990), to show existence and uniqueness it is enough that f is Lipschitz. For $a \in \mathbb{R}^m$, the partial derivatives are given by

$$\frac{\partial f_i(a)}{\partial a_i} = \frac{e^{a_i}\left(1+\sum_{k\neq i} e^{a_k}\right)}{\left(1+\sum_{k=1}^{m} e^{a_k}\right)^2} = f_i(a)\left(1-f_i(a)\right)$$

and (5.1)

$$\frac{\partial f_i(a)}{\partial a_j} = -\frac{e^{a_i+a_j}\left(1+\sum_{k\neq i} e^{a_k}\right)}{\left(1+\sum_{k=1}^{m} e^{a_k}\right)^2} = -f_i(a)f_j(a)$$

for $i \neq j$. The derivatives are continuous and bounded in absolute value by 1. Hence f is Lipschitz. In fact $\|f(a)-f(b)\| \leq m\|a-b\|$ for $a,b\in\mathbb{R}^m$. □

Before we prove Theorem 2.2 we require the following lemma which gives conditions on the integrand for convergence of a stochastic integral.

Lemma 5.1. *Let* σ *satisfy (1.3) and* $g^{(n)}$ *be a sequence of* $\mathbb{R}^m$ *valued processes such that* $g_i^{(n)}(t)$ *are bounded and continuous and converge uniformly to* 0 *in probability. Then* $\int_0^t g^{(n)}\sigma dB$ *converges uniformly on compacts to zero in probability, i.e. for each* $t>0$

$$\sup_{0\leq s\leq t}\left\|\int_0^s g^{(n)}\sigma dB\right\| \to 0$$

in probability.

Proof. Let $T_M = \inf\left\{t : \max_i \int_0^t \Sigma_{ii} dt \geq M\right\}$. By (1.3) $T_M \to \infty$ almost surely as $M \to \infty$. If we can establish the result for the stopped process $\int_0^{t\wedge T_M} g^{(n)}\sigma dB$ then we obtain the desired result upon letting $M \to \infty$. Therefore we assume that $\max_i \int_0^\infty \Sigma_{ii} dt \leq M$. First assume $g^{(n)}$ converges uniformly almost surely. By Doob's inequality

$$
\begin{aligned}
\mathrm{E}\left[\sup_t \int_0^t g^{(n)}\sigma dB\right]^2 &\leq \mathrm{E}\left[\int_0^\infty \sum_{i=1}^m g_i^{(n)}\Sigma_{ii} dt\right]^2 \\
&\leq \mathrm{E}\left[mM \sup_{t,i}\left|g_i^{(n)}\right|\right]^2
\end{aligned}
$$

which converges to 0 by the Bounded Convergence Theorem. So $\sup_t \int_0^t g^{(n)}\sigma dB$ converges to 0 in probability. Now suppose $g^{(n)}$ only converges in probability uniformly. Then for every subsequence n_k, $g^{(n_k)}$converges to zero in probability uniformly. For each of these subsequences there is another subsequence n_{k_l} for which the convergence is almost sure. From what we have shown $\sup_t \int_0^t g^{(n_{k_l})}\sigma dB$ converges to 0 almost surely. Hence $\sup_t \int_0^t g^{(n)}\sigma dB$ converges to 0 in probability. □

Theorem (2.2). *Suppose that τ is a positive constant, S, T are bounded stopping times with $0 < T - S \leq \tau$ and $\eta > 0$ is constant. Under the conditions of Theorem 2.1 the DEG(η) portfolio q converges in probability uniformly to the EG(η) portfolio π and the $LW^{(q)}$ converges in probability uniformly to $LW^{(\pi)}$ on the interval $[S, T]$.*

Proof: Let α be the solution to the SDE (2.2), f be as in Theorem 2.1, and $S < T$ be be stopping times bounded above by the constant $\tau > 0$. Recall that the DEG portfolio is given by

$$
\begin{aligned}
q(t_0) &= \xi, \\
q_i(t) &= q_i(t_{k-1}) \exp\left\{ \frac{\eta X_i(k)}{q(t_{k-1})^* X(k) + q_0(t_{k-1}) X_0(k)} \right\} V_k^{-1}, \ t_k \le t < t_{k+1}
\end{aligned}
$$

where X is the price relative process and V_k is a normalizer. Also the EG portfolio $\pi = f(\alpha)$. The idea of the proof is to consider $a_i(k) = a_i(k, \xi, \eta) = \log\left(\frac{q_i(t_k)}{q_0(t_k)}\right)$ (so that $q = f(a)$) and show that a converges uniformly to α on the interval $[S, T]$. Once this is established it follows from the continuity of f that q and its corresponding log wealth converge to π and $LW^{(\pi)}$ respectively. We first decompose a. For each i

$$
\begin{aligned}
a_i(k) &= a_{k-1} + \frac{\eta (X_i(k) - X_0(k))}{q(t_{k-1})^* X(k) + q_0(t_{k-1}) X_0(k)} \\
&= a_{k-1} + \frac{\eta \left(X_i(k) X_0(k)^{-1} - 1 \right)}{\sum_{j=1}^m q_j(t_{k-1}) \left(X_j(k) X_0(k)^{-1} - 1 \right) + 1}.
\end{aligned}
$$

Note that $X_i(k) X_0(k)^{-1} = \exp\{Z_i(t_k) - Z_i(t_{k-1})\}$. So at each step the process a is updated according to

$$
a_i(k) = a_{k-1} + g_i\left(Z(t_k) - Z(t_{k-1}), q(t_{k-1})\right) \tag{5.2}
$$

where g_i is defined for $1 \le i \le m$, $y \in \mathbb{R}^m$ and $v \in D^m$ by $g_i(y; v) = \frac{\eta(e^{y_i} - 1)}{\sum_{j=1}^m v_j (e^{y_j} - 1) + 1}$. It is important to note that since the DEG portfolio remains constant on the interval $[t_{k-1}, t_k)$, the update in (5.2) depends only on the initial value of the portfolio $q(t_{k-1})$

and the change in the discounted stock price process Z. To determine the asymptotic behavior as the mesh size tends to zero, we consider the differential of this update with respect to Z. By Ito's formula

$$
\begin{aligned}
a_i(k) &= a_{k-1} + \sum_i \int_{t_{k-1}}^{t_k} g_i^{(i)}\left(Z^k, q\right) dZ_i \\
&\quad + \frac{1}{2} \sum_i \sum_j \int_{t_{k-1}}^{t_k} g_i^{(ij)}\left(Z^k, q\right) \Sigma_{ij} dt
\end{aligned}
$$

with $Z_i^k(t) = Z_i(t) - Z_i(t_{k-1}), 1 \leq i \leq m$ and where $g_i^{(i)}(y, v) = \frac{\partial}{\partial y_i} g_i(y, v)$ and $g_i^{(ij)} = \frac{\partial^2}{\partial y_i \partial y_j} g_i(y, v)$. In Remark 5.1 which follows this proof the derivatives are presented explicitly. For our purposes it sufficient to note that each of the derivatives is continuous in first argument, bounded in absolute value by $\max_j e^{3|y_j|}$ and that for $j, j' \neq i$ we have $g_i^{(j)}(0, v) = g_i^{(jj')}(0, v) = 0$, $g_i^{(i)}(0, v) = \eta$, $g_i^{(ii)}(0, v) = \eta(1 - 2v_i)$ and $g_i^{(ij)}(0, v) = -\eta v_j$. Using the values of the partial derivatives at $y = 0$ we find that

$$
\begin{aligned}
a_i(k) &= a_{k-1} + \eta \int_{t_{k-1}}^{t_k} dZ_i + \frac{\eta}{2} \int_{t_{k-1}}^{t_k} \Sigma_{ii} \, dt \\
&\quad - \eta \sum_{j=1}^{m} \int_{t_{k-1}}^{t_k} q_j \Sigma_{ij} \, dt + R_i(t_k) - R_i(t_{k-1}) \qquad (5.3)
\end{aligned}
$$

with $1 \leq k \leq n$ and where the remainder is given by

$$
R_i(t) = \sum_i \int_S^t \hat{g}_i^{(i)} \, dZ_i + \frac{1}{2} \sum_i \sum_j \int_S^t \hat{g}_i^{(ij)} \Sigma_{ij} \, ds.
$$

and $\hat{g}_i^{(i)}$ and $\hat{g}_i^{(ij)}$ are defined piecewise by

$$\begin{aligned}\hat{g}_i^{(i)}(t) &= g_i^{(i)}\left(Z^k(t), q\right) - g_i^{(i)}(0, q)\\ \hat{g}_i^{(ij)}(t) &= g_i^{(ij)}\left(Z^k(t), q\right) - g_i^{(ij)}(0, q)\end{aligned}$$

for $t_{k-1} < t \leq t_k$, $1 \leq k \leq n$, and $1 \leq i, j, \leq m$. Notice the resemblance of the expression above for a_i and (2.2). We now show that in probability:

$$\lim_{\Delta\to 0} \sup\left\{|\alpha(t) - a(t)| : S \leq t \leq T\right\} = 0 \tag{5.4}$$

Proof of (5.4) The result follows in two steps. In the first step we prove that the remainder term R converges to 0. For this step we employ a second moment argument involving Ito's isometry. In (1.3) we have not assumed the existence of any moments of the stochastic integrals, however, we circumvent this obstacle with a localization argument. In the second step we use the method of successive approximations to prove that the integrals on the rhs of (5.3) converge to α.

Define the stopping time

$$T_M = \inf\left\{t > 0 : \max_i |Z_i(t)| = M \text{ or } \int_0^t \left[|r| + \sum_{i=1}^m (|\mu_i| + \Sigma_{ii})\right] ds = M\right\}.$$

By (1.3) $\lim T_M = \infty$ almost surely as $M \to \infty$. Thus if we can establish the result for the stopped process $\alpha(t \wedge T_M) - a(t \wedge T_M)$ (i.e. under a boundedness assumption) then we obtain the desired result upon letting $M \to \infty$. We may assume therefore that $\max_i |Z_i(t)|$ and $\int_0^t [|r| + \sum_{i=1}^m (|\mu_i| + \Sigma_{ii})]\, ds$ are bounded by $M > 0$. First we show that $\sup_{S \le t \le T} \|R(t)\| \to 0$ in probability. In terms of the Brownian integral we have

$$\begin{aligned} R_i(t) &= \sum_i \int_S^t \hat{g}_i^{(i)} \sigma_i \, dB + \sum_{i=1}^m \int_S^t \hat{g}_i^{(i)} \left((\mu_i - r) - \frac{1}{2}\Sigma_{ii} \right) ds \\ &+ \frac{1}{2} \sum_i \sum_j \int_S^t \hat{g}_i^{(ij)} \Sigma_{ij} \, ds \end{aligned}$$

By the triangle inequality and using $(a+b)^2 \le 2(a^2 + b^2)$

$$\begin{aligned} &\mathrm{E}\left[\sup_{S \le t \le T} |R_i(t)|\right]^2 \\ &\le 2\mathrm{E}\left[\sum_{i=1}^m \int_S^T \left|\hat{g}_i^{(i)}\right| \left|(\mu_i - r) - \frac{1}{2}\Sigma_{ii}\right| ds + \frac{1}{2} \sum_i \sum_{j=1} \int_S^T \left|\hat{g}_i^{(ij)}\right| |\Sigma_{ij}| \, ds\right]^2 \\ &\quad + 2\mathrm{E}\left[\sup_{S \le t \le T} \left|\sum_{i=1}^m \int_S^t \hat{g}_i^{(i)} \sigma_i dB\right|\right]^2 \\ &\le 2\mathrm{E}\left[\sum_{i=1}^m \int_S^T \left|\hat{g}_i^{(i)}\right| \left|(\mu_i - r) - \frac{1}{2}\Sigma_{ii}\right| ds + \frac{1}{2} \sum_i \sum_{j=1} \int_S^T \left|\hat{g}_i^{(ij)}\right| |\Sigma_{ij}| \, ds\right]^2 \\ &\quad + 8\mathrm{E}\left[\sum_{i=1}^m \int_S^T (\hat{g}_i^{(i)})^2 \Sigma_{ii} \, ds\right]^2 \end{aligned}$$

where the last inequality follows from Doob's inequality, Ito's isometry and the fact that $T \le \tau$. Under our assumptions Z is a bounded continuous semi-martingale and hence each of the derivatives $g_i^{(i)}(Z^k, q)$ and $g_i^{(ij)}(Z^k, q)$ are uniformly continuous as functions of $t \in [0, \tau]$ and $\int_0^\tau [|r| + \sum_{i=1}^m (|\mu_i| + \Sigma_{ii})]\, ds < M$ a.s.. Thus the expressions under the expectation, e.g. $\int_S^T (\hat{g}_i^{(i)})^2 \Sigma_{ii} ds$, converge to zero almost surely. Also the Z^k are uniformly bounded by $2M$, hence the derivatives are uniformly bounded by

e^{6M}. So by the bounded convergence theorem $\mathrm{E}\left[\sup_{S\le t\le T}|R_i(t)|\right]^2$ converges to zero. Repeating the argument for each $i = 1, \ldots, m$ we obtain $\sum_{i=1}^m \mathrm{E}\left[\sup_{S\le t\le T}|R_i(t)|\right]^2 \to 0$. Hence in probability $\sup_{S\le t\le T}\|R(t)\|^2 \to 0$.

We now use the method of successive approximations to prove that the integrals on the rhs of (5.3) converge to α. Since f is Lipschitz there exists $K > 0$ such that

$$\|f(y) - f(x)\| \le K\|y - x\|, \quad x, y \in \mathbb{R}^m.$$

Define a sequence of stopping times $T_0 = S$ and $T_l = \inf\left\{t > T_{l-1} : \int_{T_l}^t \|\Sigma\|\, ds\right\} = (2K\sqrt{m})^{-1}$ where $\|\Sigma\| = \sqrt{\sum_{i,j}\Sigma_{ij}^2}$ is the matrix norm. By the Cauchy-Schwartz inequality

$$\begin{aligned}\int_0^t \|\Sigma\|\, ds &\le m\int_0^t \max_{ij}|\Sigma_{ij}|ds\\ &\le m\int_0^t \Sigma_{ii}^{\frac{1}{2}}\Sigma_{jj}^{\frac{1}{2}}ds\\ &\le m\sqrt{\int_0^t \Sigma_{ii}ds\int_0^t \Sigma_{jj}ds}.\end{aligned}$$

Hence by (1.3) then $T_l \to \infty$ as $l \to \infty$ and there exists an L random but finite such that $T_L > T$. Define a sequence of $\mathbb{R}^m$ valued processes by

$$Y_i^{(j)}(t) = \xi + \eta\int_S^t dZ_i + \eta\frac{1}{2}\int_S^t \Sigma_{ii}ds + \int_S^t \Sigma_i f(Y^{(j-1)})ds, \quad S \le t \le T \qquad (5.5)$$

for $i = 1, \ldots, m$, $j = 1, 2, \ldots$ and with $Y^{(0)} = a$. Define

$$D^{(j,l)} = \sup\left\{\left\|Y^{(j)}(t) - Y^{(j-1)}(t)\right\| : T_{l-1} \leq t \leq T_l\right\}.$$

Then from the definition of R

$$\begin{aligned} D^{(1,l)} &= \sup_{T_{l-1} \leq t \leq T_l} \left\|Y^{(1)}(t) - a(t)\right\| \\ &= \sup_{T_{l-1} \leq t \leq T_l} \|R(t) - R(T_{l-1})\| \\ &\leq \max_l \sup_{T_{l-1} \leq t \leq T_l} \|R(t) - R(T_{l-1})\| \end{aligned}$$

which is independent of l and converges to zero in probability as $\Delta \to 0$ since L is finite and we have shown $\|R(t)\|$ converges uniformly to 0. By the definition of D, the Lipschitz property and the vector inequality $\|x\| \leq \sqrt{m}\max_i |x_i|, x \in \mathbb{R}^m$ we have

$$\begin{aligned} D^{(j,l)} &= \sup_{T_{l-1} \leq t \leq T_l} \left\| \int_{T_{l-1}}^{t} \Sigma\left(f\left(Y^{(j-1)}\right) - f\left(Y^{(j-2)}\right)\right) ds \right\| \\ &\leq \sqrt{m} D^{(j-1,l)} K \sup_{T_{l-1} \leq t \leq T_l} \int_{T_{l-1}}^{t} \|\Sigma\| ds \\ &\leq \frac{1}{2} D^{(j-1,l)} \leq \frac{1}{2^j} D^{(1,l)}. \end{aligned}$$

The final inequalities follow from the definition of the random partition formed by the stopping times T_l and by induction. Knitting together over the partition yields:

$$\sup_{S \leq t < T} \left\{\left\|Y^{(j)}(t) - Y^{(j-1)}(t)\right\|\right\} \leq \sum_{l=1}^{L} D^{(j,l)} \leq \frac{1}{2^j} \sum_{l=1}^{L} D^{(1,l)}.$$

Since the bound is the general term of a convergent sum, $\|Y^{(j)}(t) - Y^{(j-1)}(t)\|$ converges to 0 geometrically fast as $j \to \infty$. By taking limits in (5.5) it follows that the sequence of processes $Y_t^{(j)}$ converge uniformly to α_t with probability one. Thus

$$\begin{aligned}
\sup_t \left\{ \left\| \alpha(t) - \alpha^{(n)}(t) \right\| \{S \le t < T\} \right\} &= \lim_j \sup_{S \le t < T} \left\{ \|Y^{(j)}(t) - Y^{(0)}(t)\| \right\} \\
&\le \sum_{j=1}^{\infty} \sup_{S \le t < T} \left\{ \|Y^{(j)}(t) - Y^{(j-1)}(t)\| \right\} \\
&\le \sum_{l=1}^{L} D^{(1,l)}
\end{aligned}$$

which we have shown converges to 0 in probability as $\Delta \to 0$.

End of Proof of (5.4)

We now show that

$$\sup \left\{ |q_i(t) - \pi_i(t)| : S \le t \le T, 1 \le i \le m \right\} \to 0 \tag{5.6}$$

in probability and

$$\sup \left\{ \left| LW^{(q)}(t) - LW^{(\pi)}(t) \right| : S \le t \le T \right\} \to 0 \tag{5.7}$$

in probability.

Proof of (5.6) and (5.7): Since $q = f(a)$ and $\pi = f(\alpha)$ and f is uniformly continuous by (5.4) q converges uniformly to π. The log wealth generated by q is

$$LW^{(q)}(t) = \int_S^t r ds + \int_S^t q^* dZ + \frac{1}{2} \int_S^t \sum_{i=1}^{m} q_i \Sigma_{ii} ds - \frac{1}{2} \int_S^t q^* \Sigma q ds$$

Decomposing the stochastic integral we have

$$\int_S^t q^* dZ = \int_S^t q^*(\mu - r)ds + \int_S^t \sum_{i=1}^m q_i \Sigma_{ii} dt + \int_S^t q^* \sigma dB.$$

Using (1.3) it is easy to see that the each of the time integrals converge uniformly in probability to their π counterparts. For example

$$\int_S^t |q_i - \pi_i| \, \Sigma_{ii} ds \le \sup_{S \le t \le T} |q(t) - \pi_i(t)| \int_S^T \Sigma_{ii} ds \to 0$$

in probability. By Lemma 5.1 the integrals with respect to Brownian Motion converge uniformly to 0 in probability as well.

End of proof of Theorem 2.2 □

Remark 5.1.

Let g be defined for $y \in \mathbb{R}^m$ and $v \in D^m$ by $g(y;v) = \frac{\eta(e^{y_i}-1)}{\sum_{j=1}^m v_j\left(e^{y_j}-1\right)+1}$. The first order partial derivatives are:

$$\begin{aligned} g^{(i)}(y, v) = \frac{\partial g\,(y; v)}{\partial y_i} &= \frac{\eta e^{y_i}\left(1 + \sum_{j \neq i} v_j \left(e^{y_j} - 1\right)\right)}{\left(1 + \sum_{j=1}^m v_j \left(e^{y_j} - 1\right)\right)^2} \\ g^{(k)}(y, v) = \frac{\partial g\,(y; v)}{\partial y_k} &= \frac{-\eta v_k e^{y_k}\left(e^{y_i} - 1\right)}{\left(1 + \sum_{j=1}^m v_j \left(e^{y_j} - 1\right)\right)^2} \end{aligned}$$

and the second order derivatives are

$$
\begin{aligned}
\frac{\partial^2 g(y;v)}{\partial y_i \partial y_k} &= -\frac{\eta e^{y_i+y_k} v_k \left(1+\sum_{j\neq i} v_j\left(e^{y_j}-1\right)-v_i\left(e^{y_i}-1\right)\right)}{\left(1+\sum_{j=1}^m v_j\left(e^{y_j}-1\right)\right)^3} \\
\frac{\partial^2 g(y;v)}{\partial y_k \partial y_l} &= -g^{(k)}(y;v)\frac{2v_l e^{y_l}}{\left(1+\sum_{j=1}^m v_j\left(e^{z_j}-1\right)\right)} \\
\frac{\partial^2 g(y;v)}{\partial^2 y_i} &= g^{(i)}(y;v)\frac{\left(1+\sum_{j\neq i} v_j\left(e^{y_j}-1\right)-v_i\left(e^{y_i}+1\right)\right)}{\left(1+\sum_{j=1}^m v_j\left(e^{y_j}-1\right)\right)^2} \\
\frac{\partial^2 g(y;v)}{\partial^2 y_k} &= g^{(k)}(y;v)\frac{\left(1+\sum_{j\neq k} v_j\left(e^{y_j}-1\right)-v_k\left(e^{y_k}+1\right)\right)}{\left(1+\sum_{j=1}^m v_j\left(e^{y_j}-1\right)\right)^3}.
\end{aligned}
$$

where i, k, l are all distinct. Each of the derivatives above is continuous as a function of y. Also it is easy to check that $\left(1+\sum_{j=1}^m q_j\left(e^{y_j}-1\right)\right)^{-1} \leq \max_j e^{|y_j|}$. Using this bound implies that the derivatives are all bounded in absolute value by $\max_j e^{3|y_j|}$.

Theorem (2.3). *Let $\eta > 0$ be constant. For any bounded stopping times S and T with $0 < T - S \leq \tau$ and any non short-selling constant target $u \in \mathcal{T}^0(D_m)$ the EG(η,S) portfolio π satisfies*

$$
LW^{(\pi)}_{[S,T)} - LW^{(u)}_{[S,T)} = \frac{1}{2}\int_S^T (\pi-u)^* \Sigma (\pi-u)\, dt + \frac{\eta}{2}\int_S^T \pi^* \Sigma \pi dt
$$

$$
-\frac{\eta}{2}\int_S^T \sum_{i=1}^m \pi_i \Sigma_{ii} dt + \frac{d(u\|\pi(T)) - d(u\|\pi(S))}{\eta}.
$$

Proof. The proof follows by applying Ito's formula to the relative entropy function d.

First we use the definition of d, since u is constant we have

$$\begin{aligned}
&d\left(u\|\pi\left(T\right)\right)-d\left(u\|\pi\left(S\right)\right)\\
&= \log\left(\frac{u_0}{\pi_0\left(T\right)}\right)-\sum_{i=1}^{m}u_i\left[\log\left(\frac{u_i}{u_0}\right)-\log\left(\frac{\pi_i\left(T\right)}{\pi_0\left(T\right)}\right)\right]\\
&\quad -\log\left(\frac{u_0}{\pi_0\left(S\right)}\right)+\sum_{i=1}^{m}u_i\left[\log\left(\frac{u_i}{u_0}\right)-\log\left(\frac{\pi_i\left(S\right)}{\pi_0\left(S\right)}\right)\right]\\
&= \log\left(\frac{\pi_0\left(S\right)}{\pi_0\left(T\right)}\right)-u^*\left(\alpha\left(T\right)-\alpha\left(S\right)\right)\\
&= h\left(\alpha\left(T\right)\right)-h\left(\alpha\left(S\right)\right)-u^*\left(\alpha\left(T\right)-\alpha\left(S\right)\right)
\end{aligned}$$

where $h:\mathbb{R}^m\to\mathbb{R}$ is given by $h(a)=\log\left(1+\sum_{i=1}^{m}e^{a_i}\right)$. Relating to f used in the definition of π and using (5.1) we calculate the partial derivatives as:

$$\begin{aligned}
h_i\left(a\right)=\frac{\partial h\left(a\right)}{\partial a_i} &= \frac{e^{a_i}}{1+\sum_k^m e^{a_k}}=f_i(a),\\
h_{ij}\left(a\right)=\frac{\partial^2 h\left(a\right)}{\partial a_i\partial a_j} &= -f_i(a)f_j(a),\\
h_{ii}\left(a\right)=\frac{\partial^2 h\left(a\right)}{\partial^2 a_i} &= f_i(a)\left(1-f_i(a)\right).
\end{aligned}$$

Comparing with the definition of the $EG\left(\eta,S\right)$ portfolio we recognize that $h_i\left(\alpha\right)=\pi_i$, $h_{ij}\left(\alpha\right)=-\pi_i\pi_j$ and $h_{ii}\left(\alpha\right)=\pi_i\left(1-\pi_i\right)$. So by Ito's formula

$$h\left(\alpha\left(T\right)\right)=h\left(\alpha\left(S\right)\right)+\int_S^T\pi^*d\alpha-\frac{\eta^2}{2}\int_S^T\pi^*\Sigma\pi dt+\frac{\eta^2}{2}\int_S^T\sum_{i=1}^{m}\pi_i\Sigma_{ii}dt.$$

Recalling the differential for α: $d\alpha_i = \eta dZ_i + \eta\left[\frac{1}{2}\Sigma_{ii} - \Sigma_i\pi\right]dt$, we get

$$\begin{aligned}
& d\left(u\|\pi\left(T\right)\right) - d\left(u\|\pi\left(S\right)\right) \\
= & \int_S^T (\pi - u)^* \, d\alpha - \frac{\eta^2}{2}\int_S^T \pi^*\Sigma\pi dt + \frac{\eta^2}{2}\int_S^T \sum_{i=1}^m \pi_i\Sigma_{ii} dt \\
= & \eta\int_S^T (\pi - u)^* \, dZ + \frac{\eta}{2}\int_S^T \sum_{i=1}^m (\pi - u)_i \, \Sigma_{ii} dt - \eta\int_S^T (\pi - u)^* \, \Sigma\pi dt \\
& -\frac{\eta^2}{2}\int_S^T \pi^*\Sigma\pi dt + \frac{\eta^2}{2}\int_S^T \sum_{i=1}^m \pi_i\Sigma_{ii} dt \\
= & \eta\left(LW_{[S,T)}^{(\pi)} - LW_{[S,T)}^{(u)}\right) - \frac{\eta}{2}\int_S^T (\pi - u)^* \, \Sigma\left(\pi - u\right) dt \\
& -\frac{\eta^2}{2}\int_S^T \pi^*\Sigma\pi dt + \frac{\eta^2}{2}\int_S^T \sum_{i=1}^m \pi_i\Sigma_{ii} dt.
\end{aligned}$$

where the final equality follows from the general expression for the log wealth (1.6). With rearrangement of the terms and division by η we obtain the result. $\square$

Lemma 5.2. $T_M = \inf\left\{t \geq 0 : \max_i \int_0^t \Sigma_{ii} = M\right\}$ *is a stopping time with respect to* $\mathcal{G}_t = \sigma\left\{S(s) : 0 \leq s \leq t\right\}$.

Proof. We show that $\max_i \int_0^t \Sigma_{ii} ds$ is $\mathcal{G}_t$ adapted. For each partition of the interval $[0, t]$ the following sum is clearly $\mathcal{G}_t$ measurable

$$\begin{aligned}
\sum_k \left[\log(S_i(t_k)) - \log(S_i(t_{k-1}))\right]^2 &= \sum_k \left[\int_{t_{k-1}}^{t_k} \left(\mu_i - \frac{1}{2}\Sigma_{ii}\right) dt + \int_{t_{k-1}}^{t_k} \sigma_i dB\right]^2 \\
= \sum_k \left[\int_{t_{k-1}}^{t_k} \left(\mu_i - \frac{1}{2}\Sigma_{ii}\right) dt\right]^2 &+ 2\sum_k \left[\int_{t_{k-1}}^{t_k} \left(\mu_i - \frac{1}{2}\Sigma_{ii}\right) dt \int_{t_{k-1}}^{t_k} \sigma_i dB\right] \\
+ \sum_k \left[\int_{t_{k-1}}^{t_k} \sigma_i dB\right]^2 &.
\end{aligned}$$

Since the time integral has bounded variation and the Brownian integral is continuous the first two terms converge to 0 almost surely as the mesh size of the partition converges to 0. By Theorem 1.5.8 on page 32 of Karatzas and Shreve, the third term $\sum_k \left[\int_{t_{k-1}}^{t_k} \sigma_i dB\right]^2$ converges to $\int_0^t \Sigma_{ii} ds$ in probability. Hence there exists a subsequence which converges almost surely. Thus, as the almost sure limit of $\mathcal{G}_t$ measurable random variables, $\int_0^t \Sigma_{ii} ds$ is $\mathcal{G}_t$ measurable, as is the maximum over i. □

Corollary (2.6). *Let $u \in \mathcal{T}^{pc}$ be a piecewise target with n jumps. Let $S = 0$, $M > 0$ and $T_M = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii} ds = M\right\}$. Choose $\eta = \sqrt{\frac{2}{mM}}$ then the EG(η,0) portfolio π with $\pi(0) = \frac{1}{(m+1)}\mathbf{1_m}$ satisfies*

$$\begin{aligned} LW_{[0,t)}^{(\pi)} - LW_{[0,t)}^{(u)} &\geq \frac{1}{2}\int_0^T (\pi - u)^* \Sigma (\pi - u)\, dt \\ &\quad - \sqrt{\frac{mM}{2}}\left(1 + nK + \log(m+1)\right) \end{aligned}$$

where $K = \max_i \sup_{0 < s \leq T_M} \{-\log(\pi_i(s))\}$.

Proof. Let $t_1, ..., t_n$ be the jump points of u. From (2.3) we have

$$\begin{aligned} LW_{[0,t)}^{(\pi)} - LW_{[0,t)}^{(u)} &= \frac{1}{2}\int_0^t (\pi - u)^* \Sigma (\pi - u)\, dt + \frac{\eta}{2}\int_0^t \pi^* \Sigma \pi dt \\ &\quad - \frac{\eta}{2}\int_0^t \sum_{i=1}^m \pi_i \Sigma_{ii} dt + \frac{d\left(u(t-)\|\pi(t)\right) - d\left(u(0)\|\pi(0)\right)}{\eta} \\ &\quad + \eta^{-1}\sum_{t_k < t}\left[d\left(u(t_k-)\|\pi(t_k)\right) - d\left(u(t_k)\|\pi(t_k)\right)\right]. \end{aligned}$$

But $d\left(u(t_k-)\|\pi(t_k)\right) - d\left(u(t_k)\|\pi(t_k)\right) \leq \max_{0 \leq i \leq m}\{-\log(\pi_i)\}$. The result follows from the choice of η. □

Theorem (2.7). *Let $u \in \mathcal{T}^{pcl}(D_m)$. Suppose that η_t is a positive real valued process satisfying (2.1). If η_t is right continuous and bounded almost surely then for the EG portfolio π we have*

$$\begin{aligned} LW_{[S,T)}^{(\pi)} - LW_{[S,T)}^{(u)} &= \frac{1}{2}\int_S^T (\pi - u)^* \Sigma (\pi - u)\, dt + \frac{1}{2}\int_S^T \eta\pi^*\Sigma\pi dt \\ &-\frac{1}{2}\int_S^T \sum_{i=1}^m \eta\pi_i\Sigma_{ii}dt + \int_S^T d\Psi_t \end{aligned}$$

where $\Psi_t = \frac{d(u_t\|\pi_t)}{\eta_t}$.

Proof. Since $u \in \mathcal{T}^{pcl}$ there exists a sequence of piecewise constant targets whose log wealths converge to that of u. Hence there exists a sequence $u^\Delta \in \mathcal{T}^{pc}$ defined on partitions with mesh size $\Delta \to 0$ almost surely satisfying the same property. For a fixed Δ let $S = t_0 \le t_1 \le \cdots \le t_N = T$ be a partition such that u^Δ is constant on each subinterval $[t_{k-1}, t_k)$. Create the corresponding step function approximation of η by $\eta_t^\Delta = \eta_{t_k}$ for $t_k \le t < t_{k+1}$. Let π be the EG portfolio using the varying learning parameter process η and initial value $\pi_{t_0} = f(\alpha(t_0))$. Define π^Δ interatively such that $\pi^\Delta_{t_0} = f(\alpha(t_0))$, and over the interval $[t_k, t_{k+1})$ it is the EG portfolio with initial value $\pi(t_k)$ and constant learning parameter η_{t_k}. By Theorem 2.3

$$LW_{[S,T)}^{(\pi^\Delta)} - LW_{[S,T)}^{(u^\Delta)} = \frac{1}{2}\int_S^T \left(\pi^\Delta - u^\Delta\right)^* \Sigma \left(\pi^\Delta - u^\Delta\right) dt \tag{5.8}$$

$$+\frac{1}{2}\int_S^T \eta^\Delta \left(\pi^\Delta\right)^* \Sigma\pi^\Delta dt - \frac{1}{2}\int_S^T \sum_{i=1}^m \eta^\Delta\pi_i^\Delta\Sigma_{ii}dt \tag{5.9}$$

$$+\sum_k \left[\frac{d(u_{t_{k+1}}\|\pi^\Delta_{t_{k+1}})}{\eta_{t_{k+1}}} - \frac{d(u_{t_k}\|\pi^\Delta_{t_k})}{\eta_{t_k}}\right].$$

Now let $\Delta \to 0$. By definition $u^\Delta \to u$ and $LW^{(u^\Delta)} \to LW^{(u)}$.

We now prove that π^Δ converges to π. Let

$$T_M = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii} ds = M\right\}.$$

By (1.3) $\lim T_M = \infty$ almost surely as $M \to \infty$. Thus if we can establish the result for the stopped process $\pi^\Delta(t \wedge T_M)$ and $\pi(t \wedge T_M)$ then we obtain the desired result upon letting $M \to \infty$. We may assume therefore that $\max_i \int_0^t \Sigma_{ii} ds$ and the process η are bounded by the same $M > 0$. Let $\alpha(t;\eta^\Delta)_i = \log\left(\frac{\pi_i^\Delta}{\pi_0^\Delta}\right)$ then

$$\begin{aligned} \alpha_i^\Delta(t) - \alpha_i(t) &= \int_S^t \left(\eta^\Delta - \eta\right) dZ_i + \int_S^t \left(\eta^\Delta - \eta\right)\left[\frac{1}{2}\Sigma_{ii} - \Sigma_i f(\alpha)\right] ds \\ &\quad - \int_S^t \eta^\Delta \Sigma_i \left(f\left(\alpha^\Delta\right) - f(\alpha)\right) ds. \end{aligned}$$

Let $b(t) = \max_i \left|\int_S^t \left(\eta^\Delta - \eta\right) dZ_i + \int_S^t \left(\eta^\Delta - \eta\right)\left[\frac{1}{2}\Sigma_{ii} - \Sigma_i f(\alpha)\right] ds\right|$. By the right continuity of η and Lemma 5.1, $b(t)$ converges to 0 uniformly in probability. Let K be the Lipschitz constant for f, since η and Σ_{ii} are bounded we have

$$\left|\int_S^t \eta^\Delta \Sigma_i \left(f\left(\alpha^\Delta\right) - f(\alpha)\right) ds\right| \le KM^2 \int_S^t \|\alpha^\Delta(s) - \alpha(s)\| ds.$$

For the norm of the difference then

$$\begin{aligned}
\|\alpha^{\Delta}(t)-\alpha(t)\|^2 &= \sum_{i=1}^{m}\left(\alpha_i^{\Delta}(t)-\alpha_i(t)\right)^2 \\
&\leq \sum_{i=1}^{m}\left[b(t)+KM^2\int_S^t \|\alpha^{\Delta}(s)-\alpha(s)\|ds\right]^2 \\
&\leq 2mb^2(t)+2K^2M^4m(T-S)\int_S^t \|\alpha^{\Delta}(s)-\alpha(s)\|^2 ds
\end{aligned}$$

where the last inequality follows since $(x+y)^2 \leq 2x^2+2y^2$ and from Jensen's inequality. Let $\gamma = 2K^2M^4m(T-S)$ then by Gronwall's inequality

$$\|\alpha^{\Delta}(t)-\alpha(t)\|^2 \leq 2mb^2(t)+\gamma\int_0^t 2mb^2(s)\exp\{\gamma(t-s)\}ds.$$

Hence $\sup_{0\leq t\leq T}\|\alpha^{\Delta}(t)-\alpha(t)\|^2 \to 0$ uniformly in probability. Since f is uniformly continuous we also have $\sup_{0\leq t\leq T}\|\pi^{\Delta}(t)-\pi(t)\|^2 \to 0$ uniformly in probability. Using Lemma 5.1 as in the proof of Theorem 2.2 we can show that $LW^{(\pi^{\Delta})}-LW^{(\pi)}$ converges to 0 as well. Hence the lhs of (5.8) converges to the rhs of (2.7). Since u^{Δ} and π^{Δ} converge to u and π respectively we have that the first 3 terms on the rhs of (5.8) converge to the corresponding integrals in (2.7). Hence

$$\sum_k\left[\frac{d(u_{t_{k+1}}\|\pi^{\Delta}_{t_{k+1}})}{\eta_{t_{k+1}}}-\frac{d(u_{t_k}\|\pi^{\Delta}_{t_k})}{\eta_{t_k}}\right]$$

must converge in probability. We formally denote this limit as $\int_S^T d\Psi_t$. □

Lemma (2.8). *Let u be a continuous process of bounded variation taking values in the interior of D_m i.e. for all $t\geq 0$ $u_t \in A = \{u \in D_m : \sum_{i=1}^m u_i < 1, u_i > 0\}$. Then*

$u \in \mathcal{T}(D_m)$, i.e. up to time t the process u generates finite wealth. And for $\eta > 0$ constant and the EG(η) portfolio we have

$$\int_S^T d\Psi_t = \eta^{-1}\left(d\left(u(T)\|\pi(T)\right) - d\left(u(S)\|\pi(S)\right) - \sum_{i=1}^m \int_S^T \left[\log\left(\frac{u_i}{u_0}\right) - \alpha_i\right] du\right).$$

Proof. Consider the expression on the right hand side of (1.7). By the assumed smoothness the time integrals are finite for each t. Since u has bounded variation we can interpret the stochastic integral by "integration by parts" as

$$\int_0^t u^* dZ = u_T Z_T - \int_0^t Z^* du$$

where the rhs is well defined since the semi-martingale Z is continuous. Hence u is a target. Since η is constant the sum (2.8) can be decomposed as η^{-1} times

$$\begin{aligned}\sum_k \left[d\left(u_{t_{k+1}}\|\pi_{t_{k+1}}\right) - d\left(u_{t_k}\|\pi_{t_k}\right)\right] &= d(u_{T-}\|\pi_T) - d(u_S\|\pi_S) \\ &\quad - \sum_k \left[d\left(u_{t_k}\|\pi_{t_k}\right) - d\left(u_{t_{k-1}}\|\pi_{t_k}\right)\right].\end{aligned}$$

The partial derivatives of $d(u\|v)$ are

$$\frac{\partial}{\partial u_i} d(u\|v) = \log\left(\frac{u_i}{u_0}\right) - \log\left(\frac{v_i}{v_0}\right).$$

The above partial derivatives are continuous functions of u away from the boundary of D_m. Since u is continuous and restricted to the interior of D_m, on $[S, T]$ it is uniformly bounded away from the boundary of D_m. Recall that almost surely $\alpha_i = \log\left(\frac{\pi_i}{\pi_0}\right)$ is uniformly continuous on $[S,T]$. So

$$\begin{aligned}\sum_k \left[d\left(u_{t_k}\|\pi_{t_k}\right) - d\left(u_{t_{k-1}}\|\pi_{t_k}\right)\right] &= \sum_k \int_{t_{k-1}}^{t_k} \sum_{i=1}^{m} \left[\log\left(\frac{u_i}{u_0}\right) - \alpha_i\left(t_k\right)\right] du_i \\ &\to \int_S^T \sum_{i=1}^{m} \left[\log\left(\frac{u_i}{u_0}\right) - \alpha_i\left(t\right)\right] du_i\end{aligned}$$

almost surely as the mesh size $\Delta \to 0$ by uniform continuity of α_i. $\square$

Theorem (2.9). *Let u be as in Theorem 2.8 with $\int_0^\infty d|u| = K$. Let $S = 0$, $M > 0$, $c > 0$ and*

$$T_{(M,c)} = \inf\left\{t > 0 : \max_i \int_0^t \Sigma_{ii}^2 ds = M \, or \max_i \left|\log(\frac{\pi_i}{\pi_0})\right| = c\right\}$$

Choose $\eta = 2\sqrt{\frac{\log(m+1)}{mM}}$ then the EG(η,0) portfolio π with $\pi(0) = \frac{1}{(m+1)}\mathbf{1_m}$ satisfies

$$\begin{aligned}LW^{(\pi)}_{[0,T_{(M,c)})} - LW^{(u)}_{[0,T_{(M,c)})} &\geq \frac{1}{2}\int_0^T \left(\pi - u\right)^* \Sigma \left(\pi - u\right) dt + \frac{\eta}{2}\int_0^T \pi^* \Sigma \pi dt \\ &\quad -2\sqrt{\log\left(m+1\right) mM} - cK\sqrt{\frac{mM}{4\log\left(m+1\right)}}.\end{aligned}$$

Proof. As before we use that $d(u\|\frac{1}{(m+1)}\mathbf{1_m}) \leq \log(m+1)$ and note that

$$0 \geq \int \sum_{i=1}^{m} \left[\log\left(\frac{u_i}{u_0}\right)\right] du_i = \sum_{i=0}^{m} u_i \log\left(u_i\right) \geq -\log(m+1).$$

hence $\int_0^T \sum_{i=1}^m \left[\log\left(\frac{u_1}{u_0}\right)\right] du_i \geq -\log(m+1)$. By the definition of the stopping time

$$\int_0^{T_{(M,c)}} \sum_{i=1}^{m} [\alpha_i(t)] du_i \leq Kc$$

and

$$\frac{1}{2}\int_0^T \sum_{i=1}^{m} \pi_i \Sigma_{ii}\, dt < M.$$

The result follows using the choice of η with Theorem 2.7 and Lemma 2.8. $\square$

Chapter 6

Final Remarks

In Chapter 2 we were able to extend the DEG portfolio to continuous trading and obtain a lower bound in performance versus the bcrp and larger targets with a finite number of jumps and with bounded variation. These results give conditions under which the EG portfolio can achieve nearly the same exponential growth as these larger targets. At least in markets where it stays away from the boundary, the EG portfolio can track targets which are not constant but do not vary too much. An important feature of these results, as opposed to those obtained by Helmbold et.al. for the DEG portfolio, is the development of an identity containing positive terms that are large if the EG portfolio is distant from the bcrp. In Chapter 4 we have presented examples of simple oscillatory drift markets where the EG portfolio outperforms the universal portfolio and the bcrp with probability arbitrarily close to 1. A second important feature of the EG portfolio is that since it can be straight-forwardly updated on-line it is much easier to calculate than the universal portfolio.

Further examination of the behavior and properties of the solution to (2.2), theoretically or via simulation, is necessary to further delineate the market conditions under which the improved performance of the EG portfolio may hold.

The presence of learning parameter η in the EG algorithm can viewed as a boon or bane. On the one hand, the investor is allowed flexibility to control risk by choosing η large or small. We see that choosing a good η can be done by setting a level of future variation out to which the investor wants protection. One can remove this dependence by considering a doubling scheme where the portfolio is run over longer and longer epochs reinitializing each time. With Theorem (2.7), we have laid the groundwork to choose η adaptively and perhaps obtain better bounds.

Cover and Ordentlich (1996) and Helmbold et. al. working with discrete trading introduce the concept of side information. In addition to the market stock prices the investor is privy to extra information upon which to base the investment strategy. For positive integer J they define an adapted random process Y_t taking values on $\{1, 2, ..., J\}$. Then they partition according the value of Y and run J copies of the algorithm. Through examples of stock data, Helmbold et. al. exhibit simple forms of side information which can greatly increase the wealth obtained by the DEG and universal portfolios. In the continuous case we can also run J copies, however, when the process Y returns to state i there is no guarantee the stock price is the same as when it last left state i. Hence, in order to extend the results for EG portfolio to side information, we would need to modify the proofs to account for possible jumps in the stock prices.

As do Cover and Helmbold et. al., we assume that there are no transaction costs. For rebalanced portfolios in continuous time we must continuously buy and sell to maintain the desired proportion of wealth in each asset. One approach to solving this difficult problem may be to modify (1.8) to include a penalty for over trading.

Bibliography

[1] T.M. Cover. (1991) *Universal Portfolios.* Mathematical Finance, **1**, 1-29.

[2] T.M. Cover and E. Ordentlich. (1996) *Universal portfolios with side information.* IEEE Transactions on Information Theory, **2** (2).

[3] D.P. Helmbold, R.E. Schapire, Y. Singer, M.K. Warmuth (1998). *On-Line Portfolio Selection Using Multiplicative Updates*, Mathematical Finance, **8** 325-347.

[4] F. Jamshidian. (1992) *Asymptotically Optimal Portfolios*, Mathematical Finance, **2:2** 131 - 150.

[5] I. Karatzas. (1991) *Lectures on the Mathematics of Finance*, CRM Monograph Series, Vol. 8, American Mathematical Society.

[6] I. Karatzas and S.E. Shreve. (1991) *Brownian Motion and Stochastic Calculus*, Second Edition, Springer Verlag, New York.

[7] J. Kivinen and M.K. Warmuth. (1994) *Exponentiated gradient versus gradient descent for linear predictors*, Technical Report UCSC-CRL-94-16, University of Califirnia, Santa Cruz, Computer Research Laboratory.

[8] E. Ordentlich and T.M. Cover. (1998) *The cost of acheiving the best portfolio in hindsight.* Mathematics of Operations Research. **23**, 960-982.

[9] P. Protter. (1990) *Stochastic Integration and Differential Equations*, Springer - Verlag, Berlin.