

GEOMETRIC ASPECTS OF EXACT SOLUTIONS OF BELLMAN EQUATIONS OF
HARMONIC ANALYSIS PROBLEMS

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ABSTRACT

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In Chapter 1 we find the sharp constant $C = C(\tau, p, \mathbb{E}G, \mathbb{E}F)$ of the following inequality $\|(G^2 + \tau^2 F^2)^{1/2}\|_p \leq C\|F\|_p$, where G is the transform of a martingale F under a predictable sequence ε with absolute value 1, $1 < p < 2$, and τ is any real number. Thereby we solve the open problem posted by Boros–Janakiraman–Volberg.

In Chapter 2 under some assumptions on the vectors $a_1, \dots, a_n \in \mathbb{R}^k$ and the function $B : \mathbb{R}^k \rightarrow \mathbb{R}$ we find the sharp estimate of the expression $\int_{\mathbb{R}^k} B(u_1(a_1 \cdot x), \dots, u_n(a_n \cdot x)) dx$ in terms of $\int_{\mathbb{R}} u_j(y) dy, j = 1, \dots, n$. In some particular cases ($k = 1, n - 1$ and n) we show that these assumptions on B imply that there is only one Brascamp–Lieb inequality.

In Chapter 3 we find underlying PDEs on the Bellman functions B which imply inequalities such as John–Nirenberg inequality, Prekopa–Leindler inequality, Ehrhard’s inequality, Borell’s Gaussian noise “stability”, hypercontractivity of Ornstein–Uhlenbeck semigroup, logarithmic Sobolev inequality, Beckner–Sobolev inequality and Bobkov’s inequality. We also describe underlying differential geometry that arises in solving these PDEs, and we formulate some open questions.

to my fiancé Maria.

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Introduction

The current dissertation is split into 3 chapters. The first chapter solves the open problem put forward by Boros–Janakiraman–Volberg regarding sharp estimates of the perturbation for Burkholder’s martingale transform (see [8], [15]). The inequality stems from important questions concerning the L^p bounds for the perturbation of Beurling–Ahlfors operator and hence it is of interest. We refer the reader to recent works regarding martingale inequalities and estimates of Beurling–Ahlfors operator [1, 3, 4, 5, 8] and references therein. It is worth mentioning that the problem was solved by using the theory of minimal concave functions developed jointly with N. Osipov, D. Stolyarov, P. Zatitskyj, V. Vasyunin and myself. For example, such notions and objects as foliation, force functions, cup and torsion already appeared in the recent works [9], [10], [11], [12] and [13]. However, since the theory was developed in two dimensional setting, it required some additional technical work to solve the problem in three dimensional setting. In particular it includes finding minimal concave solution of homogeneous Monge–Ampère equation with Dirichlet and Neumann boundary data, and minimality was proved by constructing optimal martingale trajectories along the foliations. Concave solutions were found by detailed investigation of the important object: smooth transformation of the torsion of Dirichlet boundary data (further called force functions, see (1.3.11)) which coincides up to some positive factor (depending only on the domain

and foliation) with the trace of the Hessian of a Bellman function (or mean curvature).

Second chapter is devoted to the inequalities of Brascamp–Lieb in the Lebesgue measure case for a general function B (see also [14]). The particular case $B(x_1, \dots, x_n) = x_1^{1/p_1} \dots x_n^{1/p_n}$, which corresponds to the Brascamp–Lieb inequality is important for a number of reasons, including applications in analysis and convex geometry, and, for example, includes the sharp form of Young’s convolution inequality (established in [22], and [25]). For general function B , it turns out that B satisfies inequality of Brascamp–Lieb if under some mild assumptions on B it also satisfies some interesting concavity condition (see property L3 in Subsection 2.0.4, Chapter 2). Similar concavity condition was found recently independently by Ledoux (see [31]). Our second main contribution is that under the assumptions L1-L5 (see properties L1-L5 in Subsection 2.0.4, Chapter 2) we give complete description of such functions B in the case $k = 1, k = n - 1$ and $k = n$. In these special cases the results below imply uniqueness of Brascamp–Lieb inequalities and it sheds light to the works of E. Calren, E. Lieb and M. Loss (see [25]), and J. Bannett, A. Carbery, M. Christ and T. Tao (see [26, 27]).

Third chapter mainly contains brief overview of some old and recent isoperimetric problems, hints and new ideas about their relations to PDE and differential geometry. This chapter is not as rigorous as previous chapters because the initial purpose was to give to the reader very short overview and an attempt of the general picture which shows underlying PDEs and PDIs (partial differential inequalities), which govern these classical isoperimetric inequalities of analysis. Even though the chapter is short, it requires very careful reading since the objects (as they are written down) only make sense under some extra assumptions on the functions. For example, If there are no a priori assumptions on the functions B and f , and the reader sees the expressions of the form $\int_{\mathbb{R}} B(f(x))$, the reader should automatically

assume that we deal with those B and f such that the composition $B(f(x))$ is measurable, moreover it is integrable on the real line. The material is still under the preparation, therefore, for now we decided to avoid finding the best conditions under which the theorems of the chapter are fulfilled. However, we believe that the reader can easily find for her/himself at at least some sufficient conditions under which all the computations are justified (or one can extract such conditions them from the applications given after each theorem).

Our goal is to try to find underlying PDEs and PDIs for the following inequalities: John–Nirenberg inequality, sharp inequalities on BMO , A_p , Reverse Hölder, Gehring and the classes of functions with bounded oscillation (see [9, 10, 11, 12] and references therein), uniform convexity (see [16, 13, 28, 29, 30] and references therein). Isoperimetric, Prekopa–Leindler and Ehrhard’s inequality (see [39, 38, 37, 33, 32, 34, 35, 36] and references therein). Borell’s Gaussian noise “stability” and hypercontractivity for Ornstein–Uhlenbeck semigroup (see [31, 40, 42, 43, 44, 45, 46, 47] and references therein). Log-Sobolev, Beckner–Sobolev and Bobkov’s inequality (see [48, 49, 50, 51, 52, 53, 54, 55] and references therein). Plan of the chapter is simple. It is divided in 3 parts. First part briefly formulates these inequalities. Second part finds underlying PDEs and PDIs for these inequalities, and the third part mentions the relation to differential geometry and in it we also formulate some open questions.

Chapter 1

Inequality for Burkholder's martingale transform

1.1 History of the problem

Let I be an interval of the real line \mathbb{R} , and let $|I|$ be its Lebesgue length. By symbol \mathcal{B} we denote the σ -algebra of Borel subsets of I . Let $\{F_n\}_{n=0}^\infty$ be a martingale on the probability space $(I, \mathcal{B}, dx/|I|)$ with a filtration $\{I, \emptyset\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}$. Consider any sequence of functions $\{\varepsilon_n\}_{n=1}^\infty$ such that for each $n \geq 1$, ε_n is \mathcal{F}_{n-1} measurable and $|\varepsilon_n| \leq 1$. Let G_0 be a constant function on I ; for any $n \geq 1$, let G_n denote

$$G_0 + \sum_{k=1}^n \varepsilon_k (F_k - F_{k-1}).$$

The sequence $\{G_n\}_{n=0}^\infty$ is called the *martingale transform* of $\{F_n\}$. Obviously $\{G_n\}_{n=0}^\infty$ is a martingale with the same filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Note that since $\{F_n\}$ and $\{G_n\}$ are martingales, we have $F_0 = \mathbb{E}F_n$ and $G_0 = \mathbb{E}G_n$ for any $n \geq 0$.

In [17] Burkholder proved that if $|G_0| \leq |F_0|$, $1 < p < \infty$, then we have the sharp estimate

$$\|G_n\|_{L^p} \leq (p^* - 1) \|F_n\|_{L^p} \quad \text{for all } n \geq 0, \tag{1.1.1}$$

where $p^* - 1 = \max\{p - 1, \frac{1}{p-1}\}$. Burkholder showed that it is sufficient to prove inequality (1.1.1) for the sequences of numbers $\{\varepsilon_n\}$ such that $\varepsilon_n = \pm 1$ for all $n \geq 1$. It was also mentioned that such an estimate as (1.1.1) does not depend on the choice of filtration $\{\mathcal{F}_n\}$. For example, one can consider only the dyadic filtration. For more information on the estimate (1.1.1) we refer the reader to [17], [18].

In [20] the result was slightly generalized by Bellman function technique and Monge–Ampère equation, i.e., the estimate (1.1.1) holds if and only if

$$|G_0| \leq (p^* - 1)|F_0|. \quad (1.1.2)$$

In what follows we assume that $\{\varepsilon_n\}$ is a predictable sequence of functions such that $|\varepsilon_n| = 1$.

In [8], a perturbation of the martingale transform was investigated. Namely, under the same assumptions as (1.1.2) it was proved that for $2 \leq p < \infty$, $\tau \in \mathbb{R}$, we have the sharp estimate

$$\|(G_n^2 + \tau^2 F_n^2)^{1/2}\|_{L^p} \leq ((p^* - 1)^2 + \tau^2)^{1/2} \|F_n\|_{L^p}, \quad \text{for all } n \geq 0. \quad (1.1.3)$$

It was also claimed to be proven that the same sharp estimate holds for $1 < p < 2$, $|\tau| \leq 0.5$, and the case $1 < p < 2$, $|\tau| > 0.5$ was left open.

The inequality (1.1.3) stems from important questions concerning the L^p bounds for the perturbation of Beurling–Ahlfors operator and hence it is of interest. We refer the reader to recent works regarding martingale inequalities and estimates of Beurling–Ahlfors operator [1, 3, 4, 5, 8] and references therein.

We should mention that Burkholder's method [17] and the Bellman function approach [20], [8] have similar traces in the sense that both of them reduce the required estimate to finding a certain minimal diagonally concave function with prescribed boundary conditions. However, the methods of construction of such a function are different. Unlike Burkholder's method [17], in [20] and [8] the construction of the function is based on the Monge–Ampère equation.

1.1.1 Our main results

Firstly, we should mention that the proof of (1.1.3) presented in [8] has a gap in the case $1 < p < 2$, $0 < |\tau| \leq 0.5$ (the constructed function does not satisfy necessary concavity condition).

In the present paper we obtain the sharp L^p estimate of the perturbed martingale transform for the remaining case $1 < p < 2$ and for all $\tau \in \mathbb{R}$. Moreover, we do not require condition (1.1.2).

We define

$$u(z) \stackrel{\text{def}}{=} \tau^p(p-1) \left(\tau^2 + z^2 \right)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1.$$

Theorem 1.1.1. *Let $1 < p < 2$, and let $\{G_n\}_{n=0}^\infty$ be a martingale transform of $\{F_n\}_{n=0}^\infty$.*

Set $\beta = \frac{|G_0| - |F_0|}{|G_0| + |F_0|}$. The following estimates are sharp:

1. *If $u\left(\frac{1}{p-1}\right) \leq 0$ then*

$$\|(\tau^2 F_n^2 + G_n^2)^{1/2}\|_{L^p} \leq \left(\tau^2 + \max \left\{ \left| \frac{G_0}{F_0} \right|, \frac{1}{p-1} \right\}^2 \right)^{\frac{1}{2}} \|F_n\|_{L^p}, \quad \text{for all } n \geq 0.$$

2. If $u\left(\frac{1}{p-1}\right) > 0$ then

$$\|(\tau^2 F_n^2 + G_n^2)^{1/2}\|_{L^p}^p \leq C(\beta) \|F_n\|_{L^p}^p, \quad \text{for all } n \geq 0,$$

where $C(\beta)$ is continuous nondecreasing, and it is defined as follows:

$$C(\beta) \stackrel{\text{def}}{=} \begin{cases} \left(\tau^2 + \frac{|G_0|^2}{|F_0|^2}\right)^{p/2}, & \beta \geq s_0; \\ \frac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}, & \beta \leq -1 + \frac{2}{p}; \\ C(\beta), & \beta \in (-1 + 2/p, s_0); \end{cases}$$

where $s_0 \in (-1 + 2/p, 1)$ is the solution of the equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$.

Explicit expression for the function $C(\beta)$ on the interval $(-1 + 2/p, s_0)$ was hard to present in a simple way. The reader can find the value of the function $C(\beta)$ in Theorem 1.5.1, part (ii).

Remark 1. The condition $u\left(\frac{1}{p-1}\right) \leq 0$ holds when $|\tau| \leq 0.822$. So we also obtain Burkholder's result in the limit case when $\tau = 0$. It is worth mentioning that although the proof of the estimate (1.1.3) has a gap in [8], the claimed result in the case $1 < p < 2$, $|\tau| < 0.5$ remains true as a result of Theorem 1.1.1.

One of the important results is that we find the function (1.2.2), and the above estimates are corollaries of this result. We would like to mention that unlike [20] and [8] the argument exploited in the current paper is different. Instead of writing a lot of technical computations and checking which case is valid, we present some pure geometrical facts regarding minimal concave functions with prescribed boundary conditions, and by this way we avoid compu-

tations. Moreover, we explain to the reader how we construct our Bellman function (1.2.2) based on these geometrical facts derived in Section 1.3.

1.1.2 Plan of the Chapter 1

In Section 1.2 we formulate results about how to reduce the estimate (1.1.3) to finding of a certain function with required properties. These results are well-known and can be found in [8]. A slightly different function was investigated in [20], however, it possesses almost the same properties and the proof works exactly in the same way. We only mention these results and the fact that we look for a minimal continuous diagonally concave function $H(x_1, x_2, x_3)$ (see Definition 3) in the domain $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1|^p \leq x_3\}$ with the boundary condition $H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}$.

Section 1.3 is devoted to the investigation of the minimal concave functions in two variables. It is worth mentioning that the first crucial steps in this direction for some special cases were made in [9] (see also [10, 11]). In Section 1.3 we develop this theory for a slightly more general case. We investigate some special foliation called the *cup* and another useful object, called *force functions*.

We should note that the theory of minimal concave functions in two variables does not include the minimal diagonally concave functions in three variables. Nevertheless, this knowledge allows us to construct the candidate for H in Section 1.4, but with some additional technical work not mentioned in Section 1.3.

In section 1.5 we find the good estimates for the perturbed martingale transform. In Section 1.6 we prove that the candidate for H constructed in Section 1.4 coincides with H , and as a corollary we show the sharpness of the estimates found for the perturbed martingale transform in Section 1.5.

In conclusion, the reader can note that the hard technical lies in the construction of the minimal diagonally concave function in three variables with the given boundary condition.

1.2 Definitions and known results

Let $\mathbb{E}F \stackrel{\text{def}}{=} \langle F \rangle_I$ where

$$\langle F \rangle_J \stackrel{\text{def}}{=} \frac{1}{|J|} \int_J F(t) dt$$

for any interval J of the real line. Let F and G be real valued integrable functions. Let $G_n = \mathbb{E}(G|\mathcal{M}_n)$ and $F_n = \mathbb{E}(F|\mathcal{M}_n)$ for $n \geq 0$, where $\{\mathcal{M}_n\}$ is a dyadic filtration (see [8]).

Definition 1. *If the martingale $\{G_n\}$ satisfies $|G_{n+1} - G_n| = |F_{n+1} - F_n|$ for each $n \geq 0$, then G is called the martingale transform of F .*

Recall that we are interested in the estimate

$$\|(G^2 + \tau^2 F^2)^{1/2}\|_{L^p} \leq C \|F\|_{L^p}. \quad (1.2.1)$$

We introduce the Bellman function

$$H(\mathbf{x}) \stackrel{\text{def}}{=} \sup_{F,G} \{\mathbb{E}B(\varphi(F,G)), \mathbb{E}\varphi(F,G) = \mathbf{x}, |G_{n+1} - G_n| = |F_{n+1} - F_n|, n \geq 0\}. \quad (1.2.2)$$

where $\varphi(x_1, x_2) = (x_1, x_2, |x_1|^p)$, $B(\varphi(x_1, x_2)) = (x_2^2 + \tau^2 x_1^2)^{p/2}$, $\mathbf{x} = (x_1, x_2, x_3)$.

Remark 2. *In what follows bold lowercase letters denote points in \mathbb{R}^3 .*

Then we see that the estimate (1.2.1) can be rewritten as follows:

$$H(x_1, x_2, x_3) \leq C^p x_3.$$

We mention that the Bellman function H does not depend on the choice of the interval I . Without loss of generality we may assume that $I = [0, 1]$.

Definition 2. Given a point $\mathbf{x} \in \mathbb{R}^3$, a pair (F, G) is said to be admissible for \mathbf{x} if G is the martingale transform of F and $\mathbb{E}(F, G, |F|^p) = \mathbf{x}$.

Proposition 1. The domain of $H(\mathbf{x})$ is $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1|^p \leq x_3\}$, and H satisfies the boundary condition

$$H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}. \quad (1.2.3)$$

Definition 3. A function U is said to be diagonally concave in Ω , if it is concave in both $\Omega \cap \{(x_1, x_2, x_3) : x_1 + x_2 = A\}$ and $\Omega \cap \{(x_1, x_2, x_3) : x_1 - x_2 = A\}$ for every constant $A \in \mathbb{R}$.

Proposition 2. $H(\mathbf{x})$ is a diagonally concave function in Ω .

Proposition 3. If U is a continuous diagonally concave function in Ω with boundary condition $U(x_1, x_2, |x_1|^p) \geq (x_2^2 + \tau^2 x_1^2)^{p/2}$, then $U \geq H$ in Ω .

We explain our strategy of finding the Bellman function H . We are going to find a minimal candidate B , that is continuous, diagonally concave, with the fixed boundary condition $B|_{\partial\Omega} = (y^2 + \tau^2 x^2)^{p/2}$. We warn the reader that the symbol B denoted boundary data previously, however, in Section 1.6 we are going to use symbol B as the candidate for the

minimal diagonally concave function. Obviously $B \geq H$ by Proposition 3. We will also see that given $\mathbf{x} \in \Omega$ and any $\varepsilon > 0$, we can construct an admissible pair (F, G) such that $B(\mathbf{x}) < \mathbb{E}(F^2 + \tau^2 G^2)^{p/2} + \varepsilon$. This will show that $B \leq H$ and hence $B = H$.

In order to construct the minimal candidate B , we have to elaborate few preliminary concepts from differential geometry. We introduce notion of *foliation* and *force* functions.

1.3 Homogeneous Monge–Ampère equation and minimal concave functions

1.3.1 Foliation

Let $g(s) \in C^3(I)$ be such that $g'' > 0$, and let Ω be a convex domain which is bounded by the curve $(s, g(s))$ and the tangents that pass through the end-points of the curve (see Figure 1.1). Fix some function $f(s) \in C^3(I)$. The first question we ask is the following: how the minimal concave function $\mathbf{B}(x_1, x_2)$ with boundary data $\mathbf{B}(s, g(s)) = f(s)$ looks *locally* in a subdomain of Ω . In other words, take a convex hull of the curve $(s, g(s), f(s)), s \in I$, then the question is how the boundary of this convex hull looks like.

We recall that the concavity is equivalent to the following inequalities:

$$\det(d^2 \mathbf{B}) \geq 0, \tag{1.3.1}$$

$$\mathbf{B}''_{x_1 x_1} + \mathbf{B}''_{x_2 x_2} \leq 0. \tag{1.3.2}$$

The expression (1.3.1) is the Gaussian curvature of the surface $(x_1, x_2, \mathbf{B}(x_1, x_2))$ up to a positive factor $(1 + (\mathbf{B}'_{x_1})^2 + (\mathbf{B}'_{x_2})^2)^2$. So in order to minimize the function $\mathbf{B}(x_1, x_2)$, it is reasonable to minimize the Gaussian curvature. Therefore, we will look for a surface with

zero Gaussian curvature. Here the homogeneous Monge–Ampère equation arises. These surfaces are known as *developable surfaces* i.e., such a surface can be constructed by bending a plane region. The important property of such surfaces is that they consist of line segments, i.e., the function \mathbf{B} satisfying homogeneous Monge–Ampère equation $\det(d^2\mathbf{B}) = 0$ is linear along some *family of segments*. These considerations lead us to investigate such functions \mathbf{B} . Firstly, we define a *foliation*. For any segment ℓ in the Euclidean space by symbol ℓ° we denote an open segment i.e., ℓ without endpoints.

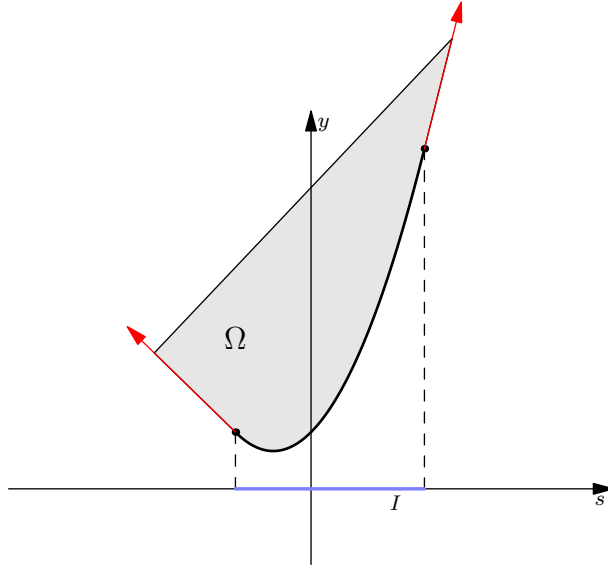


Figure 1.1 Domain Ω

Fix any subinterval $J \subseteq I$. By symbol $\Theta(J, g)$ we denote an arbitrary set of nontrivial segments (i.e. single points are excluded) in \mathbb{R}^2 with the following requirements:

1. For any $\ell \in \Theta(J, g)$ we have $\ell^\circ \in \Omega$.
2. For any $\ell_1, \ell_2 \in \Theta(J, g)$ we have $\ell_1 \cap \ell_2 = \emptyset$.
3. For any $\ell \in \Theta(J, g)$ there exists only one point $s \in J$ such that $(s, g(s))$ is one of the end-points of the segment ℓ and vice versa, for any point $s \in J$ there exists $\ell \in \Theta(J, g)$

such that $(s, g(s))$ is one of the end-points of the segment ℓ .

4. There exists C^1 smooth argument function $\theta(s)$.

We explain the meaning of the requirement 4. To each point $s \in J$ there corresponds only one segment $\ell \in \Theta(J, g)$ with an endpoint $(s, g(s))$. Take a nonzero vector with initial point $(s, g(s))$, parallel to the segment ℓ and having an endpoint in Ω . We define the value of $\theta(s)$ to be an argument of this vector. Surely argument is defined up to additive number $2\pi k$ where $k \in \mathbb{Z}$. Nevertheless, we take any representative from these angles. We do the same for all other points $s \in I$. In this way we get a family of functions $\theta(s)$. If there exists $C^1(J)$ smooth function $\theta(s)$ from this family then the requirement 4 is satisfied.

Remark 3. *It is clear that if $\theta(s)$ is $C^1(J)$ smooth argument function, then for any $k \in \mathbb{Z}$, $\theta(s) + 2\pi k$ is also $C^1(J)$ smooth argument function. Any two $C^1(J)$ smooth argument functions differ by constant $2\pi n$ for some $n \in \mathbb{Z}$.*

This remark is the consequence of the fact that the quantity $\theta'(s)$ is well defined regardless of the choices of $\theta(s)$. Next, we define $\Omega(\Theta(J, g)) = \cup_{\ell \in \Theta(J, g)} \ell^\circ$. Given a point $x \in \Omega(\Theta(J, g))$ we denote by $\ell(x)$ a segment $\ell(x) \in \Theta(J, g)$ which passes through the point x . If $x = (s, g(s))$ then instead of $\ell((s, g(s)))$ we just write $\ell(s)$. Surely such a segment exists, and it is unique. We denote by $s(x)$ a point $s(x) \in J$ such that $(s(x), g(s(x)))$ is one of the end points of the segment $\ell(x)$. Moreover, in a natural way we set $s(x) = s$ if $x = (s, g(s))$. It is clear that such $s(x)$ exists, and it is unique. We introduce a function

$$K(s) = g'(s) \cos \theta(s) - \sin \theta(s), \quad s \in J. \quad (1.3.3)$$

Note that that $K < 0$. This inequality becomes obvious if we rewrite $g'(s) \cos \theta(s) -$

$\sin \theta(s) = \langle (1, g'), (-\sin \theta, \cos \theta) \rangle$ and take into account the requirement 1 of $\Theta(J, g)$. Note that $\langle \cdot, \cdot \rangle$ means scalar product in Euclidean space. We need few more requirements on $\Theta(J, g)$.

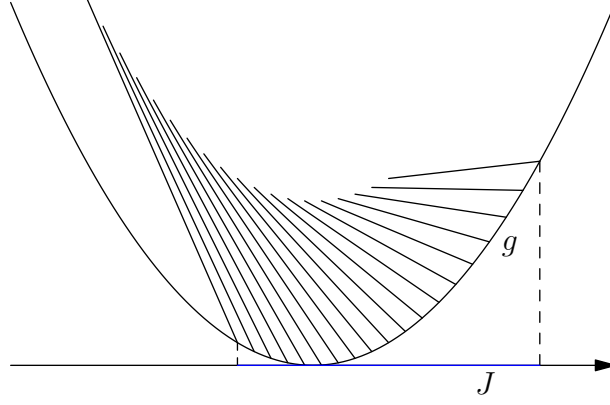


Figure 1.2 Foliation $\Theta(J, g)$

5. For any $x = (x_1, x_2) \in \Omega(\Theta(J, g))$ we have an inequality $K(s(x)) + \theta'(s(x))\|(x_1 - s(x), x_2 - g(s(x)))\| < 0$.
6. The function $s(x)$ is continuous in $\Omega(\Theta(J, g)) \cup \Gamma(J)$ where $\Gamma(J) = \{(s, g(s)) : s \in J\}$.

Note that if $\theta'(s) \leq 0$ (which happens in most of the cases) then the requirement 5 holds. If we know the endpoints of the segments $\Theta(J, g)$, then in order to verify the requirement 5 it is enough to check at those points $x = (x_1, x_2)$, where x is the another endpoint of the segment other than $(s, g(s))$. Roughly speaking the requirement 5 means the segments of $\Theta(J, g)$ do not rotate rapidly counterclockwise.

Definition 4. A set of segments $\Theta(J, g)$ with the requirements mentioned above is called foliation. The set $\Omega(\Theta(J, g))$ is called domain of foliation.

A typical example of a foliation is given in Figure 1.2.

Lemma 1. *The function $s(x)$ belongs to $C^1(\Omega(\Theta(J, g)))$. Moreover*

$$(s'_{x_1}, s'_{x_2}) = \frac{(\sin \theta, -\cos \theta)}{-K(s) - \theta' \cdot \|(x_1 - s, x_2 - g(s))\|}. \quad (1.3.4)$$

Proof. Definition of the function $s(x)$ implies that

$$-(x_1 - s) \sin \theta(s) + (x_2 - g(s)) \cos \theta(s) = 0.$$

Therefore the lemma is an immediate consequence of the implicit function theorem. \square

Let $J = [s_1, s_2] \subseteq I$, and let $(s, g(s), f(s)) \in C^3(I)$ be such that $g'' > 0$ on I . Consider an arbitrary foliation $\Theta(J, g)$ with an arbitrary $C^1([s_1, s_2])$ smooth argument function $\theta(s)$. We need the following technical lemma which describes behavior of the gradient of the function B which satisfies homogeneous Monge–Ampère equation.

Lemma 2. *The solutions of the system of equations*

$$t'_1(s) \cos \theta(s) + t'_2(s) \sin \theta(s) = 0, \quad (1.3.5)$$

$$t_1(s) + t_2(s)g'(s) = f'(s), \quad s \in J \quad (1.3.6)$$

are the following functions

$$\begin{aligned} t_1(s) &= \int_{s_1}^s \left(\frac{g''(r)}{K(r)} \sin \theta(r) \cdot t_2(r) - \frac{f''(r)}{K(r)} \sin \theta(r) \right) dr + f'(s_1) - t_2(s_1)g'(s_1), \\ t_2(s) &= t_2(s_1) \exp \left(- \int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) + \\ &\quad \int_{s_1}^s \frac{f''(y)}{K(y)} \exp \left(- \int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \cos \theta(y) dy, \quad s \in J \end{aligned}$$

where $t_2(s_1)$ is an arbitrary real number.

Proof. We differentiate (1.3.6) and combine it with (1.3.5) to obtain the system

$$\begin{pmatrix} \cos \theta & \sin \theta \\ 1 & g' \end{pmatrix} \begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -g'' \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f'' \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \frac{g''}{K} \begin{pmatrix} 0 & \sin \theta \\ 0 & -\cos \theta \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \frac{f''}{K} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}. \quad (1.3.7)$$

By solving this system of differential equations and using the fact that $t_1(s_1) + g'(s_1)t_2(s_1) = f'(s_1)$ we get the desired result. \square

Remark 4. *Integration by parts allows us to rewrite the expression for $t_2(s)$ as follows*

$$\begin{aligned} t_2(s) = & \exp \left(- \int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)} \right) + \frac{f''(s)}{g''(s)} - \\ & - \int_{s_1}^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(- \int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy. \end{aligned}$$

Definition 5. *We say that a function \mathbf{B} has a foliation $\Theta(J, g)$ if it is continuous on $\Omega(\Theta(J, g))$, and it is linear on each segment of $\Theta(J, g)$.*

The following lemma describes how to construct a function \mathbf{B} with a given foliation $\Theta(J, g)$, boundary condition $\mathbf{B}(s, g(s)) = f(s)$, such that \mathbf{B} satisfies the homogeneous Monge–Ampère equation.

Consider a function \mathbf{B} defined as follows

$$\mathbf{B}(x) = f(s) + \langle t(s), x - (s, g(s)) \rangle, \quad x = (x_1, x_2) \in \Omega(\Theta(J, g)) \quad (1.3.8)$$

where $s = s(x)$, and $t(s) = (t_1(s), t_2(s))$ satisfies the system of the equations (1.3.5), (1.3.6) with an arbitrary $t_2(s_1)$.

Lemma 3. *The function \mathbf{B} defined by (1.3.8) satisfies the following properties:*

1. $\mathbf{B} \in C^2(\Omega(\Theta(J, g))) \cap C^1(\Omega(\Theta(J, g)) \cup \Gamma)$, \mathbf{B} has the foliation $\Theta(J, g)$ and

$$\mathbf{B}(s, g(s)) = f(s) \quad \text{for all } s \in [s_1, s_2]. \quad (1.3.9)$$

2. $\nabla \mathbf{B}(x) = t(s)$, where $s = s(x)$, moreover \mathbf{B} satisfies the homogeneous Monge–Ampère equation.

Proof. The fact that \mathbf{B} has the foliation $\Theta(J, g)$, and it satisfies the equality (1.3.9) immediately follows from the definition of the function \mathbf{B} . We check the condition of smoothness. By Lemma 1 and Lemma 2 we have $s(x) \in C^2(\Omega(\Theta(J, g)))$ and $t_1, t_2 \in C^1(J)$, therefore the right-hand side of (1.3.8) is differentiable with respect to x . So after differentiation of (1.3.8) we get

$$\nabla \mathbf{B}(x) = [f'(s) - \langle t(s), (1, g'(s)) \rangle] (s'_{x_1}, s'_{x_2}) + t(s) + \langle t'(s), x - (s, g(s)) \rangle (s'_{x_1}, s'_{x_2}). \quad (1.3.10)$$

Using (1.3.5) and (1.3.6) we obtain $\nabla \mathbf{B}(x) = t(s)$. Taking derivative with respect to x the

second time we get

$$\frac{\partial^2 \mathbf{B}}{\partial x_1^2} = t'_1(s)s'_{x_1}, \quad \frac{\partial^2 \mathbf{B}}{\partial x_2 \partial x_1} = t'_1(s)s'_{x_2}, \quad \frac{\partial^2 \mathbf{B}}{\partial x_1 \partial x_2} = t'_2(s)s'_{x_1}, \quad \frac{\partial^2 \mathbf{B}}{\partial x_2^2} = t'_2(s)s'_{x_2}.$$

Using (1.3.5) we get that $t'_1(s)s'_{x_2} = t'_2(s)s'_{x_1}$, therefore $\mathbf{B} \in C^2(\Omega(\Theta(J, g)))$. Finally, we check that \mathbf{B} satisfies the homogeneous Monge–Ampère equation. Indeed,

$$\det(d^2 \mathbf{B}) = \frac{\partial^2 \mathbf{B}}{\partial x_1^2} \cdot \frac{\partial^2 \mathbf{B}}{\partial x_2^2} - \frac{\partial^2 \mathbf{B}}{\partial x_2 \partial x_1} \cdot \frac{\partial^2 \mathbf{B}}{\partial x_1 \partial x_2} = t'_1(s)s'_{x_1} \cdot t'_2(s)s'_{x_1} - t'_1(s)s'_{x_2} \cdot t'_2(s)s'_{x_1} = 0.$$

□

Definition 6. *The function $t(s) = (t_1(s), t_2(s)) = \nabla \mathbf{B}(x)$, $s = s(x)$, is called gradient function corresponding to \mathbf{B} .*

The following lemma investigates the concavity of the function \mathbf{B} defined by (1.3.8). Let $\|\tilde{\ell}(x)\| = \|(s(x) - x_1, g(s(x)) - x_2)\|$, where $x = (x_1, x_2) \in \Omega(\Theta(J, g))$.

Lemma 4. *The following equalities hold*

$$\begin{aligned} \frac{\partial^2 \mathbf{B}}{\partial x_1^2} + \frac{\partial^2 \mathbf{B}}{\partial x_2^2} &= \frac{g''}{K(K + \theta' \|\tilde{\ell}(x)\|)} \left(-t_2 + \frac{f''}{g''} \right) = \\ &= \frac{g''}{K(K + \theta' \|\tilde{\ell}(x)\|)} \times \left[-\exp \left(-\int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)} \right) \right. \\ &\quad \left. + \int_{s_1}^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(-\int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy \right]. \end{aligned}$$

Proof. Note that

$$\frac{\partial^2 \mathbf{B}}{\partial x_1^2} + \frac{\partial^2 \mathbf{B}}{\partial x_2^2} = t'_1(s)s'_1 + t'_2(s)s'_2.$$

Therefore the lemma is a direct computation and application of Equalities (1.3.4), (1.3.5), (1.3.6) and Remark 4. \square

Finally, we get the following important statement.

Corollary 1. *The function \mathbf{B} is concave in $\Omega(\Theta(J, g))$ if and only if $\mathcal{F}(s) \leq 0$, where*

$$\begin{aligned} \mathcal{F}(s) = & -\exp\left(-\int_{s_1}^s \frac{g''(r)}{K(r)} \cos \theta(r) dr\right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)}\right) \\ & + \int_{s_1}^s \left[\frac{f''(y)}{g''(y)}\right]' \exp\left(-\int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr\right) dy = \frac{f''(s)}{g''(s)} - t_2(s). \end{aligned} \quad (1.3.11)$$

Proof. \mathbf{B} satisfies the homogeneous Monge–Ampère equation. Therefore \mathbf{B} is concave if and only if

$$\frac{\partial^2 \mathbf{B}}{\partial x_1^2} + \frac{\partial^2 \mathbf{B}}{\partial x_2^2} \leq 0. \quad (1.3.12)$$

Note that

$$\frac{g''}{K(K + \theta' \|\tilde{\ell}(x)\|)} > 0.$$

Hence, according to Lemma 4, the inequality (1.3.12) holds if and only if $\mathcal{F}(s) \leq 0$. \square

Furthermore, the function \mathcal{F} will be called *force function*.

Remark 5. *The fact $t_2(s) = f''/g'' - \mathcal{F}$ together with (1.3.7) imply that the force function*

\mathcal{F} satisfies the following differential equation

$$\begin{aligned}\mathcal{F}' + \mathcal{F} \cdot \frac{\cos \theta}{K} g'' - \left[\frac{f''}{g''} \right]' &= 0, \quad s \in J \\ \mathcal{F}(s_1) &= \frac{f''(s_1)}{g''(s_1)} - t_2(s_1).\end{aligned}\tag{1.3.13}$$

We remind the reader that for an arbitrary smooth curve $\gamma = (s, g(s), f(s))$, the torsion has the following expression

$$\frac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \times \gamma''\|^2} = \frac{f'''g'' - g'''f''}{\|\gamma' \times \gamma''\|^2} = \frac{(g'')^2}{\|\gamma' \times \gamma''\|^2} \cdot \left[\frac{f''}{g''} \right]'.$$

Corollary 2. *If $\mathcal{F}(s_1) \leq 0$ and the torsion of a curve $(s, g(s), f(s))$, $s \in J$ is negative, then the function \mathbf{B} defined by (1.3.8) is concave.*

Proof. The corollary is an immediate consequence of (1.3.11). \square

Thus, we see that the torsion of the boundary data plays a crucial role in the concavity of a surface with zero Gaussian curvature. More detailed investigations about how we choose the constant $t_2(s_1)$ will be given in Subsection 1.3.2.

Let $\Theta(J, g)$ and $\tilde{\Theta}(J, g)$ be foliations with some argument functions $\theta(s)$ and $\tilde{\theta}(s)$ respectively. Let \mathbf{B} and $\tilde{\mathbf{B}}$ be the corresponding functions defined by (1.3.8), and let $\mathcal{F}, \tilde{\mathcal{F}}$ be the corresponding force functions. Note that $\mathcal{F}(s) = \tilde{\mathcal{F}}(s)$ is equivalent to the equality $t(s) = \tilde{t}(s)$ where $t(s) = (t_1(s), t_2(s))$ and $\tilde{t}(s) = (\tilde{t}_1(s), t_2(s))$ are the corresponding gradients of \mathbf{B} and $\tilde{\mathbf{B}}$ (see (1.3.6) and Corollary 1).

Assume that the functions \mathbf{B} and $\tilde{\mathbf{B}}$ are concave functions.

Lemma 5. *If $\sin(\tilde{\theta} - \theta) \geq 0$ for all $s \in J$, and $\mathcal{F}(s_1) = \tilde{\mathcal{F}}(s_1)$, then $\tilde{\mathbf{B}} \leq \mathbf{B}$ on $\Omega(\Theta(J, g)) \cap \tilde{\Omega}(\Theta(J, g))$.*

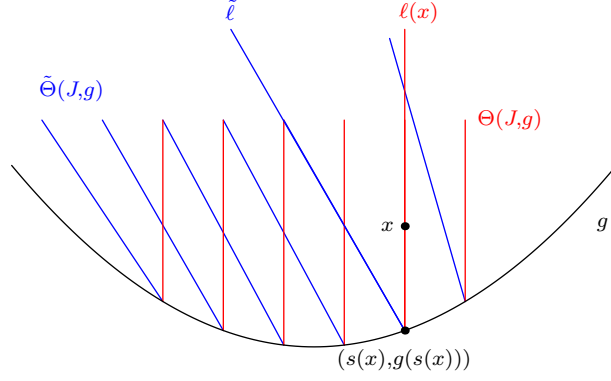


Figure 1.3 Foliations $\Theta(J, g)$ and $\tilde{\Theta}(J, g)$

In other words, the lemma says that if at initial point $(s_1, g(s_1))$ gradients of the functions $\tilde{\mathbf{B}}$ and \mathbf{B} coincide, and the foliation $\tilde{\Theta}(J, g)$ is “to the left of” the foliation $\Theta(J, g)$ (see Figure 1.3) then $\tilde{\mathbf{B}} \leq \mathbf{B}$ provided \mathbf{B} and $\tilde{\mathbf{B}}$ are concave.

Proof. Let K and \tilde{K} be the corresponding functions of \mathbf{B} and $\tilde{\mathbf{B}}$ defined by (1.3.3). The condition $K, \tilde{K} < 0$ implies that the inequality $\sin(\tilde{\theta} - \theta) \geq 0$ is equivalent to the inequality

$$\frac{\cos \tilde{\theta}}{\tilde{K}} \geq \frac{\cos \theta}{K} \quad \text{for } s \in J. \quad (1.3.14)$$

Indeed, if we rewrite (1.3.14) as $K \cos \tilde{\theta} \geq \tilde{K} \cos \theta$ then this simplifies to $-\sin \theta \cos \tilde{\theta} \geq -\sin \tilde{\theta} \cos \theta$, so the result follows. The force functions $\mathcal{F}, \tilde{\mathcal{F}}$ satisfy the differential equation

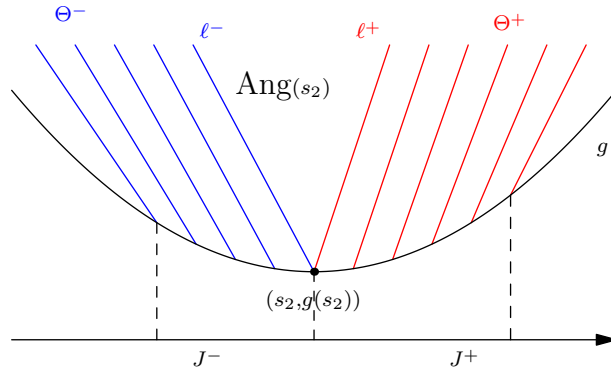


Figure 1.4 Gluing of \mathbf{B}^- and \mathbf{B}^+

(1.3.13) with the same boundary condition $\mathcal{F}(s_1) = \tilde{\mathcal{F}}(s_1)$. Then by (1.3.14) and by comparison theorems we get $\tilde{\mathcal{F}} \geq \mathcal{F}$ on J . This and (1.3.11) imply that $\tilde{t}_2 \leq t_2$ on J . Pick any point $x \in \Omega(\Theta(J, g)) \cap \tilde{\Omega}(\Theta(J, g))$. Then there exists a segment $\ell(x) \in \Theta(J, g)$. Let $(s(x), g(s(x)))$ be the corresponding endpoint of this segment. There exists a segment $\tilde{\ell} \in \tilde{\Theta}(J, g)$ which has $(s(x), g(s(x)))$ as an endpoint (see Figure 1.3).

Consider a tangent plane $L(x)$ to $(x_1, x_2, \tilde{\mathbf{B}})$ at point $(s(x), g(s(x)))$. The fact that the gradient of $\tilde{\mathbf{B}}$ is constant on $\tilde{\ell}$, implies that L is tangent to $(x_1, x_2, \tilde{\mathbf{B}})$ on $\tilde{\ell}$. Therefore

$$L(x) = f(s) + \langle (\tilde{t}_1(s), \tilde{t}_2(s)), (x_1 - s, x_2 - g(s)) \rangle,$$

where $x = (x_1, x_2)$ and $s = s(x)$. Concavity of $\tilde{\mathbf{B}}$ implies that a value of the function $\tilde{\mathbf{B}}$ at point y seen from the point $(s(x), g(s(x)))$ is less than $L(y)$. In particular $\tilde{\mathbf{B}}(x) \leq L(x)$. Now it is enough to prove that $L(x) \leq \mathbf{B}(x)$. By (1.3.8) we have

$$\mathbf{B}(x) = f(s) + \langle (t_1(s), t_2(s)), (x_1 - s(x), x_2 - g(s)) \rangle.$$

Therefore using (1.3.6), $\langle (-g', 1), (x_1 - s, x_2 - g(s)) \rangle \geq 0$ and the fact that $\tilde{t}_2 \leq t_2$ we get the desired result. \square

Let $J^- = [s_1, s_2]$ and $J^+ = [s_2, s_3]$ where $J^-, J^+ \subset I$. Consider arbitrary foliations $\Theta^- = \Theta^-(J^-, g)$ and $\Theta^+ = \Theta^+(J^+, g)$ such that $\Omega(\Theta^-) \cap \Omega(\Theta^+) = \emptyset$, and let θ^- and θ^+ be the corresponding argument functions. Let \mathbf{B}^- and \mathbf{B}^+ be the corresponding functions defined by (1.3.8), and let $t^- = (t_1^-, t_2^-)$, $t^+ = (t_1^+, t_2^+)$ be the corresponding gradient functions. Set $\text{Ang}(s_2)$ to be a convex hull of $\ell^-(s_2)$ and $\ell^+(s_2)$ where $\ell^-(s_2) \in \Theta^-$, $\ell^+(s_2) \in \Theta^+$ are the segments with the endpoint $(s_2, g(s_2))$ (see Figure 1.4). We require

that $\text{Ang}(s_2) \cap \Omega(\Theta^-) = \ell^-$ and $\text{Ang}(s_2) \cap \Omega(\Theta^+) = \ell^+$.

Let $\mathcal{F}^-, \mathcal{F}^+$ be the corresponding forces, and let \mathbf{B}_{Ang} be the function defined linearly on $\text{Ang}(s_2)$ via the values of \mathbf{B}^- and \mathbf{B}^+ on ℓ^-, ℓ^+ respectively.

Lemma 6. *If $t_2^-(s_2) = t_2^+(s_2)$, then the function \mathbf{B} defined as follows*

$$\mathbf{B}(x) = \begin{cases} \mathbf{B}^-(x), & x \in \Omega(\Theta(J^-, g)), \\ \mathbf{B}_{\text{Ang}}(x), & x \in \text{Ang}(s_2), \\ \mathbf{B}^+(x), & x \in \Omega(\Theta(J^+, g)), \end{cases}$$

belongs to the class $C^1(\Omega(\Theta^-) \cup \text{Ang}(s_2) \cup \Omega(\Theta^+) \cup \Gamma(J^- \cup J^+))$.

Proof. By (1.3.6) the condition $t_2^-(s_2) = t_2^+(s_2)$ is equivalent to the condition $t^-(s_2) = t^+(s_2)$. We recall that the gradient of \mathbf{B}^- is constant on $\ell^-(s_2)$, and the gradient of \mathbf{B}^+ is constant on $\ell^+(s_2)$, therefore the lemma follows immediately from the fact that $\mathbf{B}^-(s_2, g(s_2)) = \mathbf{B}^+(s_2, g(s_2))$. \square

Remark 6. *The fact $\mathbf{B} \in C^1$ implies that its gradient function $t(s) = \nabla \mathbf{B}$ is well defined, and it is continuous. Unfortunately, it is not necessarily true that $t(s) \in C^1([s_1, s_3])$. However, it is clear that $t(s) \in C^1([s_1, s_2])$, and $t(s) \in C^1([s_2, s_3])$.*

Finally we finish this section with the following important corollary about *concave extension* of the functions with zero gaussian curvature.

Let \mathbf{B}^- and \mathbf{B}^+ be defined as above (see Figure 1.4). Assume that $t_2^-(s_2) = t_2^+(s_2)$.

Corollary 3. *If \mathbf{B}^- is concave in $\Omega(\Theta^-)$ and the torsion of the curve $(s, g(s), f(s))$ is nonnegative on $J^+ = [s_2, s_3]$ then the function \mathbf{B} defined in Lemma 6 is concave in the domain $\Omega(\Theta^-) \cup \text{Ang}(s_2) \cup \Omega(\Theta^+)$.*

In other words the corollary tells us that if we have constructed concave function \mathbf{B}^- which satisfies homogeneous Monge–Ampère equation, and we glued \mathbf{B}^- smoothly with \mathbf{B}^+ (which also satisfies homogeneous Monge–Ampère equation), then the result \mathbf{B} is concave function provided that the space curve $(s, g(s), f(s))$ has nonnegative torsion on the interval J^+ .

Proof. By Lemma 1 concavity of \mathbf{B}^- implies $\mathcal{F}^-(s_2) \leq 0$. By (1.3.11) the condition $t_2^-(s_2) = t_2^+(s_2)$ is equivalent to $\mathcal{F}^-(s_2) = \mathcal{F}^+(s_2)$. By Corollary 2 we get that \mathbf{B}^+ is concave. Thus, concavity of \mathbf{B} follows from Lemma 6. \square

1.3.2 Cup

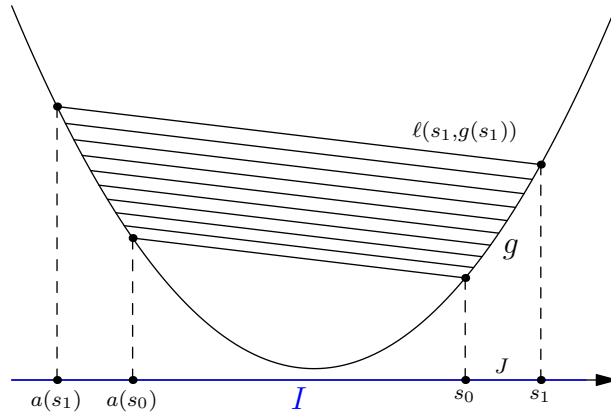


Figure 1.5 Foliation $\Theta_{\text{cup}}(J, g)$

In this subsection we are going to consider a special type of foliation which is called *Cup*. Fix an interval I and consider an arbitrary curve $(s, g(s), f(s)) \in C^3(I)$. We suppose that $g'' > 0$ on I . Let $a(s) \in C^1(J)$ be a function such that $a'(s) < 0$ on J , where $J = [s_0, s_1]$ is a subinterval of I . Assume that $a(s_0) < s_0$ and $[a(s_1), a(s_0)] \subset I$. Consider a set of open segments $\Theta_{\text{cup}}(J, g)$ consisting of those segments $\ell(s, g(s))$, $s \in J$ such that $\ell(s, g(s))$ is a segment in the plane joining the points $(s, g(s))$ and $(a(s), g(a(s)))$ (see Figure 1.5).

Lemma 7. *The set of segments $\Theta_{\text{cup}}(J, g)$ described above forms a foliation.*

Proof. We need to check the 6 requirements for a set to be the foliation. Most of them are trivial except for 4 and 5. We know the endpoints of each segment therefore we can consider the following argument function

$$\theta(s) = \pi + \arctan \left(\frac{g(s) - g(a(s))}{s - a(s)} \right).$$

Surely $\theta(s) \in C^1(J)$, so requirement 4 is satisfied. We check requirement 5. It is clear that it is enough to check this requirement for $x = (a(s), g(a(s)))$. Let $s = s(x)$, then

$$\begin{aligned} K(s) + \theta'(s) \|(a(s) - s, g(a(s)) - g(s))\| &= \frac{\langle (1, g'), (g - g(a), a - s) \rangle}{\|(g(a) - g, s - a)\|} + \\ \frac{(g' - a'g'(a))(s - a) - (1 - a')(g - g(a))}{\|(g(a) - g, s - a)\|} &= \frac{a' \cdot \langle (1, g'(a)), (g - g(a), a - s) \rangle}{\|(g(a) - g, s - a)\|} \end{aligned}$$

which is strictly negative. □

Let $\gamma(t) = (t, g(t), f(t)) \in C^3([a_0, b_0])$ be an arbitrary curve such that $g'' > 0$ on $[a_0, b_0]$. Assume that the torsion of γ is positive on $I^- = (a_0, c)$, and it is negative on $I^+ = (c, b_0)$ for some $c \in (a_0, b_0)$.

Lemma 8. *For all P such that $0 < P < \min\{c - a_0, b_0 - c\}$ there exist $a \in I^-$, $b \in I^+$ such that $b - a = P$ and*

$$\begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix} = 0. \quad (1.3.15)$$

Proof. Pick a number $a \in (a_0, b_0)$ so that $b = a + P \in (a_0, b_0)$. We denote

$$\mathcal{M}(a, b) = (a - b)(g'(b) - g'(a)) \left(\frac{g(a) - g(b)}{a - b} - g'(a) \right).$$

Note that the conditions $a \neq b$ and $g'' > 0$ imply $\mathcal{M}(a, b) \neq 0$. Then

$$\begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix} = \mathcal{M}(a, b) \left[\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f'(b) - f'(a)}{g'(b) - g'(a)} \right].$$

Thus our equation (1.3.15) turns into

$$\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f'(b) - f'(a)}{g'(b) - g'(a)} = 0. \quad (1.3.16)$$

We consider the following functions $V(x) = f(x) - f'(a)x$ and $U(x) = g(x) - g'(a)x$. Note that $U(a) \neq U(b)$ and $U' \neq 0$ on (a, b) . Therefore by Cauchy's mean value theorem there exists a point $\xi = \xi(a, b) \in (a, b)$ such that

$$\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} = \frac{V(a) - V(b)}{U(a) - U(b)} = \frac{V'(\xi)}{U'(\xi)} = \frac{f'(\xi) - f'(a)}{g'(\xi) - g'(a)}.$$

Now we define

$$W_a(z) \stackrel{\text{def}}{=} \frac{f'(z) - f'(a)}{g'(z) - g'(a)}, \quad z \in (a, b].$$

So the left hand side of (1.3.16) takes the form $W_a(\xi) - W_a(b) = 0$ for some $\xi(a, P) \in (a, b)$.

We consider the curve $v(s) = (g'(s), f'(s))$ which is a graph on $[a_0, b_0]$. The fact that the torsion of the curve $\gamma(s) = (s, g(s), f(s))$ changes sign from $+$ to $-$ at the point $c \in (a_0, b_0)$ means that the curve $v(s)$ is strictly convex on the interval (a_0, c) , and it is strictly concave on the interval (c, b_0) . We consider a function obtained from (1.3.16)

$$D(z) \stackrel{\text{def}}{=} \frac{f(z) - f(z+P) + f'(z)P}{g(z) - g(z+P) + g'(z)P} - \frac{f'(z+P) - f'(z)}{g'(z+P) - g'(z)}, \quad z \in [a_0, c]. \quad (1.3.17)$$

Note that $D(a_0) = W_{a_0}(\zeta) - W_{a_0}(a_0+P)$ for some $\zeta = \zeta(a_0, P) \in (a_0, a_0+P)$. We know that $v(s)$ is strictly convex on the interval (a_0, a_0+P) . This implies that $W_{a_0}(z) - W_{a_0}(a_0+P) < 0$ for all $z \in (a_0, a_0+P)$. In particular $D(a_0) < 0$. Similarly, concavity of $v(s)$ on $(c, c+P)$ implies that $D(c) > 0$. Hence, there exists $a \in (a_0, c)$ such that $D(a) = 0$. \square

Let a_1 and b_1 be some solutions of (1.3.15) obtained by Lemma 8.

Lemma 9. *There exists a function $a(s) \in C^1((c, b_1]) \cap C([c, b_1])$ such that $a(b_1) = a_1$, $a(c) = c$, $a'(s) < 0$, and the pair $(a(s), s)$ solves the equation (1.3.15) for all $s \in [c, b_1]$.*

Proof. The proof of the lemma is a consequence of the implicit function theorem. Let $a < b$, and consider the function

$$\Phi(a, b) \stackrel{\text{def}}{=} \begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix}.$$

We are going to find the signs of the partial derivatives of $\Phi(a, b)$ at the point $(a, b) = (a_1, b_1)$.

We present the calculation only for $\partial\Phi/\partial b$. The case for $\partial\Phi/\partial a$ is similar.

$$\begin{aligned}
\frac{\partial \Phi(a, b)}{\partial b} &= \begin{vmatrix} 1 & 0 & a - b \\ g'(a) & g''(b) & g(a) - g(b) \\ f'(a) & f''(b) & f(a) - f(b) \end{vmatrix} = \\
&= (a - b)g''(b) \left(\frac{g(a) - g(b)}{a - b} - g'(a) \right) \left[\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f''(b)}{g''(b)} \right].
\end{aligned}$$

Note that

$$(a - b)g''(b) \left(\frac{g(a) - g(b)}{a - b} - g'(a) \right) < 0,$$

therefore we see that the sign of $\partial \Phi / \partial b$ depends only on the sign of the expression

$$\frac{f(a) - f(b) - f'(a)(a - b)}{g(a) - g(b) - g'(a)(a - b)} - \frac{f''(b)}{g''(b)}. \quad (1.3.18)$$

We use the *cup equation* (1.3.16), and we obtain that the expression (1.3.18) at the point $(a, b) = (a_1, b_1)$ takes the following form:

$$\frac{f'(b) - f'(a)}{g'(b) - g'(a)} - \frac{f''(b)}{g''(b)}. \quad (1.3.19)$$

The above expression has the following geometric meaning. We consider the curve $v(s) = (g'(s), f'(s))$, and we draw a segment which connects the points $v(a)$ and $v(b)$. The above expression is the difference between the slope of the line which passes through the segment

$[v(a), v(b)]$ and the slope of the tangent line of the curve $v(s)$ at the point b . In the case as it is shown on Figure 1.6, this difference is positive. Recall that $v(s)$ is strictly convex on (a_1, c) , and it is strictly concave on (c, b_1) . Therefore, one can easily note that this expression (1.3.19) is always positive if the segment $[v(a), v(b)]$ also intersects the curve $v(s)$ at a point ξ such that $a < \xi < b$. This always happens in our case because equation (1.3.16) means that the points $v(a), v(\xi), v(b)$ lie on the same line, where ξ was determined from Cauchy's mean value theorem. Thus

$$\frac{f'(b) - f'(a)}{g'(b) - g'(a)} - \frac{f''(b)}{g''(b)} > 0. \quad (1.3.20)$$

Similarly, we can obtain that $\frac{\partial \Phi}{\partial a} < 0$, because this is the same as to show that

$$\frac{f'(b) - f'(a)}{g'(b) - g'(a)} - \frac{f''(a)}{g''(a)} > 0. \quad (1.3.21)$$

Thus, by the implicit function theorem there exists a C^1 function $a(s)$ in some neighborhood of b_1 such that $a'(s) = -\frac{\Phi'_b}{\Phi'_a} < 0$, and the pair $(a(s), s)$ solves (1.3.15).

Now we want to explain that the function $a(s)$ can be defined on $(c, b_1]$, and, moreover, $\lim_{s \rightarrow c+0} a(s) = c$. Indeed, whenever $a(s) \in (a_1, c)$ and $s \in (c, b_1)$ we can use the implicit function theorem, and we can extend the function $a(s)$. It is clear that for each s we have $a(s) \in [a_1, c)$ and $s \in (c, b_1)$. Indeed, if $a(s), s \in (a_1, c]$, or $a(s), s \in [c, b_1)$ then (1.3.15) has a definite sign (see (1.3.17)). It follows that $a(s) \in C^1((c, b_1])$, and the condition $a'(s) < 0$ implies $\lim_{s \rightarrow c+0} a(s) = c$. Hence $a(s) \in C([c, b_1])$. \square

It is worth mentioning that we did not use the fact that the torsion of $(s, g(s), f(s))$ changes sign from $+$ to $-$. The only thing we needed was that the torsion changes sign.

Let a_1 and b_1 be any solutions of equation (1.3.15) from Lemma 8, and let $a(s)$ be any function from Lemma 9. Fix an arbitrary $s_1 \in (c, b_1)$ and consider the foliation $\Theta_{\text{cup}}([s_1, b_1], g)$ constructed by $a(s)$ (see Lemma 7). Let \mathbf{B} be a function defined by (1.3.8), where

$$t_2(s_1) = \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}. \quad (1.3.22)$$

Set $\Omega_{\text{cup}} = \Omega(\Theta_{\text{cup}}([s_1, b_1], g))$, and let $\overline{\Omega_{\text{cup}}}$ be the closure of Ω_{cup} .

Lemma 10. *The function \mathbf{B} satisfies the following properties*

1. $\mathbf{B} \in C^2(\Omega_{\text{cup}}) \cap C^1(\overline{\Omega_{\text{cup}}})$.
2. $\mathbf{B}(a(s), g(a(s))) = f(a(s))$ for all $s \in [s_1, b_1]$.
3. \mathbf{B} is a concave function in $\overline{\Omega_{\text{cup}}}$.

Proof. The first property follows from Lemma 3 and the fact that $\nabla B(x) = t(s)$ for $s = s(x)$, where $s(x)$ is a continuous function in $\overline{\Omega_{\text{cup}}}$.

We are going to check the second property. We recall (see (1.3.6)) that $t_1(s) = f'(s) - t_2(s)g'(s)$. Condition (1.3.22) implies that

$$t_1(s_1) + t_2(s_1)g'(a(s_1)) = f'(a(s_1)). \quad (1.3.23)$$

Let $\mathbf{B}(a(s), g(a(s))) = \tilde{f}(a(s))$. After differentiation of this equality we get $t_1(s_1) + t_2(s_1)g'(a(s_1)) =$

$\tilde{f}'(a(s_1))$. Hence, (1.3.23) implies that $f'(a(s_1)) = \tilde{f}'(a(s_1))$. It is clear that

$$t_1(s) + t_2(s)g'(s) = f'(s),$$

$$t_1(s) + t_2(s)g'(a(s)) = \tilde{f}'(a(s)),$$

$$t_1(s)(s - a(s)) + t_2(s)(g(s) - g(a(s))) = f(s) - \tilde{f}(a(s)),$$

which implies

$$\begin{vmatrix} 1 & 1 & s - a(s) \\ g'(s) & g'(a(s)) & g(s) - g(a(s)) \\ f'(s) & \tilde{f}'(a(s)) & f(s) - \tilde{f}(a(s)) \end{vmatrix} = 0.$$

This equality can be rewritten as follows:

$$f' \cdot \begin{vmatrix} 1 & s - a(s) \\ g'(a(s)) & g(s) - g(a(s)) \end{vmatrix} - \tilde{f}'(a) \begin{vmatrix} 1 & s - a(s) \\ g' & g(s) - g(a(s)) \end{vmatrix} + (f - \tilde{f}(a))(g'(a(s)) - g'(s)) = 0.$$

By virtue of Lemma 9 we have the same equality as above except \tilde{f} is replaced by f . We subtract one from another one:

$$[f(a(s)) - \tilde{f}(a(s))] + [f'(a(s)) - \tilde{f}'(a(s))] \cdot \frac{\begin{vmatrix} 1 & s - a(s) \\ g' & g(s) - g(a(s)) \end{vmatrix}}{g'(a(s)) - g'(s)} = 0.$$

Note that

$$\frac{\begin{vmatrix} 1 & s - a(s) \\ g' & g(s) - g(a(s)) \end{vmatrix}}{g'(a(s)) - g'(s)} < 0$$

and $a(s)$ is invertible. Therefore we get the differential equation $z(u)B(u) + z'(u) = 0$ where $B \in C^1([a(b_1), a(s_1)])$, $z(u) = f(u) - \tilde{f}(u)$ and $B < 0$. The condition $z'(a(s_1)) = 0$ implies $z(a(s_1)) = 0$. Note that $z = 0$ is a trivial solution. Therefore, by uniqueness of solutions to ODEs we get $z = 0$.

We are going to check the concavity of \mathbf{B} . Let \mathcal{F} be the force function corresponding to \mathbf{B} . By Corollary 2 we only need to check that $\mathcal{F}(s_1) \leq 0$. Note that (1.3.11) and (1.3.22) imply

$$\mathcal{F}(s_1) = \frac{f''(s_1)}{g''(s_1)} - t_2(s_1) = \frac{f''(s_1)}{g''(s_1)} - \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))},$$

which is negative by (1.3.20). □

Remark 7. *The above lemma is true for all choices $s_1 \in (c, b_1)$. If we send s_1 to c then one can easily see that $\lim_{s_1 \rightarrow c+} t_2(s_1) = 0$, therefore the force function \mathcal{F} takes the following form*

$$\mathcal{F}(s) = \int_c^s \left[\frac{f''(y)}{g''(y)} \right]' \exp \left(- \int_y^s \frac{g''(r)}{K(r)} \cos \theta(r) dr \right) dy.$$

This is another way to show that the force function is nonpositive.

The next lemma shows that the regardless of the choices of initial solution (a_1, b_1) of

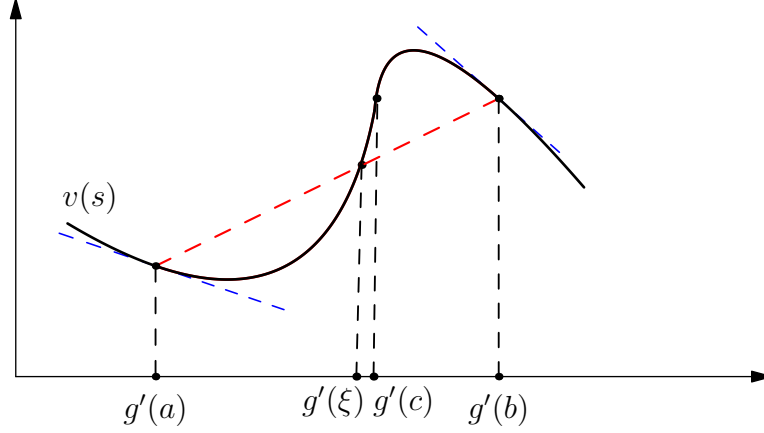


Figure 1.6 Graph $v(s)$

(1.3.15), the constructed function $a(s)$ by Lemma 9 is unique (i.e. it does not depend on the pair (a_1, b_1)).

Lemma 11. *Let pairs (a_1, b_1) , $(\tilde{a}_1, \tilde{b}_1)$ solve (1.3.15), and let $a(s), \tilde{a}(s)$ be the corresponding functions obtained by Lemma 9. Then $a(s) = \tilde{a}(s)$ on $[c, \min\{b_1, \tilde{b}_1\}]$.*

Proof. By the uniqueness result of the implicit function theorem we only need to show existence of $s_1 \in (c, \min\{b_1, \tilde{b}_1\})$ such that $a(s_1) = \tilde{a}(s_1)$. Without loss of generality assume that $\tilde{b}_1 = b_1 = s_2$. We can also assume that $\tilde{a}(s_2) > a(s_2)$, because other cases can be solved in a similar way.

Let $\Theta = \Theta_{\text{cup}}([c, s_2], g)$ and $\tilde{\Theta} = \tilde{\Theta}_{\text{cup}}([c, s_2], g)$ be the foliations corresponding to the functions $a(s)$ and $\tilde{a}(s)$. Let \mathbf{B} and $\tilde{\mathbf{B}}$ be the functions corresponding to these foliations from Lemma 10. We consider a chord T in \mathbb{R}^3 joining the points $(a(s_1), g(a(s_1)), f(a(s_1)))$ and $(s_1, g(s_1), f(s_1))$ (see Figure 1.7). We want to show that the chord T belongs to the graph of $\tilde{\mathbf{B}}$. Indeed, concavity of $\tilde{\mathbf{B}}$ (see Lemma 10) implies that the chord T lies below the graph of $\tilde{\mathbf{B}}(x_1, x_2)$, where $(x_1, x_2) \in \Omega(\tilde{\Theta})$. Moreover, concavity of \mathbf{B} , $\Omega(\tilde{\Theta}) \subset \Omega(\Theta)$ and the fact that the graph $\tilde{\mathbf{B}}$ consists of chords joining the points of the curve $(t, g(t), f(t))$

imply that the graph \mathbf{B} lies above the graph $\tilde{\mathbf{B}}$. In particular the chord T , belonging to the graph \mathbf{B} , lies above the graph $\tilde{\mathbf{B}}$. This can happen if and only if T belongs to the graph $\tilde{\mathbf{B}}$. Now we show that if $s_1 < s_2$, then the torsion of the curve $(s, g(s), f(s))$ is zero for $s \in [s_1, s_2]$. Indeed, let \tilde{T} be a chord in \mathbb{R}^3 which joins the points $(a(s_1), g(a(s_1)), f(a(s_1)))$ and $(s_2, g(s_2), f(s_2))$. We consider the tangent plane $L(x)$ to the graph $\tilde{\mathbf{B}}$ at the point $(x_1, x_2) = (a(s_1), g(a(s_1)))$. This tangent plane must contain both chords T and \tilde{T} , and it must be tangent to the surface at these chords. Concavity of $\tilde{\mathbf{B}}$ implies that the tangent plane L coincides with $\tilde{\mathbf{B}}$ at points belonging to the triangle, which is the convex hull of the points $(a(s_1), g(a(s_1)))$, $(s_1, g(s_1))$ and $(s_2, g(s_2))$. Therefore, it is clear that the tangent plane L coincides with $\tilde{\mathbf{B}}$ on the segments $\ell \in \tilde{\Theta}$ with the endpoint at $(s, g(s))$ for $s \in [s_1, s_2]$. Thus $L((s, g(s))) = \tilde{\mathbf{B}}((s, g(s)))$ for any $s \in [s_1, s_2]$. This means that the torsion of the curve $(s, g(s), f(s))$ is zero on $s \in [s_1, s_2]$ which contradicts our assumption about the torsion. Therefore $s_1 = s_2$. \square

Corollary 4. *In the conditions of Lemma 8, for all $0 < P < \min\{c - a_0, b_0 - c\}$ there exists a unique pair (a_1, b_1) which solves (1.3.15) such that $b_1 - a_1 = P$.*

The above corollary implies that if the pairs (a_1, b_1) and $(\tilde{a}_1, \tilde{b}_1)$ solve (1.3.15), then $a_1 \neq \tilde{a}_1$ and $b_1 \neq \tilde{b}_1$, and one of the following conditions holds: $(a_1, b_1) \subset (\tilde{a}_1, \tilde{b}_1)$, or $(\tilde{a}_1, \tilde{b}_1) \subset (a_1, b_1)$.

Remark 8. *The function $a(s)$ is defined on the right of the point c . We extend naturally its definition on the left of the interval by $a(s) \stackrel{\text{def}}{=} a^{-1}(s)$.*

1.4 Construction of the Bellman function

1.4.1 Reduction to the two dimensional case

We are going to construct the Bellman function for the case $p < 2$. The case $p = 2$ is trivial, and the case $p > 2$ was solved in [8]. From the definition of H it follows that

$$H(x_1, x_2, x_3) = H(|x_1|, |x_2|, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \Omega. \quad (1.4.1)$$

Also note the homogeneity condition

$$H(\lambda x_1, \lambda x_2, \lambda^p x_3) = \lambda^p H(x_1, x_2, x_3) \quad \text{for all } \lambda \geq 0. \quad (1.4.2)$$

These two conditions (1.4.1), (2.1.7), which follow from the nature of the boundary data $(x^2 + \tau^2 y^2)^{2/p}$, make the construction of H easier. However, in order to construct the function H , this information is not necessary. Further, we assume that H is $C^1(\Omega)$ smooth. Then from the symmetry (1.4.1) it follows that

$$\frac{\partial H}{\partial x_j} = 0 \quad \text{on } x_j = 0 \quad \text{for } j = 1, 2. \quad (1.4.3)$$

For convenience, as in [8], we rotate the system of coordinates (x_1, x_2, x_3) . Namely, let

$$y_1 \stackrel{\text{def}}{=} \frac{x_1 + x_2}{2}, \quad y_2 \stackrel{\text{def}}{=} \frac{x_2 - x_1}{2}, \quad y_3 \stackrel{\text{def}}{=} x_3. \quad (1.4.4)$$

We define

$$N(y_1, y_2, y_3) \stackrel{\text{def}}{=} H(y_1 - y_2, y_1 + y_2, y_3) \quad \text{on} \quad \Omega_1,$$

where $\Omega_1 = \{(y_1, y_2, y_3) : y_3 \geq 0, |y_1 - y_2|^p \leq y_3\}$. It is clear that for fixed y_1 , the function N is concave in variables y_2 and y_3 ; moreover, for fixed y_2 the function N is concave with respect to the rest of variables. The symmetry (1.4.1) for N turns into the following condition

$$N(y_1, y_2, y_3) = N(y_2, y_1, y_3) = N(-y_1, -y_2, y_3). \quad (1.4.5)$$

Thus it is sufficient to construct the function N on the domain

$$\Omega_2 \stackrel{\text{def}}{=} \{(y_1, y_2, y_3) : y_1 \geq 0, -y_1 \leq y_2 \leq y_1, (y_1 - y_2)^p \leq y_3\}.$$

Condition (1.4.3) turns into

$$\frac{\partial N}{\partial y_1} = \frac{\partial N}{\partial y_2} \quad \text{on the hyperplane} \quad y_2 = y_1, \quad (1.4.6)$$

$$\frac{\partial N}{\partial y_1} = -\frac{\partial N}{\partial y_2} \quad \text{on the hyperplane} \quad y_2 = -y_1. \quad (1.4.7)$$

The boundary condition (1.2.3) becomes

$$N(y_1, y_2, |y_1 - y_2|^p) = ((y_1 + y_2)^2 + \tau^2(y_1 - y_2)^2)^{p/2}. \quad (1.4.8)$$

The homogeneity condition (2.1.7) implies that $N(\lambda y_1, \lambda y_2, \lambda^p y_3) = \lambda^p N(y_1, y_2, y_3)$ for $\lambda \geq 0$. We choose $\lambda = 1/y_1$, and we obtain that

$$N(y_1, y_2, y_3) = y_1^p N\left(1, \frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \quad (1.4.9)$$

Suppose we are able to construct the function $M(y_2, y_3) \stackrel{\text{def}}{=} N(1, y_2, y_3)$ on

$$\Omega_3 \stackrel{\text{def}}{=} \{(y_2, y_3) : -1 \leq y_2 \leq 1, (1 - y_2)^p \leq y_3\}$$

with the following conditions:

1. M is concave in Ω_3
2. M satisfies (1.4.8) for $y_1 = 1$.
3. The extension of M onto Ω_1 via formulas (1.4.9) and (1.4.5) is a function with the properties of N (see (1.4.6), (1.4.7), and concavity of N).
4. M is minimal among those who satisfy the conditions 1,2,3.

Then the extended function M should be N . So we are going to construct M on Ω_3 . We denote

$$g(t) \stackrel{\text{def}}{=} (1 - t)^p, \quad t \in [-1, 1], \quad (1.4.10)$$

$$f(t) \stackrel{\text{def}}{=} ((1 + t)^2 + \tau^2(1 - t)^2)^{p/2}, \quad t \in [-1, 1]. \quad (1.4.11)$$

Then we have the boundary condition

$$M(t, g(t)) = f(t), \quad t \in [-1, 1]. \quad (1.4.12)$$

We differentiate the condition (1.4.9) with respect to y_1 at the point $(y_1, y_2, y_3) = (1, -1, y_3)$ and we obtain that

$$\frac{\partial N}{\partial y_1}(1, -1, y_3) = pN(1, -1, y_3) + \frac{\partial N}{\partial y_2}(1, -1, y_3) - py_3 \frac{\partial N}{\partial y_3}, \quad y_3 \geq 0.$$

Now we use (1.4.7), so we obtain another requirement for $M(y_2, y_3)$:

$$0 = pM(-1, y_3) + 2\frac{\partial M}{\partial y_2}(-1, y_3) - py_3 \frac{\partial M}{\partial y_3}(-1, y_3), \quad \text{for } y_3 \geq 0. \quad (1.4.13)$$

Similarly, we differentiate (1.4.9) with respect to y_1 at point $(y_1, y_2, y_3) = (1, 1, y_3)$ and use (1.4.6), so we obtain

$$0 = pM(1, y_3) - 2\frac{\partial M}{\partial y_2}(1, y_3) - py_3 \frac{\partial M}{\partial y_3}(1, y_3), \quad \text{for } y_3 \geq 0. \quad (1.4.14)$$

So in order to satisfy conditions (1.4.6) and (1.4.7), the requirements (1.4.13) and (1.4.14) are necessary. It is easy to see that these requirements are also sufficient in order to satisfy these conditions.

The minimum between two concave functions with fixed boundary data is a concave function with the same boundary data. Note also that the conditions (1.4.13) and (1.4.14) still fulfilled after taking the minimum. Thus it is quite reasonable to construct a candidate for $M(y_2, y_3)$ as a minimal concave function on Ω_3 with the boundary conditions (1.4.12),

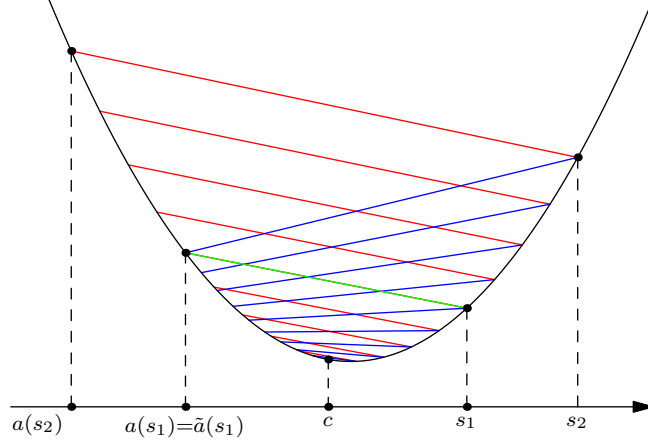


Figure 1.7 Uniqueness of the cup

(1.4.13) and (1.4.14). We remind that we should also have the concavity of the extended function $N(y_1, y_2, y_3)$ with respect to variables y_1, y_3 for each fixed y_2 . This condition can be verified after the construction of the function $M(y_2, y_3)$.

1.4.2 Construction of a candidate for M

We are going to construct a candidate \mathbf{B} for M . Firstly, we show that for $\tau > 0$, the torsion τ_γ of the boundary curve $\gamma(t) \stackrel{\text{def}}{=} (t, g(t), f(t))$ on $t \in (-1, 1)$, where f, g are defined by (1.4.10) and (1.4.11), changes sign once from $+$ to $-$. We call this point the root of a cup. We construct the cup around this point. Note that $g' < 0, g'' > 0$ on $[-1, 1)$. Therefore

$$\text{sign } \tau_\gamma = \text{sign} \left(f''' - \frac{g'''}{g''} f'' \right) = \text{sign} \left(f''' - \frac{2-p}{1-t} f'' \right) = \text{sign}(v(t)),$$

where

$$\begin{aligned} v(t) \stackrel{\text{def}}{=} & -(1 + \tau^2)^2(p - 1)t^3 + (1 + \tau^2)(3\tau^2 + \tau^2 p + 3 - 3p)t^2 + \\ & (2\tau^2 p - 9\tau^4 + \tau^4 p + 3 - 3p - 6\tau^2)t - p + 5\tau^4 + 2\tau^2 p - \tau^4 p - 10\tau^2 + 1. \end{aligned}$$

Note that $v(-1) = 16\tau^4 > 0$ and $v(1) = -8((p-1) + \tau^2) < 0$. So the function $v(t)$ changes sign from $+$ to $-$ at least once. Now, we show that $v(t)$ has only one root. For $\tau^2 < \frac{3(p-1)}{3-p}$, note that the linear function $v''(t)$ is nonnegative i.e. $v''(-1) = 8\tau^2 p(1 + \tau^2) > 0$, $v''(1) = -4(1 + \tau^2)(\tau^2 p - 3\tau^2 + 3p - 3) \geq 0$. Therefore, the convexity of $v(t)$ implies the uniqueness of the root $v(t)$ on $[-1, 1]$.

Suppose $\tau^2 < \frac{3(p-1)}{3-p}$; we will show that $v' \leq 0$ on $[-1, 1]$. Indeed, the discriminant of the quadratic function $v'(x)$ has the expression

$$D = 16\tau^2(\tau^2 + 1)^2((3-p)^2\tau^2 - 9(p-1)),$$

which is negative for $0 < \tau^2 < \frac{3(p-1)}{3-p}$. Moreover, $v'(-1) = -4\tau^2(\tau^2 p + 3\tau^2 + 3) < 0$. Thus we obtain that v' is negative.

We denote the root of v by c . It is an appropriate time to make the following remark.

Remark 9. *Note that $v(-1 + 2/p) < 0$. Indeed,*

$$v(-1 + 2/p) = \frac{(3p-2)(p^2 - 2p - 4)\tau^4 + (16 + 5p^3 - 8p^2 - 16p)\tau^2 + 8(1-p)}{p^3},$$

which is negative because coefficients of τ^4, τ^2, τ^0 are negative. Therefore, this inequality implies that $c < -1 + 2/p$.

Consider $a = -1$ and $b = 1$; the left side of (1.3.15) takes the positive value $-2^{2p-1}p(1-p)$. However, if we consider $a = -1$ and $b = c$, then the proof of Lemma 8 (see (1.3.17)) implies that the left side of (1.3.15) is negative. Therefore, there exists a unique $s_0 \in (c, 1)$ such that the pair $(-1, s_0)$ solves (1.3.15). Uniqueness follows from Corollary 4. The equation

(1.3.15) for the pair $(-1, s_0)$ is equivalent to the equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$, where

$$u(z) \stackrel{\text{def}}{=} \tau^p(p-1) \left(\tau^2 + z^2\right)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1. \quad (1.4.15)$$

Lemma 9 gives the function $a(s)$, and Lemma 10 gives the concave function $\mathbf{B}(y_2, y_3)$ for $s_1 = c$ with the foliation $\Theta_{\text{cup}}((c, s_0], g)$ in the domain $\Omega(\Theta_{\text{cup}}((c, s_0], g))$.

The above explanation implies the following corollary.

Corollary 5. *Pick any point $\tilde{y}_2 \in (-1, 1)$. The inequalities $s_0 < \tilde{y}_2$, $s_0 = \tilde{y}_2$ and $\tilde{y}_2 > s_0$ are equivalent to the following inequalities respectively: $u\left(\frac{1+\tilde{y}_2}{1-\tilde{y}_2}\right) < 0$, $u\left(\frac{1+\tilde{y}_2}{1-\tilde{y}_2}\right) = 0$ and $u\left(\frac{1+\tilde{y}_2}{1-\tilde{y}_2}\right) > 0$.*

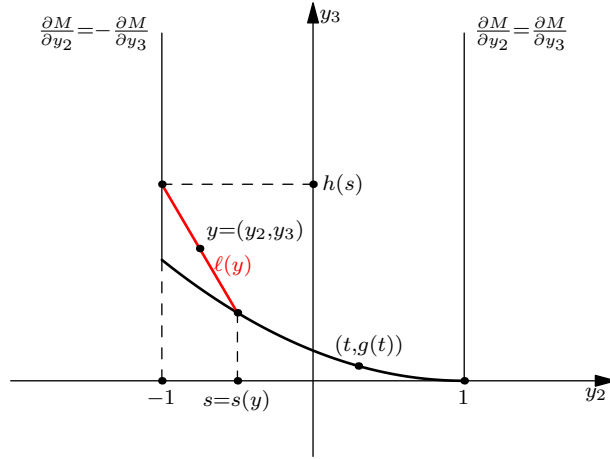


Figure 1.8 Segment $\ell(y)$

Now we are going to extend C^1 smoothly the function \mathbf{B} on the upper part of the cup. Recall that we are looking for a minimal concave function. If we construct a function with a foliation $\Theta([s_0, \tilde{y}_2], g)$ where $\tilde{y}_2 \in (s_0, 1)$ then the best thing we can do according to Lemma 6 and Lemma 5 is to minimize $\sin(\theta_{\text{cup}}(s_0) - \theta(s_0))$ where $\theta_{\text{cup}}(s)$ is an argument function of $\Theta_{\text{cup}}((c, s_0], g)$ and $\theta(s)$ is an argument function of $\Theta([s_0, \tilde{y}_2], g)$. In other words we need to

choose segments from $\Theta([s_0, \tilde{y}_2], g)$ close enough to the segments of $\Theta_{\text{cup}}((c, s_0], g)$.

Thus, we are going to try to construct the set of segments $\Theta([s_0, \tilde{y}_2])$ so that they start from $(s, g(s), f(s))$, $s \in [s_0, \tilde{y}_2]$, and they go to the boundary $y_2 = -1$ of Ω_3 .

We explain how the conditions (1.4.13) and (1.4.14) allow us to construct such type of foliation $\Theta([s_0, \tilde{y}_2], g)$ in a unique way. Let $\ell(y)$ be the segment with the endpoints $(s, g(s))$ where $s \in (s_0, \tilde{y}_2)$ and $(-1, h(s))$ (see Figure 1.8).

Let $t(s) = (t_1(s), t_2(s)) = \nabla \mathbf{B}(y)$ where $s = s(y)$ is the corresponding gradient function. Then (1.4.13) takes the form

$$0 = p\mathbf{B}(-1, h(s)) + 2t_1(s) - ph(s)t_2(s). \quad (1.4.16)$$

We differentiate this expression with respect to s , and we obtain

$$2t'_1(s) - ph(s)t'_2(s) = 0. \quad (1.4.17)$$

Then according to (1.3.5) we find the function $\tan \theta(s)$, and, hence, we find the quantity $h(s)$

$$\tan \theta(s) = -\frac{ph(s)}{2} \Leftrightarrow \frac{h(s) - g(s)}{s + 1} = \frac{ph(s)}{2}.$$

Therefore,

$$h(s) = \frac{2g(s)}{p} \left(\frac{1}{y_p - s} \right) \quad \text{where} \quad y_p \stackrel{\text{def}}{=} -1 + \frac{2}{p}. \quad (1.4.18)$$

We see that the function $h(s)$ is well defined, it increases, and it is differentiable on $-1 \leq$

$s < y_p$. So we conclude that if $s_0 < y_p$ then we are able to construct the set of segments $\Theta([s_0, y_p], g)$ that pass through the points $(s, g(s))$, where $s \in [s_0, y_p)$ and through the boundary $y_2 = -1$ (see Figure 1.9).

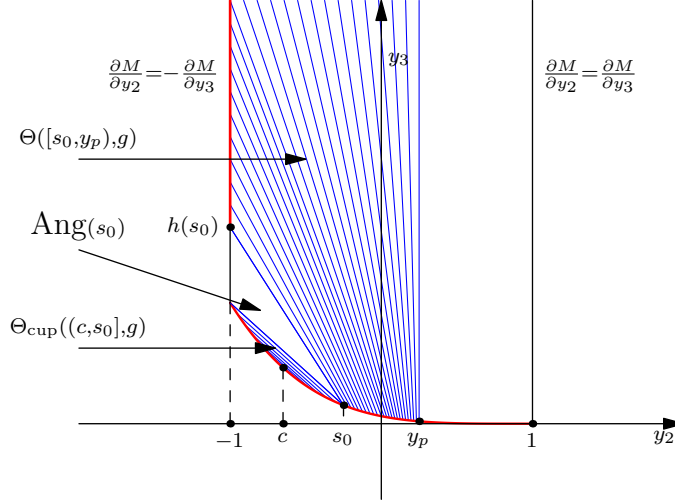


Figure 1.9 Foliations $\Theta_{\text{cup}}((c, s_0], g)$ and $\Theta([s_0, y_p], g)$

It is easy to check that $\Theta([s_0, y_p], g)$ is a foliation. So choosing the value $t_2(s_0)$ of \mathbf{B} on $\Omega(\Theta([s_0, y_p], g))$ according to Lemma 6, then by Corollary 3 we have constructed the concave function \mathbf{B} in the domain $\Omega(\Theta_{\text{cup}}((c, s_0], g)) \cup \text{Ang}(s_0) \cup \Omega(\Theta([s_0, y_p], g))$.

It is clear that the foliation $\Theta([s_0, y_p], g)$ exists as long as $s_0 < y_p$. Note that $\frac{1+y_p}{1-y_p} = \frac{1}{p-1}$. Therefore, Corollary 5 implies the following remark.

Remark 10. *The inequalities $s_0 < y_p$, $s_0 = y_p$ and $s_0 > y_p$ are equivalent to the following inequalities respectively: $u\left(\frac{1}{p-1}\right) < 0$, $u\left(\frac{1}{p-1}\right) = 0$ and $u\left(\frac{1}{p-1}\right) > 0$.*

At the point y_p the segments from $\Theta([s_0, y_p], g)$ become vertical. After the point $(y_p, g(y_p))$ we should consider vertical segments $\Theta([y_p, 1], g)$ (see Figure 1.10), because by Lemma 5 this corresponds to the minimal function. Surely $\Theta([y_p, 1], g)$ is the foliation. Again, choosing the value $t_2(y_p)$ of \mathbf{B} on $\Omega(\Theta([y_p, 1], g))$ according to Lemma 6, then by Corollary 3 we have

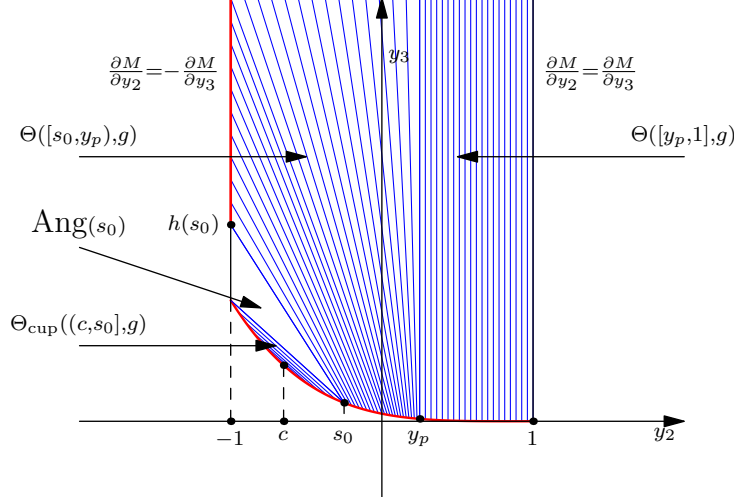


Figure 1.10 Case $u\left(\frac{1}{p-1}\right) < 0$

constructed the concave function \mathbf{B} on Ω_3 . Note that if $s_0 \geq y_p$ (which corresponds to the inequality $u\left(\frac{1}{p-1}\right) > 0$) then we do not have the foliation $\Theta([s_0, y_p], g)$. In this case we consider only vertical segments $\Theta([s_0, 1], g)$ (see Figure 1.11), and again choosing the value $t_2(s_0)$ of \mathbf{B} on $\Omega(\Theta([s_0, 1], g))$ according to Lemma 6 then by Corollary 3 we construct a concave function \mathbf{B} on Ω_3 . We believe that $\mathbf{B} = M$.

We still have to check the requirements (1.4.13) and (1.4.14). The crucial role is played by symmetry of the boundary data of N . Further, the given proofs work for both of the cases $y_p < s_0$ and $y_p \geq s_0$. Therefore, we do not consider them separately.

The requirement (1.4.14) follows immediately. Indeed, the condition (1.3.8) at the point $y = (1, y_3)$ (note that in (1.3.8) instead of $x = (x_1, x_2)$ we consider $y = (y_2, y_3)$) implies that $\mathbf{B}(1, y_3) = f(1) + t_2(1)(y_3 - g(1))$. Therefore, the requirement (1.4.14) takes the form $0 = pf(1) - 2t_1(1)$. Using (1.3.6), we obtain that $t_1(1) = f'(1)$. Therefore, we see that $pf(1) - 2t_1(1) = pf(1) - 2f'(1) = 0$.

Now, we are going to obtain the requirement (1.4.13) which is the same as (1.4.16). The quantities t_1, t_2 of \mathbf{B} with the foliation $\Theta([s_0, y_p], g)$ satisfy the condition (1.4.17) which was

obtained by differentiation of (1.4.16). So we only need to check the condition (1.4.16) at the initial point $s = s_0$. If we substitute the expression of \mathbf{B} from (1.3.8) into (1.4.16), then (1.4.16) turns into the following equivalent condition:

$$t_1(s)(s - y_p) + t_2(s)g(s) = f(s). \quad (1.4.19)$$

Note that (1.3.6) allows us to rewrite (1.4.19) into the equivalent condition

$$t_2(s) = \frac{f(s) - (s - y_p)f'(s)}{g(s) - (s - y_p)g'(s)}. \quad (1.4.20)$$

And as it was mentioned above we only need to check condition (1.4.20) at the point $s = s_0$, i.e.

$$t_2(s_0) = \frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)}. \quad (1.4.21)$$

On the other hand, if we differentiate the boundary condition $\mathbf{B}(s, g(s)) = f(s)$ at the points $s = s_0, -1$, then we obtain

$$t_1(s_0) + t_2(s_0)g'(-1) = f'(-1),$$

$$t_1(s_0) + t_2(s_0)g'(s_0) = f'(s_0).$$

Thus we can find the value of $t_2(s_0)$:

$$t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}. \quad (1.4.22)$$

So these two values (1.4.22) and (1.4.21) must coincide. In other words we need to show

$$\frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)} = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}. \quad (1.4.23)$$

It will be convenient for us to work with the following notations for the rest of the current subsection. We denote $g(-1) = g_-$, $g'(-1) = g'_-$, $f(-1) = f_-$, $f'(-1) = f'_-$, $g(s_0) = g$, $g'(s_0) = g'$, $f(s_0) = f$, $f'(s_0) = f'$. The condition (1.4.23) is equivalent to

$$\begin{aligned} s_0 &= \frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} + y_p = \\ &= \left(\frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} - 1 \right) + \frac{2}{p}. \end{aligned} \quad (1.4.24)$$

On the other hand, from (1.3.15) for the pair $(-1, s_0)$ we obtain that

$$s_0 = \left(\frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} - 1 \right) + \frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-}.$$

So, from (1.4.24) we see that it suffices to show that

$$\frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-} = \frac{2}{p}.$$

We note that $g'_- = -(p/2)g_-$, $f'_- = -(p/2)f_-$, hence $g'_-f_- = f'_-g_-$. Therefore, we have

$$\frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-} = \frac{f'g_- - g'f_-}{g'f'_- - f'g'_-} = \frac{2}{p}.$$

1.4.3 Concavity in another direction

We are going to check the concavity of the extended function N via \mathbf{B} in another direction. It is worth mentioning that the both of the cases $y_p < s_0$, $y_p \geq s_0$ do not play any role in the following computations, therefore we consider them together. We define a candidate for N as

$$N(y_1, y_2, y_3) \stackrel{\text{def}}{=} y_1^p \mathbf{B}(1, y_2/y_1, y_3/y_1^p) \quad \text{for} \quad \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \in \Omega_3, \quad (1.4.25)$$

and we extend N to the Ω_1 by (1.4.5). Then, as it was already discussed, $N \in C^1(\Omega_1)$. We need the following technical lemma:

Lemma 12.

$$N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = -t'_2 s'_{y_3} p(p-1) y_1^{p-2} \left(st_1 + gt_2 - f + \frac{y_2}{y_1} t_1 \cdot \left(\frac{2}{p} - 1 \right) \right)$$

$$\text{where } s = s \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \text{ and } \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \in \text{int}(\Omega_3) \setminus \text{Ang}(s_0).$$

As it was mentioned in Remark 6, the gradient function $t(s)$ is not necessarily differentiable at point s_0 , this is the reason of the requirement $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \in \text{int}(\Omega_3) \setminus \text{Ang}(s_0)$ in the lemma. However, from the proof of the lemma, the reader can easily see that $N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = 0$ whenever the points $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right)$ belong to the interior of the domain $\text{Ang}(s_0)$.

Proof. Definition of the candidate N (see (1.4.25)) implies $N''_{y_3 y_3} = t'_2(s) s'_{y_3}$, $N''_{y_3 y_1} = t'_2 s'_{y_1}$,

$$N'_{y_1} = y_1^{p-1} \left(p\mathbf{B} \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) - t_1 \frac{y_2}{y_1} - pt_2 \frac{y_3}{y_1^p} \right). \quad (1.4.26)$$

Condition (1.3.8) implies

$$\mathbf{B} \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) = f(s) + t_1 \cdot \left(\frac{y_2}{y_1} - s \right) + t_2 \cdot \left(\frac{y_3}{y_1^p} - g(s) \right).$$

We substitute this expression for $\mathbf{B} \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right)$ into (1.4.26), and we obtain:

$$N'_{y_1} = y_1^{p-1} \left(pf + \frac{y_2}{y_1} t_1 (p-1) - pst_1 - pgt_2 \right). \quad (1.4.27)$$

Condition $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p} \right) \in \text{int}(\Omega_3) \setminus \text{Ang}(s_0)$ implies the equality $N''_{y_1 y_3} = N''_{y_3 y_1}$ which in turn gives

$$t'_2 s'_{y_1} = y_1^{p-1} \left(pf' + \frac{y_2}{y_1} t'_1 (p-1) - (pst_1 + pgt_2)'_s \right) s'_{y_3}.$$

Hence

$$t'_2 \cdot (s'_{y_1})^2 = y_1^{p-1} \left(pf' + \frac{y_2}{y_1} t'_1 (p-1) - (pst_1 + pgt_2)'_s \right) s'_{y_3} s'_{y_1}. \quad (1.4.28)$$

We keep in mind this identity and continue our calculations

$$N''_{y_1 y_1} = (p-1)y_1^{p-2} \left(pf + \frac{y_2}{y_1} t_1 (p-2) - pst_1 - pgt_2 \right) +$$

$$y_1^{p-1} \left(pf' + \frac{y_2}{y_1} t'_1 (p-1) - (pst_1 + pgt_2)'_s \right) s'_{y_1}.$$

So, finally we obtain

$$N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = t'_2 \left(N''_{y_1 y_1} s'_{y_3} - t'_2 (s'_{y_1})^2 \right).$$

Now we use the identity (1.4.28), and we substitute the expression $t'_2 (s'_{y_1})^2$:

$$N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = t'_2 s'_{y_3} \left(N''_{y_1 y_1} - y_1^{p-1} \left(pf' + \frac{y_2}{y_1} t'_1 (p-1) - (pst_1 + pgt_2)'_s \right) s'_{y_1} \right) =$$

$$t'_2 s'_{y_3} (p-1) y_1^{p-2} \left(pf + \frac{y_2}{y_1} t_1 (p-2) - pst_1 - pgt_2 \right) =$$

$$- t'_2 s'_{y_3} p (p-1) y_1^{p-2} \left(st_1 + gt_2 - f + \frac{y_2}{y_1} t_1 \cdot \left(\frac{2}{p} - 1 \right) \right).$$

□

Now we are going to consider several cases when the points $(y_2/y_1, y_3/y_1^p)$ belong to the different subdomains in Ω_3 . Note that we always have $N''_{y_3 y_3} \leq 0$, because of the fact that \mathbf{B} is concave in Ω_3 and (1.4.25). So we only have to check that the determinant of the Hessian N is negative. If the determinant of the Hessian is zero, then it is sufficient to ensure that $N''_{y_3 y_3}$ is strictly negative, and if $N''_{y_3 y_3}$ is also zero, then we need to ensure that N''_{y_1, y_1} is nonpositive.

Domain $\Omega(\Theta[s_0, y_p])$.

In this case we can use the equality (1.4.19), and we obtain that

$$st_1 + gt_2 - f = y_p t_1.$$

Therefore

$$N''_{y_1 y_1} N''_{y_3 y_3} - (N''_{y_1 y_3})^2 = -t'_2 s'_{y_3} p(p-1) y_1^{p-2} t_1 y_p \left(1 + \frac{y_2}{y_1}\right) \geq 0.$$

because $t_1 \geq 0$. Indeed, $t_1(s)$ is continuous on $[c, 1]$, where c is the root of the cup and $\mathbf{B}''_{y_2 y_2} = t'_1 s'_{y_2} \leq 0$, therefore, because of the fact $s'_{y_2} > 0$, it suffices to check that $t_1(1) \geq 0$ which follows from the following inequality

$$t_1(1) = f'(1) - t_2(1)g'(1) = f'(1) > 0.$$

Domain of linearity $\text{Ang}(s_0)$.

This is the domain which is obtained by the triangle ABC , where $A = (-1, g(-1))$, $B = (s_0, g(s_0))$, and $C = (-1, h(s_0))$ if $s_0 < y_p$ and by the infinity domain of linearity, which is rectangular type, and which lies between the chords AB , BC' , where $C' = (s_0, +\infty)$ and AC'' , where $C'' = (-1, +\infty)$ (see Figure 1.11).

Suppose the points $(y_2/y_1, y_3/y_1^p)$ belong to the interior of $\text{Ang}(s_0)$. Then the gradient function $t(s)$ of \mathbf{B} is constant, and moreover $s\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right)$ is constant. The fact that the determinant of the Hessian is zero in the domain of linearity (note that $s'_{y_3} = 0$) implies that

we only need to check $N''_{y_1 y_1} < 0$. Equality (1.4.27) implies

$$\begin{aligned} N''_{y_1 y_1} &= (p-1)y_1^{p-2} \left(pf + \frac{y_2}{y_1} t_1 (p-2) - ps_0 t_1 - pgt_2 \right) \leq \\ &(p-1)y_1^{p-2} (pf - ps_0 t_1 - pgt_2 - t_1 (p-2)) = 0. \end{aligned}$$

The last equality follows from (1.4.19). The above inequality turns into the equality if and only if $\frac{y_2}{y_1} = s_0$, this is the boundary point of $\text{Ang}(s_0)$.

Domain of vertical segments.

On the vertical segments determinant of the Hessian is zero (for example, because the vertical segment is vertical segment in all directions) and $\mathbf{B}''_{y_3 y_3} = 0$, therefore, we must check that $N''_{y_1 y_1} \leq 0$. We note that $s(y_2, y_3) = y_2$, therefore,

$$N''_{y_1 y_1} = y_1^{p-2} \times \left[(p-1) (pf + st_1 (p-2) - pst_1 - pgt_2) - s (pf' - t'_1 s - t_1 p - pg't_2) \right].$$

However, from (1.3.6) we have $pf' - t_1 p - pg't_2 = 0$, therefore,

$$N''_{y_1 y_1} = y_1^{p-2} \times \left[(p-1) (pf - 2st_1 - pgt_2) + s^2 t'_1 \right].$$

Condition $t'_1 \leq 0$ implies that it is sufficient to show $pf - 2st_1 - pgt_2 \leq 0$. We use (1.3.6), and we find $t_1 = f' - g't_2$. Hence,

$$pf - 2st_1 - pgt_2 = pf - gpt_2 - 2s(f' - g't_2) = pf - 2sf' - t_2(gp - 2sg').$$

Note that $gp - 2sg' \geq 0$ (because $s \geq 0$ and $g' \leq 0$), and we recall that from (1.3.6) and

the fact that on the vertical segments t_2 is constant, since we have $\cos \theta(s) = 0$ (see the expression of t_2 from Lemma 2), so t_2 is constant and hence $0 \geq t'_1 = f'' - g''t_2$, therefore, we have $t_2 \geq f''/g''$. Therefore,

$$pf - 2sf' - t_2(gp - 2sg') \leq pf - 2sf' - \frac{f''}{g''}(gp - 2sg').$$

Now we recall the values (1.4.12), (1.4.11), and after direct calculations we obtain

$$pf - 2sf' - \frac{f''}{g''}(gp - 2sg') = \frac{f(1-s^2)p(p-2)(\tau^2(1+s)^2 + (1-s)^2 + 2\tau^2(1-s^2))}{(p-1)((1+s)^2 + \tau^2(1-s)^2)^2} \leq 0.$$

Domain of the cup $\Omega(\Theta_{\text{cup}}((c, s_0], g))$.

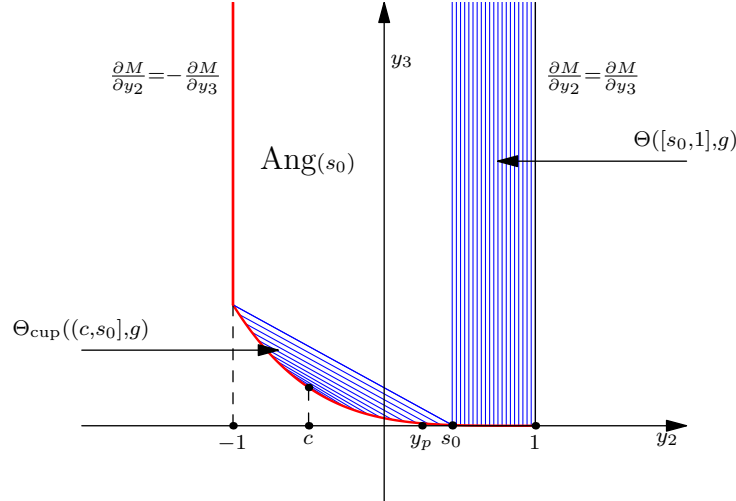


Figure 1.11 Case $u\left(\frac{1}{p-1}\right) \geq 0$

The condition that $N''_{y_3 y_3}$ is strictly negative in the cup implies that we only need to show $st_2 + gt_3 - f + \frac{y_2}{y_1}t_1(\frac{2}{p} - 1) \geq 0$, where $s = s(y_2/y_1, y_3/y_1^p)$ and the points $y = (y_2/y_1, y_3/y_1^p)$ lie in the cup. We can think that $y_2/y_1 \rightarrow y_2$ and $y_3/y_1^p \rightarrow y_3$ and $s(y_2/y_1, y_3/y_1^p) \rightarrow s(y_2, y_3)$, and we can think that the points (y_2, y_3) lie in the cup. Therefore it suffices to show that

$st_2 + gt_3 - f + y_2t_1(\frac{2}{p} - 1) \geq 0$, where $y = (y_2, y_3) \in \Omega(\Theta_{\text{cup}}((c, s_0], g))$. On a segment with the fixed endpoint $(s, g(s))$ the expressions $s, t_1, g(s), t_2, f(s)$ are constant, except of y_2 , so the expression $st_1 + gt_2 - f + y_2t_1(\frac{2}{p} - 1)$ is linear with respect to the y_2 on the each segment of the cup. Therefore, the worst case appears when $y_2 = a(s)$ ($a(s)$ – is the left end (it is abscissa) of the given segment). This is true because $t_1 \geq 0$ (as it was already shown) and $(\frac{2}{p} - 1) \geq 0$. So, as the result, we derive that it is sufficient to prove the inequality

$$st_1 + gt_2 - f + a(s)t_1 \cdot \left(\frac{2}{p} - 1\right) = t_1(s - a(s)) + gt_2 - f + \frac{2a(s)}{p}t_1 \geq 0. \quad (1.4.29)$$

We use the identity (1.3.8) at the point $y = (a(s), g(a(s)))$, and we find that

$$t_1(s - a(s)) + gt_2 - f = g(a(s))t_2 - f(a(s)).$$

We substitute this expression into (1.4.29) then we will get that it suffices to prove the inequality:

$$g(a(s))t_2 - f(a(s)) + \frac{2a(s)}{p}t_1 \geq 0. \quad (1.4.30)$$

We differentiate condition $\mathbf{B}(a(s), g(a(s))) = f(s)$ with respect to s . Then we find the expression for $t_1(s)$, namely $t_1(s) = f'(a(s)) - t_2(s)g'(a(s))$. After substituting this expression into (1.4.30) we obtain that:

$$g(a(s))t_2 - f(a(s)) + \frac{2a(s)}{p}t_1 = \frac{1+z}{g'(z)} \left(\frac{(z-1)(\tau^2+1)f(z)}{((1+z)^2 + \tau^2(1-z)^2)g'(z)} - t_2(s) \right),$$

where $z = a(s)$. So it suffices to show that

$$\frac{(z-1)(\tau^2+1)f(z)}{((1+z)^2+\tau^2(1-z)^2)g'(z)} - t_2(s) \leq 0 \quad (1.4.31)$$

because g' is negative. We are going to show that the condition (1.4.31) is sufficient to check at the point $z = -1$. Indeed, note that $(t_2)'_z \geq 0$ on $[-1, c]$, where c is the root of the cup, and also note that

$$\begin{aligned} & \left(\frac{(z-1)(\tau^2+1)f}{((1+z)^2+\tau^2(1-z)^2)g'} \right)'_z = \\ & \frac{\tau^2+1}{p}(p-2)(1-z)^{-(p-1)}[(1+z)^2+\tau^2(1-z)^2]^{p/2-2}2(1+z) \leq 0. \end{aligned}$$

The condition (1.4.31) at the point $z = -1$ turns into the following condition

$$t_2(s_0) - \frac{\tau^{p-2}(\tau^2+1)}{p} \geq 0.$$

Now we recall (1.3.21) and $t_2(s_0) = (f'(-1) - f'(s_0))/(g'(-1) - g'(s_0))$, therefore we have

$$t_2(s_0) - \frac{\tau^{p-2}(\tau^2+1)}{p} \geq \frac{f''(-1)}{g''(-1)} - \frac{\tau^{p-2}(\tau^2+1)}{p} = \frac{\tau^p(p-1)^2 + \tau^{p-2}}{p(p-1)} > 0.$$

Thus we finish this section by the following remark.

Remark 11. *We still have to check the cases when the points $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right)$ belong to the boundary of $\text{Ang}(s_0)$ and the vertical rays $y_2 = \pm 1$ in Ω_3 . The reader can easily see that in this case concavity of N follows from the observation that $N \in C^1(\Omega_1)$. Symmetry of N covers the rest of the cases when $\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \notin \Omega_3$.*

Thus we have constructed the candidate N .

1.5 Sharp constants via foliation

1.5.1 Main theorem

We remind the reader the definition of the functions $u(z)$, $g(s)$, $f(s)$ (see (1.4.15), (1.4.10), (1.4.11)), the value $y_p = -1 + 2/p$ and the definition of the function $a(s)$ (see Lemma 9, Lemma 11 and Remark 8).

Theorem 1.5.1. *Let $1 < p < 2$, and let G be the martingale transform of F and let $|\mathbb{E}G| \leq \beta|\mathbb{E}F|$. Set $\beta' = \frac{\beta-1}{\beta+1}$.*

(i) *If $u\left(\frac{1}{p-1}\right) \leq 0$ then*

$$\mathbb{E}(\tau^2 F^2 + G^2)^{p/2} \leq \left(\tau^2 + \max \left\{ \beta, \frac{1}{p-1} \right\}^2 \right)^{\frac{p}{2}} \mathbb{E}|F|^p. \quad (1.5.1)$$

(ii) *If $u\left(\frac{1}{p-1}\right) > 0$ then*

$$\mathbb{E}(\tau^2 F^2 + G^2)^{p/2} \leq C(\beta') \mathbb{E}|F|^p,$$

where $C(\beta')$ is continuous, nondecreasing, and it is defined by the following way:

$$C(\beta') \stackrel{\text{def}}{=} \begin{cases} (\tau^2 + \beta^2)^{p/2}, & \beta' \geq s^*; \\ \frac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}, & \beta' \leq -1 + \frac{2}{p}; \\ \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}, & R(s_1, \beta') = 0 \text{ for } s_1 \in (\beta', s_0); \end{cases}$$

where $s_0 \in (-1 + 2/p, 1)$ is the solution of the equation $u\left(\frac{1+s_0}{1-s_0}\right) = 0$, and the function $R(s, z)$ is defined as follows

$$R(s, z) \stackrel{\text{def}}{=} -f(s) - \frac{f'(a(s))g'(s) - f'(s)g'(a(s))}{g'(s) - g'(a(s))} (z - s) + \frac{f'(s) - f'(a(s))}{g'(s) - g'(a(s))} g(s) = 0, \quad z \in [-1 + 2/p, s^*], \quad s \in [z, s_0].$$

The value $s^* \in [-1 + 2/p, s_0]$ is the solution of the equation

$$\frac{f'(s^*) - f'(a(s^*))}{g'(s^*) - g'(a(s^*))} = \frac{f(s^*)}{g(s^*)}. \quad (1.5.2)$$

Proof. Before we investigate some of the cases mentioned in the theorem, we should make the following observation. The inequality of the type (1.5.1) can be restated as follows

$$H(x_1, x_2, x_3) \leq Cx_3, \quad (1.5.3)$$

where H is defined by (1.2.2) and $x_1 = \mathbb{E}F$, $x_2 = \mathbb{E}G$, $x_3 = \mathbb{E}|F|^p$. In order to derive the

estimate (1.5.1) we have to find the sharp C in (1.5.3). Because of the property (1.4.1) we can assume that both of the values x_1, x_2 are nonnegative. So non-negativity of x_1, x_2 and the condition $|\mathbb{E}G| \leq \beta|\mathbb{E}F|$ can be reformulated as follows

$$-\frac{x_1 + x_2}{2} \leq \frac{x_2 - x_1}{2} \leq \left(\frac{\beta - 1}{\beta + 1}\right) \left(\frac{x_1 + x_2}{2}\right). \quad (1.5.4)$$

The condition (1.5.4) with (1.5.3) in terms of the function N and the variables y_1, y_2, y_3 means that we have to find the sharp C such that

$$N(y_1, y_2, y_3) \leq Cy_3 \quad \text{for} \quad -y_1 \leq y_2 \leq \left(\frac{\beta - 1}{\beta + 1}\right) y_1, \quad \mathbf{y} \in \Omega_2.$$

Because of (1.4.9) the above condition can be reformulated as follows

$$\mathbf{B}(y_2, y_3) \leq Cy_3 \quad \text{for} \quad -1 \leq y_2 \leq \left(\frac{\beta - 1}{\beta + 1}\right), \quad y_3 \geq g(y_2), \quad (1.5.5)$$

where $\mathbf{B}(y_2, y_3) = N(1, y_2, y_3)$. So our aim is to find the sharp C , in other words the value $\sup_{y_1, y_2} \mathbf{B}/y_3$ where the supremum is taken from the domain mentioned in (1.5.5). Note that the quantity $\mathbf{B}(y_2, y_3)/y_3$ increases with respect to the variable y_2 . Indeed, $(\mathbf{B}(y_2, y_3)/y_3)'_{y_2} = t_1(s(y))/y_3$, where the function $t_1(s)$ is nonnegative on $[c_0, 1]$ (see the end of the proof of the concavity condition in the domain $\Omega(\Theta[s_0, y_p])$). Note that as we increase the value y_2 then the range of y_3 also increases. This means that the supremum of the expression \mathbf{B}/y_3 is attained on the subdomain where $y_2 = (\beta - 1)/(\beta + 1)$. It is worth mentioning that since the quantity $(\beta - 1)/(\beta + 1) \in [-1, 1]$ increases as β increases and because of the observation made above we see that the value C increases as the β' increases.

1.5.2 Case $y_p \leq s_0$.

We are going to investigate the simple case (i). Remark 10 implies that $s_0 \leq y_p$, in other words, the foliation of vertical segments is $\Theta([y_p, 1], g)$ where the value $\theta(s)$ on $[y_p, 1]$ is equal to $\pi/2$. This means that $t_2(s)$ is constant on $[y_p, 1]$ (see Lemma 2), and it is equal to $f(y_p)/g(y_p) = (\tau^2 + \frac{1}{(p-1)^2})^{p/2}$ (see (1.4.20)).

If $\frac{\beta-1}{\beta+1} \geq y_p$, or equivalently $\beta \geq \frac{1}{p-1}$, then the function \mathbf{B} on the vertical segment with the endpoint $(\beta', g(\beta'))$ where $\frac{\beta-1}{\beta+1} = \beta' \in [y_p, 1]$ has the expression (see (1.3.8))

$$\mathbf{B}(\beta', y_3) = f(\beta') + \frac{f(y_p)}{g(y_p)}(y_3 - g(\beta')), \quad y_3 \geq g(\beta').$$

Therefore,

$$\frac{\mathbf{B}(\beta', y_3)}{y_3} = \frac{f(y_p)}{g(y_p)} + \frac{g(\beta')}{y_3} \left(\frac{f(\beta')}{g(\beta')} - \frac{f(y_p)}{g(y_p)} \right), \quad y_3 \geq g(\beta'). \quad (1.5.6)$$

The expression $f(s)/g(s)$ is strictly increasing on $(-1, 1)$, therefore, the expression (1.5.6) attains its maximal value at the point $y_3 = g(\beta')$. Thus, we have

$$\begin{aligned} \frac{\mathbf{B}(y_2, y_3)}{y_3} &\leq \frac{\mathbf{B}(\beta', y_3)}{y_3} \leq \frac{\mathbf{B}(\beta', g(\beta'))}{g(\beta')} = \frac{f(\beta')}{g(\beta')} = \left(\tau^2 + \beta^2 \right)^{p/2} \\ &\text{for } -1 \leq y_2 \leq \beta', \quad y_3 \geq g(y_2). \end{aligned}$$

If $\frac{\beta-1}{\beta+1} < y_p$, or equivalently $\beta < \frac{1}{p-1}$, then we can achieve such value for C which was achieved at the moment $\beta = \frac{1}{p-1}$, and since the function $C = C(\beta')$ increases as β' increases this value will be the best. Indeed, it suffices to look at the foliation (see Figure 1.10). For

any fixed y_2 we send y_3 to $+\infty$, and we obtain that

$$\begin{aligned} \lim_{y_3 \rightarrow \infty} \frac{\mathbf{B}(y_2, y_3)}{y_3} &= \lim_{y_3 \rightarrow \infty} \frac{f(s) + t_1(s)(y_2 - s) + t_2(s)(y_3 - g(s))}{y_3} = \\ \lim_{y_3 \rightarrow \infty} t_2(s(y)) &= t_2(y_p) = \left(\tau^2 + \frac{1}{(p-1)^2} \right)^{p/2}. \end{aligned}$$

1.5.3 Case $y_p > s_0$.

As it was already mentioned, the condition in the case (ii) is equivalent to the inequality $s_0 > y_p$ (see Remark 10). This means that the foliation of the vertical segments is $\Theta([s_0, 1], g)$ (see Figure 1.11). We know that $C(\beta')$ is increasing. We remind that we are going to maximize the function $\frac{\mathbf{B}(y_2, y_3)}{y_3}$ on the domain mentioned in (1.5.5). It was already mentioned that we can require $y_2 = \left(\frac{\beta-1}{\beta+1} \right) = \beta'$. For such fixed $y_2 = \beta' \in [-1, 1]$ we are going to investigate the monotonicity of the function $\frac{\mathbf{B}(\beta', y_3)}{y_3}$. We consider several cases. Let $\beta' \geq s_0$. We differentiate the function $\mathbf{B}(\beta', y_3)/y_3$ with respect to the variable y_3 , and we use the expression (1.3.8) for \mathbf{B} , so we obtain that

$$\frac{\partial}{\partial y_3} \left(\frac{\mathbf{B}(\beta', y_3)}{y_3} \right) = \frac{t_2(\beta')y_3 - \mathbf{B}(\beta', y_3)}{y_3^2} = \frac{-f(\beta') + t_2(\beta')g(\beta')}{y_3^2}.$$

Recall that $t_2(s) = t_2(s_0)$ for $s \in [s_0, 1]$, therefore, direct calculations imply

$$t_2(\beta') = \frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)} < \frac{f(s_0)}{g(s_0)} \leq \frac{f(\beta')}{g(\beta')}, \quad \beta' \geq s_0.$$

This implies that

$$C(\beta') = \sup_{y_3 \geq g(\beta')} \frac{\mathbf{B}(\beta', y_3)}{y_3} = \frac{\mathbf{B}(\beta', y_3)}{y_3} \Big|_{y_3=g(\beta')} = \frac{f(\beta')}{g(\beta')} = (\tau^2 + \beta^2)^{p/2}.$$

Now we consider the case $\beta' < s_0$.

For each point $y = (\beta', y_3)$ that belongs to the line $y_2 = \beta'$ there exists a segment $\ell(y) \in \Theta((c, s_0], g)$ with the endpoint $(s, g(s))$ where $s \in [\max\{\beta', a(\beta')\}, s_0]$. If the point y belongs to the domain of linearity $\text{Ang}(s_0)$, then we can choose the value s_0 , and consider any segment with the endpoints y and $(s_0, g(s_0))$ which surely belongs to the domain of linearity. The reader can easily see that as we increase the value y_3 the value s increases as well. So,

$$\frac{\partial}{\partial y_3} \left(\frac{\mathbf{B}(\beta', y_3)}{y_3} \right) = \frac{t_2(s)y_3 - \mathbf{B}(\beta', y_3)}{y_3^2} = \frac{-f(s) - t_1(s)(\beta' - s) + t_2(s)g(s)}{y_3^2}.$$

Our aim is to investigate the sign of the expression $-f(s) - t_1(s)(\beta' - s) + t_2(s)g(s)$ as we variate the value $y_3 \in [g(\beta'), +\infty)$. Without loss of generality we can forget about the variable y_3 , and we can variate only the value s on the interval $[\max\{\alpha(\beta'), \beta'\}, s_0]$.

We consider the function $R(s, z) \stackrel{\text{def}}{=} -f(s) - t_1(s)(z - s) + t_2(s)g(s)$ with the following domain $-1 \leq z \leq s_0$ and $s \in [\max\{\alpha(z), z\}, s_0]$ (see Figure 1.12). As we already have seen $R(s_0, s_0) < 0$. Note that $R(s_0, -1) > 0$. Indeed, note that $R(s_0, -1) = t_2(s_0)g(-1) - f(-1)$.

This equality follows from the fact that

$$f(s_0) - f(-1) = t_1(s_0)(s_0 + 1) + t_2(s_0)(g(s_0) - g(-1)),$$

which is consequence of Lemma 10. So, (1.4.22) and (1.3.21) imply

$$t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} > \frac{f''(-1)}{g''(-1)} \geq \frac{f(-1)}{g(-1)}.$$

Note that the function $R(z, s_0)$ is linear with respect to z . So on the interval $[-1, s_0]$ it has the root $y_p = -1 + 2/p$. Indeed,

$$\frac{-f(s_0) + t_2(s_0)g(s_0) + t_1(s_0)s_0}{t_1(s_0)} = y_p.$$

The last equality follows from (1.4.22), (1.4.24) and (1.3.6). We need few more properties of the function $R(s, z)$. Note that for each fixed z , the function $R(s, z)$ is nonincreasing on $[\max\{\alpha(z), z\}, s_0]$. Indeed

$$R'_s(s, z) = -f'(s) - t'_1(s)(z - s) + t_1(s) + t'_2(s)g(s) + t_2(s)g'(s). \quad (1.5.7)$$

We take into account the condition (1.3.6), so the expression (1.5.7) simplifies as follows

$$R'_s(s, z) = t'_2(s)g(s) + t'_1(s)(s - z).$$

We remind the reader equality (1.3.5) and the fact that $t'_2(s) \leq 0$. Therefore we have $R'_s(s, z) = y_3 t'_2(s)$ where $y_3 = y_3(s) > 0$. Thus we see that $R(s, \beta') \geq 0$ for $\beta' \leq y_p$.

So if the function $R(\cdot, z)$ at the right end on its domain $[\max\{\alpha(z), z\}, s_0]$ is positive, this will mean that the function \mathbf{B}/y_3 is increasing, hence, the constant $C(\beta')$ will be equal to

$$C(\beta') = \lim_{y_3 \rightarrow \infty} \frac{\mathbf{B}(z, y_3)}{y_3} = t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}$$

(this follows from (1.4.22) and the structure of the foliation). Since $u\left(\frac{1+s_0}{1-s_0}\right) = 0$ and (1.4.23) direct computations show that

$$\frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} = \frac{\tau^p}{1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}}. \quad (1.5.8)$$

So it follows that if $\beta' \leq y_p$ then (1.5.8) is the value of $C(\beta')$.

If the function $R(\cdot, z)$ on the left end of its domain is nonpositive this will mean that the function \mathbf{B}/y_3 is decreasing, so the sharp constant will be the value of the function $\mathbf{B}(z, y_3)/y_3$ at the left end of its domain

$$C(\beta') = \frac{\mathbf{B}(z, y_3)}{y_3} \Big|_{y_3=g(z)} = \frac{f(z)}{g(z)} = (\tau^2 + \beta^2)^{p/2}. \quad (1.5.9)$$

We recall that c is the root of the cup and $c < y_p$ (see Remark 9). We will show that the function $R(z, s)$ is decreasing on the boundary $s = z$ for $s \in (y_p, s_0]$. Indeed, (1.3.6) implies

$$(R(s, s))'_s = -f'(s) + t'_2(s)g(s) + t_2(s)g'(s) = -t_1(s) + t'_2(s)g(s) < 0.$$

The last inequality follows from the fact that $t'_2(s) \leq 0$ and $t_1(s) > 0$ on $(c, 1]$. Surely $R(y_p, y_p) > R(s_0, y_p) = 0$, and we recall that $R(s_0, s_0) < 0$, so there exists unique $s^* \in [y_p, s_0]$ such that $R(s^*, s^*) = 0$. This is equivalent to (1.5.2). So it is clear that $R(z, z) \leq 0$ for $z \in [s^*, s_0]$. Therefore $C(\beta')$ has the value (1.5.9) for $\beta' \geq s^*$.

The only case remains is when $\beta' \in [y_p, s^*]$. We know that $R(z, z) \geq 0$ for $z \in [y_p, s^*]$ and $R(s_0, z) \leq 0$ for $z \in [y_p, s^*]$. The fact that for each fixed z the function $R(s, z)$ is decreasing implies the following: for each $z \in [y_p, s^*]$ there exists unique $s_1(z) \in [z, s_0]$ such

that $R(z, s_1(z)) = 0$. Therefore, for $\beta' \in [y_p, s^*]$ we have

$$C(\beta') = \frac{\mathbf{B}(\beta', y_3(s_1(\beta')))}{y_3(s_1(\beta'))}, \quad (1.5.10)$$

where the value $s_1(\beta')$ is the root of the equation $R(s_1(\beta'), \beta') = 0$. Recall that

$$R(s_1(\beta'), \beta') = t_2(s_1)y_3(s_1) - \mathbf{B}(\beta', y_3(s_1)) = -f(s_1) - t_1(s_1)(\beta' - s_1) + t_2(s_1)g(s_1). \quad (1.5.11)$$

So the expression (1.5.10) takes the form

$$C(\beta') = t_2(s_1) = \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}.$$

Finally, we remind the reader that

$$\begin{aligned} t_2(s) &= \frac{f'(s) - f'(a(s))}{g'(s) - g'(a(s))}, \\ t_1(s) &= \frac{f'(a(s))g'(s) - f'(s)g'(a(s))}{g'(s) - g'(a(s))}. \end{aligned}$$

for $s \in (c, s_0]$, and we finish the proof of the theorem. \square

1.6 Extremizers via foliation

We set $\Psi(F, G) = \mathbb{E}(G^2 + \tau^2 F^2)^{2/p}$. Let N be the candidate that we have constructed in Section 1.4 (see (1.4.25)). We define the candidate B for the Bellman function H (see

(1.2.2)) as follows

$$B(x_1, x_2, x_3) = N\left(\frac{x_1 + x_2}{2}, \frac{x_2 - x_1}{2}, x_3\right), \quad (x_1, x_2, x_3) \in \Omega.$$

We want to show that $B = H$. We already know that $B \geq H$ (see Lemma 3). So, it remains to show that $B \leq H$. We are going to do this as follows: for each point $\mathbf{x} \in \Omega$ and any $\varepsilon > 0$ we are going to find an admissible pair (F, G) such that

$$\Psi(F, G) > B(\mathbf{x}) - \varepsilon. \quad (1.6.1)$$

Up to the end of the current section we are going to work with the coordinates (y_1, y_2, y_3) (see (1.4.4)). It will be convenient for us to redefine the notion of admissibility of the pair.

Definition 7. *We say that a pair (F, G) is admissible for the point $(y_1, y_2, y_3) \in \Omega_1$, if G is the martingale transform of F and $\mathbb{E}(F, G, |F|^p) = (y_1 - y_2, y_1 + y_2, y_3)$.*

So in this case condition (1.6.1) in terms of the function N takes the following form: for any point $\mathbf{y} \in \Omega_1$ and for any $\varepsilon > 0$ we are going to find an admissible pair (F, G) for the point \mathbf{y} such that

$$\Psi(F, G) > N(\mathbf{y}) - \varepsilon. \quad (1.6.2)$$

We formulate the following obvious observations.

Lemma 13. *The following statements hold:*

1. *A pair (F, G) is admissible for the point $\mathbf{y} = (y_1, y_2, y_3)$ if and only if $(\tilde{F}, \tilde{G}) = (\pm F, \mp G)$ is admissible for the point $\tilde{\mathbf{y}} = (\mp y_2, \mp y_1, y_3)$; moreover, $\Psi(\tilde{F}, \tilde{G}) = \Psi(F, G)$.*

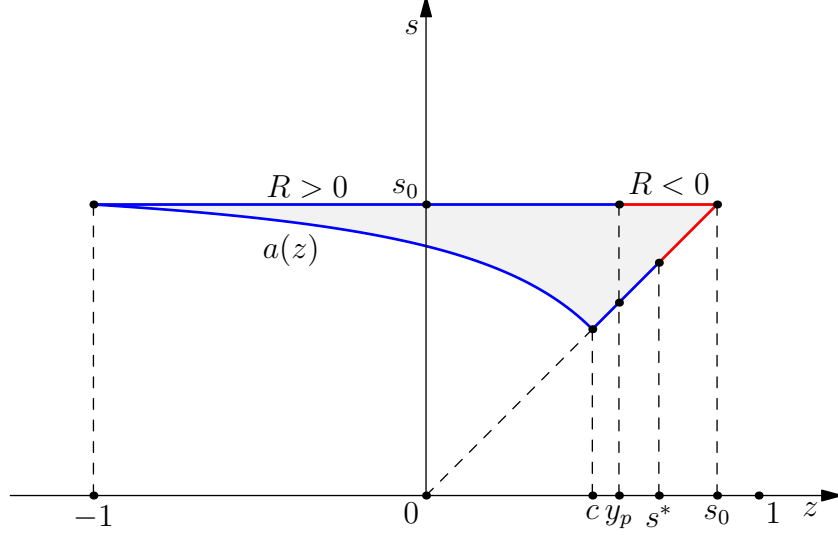


Figure 1.12 Domain of $R(s, z)$

2. A pair (F, G) is admissible for the point $\mathbf{y} = (y_1, y_2, y_3)$, if and only if $(\tilde{F}, \tilde{G}) = (\lambda F, \lambda G)$ (where $\lambda \neq 0$) is admissible for the point $\tilde{\mathbf{y}} = (\lambda y_1, \lambda y_2, |\lambda|^p y_3)$; moreover, $\Psi(\tilde{F}, \tilde{G}) = |\lambda|^p \Psi(F, G)$.

Definition 8. The pair of functions (F, G) is called an ε -extremizer for the point $\mathbf{y} \in \Omega_1$ if (F, G) is admissible for the point \mathbf{y} and $\Psi(F, G) > N(\mathbf{y}) - \varepsilon$.

Lemma 13, homogeneity, and symmetry of N imply that we only need to check (1.6.2) for the points $\mathbf{y} \in \Omega_1$ where $y_1 = 1$ ($y_2, y_3 \in \Omega_3$). In other words, we show that $\Psi(F, G) > \mathbf{B}(y_2, y_3) - \varepsilon$ for some admissible (F, G) for the point $(1, y_2, y_3)$ where $(y_2, y_3) \in \Omega_3$. Further, instead of saying that (F, G) is an admissible pair (or ε -extremizer) for the point $(1, y_2, y_3)$ we just say that it is an admissible pair (or an ε -extremizer) for the point (y_2, y_3) . So we only have to construct ε -extremizers in the domain Ω_3 .

It is worth mentioning that we construct ε -extremizers (F, G) such that G will be the martingale transform of F with respect to some filtration other than dyadic. A detailed explanation on how to pass from one filtration to another the reader can find in [19].

We need a few more observations. For $\alpha \in (0, 1)$ we define the α -concatenation of the pairs (F, G) and (\tilde{F}, \tilde{G}) as follows

$$(F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha(x) = \begin{cases} (F, G)(x/\alpha) & x \in [0, \alpha], \\ (\tilde{F}, \tilde{G})((x - \alpha)/(1 - \alpha)) & x \in [\alpha, 1]. \end{cases}$$

Clearly $\Psi((F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha(x)) = \alpha\Psi(F, G) + (1 - \alpha)\Psi(\tilde{F}, \tilde{G})$.

Definition 9. Any domain of the type $\Omega_1 \cap \{y_1 = A\}$ where A is some real number is said to be a positive domain. Any domain of the type $\Omega_1 \cap \{y_2 = B\}$ where B is some real number is said to be a negative domain.

The following lemma is obvious.

Lemma 14. If (F, G) is an admissible pair for a point \mathbf{y} and (\tilde{F}, \tilde{G}) is an admissible pair for a point $\tilde{\mathbf{y}}$ such that either of the following is true: $\mathbf{y}, \tilde{\mathbf{y}}$ belong to a positive domain, or $\mathbf{y}, \tilde{\mathbf{y}}$ belong to a negative domain, then $(F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha$ is an admissible pair for the point $\alpha\mathbf{y} + (1 - \alpha)\tilde{\mathbf{y}}$.

Let (F, G) be an admissible pair for a point \mathbf{y} , and let (\tilde{F}, \tilde{G}) be an admissible pair for a point $\tilde{\mathbf{y}}$. Let $\hat{\mathbf{y}}$ be a point which belongs to the chord joining the points \mathbf{y} and $\tilde{\mathbf{y}}$.

Remark 12. It is clear that if (F^+, G^+) is admissible for a point (y_2^+, y_3^+) and (F^-, G^-) is admissible for a point (y_2^-, y_3^-) then an α -concatenation of these pairs is admissible for the point $(y_2, y_3) = \alpha \cdot (y_2^+, y_3^+) + (1 - \alpha) \cdot (y_2^-, y_3^-)$.

Now we are ready to construct ε -extremizers in Ω_3 . The main idea is that these functions Ψ and \mathbf{B} are very similar: they obey almost the same properties. Moreover, foliation plays crucial role in the contraction of ε -extremizers.

1.6.1 Case $s_0 \leq y_p$.

We want to find ε -extremizers for the points in Ω_3 .

Extremizers in the domain $\Omega(\Theta_{\text{cup}}((c, s_0], g))$.

Pick any $y = (y_2, y_3) \in \Omega(\Theta_{\text{cup}}((c, s_0], g))$. Then there exists a segment $\ell(y) \in \Theta_{\text{cup}}((c, s_0], g)$. Let $y^+ = (s, g(s))$ and $y^- = (a(s), g(a(s)))$ be the endpoints of $\ell(y)$ in Ω_3 . We know ε -extremizers at these points y^+, y^- . Indeed, we can take the following ε -extremizers $(F^+, G^+) = (1 - s, 1 + s)$ and $(F^-, G^-) = (1 - a(s), 1 + a(s))$ (i.e. constant functions). Consider an α -concatenation $(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha$, where α is chosen so that $y = \alpha y^+ + (1 - \alpha)y^-$. We have

$$\begin{aligned} \Psi[(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha] &= \alpha \Psi(F^+, G^+) + (1 - \alpha) \Psi(F^-, G^-) > \\ \alpha \mathbf{B}(y^+) + (1 - \alpha) \mathbf{B}(y^-) - \varepsilon &= \mathbf{B}(y) - \varepsilon. \end{aligned}$$

The last equality follows from the linearity of \mathbf{B} on $\ell(y)$.

Extremizers on the vertical line $(-1, y_3)$, $y_3 \geq h(s_0)$.

Now we are going to find ε -extremizers for the points $(-1, y_3)$ where $y_3 \geq h(s_0)$. We use a similar idea mentioned in [20] (see proof of Lemma 3). We define the functions (F, G) recursively:

$$G(t) = \begin{cases} -w & 0 \leq t < \varepsilon; \\ \gamma \cdot g\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & \varepsilon \leq t \leq 1 - \varepsilon; \\ w & 1 - \varepsilon < t \leq 1; \end{cases}$$

$$F(t) = \begin{cases} d_- & 0 \leq t < \varepsilon; \\ \gamma \cdot f\left(\frac{t-\varepsilon}{1-2\varepsilon}\right) & \varepsilon \leq t \leq 1-\varepsilon; \\ d_+ & 1-\varepsilon < t \leq 1; \end{cases}$$

where the nonnegative constants w, d_-, d_+, γ will be obtained from the requirement $\mathbb{E}(F, G, |F|^p) = (2, 0, y_3)$ and the fact that G is the martingale transform of F . Surely $\langle G \rangle_{[0,1]} = 0$. Condition $\langle F \rangle_{[0,1]} = 2$ means that

$$(d_- + d_+)\varepsilon + 2\gamma(1-2\varepsilon) = 2. \quad (1.6.3)$$

Condition $\langle |F|^p \rangle_{[0,1]} = y_3$ implies that

$$y_3 = \frac{\varepsilon(d_+^p + d_-^p)}{1 - (1-2\varepsilon)\gamma^p}. \quad (1.6.4)$$

Now we use the condition $|F_0 - F_1| = |G_0 - G_1|$. In the first step we split the interval $[0, 1]$ at the point ε with the requirement $F_0 - F_1 = G_0 - G_1$, from which obtain $w = 2 - d_-$. In the second step we split at the point $1 - \varepsilon$ with the requirement $F_1 - F_2 = G_2 - G_1$, obtaining $w = 2\gamma - d_+$. From these two conditions we obtain $d_- + d_+ = 2(1 + \gamma) - 2w$, and by substituting in (1.6.3) we find the γ

$$\gamma = 1 + \frac{\varepsilon w}{1 - \varepsilon}.$$

Now we investigate what happens as ε tends to zero. Our aim will be to focus on the limit

value $\lim_{\varepsilon \rightarrow 0} w = w_0$. We have $1 - (1 - 2\varepsilon)\gamma^p \approx \varepsilon(2 - wp)$. So (1.6.4) becomes

$$y_3 = \frac{\varepsilon(d_+^p + d_-^p)}{1 - (1 - 2\varepsilon)\gamma^p} \rightarrow \frac{2(2 - w_0)^p}{2 - w_0 p}. \quad (1.6.5)$$

Note that for $w_0 = 1 + s$ equation (1.6.5) is the same as (1.4.18). By direct calculations we see that as $\varepsilon \rightarrow 0$ we have

$$\langle (G^2 + \tau^2 F^2)^{p/2} \rangle_{[0,1]} = \frac{\varepsilon[(w^2 + \tau^2 d_-^2)^{p/2} + (w^2 + \tau^2 d_+^2)^{p/2}]}{1 - (1 - 2\varepsilon)\gamma^p} \rightarrow \frac{2f(w_0 - 1)}{2 - w_0 p}.$$

Now we are going to calculate the value $\mathbf{B}(-1, h(s))$ where $h(s) = y_3$. From (1.4.16) we have

$$\mathbf{B}(-1, h(s)) = h(s)t_2(s) - \frac{2}{p}t_1(s).$$

By using (1.3.6) we express t_1 via t_2 , also because of (1.4.18) and (1.4.21) we have

$$\begin{aligned} \mathbf{B}(-1, h(s)) &= h(s)t_2(s) - \frac{2}{p}t_1(s) = h(s)t_2 - \frac{2}{p}(f' - t_2 g') = \\ t_2(h(s) + \frac{2}{p}g') - f' \frac{2}{p} &= \frac{f(s) - (s - y_p)f'(s)}{g(s) - (s - y_p)g'(s)} \left(\frac{2g}{p(y_p - s)} + \frac{2}{p}g' \right) - f' \frac{2}{p} = \frac{2}{p} \left[\frac{f(s)}{y_p - s} \right] = \frac{2(2 - w_0)^p}{2 - w_0 p}. \end{aligned}$$

Thus we obtain the desired result

$$\langle (G^2 + \tau^2 F^2)^{p/2} \rangle_{[0,1]} \rightarrow \mathbf{B}(-1, y_3) \quad \text{as } \varepsilon \rightarrow 0.$$

Extremizers in the domain $\Omega(\Theta([s_0, y_p], g))$.

Pick any point $y = (y_2, y_3) \in \Omega(\Theta([s_0, y_p], g))$. Then there exists a segment $\ell(y) \in \Theta([s_0, y_p], g)$. Let y^+ and y^- be the endpoints of this segment such that $y^+ = (-1, \tilde{y}_3)$ for

some $\tilde{y}_3 \geq h(s_0)$ and $y^- = (\tilde{s}, g(\tilde{s}))$ for some $\tilde{s} \in [y_p, s_0]$. We remind the reader that we know ε -extremizers for the points $(s, g(s))$ where $s \in [s_0, 1]$, and we know ε -extremizers on the vertical line $(-1, y_3)$ where $y_3 \geq h(s_0)$. Therefore, as in the case of a cup, taking the appropriate α -concatenation of these ε -extremizers and using the fact that \mathbf{B} is linear on $\ell(y)$, we find an ε -extremizer at point y .

Extremizers in the domain $\text{Ang}(s_0)$.

Pick any $y = (y_1, y_2) \in \text{Ang}(s_0)$. There exist the points $y^+ \in \ell^+$, $y^- \in \ell^-$, where $\ell^+ = \ell^+(s_0, g(s_0)) \in \Theta([s_0, y_p], g)$ and $\ell^- = \ell^-(s_0, g(s_0)) \in \Theta((c, s_0], g)$, such that $y = \alpha y^+ + (1 - \alpha)y^-$ for some $\alpha \in [0, 1]$. We know ε -extremizers at the points y^+ and y^- . Then by taking an α -concatenation of these extremizers and using the linearity of \mathbf{B} on $\text{Ang}(s_0)$ we can obtain an ε -extremizer at the point y .

Extremizers in the domain $\Omega(\Theta([y_p, 1], g))$.

Finally, we consider the domain of vertical segments $\Omega(\Theta[y_p, 1], g)$. Pick any point $y = (y_2, y_3) \in \Omega(\Theta[y_p, 1])$. Take an arbitrary point $y^+ = (-1, y_3^+)$ where y_3^+ is sufficiently large such that $y = \alpha y^+ + (1 - \alpha)y^-$ for some $\alpha \in (0, 1)$ and some $y^- = (y_2^-, y_3^-)$ such that $(1, y_2^-, y_3^-) \in \partial\Omega_1$. Surely, y^+, y^- belong to a positive domain. Condition $(1, y_2^-, y_3^-) \in \partial\Omega_1$ implies that we know an ε -extremizer (F^-, G^-) at the point y^- (these are constant functions). We also know an ε -extremizer at the point y^+ . Let $(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha$ be an α -concatenation of these extremizers. Then

$$\Psi[(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha] > \alpha \mathbf{B}(y^+) + (1 - \alpha) \mathbf{B}(y^-) - \varepsilon.$$

Note that the condition $y = \alpha y^+ + (1 - \alpha)y^-$ implies that

$$\alpha = \frac{y_3 - \frac{y_2}{y_2^-} y_3^-}{y_3^+ + \frac{y_3^-}{y_2^-}}.$$

Recall that $\mathbf{B}(y_2, g(y_2)) = f(y_2)$ and $\mathbf{B}(y^+) = f(s) + t_1(s)(-1 - s) + t_2(s)(y_3^+ - g(s))$, where $s \in [s_0, y_p]$ is such that a segment $\ell(s, g(s)) \in \Theta([s_0, y_p], g)$ has an endpoint y^+ .

Note that as $y_3^+ \rightarrow \infty$ all terms remain bounded; moreover, $\alpha \rightarrow 0$, $y^- \rightarrow (y_2, g(y_2))$ and $s \rightarrow y_p$. This means that

$$\begin{aligned} \lim_{y_3^+ \rightarrow \infty} \alpha \mathbf{B}(y^+) + (1 - \alpha) \mathbf{B}(y^-) - \varepsilon = \\ \lim_{y_3^+ \rightarrow \infty} t_2(s) y_3^+ \left(\frac{y_3 - \frac{y_2}{y_2^-} y_3^-}{y_3^+ + \frac{y_3^-}{y_2^-}} \right) + f(y_2) - \varepsilon = t_2(y_p)(y_3 - g(y_2)) + f(y_2) - \varepsilon. \end{aligned}$$

We recall that $t_2(s) = t_2(y_p)$ for $s \in [y_p, 1]$. Then

$$\mathbf{B}(y) = f(y_2) + t_2(s(y))(y_3 - g(y_2)) = f(y_2) + t_2(y_p)(y_3 - g(y_2)).$$

Thus, if we choose y_3^+ sufficiently large then we can obtain a 2ε -extremizer for the point y .

1.6.2 Case $s_0 > y_p$.

In this case we have $s_0 \geq y_p$ (see Figure 1.11). This case is a little bit more complicated than the previous one. Construction of ε -extremizers (F, G) will be similar to the one presented in [21].

We need a few more definitions.

Definition 10. Let (F, G) be an arbitrary pair of functions. Let $(y_2, g(y_2)) \in \Omega_3$ and let J be a subinterval of $[0, 1]$. We define a new pair (\tilde{F}, \tilde{G}) as follows:

$$(\tilde{F}, \tilde{G})(x) = \begin{cases} (F, G)(x) & x \in [0, 1] \setminus J \\ (1 - y_2, 1 + y_2) & x \in J. \end{cases}$$

We will refer to the new pair (\tilde{F}, \tilde{G}) as putting the constant $(y_2, g(y_2))$ on the interval J for the pair (F, G)

It is worth mentioning that sometimes the new pair (\tilde{F}, \tilde{G}) we will denote by the same symbol (F, G) .

Definition 11. We say that the pairs (F_α, G_α) , $(F_{1-\alpha}, G_{1-\alpha})$ are obtained from the pair (F, G) by splitting at the point $\alpha \in (0, 1)$ if

$$\begin{aligned} (F_\alpha, G_\alpha) &= (F, G)(x \cdot \alpha) \quad x \in [0, 1]; \\ (F_{1-\alpha}, G_{1-\alpha}) &= (F, G)(x \cdot (1 - \alpha) + \alpha) \quad x \in [0, 1]; \end{aligned}$$

It is clear that $\Psi(F, G) = \alpha\Psi(F_\alpha, G_\alpha) + (1 - \alpha)\Psi(F_{1-\alpha}, G_{1-\alpha})$. Also note that if (F_α, G_α) , $(F_{1-\alpha}, G_{1-\alpha})$ are obtained from the pair (F, G) by splitting at the point $\alpha \in (0, 1)$, then (F, G) is an α -concatenation of the pairs (F_α, G_α) , $(F_{1-\alpha}, G_{1-\alpha})$. Thus, such operations as splitting and concatenation are opposite operations.

Instead of explicitly presenting an admissible pair (F, G) and showing that it is an ε -extremizer, we present an algorithm which constructs the admissible pair, and we show that the result is an ε -extremizer.

By the same explanations as in the case $s_0 \leq y_p$, it is enough to construct an ε -extremizer

(F, G) on the vertical line $y_2 = -1$ of the domain Ω_3 . Moreover, linearity of \mathbf{B} implies that for any $A > 0$, it is enough to construct ε -extremizers for the points $(-1, y_3)$, where $y_3 \geq A$. Pick any point $(-1, y_3)$ where $y_3 = y_3^{(0)} > g(-1)$. Linearity of \mathbf{B} on $\text{Ang}(s_0)$ and direct calculations (see (1.3.8), (1.4.22)) show that

$$\mathbf{B}(-1, y_3) = f(-1) + t_3(s_0)(y_3 - g(-1)) = f(-1) + (y_3 - g(-1)) \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}. \quad (1.6.6)$$

We describe the first iteration. Let (F, G) be an admissible pair for the point $(-1, y_3)$, whose explicit expression will be described during the algorithm. For a pair (F, G) we put a constant $(s_0, g(s_0))$ on an interval $[0, \varepsilon]$ where the value $\varepsilon \in (0, 1)$ will be given later. Thus we obtain a new pair (F, G) which we denote by the same symbol. We want (F, G) to be an admissible pair for the point $(-1, y_3)$. Let the pairs $(F_\varepsilon, G_\varepsilon)$, $(F_{1-\varepsilon}, G_{1-\varepsilon})$ be obtained from the pair (F, G) by splitting at point ε . It is clear that $(F_\varepsilon, G_\varepsilon)$ is an admissible pair for the point $(s_0, g(s_0))$. We want $(F_{1-\varepsilon}, G_{1-\varepsilon})$ to be an admissible pair for the point $P = (\tilde{y}_2, \tilde{y}_3)$ so that

$$(-1, y_3) = \varepsilon(s_0, g(s_0)) + (1 - \varepsilon)P. \quad (1.6.7)$$

Therefore we require

$$P = \left(\frac{-1 - \varepsilon s_0}{1 - \varepsilon}, \frac{y_3 - \varepsilon g(s_0)}{1 - \varepsilon} \right). \quad (1.6.8)$$

So we make the following simple observation: if $(F_{1-\varepsilon}, G_{1-\varepsilon})$ were an admissible pair for the point P , then (F, G) (which is an ε -concatenation of the pairs $(1 - s_0, 1 + s_0)$ and $(F_{1-\varepsilon}, G_{1-\varepsilon})$) would be an admissible pair for the point $(-1, y_3)$. Explanation of this obser-

vation is simple: note that these pairs $(F_{1-\varepsilon}, G_{1-\varepsilon})$ and $(1-s_0, 1+s_0)$ are admissible pairs for the points P and $(s_0, g(s_0))$ which belong to a positive domain (see (1.6.7)); therefore, the rest immediately follows from Lemma 14. So we want to construct the admissible pair $(F_{1-\varepsilon}, G_{1-\varepsilon})$ for the point (1.6.8).

We recall Lemma 13 which implies that the pair $(F_{1-\varepsilon}, G_{1-\varepsilon})$ is admissible for the point $\left(1, \frac{-1-\varepsilon s_0}{1-\varepsilon}, \frac{y_3-\varepsilon g(s_0)}{1-\varepsilon}\right)$ if and only if the pair (\tilde{F}, \tilde{G}) where

$$(F_{1-\varepsilon}, -G_{1-\varepsilon}) = \frac{1+\varepsilon s_0}{1-\varepsilon}(\tilde{F}, \tilde{G})$$

is admissible for a point $W = \left(1, \frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{(y_3-\varepsilon g(s_0))}{(1+\varepsilon s_0)^p} \cdot (1-\varepsilon)^{p-1}\right)$. So, if we find the admissible pair (\tilde{F}, \tilde{G}) then we automatically find the admissible pair (F, G) .

Choose ε small enough so that $\left(\frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{(y_3-\varepsilon g(s_0))}{(1+\varepsilon s_0)^p} \cdot (1-\varepsilon)^{p-1}\right) \in \Omega_3$ and

$$\left(\frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{(y_3-\varepsilon g(s_0))}{(1+\varepsilon s_0)^p} \cdot (1-\varepsilon)^{p-1}\right) = \delta(s_0, g(s_0)) + (1-\delta)(-1, y_3^{(1)})$$

for some $\delta \in (0, 1)$ and $y_3^{(1)} \geq g(-1)$. Then

$$\begin{aligned} \delta &= \frac{\varepsilon}{1+\varepsilon s_0} = \varepsilon + O(\varepsilon^2) \\ y_3^{(1)} &= \frac{\frac{(y_3-\varepsilon g(s_0))}{(1+\varepsilon s_0)^p} \cdot (1-\varepsilon)^{p-1} - \frac{\varepsilon}{1+\varepsilon s_0} g(s_0)}{1 - \frac{\varepsilon}{1+\varepsilon s_0}} = y_3(1 - \varepsilon(p + ps_0 - 2)) - 2\varepsilon g(s_0) + O(\varepsilon^2). \end{aligned} \tag{1.6.9}$$

For the pair (\tilde{F}, \tilde{G}) we put a constant $(s_0, g(s_0))$ on the interval $[0, \delta]$. We split the new pair (\tilde{F}, \tilde{G}) at point δ so we get the pairs $(\tilde{F}_\delta, \tilde{G}_\delta)$ and $(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$. We make a similar observation as above. It is clear that if we know the admissible pair $(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$ for the point

$(-1, y_3^{(1)})$ then we can obtain an admissible pair (\tilde{F}, \tilde{G}) for the point $\left(\frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{(y_3-\varepsilon g(s_0))}{(1+\varepsilon s_0)^p} \cdot (1-\varepsilon)^{p-1}\right)$. Surely (\tilde{F}, \tilde{G}) is a δ -concatenation of the pairs $(1-s_0, 1+s_0)$ and $(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$.

We summarize the first iteration. We took $\varepsilon \in (0, 1)$, and we started from the pair $(F^{(0)}, G^{(0)}) = (F, G)$, and after one iteration we came to the pair $(F^{(1)}, G^{(1)}) = (\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta})$. We showed that if $(F^{(1)}, G^{(1)})$ is an admissible pair for the point $(1, y_3^{(1)})$, then the pair $(F^{(0)}, G^{(0)})$ can be obtained from the pair $(F^{(1)}, G^{(1)})$; moreover, it is admissible for the point $(1, y_3^{(0)})$.

Continuing these iterations, we obtain the sequence of numbers $\{y_3^{(j)}\}_{j=0}^N$ and the sequence of pairs $\{(F^{(j)}, G^{(j)})\}_{j=0}^N$. Let N be such that $y_3^{(N)} \geq g(-1)$. It is clear that if $(F^{(N)}, G^{(N)})$ is an admissible pair for the point $(-1, y_3^{(N)})$ then the pairs $\{(F^{(j)}, G^{(j)})\}_{j=0}^{N-1}$ can be determined uniquely, and, moreover, $(F^{(j)}, G^{(j)})$ is an admissible pair for the point $(-1, y_3^{(j)})$ for all $j = 0, \dots, N-1$.

Note that we can choose sufficiently small $\varepsilon \in (0, 1)$, and we can find $N = N(\varepsilon)$ such that $y_3^{(N)} = g(-1)$ (see (1.6.9), and recall that $s_0 > y_p$). In this case the admissible pair $(F^{(N)}, G^{(N)})$ for the point $(-1, y_3^{(N)}) = (-1, g(-1))$ is a constant function, namely, $(F^{(N)}, G^{(N)}) = (2, 0)$. Now we try to find N in terms of ε , and we try to find the value of $\Psi(F^{(0)}, G^{(0)})$.

Condition (1.6.9) implies that $y_3^{(1)} = y_3^{(0)}(1 - \varepsilon(p + ps_0 - 2)) - 2\varepsilon g(s_0) + O(\varepsilon^2)$. We denote $\delta_0 = p + ps_0 - 2 > 0$. Therefore, after the N -th iteration we obtain

$$y_3^{(N)} = (1 - \varepsilon\delta_0)^N \left(y_3^{(0)} + \frac{2g(s_0)}{\delta_0} \right) - \frac{2g(s_0)}{\delta_0} + O(\varepsilon).$$

The requirement $y_3^{(N)} = g(-1)$ implies that

$$(1 - \varepsilon\delta_0)^{-N} = \frac{y_3^{(0)} + \frac{2g(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} + O(\varepsilon).$$

This implies that $\limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot N = \limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot N(\varepsilon) < \infty$. Therefore, we get

$$e^{\varepsilon\delta_0 N} = \frac{y_3^{(0)} + \frac{2g(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} + O(\varepsilon). \quad (1.6.10)$$

Also note that

$$\begin{aligned} \Psi(F^{(0)}, G^{(0)}) &= \Psi(F, G) = \varepsilon\Psi(F_\varepsilon, G_\varepsilon) + (1 - \varepsilon)\Psi(F_{1-\varepsilon}, G_{1-\varepsilon}) = \\ \varepsilon f(s_0) + (1 - \varepsilon)\Psi(F_{1-\varepsilon}, G_{1-\varepsilon}) &= \varepsilon f(s_0) + (1 - \varepsilon) \left(\frac{1 + \varepsilon s_0}{1 - \varepsilon} \right)^p \Psi(\tilde{F}, \tilde{G}) \\ &= \varepsilon f(s_0) + (1 - \varepsilon)(1 - \varepsilon) \left(\frac{1 + \varepsilon s_0}{1 - \varepsilon} \right)^p \left[\delta f(s_0) + (1 - \delta)\Psi(\tilde{F}_{1-\delta}, \tilde{G}_{1-\delta}) \right] \\ &= 2\varepsilon f(s_0) + (1 + \varepsilon\delta_0)\Psi(F^{(1)}, G^{(1)}) + O(\varepsilon^2). \end{aligned}$$

Therefore, after the N -th iteration (and using the fact that $\Psi(F^{(N)}, G^{(N)}) = f(-1)$) we obtain

$$\begin{aligned} \Psi(F^{(0)}, G^{(0)}) &= (1 + \varepsilon\delta_0)^N \left(f(-1) + \frac{2f(s_0)}{\delta_0} \right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon) = \\ e^{\varepsilon\delta_0 N} \left(f(-1) + \frac{2f(s_0)}{\delta_0} \right) &- \frac{2f(s_0)}{\delta_0} + O(\varepsilon). \end{aligned} \quad (1.6.11)$$

The last equality follows from the fact that $\limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot N(\varepsilon) < \infty$.

Therefore (1.6.10) and (1.6.11) imply that

$$\begin{aligned}\Psi(F^{(0)}, G^{(0)}) &= \left(\frac{y_3^{(0)} + \frac{2g(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} \right) \left(f(-1) + \frac{2f(s_0)}{\delta_0} \right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon) = \\ &f(-1) + (y_3^{(0)} - g(-1)) \left(\frac{f(-1) + \frac{2f(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} \right) + O(\varepsilon).\end{aligned}$$

Now we recall (1.6.6). So if we show that

$$\frac{f(-1) + \frac{2f(s_0)}{\delta_0}}{g(-1) + \frac{2g(s_0)}{\delta_0}} = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} \quad (1.6.12)$$

then (1.6.12) will imply that $\Psi(F^{(0)}, G^{(0)}) = \mathbf{B}(-1, y_3^{(0)}) + O(\varepsilon)$. So choosing ε sufficiently small we can obtain the extremizer $(F^{(0)}, G^{(0)})$ for the point $(-1, y_3)$. Therefore, we need only to prove equality (1.6.12). It will be convenient to make the following notations: set $f_- = f(-1)$, $f'_- = f'(-1)$, $f = f(s_0)$, $f' = f'(s_0)$, $g_- = g(-1)$, $g'_- = g'(-1)$, $g = g(s_0)$ and $g' = g'(s_0)$. Then the equality (1.6.12) turns into the following one

$$\frac{\delta_0}{2} = \frac{fg'_- - fg' - f'_-g + f'g}{g'f_- - f'g_-}. \quad (1.6.13)$$

This simplifies into the following one

$$s_0 - y_p = \frac{2}{p} \cdot \frac{fg'_- - fg' - f'_-g + f'g}{g'f_- - f'g_-} = \frac{fg'_- - fg' - f'_-g + f'g}{-g'f'_- + f'g'_-}$$

which is true by (1.4.24).

Chapter 2

Hessian of Bellman functions and uniqueness of Brascamp–Lieb inequality

2.0.3 Brascamp–Lieb inequality

The classical Young’s inequality for convolutions on the real line asserts that for any $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ where $p, q \geq 1$, we have an inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \quad (2.0.1)$$

if and only if

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \quad (2.0.2)$$

In what follows $f * g$ denotes convolution i.e. $f * g(y) = \int_{\mathbb{R}} f(x)g(y-x)dx$. The necessity of (2.0.2) follows immediately: by stretching the functions f and g as $f_{\lambda}(x) = \lambda^{1/p}f(\lambda x)$, $g_{\lambda}(x) = \lambda^{1/q}g(\lambda x)$ corresponding norms do not change. Since $\|f_{\lambda} * g_{\lambda}\|_r = \lambda^{\frac{1}{p} + \frac{1}{q} - 1 - \frac{1}{r}} \|f * g\|_r$ we obtain (2.0.2). Beckner (see [22]) found the sharp constant $C = C(p, q, r)$ of the inequality

$\|f * g\|_r \leq C \|f\|_p \|g\|_q$. At the same time (see [23]) Brascamp and Lieb derived more general inequality, namely, let a_1, \dots, a_n be the vectors of \mathbb{R}^k where $1 \leq k \leq n$, let $u_j \in L^1(\mathbb{R})$ be nonnegative functions, where $1 \leq p_j \leq \infty$ and $\sum_{j=1}^n \frac{1}{p_j} = k$, then we have a sharp inequality

$$\int_{\mathbb{R}^k} \prod_{j=1}^n u_j^{1/p_j}(\langle a_j, x \rangle) dx \leq D(p_1, \dots, p_n) \prod_{j=1}^n \left(\int_{\mathbb{R}} u_j(x) dx \right)^{1/p_j}, \quad (2.0.3)$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product in Euclidian space,

$$D(p_1, \dots, p_n) = \sup_{b_1, \dots, b_n > 0} \int_{\mathbb{R}^k} \prod_{j=1}^n g_j^{1/p_j}(\langle a_j, x \rangle) dx \quad (2.0.4)$$

and $g_j(x) = b_j^{1/2} e^{-\pi x^2 b_j}$. In other words, supremum in (2.0.3) is achieved by centered, normalized (i.e. $\|g_j\|_1 = 1$) gaussian functions. Usually inequality (2.0.3) is written as follows:

$$\int_{\mathbb{R}^k} \prod_{j=1}^n w_j(\langle a_j, x \rangle) dx \leq D(p_1, \dots, p_n) \prod_{j=1}^n \|w_j\|_{p_j}.$$

for $u_j \in L^{p_j}(\mathbb{R})$. Surely the above inequality becomes the same as (2.0.3) after introducing the functions $w_j(x) = u_j^{1/p_j}(y)$.

It is clear that Brascamp–Lieb inequality (2.0.3) implies sharp Young’s inequality for convolutions. Indeed, just take $n = 3, k = 2, a_1 = (1, 0), a_2 = (1, -1), a_3 = (0, 1)$ and use duality argument.

The next natural question which arose was the following one: what conditions should the vectors a_j and the numbers p_j satisfy in order for the constant $D(p_1, \dots, p_n)$ to be finite. It turns out that the answer has simple geometrical interpretation which was first

found by Barthe (see [24]): we consider all different k -tuples of vectors $(a_{j_1}, \dots, a_{j_k})$ such that they create basis in \mathbb{R}^k . All we need from these tuples are the numbers j_1, \dots, j_k . Each k -tuple defines a unique vector $v \in \mathbb{R}^n$ with entries 0 and 1 so that j_i -th component is 1 ($i = 1, \dots, k$) and the rests are zeros. Finally we take convex hull of the vectors v and denote it by K . The constant $D(p_1, \dots, p_n)$ is finite if and only if $(\frac{1}{p_1}, \dots, \frac{1}{p_n}) \in K$. In other words, in order to make the set K large we want the vectors a_1, \dots, a_n to be *more linearly independent*. Later the proof of the Brascamp–Lieb inequality (2.0.3) was simplified (see [25]) by heat flow method. The idea of the method is quite similar to *Bellman function technique* which we are going to discuss in the current article. The same idea was used in [26] in order to derive *general rank Brascamp–Lieb inequality* (see also [27]): let $B_j : \mathbb{R}^k \rightarrow \mathbb{R}^{k_j}$ be a surjective linear maps, $u_j : \mathbb{R}^{k_j} \rightarrow \mathbb{R}_+$, $k_j, k \in \mathbb{N}$, and $p_j \geq 1$ are such that $\sum_{j=1}^n \frac{k_j}{p_j} = k$ then we have a sharp inequality

$$\int_{\mathbb{R}^k} u_1^{1/p_1}(B_1 x) \cdots u_n^{1/p_n}(B_n x) dx \leq C \left(\int_{\mathbb{R}^{k_1}} u_1 \right)^{1/p_1} \cdots \left(\int_{\mathbb{R}^{k_n}} u_n \right)^{1/p_n}$$

where

$$C = \sup_{A_1, \dots, A_n > 0} \int_{\mathbb{R}^k} G_1^{1/p_1}(B_1 x) \cdots G_n^{1/p_n}(B_n x) dx \quad (2.0.5)$$

and $G_j(y) = e^{-\pi \langle A_j y, y \rangle} (\det A_j)^{1/2}$. Supremum in (2.0.5) is taken over all positive definite $k_j \times k_j$ matrices A_j . One of the main result obtained in [26] describes finiteness of the number C , namely, C is finite if and only if

$$\dim(V) \leq \sum_{j=1}^n \frac{\dim(B_j V)}{p_j} \quad \text{for all subspaces } V \subset \mathbb{R}^k.$$

After this result the original inequality (2.0.3) got a name *rank 1 Brascamp–Lieb inequality*.

If $k = 1$ the inequality (2.0.3) becomes usual multilinear Hölder's inequality

$$\int_{\mathbb{R}} \prod_{j=1}^n u_j^{1/p_j}(x) dx \leq \prod_{j=1}^n \left(\int_{\mathbb{R}} u_j \right)^{1/p_j} \iff \sum_j \frac{1}{p_j} = 1. \quad (2.0.6)$$

From the *Bellman function point of view* the multilinear Hölder's inequality holds because the following function

$$B(x_1, \dots, x_n) = x_1^{1/p_1} \dots x_n^{1/p_n} \quad (2.0.7)$$

is concave in the domain $x_j \geq 0$ for $\sum_{j=1}^n \frac{1}{p_j} \leq 1$ (we assume that $p_j > 0$).

This Bellman function point of view asks us to look for the description of functions B such that

$$\int_{\mathbb{R}^k} B(u_1(\langle a_1, x \rangle), \dots, u_n(\langle a_n, x \rangle)) dx \text{ is estimated in terms of } \left\{ \int_{\mathbb{R}} u_i(x) dx \right\}_{i=1}^n. \quad (2.0.8)$$

Function $B(x_1, \dots, x_n) = x_1^{1/p_1} \dots x_n^{1/p_n}$, $\sum_{j=1}^n \frac{1}{p_j} = 1$, is an example of such a function for $k = 1$. But for $k = 1$ one can easily get the full description of “Bellman functions” that give inequality (2.0.9) below.

The equality $\sum_{j=1}^n \frac{1}{p_j} = 1$ was needed because the function $B(x_1, \dots, x_n)$ has to be homogeneous of degree 1 i.e., $B(\lambda \mathbf{x}) = \lambda B(\mathbf{x})$. This allows us to write integral over the real line. Indeed, if the nonnegative functions u_j are integrable then Jensen's inequality implies

$$\frac{1}{|I|} \int_I B(u_1, \dots, u_n) dx \leq B\left(\frac{1}{|I|} \int_I u_1 dx, \dots, \frac{1}{|I|} \int_I u_n dx\right)$$

where I is any subinterval of the real line. Since the function B is 1-homogeneous we can rewrite the above inequality as follows

$$\int_I B(u_1, \dots, u_n) dx \leq B \left(\int_I u_1 dx, \dots, \int_I u_n dx \right).$$

Take $I = [-R, R]$ and send R to infinity. B is continuous, so that

$$B \left(\int_I u_1(x) dx, \dots, \int_I u_n(x) dx \right) \rightarrow B \left(\int_{\mathbb{R}} u_1 dx, \dots, \int_{\mathbb{R}} u_n dx \right).$$

Continuity of B and monotone convergence theorem implies that

$$\int_{\mathbb{R}} B(u_1(x), \dots, u_n(x)) dx \leq B \left(\int_{\mathbb{R}} u_1 dx, \dots, \int_{\mathbb{R}} u_n dx \right) \quad (2.0.9)$$

It is worth to formulate the following lemma. Set $\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_j \geq 0\}$.

Lemma 15. *Let u_j be nonnegative integrable functions $j = 1, \dots, n$ on the real line. If B is 1-homogeneous concave function on \mathbb{R}_+^n , then (2.0.9) holds. Equality is achieved in (2.0.9) if (u_1, \dots, u_n) are all proportional.*

Proof. As we just saw, the proof follows from showing that $\int_I B(u_1, \dots, u_n) \rightarrow \int_{\mathbb{R}} B(u_1, \dots, u_n)$.

We are going to find now a summable majorant. Take any point \mathbf{x}_0 from the interior of \mathbb{R}_+^n .

Consider any subgradient $v = (v_1, \dots, v_n)$ at point \mathbf{x}_0 i.e. $B(\mathbf{x}) \leq \langle v, \mathbf{x} - \mathbf{x}_0 \rangle + B(\mathbf{x}_0)$. Take

$\mathbf{x} = \lambda \mathbf{x}_0$ and use the homogeneity of B . Thus we obtain $(\lambda - 1)B(\mathbf{x}_0) \leq (\lambda - 1)\langle v, \mathbf{x}_0 \rangle$ for

any $\lambda \geq 0$. This means that $B(\mathbf{x}_0) = \langle v, \mathbf{x}_0 \rangle$ and, therefore, $B(\mathbf{x}) \leq \langle v, \mathbf{x} \rangle$. On the other

hand let $e_j = (0, \dots, 1, \dots, 0)$ be a basis vectors (j -th component entry is 1 and the rests

are zero). Consider any point $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_+^n$. Concavity and homogeneity of B

implies that $B(x) \geq \sum_{j=1}^n x_j B(e_j)$. So we obtain the majorant

$$|B(\mathbf{x})| \leq \max \left\{ \sum_{j=1}^n x_j |B(e_j)|, \sum_{j=1}^n x_j |v_j| \right\} \quad \text{for any } \mathbf{x} \in \mathbb{R}_+^n.$$

Plugging u_j for x_j we see that the use of Lebesgue's dominated convergence theorem is justified. \square

The above Lemma says that homogeneity and concavity of the function implies the inequality (2.0.9). The converse is also true.

Now the following question becomes quite natural:

Question. Assume $a_1, \dots, a_n \in \mathbb{R}^k$, B is continuous function defined on \mathbb{R}_+^n and u_j , $j = 1, \dots, n$ are nonnegative integrable functions. What is the sharp estimate of the expression

$$\int_{\mathbb{R}^k} B(u_1(\langle a_1, \mathbf{x} \rangle), \dots, u_n(\langle a_n, \mathbf{x} \rangle)) d\mathbf{x} \tag{2.0.10}$$

in terms of $\int_{\mathbb{R}} u_j$?

In other words, along with Young's functions

$$B(x_1, \dots, x_n) = x_1^{1/p_1} \dots x_n^{1/p_n}, \quad \sum \frac{1}{p_j} = k,$$

what can be other Brascamp–Lieb Bellman functions that would give us sharp estimates of (2.0.10)?

We give partial answer on this question. It turns out that if one requires function B is homogeneous of degree k and in addition it satisfies some mild assumptions (smoothness and exponential integrality given below), then we can find the sharp estimate of the expression

(2.0.10) in terms of $\int_{\mathbb{R}} u_j$, if B satisfies an interesting concavity condition. In a trivial case $k = 1$ our theorem gives us of course inequality (2.0.9).

In the trivial case $k = 1$ we already saw that the interesting concavity condition mentioned above is precisely the usual concavity of B . In another trivial case $k = n$, the interesting concavity condition mentioned above becomes “separate concavity” of B in each of its variables.

For $1 < k < n$ our concavity condition is, in fact, some compromise between these two concavities.

As we will see $k = n - 1$ and $k = n$ this concavity condition (plus k -homogeneity and mild regularity) imply that Brascamp–Lieb Bellman functions B can be only the standard ones: $B(x_1, \dots, x_n) = x_1^{1/p_1} \cdots x_n^{1/p_n}$, $\sum \frac{1}{p_j} = k$.

Before we start formulating our results, we will explain that there are *many* 1-homogeneous concave functions B on \mathbb{R}_+^n .

Lemma 16. *Function B is continuous, concave and homogeneous of degree 1 on \mathbb{R}_+^n if and only if there exists continuous, concave function $\tilde{B}(\mathbf{y})$ on \mathbb{R}_+^{n-1} such that $\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \tilde{B}(\lambda \mathbf{y})$ exists, it is continuous with respect to \mathbf{y} and $B(x_1, \dots, x_n) = x_1 \tilde{B}\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right)$*

Proof. Indeed, if B is continuous, concave and homogeneous of degree 1 then $B(x_1, \dots, x_n) = x_1 B\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right)$ and the function $\tilde{B} = B(1, y_1, \dots, y_{n-1})$ is continuous and concave in \mathbb{R}_+^{n-1} . Moreover, for each $\mathbf{y} \in \mathbb{R}_+^{n-1}$, $\lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \tilde{B}(\lambda \mathbf{y})$ exists and it is continuous with respect to \mathbf{y} .

Assume \tilde{B} satisfies the conditions of the Lemma. Consider

$$B(x_1, \dots, x_n) = x_1 \tilde{B}\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right).$$

It is clear that B is continuous on \mathbb{R}_+^n and it is homogeneous of degree 1. We will show that B is concave in the interior of \mathbb{R}_+^n and hence by continuity it will be concave on the closure as well. Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. Let $\bar{\mathbf{x}} = (x_2, \dots, x_n), \bar{\mathbf{y}} = (y_2, \dots, y_n)$ and $\alpha + \beta = 1$ for $\alpha, \beta \in [0, 1]$. Then we have

$$\begin{aligned} B(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_1 + \beta y_1) B\left(1, \alpha \frac{\bar{\mathbf{x}}}{\alpha x_1 + \beta y_1} + \beta \frac{\bar{\mathbf{y}}}{\alpha x_1 + \beta y_1}\right) \geq (\alpha x_1 + \beta y_1) \times \\ &\left[\alpha B\left(1, \frac{\bar{\mathbf{x}}}{\alpha x_1 + \beta y_1}\right) + \beta B\left(1, \frac{\bar{\mathbf{y}}}{\alpha x_1 + \beta y_1}\right) \right] = \alpha B(\mathbf{x}) + \beta B(\mathbf{y}). \end{aligned}$$

□

2.0.4 Bellman function in Brascamp–Lieb inequality

In what follows we assume that $B \in C(\mathbb{R}_+^n) \cap C^2(\text{int}(\mathbb{R}_+^n))$. In order for the quantity (2.0.10) to be finite it is necessary to assume that $1 \leq k \leq n$. Fix some vectors $a_j = (a_{j1}, \dots, a_{jn}) \in \mathbb{R}^k$ and $k \times k$ symmetric matrix C such that $\langle Ca_j, a_j \rangle > 0$ for $j = 1, \dots, n$. Let A be a $k \times n$ matrix constructed by columns a_j i.e. $A = (a_1, \dots, a_n)$. We denote A^* transpose matrix of A .

Let $u_j : \mathbb{R} \rightarrow \mathbb{R}_+$ be such that $0 < \int_{\mathbb{R}} u_j < \infty$. Let $u_j(y, t)$ solves the heat equation $\frac{\partial u_j}{\partial t} - \langle Ca_j, a_j \rangle \frac{\partial^2 u_j}{\partial y^2} = 0$ with the initial value $u_j(y, 0) = u_j(y)$. Let $\text{Hess } B(\mathbf{y})$ denotes Hessian matrix of the function B at point \mathbf{y} .

For two square matrices of the same size $P = \{p_{ij}\}$ and $Q = \{q_{ij}\}$, let $P \bullet Q = \{p_{ij}q_{ij}\}$ be Hadamard product. Denote by symbol

$$\mathbf{u}(x, t) = (u_1(\langle a_1, x \rangle, t), \dots, u_n(\langle a_n, x \rangle, t))$$

and denote

$$\mathbf{u}'(x, t) = (u'_1(\langle a_1, x \rangle, t), \dots, u'_n(\langle a_n, x \rangle, t)),$$

where $u'_j(\langle a_j, x \rangle, t) = \frac{\partial u_j(y, t)}{\partial y} \Big|_{y=\langle a_j, x \rangle}$.

Lemma 17. *For any $0 < t < \infty$ and any $x \in \mathbb{R}^k$ we have*

$$\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) B(\mathbf{u}(\mathbf{x}, t)) = -\langle (A^* C A) \bullet \text{Hess } B(\mathbf{u}(\mathbf{x}, t)) \mathbf{u}'(\mathbf{x}, t), \mathbf{u}'(\mathbf{x}, t) \rangle. \quad (2.0.11)$$

Proof. First we show that the functions $u_\ell(\langle a_\ell, \mathbf{x} \rangle, t)$ satisfy the following heat equation

$$\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) u_\ell(\langle a_\ell, \mathbf{x} \rangle, t) = 0, \quad \text{for any } \ell = 1, \dots, n.$$

Indeed, let $u''_j(\langle a_j, \mathbf{x} \rangle, t) = \frac{\partial^2 u_j(y, t)}{\partial y^2} \Big|_{y=\langle a_j, \mathbf{x} \rangle}$. Then

$$\begin{aligned} \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u_\ell(\langle a_\ell, \mathbf{x} \rangle, t) &= \sum_{i,j=1}^k c_{ij} \frac{\partial}{\partial x_i} (a_{\ell j} u'_\ell(\langle a_\ell, \mathbf{x} \rangle, t)) = \\ &= \sum_{i,j=1}^k c_{ij} a_{\ell j} a_{\ell i} u''_\ell(\langle a_\ell, \mathbf{x} \rangle, t) = \langle C a_\ell, a_\ell \rangle u''_\ell(\langle a_\ell, \mathbf{x} \rangle, t) = \frac{\partial}{\partial t} u_\ell(\langle a_\ell, \mathbf{x} \rangle, t). \end{aligned}$$

Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ and $u_\ell = u_\ell(\langle a_\ell, \mathbf{x} \rangle, t)$. Then

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) B(\mathbf{u}) &= \sum_{\ell=1}^n \frac{\partial B}{\partial u_\ell} \frac{\partial u_\ell}{\partial t} - \sum_{i,j=1}^k c_{ij} \frac{\partial}{\partial x_i} \left(\sum_{\ell=1}^n \frac{\partial B}{\partial u_\ell} \frac{\partial u_\ell}{\partial x_j} \right) = \\
\sum_{\ell=1}^n \frac{\partial B}{\partial u_\ell} \left(\sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u_\ell \right) &- \sum_{i,j=1}^k c_{ij} \left(\sum_{\ell,m=1}^n \frac{\partial^2 B}{\partial u_m \partial u_\ell} \frac{\partial u_m}{\partial x_i} \frac{\partial u_\ell}{\partial x_j} + \sum_{\ell=1}^n \frac{\partial B}{\partial u_\ell} \frac{\partial^2 u_\ell}{\partial x_i \partial x_j} \right) = \\
- \sum_{i,j=1}^k \sum_{\ell,m=1}^n c_{ij} \frac{\partial^2 B}{\partial u_m \partial u_\ell} \frac{\partial u_m}{\partial x_i} \frac{\partial u_\ell}{\partial x_j} &= - \sum_{i,j=1}^k \sum_{\ell,m=1}^n c_{ij} \frac{\partial^2 B}{\partial u_m \partial u_\ell} a_{mi} a_{\ell j} u'_m u'_\ell = \\
- \sum_{\ell,m=1}^n \langle C a_m, a_\ell \rangle \frac{\partial^2 B}{\partial u_m \partial u_\ell} u'_m u'_\ell &= - \langle (A^* C A) \bullet \text{Hess } B(\mathbf{u}) \mathbf{u}', \mathbf{u}' \rangle.
\end{aligned}$$

□

Remark 13. Let us denote by $\Delta_C := \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$. We used above that

$$\left(\frac{\partial}{\partial t} - \Delta_C \right) u_\ell(\langle a_\ell, \mathbf{x} \rangle, t) = 0.$$

This is exactly the equality that implies

$$\left(\frac{\partial}{\partial t} - \Delta_C \right) B(u_\ell(\langle a_\ell, \mathbf{x} \rangle, t)) = - \langle (A^* C A) \bullet \text{Hess } B(\mathbf{u}(\mathbf{x}, t)) \mathbf{u}'(\mathbf{x}, t), \mathbf{u}'(\mathbf{x}, t) \rangle. \quad (2.0.12)$$

In other words, we look at the natural “energy” of the problem $\int B(u_\ell(\langle a_\ell, \mathbf{x} \rangle, t)) d\mathbf{x}$ at time t , and differentiate it in t . Replacing d/dt by $d/dt - \Delta_C$ does not change the result because when we integrate the above equality over \mathbf{x} variables, we should expect the term $\int \Delta_C B(u_\ell(\langle a_\ell, \mathbf{x} \rangle, t)) d\mathbf{x}$ to disappear (and this is exactly what happens below). But the definite sign in the right hand side of (2.0.11) guarantees us now the monotonicity property of the energy.

So composing of the special heat flow $e^{-t\Delta_C}$ and special function B seems like a good idea

exactly because of the monotonicity formula, which we are going to obtain shortly below.

Further we make several assumptions on the function B . The assumption $L3$ is exactly the concavity we were talking about above.

$$L1. \ B \in C(\mathbb{R}_+^n) \cap C^2(\text{int}(\mathbb{R}_+^n)).$$

$$L2. \ B(\lambda \mathbf{y}) = \lambda^k B(\mathbf{y}) \text{ for all } \lambda \geq 0 \text{ and } \mathbf{y} \in \mathbb{R}_+^n.$$

$$L3. \ \text{There exists } k \times k \text{ symmetric matrix } C \text{ such that } (A^*CA) \bullet \text{Hess } B(\mathbf{y}) \leq 0 \text{ for } \mathbf{y} \in \text{int}(\mathbb{R}_+^n), \text{ and } \langle Ca_j, a_j \rangle > 0 \text{ for all } j = 1, \dots, n.$$

$$L4. \ B \geq 0 \text{ and } B \text{ is not identically } 0.$$

$$L5.$$

$$\int_{\mathbb{R}^k} B(e^{-\langle a_1, \mathbf{x} \rangle^2}, \dots, e^{-\langle a_n, \mathbf{x} \rangle^2}) d\mathbf{x} < \infty. \quad (2.0.13)$$

We make several observations: properties $L3$ and $L4$ imply that the function B is separately concave (i.e. concave with respect to each variable) and increasing with respect to each variable, moreover, $B > 0$ in $\text{int}(\mathbb{R}_+^n)$. The above properties imply that

$$\int_{\mathbb{R}^k} B(b_1 e^{-\delta_1 \langle a_1, \mathbf{x} \rangle^2}, \dots, b_n e^{-\delta_n \langle a_n, \mathbf{x} \rangle^2}) d\mathbf{x} < \infty \quad (2.0.14)$$

for any positive numbers $b_j, \delta_j > 0$.

Consider the following class of functions $E(\mathbb{R})$: $u \in E(\mathbb{R})$ if and only if there exist constants $b, \delta > 0$ such that $|u(y)| \leq b e^{-\delta y^2}$. It is clear that if $u \in E(\mathbb{R})$ then $u(y, t) \in E(\mathbb{R})$ for any $t \geq 0$ where $u(y, t)$ denotes heat extension of $u(y)$ i.e. $u(y, t) = u(y, 0)$ and

$\frac{\partial}{\partial t}u(y, t) = \sigma \frac{\partial^2}{\partial y^2}u(y, t)$ with some $\sigma > 0$. Note that $E(\mathbb{R})$ contains the functions with compact support. Also note that if nonnegative functions u_j belong to the class $E(\mathbb{R})$ then the following function

$$\mathbf{B}(t) = \int_{\mathbb{R}^k} B(u_1(\langle a_1, x \rangle, t), \dots, u_n(\langle a_n, x \rangle, t)) d\mathbf{x}. \quad (2.0.15)$$

is finite for any $t \geq 0$.

Lemma 18. *Let u_j be nonnegative functions from $E(\mathbb{R})$. Then for any $t \in (0, \infty)$ we have*

$$\lim_{r \rightarrow \infty} \int_{V_r} \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j} B(\mathbf{u}(\mathbf{x}, t)) dx = 0 \quad \text{where} \quad V_r = \{x \in \mathbb{R}^k : \|x\| \leq r\}.$$

Proof. Let $F(x) = B(e^{-\langle a_1, x \rangle^2}, \dots, e^{-\langle a_n, x \rangle^2})$. Let $x = r\sigma$ where $\sigma \in \mathbb{S}_1^{k-1}$. Since B is increasing with respect to each components, for each σ the function $F(r\sigma)$ is decreasing with respect to r . Therefore the function $\tilde{F}(r) = \int_{\mathbb{S}_1^{k-1}} F(r\sigma) d\sigma_1$ is decreasing. Here σ_r denotes surface measure of the sphere \mathbb{S}_r^{k-1} or radius r . Since $\int_0^\infty \tilde{F}(r) r^{k-1} dr < \infty$ we obtain

$$R^k \tilde{F}(2R) \leq R \min_{R \leq r \leq 2R} \tilde{F}(r) r^{k-1} \leq \int_R^{2R} \tilde{F}(r) r^{k-1} dr.$$

This implies that $\lim_{r \rightarrow \infty} r^k \tilde{F}(r) = 0$.

By Stokes' formula we have

$$\int_{V_r} \frac{\partial^2}{\partial x_i \partial x_j} B(\mathbf{u}(\mathbf{x}, t)) = \int_{\partial V_r} \frac{\partial}{\partial x_j} B(\mathbf{u}(\mathbf{x}, t)) n_i d\sigma_r$$

where n_i is the i -th component of the unit normal vector to the boundary of the ball V_r .

Homogeneity of B implies that $\sum_{j=1}^n \frac{\partial}{\partial y_j} B(\mathbf{y}) y_j = k B(\mathbf{y})$. Since $\frac{\partial}{\partial y_j} B(\mathbf{y}) \geq 0$ we obtain estimate $\frac{\partial}{\partial y_j} B(\mathbf{y}) y_j \leq k B(\mathbf{y})$. Also we note that for each $t > 0$ there exists a constant L depending on the parameters t, u_j such that $\left| \frac{\partial}{\partial y} u_j(y, t) \right| \leq L y u_j(y, t)$ for all $y \in \mathbb{R}$.

So we obtain that

$$\begin{aligned} \left| \int_{\partial V_r} \frac{\partial}{\partial x_j} B(\mathbf{u}(\mathbf{x}, t)) n_i d\sigma_r \right| &\leq \sum_{\ell=1}^n \int_{\partial V_r} \frac{\partial B}{\partial u_\ell} \left| \frac{\partial u_\ell(\langle a_\ell, \mathbf{x} \rangle, t)}{\partial x_j} \right| d\sigma_r \leq \\ &c \int_{\partial V_r} r B(\mathbf{u}(\mathbf{x}, t)) d\sigma_r \leq C_1 r^k \tilde{F}(C_2 r) \end{aligned}$$

where constants C_1, C_2 do not depend on r . Since B is homogeneous and it is increasing with respect to each components the last inequality follows from the observation $B(\mathbf{u}(\mathbf{x}, t)) \leq C_3 F(C_2 x)$ where C_3, C_2 do not depend on x . So the lemma follows. □

Remark 14. *Lemma 18 holds even if we take supremum with respect to t over any compact subset of $(0, \infty)$.*

Corollary 6. *The function $\mathbf{B}(t)$ is increasing for $t > 0$, and it is constant if and only if $(A^* C A) \bullet \text{Hess } B(\mathbf{u}(\mathbf{x}, t)) \mathbf{u}'(\mathbf{x}, t) = \mathbf{0}$ for all $x \in \mathbb{R}^n$ and any $t > 0$.*

Proof. First we integrate (2.0.11) with respect to t (over any closed interval $[t_1, t_2] \subset (0, \infty)$) and then we integrate over the balls V_r . Thus the corollary is immediate consequence of Lemmas 17, 18 and Remark 14. □

Thus we obtain an inequality $\mathbf{B}(t_1) \leq \mathbf{B}(t_2)$ for $0 < t_1 \leq t_2 < \infty$ and we want to pass to the limits.

Lemma 19. *Let B satisfies assumptions L1 – L5 and let $u_j \in E(\mathbb{R})$ be nonnegative (not identically zero) functions. Then the following equalities hold*

$$\lim_{t \rightarrow 0} \mathbf{B}(t) = \int_{\mathbb{R}^k} B(u_1(\langle a_1, \mathbf{x} \rangle), \dots, u_n(\langle a_n, \mathbf{x} \rangle)) d\mathbf{x}, \quad (2.0.16)$$

$$\lim_{t \rightarrow \infty} \mathbf{B}(t) = \int_{\mathbb{R}^k} B \left(\frac{e^{-\frac{\langle a_1, \mathbf{x} \rangle^2}{4\langle Ca_1, a_1 \rangle}}}{\sqrt{\pi \langle Ca_1, a_1 \rangle}} \int_{\mathbb{R}} u_1 dx, \dots, \frac{e^{-\frac{\langle a_n, \mathbf{x} \rangle^2}{4\langle Ca_n, a_n \rangle}}}{\sqrt{\pi \langle Ca_n, a_n \rangle}} \int_{\mathbb{R}} u_n dx \right) d\mathbf{x}. \quad (2.0.17)$$

Proof. Take any nonnegative (not identically zero) functions $u_j \in E(\mathbb{R})$. Then there exist positive numbers β_j, δ_j such that $u_j(y) \leq \beta_j e^{-\delta_j y^2}$ for all $j = 1, \dots, n$. Note that

$$\begin{aligned} u_j(y, t) &= \frac{1}{(4\pi t \langle Ca_j, a_j \rangle)^{1/2}} \int_{\mathbb{R}} u_j(x) e^{-\frac{(y-x)^2}{4t \langle Ca_j, a_j \rangle}} dx \leq \\ &= \frac{\beta_j}{\sqrt{1 + 4t\delta_j \langle Ca_j, a_j \rangle}} e^{-\frac{y^2 \delta_j}{1 + 4t\delta_j \langle Ca_j, a_j \rangle}} \end{aligned}$$

So the first limit (2.0.16) follows immediately from Lebesgue's dominated convergence theorem. For the second limit (2.0.17) we use homogeneity of the function B . So by changing variable $\mathbf{x} = \mathbf{y}\sqrt{t}$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^k} B \left(\dots, \frac{1}{(4\pi t \langle Ca_j, a_j \rangle)^{1/2}} \int_{\mathbb{R}} u_j(x) e^{-\frac{(\langle a_j, \mathbf{x} \rangle - x)^2}{4t \langle Ca_j, a_j \rangle}} dx, \dots \right) d\mathbf{x} = \\ &\int_{\mathbb{R}^k} B \left(\dots, \frac{e^{-\frac{\langle a_j, \mathbf{y} \rangle^2}{4\langle Ca_j, a_j \rangle}}}{(4\pi \langle Ca_j, a_j \rangle)^{1/2}} \int_{\mathbb{R}} u_j(x) e^{-\frac{2\sqrt{t} \langle a_j, \mathbf{y} \rangle x - x^2}{4t \langle Ca_j, a_j \rangle}} dx, \dots \right) d\mathbf{y}. \end{aligned}$$

It is clear that that for each fixed \mathbf{y} integrand tends to

$$B \left(\dots, \frac{e^{-\frac{\langle a_j, \mathbf{y} \rangle^2}{4\langle Ca_j, a_j \rangle}}}{(4\pi\langle Ca_j, a_j \rangle)^{1/2}} \int_{\mathbb{R}} u_j(x) dx, \dots \right).$$

Since $u_j(x) \leq b_j e^{-\delta_j x^2}$ we obtain

$$u_j(x) e^{\frac{2\sqrt{t}\langle a_j, \mathbf{y} \rangle x}{4t\langle Ca_j, a_j \rangle}} \leq b_j e^{-\frac{\delta_j}{2}x^2} e^{\max_{x \geq 0} [-\frac{\delta_j}{2}x^2 + \alpha_j(t)x]},$$

where $\alpha_j(t) := \frac{\langle a_j, \mathbf{y} \rangle}{2\sqrt{t}\langle Ca_j, a_j \rangle}$. Hence

$$u_j(x) e^{\frac{2\sqrt{t}\langle a_j, \mathbf{y} \rangle x}{4t\langle Ca_j, a_j \rangle}} \leq b_j e^{-\frac{\delta_j}{2}x^2} e^{\frac{1}{2}\frac{\alpha_j(t)^2}{\delta_j}} = b_j e^{-\frac{\delta_j}{2}x^2} e^{\frac{\langle a_j, \mathbf{y} \rangle^2}{8t\delta_j\langle Ca_j, a_j \rangle^2}},$$

Now we can apply Lebesgue's dominated convergence theorem twice.

The last display estimate gives us a summable majorant for the integration in x . On the other hand,

$$e^{-\frac{\langle a_j, \mathbf{y} \rangle^2}{4\langle Ca_j, a_j \rangle}} e^{\frac{\langle a_j, \mathbf{y} \rangle^2}{8t\delta_j\langle Ca_j, a_j \rangle^2}} \leq e^{-\frac{\langle a_j, \mathbf{y} \rangle^2}{8\langle Ca_j, a_j \rangle}}$$

for all $t \geq t_C$. Thus we get the uniform in t estimate for the j th argument of function B :

$$\frac{\gamma_j e^{-\frac{\langle a_j, \mathbf{y} \rangle^2}{8\langle Ca_j, a_j \rangle}}}{(4\pi\langle Ca_j, a_j \rangle)^{1/2}},$$

where $\gamma_j := \int_{\mathbb{R}} b_j e^{-\frac{\delta_j}{2}x^2} dx$. Therefore we have the summable majorant (that it is summable

follows from L3)

$$B \left(\dots, \frac{\gamma_j e^{-\frac{\langle a_j, \mathbf{y} \rangle^2}{8\langle Ca_j, a_j \rangle}}}{(4\pi\langle Ca_j, a_j \rangle)^{1/2}}, \dots \right),$$

and the lemma is proved. \square

Corollary 6 and Lemma 19 imply the following theorem.

Theorem 2.0.1. *Let B satisfies assumptions L1 – L5 and let $u_j \in E(\mathbb{R})$ be nonnegative (not identically zero) functions. Then we have*

$$\begin{aligned} \int_{\mathbb{R}^k} B(u_1(\langle a_1, \mathbf{x} \rangle), \dots, u_n(\langle a_n, \mathbf{x} \rangle)) d\mathbf{x} \leq \\ \int_{\mathbb{R}^k} B \left(\frac{e^{-\frac{\langle a_1, \mathbf{x} \rangle^2}{\langle Ca_1, a_1 \rangle}}}{\sqrt{\pi\langle Ca_1, a_1 \rangle}} \int_{\mathbb{R}} u_1, \dots, \frac{e^{-\frac{\langle a_n, \mathbf{x} \rangle^2}{\langle Ca_n, a_n \rangle}}}{\sqrt{\pi\langle Ca_n, a_n \rangle}} \int_{\mathbb{R}} u_n \right) d\mathbf{x}. \end{aligned} \quad (2.0.18)$$

Equality holds if and only if

$$(A^*CA) \bullet \text{Hess } B(\mathbf{u}(\mathbf{x}, t)) \mathbf{u}'(\mathbf{x}, t) = \mathbf{0} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \quad \text{and any } t > 0. \quad (2.0.19)$$

Remark 15. *So any function satisfying our strange concavity condition L3, homogeneity condition L2 and some mild conditions L1, L4, L5 gives a certain Brascamp–Lieb inequality. Our next goal will be to show that in interesting cases the finiteness of (2.0.13) implies that there is basically only one such B .*

In the *Bellman function technique* theorems of the above type are known as a first part of the Bellman function method which is usually simple. Any function B that satisfies properties L1 – L5 will be called Bellman function of Brascamp–Lieb type.

The difficult technical part is how to find such Bellman functions. It is worth mentioning

that the property $L3$ in principle requires solving partial differential inequalities. We are going to give partial answer on this question in the following section.

2.1 How to find the Bellman function

Definition 12. Let $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \text{int}(\mathbb{R}_+^n)$ and let $D(\mathbf{y})$ be a diagonal square matrix such that on the diagonal it has the terms $\frac{y_j}{\langle Ca_j, a_j \rangle}, j = 1, \dots, n$.

Theorem 2.1.1. If the function B satisfies assumptions $L1 - L5$ then we have

$$AD(\mathbf{y})[A^*CA \bullet \text{Hess } B(\mathbf{y})] = \mathbf{0} \quad \text{for all } \mathbf{y} \in \text{int}(\mathbb{R}_+^n). \quad (2.1.1)$$

Remark 16. Equality (2.1.1) is a second order partial differential equation on B . However, assumptions $L1 - L5$ are either of quantitative nature, or in the form of partial differential inequalities. So it is quite surprising that based only on assumptions $L1 - L5$ one can expect equality (2.1.1).

The proof of the above equality is interesting in itself.

Proof. We saw in the previous section that assumptions $L1 - L5$ imply the inequality (2.0.18).

One can easily observe that the following functions

$$u_j(y) = b_j \frac{e^{-\frac{y^2}{\langle Ca_j, a_j \rangle}}}{\sqrt{\pi \langle Ca_j, a_j \rangle}}, \quad b_j > 0.$$

give equality in the inequality (2.0.18). Since $\mathbf{u}'(\mathbf{x}, t) = \frac{-2}{4t+1} D(\mathbf{u}(\mathbf{x}, t)) A^* \mathbf{x}$, Theorem 2.0.1

implies that

$$A^*CA \bullet \text{Hess } B(\mathbf{u}(\mathbf{x}, t))D(\mathbf{u}(\mathbf{x}, t))A^*\mathbf{x} = \mathbf{0}$$

Choose any $\mathbf{x} \in \mathbb{R}^k$, any $\mathbf{y} \in \text{int}(\mathbb{R}_+^n)$ and any $t > 0$. We can find $b_1, \dots, b_n > 0$ such that

$$u_j(\langle a_j, \mathbf{x} \rangle, t) = b_j \frac{e^{-\frac{\langle a_j, \mathbf{x} \rangle^2}{\langle Ca_j, a_j \rangle(4t+1)}}}{\sqrt{\pi \langle Ca_j, a_j \rangle(4t+1)}} = y_j, \quad j = 1, \dots, n.$$

Hence we obtain

$$[A^*CA \bullet \text{Hess } B(\mathbf{y})]D(\mathbf{y})A^*\mathbf{x} = \mathbf{0}, \quad \forall \mathbf{x} \in \mathbb{R}^k, \quad \forall \mathbf{y} \in \text{int}(\mathbb{R}_+^n).$$

So equality (2.1.1) follows. □

Theorem 2.1.1 implies the following corollary.

Corollary 7. *For any $\mathbf{y} \in \text{int}(\mathbb{R}_+^n)$ we have*

$$\text{rank}(A^*CA \bullet \text{Hess } B(\mathbf{y})) \leq n - k. \tag{2.1.2}$$

The above corollary immediately follows from the fact that $\text{rank}(AD(\mathbf{y})) = \text{rank}(A) = k$ and, for example, from the Sylvester's rank inequality.

Thus for each fixed n we have a range of admissible dimensions $1 \leq k \leq n$. For the boundary cases $k = 1$ and $k = n$, we find the Bellman function with the properties $L1 - L5$. For the intermediate cases $1 < k < n$ we partially find the function B .

2.1.1 Case $k = 1$. Jointly concave and homogeneous function

We want to see that in this case $L1 - L5$ gives us precisely convex and 1-homogeneous functions. In the case $k = 1$ we have $A = (a_1, \dots, a_n) \in \mathbb{R}^n$. Since the condition $\langle Ca_j, a_j \rangle > 0$ must hold, the 1×1 matrix C must be a positive number and $a_j \neq 0$ for all $j = 1, \dots, n$. The fact that B is homogeneous of degree 1 and B is increasing with respect to each variable immediately imply $L5$. The only property we left to ensure is $L3$. For $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ let $d(\mathbf{v})$ denotes $n \times n$ diagonal matrix with entries v_j on the diagonal.

$$A^*CA \bullet \text{Hess } B(\mathbf{y}) = C \cdot A^*A \bullet \text{Hess } B(\mathbf{y}) = C \cdot d(A)\text{Hess } B(\mathbf{y})d(A) .$$

So the inequality $A^*CA \bullet \text{Hess } B(\mathbf{y}) \leq 0$ is equivalent to the inequality $\text{Hess } B(\mathbf{y}) \leq 0$, because C is just a number. Thus we obtain the following lemma.

Lemma 20. *If the function B satisfies assumptions $L1 - L5$ then $a_j \neq 0$ for all j , C is any positive number and $B \in C(\mathbb{R}_+^n) \cap C^2(\text{int}(\mathbb{R}_+^n))$ is a concave homogeneous function of degree 1. Conversely, if $a_j \neq 0$ for all j and $B \in C(\mathbb{R}_+^n) \cap C^2(\text{int}(\mathbb{R}_+^n))$ is a nonnegative, not identically zero, concave, homogeneous function of degree 1 then B satisfies assumptions $L1 - L5$.*

The above lemma gives complete characterization of the Bellman function in the case $k = 1$, and the inequality (2.0.18) is the same as inequality (2.0.9) (see Lemma 15).

2.1.2 Case $k = n$. $B(\mathbf{y}) = \text{Const} \cdot y_1 \cdots y_n$

We show that in the case $k = n$ the assumptions $L1 - L5$ are satisfied if and only if $B(\mathbf{y}) = My_1 \cdots y_n$ where M is a positive number. We present 3 different proofs (according

to their chronological order), each of them uses different assumptions on B in necessity part. Sufficiency follows immediately. Indeed, if $B = My_1 \cdots y_n$ then all the assumptions $L1 - L5$ are satisfied except that one has to check existence of the symmetric matrix C such that $A^*CA \bullet \text{Hess } B \leq 0$ and $\langle Ca_j, a_j \rangle > 0$. But it is enough to take $C = (AA^*)^{-1}$. Now we go to proving necessity.

2.1.2.1 First proof

As we already mentioned the assumptions $L1 - L5$ imply that $B \in C^2(\text{int}(\mathbb{R}_+^n)) \cap C(\mathbb{R}_+^n)$ is nonnegative, separately concave, and it is homogeneous of degree n . We need to show that such B then must have the form $B(\mathbf{y}) = My_1 \cdots y_n$. To show this, we consider a function G such that $G(\ln z_1, \dots, \ln z_n) = \frac{B(z_1, \dots, z_n)}{z_1 \cdots z_n}$ for $z_j > 0$. Homogeneity of order 0 of B implies that $\text{div } G = 0$, and concavity of B with respect to each variable implies that $\frac{\partial G}{\partial y_j} + \frac{\partial^2 G}{\partial y_j^2} \leq 0$ for $j = 1, \dots, n$. After summation of the last inequalities we obtain that G is superharmonic function on \mathbb{R}^n . But then it is easy to check that if $\Delta G \leq 0$, then $\Delta g \geq 0$, where $g := e^{-G}$. We get a bounded subharmonic function $g, 0 \leq g \leq 1$, in the whole space. It is well known that then g must be constant. This implies that G is a constant.

2.1.2.2 Second proof

The second proof immediately follows from the following lemma which does not use any assumptions regarding smoothness of B .

Lemma 21. *If a function B defined on \mathbb{R}_+^n is nonnegative on the boundary of \mathbb{R}_+^n , and it is separately concave and homogeneous of degree n then $B(y_1, \dots, y_n) = My_1 \cdots y_n$ for some real number M .*

Proof. The idea is almost as follows: we are going to construct superharmonic function in the bounded domain such that it is nonnegative on the boundary and it achieves zero value at an interior point of the domain. This implies that the constructed function is identically zero.

Consider a function $G(y_1, \dots, y_n) = B(\mathbf{y}) - B(1, \dots, 1)y_1 \cdots y_n$. Take any cube $Q = [0, R]^n$ where $R > 1$. The function G is separately concave and it is zero on the diagonal of the cube Q i.e. $G(y, \dots, y) = 0$ for $y \in [0, R]$. G is nonnegative on the whole boundary of the cube Q . Indeed, G is zero at the point (R, \dots, R) and it is nonnegative at point $(R, \dots, R, 0)$, so separate concavity implies that G is nonnegative on the set (R, \dots, R, t) where $t \in [0, R]$. Similar reasoning implies that G is nonnegative on the whole boundary of the cube Q .

Suppose now G is not zero at some interior point of the cube Q , say at point W . Take any interior point A_0 of the cube Q such that $G(A_0) = 0$. Take a sequence of points A_1, \dots, A_n belonging to the interior of Q such that the segments $A_j A_{j+1}$ (the segment in \mathbb{R}^n with the endpoints A_j, A_{j+1}) are collinear to one of the vector $e_k = (0, \dots, 1, \dots, 0)$ (on the k -th position we have 1 and the rest of the components are zero) for all $j = 0, \dots, n-1$, and the same is true for the segment $A_n W$. Then clearly G is zero on the segment $A_0 A_1$. Indeed, It is zero at point A_0 . Take a line joining the points A_0, A_1 . This line intersects the boundary of the cube Q , and G is concave on the line. Since G is nonnegative at the points of the intersection and it is zero at point A_0 we obtain that G is zero on the part of the line which lies in the cube Q . In particular, it is zero at A_1 . By induction we obtain that G is zero at the points A_2, \dots, A_n, W . So the lemma follows.

□

2.1.2.3 Third proof

In this proof let us assume that B is infinitely differentiable in $\int(\mathbb{R}_+^n)$. The assumptions $L1 - L5$ imply that B must be a separately concave. Therefore for the assumption $L3$ we can choose $C = (AA^*)^{-1}$. Then (2.1.2) implies that $\frac{\partial^2 B}{\partial y_j^2} = 0$ for all $j = 1, \dots, n$. We claim that if B satisfies the system of differential equations $\frac{\partial^2 B}{\partial y_j^2} = 0$ for all $j = 1, \dots, n$ then it has a form

$$c_0 + \sum_{k=1}^n \left(\sum_{i_p \neq i_q, i_1, \dots, i_k=1}^n c_{i_1 \dots i_k} \prod_{j=1}^k y_{i_j} \right) \quad (2.1.3)$$

where the second summation is taken over the pairwise different indexes. Indeed, proof is by induction over the dimension n . If $n = 1$ the claim is trivial. Since $\frac{\partial^2 B}{\partial y_1^2} = 0$ we have $B(\mathbf{y}) = y_1 B_1(y_2, \dots, y_n) + B_2(y_2, \dots, y_n)$. The condition $\frac{\partial^2 B_1}{\partial y_j^2} = \frac{\partial^3}{\partial y_1 \partial^2 y_j} B = 0$ for $j = 2, \dots, n$ implies that B_1 satisfies hypothesis of the claim. On the other hand, $B_2 = B(0, y_2, \dots, y_n)$, and so B_2 has less variables, but satisfies the same system of differential equations.

Homogeneity of B implies that $B(\mathbf{y}) = cy_1 \cdots y_n$.

Remark 17. *The second proof is a modification of the proof shown to us by Bernd Kirchheim, we express our gratitude to him.*

2.1.3 Case $k = n - 1$. Young's function.

Theorem 2.1.2. *If B satisfies assumptions $L1 - L5$ and $B_{y_i y_j} \neq 0$ in $\text{int}(\mathbb{R}_+^n)$ for all $i, j = 1, \dots, n$ then $B(\mathbf{y}) = My_1^{\alpha_1} \cdots y_n^{\alpha_n}$ for some $M > 0$ and $0 < \alpha_j < 1$, $j = 1, \dots, n$*

In the end of the section we present F. Nazarov's examples which show that if we remove the condition $B_{x_i x_j} \neq 0$ in the Theorem 2.1.2 then the conclusion of the theorem does not

hold. It is also worth mentioning that in the classical case when $n = 3$ and $k = 2$ we obtain that under the assumptions $L1 - L5$ and $B_{y_i y_j} \neq 0$ there are *only* Young's inequalities for convolution.

Proof. Equality (2.1.1) is the same as

$$\sum_{j=1}^n y_j a_{js} B_{y_\ell y_j} \frac{\langle Ca_\ell, a_j \rangle}{\langle Ca_j, a_j \rangle} = 0, \quad \forall \ell = 1, \dots, n, \quad \forall s = 1, \dots, k. \quad (2.1.4)$$

We introduce a vector function $P(\mathbf{x}) = (p^1(\mathbf{x}), \dots, p^n(\mathbf{x}))$, where $\mathbf{x} \in \mathbb{R}^n$, such that

$$P(\ln y_1, \dots, \ln y_n) = \nabla B(\mathbf{y}).$$

Then equality (2.1.4), the fact that $B_{y_i y_j} = B_{y_j y_i}$ and homogeneity of B combined imply the following

$$\langle \nabla p^\ell, w_{\ell s} \rangle = 0, \quad \forall \ell = 1, \dots, n, \quad \forall s = 1, \dots, k; \quad (2.1.5)$$

$$e^{-x_j} p_{x_j}^i = e^{-x_i} p_{x_i}^j, \quad i, j = 1, \dots, n; \quad (2.1.6)$$

$$\operatorname{div} p^\ell = (k - 1) p^\ell, \quad \ell = 1, \dots, n. \quad (2.1.7)$$

where

$$w_{\ell s} = \left(a_{1s} \frac{\langle Ca_\ell, a_1 \rangle}{\langle Ca_1, a_1 \rangle}, \dots, a_{ns} \frac{\langle Ca_\ell, a_n \rangle}{\langle Ca_n, a_n \rangle} \right), \quad \forall \ell = 1, \dots, n, \quad \forall s = 1, \dots, k. \quad (2.1.8)$$

Now we show that the assumptions $B_{y_i y_j} \neq 0$ imply that $\langle Ca_i, a_j \rangle \neq 0$ for all $i, j = 1, \dots, n$.

Indeed, suppose that $\langle Ca_{i_0}, a_{j_0} \rangle = 0$ for some i_0 and j_0 . Assumption $L3$ implies that

$i_0 \neq j_0$. Since C is symmetric we get that $\langle Ca_{j_0}, a_{i_0} \rangle = 0$. Corollary 7 says now that $\text{rank}(A^*CA \bullet \text{Hess } B(\mathbf{y})) \leq 1$. So the determinant of any 2×2 submatrix of $A^*CA \bullet \text{Hess } B = \left\{ \langle Ca_i, a_j \rangle B_{y_i y_j} \right\}_{i,j}$ (2-minor) is zero. Consider 2×2 submatrix of $A^*CA \bullet \text{Hess } B$ with the following entries: $(i_0, i_0), (i_0, j_0), (j_0, i_0)$ and (j_0, j_0) . Since its determinant is zero and we assumed $\langle Ca_{i_0}, a_{j_0} \rangle = 0$, we get that $\langle Ca_{i_0}, a_{i_0} \rangle \langle Ca_{j_0}, a_{j_0} \rangle = 0$. This contradicts to our assumption *L3*.

Thus we obtain that for each fixed ℓ the vectors $w_{\ell s}$, $s = 1, \dots, n$, span $k = n - 1$ dimensional subspace W_ℓ . Therefore, equality (2.1.5) implies that $\nabla p^\ell(\mathbf{x}) = \lambda^\ell(\mathbf{x}) v^\ell$ where $\lambda^\ell(\mathbf{x})$ is a nonvanishing scalar valued function in $\text{int}(\mathbb{R}_+^n)$, $v^\ell \perp W_\ell$ and none of the components of v^ℓ is zero.

The equality (2.1.7) implies that we can choose v^ℓ so that $\langle v^\ell, \mathbf{1} \rangle = k - 1$ (here $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$) and so that $\lambda^\ell(\mathbf{x}) = p^\ell(\mathbf{x})$ for all $\ell = 1, \dots, n$.

Hence the equation $\nabla p^\ell(\mathbf{x}) = p^\ell(\mathbf{x}) v^\ell$ easily implies that $p^\ell(\mathbf{x}) = e^{\langle v^\ell, \mathbf{x} \rangle} p^\ell(\mathbf{0})$ for all ℓ . The equalities (2.1.6) imply that

$$v^\ell = (q_1, \dots, q_{\ell-1}, q_\ell - 1, q_{\ell+1}, \dots, q_n), \quad \forall \ell = 1, \dots, n.$$

where $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}^n$. It also follows that $P(\mathbf{0}) = k\mathbf{q}$ for some number $k \neq 0$. Thus we get that $B_{y_\ell} = k q_\ell y_1^{q_1} \cdots y_n^{q_n} / y_\ell$ and this proves Theorem 2.1.2. \square

2.1.3.1 Example of necessity of the assumption $B_{y_i y_j} \neq 0$ in Theorem 2.1.2

Let $n = 3, k = 2$ and $B(x_1, x_2, x_3) = \varphi(x_1, x_2)x_3$ where $\varphi \in C^2(\text{int } \mathbb{R}_+^2) \cap C(\mathbb{R}_+^2)$ is an arbitrary concave function and homogeneous of degree 1. Let

$$A = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} \\ 1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^*CA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since φ satisfies homogeneous Monge–Ampère equation we have $A^*CA \bullet \text{Hess } B \leq 0$. Clearly all the assumptions $L1 - L5$ are satisfied.

2.1.3.2 Theorem 2.1.2 does not hold in the case $1 < k < n - 1$

It turns out that even if $B_{y_i y_j} \neq 0$ and $1 < k < n - 1$ then it is not necessarily true that a function B which satisfies assumptions $L1 - L5$ has a form $B = My_1^{\alpha_1} \cdots y_n^{\alpha_n}$. This means that Theorem 2.1.2 cannot be improved. We give an example in a general case.

Assume that $1 < k < n - 1$ and $n > 3$ (case $n = 3$ was already discussed above). Take arbitrary nonnegative $\varphi \in C^2(\text{int } (\mathbb{R}_+^2)) \cap C(\mathbb{R}_+^2)$ so that φ is a concave function and homogeneous of degree one. We choose φ so that it has nonzero second derivatives. We

consider the following function

$$B(\mathbf{y}) = y_1^{\alpha_1} \cdots y_{n-2}^{\alpha_{n-2}} \cdot \varphi(y_{n-1}, y_n), \quad \mathbf{y} \in \mathbb{R}_+^n. \quad (2.1.9)$$

Let $a_{n-1} = a_n = (0, \dots, 0, 1) \in \mathbb{R}^k$ and let $a_1 = (\tilde{a}_1, 0), \dots, a_{n-2} = (\tilde{a}_{n-2}, 0) \in \mathbb{R}^k$. We choose vectors $\tilde{a}_1, \dots, \tilde{a}_{n-2} \in \mathbb{R}^{k-1}$ in the following way. First of all they span \mathbb{R}^{k-1} . Intersection of the interior of the convex hull K (described in the Introduction and constructed by the vectors $\tilde{a}_1, \dots, \tilde{a}_{n-2}$) with the hyperplane $\{y_1 + y_2 + \dots + y_{n-2} = k-1\}$ is nonempty. We choose a point $(\alpha_1, \dots, \alpha_{n-2})$ from this intersection.

Then there exists $(k-1) \times (k-1)$ symmetric, positive semidefinite matrix \tilde{C} such that $\tilde{A}^* \tilde{C} \tilde{A} \bullet \text{Hess } \tilde{B} \leq 0$ where $\tilde{A} = (\tilde{a}_1, \dots, \tilde{a}_{n-2})$ and

$$\tilde{B}(y_1, \dots, y_{n-2}) = y_1^{\alpha_1} \cdots y_{n-2}^{\alpha_{n-2}}.$$

Moreover, we have $\langle \tilde{C} \tilde{a}_j, \tilde{a}_j \rangle > 0$ for $j = 1, \dots, n-2$. The existence of such a matrix \tilde{C} follows from the solution of the Euler–Lagrange equation for the right side of (2.0.18) (see [25], Theorem 5.2), see also Subsection 2.1.4 below. It is clear that the function B satisfies all properties $L1 - L5$ except one has to check the property $L3$. We choose C as follows

$$C = \begin{pmatrix} \tilde{C} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

Function B from (2.1.9) satisfies $L3$ (and of course it can easily be made to satisfy all other properties $L1 - L5$), but it is not a Young function.

2.1.4 Case of Young's function

In this subsection we consider classical case when $B(\mathbf{y}) = y_1^{1/p_1} \cdots y_n^{1/p_n}$ where $1 \leq p_j < \infty$. Assumptions $1 \leq p_j$ follows from the assumption $L3$ (which implies in particular that the function B is separately concave) and the assumption $p_j < \infty$ was made because otherwise we have a function of less variables $m < n$. Note that we also must require that $\sum \frac{1}{p_j} = k$. This function satisfies all assumptions of $L1 - L5$ except of $L3$. We try to understand for which matrix A and numbers p_j there is a matrix C mentioned in the assumption $L3$. The answer on this question was obtained in [25] by using Euler–Lagrange equation.

We will obtain equation on the matrix C .

Note that $\text{Hess } B = B \cdot \left\{ \frac{1}{p_i p_j y_i y_j} \right\} - B \cdot \left\{ \frac{\delta_{ij}}{p_i y_i^2} \right\}$ where $\delta_{ij} = 1$ if $i = j$, and otherwise it is zero. Therefore equality (2.1.1) becomes

$$A \left\{ \frac{y_i}{\langle C a_i, a_i \rangle} \right\} \left[A^* C A \bullet \left(B \cdot \left\{ \frac{1}{p_i p_j y_i y_j} \right\} - B \cdot \left\{ \frac{\delta_{ij}}{p_i y_i^2} \right\} \right) \right] = 0$$

After simplification we obtain

$$A \left\{ \frac{1}{p_i \langle C a_i, a_i \rangle} \right\} A^* C = I_{k \times k} \quad (2.1.10)$$

Notice that the rank of $A \left\{ \frac{1}{\sqrt{p_i \langle C a_i, a_i \rangle}} \right\}$ is k because the rank of A is k . Then $k \times k$ matrix $A \left\{ \frac{1}{p_i \langle C a_i, a_i \rangle} \right\} A^*$ is invertible by Binet–Cauchy formula. Then we can find C from (2.1.10) by the following obvious formula

$$C = \left(A \text{diag} \left\{ \frac{1}{p_i \langle C a_i, a_i \rangle} \right\} A^* \right)^{-1} \quad (2.1.11)$$

if we can solve the following system of non-linear equations defining $\langle Ca_j, a_j \rangle$, $j = 1, \dots, n$:

$$\langle Ca_j, a_j \rangle = \left\langle \left(A \operatorname{diag} \left\{ \frac{1}{p_i \langle Ca_i, a_i \rangle} \right\} A^* \right)^{-1} a_j, a_j \right\rangle. \quad (2.1.12)$$

Using the notations

$$s_j^2 := \frac{1}{p_j \langle Ca_j, a_j \rangle}, \quad j = 1, \dots, n,$$

we readily transfer (2.1.12) to

$$\frac{1}{p_j} = s_j^2 \left\langle \left(A \operatorname{diag} \left\{ s_j^2 \right\} A^* \right)^{-1} a_j, a_j \right\rangle, \quad j = 1, \dots, n, \quad (2.1.13)$$

which is precisely equation (3.12) of [25]. In [24], [25] it is proved that for $\{\frac{1}{p_j}\}_{j=1}^n$ in the interior of the convex set K from [24], [25] this system (2.1.13) has a solution. In particular, C as in (2.1.11) does exist.

Notice also, that the Young's functions found by Brascamp–Lieb [23] and corresponding to the interior of the convex set K from [24], [25], do satisfy all properties $L1 - L5$. Only $L3$ is interesting because we need to show that there exists a certain matrix C . We just found a certain C in (2.1.11) (when the system (2.1.12) has a solution). This matrix C will satisfy $L3$ when B is the Young's function $B(\mathbf{y}) = y_1^{1/p_1} \dots y_n^{1/p_n}$ where $1 < p_j < \infty$, $\sum_{j=1}^n 1/p_j = k$. In fact, $A^*CA \bullet \operatorname{Hess}B(\mathbf{y}) \leq 0$ for such a B is equivalent to

$$\operatorname{diag} \left\{ \frac{1}{y_j p_j} \right\} A^*CA \operatorname{diag} \left\{ \frac{1}{y_j p_j} \right\} \leq \operatorname{diag} \left\{ \frac{\langle Ca_j, a_j \rangle p_j}{y_j^2 p_j^2} \right\}.$$

This is immediately equivalent to

$$A^*CA \leq \operatorname{diag}\{1/s_j^2\}.$$

But denoting $S = \text{diag}\{s_j\}$ we make this inequality $(AS)^*C(AS) \leq I_{n \times n}$, which holds because $(AS)^*C(AS)$ is an orthogonal projection onto the span of the columns of $S(A^*)$.

So, we repeat, that the Young's functions found by Brascamp–Lieb [23] and corresponding to the interior of the convex set K from [24], [25], do satisfy all properties $L1 - L5$. But it is more interesting that, as we have shown above, in certain situations all functions satisfying $L1 - L5$ must be of the form of a Young function found by Brascamp and Lieb.

Chapter 3

Harmonic analysis, PDE and differential geometry

3.0.5 Short review of some harmonic analysis problems

3.0.5.1 John–Nirenberg inequality

For a measurable function f , and measurable set K we set

$$\langle f \rangle_K \stackrel{\text{def}}{=} \frac{1}{|K|} \int_K f.$$

We say that the measurable function f belongs to $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} \stackrel{\text{def}}{=} \sup_{Q \subset \mathbb{R}^n} \langle |f - \langle f \rangle_Q|^2 \rangle_Q^{1/2} < \infty,$$

where the supremum is take over all hypercubes Q . Theorem about equivalence of BMO norms states that

Theorem 3.0.3. *For any $p \in (0, \infty)$ there exist $c_1, c_2 > 0$ such that:*

$$c_1 \|f\|_{BMO} \leq \sup_{Q \subset \mathbb{R}^n} \langle |f - \langle f \rangle_Q|^p \rangle_Q^{1/p} \leq c_2 \|f\|_{BMO}.$$

The celebrated John–Nirenberg inequality describes growth of the distributions of the function from BMO :

Theorem 3.0.4. *There exist constants $c_1, c_2 > 0$ such that for any $f \in BMO$ we have*

$$\frac{1}{|Q|} \left| \left\{ x : |f(x) - \langle f \rangle_Q| > \lambda \right\} \right| \leq c_1 e^{-c_2 \frac{\lambda}{\|f\|_{BMO}}}.$$

John–Nirenberg inequality in integral form states that:

Theorem 3.0.5. *There exists $\varepsilon_0 > 0$ and a positive function $C(\varepsilon)$, $0 < \varepsilon < \varepsilon_0$ such that*

$$\langle e^\varphi \rangle_I \leq C(\varepsilon) e^{\langle \varphi \rangle_I} \quad \text{for all } \varphi \in BMO(I) : \|\varphi\|_{BMO} \leq \varepsilon.$$

Interesting question is to find the best possible $C(\varepsilon)$ and ε_0 .

Closely related classes to BMO are A_p classes, reverse Hölder classes and Gehring classes.

3.0.5.2 Uniform convexity

Let I be an interval of the real line. For an integrable function f over I , we set $\|f\|_p \stackrel{\text{def}}{=} \langle |f|^p \rangle_I^{1/p}$. We recall the definition of uniform convexity of a normed space $(X, \|\cdot\|)$ (see [29]).

Definition 13. (Clarkson '36) *X is uniformly convex if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\left\| \frac{x+y}{2} \right\| \leq 1 - \delta$.*

Modulus of convexity of the normed space X is defined as follows:

$$\delta_X(\varepsilon) = \inf \left\{ \left(1 - \frac{\|f+g\|}{2} \right) : \|f\| = 1, \|g\| = 1, \|f - g\| \geq \varepsilon \right\}.$$

Remark 18. $(X, \|\cdot\|)$ space is uniformly convex iff $\delta_X(\varepsilon) > 0$.

O. Hanner (see [28]) gave an elegant proof of finding the constant $\delta_{L^p}(\varepsilon)$ in $L^p([0, 1])$ space for $p \in (1, \infty)$ in 1955. He proved two necessary inequalities (further called Hanner's inequalities) in order to obtain constant $\delta_{L^p}(\varepsilon)$. Namely,

$$\|f + g\|_p^p + \|f - g\|_p^p \geq (\|f\|_p + \|g\|_p)^p + \left| \|f\|_p - \|g\|_p \right|^p, \quad p \in [1, 2], \quad (3.0.1)$$

and the inequality (3.0.1) is reversed if $p \geq 2$. Hanner mentions in his note [28] that his proof is a reconstruction of some Beurling's ideas given at a seminar in Upsala in 1945. In [30] non-commutative case of Hanner's inequalities was investigated. Namely, Hanner's inequality holds for $p \in [1, 3/4] \cup [4, \infty)$, and the case $p \in (3/4, 4)$ (where $p \neq 2$) was left open.

3.0.5.3 Brunn–Minkowski and isoperimetric inequalities

Let A and B be nonempty compact subsets of \mathbb{R}^n .

Theorem 3.0.6. *The following sharp Brunn–Minkowski inequality holds*

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n},$$

where $n \geq 1$ and $|A|$ denotes Lebesgue measure of the set A .

The Brunn–Minkowski inequality is a consequence of its multiplicative version:

Theorem 3.0.7. *Let $\lambda \in (0, 1)$. Then for any compact measurable sets $U, V \subset \mathbb{R}^n$ we have*

$$|\lambda U + (1 - \lambda)V| \geq |U|^\lambda |V|^{1-\lambda}. \quad (3.0.2)$$

Indeed, if one sets $U\lambda = A$ and $(1 - \lambda)V = B$ then inequality (3.0.2) takes the form

$$|A + B| \geq \frac{|A|^\lambda |B|^{1-\lambda}}{\lambda^\lambda (1 - \lambda)^{1-\lambda}}. \quad (3.0.3)$$

By maximizing the right hand side of (3.0.3) over $\lambda \in (0, 1)$ we obtain the Brunn–Minkowski inequality.

Brunn–Minkowski inequality implies the classical isoperimetric inequality:

Theorem 3.0.8. *Among all simple closed surfaces with given surface area, the sphere encloses a region of maximal volume. In other words*

$$|\partial A| \geq n|A|^{1-\frac{1}{n}}|B(0, 1)|^{\frac{1}{n}}.$$

Where $|\partial A|$ means surface area of the boundary of the body A . $|A|$ denotes volume of the body and $B(0, 1)$ denotes the ball of radius 1 at center 0.

Indeed, let us sketch the proof: Since $|A + B(0, \varepsilon)| = |A| + \varepsilon|\partial A| + O(\varepsilon^2)$, we have

$$|\partial A| = \lim_{\varepsilon \rightarrow 0} \frac{|A + B(0, \varepsilon)| - |A|}{\varepsilon} \geq \lim_{\varepsilon \rightarrow 0} \frac{(|A|^{1/n} + |B(0, \varepsilon)|^{1/n})^n - |A|}{\varepsilon} = n|A|^{1-\frac{1}{n}}|B(0, 1)|^{\frac{1}{n}}.$$

3.0.5.4 Sobolev inequality

It is known that the classical isoperimetric inequality is equivalent to its functional version, to Sobolev inequality on \mathbb{R}^n with optimal constant

$$\int_{\mathbb{R}^n} |\nabla f| \geq n|B(0, 1)|^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} \right)^{1-\frac{1}{n}}. \quad (3.0.4)$$

Indeed, testing (3.0.4) over characteristic functions $f(x) = \mathbb{1}_A(x)$ we obtain implication in one direction. Opposite direction follows from Coarea formula: assume $f \geq 0$ is sufficiently nice compactly supported function. Then by coarea formula we have

$$\int_{\mathbb{R}^n} |\nabla f| dx = \int_0^\infty |\{x : f(x) = t\}| dt \geq n|B(0, 1)|^{\frac{1}{n}} \int_0^\infty |\{x : f(x) \geq t\}|^{1-\frac{1}{n}} dt.$$

It is left to show that

$$\left(\int_0^\infty |\{x : f(x) \geq t\}|^{\frac{n-1}{n}} dt \right)^{\frac{n}{n-1}} \geq \frac{n}{n-1} \int_0^\infty |\{x : f(x) \geq t\}| t^{\frac{1}{n-1}} dt$$

This follows from the following observation

$$F \left(\int_0^\infty \varphi \right) = \int_0^\infty \frac{d}{dt} F \left(\int_0^t \varphi \right) dt = \int_0^\infty F' \left(\int_0^t \varphi \right) \varphi dt \geq \int_0^\infty F'(t\varphi(t)) \varphi(t) dt,$$

where φ is decreasing and F' is increasing ($F(t) = t^{\frac{n}{n-1}}$, $\varphi(t) = |\{x : f(x) \geq t\}|^{\frac{n-1}{n}}$). So the claim follows.

3.0.5.5 Prekopa–Leindler inequality

Multiplicative Brunn–Minkowski inequality follows from its functional version, so called Prekopa–Leindler inequality.

Theorem 3.0.9. *Let h, f, g be positive measurable functions and $\lambda \in (0, 1)$. If*

$$h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda} \tag{3.0.5}$$

Then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^\lambda \left(\int_{\mathbb{R}^n} g \right)^{1-\lambda}.$$

If one takes $h(x) = \mathbb{1}_{\lambda U + (1-\lambda)V}(x)$, $f(x) = \mathbb{1}_U(x)$ and $g(x) = \mathbb{1}_V(x)$ then clearly the assumption (3.0.5) is satisfied and one obtains multiplicative version of Brunn–Minkowski inequality. It is not quite clear what will be the functional analog of original Brunn–Minkowski inequality.

Straightforward generalization of Prekopa–Leindler inequality takes the following form:

Theorem 3.0.10. *Let $f_j : \mathbb{R}^n \rightarrow R_+$ be integrable functions, and let $\sum_{j=1}^m \lambda_j = 1$, $0 < \lambda_j < 1$. If*

$$h \left(\sum_{j=1}^m \lambda_j x_j \right) \geq \prod_{j=1}^m f(x_j)^{\lambda_j},$$

then

$$\int_{\mathbb{R}^n} h \geq \prod_{j=1}^m \left(\int_{\mathbb{R}^n} f_j \right)^{\lambda_j}.$$

The above inequality can be treated as reverse to Hölder’s inequality:

$$\int_{\mathbb{R}^n} \sup \left\{ \prod_{j=1}^m f(x_j)^{\lambda_j} : \sum x_j \lambda_j = z \right\} dz \geq \prod_{j=1}^m \left(\int_{\mathbb{R}^n} f_j \right)^{\lambda_j} \geq \prod_{j=1}^m \int_{\mathbb{R}^n} f_j(x_j)^{\lambda_j}.$$

where integral in the left hand side is understood as upper Lebesgue integral.

One of the other applications of Prekopa–Leindler inequality in probability is that:

Corollary 8. *If $F(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^+$ is log-concave distribution i.e.,*

$$F(\lambda u + (1 - \lambda)v) \geq F(u)^{1-\lambda} F(v)^\lambda \quad \text{for all } u, v \in \mathbb{R}^{n+m},$$

then $H(x) = \int_{\mathbb{R}^m} F(x, y) dy$ is log-concave distribution.

The corollary immediately follows from application of Prekopa–Leindler inequality to the functions $F(x, \lambda y_1 + (1 - \lambda)y_2)$, $F(x, y_1)$ and $F(x, y_2)$.

3.0.5.6 Borell–Brascamp–Lieb inequality

We also mention Borell–Brascamp–Lieb inequality since it generalizes Prekopa–Leindler inequality

Theorem 3.0.11. *Let h, f, g be nonnegative functions, $0 < \lambda < 1$ and $-\frac{1}{n} \leq p \leq \infty$.*

Suppose

$$h(\lambda x + (1 - \lambda)y) \geq M_p(f(x), g(y), \lambda),$$

where

$$M_p(a, b, \lambda) \stackrel{\text{def}}{=} (\lambda a^p + (1 - \lambda)b^p)^{1/p}, \quad M_0 \stackrel{\text{def}}{=} (a, b, \lambda) = a^\lambda b^{1-\lambda}.$$

Then

$$\int_{\mathbb{R}^n} h \geq M_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g, \lambda \right).$$

3.0.5.7 Ehrhard's inequality

The condition of Preko–Leindler type appears in Ehrhards inequality:

Theorem 3.0.12. *Let $d\gamma(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}} dx$ be the Gaussian measure. And let $\Phi(x) = \int_{-\infty}^x d\gamma$. Then for any measurable compact sets $A, B \subset \mathbb{R}^n$ and any numbers $\lambda, \mu \geq 0$, such that $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ we have*

$$\Phi^{-1}(|\lambda A + \mu B|_\gamma) \geq \lambda \Phi^{-1}(|A|_\gamma) + \mu \Phi^{-1}(|B|_\gamma),$$

where $|A|_\gamma$ denotes Gaussian measure of A i.e., $|A|_\gamma = \int_A d\gamma$.

The inequality initially was stated for convex sets A and B . Later it was improved in the sense that only one of them has to be convex and it was conjectured that the inequality is true in general for any measurable sets, and the conjecture was proved recently. Ehrhrad's inequality is consequence of its functional version:

Theorem 3.0.13. *Let $h, f, g : \mathbb{R}^n \rightarrow [0, 1]$ be functions such that*

$$\Phi^{-1}(h(\lambda x + \mu y)) \geq \lambda \Phi^{-1}(f(x)) + \mu \Phi^{-1}(g(y)), \quad \text{for all } x, y \in \mathbb{R}^n,$$

where $\lambda, \mu \geq 0$, $\lambda + \mu \geq 1$ and $|\lambda - \mu| \leq 1$ then

$$\Phi^{-1}\left(\int_{\mathbb{R}^n} h d\gamma\right) \geq \lambda \Phi^{-1}\left(\int_{\mathbb{R}^n} f d\gamma\right) + \mu \Phi^{-1}\left(\int_{\mathbb{R}^n} g d\gamma\right).$$

3.0.5.8 Borell's Gaussian noise "stability"

Let $\gamma_n = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}$ be a standard Gaussian measure on \mathbb{R}^n and let $\Phi = \int_{-\infty}^x d\gamma_1$. Borell's Gaussian noise "stability" states that

Theorem 3.0.14. *If A, B are measurable subsets of \mathbb{R}^n . Then if $X = (X_1, \dots, X_n), Y = (Y_1, \dots, Y_n)$ are independent Gaussian standard random variables, and $p \in (0, 1)$ then*

$$\mathbb{P}(X \in A, pX + \sqrt{1-p^2}Y \in B) \leq \mathbb{P}(X_1 \leq \Phi^{-1}(\gamma_n(A)), pX_1 + \sqrt{1-p^2}Y_1 \leq \Phi^{-1}(\gamma_n(B))).$$

The functional version of the above inequality can be stated as follows:

Theorem 3.0.15. *Let $p \in (0, 1)$, $f, g : \mathbb{R}^n \rightarrow (0, 1)$ and let*

$$B(u, v) = \mathbb{P}(X_1 \leq \Phi^{-1}(u), pX_1 + \sqrt{1-p^2}Y_2 \leq \Phi^{-1}(v)).$$

Then

$$\int_{\mathbb{R}^{2n}} B\left(f(x), g(px + \sqrt{1-p^2}y)\right) d\gamma d\gamma \leq B\left(\int_{\mathbb{R}^n} f d\gamma, \int_{\mathbb{R}^n} g d\gamma\right).$$

3.0.5.9 Hypercontractivity

Let

$$P_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y) d\gamma(y)$$

be Ornstein–Uhlenbeck semigroup where $t \geq 0$. The hypercontractivity for Ornstein–Uhlenbeck semigroup means that

Theorem 3.0.16. *Let $p, q > 1$ be such that $\frac{q-1}{p-1} \geq e^{-2t}$. Then*

$$\|P_t f\|_{L^p(d\gamma)} \leq \|f\|_{L^q(d\gamma)}.$$

3.0.5.10 Logarithmic Sobolev inequalities

Let $d\gamma(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{k/2}} dx$. Logarithmic Sobolev inequality can be stated as follows

Theorem 3.0.17. *For any positive function f on \mathbb{R}^k we have*

$$\int_{\mathbb{R}^k} f^2 \ln f^2 d\gamma - \left(\int_{\mathbb{R}^k} f^2 d\gamma \right) \ln \left(\int_{\mathbb{R}^k} f^2 d\gamma \right) \leq 2 \int_{\mathbb{R}^k} |\nabla f|^2 d\gamma. \quad (3.0.6)$$

3.0.5.11 Beckner–Sobolev inequality

W. Beckner proved the following Sobolev inequality for Gaussian measure:

Theorem 3.0.18. *Let $1 \leq p \leq 2$. Then*

$$\|f\|_{L^2(d\gamma)}^2 - \|f\|_{L^p(d\gamma)}^2 \leq (2-p) \|\nabla f\|_{L^2(d\gamma)}^2.$$

3.0.5.12 Lévy–Gromov’s isoperimetric inequality

Let M be a compact connected Riemannian manifold of dimension $n \geq 2$, and of Ricci curvature bounded below by $R > 0$. Let μ be normalized Riemannian measure on M . Let $\sigma(r)$ be normalized volume of a geodesic ball of radius $r \geq 0$ on the n -sphere with curvature $R > 0$. Lévy–Gromov’s isoperimetric inequality states that:

Theorem 3.0.19. *For every set A in M with smooth boundary ∂A we have*

$$\sigma'(\sigma^{-1}(\mu(A))) \leq \mu_s(\partial A),$$

where $\mu_s(\partial A)$ stands for the surface measure of the boundary ∂A .

3.0.5.13 Bobkov's inequality

In particular since the spherical measures converge to Gaussian distributions one expects to obtain Lévy–Gromov's isoperimetric inequality in infinitely dimensional setting

Theorem 3.0.20. *Let $\Phi(x) \int_{-\infty}^x d\gamma$ and $\varphi(x) = \Phi'(x)$. Then for every Borel set $A \in \mathbb{R}^n$ with smooth boundary,*

$$\varphi(\Phi^{-1}(\gamma(A))) \leq \gamma_s(\partial A). \quad (3.0.7)$$

The celebrated functional version of the above inequality was obtained and proved by Bobkov.

Theorem 3.0.21. *Let $U(x) = \varphi(\Phi^{-1}(x))$. Then for any differentiable $f : \mathbb{R}^n \rightarrow (0, 1)$*

$$U \left(\int_{\mathbb{R}^n} f d\gamma \right) \leq \int_{\mathbb{R}^n} \sqrt{U^2(f) + |\nabla f|^2} d\gamma. \quad (3.0.8)$$

Testing (3.0.8) on the characteristic functions $f = \mathbb{1}_A$ and noticing that $U(0) = U(1) = 0$ gives the desired result (3.0.7).

It is worth mentioning Brascamp–Lieb inequality. We refer the reader to the Chapter 2.

3.0.6 Relation to PDEs

3.0.6.1 Prekopa–Leindler, Ehrhard’s inequality and its underlying PDE

One can see that in general inequalities of these types (Prekopa–Leindler, Ehrhard) can be formulated as follows. Let $B(x_1, \dots, x_m)$ be a smooth real valued function defined on $\Omega \subset \mathbb{R}^m$. And let $A_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n_j}$ be matrices $j = 1, \dots, m$. Let $(u_1, \dots, u_m) : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \rightarrow \Omega$ be smooth functions.

Question. *Under what conditions on B it is true that whenever*

$$B(u_1(A_1x), \dots, u_m(A_mx)) \geq 0, \quad \text{for all } x \in \mathbb{R}^n \quad (3.0.9)$$

we have that

$$B\left(\int_{\mathbb{R}^{n_1}} u_1 d\gamma_{n_1}, \dots, \int_{\mathbb{R}^{n_m}} u_m d\gamma_{n_m}\right) \geq 0. \quad (3.0.10)$$

where $d\gamma_k(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{k/2}} dx$ denotes k dimensional Gaussian measure.

Note that for the function $u : \mathbb{R}^k \rightarrow \mathbb{R}$ its heat extension can be written as follows

$$P_t u(x) = u(x, t) = \int_{\mathbb{R}^k} u(x + \sqrt{2t} y) d\gamma_k(y) \quad t \geq 0.$$

Notice that inequality (3.0.9) must remain true under the shifts and dilation of the variable

x . In particular this implies that (3.0.10) can be written as follows

$$C(x, t) \stackrel{\text{def}}{=} B(u_1(A_1x, t), \dots, u_m(A_mx, t)) = \\ B\left(\int_{\mathbb{R}^{n_1}} u_1(A_1x + \sqrt{2t}y_1)d\gamma_{n_1}(y), \dots, \int_{\mathbb{R}^{n_m}} u_m(A_mx + \sqrt{2t}y_m)d\gamma_{n_m}(y)\right) \geq 0.$$

Remark 19. *The same is true for Ornstein–Uhlenbeck extensions.*

Now the question is under what conditions on $C(x, t)$ the inequality $C(x, 0) \geq 0$ implies that $C(x, t) \geq 0$ for all $t > 0$. Further we always assume that

$$\liminf_{|x| \rightarrow \infty} \left(\inf_{T \geq t \geq 0} C(x, t) \right) \geq 0 \quad \text{for all } T > 0.$$

There is a simple way to check this condition: maximum principle for elliptic operator. If the action of an elliptic operator is nonpositive then the infimum is attained on the boundary. In other words, if there exists positive semi-definite matrix $\{a_{ij}(x, t)\}_{i,j=1}^n$ such that

$$\left(\sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^n b_j(x, t) \frac{\partial}{\partial x_j} - \frac{\partial}{\partial t} \right) C(x, t) \leq 0 \quad (3.0.11)$$

for some vector $b(x, t) = (b_1(x, t), \dots, b_n(x, t))$ then infimum is attained on the boundary $t = 0$. Indeed, the condition (3.0.11) implies that “whenever we are on the hill we go up”.

In other words if whenever $\text{Hess}_x C(x, t) \geq 0$ and $\nabla_x C(x, t) = 0$ then $\frac{\partial}{\partial t} C(x, t) \geq 0$.

Now the last conclusion gives the desired result. Indeed (see also [32]), take any $\varepsilon > 0$ and set $C_\varepsilon(x, t) = C(x, t) + \varepsilon t$. If $C_\varepsilon(x, t) < 0$ at some point then since $\text{Hess}_x C_\varepsilon(x, t) = \text{Hess}_x C(x, t) \geq 0$ and $\nabla_x C_\varepsilon(x, t) = \nabla_x C(x, t) = 0$ at that point then we must have $\frac{\partial}{\partial t} C_\varepsilon(x, t) = 0$. This means that $\frac{\partial}{\partial t} C(x, t) = -\varepsilon < 0$, but this contradicts to the fact

that whenever we are on the hill we must go up.

Definition 14. *We say that the function $C(x, t)$ satisfies ellipticity if (3.0.11) holds for some nonnegative matrix $\{a_{ij}(x, t)\}_{i,j=1}^n$ and the vector $\{b_j(x, t)\}_{j=1}^n$. We say that the function $C(x, t)$ satisfies hill property if whenever $\text{Hess}_x C(x, t) \geq 0$ and $\nabla_x C(x, t) = 0$ then $\frac{\partial}{\partial t} C(x, t) \geq 0$.*

Remark 20. *As we already mentioned ellipticity implies the hill property, however the converse is not true. Ehrhard's inequality is an example when the desired elliptic operator does not exist, however the hill property holds.*

In the following particular case we will describe all functions B for which $C(x, t)$ satisfies hill property and thus we will answer on our question. We consider the following case

$$B(u_1(a_1x + a_2y), u_2(x), u_3(y)) \geq 0, \quad (3.0.12)$$

where a_1, a_2 are real numbers and $x, y \in \mathbb{R}^n$. If B is sufficiently nice then one can rewrite the pointwise inequality (3.0.12) as follows

$$u_1(a_1x + a_2y) \geq H(u_2(x), u_3(y))$$

for some function H .

Theorem 3.0.22. *Let Ω be a rectangular domain in \mathbb{R}^2 . Let $a_1, a_2 \in \mathbb{R}$ be such that $|a_1| + |a_2| \geq 1$ and $1 \geq ||a_1| - |a_2||$. Let the function $H(x, y) : \Omega \rightarrow \mathbb{R}$ be such that $H_1, H_2 \neq 0$ and*

$$H_1 H_2 H_{12} (1 - a_1^2 - a_2^2) + a_1^2 H_2^2 H_{11} + a_2^2 H_1^2 H_{22} \geq 0. \quad (3.0.13)$$

If the real-valued functions u_1, u_2, u_3 are such that $(u_2, u_3) : \mathbb{R}^n \rightarrow \Omega$ and

$$u_1(a_1x + a_2y) \geq H(u_2(x), u_3(y)) \quad \text{for all } x, y \in \mathbb{R}^n,$$

then

$$\int_{\mathbb{R}^n} u_1 d\gamma \geq H \left(\int_{\mathbb{R}^n} u_2 d\gamma, \int_{\mathbb{R}^n} u_3 d\gamma \right).$$

Proof. It is enough to prove the theorem for the case $n = 1$. Indeed, for arbitrary n proof goes by induction. Consider the functions

$$h(z) = u_1(z, a_1x_2 + a_2y_2, \dots, a_1x_n + a_2y_n)$$

$$f(x_1) = u_2(x_1, x_2, \dots, x_n)$$

$$g(y_1) = u_3(y_1, \dots, y_n)$$

Then the theorem implies that

$$\int_{\mathbb{R}} h(z) d\gamma(z) \geq H \left(\int_{\mathbb{R}} u_2(x_1) d\gamma(x_1), \int_{\mathbb{R}} u_3(y_1) d\gamma(y_1) \right)$$

After that we apply the theorem to the new functions

$$\tilde{h}(z) = \int_{\mathbb{R}} u_1(x, z, a_1x_3 + a_2y_3, \dots, a_1x_n + a_2y_n) d\gamma(x)$$

$$\tilde{f}(x_2) = \int_{\mathbb{R}} u_2(x, x_2, x_3, \dots, x_n) d\gamma(x)$$

$$\tilde{g}(y_2) = \int_{\mathbb{R}} u_3(x, y_2, \dots, y_n) d\gamma(x)$$

and by iterating this process we see that it is sufficient to prove the theorem for the case $n = 1$.

We will check the hill property. We would like to show that the following quantity is nonnegative

$$\frac{\partial}{\partial t} (u_1(a_1x + a_2y, t) - H(u_2(x, t), u_3(y, t))) = u_1'' - H_1u_2'' - H_2u_3'', \quad (3.0.14)$$

under the assumptions that

$$\begin{pmatrix} u_1''a_1^2 - H_{11}(u_2')^2 - H_1u_2'' & u_1''a_1a_2 - H_{12}u_2'u_3' \\ u_1''a_1a_2 - H_{12}u_2'u_3' & u_1''a_2^2 - H_{22}(u_3')^2 - H_2u_3'' \end{pmatrix} \geq 0, \quad (3.0.15)$$

and

$$a_1u_1' - H_1u_2' = 0 \quad \text{and} \quad a_2u_1' - H_2u_3' = 0. \quad (3.0.16)$$

Expressing u_2', u_3' from (3.0.16) and substituting into (3.0.15) gives

$$\begin{pmatrix} u_1''a_1^2 - \frac{H_{11}}{H_1^2}a_1^2(u_1')^2 - H_1u_2'' & u_1''a_1a_2 - \frac{H_{12}}{H_1H_2}a_1a_2(u_1')^2 \\ u_1''a_1a_2 - \frac{H_{12}}{H_1H_2}a_1a_2(u_1')^2 & u_1''a_2^2 - \frac{H_{22}}{H_2^2}a_2^2(u_1')^2 - H_2u_3'' \end{pmatrix} \geq 0.$$

Further we assume that $H_1, H_2 \neq 0$. We will treat derivatives $u_1', u_1'', u_2'', u_3''$ as independent variables. We multiply the above matrix by $(u_2')^{-2}$ (and (3.0.14) as well) and we introduce the new variables $\frac{u_1''}{(u_1')^2} = x$, $\frac{H_1u_2''}{(u_1')^2} = ya_1^2$ and $\frac{H_2u_3''}{(u_1')^2} = za_2^2$ (further we assume that $a_1, a_2 \neq 0$). Then positive definiteness of the above matrix is the same as positive definiteness of the following matrix (after conjugating by 2×2 diagonal matrix with the elements on the diagonal

$a_1, a_2)$

$$\begin{pmatrix} x - \frac{H_{11}}{H_1^2} - y & x - \frac{H_{12}}{H_1 H_2} \\ x - \frac{H_{12}}{H_1 H_2} & x - \frac{H_{22}}{H_2^2} - z \end{pmatrix} \geq 0. \quad (3.0.17)$$

Desired expression (3.0.14) takes the following form

$$x - ya_1^2 - za_2^2 = (x, y, z) \cdot (1, -a_1^2, -a_2^2). \quad (3.0.18)$$

We denote $\frac{H_{11}}{H_1^2} = p$, $\frac{H_{22}}{H_2^2} = q$, and $\frac{H_{12}}{H_1 H_2} = r$. Then Condition (3.0.17) is equivalent to the following two inequalities

$$(x, y, z) \cdot (2, -1, -1) \geq p + q;$$

$$yz - xy - xz + pz + qy - x(p + q - 2r) + pq - r^2 \geq 0.$$

We shift coordinates $x \mapsto x + r$, $y \mapsto y + r - p$, $z \mapsto z + r - q$ and we obtain that the above conditions take the following form:

$$(x, y, z) \cdot (2, -1, -1) \geq 0; \quad (3.0.19)$$

$$yz - xy - xz \geq 0; \quad (3.0.20)$$

and the desired inequality becomes

$$(x, y, z) \cdot (1, -a_1^2, -a_2^2) + r - a_1^2(r - p) - a_2^2(r - q) \geq 0. \quad (3.0.21)$$

Testing (3.0.21) for $x = 0, y = 0, z = 0$ we obtain the necessary condition that

$$r - a_1^2(r - p) - a_2^2(r - q) \geq 0.$$

By stretching the point $(x, y, z) \mapsto \lambda(x, y, z)$ (where $\lambda > 0$) we see that conditions (3.0.19, 3.0.20) remain true and we can get rid off the term $r - a_1^2(r - p) - a_2^2(r - q)$. So it remains to understand when the quantity $(x, y, z) \cdot (1, -a_1^2, -a_2^2)$ is nonnegative under the assumptions (3.0.19, 3.0.20). Since the cone (3.0.19, 3.0.20) must lie below the hyperplane $(x, y, z) \cdot (1, -a_1^2, -a_2^2) = 0$ we obtain that we must have

$$|a_1| + |a_2| \geq 1;$$

$$1 \geq ||a_1| - |a_2||.$$

□

Corollary 9. *If a function H satisfies conditions of Theorem 3.0.22 and in addition it is 1-homogeneous then the conclusion of the theorem holds for Lebesgue measure, i.e.,*

$$\int_{\mathbb{R}^n} u_1 dx \geq H \left(\int_{\mathbb{R}^n} u_2 dx, \int_{\mathbb{R}^n} u_3 dx \right)$$

Corollary 10. *If H is convex then conclusion of Theorem 3.0.22 holds.*

Corollary 11. *If $H(x, y) = x^{a_1} y^{a_2}$ then*

$$H_1 H_2 H_{12} (1 - a_1^2 - a_2^2) + a_1^2 H_2^2 H_{11} + a_2^2 H_1^2 H_{22} = x^{3a_1-2} y^{3a_2-2} a_1^2 a_2^2 (1 - a_1 - a_2)$$

so the hill property holds if $a_1 + a_2 \geq 1$ and $||a_1| - |a_2|| \leq 1$.

Corollary 12. Set $\Phi = \int_{-\infty}^x \varphi(s)ds$, where $\varphi \in C^1$ is positive and integrable function at negative infinity, and let $\Phi^{-1}(x)$ be inverse function of Φ . Take

$$H(x, y) = \Phi(a_1\Phi^{-1}(x) + a_2\Phi^{-1}(y)) \quad \text{where } (x, y) \in (0, 1)^2.$$

Denote $\Lambda = a_1\Phi^{-1}(x) + a_2\Phi^{-1}(y)$. Then direct computation shows that

$$\begin{aligned} H_1 &= \frac{a_1\varphi(\Lambda)}{\varphi(\Phi^{-1}(x))}; \\ H_2 &= \frac{a_2\varphi(\Lambda)}{\varphi(\Phi^{-1}(y))}; \\ H_{11} &= \frac{a_1^2\varphi'(\Lambda)}{\varphi(\Phi^{-1}(x))^2} - \frac{a_1\varphi(\Lambda)\varphi'(\Phi^{-1}(x))}{\varphi(\Phi^{-1}(x))^3}; \\ H_{12} &= \frac{a_1a_2\varphi'(\Lambda)}{\varphi(\Phi^{-1}(x))\varphi(\Phi^{-1}(y))}; \\ H_{22} &= \frac{a_2^2\varphi'(\Lambda)}{\varphi(\Phi^{-1}(y))^2} - \frac{a_2\varphi(\Lambda)\varphi'(\Phi^{-1}(y))}{\varphi(\Phi^{-1}(y))^3}. \end{aligned}$$

So we obtain

$$\begin{aligned} H_1H_2H_{12}(1 - a_1^2 - a_2^2) + a_1^2H_2^2H_{11} + a_2^2H_1^2H_{22} &= \frac{a_1^2a_2^2\varphi(\Lambda)^3}{\varphi(\Phi^{-1}(x))^2\varphi(\Phi^{-1}(y))^2} \times \\ &\left(\frac{\varphi'(a_1\Phi^{-1}(x) + a_2\Phi^{-1}(y))}{\varphi(a_1\Phi^{-1}(x) + a_2\Phi^{-1}(y))} - a_1\frac{\varphi'(\Phi^{-1}(x))}{\varphi(\Phi^{-1}(x))} - a_2\frac{\varphi'(\Phi^{-1}(y))}{\varphi(\Phi^{-1}(y))} \right). \end{aligned}$$

Denote $\Phi^{-1}(x) = u, \Phi^{-1}(y) = v$. Then we see that if the logarithmic derivative of the density φ satisfies concavity condition

$$f(a_1u + a_2v) \geq a_1f(u) + a_2f(v),$$

where $f(x) = \frac{\varphi'(x)}{\varphi(x)}$ then the hill property holds. In particular this implies Ehrhard's inequality

if $\varphi = e^{-|x|^2/2}$ and it implies Prekopa–Leindler inequality if $\varphi = e^x$.

3.0.6.2 log-Sobolev inequality, Beckner–Sobolev inequality, Bobkov’s inequality and its underlying PDE

The inequalities mentioned in the title are related to some PDEs which we are going to describe shortly. Firstly we will need some preliminaries from Bakry–Émery “Gamma calculus”. Complete description of the required material the reader can find in [48].

Mostly we will be working with Ornstein–Uhlenbeck semigroup P_t on the Euclidean space whose generator is $L = \Delta - x \cdot \nabla$, however the most part of the results (thanks to the Bochner–Lichnerowicz formula) can be pushed forward to the diffusion semigroups P_t on the some weighted Riemannian manifolds (M, g) with uniformly bounded below Ricci curvature.

We remind that the differential operator L is a diffusion operator if for every C^∞ function Ψ on \mathbb{R}^k and every finite family $F = (f_1, \dots, f_k)$ from some suitable algebra \mathcal{A} we have

$$L\Psi(F) = \sum_j \frac{\partial \Psi}{\partial f_j} Lf_j + \sum_{i,j} \frac{\partial^2 \Psi}{\partial f_i \partial f_j} \Gamma(f_i, f_j),$$

where the so-called carré du champ operator Γ is defined as follows

$$2\Gamma(f, g) = L(fg) - Lf \cdot g - f \cdot Lg.$$

For example, if

$$L = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(x) \frac{\partial}{\partial x_j},$$

where $a_{ij}(x)$ is symmetric matrix then

$$\Gamma(f, g) = \sum_{i,j} a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

for $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$. One can define iterated carré du champ operator

$$2\Gamma_2(f, g) = L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf).$$

Sometimes we will write just $\Gamma(f, f) = \Gamma(f)$ and $\Gamma_2(f, f) = \Gamma_2(f)$. It is important to notice that for the Ornstein–Uhlenbeck generator $L = \Delta - x \cdot \nabla$ we have

$$\begin{aligned} \Gamma(u_i, u_j) &= \nabla u_i \nabla u_j; \\ \Gamma_2(u_j) &= |\nabla u_j|^2 + |\text{Hess } u_j|_{HS}^2; \\ \Gamma(u_i, \Gamma u_j) &= 2\langle \text{Hess } u_j \nabla u_j, \nabla u_i \rangle; \\ \Gamma(\Gamma u_i, \Gamma u_j) &= 4(\text{Hess } u_i \nabla u_i)^T \text{Hess } u_j \nabla u_j. \end{aligned}$$

where $|\text{Hess } u_j|_{HS}$ denotes Hilbert–Schmidt norm of the matrix $\text{Hess } u_j$. For the C^2 function $M : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, say $M(x_1, \dots, x_n, y_1, \dots, y_n)$ by $\left(\frac{\nabla_y M}{y}\right)_{2n \times 2n}$ we denote $2n \times 2n$ diagonal matrix so that in the first $n \times n$ block it has entries $\frac{M_{y_i}}{y_i}$ and the rest is zero. For the vector v we write $v \leq 0$ if each component of v is nonpositive.

Theorem 3.0.23. *Let Ω be a convex subset of \mathbb{R}^n . Let $(u_1, \dots, u_n) : \mathbb{R}^k \rightarrow \Omega$ be sufficiently nice. Let $M : \Omega \times \mathbb{R}_+^n \rightarrow \mathbb{R}$, say $M(x_1, \dots, x_n, y_1, \dots, y_n)$ where $y_j \geq 0$, satisfy*

$$\text{Hess } M + \left(\frac{\nabla_y M}{y}\right)_{2n \times 2n} \leq 0, \quad \text{and} \quad \nabla_y M \leq 0, \quad (3.0.22)$$

then

$$\int_{\mathbb{R}^k} M(u_1, \dots, u_n, |\nabla u_1|, \dots, |\nabla u_n|) d\gamma \leq M \left(\int_{\mathbb{R}^k} u_1 d\gamma, \dots, \int_{\mathbb{R}^k} u_n d\gamma, 0, \dots, 0 \right). \quad (3.0.23)$$

If the inequalities (3.0.22) are reversed then we have reversed inequality in (3.0.23). In the case $n = 1$ we do not need the condition $\nabla_y M \leq 0$ (or $\nabla_y M \geq 0$).

Proof. Let $u_j : \mathbb{R}^k \rightarrow \mathbb{R}$ for all $j = 1, \dots, n$. Take

$$B(x_1, \dots, x_n, y_1^2, \dots, y_n^2) = M(x_1, \dots, x_n, y_1, \dots, y_n).$$

Then we are going to show that $P_t B - B(P_t) \leq 0$ for all $t \geq 0$ where

$$P_t B = P_t B(u_1, \dots, u_n, \Gamma u_1, \dots, \Gamma u_n)$$

and $B(P_t) = B(P_t u_1, \dots, P_t u_n, \Gamma P_t u_1, \dots, \Gamma P_t u_n)$ then by sending $t \rightarrow \infty$ and noticing that $\Gamma P_t u = e^{-2t} P_t \Gamma u$ we obtain the desired result thanks to the ergodicity of the semigroup P_t i.e., $P_t u \rightarrow \int u d\gamma$.

Remark 21. The condition $P_t B - B(P_t) \leq 0$ can be verified by showing that the function $C(x, t) = P_t B - B(P_t)$ satisfies ellipticity (or even hill property see Definition 14) i.e., $(L - \frac{\partial}{\partial t})(B(P_t) - P_t B) \leq 0$ for some elliptic operator L . We will check its ellipticity by choosing L to be Ornstein–Uhlenbeck generator. Note that

$$P_t B - B(P_t) = \int_0^t \frac{d}{ds} P_s B(P_{t-s}) ds = \int_0^t P_s \left(L - \frac{\partial}{\partial t} \right) B(P_{t-s}).$$

Therefore the reader can notice that since P_s is positive operator, the next computations are

the same as checking ellipticity for the function $B(P_t) - P_t B$ where L is Orsntein–Uhlenbeck generator.

$$\begin{aligned}
& P_t B(u_1, \dots, u_n, \Gamma u_1, \dots, \Gamma u_n) - B(P_t u_1, \dots, P_t u_n, \Gamma P_t u_1, \dots, \Gamma P_t u_n) = \\
& = \int_0^t \frac{d}{ds} P_s B(P_{t-s} u_1, \dots, P_{t-s} u_n, \Gamma P_{t-s} u_1, \dots, \Gamma P_{t-s} u_n) = \\
& \int_0^t P_s \left(LB - \sum_{j=1}^n \frac{\partial B}{\partial u_j} L P_{t-s} u_j - \sum_{j=1}^n \frac{\partial B}{\partial v_j} 2\Gamma(L P_{t-s} u_j, P_{t-s} u_j) \right) = \\
& \int_0^t P_s \left(\sum_{j=1}^n \frac{\partial B}{\partial u_j} L P_{t-s} u_j + \sum_{j=1}^n \frac{\partial B}{\partial v_j} L \Gamma P_{t-s} u_j + \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial u_j} \Gamma(P_{t-s} u_i, P_{t-s} u_j) + \right. \\
& \sum_{i,j=1}^n \frac{\partial^2 B}{\partial v_i \partial v_j} \Gamma(\Gamma P_{t-s} u_i, \Gamma P_{t-s} u_j) + 2 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial v_j} \Gamma(P_{t-s} u_i, \Gamma P_{t-s} u_j) \\
& \left. - \sum_{j=1}^n \frac{\partial B}{\partial u_j} L P_{t-s} u_j - \sum_{j=1}^n \frac{\partial B}{\partial v_j} 2\Gamma(L P_{t-s} u_j, P_{t-s} u_j) \right) = \\
& \int_0^t P_s \left(\sum_{j=1}^n \frac{\partial B}{\partial v_j} \Gamma^2(P_{t-s} u_j) + \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial u_j} \Gamma(P_{t-s} u_i, P_{t-s} u_j) + \right. \\
& \left. \sum_{i,j=1}^n \frac{\partial^2 B}{\partial v_i \partial v_j} \Gamma(\Gamma P_{t-s} u_i, \Gamma P_{t-s} u_j) + 2 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial v_j} \Gamma(P_{t-s} u_i, \Gamma P_{t-s} u_j) \right).
\end{aligned}$$

Now notice that

$$\begin{aligned}
& \sum_{j=1}^n \frac{\partial B}{\partial v_j} \Gamma^2(P_{t-s}u_j) + \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial u_j} \Gamma(P_{t-s}u_i, P_{t-s}u_j) + \\
& \sum_{i,j=1}^n \frac{\partial^2 B}{\partial v_i \partial v_j} \Gamma(\Gamma P_{t-s}u_i, \Gamma P_{t-s}u_j) + 2 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial v_j} \Gamma(P_{t-s}u_i, \Gamma P_{t-s}u_j) = \\
& \sum_{j=1}^n \frac{\partial B}{\partial v_j} (2|\nabla P_{t-s}u_j|^2 + 2|\text{Hess } P_{t-s}u_j|_{HS}^2) + \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial u_j} \nabla P_{t-s}u_i \nabla P_{t-s}u_j + \\
& 4 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial v_i \partial v_j} (\text{Hess } P_{t-s}u_i \nabla P_{t-s}u_i)^T \text{Hess } P_{t-s}u_j \nabla P_{t-s}u_j + \\
& 4 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial v_j} \langle \text{Hess } P_{t-s}u_j \nabla P_{t-s}u_j, \nabla P_{t-s}u_i \rangle.
\end{aligned}$$

We will check that the last expression is nonpositive. We need the following technical lemma

Lemma 22. *For the given x the image of Ax as A runs over all matrices such that $\|A\|_{HS} = r$ is the set $B(0, \|x\|r)$.*

Proof. If $x = 0$ there is nothing to prove. Assume $x \neq 0$. It is clear that $\{Ax : \|A\|_{HS} = r\} \subset B(0, \|x\|r)$ because $\|Ax\| \leq \|A\|\|x\| \leq \|A\|_{HS}\|x\| = r\|x\|$. Let $y \in B(0, \|x\|r)$. Take the matrix $A^+ = \frac{yx^T}{\|x\|^2}$. Clearly $A^+x = y$ and

$$\|A^+\|_{HS} = (Tr[A^+(A^+)^T])^{1/2} = \frac{1}{\|x\|^2} (Tr[yx^Txy^T])^{1/2} = \frac{\|y\|}{\|x\|} \leq r.$$

Consider the matrix $\tilde{A} = I - \frac{xx^T}{\|x\|^2}$. Then $\tilde{A}x = 0$. Take the matrix $f(\lambda) = A^+ + \lambda\tilde{A}$. Clearly $f(\lambda)x = y$ for all $\lambda \in \mathbb{R}$ and $\|f(\lambda)\|_{HS} = (Tr[A^+(A^+)^T] + 2\lambda Tr[A^+\tilde{A}^*] + \lambda^2 Tr[\tilde{A}\tilde{A}^*])^{1/2}$. And $\|f(0)\|_{HS} \leq r$, and $\lim_{\lambda \rightarrow \infty} \|f(\lambda)\|_{HS} \rightarrow \infty$. Hence there exists λ^* such that $\|f(\lambda^*)\|_{HS} = r$. \square

The above lemma implies that we should study the sign of the expression

$$\begin{aligned} & \sum_{j=1}^n \frac{\partial B}{\partial v_j} (2|\nabla P_{t-s}u_j|^2 + 2|\text{Hess } P_{t-s}u_j|_{HS}^2) + \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial u_j} \nabla P_{t-s}u_i \nabla P_{t-s}u_j + \\ & 4 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial v_i \partial v_j} \langle z_i, z_j \rangle |\nabla P_{t-s}u_i| |\nabla P_{t-s}u_j| + 4 \sum_{i,j=1}^n \frac{\partial^2 B}{\partial u_i \partial v_j} |\nabla P_{t-s}u_j| \langle z_j, \nabla u_i \rangle \end{aligned}$$

for any vectors z_i so that they satisfy the condition $\|z_i\| \leq |\text{Hess } u_i|_{HS}$. Going back to the function $B(x_1, \dots, x_n, y_1^2, \dots, y_n^2) = M(x_1, \dots, x_n, y_1, \dots, y_n)$ we obtain

$$\begin{aligned} & \sum_{j=1}^n \frac{\partial M}{\partial y_j} |\nabla P_{t-s}u_j| + \sum_{j=1}^n \frac{\partial M}{\partial y_j} \frac{|\text{Hess } P_{t-s}u_j|_{HS}^2 - \|z_j\|^2}{|\nabla P_{t-s}u_j|} + \sum_{i,j=1}^n \frac{\partial^2 M}{\partial x_i \partial x_j} \nabla P_{t-s}u_i \nabla P_{t-s}u_j + \\ & \sum_{i,j=1}^n \frac{\partial^2 M}{\partial y_i \partial y_j} \langle z_i, z_j \rangle + 2 \sum_{i,j=1}^n \frac{\partial^2 M}{\partial x_i \partial y_j} \langle z_j, \nabla P_{t-s}u_i \rangle \leq \\ & \sum_{j=1}^n \frac{\partial M}{\partial y_j} |\nabla P_{t-s}u_j| + \sum_{i,j=1}^n \frac{\partial^2 M}{\partial x_i \partial x_j} \nabla P_{t-s}u_i \nabla P_{t-s}u_j + \sum_{i,j=1}^n \frac{\partial^2 M}{\partial y_i \partial y_j} \langle z_i, z_j \rangle + \\ & 2 \sum_{i,j=1}^n \frac{\partial^2 M}{\partial x_i \partial y_j} \langle z_j, \nabla P_{t-s}u_i \rangle. \end{aligned}$$

Let $w_1 = \nabla P_{t-s}u_1, \dots, w_n = \nabla P_{t-s}u_n, w_{n+1} = z_1, \dots, w_{2n} = z_n$ be columns. And let W be the corresponding matrix constructed by these columns. Let w^j be the columns of the transpose matrix W . Then the above expression can be written as

$$\left(\sum_j w^j \right)^T \left[\text{Hess } M + \left(\frac{\nabla_y M}{y} \right)_{2n \times 2n} \right] \left(\sum_j w^j \right).$$

The last expression is nonpositive and since the operator P_t is positive semidefinite we obtain the desired result. \square

Corollary 13. *Take*

$$M(x, y) = x \ln x - \frac{1}{2} \frac{y^2}{x}.$$

Then clearly M satisfies conditions of Theorem 3.0.23, i.e., $\Omega = \mathbb{R}_+$,

$$\begin{pmatrix} M_{11} + \frac{1}{y} M_2 & M_{12} \\ M_{12} & M_{22} \end{pmatrix} \leq 0, \quad \text{and} \quad M_2 \leq 0.$$

Therefore

$$\int_{\mathbb{R}^k} \left(u \ln u - \frac{1}{2} \frac{|\nabla u|^2}{u} \right) d\gamma = \int_{\mathbb{R}^k} M(f, |\nabla u|) d\gamma \leq M \left(\int_{\mathbb{R}^k} u d\gamma, 0 \right) = \left(\int_{\mathbb{R}^k} u d\gamma \right) \ln \left(\int_{\mathbb{R}^k} u d\gamma \right).$$

This is log-Sobolev inequality (3.0.6) after introducing the function $u(x) = f(x)^2$.

Corollary 14. *Take*

$$M(x, y) = x^{\frac{2}{p}} - \frac{2-p}{p^2} x^{\frac{2}{p}-2} y^2,$$

and $\Omega = \mathbb{R}_+$. Clearly M satisfies condition of Theorem 3.0.23. This function gives Beckner–Sobolev inequality for $f \geq 0$.

Corollary 15. *Take*

$$M(x, y) = \sqrt{U^2(x) + y^2},$$

where $\Omega = (0, 1)$, $U(x) = \varphi(\Phi^{-1}(x))$, $\Phi(x) = \int_{-\infty}^x d\gamma$ and $\varphi(x) = \Phi'(x)$. Clearly M satisfies

opposite inequalities of Theorem 3.0.23. Therefore it gives Bobkov's inequality

$$\int_{\mathbb{R}^k} M(f, |\nabla f|) d\gamma \geq M\left(\int_{\mathbb{R}^k} f d\gamma, 0\right).$$

Thus we see that some special functions which satisfy underlying PDE with prescribed obstacle conditions solve the problems.

3.0.6.3 Brascamp–Lieb, hypercontractivity, Borell's Gaussian noise “stability” and its underlying PDE

Let $L = \sum_{i,j=1}^k c_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ be an elliptic operator (i.e., $C = \{c_{ij}\}_{i,j=1}^k \geq 0$). Let $a_j = (a_{j1}, \dots, a_{jk}) \in \mathbb{R}^k$ for $j = 1, \dots, n$ so that the matrix $A = (a_1, \dots, a_n)$ has full rank ($n \geq k$). Let P_t be a semigroup with generator L and let

$$P_t f(y) = \int_{\mathbb{R}^k} f(x) p_t^C(y, x) dx, \quad \text{for all } t \geq 0.$$

Note that $p_t^{id}(y, x) = \frac{1}{\sqrt{4\pi t}} e^{-|x-y|^2/4t}$. Also note that if $u : \mathbb{R} \rightarrow \mathbb{R}$, and $g_j(x) = u_j(a_j \cdot x)$ then

$$P_t g_j(x) = \int_{\mathbb{R}} u(a_j \cdot y + x \sqrt{2t \langle C a_j, a_j \rangle}) d\gamma(x).$$

Theorem 3.0.24. *If $A^T C A \bullet \text{Hess } B \leq 0$ then*

$$\begin{aligned} & \int_{\mathbb{R}^k} B(u_1(a_1 \cdot x), \dots, u_n(a_n \cdot x)) p_t^C(y, x) dx \leq \\ & B\left(\int_{\mathbb{R}} u_1(a_1 \cdot y + x \sqrt{2t \langle C a_1, a_1 \rangle}) d\gamma, \dots, \int_{\mathbb{R}} u_n(a_n \cdot y + x \sqrt{2t \langle C a_n, a_n \rangle}) d\gamma\right). \end{aligned}$$

If $A^T C A \bullet \text{Hess } B \geq 0$ then the inequality is reversed.

Proof. First we refer the reader to Remark 21.

$$\begin{aligned} P_t B(u_j) - B(P_t u_j) &= \int_0^t \frac{d}{ds} P_s B(P_{t-s} u_j) = \int_0^t P_s \sum_{i,j,p,q} \left(\frac{\partial^2 B}{\partial u_i \partial u_j} c_{pq} \frac{\partial P_{t-s} u_i}{\partial x_p} \frac{\partial P_{t-s} u_j}{\partial x_q} \right) = \\ &= \int_0^t P_s \sum_{i,j,p,q} \left(\frac{\partial^2 B}{\partial u_i \partial u_j} c_{pq} a_{ip} a_{jq} P_{t-s} u'_i P_{t-s} u'_j \right) = \int_0^t P_s (\langle A^* C A \bullet \text{Hess } B P_{t-s} u', P_{t-s} u' \rangle). \end{aligned}$$

□

Corollary 16. *If we choose $C = id$, $t = 1/2$ and $y = 0$ then we obtain that if $A^* A \bullet \text{Hess } B \leq 0$ then*

$$\int_{\mathbb{R}^k} B(u_1(a_1 \cdot x), \dots, u_n(a_n \cdot x)) d\gamma \leq B \left(\int_{\mathbb{R}} u_1(x|a_1|) d\gamma, \dots, \int_{\mathbb{R}} u_n(x|a_n|) d\gamma \right).$$

Corollary 17. *The similar reasoning shows that if A_j are $k_j \times k$ matrices i.e., $A_j : \mathbb{R}^k \rightarrow \mathbb{R}^{k_j}$ then*

$$\begin{aligned} &\int_{\mathbb{R}^k} B(u_1(A_1 x), \dots, u_n(A_n x)) p_t^C(y, x) dx \leq \\ &B \left(\int_{\mathbb{R}^{k_1}} u_1(x) p_t^{A_1 C A_1^*}(A_1 y, x) dx, \dots, \int_{\mathbb{R}^{k_n}} u_n(x) p_t^{A_n C A_n^*}(A_n y, x) dx \right). \end{aligned}$$

If

$$A C A^* \bullet \text{Hess } B \leq 0 \tag{3.0.24}$$

where $A^* = (A_1^*, \dots, A_n^*)$ and Hadamard product is understood as $\left\{ A_j C A_i^* \frac{\partial^2 B}{\partial x_i \partial x_j} \right\}_{i,j=1}^k$.

Before we continue to the applications we have to mention some elementary facts from

the linear algebra. If A is $n \times n$ positive semidefinite matrix then by spectral decomposition we have

$$A = \sum_j \lambda_j v_j \otimes v_j, \quad v_j \in \mathbb{R}^n, \quad \lambda_j \geq 0.$$

Proposition 4. *If by using the $n \times n$ matrix A one constructs k copies of the matrix A , i.e., block matrix $kn \times kn$ then it is positive semidefinite if and only if A is positive semidefinite.*

Proof. In one direction claim is obvious, in another direction claim follows from the spectral decomposition. □

Proposition 5. *Let $M \geq 0$ be $kn \times kn$ matrix constructed by k copies of the matrix $A \geq 0$. Let $U \geq 0$ be $k \times k$ matrix. Then $U \otimes A \geq 0$.*

Proof. The claim follows by linearity. It is enough to consider only the case $A = vv^T$ and $U = uu^T$ and M is k copy of vv^T . In this case

$$M \otimes U = (vu_1, vu_2, \dots, vu_k) \otimes (vu_1, vu_2, \dots, vu_k) \geq 0.$$

So the claim follows. □

Corollary 18. *In Borell's Gaussian noise "stability", take $A_1 = (I_n, 0_n)$, $A_2 = (pI_n, (1 - p^2)^{1/2}I_n)$, $C = I_n$. Then*

$$A_1 A_1^* = I_n;$$

$$A_1 A_2^* = A_2 A_1^* = pI_n;$$

$$A_2 A_2^* = I_n.$$

Therefore the condition $ACA^* \bullet \text{Hess } B \leq 0$ becomes

$$\begin{pmatrix} B_{11} & pB_{12} \\ pB_{12} & B_{22} \end{pmatrix} \otimes I_n \leq 0.$$

So in order to prove the functional version of Borell's Gaussian noise "stability" it is enough to check that the symmetric function

$$B(u, v) = \mathbb{P}(X_1 \leq \Phi^{-1}(u), pX_1 + \sqrt{1-p^2}Y_2 \leq \Phi^{-1}(v)),$$

satisfies the condition

$$\begin{pmatrix} B_{11} & pB_{12} \\ pB_{12} & B_{22} \end{pmatrix} \leq 0.$$

Note that

$$\begin{aligned}
B(u, v) &= \mathbb{P} \left(X_1 \leq \Phi^{-1}(u), Y_2 \leq \frac{\Phi^{-1}(v) - pX}{\sqrt{1-p^2}} \right) = \\
&= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\frac{\Phi^{-1}(v) - ps}{\sqrt{1-p^2}}} \varphi(t) \varphi(s) dt ds = \int_{-\infty}^{\Phi^{-1}(v)} \int_{-\infty}^{\frac{\Phi^{-1}(u) - ps}{\sqrt{1-p^2}}} \varphi(t) \varphi(s) dt ds; \\
B_1 &= \int_{-\infty}^{\frac{\Phi^{-1}(v) - p\Phi^{-1}(u)}{\sqrt{1-p^2}}} \varphi(t) dt; \\
B_{11} &= \varphi \left(\frac{\Phi^{-1}(v) - p\Phi^{-1}(u)}{\sqrt{1-p^2}} \right) \frac{-p}{(1-p^2)^{1/2} \varphi(\Phi^{-1}(u))}; \\
B_{12} &= \varphi \left(\frac{\Phi^{-1}(v) - p\Phi^{-1}(u)}{\sqrt{1-p^2}} \right) \frac{1}{(1-p^2)^{1/2} \varphi(\Phi^{-1}(v))}; \\
B_2 &= \int_{-\infty}^{\frac{\Phi^{-1}(u) - p\Phi^{-1}(v)}{\sqrt{1-p^2}}} \varphi(t) dt; \\
B_{22} &= \varphi \left(\frac{\Phi^{-1}(u) - p\Phi^{-1}(v)}{\sqrt{1-p^2}} \right) \frac{-p}{(1-p^2)^{1/2} \varphi(\Phi^{-1}(v))};
\end{aligned}$$

It is clear that $B_{11}, B_{22} \leq 0$ and

$$B_{11}B_{22} - p^2B_{12}^2 = 0.$$

Thus we obtain Borell's Gaussian noise "stability".

Corollary 19. *With the same tensor trick we see that we have inequality*

$$\int_{\mathbb{R}^{2n}} B(\varphi(x), \psi(e^{-t}x + \sqrt{1-e^{-2t}}y)) d\gamma \leq B \left(\int_{\mathbb{R}^n} \varphi d\gamma, \int_{\mathbb{R}^n} \psi d\gamma \right).$$

If the function B satisfies the following condition

$$\begin{pmatrix} B_{11} & e^{-t}B_{12} \\ e^{-t}B_{12} & B_{22} \end{pmatrix} \leq 0.$$

Taking $B(u, v) = u^{1/a}v^{1/b}$ we see that the above condition holds if and only if $1 \leq a, 1 \leq b$ and

$$(a-1)(b-1) - e^{-2t} \geq 0.$$

This means that if we denote $\varphi = f^a, \psi = g^b$ then we have inequality

$$\int_{\mathbb{R}^n} f \cdot P_t g d\gamma = \int_{\mathbb{R}^{2n}} f(x) g(e^{-t}x + \sqrt{1-e^{-2t}} y) d\gamma \leq \left(\int_{\mathbb{R}^n} f^a d\gamma \right)^{1/a} \left(\int_{\mathbb{R}^n} g^b d\gamma \right)^{1/b}.$$

Taking supremum over $f \in L^a(d\gamma)$ we obtain hypercontractivity for Ornstein–Uhlenbeck semigroup P_t :

$$\|P_t g\|_{L^q(d\gamma)} \leq \|g\|_{L^p(d\gamma)},$$

where $q = \frac{a}{a-1}$ is dual exponent to a , $p = b$ under the condition $(a-1)(b-1) - e^{-2t} \geq 0$ which can be rewritten in terms of $p, q \geq 1$ as follows

$$\frac{p-1}{q-1} - e^{-2t} \geq 0.$$

3.0.6.4 Extremal problems on BMO , A_p classes, Gehring classes and its underlying PDE

PDE in this subsection corresponds to homogeneous Monge–Ampère equation (surfaces with zero Gaussian curvature). These extremal problems correspond to finding minimal concave function over an obstacle. We will start with the simplest example which correspond to the convex domains.

Let $\Omega \subset \mathbb{R}^n, m : \Omega \rightarrow \mathbb{R}^k$ and $H : \Omega \rightarrow \mathbb{R}$. Let $\Omega(I)$ denotes class of vector-valued functions $\varphi : I \rightarrow \Omega$, and let $\text{conv}(\Omega)$ be the convex hull of the set Ω . We define the Bellman function as follows

$$B(x) = \sup_{\varphi \in \Omega(I)} \{ \langle H(\varphi) \rangle_I : \langle m(\varphi) \rangle_I = x \}.$$

Theorem 3.0.25. *The following properties hold:*

1. B is defined on the convex set $\text{conv}[m(\Omega)]$;
2. $B(m(y)) \geq H(y)$ for all $y \in \Omega$;
3. B is concave function;
4. B is minimal among those who satisfy properties 1,2 and 3.

Proof. First we show the property 1. Let $\text{Dom } B$ denotes the domain where B is defined. Since $m(\varphi) \in \text{conv}[m(\Omega)]$ we have $\langle m(\varphi) \rangle_I \in \text{conv}[m(\Omega)]$. Therefore $\text{Dom } B \subseteq \text{conv}[m(\Omega)]$. Now we show the opposite inclusion. Carathéodory's theorem implies that for any $x \in \text{conv}[m(\Omega)]$ we have $x = \sum_{j=1}^{n+1} a_j x_j$, where $a_j \geq 0$, $\sum_{j=1}^{n+1} a_j = 1$ and $x_j \in m(\Omega)$. Let the points y_j be such that $m(y_j) = x_j$. We choose φ so that $|\{t \in I : \varphi(t) = y_j\}| = a_j |I|$. Then $\varphi \in \Omega(I)$.

Hence,

$$\langle m(\varphi) \rangle_I = \frac{1}{|I|} \int_I m(\varphi(t)) dt = \sum_{j=1}^{n+1} \frac{1}{|I|} \int_{\{t: \varphi(t)=y_j\}} m(\varphi(t)) dt = \sum_{j=1}^{n+1} \frac{1}{|I|} x_j |I| a_j = x$$

Now we show the property 2. Let $\varphi_0(t) = y$, $t \in I$. Then $\langle m(\varphi_0) \rangle_I = m(y)$. Thus

$$B(m(y)) = \sup_{\varphi \in \Omega(I): \langle m(\varphi) \rangle_I = m(y)} \langle H(\varphi) \rangle_I \geq \langle H(\varphi_0) \rangle_I = H(y).$$

Now we show the property 3. It is enough to show that $B(\theta x + (1 - \theta)y) \geq \theta B(x) + (1 - \theta)B(y)$ for all $x, y \in \text{conv}[m(\Omega)]$ and $\theta \in [0, 1]$. There exist functions $\varphi, \psi \in \Omega(I)$ such that $\langle m(\varphi) \rangle_I = x$, $\langle m(\psi) \rangle_I = y$ and

$$\langle H(\varphi) \rangle_I \geq B(x) - \varepsilon, \quad \langle H(\psi) \rangle_I > B(y) - \varepsilon.$$

We split interval I by two disjoint subintervals I_1 and I_2 so that $|I_1| = \theta|I|$. Let $L_j : I_j \rightarrow I$ be a linear bijections. We consider the concatenation as follows

$$\eta(t) = \begin{cases} \varphi(L_1(t)), & t \in I_1, \\ \psi(L_2(t)), & t \in I_2. \end{cases}$$

Clearly $\eta(t) \in \Omega(I)$, $\langle m(\eta) \rangle_I = \theta x + (1 - \theta)y$ and

$$B(\theta x + (1 - \theta)y) \geq \langle H(\eta) \rangle_I = \theta \langle H(\varphi) \rangle_I + (1 - \theta) \langle H(\psi) \rangle_I > \theta B(x) + (1 - \theta)B(y) - \varepsilon.$$

Now we show the property 4. Let G satisfies properties 1,2 and 3. Then Jensen's

inequality implies that for any $\varphi \in \Omega(I)$ we have

$$\langle H(\varphi) \rangle_I \leq \langle G(m(\varphi)) \rangle_I \leq G(\langle m(\varphi) \rangle_I) = G(x)$$

□

Corollary 20. *The definition of the modulus of uniform convexity tells us to consider the following function*

$$B_\theta(x_1, x_2, x_3) = \sup_{f, g} \{ \langle |\theta f + (1 - \theta)g|^p \rangle_I, \langle (|f|^p, |g|^p, |f - g|^p) \rangle_I = (x_1, x_2, x_3) \}. \quad (3.0.25)$$

Theorem 3.0.25 implies that B is a minimal concave function over the obstacle. It is clear that

$$\delta_{L^p}(\varepsilon) = 1 - \sup_{2^p \geq x_3 \geq \varepsilon^p} (B_{1/2}(1, 1, x_3))^{1/p}.$$

There are different modulus of uniform convexity. One of them corresponds to the complex case. Let $(X, \|\cdot\|)$ be a normed space. For $0 < p < \infty$ we set

$$h_p^X(\varepsilon) = \inf \left\{ 1 - \|x\| : \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta \leq 1, \|y\| = \varepsilon \right\}.$$

Corollary 21. *Theorem 3.0.25 implies that*

$$B(x, y) = \sup_{f, g} \left\{ \langle |f|^p \rangle_I : \left\langle \frac{1}{2\pi} \int_0^{2\pi} |f + e^{i\theta}g|^p \right\rangle_I = x, \langle |g|^p \rangle_I = y \right\},$$

corresponds to the minimal concave function over the obstacle. By knowing B one finds the

value $h_p^{L^p}(\varepsilon)$ i.e.,

$$h_p^{L^p}(\varepsilon) = 1 - \left(\sup_{0 \leq x \leq 1} B(x, \varepsilon^p) \right)^{1/p}.$$

Corollary 22. *It is worth mentioning that Hölder's inequality corresponds to the function*

$$B(x_1, \dots, x_n) = x_1^{1/p_1} \dots x_n^{1/p_n},$$

which coincides with its obstacle. Minkowski inequality corresponds to the function

$$B(x, y) = (x^{1/p} + y^{1/p})^p,$$

which again coincides with its obstacle.

It turns out that if one imposes extra condition — “boundedness of the mean oscillation” on the class of functions $\Omega(I)$, one again obtains minimal concave function over an obstacle but in a different domain (not necessarily convex domain as it was before). Let us illustrate example on the BMO class.

Let Ω_ε be a parabolic strip

$$\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq x^2 + \varepsilon^2\}, \quad (3.0.26)$$

and let $\gamma(t) = (t, t^2)$. Then

$$BMO_\varepsilon(J) = \{\varphi : \langle \gamma(\varphi) \rangle_I \in \Omega_\varepsilon \ \forall I \subset J\}.$$

In other words for general φ the points $\gamma(\varphi(x)), x \in J$ belong to the parabola $\Gamma = \{(x, y) : y = x^2\}$ (a convex curve) and the integral average (or convex hull of these points) $\langle \gamma(\varphi) \rangle_I$ always will belong to the convex full of the parabola i.e., $\text{conv}(\Gamma) = \{(x, y) : y \geq x^2\}$. Therefore extremal problems on this class of functions would correspond to the minimal concave functions on the convex hull of the parabola (which is convex set). However, $BMO_\varepsilon(J)$ class can be seen as those functions $\varphi \in L^1$ so that the convex hull of the points $\gamma(\varphi)$ belong to the parabolic strip Ω_ε and therefore extremal problems on this class of functions correspond to the minimal concave functions on the parabolic strip.

Theorem 3.0.26. . *Let $f \in C^3(\mathbb{R})$ be sufficiently nice. Then*

$$B_\varepsilon(x, y) = \sup_{\varphi} \{ \langle f(\varphi) \rangle_I : \varphi \in BMO_\varepsilon, \langle \gamma(\varphi) \rangle_I = (x, y) \}$$

is a minimal concave function defined in the parabolic strip Ω_ε with the obstacle condition $B(\gamma(t)) \geq f(t)$. If the sign of the torsion of the space curve $(t, t^2, f(t))$ changes finitely many times then there we have the finite algorithm which finds the function B .

For the proof of the theorem we refer the reader to [9, 10, 11, 12]. What is more we also describe evolution of the function B_ε as ε changes. The important part of the theorem is that we find the function B , and the fact that it is minimal concave function is just the corollary of this result.

In other words one can think about the problem as follows. The function $B(t, t^2)$ is given and it is equal to $f(t)$. This is what we call obstacle condition, or boundary condition. Now the question is to find its minimal concave extension in the parabolic strip Ω_ε , and the result we will denote by the same function $B(x, y)$. This means that we are looking for the function

$$B_\varepsilon(x, y) = \sup_{\varphi} \{ \langle B(\gamma(\varphi)) \rangle_I : \varphi \in BMO_\varepsilon, \langle \gamma(\varphi) \rangle_I = (x, y) \},$$

where $B(\gamma(t)) = f(t)$. Before we continue let us mention some applications.

Corollary 23. *For every $p \in (0, \infty)$ the p -(quasi)norm on the $BMO(I)$ is equivalent to the 2-norm:*

$$c_p \|\varphi\|_{BMO(I)} \leq \sup_{J \subset I} \langle |\varphi - \langle \varphi \rangle_J|^p \rangle_J^{1/p} \leq C_p \|\varphi\|_{BMO(I)}. \quad (3.0.27)$$

In term of the Bellman function this is the same as

$$\text{if } B(t, t^2) = |t|^p, \quad \text{then } \left(\sup_{\ell \leq \varepsilon^2} B_\varepsilon(0, \ell) \right)^{1/p} \leq C_p \varepsilon, \quad \forall \varepsilon > 0.$$

$$\text{if } B(t, t^2) = -|t|^p, \quad \text{then } \left(\inf_{\ell \leq \varepsilon^2} -B_\varepsilon(0, \ell) \right)^{1/p} \geq c_p \varepsilon, \quad \forall \varepsilon > 0.$$

which is true by Theorem 3.0.26.

What is also important in these corollaries is that these inequalities (3.0.27) are equivalent to the statements about estimates for the minimal concave extensions $B_\varepsilon(x, y)$. This means that if one tries to prove these inequalities (3.0.27), one implicitly tries to obtain estimates for the functions B .

Corollary 24. *There exist some constants $c_1, c_2 > 0$ such that for all $\varphi \in BMO(I)$ we have*

John–Nirenberg inequality

$$\frac{1}{|I|} |\{t \in I : |\varphi(t) - \langle \varphi \rangle_I| > \lambda\}| \leq c_1 \exp \left(-\frac{c_2 \lambda}{\|\varphi_{BMO(I)}\|} \right).$$

In terms of the Bellman function it is the same as to show that

$$\text{if } B(t, t^2) = \mathbb{1}_{(-\infty, -\lambda) \cup (\lambda, \infty)}, \quad \text{then } \sup_{\ell \leq \varepsilon^2} B_\varepsilon(0, \ell) \leq c_1 e^{-c_2 \lambda / \varepsilon}, \quad \forall \varepsilon, \lambda > 0,$$

which is true by Theorem 3.0.26.

Corollary 25. *There exist $\varepsilon_0 > 0$ and a positive function $C(\varepsilon)$, $0 < \varepsilon < \varepsilon_0$, such that*

$$\langle e^\varphi \rangle_I \leq C(\varepsilon) e^{\langle \varphi \rangle_I}, \quad \text{for all } \varphi \in BMO_\varepsilon(I).$$

This is the same as to show that

$$\text{if } B(t, t^2) = e^t, \quad \text{then } B_\varepsilon(x, y) \leq C(\varepsilon) e^x, \quad 0 < \varepsilon < \varepsilon_0, \quad (x, y) \in \Omega_\varepsilon,$$

which is true by Theorem 3.0.26.

One can find similar characterization for the Reverse Hölder classes, Gehring classes, A_p a classes etc. We formulate our abstract theorem and then we mention that these examples become corollaries of our theorem. It is important to notice that $BMO_\varepsilon(I)$ defines parabolic strip Ω_ε which uniquely reconstructs the class $BMO_\varepsilon(I)$. On the other hand notice that the parabolic strip Ω_ε is difference of two convex domains, of the convex hull of the parabola $\{y = x^2\}$ and the convex hull of another parabola $\{y = x^2 + \varepsilon^2\}$, and second convex set $\text{conv}(\{y = x^2 + \varepsilon^2\})$ belongs to first convex set $\text{conv}(\{y = x^2\})$. This idea suggest to

introduce the following class of functions. Let $\gamma_1, \gamma_2 \in C^3$ be a strictly convex curves in \mathbb{R}^2 such that $\text{conv}(\gamma_2) \subseteq \text{conv}(\gamma_1)$. Set

$$A(\gamma_1, \gamma_2, I) \stackrel{\text{def}}{=} \{\varphi : \forall J \subset I \langle \gamma_1(\varphi) \rangle_J \in \text{conv}(\gamma_1) \setminus \text{conv}(\gamma_2)\}.$$

In some sense $A(\gamma_1, \gamma_2, I)$ represents class of functions with “small mean oscillation”. Under some mild assumptions on γ_1, γ_2, f we have the following theorem

Theorem 3.0.27. *If the space curve $(\gamma_1(t), f(t))$ has nowhere vanishing curvature and its torsion changes sign finitely many times then we present an algorithm which finds expression for the function B where*

$$B(x, y) \stackrel{\text{def}}{=} \sup_{\varphi \in A(\gamma_1, \gamma_2, I)} \{\langle f(\varphi) \rangle_I : \langle \gamma_1(\varphi) \rangle_I = (x, y)\}.$$

$B(x, y)$ is minimal concave function defined in $\text{conv}(\gamma_1) \setminus \text{conv}(\gamma_2)$ such that $B(\gamma_1) = f$.

For the proof we refer the reader to [9, 10, 11, 12]. It is worth mentioning that we can describe dynamics of the function B (which we call evolution) if the domain $\text{conv}(\gamma_2)$ increases by inclusion. Note that

$$A((t, t^2), (t, t^2 + \varepsilon^2), I) = BMO_\varepsilon(I).$$

Let $p_1 > p_2$ and $Q \geq 1$. Then

$$A((t^{1/p_2}, t^{1/p_1}), (Qt^{1/p_2}, t^{1/p_1}), I) = A_{p_1, p_2}(Q),$$

where

$$A_{p_1, p_2}(Q) \stackrel{\text{def}}{=} \{\varphi : [\varphi]_{A_{p_1, p_2}} = \sup_{J \subset I} \langle \varphi^{p_1} \rangle_J^{1/p_1} \langle \varphi^{p_2} \rangle_J^{-1/p_2} \leq Q\}.$$

If $p \in (1, \infty)$ then $A_{1, -\frac{1}{p-1}} = A_p$ where A_p stands for the classical Muckenhoupt class. When $p_2 = 1$ and $p_1 > 1$, the class A_{p_1, p_2} coincides with the so-called Gehring class (sometimes the Gehring class is called reverse-Hölder class).

Finally we finish this subsection by mentioning underlying PDE in these extremal problems. The concave function B has prescribed boundary condition $f(t)$ and it satisfies homogeneous Monge–Ampère equation i.e.,

$$\begin{aligned} B(\gamma_1(t)) &= f(t); \\ \text{Hess } B &\leq 0 \quad \text{in} \quad \text{conv}(\gamma_1) \setminus \text{conv}(\gamma_2); \\ \det(\text{Hess } B) &= 0 \quad \text{in} \quad \text{conv}(\gamma_1) \setminus \text{conv}(\gamma_2). \end{aligned} \tag{3.0.28}$$

3.0.7 Relation to differential geometry

If one tries to obtain sharp inequalities in the theorems 3.0.22, 3.0.23, 3.0.24 and 3.0.27 then besides of solving the corresponding partial differential inequalities (3.0.13), (3.0.22), (3.0.24)

and (3.0.28) one also has to solve corresponding partial differential equations

$$H_1 H_2 H_{12} (1 - a_1^2 - a_2^2) + a_1^2 H_2^2 H_{11} + a_2^2 H_1^2 H_{22} = 0. \quad (3.0.29)$$

$$\text{Hess } M + \left(\frac{\nabla_y M}{y} \right)_{2n \times 2n} \leq 0 \quad \text{and} \quad \det \left[\text{Hess } M + \left(\frac{\nabla_y M}{y} \right)_{2n \times 2n} \right] = 0. \quad (3.0.30)$$

$$ACA^* \bullet \text{Hess } B \leq 0 \quad \text{and} \quad \det(ACA^* \bullet \text{Hess } B) = 0. \quad (3.0.31)$$

$$\text{Hess } B \leq 0 \quad \text{and} \quad \det(\text{Hess } B) = 0. \quad (3.0.32)$$

And find the minimal (or maximal if the inequalities are reversed) solution if possible with prescribed obstacle conditions.

3.0.7.1 Developable surfaces, concavity and the torsion of the space curve

Solution of Theorem 3.0.27 includes finding the following function B : given $f(t)$ and the strictly convex curves γ_1, γ_2 on the plane \mathbb{R}^2 such that $\text{conv}(\gamma_2) \subset \text{conv}(\gamma_1)$, find B in the domain $\text{conv}(\gamma_1) \setminus \text{conv}(\gamma_2)$ such that

$$\begin{aligned} B(\gamma_1(t)) &= f(t); \\ \text{Hess } B &\leq 0 \quad \text{in} \quad \text{conv}(\gamma_1) \setminus \text{conv}(\gamma_2); \\ \det(\text{Hess } B) &= 0. \end{aligned} \quad (3.0.33)$$

And among all possible solutions choose the minimal one. This means that in general we do not have uniqueness results for B . This is so because the domain of B is not convex.

These surfaces $(x, y, B(x, y))$ are known as developable surfaces because they have almost everywhere zero Gaussian curvature. It turns out that torsion of the boundary data $(\gamma_1(t), f(t))$ appears as the main object in studying concavity of the surfaces with zero Gaus-

sian curvature (see Chapter 1, see also [9, 10, 11, 12]). It is worth mentioning that some flaw of 4 vertex theorem and bitangent line (see [56, 58, 58]) appeared in the work [13] in order to find the function B in uniform convexity.

Some abstract works have been done regarding existence of such solutions in the works of Caffarelli, Nirenberg and Spruck (see [2]). The main difference between these results is that we explicitly construct solutions for each given boundary data (which as a corollary gives answer about existence and uniqueness of the solution), however we do it only in two dimensional case whereas in the work of [2] the proof of existence of smooth solutions (for the smooth boundary data) in any dimension is given. Theory in [9] was developed also for piecewise continuous boundary data $f(t)$, and in particular it explicitly shows what happens with the continuity of the global concave solutions of homogeneous Monge–Ampère equation once we remove smoothness of the boundary data. We also developed dynamic for the solutions once we change smoothly domain of the function B . This is what we call evolution of the Bellman function over its domain.

There are still some open problems left:

Problem 1. *Find the function B in some adequate way if the torsion of the space curve $(\gamma_1(t), f(t))$ changes sign infinitely many times.*

Problem 2. *Find the similar characterizations in the high dimensions $n \geq 3$.*

3.0.7.2 Modified Monge–Ampère equation

Finding sharp inequalities in Theorem 3.0.24 includes solving the following differential equations

$$A^*CA \bullet \text{Hess } B \leq 0 \quad \text{and} \quad \det(A^*CA \bullet \text{Hess } B) = 0,$$

where $B \in C^2$ is given in some parallelepiped, for example, say \mathbb{R}_+^k , $A = (a_1, \dots, a_n)$ is $k \times n$ matrix with full rank.

If B satisfies assumptions L1-L5 (see Chapter 2) then we gave complete characterization of such functions B in the case $k = 1, k = n$ and in the case $k = n - 1$ if in addition $B_{ij} \neq 0$.

General questions still remains open. Let A be $k \times n$ matrix.

Problem 3. *Describe all possible solutions of the partial differential inequalities*

$$A^*A \bullet \text{Hess } B \leq 0 \quad \text{and} \quad \det(A^*A \bullet \text{Hess } B) = 0.$$

Let us consider the following particular. Let $B \in C^2$ be given in some rectangular domain. Let $n = k = 2$ and take $A = (a_1, a_2)$ where $a_1, a_2 \neq 0$. Then we must have

$$A^*A \bullet \text{Hess } B = \begin{pmatrix} |a_1|^2 B_{11} & a_1 \cdot a_2 B_{12} \\ a_1 \cdot a_2 B_{12} & |a_2|^2 B_{22} \end{pmatrix} \leq 0 \quad \text{and} \quad \det(A^*A \bullet \text{Hess } B) = 0.$$

If $a_1 \cdot a_2 = 0$ then B has to be separate concave functions such that $B_{11}B_{22} = 0$ and these are the all possible solutions. Therefore we assume that $a_1 \cdot a_2 \neq 0$. Then we see that B

must be separate concave function and moreover

$$\frac{|a_1|^2|a_2|^2}{|a_1 \cdot a_2|^2} B_{11} B_{22} - B_{12}^2 = 0.$$

So in the case $n = k = 2$ the problem reduces to the following one

Problem 4. *Let $|c| \in [1, \infty)$ and let $B \in C^2$ be given on some rectangular domain in \mathbb{R}^2 .*

Characterize all possible separately concave functions B such that

$$c^2 B_{11} B_{22} - B_{12}^2 = 0. \tag{3.0.34}$$

The trivial case $|c| = 1$ corresponds to developable surface and the characterization of these surfaces are mostly known. For general $|c| > 1$ we can give local characterization. Namely, we will show that the above equation can be reduced to the following one

$$\frac{\partial f}{\partial \bar{z}} = \bar{f}$$

for some appropriate f (see below).

For separately concave $B(x, y)$ set $B_{xx} = -p^2, B_{yy} = -q^2$. Then equation (3.0.34) implies that $B_{xy} = cpq$. We also have

$$-2pp_y = cqp_x + cpq_x, \tag{3.0.35}$$

$$-2qq_x = cqp_y + cpq_y. \tag{3.0.36}$$

Further we assume that $p, q \neq 0$. Assume that the locally the map $p, q : (x, y) \rightarrow \mathbb{R}^2$ is

invertible, and let (x, y) be its inverse map. Then

$$\begin{pmatrix} p_x & p_y \\ q_x & q_y \end{pmatrix} = \begin{pmatrix} x_p & x_q \\ y_p & y_q \end{pmatrix}^{-1} = \frac{1}{\det(\text{Jacob}(x, y))} \cdot \begin{pmatrix} y_q & -x_q \\ -y_p & x_p \end{pmatrix}.$$

Therefore equations (3.0.35) and (3.0.36) take the following form

$$2px_q = cqy_q - cpy_p,$$

$$2qy_p = -cqx_q + cpx_p.$$

This can be written as follows

$$2(px)_q = c(qy)_q - c(py)_p,$$

$$2(qy)_p = -c(qx)_q + c(px)_p.$$

We set $\tilde{U}(p, q) = px(p, q)$ and $\tilde{V}(p, q) = qy(p, q)$. Then we obtain

$$2\tilde{U}_q = c\tilde{V}_q - c \left(\frac{\tilde{V}p}{q} \right)_p,$$

$$2\tilde{V}_p = -c \left(\frac{\tilde{U}q}{p} \right)_q + c\tilde{U}_p.$$

After the logarithmic substitution $U(p, q) = M(\ln p, \ln q)$ and $V(p, q) = N(\ln p, \ln q)$ then we

obtain the linear equation

$$2M_2 = c(N_2 - N - N_1),$$

$$2N_1 = c(-M - M_2 + M_1).$$

By setting $k = 2/c \in (-2, 2)$, this can be rewritten as follows

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -k & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + \begin{pmatrix} 0 & -k \\ 1 & -1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}.$$

We need the following technical lemma.

Lemma 23. *If the vector function $\vec{N}(x, y) = (N, M) : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies the following first order system of linear differential equations*

$$\vec{N} = P\vec{N}_1 + Q\vec{N}_2$$

for some invertible 2×2 matrices P, Q where

$$QP^{-1} = \begin{pmatrix} -2t & \delta^2 \\ -1 & 0 \end{pmatrix}$$

for some $t \in (-\delta, \delta)$, $\delta > 0$ then after making change of variables $\vec{N}(\vec{x}) = B\vec{U}(A\vec{x})$ where

$$B = \begin{pmatrix} t & \sqrt{\delta^2 - t^2} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A^T = \frac{1}{2}P^{-1} \begin{pmatrix} -1 & -\frac{t}{\sqrt{\delta^2 - t^2}} \\ 0 & -\frac{1}{\sqrt{\delta^2 - t^2}} \end{pmatrix},$$

and we obtain

$$\frac{\partial f}{\partial \bar{z}} = \bar{f},$$

where $f = U + iV$.

Proof. Set $P = (P_1, P_2)$, $Q = (Q_1, Q_2)$ where P_i, Q_j are columns.

$$\begin{pmatrix} N \\ M \end{pmatrix} = N_1 P_1 + N_2 P_2 + M_1 Q_1 + M_2 Q_2.$$

Now let $N(x, y) = \tilde{N}(\alpha_1 x + \alpha_2 y, \beta_1 x + \beta_2 y)$ then

$$N_1 = \alpha_1 \tilde{N}_1 + \beta_1 \tilde{N}_2;$$

$$N_2 = \alpha_2 \tilde{N}_1 + \beta_2 \tilde{N}_2;$$

$$M_1 = \alpha_1 \tilde{M}_1 + \beta_1 \tilde{M}_2;$$

$$M_2 = \alpha_2 \tilde{M}_1 + \beta_2 \tilde{M}_2.$$

So we obtain

$$\begin{pmatrix} \tilde{N} \\ \tilde{M} \end{pmatrix} = (P_1\alpha_1 + P_2\alpha_2)\tilde{N}_1 + (P_1\beta_1 + P_2\beta_2)\tilde{N}_2 + \\ (Q_1\alpha_1 + Q_2\alpha_2)\tilde{M}_1 + (Q_1\beta_1 + Q_2\beta_2)\tilde{M}_2.$$

Finally we set

$$\tilde{N} = a_1U + b_1V$$

$$\tilde{M} = a_2U + b_2V.$$

and

$$B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Thus we obtain

$$\begin{pmatrix} U \\ V \end{pmatrix} = \\ B^{-1}[a_1(P_1\alpha_1 + P_2\alpha_2) + a_2(Q_1\alpha_1 + Q_2\alpha_2)]U_1 + \\ B^{-1}[a_1(P_1\beta_1 + P_2\beta_2) + a_2(Q_1\beta_1 + Q_2\beta_2)]U_2 + \\ B^{-1}[b_1(P_1\alpha_1 + P_2\alpha_2) + b_2(Q_1\alpha_1 + Q_2\alpha_2)]V_1 + \\ B^{-1}[b_1(P_1\beta_1 + P_2\beta_2) + b_2(Q_1\beta_1 + Q_2\beta_2)]V_2.$$

And we would like to see that

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

This can hold if and only if

$$\begin{aligned} (P\alpha, Q\alpha) &= \frac{1}{2} B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B^{-1} = \frac{1}{2} B I^+ B^{-1}; \\ (P\beta, Q\beta) &= \frac{1}{2} B \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} B^{-1} = \frac{1}{2} B I^- B^{-1}, \end{aligned}$$

where

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}; \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$, and let

$$\begin{aligned} B_1 &= \frac{1}{2} \left(B I^+ B^{-1} e_1, B I^- B^{-1} e_1 \right), \\ B_2 &= \frac{1}{2} \left(B I^+ B^{-1} e_2, B I^- B^{-1} e_2 \right). \end{aligned}$$

Then the above conditions hold iff

$$P(\alpha, \beta) = B_1;$$

$$Q(\alpha, \beta) = B_2.$$

Therefore if B is chosen in such a way that $QP^{-1} = B_2B_1^{-1}$ then by setting $(\alpha, \beta) = P^{-1}B_1$ we obtain the desired result. But note that if a and b are corresponding rows of the matrix B then

$$B_2B_1^{-1} = \begin{pmatrix} -2\frac{a \cdot b}{|b|^2} & \frac{|a|^2}{|b|^2} \\ -1 & 0 \end{pmatrix},$$

so the claim follows. □

In our case of P, Q we have

$$QP^{-1} = \begin{pmatrix} -k & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{c} & 1 \\ -1 & 0 \end{pmatrix},$$

therefore we can apply the lemma and we see that taking $t = 1/c \in (-1, 1)$ and $\delta = 1$ we have

$$B = \begin{pmatrix} t & \sqrt{1-t^2} \\ 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{4t\sqrt{1-t^2}} & -\frac{1}{4}\frac{2t^2-1}{t\sqrt{1-t^2}} \end{pmatrix}.$$

This means that if we set

$$\begin{pmatrix} N(x, y) \\ M(x, y) \end{pmatrix} = \begin{pmatrix} t & \sqrt{1-t^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U\left(-\frac{y}{2}, \frac{x}{4t\sqrt{1-t^2}} - \frac{y}{4} \frac{2t^2-1}{t\sqrt{1-t^2}}\right) \\ V\left(-\frac{y}{2}, \frac{x}{4t\sqrt{1-t^2}} - \frac{y}{4} \frac{2t^2-1}{t\sqrt{1-t^2}}\right) \end{pmatrix},$$

where by setting $z = x + iy$ for the function $f(z, \bar{z}) = U(x, y) + iV(x, y)$ we have

$$\frac{\partial f}{\partial \bar{z}} = \bar{f}.$$

It is known that all C^1 solutions of the above equation are real analytic and they can be represented in terms of power series

$$f(z) = \sum_{k=0}^{\infty} c_k J^{(k)}(z\bar{z}) z^k + \bar{c}_k J^{(k+1)}(z\bar{z}) \bar{z}^{k+1},$$

where $J(r)$ is modified Bessel I -functions whose series representation is

$$f(r) = \sum_{j=0}^{\infty} \frac{r^j}{(j!)^2}.$$

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