SUPPLEMENT

TO THE THESIS

# AN EXPERIMENT USING PROGRAMED MATERIAL IN TEACHING A NONCREDIT ALGEBRA COURSE AT THE COLLEGE LEVEL

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

Elaine Vivian Alton

1965

SUPPLEMENT

TO THE THESIS

# AN EXPERIMENT USING PROGRAMED MATERIAL IN TEACHING A NONCREDIT ALGEBRA COURSE AT THE COLLEGE LEVEL

By

Elaine Vivian Alton

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

College of Education

1965

# Preface

Increasing enrollments have brought to colleges a number of students who are inadequately prepared to complete college algebra successfully. These students need to relearn many of the basic ideas of elementary algebra.

This material was written for the student who needs to relearn and review these basic ideas. While most students will have some previous background in algebra, it is assumed only that students are able to do addition, subtraction, multiplication and division with positive integers. Therefore a review of arithmetic as well as a complete treatment of the basic ideas of algebra through quadratic equations are included. Thus this material may be used by students who have no previous background in algebra.

The answers to the problems are more complete than those usually found in a textbook or workbook. This is to give a student additional help with the intermediate steps of a complex problem.

TABLE OF CONTENTS


| | | Page |
|---|---|---|
| PREFACE | | iii |
| INSTRUCTIONS | | 1 |
| CHAPTERS | | |
| 1. | REVIEW OF ARITHMETIC | 2 |
| 2. | SIGNED NUMBERS | 44 |
| 3. | OPERATIONS WITH POLYNOMIALS | 69 |
| 4. | FACTORING | 104 |
| 5. | FRACTIONS | 144 |
| 6. | LINEAR AND FRACTIONAL EQUATIONS | 174 |
| 7. | EXPONENTS AND RADICALS | 207 |
| 8. | QUADRATIC EQUATIONS | 234 |
| TEST A | | 252 |
| TEST B | | 254 |
| TEST C | | 256 |
| TEST D | | 257 |

## Instructions

A series of questions and incomplete statements appear on the following pages. You are to answer the questions or fill in the blank or blanks. The information needed is contained in the statement or in previous statements.

Read each frame completely before writing your answer. You must read carefully to make sure your answer is the correct one.

Clues such as underlined words and the first letter of missing words are often given. Pay attention to these.

Compare your answer to the correct one. The correct one is given to the left of the next statement.

It is recommended that you use an index card or a piece of paper to hide the answer until you have decided what the answer should be. You will probably learn more as reading an answer is not the same as constructing one yourself. The eye is quicker than the will power of the most honest.

If you make an error, make sure that the given answer makes sense to you before proceeding to the next frame.

For continuity of thought, take a few minutes at the beginning of each work session to reread a few frames immediately preceding your last answer.

## Chapter 1 - Review of Arithmetic

The algebra presented in this book deals with the applications of the processes of arithmetic to symbols and letters which represent numbers. In order to do algebra, you will need an understanding of the arithmetic processes. You will also need to do these processes with a certain amount of speed. The only way to obtain understanding, accuracy and speed is through practice. Therefore, it is necessary to review the arithmetic processes.

| | |
|---|---|
| | 1. The numbers you first dealt with in arithmetic were the numbers 1, 2, 3, ... These are called the counting numbers or the <u>positive</u> <u>integers</u>. 2, 423 and 17 are examples of ________ ________. |
| 1. positive integers | 2. How many positive integers are there? |
| 2. an infinite number | 3. Is 0 a positive integer? |
| 3. No. 0 is an integer but is <u>not</u> a positive integer. | 4. Is 0 an integer? |
| 4. Yes. (See answer to frame 3.) | 5. You dealt with four fundamental operations in arithmetic. These operations are addition, ________, ____________, and __________. |
| 5. subtraction, multiplication, and division. | 6. We deal with these same operations in algebra. These operations obey certain basic laws and we shall find out about these laws as well as reviewing arithmetic techniques. In addition, the numbers to be added are called <u>terms</u>.<br><br>In 2+5+7, the 5 is called a _____. |
| 6. term | 7. The result of addition is called the <u>sum</u>.<br><br>2+5 can be read "find the _____ of 2 and 5". |

7. sum

8. 2+5 also represents the sum of 2 and 5. Most of you would probably say that 7 was the sum of 2 and 5. However, 7 is another numeral which represents the same number as 2+5 represents. Using 4 and one other number, write another numeral which represents the same number as 2+5 represents.

8. 4+3 or 11-4

9. 4+3 represents the ______ of 3 and 4.

9. sum

10. Write a single numberal which represents the sum of 3 and 4.

10. 7

11. We often use () parentheses, brackets[], and braces to indicate that a group of terms is to be treated as a single number. For example, 5+(3-1) indicates that we are to add ___________ (how many) numbers?

11. two

12. In 5+(3-1), how many terms are there?

What are the terms?

12. two terms

terms are (3-1) and 5 (these can be given in any order)

13. It is customary in arithmetic to use the symbol "x" to mean multiplication. It is more convenient in algebra to use either a dot · or () to indicate multiplication. Instead of writing 2 x 3, we can write 2·3, 2(3) or (2)(3). All of these mean

2 ______________ 3.

13. multiplied by
or
times

14. The result of multiplication is called the <u>product</u>. What is the product of 7 and 9?

14. $7 \cdot 9$ or $7(9)$ or $(7)(9)$ or 63

15. Most of you probably wrote 63 as the product of 7 and 9. Note that $7 \cdot 9$, $7(9)$ and $(7)(9)$ are different symbols which represent the product of 7 and 9. 63 is another symbol which also represents this product. Write the product of 4 and 5 in three different ways.

15. $4 \cdot 5$, $4(5)$, $(4)(5)$, 20

any three of the above

16. Express 12 as the product of two numbers each of which is less than 5.

16. $4 \cdot 3$, or $4(3)$ or $(3)(4)$
the 4 and the 3 can be listed in any order.

17. In $3 \cdot 4$, the 4 and the 3 are known as factors. <u>Factors</u> are numbers which are multiplied together or are divisors of the product.
What are the factors of 14?

17. 2 and 7
or
1 and 14

18. How many factors are there in $3 \cdot 2(4-1)$?

List the factors.

18. three factors
the factors are 2, $(4-1)$ and 3

19. In $2+3$, is the 2 a factor? Give a reason for your answer.

19. No because factors are associated with products and this is a sum.
or
No as no multiplication is indicated.

20. What is the 2 in $2+3$ called?

20. a term
(Refer to frame 6 if necessary)

21. In $2+3 \cdot 5$, there are how many terms? What are they?

21. two terms
the terms are $3 \cdot 5$ and 2

22. In $2+3 \cdot 5$, is the 5 a factor of the whole quantity? Give a reason for your answer.

22. No. The 5 is a factor of the second term. It is <u>not</u> a factor of the whole expression as it is <u>not</u> multiplied by $2+3$.

23. In $(2+3)5$, is the 5 a factor of the whole quantity?

23. Yes because it is multiplied by 2+3 or because it is a divisor of (2+3)5.

24. Note carefully the difference between 2+3·5 and (2+3)5. When we talk of factors, we usually mean factors of the complete expression.
In 4·3+5(1+6)+13(9-2)(11+7), the second term is ________.

24. 5(1+6)

25. If we add or multiply any two positive integers together, it is evident that the order of the numbers is immaterial. This is known as the commutative law.

For example, 4+7=7+4 illustrates the ____________ law for addition.

25. commutative

26. Complete the following statement to show the commutative law of multiplication.

3(5) = __________

26. 3(5) = 5(3)
only the order of the two numbers is changed

27. We postulate the commutative law when the numbers are positive integers. After the operations of addition and multiplication are defined for negative integers, rational numbers and irrational numbers, we can prove the commutative law true for all real numbers. For now, we will only deal with the commutative law for positive integers.

Commutative Law
If a and b are positive integers, then a+b=b+a and a·b=b·a.

State the commutative law in words.

27. Given two nonnegative integers to add or multiply, the order of the two numbers is immaterial.

28. 4 + (5-2) indicates that __________ numbers are to be
(how many)
added?

By the commutative law of addition
4 + (5-2) = ______________

28. two

(5-2) + 4
Remember change only the order of two numbers.

29. By the commutative law of multiplication

3(4+6) = ______________

29. (4+6)3 or (4+6)(3)

You must use () or brackets [] or braces around the 4+6 to show that you are considering this as one number.

30. (4+6) in 3(4+6) is called a

______________.

30. factor
(see frame 17 if necessary)

31. No matter how many numbers you are given to add or multiply, only 2 numbers are added or multiplied at one time. We need to know what it means when there are more than 2 numbers to be added or multiplied. Consider 2+7+8.
We could add the 2 and the 7 and then add the result to 8. Using parentheses to show this:
2+7+8 = (2+7)+8.

Without changing the order of the numbers, how else could you find the value of 2+7+8?

31. Add 7 and 8 and then add this sum to 2.

32. Expressed using parentheses, we have 2+7+8 = 2+(7+8).
In other words, no matter how we group the numbers in addition, the sum is the same. This is called the associative law and the associative law applies to both addition and multiplication. The associative law is postulated true for positive integers.

Associative Law
If a, b and c are positive integers, then a+b+c = (a+b)+c = a+(b+c) and a·b·c = (a·b)c = a(b·c).

By the associative law of multiplication,

$2 \cdot 5 \cdot 7 = (2 \cdot 5)7 =$ ________

32. $2(5 \cdot 7)$
Note that we change only the way the numbers are grouped together <u>not</u> the order of the numbers.

33. By the associative law of addition,

$4+5+3 =$ ________ $=$ ________

33. $(4+5)+3 = 4+(5+3)$

in either order

34. State whether the following are true because of the commutative law or the associative law. Specify the operation (addition or multiplication).

(a) $3+(6+4)=(6+4)+3$ ________

(b) $7+(8+4)=(7+8)+4$ ________

(c) $(3 \cdot 6) \cdot 5 = 3(6 \cdot 5)$ ________

(d) $2(7+9)=(7+9) \cdot 2$ ________

(e) $2(3 \cdot 5)=(3 \cdot 5)(2)$ ________

34. (a) commutative law for addition
(b) associative law for addition
(c) associative law for multiplication
(d) commutative law for multiplication
(e) commutative law for multiplication

35. To show that $5+(6+4) = (5+4)+6$, we need to use both the commutative and associative laws of addition.

$5+(6+4) = 5+(4+6)$ by the commutative law of addition
$5+(4+6) = (5+4)+6$ by the associative law of addition
Therefore, $5+(6+4) = (5+4)+6$

Show that $(2 \cdot 5)8 = (8 \cdot 2)5$. State reasons.

35. $(2 \cdot 5)8 = 8(2 \cdot 5)$ by the commutative law of multiplication
$8(2 \cdot 5) = (8 \cdot 2)5$ by the associative law of multiplication.
Therefore, $(2 \cdot 5)8 = (8 \cdot 2)5$.

36. Show that $(3 \cdot 6)9 = 9(6 \cdot 3)$. State reasons.

37. The distributive law connects the operations of addition and multiplication.

Distributive Law
If a, b and c are positive integers, then
$a(b+c) = a\cdot b + a\cdot c$.

For example, $2(7+3) = 2\cdot7+2\cdot3$.
By the distributive law,
$3(5+4) =$ ______________.

36. $(3\cdot6)9 = 9(3\cdot6)$ by the comm. law of multiplication.
$9(3\cdot6) = 9(6\cdot3)$ by the comm. law of multiplication.
Therefore, $(3\cdot6)9 = 9(6\cdot3)$.

38. $8(6+9) = 8\cdot6+8\cdot9$ because of the __________________ law.

37. $3\cdot5+3\cdot4$

39. $8(6+9) = (6+9)8$ by the ____________ law of multiplication.

38. distributive

40. Thus both $8(6+9)$ and $(6+9)8$ are equal to $8\cdot6+8\cdot9$.

Using the commutative law of multiplication first and then the distributive law, complete

$13(7+8) =$ ____________ = ____________

39. commutative

41. An equality is true whether read from left to right or from right to left. Thus the distributive law not only states that $a(b+c) = a\cdot b+a\cdot c$ but also that $a\cdot b+a\cdot c = a(b+c)$.
Hence, $8\cdot5+8\cdot4 =$ ______________

40. $(7+8)13$ by the comm. law

$= 13\cdot7+13\cdot8$ or $7\cdot13+8\cdot13$ by the distributive law.

42. Complete using the distributive law.

(a) $5(1+3) =$ ________________

(b) $5\cdot2+5\cdot3 =$ ________________

(c) $12\cdot4+12\cdot5 =$ ________________

41. $8(5+4)$

(d) 4· __________ =

____ ·3+ ______ ·5

(e) 3· ____ + ____ ·2 =

5· __________

42. (a) 5·1+5·3
(b) 5(2+3)
(c) 12(4+5)
(d) 4·(3+5) = 4·3+4·5
Note - if you wrote 4·3+5 on the left side of the equation, it is wrong. 4·3+5 states that only 3 is multiplied by 4. You need the number (3+5) to be multiplied by 4.
(e) 3·5+5·2 = 5·(3+2)

43. By the distributive law, 2(7+3) = 2·7+2·3. This means that when 2(7+3) is evaluated we must get the same result as when 2·7+2·3 is evaluated.

2(7+3) means to ______________ (what operation) 2 and (7+3)?

2·7+2·3 means to ______________ (what operation ______________ on what numbers)

43. multiply

add 2·7 and 2·3

44. (7+3) is another symbol representing 10, therefore

2(7+3) = 2· ____ = __________

44. 2(7+3) = 2·10 = 20

45. 2·7 is another symbol representing 14.
2·3 is another symbol representing 6.

Therefore, 2·7+2·3= __________ =

__________.

45. 14+6 = 20

46. In 8+4(5+6), what one operation connects all the numerals?

46. addition

47. What terms are to be added if all the numerals are to be used?

47. 8 and
4(5+6)

48. Then we must evaluate 4(5+6) before we can add it to 8. 4(5+6) is another symbol representing

______________.

48. 44

to evaluate $4(5+6)$, we can say this is $4\cdot 11$ which equals 44
or
we can use the distributive law so $4(5+6) = 4\cdot 5+4\cdot 6$ and this equals $20+24$ which equals 44.

49. Hence, $8+4(5+6) = 8+$_____=________.

49. $8+44 = 52$

50. To simplify $(2+4)(19+2)$, first determine the <u>one</u> operation which connects all the numbers.

This is ________________.

50. multiplication

51. Numbers which are to be multiplied are called _____________. (see frame 17 if necessary)

51. factors

52. The factors in $(2+4)(19+2)$ are _____________ and _____________.

52. $(2+4)$ and $(19+2)$

53. Thus $(2+4)(19+2) =$ _____________ and this equals _____________.

53. $(6)(21) = 126$

54. $(2+4)(19+2)$ can also be evaluated using the distributive law. $(2+4)(19+2) = 2(19+2)+4(19+2)$ by the distributive law.

54. $2\cdot 19+2\cdot 2+4\cdot 19+4\cdot 2 =$
$38+4+76+8 =$
$126$

55. Use the distributive law to simplify $(3+4)(5+6+2)$.

55. $(3+4)(5+6+2) =$
$3(5+6+2)+4(5+6+2) =$
$3\cdot 5+3\cdot 6+3\cdot 2+4\cdot 5+4\cdot 6+4\cdot 2 =$
$15+18+6+20+24+8 = 91$
or
$(3+4)(5+6+2) =$
$3(5+6+2)+4(5+6+2) =$
$3\cdot 13 + 4\cdot 13 =$
$39+52 = 91$

56. Simplify the following.

(a) $2+3(4+5)$

(b) $(2+3)(4+5)$

(c) $2\cdot 3+4\cdot 5$

(d) $(2\cdot 3+4)\cdot 5$

56. (a) $2+3(4+5) = 2+3(9) =$
$2+27 = 29$
or
$2+3(4+5) = 2+3\cdot4+3\cdot5 =$
$2+12+15 = 29$

(b) $(2+3)(4+5) = 5(9) = 45$
or
$(2+3)(4+5) = 2(4+5)+3(4+5) =$
$2\cdot9+3\cdot9 = 18+27 = 45$

(c) $2\cdot3+4\cdot5 = 6+20 = 26$

(d) $(2\cdot3+4)5 = (6+4)5 =$
$10\cdot5 = 50$
or
$(6+4)5 = 6\cdot5+4\cdot5 = 30+20 =$
$50$

57. Note that in the last frame, the four problems all contained exactly the same numbers 2, 3, 4 and 5. However, they were grouped and combined in different ways and so none of the results were equal. <u>Make sure you understand the differences in these</u>.

$3^2$ is read "3 squared" and means $(3)(3)$ or $3\cdot3$.

$3^3$ is read "3 cubed" and means

__________ or that there are

__________ factors of 3.
(how many)

57. $3(3)(3)$

three factors

58. $3^4$ is read "3 raised to the fourth power" and means ______________.

58. $3\cdot3\cdot3\cdot3$ or that there are four factors of 3.

59. $2(2)(2)(2)$ can be written _______

which is read ____________________.

59. $2^4$
2 raised to the fourth power

60. In $2^4$, the 4 is called an <u>exponent</u> and the 2 is called the <u>base</u>. An <u>exponent</u> shows the number of times the base is to be used as a factor providing that the exponent is a positive integer.

In $4^6$, the 6 is called the _______.

60. exponent

61. In $5^3$, the 5 is called the _______.

61. base

62. $2^3$ means __________ and this equals

____________.

62. $2\cdot2\cdot2 = 8$

63. Evaluate the following.

(a) $3^2 =$ _______ $=$ _______

(b) $2^5 =$ _______ $=$ _______

(c) $4^3$ = ________ = ________

(d) $5^2$ = ________ = ________

(e) $3^3$ = ________ = ________

63. (a) $3\cdot3 = 9$
(b) $2\cdot2\cdot2\cdot2\cdot2 = 32$
(c) $4\cdot4\cdot4 = 64$
(d) $5\cdot5 = 25$
(e) $3\cdot3\cdot3 = 27$

64. Write $2(2)(2)(5)(5)$ using exponents.

64. $2^3\cdot5^2$

65. $2^3\cdot5^2$ = __________ = __________

65. $8\cdot25 = 200$

66. $2^3+3^2$ = __________ = __________

66. $8+9 = 17$

67. In $(2\cdot5)^4$, the 4 is called an

____________.

67. exponent

68. In $(2\cdot5)^4$, the 4 indicates that __________ is to be used as a factor 4 times.

68. $(2\cdot5)$

69. Thus $(2\cdot5)^4 = (2\cdot5)(2\cdot5)(2\cdot5)(2\cdot5)$.

Finish evaluating $(2\cdot5)^4$.

69. $= 10\cdot10\cdot10\cdot10 = 10{,}000$

70. $(2+3)^3$ means ________________.

70. $(2+3)$ used as a factor three times
or $(2+3)(2+3)(2+3)$

71. Thus, $(2+3)^3 = (2+3)(2+3)(2+3)$ =

____________ = ____________

71. $5(5)(5) = 125$

72. Is $(2+3)^3$ equal to $2^3+3^3$? Give a reason for your answer.

72. No.
$(2+3)^3 = (2+3)(2+3)(2+3)$

$2^3+3^3 = 2\cdot2\cdot2+3\cdot3\cdot3$
and these two quantities <u>don't</u> equal the same number.
or
$(2+3)^3$ means to add 2 and 3 and then cube the result.

73. We shall use the symbol $\neq$ to mean unequal.

So we may write $(2+3)^3$

__________ $2^3+3^3$.

$2^3+3^3$ means to cube 2 and then cube 3 and then to add the results.

73. $\neq$

74. Does $3\cdot 2^4$ equal $(3\cdot 2)^4$? Give a reason for your answer.

74. No. $3\cdot 2^4 = 3\cdot 2\cdot 2\cdot 2\cdot 2$ and $(3\cdot 2)^4=(3\cdot 2)(3\cdot 2)(3\cdot 2)(3\cdot 2)$
The same factors are not used in both cases.
or
$3\cdot 2^4 = 3(16) = 48$
and
$(3\cdot 2)^4= 6^4 = 1{,}296$ and these aren't equal.

75. Complete the following.

(a) $3\cdot 4^2 =$ ________ = ________

(b) $4(1)^3 =$ ________ = ________

(c) $(4+5)^2 =$ ________ = ________

(d) $3\cdot 2^3+3^2\cdot 10 =$ ________=________

(e) $(3\cdot 2^3+3^2)10 =$ ________=________

(f) $(4+3)(1+4)^3 =$ ________=________

75. (a) $3\cdot 16 = 48$
(b) $4(1) = 4$
(c) $(4+5)(4+5) = 9\cdot 9 = 81$
(d) $3\cdot 8+9\cdot 10 = 24+90 = 114$
(e) $(3\cdot 8+9)10 = (24+9)10 = 330$
(f) $(7)(5)^3 = 7(125) = 875$

76. Use either a = or $\neq$ to make the following true.

(a) $1^5$ ______ 5

(b) $(2+3)^2$ ______ $5^2$

(c) $2\cdot 5^2$ ______ 50

(d) $7^3$ ______ $7\cdot 1^3$

(e) $4^3+5^3$ ______ $(4+5)^3$

76. (a) $\neq$
(b) =
(c) =
(d) $\neq$
(e) $\neq$

77. So far we have dealt with the operations of addition and multiplication with positive integers. Let us now consider these two operations and the number 0. Is 0 an integer?

77. Yes (see frame 3 if necessary)

78. 0 is an integer but is not a positive integer. The positive integers and zero comprise the set of <u>nonnegative</u> integers.

<u>Properties of 0</u>

If a is a nonnegative integer, then a+0 = 0 and a·0 = 0.

Thus, 156 + 0 = ________.

78. 156

79. 481·0 = ________.

79. 0

80. State whether the following are true or false.

(a) 16·0 = 16

(b) 16+0 = 16

(c) 0+0 = 0

(d) 0·0 = 0

80. (a) false
(b) true
(c) true
(d) true

81. The commutative law holds for the nonnegative integers. By the commutative law, 56+0 = ________.

81. 0+56

82. By the commutative law 34·0 =

________.

82. 0·34

83. If two nonnegative integers are multiplied and the product is 0, what can you say about the two numbers?

83. One or else both of the numbers is 0.

84. This can be stated:
If a and b are nonnegative integers, and if a·b = 0, then either a or b or both must be 0.

0 is known as the <u>identity</u> element of addition because when 0 is added to any nonnegative integer the sum is that same nonnegative integer.

If there is an identity element for multiplication, it must be a nonnegative integer such that when it is multiplied by any nonnegative integer, the product is that nonnegative integer.
In other words, if a is a nonnegative integer and if there is an identity element for multiplication, then a·(the identity element) = a.
Is there an identity element for multiplication and if so, what is it?

84. Yes.
identity element is 1.

85. If the product of 342 and some number is 342, what is that number?

85. 1

86. The operation of subtraction is defined in terms of addition.
<u>Subtraction</u>
If a and b are nonnegative integers, then a-b is the difference of a and b and a-b = c providing c is a nonnegative integer and providing c+b = a.
For example, the difference of 6 and 4 is written 6-4 and this exists providing there is a nonnegative integer which can be added to 4 and give 6.

Write the difference of 2 and 3.

__________.

Does this exist?

86. 2-3
no this doesn't exist as there is <u>no</u> nonnegative integer which can be added to 3 to give 2.

87. Write the difference of 45 and 27.

__________.

87. 45-27

88. Complete the following.

(a) 45-27 = ________ because

________________.

(b) _____-46 = 19 because

________________.

88. (a) 18 because 18+27 = 45
(b) 65 because 19+46 = 65

89. The difference of 6 and 4 can be written as 6-4.
It can also be written as

$$\begin{array}{r} 6 \\ -4 \\ \hline 2 \end{array}$$

The number to be subtracted is called the <u>subtrahend</u>.
The subtrahend in this case is 4.
The 6 in this case is called the <u>minuend</u>.

In 31-19 = 12, the 19 is called

the ______________.

89. subtrahend

90. If you have been in the habit of choosing the larger of two numbers as the minuend, you must break this habit. The larger number <u>isn't</u> always the minuend. The difference of 4 and 7 is written 4-7.

The difference of 7 and 4 is written 7-4.

Do both of these represent the same quantity? Give a reason for your answer.

90. no. 4-7 isn't defined as there is no nonnegative integer which when added to 7 equals 4.
7-4 = 3 because 3+4 = 7

91. Is subtraction commutative? Why?

91. no because the difference of 4 and 7 isn't equal to the difference of 7 and 4.

92. 7-4 can also be read "subtract 4 from 7".

In order to disprove a statement, you only need one counterexample, i.e., you need only one case that isn't true.

Write "subtract 14 from 29".

92. 29-14

93. 7-4 also means "take 4 from 7".

Write "take 8 from 4".

93. 4-8

94. Write symbols expressing the following and tell whether each exists.

(a) the difference of 54 and 67.

(b) the product of 43 and 0.

(c) take 9 from 16.

(d) the sum of 42 and 68.

(e) subtract 3 from 2.

94. (a) 54-67
this doesn't exist
(b) 43(0) or 0(43)
this does exist
(c) 16-9
this does exist
(d) 43+68 or 68+43
this does exist
(e) 2-3
this doesn't exist

95. In part b of the last frame why could my answer be written either as 43(0) or as 0(43)?

95. Multiplication is commutative, i.e., when two numbers are multiplied, they can be used in any order.

96. Division is defined in terms of multiplication.

<u>Division</u>

If a, b and c are nonnegative integers and $b \neq 0$, then the quotient of a and b (written $a \div b$ or $\frac{a}{b}$) is defined as c, (written $\frac{a}{b} = c$) only if $c \cdot b = a$. For example, $\frac{6}{3} = 2$ because $2(3) = 6$.

Complete: $\frac{39}{13}$ = _____ because

____________________.

96. 3 because 3(13) = 39

97. We may write the quotient of two numbers, however the quotient doesn't always have to exist. For example, the quotient of 4 and 8 can be written as $\frac{4}{8}$ but this doesn't exist as there is no nonnegative integer which when multiplied by 8 equals 4. Don't forget at this point the only numbers we can use are positive integers and 0.

Find the quotient of 0 and 4. Does this exist?

97. $\frac{0}{4}$ is the required quotient

This exists if there is a nonnegative integer which when multiplied by 4 equals 0. That is, (some nonnegative integer)(4) = 0. 0 is the required number so the quotient of 0 and 4 exists and is 0.

98. Write the following quotients. State which ones do not exist. For the ones which do exist, find the nonnegative integer which equals the quotient.

(a) the quotient of 56 and 4

(b) the quotient of 15 and 25

(c) the quotient of 0 and 24

(d) the quotient of 72 and 12

(e) the quotient of 36 and 8

(f) the quotient of 5 and 0

98. (a) $\frac{56}{4}$ This equals 14.

(b) $\frac{15}{25}$ This doesn't exist because there is no integer multiplied by 25 which equals 15.

99. Is division commutative? Give a reason for your answer.

(c) $\frac{0}{24}$ which equals 0.

(d) $\frac{72}{12}$ which equals 6.

(e) $\frac{36}{8}$ which doesn't exist because there is no integer multiplied by 8 which equals 36.

(f) $\frac{5}{0}$ which doesn't exist because every number multiplied by 0 equals 0
or
because there isn't any number multiplied by 0 which equals 5.

99. no. The quotient of 4 and 2 doesn't mean the same as the quotient of 2 and 4.

100. In frame 98, we stated that the quotient of 15 and 25 didn't exist. This is true since we were only concerned with the positive integers and zero. $\frac{15}{25}$ represents the quotient of 15 and 25, and this isn't an integer. We shall enlarge our set of numbers so that $\frac{15}{25}$ is defined.
You dealt in arithmetic with fractions and whole numbers. We are going to call these <u>rational numbers</u>.

<u>Definition</u>

A rational number is a number of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

$\frac{15}{25}$ is now defined to be a ______________ number.

100. rational

101. Are the positive integers part of the rational numbers? They are if they can be expressed as the quotient of two integers where the denominator isn't 0.

The integer 2 can be expressed as 2 divided by what integer?

101. 1

102. Can every positive integer be expressed as the quotient of itself and one?

102. yes

103. Can 0 be expressed as the quotient of itself and some other integer? If so, what integer?

103. yes
0 can be expressed as the quotient of 0 and any positive integer. It cannot be expressed as the quotient of 0 and 0 because by the definition in frame 100, the denominator can't be zero.

104. Then every integer can be expressed as the quotient of two integers where the denominator isn't zero and so every non-negative integer is also a

____________ number.

104. rational

105. Is every rational number also an integer? That is, can the quotient of every two integers be expressed as an integer?

105. no

106. Give an example of a rational number which is not an integer.

106. $\frac{1}{2}$, $\frac{5}{4}$, or any fraction where the numerator isn't a multiple of the denominator. There are an infinite number of possibilities.

107. Let us examine $\frac{4}{0}$ to see why we must exclude division by 0.
In any division problem such as 6 divided by 2, the quotient $\frac{6}{2}$ can be expressed as 3 because $3(2) = 6$.
So if the quotient $\frac{4}{0}$ exists, then $\frac{4}{0}$ must equal some number a such that when a is multiplied by 0 we will get 4, that is $4(0) = 4$.

We already know that any number multiplied by 0 is ________.

107. 0

108. Therefore there is <u>no</u> number, a, such that $\frac{4}{0} = a$ because there is <u>no</u> number, a, such that $a(0) = 4$.

Therefore, $\frac{4}{0}$ (is, is not) a number.
choose one

108. is not

109. Which of the following represent rational numbers?

$\frac{7}{4}$, 15, $\frac{0}{5}$, $\frac{24}{4}$, 0, $\frac{6}{0}$, $\frac{0}{0}$

109. all except $\frac{6}{0}$ and $\frac{0}{0}$

110. Which of the rational numbers in frame 109 are also integers?

110. 15, $\frac{0}{5}$ because it is another symbol for 0, $\frac{24}{4}$ which is another symbol for 6, and 0.

111. Let us now review the meaning of fractions and the operations on rational numbers.

We defined a rational number as a number of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

So any rational number can be thought of as the quotient of two numbers.
$\frac{a}{b}$ can also be considered as $a \cdot \frac{1}{b}$ where $b \neq 0$. Thus, $\frac{10}{6}$ means

________________.

111. $10 \cdot \frac{1}{6}$

112. Since the integers are rational numbers, the operations on rational numbers are defined so that the basic laws of commutativity, associativity and distribution hold with all rational numbers.
<u>Multiplication of rational numbers</u>
If a, b, c and d are integers and $b \neq 0$ and $d \neq 0$, then
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

In other words, multiply the numerators to get the new numerator and multiply the denominators to get the new denominator.
Why can't b and d equal 0?

112. division by 0 isn't defined

113. Then to multiply $\frac{5}{7}$ by $\frac{3}{2}$, we'd have $\frac{5}{7} \cdot \frac{3}{2} =$ ____________.

113. $\frac{5 \cdot 3}{7 \cdot 2} = \frac{15}{14}$

114. Is $\frac{3}{2} \cdot \frac{5}{7} = \frac{5}{7} \cdot \frac{3}{2}$ ? Why?

114. yes because as stated in frame 111, the commutative law for multiplication is to hold.

115. Multiply: (a) $\frac{24}{5} \cdot \frac{7}{5}$

(b) $\frac{3}{5} \cdot \frac{11}{2}$

(c) $\frac{5}{3} \cdot \frac{14}{9}$

(d) $\frac{13}{7} \cdot \frac{3}{4} \cdot \frac{5}{8}$

(e) $\frac{3}{2} \cdot 11$

115. (a) $\frac{24 \cdot 7}{5 \cdot 5} = \frac{168}{25}$

(b) $\frac{3 \cdot 11}{3 \cdot 2}$ $\frac{33}{10}$

(c) $\frac{5 \cdot 14}{3 \cdot 9} = \frac{70}{27}$

(d) remember the associative law applies here
$(\frac{13}{7} \cdot \frac{3}{4}) \frac{5}{8} = \frac{39}{28} \cdot \frac{5}{8} = \frac{195}{224}$

(e) 11 can be considered as $\frac{11}{1}$

$\frac{3}{2} \cdot 11 = \frac{3}{2} \cdot \frac{11}{1} = \frac{33}{2}$

116. <u>Equivalent fractions</u>
If a, b and c are rational numbers, and $b \neq 0$ and $c \neq 0$, then $\frac{a}{b} = \frac{ac}{bc}$ and a/b and ac/bc are called equivalent fractions.
In other words, the numerator and denominator of a fraction may be multiplied by a nonzero number and the value of the fraction will remain unchanged.
Change $\frac{2}{3}$ to an equivalent fraction having a denominator of 60.
State the process you used.

116. $\frac{2}{3} = \frac{40}{60}$
both numerator and denominator were multiplied by 20.
Note - it is <u>wrong</u> to say multiply by 20. This changes the value and <u>doesn't</u> give an equivalent fraction.

117. $\frac{40}{60}$ is a different rational number from $\frac{2}{3}$. However, it has the same value as 2/3.
How many rational numbers are there which are equivalent to 2/3?

117. an infinite number

118. Change the following to equivalent fractions by filling in the missing parts.

(a) $\frac{5}{4} = \frac{\quad}{56}$

(b) $\frac{5}{3} = \frac{45}{\quad}$

(c) $\frac{7}{8} = \frac{\quad}{72}$

(d) $\frac{13}{4} = \frac{52}{\quad}$

118. (a) 70
(b) 27
(c) 63
(d) 16

119. Change the following fractions to equivalent fractions all having the denominator of 24.

$\frac{3}{8}, \frac{7}{12}, \frac{7}{6}, \frac{11}{4}, \frac{13}{3}$

119. $\frac{3}{8} = \frac{9}{24}$, $\frac{7}{12} = \frac{14}{24}$,
$\frac{7}{6} = \frac{28}{24}$, $\frac{11}{4} = \frac{66}{24}$
$\frac{13}{3} = \frac{104}{24}$

120. What did you do to determine that $\frac{9}{24}$ was equivalent to $\frac{3}{8}$ ?

120. Multiplied both numerator and denominator of 3/8 by 3. NOT multiplied by 3.

121. We can now express $\frac{\frac{2}{3}}{\frac{5}{4}}$ as the quotient of two integers. We can change to an equivalent fraction by multiplying both numerator and denominator by the same number. If we multiply both numerator and denominator by 12, we will obtain integers in both places.

$$\frac{\frac{2}{3}}{\frac{5}{4}} = \frac{\frac{2}{3}(12)}{\frac{5}{4}(12)} = ?$$

121. $\frac{8}{15}$

122. By using this procedure, it is always possible to express the quotient of two rational numbers as the quotient of two integers or as a rational number. If you are given two fractions to determine whether or not they are equivalent fractions, you can make use of the following.

$\frac{a}{b} = \frac{c}{d}$ only if b and d aren't equal to 0 and ad = bc.

Using this, determine whether 11/13 and 9/17 are equivalent fractions.

122. no
$11(17) \neq 9(13)$

123. Are 8/12 and 2/3 equivalent fractions? Why?

123. yes because 8(3) = 12(2).

124. If we consider this last case, $\frac{8}{12} = \frac{2}{3}$, you will note that we have equivalent fractions when both the numerator and denominator are divided by the same nonzero number.
This comes directly from our definition of equivalent fractions.

The definition stated that $\frac{a}{b}$ and $\frac{ac}{bc}$ were equivalent fractions if a, b and c were rational numbers and b and c weren't equal to 0.

All equalities are true whether read from left to right or from right to left, so $\frac{a}{b} = \frac{ac}{bc}$ and $\frac{ac}{bc} = \frac{a}{c}$ when b and c don't equal 0.
Fill in the missing parts so that the following involve equivalent fractions.

(a) $\frac{7}{8} = \frac{\_\_}{32}$

(b) $\frac{16}{24} = \frac{4}{\_\_}$

(c) $\frac{5}{2} = \frac{\_\_}{1}$

(d) $\frac{18}{48} = \frac{\_\_}{8}$

(e) $\frac{18}{48} = \frac{6}{\_\_}$

(f) $\frac{18}{48} = \frac{\_\_}{12}$

124. (a) 28
(b) 6
(c) 5/2
(d) 3
(e) 16
(f) 18/4 or 9/2

125. In the last frame in parts d, e and f we had the same rational number $\frac{18}{48}$ and were to find equivalent fractions or were to find other rational numbers which had the same value as the given one. In part d we obtained $\frac{3}{8}$, in part e we obtained $\frac{6}{16}$, and in

part f we obtained $\frac{9/3}{12}$. Since all of these are equivalent to 18/48, they must be equivalent to each other.
Given a rational number, how many other rational numbers are there which are equivalent to it?

125. an infinite number

126. $\frac{3}{8}$ is a fraction in its lowest terms. A fraction in its lowest terms is one where the numerator and denominator are integers and have no common factors.

To reduce a fraction to its lowest terms, divide both numerator and denominator by their common factors.

What is the largest common factor of 16 and 24?

126. 8

127. Then to reduce $\frac{16}{24}$ to its lowest terms, we should divide both numerator and denominator by 8. Thus $\frac{16}{24}$ is equivalent to $\frac{2}{3}$ and since 2 and 3 have no common factors, 2/3 is in its lowest terms. Find a fraction equivalent to $\frac{36}{108}$ which is in its lowest terms.

127. $\frac{1}{3}$

128. You may find it difficult to decide if some numbers have common factors or what the largest common factor may be. You <u>don't</u> have to reduce a fraction by using only one step.

In reducing $\frac{36}{108}$, you can see that both of these are even numbers and every even number must be divisible by 2. Dividing both numerator and denominator by 2, we obtain $\frac{18}{54}$. We could repeat the same process as both 18 and 54 are even numbers. Dividing both numerator and denominator by 2, we now get $\frac{9}{27}$. Both of these numbers are divisible by 9 and so we get 1/3 which is in its lowest terms.

Reduce $\frac{144}{120}$

128. $\frac{144}{120} = \frac{6}{5}$

129. Express $\frac{57}{76}$ as an equivalent fraction in its lowest terms. It may appear that 57 and 76 have no common factor and in this case express each integer as factors without considering whether they are common factors.

Factors of 57 are ________

Factors of 76 are ________

129. $57 = 3 \cdot 19$
$76 = 2 \cdot 2 \cdot 19$ or $4 \cdot 19$ or $2 \cdot 38$

130. Then $\frac{57}{76} = \frac{3 \cdot 19}{2 \cdot 2 \cdot 19}$ or $\frac{3 \cdot 19}{4 \cdot 19}$ and in either case it is easy to see that both the numerator and denominator have a common factor of 19, or $\frac{57}{76} = \frac{3}{4}$.
Write equivalent fractions for each of the following making sure each result is in its lowest terms.

(a) $\frac{39}{52}$ (b) $\frac{72}{24}$ (c) $\frac{198}{154}$ (d) $\frac{68}{119}$

130. (a) $\frac{3}{4}$

(b) 3 or $\frac{3}{1}$

(c) $\frac{9}{7}$

(d) $\frac{4}{7}$

131. In $\frac{2+7}{5-2}$ is 2 a common factor of the numerator and denominator?

131. no

132. In the numerator of 2+7, the 2 is called a ____________.

132. term

133. Then to write a fraction equivalent to $\frac{2+7}{5-2}$ what must we do before we can decide whether the numerator and denominator have a common factor?

133. combine the terms in the numerator and denominator

134. $\frac{2+7}{5-2} = \frac{9}{3}$ and this can be reduced to 3 or $\frac{3}{1}$.

Reduce the following.

(a) $\frac{10+6}{30-6}$ (b) $\frac{10 \cdot 6}{30 \cdot 6}$ (c) $\frac{2(6)+1}{2(20)-1}$

(d) $\frac{72}{48}$ (e) $\frac{3^2+1}{4(3)-2}$

134. (a) $\frac{16}{24} = \frac{2}{3}$

(b) In this problem, the 6 is a factor in both the numerator and denominator, so you can immediately divide both numerator and denominator by the common factors.

$\frac{10 \cdot 6}{30 \cdot 6} = \frac{1}{3}$

(c) the 2 <u>is</u> <u>not</u> a factor here

$\frac{2(6)+1}{2(20)-1} = \frac{13}{39} = \frac{1}{3}$

135. <u>Addition of rational numbers</u>

If $b \neq 0$ and if a, b and c are rational numbers, then

$$\frac{a}{b} + \frac{c}{b} = \frac{1}{b} \cdot a + \frac{1}{b} \cdot c = \frac{1}{b}(a+c) = \frac{a+c}{b}$$

In this definition, note that both fractions have the same denominator. Since $\frac{a}{b}$ is defined to mean 1/b multiplied by a, we can use the distributive law to find the sum.

Find the sume of 5/3, 2/3 and 17/3.

(d) $\frac{3}{2}$

(e) the 3 is not a factor here.

$$\frac{3^2+1}{4(3)-2} = \frac{10}{10} = 1$$

135. $\frac{1}{3}(5+2+17) = \frac{24}{3} = 8$

136. Answers are usually expressed as equivalent rational numbers where the fractions are in their lowest terms.

Evaluate the following.

(a) $\frac{14}{9} + \frac{7}{9} + \frac{11}{9}$

(b) $\frac{7}{5} + \frac{4}{5} + \frac{9}{5}$

(c) $\frac{11}{2} \cdot \frac{1}{2}$

136. (a) $\frac{1}{9}(14+7+11) = \frac{32}{9}$

(b) $\frac{1}{5}(7+4+9) = \frac{20}{5} = 4$

(c) Note that this is multiplication and not addition.

$\frac{11}{4}$

137. According to this definition, we can't add rational numbers unless the denominators are the same. To add 3/4 and 5/7, we must first change each of them to equivalent fractions having the same denominator. Then we can add them by the above definition. When we change fractions to equivalent fractions having the same denominator, we generally find the least common denominator or the smallest number which is evenly divisible by each of the original denominators. We could use any number which is evenly divisible by the original denominators as a common denominator but this would give us larger numbers to work with. The least common denominator for

3/4 and 4/7 is ________.

137\. 28

138\. Change the fractions 3/4 and 5/7 to equivalent fractions and add.

$\frac{3}{4} + \frac{5}{7} = \qquad =$

138\. $\frac{21}{28} + \frac{20}{28} = \frac{41}{28}$

139\. Remember to reduce the final answer to its lowest terms. Complete the following.

(a) $\frac{7}{12} + \frac{4}{3} =$

(b) $\frac{7}{18} + \frac{5}{36} =$

(c) $\frac{11}{24} + \frac{7}{18} =$

139\. (a) $\frac{7}{12} + \frac{16}{12} = \frac{23}{12}$

(b) $\frac{14}{36} + \frac{5}{36} = \frac{19}{36}$

(c) The least common denominator here is 72.

$\frac{33}{72} + \frac{28}{72} = \frac{61}{72}$

140\. You may have had some difficulty in finding the least common denominator (LCD) in part c. So far you have obtained the LCD by trial and error. We can use the following definition to find the LCD more easily. However, first we must know what prime factors are.

Prime numbers are numbers whose only integer factors are itself and 1.
For example 6 is not a prime number as the factors of 6 are 3 and 2 or are 6 and 1. The only integer factors of 6 are not itself and 1.

2 is a prime number because the only integer factors of 2 are itself and 1.
List the prime numbers that are found between 2 and 35 inclusive.

140\. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31

141\. What are the prime factors of

(a) 42 (b) 63 (c) 56

141. (a) 2, 3 and 7
(b) 3, 7 and 3
(c) 2, 2, 7 and 2

142. You may have listed the prime factors of 42 as 3, 7 and 2. Even though you have listed them in a different order, they are the same factors. A theorem states that there is only one set of prime factors for any integer. Let us now consider finding the LCD for fractions with denominators of 10 and 15. First determine the prime factors of the given denominators.

142. $10 = 2 \cdot 5$
$15 = 5 \cdot 3$

143. The LCD is composed of every different prime factor in each of the original denominators and each factor appears the maximum number of times it appears in any single denominator.
Thus to find the LCD for fractions with denominators of 10 and 15, we use every different factor as many times as it appears in any single denominator.

What factors would appear in the LCD if 10 and 15 were the original denominators?

143. factors of 2, 5 and 3

144. Do any of these factors appear more than once in any one of the original denominators?

144. no

145. Then the LCD for fractions with denominators of 10 and 15 is $2 \cdot 3 \cdot 5$ or is 30.

Find the LCD for fractions with denominators of 6 and 75.

145. prime factors of 6 are 2 and 3
prime factors of 75 are 5, 3 and 5

146. What is the LCD for fractions with denominators of 45, 50 and 6?

LCD has factors of 2, 3, 5 and 5. It has two factors of 5 as 5 is a factor twice in one of the original denominators.
LCD = $2 \cdot 3 \cdot 5 \cdot 5$

146. prime factors of 6 are 2 and 3
prime factors of 50 are 5, 5 and 2
prime factors of 45 are 3, 5 and 3
LCD = $2 \cdot 3 \cdot 5 \cdot 5 \cdot 3$

147. Find the following sums.

(a) $\frac{1}{9} + \frac{7}{18} + \frac{5}{12}$

(b) $\frac{8}{39} + \frac{4}{65}$

(c) $4 + \frac{5}{12} + \frac{13}{36}$

(d) $\frac{13}{21} + \frac{11}{28}$

(e) $\frac{43}{4} + \frac{5}{6} + \frac{28}{9}$

147. (a) LCD is $3 \cdot 3 \cdot 2 \cdot 2$ or 36
$\frac{4+14+15}{36} = \frac{33}{36} = \frac{11}{12}$

(b) LCD is $3 \cdot 5 \cdot 13$ or 195
$\frac{40}{195} + \frac{12}{195} = \frac{52}{195} = \frac{4}{15}$

(c) LCD is 36.
$\frac{144}{36} + \frac{15}{36} + \frac{13}{36} = \frac{172}{36} = \frac{43}{9}$

(d) LCD is $3 \cdot 4 \cdot 7$ or 84
$\frac{52}{84} + \frac{33}{84} = \frac{85}{84}$

(e) LCD is 36
$\frac{387}{36} + \frac{30}{36} + \frac{112}{36} = \frac{529}{36}$

148. Let us consider part b of the last frame again. In order to get the LCD, we found the prime factors and then used each different one the maximum number of times it appears in any single denominator.
Our LCD this time was $3 \cdot 5 \cdot 13$ or 195.
Suppose we hadn't multiplied our factors together but had used $3 \cdot 5 \cdot 13$ as our LCD]

$\frac{8}{39} \quad \frac{4}{65} \quad \frac{40}{3 \cdot 5 \cdot 13} \quad \frac{12}{3 \cdot 5 \cdot 13} \quad \frac{52}{3 \cdot 5 \cdot 13}$

and now we could reduce this to its lowest terms by finding out which of the factors of the denominator are also factors of the numerator. 13 is a factor of both and dividing both numerator and denominator by this we get $\frac{4}{3 \cdot 5}$ or 4/15.

Find the LCD of the following fractions and add them keeping the denominator in factored form.

$\frac{10}{15} + \frac{1}{6} + \frac{1}{14}$

148. LCD is $3\cdot 5\cdot 2\cdot 7$

$\frac{10\cdot 2\cdot 7}{3\cdot 5\cdot 2\cdot 7} + \frac{7\cdot 5}{3\cdot 5\cdot 2\cdot 7} + \frac{3\cdot 5}{3\cdot 5\cdot 2\cdot 7} =$

$\frac{140\cdot 35\cdot 15}{3\cdot 5\cdot 2\cdot 7} = \frac{190}{3\cdot 5\cdot 2\cdot 7} = \frac{19}{3\cdot 7} =$

$\frac{19}{21}$

149. Subtraction of rational numbers can only be done when the fractions have the same denominators. Again we usually get the LCD when the denominators are different as this gives us smaller numbers to work with.

<u>Subtraction of rational numbers.</u>

If a, b and c are rational numbers and $b \neq 0$, then

$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$

Perform the indicated operations and simplify.

(a) $\frac{5}{6} - \frac{1}{2}$

(b) $\frac{1}{6} - \frac{1}{14}$

(c) $4 - \frac{7}{12}$

149. (a) LCD is 6.

$\frac{5}{6} - \frac{3}{6} = \frac{5-3}{6} = \frac{2}{6} = \frac{1}{3}$

(b) LCD is $2\cdot 3\cdot 7$ or 42

$\frac{7}{42} - \frac{3}{42} = \frac{4}{42} = \frac{2}{21}$

(c) LCD is 12.

$\frac{48}{12} - \frac{7}{12} = \frac{41}{12}$

150. <u>Division of rational numbers.</u>

If a, b, c and d are rational numbers, and b, c and d are not 0, then

$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

$\frac{2/3}{4/5} =$

150. $\frac{2/3}{4/5} = \frac{2}{3} \cdot \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$

151. The denominator of the original fraction was 4/5. We multiplied the numerator by 5/4. These numbers - 4/5 and 5/4 are called the <u>reciprocals</u> of each other.

<u>Reciprocal</u>

If a and b are rational numbers, then a and b are reciprocals if $a \cdot b = 1$.

Give the reciprocals of each of the following.

(a) 2 (b) $\frac{1}{3}$ (c) $\frac{4}{7}$

(d) $\frac{21}{2}$ (e) 0

151. (a) $\frac{1}{2}$
(b) 3
(c) 7/4
(d) 2/21
(e) there is no reciprocal in this case.

A reciprocal is a number which when multiplied by the original number equals 1. What number multiplied by 0 equals 1? There is no such number. (Refer to frames 107 - 109 if necessary.)

152. If we look again at the definition of division, we see that dividing by a number is the same as multiplying by the reciprocal of the number.

$\frac{2/3}{4/5}$ means to multiply 2/3 by the reciprocal of 4/5

$\frac{2/7}{12/63} =$

152. $\frac{2}{7} \cdot \frac{63}{12} = \frac{2 \cdot 63}{7 \cdot 12} = \frac{3}{2}$

153. To multiply two rational numbers, remember we take the product of the numerators of the fractions and divide by the product of the denominators of the fractions. To multiply $\frac{2}{7}$ by $\frac{63}{12}$ using the definition of multiplication, we would have a product of $\frac{126}{84}$.

This fraction is not in its lowest terms. To write an equivalent fraction which is in its lowest terms, we must divide

both numerator and denominator by their common factors. It may take several steps to find all the common factors of 126 and 84. Since we obtain equivalent fractions by dividing both numerator and denominator by a common factor we could do this before we multiply.

For example, $\frac{2}{7} \cdot \frac{63}{12} = \frac{2(63)}{7(12)} = \frac{2 \cdot 7 \cdot 9}{7 \cdot 2 \cdot 6}$

Now it is evident that both the numerator and denominator have common factors of 14 and that the equivalent fraction of 9/6 has a common factor of 3 in both numerator and denominator giving a final fraction of 3/2.

Multiply 27/8 by 28/81.

153. $\frac{27 \cdot 28}{8 \cdot 81} = \frac{3 \cdot 3 \cdot 3 \cdot 7 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{7}{2 \cdot 3} = \frac{7}{6}$

(Note - we usually get the prime factors when we are to find the largest common factor.)

154. Perform the indicated operations. Express all fractions in lowest terms.

(a) $\frac{21}{16} \div \frac{28}{9}$

(b) $\frac{39}{72} \div \frac{52}{24}$

(c) $\frac{3}{5} \cdot \frac{5}{7} \cdot \frac{105}{60}$

(d) $\frac{1}{3} \cdot \frac{21}{16} \div \frac{1}{32}$

(e) $\frac{64}{21} \div \frac{42}{9} \cdot \frac{10}{27}$

(f) $\frac{64}{21} \div (\frac{42}{9} \cdot \frac{10}{27})$

154. (a) $\frac{21(9)}{16(28)} = \frac{27}{64}$
(b) $\frac{39}{72} \cdot \frac{24}{52} = \frac{1}{4}$
(c) 3/4
(d) $\frac{1}{3} \cdot \frac{21}{16} \cdot \frac{32}{1} = 14$
(e) $\frac{64}{21} \cdot \frac{9}{42} \cdot \frac{10}{27} = \frac{320}{1323}$
(f) Be careful here. This is different from part c. This says to find the value of the number in parentheses and then to divide by it.
$\frac{42}{9} \cdot \frac{10}{27} = \frac{14(10)}{3(27)}$
The given problem becomes then
$\frac{64}{21} \div \frac{14(10)}{3(27)} = \frac{64}{21} \cdot \frac{3(27)}{14(10)} = \frac{432}{245}$

155. You often dealt with mixed numbers in arithmetic. Mixed numbers are numbers such as $6\frac{1}{2}$ or numbers

which consist of a whole number and a fraction. We need no further rules of computation when mixed numbers are involved as mixed numbers can be changed to fractions or can be expressed as the quotient of two integers.

For example, $6\frac{1}{2} = 6 + \frac{1}{2}$ and by the addition of rational numbers this equals 13/2.

Express each of the following as fractions.
(a) $4\frac{3}{5}$ (b) $5\frac{1}{13}$

(c) 3 6/11 (This means the same as $3\frac{6}{11}$)

155. (a) $\frac{23}{5}$
(b) $\frac{66}{13}$
(c) $\frac{39}{6}$

156. Since mixed numbers can be expressed as fractions or as the quotient of two integers, mixed numbers are _________ numbers.

156. rational

157. Then to divide $18\frac{1}{3}$ by $4\frac{1}{6}$, we would first express those numbers as fractions and then do the division.

$18\frac{1}{3} \div 4\frac{1}{6} =$

157. $\frac{55}{3} \div \frac{25}{6} = \frac{55}{3} \cdot \frac{6}{25} = \frac{22}{5}$

158. Perform the indicated operations and simplify.
(a) $3\frac{1}{4} \cdot \frac{10}{39}$

(b) $\frac{1}{4} \div 1\frac{1}{8}$

(c) $3\frac{1}{3} \div \frac{10}{3}$

(d) $1\frac{1}{3} \cdot 15 \div 1\frac{5}{7}$

(e) $3\frac{1}{3} + 1\frac{7}{12}$

(f) $\frac{13}{6} + 1\frac{2}{15}$

158. (a) $\frac{13}{4} \cdot \frac{10}{39} = \frac{5}{6}$
(b) $\frac{1}{4} \div \frac{9}{8} = \frac{1 \cdot 8}{4\ 9} = \frac{2}{9}$
(c) $\frac{10}{3} \div \frac{10}{3} = \frac{10 \cdot 3}{3\ 10} = 1$
(d) $\frac{4}{3} \cdot 15 \cdot \frac{7}{12} = \frac{35}{3}$
(e) $\frac{10}{3} + \frac{19}{12} = \frac{40}{12} + \frac{19}{12} = \frac{59}{12}$
Note - this problem involves <u>addition</u> so you need fractions having a common denominator.
(f) $\frac{13}{6} + \frac{17}{15} = \frac{65}{30} + \frac{34}{30} = \frac{99}{30} = \frac{33}{10}$

159. Decimals can also be converted to fractions. For example, .34 is read <u>thirty-four hundredths</u> or can be expressed as a fraction with a numerator of 34 and a denominator of 100.

$.34 = \frac{34}{100} = \frac{17}{50}$

Express .035 as a fraction.

159. this is read thirty-five <u>thousandths</u>.

$.035 = \frac{35}{1000} = \frac{7}{200}$

160. To convert 3.4 to a fraction, remember 3.4 means 3 + .4 and convert the .4 to a fraction and then add 3 to this result.

$3.4 = 3+.4 = 3 + \frac{4}{10} =$

160. $\frac{34}{10} = \frac{17}{5}$

161. Since .34 and 3.4 can be expressed as the quotient as two integers, .34 and 3.4 are examples of

________________ numbers.

161. rational

162. The symbol % is used to represent $\frac{1}{100}$. Thus we mean 13 $(\frac{1}{100})$ when we write 13%.

What fraction does 42% equal?

162. $\frac{42}{100}$ or $\frac{21}{50}$

163. Convert the following fractions.

(a) 2.6

(b) .23

(c) 2%

(d) 2.3%

163. (a) $2.6 = \frac{26}{10} = \frac{13}{5}$

(b) $\frac{23}{100}$

(c) $2(\frac{1}{100}) = \frac{2}{100} = \frac{1}{50}$

(d) $2.3\ (\frac{1}{100}) = \frac{23}{10} \cdot \frac{1}{100} = \frac{23}{1000}$

164. In part d of the last frame you were to convert 2.3% to a fraction.

Since % represents $\frac{1}{100}$, 2.3% = $2.3\ (\frac{1}{100})$.

In the answer to part d, 2.3 was converted to a fraction and then the two fractions were multiplied. Instead of following this procedure, we could have multiplied 2.3 by

$\frac{1}{100} \cdot 2.3\ (\frac{1}{100}) = \frac{2.3}{100}$.

It is often not convenient to leave a fraction with a decimal in the numerator or denominator and so in this case, we will change to an equivalent fraction where the numerator is an integer. Multiplying both numerator and denominator by 10, $\frac{2.3}{100} = \frac{23}{1000}$.

Write equivalent fractions for each of the following.

(a) $\frac{.33}{10}$ (b) $\frac{.03}{100}$ (c) $\frac{23.01}{10}$

164. (a) multiply both numerator and denominator by 100 getting $\frac{33}{1000}$

(b) $\frac{.03}{100} = \frac{3}{10000}$

(c) $\frac{23.01}{10} = \frac{2301}{1000}$

165. Decimals and percents (%) can be written in fractional form. Thus decimals and percents are kinds of

_______________ numbers.

165. rational

166. Rational numbers are numbers

___________________________.

166. of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$
or
which can be expressed as the quotient of two integers where the denominator is not 0.

167. To find the value of 12%(20) we must first express 12% as a fraction and then do the multiplication.

$12\%(20) = 12(\frac{1}{100})(20) =$

$\frac{12}{100}(20) = \frac{12}{5}$

We usually express results as fractions in their lowest terms.

To divide 20 by 12%, we also must first express 12% as a fraction before doing the division.

$20 \div 12\% = 20 \div (\quad ? \quad) = ?$

167. $20 \div \frac{12}{100} = 20(\frac{100}{12}) = \frac{500}{3}$

168. Find the value: 4% of 42.
[4% of 42 means the same as 4%(42).]

168. $\frac{4}{100}(42) = \frac{4(42)}{100} = \frac{42}{25}$

169. Evaluate:

(a) 50% of 132

(b) 5% of 132

(c) .5% of 132

(d) $125 \div 15\%$

(e) $125 \div 1.5\%$

(f) $125 \div .15\%$

169. (a) $\frac{50}{100}(132) = 66$

(b) $\frac{5}{100}(132) = \frac{33}{5}$

(c) $\frac{.5}{100}(132) = \frac{5}{1000}(132) = \frac{33}{50}$

(d) $125 \div \frac{15}{100} = 125(\frac{100}{15}) = \frac{2500}{3}$

(e) $125 \div \frac{1.5}{100} = 125 \div \frac{15}{1000} = \frac{25000}{3}$

(f) $125 \div \frac{.15}{100} = 125 \div \frac{15}{10000} = \frac{250000}{3}$

170. When we were asked to find 50% of 132, we wanted a number N which equaled the product of 50% and 132, or $N = \frac{50}{100}(132)$.

r% of a number c is a number N such that $\frac{r}{100}(c) = N$.

In finding 30% of 15, we know the values of two of the three quantities in the statement

$\frac{r}{100}(c) = N$.

We know the values of _____ and __________.

170. r and c

171. r = 30 and c = 15, when finding 30% of 15.

Since we know the values of two of the quantities, we can find the value of the third quantity by performing the indicated operation.

What would you do to find the value of N in the above problem?

Find N.

171. Multiply $\frac{50}{100}$ by 132

N = 66

172. In the formula $\frac{r}{100}(c) = N$, we have a case where the product of two numbers equals a third number. $\frac{r}{100}$ and c are called factors.

(see frame 17)

Suppose we knew the value of only one of the factors and the value of the product, how could we find out the value of the other factor? Be specific.

172. Divide the product by the one factor.

173. If we know that 12% of some number is 24, what quantity in the formula $\frac{r}{100}(c) = N$ do we need to find and how would we find it?

173. we need to find the value of c
we can divide the value of N by the value of $\frac{r}{100}$.

174. Problem: 12% of some number is 24.

Set up the division problem to solve for the number.

174. Since $\frac{12}{100}(c) = 24$ then

$c = 24 \div \frac{12}{100}$

or

$c = \frac{24}{\frac{12}{100}}$

175. We made use of the principle that if the product of two numbers is a third number, then the third number divided by one of the factors must equal the other factor.

Find the value of a number such that 12% of it is 24.

175. $c = 24 \div \frac{12}{100}$

$c = 24 \cdot \frac{100}{12}$

$c = 200$

176. (a) Find a number such that 25% of it is 18.

(b) Find a number such that 120% of it is 72.

(c) Find a number such that it is 22% of 50.

(d) 39 is what % of 52?

176. (a) In $\frac{r}{100}(c) = N$, c is what we wish to find.

$\frac{25}{100}(c) = 18$, so $c =$
$18 \div \frac{25}{100}$ or $c = 72$

177. How did we decide in part d of the last frame that r = 75%?
We had the statement $\frac{r}{100} = \frac{3}{4}$.
We know that two fractions are equal if the numerators and denominators are equal - in other words, if a = c and b = d and b and d aren't 0 then $\frac{a}{b} = \frac{c}{d}$.

(b) in $\frac{r}{100}(c) = N$, we wish to find c. $\frac{120}{100}(c) = 72$ so $c = 72 \div \frac{120}{100}$ and $c=60$.

(c) in $\frac{r}{100}(c) = N$, we wish to find N. $\frac{22}{100}(50) = N$ or $N = 11$.

(d) in $\frac{r}{100}(c) = N$, we wish to find r. $\frac{r}{100}(52) = 39$.

Thus $\frac{r}{100} = 39 \div 52$

or $\frac{r}{100} = \frac{3}{4}$ and $r = 75\%$

So if in the statement $\frac{r}{100} = \frac{3}{4}$, we find a fraction equivalent to 3/4 with a denominator of 100, then the two fractions will have the same denominator and so the numerators must be equal.

$\frac{3}{4} = \frac{?}{100}$ so $\frac{r}{100} = \frac{?}{100}$ and r = ______

177. $\frac{3}{4} = \frac{75}{100}$

$\frac{r}{100} = \frac{75}{100}$ and $r = 75$.

178. We also know that two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal if b and d aren't 0 and if $ad = bc$.

Applying this statement to

$\frac{r}{100} = \frac{3}{4}$, we get $4(r) = 3(100)$.

This states that 4 multiplied by some number r is equal to 300. How would we find the value of r?

178. r must equal 300 divided by 4.

179. What percent of 180 is 36?

179. In $\frac{r}{100}(c) = N$, we wish to find r. $\frac{r}{100}(180) = 36$ or

$\frac{r}{100} = \frac{36}{180}$

$r = \frac{36(100)}{180}$ or $r = 20$

So, 20% of 180 equals 36.

180. In frames 167 through 179 we have used the same relationship to solve these problems regardless of which quantities were given.

We used the relationship $\frac{r}{100}(c)=N$

to find a number N such that it is r% of a number c. If values of N and r or if values of N and c were given, we also used that if the product of two numbers equals a

number N, then that number divided by one of the factors must equal the other factor.

Perform the indicated operations and simplify.

(a) $\frac{27}{10} + \frac{13}{15} =$

(b) $3\,\frac{3}{8} + 1\,\frac{7}{16}$

(c) $\frac{24}{9} \cdot \frac{65}{6} \div \frac{39}{54}$

(d) $\frac{36}{63} \div \frac{28}{49}$

(e) 6% of 150

180. (a) LCD is 30.

$\frac{81}{30} + \frac{26}{30} = \frac{107}{30}$

(b) $\frac{27}{8} + \frac{23}{16} = \frac{54}{16} + \frac{23}{16} = \frac{77}{16}$

(c) $\frac{24}{9} \cdot \frac{65}{6} \cdot \frac{54}{39} = 40$

(d) $\frac{36}{63} \cdot \frac{49}{28} = 1$

(e) $\frac{6}{100}(150) = 9$

## Chapter 2 - Signed Numbers

We stated in frame 86 of the last chapter that the difference of 2 and 3 was written 2 - 3. We also stated that this doesn't exist as long as we have only the positive rational numbers and 0. In this chapter we shall introduce the negative rational numbers and consider the operations of addition, subtraction, multiplication and division with both positive and negative numbers.

1. A number preceded by a "-" (negative) sign is called a negative number.

   $-\frac{1}{2}$, $-3$, and $-\frac{7}{4}$ are examples of ____________ numbers.

1. negative

2. A number with a "+" (positive) sign before it is called a positive number.

   $+3$, $+\frac{7}{8}$ and $+\frac{3}{4}$ are examples of ____________ numbers.

2. positive

3. When we talk about negative numbers we must always use the "-" or negative sign. To denote positive numbers, we don't have to use the "+" or positive sign. Thus "2" and "+2" represent the same number.

   Do "4" and "-4" represent the same number?

3. no

4. Consider "4" and "-4". Which represents a positive number?

4. 4

5. In Chapter 1 we talked about the nonnegative rational numbers.

   A rational number is ____________ ______________________________.

5. a number which can be expressed as the quotient of two integers where the denominator isn't 0
or

6. In your definition, be sure you stated that the denominator couldn't equal 0. Why can't the denominator equal 0?

a number of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

6. because division by 0 is undefined
or
because you can't divide by 0
or
because something divided by 0 doesn't represent a number.

7. Some rational numbers can be expressed in the form $\frac{a}{1}$.

Rational numbers of this form are called ____________.

7. integers

8. The positive integers are 1, 2, 3, ... (the three dots mean and so forth)

-1, -2, -3, ... compose the set of __________ __________ .
(two words)

8. negative integers

9. Do the sets of positive integers and negative integers include <u>all</u> the integers?

9. no

10. What one(s) isn't included in either of these two sets?

10. 0
Note: 0 is sometimes referred to as an unsigned number as 0 is neither positive nor negative.

11. The <u>set of integers</u> is composed of all integers, i.e., the positive integers, zero and the negative integers make up the set make up the set of integers. Which of the following are sets of integers?

(a) $2, -4, \frac{25}{3}$

(b) $0, -3, 5^2$

(c) $1\frac{1}{3}, .34, 60\%$

(d) $-2, \frac{5}{0}, 1$

(e) $\frac{0}{5}, 15, -\frac{24}{4}$

11. parts b and e.
In part e, $\frac{0}{5} = 0$
and $-\frac{24}{4} = -6$, so $\frac{0}{5}$, 15 and $-\frac{24}{4}$ represent integers.

12. Did you say that part d of the last frame represented a set of integers?
$\frac{5}{0}$ isn't an integer.
In fact, we can make an even stronger statement about $\frac{5}{0}$.

What can we say about $\frac{5}{0}$ ?

12. $\frac{5}{0}$ isn't any number. (See frames 107 - 108 in Chapter 1 if necessary)

13. The set of integers are part of the set of ________ numbers.

13. rational

14. $-6$, $-\frac{3}{4}$, $\frac{4}{3}$, 11, $-6\frac{1}{2}$, $-15\%$, 4.3, and 0 are all examples of

________ numbers. Why?

14. rational numbers because they can be expressed as the quotient of two integers where the denominator isn't 0. They are rational numbers and not integers because of 4/3, -15%, and others.

15. In Chapter 1 - frame 84, we called 0 the identity element of addition because when 0 is added to any number the sum is that number. We could write $a + 0 = a$. We only stated that this was true for the nonnegative integers. It can be shown that 0 is the identity element of addition for all rational numbers. We shall accept this without proof.

Thus, we can say that $(-6) + 0 =$

________

15. -6

16. In Chapter 1, we also stated the commutative, associative and distributive laws for nonnegative integers. We could prove that these laws hold for negative integers and for rational numbers which aren't integers and therefore for all rational numbers. We will accept these laws for all rational numbers without proof. By the commutative law,

$(-6) + 0 =$ ________

16. $0 + (-6)$ The commutative law states that the order in which two numbers are added or multiplied is immaterial. (frame 27 Chapter 1)

17. In frame 15, we have $(-6) + 0 = -6$ because 0 is the identity element of addition.

In frame 16, we have $(-6) + 0 = 0 + (-6)$ by the commutative law of addition.

Thus we may say that $0 + (-6) = -6$.

Find the following sums:
(a) (-17) + 0 = ____________

(b) 0 + 24 = ___________

(c) 0 + 0 = ___________

(d) 0 + (-21) = ___________

17. (a) -17
(b) 24
(c) 0
(d) -21

18. Sometimes we shall be interested in considering the value of a signed number without considering the sign of the number. This is called the <u>absolute value</u> of a number.
The absolute value of -2 is 2.
The absolute value of +2 is 2.

What is the absolute value of -6?

18. 6

19. Find the absolute value of:

(a) $-\frac{3}{4}$

(b) $-.03$

(c) $+\frac{7}{3}$

(d) $-425$

(e) 42

19. (a) $\frac{3}{4}$

(b) .03

(c) $\frac{7}{3}$

(d) 425

(e) 42

20. What is the absolute value of $+\frac{1}{2}$ ?

What other number has the <u>same</u> absolute value as $+\frac{1}{2}$ ?

20. absolute value of $+\frac{1}{2}$ is $\frac{1}{2}$ .
$-\frac{1}{2}$ has the same <u>absolute value</u> as $+\frac{1}{2}$ .

21. What number has the same absolute value as -7?

21. +7

22. The absolute value of a number b is written $|b|$.
$|-4|$ is read ______________.

22. the absolute value of -4.

23. $|3 - 2|$ is read ______________.

23. the absolute value of <u>the</u> number (3 - 2).

24. To find the value of $|3 - 2|$, we must first find the value of 3 - 2 and then find the absolute value of this.

$|3 - 2| = |\_\_\_\_\_\_| = \_\_\_\_\_\_$

24. $|3 - 2| = |1| = 1$

25. Does $|3| - |2|$ tell us to follow the same procedure as $|3 - 2|$ ?

25. no

26. What procedure are we asked to do in $|3| - |2|$ ?

26. find the absolute value of 3, then find the absolute value of 2 and then find the difference of the results.

27. Express the following without absolute value signs and simplify.

(a) $|+4| + |-4|$

(b) $|+4| - |-4|$

(c) $|-\frac{1}{2}| + |-\frac{1}{4}|$

(d) $|\frac{1}{2} - \frac{1}{4}|$

(e) $|2 + 3(4)|$

27. (a) $4 + 4 = 8$
(b) $4 - 4 = 0$
(c) $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
(d) $\left|\frac{1}{4}\right| = \frac{1}{4}$
(e) $|2 + 12| = |14| = 14$

28. Two numbers have a sum of 0 when one number is a positive number and the other number is a negative number and both have the same absolute value.

For example, the sum of +4 and -4 is 0 because +4 is a positive number and -4 is a negative number and both numbers have the same absolute value.

The sum of -31 and 31 is ______

28. 0

29. The sum of -31 and 31 can be written as -31 + 31 where the + is used to mean <u>add</u> and is not the sign of the number.

What number added to 16 gives a sum of 0?

In other words, 16 + ______ = 0.

29. -16

30. Complete:
(a) $(-\frac{1}{2})$ + ________ = 0

(b) ______ + (-14) = 0

(c) ______ + 35 = 0

(d) -10 + 10 = ______

(e) 16 + (-16) = ______

30. (a) $\frac{1}{2}$ or $+\frac{1}{2}$
(b) 14 or +14
(c) -35
(d) 0
(e) 0

31. The numbers 16 and -16 are called the <u>additive inverses of each other</u>.

Definition: One number is called the additive inverse of another if their sum is 0.

What is the additive inverse of $-\frac{1}{2}$ ? Why?

31. $\frac{1}{2}$ or $+\frac{1}{2}$ because the sum of $-\frac{1}{2}$ and $+\frac{1}{2}$ is 0

32. What is the additive inverse of 29? Why?

32. -29 because the sum of 29 and -29 is 0

33. Numbers which are additive inverses of each other are also called the <u>negatives</u> of each other.

The additive inverse of 3 is ________ and so -3 is the

________ of 3.

33. -3
negative

34. What must be true of a number and its negative?

34. their sum must be 0

35. The negative of -14 is ______

because ____________________.

35. 14 or +14 because the sum of -14 and 14 is 0

36. What is the negative (the additive inverse) of each of the following?

(a) $\frac{1}{2}$

(b) 34

(c) -23

(d) $-\frac{1}{4}$

(e) $-\frac{4}{3}$

(f) 0

(g) (5+2)

(h) - (4-3)

36. (a) $-\frac{1}{2}$

(b) -34

(c) 23

(d) $\frac{1}{4}$

(e) $\frac{4}{3}$

(f) 0

(g) - (5+2)

(h) + (4-3)

37. In part f of the preceding frame 0 is the correct answer. We do not use a "-" sign as 0 is neither positive nor negative.

In parts g and h we have only one number given. This is shown by the use of parentheses. Parentheses are used to show that several terms are to be considered as one number. (5+2) is the same as + (5+2) or is a positive number and the negative of this number would be - (5+2).

What is the negative of -(-2+4)?

37. +(-2+4) or (-2+4) because -(-2+4) + (-2+4) = 0

38. Note that we have used the word negative in two different ways. First, we talked of a negative number. In frame 36 which of the given numbers were negative numbers?

38. -23, $-\frac{1}{4}$, $-\frac{4}{3}$, -(4-3)

39. Negative numbers are numbers preceded by a "-" (negative) sign.
Secondly, we talked of the negative of a number. Another name for the negative of a number is the

__________ ______________.

39. additive inverse (see frame 31)

40. In frame 36 part a, we found the negative of $\frac{1}{2}$ was $-\frac{1}{2}$. Here the given number is a positive number and its negative is a negative number.

Will this be true for the negatives of all positive numbers?

40. yes

41. In frame 36, we found that the negative of -23 was 23. In this case we were given a negative number and the negative of this number was a positive number.

What can you say about the negative of a negative number?

41. it will always be a positive number

42. In symbols, the statement "23 is the negative of -23" would be written: $23 = - -23$ or $23 = -(-23)$.

Write in symbols: 35 is the negative of -35.

42. $35 = - -35$ or $35 = -(-35)$

43. If b represents some number, then -b represents the negative of this number.

Does -b represent a negative number? Explain.

43. not necessarily

-b is a negative number if b is positive and -b is a positive number if b is a negative number

44. Now we can give the definition of absolute value.
The <u>absolute value</u> of a number b (written $|b|$) is:

$|b| = b$ if b is a positive number or is 0

$|b| = -b$ if b is a negative number

Thus, $|-3| = - -3$ as the given number -3 is a negative number.

$|-5| =$ ________ $=$ ________

44. $- -5 = 5$

45. In other words, the absolute value of a positive number or zero is the number itself and the absolute value of a negative number is the negative of the number.
Complete:
(a) $|-\frac{4}{3}| =$ ________ $=$ ________

(b) $|13| =$ ________

(c) $|0| =$ ________

45. (a) $- -\frac{4}{3} = \frac{4}{3}$
(b) 13
(c) 0

46. (a) If a represents a positive number, then $|a| =$ ______ and this is a (positive, negative) number
choose one

(b) If c represents a negative number, then $|c| =$ ______ and this is a (positive, negative) number
choose one

46. (a) a
positive

(b) -c
positive

47. We will accept without proof that the commutative, associative and distributive laws hold for all rational numbers. Therefore, we may make use of these laws in doing computations involving both positive and negative rational numbers.

Addition of Signed Numbers

(a) To add two numbers having the same sign, add the absolute values and use the common sign.

(b) To add two numbers having different signs, take the difference of the absolute values and use the sign of the larger number.

For example, add -2
-3

We will use part a above since these numbers have the same sign.

$|-2|$ means ______________.

$|-2|$ = ______. $|-3|$ = ______.

47. absolute value of -2
2
3

48. ∴ (∴ means therefore) adding the absolute values and using the common sign, the sum of -2 and -3 is __________.

48. -5

49. (a) Find the sum of $-\frac{1}{2}$ and $-\frac{1}{4}$.

(b) Add: 2
$6\frac{3}{5}$

49. (a) $-\frac{3}{4}$

(b) $8\frac{3}{5}$

50. To find the sum of -3 and +2, we need to use part b of frame 47 since the numbers have different signs.
We first have to ______________.

50. find the absolute values of the numbers and then take the difference of the absolute values.

51. The sum of -3 and 2 is ________.
We used the "-" sign because

______________________________.

DON'T say subtract them as you will later confuse the operations of addition and subtraction.

51. -1
because we use the sign of the number with the larger absolute value

52. Add: 

| |
|---|
| -16 |
| 20 |

Explain how you found the sum.

52. sum is 4 or +4

found the absolute values, then found the <u>difference</u> of these, and used the sign of the 20 as it has the larger absolute value

53. Add:

| | | | |
|---|---|---|---|
| 16 | 0 | -14 | -21 |
| -43 | -3 | 7 | -13 |

53. -27
- 3
- 7
-34

54. The sum of $-6\frac{1}{2}$ and 15 is

__________.

54. $8\frac{1}{2}$

55. Complete the following.
(a) -13 ______ 9 = -4

(b) 2 ______ -11 = -9

(c) 5 + _____ = -6

(d) ______ + -9 = -15

(e) ______ + -23 = -11

(f) 4 + ________ = 0

(g) ______ + $\frac{3}{5}$ = 0

55. (a) +
(b) +
(c) -11
(d) -6
(e) 12 or +12
(f) -4
(g) $-\frac{3}{5}$

56. In 4 + -4 = 0, 4 and -4 are known as the _________ of each other. (See frame 33 if necessary)

56. negatives
or
additive inverses

57. $\frac{1}{2}$ is the negative of ______.

57. $-\frac{1}{2}$

58. What must always be true of a number and its negative?

58. The sum of a number and its negative is always 0.

59. If we have more than two numbers to add, we can make use of the associative law.

For example, Add:

```
 -12
   7
 -10
_____
```

We can add -10 and 7, getting a sum of ________, and then add this to -12, getting ________.

59. -3
-15

60. The example, Add:

```
 -12
   7
 -10
_____
```

can also be written as:
-12 + 7 + -10.

Do the + signs here tell us to add or do they tell us that we have positive numbers?

60. Both of them tell us to add.

61. Note that if there is only one sign between two numbers, it tells the operation to perform. If there are two signs between two numbers, the first tells the operation and the second is the sign of the number.
In 8 + -12, what operation is to be performed?

61. addition

62. In 8 + -12, ________ is a negative number and _______ is a positive number.

62. -12
8 (Remember positive numbers can be written without the + sign)

63. Add:

```
 -12
   7
 -10
_____
```

We said you could have added the -10 and 7 and then added the result to -12.
This makes use of the ___________ law.

63. associative

64. We also could have added -12 and 7 and then added the result to -10.

The associative law states that (-12 + 7) + -10 = -12 + (7 + -10), or in other words that the grouping of terms in addition is immaterial.

Use the associative law to complete: -22 + 50 + -28 =

__________ = __________

64. (-22 +50) + -28 = 28 + -28 = 0
    or
    -22 + (50 + -28) = -22 + 22 = 0

65. Complete the following.

(a) - 3/4 + -9/4 + 5/4 =

(b) 14 + -3 + -2 + 12 + -8 + 16 + -7 + 15 =

(c) 12 + -15 + -13 + 20 + -55 + -15 + 4 + 6 =

65. (a) - 7/4
    (b) 37
    (c) -56

66. |-5| means _______________.

66. the absolute value of -5 (see frame 44 if necessary)

67. |-5| + |-9| + 2 = __________ = __________

67. 5 + 9 + 2 = 16

68. |-3 +1| + -7 = _______ = _______

68. |-2| + -7 = 2 + -7 = -5

69. Example: find the sum of the absolute values of -2, 13, -21, -5.

In this problem, you are asked to do two things. One is to find the sum or to add and the other is to find the absolute values. In a case such as this, you have to decide which you are told to do first.

Which do you do first in this case?

69. find the absolute values of each number

70. Find the sum of the absolute values of -2, 13, -21 and -5.

70. |-2| + |13| + |-21| + |-5| =

    2 + 13 +21 +5 = 41

71. If you are asked to find the absolute value of the sum of -2, 13, -21 and -5, are you asked to do the same process as in the above problem?

71. no

72. Explain the procedure you are asked to use.

72. Add the given numbers and then find the absolute value of the result.

73. Find the absolute value of the sum of -2, 13, -21 and -5.

73. $|-2 + 13 + -21 + -5| =$

$|-15| =$ 74.

15

74. You must always read a problem carefully to see exactly what you are supposed to do. Sometimes it will make a difference in the answer you get.

(a) Find the absolute value of the sum of -32 and 21.

(b) Find the sum of the absolute values of -32 and 21.

74. (a) $|-32 + 21| = |-11| = 11$

(b) $|-32| + |21| = 32 + 21 = 53$

75. In frame 86 of Chapter 1, we defined subtraction for nonnegative integers. We can define subtraction of <u>all</u> rational numbers in the same way.

<u>Subtraction</u>

If a and b are two rational numbers, then a-b represents their difference and a-b = c only if c+b = a.

By this definition, 3 - 4 = -1 because -1 + 4 = 3.

-6 - ______ = 2 because

2 + ______ = -6

75. -8 belongs in both places

76. Complete the following.

(a) -6 - _____ = -10 because

________________________.

(b) 0 - -3 = _____ because

________________________.

(c) -3 - _____ = -3 because

________________________.

76. (a) 4 or +4 because -10 + 4 = -6
(b) 3 or +3 because 3 + -3 = 0
(c) 0 because -3 + 0 = -3

77. Doing subtraction by using the definition is not a very rapid process. It can be proven that a - b = a + (-b) or that subtracting two numbers equals the sum of the first number and the negative of the second.

Then -5 - -11 = -5 + ________

77. +11 or 11

78. We have two signs preceding the 11. The first one tells us to

______________.

78. subtract

79. What does the second sign mean?

79. It tells us that we have a negative number

80. -5 - -11 tells us to subtract -11 from -5.

According to frame 77, this can be done by adding the negative of -11 to -5.
or -5 - -11 = -5 + +11

Note that we found the negative of -11 and also changed to the operation of addition.

-5 - +7 = _____________

80. -5 + -7

81. To complete the problem -5 - +7 = -5 + -7, we must now

_____________________________
(what operation on what numbers?)
getting ______________.

81. <u>add</u> -5 and -7
we get -12

82. 13 - -4 = __________ = __________

82. 13 + +4 = 17

83. In -3 - 4, there is only one sign between the numbers.
This tells us to ______________.

83. subtract

84. In -3 - 4, we are to subtract

________ from _________.

84. 4 or +4 from -3

85. We are <u>not</u> to subtract a -4. If you said -4, you have considered -3 - -4 <u>Not</u> -3 - 4.

There is only one "-" sign between the two numbers and if you say that it means to subtract, you have used it and there is no other sign written between those numbers.

Making use of the fact that the difference of two numbers equals the sum of the first number and the negative of the second number or that a-b = a + (-b), we can write -3 - 4 = 3 + __________ = __________

85. -3 - 4 = -3 + -4 = -7

86. Change to an equivalent addition problem and simplify.

-16 - -5 = __________ = __________

86. -16 + +5 = -11

Remember you are adding two numbers which have different signs so you find the difference of their absolute values and use the sign of the number with the largest absolute value. (See frame 47 if you need to review the addition of signed numbers.)

87. In -14 + -6, what operation is to be performed and on what numbers are you to operate?

87. addition
add -14 and -6

88. In -8 - 11, what operation is to be performed and on what numbers are you to operate?

88. subtract

subtract 11 from -8
or
subtract +11 from -8

89. Evaluate the following.

(a) -7 - -4 =

(b) -8 - 3 =

(c) 0 - -2 =

(d)-4 - -9 =

(e) 13 - -21 =

(f) -18 + -12 =

(g) -26 + 17 =

(h) 35 - -6 =

(i) 42 - 59 =

(j) 34 + -16 =

89. (a) -7 + +4 = -3
(b) -8 + -3 = -11
(c) 0 + +2 = 2
(d) -4 + +9 = 5
(e) 13 + +21 = 34
(f) -30 Careful here. Did you notice that this was addition?
(g) -9
(h) 35 + +6 = 41
(i) 42 + -59 = -17
(j) 18

90. If you are to combine -7 + -5 - -4, you can use the associative law if the <u>only operation</u> to be used is <u>addition</u>.
In -7 + -5 - -4 you are to add the first two numbers but then you are to subtract -4.

Write -7 + -5 - -4 so that it involves only the operation of addition.

-7 + -5 - -4 = -7 + -5 + ______

90. -7 + -5 - -4 = -7 + -5 + <u>+4</u>

91. Now use the associative law to find a single number which expresses the value of -7 + -5 - -4.

91. -7 + -5 - -4 = -7 + -5 + +4 = (-7 + -5) + 4 = -12 + 4 = -8
or
-7 + -5 + +4 = -7 +(-5 + +4) = -7 + -1 = -8

92. Complete the following.
(a) -19 - -6 =
(b) -11 - 15 =
(c) -16 - 5 - -2 =
(d) 43 + -32 - -57 =
(e) (-17 + 8) - (5 - 14) =

92. (a) -19 + +6 = -13
(b) -11 + -15 = -26
(c) -16 + -5 + +2 = (-16 + -5)+2 = -21 + 2 = -19
(d) (43 + -32) + +57 = 11 + 57 = 68
(e) Remember parentheses are used to indicate that a group of terms are to be used as a single number, so first we must evaluate (-17 + 8). Then we must evaluate (5 - 15) and then subtract these numbers.
(-17 + 8) - (5 - 14) =
-9 - -9 = -9 + +9 = 0

93. Find the following values.
(a) - (-9 - 6) + (7 - -8) =
(b) - (12 - 25) + (4 - 11) - (-21 + 32) =
(c) -41 - (11 - 32) + (-14 - -20) =

93. (a) -(-9 - 6) + (7 - -8) = - -15 + 15 = 30
(b) - (12 - 25) + (4 - 11) - (-21 + 32) = - -13 + -7 - 11 = + +13 + -7 + -11 = -5

94. <u>Multiplication of signed numbers</u>
To multiply two signed numbers, multiply the absolute values. The product is positive if the numbers have the same sign. The

(c) $-41 - (11 - 32) + (-14 - -20) =$
$-41 - -21 + 6 = 26$

product is negative if the numbers have different signs.

If we are to multiply -6 and -3, we would first find the absolute values of these numbers. The absolute values are 6 and 3 respectively.
The product of the absolute values is 18. Since the two numbers have the same sign, the product is a positive number.
Hence, the product of -6 and -3 is 18.

What is the product of 2 and -3?

94. -6
This time the product is a negative number since the two numbers have different signs.

95. Multiply:

| | | | | | | |
|---|---|---|---|---|---|---|
| -5 | -11 | $\frac{1}{2}$ | 9 | -7 | 0 | -4 |
| -6 | 6 | -8 | -7 | -6 | -3 | 0 |

95. 30, -66, -4, -63, 42, 0, 0

96. Why were the products in the last two problems equal to 0?

96. because any number multiplied by 0 equals 0

97. If the product of two numbers is zero, what can you say about the numbers?

97. one of the numbers or else both of them must equal zero

Don't forget the case where both numbers equal 0.

98. We can state: if a and b are rational numbers such that $ab = 0$, then at least one of the factors is 0.

What does it mean to say "at least one of the factors is 0"?

98. either a or b or both must equal 0

99. Remember we agreed not to use x to mean multiplication. Instead we will use either a dot or parentheses. It will be much less confusing if we use parentheses when we are dealing with negative numbers and mean multiplication.

(a) $-5(-7) =$ ________

(b) $9\ (\_\_\_\_) = -63$

(c) $-2(\_\_\_\_) = -2$

(d) $(-\frac{1}{2})(-14) =$ ________

99. (a) 35
(b) -7
(c) 1
(d) 7 or +7

100. In (-5)(-7) the -5 is called a

____________.

100. factor

101. Since (-5)(-7) = 35, we could say that the factors of 35 are -5 and -7.
List three other different sets of factors of 35.

____________

____________

____________

101. 5 and 7
-1 and -35
1 and 35
These can be listed in any order.

102. What are 4 different sets of factors of -6?

102. -3 and 2
3 and -2
-1 and 6
1 and -6

103. $2^3$ is a shortcut way of writing

__________________.

103. 2(2)(2)
or
three factors of 2

104. $(-2)^3$ is a shortcut way of writing ________________.

104. (-2)(-2)(-2)

105. $-2^3$ does not mean the same as $(-2)^3$.
$-2^3$ is the same as $(-1)(2^3)$. You will note that only the two is raised to the third power.

Written in terms of factors, $-2^3 = (-1)(2)(2)(2)$.

To evaluate this, we make use of the associative law and find that

$-2^3 = (-1)(2)(2)(2) =$ __________

105. -8

106. We said that $(-2)^3 = (-2)(-2)(-2)$ and this also can be evaluated using the associative law.

$(-2)^3 = (-2)(-2)(-2) =$ __________

106. -8

107. Some of you will wonder why the emphasis on the meaning of the two quantities, after all the answer came out the same.
It did in this case, but this doesn't always happen. Even if it does, the important thing is to do exactly what the symbols tell you to do.

$(-3)^2$ means ________ and this

equals __________.

107. $(-3)(-3) = 9$

108. $-3^2$ means ____________ and this

equals ____________.

108. $(-1)(3)(3) = -9$
Note that the results of $(-3)^2$ and $-3^2$ are not equal.

109. Look carefully at $(-3)^2$ and $-3^2$.

$(-3)^2$ tells us to square the number -3.

$-3^2$ tells us to square the number 3 and then to multiply the result by -1.

$-2^4 =$ _____________ = _______

109. $(-1)(2)(2)(2)(2) = -16$

110. Evaluate.

(a) $(-5)^3$

(b) $-3^4$

(c) $(-3)^4$

(d) $-4^2$

(e) $(-4)^2$

110. (a) $(-5)(-5)(-5) = -125$
(b) $(-1)(3)(3)(3)(3) = -81$
(c) $(-3)(-3)(-3)(-3) = 81$
(d) $(-1)(4)(4) = -16$
(e) $(-4)(-4) = 16$

111. $1^5$ means ____________ and this

equals ________.

111. $1(1)(1)(1)(1) = 1$
or
five factors of 1 which equals 1

112. $(-1)^5$ means ___________ and this

equals _________.

112. $(-1)(-1)(-1)(-1)(-1) = 1$

113. Simplify the following:

(a) $3(-4)(\frac{1}{2})(9)(-1) =$

(b) $(-2)^3(3)^3 =$

(c) $(-9+7)^5 =$

(d) $(-3)^3(2)^3 =$

(e) $-5^2(-2)^4 =$

(f) $(-3 + -6)^2 =$

113. (a) 54
(b) $(-8)(9) = -72$
(c) $(-2)^5 = -32$
(d) $(-27)(8) = -216$
(e) $(-1)(25)(16) = -400$
(f) $(-9)^2 = 81$

114. Do -2-3 and -2(-3) mean the same? Explain.

114. no
-2-3 means to subtract +3 from -2.
-2(-3) means to multiply -2 and -3.

115. To simplify 2 -3(-4), what procedure would you follow?

115. multiply 3 by -4 and then subtract the result from 2.

116. 2 -3(-4)

= 2 - -12 (Multiplying 3 by -4)

= 2 + +12 = 14 (Performing the subtraction)

Could 2 - 3(-4) be considered as 2 + (-3)(-4)? Explain.

116. yes because subtracting two numbers is equal to the sum of the first and the negative of the second. In 2 - 3(-4), the second number is 3(-4) and its negative is (-3)(-4). Hence, 2 -3(-4) = 2 + (-3)(-4).

117. Then in evaluating 2 - 3(-4), we could have multiplied -3 and -4 and then added the results.

Simplify: $4^2 + 9(-2) +1$

117. 16 + -18 + 1 = -1

118. Simplify: $(-3 + -6)(2^3 + -3)$

118. (-9)(5) = -45

119. $(|3 + -6|)^3 =$ ____________ =

____________

119. $(|-3|)^3 = (3)^3 = 27$

120. $[(-8)(7) + 62][19 - (-11)(-3)] =$

120. $[-56 + 62][19 - +33] =$
$(6)(-14) = -84$

121. $-\frac{1}{2}(-2 - 12)(-8[15 - 17] + -6) =$

121. $-\frac{1}{2}(-14)(-8[-2] + -6) =$
$-\frac{1}{2}(-14)(16 + -6) =$
$7(10) = 70$

122. In Chapter 1 frame 96, we defined division of nonnegative integers in terms of multiplication. We can use the same definition for all rational numbers.

Division

If a and b are rational numbers and $b \neq 0$, the quotient of a and b is written $\frac{a}{b}$ and $\frac{a}{b} = c$ only if $c \cdot b = a$.

Why can't b equal zero?

122. because division by 0 is not defined
or
because we can't divide by 0

123. Then to divide -18 by 2, we must find a number such that when it is multiplied by 2 we get -18.

$\frac{-18}{2} =$ ______________ because ____________________.

123. -9 because $(-9)(2) = -18$

124. -39 divided by -13 = __________ because ______________.

-54 divided by __________ = -6 because ________________.

72 divided by _________ = -8 because ________________.

124. 3 because $3(-13) = -39$
9 because $9(-6) = -54$
-9 because $-9(-8) = 72$

125. By the examples in the last frame, you can see that the division of signed numbers follows the same rule as that for multiplication of signed numbers.

Division of signed numbers

To divide two signed numbers, divide the absolute values. The result is positive if the numbers have the same sign. The result is negative if the numbers have different signs.

Find the following quotients.

$\frac{-68}{-17}$ $\frac{-96}{16}$ $\frac{135}{-9}$ $\frac{0}{-2}$ $\frac{-42}{8}$

$\frac{7}{-21}$ $\frac{120}{-52}$ $\frac{14}{0}$

125. 4
-6
-15
0 (Every fraction with a zero numerator and a number which is not zero in the denominator equals 0)

$-5\frac{1}{4}$ or $-\frac{21}{4}$

$-\frac{1}{3}$

$-\frac{30}{13}$

no number (Remember division by 0 is not defined)

126. Are "find the quotient of -68 and -17" and "find the quotient of -17 -17 and -68" equivalent statements? Explain.

126. no. the first means $\frac{-68}{-17}$
the second means $\frac{-17}{-68}$

127. Since the order of two numbers is important in division, we may say that division is not __________.

127. commutative
(To show falseness, one counter example is sufficient).

128. (a) __________ and __________ of rational numbers are commutative operations.

(b) __________ and __________ of rational numbers are not commutative operations.

128. (a) multiplication and addition
(b) subtraction and division

129. Perform the indicated operations and simplify.

(a) $\frac{-52 + 10}{6}$

(b) $\frac{19 - -16}{11 - 18}$

(c) $\frac{(13 - 20)(19 + 3)}{(-8 - 3)(16 - 51)}$

129. (a) $\frac{-42}{6} = -7$

(b) $\frac{35}{-7} = -5$

(c) $\frac{(-7)(22)}{(-11)(-35)} = -\frac{2}{5}$

130. If we are given any two rational numbers, then one of three conditions exists. Either the two numbers are equal or the first number is larger than the second one or the first number is smaller than the second one.

The signs of inequality are $>$ and $<$.

If a and b represent two rational numbers such that a is larger than b, we may write $a>b$. Of course, if a is larger than b, then b is smaller than a and this is written $b<a$.
Complete the following using $>$ and $<$.

(a) 6 ____ 0

(b) 1 ____ 3

(c) $\frac{1}{2}$ ____ $\frac{1}{4}$

130. (a) $>$
(b) $<$
(c) $>$

131. As long as both numbers are positive numbers, we can tell very easily which number is the larger. If you were more familiar with negative numbers, you could probably choose the larger one just as easily.
However, we will use the following definition so that we won't have to guess.

Definition: Given two rational numbers a and b, we say that $a<b$ or $b>a$, if b - a is positive.

To decide whether -4 is larger or smaller than -2, we shall evaluate the difference of -4 and -2 and we shall also evaluate the difference of -2 and -4.

-4 - -2 = ________

-2 - -4 = ________

131. -4 - -2 = -2
-2 - -4 = 2

132. The difference of -2 and -4 is positive, which according to our definition means that -2 is the larger of the two numbers, or that $-2>-4$.

Which is the larger: -1 or -5? Write a statement showing this relationship.

132. $-1$ is the larger because $-1 - -5$ is a positive number.

$-1 > -5$ or $-5 < -1$

133. Complete the following using $=$, $>$ or $<$.

(a) $2$ ______ $4$

(b) $\frac{3}{6}$ ______ $\frac{1}{2}$

(c) $\frac{5}{6}$ ______ $\frac{4}{6}$

(d) $-\frac{14}{3}$ ______ $-\frac{11}{3}$

133. (a) $<$
(b) $=$
(c) $>$
(d) $<$

134. Suppose that we were to determine the larger of 5/6 and 7/9. Using the definition, we would evaluate $\frac{5}{6} - \frac{7}{9}$ and $\frac{7}{9} - \frac{5}{6}$ to find out which difference was positive. In order to find out the value of $\frac{5}{6} - \frac{7}{9}$ what procedure would you have to follow?

134. In order to add or subtract fractions you need the same denominator so each fraction must be changed to an equivalent fraction having the same denominator.

135. Simplify $\frac{5}{6} - \frac{7}{9}$ .

135. LCD is 18.
$\frac{5}{6} - \frac{7}{9} = \frac{15}{18} - \frac{14}{18} = \frac{1}{18}$

136. Since the difference is a positive number, is $\frac{5}{6}$ or $\frac{7}{9}$ the larger?

136. 5/6

137. Note that if both fractions are positive numbers, the larger fraction has the larger numerator when the denominators are the same.

Using $=$, $>$ or $<$ complete the following.

(a) $\frac{3}{5}$ ______ $\frac{39}{65}$

(b) $\frac{3}{22}$ ______ $\frac{10}{77}$

(c) $-\frac{7}{60}$ ______ $-\frac{11}{84}$

(d) $|-3|$ _____ $|-2|$

(e) $|3|$ _____ $|-5|$

(f) $\left|-\frac{3}{2}\right|$ _____ $\frac{9}{6}$

(g) $\left|-\frac{7}{9}\right|$ _____ $\frac{5}{7}$

137. (a) =
(b) >
(c) >
(d) >
(e) <
(f) =
(g) >

## Chapter 3 - Operations with Polynomials

In the last two chapters we studied the rational numbers and the operations of addition, subtraction, multiplication and division on these numbers. In this chapter the operations of addition, subtraction, multiplication and division on polynomials are studied. In this chapter and in fact in the next three chapters, all the letters that we use will be understood to denote rational numbers.

1. We will use letters to stand for rational numbers. If you are given that p is a rational number, does p represent any specific rational number?

Answer 1. no

2. 15p is a symbol which we will use to mean 15 multiplied by the value of the number p. If p stands for the number 3, then 15p represents the number 15(3) or 45. If p represents the number - 1/5, then 15p represents the number

____________ or ___________.

Answer 2. $15(-\frac{1}{5})$ or -3

3. In 15p, the p is called a <u>variable</u>.

A variable is a symbol which represents one or more numbers in a given set of numbers.

If p represents any integer less than 50 which is divisible by 10, list the possible values of p.

possible values of p are ________

Answer 3. 10, 20, 30, 40

4. Since p represents one or more numbers in a given set, p is

called a ______________.

Answer 4. variable

5. If p represents any of the integers 10, 20, 30 or 40, what does 15p represent?

Answer 5. 15p represents 150 if p is 10
15p " 300 if p is 20
15p " 450 if p is 30
15p " 600 if p is 40

6. $b^2$ is read "b squared" and means (b)(b) or b·b.

$b^3$ is read "b cubed" and means

__________ or that there are

_______ factors of b.
how many

$b^4$ is read "b to the fourth power"

and means ____________.

6. $b^3$ means b·b·b.

three factors of b

$b^4$ means b·b·b·b

7. (x)(x)(x)(x)(x) can be written

_________ which is read

______________________________.

7. $x^5$

x to the fifth power

8. (2)(2)(2)(2) can be written

______________________________.

8. $2^4$

9. $3^2$ means ____________.

9. 3(3)

10. In $x^6$, the 6 is called an <u>exponent</u> and x is called the <u>base</u>.

In $a^5b^3$, the 3 is called the

____________ of the base _____.

10. exponent of the base b.

11. $a^5b^3$ means ____________________.

11. a(a)(a)(a)(a)(b)(b)(b)
or
there are 5 factors of a and 3 factors of b

12. $4x^5y^2$ means that there is _____

factor of 4, _____ factors of x

and _____ factors of y.

12. one

five

two

13. $pq^3$ means that there is _____

factor of p and ______ factors

of q.

13. one

three

14. $(ab^2)^4$ means that there are <u>four</u>

<u>factors</u> of ______.

14. $ab^2$

15. Note that in the above, the exponent applies to the letter, symbol or number which immediately precedes it and to no other letter, number or symbol.

If n is a positive integer, then

$a^n$ is defined to be ____________.

15. n factors of a
or
a•a•a•...•a

16. 15p, $-11m^3n^2$ and $\frac{4}{3}xy^2$ are

examples of monomials.

A monomial is the product of a rational number and a variable (or variables) raised to a positive integral power (or powers).

$-8r^3s^6$ is a ____________.

16. monomial

17. Is $5(x-y)(a+b)^2$ a monomial? Give a reason for your answer.

17. yes
Check the definition. Is this a product of a rational number and variables raised to positive integer powers? Remember () are used to indicate that a group of terms is to be used as a single number. Thus one variable is (x-y) and this has an exponent of 1 and the variable (a+b) has an exponent of 2.

18. Is $\frac{y^2}{x}$ a monomial? Give a reason

for your answer.

18. No. $y^2$ is divided by x and not multiplied by it.

19. Choose the monomials from the following.

$\frac{16}{3}x^4$, $3p^{-2}$, $-2mn^5$, $-3x+8p$,

$-abc$

19. $\frac{16}{3}x^4$, $-2mn^5$, and $-abc$ are monomials.

$3p^{-2}$ doesn't have a positive integer exponent.

$-3x+8p$ is not a product

20. In $-2mn^5$, the $-2$ is called a coefficient.
A coefficient is the numerical factor in a monomial.

In $-\frac{1}{2}a^3$, the coefficient is

________.

20. $-\frac{1}{2}$

21. In $-\frac{1}{2}a^3$, the 3 is called an

______________.

| | |
|---|---|
| 21. exponent or power | 22. In $-11m^2nt^3$, -11 is called the ______________. |
| 22. coefficient | 23. In $y^4$ there is no coefficient written however this coefficient is understood to be 1. Thus<br><br>$y^4$ and $1y^4$ are regarded as equivalent.<br><br>What is the coefficient of $-a^2b$? |
| 23. -1 | 24. Name the coefficient of each of the following.<br><br>$-2ab$, $6x^2$, $5(c-d)$,<br><br>$x^2y$, $-\frac{1}{4}u^2z^3py^2$, $-w^5$ |
| 24. -2, 6, 5, 1, $-\frac{1}{4}$, -1 | 25. Since we are using letters to represent numbers, then addition and multiplication of monomials must satisfy the same rules as addition and multiplication of rational numbers. These laws<br><br>are the c______________,<br><br>a______________ and d______________ laws. |
| 25. commutative<br>associative<br>distributive | 26. The ______________ law states that the order of two numbers in addition or multiplication is immaterial. The ______________ law connects the two operations of addition and multiplication. |
| 26. commutative<br>distributive | 27. The addition of two numbers such as 4x and 7x may be done by applying the distributive law.<br><br>$4x + 7x = x(4+7) = x(11) = 11x$<br><br>We generally write the numerical symbol first.<br><br>Using the distributive law add 3p and 9p. |

27. $3p+9p = p(3+9) = p(12) = 12p$

28. Why does $p(12)$ equal $12p$?

28. by the commutative law of multiplication

29. Use the distributive law to find the following sum.

$-4w^2 + 9w^2 + -2w^2 =$ ____________

29. $w^2(-4 + 9 + -2) = w^2(3) = 3w^2$

30. Find the sum of $4y$ and $y$ using the distributive law.

30. $4y + y = y(4+1) = y(5) = 5y$
Don't forget $y$ is the same as $1y$.

31. Find the sum of $6r^2s$, $8r^2s$ and $-2r^2s$.

31. $6r^2s+8r^2s+ -2r^2s =$

$r^2s(6+8+ -2) = 12r^2s$

32. Example:
Add $2a$, $3b$, $4a$, $8a$ and $5b$

$2a + 3b + 4a + 8a + 5b = ?$

You will notice in this case that there is no number distributed over the whole sum. It is therefore necessary to rearrange the terms and group them so that in each group there is a number distributed over the whole group. We can do this as addition of rational numbers is commutative and associative.

$2a+3b+4a+8a+5b = 2a+4a+8a+3b+5b$

Now a is distributed over the first three terms and b is distributed over the last two terms. Applying the distributive law to the first three terms and then to the last two terms, we get

$2a+4a+8a+3b+5b = a(2+4+8)+b(3+5)$
and this equals $14a + 8b$.

Add $3xy^2$, $2x^2y$, $x^2y$, $7xy^2$ and $4x^2y$.

32. $3xy^2+7xy^2+2x^2y+x^2y+4x^2y =$

$xy^2(3+7)+x^2y(2+1+4) =$

$10xy^2+7x^2y$

33. $3xy^2$ and $7xy^2$ are called similar monomials.

Monomials which have exactly the same factors except for coefficients are called similar monomials.

Are $3p^2$ and $-7p^2$ similar monomials? Why?

34. To add similar monomials, add the coefficients and multiply the result by the common factor.

Thus to add $-8mn$ and $12mn$, we can add the coefficients of $-8$ and $12$, getting $4$, and multiply this result by the common factor of $mn$, getting $4mn$. This is exactly what we do when we use the distributive law.

$-8mn+12mn = mn(-8+12) = 4mn$.

Add $7mn^2$, $-2m^2n$, $6m^2n$, $3m^2n^2$, $-11mn^2$ and $-m^2n$.

33. No, $p^2$ and $p^3$ are not the same factor.

35. $-4mn^2+3m^2n+3m^2n^2$ is a <u>polynomial</u> in two variables.

A polynomial consists of one or more monomials used as terms.

Terms are connected by the operation of ______________.

34. $7mn^2+ -11mn^2+ -2m^2n+6m^2n+ -m^2n+3m^2n^2 = mn^2(7+ -11)+ m^2n(-2+6+ -1)+3m^2n^2 = -4mn^2 + 3m^2n + 3m^2n^2$

36. $3m^4+5m^2-6m$ is a ____________ in one variable.

35. addition

37. $3p^2-4p^3$ is also a polynomial in one variable. This may not seem to fit the definition as the terms are not added but remember

$3p^2-4p^3 = 3p^2+ -4p^3$.

The degree of a polynomial is the largest power of a particular variable.

In $3p^2-4p^3$, the variable is ____________, and the largest exponent of this variable is ____________, so $3p^2-4p^3$ is a polynomial of degree ________.

36. polynomial

38. What is the degree of

$-6y^5+13y^3-8y^9-y^6$ ?

37. variable is p
highest power is 3
degree of polynomial is 3

38. 9

39. We will agree that if a term has no variable, it is of degree zero.

If we consider the term 6, it is of

degree _____ because ___________.

39. 0 because there is no variable present

40. Given the polynomial in x,

$3x^5+4x^3-3x^2+7$, what is the coefficient of the second term?

40. the second term is $4x^3$ its coefficient is 4 or +4

41. The polynomial $3x^5+4x^3-3x^2+7$ is arranged in descending order.

This means that the term with the largest exponent is written first, then the term with the next largest exponent, etc. until finally we have the term without the variable (i.e. the term of degree 0).

Arrange in descending order:

$3x-7x^4+5x^2+6+2x^7$

41. $2x^7-7x^4+5x^2+3x+6$

42. Be sure you have a + or a - between each term. If you wrote $5x^2\ 3x$, this means to perform the operation of _______________.

42. multiplication

43. A polynomial may be arranged in ascending order.

How would a polynomial be written if it is in ascending order?

43. the term with no variable is written first, then the term with the variable to the first power, then the term with the variable squared, etc.

44. In the polynomial $3x^5+4x^3-3x^2+7$, the terms containing $x^4$ and x are missing. It is often convenient to to consider the polynomial with these terms present.

Since any number multiplied by 0 equals 0, we can write

$3x^5+4x^3-3x^2+7$ as $3x^5+0\cdot x^4+4x^3$

$-3x^2+0\cdot x+7$

In other words, we consider the coefficients of the missing terms to be ______.

44. 0

45. Considering the coefficients of these terms to be zero, doesn't change the polynomial because

______________________________.

45. 0 multiplied by any number is 0.

Remember $x^4$ and x represent numbers

46. We can rewrite any polynomial or rational number in a different form providing the value remains the same.

In $2p^3+5p+7p^5-3$, the coefficient of the fifth degree term is _____.

The coefficient of the fourth power term is ______.

46. 7 or +7

the fourth power term is missing, so the coefficient is 0

47. To add polynomials, we make use of the commutative, associative and distributive laws.

For example: Add $3y^2+y^3-7$ and

$-4y^3+6y^2-2y+3$.

This is: $(3y^2+y^3-7)+(-4y^3+6y^2-2y+3)$
Using both the associative and commutative laws, this equals:

$(y^3+ -4y^3)+(3y^2+6y^2)+(-2y)+(-7+3)$.
Using the distributive law, we get:

$(1-4)y^3+(3+6)y^2+(-2y)+(-7+3)$ which

equals $-3y^3+9y^2-2y-4$.

Add: $-5p^6+7p^2-4p^3$ and $3p^6+2p^4-7p^2$.

47. $(-5p^6+7p^2-4p^3)+(3p^6+2p^4-7p^2)=$

$(-5p^6+3p^6)+(7p^2+-7p^2)+(-4p^3)$

$+(2p^4)$ by the commutative and associative laws.
By the distributive law, we get,

$(-5+3)p^6+(7+ -7)p^2+(-4p^3)+(2p^4)$

which equals $-2p^6-4p^3+2p^4$

48. Find the sum of:

(a) $3c-4b+7c+d$ and $-5a+12b-11c-4d$

(b) $-x^3+4x-8$ and $9x^2-5x^4+6-7x^3-2x$

48. (a) $-2a+8b-4c-3d$

(b) $-8x^3+2x-2+9x^2-5x^4$

49. To the sum of $2x^4+x^2-x$ and

$-5x^2-7x^3+3$ add the sum of

The terms may be arranged in any order. Why may we say that $-2a+8b-4c-3d$ equals $8b-2a-3d-4c$

$-x^3-8$ and $4x+9x^2-5x^4+6$.

49. sum of $2x^4+x^2-x$ and $-5x^2-7x^3+3$ is $2x^4-4x^2-7x^3-x+3$.

sum of $-x^3-8$ and $4x+9x^2-5x^4+6$ is $-x^3-2+4x+9x^2-5x^4$.

$(2x^4-4x^2-7x^3-x+3)+(-x^3-2+4x+9x^2-5x^4)$

$= -3x^4+5x^2-8x^3+3x+1$

50. What is the coefficient of $-(-2x+y+5p)$?

50. -1

Remember that () are used to indicate that a group of terms is to be used as a single number.

51. $-(-2x+y+5p)$ can be written without parentheses by using the distributive law.

By the distributive law, $-(-2x+y+5p)$ = _______________

51. $-1(-2x+y+5p) = 2x-y-5p$

52. Write $+(5a-3b+4c)$ without parentheses.

$+(5a-3b+4c)$ = _______________

52. $5a-3b+4c$

$+(5a-3b+4c)$ means to multiply by 1 or +1 and any number multiplied by 1 equals itself.

53. $3a+4b-(2a-8b)$ = _______________ when written without parentheses.

53. $3a+4b-2a+8b$ because $-(2a-8b)$ is $-1(2a-8b)$ and by the distributive law this becomes $-1\cdot 2a+(-1)(-8b) = -2a+8b$.

54. Insert a quantity inside parentheses which are preceded by a - sign so that the entire quantity equals $-3p+4q$.

$-(\qquad\qquad) = -3p+4q$

54. $-(3p-4q)$

check this by multiplying $-1$ by $3p-4q$ to see if you get $-3p+4q$.

55. If we are to subtract $3x-2$ from $-2x-4$, we would use parentheses around these two quantities to indicate that the whole quantity was to be treated as one number. We would write: $(-2x-4)-(3x-2)$.

Without the parentheses, this quantity equals _______________.

55. $-2x-4-3x+2$
because
$(-2x-4) = +1(-2x-4) = -2x-4$
and
$-(3x-2) = -1(3x-4) = -3x+4$

56. If we combined similar terms in $-2x-4-3x+2$, we would obtain

______________.

56. $-5x-2$

57. Write: Subtract $4mn-5m^2$ from $-3mn+3m^2$.

57. $(-3mn+3m^2) - (4mn-5m^2)$

58. Remove the parentheses and complete the subtraction.

58. $-3mn+3m^2-4mn+5m^2 = -7mn+8m^2$

59. Write: the difference between $x^2+4x-2$ and $-3x^2+6x+7$

59. $(x^2+4x-2) - (3x^2+6x+7)$

You must have the second set of parentheses as you are to subtract the whole quantity of $-3x^2+6x+7$.

60. Complete the subtraction in the preceding frame being sure to combine terms.

60. $x^2+4x-2+3x^2-6x-7 = 4x^2-2x-9$

61. Find the sum of $-11a+8b-2c$ and $a-13b-19c$ and then find the difference between this result and $9a+b+c$.

61. sum of the first two quantities is $-10a-5b-21c$.

The subtraction is written $(-10a-5b-21c)-(9a+b+c)$.
This equals $-10a-5b-21c-9a-b-c$ or $-19a-6b-22c$.

62. What number added to $-3p^3+8q^2$ equals 0?

62. You are asked to find this missing number:

$(-3p^3+8q^2) + ( \quad ? \quad ) = 0$

If the sum of two numbers is a third number, then the sum less one of the numbers must equal the other number.
That is, if $a+b=c$ then $c-a$ must equal b.
Thus, in the given problem,

$0 - (-3p^3+8q^2)$ is the required

number and this is $3p^3 - 8q^2$.

63. In frames 31 and 33 of Chapter 2, we defined two numbers whose sum

was 0 as the ______________ or the

______________ of each other.

63. additive inverses or negatives

64. Since $3p^3-8q^2$ when added to $-3p^3+8q^2$ equals 0, then $3p^3-8q^2$ is the __________ of $-3p^3+8q^2$.

64. negative or additive inverse

65. A number and its negative always

______________________________.

65. have a sum of 0

66. What is the negative of $-4y+y^2-7$?

66. $-(-4y+y^2-7)$ or $4y-y^2+7$

67. How can you check to see if this is the negative of $-4y+y^2-7$?

67. It is the negative if the sum of it and the original number is 0.

68. Write a quantity equal to the given quantity by enclosing some terms in parentheses preceded by a - sign.

(a) $-6x+4y+9p$ = __________

(b) $-a-2b-7d$ = __________

(c) $3m+12n-2q$ = __________

68. (a) $-(6x-4y-9p)$
(b) $-(a+2b+7d)$
(c) $-(-3m-12n+2q)$

69. Write a quantity equal to $-6x+4y+9p$ by enclosing some terms in parentheses preceded by a + sign.

$-6x+4y+9p$ = __________

69. $+(-6x+4y+9p)$

70. Remove the parentheses in the following and combine similar terms. Brackets and braces are treated the same as parentheses.

(a) $8a-(-2a+3b)+(3a-5b)$

(b) $-3a-[-3-(4a-5)]$ Remove <u>only</u> the brackets or the parentheses the first time. In the second step, remove the other. Follow this procedure when one set of parentheses is inside another.

(c) $x-(-x+3y-[-6x-2y]-x+y)$

70. (a) $8a+2a-3b+3a-5b = 13a-8b$

(b) Removing the parentheses, we get $-3a-[-3-4a+5]$.

71. In each of the following, you are given three polynomials. Find the sum of all three and then subtract the result from the first one.

Then removing the brackets, we get $-3a+3+4a-5$ or $a-2$.

Removing the brackets first, we get, $-3a+3+(4a-5)$. Then remove the parentheses getting $-3a+3+4a-5$ or $a-2$.

Remember inside the brackets, there are two terms: $-3$ and $(4a-5)$.

(c) $x-(-x+3y+6x+2y-x+y) =$ $x+x-3y-6x-2y+x-y = 3x-6y$

(a) $7y^3+2y^2-y$, $-12y+5-7y^2$, $3y^4+3y-2$.

(b) $p^5-7p^3q^2-5p^2q^3$, $3p^4q+9pq^4+8p^2q^3$, $-3p^5-6pq^4$.

(c) $-11ab+bc-3ac$, $5bc+7ab$, $5ac-3abc-9bc$.

71. (a) sum is $3y^4+7y^3-5y^2-10y+3$. difference is written:

$(7y^3+2y^2-y)$ - (the above sum).

This equals $-3y^4+7y^2+9y-3$

(b) sum is

$-2p^5+3p^4q-7p^3q^2+3p^2q^3+3pq^4$

difference is written:

$(p^5-7p^3q^2-5p^2q^3)$-(the above sum)

Removing parentheses and combining we get

$3p^5-3p^4q-8p^2q^3-3pq^4$.

(c) sum is $-4ab-4bc+2ac-3abc$. difference is: $(-11ab+bc-3ac)$-(the above sum) and this equals $-7ab+5bc-5ac+3abc$.

72. We are now ready to consider products of monomials and polynomials. First, we must develop some of the laws of exponents.

In frame 15, we defined $a^n$ to mean $a\cdot a\cdot a\cdot\ldots$ for n factors providing n is a positive integer.

Thus, $(p^2)(p^3)$ means $(p\cdot p)(p\cdot p\cdot p)$. This represents 5 factors of p and can be written as $p^5$.

$(m^4)(m^3) =$ __________ = __________

72. $(m\cdot m\cdot m\cdot m)(m\cdot m\cdot m) = m^7$

73. In general, $(a^m)(a^n) = a^{m+n}$ if m and n are _________ (what kind) numbers. (Remember $a^n$ is defined only if n is ____________.) See frame 15 if necessary.

73. m and n must be <u>positive integers</u> as $a^n$ has been defined so far only when n is a positive integer.

74. In $m^4$, m is called the __________.

74. base

75. Note that the law of exponents given in frame 72 involves the <u>same</u> base and this law holds only when the bases are the same.

$x^5(x^9) =$ __________

75. $x^{5+9} = x^{14}$

76. Does $x^3(y^2) = xy^5$? Give a reason for your answer.

76. No. The bases are <u>not</u> the same so the law given in frame 72 can't be used.

Try $x = 2$ and $y = 1$

$x^3(y^2) = x \cdot x \cdot x \cdot y \cdot y \cdot$ or there are three factors of x and two factors of y.

$xy^5 = x \cdot y \cdot y \cdot y \cdot y \cdot y$ or there are five factors of y and one factor of x.

Since the same factors are not involved, the two cannot be equal.

77. To multiply $p^2r^3$ by $p^5r^6$ we have to use the commutative and associative laws.

$p^2r^3(p^5r^6) = p^2p^5(r^3r^6)$ by these laws.

Now we can use the law $(a^m)(a^n) = a^{m+n}$) as we have the same base and positive exponents.

Thus, $p^2p^5(r^3r^6) =$ _____ $=$ _____

77. $p^{2+5}r^{3+6} = p^7r^9$

78. $(-3x^2y^4z)(9x^3y^2z^4)$ can be multiplied together in the same manner. Each of these quantities consists of a single term and so each is called a __________.

78. monomial

79. In multiplying monomials, we use the commutative and associative laws to change the order of the factors and to regroup the factors. Then we can use the laws of multiplication.

79. $-27x^5y^6z^5$

80. Multiply $-2pq^3r^2$ by $-4p^2q^5r^2$.

80. $(-2)(-4)(p \cdot p^2)(q^3 \cdot q^5)$

$(r^2 \cdot r^2) = 8p^3q^8r^4$

81. In $-2pq^3r^2$, the coefficient is _____. The exponent of p is _____.

81. coefficient is -2
exponent of p is 1

82. (-m np )(12np) = _____ = _____

82. $(-1)(12)(m^2)(n \cdot n)(p^5 \cdot p) = -12m^2n^2p^6$

83. In frame 72, we stated that $a^m \cdot a^n = a^{m+n}$ when m and n are positive integers.

Can we use this law to multiply $3^5$ by $3^7$? Why?

83. yes because the bases are the same and the exponents are positive integers.

84. $3^5 \cdot 3^7$ = ________ = ________

84. $3^{5+7} = 3^{12}$

85. The product is $3^{12}$. This makes sense if we interpret the meaning of $3^5 \cdot 3^7$.

$3^5 = 3(3)(3)(3)(3)$ or means that there are 5 factors of 3.

$3^7 = 3(3)(3)(3)(3)(3)(3)$ or means that there are 7 factors of 3.

Hence, $3^5 \cdot 3^7$ means that 5 factors of 3 are multiplied by 7 factors of 3 which means that 12 factors of 3 are multiplied together.

Twelve factors of 3 can be written as $3^{12}$.

In $4^3 \cdot 4^6$, how many factors of 4 are present?

$4^3 \cdot 4^6$ = ________ (Write using an exponent)

85. 9 factors of 4

$4^9$

86. $2^5(2^3)$ = ________ (Express using exponents)

86. $2^8$

87. Does $7^3 \cdot 5^6 = 35^9$? Why?

87. No because since we don't have the same base we <u>cannot</u> use the law given in frame 72.

If you're in doubt, multiply it out and compare the results.

88. $(3xy^7z^3)(3^5x^2yz^2)$ = ________ = ________

88. $(3 \cdot 3^5)(x \cdot x^2)(y^7 \cdot y)(z^3 \cdot z^2) =$

$3^6x^3y^8z^5$ or $729x^3y^8z^5$

89. $2^4 = (+2)^4$ and these both mean ________.

$(-2)^4$ means ________.

89. $2(2)(2)(2)$ or $+2(+2)(+2)(+2)$

$(-2)(-2)(-2)(-2)$

90. $-2^4$ does not equal either of the above.

$-2^4$ means the same as $-(2)^4$ and both of these mean $-1(2)(2)(2)(2)$ or this can be written $-2(2)(2)(2)$.

$-3^2$ means ______ and this equals ______.

90. $-3^2$ means $-3(3)$ or $-(3)(3)$ or $-1(3)(3)$ and any of these equal $-9$

91. $(-3)^2$ means ______ and this equals ______.

91. $(-3)(-3)$ and this equals 9

92. Evaluate $-3^4$

92. $-3^4 = -(3)(3)(3)(3) = -81$

93. Evaluate: (a) $-3^3(2^3)$

(b) $(-3)^3(-2)^3$

93. (a) $-(3)(3)(3)(2)(2)(2) = -(27)(8) = -216$

(b) $(-3)(-3)(-3)(-2)(-2)(-2) = (-27)(-8) = 216$

94. If it is possible, express each of the following results using exponents. If the result can't be expressed as a number to a power, evaluate it.

(a) $5^3 \cdot 5^5$

(b) $(-7)^4(-7)^5$

(c) $(-3)^2(-2)^3$

(d) $(-3)^2(-2)^2$

(e) $-3^2(2)^3$

(f) $-(-4)^3(-1)^4$

(g) $(-2)^5(-1)^7$

(h) $5(-7)^2(-\frac{1}{2})^3$

94. (a) $5^{3+5} = 5^8$

(b) $(-7)^{4+5} = (-7)^9$

(c) $(9)(-8) = -72$

(d) $(9)(4) = 36$

(e) $-(9)(8) = -72$

(f) $-(-64)(1) = 64$

95. What does $(-5x^2y)^3$ mean?

(g) $(-32)(-1) = 32$

(h) $5(49)(-\frac{1}{8}) = -\frac{245}{8}$

95. $(-5x^2y)(-5x^2y)(-5x^2y)$

96. Complete the multiplication.

$(-5x^2y)(-5x^2y)(-5x^2y) =$ ________ =

________

96. $-5(-5)(-5)(x^2 \cdot x^2 \cdot x^2)(y \cdot y \cdot y) =$

$-125\ x^6y^3$ or $(-5)^3x^6y^3$

97. $(2m^4n^3z)^4$ means ____________.

This equals ______________.

97. $(2m^4n^3z)(2m^4n^3z)(2m^4n^3z)$

$(2m^4n^3z)$

this equals $16m^{12}n^{12}z^4$

98. Let us see if there is a law we can use to raise a quantity to a power.

Examine $(m^3)^2$. $(m^3)^2$ means

____________.

98. $(m^3)(m^3)$

99. In words, we could say that raising $m^3$ to the second power is equivalent to multiplying $m^3$ by $m^3$.

$(m^3)(m^3)$ can be evaluated by the law given in frame 72.

So $m^3(m^3) = m^{3+3} = m^6$.

Note that $m^{3+3}$ is the same as

$m^{2(3)}$ because $3+3 = 2(3)$.

In other words, when a monomial is raised to a power, <u>multiplying</u> the exponents gives the same result as writing the meaning of the quantity and then doing the indicated multiplication.

$(p^5)^4 =$ ______________

99. $p^{4(5)} = p^{20}$ or

$(p^5)(p^5)(p^5)(p^5) =$

$p^{5+5+5+5} = p^{20}$

100. Stated in general terms, if m and n are <u>positive integers</u> then

$(a^m)^n = a^{mn}$.

$(x^{11})^3 =$ __________ = __________

$(p)^5 =$ __________ = __________

100. $(x^{11})^3 = x^{3(11)} = x^{33}$

$(p)^5 = (p^1)^5 = p^{5(1)} = p^5$

101. $(x^{11})^3$ means to do what operation?

101. to cube $x^{11}$ or

to raise $x^{11}$ to the third power or

$x^{11}(x^{11})(x^{11})$.

102. If you said that you were to multiply meaning that you were to multiply 11 by 3, you are WRONG.

When exponents are multiplied, you have performed the operation of <u>raising to powers</u>.

To multiply $x^{11}$ by $x^3$, that is

$x^{11}(x^3)$ what did you do with the exponents?

102. added the exponents

103. What gives you the right to say

that $x^{11}(x^3) = x^{11+3} = x^{14}$?

103. because $a^m \cdot a^n = a^{m+n}$ if m and n are positive exponents.

Note: the bases must be the same to apply this law.
or
there are eleven factors of x multiplied by three factors of x which gives fourteen factors of x which can be written $x^{14}$.

104. This law states that we may add the exponents when the bases are the same in order to perform the operation of ____________.

104. multiplication

105. $(3^4)^6 =$ ____________

$2^5 \cdot 2^9 =$ ____________

105. $3^{4(6)} = 3^{24}$

106. In evaluating $(3p^2qr^3)^4$, we can write out the meaning of this and proceed to multiply the factors.

For example: $3p^2qr^3)$ means

$(3p^2qr^3)(3p^2qr^3)(3p^2qr^3)$. By the associative and commutative laws this can be written

$3(3)(3)(p^{2 \cdot 2 \cdot 2})(q \cdot q \cdot q)(r^3 \cdot r^3 \cdot r^3)$

which equals $3^3(p^2)^3(q)^3(r^3)^3$

or $27\ p^6q^3r^9$.

Instead of writing out the meaning of the given statement, we can apply the law $(a^m)^n = a^{mn}$ when m and n are positive exponents after we have applied another law.

We need the fact that $(ab)^n = a^n b^n$.

By this law we can write that

$(x^2yz^3)^4 =$ ____________.

106. $(x^2yz^3)^4 = (x^2)^4(y)^4(z^3)^4$

107. Note that the law $(ab)^n = a^n b^n$ only applies when n is a positive integer and when the quantity raised to the power is composed of factors.

Can we write that $(a^2+b)^3 =$

$(a^2)^3 + (b)^3$? Why?

107. NO. The law $(ab)^n = a^n b^n$ applies only when the base is composed of factors.

The base $(a^2+b)$ is not composed of factors as factors must be multiplied.

Try a = 2 and b = 1. You can see the two results are not equal.

108. First apply the law $(ab)^n = a^n b^n$ and then apply the law $(a^m)^n = a^{mn}$ to find the value of

$(3a^7bc^4)^5$.

108. $(3)^5(a^7)^5(b)^5(c^4)^5 =$

$-3^5a^{35}b^5c^{20}$ or $243a^{35}b^5c^{20}$

109. Evaluate $(-2p^3q^4)^2$.

109. $(-2)^2(p^3)^2(q^4)^2 = 4p^6q^8$

110. Did you write $-2^2(p^3)^2(q^4)^2$ for the last problem?

If you did it is wrong. The error is in the coefficient.

The correct coefficient is $(-2)^2$ as an exponent affects the letter, number or symbol which immediately precedes it and in this case, the exponent 2 affects everything in the parentheses, and the -2 is inside the parentheses. Remember $-2^2$ and $(-2)^2$ do not mean the same and are not equal.

Evaluate $(-3x^2)^4$.

110. $(-3)^4(x^2)^4 = 81x^8$

111. Evaluate: (a) $(-5p^3q^6r^5)^2$

(b) $(-a^3b^2c^5)^6$

(c) $(-y^4z^8)^4$

111. (a) $(-5)^2(p^3)^2(q^6)^2(r^5)^2 = 25p^6q^{12}r^{10}$

(b) $(-1)^6(a^3)^6(b^2)^6(c^5)^6 = a^{18}b^{12}c^{30}$

(c) $(-1)^4(y^4)^4(z^8)^4 = y^{16}z^{32}$

Remember in parts b and c that $-a$ means $-1(a)$.

112. Evaluate $-(3mn^2)^2(-2m^3n^4)$.

Write out the meaning of this statement before you try to simplify.

112. $-(3mn^2)(3mn^2)(-2m^3n^4)$ or

$(-1)(3mn^2)(3mn^2)(-2m^3n^4)$

These equal $(-1)(3)(3)(-2)(m\cdot m\cdot m^3)(n^2\cdot n^2\cdot n^4)$

or $18m^5n^8$

113. Instead of writing out the meaning of $-(3mn^2)^2(-2m^3n^4)$ we could square $3mn^2$ and then perform the multiplication.

Doing this, $-(3mn^2)^2(-2m^3n^4) =$

$-(9m^2n^4)(-2m^3n^4)$ and this is

$(-1)(9)(-2)(m^2\cdot m^3)(n^4\cdot n^4)$ or

$18m^5n^8$.

$-(-p^2q)^3(-3pq^2)^2 =$ __________ $=$

__________

113. $-(-p^6q^3)(9p^2q^4) = 9p^8q^7$

114. Perform the following operations and simplify.

(a) $(3^5)^4$ Leave in exponent form.

(b) $[(-2)^3]^5$ Leave in exponent form.

(c) $(5)^2(5^3)^2$ Leave in exponent form.

(d) $(3^4p^3q^5r)^2(3p^2qr^5)^3$

(e) $-(-2xy^3)^7(-x^3y^4)^3$

114. (a) $3^{20}$

(b) $(-2)^{15}$

(c) $5^2 \cdot 5^6 = 5^8$

(d) $(3^8p^6q^{10}r^2)(3^3p^6q^3r^{15}) =$

$3^{11}p^{12}q^{13}r^{17}$

(e) $-([-2]^7x^7y^{21})(-x^9y^{12}) =$

$-(-128x^7y^{21})(-x^9y^{12}) =$

$-128x^{16}y^{33}$

115. So far we have multiplied together quantities which consisted of factors. Now let us consider multiplication when terms are involved.

We multiplied $3(4+7)$ by two different methods.

In one case, we said $3(4+7) = 3(11) = 33$.

In the other case, we said $3(4+7) = 3 \cdot 4 + 3 \cdot 7$

Why can we make this last statement?

115. by the distributive law

116. The distributive law states that $a(b+c) = ab + ac$ providing a, b and c are rational numbers.

Using the distributive law,

$3a(a^2+3a) =$ ____________

116. $3a(a^2)+3a(3a)$

117. Complete your answer to the last frame.

117. $3a^3+9a^2$

118. $-2p^3(3p^2+7p) =$ ________ = ________

118. $-2p^3(3p^2)+(-2p^3)(7p) =$

$-6p^5-14p^4$

You may have written $-6p^5+ -13p^4$ as your answer. This is correct. However, we generally use only one sign between terms since adding a negative is the same as subtracting a positive, we can write

$-6p^5-14p^4$.

119. The distributive law can be extended to read

$a(b+c+d) = ab+ac+ad+ae$
$a(b+c+d+e) = ab+ac+ad+ae$
.
.
.
$a(b+c+d+\ldots+n) = ab+ac+ad+\ldots+an$

Using the distributive law,

$-x^3y^2(4x^2-3xy+5y^3) =$ ____________

119. $-x^3y^2(4x^2)-x^3y^2(-3xy)$

$-x^3y^2(5y^3) = -4x^5y^2+3x^4y^3$

$-5x^3y^5$

120. $9mn^3(3m^2n+4mn^3-m^3) =$ ____________

120. $9mn^3(3m^2n)+9mn^3(4mn^3)+9mn^3(-m^3) = 27m^3n^4+36m^2n^6-9m^4n^3$

121. $-pq^2(p^5-3p^4q)+2p^3q(-3p^2q^2-pq^2) =$ __________ = __________

121. $-p^6q^2+3p^5q^3-6p^5q^3-2p^4q^3$ by the distributive law.

Combining similar terms, we have

$-p^6q^2-3p^5q^3-2p^4q^3$

122. Simplify the following.

(a) $(2xy^3z^5)(-3x^2y^2z^6)$

(b) $-3xw^3(4x^2-7xw^2+5w^4-9)$

(c) $(-2p^5q^2w)^4$

(d) $-3xy^2(x^3+x^2y)+2y^3(-4x^3+4y^2)$

122. (a) $-6x^3y^5z^{11}$
you don't need to use the distributive law here as only factors are involved.

(b) Use the distributive law here.

$-12x^3w^3+21x^2w^5-15xw^7+27xw^3$

(c) Use the law $(ab)^n = a^nb^n$ to get

$(-2)^4(p^5)^4(q^2)^4(w)^4$.

Now use that $(a^m)^n = a^{mn}$ and get

$16p^{20}q^8w^4$

(d) Use the distributive law.

$-3x^4y^2-3x^3y^3-8x^3y^3+8y^5$
which equals

$-3x^4y^2-11x^3y^3+8y^5$

123. The distributive law can also be extended to $(a+b)(c+d)$

Let $(a+b) = N$, then $(a+b)(c+d)$ becomes $N(c+d)$. By the distributive law $N(c+d) = Nc+Nd$.

Since $N = a+b$, $Nc+Nd$ becomes $(a+b)c + (a+b)d$ which equals $ac+bc+ad+bd$.

Using the distributive law,

$(2a+3)(b+3c) =$ __________

123. $(2a+3)b+(2a+3)3c = 2ab+3b+6ac+9c$

124. $(3a-2)(4a+1) =$ __________ = __________

124. $(3a-2)4a+(3a-2)1 =$

$12a^2-8a+3a-2 = 12a^2-5a-2$

125. $(3p-2q)(p^2+4pq-3q^2) =$ __________ = __________

125. $(3p-2q)p^2+(3p-2q)4pq+$

$(3p-2q)(-3q^2) = 3p^3-2qp^2$

$+12p^2q-8pq^2-9pq^2+6q^3 =$

$3p^3+10p^2q-17pq^2+6q^3$

126. Since multiplication is commutative $(3p-2q)(p^2+4pq-3q^2) = (p^2+4pq-3q^2)(3p-2q)$.

By the distributive law, $(p^2+4pq-3q^2)(3p-2q)$ equals $(p^2+4pq-3q^2)3p+(p^2+4pq-3q^2)(-2q)$. When this is expanded and similar terms are combined, we get the same result as in frame 124. You can use either expansion.

$(5x-3y)(x^3-4x^2y-5xy^2+3y^3) =$

__________ = __________

126. $5x(x^3-4x^2y-5xy^2+3y^3)$

$-3y(x^3-4x^2y-5xy^2+3y^3) =$

$5x^4-23x^3y-13x^2y^2+30xy^3-9y^4$
(All the steps are not given in this answer, however enough are given so that you can determine whether you used a correct procedure and where you made errors if any occurred.)

127. $(m-2n)^2$ means __________

127. $(m-2n)(m-2n)$

128. Use the distributive law to expand $(m-2n)(m-2n)$.

$(m-2n)(m-2n) =$ ________ = ________

128. $(m-2n)m+(m-2n)(-2n) =$

$m^2-4mn+4n^2$

129. $(3p+4r)^2$ means __________.

This equals ________ = ________.

129. $(3p+4r)(3p+4r)$.
This equals $(3p+4r)3p+$ $(3p+4r)4r = 9p^2+24pr+16r^2$

130. $(2x+y)^3$ means __________.

130. $(2x+y)(2x+y)(2x+y)$

131. This can be written $[(2x+y)(2x+y)](2x+y)$ because of the __________ law.

131. associative

132. $[(2x+y)(2x+y)](2x+y) =$ ________ =

________ = ________

132. $[(2x+y)2x+(2x+y)y](2x+y)=$

$[4x^2+2xy+2xy+y^2](2x+y) =$

133. Perform the indicated operations and simplify.

$(4x^2+4xy+y^2)2x+(4x^2+4xy+y^2)y =$

$8x^3+12x^2y+6xy^2+y^3$

(a) $(4a-3b)(a^2-ab-2b^2)$

(b) $(3x-2)(2x+3)$

(c) $(2y-3)^2$

(d) $(9p^4+6p^2q^3+4q^6)(3p^2-2q^3)$

(e) $(7a^4+8b^3)(7a^4-8b^3)$

133. (a) $4a(a^2-ab-2b^2)-3b$

$(a^2-ab-2b^2)$

Multiply and collect similar terms.

Product is $4a^3-7a^2b-5ab^2+6b^3$

(b) $(3x-2)2x+(3x-2)3 =$

$6x^2+5x-6$

(c) $(2y-3)(2y-3) = 4y^2-12y+9$

(d) $(9p^4+6p^2q^3+4q^6)3p^2+$

$(9p^4+6p^2q^3+4q^6)(-2q^3)$

$27p^6-8q^9$

(e) $(7a^4+8b^3)7a^4+(7a^4+8b^3)$

$(-8b^3) = 49a^8-64b^6$

134. If $(3x-2)(2x+3) = 6x^2+5x-6$, then we may say that $(3x-2)$ is a

__________ of $6x^2+5x-6$.

134. factor

135. $a(a+2) =$ __________.

a is a ________ of $a^2+2a$.

135. $a^2+2a$
factor

136. Name another factor of $a^2+2a$.

136. $a+2$

137. If m is one factor of $m^4$, the other factor is _____.

137. $m^3$

138. If $a^3$ is one factor of $a^5$, then the other factor is ________ because ______________.

138. $a^2$ because $a^3 \cdot a^2 = a^5$

139. Since $a^3 \cdot a^2 = a^5$, then $\frac{a^5}{a^3} =$ _____.

139. $a^2$

140. $\frac{y^{12}}{y^7}$ = ________ because ________.

140. $y^5$ because $y^7 \cdot y^5 = y^{12}$

141. In general terms, if m and n are positive integers and if $m > n$, then

$\frac{a^m}{a^n} = a^{m-n}$. $\frac{p^{13}}{p^4}$ = ________

141. $p^9$

142. To divide $\frac{x^7y^5}{x^3y^2}$, we consider this as $\frac{x^7}{x^3} \cdot \frac{y^5}{y^2}$ and now we can apply the law given in frame 140.

$\frac{x^7}{x^3} \cdot \frac{y^5}{y^2} = x^4y^3$

$\frac{a^8b^5c^2}{a^4b^2c^2}$ = ________

142. $a^4b^3$

Remember any number divided by itself is 1, so

$\frac{c^2}{c^2} = 1.$

143. $\frac{-32b^9c^{12}}{4b^6c^3}$ = ________

143. $-8b^3a^9$

144. To divide $x^2$ by $x^5$ that is $\frac{x^2}{x^5}$, we cannot use the law stated in frame 140. Why can't we use this law?

144. because the exponent in the numerator isn't larger than the exponent in the denominator.

If we use the law given in frame 140, we get $x^{-3}$, what does this mean? At this point, we don't know as $x^{-3}$ hasn't been defined yet.

145. We will write another expression which is equal to the given one.

$\frac{x^2}{x^5} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x}$

The numerator and denominator have two factors of x in common. Or we may state that x(x) is a common factor of both the numerator and denominator.

Dividing both numerator and denominator by the common factor, we have $\frac{x\cdot x \div x\cdot x}{x\cdot x\cdot x\cdot x\cdot x \div x\cdot x} = \frac{1}{x\cdot x\cdot x} = \frac{1}{x^3}$.

Thus, $\frac{x^2}{x^5} = \frac{1}{x^3}$ because $\frac{1}{x^3}(x^5) = x^2$.

$\frac{y^5}{y^9} =$ __________ because __________.

145. $\frac{1}{y^4}$ because $\frac{1}{y^4}(y^9) = y^5$

146. In more general terms, if m and n are positive integers and $m<n$, then

$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$

(a) $\frac{a^{12}b^4}{a^5b^7}$

(b) $\frac{xy^4}{x^5y^9}$

146. (a) $\frac{a^{12}}{a^5} \cdot \frac{b^4}{b^7} = a^7 \cdot \frac{1}{b^3} = \frac{a^7}{b^3}$

(b) $\frac{1}{x^4y^5}$

147. $\frac{26x^4y^2z^3}{-13x^2y^6z} =$

147. $\frac{-2x^2z^2}{y^4}$

148. $2a(3a-4) = 6a^2-8a$.

Since division is the inverse of multiplication, if $6a^2-8a$ were divided by 2a, the quotient would be __________ or $\frac{6a^2-8a}{2a} =$ __________.

148. 3a-4 (This goes in both blanks)

149. To divide $\frac{6a^2-8a}{2a}$, all the terms in the numerator must be divided by the monomial denominator.

$\frac{6a^2-8a}{2a} = \frac{6a^2}{2a} - \frac{8a}{2a} = 3a-4.$

$\frac{2a^2+6ab-4ab^2}{-2a}$ = ________ = ________

149. $\frac{2a^2}{-2a} + \frac{6ab}{-2a} - \frac{4ab^2}{-2a} =$

$-a-3b+2b^2$

150. $\frac{-14x^2y^3+84x^3y^4}{-7x^2y}$ = ________ = ________

150. $\frac{-14x^2y^3}{-7x^2y} + \frac{84x^3y^4}{-7x^2y} =$

$2y^2-12xy^3$

151. How would you check to see if your answer to the last problem was correct?

151. Find the product of $-7x^2y$ $2y^2-12xy^3$.

152. $\frac{39m^3n^5-52m^2n^3+26mn}{13mn}$ = __________ = __________.

Check your result.

152. $\frac{39m^3n^5}{13mn} - \frac{52m^2n^3}{13mn} + \frac{26mn}{13mn} =$

$3m^2n^4-4mn^2+2$

Check:
$13mn(3m^2n^4-4mn^2+2)$ equals the numerator of the given fraction.

153. Since $\frac{39m^3n^5-52m^2n^3+26mn}{13mn} =$ $3m^2n^4-4mn^2+2$, we can say that $13mn$ and $3m^2n^4-4mn^2+2$ are ________ of $39m^3n^5-52m^2n^3+26mn$.

153. factors

154. If $-3p^2q$ is one factor of $27p^3q^2-126p^2q^5-3p^2q$, then the other factor is __________.

154. $\frac{27p^3q^2-126p^2q^5-3p^2q}{-3p^2q} =$

$\frac{27p^3q^2}{-3p^2q} - \frac{126p^2q^5}{-3p^2q} - \frac{3p^2q}{-3p^2q} =$

$-9pq+42q^4+1$

155. Check *your* result.

If your result was different from the correct one, note that it will not check.

To get the 1, we divided a number $-3p^2q$ by itself. What is the quotient when a number is divided by itself?

155. 1 *except* when 0 is divided by 0. Keep in mind that division by 0 is impossible.

The check should be
$-3p^2q(-9pq+42q^4+1) =$
$27p^3q^2-126p^2q^5-3p^2q$

156. $\frac{m^4}{m^4} = 1$ only if m __________.

156. m isn't 0.

Remember division by 0 isn't defined.

157. If $4x^2y^3$ is one factor of $64x^3y^3-100x^2y^6-4x^2y^3$, the other factor is ____________. Check your result.

157. $\frac{64x^3y^3}{4x^2y^3} - \frac{100x^2y^3}{4x^2y^3} - \frac{4x^2y^3}{4x^2y^3} =$ $16x-25y^3-1$.

Check: $4x^2y^3(16x-25y^3-1) =$ $64x^3y^3-100x^2y^3-4x^2y^3$

158. If $-6x^3y^2$ is one factor of $96xy^3+78x^5y^2-3x^4y^4$, the other factor is ____________. Check your result.

158. $\frac{96xy^3}{-6x^3y^2} + \frac{78x^5y^2}{-6x^3y^2} - \frac{3x^4y^4}{-6x^3y^2} =$

$-\frac{16y}{x^2} - 13x^2 = \frac{1}{2}xy^2$

Check:
$-6x^3y^2(-\frac{16y}{x^2} - -13x^2 + \frac{1}{2}xy^2) =$ $96xy^3+78x^5y^2-3x^4y^4$

159. If $4x^2$ is one factor of $56x^4y+64x^3-4x^2$, what is the other factor? What operation did you use to obtain this factor?

159. $14x^2y+16x-1$

divided $56x^4y+64x^3-4x^2$ by $4x^2$

160. If a-2 is one factor of $3a^2-5a-2$, how would you obtain the other factor?

160. Divide $3a^2-5a-2$ by a-2

161. When you are to divide a polynomial which consists of more than one term, the division is set up as a long division problem. Both polynomials must be arranged in the same order. That is, both polynomials must be in descending order or else both polynomials must be in ascending order.

Both $3a^2-5a-2$ and a-2 are in

____________ order.

161. descending

162. The division problem would look like this: $a-2 \overline{)3a^2-5a-2}$

Now divide the first terms of each polynomial.

In this case, divide $3a^2$ by a, so the first term in the quotient is

________.

162. 3a

163. The problem now is: $$\begin{array}{r} 3a \\ a-2\overline{)3a^2-5a-2} \end{array}$$

This is like long division in arithmetic. If we were to divide 542 by 36, we would estimate the quotient, $\begin{array}{r} 1 \\ 36\overline{)542} \end{array}$, and then multiply the partial quotient by the number we are dividing by and then find the difference of these numbers.

$$\begin{array}{r} 1 \\ 36\overline{)542} \\ \underline{36} \\ 18 \end{array}$$

We do exactly the same process when dividing polynomials.

In $\begin{array}{r} 3a \\ a-2\overline{)3a^2-5a-2} \end{array}$, what will we do with the da?

Our problem will then be ________?

163. Multiply 3a by a-2

$$\begin{array}{r} 3a \\ a-2\overline{)3a^2-5a-2} \\ \underline{3a^2-6a} \end{array}$$

164. The next thing to do is ________.

Do this.

164. subtract

$$\begin{array}{r} 3a \\ a-2\overline{)3a^2-5a-2} \\ \underline{3a^2-6a} \\ +a \end{array}$$

165. In $\begin{array}{r} 1 \\ 36\overline{)542} \\ \underline{36} \\ 18 \end{array}$, you next ________.

165. brought down the next number
or brought down the 2.

166. You do the same thing when dividing polynomials.

$$\begin{array}{r} 3a \\ a-2\overline{)3a^2-5a-2} \\ \underline{3a^2-6a} \\ a-2 \end{array}$$

You then repeated the above process until you had no more numbers to bring down.

Look at the above problem. You will now ____________.
(If necessary look at frames 161-165.)

166. divide a by a

167. The quotient is __________.

Complete the division.

167. 1

$$\begin{array}{r@{\,}l} & \underline{3a+1} \\ a-2 \,\big) & 3a^2-5a-2 \\ & \underline{3a^2-6a} \\ & \quad +a-2 \\ & \quad \underline{a-2} \end{array}$$

168. The remainder in this division is __________.

168. 0

169. Since the remainder is 0, we can say that $a-2$ is a factor of

$3a^2-5a-2$.

In order to divide $11a^2-2a-8+6a^3$ by $2a+5$, we must first __________.

169. arrange both polynomials in the same order. (See frame 160.)

170. Set up the division arranging both polynomials in descending order.

170. $2a+5 \,\big)\overline{6a^3+11a^2-2a-8}$

171. We first divide __________, and then __________.

Then we __________, and bring down the next number. This process is repeated until the division is completed.

171. $6a^3$ by $2a$
multiply $2a+5$ by this result
subtract

172. Do the division:

$2a+5 \,\big)\overline{6a^3+11a^2-2a-8}$

172.

$$\begin{array}{r@{\,}l} & \underline{3a^2-2a+4} \\ 2a+5 \,\big) & 6a^3+11a^2-2a-8 \\ & \underline{6a^3+15a^2} \\ & \quad -4a^2-2a \\ & \quad \underline{-4a^2-10a} \\ & \qquad\qquad 8a-8 \\ & \qquad\qquad \underline{8a+20} \\ & \qquad\qquad\quad -28 \end{array}$$

173. We write the remainder just as we did in arithmetic.

The remainder here is $-28$ and is written $\frac{-28}{2a+5}$.

In other words, $6a^3+11a^2-2a-8$ divided by $2a+5$ equals

$3a^2-2a+4+\frac{-28}{2a+5}$

Is $2a+5$ a factor of $6a^3+11a^2-2a-8$?

Give a reason.

173. No because there is a remainder. (See frame 168.)

174. How would you check this problem?

174. Multiply $2a+5$ by $3a^2-2a+4$ and then <u>add</u> $-28$ to the product.

This should equal

$6a^3+11a^2-2a-8$

175. Do the check.

175. $(2a+5)(3a^2-2a+4) =$

$6a^3+11a^2-2a+20$

This $+ -28 = 6a^3+11a^2-2a-8$ which is the required quantity.

176. Divide $3a^3-16-12a+4a^2$ by $3a+4$.

Check your result.

176. Arrange in order first.

$$\begin{array}{r} a^2 \quad -4 \\ 3a+4 \,/\, \overline{3a^3+4a^2-12a-16} \\ \underline{3a^3+4a^2} \qquad\qquad \\ -12a-16 \\ \underline{-12a-16} \end{array}$$

Check: $(a^2-4)(3a+4) =$

$3a^3+4a^2-12a-16$

177. When you found the difference between $3a^3+4a^2$ and the product of $a^2$ and $3a+4$, the difference was 0. In this case you have to bring down more than one number as you would in the case of dividing 3742 by 18.

$$\begin{array}{r} 2\phantom{42} \\ 18 \,/\, \overline{3742} \\ \underline{36}\phantom{42} \\ 14\phantom{2} \end{array}$$

Here 14 is not divisible by 18, so a 0 goes in the quotient and the next number is brought down.

$$\begin{array}{r} 20\phantom{2} \\ 18 \,/\, \overline{3742} \\ \underline{36}\phantom{42} \\ 142 \end{array}$$

Divide: $6a^2+23a+34-29a^2$ by $2a-5$

177. Arrange in order first.

$$\begin{array}{r} 3a^2-7a-6 \qquad \\ 2a-5 \,/\, \overline{6a^3-29a^2+23a+34} \\ \underline{6a^3-15a^2} \qquad\qquad\quad \\ -14a^2+23a \qquad \\ \underline{-14a^2+35a} \qquad \\ -12a+34 \\ \underline{-12a+30} \\ 4 \end{array}$$

178. Write the complete quotient of the last problem including the remainder.

Quotient is _______________.

178. $3a^2-7a-6+\frac{4}{2a-5}$

179. Is $2a-5$ a factor of $6a^3+23a+34-29a^2$? Why?

179. No because there is a remainder.

180. If a polynomial contains more than one variable, consider only <u>one</u> variable when arranging it in order.

For example, to arrange

$18p^3-32pq^2+16q^3-9p^2$ in descending order, choose one of the variables and arrange the polynomial in descending order for that variable.

Choosing the variable p and arranging the polynomial in descending order, we get

____________________.

180. $18p^3-9p^2q-32pq^2+16q^3$

181. Is $3p+4q$ a factor of

$18p^3-32pq^2+16q^3-9p^2q$?

181. Divide to find out. Be sure that both polynomials are arranged in the same order.

$$\begin{array}{r|l}
 & 6p^2 - 11pq + 4q^2 \\ \hline
3p+4q & 18p^3-9p^2q-32pq^2+16q^3 \\
 & \underline{18p^3+24p^2q} \\
 & \quad -33p^2q-32pq^2 \\
 & \quad \underline{-33p^2q-44pq^2} \\
 & \qquad 12pq^2+16q^3 \\
 & \qquad \underline{12pq^2+16q^3}
\end{array}$$

Yes it is a factor.

182. In the polynomial $a^3-4a+1$, the $a^2$ term is missing.

It is often necessary to consider missing terms when we are dividing.

The coefficient of any missing term is ______________.

182. 0

183. Write the polynomial $a^3-4a+1$ putting in the $a^2$ term.

183. $a^3+0\cdot a^2-4a+1$

184. Divide $a^3+0\cdot a^2-4a+1$ by $a+1$.

184.

$$\begin{array}{r|l}
 & a^2-a-3 \\ \hline
a+1 & a^3+0\cdot a^2-4a+1 \\
 & \underline{a^3+\ a^2} \\
 & \ -\ a^2-4a \\
 & \ \underline{-\ a^2-\ a} \\
 & \qquad -3a+1 \\
 & \qquad \underline{-3a-3} \\
 & \qquad\qquad 4
\end{array}$$

Quotient is $a^2-a-3+\frac{4}{a+1}$

185. You could divide $a^3+0\cdot a^2-4a+1$ by $a+1$ by arranging both polynomials in ascending order.

The division then would be

$$1+a \;\overline{)\;1-4a+0\cdot a^2+a^3}$$

Do this division.

185.

$$
\begin{array}{r l}
 & 1-5a+5a^2 \\
1+a \,/ & 1-4a+0\cdot a^2+a^3 \\
 & \underline{1+a} \\
 & -5a+0\cdot a^2 \\
 & \underline{-5a-5a^2} \\
 & 5a^2+a^3 \\
 & \underline{5a^2+5a^3} \\
 & -4a^3
\end{array}
$$

Quotient is $1-5a+5a^2+\frac{-4a^3}{1+a}$

186. In frames 183 and 184, we've divided the same polynomials and gotten different results. However, <u>both</u> will check.

Check the results for frames 183 and 184.

186. From frame 183.
$(a+1)(a^2-a-3) = a^3-4a-3$
Adding 4 to this we get $a^3-4a+1$ which is the required quantity.

From frame 184.
$(1+a)(1-5a+5a^2) = 1-4a+5a^3$
Adding this to $-4a^3$ we get the required quantity of $1-4a+a^3$

187. You will get different quotients if the polynomials are arranged in different orders when one polynomial is <u>not</u> a factor of the other polynomial.

Both results will check and thus both quotients are correct.

If you are to divide $5x^3-x^2+3$ by $x+2$, it is more desirable to arrange both polynomials in descending order, as then $5x^3$ divided by x gives a number which doesn't involve a fraction.

If ascending order is used, the problem is $2+x \,/\, 3+0\cdot x-x^2+5x^3$ and the first step involves dividing 3 by 2. The result involves a fraction which then must be multiplied by 2+x.

$$
\begin{array}{r l}
 & 1\frac{1}{2} \\
2+x \,/ & 3+0\cdot x-x^2+5x^3 \\
 & \underline{3+1\frac{1}{2}x} \\
 & -1\frac{1}{2}x-x^2
\end{array}
$$

You can see that the rest of the problem will involve fractions as now we must divide $-1\frac{1}{2}x$ by 2.

Do the division, first arranging in descending order and then arranging in ascending order. Check both results.

187.

$$x+2 \overline{\big)\, 5x^3 - x^2 + 0\cdot x + 3}$$ quotient: $5x^2-11x+22+\frac{-41}{x+2}$

Check:
$(x+2)(5x^2-11x+22) + -41 =$
$5x^3-x^2+3.$

$$2+x \overline{\big)\, 3+0\,x - x^2 + 5x^3}$$

Quotient here is

$\frac{3}{2} - \frac{3}{4}x - \frac{1}{8}x^2 + \frac{5\frac{1}{8}x^3}{2+x}$

Check:
$(x+2)(\frac{3}{2} - \frac{3}{4}x - \frac{1}{8}x^2) + 5\frac{1}{8}x^3 =$
$5x^3-x^2+3$

188. Divide:

(a) $12a^4-10a^3b+4a^2b^2-ab^3$ by $2a-b$

(b) $4p-2p^3+21-7p^2$ by $p+3$

(c) $2q^4-2p^2q^2-3p^4+10p^3q+5pq^3$ by $q^2-p^2+3pq$

(d) $64m^6-27q^3$ by $4m^2-3q$

188. (a) $6a^3-2a^2b+ab^2$

(b) $-2p^2-p+7$

(c) $2q^2-pq+3p^2$

(d) $16m^4+12m^2q+9q^2$

There are several missing terms here, and these all have coefficients of 0.

You can supply as many missing terms as you need.

189. Is $w-2$ a factor of $w^3-8$? Why?

189. Yes because when $w^3-8$ is divided by $w-2$, we get $w^2+2w+4$ and there isn't any remainder.

190. Write $w^3-8$ as a product.

190. $(w-2)(w^2+2w+4)$

Remember if $\frac{a}{b} = c$ then $bc = a$.

191. Determine whether the first polynomial is a factor of the second polynomial in each of the following.

(a) $3k+2$; $27k^3+8+36k+54k^2$

(b) $16a^4-12a^2b^2+9b^4$; $64a^6+27b^6$

(c) $2x-3$; $4x^3-17x+10$

(d) $2x-5y$; $18x^3-8xy^2-45x^2y+20y^3$

(e) $7-4c+3c^2$; $24c^2+18c-7+18c^4-14c^3$

191. (a) yes, quotient is $9k^2+12k+4$

(b) yes, quotient is $4a^2+3b^2$

(c) no, quotient is

$$2x^2+3x-4+\frac{-2}{2x-3}$$

(d) yes, quotient is $9x^2-4y^2$

(e) no, quotient is

$$-1+2c+5c^2+\frac{3c^4}{7-4c+3c^2}$$

Note: If the polynomials are both in ascending order in part e, you get the above result. If you used descending order, you have a different quotient. Check it to see if it is correct.

192. Write the <u>second</u> polynomials of parts a, b, and d of frame 190 as products.

192. (a) $(3k+2)(9k^2+12k+4)$

(b) $(4a^2+3b^2)(16a^4-12a^2b^2+9b^4)$

(d) $(2x-5y)(9x^2-4y^2)$

193. Perform the indicated operations and simplify.

(a) $(2x+5)^2-(2x-5)^2$

(b) $(3a^2-4a+8)(2a-1)-(a^3-2a+5)(a+2)$

(c) $(3y-8)^2 - \dfrac{y^4+y^2+1}{y^2+y+1}$

193. (a) $(4x^2+20x+25)-(4x^2-20+25) = 40x$

(b) $(6a^3-11a^2+20a-8) - (a^4+2a^3-2a^2+a+10) = -a^4+4a^3-9a^2+19a-18$

(c) $(9y^2-48y+64)-(y^2-y+1) = 8y^2-47y+63$

Note: $\dfrac{y^4+y^2+1}{y^2+y+1}$ must be done by long division

194. Remember -- if the result of an operation is a polynomial which contains more than one term and this result is then to be subtracted from another polynomial, enclose this result in parentheses so that you will remember to subtract the entire quantity.

For example:

$(2x+5)^2-(2x-5)^2 = 4x^2+20x+25-(4x^2-20x+25)$

or $4x^2+20x+25\underline{-4x^2+20x-25}$

It doesn't equal

$4x^2+20x+25-4x^2-20x+25$

Perform the indicated operations and simplify.

(a) $(-x^3y^4)^3$

(b) $-3m^4y^5(2mn^4)^3$

(c) $3p^3q(-4p^2-2pq^3+\frac{1}{2}q^5)$

(d) $(4p-3q)^2$

(e) $3(b^3-5b^2+7b-1)-(2b^2-b)(b+4)$

(f) $\frac{56m^4p^3-7m^2p+98m^7p^6}{-7m^2p}$

(g) $\frac{15x^3-42x+8+7x^2}{3x^2+2x-8}$

194. (a) $-x^9y^{12}$

(b) $-3m^4y^5(8m^3n^{12}) = 24m^7y^5n^{12}$

(c) $-12p^5q-6p^4q^4+\frac{3}{2}p^3q^6$

(d) $16p^2-24pq+9q^2$

(e) $3b^3-15b^2+21b-3-(2b^3+7b^2-4b) =$

$b^3-22b^2+25b-3$

(f) $-8m^2p^2+1-14m^5p^5$

(g) You need to use long division here. Be sure to arrange in order first.

Quotient is $5x-1$.

## Chapter 4 - Factoring

In this chapter, we shall learn to factor certain kinds of polynomials. The problem of factoring is to write a polynomial as the product of polynomials.

1. In Chapter 3, frame 154, you were told that $3p^2q$ was one factor of

   $27p^3q^2-126p^2q^5-3p^2q$. How would you find the other factor?

1. Divide $27p^3q^2-126p^2q^5-3p^2q$ by $-3p^2q$

2. If polynomials can be expressed as the product of two or more factors, then factors are concerned with

   __________________.
   what operation

2. multiplication

3. If $-3p^2q$ is one factor of

   $27p^3q^2-126p^2q^5-3p^2q$, the other

   factor is ____________.

3. $-9pq+42q^3+1$

4. $27p^3q^2-126p^2q^5-3p^2q$ expressed as a

   a product is ____________.

4. $-3p^2q(-9pq+42q^3+1)$

5. If $4x$ is one factor of $4x^2-8x$, express $4x^2-8x$ as a product.

   $4x^2-8x =$ ________________

5. $4x(x-2)$

6. $4x$ is called a <u>common factor</u> of $4x^2-8x$. Each term in $4x^2-8x$ can be divided by $4x$ and no fractions are obtained in the result. $4x$ is called the <u>largest common factor</u> of $4x^2-8x$. If a polynomial has a common factor, the polynomial can be expressed as the product of a monomial and another polynomial. If the monomial is the <u>largest common factor</u>, then this second polynomial is irreducible to a monomial multiplied by another polynomial.

   What is the largest common factor of $9y^2-6y$?

6. $3y$

7. Express $9y^2-6y$ as a product.

7. $3y(3y-2)$

8. Since $3y-2$ is not reducible to the product of a monomial and a polynomial, we say $3y$ is the <u>largest common factor</u> of $9y^2-6y$.

What is the largest common factor of $27x^2y-54xy$?

8. $27xy$

9. Express $27x^2y-54xy$ in factored form.

9. $27xy(x-2)$

10. Note that when $27xy(x-2)$ is expanded, we get $27x^2y-54xy$ or the same thing we started with. What is the largest common factor of

$-42p^2q^3+84p^5q^2+49pq^2$ ?

10. Either $7pq^2$ or $-7pq^2$

11. Factor $-42p^2q^3+84p^5q^2+49pq^2$ using $7pq^2$ as one of the factors.

11. $7pq^2(-6pq+12p^4+7)$

12. Factor $27m^3-54m^5+63m^2$.

12. $9m^2(3m-6m^3+7)$
or
$-9m^2(-3m+6m^3-7)$

13. Note that in the second factor there are the same number of terms as in the original polynomial.

$27m^3-54m^5+63m^2$ has <u>3</u> terms. The common factor of $9m^2$ must divide each of those terms and so we get <u>3</u> terms in the other factor.

Factor: (a) $-16m^5+64m^3-48m^4n$

(b) $14xy^3-70x^3y^5+28x^2y^4$

13. (a) $16m^3(-m^2+4-3m)$ or

$-16m^3(m^2-4+3m)$
If only one possible set of factors is given, it doesn't mean that your result is incorrect. Multiply and check to see if your result is also correct.

(b) $14xy^3(1-5x^2y^2+2xy)$
Be sure that you have <u>3</u> terms in the one factor.

14. To factor $3(a-b)+a(a-b)$ we use the same technique.

Is there a common factor for both terms in $3(a-b)+a(a-b)$? Remember () are used to consider a group of terms as one number.

What is it?

14. yes there is a factor which divides both terms.

It is $(a-b)$.

15. Then a common factor of $3(a-b)+a(a-b)$ is $(a-b)$. If there is no other number which divides both terms this is the largest common factor.

To get the other factor, we must

__________________.
what operation

15. divide $3(a-b)+a(a-b)$ by $(a-b)$.

16. $\frac{3(a-b)+a(a-b)}{(a-b)}$ gives what other factor?

16. $\frac{3(a-b)}{(a-b)} + \frac{a(a-b)}{(a-b)} = 3 + a$

So $(3+a)$ is the other factor.

17. Expressed as a product, $3(a-b)+a(a-b) = (a-b)(3+a)$.

How would you check to see if this is correct? Be specific.

17. See if $(a-b)(3+a)$ equals $3(a-b)+a(a-b)$.
or
multiply $(a-b)$ by $(3+a)$ and multiply $3(a-b)+a(a-b)$ and see if the results are the same.

18. What is the largest common factor of $2x(x-4y)-3y(x-4y)$?

18. $(x-4y)$

19. Factor $2x(x-4y)-3y(x-4y)$.

19. $(x-4y)(2x-3y)$

20. If you wrote $(x-4y)2x-3y$, you are <u>wrong</u>.

Consider $(x-4y)2x-3y$, how much is $(x-4y)$ to be multiplied by?

20. only by $2x$

21. In $(x-4y)(2x-3y)$, how much is $(x-4y)$ to be multiplied by?

21. by $(2x-3y)$

22. Obviously then $(x-4y)2x-3y$ and $(x-4y)(2x-3y)$ <u>can't</u> equal the same quantity.
Are both of these quantities expressed as factors?
Which one(s) is expressed in factored form?

22. No.
Only $(x-4y)(2x-3y)$ is in factored form.

23. Factor $2x(3x-y)+4y(3x-y)-3z(3x-y)$.

23. Common factor is $(3x-y)$.
$(3x-y)(2x+4y-3z)$

24. Factor $3p(p+q)+(p+q)$.

24. The common factor here is $(p+q)$.
$(p+q)(3p+1)$
Remember $+(p+q) = +1(p+q)$.

25. Factor $3x(a-b-c)+9y(a-b-c)$. Be sure that there are no common factors in any of the factors of your result.

25. $3(a-b-c)(x+3y)$

If you had $(a-b-c)(3x+9y)$, note that $3x+9y$ has a common factor of 3 and you should factor again.

If you had $(a-b-c)3(x+3y)$, is your result the same as the above?
Why?

26. Factor completely.

(a) $27x^3y^2(x^3-y^2)+9x^2y^5(x^3-y^2)-3x^2y(x^3-y^2)$

(b) $4x^2(x^2+3x)+8xy(x^2+3x)$

26. (a) $3x^2y(x^3-y^2)(9xy+3y^4-1)$

If you had $(x^3-y^2)(27x^3y^2+9x^2y^5-3x^2y)$ you need to factor again.

(b) $(x^2+3x)(4x^2+8xy) =$
$x(x+3)4x(x+2y) =$
$4x^2(x+3)(x+2y)$

27. In the polynomial $ac+bc+ad+bd$, there is no number distributed over the entire sum. However, notice that c is distributed over the first two terms and d is distributed over the last two terms.

Using parentheses to group the first two terms and the last two terms and <u>keeping the same value</u> we have:

$ac+bc+ad+bd = (ac+bc) + (ad+bd)$.

This can be rewritten as $c(a+b)+d(a+b)$. The terms here have a common factor of ________.

Factor $c(a+b)+d(a+b)$.

27. common factor is $(a+b)$.

$c(a+b)+d(a+b) = (a+b)(c+d)$.

28. So even if we don't have a number distributed over the entire quantity, we can sometimes factor the polynomial providing we group the terms so that the resulting terms have a common factor.

Group the expression $ab-ac+xb-cx$ so that each group has a common factor and the value is equal to that of the given quantity.

28. $(ab-ac)+(xb-cx)$
or
$(ab+xb)+(-ac-cx)$

29. If you didn't look at the answer to the last frame look now. Notice that there are several ways of

or
$(ab+xb)-(ac+cx)$

grouping the terms so that each group has a common factor and the entire quantity equals the original one.

Factor: $ab-ac+xb-cx = (ab-ac)+(bx-cx) =$ __________ = __________

29. $a(b-c)+x(b-c) = (b-c)(a+x)$

30. Could we have written $(a+x)(b-c)$ instead of $(b-c)(a+x)$? Why?

30. yes because multiplication is commutative.

31. Suppose we had group $ab-ac+xb-cx$ as $(ab+xb)+(-ac-cx)$.

The first group of terms has a common factor of b and consider c as the common factor of the second group.
We then would have:
$(ab+xb)+(-ac-cx) = b(a+x)+c(-a-x)$.

What is the common factor in $b(a+x)+c(-a-x)$?

31. there is no common factor.

32. Thus, we cannot factor $b(a+x)+c(-a-x)$ as there is no common factor.

However, we could consider $-c$ as the common factor of the second group and then we have:

$(ab+xb)+(-ac-cx) = b(a+x) - c(a+x)$.

This has a common factor of $(a+x)$ so we may write $ab-ac+xb-cx = b(a+x)-c(a+x) = (a+x)(b-c)$.

Factor $p^2-pq+qr-pr$. Show all steps.

32. $(p^2-pq)+(qr-pr) = p(p-q)-r(-q+p) = (p-q)(p-r)$

Note $(p-q)$ and $(-q+p)$ are the same.

33. There are two conditions which must always be met. The quantities must be equal and there must be a common factor.

You were given $p^2-pq+qr-pr$. Is $p(p-q)+r(q-p)$ equal to this?

33. yes

34. Therefore the first condition is met. Is there a common factor in $p(p-q)+r(q-p)$?

34. no

35. Then the second condition is <u>not</u> met.
Is $p(p-q)+r(q-p)$ in factored form? Why?

35. No.
Factors are connected by multiplication.

36. Factor $3a^2-3ab-4ac+4bc$.

36. $(3a^2-3ab)+(-4ac+4bc) = 3a(a-b)-4c(a-b) = (a-b)(3a-4c)$

37. If you had $3a(a-b)+4c(-a+b)$, this equals the original but <u>can't be factored</u> as there is no common factor.

Factor $2xy-4xz-3wy+6wz$.

37. $(2xy-4wz)+(-3wy+6wz) = 2x(y-2z)-3w(y-2z) = (y-2z)(2x-3w)$

38. Suppose you had written that $2xy-4xz-3wy+6wz = (2xy-4xz)-(3wy+6wz)$.
What is wrong with this?

38. $(2xy-4xz)-(3wy+6wz)$ <u>doesn't</u> equal the other quantity.

$(2xy-4wz)-(3wy+6wz)$ equals $2xy-4wz-3wy-\underline{6wz}$

39. Factor the following. Be sure that you remove <u>all</u> the common factors.

(a) $3p+3q+ap+aq$

(b) $4x^2y+8xy^2+3abx+6aby$

(c) $2x^3-2x^2y+5xy-5y^2$

(d) $3a^2-3ab-4az+4bz$

(e) $4pr+2p^2-6qr+3pq$

(f) $6p^2r-3p^3+12pqr-6p^2q$

(g) $4p^3-8p^2+p-2$

(h) $9x^3-27x^2-x+3$

(i) $2p^3+2r-4p^2-pr$

39. (a) $3(p+q)+a(p+q)=(p+q)(3+a)$
(b) $4xy(x+2y)+3ab(x+2y) = (x+2y)(4xy+3ab)$
(c) $2x^2(x-y)+5y(x-y)= (x-y)(2x^2+5y)$
(d) $3a(a-b)-4z(a-b)=(a-b)(3a-4z)$
(e) $2p(2r+p)-3q(2r-p)$
There is <u>no</u> common factor here so try another grouping of terms. It happens in this case, that <u>no</u> grouping works, so this <u>cannot</u> be factored.

40. If $4x^2$ is one factor of $56x^4y+64x^3-4x^2$, the other factor is ______________.

What operation did you use to obtain this factor?

(f) $2p^2(2r-p)+6pq(2r-p) =$
$(2r-p)(2p^2+6pq) = (2r-p)2p$
$(p+3q)$
or $2p(2r-p)(p+3q)$

(g) $4p^2(p-2)+1(p-2)=(p-2)(4p^2+1)$

(h) $9x^2(x-3)-1(x-3)=(x-3)(9x^2-1)$

(i) In order to group and get terms having a common factor, you have to consider this as

$(2p^3-4p^2)+(-pr+2r)$ or as
$(2p^3-pr)+(-4p^2+2r)$.

$(2p^3-4p^2)+(-pr+2r) =$
$2p^2(p-2)-r(p-2) =$
$(p-2)(2p^2-r)$

40. $14x^2y+16x-1$

divided $56x^4y+64x^3-4x^2$ by $4x^2$

41. If a-2 is one factor of $3a^2-5a-2$, how would you obtain the other factor?

41. divide $3a^2-5a-2$ by a-2

42. If a-2 is one factor of $3a^2-5a-2$, what is the other factor?

42. 3a+1

43. How would you check to see if this is correct?

43. multiply 3a+1 by a-2

44. We multiply 3a+1 by a-2 using the distributive law.

$(3a+1)(a-2) = 3a(a-2)+1(a-2)$

Each factor contains ______ terms?
(how many)

44. 2

45. A polynomial containing two terms is called a binomial.

$3xy-4x^2$ ______ a binomial.

$2a^3-3ab^2+7b^3$ ______ a binomial.

45. is

is not

46. The product of two binomials occurs very frequently in algebra and therefore it is useful to use a shortcut in order to get the product more quickly. However, remember if you forget the shortcut, you can always use the distributive law.

In $(a-b)(2a+b) = 2a^2-ab-b^2$, notice that the first term in the product, $2a^2$, is the product of the first terms in each binomial.

Next notice that the last term in the product, $-b^2$, is the product of the last terms in each binomial.

In $(3x-4y)(2a-5y)$, the first term

in the product is ________ and

the last term in the product is

________.

46. $6ax$

$+20y^2$

47. Expanding $(3x-4y)(2a-5y)$ by the distributive law, we get $3x(2a-5y)-4y(2a-5y) = 6ax-15xy-8ay+20y^2$.

So far by the shortcut, we have

$(3x-4y)(2a-5y)=6ax$ ________ $+20y^2$

We are missing $-15xy-8ay$. Notice that $-15xy$ is the product of the first term of the first binomial and the second term of the second binomial.

Also notice that $-8ay$ is the product of the last term of the first binomial and the first term of the second binomial.

Thus to multiply $(2a+3b)(5a-4d)$, by the shortcut, we would first multiply $2a$ by $5a$ getting $10a^2$.

Then we multiply the first term of the first binomial by the last term of the second binomial getting ___.

Then we multiply the last terms of each binomial getting $-12bd$.

47. $-8ad$

$+15ab$

48. Thus $(2a+3b)(5a-4d) = 10a^2-8ad+15ab-12bd$.

If we expanded $(2a+3b)(5a-4d)$ by the distributive law we would get the same result.

$(-2c+3d)(5a-b) =$ ______________

48. $-10ac+2bc+15ad-3bd$

49. The product of the first term of the first binomial and the last term of the second binomial and the product of the last term of the first binomial and the first term of the second binomial are referred to as cross products.

$(x^2-3y^2)(4x^2+y^2) =$ ______________

49. $4x^4+x^2y^2-12x^2y^2-3y^4$

We now have some similar terms which we can combine getting

$4x^4-11x^2y^2-3y^4$

50. When the binomials have similar terms, we shall always have similar terms in the product which we can combine.

Whether we multiply using the distributive law or by the short-cut, notice that each term of the first binomial is multiplied by each term of the second binomial.

Remember this shortcut only applies when multiplying two binomials.

Find the following products.

(a) $(2p-5q)(3p+3a)$

(b) $(7m-4)(5m+2)$

(c) $(4p+9)(3p+6)$

(d) $(5x-3y)^2$

(e) $(a-b)(a+b)$

(f) $(a-b)^2$

(g) $(3m-4n)(3m+4n)$

(h) $(5x^2+6y)(5x^2-6y)$

50. (a) $6p^2-15pq+6ap-15aq$
(b) $35m^2-20m+14m-8 = 35m^2-6m-8$
(c) $12p^2+24p+27p+54=12p^2+51p+54$
(d) Write this as $(5x-3y)(5x-3y)$ product is $25x^2-30xy+9y^2$
(e) $a^2-ab+ab-b^2 = a^2-b^2$
(f) Write this as $(a-b)(a-b)$. product is $a^2-2ab+b^2$
(g) $9m^2+12mn-12mn-16n^2 = 9m^2-16n^2$

51. In parts e, g and h of the last frame, the product contained only two terms. That is the two cross products had the same absolute values. Since these terms had different signs, their sum was 0.

This will always happen if the two binomials are the sum and difference of the same two numbers.

(h) $25x^4-30x^2y+30x^2y-36y^2 = 25x^4-36y^2$

For example, $(a-b)(a+b)$ is the difference of a and b multiplied by the sum of a and b, so the product will be $a^2-b^2$.

(a) $(4m+5p)(4m-5p) =$ __________

(b) $(3a+2b)(3a-b) =$ __________

51. (a) $16m^2-25p^2$

(b) $9a^2+3ab-2b^2$
Be careful here. This isn't the sum of two numbers multiplied by the difference of the same two numbers.

52. Let us consider the product of the sum and difference of the number 4m and 5p.

$(4m+5p)(4m-5p) = 16m^2-25p^2$

$16m^2$ is the square of 4m and $25p^2$ is the square of 5p. In other words, the first term of the product is the square of the first term of either binomial and the last term of the product is the square of the last term of either binomial. The product is the difference of these squares.

$(3m^3-5n^2)(3m^3+5n^2) =$ __________

52. $9m^6-25n^4$

53. Complete the following.

(a) $(4p^6+3q^5)(4p^6-3q^5)$

(b) $(\frac{1}{2}x^2-\frac{1}{2}y^2)(\frac{1}{2}x^2+\frac{1}{2}y^2)$

(c) $(3p+q^3)(3p-4q^3)$

(d) $(3b-4c)(3b-4c)$

53. (a) $16p^{12}-9q^{10}$

(b) $\frac{1}{4}x^4-\frac{1}{4}x^4$

(c) $9p^2-9pq^3-4q^6$
Be careful here. This isn't the sum and difference of the same two numbers.

(d) $9b^2-24bc+16c^2$
Be careful here too.

54. Since $(3m^3-5n^2)(3m^3+5n^2) = 9m^6-25n^4$, we can say that $3m^3-5n^2$ and $3m^3+5n^2$ are __________ of $9m^6-25n^4$.

54. factors

55. So, if we are asked to factor $9m^6-25n^4$, or to write $9m^6-25n^4$ as a product, we can write

$$9m^6-25n^4 = (3m^3-5n^2)(3m^3+5n^2).$$

The expression $9m^6-25n^4$ is the difference of two squares.
The factors of this expression are the sum and difference of the square roots.
The only thing now is to be able to determine the square root of a number.

The square root of a number is

______________________________.

55. a number which when

56. In symbols, $\sqrt{N} = a$ only if $a(a)=N$.

Find the square root of:

(a) 64 (b) 16 (c) $m^8$

(d) $25p^6$ (e) $9p^2q^{10}r^6$

56. (a) 8
(b) 4
(c) $m^4$
(d) $5p^3$
(e) $3pq^5r^3$

57. Sometimes you will wish to find the square root of a number such as 2304. There is an arithmetic process by which this can be done. However, we will factor the number and then make use of the definition given in frame 56.

Factor 2304. Choose any number which will divide 2304. (A number which divides another gives a quotient with no remainder.)

2304 is an even number so 2 must divide it.

$2304 = 2 \cdot 1152$

Continue this process and factor 2304 until you get a number which you recognize as a perfect square.

$2304 = 2 \cdot$ ______________

57. $2304 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 144$
or
$2304 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 36$

58. The definition in frame 56 states $\sqrt{N} = a$ if $a \cdot a = N$. Therefore, in $2 \cdot 2 \cdot 2 \cdot 2 \cdot 144$, $2 \cdot 2$ must represent a perfect square whose square root is 2.

So, $\sqrt{(2 \cdot 2)(2 \cdot 2)(144)} = 2 \cdot 2 \cdot 12$ is 48

Use this to find $\sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 36}$

58. $\sqrt{(2\cdot2)(2\cdot2)(2\cdot2)(36)}$ =

$2\cdot2\cdot2\cdot6 = 48$

59. Find the square root of:

(a) 324 (b) 784

(c) 1936 (d) 3969

59. (a) $\sqrt{2\cdot2\cdot81} = 2\cdot9 = 18$

(b) $\sqrt{2\cdot2\cdot2\cdot2\cdot49} = 2\cdot2\cdot7 = 28$

(c) $\sqrt{2\cdot2\cdot2\cdot2\cdot121} = 2\cdot2\cdot11 = 44$

(d) $\sqrt{3\cdot3\cdot3\cdot3\cdot49} = 3\cdot3\cdot7$ 63

60. Let us once more consider factoring polynomials such as $25x^4-64y^8$.

In frame 52, we said that if we multiplied together the sum of two numbers and the difference of the same two numbers the product would consist of the difference of the squares of these two numbers.

For example, $(x-3)(x+3) = x^2-9$

Thus we can say that if we have a polynomial which is the difference of two squares, then its factors are the sum and difference of the square roots.

For example, the factors of $x^2-16$ are $(x-4)$ and $(x+4)$, where x is the square root of $x^2$ and 4 is the square root of 16.

To factor $25x^4-64y^8$, we must first determine if both of these terms are squares.

$\sqrt{25x^4}$ = ____________. Why?

$\sqrt{64y^8}$ = ____________. Why?

60. $5x^2$ because $5x^2(5x^2) = 25x^4$

$8y^4$ because $8y^4(8y^4) = 64y^8$

61. Then the factors of $25x^4-64y^8$ will be the sum and difference of the square roots.

Factor $25x^4-64y^8$.

61. $(5x^2+8y^4)(5x^2-8y^4)$

62. Factor each of the following. Be sure and check to see if each of the terms is a square.

(a) $w^2-49$

(b) $4x^2-81y^2$

(c) $9p^6-16q^{12}$

(d) $m^6-12$

(e) $x^2y^6-9$

(f) $9x^2+4$

62. (a) $(w-7)(w+7)$
(b) $(2x+9y)(2x-9y)$
(c) $3p^3+4q^6)(3p^3-4q^6)$
(d) This can't be factored as it isn't the difference of two squares. 12 isn't a square.
(e) $(xy^3-3)(xy^3+3)$
(f) This can't be factored. Both terms are squares but we don't have the difference of them.

Note: Some of these non-factorable polynomials may be factorable at a later time. However, they are not factorable by the rules we've had.

63. Factor $16a^{10}-256b^8$.

63. Factoring this as the difference of two squares, we get

$(4a^5+16b^4)(4a^5-16b^4)$

64. Let us consider the factor $4a^5-16b^4$. Notice that there is a common factor here. Or that both of these terms may be divided by the same number.

The common factor here is ______.

64. 4

65. Then $4a^5-16b^4$ can be written as

$4(a^5-16b^4)$.

Is there a common factor in

$4a^5+16b^4$?

If so, what is it?

65. yes.

4

66. Then $(4a^5+16b^4)(4a^5-16b^4) =$

$4(a^5+4b^4)4(a^5-4b^4)$.

Putting the numerical factors together, we get

$16(a^5+4b^4)(a^5-4b^4)$.

We shall factor polynomials completely which means that none of the polynomial factors will be factorable by any of the methods that we know.

So far, we know how to factor what kinds of polynomials?

66. ones where there is a common factor and polynomials which are the difference of two squares.

67. In $16a^{10}-256b^8$, we could have removed the common factor first. Both of these terms are divisible by 16. So

$$16a^{10}-256b^8 = 16(a^{10}-16b^8).$$

Then $a^{10}-16b^8$ can be factored as the difference of two squares. We would obtain exactly the same set of factors although the factors might be arranged in a different order.

Factor $16p^4-4q^6$.

67. Removing the common factor first we get $4(4p^4-q^6)$.

Factoring $4p^4-q^6$ as the difference of two squares,

$$16p^4-4q^6 = 4(2p^2+q^3)(2p^2-q^3).$$

This could have been factored as the difference of two squares getting

$(4p^2-2q^3)(4p^2+2q^3)$. Each of these factors has a common factor and when we remove this we get the same result as above.

68. In some cases we must remove the common factor first.

Consider $3p^2-75$. These terms are <u>not</u> squares, so we must see if we can factor by any other means. Notice that there is a common factor.

Remove the common factor and then factor again if possible. Repeat this process until none of the factors can be factored again.

$3p^2-75 =$ __________ $=$ __________

68. $3(p^2-25) = 3(p-5)(p+5)$

69. Factor completely.

(a) $64-49p^2q^2$

(b) $4a^6-24a^3$

(c) $4x^2+y^2$

(d) $3x^5y-48xy^3$

(e) $189m^8n^3-105m^6n$

(f) $45a^4b^5-63a^3b^7c+9a^2b^3$

(g) $4y(p-q)+2(p-q)-x(p-q)$

(h) $2x^3-8x-3x^2y+12y$

(i) $x^3-2x^2-4x+8$

69. (a) $(8-7pq)(8+7pq)$
(b) $4a^3(a^3-6)$
(c) This can't be factored.
(d) $3xy(x^2+4y)(x^2-4y)$
(e) $21m^6n(9m^2n^2-5)$

(f) $9a^2b^3(5a^2b^2-7ab^4c+1)$
(g) $(p-q)(4y+2-x)$
(h) $2x(x^2-4)-3y(x^2-4) =$
$(2x-3y)(x^2-4) =$
$(2x-3y)(x-2)(x+2)$
(i) $x^2(x-2)-4(x-2) =$
$(x-2)(x^2-4) = (x-2)(x+2)(x-2)$

70. We obtained the factors $(x-2)(x+2)(x-2)$ as the result to part i of the last frame.

Can this be written as $(x+2)(x-2)^2$?

Is $(x+2)(x-2)^2$ in factored form?

70. yes

yes

71. Is $(x-3)^2-p^2$ the difference of two squares?

To help you decide you need to evaluate

$\sqrt{(x-3)^2}$ and $\sqrt{p^2}$

71. yes, it is the difference of two squares.

$\sqrt{(x-3)^2} = x-3$ because $(x-3)(x-3) = (x-3)^2$

$\sqrt{p^2} = p$ because $p \cdot p = p^2$

72. Then the factors of $(x-3)^2-p^2$ are the sum and the difference of the square roots.

The sum of the square roots is ____________.

The difference of the square roots is ____________.

72. sum is $(x-3)+p$
difference is $(x-3)-p$

73. Therefore, $(x-3)^2-p^2 =$ __________ when expressed in factored form.

73. $[(x-3)+p][(x-3)-p]$

74. Factor $x^2-(y+1)^2$.

74. $\sqrt{x^2} = x$

$\sqrt{(y+1)^2} = (y+1)$

sum of square roots is $x+(y+1)$
difference of square roots is $x-(y+1)$

$x^2-(y-1)^2 = [x+(y+1)][x-(y+1)]$

75. Factor $9b^6-(2a-7)^2$.

75. sum of square roots is $3b^3+(2a-7)$.
difference of square roots is $3b^3-(2a-7)$.

$9b^6-(2a-7)^2 = [3b^3-(2a-7)]$

$[3b^3+(2a-7)]$

76. Factor $16m^4-(5+3n)^2$.

76. $[4m^2+(5+3n)][4m^2-(5+3n)]$

77. When parentheses and brackets are used, it is sometimes necessary to remove whichever set is innermost.
Write the answer to the last frame using only the brackets.

77. $[4m^2+5+3n][4m^2-5-3n]$

78. When we factored $16m^4-(5+3n)^2$, we wrote the sum and difference of the square roots. Note that when we did this, we considered the square root of $(5+3n)^2$ as $(5+3n)$ <u>not</u> $5+3n$.
Whenever the polynomial consists of more than one term and it is to be connected to another number by one of the operations addition, subtraction, multiplication or division, we shall treat this polynomial as one number and enclose it in parentheses. If the result then contains more than one symbol of grouping, we generally write the result so it contains at most one symbol of grouping.

Find the factors of $25b^8-(5a^2-10)^2$ and write the result so it contains at most one symbol of grouping.

78. $(5b^4+[5a^2-10])(5b^4-[5a^2-10])=$

$(5b^4+5a^2-10)(5b^4-5a^2+10)$

79. Look at the factors in the answer to the last frame.

Can either or both of these be factored again? If further factoring can be done, how would you proceed?

79. Both of these factors can be factored again.
Both have a common factor of 5.

80. Factor $25b^8-(5a^2-10)^2$ completely.

80. $(5b^4+5a^2-10)(5b^4-5a^2+10) =$

$5(b^4+a^2-2)5(b^4-a^2+2) =$

$25(b^4+a^2-2)(b^4-a^2+2)$

81. When there are several numerical factors, we usually multiply them so that the final result has only one numerical factor. This is not absolutely necessary, however.

Factor the following completely. Use only one symbol of grouping in the final answer.

(a) $x^8-81$

(b) $(2p-5)^2-36$

(c) $144-(3p+6)^2$

(d) $(x-3)^2-(y+4)^2$

(e) $(2y+7)^2-(3y+8)^2$

(f) $(3p-1)^2+36$

(g) $4x^2(y^2-1)-9(y^2-1)$

(h) $p^6-9p^4-16p^2+144$

(i) $x^9-25$

81. (a) $(x^4+9)(x^2+3)(x^2-3)$
(b) $(2p-5)-6\ (2p-5+6) =$
$(2p-5-6)(2p-5+6) =$
$(2p-11)(2p+1)$
(c) $(12+3p+6)(12-3p+6) =$
$(12+3p+6)(12-3p-6) =$
$(18+3p)(6-3p)$
There is a common factor of 3 in both factors.
$(18+3p)(6-3p)=9(6+p)(2-p)$
(d) $(x-3)+(y+4)\ (x-3)(y+4) =$
$x-3+y+4\ x-3-y-4 =$
$(x+y+1)(x-y-7)$
(e) $(2y+7)+(3y+8)\ (2y+7)-(3y+8)=$
$2y+7+3y+8\ \ 2y+7-3y-8 =$
$(5y+15)(-y-1)$
The factor of $5y+15$ has a common factor of 5.
$(5y+15)(-y-1)=5(y+3)(-y-1)$
(f) This can't be factored as it isn't the difference of two squares.
(g) This isn't the difference of two squares, but it can be factored as the two terms have a common factor of $(y^2-1)$.

82. $x^3-8$ is <u>not</u> the difference of two squares.

$x^3-8$ is the difference of two cubes.

In order to determine whether a number is a cube, we must first have the definition of a cube root.

The cube root of N (written $\sqrt[3]{N}$) equals a only if $a \cdot a \cdot a = N$

What is the cube root of 8? Why?

$\sqrt[3]{x^3}$ = ______ because ____________.

$(y^2-1)(4x^2-9)$.
Both of these factors can be factored as the difference of squares. Don't forget that the square root of 1 is 1.

$(y^2-1)(4x^2-9) =$
$(y+1)(y-1)(2x-3)(2x+3)$

(h) $p^4(p^2-9)-16(p^2-9) =$
$(p^2-9)(p^4-16) =$
$(p-3)(p+3)(p^2+4)(p-2)(p+2)$

(i) This can't be factored. $x^9$ is not a square. Don't forget you are dealing with $x^9$ not 9.

82. 2 because $2 \cdot 2 \cdot 2 = 8$

x because $x \cdot x \cdot x = x^3$

83. We shall use much the same method to find cube roots as we did to find square roots.
First factor the number and then use the definition.

To find the cube root of 64, first factor the number. You might say $64 = 8 \cdot 8$. If you recognize that 8 is a cube, you can take the cube root of 64.

$\sqrt[3]{64} = \sqrt[3]{8 \cdot 8} = 2 \cdot 2 = 4$

You might not recognize 8 as being a cube. In that case continue factoring until you have numbers whose only integer factors are the number and 1.

Express 64 as the product of integers whose only integer factors are itself and 1.

83. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

84. Since by definition, $\sqrt[3]{N} = a$ only if $a \cdot a \cdot a = N$, $2 \cdot 2 \cdot 2$ must be a cube.

$\sqrt[3]{(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)} = 2 \cdot 2 = 4$.

Numbers whose only integer factors are itself and 1 are called <u>prime numbers</u>.

In frame 84 then we expressed 64 as the product of prime numbers.

Express each of the following as the product of prime numbers.

(a) 42 (b) 63 (c) 56

84. (a) $7 \cdot 2 \cdot 3$
(b) $3 \cdot 7 \cdot 3$
(c) $2 \cdot 2 \cdot 7 \cdot 2$

The order of the factors is immaterial.

85. To find the $\sqrt[3]{216}$, first ______.

85. factor 216

86. The question might arise as to how we factor this number. Certainly $216 = 2 \cdot 108$, and 2 and 108 are factors of 216.
However, we must consider why we wished to factor 216. Our original problem was to find the cube root of 216.
Whenever we are to find the cube root of a number, we factor it until we have prime factors or until one or more of the factors is a number you recognize as a cube.

Factor 216 so that we may then find the cube root of it.

86. $2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3$
or
$8 \cdot 27$
or
$8 \cdot 3 \cdot 3 \cdot 3$

87. Now find the cube root of your set of factors.

87. $\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3} = 2 \cdot 3 = 6$

$\sqrt[3]{8 \cdot 27} = 2 \cdot 3 = 6$

88. $\sqrt[3]{729}$ = __________ = __________

88. $\sqrt[3]{(3 \cdot 3 \cdot 3)27} = 3 \cdot 3 = 9$

89. $\sqrt[3]{3375}$ = __________ = __________

89. $\sqrt[3]{(5 \cdot 5 \cdot 5)(3 \cdot 3 \cdot 3)} = 5 \cdot 3 = 15$

90. (a) $\sqrt[3]{1728}$ = __________

(b) $\sqrt[3]{4096}$ = __________

(c) $\sqrt{1764}$ = __________

90. (a) $\sqrt[3]{3 \cdot 3 \cdot 3 \cdot 8 \cdot 8} = 3 \cdot 2 \cdot 2 = 12$

(b) $\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 8 \cdot 8} = 2 \cdot 2 \cdot 2 \cdot 2$
$= 16$

91. If you are going to do much algebra, you would find it useful to know the squares of the numbers 1 through 25 and the cubes of the numbers 1 through 7.

(c) Remember this is a <u>square</u> root and $\sqrt{N} = a$ only if $a \cdot a = N$.
$\sqrt{2\cdot2\cdot3\cdot3\cdot49} = 2\cdot3\cdot7 = 42$

To factor a polynomial such as $x^3-8$, first determine whether this is the difference of two squares or the difference of two cubes.

That is, find $\sqrt{x^3}$ and the $\sqrt{8}$ if you can. If both of these do not have square roots, then find

$\sqrt[3]{x^3}$ and $\sqrt[3]{8}$ if you can.
In this case, $x^3$ and 8 are both (squares, cubes)
choose one

91. cubes

92. One factor of the difference of two cubes is the difference of the cube roots.

Therefore, one factor of $x^3-8$ is

__________.

92. x-2

93. Since $x^3-8$ has a factor of x-2 or

$x^3-8 = (x-2)($_______$)$, how would you find the other factor?

93. divide $x^3-8$ by x-2.
Remember if $a\cdot b = c$, then $\frac{c}{a} = b$.

94. Find the other factor and then express $x^3-8$ as a product.

94. The other factor is $x^2+2x+4$.

$x^3-8 = (x-2)(x^2+2x+4)$

95. Does $8p^6-27q^9$ represent the difference of two squares or the difference of two cubes?

95. difference of two cubes

96. Find $\sqrt[3]{8p^6}$ and $\sqrt[3]{27q^9}$ and tell why they are the correct cube roots.

96. $2p^2$ because $2p^2(2p^2)(2p^2)=8p^6$

$3q^3$ because $3q^3(3q^3)(3q^3)=27q^9$

97. One factor of $8p^6-27q^9$ is ________ because one factor of the difference of two cubes is

________________________.
(See frame 93 if necessary.)

97. $2p^2-3q^3$ because one factor of the difference of two cubes is the difference of the cube roots.

98. If one factor of $8p^6-27q^9$ is $2p^2-3q^3$, the other factor is

________________.

Express $8p^6-27q^9$ as a product.

98. other factor is $4p^4+6p^2q^3+9p^6$.

$8p^6-27q^9=(2p^2-3q^3)(4p^4+6p^2q^3+9p^6)$

99. Express $216-p^3q^{12}$ as a product.

99. This is the difference of two cubes.
cube root of 216 is 6, cube root of $p^3q^{12}$ is $pq^4$.

$216-p^3q^{12}=(6-pq^4)(36+6pq^4+p^2q^8)$

100. The second factor in these examples is a polynomial which has <u>three</u> terms. A polynomial which has three terms is called a <u>trinomial</u>.

In $(6-pq^4)(36+6pq^4+p^2q^8)$, notice that the first term of the trinomial, 36, is the square of the first term of the binomial, 6.
The second term of the trinomial, $+6pq^4$, is the product of the two terms in the binomial, 6 and $-pq^4$, with the opposite sign.
The third term of the trinomial, $+p^2q^8$, is the square of the last term of the binomial, $-pq^4$.

This is true when we factor the difference of two cubes.

Factor $64y^3-27$.

100. $64y^3-17=(4y-3)(16y^2+12y+9)$

The binomial factor of $4y-3$ is the difference of the cube roots. In the trinomial, $16y^2$ is the square of $4y$; $+12y$ is the product of $4y$ and $-3$ with the opposite sign; and $+9$ is the square of $-3$.

101. $y^6-25 =$ ____________ when expressed as a product.

101. <u>Be careful here</u>. $y^6$ has a cube root of $y^2$ but 25 <u>doesn't</u> have a cube root.
NOW examine them to see if they both have square roots.

$\sqrt{y^6} = y^3$ and $\sqrt{25} = 5$.

$y^6-25 = (y^3+5)(y^3-5)$

102. If you have forgotten how to factor the difference of two squares, go back to frame 61.

In this last frame, we had a number $y^6$ which was both a perfect square and a perfect cube. You must examine both terms before proceeding to factor. If both terms are not cubes or if both terms are not squares, then you cannot factor unless there is a common factor.

Factor: (a) $8p^6-64q^9$

(b) $4m^6-64n^{12}$

(c) $8x^6-49$

102. (a) This is the difference of two cubes. 8 and $q^9$ are cubes. $p^6$ and 64 are both cubes and squares. All the numbers must be squares or all the numbers must be cubes.
$(2p^2-4q^3)(4p^4+8p^2q^3+16q^6)$
Notice that both of these factors have a common factor. 2 is the common factor of the binomial and 4 is the common factor of the trinomial. Removing these, we get
$8(p^2-2q^3)(p^4+2p^2q^3+4q^6)$

(b) This is the difference of two squares. 4 is a square. $m^6$, 64 and $n^{12}$ are both squares and cubes. All the numbers must meet the same condition.

$4m^6-64n^{12} = (2m^3-6n^6)(2m^3+6n^6)$.
Removing a common factor of 2 from each factor we get
$(m^3-4n^6)(m^3+4n^6)$

(c) This can't be factored. 8 is a cube, $x^6$ is both a cube and a square, and 49 is a square. There is no common factor.

103. In $4m^6-64m^{12}$, we factored this as the difference of two squares and then found a common factor in each of the factors we had. We could have removed the common factor first, and

$4p^6-64m^{12} = 4(p^6-16n^{12})$.

Then $p^6-16n^{12}$ can be factored as

the difference of two __________

and equals ____________.

103. squares

$p^6-16n^{12} = (p^3-4n^6)(p^3+4n^6)$

104. We would get the same factors for $4m^6-64n^{12}$ as we had in the answer to frame 103 part b. It makes no difference in which way we do the factoring, however if there is a common factor, it is usually to your advantage to remove it first as then you have smaller numbers to deal with.

Factor $3x^3-81$.

104. $3(x-3)(x^2+3x+9)$

105. The sum of two cubes can be factored into a binomial and a trinomial.
The binomial factor is the sum of the cube roots of each of the terms.

The trinomial factor is formed in exactly the same way as it is formed when we factor the difference of two cubes. (Refer to frame 100 if necessary.)

For example, to factor $8m^3+27$, we first check to make sure that all the terms are cubes.

Then we can form the two factors. The binomial factor for $8m^3+27$

is ____________.

105. $2m+3$

106. The first term of the trinomial factor is _______ and is found by ____________.

106. $4m^2$
found by squaring the first term of the binomial factor
or
by squaring $2m$.

107. The second term of the trinomial factor is _______ and is found by _______________.

107. $-6m$
found by finding the product of the two terms in the binomial and using the opposite sign
or
by multiplying $2m$ by $+3$ and using a - sign.

108. The last term of the trinomial factor is _______ and is found by _______________.

108. $+9$
found by squaring the last term of the binomial factor
or
by squaring $+3$.

109. Thus, $8m^3+27 =$ ____________ when expressed as a product.

109. $(2m+3)(4m^2-6m+9)$

110. Factor $216+125x^3$.

110. This is the sum of two cubes.
$(6+5x)(36-30x+25x^2)$

111. We have had four kinds of polynomials which we may factor thus far. They are __________,

____________, ____________,

____________.

111. difference of two squares
sum of two cubes
difference of two cubes

112. The sum of two squares is not factorable. We will discuss this in more detail later.

and polynomials which have a common factor.

Remember this last category includes polynomials which don't have a common factor in the original but which do have a common factor when terms are grouped together.

Factor the following completely.

(a) $216-27x^3y^6$

(b) $4x^6-64$

(c) $x^5-16x^3+x^2-16$

(d) $27x^8-y^9$

(e) $125a^{12}-b^9c^6$

(f) $27+8y^6$

(g) $y^3(4x^2-1)+8(4x^2-1)$

(h) $y^5-4y^3-64y^2+256$

(i) $y^5-4y^3+64y^2-256$

(j) $m^6+16$

112. (a) $(6-3xy^2)(36+18xy^2+9x^2y^4)=$
$27(2-xy^2)(4+2xy^2+x^2y^4)$
(b) $4(x^3-4)(x^3+4)$
(c) $x^3(x^2-16)+1(x^2-16) =$
$(x^2-16)(x^3+1) =$
$(x+4)(x-4)(x+1)(x^2-x+1)$
(d) can't be factored
(e) $(5a^4-b^3c^2)(25a^8+5a^4b^3c^2+b^6c^4)$
(f) $(3+2y^2)(9-6y^2+4y^4)$
(g) $(4x^2-1)(y^3+8) =$
$(2x+1)(2x-1)(y+2)(y^2-2y+4)$
(h) $y^3(y^2-4)-64(y^2-4) = (y^2-4)(y^3-64)$
$= (y-2)(y+2)(y-4)(y^2+4y+16)$
(i) $y^3(y^2-4)+64(y^2-4) =$
$(y^2-4)(y^3+64) =$
$(y-2)(y+2)(y+4)(y^2-4y+16)$
(j) can't be factored

113. $(2x-3)^3-y^3$ is the difference of two cubes and so can be factored.

$\sqrt[3]{(2x-3)^3} =$ ____________
because ________________.

$\sqrt[3]{y^3} =$ ____________ because
________________________.

113. $(2x-3)$ because three factors of $(2x-3)$ equals $(2x-3)^3$.

$y$ because $y \cdot y \cdot y = y^3$

114. Factor $(2x-3)^3-y^3$

114. $[(2x-3)-y][(2x-3)^2+y(2x-3)+y^2]$

115. Express this result using only one sign of grouping.

115. $[2x-3-y][4x^2-12x+9+2xy-3y+y^2]$

116. Factor $8y^3+(3y-5)^6$.

116. $\sqrt[3]{8y^3} = 2y$

$\sqrt[3]{(3y-5)^6} = (3y-5)^2$

117. Express this result using only one symbol of grouping

$[2y+(3y-5)^2][4y^2-2y(3y-5)^2+(3y-5)^4]$

117. $[2y+9y^2-30y+25][4y^2-2y(9y^2-30y+25)+81y^4-540y^3+1350y^2-1500y^2-1500y+625] =$
$(9y^2-28y+25)(81y^4-558y^3+1414y^2-1550y+625)$

118. Factor the following. Express your results using only one symbol of grouping.

(a) $(2y+5)^3+(y+1)^3$

(b) $64y^9-(4y-3)^3$

(c) $(2x^3-5)^3-(4x^3+3)^3$

118. (a) $[(2y+5)+(y+1)][(2y+5)^2-(2y+5)(y+1)+(y+1)^2]$
$= [3y+6][7y^2+29y+31] =$
$3(y+2)(7y^2+29y+31)$

Don't forget to remove common factors if there are any or to refactor by any of the other methods we've had if they apply.

(b) $[4y^3-(4y-3)][16y^6+4y^3(4y-3)+(4y-3)^2] =$
$[4y^3-4y+3][16y^6+16y^4-12y^3+16y^2-24y+9]$

(c) $[(2x^3-5)-(4x^3+3)][(2x^3-5)^3+(2x^3-5)(4x^3+3)+(4x^3+3)^2] =$
$[-2x^3-8][28x^6-10x^3+19] =$
$2(-x^3-4)(28x^6-10x^3+19)$

119. In this last frame you multiplied several binomials together. These binomials had similar terms. For example, $(2y+5)$ is an instance of multiplying two binomials which are similar.

$2y+5$ and $y^2+2$ are <u>not</u> similar. They are of different degrees.

$3a-b$ and $2a+3c$ are <u>not</u> similar. Even though they are of the same degree, the terms in $3a-b$ are <u>not</u> similar to those in $2a+3c$.

When you multiply two binomials which are similar such as $2y+5$ and $y+1$, how many terms are there in the product?

119. 3 terms

120. In $(2y+5)(y+1)$, the product is $2y^2+5y+2y+5$ which equals $2y^2+7y+5$. This is a trinomial. Why?

120. because it has 3 terms

121. Since $(2y+5)(y+1) = 2y^2+7y+5$, the factors of $2y^2+7y+5$ are ________ and ________.

121. $2y+5$ and $y+1$ (Either order)

122. In order for a trinomial to be factorable, it must be of <u>quadratic form</u>. To determine whether a trinomial is of quadratic form, arrange it in either ascending or descending order and put in any missing terms.
For example, $1+x^6-x^3$ could be written

$x^6+0\cdot x^5+0\cdot x^4-x^3+0\cdot x^2+0\cdot x+1.$

If the terms are arranged in descending order, the last term will be a number without a variable. If the terms are arranged in ascending order, the first term will be a number without a variable.

If the polynomial is of quadratic form, there will be the same number of missing terms between each successive pair of given terms.
In the above case, there are two missing terms between $x^6$ and $x^3$ and there are also two missing terms between $x^3$ and 1, so this polynomial is of quadratic form.
Which of the following are of quadratic form?

(a) $x^4+4-3x$

(b) $x^4+2x^2-3$

(c) $x^8-2-3x^4$

(d) $x-3x^2+2$

(e) $x^8-6-7x^5$

122. b, c and d

123. Not all trinomials of quadratic form are factorable. If a trinomial is of quadratic form and is factorable, it will factor into two binomials.

For example, $x-3x^2+2$ is of quadratic form and equals $(3x+2)(-x+1)$ because $(3x+2)(-x+1)$ has a product of $-3x^2+x+2$.

In multiplying two binomials, we follow certain steps. We shall review these as we will reverse them in order to factor the trinomials.
List the steps used to get each term of the product $(3x-4)(5x+2)$.

123. the product is $15x^2-14x-8$.

(a) multiply the first terms of each binomial to get $15x^2$
(b) find the two cross products and add them to get $-14x$
(c) multiply the last terms of each binomial to get $-8$.

Review frames 46 and 47 if necessary.

124. Then to factor $x^2-3x+2$, we will factor the $x^2$ to get the first term of each binomial.

$x^2-3x+2 = (x\quad)(x\quad)$.

We can't use the second term of the trinomial, $-3x$, at this time as it is the result of adding the two cross products and in order to find the cross products we need both terms of each binomial.

The last term of each binomial comes from what term in the trinomial?

124. From the last term of the trinomial which is +2 in this case.

125. When we factor the last term of a trinomial, we will consider only the absolute value of the number and then put in the algebraic signs later.

Thus, in this case, we shall factor 2. The only possible factors of 2 are 2 and 1, so now we have

$x^2-3x+2 = (x\quad 2)(x\quad 1)$.

Now consider the cross products. In this case the cross products

are ____________ and ____________.

125. x and 2x (Either order)

126. Our problem now is to put in the correct signs so that the cross products equal the second term of the trinomial which is $-3x$ in this case. These signs must also be such that the product of the last terms in each binomial equals the last term of the trinomial or +2 in this case.

Supply the necessary signs so

that ____x + ____2x = $-3x$.

126. $-x + -2x = -3x$

127. Our problem then has become $x^2-3x+2 = (x-2)(x-1)$.

Now we must check to see if the product of the last terms in each

binomial equals the last term in the trinomial.

Does -2 multiplied by -1 eqyal +2? Yes, it does. Thus (x-2) and (x-1) are the correct factors.

To factor $x^2-4x-12$, we first determine that it is a trinomial in quadratic form and then we know that if it is factorable, it will factor into two binomials. Once we have determined this, we will ____________ and then ____________.

127. factor 2x and then factor 12.

128. Put in the first terms of each binomial.

$x^2-4x-12 = (\_\_\ \ )(\_\_\ \ )$

128. (x )(x )

129. Now we must determine the factors of 12.

What are the factors of 12?

129. There are several pairs of factors of 12.
6 and 2
3 and 4
1 and 12

130. Our problem now is to choose the correct pair of factors of 12.

We have a choice of:

(x  1)(x  12)

(x  2)(x  6)

(x  3)(x  4)

The second term of the trinomial which is -4x in this case is the result of adding the cross products of the binomials.
Thus we must find the cross products for our 3 possibilities.

For (x  1)(x  12), these products are ___________.

For (x  2)(x  6), these products are ___________.

For (x  3)(x  4), these products are ___________.

130. x and 12x

2x and 6x

4x and 3x

131. We must now consider the signs we are going to use.

In the given trinomial, $x^2-4x-12$, the last term is -12. In order for the product of the last terms in each binomial to be a negative number, what must be true of the numbers which are multiplied?

131. The numbers must have different signs
or
one number must be positive and the other must be negative

132. For (x 1)(x 12), we must consider (x-1)(x+12) and also (x+1(x-12) as both -1(+12) and 1(-12) have a product of -12. Our trinomial was $x^2-4x-12$. Now we only need to consider whether or not we can get the -4x as we chose the first terms in each of the binomial factors so that we could get $x^2$ and we also chose the last terms in each of the binomial factors so that their product was -12.

Do either (x-1)(x+12) or (x+1)(x-12) equal the given polynomial $x^2-4x-12$?

132. No.
The sum of the cross products in the first case is +11x and in the second case is -11x.

133. Since neither (x-1)(x+12) nor (x+1)(x-12) are the correct factors for $x^2-4x-12$, what should we do now?

133. Consider (x 2)(x 6) to see if we can put in the correct signs to get our given polynomial. If neither possibility gives us the required result, we will then consider (x 4)(x 3). If neither possibility works here, we can say that $x^2-4x-12$ is not factorable or can't be factored.

134. Find the correct factors of

$x^2-4x-12$.

134. (x+2)(x-6)

135. List all of the possible sets of factors that there are for

$x^2-6x-24$.

135. First factor $x^2$ and then factor 24.
Remember that the last term of the trinomial is negative, so we need numbers with different signs.

136. To decide which is the correct set of factors, we must

____________________________.

Possible factors: (x-1)(x+24)
(x+1)(x-24)
(x-2)(x+12)
(x+2)(x-12)
(x-3)(x+8)
(x+3)(x-8)
(x-4)(x+6)
(x+4)(x-6)

136. decide which pair of factors gives -6x which is the second term of the given polynomial.

137. Factor $x^2-6x-24$.

137. None of the possibilities listed in frame 135 give $x^2-6x-24$ and since these are the only possibilities, $x^2-6x-24$ <u>cannot</u> be factored.

138. Sometimes the factoring of a trinomial of the quadratic form is referred to as the trial and error method. It is obvious why this is so.

Factor $x^2-6x-16$.

138. Possible factors are
(x-1)(x+16)
(x+1)(x-16)
(x+2)(x-8)
(x-2)(x+8)
(x-4)(x+4)

Correct set of factors is (x-8)(x+2)

139. To factor $x^4+18-11x^2$, what do we need to do before starting to form the possible binomial factors?

139. Arrange it in order and check to see if this is of the quadratic form.

140. Factor $x^4+18-11x^2$.

140. $x^4-11x^2+18$ is of quadratic form since there is one missing term between each pair of successive terms when the polynomial is arranged in order.

Note that since the last term is positive, the last terms in each of the binomials must have the same sign.

Possible factors: $(x^2-1)(x^2-18)$
$(x^2+1)(x^2+18)$
$(x^2-2)(x^2-9)$
$(x^2+2)(x^2+9)$
$(x^2-3)(x^2-6)$
$(x^2+3)(x^2+6)$

Correct factors are $(x^2-2)(x^2-9)$.

141. In frame 66, we stated that when we factored, we usually expressed the result in terms of factors which could not be factored again - at least by any of the methods we have had.

Look at our last result.

$x^4-11x^2+18 = (x^2-2)(x^2-9)$.

Can either or both of these factors be factored again and if so, which one(s) and why did you decide it could be factored again?

141. $x^2-9$ can be factored again because it is the difference of two squares.

$x^2-2$ <u>can't</u> be factored again.

142. Complete:

$x^4-11x^2+18 = (x^2-2)(x^2-9) =$

________________

142. $(x^2-2)(x-3)(x+3)$

143. Factor $12+x-x^2$. Do this twice - once when the terms are arranged in descending order and once when the terms are arranged in ascending order.

143. In ascending order we have $12+x-x^2 = (4-x)(3+x)$

Arranged in descending order, we have that $12+x-x^2 = x^2+x+12$.

$-x^2+x+12 = (-x+4)(x+3)$
or
$-x^2+x+12 = (x-4)(-x-3)$

144. Since we factored the same polynomial in each case, all our results must be equal.

That is, $(4-x)(3+x) = (-x+4)(x+3) = (x-4)(-x-3)$.

Consider $(4-x)(3+x)$ and $(-x+4)(x+3)$. It is quite obvious that these are equal since $4-x$ equals $-x+4$ as these are exactly the same except for the order in which they are written and certainly $3+x$ and $x+3$ are the same.

Now consider $(-x+4)(x+3)$ and $(x-4)(-x-3)$. The factors $-x+4$ and $x-4$ are <u>not</u> equal and $x+3$ and $-x-3$ are <u>not</u> equal.

Even though the individual factors are not equal, we have said that $(-x+4)(x+3)$ and $(x-4)(-x-3)$ or that their products are equal.

How would you determine if this is a true statement? That is, does $(-x+4)(x+3)$ equal $(x-4)(-x-3)$?

144. Find the products or multiply the factors to see if we obtain the same polynomial in each case.

145. $(-x+4)(x+3) =$ ____________ and

$(x-4)(-x-3) =$ ____________.

Are these the same polynomial?

145. $-x^2+x+12$ is obtained in both cases and it is the same polynomial.

146. There sometimes are several different sets of factors for a given polynomial. If you obtain a different set than the ones given in the answer, how can

you check to see if yours are correct?

146. Multiply them together to see if you get the original polynomial.

147. There is another technique that we may use.

We had (-x+4)(x+3) and (x-4)(-x-3). Consider the first factor in each case. In -x+4) and (x-4) what similarities and differences do you note?

147. The numbers x and 4 are the same but each one has the opposite sign
or
all of the signs in one factor are the opposite of the signs in the other factor for the numbers x and 4.

148. If we factored (-x+4), what could we use as a common factor so that we would get (?)(x-4)?

148. use a factor of -1 because

-1(x-4) = (-x+4)

149. Consider the second factor in each of (-x+4)(x+3) and (x-4)(-x-3).

Factor (x+3) so that it equals some number multiplied by (-x-3).

149. (x+3) = -1(-x-3)

150. Then (-x+4)(x+3) can be expressed as -1(x-4)(-1)(-x-3).

By rearranging the order of these factors, we have -1(-1)(x-4)(-x-3) and if we multiply the numerical factors we get +1(x-4)(-x-3) or (x-4)(-x-3).

In frames 31 and 33 of Chapter 2, we defined the additive inverse or the negative of a number.

To review, the additive inverse or the negative of a number is a number such that the sum of it and the original number is 0.

Could we say that (-x+4) and (x-4) are the negatives of each other? Why?

150. yes because their sum is 0.

151. The product of <u>two</u> numbers is always equal to the product of the negatives of these two numbers.

Let us consider the numbers -2 and 3. What are the negatives of these numbers?

151. 2 and -3

152. According to our statement in frame 152, the product of our two original numbers, -2 and 3, should equal the product of their negatives which are 2 and -3. Does the product of -2 and 3 equal the product of 2 and -3?

152. yes.

153. Consider the polynomials $(x-2)$ and $(x^2+3)$.

The negative of $(x-2)$ is ______.

The negative of $(x^2+3)$ is ______.

153. $(-x+2)$ or $(2-x)$

$(-x^2-3)$

154. Multiply and see if the product of the two original polynomials equals the product of their negatives.

154. yes
product is $x^3-2x^2+3x-6$ in both cases

155. Use the statement in frame 152 to write an equivalent set of factors for $(-x+3)(x-2)$.

155. $(x-3)(-x+2)$

$(x-3)$ is the negative of $(-x+3)$
$(-x+2)$ is the negative of $(x-2)$

156. To return to our factoring problem of $12+x-x^2$.

We can say that $(4-x)(3+x)$ equals $(x-4)(-x-3)$ because the product of two numbers equals the product of the negatives of these two numbers.

Warning - You must have the negatives of both numbers. We have only talked of <u>two</u> factors.

Is $(-3+x)(3+x)$ equal to $(3-x)(3-x)$? Give a reason for your answer.

156. NO.
$(3-x)$ is <u>not</u> the negative of $(3+x)$.

157. Find the products and verify the answer to the last frame.

$(-3+x)(3+x) =$ ______________

$(3-x)(3-x) =$ ______________

157. $(-3+x)(3+x) = -9+x^2$

$(3-x)(3-x) = 9-6x+x^2$

158. Does $(3-x)(2+x)$ equal $(x-3)(x-2)$? Give a reason for your answer.

Find the products and verify your answer.

158. NO.
$(x-2)$ is <u>not</u> the negative of $(2+x)$.
$(3-x)(2+x) = 6+x-x^2$

$(x-3)(x-2) = x^2-5x+6$

159. Factor completely. (Refer to frames 124 - 143 if necessary.)

(a) $x^2-x-6$

(b) $32-12x+x^2$

(c) $x^4-6+5x^2$

(d) $-96+x^6-4x^3$

(e) $x^2+18x+32$

(f) $x^2+96+20x$

(g) $x^2-6x+7$

(h) $x^2-18x+80$

(i) $80+x^4-21x^2$

159. (a) $(x-3)(x+2)$
(b) $(8-x)(4-x)$ or $(x-8)(x-4)$
(c) Arrange in order first.
$x^4+5x^2-6 = (x^2+6)(x^2-1)$.
Factor again to get
$(x^2+6)(x-1)(x+1)$
(d) Arrange in order first.
$x^6-4x^3-96 = (x^3-12)(x^3+8)$
$x^3+8$ is the sum of two cubes.
Factoring again, we get
$(x^3-12)(x+2)(x^2-2x+4)$
(e) $(x+16)(x+2)$
(f) $(x+12)(x+8)$
(g) can't be factored
(h) $(x-10)(x-8)$
(i) Arrange in order.
$x^4-21x^2+80 = (x^2-16)(x^2-5)$
$x^2-16$ is the difference of two squares.
Final result:
$(x-4)(x+4)(x^2-5)$

160. Suppose you were to find the factors of $2x^2-13x+15$.

This is a trinomial of quadratic form, so you should try to factor it. If it is factorable, you will obtain two binomial factors.

The first term in each binomial must satisfy what condition?

$x^4-21x^2+80 = (x^2-16)(x^2-5)$

$x^2-16$ is the difference of two squares.

final result: $(x-4)(x+4)(x^2-5)$

160. Their product must equal the first term of the trinomial of $2x^2$ in this problem.

161. Fill in the first terms in each binomial.

$2x^2-13x+15 = (\_\_\ \ )(\_\_\ \ )$

161. (2x  )(x  )

162. Now we must determine what numbers will be the last terms in each binomial.

Since the last term in the trinomial is +15, we can use +1 and +15, -1 and -15  +3 and +5 or -3 and -5. Remember since we have a +15, both of the factors must have the same sign.

The first terms in the binomials have different coefficients. Therefore, if we use the factors of +1 and +15, we must try both $(2x+1)(x+15)$ and $(2x+15)(x+1)$. You can see that the sum of the cross products is different in each of these.

Are either $(2x+1)(x+15)$ or $(2x+15)(x+1)$ the correct set of factors for $2x^2-13x+15$?

162. No.

163. What other possibilities are there?

Which one, if any, is the required one?

163. Other possibilities are:
$(2x-1)(x-15)$
$(2x-15)(x-1)$
$(2x-5)(x-3)$
$(2x-3)(x-5)$
$(2x+5)(x+3)$
$(2x+3)(x+5)$

correct set is $(2x-3)(x-5)$

164. Factor $6x^2+5x-4$.

here are many possibilities ere as $6x^2$ has two sets of ctors and -4 has two sets factors.

$x^2+5x-4 = (3x+4)(2x-1)$

165. There is one fact that will make less of a trial and error process when factoring a trinomial.

If the trinomial has a common factor, remove it first. Then the resulting trinomial will have no common factor and hence its factors will not have a common factor.

For example, when factoring $6x^2+5x-4$, suppose we use as factors of $6x^2$, $2x$ and $3x$.

So far, we have
$6x^2+5x-4 = (2x \quad)(3x \quad)$.

Now we will use the factors of -4 in order to complete the binomials.
Let us consider -2 and +2 as the factors of -4. We cannot use the -2 with the 2x, having (2x-2), as then this factor has a common factor and our given polynomial of $6x^2+5x-4$, doesn't have a common factor.
Likewise, we cannot use the +2 with the 2x getting (2x+2) for the same reason.
Then -2 and +2 are not the factors of -4 when we consider 2x and 3x as the factors of $6x^2$.

Keeping the factors of $6x^2$ the same - namely 2x and 3x, we will try -1 and +4 as the factors of -4.
If -1 and +4 are the correct ones, in which of the binomials, $(2x \quad)(3x \quad)$, must the +4 be put?

the second one

h the 3x

166. It can't go with the 2x giving a factor of 2x+4 as this now has a common factor and this can't be as long as the original polynomial doesn't have a common factor.

Factor $2x^2-xy-6y^2$.

$x+3y)(x-2y)$

167. Factor completely.

(a) $6p^2+7p-5$

(b) $63m^2+2-25m$

(c) $24x^3y^2-50x^2y^2+24xy^2$

(d) $x^4-13x^2+36$

(e) $81x^6-1-80x^3$

(f) $28m^4-27m^2n-10n^2$

(g) $24x^2+23x-12$

(h) $4p^4-(7p-4)^2$

(i) $x^2(x-3)-(x-3)(4x-3)$

) $(3p+5)(2p-1)$

) $(7m-2)(9m-1)$

) $2xy^2(4x-3)(3x-4)$

) $(x^2-9)(x^2-4) =$
$(x+3)(x-3)(x-2)(x+2)$

) $(81x^3+1)(x^3+1) =$
$(81x^3+1)(x+1)(x^2-x+1)$

) $(4m^2-5n)(7m^2+2n)$

) $(8x-3)(3x+4)$

) $[2p^2-(7p-4)][2p^2+(7p-4)] =$
$(2p^2-7p+4)(2p^2+7p-4) =$
$(2p^2-7p+4)(2p-1)(p+4)$

$(x-3)(x^2-[4x-3]) =$
$(x-3)(x^2-4x+3) =$
$(x-3)(x-3)(x-1)$

168. $(a+b)^2-2(a+b)-8$ is a trinomial in $(a+b)$ if we do not remove the parentheses. We can factor it as we do any other trinomial. First terms in each binomial are the factors of $(a+b)^2$, or $(a+b)$ and $(a+b)$.

$(a+b)^2-2(a+b)-8=[(a+b) \quad ][(a+b) \quad ]$

Next we put in factors of 8 remembering to use numbers with different signs as we want the product to be -8. Last we examine the cross products so that we obtain the correct second term of the trinomial or $-2(a+b)$ here.

Finish factoring.

$+b)-4][(a+b)+2]$

169. Let us consider the cross products here.

These are __________ and ________.

$a+b)$ and $+2(a+b)$

170. These are similar terms since the factors other than the numerical ones are the same, so we can add by adding the coefficients and multiplying the result by the

common factor, getting $-2(a+b)$.

Write these factors using only one symbol of grouping.

170. $(a+b-4)(a+b+2)$

171. Factor $2(x-y)^2-3(x-y)+1$.

171. $[2(x-y)-1][(x-y)-1] =$

$[2x-2y-1][x-y-1]$

172. It is usually a good idea to keep several signs of grouping in the first step of a problem where you have a group of terms used as a single number in the original. If it is necessary to have only one symbol of grouping in your final answer, then remove the other symbols of grouping in succeeding steps. You will probably make fewer errors this way.

Factor $4-3(2x+y)-(2x+y)^2$. Use only one symbol of grouping in your final answer.

172. $[4+(2x+y)][1-(2x+y)] =$

$[4+2x+y][1-2x-y]$

173. Factor completely using only one symbol of grouping in your final answer.

(a) $6(x-y)^2-5(x-y)-6$

(b) $3(2a+b)^2+8b(2a+b)-3b^2$

(c) $15(x-2y)^2+14(x-2y)-8$

173. (a) $[3(x-y)+2][2(x-y)-3] =$
$[3x-3y+2][2x-2y-3]$

(b) $[3(2a+b)-b][(2a+b)+3b] =$
$[6a+3b-b][2a+b+3b] =$
$(6a+2b)(2a+4b) =$
$4(3a+b)(a+2b)$

(c) $[5(x-2y)-2][3(x-2y)+4] =$
$(5x-10y-2)(3x-6y+4)$

174. Factor completely. These involve all of the different kinds of factoring that we have had.

(a) $6a^2b^2-66a^3b^4-72a^4b^6$

(b) $6x^4-7x^2-1$

(c) $125+64y^6$

(d) $27a^3-75ab^6c^4$

(e) $12-23(p+q)+10(p+q)^2$

(f) $3x^{17}+9x^9-12x$

(g) $(2x^2-8t^2)(2-5x)+3x^2(2x^2-8t^2)$

(h) $x^7-8x^4-4x^3+32$

(i) $16-(2x-3y)^2$

(j) $64-(3p+q)^3$

(k) $x^3+16$

(l) $4m^2+9$

(m) $10-3(r-s)-27(r-s)^2$

(n) $2x^6-8x^4-x^2+4x$

174. (a) $6a^2b^2(1-12ab^2)(1+ab^2)$

(b) can't be factored

(c) $(5+4y^2)(25-20y^2+16y^4)$

(d) $3a(3a-5b^3c^2)(3a+5b^3c^2)$

(e) $[3-2(p+q)][4-5(p+q)] =$
$(3-2p-2q)(4-5p-5q)$

(f) $3x(x^8-1)(x^8+4) =$
$3x(x^4+1)(x^2+1)(x-1)(x+1)(x^8+4)$

(g) $(2x^2-8t^2)(2-5x+3x^2) =$
$2(x-2t)(x+2t)(2+x)(1-3x)$

(h) $x^4(x^3-8)-4(x^3-8) =$
$(x^3-8)(x^4-4) =$
$(x-2)(x^2+2x+4)(x^2+2)(x^2-2)$

(i) $[4-(2x-3y)][4+(2x-3y)] =$
$(4-2x+3y)(4+2x-3y)$

(j) $[4-(3p+q)][16+4(3p+q)+(3p+q)^2] =$
$(4-3p-q)(16+12p+4q+9p^2+6pq+q^2)$

(k) can't be factored

(l) can't be factored

(m) $[5-9(r-s)][2+3(r-s)] =$
$(5-9r+9s)(2+3r-3s)$

(n) can't be factored

If you have different factors than the ones given here, multiply and see if they equal the original.

all the steps are given in
se answers, so be sure to
pletely factor each poly-
ial. When factoring, do
stop after factoring once,
not stop after factoring
ce, etc. There are no
cific number of times to
tor. Each time that you
tor, look at the resulting
tors and decide if any can
factored again. If so,
tor them and continue this
cess until none of the factors
be factored further.

Chapter 5 - Fractions

In this chapter, we shall define rational functions and consider
e operations of addition, subtraction, multiplication and division
rational functions.

1. We are now ready to start our study of fractions involving polynomials. In order to work with fractions which involve polynomials, it is necessary to be able to perform the operations with rational numbers. We will first review the meaning of fractions, their properties and the operations of addition, subtraction, multiplication and division on rational numbers. (Students needing additional review on these topics, should go over frames 112 - 154 in Chapter I.)

First, if a and b represent any rational numbers, then if $b \neq 0$

$\frac{a}{b} = a \cdot \frac{1}{b}$

$\frac{2/3}{5/4}$ means ____________

. $\cdot \frac{1}{5/4}$ or $\frac{1}{5/4} \cdot \frac{2}{3}$

r $\frac{2}{3} \cdot \frac{4}{5}$

2. $\frac{10}{6}$ means ____________

0 $\cdot \frac{1}{6}$ or $\frac{1}{6} \cdot 10$

3. $\frac{1}{6} \cdot 10$ is often read "1/6 of 10".

The word "of" means the operation of multiplication when used in this way.

Express in symbols: one-third of 9

Express this same statement as a fraction of the form $\frac{a}{b}$ using the numbers one-third and 9.

$\cdot$ 9

Tou might have said 3. This is
orrect as when the multipli-
ation is performed, the product
; 3.)
3 See frame 1 if necessary.

4. <u>Multiplication of fractions.</u>
If a, b, c and d are real numbers and $b \neq 0$ and $d \neq 0$, then

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

In other words, multiply the numerators to get the new num-

r $\frac{a}{b}$ means $a \cdot \frac{1}{b}$, and so

must mean $\frac{a}{b}$.

on by 0 can't be done

on by 0 isn't defined

(b) $\frac{168}{25}$ (c) $\frac{429}{70}$

$\frac{2(a+b)}{5(a-2b)} = \frac{2a+2b}{5a-10b}$

ause then a-2 would equal
d we can't have a fraction
a denominator of zero.

ecause then a+3 would equal
nd we can't divide by zero.

d b cannot have the same
e.
. and b have the same value,
a-b would equal 0 and we
t have a denominator equal
).

$\frac{)}{-2} = \frac{0}{5} = 0$

erator and multiply the denominators to get the new denominator.

Do you need to have the same denominators in order to multiply fractions?

5. Why must b and d <u>not</u> equal 0?

6. Multiply: (a) $\frac{5}{7} \cdot \frac{3}{2}$

(b) $\frac{24}{5} \cdot \frac{7}{5}$

(c) $\frac{3}{5} \cdot \frac{11}{2} \cdot \frac{13}{7}$

(d) $\frac{2}{a-2b} \cdot \frac{a+b}{5}$

7. In part d of frame 6, any other answer than the two given is wrong. Check your answer now if you haven't done so before.
When letters are used to represent numbers, we must make sure that we don't use any values which will make the denominators equal to 0.

In $\frac{2}{a-2}$, a can't equal ______

because ______________________.

8. In $\frac{32}{a+3}$, a can't equal ______

because ______________________.

9. In $\frac{4x}{a-b}$, what can you say about the values of a and b?

10. In $\frac{4x}{a-b}$, suppose a = 3, b = -2 and x = 0. Then $\frac{4x}{a-b}$ = ______.

11. Any of the preceding answers are correct. Remember the definition of division. $\frac{a}{b} = c$ only if $c \cdot b = a$.

$\frac{0}{5} = 0$ because $0 \cdot 5 = 0$.

It is ONLY 0 IN THE DENOMINATOR

that is not permissible.

In $\frac{-3a}{p+q}$, what values may a have?

In the same fraction, what values may p and q have?

11. a may equal any real number.

p and q must not equal the negatives of each other. For example, if p = 2, then $q \neq -2$ or if p = -3, then $q \neq 3$, because then p+q would equal 0 and the denominator of a fraction can't equal 0.

12. Evaluate each of the following if a = 2, b = -4, c = 0 and d = 1.

(a) $\frac{b}{a} + \frac{c}{d}$

(b) $\frac{d}{2a+b}$

(c) $\frac{b+4d}{a-3d}$

(d) (c+2b)(b+3a-2d)

12. (a) $\frac{-4}{2} - \frac{0}{1} = -2 + 0 = -2$

(b) $\frac{1}{2(2)+-4} = \frac{1}{4-4} = \frac{1}{0}$ and this doesn't represent a number or this isn't defined or division by 0 isn't defined.

(c) $\frac{-4+4(1)}{2-3(1)} = \frac{0}{-1} = 0$

(d) $(0+-8)(-4+6-2) = -8(0) = 0$
Remember a number multiplied by 0 equals 0.

13. In frame 4, we defined multiplication of rational numbers in the following way.

$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ if $b \neq 0$ and $d \neq 0$.

Multiply the following.

(a) $\frac{3x^2}{-4a} \cdot \frac{2xy^5}{5bc} \cdot \frac{13y^4}{-7a^2b}$

(b) $\frac{p-q}{2p+3q} \cdot \frac{2p+5q}{3p-q}$

(c) $\frac{3a^2(a+b)}{2a-b} \cdot \frac{3a+b}{4b} \cdot \frac{5}{b}$

13. (a) $\frac{18x^3y^9}{140a^3b^2c}$ Some of you may have written $\frac{9x^3y^9}{70a^3b^2c}$.
Both of these are correct at this time.

(b) $\frac{(p-q)(2p+5q)}{(2p+3q)(3p-q)} = \frac{2p^2+3pq-5q^2}{6p^2+7pq-3q^2}$

(c) $\frac{15a^2(a+b)(3a+b)}{4ab^2(2a-b)} = \frac{45a^4+60a^3b+15a^2b^2}{8a^2b^2-4ab^3}$

14. In parts b and c of the last frame, the answers could be left either in factored form or could be multiplied out. In our future work, you will often find that it is more convenient to use the factored form.

In frame 116 of Chapter I, we defined equivalent fractions.

$\frac{a}{b} = \frac{ac}{bc}$ if b and c don't equal 0.

a/b and ac/bc are called equivalent fractions. In other words, if the numerator and denominator of a fraction are multiplied by the same number, the resulting fraction is equivalent to the given one providing we didn't multiply by 0.

Change to equivalent fractions.
Fill in the missing parts.

(a) $\frac{2}{3} = \frac{\quad}{60}$

(b) $\frac{5}{3} = \frac{45}{\quad}$

(c) $\frac{5}{4} = \frac{\quad}{56}$

(d) $\frac{8}{7} = \frac{72}{\quad}$

14. (a) 40
    (b) 27
    (c) 70
    (d) 63

15. In making the statement $\frac{2}{3} = \frac{40}{60}$ ,

what operation did you perform?

15. multiplied <u>both</u> numerator and denominator by 20
or
multiplied the 2 and 3 by 20.

16. Make sure you said to multiply both the numerator and denominator by 20. It is wrong if you said to multiply by 20. Let us consider these two statements.

Multiply both numerator and denominator by 20.

$\frac{2 \cdot 20}{3 \cdot 20} = \frac{40}{60}$

Multiply by 20.

$\frac{2}{3}(20) = \frac{2}{3} \cdot \frac{20}{1} = \frac{40}{3}$

You can see that the two results are not equal.

Write an equivalent fraction for 5/7 which has a denominator of 56. State what you did to get this fraction.

16. $\frac{40}{56}$

Multiplied both numerator and denominator by 8.

NOT multiply by 8.

17. Consider the fraction $\frac{2}{a-3}$ .

This is a well defined rational number if the denominator is not equal to 0. What value of a would make the denominator equal to 0?

17. a = 3 makes the denominator equal to 0.

18. We will not state every time that the denominator is not equal to 0, but this is always implied since we are talking about rational numbers and <u>division by 0 does not give a number</u>.
So, if we consider values of a where a is any real number except

3, $\frac{2}{a-3}$ is a well defined number.
We can find equivalent fractions by the same process we used in frames 14 - 16. That is, we can find equivalent fractions by multiplying both numerator and denominator by the same number or by dividing both numerator and denominator by the same number.

$$\frac{2}{a-3} = \frac{?}{3(a-3)}$$

Give a reason for your answer.

18. $\frac{6}{3(a-3)}$ because both numerator and denominator were multiplied by the same number which was 3.

19. Change $\frac{3a}{4+a}$ to an equivalent fraction having $6a^2$ for its numerator. State for what values of a the original fraction was defined.

19. $\frac{3a}{4+a} = \frac{6a^2}{2a(4+a)}$

$\frac{3a}{4+a}$ is defined for all real numbers except $a = -4$.

20. Change each of the following to equivalent fractions by filling in the missing parts. State what you did in each case.

(a) $\frac{p+3q}{6} = \frac{\phantom{xxxx}}{12p}$

(b) $\frac{a-3}{3a} = \frac{4(a-3)}{\phantom{xxxx}}$

(c) $\frac{2}{m+3n} = \frac{\phantom{xxxx}}{(m+3n)(m-n)}$

(d) $\frac{2a-b}{2a(a+b)} = \frac{(2a-b)(3a+2b)}{\phantom{xxxx}}$

(e) $\frac{4(x-3y)(x+2y)}{x(x+2y)(2x-y)} = \frac{\phantom{xxxx}}{x(2x-y)}$

20. (a) $2p(p+3q)$, multiplied both numerator and denominator by $2p$.
(b) $12a$, multiplied both numerator and denominator by 4.
(c) $2(m-n)$, multiplied both numerator and denominator by $m-n$.
(d) $2a(a+b)(3a+2b)$, multiplied both numerator and denominator by $3a+2b$.
(e) $4(x-3y)$, divided both numerator and denominator by $x+2y$.

21. How would you check to see if $\frac{2}{m+3n} = \frac{2(m-n)}{(m+3n)(m-n)}$ ?

to see if $\frac{4(x-3y)(x+2y)}{x(x+2y)(2x-y)} = \frac{4(x-3y)}{x(2x-y)}$?

21. If $2(m+3n)(m-n) = (m+3n)(m-n)2$ then these are equivalent fractions.
Remember, $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$ and b and d don't equal 0.
Refer to frame 122, Chapter I if necessary.

If $4(x-3y)(x+2y)x(2x-y) = x(x+2y)(2x-y)4(x-3y)$ then these are equivalent fractions.

Note that in the above cases, the factors are the same, but occur in different order. The commutative and associative laws of multiplication allow us to say that the two results are equal.

22. In other words, we may divide both numerator and denominator by a <u>common factor</u> or we may multiply both numerator and denominator by a common factor to get equivalent fractions.
If the numerator and denominator have <u>no</u> common factors, then we say the fraction is in its lowest terms.
Reduce each of the following to lowest terms.

(a) $\frac{3x^2(2x+5y)}{6x(3+2y)}$

(b) $\frac{-42a^2b^3}{24a^4b}$

(c) $\frac{6pq^2(2p-5q)(p+3q)}{-3q(p+3q)(2p+5q)}$

22. (a) Divide both numerator and denominator by 3x getting $\frac{x(2x+5y)}{2(3+2y)}$
(b) divide both numerator and denominator by $6a^2b$ getting $\frac{-7b^2}{4a^2}$
(c) divide both numerator and denominator by $3q(p+3q)$ getting $\frac{2pq(2p-5q)}{-(2p+5q)}$

23. In the answers to parts b and c, if you have the "-" sign in a different place, your answer may be correct. Check it to see if $ad = bc$ when the fractions are considered as a/b and c/d.
Let us consider another useful technique in determining equivalent fractions.
How would you decide if $\frac{-16}{21}$ is equal to $\frac{16}{-21}$ ?

23. Since two fractions a/b and c/d are equal if $ad = bc$, see if $ad = bc$ when $\frac{16}{-21}$ is considered as a/b and $\frac{-16}{21}$ is considered as as c/d.

24. $-16(-21)$ equals $16(21)$ so $\frac{16}{-21} = \frac{-16}{21}$.
Any two fractions where the numerators and denominators are the <u>negatives</u> of each other will be equivalent fractions.
(Review frames 31 - 37 of Chapter II if you don't remember what the negative of a number is.)

$\frac{-3}{2} = \frac{\phantom{3}}{-2}$ because ____________ .

24. $\frac{3}{-2}$ because $3(2) = -2(-3)$ or because the numerators and and denominators of the two fractions are the negatives of

25. $\frac{(-4)(5)}{(-3)(7)} = \frac{(4)(5)}{(3)(7)}$ because $(-4)(5)(3)(7) = (-3)(7)(4)(5)$.

You will notice that $(4)(5)$ is

each other.

the negative of $(-4)(5)$. Using the negative of one <u>factor</u> produces the negative of the entire quantity.

What is the negative of $(-3)(-4)(-2)$?

25. the negative is $(3)(-4)(-2)$ OR is $(-3)(4)(-2)$ OR is $(-3)(-4)(2)$.

26. $\frac{(-2)(-3)}{(-5)(7)} =$ ______
Fill in the blanks with the negatives of the given quantities.

26. $\frac{(2)(-3)}{(5)(7)}$ or $\frac{(-2)(3)}{(-5)(-7)}$

27. Check the answers given in the last problem if you haven't done so before. Note that both of them give the same value and this is the same value as the original quantity.
So far we have:
$\frac{(-2)(-3)}{(-5)(7)} = \frac{(2)(-3)}{(5)(7)} = \frac{(-2)(3)}{(-5)(-7)}$.

Now consider $\frac{(2)(3)}{(-5)(7)}$. Is the value of this equal to the value of the fractions above?
How do you know?

27. yes. Work out the fractions. All have a value of $-6/35$.

28. In this case, we didn't use the negative of both the numerator and denominator. We changed the signs of two <u>factors</u> or we could say that we multiplied two factors by -1.
Multiplying two factors by -1 keeps the same value as multiplying two factors by -1 is equivalent to multiplying by +1. Multiplying by +1 doesn't change the value.

For example, $(-2)(3) = (-2\cdot-1)(3\cdot-1) = (2)(-3)$.

Write four different expressions which are equivalent to $\frac{(-4)(5)}{(-3)(7)}$ by changing signs.

28. $\frac{(4)(-5)}{(-3)(7)} = \frac{(4)(5)}{(3)(7)} = \frac{(-4)(-5)}{(3)(7)}$
$= \frac{(-4)(-5)}{(-3)(-7)} = \frac{(4)(5)}{(-3)(-7)} = \frac{(-4)(5)}{(3)(-7)}$

29. Remember we always multiply two <u>factors</u> by -1 to keep the same value.
Can we write that $\frac{4-2}{5-2} = \frac{-4+2}{-5+2}$ ? Why?

29. They are equal if $(4-2)(-5+2) = (-4+2)(5-2)$.
This becomes $(2)(-3) = (-2)(3)$ which is a true statement.
The given statement is correct.

30. You may be saying that too many signs were changed. However, remember that two <u>factors</u> must be multiplied by -1 to keep the same value.

In $\frac{4-2}{5-2}$, is 4 a factor?

30. No, it is a term. Factors are associated with multiplication.

31. Then to find the negative of (4-2) we must find the value of -1(4-2) which is -4+2.
The negative of 5-2 is ______.

So, $\frac{4-2}{5-2} = \frac{-4+2}{\phantom{xx}}$

31. negative of 5-2 is -1(5-2) or -5+2.
$\frac{4-2}{5-2} = \frac{-4+2}{-5+2} = \frac{-2}{3} = \frac{2}{-3} = -\frac{2}{3}$

32. $\frac{(-2)(-3-4)}{(-5+3)(-3)} =$ ______ = ______

32. $\frac{(2)(3+4)}{(-5+3)(-3)} = \frac{(-2)(-3-4)}{(5-3)(3)}$
$= \frac{(2)(-3-4)}{(-5+3)(3)} = \frac{(-2)(3+4)}{(5-3)(-3)}$
$= \frac{(2)(-3-4)}{(5-3)(-3)} = \frac{(-2)(3+4)}{(-5+3)(3)}$

Any two of these are the answers to frame 32.

33. Does $\frac{-7b^2}{4a^2}$ equal $\frac{7b^2}{-4a^2}$? Why?

33. Yes because $(-7b^2)(-4a^2) = (7b^2)(4a^2)$ or because the numerators and denominators of the two fractions are the negatives of each other or because two <u>factors</u> in the fraction have been multiplied by -1.

34. In other words, if we multiply two <u>factors</u> by -1, we obtain an equivalent fraction.
$\frac{2pq(2p-5q)}{-p(2p+5q)} = \frac{?}{p(2p+5q)}$

34. numerator is $-2pq(2p-5q)$ or is $2pq(-2p+5q)$.
As long as two <u>factors</u> are multiplied by -1, then the value remains the same. In this case, we multiplied one factor of the denominator by -1 so we must multiply one factor of the numerator by -1 to have equivalent fractions.

35. Write two other fractions which are equal to the given fraction and make use of changing the appropriate signs.

$\frac{3a^2(3a-b)(a+4b)}{b(2a+b)}$

35. $\frac{-3a^2(-3a+b)(a+4b)}{b(2a+b)} =$

36. Note the answers given to the last problem. Any of these are

$$\frac{-3a^2(3a-b)(-a-4b)}{b(2a+b)} = \frac{3a^2(-3a+b)(-a-4b)}{b(2a+b)} = \frac{3a^2(3a-b)(a+4b)}{b(-2a-b)}$$
$$= \frac{-3a^2(3a-b)(a+4b)}{-b(2a+b)} = \frac{-3a^2(3a-b)(a+4b)}{b(-2a-b)}$$
$$= \frac{3a^2(-3a+b)(a+4b)}{-b(2a+b)}$$

correct. There are also three more possible answers. What are these?

36. $\frac{3a^2(-3a+b)(a+4b)}{b(-2a-b)} =$
$$\frac{3a^2(3a-b)(-a-4b)}{-b(2a+b)} = \frac{3a^2(3a-b)(-a-4b)}{b(-2a-b)}$$

37. We do not multiply factors by -1 in every case. This is only done when it will help us in handling the quantities.
In the fraction $\frac{(a-3)}{(3-a)}$, note that all the signs of the factor in the numerator are the opposite of those of the factor in the denominator. In other words, the numerator is the negative of the denominator or the denominator is the negative of the numerator.

If we multiply the factor of the numerator by -1, we get -a+3. This is exactly the same as the factor in the denominator. However, we have only multiplied one factor by -1, so we do not have the same value. We must multiply another factor by -1 in order to have a value equal to the original.

$\frac{(a-3)}{(3-a)} = \frac{(-a+3)}{-1(3-a)}$. Now the two fractions have the same value. Let a = 2 and evaluate both fractions. What values do you get? Are these equal?

37. $\frac{(a-3)}{(3-a)} = \frac{(2-3)}{(3-2)} = \frac{-1}{1} = -1$
$$\frac{(-a+3)}{-1(3-a)} = \frac{(-2+3)}{-1(3-2)} = \frac{1}{-1(1)} = -1$$

38. Consider $\frac{(-a+3)}{-1(3-a)}$. Both the numerator and denominator have a common factor which is ______.

Dividing both numerator and denominator by this common factor, we get __________.

38. common factor is -a+3 or 3-a. Note that -a+3 and 3-a are the same factor.
Dividing both numerator and

39. Consider the fraction
$$\frac{3p(p+q)(2p-3q)}{-9(3p-2q)(-p-q)}.$$
If we are to reduce this fraction

denominator by -a+3, we get 1/-1 which equals -1.

to its lowest terms, we must divide both the numerator and denominator by the same number or we must divide both numerator and denominator by a common factor. As the fraction is given, what common factor is found in both numerator and denominator?

39. common factor is 3

40. Divide both numerator and denominator by this common factor and write the result.

40. $\frac{p(p+q)(2p-3q)}{-3(3p-2q)(-p-q)}$

41. Look at this result.

$$\frac{p(p+q)(2p-3q)}{-3(3p-2q)(-p-q)}$$

Are there any factors which are the negatives of each other? If so, what are they?

41. yes, p+q and -p-q are negatives of each other.
That is, $p+q = -1(-p-q)$ and $-p-q = -1(p+q)$.

42. If two factors are the negatives of each other, it is to our advantage to change the appropriate signs so that the <u>same</u> factor appears in both the numerator and denominator.
Write a fraction equivalent to

$$\frac{p(p+q)(2p-3q)}{-3(3p-2q)(-p-q)}$$

having the factor -p-q in both numerator and denominator.

42. $\frac{p(-p-q)(2p-3q)}{3(3p-2q)(-p-q)}$ or $\frac{-p(-p-q)(2p-3q)}{-3(3p-2q)(-p-q)}$ or $\frac{p(-p-q)(-2p+3q)}{-3(3p-2q)(-p-q)}$ or $\frac{p(-p-q)(2p-3q)}{-3(-3p+2q)(-p-q)}$

43. In each of the possible answers to the last frame, <u>only two factors</u> were multiplied by -1. Go back and check to make sure that this is true.

Now you have a fraction where the numerator and denominator have a common factor. Write an equivalent fraction in its lowest terms.

43. $\frac{p(2p-3q)}{3(3p-2q)}$ or $\frac{-p(2p-3q)}{-3(3p-2q)}$ or $\frac{p(-2p+3q)}{-3(3p-2q)}$ or $\frac{p(2p-3q)}{-3(-3p+2q)}$

44. Any of these are correct, so therefore they must all be equivalent. By assigning values to p and q, you can check to make sure that the fractions are equivalent.
<u>Remember</u>, don't assign values to p and q which will make the denominator equal to 0.
Why can't the denominator equal 0?

44. division by 0 is undefined
or
can't divide by 0

45. Express the following as equivalent fractions in their lowest terms.

(a) $\dfrac{-3a(-a-2)}{a^3b(a+2)}$

(b) $\dfrac{(a-b)(3a+b)}{b(b-a)}$

(c) $\dfrac{(x-y)(2x-3y)}{(y-x)(3y-2x)}$

45. (a) $= \dfrac{-3a(a+2)}{-a^3b(a+2)}$ Now both numerator and denominator have common factors of $-a(a+2)$ and so equals $\dfrac{3}{a^2b}$.

(b) $= \dfrac{(a-b)(3a+b)}{-b(a-b)}$ Now both numerator and denominator have common factors of a-b and so equals $\dfrac{3a+b}{-b}$.

(c) $= \dfrac{(x-y)(2x-3y)}{-1(x-y)(3y-2x)}$ Now divide both numerator and denominator by the common factor of x-y, getting $\dfrac{2x-3y}{-1(3y-2x)}$.
Now notice that 2x-3y and 3y-2x are also negatives of each other. Then, the above fraction equals $\dfrac{2x-3y}{+1(2x-3y)}$ and dividing both numerator and denominator by the common factor, we get 1/1 or 1.

46. In these last problems, you may have changed different signs from those indicated in the answers. It makes no difference which signs you changed as long as you multiplied two factors by -1.

Check and make sure that you have multiplied two factors by -1.

You will note in part c, four factors were multiplied by -1. The value of the fraction will remain the same if an even number of factors are multiplied by -1 as the total effect is that of multiplying by +1.

We have been careful to mention that both numerator and denominator must be divided by a common factor.
If we are to reduce the following fraction to lowest terms, we don't have factors, we have terms.

$\dfrac{2x^2 - 4x}{3x - 6}$ What could we do in order to find out whether the numerator and denominator have any common factors?

46. factor the numerator and denominator

47. Factoring the numerator and denominator,

$$\frac{2x^2 - 4x}{3x - 6} = \underline{\hspace{3cm}}$$

47. $\dfrac{2x(x-2)}{3(x-2)}$

48. Now the numerator and denominator of this fraction have a common factor of x-2 and dividing both numerator and denominator by this factor, we get ___________ .

48. $\frac{2x}{3}$ This fraction is in its lowest terms.

49. Reduce to lowest terms and state the procedure you used.

$$\frac{6a^4+3a^3b-9a^2b^2}{2a^2+3ab}$$

49. Factor both numerator and denominator getting

$$\frac{3a^2(a-b)(2a+3b)}{a(2a+3b)}$$

Divide both numerator and denominator by the common factor of $a(2a+3b)$ getting $3a(a-b)$.

50. Factoring a polynomial always means to express as <u>prime</u> factors.

Express each of the following as equivalent fractions in their lowest terms.

(a) $\frac{x^2-5x+6}{x^2-2x-3}$

(b) $\frac{ax-ay+bx-by}{cx-cy+dx-dy}$

(c) $\frac{27x^4y^2-18x^3y^3}{6xy^3-9x^2y^2}$

(d) $\frac{x^3-8}{x^2-4}$

(e) $\frac{4x^5-4x^3-108x^2+108}{(8x-8x^2)(x^2+3x+9)}$

(f) $\frac{24a^5b^6-36a^6b^5}{18a^8b^6}$

(g) $\frac{120x^7y^3z^2}{75x^5y^6z^2}$

(h) $\frac{2ab-cb-4ad+2cd}{2cb-4ab+cd-2ad}$

(i) $\frac{(x+2)x+1}{(x+1)x-2}$

50. (a) $\frac{(x-3)(x-2)}{(x-3)(x+1)} = \frac{x-2}{x+1}$

(b) $\frac{(a+b)(x-y)}{(c+d)(x-y)} = \frac{a+b}{c+d}$

(c) $\frac{9x^3y^2(3x-2y)}{3xy^2(2y-3x)} = \frac{3x^2(3x-2y)}{(2y-3x)} =$

$\frac{-3x^2(2y-3x)}{2y-3x} = -3x^2$

Don't forget about multiplying two factors by -1 in order to keep the same value.

(d) $\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \frac{x^2+2x+4}{x+2}$

(e) $\frac{(4x^3-108)(x^2-1)}{8x(1-x)(x^2+3x+9)} =$

51. Let us examine the fraction in part i of the last frame again.

You were given $\frac{(x-2)x+1}{(x+1)x-2}$.

The numerator of $(x-2)x+1$ tells you to multiply $(x-2)$ by $x$ and then add 1.

If you were told to multiply $(x-2)$ by $(x+1)$, it would be written $(x-2)(x+1)$. Notice that this is <u>not</u> the same as what you were given.

In order to reduce a fraction to its lowest terms, both numerator and denominator must be divided

$$\frac{4(x-3)(x^2+3x+9)(x-1)(x+1)}{8x(1-x)(x^2+3x+9)} =$$

$$\frac{4(x-3)(x-1)(x+1)}{8x(1-x)} = \frac{4(x-3)(x-1)(x+1)}{-8x(x-1)}$$

$$= \frac{(x-3)(x+1)}{-2x}$$

You might not have exactly the same answer, just make sure it is equivalent to this one. If you are in doubt, substitute a numerical value for x.

(f) $\frac{6a^5b^5(4b-6a)}{18a^8b^6} = \frac{4b-6a}{3a^3b}$

(g) divide both numerator and denominator by $15x^5y^3z^2$ as these are already factors. Answer is $8x^2/5y^3$

(h) $\frac{b(2a-c)-2d(2a-c)}{2b(c-2a)+d(c-2a)} =$

$$\frac{(b-2d)(2a-c)}{(2b+d)(c-2a)} = \frac{(-b+2d)(-2a+c)}{(2b+d)(c-2a)}$$

$$= \frac{-b+2d}{2b+d}$$

(i) WARNING (x-2) is not a factor of the whole numerator. (x-2) is only a factor of the first term of the numerator.

$$\frac{x^2-2x+1}{x^2+x-2} = \frac{(x-1)(x-1)}{(x+2)(x-1)} = \frac{x-1}{x+2}$$

by the common factor.

Is x-2 a factor of (x-2)x+1?

In the denominator (x+1)x-2, is x+1 a factor?

Is x-2 a factor of the denominator?

Is x+1 a factor of the numerator?

51. The answer is no to all of these questions. For x+1 to be a factor of the denominator, it must be multiplied by everything else in the denominator. This is true of any factor. In (x+1)x-2, x+1 is a factor of only the first term, not of the complete denominator.

52. Express each of the following as equivalent fractions in their lowest terms.

(a) $\frac{(2x+3)x+1}{(2x+3)(x+1)}$

(b) $\frac{(x-3)(x+2)}{x^2+2x}$

(c) $\frac{(3p-5)(p-2)}{(3p-5)p-2}$

(d) $\frac{a^6-b^6}{a^4-b^4}$

(e) $\frac{(3x^2+2xy-5y^2)(x+y)}{(3x^2+8xy+5y^2)(x-y)}$

52. (a) $\frac{2x^2+3x+1}{(2x+3)(x+1)} = \frac{(2x+1)(x+1)}{(2x+3)(x+1)}$

53. If we were to multiply 2/9 by 12/16 or $\frac{2}{9} \cdot \frac{12}{16}$, using the law

$= \frac{2x+1}{2x+3}$

(b) $\frac{(x-3)(x+2)}{x(x+2)} = \frac{x-3}{x}$

(c) $\frac{(p-2)(3p-5)}{3p^2-5p-2} = \frac{(p-2)(3p-5)}{(3p+1)(p-2)}$

$= \frac{3p-5}{3p+1}$

(d) $\frac{(a^2-b^2)(a^4+a^2b^2+b^4)}{(a^2-b^2)(a^2+b^2)} =$

$\frac{a^4+a^2b^2+b^4}{a^2+b^2}$

(e) $\frac{(3x+5y)(x-y)(x+y)}{(3x+5y)(x+y)(x-y)} = 1$

Note - the answer is not 0. You are dividing a number by itself and the result is 1.

for the multiplication of fractions, we would get $\frac{2\cdot 12}{9\cdot 16} = \frac{24}{144}$.

This fraction is not in its lowest terms. To put it in its lowest terms, we must divide both numerator and denominator by their common factors. This means dividing both numerator and denominator by 24 getting 1/6.

Since we can always get equivalent fractions by dividing both numerator and denominator by a common factor, we could do this before we multiply.

For example, $\frac{2}{9}\cdot\frac{12}{16} = \frac{2}{9}\cdot\frac{3}{4} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

This is the same result as when 24/144 is reduced to lowest terms.

Multiply: 27/8 by 28/81.
Remove common factors before you multiply.

53. $\frac{27}{8} \cdot \frac{28}{81} = \frac{1}{2} \cdot \frac{7}{3} = \frac{7}{6}$

54. Find the following products. Express in lowest terms.

(a) $\frac{1}{3} \cdot \frac{21}{16} \cdot 32$

(b) $\frac{8}{3} \cdot \frac{39}{56} \cdot \frac{7}{13}$

54. (a) Remember 32 is an integer, and any integer can be written with a denominator of 1.

$\frac{1}{3} \cdot \frac{21}{16} \cdot \frac{32}{1} = 14$

(b) Product is 1. Remember when both numerator and denominator are divided by a factor which is the same as the numerator and denominator or when a number is divided by itself, the result is 1.

55. When the numerator and denominator of a fraction are polynomials, we can divide by the common factors and then multiply or multiply and then reduce to lowest terms. Usually the first procedure gives us lower degree polynomials to deal with.

For example, $\frac{x^2-2x}{x^2-5x+6} \cdot \frac{2x^2-7x+3}{x^3-4x^2}$

Before we can reduce to lowest terms, we must first ________.

55. factor the polynomials

56. Factor these polynomials, divide both numerator and denominator by their common factors and multiply the resulting two fractions.

$\frac{x^2-2x}{x^2-5x+6} \cdot \frac{2x^2-7x+3}{x^3-4x^2} =$

56. $\frac{x(x-2)}{(x-2)(x-3)} \cdot \frac{(2x-1)(x-3)}{x^2(x-4)}$

$= \frac{2x-1}{x(x-4)}$

Both numerator and denominator can be divided by $x(x-2)(x-3)$.

57. (a) $\frac{34ab^3c^2}{(a-8)(a+8)} \cdot \frac{(2a+3)(a-8)}{51a^4b^7c}$

$= \frac{2c}{(a+8)} \cdot \frac{2a+3}{3a^3b^4} = \frac{2c(2a+3)}{3a^3b^4(a+8)}$

(b) $\frac{(a-4)(a^2+4a+16)}{(a-4)(a-4)} \cdot \frac{a(a-4)}{a(a^2+4a+16)}$

$= 1$

(c) $\frac{(6-5x)(6+5x)}{(5x-6)(x+1)} \cdot \frac{x-2}{(5x+6)(x-2)(x+2)}$

$= \frac{6-5x}{(5x-6)(x+1)(x+2)}$ and now multiplying two factors by -1, this becomes

$\frac{(-6+5x)}{(5x-6)(-x-1)(x+2)} = \frac{1}{(-x-1)(x+2)}$

There are other possible answers here. Just make sure that any other is equivalent to this one.

58. (a) $\frac{2}{5xy^2z^2}$

(b) 1

(c) $\frac{3p+5}{-p}$ Don't forget to multiply two factors by -1.

(d) $\frac{(2x-1)(x+1)}{(x-1)(2x+5)}$

(e) $\frac{(x-3)^2(2x+1)}{(x^2-7x-6)(x-1)}$

don't forget, x-1 is not a factor of the first numerator.

(f) 1

(g) $-\frac{(2x+1)(2x+5)}{x-1}$

or equivalent.

57. Find the following products. Express results in lowest terms.

(a) $\frac{34ab^3c^2}{a^2-64} \cdot \frac{2a^2-13a-24}{51a^4b^7c}$

(b) $\frac{a^3-64}{a^2-8a+16} \cdot \frac{a^2-4a}{a^3+4a^2+16a}$

(c) $\frac{36-25x^2}{5x^2-x-6} \cdot \frac{x-2}{5x^3+6x^2-20x-24}$

58. (a) $\frac{33x^3y^2z}{72x^5y^3} \cdot \frac{48x^7y^8z^3}{55x^6y^9z^6} =$

(b) $\frac{3a+6b}{4a+8b} \cdot \frac{8a-12b}{6a-9b} =$

(c) $\frac{p^2-16}{4-3p-p^2} \cdot \frac{3p^2+2p-5}{p^2-4p} =$

(d) $\frac{2x^2+x-1}{3x^2-13x+14} \cdot \frac{x^2-4}{x^2+x-2} \cdot \frac{3x^2+2x-21}{2x^2+11x+15}$

(e) $\frac{x(x-1)-4(x-1)+2}{(x-1)(x-2)} \cdot \frac{2x^2-5x-3}{x^2-7x-6} =$

(f) $\frac{x^3+64}{x^3y+4x^2y} \cdot \frac{2y^2-y-10}{y^3+2y^2+4y+8} \cdot \frac{x^2y^3+4x^2y}{x^2(2y-5)+(16-4x)(2y-5)}$

(g) $\frac{(2x-5)x-3}{(2x-5)(x-1)} \cdot \frac{4x^2+12x+5}{9-x^2} \cdot \frac{2x^2+x-15}{2x+1}$

59. Notice that the answers to the problems in the last frame are in factored form. You can multiply the factors together or you can leave the answers in factored form.

Addition of fractions.

If $b \neq 0$ and a, b and c are real numbers, then

$\frac{a}{b} + \frac{c}{b} = \frac{1}{b} \cdot a + \frac{1}{b} \cdot c = \frac{1}{b}(a+c) = \frac{a+c}{b}$

In the above definition, both fractions had the same denominator.

What would we have to do if we were to add 3/4 and 5/7?

59. Change both fractions to equivalent fractions having the same denominator.
or
get the least common denominator

60. In frame 143 of Chapter I, we defined the least common denominator (LCD) as follows.

The LCD is composed of every different prime factor in each of the original denominators and each factor is found in the LCD the maximum number of times it appears in any single denominator.

To find the LCD for 24 and 18, first express these as prime factors.

24 = __________

18 = __________

60. Review the meaning of prime factors in frame 140, Chapter I, if necessary.

$24 = 2 \cdot 2 \cdot 2 \cdot 3$
$18 = 2 \cdot 3 \cdot 3$

61. The different factors which are found here are 2 and 3.
The maximum number of times 2 is a factor in 24 is three times.
2 is a factor of 18 only once.
Therefore 2 is a factor in the LCD three times.

What is the maximum number of times 3 is a factor in any one of the denominators?

61. twice

62. Then the LCD contains two factors of 3.
The LCD for fractions with denominators of 24 and 18 contains three factors of 2 and two factors of 3, or the LCD is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ which equals 72.

Given three fractions with denominators of 48, 18 and 27, find the least common denominator.

62. $48 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$18 = 2 \cdot 3 \cdot 3$
$27 = 3 \cdot 3 \cdot 3$
Therefore, the LCD must have four factors of 2 and three factors of 3 or the LCD is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$ or is 432.

63. The LCD contains every different factor, so in this case must have a 2 and a 3. It also contains each factor the maximum number of times it appears in any single denominator, so 2 must be used as a factor four times and 3 must be used as a factor three times.

Let us consider adding $\frac{3a}{4b}$ and $\frac{2c}{b}$.
These fractions have different denominators, so we must ______.

63. find the LCD
or
change to equivalent fractions having the same denominators.

64. To find the LCD for rational functions, we follow the same procedure as in frames 60 - 62.

First, find the prime factors of each denominator and then include every different factor the maximum number of times it appears in any single denominator.

The LCD for $\frac{3a}{4b}$ and $\frac{2c}{b}$ is ______.

64. 4b

65. Now change to equivalent fractions, and find the sum.

$\frac{3a}{4b} + \frac{2c}{b} = \qquad\qquad =$

65. $\frac{3a}{4b} + \frac{2c}{b} = \frac{3a}{4b} + \frac{8c}{4b} = \frac{3a+8c}{4b}$

66. $\frac{3a}{4b}$, $\frac{8c}{4b}$ and $\frac{3a+8c}{4b}$ are examples of <u>rational functions</u>.

<u>Definition.</u>
A rational function is the quotient of two polynomials.

Which of the following are rational functions?

$$\frac{3q^2-7bc}{(x-3)(x+5)}, \frac{\sqrt{x+4}}{3x}, \frac{3a^3b^2c}{a^3-64}$$

66. The first and third are examples of rational functions.

The second one is <u>not</u> a rational function as the numerator is <u>not</u> a polynomial.

Note: Check the definitions in frames 16 and 35 of Chapter III if you have forgotten what a polynomial is.

67. Add. $\frac{1}{2x} + \frac{5}{3x} - \frac{3x-4}{12x}$

67. LCD is 12x.

$\frac{6}{12x} + \frac{20}{12x} - \frac{3x-4}{12x} =$

$\frac{1}{12x}(6+20-[3x-4]) = \frac{1}{12x}(6+20-3x+4)$

$= \frac{-3x+30}{12x} = \frac{3(-x+10)}{12x} = \frac{-x+10}{4x}$

68. Check your answer to the last frame if you have not done so before. Note the numerator of the last fraction. This fraction is to be subtracted, so the <u>entire numerator</u> must be subtracted. This changes the sign of <u>all</u> the terms in the numerator of the fraction preceded by the "-" sign.

$\frac{7a}{3x} + \frac{2}{5} - \frac{2a+5}{15x} =$

68. $\frac{35a+6x-(2a+5)}{15x} = \frac{35a+6x-2a-5}{15x} =$
$\frac{33a+6x-5}{15x}$

69. In the preceding problems, the denominators of the given fractions were in factored form. If fractions with the denominators $a^2-3a$ and $a^2-9$ were given, how would you determine what factors whould go in the LCD?

69. Factor each polynomial and then form the LCD by using each different factor the maximum number of times it is found in each single denominator.

70. For $\frac{6}{a^2-3a}$ and $\frac{3}{a^2-9}$ what is the LCD?

70. $a^2-3a = a(a-3)$

$a^2-9 = (a-3)(a+3)$

Therefore, the LCD is $a(a-3)(a+3)$

71. In order to add these two fractions, we must now change them to equivalent fractions having the same denominators.
$\frac{6}{a(a-3)} = \frac{6(a+3)}{a(a-3)(a+3)}$
since we may multiply both numerator and denominator by the same number as long as that number isn't 0.

$\frac{3}{(a-3)(a+3)} = \frac{?}{a(a-3)(a+3)}$ Why?

71. 3a belongs in the numerator because the numerator of the given fraction must be multiplied by a since the denominator has been multiplied by a.

72. So $\frac{6}{a^2-3a} + \frac{3}{a^2-9} =$
$\frac{6(a+3)}{a(a-3)(a+3)} + \frac{3a}{a(a-3)(a+3)}$
$=$ $=$

72. $\frac{6a+18+3a}{a(a-3)(a+3)} = \frac{9a+18}{a(a-3)(a+3)}$

73. Is this fraction in its lowest terms?

A fraction in its lowest terms means that the numerator and denominator have ______________.

73. yes

have no common factors.

Note. $9a+18$ can be factored and equals $9(a+2)$. However, this is not necessary since neither of these factors is the same as any of those in the denominator.

74. Look at the answer to frame 72. The denominator has been left in factored form. You may multiply the factors together. Either way is acceptable.
However, it is more convenient to leave the denominator in factored form in case the fraction can be reduced to lower terms. Notice that the denominators have been written in factored form throughout the problem in frame 72.
Look at frame 148, Chapter I, to

see a numerical example where the denominators were left in factored form throughout the problem.

What is the LCD for the following fractions?

$$\frac{x^2+2xy+y^2}{xy+3y^2},\ \frac{2x+y}{x+3y},\ \frac{x}{y}$$

74. $y(x+3y)$

75. Using the fractions in the last frame, change them to equivalent fractions having the denominator of $y(x+3y)$.

75. $\frac{x^2+2xy+y^2}{y(x+3y)},\ \frac{y(2x+y)}{y(x+3y)},\ \frac{x(x+3y)}{y(x+3y)}$

76. Complete: $\frac{x^2+2xy+y^2}{xy+3y^2} - \frac{2x+y}{x+3y} + \frac{x}{y} =$

76. $\frac{x^2+2xy+y^2}{y(x+3y)} - \frac{y(2x+y)}{y(x+3y)} + \frac{x(x+3y)}{y(x+3y)}$

$\frac{x^2+2xy+y^2-y(2x+y)+x(x+3y)}{y(x+3y)} =$

$\frac{x^2+2xy+y^2-2xy-y^2+x^2+3xy}{y(x+3y)} = \frac{2x^2+3xy}{y(x+3y)}$

77. Perform the indicated operations and simplify.

(a) $\frac{p-3}{3} + \frac{p+3}{6} - \frac{2p-4}{8} =$

(b) $\frac{4x}{3y} - \frac{4x+2y}{x-3y} =$

(c) $\frac{4}{y^2-1} + \frac{y}{y-1} - \frac{y}{y+1}$

(d) $\frac{1}{m} + \frac{2m}{m-3} - \frac{2m^3+m+1}{m^3-3m^2}$

(e) $\frac{x+y}{x-2y} - \frac{5xy-3y^2}{x^2+xy-6y^2}$

77. (a) LCD is 24.

$\frac{8(p-3)+4(p+3)-3(2p-4)}{24} = \frac{6p}{24} = \frac{p}{4}$

(b) LCD is $3y(x-3y)$

$\frac{4x(x-3y)-3y(4x+2y)}{3y(x-3y)} = \frac{4x^2-24xy-6y^2}{3y(x-3y)}$

(c) LCD is $(y-1)(y+1)$

$\frac{4+y(y+1)-y(y-1)}{(y-1)(y+1)} = \frac{4+2y}{(y-1)(y+1)}$

(d) LCD is $m^2(m-3)$

$\frac{m(m-3)+2m^3-(2m^3+m+1)}{m^2(m-3)} = \frac{m^2-4m-1}{m^2(m-3)}$

(e) LCD is $(x-2y)(x+3y)$

$\frac{(x+y)(x+3y)-(5xy-3y^2)}{(x-2y)(x+3y)} =$

$\frac{x^2+4xy+3y^2-5xy+3y^2}{(x-2y)(x+3y)} =$

78. Check your procedures in parts d and e if you have not done so before. Notice that in the case of the last fractions in both problems, the fraction was preceded by a negative sign and the numerator did not have to be changed as the denominator was was the same as the LCD. However, when the distributive law was applied, the numerator was placed in parentheses preceded by a minus sign or in other words, the numerator was multiplied by a -1.

Caution. In situations such as the ones described above, use parentheses or be sure to change the signs of all terms in the

$= \frac{x^2-xy+6y^2}{(x-2y)(x+3y)}$

numerator.

$\frac{2x+y}{x+y} - \frac{x-2y}{3x-4y} =$

78. $\frac{(2x+y)(3x-4y)-(x+y)(x-2y)}{(x+y)(3x-4y)} =$

$\frac{6x^2-5xy-4y^2-x^2+xy+2y^2}{(x+y)(3x-4y)} =$

$\frac{5x^2-4xy-2y^2}{(x+y)(3x-4y)} =$

79. Check your multiplication in the last frame. $-(x+y)(x-2y)$ means $-1(x+y)(x-2y)$, that is, you have three factors to multiply together.

$\frac{y^2}{y^2-4} - \frac{2y+1}{y-2} + \frac{y+3}{y+2}$

79. $\frac{y^2-(2y+1)(y+2)+(y+3)(y-2)}{(y-2)(y+2)} =$

$\frac{y^2-2y^2-5y-2+y^2+y-6}{(y-2)(y+2)} = \frac{-4y-8}{(y-2)(y+2)}$

$= \frac{-4(y+2)}{(y-2)(y+2)} = \frac{-4}{y-2}$

80. Remember all fractions should be reduced to their lowest terms.

Watch your multiplication in $-(2y+1)(y+2)$.

Consider the LCD for the following problem.

$\frac{2}{a^2-4a+4} + \frac{a-1}{a^2-4}$

The LCD here is ____________ .

80. $a^2-4a+4 = (a-2)(a-2)$
$a^2-4 = (a-2)(a+2)$

LCD is $(a-2)(a-2)(a+2)$

If necessary, check frames 60 and 64 for the definition of the LCD.

81. The LCD is $(a-2)(a-2)(a+2)$. The factor of $a-2$ appears twice in one of the original denominators and so must be included twice in the LCD.
If you had $(a-2)(a+2)$, this is <u>not</u> evenly divisible by $(a-2)(a-2)$.

$\frac{(a-2)(a+2)}{(a-2)(a-2)} = \frac{a+2}{a-2}$

So, $(a-2)(a+2)$ is <u>not</u> the LCD.

Complete: $\frac{2}{a^2-4a+4} + \frac{a-1}{a^2-4} =$

81. $\frac{2(a+2)+(a-1)(a-2)}{(a-2)(a-2)(a+2)} =$

$\frac{2a+4+a^2-3a+2}{(a-2)(a-2)(a+2)} = \frac{a^2-a+6}{(a-2)(a-2)(a+2)}$

82. $\frac{2x+3}{x^2+6x+9} + \frac{5-2x}{9-x^2}$

82. LCD is $(x+3)(x+3)(3-x)$.

$\frac{(2x+3)(3-x)+(5-2x)(x+3)}{(x+3)(x+3)(3-x)} =$

$\frac{-2x^2+3x+9-2x^2-x+15}{(x+3)(x+3)(3-x)} =$

83. Now, consider $\frac{p+1}{3-p} + \frac{p}{p-3}$ .

Notice that the denominators of the two fractions are the negatives of each other, that is, $-1(3-p) = -3+p$.

$$\frac{-4x^2+2x+24}{(x+3)(x+3)(3-x)}$$

All the signs of each term in 3-p are the opposite of all the signs of the terms of p-3. In this case, it is to our advantage to multiply two factors by -1 in one fraction getting an equivalent fraction. This will give us the same denominator for each fraction.

$$\frac{p+1}{3-p} = \frac{?}{-3+p}$$

83. -(p+1) or -p-1

84. Instead of writing $\frac{-p-1}{-3+p}$, we could have written $-\frac{p+1}{-3+p}$.
In $-\frac{p+1}{-3+p}$, we have multiplied two factors of $\frac{p+1}{3-p}$ by -1. One was the factor in the denominator and the other was the factor in front of the fraction.
$\frac{p+1}{3-p}$ can be considered as $(1)\frac{p+1}{3-p}$.
Therefore, we can write:
$\frac{p+1}{3-p} = -\frac{p+1}{-3+p}$ or $\frac{p+1}{3-p} = \frac{-p-1}{-3+p}$.

Complete:
$$\frac{p+1}{3-p} + \frac{p}{p-3} = -\frac{p+1}{-3+p} + \frac{p}{p-3} =$$

84. $\frac{-(p+1)+p}{p-3} = \frac{-1}{p-3}$

85. Suppose the answer were given as $\frac{1}{3-p}$, is this equal to $\frac{-1}{p-3}$? Why?

85. yes because two factors of the fraction -1/(p-3) have been multiplied by -1.

86. Write another fraction which is equal to both of these.
$$\frac{1}{3-p} = \frac{-1}{p-3} =$$

86. $-\frac{1}{p-3}$

87. Write one fraction equal to
$$\frac{a}{a-2} + \frac{2}{2-a}$$
which has a denominator of a-2.

87. $\frac{a}{a-2} - \frac{2}{a-2} = \frac{a-2}{a-2} = 1$
or
$\frac{a}{a-2} + \frac{-2}{a-2} = \frac{a+ -2}{a-2} = 1$

88. Perform the indicated operations in each of the following and express the answers in simplest form.
(a) $\frac{14p-15q}{5p-6q} + \frac{4p-3q}{6q-5p}$

$\dfrac{-4x^2+2x+24}{(x+3)(x+3)(3-x)}$

All the signs of each term in 3-p are the opposite of all the signs of the terms of p-3. In this case, it is to our advantage to multiply two factors by -1 in one fraction getting an equivalent fraction. This will give us the same denominator for each fraction.

$\dfrac{p+1}{3-p} = \dfrac{?}{-3+p}$

83. -(p+1) or -p-1

84. Instead of writing $\dfrac{-p-1}{-3+p}$, we could have written $-\dfrac{p+1}{-3+p}$.
In $-\dfrac{p+1}{-3+p}$, we have multiplied two factors of $\dfrac{p+1}{3-p}$ by -1. One was the factor in the denominator and the other was the factor in front of the fraction.
$\dfrac{p+1}{3-p}$ can be considered as $(1)\dfrac{p+1}{3-p}$.
Therefore, we can write:
$\dfrac{p+1}{3-p} = -\dfrac{p+1}{-3+p}$ or $\dfrac{p+1}{3-p} = \dfrac{-p-1}{-3+p}$.

Complete:
$\dfrac{p+1}{3-p} + \dfrac{p}{p-3} = -\dfrac{p+1}{-3+p} + \dfrac{p}{p-3} =$

84. $\dfrac{-(p+1)+p}{p-3} = \dfrac{-1}{p-3}$

85. Suppose the answer were given as $\dfrac{1}{3-p}$, is this equal to $\dfrac{-1}{p-3}$ ? Why?

85. yes because two factors of the fraction -1/(p-3) have been multiplied by -1.

86. Write another fraction which is equal to both of these.
$\dfrac{1}{3-p} = \dfrac{-1}{p-3} =$

86. $-\dfrac{1}{p-3}$

87. Write one fraction equal to
$\dfrac{a}{a-2} + \dfrac{2}{2-a}$
which has a denominator of a-2.

87. $\dfrac{a}{a-2} - \dfrac{2}{a-2} = \dfrac{a-2}{a-2} = 1$
or
$\dfrac{a}{a-2} + \dfrac{-2}{a-2} = \dfrac{a+ -2}{a-2} = 1$

88. Perform the indicated operations in each of the following and express the answers in simplest form.
(a) $\dfrac{14p-15q}{5p-6q} + \dfrac{4p-3q}{6q-5p}$

(b) $\frac{2cd}{c^2-d^2} - \frac{2c-d}{d-c}$

(c) $\frac{y+2x}{2x^2+xy-3y^2} + \frac{2}{y^2+xy-2x^2}$

(d) $\frac{2x^2+8xy+6y^2}{x^3+y^3} \cdot \frac{3y-x}{x^2-9y^2}$

88. (a) $\frac{14p-15q}{5p-6q} - \frac{4p-3q}{5p-6q} =$

$\frac{14p-15q-4p+3q}{5p-6q} = \frac{10p-12q}{5p-6q} =$

$\frac{2(5p-6q)}{5p-6q} = 2$

(b) $\frac{2cd}{(c-d)(c+d)} + \frac{2c-d}{c-d} =$

$\frac{2cd+(2c-d)(c+d)}{(c-d)(c+d)} = \frac{2c^2+3cd-d^2}{(c-d)(c+d)}$

(c) $\frac{y+2x}{(2x+3y)(x-y)} + \frac{2}{(y+2x)(y-x)}$

$= \frac{y+2x}{(2x+3y)(x-y)} - \frac{2}{(y+2x)(x-y)} =$

$\frac{(y+2x)(y+2x)-2(2x+3y)}{(2x+3y)(x-y)(y+2x)} =$

$\frac{y^2+4xy+4x^2-4x-6y}{(2x+3y)(x-y)(y+2x)}$

(d) Be careful here. This is multiplication, not addition. You don't need a LCD in multiplication.

$\frac{2(x+3y)(x+y)}{(x+y)(x^2-xy+y^2)} \cdot \frac{-(-3y+x)}{(x-3y)(x+3y)}$

$= \frac{-2}{x^2-xy+y^2}$

89. $\frac{x}{xp-yp-x^2+xy} - \frac{y}{y^2-py-xy+xp} + \frac{p}{xp-p^2-xy+yp}$

89. $\frac{x}{(x-y)(p-x)} - \frac{y}{(y-p)(y-x)} + \frac{p}{(x-p)(p-y)} =$

$\frac{x}{(x-y)(p-x)} + \frac{y}{(y-p)(-y+x)} + \frac{p}{(-x+p)(-p+y)} =$

$\frac{x(y-p)+y(p-x)+p(x-y)}{(x-y)(p-x)(y-p)} =$

$\frac{0}{(x-y)(p-x)(y-p)} = 0$

90. There are two things to note in the last frame. First, you may not have multiplied the same factors by -1, but the value of each fraction would be equal to those in the answer.
Secondly, when the terms in the numerator are combined, the sum is 0 and if the numerator of a fraction is 0 and the denominator is not 0, then the value of the fraction is 0.

What is the value of $\frac{(x-y)(x-p)}{(x-p)(x-y)}$ ?

90. 1

NOT 0. This problem asks you to divide both numerator and denominator by the same number, and when a number is divided by itself, the result is 1.

91. Complete the following.

(a) $\frac{b}{b+1} + \frac{1}{b+1}$

(b) $\frac{b-1}{b+1} + \frac{2}{b+1}$

(c) $\frac{6}{2b+2} - \frac{3}{b+1}$

91. (a) $\frac{b+1}{b+1} = 1$

(b) $\frac{b-1+2}{b+1} = \frac{b+1}{b+1} = 1$

(c) $\frac{6}{2(b+1)} - \frac{3}{b+1} = \frac{6-3(2)}{2(b+1)} = 0$

92. Division of fractions.

If a, b, c and d are real numbers, and b, c and d are not 0, then

$$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} .$$

This is true because $\frac{ad}{bc} \cdot \frac{c}{d} = \frac{a}{b}$ .

We learned some time ago that division is the inverse of _____.

We also learned that dividing by 2 gave the same result as multiplying by _____ .

92. multiplication

1/2

93. In other words, dividing by a number gives the same result as multiplying by the reciprocal of a number.

If c and d aren't 0, the reciprocal of $\frac{c}{d}$ is __________ .

What must always be true of a number and its reciprocal?

93. $\frac{d}{c}$

A number multiplied by its reciprocal equals 1.

94. Then the definition given in frame 92, makes use of the reciprocal of the number in the denominator of the fraction and also uses the fact that multiplication is the inverse of division.

$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$ because dividing by a number other than 0 gives the same result as multiplying by the reciprocal of that number.

$\frac{2}{7} \div \frac{12}{63} =$ $\qquad$ $=$

94. $\frac{2}{7} \cdot \frac{63}{12} = \frac{3}{2}$

95. Perform the indicated operations and simplify.

(a) $\frac{39}{72} \div \frac{52}{24}$

(b) $\frac{64}{21} \div \frac{42}{9} \cdot \frac{10}{27}$

(c) $\frac{64}{21} \div (\frac{42}{9} \cdot \frac{10}{27})$

95. (a) $\frac{39}{72} \cdot \frac{24}{52} = \frac{1}{4}$

(b) $\frac{64}{21} \cdot \frac{9}{42} \cdot \frac{10}{27} = \frac{32(10)}{(21)(21)(3)}$

(c) Be careful here. This is different from part b. This says to find the value of the number in parentheses and then to divide by it.

$\frac{42}{9} \cdot \frac{10}{27} = \frac{14(10)}{3(27)}$

Our problem now becomes

$\frac{64}{21} \div \frac{14(10)}{3(27)} = \frac{64}{21} \cdot \frac{3(27)}{14(10)} =$

$\frac{16(27)}{7(7)(5)} = \frac{432}{245}$

96. Take a careful look at parts b and c of the preceding frame. These two problems are not the same. Make sure you understand the difference between them.
Also note that the answer to part b was left in factored form. The answer to part c could have been left in factored form.
Common factors in the numerator and denominator can be found more quickly if factors are used
If the fractions contain polynomials, the rule for the division of fractions given in frame 92 applies as long as the denominators aren't 0.
Why can't the denominators equal 0?

96. division by 0 isn't defined
or
can't divide by 0

97. In $\frac{3}{a-1} \div \frac{4}{7}$, a can't have the value of _____ .

97. 1

98. In $\frac{2}{x+3} \div \frac{1}{x-2}$, x can't have the values of __________ .

98. -3 and 2

99. In $\frac{3}{p+1} \div \frac{p+2}{2}$, p can't have the values of ______ and ______.
(You have to complete the division in order to get both values.)

99. -1 and -2

100. Rational functions (fractions containing polynomials) are divided in the same way as fractions containing rational numbers.
Namely, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$
providing b, c and d $\neq$ 0.

(a) $\frac{8a^2}{5w^2} \div \frac{14a}{35w^4} =$

(b) $\frac{a^2+ab-2b^2}{15ab} \div \frac{a^2-4b^2}{25ab^2} =$

100. (a) $\frac{8a^2}{5w^2} \cdot \frac{35w^4}{14a} = \frac{4aw^2}{1} = 4aw^2$

(b) $\frac{(a+2b)(a-b)}{15ab} \cdot \frac{25ab^2}{(a-2b)(a+2b)}$

$= \frac{5b(a-b)}{3(a-2b)}$

101. Remember that in changing to equivalent fractions, we multiply or divide both numerator and denominator by the same factor. Therefore, we must factor the polynomials before we can reduce the fractions.

Perform the indicated operations and simplify.

(a) $\frac{x^2+6x-16}{x^2-9x+14} \div \frac{x^2+10x+16}{x^2-8x+7}$

(b) $\frac{a^2-7a+10}{3a^5b} \div \frac{(25-a^2)(a-2)^2}{24a^4b^3}$

(c) $\frac{(y+2)^4(y-2)^2}{(y^4-16)(y^2-4)} \cdot \frac{2y^3+8y}{y^2+4y+4} \div 4y^3$

(d) $\frac{x^3+y^3}{xy} \div \frac{x^2-xy+y^2}{3x+2y} \cdot \frac{x^3y^3}{3x^2+5xy+2y^2}$

(e) $\frac{p^2-3p}{p^2-3p+9} \div \left[\frac{(p+3)^2}{p^2-9} \cdot \frac{(p-3)^3}{p^3+27}\right]$

(f) $\frac{6a^2-11ab+3b^2}{6a^2-5ab+b^2} \div \frac{9b^2-4a^2}{3b^2-4ab-4a^2}$

(g) $\frac{(a-3)3a+1(a+5)}{(3a+1)(a-3)} \div \frac{(a-1)(3a-5)}{(a-1)6a-1(a+3)}$

(h) $\frac{c-d}{c} - \frac{d}{c+d} - \frac{d^2}{c^2+cd}$

(i) $\frac{x}{x^2+3x+9} + \frac{x-2}{x^2-3x} - \frac{1}{x^3-27}$

(j) $\frac{a}{a^2-a-12} + \frac{3}{a^2-9} - \frac{a+1}{12-7a+a^2}$

101. (a) $\frac{(x+8)(x-2)}{(x-7)(x-2)} \cdot \frac{(x-7)(x-1)}{(x+8)(x+2)} = \frac{x-1}{x+2}$

(b) $\frac{(a-5)(a-2)}{3a^5b} \cdot \frac{24a^4b^3}{(5-a)(5+a)(a-2)^2}$

$= \frac{-8b^2}{a(5+a)(a-2)}$

Remember to change the signs of two factors. Your answer may be different than this, just make sure it is equivalent to this.

(c) $\frac{(y+2)^4(y-2)^2}{(y^2+4)(y+2)^2(y-2)^2} \cdot$

$\frac{2y(y^2+4)}{(y+2)(y+2)} \cdot \frac{1}{4y^3} =$

102. A complex fraction is a fraction which has fractions in the numerator or denominator or in both places.

$\frac{2/3}{5}$, $\frac{24}{\frac{16}{5}}$, and $\frac{\frac{2}{3}}{\frac{27}{18}}$ are examples of complex fractions.

$\frac{\frac{2}{3}}{5}$ is the same as writing

$\frac{2}{3} \div 5$ and so would equal $\frac{2}{3} \cdot \frac{1}{5}$

which equals $\frac{2}{15}$. Find the values of the other two complex fractions given above.

$\frac{1}{2y^2}$

(d) $\frac{(x+y)(x^2-xy+y^2)}{xy} \cdot \frac{3x+2y}{x^2-xy+y^2} \cdot \frac{x^3y^3}{(3x+2y)(x+y)} =$

$\frac{x^2y^2}{1} = x^2y^2$

(e) $\frac{p(p-3)}{p^2-3p+9} \div \left[\frac{(p+3)^2}{(p-3)(p+3)} \cdot \frac{(p-3)^3}{(p+3)(p^2-3p+9)}\right]$

$= \frac{p(p-3)}{p^2-3p+9} \div \frac{(p-3)^2}{p^2-3p+9} = \frac{p}{p-3}$

(f) $\frac{(3a-b)(2a-3b)}{(3a-b)(2a-b)} \cdot \frac{(3b+2a)(b-2a)}{(3b-2a)(3b+2a)}$

$= 1$ (Don't forget to multiply two factors by -1.)

(g) $\frac{3a^2-8a+5}{(3a+1)(a-3)} \div \frac{(a-1)(3a-5)}{6a^2-7a-3} =$

$\frac{(3a-5)(a-1)}{(3a+1)(a-3)} \cdot \frac{(2a-3)(3a+1)}{(3a-5)(a-1)} =$

$\frac{(2a-3)}{a-3}$

(h) This is <u>addition</u> so you must get an LCD. LCD is c(c+d).

$\frac{(c-d)(c+d)-cd-d^2}{c(c+d)} = \frac{c-2d}{c}$

(i) LCD is $x(x-3)(x^2+3x+9)$.

$\frac{x^2(x-3)+(x-2)(x^2+3x+9)-x}{x(x-3)(x^2+3x+9)} =$

$\frac{2x^3-2x^2+2x-18}{x(x-3)(x^2+3x+9)}$

(j) LCD is $(a-4)(a+3)(a-3)$.

$\frac{a(a-3)+3(a-4)-(a+1)(a+3)}{(a-4)(a+3)(a-3)} =$

$\frac{-4a-15}{(a-4)(a+3)(a-3)}$

102. $\frac{24}{\frac{16}{5}} = \frac{24}{1} \cdot \frac{5}{16} = \frac{15}{2}$

$\frac{\frac{2}{3}}{\frac{27}{18}} = \frac{2}{3} \cdot \frac{18}{27} = \frac{4}{9}$

103. 4 by 22

103. Given the fraction, $\frac{2+4}{22-4}$, we could write it as $(2+4) \div (22-4)$. We <u>can't</u> write it as $2+4 \div 22-4$.

This last statement tells us to divide ______ by ______.

104. Given the fraction, $\frac{2+4}{22-4}$, it means to divide ______ by ______ and

this can be simplified to read divide ____ by ____.

104. (2+4) by (22-4)

simplified to 6 divided by 18 or to 1 divided by 3

105. Write $\dfrac{(2+4)}{\frac{22-4}{2}}$

as one number divided by another and simplify.

105. $(2+4) \div (\frac{22-4}{2})$ or $\frac{6}{9} = \frac{2}{3}$

106. Write $(x^2 + \frac{3}{x}) \div (1 + \frac{x}{2})$

in fractional form. DO NOT simplify.

106. $\dfrac{x^2 + \frac{3}{x}}{1 + \frac{x}{2}}$

107. Write another expression equivalent to

$$x^2 + \frac{3}{x} \div 1 + \frac{x}{2}$$

107. $x^2 + \frac{3/x}{1} + \frac{x}{2}$

Notice that this is not the same as the answer to the last frame.

108. Simplify: $\dfrac{2 + \frac{3}{4}}{\frac{1}{8} + 1}$

In this problem, you must first do what operation? If you can't decide, rewrite the expression as something divided by something. Do the simplifying.

108. First combine the terms in the numerator and denominator. This equals

$(2 + \frac{3}{4}) \div (\frac{1}{8} + 1)$ or equals

$\frac{11}{4} \div \frac{9}{8} = \frac{22}{9}$

109. $\dfrac{\frac{3}{x} - 4}{9 - 16x^2} = \qquad =$

109. $(\frac{3}{x} - 4) \div (9-16x^2) =$

$\frac{3-4x}{x} \cdot \frac{1}{(3-4x)(3+4x)} = \frac{1}{x(3+4x)}$

110. In the case of any complex fraction, the numerator and denominator must be expressed as one fraction before any further simplification can be done. You may express it as one quantity divided by another quantity if you want to. Be sure to use parentheses in the proper places. If a problem is given with parentheses in it, be sure and pay attention to the signs of grouping and what they signify.
Perform the indicated operations and simplify.

(a) $(2 - \frac{5}{x} + \frac{2}{x^2}) \div (1 - \frac{4}{x^2})$

(b) $$\frac{\frac{6}{x^2} + \frac{5}{x} + 1}{3 + \frac{2}{x}}$$

(c) $$\frac{1 - \frac{1}{x^3}}{1 + \frac{1}{x-2}}$$

110. (a) $(\frac{2x^2-5x+2}{x^2}) \div (\frac{x^2-4}{x^2}) =$

$\frac{(2x-1)(x-2)}{x^2} \cdot \frac{x}{(x-2)(x+2)} = \frac{2x-1}{x+2}$

(b) $\frac{6+5x+x^2}{x^2} \div \frac{3x+2}{x} =$

$\frac{(3+x)(2+x)}{x^2} \cdot \frac{x}{3x+2} =$

$\frac{(3+x)(2+x)}{x(3x+2)}$

(c) $\frac{x^3-1}{x^3} \div \frac{x-2+1}{x-2} =$

$\frac{(x-1)(x^2+x+1)}{x^3} \cdot \frac{x-2}{x-1} =$

$\frac{(x^2+x+1)(x-2)}{x^3}$

111. In part c of the last frame, the denominator became $\frac{x-2+1}{x-2}$. Note that we must combine similar terms before we can decide what factor we have and so we write $\frac{x-1}{x-2}$.

$$\frac{x + 2 - \frac{18}{x-5}}{\frac{x^2-16}{x^2-25}} =$$

111. $\frac{(x=2)(x-5)-18}{x-5} \div \frac{x^2-16}{x^2-25} =$

$\frac{x^2-3x-10-18}{x-5} \cdot \frac{(x-5)(x+5)}{(x-4)(x+4)} =$

$\frac{x^2-3x-28}{x-5} \cdot \frac{(x-5)(x+5)}{(x-4)(x+4)} =$

$\frac{(x-7)(x+4)}{x-5} \cdot \frac{(x-5)(x+5)}{(x-4)(x+4)} =$

$\frac{(x-7)(x+5)}{x-4}$

112. Note the numerator of the last fraction. In going from the second to the third step, we had to combine similar terms to get $x^2-3x-28$ before factoring. You will often find that you don't do the steps exactly in the same order as in the answer, but be sure yours are equivalent. You will often find some steps are missing in the answers, but you ought to be able to supply these by this time.

$$\frac{\frac{2b}{a-b} - 1}{2 - \frac{a+b}{a-b}}$$

112. $\frac{2b-(a-b)}{a-b} - \frac{2(a-b)-(a+b)}{a-b} =$

$\frac{3b-a}{a-b} \cdot \frac{a-b}{a-3b} = -1$

Remember to multiply two factors by -1.

113. Perform the indicated operations and simplify.

(a) $\frac{a^4-b^4}{a^2} \cdot \frac{a^2b-b^3+a^2c-cb^2}{ab+ac}$

(b) $\frac{6x^3y^4}{x^3-8y^3} \cdot \frac{x^3-4xy^2}{27x^5y^2}$

(c) $\frac{a+c}{a-c} \cdot \frac{c^2-a^2}{3a+3c}$

(d) $\frac{(x-y)^4}{3x-y} - \frac{x^2-2xy+y^2}{6x^2+xy-y^2}$

(e) $\frac{x+y}{x-y} - \frac{3x-y}{x-y}$

(f) $\frac{a}{a+b} + \frac{b}{a-b}$

(g) $\frac{1-x}{1-x} + x - \frac{1-x^2}{1-x}$

(h) $\frac{5}{4m^2-1} - \frac{1}{2m-1} + \frac{4}{2m+1}$

(i) $\frac{2x^2+3x-12}{x^3-4x^2-9x+36} + \frac{1}{3-x}$

(j) $\frac{1+\frac{a}{b}}{a+b}$

(k) $\frac{3a+\frac{8a-1}{a-2}}{3a-\frac{2a-1}{a+2}}$

113. (a) $\frac{(a^2+b^2)(a-b)(a+b)}{a^2} \cdot \frac{(b+c)(a^2-b^2)}{a(b+c)}$

$\frac{(a^2+b^2)(a^2-b^2)}{a^3}$

(b) $\frac{6x^3y^4 \cdot x(x-2y)(x+2y)}{(x-2y)(x^2+2xy+4y^2)27x^5y^2} =$

$\frac{2y^2(x+2y)}{9x(x^2+2xy+4y^2}$

(c) $\frac{-(c+a)}{3}$ Two factors must be multiplied by -1.

(d) $\frac{(x-y)^4(6x^2+xy-y^2)}{(3x-y)(x-y)(x-y)} = (x-y)^2(2x+y)$

(e) $\frac{(x+y)-(3x-y)}{x-y} = \frac{-2x+2y}{x-y} = -2$

(f) $\frac{a(a-b)+b(a+b)}{(a-b)(a+b)} = \frac{a^2+b^2}{(a-b)(a+b)}$

(g) $\frac{1-x+x(1-x)-1+x^2}{1-x} = 0$

(h) $\frac{5-(2m+1)+4(2m-1)}{(2m-1)(2m+1)} = \frac{6m}{(2m-1)(2m+1)}$

(i) $\frac{2x^2+3x-12-(x-4)(x+3)}{(x-4)(x-3)(x+3)} =$

$\frac{x^2+4x}{(x-4)(x-3)(x+3)}$

(j) $\frac{b+a}{b} \cdot \frac{1}{a+b} = \frac{1}{b}$

(k) $\frac{3a(a-2)+8a-1}{a-2} \cdot \frac{a+2}{3a(a+2)-(2a-1)} =$

$\frac{(3a-1)(a+2)}{(a-2)(3a+1)}$

# Chapter 6 - Linear and Fractional Equations

This chapter deals with the solution of first degree equations. We will also be concerned with analyzing word statements, translating these word statements into symbols, obtaining equations which express relationships given in the word statements and solving these equations. Knowledge of the techniques and skills presented in the first five chapters is necessary before proceeding with this chapter.

1. An equation is a statement of equality between two quantities.

$x+2y = x+3y-y$ is an example of an equation. This equation is called an <u>identity</u> because it is true for all values of x and y.

We have some identities which are true for all values of the variable with one or two exceptions. For example, $\frac{x}{x-3} + \frac{1}{x-3} = \frac{x+1}{x-3}$ is true for all values of x except 3. Why can't $x = 3$?

1. because if $x = 3$ the denominator would equal 0 and division by 0 isn't defined

2. We are going to be dealing with the solution of <u>conditional equations</u>. These are equations which are true for only one, two or some finite number of values. They impose a condition or conditions on the variable.
$x - 2 = 5$ is a conditional equation. There is only one value of x which makes this statement true.

Actually we can substitute any real number in place of the variable, but if we let x take on any value other than 7, we get a false statement. For example, if $x = 3$ then $x-2=5$ becomes $3-2=5$ which is <u>not</u> true.

$x-2=5$ is an example of a <u>linear</u> equation.
A <u>linear equation</u> is an equation in which the variable appears only to the first power after similar terms are combined.
Which of the following are linear equations?
(a) $2(x-2) + 7 = 3x-1$
(b) $x(x+2) - 3 = 0$

(c) $\frac{x-2}{2} - \frac{3}{4} = \frac{1}{2}$

(d) $(2x-1)(2x+3) - (4x^2+3x) = 0$

2. a, c and d are linear equations.

Be sure you did the multiplication and combined similar terms before you decided. While it isn't necessary in part c to get the LCD before you decided whether or not the equation was a linear one, you must get the LCD in more complex problems.

3. Two equations are equivalent if they have the same solutions.

It may be verified by direct substitution that a=2 is a solution of 2a=4 and of 3a-6=0. Thus 2 is a solution for both equations. A solution to an equation is called the root of the equation.

3a-6=0 is a ________ equation and 2 is a ____ of this equation.

3. linear

root

4. In the last frame, 2 was a root of both the equation 2a=4 and of 3a-6=0. If 2 is the only solution of these two equations or if some other number is also the solution of both equations, then these equations are equivalent equations.

2a=4 and 3a-6=0 are both linear equations and linear equations have at most one solution or root. Thus, 2a=4 and 3a-6=0 are e_____________ equations.

4. equivalent

Remember - equivalent equations must have exactly the same solution or solutions.

5. In each of the following, you are given two equations and a value of the variable. These are all linear equations so have at most _______ root.

5. one

6. Determine by substitution if the given value of the variable is a root of the equation and state whether the two equations are equivalent equations.

(a) $3x-2 = 2x+1$; $4(x-3) = 0$; $x=3$.

(b) $x(x-1) - (x-2)(x+3) = 2$; $\frac{3}{x-2} + \frac{x}{2} = x$; $x = 2$.

(c) $4x = x+9$; $9x-4 = 3x-22$; $x=3$.

6. only the equations in part a are equivalent equations
   (b) 2 is a root of the first equation, but not of the second one. In the second one 2 makes the first denominator equal to 0 and division by 0 is undefined.
   (c) 3 is a root of the first equation but not of the second one.

7. In solving equations, it is often convenient to find an equivalent equation, so we must be able to determine when we have an equivalent equation.

Equivalent equations.
1. An equation equivalent to the given equation is obtained when the same quantity is added to or subtracted from both sides of the equation.
2. An equation equivalent to the given equation is obtained when both sides of the equation are multiplied or divided by the same nonzero number.

Given the equation $x+2 = 5$, is $x = 3$ equivalent to it? Why?

7. yes because the same number, -2, has been added to both sides of the equation
or
because 2 has been subtracted from both sides of the equation.

8. Given the equation $6x-18 = 15$, is $2x-6 = 5$ equivalent to it? Why?

8. yes because both sides of the equation were divided by 3.

9. Given the equation $\frac{1}{2x} = 3$ is $1 = 6x$ equivalent to it? Why?

9. It is not equivalent if $x = 0$, as then the denominator of the left side of the original equation is 0 and a fraction with 0 in the denominator isn't defined. These two equations are equivalent for all values of x except $x = 0$ because both sides of the equation have been multiplied by the same nonzero number.

10. Given the equation $1-x = 4$, is the equation $4-4x = 16$ equivalent to it? Why?
    (b) is $(1-x)(2+x) = 4(2+x)$ equivalent to it? Why?
    (c) is $\frac{1}{2} - x = 2$ equivalent to it? Why?
    (d) is $\frac{1-x}{x+2} = \frac{4}{x+2}$ equivalent to it? Why?

10. (a) yes because both sides have been multiplied by the nonzero number of 4.
    (b) yes if $x \neq -2$ because then both sides of the equation have been multiplied by the nonzero quantity of $x+2$.

11. In solving linear equations, we change the given equation to an equivalent equation until we have arranged the equation so that all the terms containing the variable are on one side and all other terms are on the other side.

(c) no because the complete left side of the equation hasn't been divided by 2.
(d) yes if $x \neq -2$ because then both sides of the equation have been divided by the nonzero quantity of x+2.

Then we continue changing to equivalent equations until we get 1 multiplied by the variable equal to a number. This number is the root of the equation provided substitution of this value makes the original equation true. For example, to solve $5(x-2) + 1 = 3x+7$, we could first remove the parentheses, getting $5x-10+1 = 3x+7$.
Adding $-3x$ to both sides of this equation, we get the equivalent equation $2x-10+1 = 7$.
Adding 9 or subtracting -9 from both sides of this equation, we get the equivalent equation $2x=16$.
Dividing both sides of this equation by 2, we get the equivalent equation $x = 8$.

Substituting 8 for x in the original equation gives a true statement, so $x = 8$ is a root of the equation $5(x-2) + 1 = 3x+7$.

Notice that in the solution given above, we have equivalent equations in all cases since we added the same quantity to both sides of the equation and divided both sides of the equation by the same nonzero constant.

Using the equation $5(x-2) + 1 = 3x+7$ find equivalent equations by first adding -1 to both sides of the equation and then by dividing both sides of the equation by 5.

11. $5(x-2) + 1 = 3x+7$
Adding -1 to both sides, we get $5(x-2) = 3x+6$.
Dividing both sides by 5, we get
$x-2 = \frac{3x}{5} + \frac{6}{5}$

12. Take the last equation obtained to frame 11 and find equivalent equations by first adding $\frac{-3x}{5}$ to both sides and then by adding 2 to both sides of the equation.

12. Adding $\frac{-3}{5}x$ to both sides, we get $\frac{2}{5}x - 2 = \frac{6}{5}$.
Adding 2 to both sides, we get $\frac{2}{5}x = \frac{16}{5}$.

13. We now have the equation $\frac{2}{5}x = \frac{16}{5}$ or $\frac{2x}{5} = \frac{16}{5}$. What two processes can we do to find out the value of x? Will you obtain equivalent equations in each case? Why?

13. Divide both sides of the equation by 2 and multiply both sides of the equation by 5. (These operations can be done in either order.)

Yes, equivalent equations are obtained in both cases because multiplying and dividing both sides of an equation by the same nonzero quantity gives equivalent equations.

14. Perform these two operations and obtain the equivalent equations. Is the value obtained for x a root to the equation? Why?

14. Multiplying both sides by 5 gives $2x = 16$.
Dividing both sides by 2 gives $x = 8$.

$x = 8$ is a root because when it is substituted in the original equation, a true statement is obtained.

15. This root of the equation and any root of the equation can always be checked by direct substitution in the original equation. When the value of the variable is substituted in the original equation, a true statement must be obtained.

The equation $5(x-2)+1=3x+7$ was solved in frame 11 by obtaining certain equivalent equations. In frames 11 - 14, you were told to find other equivalent equations starting with this same equation. Note that the same root was obtained in both cases.

What can you say about solving equations by obtaining equivalent equations.

15. It makes no difference the order in which operations are done as long as the same operation is done to both sides of the equation and equivalent equations are obtained.

16. Find the root of the following equations and check your result. In parts a and b state the process you used to obtain equivalent equations.

(a) $3a-4 = 5a+6$

(b) $\frac{3x}{4} - 3 = \frac{1}{2}x + 1$

(c) $3(x-4) - 4(2x+1) = 4$

(d) $(2x-1)(3x+2)-(6x+1)(x-3) = 13$

(e) $3p(2p+3) - 4(p-2) = 6p^2-2(p+3)$

(f) $\frac{2x-17}{3} - \frac{8x+11}{6} = \frac{5x}{6}$

(g) $\frac{1}{y} - \frac{5}{y} = 6$

(h) $\frac{a-3}{a+2} - \frac{a-2}{a+8} = 0$

16. (a) Adding 4 to both sides and subtracting 5a from both sides we get $-2a=10$. Dividing both sides by $-2$, we get $a = -5$.
(b) Subtracting $\frac{x}{2}$ from both sides and adding 3 to both sides, we get $\frac{1}{4}x=4$. Multiplying both sides by 4, we get $x=16$.
(c) $3x-12-8x-4=4$
$-5x=20$, $x = -4$.
(d) $6x^2+x-2-(6x^2-17x-3)=13$
$18x=12$, $x = 2/3$.
(e) $6p^2+9p-4p+8=6p^2-2p-6$
$7p= -14$, $p = -2$.
(f) $4x-34-8x-11=5x$
$-9x=45$, $x = -5$.
(g) $1-5=6y$ if $y \neq 0$.
$y = -2/3$, if $y \neq 0$.
(h) $(a-3)(a+9)-(a-2)(a+2)=0$ if $a \neq -2$ and $a \neq -8$.
$5a=20$, $a = 4$.

17. In part g of the last frame, why can't $y = 0$?

In part h of the last frame, why can't $a = -2$ or $-8$?

In part h of the last frame, if $a \neq -2$ and $a \neq -8$, why is the given equation equivalent to $(a-3)(a+8)-(a-2)(a+2)=0$?

Why does the left side of the equation change and the right side of the equation remain the same?

17. y can't equal 0 because this gives a denominator of 0 and division by 0 is undefined.

same reason as above for $a \neq -2$ $a \neq -8$ in part h.

because you obtain an equivalent equation when both sides of an equation are multiplied by the same nonzero constant.

Both sides of the equation have been multiplied by $(a+8)(a+2)$. On the right side of the equation this is multiplied by 0 and any number multiplied by 0 equals 0.

18. We don't always get equivalent equations when both sides of an equation are multiplied or divided by the same number. If the number is equal to 0, then we <u>don't</u> get equivalent equations. Sometimes, when both sides of an equation are multiplied by a polynomial, a value of the variable which will not check in the original equation and so then there is <u>no root</u> or <u>no solution</u> to the equation.

For example, $\frac{1}{x-2} = \frac{4}{x^2-4}$.
If we multiply both sides of the equation by $x+2$, we obtain $x+2 = 4$ and adding $-2$ to both sides of this equation gives $x = 2$.

Substituting 2 for x in the original equation gives $\frac{1}{0}$ on the left side of the equation. This is <u>not</u> a number, so there is <u>no solution</u> to the equation.
Solve for a: $\frac{a-1}{2a+4} - \frac{a-3}{3a+6} = \frac{1}{6a+12}$

18. $3(a-1)-2(a-3) = 1$ gives $a = -2$.

Don't forget to first get the LCD for the three fractions and then multiply both sides of the equation by this quantity. Be sure to factor the given denominators in order to find the LCD.

When both sides of an equation are multiplied by a polynomial an equivalent equation is not always obtained. This happens in the problem just solved. Substitution of -2 for a in the original equation gives a denominator equal to 0 and this doesn't represent a number. Therefore, there is no root or no solution to the equation given in frame 18.

19. Equations which contain several letters can be solved in the same way as the above equations. Namely, change to equivalent equations until you have a value for the variable. The variable you are solving for must be only to the first power after similar terms are collected as we only know how to solve linear equations.

Solve for a: $ax+b = c$

19. Adding -b or subtracting b from both sides gives $ax = c-b$.

Dividing both sides of the equation by x where s $\neq 0$ gives

$$a = \frac{c-b}{x}$$

Why can't x equal 0?

20. The terms which didn't contain the variable (the variable is a in this equation) were put on one side of the equation and all terms containing the variable were put on the other side.
To check and see if your result is a root to the equation $ax+b = c$, what would you do?

Perform the check.

20. Substitute $\frac{c-b}{x}$ for a in the given equation.
Substituting on the left side,

$$\left(\frac{c-b}{x}\right) x + b = \frac{(c-b)x}{x} + b = c-b+b = c.$$

This is the same value as is found on the right side of the equation so $\frac{c-b}{x}$ is the root of the equation providing $x \neq 0$.

21. Solve the following equations and check.

(a) $\frac{4}{x-3} - \frac{x-2}{x+3} = \frac{-21-x^2}{x^2-9}$ for x

(b) $\frac{a}{x} + \frac{b}{x} = 3$ for a

(c) $a = (n-1)d + L$ for n

(d) solve the equation in part c for d.

(e) $\frac{4}{2x^2+5x+2} - \frac{6}{x^2-4} = \frac{7}{(2x+1)(x+2)}$ for x.

(f) $(xy+7)(xy-2) = xy(xy+1)$ for y.

21. (a) no solution because you get a denominator of 0.
(b) $a = 3x-b$ if $x \neq 0$.
(c) Remove the parentheses first.

22. Note that in part e you got the equation $-15x = 0$. This is solved by dividing both sides of the equation by -15. $\frac{-15x}{-15} = \frac{0}{-15}$

$n = \frac{a-L+d}{d}$

(d) $d = \frac{a-l}{n-l}$

(e) LCD is $(2x+1)(x+2)(x-2)$. Both sides of the equation can be multiplied by this if $x \neq 2$, $-2$ and $-1/2$.
$4(x-2)-6(2x+1) = 7(x-2)$
$-15x = 0$, $x = 0$.
(f) $x^2y^2+5xy-14 = x^2y^2+xy$
$4xy = 14$
$y = \frac{14}{4x} = \frac{7}{2x}$

gives $x = 0$.

Suppose we were to solve the equation $ax - bx = a-b$, for x. The equation is already arranged so that all the terms containing the variable (this is x here) we are to solve for are on one side of the equation and all other terms are on the other side. Consider the left side of the equation: $ax-bx$. What process could you use to separate the x from these terms?

22. factor the expression.
$ax-bx = x(a-b)$

23. Then $ax-bx = a-b$ can be written as an equivalent equation of $x(a-b) = a-b$.

What process would you use to find the value of x?

Solve for x and check. Be sure to state any values of a and b which could not be used.

23. If a and b do not have the same value, then you can divide both sides of the equation by a-b. This gives $x = 1$ which checks in the original equation and so is a root to the equation.

24. Solve and check the following equations.
(a) $5x+3(-x-7) = -x+24$
(b) $\frac{2}{3}(12-9a) = 2$
(c) $\frac{p}{3} - 4 = \frac{2p}{5}$
(d) $\frac{p-5}{15} - \frac{4p+1}{3} = -p-1 - \frac{2p+5}{9}$
(e) $P = 2x+2y$, solve for y.
(f) $A = 2\pi r$, solve for r.
(g) $ab-2(a+3) = 3ab$, sove for b.
(h) $cx+d = dx-c$, solve for x.
(i) $\frac{a+2}{a-3} - \frac{a-2}{a+3} = \frac{30}{a^2-9}$
(j) $\frac{x}{b} - \frac{x}{a} = \frac{a}{b} - \frac{b}{a}$, solve for x.
(k) $\frac{a-5}{a+1} - \frac{2a^2-2}{a^2-7a-8} = -\frac{a+3}{a-8}$
(l) $\frac{2m-1}{m+2} - \frac{m-3}{m-2} = 1$
(m) $\frac{x-2}{x-8} - \frac{18}{x-8} = 3$

24. DON'T FORGET to do the checks for each problem.
   (a) $x = 15$
   (b) $a = 1$
   (c) $p = -60$
   (d) $p = 20$
   (e) $y = \frac{P-2x}{2}$
   (f) $r = \frac{A}{2\pi}$
   (g) $b = \frac{-(a+3)}{a}$
   (h) $x = \frac{-d-c}{c-d}$
   (i) no solution
   (j) $x = a+b$
   (k) $a = 5$
   (l) $m = 3$
   (m) $x = 2$

25. (a) $t = \frac{rs}{s-r}$
   (b) $s = \frac{rt}{t-r}$

   Parts a and b are true only if r, s and $t \neq 0$.
   Also, in part a, s and r can't have the same value. In part b, t and r can't have the same value. Why?
   (c) $w = -14/3$
   (d) $x = 11/2$
   (e) no solution
   (f) $R = \frac{-Wr}{P-W}$ only if P and W aren't equal
   (g) $x = -(c+d)$ if c and d aren't 0 and aren't equal.
   (h) $a = \frac{6b}{-2b-3}$ if $b \neq -3/2$.

25. Solve and check the following equations.
   (a) $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$, solve for t.
   (b) Solve the equation in part a for s.
   (c) $\frac{w}{w^2-16} - \frac{w+3}{w-4} = -1$
   (d) $\frac{3}{x^2-5x+6} - \frac{2}{x^2-4} = \frac{5}{x^2-x-6}$
   (e) $\frac{x+2}{2x^2-3x} + \frac{x-1}{2x^2-5x+3} = \frac{1}{x}$
   (f) $P = \frac{W(B-r)}{R}$, for R.
   (g) $\frac{x}{c} - \frac{x}{d} = \frac{c}{d} - \frac{d}{c}$, for x.
   (h) $ab-3(a+2b) = 3ab$, solve for a.

26. Make sure you understand why the limitations on the values of certain variables apply in the last frame. You may never have a denominator of 0 as division by 0 is not defined. This condition applies to the given equation and to the solution.

   If you are given $\frac{1}{x} = 3$, even though it is not stated that $x \neq 0$, it is understood that this condition applies. We can never use any value of a variable which gives a denominator of 0.

   The main purpose of learning to handle algebraic expressions is to solve equations. In many cases, we will have a relationship stated in words, and we must translate this into symbols and obtain an equation expressing this relationship and then solve the equation.

   The main problem in translating word statements into an algebraic statements seems to be in recognizing what the symbols stand for. Therefore, we will spend some time in translating word

statements into algebraic symbols.

If we were to write the sum of two numbers, we would first have to know what numbers we had. Thus to represent the sum of two numbers, you first must represent the two numbers.

If you are just told that you have two numbers, they may be represented as follows.
Let n = one number
Let m = the other number

You will notice here that we chose two <u>different</u> symbols to represent the two numbers as we don't know what relationship holds between the two numbers.

Now we can use n+m to represent the sum of the two numbers.

Let m and n represent two numbers. Represent
(a) the product of m and n.
(b) the difference of m and n.
(c) the quotient of m and n.
(d) subtract m from n.
(e) the product of a number 5 less than n and 2.
(f) a number 6 more than twice n.
(g) a number which is twice the difference of m and n.
(h) a number which is three times the difference of n and 5.

26. (a) mn
(b) m - n
(c) $\frac{m}{n}$
(d) n - m
(e) (n-5)2 or 2(n-5)
NOT n-5·2. This says n minus the product of 5 and 2.

NOT 2·n-5, this says only 2 multiplied by n and then to subtract 5.
(f) 2n+6 or 6+2n
(g) 2(m-n) or (m-n)(2)
(h) 3(n-5) or (n-5)(3)

27. If you are not sure of the relationship expressed when letters are used, state the problem using numbers. Decide <u>what process</u> you used when the numbers were in the problem and then do the <u>same process</u> using the appropriate letters.

For example, If a car goes m miles per hour, how far does it go in h hours?
This is the same type problem as "A car goes 40 miles per hour, how far does it go in 3 hours?

statements into algebraic symbols.

If we were to write the sum of two numbers, we would first have to know what numbers we had. Thus to represent the sum of two numbers, you first must represent the two numbers.

If you are just told that you have two numbers, they may be represented as follows.
  Let n = one number
  Let m = the other number

You will notice here that we chose two different symbols to represent the two numbers as we don't know what relationship holds between the two numbers.

Now we can use n+m to represent the sum of the two numbers.

Let m and n represent two numbers. Represent
(a) the product of m and n.
(b) the difference of m and n.
(c) the quotient of m and n.
(d) subtract m from n.
(e) the product of a number 5 less than n and 2.
(f) a number 6 more than twice n.
(g) a number which is twice the difference of m and n.
(h) a number which is three times the difference of n and 5.

26. (a) mn
(b) m - n
(c) $\frac{m}{n}$
(d) n - m
(e) (n-5)2 or 2(n-5)
NOT n-5·2. This says n minus the product of 5 and 2.

NOT 2·n-5, this says only 2 multiplied by n and then to subtract 5.
(f) 2n+6 or 6+2n
(g) 2(m-n) or (m-n)(2)
(h) 3(n-5) or (n-5)(3)

27. If you are not sure of the relationship expressed when letters are used, state the problem using numbers. Decide what process you used when the numbers were in the problem and then do the same process using the appropriate letters.

For example, If a car goes m miles per hour, how far does it go in h hours?
This is the same type problem as "A car goes 40 miles per hour, how far does it go in 3 hours?

Decide what process is necessary in order to solve the second problem - don't worry about what the answer is.
What relationship would you use to solve the second problem?

27. distance = rate per hour multiplied by time in hours.

28. So we would multiply 40 miles per hour by 3 hours or we would multiply the rate by the time.

Use the same principle to represent the number of miles traveled if a car travels m miles per hour for h hours.

28. multiply m which is the hourly rate by h which is the number of hours.
The distance is mh miles.

29. If you travel at x miles per hour for 1 hour and 30 minutes, represent the distance traveled.

29. Multiply x which is the hourly rate by 1 1/2 hours or $\frac{3}{2}$ hours which is the number of hours traveled.
The distance is $\frac{3x}{2}$ miles or $\frac{3}{2}x$ miles.

30. Suppose you wanted the distance traveled when you travel at y miles per hour for 30 minutes. What would you have to do before you multiplied the rate by the time?

30. Change the time from minutes to hours because the rate is given as miles per hour.

31. Represent the distance traveled when you travel y miles per hour for 30 minutes.

31. $\frac{y}{2}$ or $\frac{1}{2}y$

32. Represent the cost of a pounds of coffee which sells bor b cents a pound.
Represent this cost in cents.

32. You multiply the number of pounds by the cost per pound. Since the cost is given in cents, ab represents the cost in cents of the coffee.

33. Suppose you needed the cost expressed in dollars of a pounds which sells for b cents a pound. The result to frame 32 which is ab cents would have to be expressed in dollars.

Decide how you would express 35 cents or 190 cents in dollars and then do the same process and express ab cents in dollars.

33. To change 35 cents or 190 cents to dollars, you would divide by 100 since this is the number of

34. You could express $\frac{ab}{100}$ in decimal form instead of in fractional form.

cents in one dollar.
NOTE - determine the process not the result.

Now change ab cents to dollars using the same process.

$\frac{ab}{100}$ dollars

$\frac{ab}{100} = \frac{1}{100}ab$ and $\frac{1}{100} = .01$ so

$\frac{ab}{100} = .01ab$

You cannot write .ab as this has no meaning. A decimal point only has meaning when the number is expressed with numerical digits.

Represent each of the following.
(a) the number of cents in d dollars.
(b) the number of dollars in (3x) cents.
(c) the cost in cents of a pencils at (x-3) cents each.
(d) the cost in dollars of x pounds of sugar at y cents per pound.
(e) the cost in cents of w pounds of sugar which sells at 3 pounds for 25 cents.
(f) the cost of x roses which sell at d dollars a dozen.
(g) the number of miles traveled by a car which travels p miles per hour for m minutes.
(h) the rate of a car which travels 80 miles in x hours.
(i) the number of hours traveled when a car goes d miles at x miles per hour.

34. (a) 100d cents
(b) $\frac{3x}{100}$ or .03x dollars
(c) a(x-3) or (x-3)a cents
(d) $\frac{xy}{100}$ or .01xy dollars.
(e) you need the cost of one pound of sugar which is $\frac{25}{3}$ cents.
So the cost in cents of w pounds is $\frac{25w}{3}$ or $\frac{25}{3}$ w.
(f) cost in dollars of one rose is d/12. Thus, the cost of x roses is $\frac{d}{12}$ x or $\frac{dx}{12}$ dollars.
(g) The time needs to be expressed in hours. m minutes equals m/60 hours.

35. 3,4,5,6 are examples of consecutive numbers.

Notice that one is added to each number in order to get the next consecutive number.

3, 5, 7, 9 are examples of consecutive odd numbers.
2, 4, 6, 8 are examples of consecutive even numbers.

Consecutive odd and consecutive even numbers differ by 2 or in other words, add 2 to a number in order to get the next consecutive odd or consecutive even number. The resulting number is odd or even depending on whether the original number is odd or even.

distance is $\frac{m}{60}$ p or $\frac{mp}{60}$ miles.

(h) We know that distance = rate multiplied by time. Here we know the distance and the time, so how would we find the rate?

rate = distance divided by time.

In this problem, $r = \frac{80}{x}$ .

(i) number of hours = $\frac{d}{x}$

Does n+2 represent an even number sometimes or does it always represent an even number? Give a reason for your answer.

35. <u>sometimes</u>

n+2 is even if n is even.
n+2 is odd if n is odd.

36. If n+3 represents an even number, what can you say about the value of n?

Write the next higher consecutive number. Is this an odd or even number?

36. If n+3 is even, then n must be odd.
n+4 is the next higher consecutive number.
n+3 represents an even number so n+4 represents an odd number.

37. If n+3 represents an even number, represent the two consecutive numbers which immediately <u>precede</u> n+3.

Represent the two consecutive <u>even</u> numbers which immediately precede n+3.

37. n+2 and n+1 are the two consecutive numbers which precede n+3. Remember consecutive numbers differ by 1.

n+1 and n-1 are the consecutive even numbers which immediately precede n+3. Remember consecutive even numbers differ by 2.

38. If w represents John's age now Susan is 4 years older than John, represent
(a) John's age 7 years ago.
(b) Susan's age in 7 years.
(c) three times Susan's age in 3 years.

38. (a) w - 7
(b) w + 11. w + 4 represents Susan's age now.
(c) 3(w + 7)

39. Represent the following.
(a) The sum of two numbers is 7. One of the numbers is y, what is the other one?
(b) The product of two numbers is 12. One of the numbers is w, what is the other one?
(c) One number is t. Another number is 5 more than twice that number. The second number is ________.

40. When finding a value which satisfies a statement problem, we must first discover a relationship which holds between the quantities we have represented.

For example, given the following.
Five more than twice a number is three less than three times a number. Find the number.

First we must represent what we wish to find. In every statement problem, you are asked to find some quantity. This time we are asked to find a number.
So, the first thing we put down is

Let n = a number

Now we must represent the relationships given in the problem.

One relationship is 5 more than twice the number. This can be represented as ________ .

Another relationship is three less than three times the number. This can be represented by ____.

39. (a) 7 - y
(b) 12/w
(c) 2t+5 or 5+2t

41. Now we must write an equation which used the relationship given between these tow quantities.

What are you told is true of the quantities 2n+5 and 3n-3 in the problem?

Write an equation using this relationship.

40. 2n+5

3n-3. NOT 3-3n.

42. Solve this equation to obtain the required number.

41. These quantities are equal.

2n+5 = 3n-3.

43. The procedure outlined in frames 40 - 42 is the procedure used in solving all statement problems.

First, represent the quantity or quantities you are asked to find.
Second, represent any relationships which are given in the

42. the required number is 8.

problem or which you can deduce from the information given in the problem. Sometimes there is a formula which gives the relationship between certain quantities. Make use of such a formula if there is one.

Third, write an equation expressing a relationship which exists between quantities you have represented.
REMEMBER an equation is an equality.

Fourth, solve the equation to get the required value or values.

In order to do steps 2 and 3, you must read the problem carefully. You must also learn to interpret your algebraic expressions in words to see if they express the same relationship as the one given in the statement of the problem.

Suppose we were to set up the equation and solve the following.
Five times a number is 10 more than twice that number increased by 3. Find the number.

First, you would ____________.

43. represent the quantity you wish to find
or
represent a number as that is what you are told to find.

44. Let p = a number. (You can use any letter you choose. To see if you have the same quantities as those given, substitute your letter for p.)

Next you would __________.
Do this.

44. Represent the relationships given in the problem.
five times a number is 5p.
twice a number is 2p.

45. Now we are ready to write an equation which will express an equality between these quantities or which involves these quantities. This relationship will be given in the statement of the problem or can be deduced from the conditions of the problem.

We are told that the number 5p is more than the number 2p increased by 3.

So, comparing 5p and 2p+3, which represents the larger number?

45. 5p

46. Then to express an equality, what must we do with the 10? Be specific.

Write the equation which expresses the given relationship.

46. You can't say add. You must say what you added to what. You can add 10 to the number 2p+3 because 2p+3 represents the smaller number and adding 10 to it will make it equal to the larger number.
equation would be $5p = 2p+3+10$.
or you could subtract 10 from 5p. equation then: $5p-10 = 2p+3$.

47. Remember you are comparing the numbers 5p and 2p+3. You are going to complete the following so that you have an equality expressing the relationship given in the problem.

5p __________ 2p+3___________

Since 5p represents the larger number, you could subtract something from it to get a smaller quantity. Or since 2p+3 represents the smaller number, we could add something to it so it would equal the larger quantity.

Solve the equation to find the required number.

47. The number is $\frac{13}{3}$.

48. Set up the equation you could use to solve the following.

Find a number such that twice the number is 9 more than 1 1/3 times the number.

48. Your first statement must tell what letter you are using and what it represents.
For example, let n = a number.
You are comparing twice a number or 2n with 1 1/3 times a number which is represented as $\frac{4}{3}n$ or as $\frac{4n}{3}$ or as $1\frac{1}{3}n$.

Pick out the larger quantity and then decide how to get an equality.

49. The last equation of $2n - \frac{4}{3}n = 9$, states that the difference of two numbers is 9. This is certainly true as one number was given as being 9 larger than the other number, so their difference must be 9.

Solve the equation and find the required number.

Any of the following equations express a correct relationship between the quantities which are compared in the problem.

$2n-9 = \frac{4}{3}n$ or $2n = \frac{4n}{3} + 9$

or $2n - \frac{4}{3}n = 9$.

49. The number is 13 1/2 or $\frac{27}{2}$.

50. To check statement problems, you must go back to the given statement. You can't check in the equation as the equation wasn't given to you. The equation was something you decided on.

Going back to the statement - if I find twice the number or $2 \cdot \frac{27}{2}$ and also find $\frac{4}{3}$ of the number or $\frac{4}{3} \cdot \frac{27}{2}$, is the first result 9 larger than the second result? If it is, then the problem checks.

Set up the equation for:
The sum of three numbers is 57. The second number is 7 more than the first number and the third number is twice the second number. Find the three numbers.

50. Let w = the first number
$w + 7$ = the second number
$2(w + 7)$ = the third number

$w + (w+7) + 2(w+7) = 57$

51. The sum of the three numbers is represented by adding the three numbers.

Solve the equation and be sure to answer the question asked in the problem.

51. $4w + 21 = 57$

$w = 9$
$w+7 = 16$
$2(w+7) = 32$

All of these are the required numbers.

You were asked to find three numbers in this problem.

52. Set up the equation for the following.
George is 7 years younger than Jim. Jim's age two years ago was twice George's age then. Find their present ages.

52. Be specific here when you represent the quantities you are dealing with.
You are dealing with two ages for each boy. State specifically

53. You may have a different equation than the one given in the answer to the last frame. This will be true if you represented different relationships. Your equation

what you mean.
For example, let a = Jim's <u>present</u> age
IT IS WRONG to say let z = Jim.
z represents something about Jim not the boy himself.

Solution: let a = Jim's present age
then a-7 = George's present age
Two years ago both boys were two years younger. So,
a-2 = Jim's age two years ago
a-9 = George's age two years ago
Jim's age 2 years ago = 2(George's age 2 years ago)
So, a-2 = 2(a-9)

will be correct if you used the relationships given in the statement of the problem.

The following is another way of representing the information in the last problem.

Solution: let y = George's present age
then y+7 = Jim's present age

y-2 = George's age two years ago
y+5 = Jim's age two years ago

Jim's age 2 years ago = 2(George's age 2 years ago)
So, y+5 = 2(y-2)

This equation is correct for the above representation of the information given in the statement. There are other correct possibilities. If you have something else, make sure that when it is interpreted in words, you have the relationship given in the problem.

Solve your equation. Be sure to answer the question asked in the problem.

53. Jim is 16 now and George is 9 now.

54. Set up the equations for each of the following.
(a) 72 is divided into two parts such that one part is 18 more than twice the other part. Find the two parts.
(b) The sum of three consecutive integers is 144. Find the three numbers.
(c) A man is twice as old as his son. Ten years ago, the man was four times as old as his son was then. Find their present ages.
(d) Van had twice as many marbles as Keith. Van lost 4 of his to Keith. Then Van had 26 less than three times as many as Keith. How many marbles did each boy have to begin?

54. (a) let b = the larger part
then 72-b = the smaller part
b-18 = 2(72-b) or b = 2(72-b)+18
or
You could have used the following representation.
let y = the smaller part
then 2y+18 = the larger part
Since the two parts must total 72,
y + (2y+18) = 72.

(b) let n = the first integer
n+1 = the next higher integer
n+2 = the next higher integer
n + (n+1) + (n+2) = 144

(c) let a = son's present age
then 2a = man's present age
a-10 = son's age 10 years ago
2a-10 = man's age 10 years ago
2a-10 = 4(a-10)

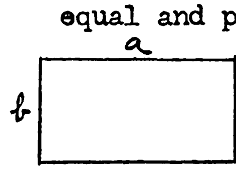
(d) let x = number of marbles Keith had to begin
then 2x = number of marbles Van had to begin
After the game, Keith had x+4 marbles and Van had 2x-4 marbles.
2x-4 + 26 = 3(x+4)

55. Finish solving each of the problems given in the preceding frame. Make sure you answer the question asked in each problem.

55. (a) larger part is 54 and the smaller part is 18.
(b) integers are 47, 48 and 49.
(c) the man is 30 and the son is 15.
(d) Van had 20 marbles at the beginning and Keith had 10.

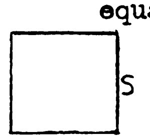
56. Sometimes we deal with geometric figures and need to know their dimensions, area or perimeter. The common figures we deal with are the rectangle, square, triangle, circle and cube.

Rectangle - a four-sided figure where the opposite sides are equal and parallel.




Perimeter = distance around.
Perimeter = 2a+2b
Area = ab

Square - a rectangle with all sides equal.




Perimeter = 4s
Area = $s^2$

Write the equations which express the following relationships.

(a) One dimension of a rectangle is four times the other dimension. If the perimeter

is 34 feet, find the dimensions.

(b) One dimension of a rectangle is one less than five times the other dimension. If the area is 52 sq. ft., find the dimensions.

(c) Find the area of a square if the perimeter is 56 feet.

56. (a) let x = the length of one side

then 4x+2 = the length of the other side

$2x + 2(4x+2) = 34$

(b) let y = length of one side

then 5y-1 = length of other side

$y(5y-1) = 52$

(c) let a = lenth of one side

then 4a = perimeter

$4a = 56$

The area is found by squaring one side and can be found after a value for a is obtained.

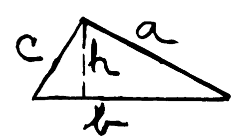
57. The equation obtained in part b of the last frame is not a linear equation, and so you are not able to solve it at this time. You should be able to solve both of the other equations.

Triangle - a three-sided polygon.

Right triangle - a triangle having a right angle.

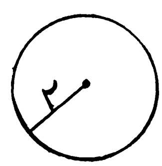
The perimeter of any figure is the distance around the figure.

Perimeter of a triangle = a+b+c




Area of triangle = $\frac{1}{2}$ bh where b is the length of one side of the triangle and h is the length of the perpendicular to that side from the opposite vertex.

Circle - the perimeter of a circle is called the circumference and $C = 2\pi r$ .




Area of circle $= \pi r^2$.

r stands for the radius of the circle.

diameter = 2r.

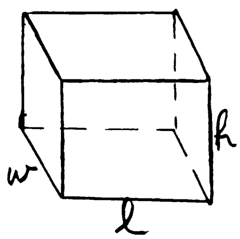
(a) Find the circumference of a circle of radius 4".

(b) Find the area of a circle of radius 1 1/2 feet.

(c) One side of a triangle is twice the other side and the third side is 6 less than the first side. If the perimeter is 38", find the dimensions.

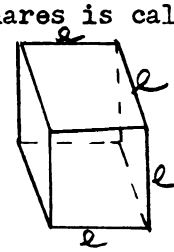
57. (a) $C = 8\pi$ in.
(b) $A = \frac{9}{4}\pi$ sq. ft.
(c) let p = length of first side
then 2p = length of second side
and p-6 = length of third side
$p + 2p + (p-6) = 38$
sides are 11", 22", and 5".

58. A rectangular solid called a <u>parallelpiped</u> is a solid where all of the faces are rectangles.




Parallelpiped
Volume = lwh
(l represents the length)
Surface Area = the sum of the areas of the faces and equals 2wh+2lw+2lh.

A parallelpiped whose faces are squares is called a <u>cube</u>.




What is the volume of a cube?

What is the surface area of a cube?

58. Volume of a cube = $e^3$ where e represents the length of one edge.
Surface area of a cube is $6e^2$ as there are 6 faces to a cube and each face is a square.

59. (a) Find the volume of a rectangular box which is 6" long, 3" wide and 2" high.
(b) If one edge of a cube is (2x-1) units long, represent the volume of the cube.
(c) The length of a rectangular box is twice the width and the height is half of the width. Represent the surface area of the box if the box has a cover.
(d) Represent the surface area of the box given in part c if the box is open.

59. (a) V = 36 cu. in.
(b) $V = (2x-1)^3 = 8x^3-12x^2+6x-1$
(c) let w = width
then 2w = length
and $\frac{1}{2}w$ = height
$V = w(2w)(\frac{w}{2}) = w^3$

Surface Area = $2(\frac{w}{2})(w)+2(w)(2w)+2(\frac{w}{2})(2w)$

Surface Area = $w^2+2w^2+4w^2=7w^2$.
(d) without a cover, the surface area = $2(\frac{w}{2})(w) + (2w)(w) + 2(\frac{w}{2})(2w)$
or includes the area of only five faces.
Surface Area = $5w^2$

60. Set up the equations you would use to solve the following.
(a) A man wishes to enclose a field by using 740 yds. of fence. If he wants the field to be 10 yds. longer than twice the width, what should be the dimensions of the field?
(b) A salesman gets a salary of \$22 plus a commission of 45% on all sales over \$50. If he earned \$89.50, what was the amount of his sales?
(c) The length of a rectangle is 6 less than three times the width. If the length were decreased by 1, the perimeter would be increased by 2. Find the dimensions of the original rectangle.

(d) Linen sells for twice as much a yard as rayon. For \$30.60, I can buy 5 yards of linen and 7 yards of rayon. How much does each cost per yard?

60. (a) let w = width
then 10+2w = length
Perimeter = 2(w)+2(10+2w)
So, 2(w)+2(10+2w) = 740

(b) let q = total amount of sales
He received commission on all but \$50 or on q-50 dollars.
commission is .45(q-50) dollars.
So, 22+.45(q-50) = 89.50

The above equation expresses the various amounts of money in dollars. You could have expressed the amounts of money in cents. The equation then would be 2200 + 45(q-50) = 8950. Note that both sides of the first equation were multiplied by 100 to get the second equation and so the two equations are equivalent. Why? See frame 7 if you don't know why.

(c) Let w = width of the original rectangle
Then 3w-6 = length of the original rectangle
w-1 = width of new rectangle
3w-4 = length of new rectangle

perimeter of original rectangle is 2(w) + 2(3w-6).
perimeter of new rectangle is 2(w-1) + 2(3w-4).

The periemter of the original rectangle is increased when the dimensions are changed, so the perimeter of the original rectangle plus the increase equals the perimeter of the new rectangle.
2(w)+2(3w-6) + 2 = 2(w-1)+2(3w-4)

61. Solve the following.

(a) One number is 2 less than six times another. When the larger number is divided by the smaller one, a quotient of 5 and a remainder of 3 is obtained. Find the two numbers.

(b) Evelyn bought a dress, a suit and a pair of shoes. The suit cost two and a half times as much as the dress and the shoes cost three-fifths as much as the dress. How much did she spend for each if the total bill was \$102.50?

(c) Jim left Lansing at 7 am and drove 50 miles per hour. George left the same place at 9 am and drove at 60 miles per hour. How long would it take George to overtake Jim?
(Hint: what must be true when George overtakes Jim?

(d) A boy started walking home at the same time his brother started out from home on his bicycle to find him. The boy on the bicycle traveled one mile per hour faster tham twice the speed of the boy walking. If they met after 2 hours and the boy was 10 miles from home originally, at what speeds did they travel? When the boys met, how far from home were they?

61. (a) Let x = the smaller number
Then 6x-2 = the larger number

$$\frac{6x-2}{x} = 5 + \frac{3}{x}$$

62. So far we have solved linear equations in one variable. For example, we have solved equations such as 3x-4(x+2) = 2x-7.

The numbers are 5 and 28.
Don't forget that the remainder in division is added to the quotient and is placed over the divisor.
Example: 13 divided by b = 6 1/2 or 6 + 1/2.

(b) let c = cost of dress
Then $2\frac{1}{2}c$ = cost of suit
and $\frac{3}{5}c$ = cost of shoes

$$c + \frac{5}{2}c + \frac{3}{5}c = 102.50$$

dress cost \$25, suit cost \$62.50 and shoes cost \$15.

(c) The distance each man traveled must be equal to each other if one man overtakes the other.
let t = number of hours Jim traveled
then t-2 = number of hours George traveled
Jim went 50t miles and George went 60(t-2) miles.
So, 50t = 60(t-2).
The men met after 12 hours.

(d) let r = speed of boy walking
Then 2r+1 = rate of boy on bike
2r = distance traveled by boy walking
2(2r+1) = distance traveled by boy on bike
Since the total distance is 10 miles, 2r + 2(2r+1) = 10.
The boy walked at the rate of 1 1/3 miles per hour. The boy on the bike traveled at the rate of 3 2/3 miles per hour.
They were 7 1/3 miles from home when they met.

We have also dealt with equations which contained more than one variable, and have solved these for one variable in terms of the others. The variable we solved for could only be to the first power so we were still solving linear equations.
For example, ax+bx = c is a linear equation in a and $a = \frac{c-bx}{x}$

providing $x \neq 0$.

Now we will consider linear equations in two variables.

ax+by+c = 0 is a linear equation in x and y if a, b and c are constants.

The solution to a linear equation in two variables consists of a pair of values - one value for each variable.
The solution of a linear equation in one variable consists of a single value. In the two examples given above, there is only one value of x which satisfies the given equation.

Consider the equation x+y = 5.
A solution to this equation consists of one value for x and a corresponding value for y, such that the sum of these is 5.
x = 2 and y = 3 is one solution

Name one other solution to x+y = 5.
How many solutions to x+y = 5 are there?

62. x = 3 and y = 2 is a <u>different</u> solution.

There are an <u>infinite</u> number of solutions as you can name any two numbers whose sum is 5.

63. If we are given <u>two</u> linear equations in <u>two</u> variables (we will refer to this situation as a <u>system</u> of equations) our problem is to find <u>one pair</u> of values - one for each variable - which will satisfy <u>both</u> equations.
For example: Given the equations

$$\begin{cases} 2x+y = 3 \\ x-2y = 4 \end{cases}$$

to find the common solution, means we are looking for a pair of values - one for x and a corresponding one for y - which satisfy both equations.

Verify that x = 2 and y = -1 is the common solution for these equations.

63. You can verify that $x = 2$ and $y = -1$ is the common solution by substituting these values in both equations.

$2(2) + -1 = 4-1 = 3$ so these values check in the first equation.
$2 - 2(-1) = 2+2 = 4$ so these values check in the second equation.
Since these values check in both equations, they are the common solution.

64. The procedure of solving these equation is called finding the simultaneous solution or finding the common solution.

Simultaneous solution by substitution. Solve one of the equations for one of the variables in terms of the other and substitute this into the other equation.

$$\begin{cases} 2x+y = 3 \\ x-2y = 4 \end{cases}$$

Choose one of the equations and solve it for one of the variables.
It really makes no difference which equation is used or which variable is solved for, but you are to solve for x in the second equation.

64. $x - 2y = 4$, so $x = 4 + 2y$

65. Since this value of x must satisfy both equations in order to have a common solution, we may now substitute it in the other equation in place of x.

What equation do you obtain when you do this?

65. $2(4+2y)+y = 3$

66. The equation we obtain will be a linear equation in one variable, which we can solve using previous methods.

Solve the equation obtained in frame 65.

66. $y = -1$

67. This one value is not the common solution of the given pair of equations.
What must you do now?
What is the simultaneous or common solution of these equations?

67. You must find a value for x.

common solution is x = 2 and y = -1. You must specify the variable as well as the value.

68. You will sometimes find the common solution written as (2,-1). This notation is referred to as an ordered pair. The first number of the pair is the value of x and the second one is the value of y.

What values of x and y are represented by (-1,2)?

68. x = -1 and y = 2.

69. You can see that the ordered pair (2,-1) doesnot represent the same values as the ordered pair (-1,2).

Find the common solution for

$$\begin{cases} 2x - 3y = 13 \\ -3x + y = -9 \end{cases}$$

We must first ________________, then ______________ and then _____________.

69. solve one equation for one variable, then substitute this value in the other equation and then solve for both variables.

70. Do the indicated process.

Since there is no preference as to which variables you solve for now in which equation you use, choose the equation and variable where it is easiest to solve.
In this problem, the easiest to solve would be ________ in the __________ equation.

70. The easiest to solve is for y in the second equation because the coefficient of y is 1.
From the second equation, y = -9+3x. Substituting this in the first equation, we obtain 2x - 3(-9+3x) = 13.
Solving, we get x = 2 and then solving for y we get y = -3.
Common solution can be written as the ordered pair (2,-3).

71. (2,-3) is the common solution to these two equations, so in order to check this, what must be done?

Perform the check.

71. Be sure that you said substitute these values in both equations. If the values do not check in both equations, either the solution is wrong or you have made a mistake in the check.

72. Find the common solution for

$$\begin{cases} \frac{1}{2}x + \frac{2}{3}y = 2\frac{1}{2} \\ 6x - y = 3 \end{cases}$$

(Hint - remember you can multiply both sides of an equation by a constant getting an equivalent equation.)

72. Multiply both sides of the first equation by 6 to get the equivalent equation $3x+4y = 15$. Now choose one equation and solve for one of the variables. The easiest to solve is for y in the second equation because its coefficient is -1. Doing this we get $y = 6x-3$. Now substitute this in the other equation. $3x+4y = 15$ becomes $3x+4(6x-3) = 15$. Solve this equation for x and then determine the value for y. Solution is (1,3). Remember these values must check in both equations and this means in both original equations.

73. In the last frame, we found an equivalent equation for one of the given equations. Thus we found the solution for

$$\begin{cases} 3x+4y = 15 \\ 6x- y = 3 \end{cases}$$

Since the above system is equivalent to the system given in frame 72, we can say that the the solution to both systems is the same. Solve and check.

(a) $\begin{cases} 5x = -4y \\ 2x - y = 26 \end{cases}$

(b) $\begin{cases} 2x - 4y = -26 \\ x + 3y = 17 \end{cases}$

(c) $\begin{cases} \frac{3}{5}x = 1 - .8y \\ x - y + 2 = 0 \end{cases}$

(d) $\begin{cases} ax + 3y - 8a = 0 \\ y + 4a = ax \end{cases}$
Solve for x and y.

(e) $\begin{cases} 7x + 5y = -2 \\ 6x - 3y = 8 \end{cases}$

73. (a) (8,-10)
(b) (-1,6)
(c) (-3/7, 11/7)
(d) (5,a)
(e) (2/3, -4/3)

74. In any system consisting of two equations in two variables, we can add or subtract the given equations and obtain another equation of the same system.

Consider part e of the last frame.

$$7x+5y = -2$$
$$6x-3y = 8$$

Adding these two equations, we get $13x+2y = 6$. This represents another equation which will have the same common solution as the ones you solved. Check this fact by substituting the common solution into $13x+2y = 6$.

If we had equations where the coefficients of one of the variables were the same, then when we added or subtracted the equations we would get an equation which contains only one variable.

For example, take the equations

$$\begin{cases} 2x-3y = 4 \\ 4x+3y = 14. \end{cases}$$

Adding these, we obtain the equation $6x = 18$. This equation contains only one variable and can be solved for this variable using previous methods.
In this example, the coefficients of y were such that the terms containing y were eliminated when we added the equations.

In the case of $\begin{cases} 7x+5y = -2 \\ 6x-3y = \ \ 8 \end{cases}$
we don't eliminate either variable when we add or subtract the equations. We can change each equation to an equivalent equation so that the coefficients of one variable will have the same absolute value in both equations.

Change these two equations to equivalent equations such that the coefficients of x will be the same.

74. We obtain equivalent equations by multiplying or dividing both sides of an equation by a non-zero number.
If we multiply both sides of the first equation by 6 and both sides of the second equation by 7, we will obtain equivalent equations where the coefficients of x in both equations are the same.
We get $42x+30y = -12$
$42x-21y = \ \ 56$

75. Solving this system will give us the same solution as solving the given system.
Now, since we wish to have a linear equation in only one variable, what can we do with these two equations?

75. Subtract the equations.

76. Write the equation you obtain by this process and finish solving.

76. $51y = -68$

$y = -68/51 = -4/3$

Solution is $\left(\frac{2}{3}, -\frac{4}{3}\right)$

77. Solve the following system by making the coefficients of y the same. Check.

$$\begin{cases} 3x-8y \ = -13 \\ 2x-12y = -\ 7 \end{cases}$$

77. To make the coefficients of y alike, the smallest integer to

78. In the problem given in the last frame, if you had multiplied both

multiply both sides of the first equation by is 3 and is 2 for the second equation. This gives coefficients of -24 for y in both equations. You can multiply both sides of the first equation by 12 and both sides of the second equation by 8, and then all the coefficients will be larger but in the same ratio as the ones given below.

$$\begin{cases} 9x-24y = -39 \\ 4x-24y = -14 \end{cases}$$

Solution is $(-5,-\frac{1}{4})$.
Remember that these values <u>must</u> check in <u>both</u> equations.

sides of the first equation by -3 and both sides of the second equation by 2, then you would have had to add the two equations in order to eliminate the terms containing y.
Solve each of the following for x and y and check.

(a) $\begin{cases} 2x+3y = -15 \\ 3x-5y = 63 \end{cases}$

(b) $\begin{cases} 4x+3y = 7 \\ 5x-\frac{1}{2}y = -4 \end{cases}$

(c) $\begin{cases} 2x-3ay = 5b \\ 7x+6ay = 12b \end{cases}$

(d) $\begin{cases} ax+by = b \\ bx-ay = a \end{cases}$

(e) $\begin{cases} cx-dy = -1 \\ dx-cy = 1 \end{cases}$

78. (a) $(6,-9)$
(b) $(-\frac{1}{2},3)$
(c) $(2b, -b/3a)$
(d) Exactly the same procedure is used here as in the above problems. Obtain an equivalent system of equations by multiplying both sides of each equation by some non-zero number so that the coefficients of one of the variables is the same in both equations.
To make the coefficients of y the same, multiply both sides of the first equation by a and both sides of the second equation by b.
Solution $\left(\frac{2ab}{a^2+b^2}, \frac{-a^2+b^2}{a^2+b^2}\right)$
(e) You will need to use the fact that when two factors are multiplied by -1, the value of the fraction is unchanged.
Solution $\left(\frac{-1}{c-d}, \frac{-1}{c-d}\right)$

79. We have dealt so far with two linear equations in two variables which have a common solution. Not all such equations have a common solution.

Linear equations in two variables which have no common solution are called <u>inconsistent</u> equations.

For example, if we tried to solve

$$\begin{cases} 3x+2y = 5 \\ 6x+4y = 2 \end{cases}$$

we would find that no matter which method of solution we used, we would get an equation which contains no variable and which is of the form, 0 = some nonzero number. An equation of this form can never be true.
Let us multiply both sides of the first equation by -2. We would get the equivalent equation $-6x-4y = -10$. Now if we add this to the second equation, we get $0 = -8$. This equation is <u>never</u> true. Therefore, the equations have <u>no</u> common solution, and are called _______ equations.

79. inconsistent

80. Sometimes we have two equations in two variables and they have an unlimited number of solutions. Equations which have an unlimited number of solutions are called <u>dependent</u> equations.

When dependent equations are solved by either of the methods we know, we obtain an equation which is an identity.

Identities are equations which are ______ true.

80. always

Identities are true for <u>all</u> values of the variables.

81. Solve the system $\begin{cases} 4x-3y = 9 \\ 6y-8x+18 = 0 \end{cases}$ by substitution.

81. In substitution, you solve one of the equations for one of the variables in terms of the other variable and then substitute into the other equation. Solving the first equation for $x$ we get $x = \frac{9+3y}{4}$.
Substituting this in the other equation, we get
$6y-8(\frac{9+3y}{4})+18 = 0$. This simplifies to $0 = 0$.

82. Since we obtain an identity when we solve these equations and identities are true for all values of the variable, we have a system of equations which has an unlimited number of solutions and a system of equations which has an unlimited number of solutions is known as a ________ system.

82. dependent

83. Solve the following sets of equations and state whether they have a common solution and so are consistent equations or whether they are inconsistent or dependent.

(a) $\begin{cases} x+y+2 = 0 \\ 1-6y = 6x \end{cases}$

(b) $\begin{cases} 3(2x-5)+5(3y+1) = 2 \\ -\frac{3}{x-3} = \frac{6}{y+2} \end{cases}$

(c) $\begin{cases} \frac{x-3y+8}{2x+5y-2} = 2 \\ 6x-3y+5 = 0 \end{cases}$

(d) $\begin{cases} 3x-2y-7 = 0 \\ 42+12y = 18x \end{cases}$

(e) $\begin{cases} 3x = 5-2y \\ 4y = 10-6x \end{cases}$

(f) $\begin{cases} 5x-3y-7 = 0 \\ 10x = 6y+2 \end{cases}$

83. (a) inconsistent
(b) consistent, solution (2,0).
(c) consistent, solution (-1/3,1)
(d) dependent
(e) dependent
(f) inconsistent

84. Sometimes it is more convenient to solve word problems by using two variables and two equations than to use one variable and one equation. The procedure is exactly the same as when one variable is used. First, represent the quantities you are asked to find. Then analyze the problem, representing the other relationships given and write two equalities if you use two variables. Then solve to find the common solution.

To solve the following.
The sum of two numbers is 70. One number is one more than twice the other. Find the two number.

Solution: let x = one number
let y = other number
$x+y = 70$ expresses the relationship that the sum of the two numbers is 70.
$x = 2y+1$ expresses the relationship that one number is 1 more than twice the other.

Find the common solution of these two equations.

84. the numbers are 23 and 47.

85. Set up the equations you would use to solve the following. Use <u>two</u> variables.

The first number is 9 more than twice the second number. Three times the second number is 6 more than the first number. Find the two numbers.

85. let x = the first number
let y = the second number
Equations are $\begin{cases} x-9 = 2y \\ 3y = x+6 \end{cases}$

86. Your equations may be set up differently from these, but they must express the same relationships. Make sure than when the equations are interpreted in words, your equations express the same rela-

tionships as those given in the problem.
Solve the equations you got in frame 85 and find the two numbers.

86. first number is 39 and the second number is 15.

87. Set up the equations you would use to solve each of the following. Use two variables.

(a) If 12 yards of cotton cost the same as 5 yards of rayon and if 3 yards of rayon and 5 yards of cotton cost $6.10, what is the price per yard of each material?

(b) The charges for a telegram consist of a flat rate for the first ten words and an additional charge for each additional word. If it costs $.98 to send a telegram of 17 words and $1.26 to send a telegram of 24 words, what is the flat rate and what is the cost of each additional word?

(c) In a certain rectangle, the length is 4 less than three times the width. If the width is increased by 2 and the length is decreased by 3, then the perimeter is decreased by 2. Find the dimensions of the rectangle.

(d) A merchant bought some radios. He got 8 of one kind and 4 of another kind for $128. The next time he ordered, he received 9 of the first kind and 3 of the second kind for $4 more. What was the price of each kind of radio?

(e) A man traveled 280 miles. If he had traveled 10 miles per hour faster, it would have taken him 40 minutes less time. What was his rate and time?

(f) In a number such as 23, the 2 is called the tens digit and the 3 is called the units digit. To form the number, the tens digit is multiplied by ten and the units digit is multiplied by 1 and the results are added. Thus if the tens digit

87. (a) Let a = cost per yd. rayon
Let b = cost per yd. of cotton
$12b = 5a$
$3a + 5b = 6.10$

Both a and b represent the cost per yd. expressed in dollars. If your variables represent the cost in cents, your second equation will have 610 instead of 6.10.

(b) let x = flat rate
let y = cost of each additional word
$x + 7y = 98$ when x and y are
$x + 14y = 126$ expressed in cents.

(c) let w = width
let l = length
perimeter is $2w+2l$
new width is $w+2$
new length is $l-3$
new perimeter is $2(w+2)+2(l-3)$
$2w+2l-2 = 2(w+2)+2(l-3)$
$l = 3w-4$

(d) let p = cost in dollars of each radio of 1st kind
let q = cost in dollars of each radio of snd kind.
$8p+4q = 128$
$9p+3q = 132$

(e) let r = number of miles per hour at which man traveled
let t = number of hours
$rt = 280$
$(r+10)(t- 2/3) = 280$
(f) let t = tens digit and u = the units digit.
number is $10t+u$.

of a number is 7 and the units digit is 4, the number is $7(10)+4$ or 74.

Let t = the tens digit and u = the units digit and represent the number which this forms.

Write the number where the hundreds digit = h, the tens digit = t and the units digit = u.

88. Solve each of the preceding systems of equations.

Solve each of the following problems. You may use either one variable or two variables.

(a) In a certain two digit number, the sum of the digits is 11. The number is 20 less than twice the number which is formed by reversing the digits. Find the number.

Hint - when the digits are reversed, the tens digit becomes the units digit and the units digit becomes the tens digit. Thus the number 24 becomes the number 42.

(b) Sue's present age is one year more than 3 times her age five years ago. Her age in five years will be twice what it was last year. What is her present age?

(c) A boy started for a place 23 miles away. He started by walking and walked at the rate of $1\frac{1}{2}$ miles per hour. Later he was offered a ride and completed his journey at the rate of 40 miles per hour. If it took him $2\frac{1}{2}$ hours to get to his destination, how far did he walk?

let h = hundreds digit
let t = tens digit
let u = units digit
number is 100h+10t+u

88. Solutions to systems of equations in frame 87.
(a) rayon cost $1.20 a yd. and cotton cost $.50 a yd.
(b) the flat rate was 70¢ and the cost of each additional word was 4¢.
(c) length is 17 and width is 7.
(d) the first kind cost $12 each and the second kind cost $8 each.
(e) In solving these two equations, first do the multiplication in the second one. Then subtract the two equations. This will give you another equation which belongs to the system and when this is solved with one of the original ones, you will get the required solution.
This equation is $10t + \frac{2}{3}r - \frac{20}{3} = 0$.
Solve this with rt = 280 using substitution. Remember the object is to get one equation in one variable. However, when you do this, you do not get a linear equation so you cannot complete the solution at this time.

The solutions for the three additional problems in frame 88 are as follows.
(a) let t = tens digit
let u = units digit
$t + u = 11$
$10t + u + 20 = 2(10u + t)$
the number is 74.

(b) let a = Sue's present age
a-5 = her age 5 yrs. ago
a+5 = her age in 5 yrs.
$a - 1 = 3(a-5)$
$a + 5 = 2(a-1)$
Sue's present age is 7 yrs.

(c) let h = number of hours walked
then $2\frac{1}{2} - h$ = number of hours rode
$1\frac{1}{2}h + 40(2\frac{1}{2} - h) = 23$
He walked 3 miles.
Note - h isnot what is asked for.

## Chapter 7 - Exponents and Radicals

In this chapter, we shall define the meaning of fractional and negative exponents. We will also define complex numbers.

1. First, we will review the definitions and laws with positive integral exponents.

   If n is a positive integer, then $a^n$ means $a \cdot a \cdot a \cdot \ldots$ for n factors.

   If m and n are positive integers, then $a^m \cdot a^n = a^{m+n}$.

   If m and n are positive integers and $m > n$ and $a \neq 0$, then $\frac{a^m}{a^n} = a^{m-n}$.

   If m and n are positive integers, then $(a^m)^n = a^{mn}$.

   Simplify the following.

   (a) $(m^3n^4p)(m^2np^5) =$

   (b) $(3^2a^4b^5)^3 =$

   (c) $\frac{x^4y^5}{x^2y^4} =$

   (d) $\frac{(x^3p^2)^4(x^2y)}{(xp^2)^3} =$

1. (a) $m^5n^5p^6$

   (b) $3^6a^{12}b^{15}$

   (c) $x^2y$

   (d) $x^{11}p^2y$

2. There are times when we need to know the meaning of an exponent of 0, and also the meaning of a negative exponent. We shall use the following definitions.

   $a^0 = 1$ providing $a \neq 0$.

   $a^{-n} = \frac{1}{a^n}$ providing $a \neq 0$.

   (a) $x^0 =$ ____, (b) $12^0 =$ ____,

   (c) $2p^0 =$ ____, (d) $(2p)^0 =$ _____,

   (e) $2^{-1} =$ _____, (f) $2^{-3} =$ _____.

2. (a) 1
   (b) 1
   (c) 2 Be careful here. An

3. Express the following with positive exponents and combine.
   (a) $x^2y^0$

exponent affects only the letter, number or symbol which immediately precedes it, so the 0 affects only the p.
$2p^0 = 2^1 \cdot p^0 = 2 \cdot 1 = 2$.

(b) $3x^{-2}y^3$

(c) $(2x^{-3})^0(2^{-1}x^3)$

(d) $3x^{-4}+(3x)^{-2}$

(e) $p^{-1}+q^{-1}$

(f) $(p+q)^{-1}$

(g) $a^{-2}+a^{-1}b^{-1}+b^{-2}$

3. (a) $x^2$
   (b) $\frac{3y^3}{x^2}$
   (c) $\frac{x^3}{2}$ or $\frac{1}{2}x^3$
   (d) $\frac{3}{x^4} + \frac{1}{9x^2} = \frac{27+x^2}{9x^4}$
   (e) $\frac{1}{p} + \frac{1}{q} = \frac{q+p}{pq}$
   (f) $\frac{1}{p+q}$
   (g) $\frac{1}{a^2} + \frac{1}{a} \cdot \frac{1}{b} + \frac{1}{b^2} = \frac{b^2+ab+a^2}{a^2b^2}$

4. Definition.

$a^{1/n} = \sqrt[n]{a}$ when the nth root of a exists.

This defines the meaning of fractional exponents providing we remember the definition of a root. $\sqrt[3]{N} = a$ providing $a \cdot a \cdot a = N$, therefore, $\sqrt[n]{N} = a$ providing

____________________ .

4. n factors of a must equal N.

5. $8^{1/3}$ means the same as ________

and this equals ________.

5. $\sqrt[3]{8}$ which equals 2.

6. Find the value of each of the following.

(a) $16^{1/2}$ (b) $16^{1/4}$

(c) $64^{1/3}$ (d) $64^{-1/3}$

(e) $(9^{-1/2})(81^{1/2})$ (f) $81^{1/4}$

(g) $32^{-1/5}$ (h) $(-32)^{1/5}$

(i) $-32^{1/5}$ (j) $-32^{-1/5}$

6. (a) 4
   (b) 2
   (c) 4
   (d) $\frac{1}{64^{1/3}} = \frac{1}{4}$

7. Carefully note the difference in the last four parts of the last frame. In both parts g and j, the only number raised to the $-1/5$ power was 32. This is also true in part i. Then in parts i and j, the result

(e) (1/3)(9) = 3
(f) 3
(g) $\frac{1}{32^{1/5}} = \frac{1}{2}$
(h) -2
(i) -2
(j) $-\frac{1}{32^{1/5}} = -\frac{1}{2}$

is multiplied by -1. However, in part h, the number raised to the -1/5 power is -32. This doesn't seem to make any difference in the answers to these problems, but it necessary to note these differences carefully as it does make a difference in some cases.
By defining exponents of 0, negative numbers and fractions in this way, we can use the same laws as those given in frame 1 for positive integral exponents.

Thus, $a^{1/2} \cdot a^{1/4} = a^{1/2+1/4} = a^{3/4}$.

$\frac{a^{3/5}}{a^{1/10}} =$

$(a^{1/4})^2 =$

7. $a^{3/5\ -\ 1/10} = a^{5/10} = a^{1/2}$

$a^{(1/4)2} = a^{2/4} = a^{1/2}$

8. Since we have defined the meaning of negative exponents, we can now define division in the following way.

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{when } m \neq n \\ a^0 & \text{when } m = n \end{cases}$$

When $m < n$, then m-n is a negative number, but by using the definition of negative exponents, we can write an equivalent quantity having positive exponents.

Simplify the following. Write the final answers with positive exponents.

(a) $25^{1/2}x^{2/3}y^{-2} \cdot 100^{-1/2}x^{1/2}y$

(b) $(36p^{-4}q^{5/3}r^3)^{1/2}$

(c) $\frac{64^{-1}a^2b^{3/4}c^{-3}}{4^{1/2}a^{-5}bc^{-3}}$

(d) $\frac{x^{-2}+y^{-2}}{(x+y)^{-2}}$

8. (a) You can multiply by adding exponents <u>only</u> if the bases are the same. The numerical bases in this case are <u>not</u> the same.
$5x^{2/3}y^{-2} \cdot \frac{1}{10}\overline{x^{1/2}y} = \frac{1}{2}x^{7/6}y^{-1}$
$= \frac{1}{2}x^{7/6}(1/y) = \frac{x^{7/6}}{2y}$

9. In the last frame, we had $x^{2/3}$. We have defined $x^{1/n}$ as $\sqrt[n]{x}$ providing the nth root of x exists. The nth root of x <u>doesn't</u> exist at this time, if the nth root <u>isnot</u> a real number.

(b) $36^{1/2}(p^{-4})^{1/2}(q^{5/3})^{1/2}(r^3)^{1/2}$

$= 6p^{-2}q^{5/6}r^{3/2} = \frac{6q^{5/6}r^{3/2}}{p^2}$

(c) When the negative exponents are changed to positive ones, the numerator becomes $\frac{a^2b^{3/4}}{64c^3}$.

The denominator becomes $\frac{2b}{a^5c^3}$.

Dividing the numerator by the denominator, we get $\frac{a^7}{128b^{1/4}}$.

(d) The numerator becomes $\frac{1}{x^2} + \frac{1}{y^2} = \frac{y^2+x^2}{x^2y^2}$. The denominator becomes $\frac{1}{(x+y)^2}$.

Dividing, we get $\frac{(x+y)^2(y^2+x^2)}{x^2y^2}$.

For example, $\sqrt{-4}$ <u>isnot</u> a real number because we can find <u>no</u> real number such that when it is multiplied by itself, we get -4.

Is $\sqrt[3]{-8}$ a real number?

Is $\sqrt[4]{-16}$ a real number?

9. $\sqrt[3]{-8} = -2$ so yes it is a real number.

There is no real number such that $a \cdot a \cdot a \cdot a = -16$ so $\sqrt[4]{-16}$ is <u>not</u> a real number.

10. $x^{1/n}$ or $\sqrt[n]{x}$ exists when x is a real number. n can be an integer greater than 0.
This also exists when x is a negative number and n is an <u>odd</u> integer.

$x^{p/q}$ will be a real number when __________ or when ________ .

10. x is positive and p and q are any real numbers <u>except</u> q = 0
or
when x is negative and q is an integer.

11. $x^{p/q} = (x^{1/q})^p = (x^p)^{1/q} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p$ providing the qth root of x and the qth root of $x^p$ are real numbers.

$27^{2/3} = (27^{1/3})^2 =$ ________ =

________ = ________. Express the final answer without exponents.

11. $(27^2)^{1/3} = 3^2 = 9$.

You can see that it is easier to find the cube root of 27 and then square the result than to square 27 and then take the cube root of the result. However, both procedures are correct.

12. Evaluate each of the following.

(a) $8^{2/3}$ (b) $8^{5/3}$ (c) $(-8)^{5/3}$

(d) $4^{-3/2}$ (e) $32^{-3/5}$ (f) $(-32)^{-3/5}$

(g) $(-125)^{2/3}$ (h) $-125^{2/3}$ (i) $-16^{-3/2}$

(j) $16^{3/4}$ (k) $(-27)^{-4/3}$ (l) $-27^{-4/3}$

12. (a) $(\sqrt[3]{8})^2 = 4$

(b) $(\sqrt[3]{8})^5 = 32$

(c) $(\sqrt[3]{-8})^5 = (-2)^5 = -32$

(d) $\frac{1}{(\sqrt{4})^3} = \frac{1}{8}$

(e) $(\sqrt[5]{32})^{-3} = \frac{1}{2^3} = \frac{1}{8}$

(f) $(\sqrt[5]{-32})^{-3} = -\frac{1}{8}$

(g) $(\sqrt[3]{-125})^2 = (-5)^2 = 25$

(h) $-(\sqrt[3]{125})^2 = -25$

(i) $-(\sqrt{16})^{-3} = -(4)^{-3} = -\frac{1}{64}$

(j) $(\sqrt[4]{16})^3 = (2)^3 = 8$

(k) $(\sqrt[3]{-27})^{-4} = (-3)^{-4} = \frac{1}{81}$

(l) $-(\sqrt[3]{27})^{-4} = -(3)^{-4} = -\frac{1}{81}$

13. Note carefully the differences between parts g and h. Also note the differences between parts k and l. If you made errors in the last frame, it would be advisable to write out the meanings of these in terms of the radicals. Remember an exponent affects only the letter, number or symbol which immediately precedes it. Thus, in part h, only 125 is raised to the power, and in part g, -125 is raised to the power.

As stated before, the laws of exponents as given in frame 1, apply whether the exponents are integers or fractions and regardless of whether they are positive or negative.
Simplify each of the following. Express with positive exponents.

(a) $(3a^{1/2}b^{-3}c^2)(4a^{3/2}b^{-5}c^{-3})$

(b) $(4ab^{-2}c^3)^{1/2}$

(c) $\frac{(5x^{1/2}y^2)^3}{-10x^3y^{-2}}$

(d) $(-p^2q^{-3}r^5)^{-5/3}$

(e) $\frac{(-2a^pb^{-3q})^3}{-2^4a^{-2p}b^q}$

(f) $(\frac{81a^{-6}}{b^{-1}})^{-1/2} (\frac{b^2}{27a^{-2}})^{-1}$

13. You can remove the negative exponents first and then do the operations or you can do the operations and then remove the negative exponents when the operations are multiplication, division and raising to powers.

(a) $12a^2b^{-8}c^{-1} = 12a^2 \cdot \frac{1}{b^8} \cdot \frac{1}{c}$

$= \frac{12a^2}{b^8c}$

(b) $4^{1/2}a^{1/2}b^{-1}c^{3/2} = \frac{2a^{1/2}c^{3/2}}{b}$

14. If we are to express $(x^{-2}+y)^{-1}$ with positive exponents, it is usually to our advantage to remove the negative exponents first. However, be sure to remove only one negative exponent at a time.

$(x^{-2}+y)^{-1} = \frac{1}{x^{-2}+y} = \frac{1}{\frac{1}{x^2}+y}$

This is a complex fraction and to simplify it, we must _______.

Do the simplification.

(c) $\frac{125x^{3/2}y^6}{-10x^3y^{-2}} = -\frac{25y^8}{2x^{3/2}}$

(d) $(-1)^{5/3}p^{-10/3}q^5r^{-25/3} = \frac{-1q^5}{p^{10/3}r^{25/3}}$

(e) $\frac{-8a^{3p}b^{-9q}}{-16a^{-2p}b^q} = \frac{a^{5p}}{2b^{10q}}$

(f) $\frac{81^{-1/2}a^3}{b^{1/2}} \cdot \frac{b^{-2}}{27^{-1}a^2} = \frac{27a}{9b^{5/2}} = \frac{3a}{b^{5/2}}$

14. Multiply both numerator and denominator by the same number namely $x^2$.
or
express the denominator as one fraction and divide.
$\frac{x^2}{1+x^2y}$

15. You must note that $(x^{-2}+y)^{-1}$ doesnot equal $x^2+y^{-1}$.

You cannot multiply the exponents to raise to powers unless the the base is composed of factors.

Perform the indicated multiplications and then express the products with positive exponents.

(a) $(x^{-2}+y^{-2})(x^{-2}-y^{-2})$

(b) $(x^{-1/4}+y^{1/3})(x^{-1/4}-y^{1/3})$

(c) $(x^{-2}-3y^{-2})(x^{-4}+3x^{-2}y^{-2}+9y^{-4})$

(d) $(2x^{-2/3}+y^{-1/3})(4x^{-4/3}-2x^{-2/3}y^{-1/3}+y^{-2/3})$

(e) $(x^{-3/2}-y^{-3/2})(x^{-3/2}+y^{-3/2})$

15. (a) $x^{-4}-y^{-4} = \frac{y^4-x^4}{x^4y^4}$

(b) $x^{-1/2}-y^{-2/3} = \frac{1-y^{2/3}x^{1/2}}{x^{1/2}}$

(c) $x^{-6}-27y^{-6} = \frac{y^6-27x^6}{x^6y^6}$

(d) $8x^{-2}+y^{-1} = \frac{8y+x^2}{x^2y}$

(e) $x^{-3}-y^{-3} = \frac{y^3-x^3}{x^3y^3}$

16. You will note that in parts a, b and e of the last frame you were multiplying two quantities - one of which was the sum of two numbers and the other was the difference of the same two numbers.

When two quantities of this sort are multiplied together, the result is the difference of the squares of each of the terms. Thus, in part a, $x^{-4}$ is the square of $x^{-2}$ and $y^{-4}$ is the square of $y^{-2}$. Consider parts c and d. What relation do the terms in the product have to the terms in the binomial which is given as one of the factors?

16. They are the cubes of these terms.
In part c, $x^{-6}$ is the cube of of $x^{-2}$ and $y^{-6}$ is the cube of $y^{-2}$.

17. Then we can consider $x^{-6}-27y^{-6}$ as the difference of two cubes and factor it as we would the difference of any two cubes.

Consider x-y as the difference of two cubes and write its factors.

17. $(x^{1/3}-y^{1/3})(x^{2/3}+x^{1/3}y^{1/3}+y^{2/3})$

18. We can also consider x-y as the difference of two squares if x and y are positive numbers.

Consider x-y as the difference of two squares and write its factors.

18. $(x^{1/2}-y^{1/2})(x^{1/2}+y^{1/2})$

19. Why do x and y have to be positive numbers in order to write the factors of x-y considering this as the difference of two squares?

19. the square roots of negative numbers are <u>not</u> real numbers.

20. In factoring x-y as the difference of two cubes, do we have to limit the values of x and y? Give a reason for your answer.

20. No because cube roots of both positive and negative numbers are real numbers.

21. Factor each of the following in two ways. First consider each as the difference of two squares and second consider each as the difference of two cubes. <u>Donot</u> change to positive exponents.

(a) $x^{-4}-64y^{-4}$

(b) $x^{-3}-y^{-2}$

(c) $x^{-6}-y^{6}$

21. (a) as squares
$(x^{-2}+8y^{-2})(x^{-2}-8y^{-2})$
When factored as cubes,
$(x^{-4/3}-4y^{-4/3})(x^{-8/3}+4x^{-4/3}y^{-4/3}+16y^{-8/3})$
(b) as squares
$(x^{-3/2}-y^{-1})(x^{-3/2}+y^{-1})$
as cubes
$(x^{-1}-y^{-2/3})(x^{-2}+x^{-1}y^{-2/3}+y^{-4/3})$
(c) as squares
$(x^{-3}-y^{-3})(x^{-3}+y^{-3})$
as cubes
$(x^{-2}-y^{2})(x^{-4}+x^{-2}y^{2}+y^{4})$

22. Remember the sum of two cubes can be factored but that the sum of two squares <u>cannot</u> be factored. You should also look for common factors and try to factor trinomials of the quadratic type.

Factor each of the following. The numbers in each case give a clue as to how to factor each one.
(a) $27x^{-6}+8y^{-3}$
(b) $x^{-2}-4y^{-3}$
(c) $x^{-5}-16x^{-3}$ Hint: remove common factor first so that exponents in one factor are positive.
(d) $3x^{-2}-13x^{-1}-30$

22. (a) Do as cubes.
$(3x^{-2}+2y^{-1})(9x^{-4}+6x^{-2}y^{-1}+4y^{-2})$

(b) $(x^{-1}+2y^{-3/2})(x^{-1}-2y^{-3/2})$
when considered as squares.
(c) remove a common factor of $x^{-5}$.
$x^{-5}(1-16x^2) = x^{-5}(1-4x)(1+4x)$
(d) $(3x^{-1}+5)(x^{-1}-6)$

23. Reduce the following fractions to lowest terms by first factoring. Express final results with positive exponents.

(a) $\dfrac{x^{-9}+27y^{-3}}{x^{-3}+3y^{-1}}$

(b) $\dfrac{16x^{-8}-25y^{-4}}{4x^{-4}+5y^{-2}}$

(c) $\dfrac{x^{-5}+2x^{-4}}{x^{-7}-4x^{-5}}$

(d) $\dfrac{x^{1/2}+y^{1/2}}{x-y}$

(e) $\dfrac{x^{-2}+y^{2}}{x^{-2/3}+y^{2/3}}$

23. (a) $x^{-6}-3x^{-3}y^{-1}+9y^{-2} = \dfrac{y^2-3x^3y+9x^6}{x^6y^2}$

(b) $4x^{-4}-5y^{-2} = \dfrac{4y^2-5x^4}{x^4y^2}$

(c) $\dfrac{x^{-5}}{x^{-7}(1-2x)} = \dfrac{x^2}{1-2x}$

(d) $\dfrac{1}{x^{1/2}-y^{1/2}}$

(e) $x^{-4/3}+x^{-2/3}y^{2/3}+y^{4/3} = \dfrac{1+x^{2/3}y^{2/3}+y^{4/3}x^{4/3}}{x^{4/3}}$

24. We can solve equations involving exponents if they are of the form $a^m = a^n$. In this case, we have an equality involving exponentials and since the bases are the same, the exponents must be equal. Thus if $2^x = 2^5$, x must equal 5.

Also, if the exponents are equal, the bases must be equal. Thus, if $x^3 = 2^3$, than x equals 2.

In other words, exponentials are equal if the bases are the same and the exponents are equal.
Solve for x in each of the following.

(a) $x^3 = 2^3$ (b) $3^{2-x} = 3^5$

(c) $4^{2x} = 4^{x-2}$ (d) $x^5 = 32$

(e) $x^3 = 27$ (f) $5^{3x-2} = 5^{2(x-1)}$

24. (a) $x = 2$
(b) $2-x = 5$, $x = -3$.
(c) $2x = x-2$, $x = -2$.
(d) Here we don't have an equation of the form $a^m=a^n$, but $32 = 2^5$, so $x^5 = 32$ becomes $x^5=2^5$ and $x=2$.

(e) Write 27 as $3^3$. If $x^3=3^3$, then $x = 3$.
(f) $3x-2 = 2(x-1)$, so $x = 0$.

25. If equations are not given in the form $a^m = a^n$, then we must put them in this form before solving. For example, to solve $3^2 \cdot 3^x = 27$, we must use the laws of exponents on the left side of the equation and express the right side of the equation as a power of three so that we have the same base on both sides of the equation.

$3^2 \cdot 3^x =$ __________ .
$27 =$ __________.

25. $3^2 \cdot 3^x = 3^{2+x}$
$27 = 3^3$

26. Now we can write $3^{2+x}=3^3$. Solve for x.

26. $2+x = 3$, $x = 1$.

27. Solve for x: $4 = (2^{x-2})^2$.

27. 4 has to be expressed as a power of 2 or as $2^2$. On the right side of the equation we have to raise to a power and obtain $2^{2x-4}$. The equation now reads $2^2=2^{2x-4}$. So $2 = 2x-4$, and $x = 3$.

28. Solve the following.

(a) $2^{x-1} \cdot 2^{3x} = 8$

(b) $(5^{3x})^4 \cdot 5^3 = 25$

(c) $4^2 \cdot 2^{3+x} = 16$

(d) $27 \cdot 3^{2x} = 9^{2x-1}$

28. (a) $2^{x-1+3x} = 2^3$
$x-1+3x = 3$, $x = 1$

(b) $5^{12x} \cdot 5^3 = 5^2$
$12x+3 = 2$, $x = -1/12$

(c) $(2^2)^2 \cdot 2^{3+x} = 2^4$, $2^4 \cdot 2^{3+x} = 2^4$,
$4+3+x=4$, $x = -3$.

(d) $3^3 \cdot 3^{2x} = (3^2)^{2x-1}$, $3^{3+2x} = 3^{4x-2}$
$3+2x = 4x-2$, $x = 5/2$

29. To simplify the following, we would first express with positive exponents and then combine.

Simplify:
$(3y+1)^{1/2}(2x-3)^{-1/3} + (3y+1)^{-1/2}(2x-3)^{2/3}$

29. $\dfrac{(3y+1)^{1/2}}{(2x-3)^{1/3}} + \dfrac{(2x-3)^{2/3}}{(3y+1)^{1/2}} =$

The LCD is $(2x-3)^{1/3}(3y+1)^{1/2}$

$$\frac{(3y+1)^{1/2}(3y+1)^{1/2} + (2x-3)^{2/3}(2x-3)^{1/3}}{(2x-3)^{1/3}(3y+1)^{1/2}}$$

30. Let us consider the first term in the numerator of the answer to the last frame to see if it can be simplified further. $(3y+1)^{1/2}(3y+1)^{1/2}$ is a situation where we can apply the law of multiplication for exponents.

This law states that $a^m \cdot a^n = a^{m+n}$ or that the product of two exponentials which have the same base is that base to the sum of the powers.
So, $(3y+1)^{1/2}(3y+1)^{1/2} =$ ______.

30. $(3y+1)^1$ or $3y+1$

31. $(2x-3)^{2/3}(2x-3)^{1/3} =$ ______.

31. $(2x-3)^1$ or $2x-3$

32. Then the fraction given as the answer in frame 29 can be rewritten as an equivalent fraction which involves no fractional exponents in the numerator.

$$\frac{(3y+1)^{1/2}(3y+1)^{1/2} + (2x-3)^{2/3}(2x-3)^{1/3}}{(3y+1)^{1/2}(2x-3)^{1/3}}$$

= ______ = ______.

32. $\dfrac{3y+1+2x-3}{(3y+1)^{1/2}(2x-3)^{1/3}} =$

$\dfrac{3y+2x-2}{(3y+1)^{1/2}(2x-3)^{1/3}}$

33. Simplify each of the following so that the answers are expressed as a single fraction and with positive exponents.

(a) $(5x-2)^{-1/4}(x+4)^{1/2}+(5x-2)^{3/4}(x+4)^{-1/2}$

(b) $(2y+7)^{1/3}(3y-1)^{2/5}+(2y+7)^{-2/3}(3y-1)^{-3/5}$

(c) $3(x^2-1)^{-1/2}(2x^2+3)-(x^2-1)^{1/2}$

(d) $2y(3y-4)^{-2/3}-3(3y-4)^{1/3}$

(e) $2x^2(x-2y)^{-2/5}-(x-2y)^{3/5}$

33. (a) $\dfrac{(x+4)^{1/2}}{(5x-2)^{1/4}}+\dfrac{(5x-2)^{3/4}}{(x+4)^{1/2}} =$

$\dfrac{6x+2}{(5x-2)^{1/4}(x+4)^{1/2}}$

(b) $\dfrac{6y^2+19y-6}{(2y+7)^{2/3}(3y-1)^{3/5}}$

(c) $\dfrac{3(2x^2+3)-(x^2-1)}{(x^2-1)^{1/2}} = \dfrac{5x^2+10}{(x^2-1)^{1/2}}$

(d) $\dfrac{2y-3(3y-4)}{(3y-4)^{2/3}} = \dfrac{12-7y}{(3y-4)^{2/3}}$

(e) $\dfrac{2x^2-(x-2y)}{(x-2y)^{2/5}} = \dfrac{2x^2-x+2y}{(x-2y)^{2/5}}$

34. We are now going to consider the radical form instead of using fractional exponents. Sometimes it is more convenient to use the radical form than the exponential form.

We defined $a^{1/n}$ to be the nth root of a providing this was a real number.

That is, $a^{1/n} = \sqrt[n]{a}$ .

We shall need some other laws concerning radicals. These laws can be derived from the laws of exponents.

$\sqrt[n]{a^n} = a$ because $\sqrt[n]{a^n} = (a^n)^{1/n} =$

$a^{n/n} = a^1 = a.$

$\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n}.$

$\sqrt[3]{y^3} =$ ________

$\sqrt[4]{3\cdot 4} =$ ________

34. $\sqrt[3]{y^3} = y$

$\sqrt[4]{3\cdot 4} = 3^{1/4}\cdot 4^{1/4}$

35. We can apply these two laws to reduce radicals.

For example, $\sqrt[3]{24} = \sqrt[3]{2\cdot 2\cdot 2\cdot 3} =$

$\sqrt[3]{2^3\cdot 3} = \sqrt[3]{2^3}\cdot\sqrt[3]{3} = 2\sqrt[3]{3}$ .

$\sqrt{12} =$ ________ .

35. $\sqrt{2\cdot 2\cdot 3} = \sqrt{2^2\cdot 3} = \sqrt{2^2}\cdot\sqrt{3} = 2\sqrt{3}$

36. Reduce the following.

(a) $\sqrt{75}$ (b) $\sqrt[3]{16}$ (c) $\sqrt[4]{32}$

(d) $\sqrt[3]{54}$ (e) $\sqrt{300}$ (f) $\sqrt{108}$

(g) $\sqrt[3]{375}$ (h) $\sqrt[5]{192}$ (i) $\sqrt[4]{162}$

(j) $\sqrt{98}$ (k) $\sqrt{432}$ (l) $\sqrt[3]{1728}$

36. (a) $\sqrt{5^2 \cdot 3} = 5\sqrt{3}$
(b) $\sqrt[3]{2^3 \cdot 2} = 2\sqrt[3]{2}$
(c) $\sqrt[4]{2^4 \cdot 2} = 2\sqrt[4]{2}$
(d) $\sqrt[3]{3^3 \cdot 2} = 3\sqrt[3]{2}$
(e) $\sqrt{10^2 \cdot 3} = 10\sqrt{3}$
(f) $\sqrt{2^2 \cdot 3^2 \cdot 3} = 2 \cdot 3\sqrt{3} = 6\sqrt{3}$
(g) $\sqrt[3]{5^3 \cdot 3} = 5\sqrt[3]{3}$
(h) $\sqrt[5]{2^5 \cdot 6} = 2\sqrt[5]{6}$
(i) $\sqrt[4]{3^4 \cdot 2} = 3\sqrt[4]{2}$
(j) $\sqrt{7^2 \cdot 2} = 7\sqrt{2}$
(k) $\sqrt{4^2 \cdot 3^2 \cdot 3} = 12\sqrt{3}$
(l) $\sqrt[3]{2^3 \cdot 2^3 \cdot 3^3} = 2 \cdot 2 \cdot 3 = 12$

37. The same principles can be applied in reducing radicals such as $\sqrt{a^7b^3}$.

$\sqrt{a^7b^3} = \sqrt{a^2 \cdot a^2 \cdot a^2 \cdot a \cdot b^2 \cdot b} =$

$\sqrt{a^2}\sqrt{a^2}\sqrt{a^2}\sqrt{a}\sqrt{b^2}\sqrt{b} = a \cdot a \cdot a \cdot b\sqrt{a}\sqrt{b}$
$= a^3b\sqrt{ab}$

Reduce the following.

(a) $\sqrt{a^5b^6}$ (b) $\sqrt{x^{11}y^8}$
(c) $\sqrt[3]{a^4b^6}$ (d) $\sqrt[3]{x^{12}y^8z^4}$
(e) $\sqrt[4]{p^3q^6}$ (f) $\sqrt{48xy^4z^5}$
(g) $\sqrt[3]{48xy^4z^5}$ (h) $\sqrt[5]{x^{10}y^{12}z^8}$

37. (a) $\sqrt{a^2 \cdot a^2 \cdot a \cdot b^2 \cdot b^2 \cdot b^2} = a^2b^3\sqrt{a}$
(b) $\sqrt{x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot x \cdot y^2 \cdot y^2 \cdot y^2 \cdot y^2}$
$= x^5y^4\sqrt{x}$
(c) $\sqrt[3]{a^3 \cdot a \cdot b^3 \cdot b^3} = ab^2\sqrt[3]{a}$
(d) $\sqrt[3]{x^3 \cdot x^3 \cdot x^3 \cdot x^3 \cdot y^3 \cdot y^3 \cdot y^2 \cdot z^3 \cdot z}$
$= x^4y^2z\sqrt[3]{y^2z}$
(e) $\sqrt[4]{p^3 \cdot q^4 \cdot q^2} = q\sqrt[4]{p^3q^2}$
(f) $\sqrt{4^2 \cdot 3 \cdot x \cdot y^2 \cdot y^2 \cdot z^2 \cdot z^2 \cdot z} =$
$4y^2z^2\sqrt{3xz}$
(g) $\sqrt[3]{2^3 \cdot 6 \cdot x \cdot y^3 \cdot y \cdot z^3 \cdot z^2} = 2yz\sqrt[3]{6xyz^2}$
(h) $\sqrt[5]{x^{10} \cdot y^{10} \cdot y^2 \cdot z^5 \cdot z^3} = x^2y^2z\sqrt[5]{y^2z^3}$

38. If we are to reduce $\sqrt{x^2+4xy+4y^2}$ then we must use the same laws as we used in the last two frames.

We know that $\sqrt[n]{a^n} = a$, and in this case we are looking for a square root. Thus, we want $\sqrt{a^2}$ so that we can apply the law.

We also know that $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

Can we apply these <u>laws immediately</u> to simplify $\sqrt{x^2+4xy+4y^2}$ ? Give a reason for your answer.

38. No. We have to have <u>factors</u> in order to apply

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

39. What must be done first to simplify $\sqrt{x^2+4xy+4y^2}$ ?

39. factor the polynomial

40. Simplify $\sqrt{x^2+4xy+4y^2}$.

40. $\sqrt{(x+2y)(x+2y)} = \sqrt{(x+2y)^2}$
$= x+2y$

41. Reduce $\sqrt{(x^2-9y^2)(x-3y)}$.

41. $\sqrt{(x-3y)(x+3y)(x-3y)} =$

$\sqrt{(x-3y)^2(x+3y)} = (x-3y)\sqrt{x+3y}$

42. Note carefully the answer to the last frame.
$(x-3y)\sqrt{x+3y}$ is not the same as $x-3y\sqrt{x+3y}$ .
What is the difference in these two quantities?

42. $(x-3y)\sqrt{x+3y} = x\sqrt{x+3y} - 3y\sqrt{x+3y}$

or the quantity x-3y is multiplied by the square root.

In $x-3y\sqrt{x+3y}$ only 3y is multiplied by the square root.

43. Simplify.

(a) $\sqrt{4x^2-28x+49}$

(b) $\sqrt{(3x-5)^2(2x+1)^2}$

(c) $\sqrt{x^3-4x^2-16x+64}$

43. (a) $\sqrt{(2x-7)^2} = 2x-7$

(b) $\sqrt{(3x-5)^2}\sqrt{(2x+1)^2} = (3x-5)(2x+1)$

(c) $\sqrt{x^2(x-4)-16(x-4)} =$

$\sqrt{(x-4)(x^2-16)} = \sqrt{(x-4)(x-4)(x+4)}$

$= (x-4)\sqrt{x+4}$

44. To deal with fractions which occur under the radical sign, we need one further law.
$(\frac{a}{b})^{1/n} = \frac{a^{1/n}}{b^{1/n}}$

By applying this law, $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}}$
$= \frac{\sqrt{3}}{2}$ .
Simplify: $\sqrt{\frac{18}{25}}$ .

44. $\frac{\sqrt{18}}{\sqrt{25}} = \frac{\sqrt{3^2 \cdot 2}}{5} = \frac{3\sqrt{2}}{5}$

45. Simplify each of the following.

(a) $\sqrt{\frac{252}{98}}$

(b) $\sqrt{\frac{72a^3b^4}{x^2y^6}}$

(c) $\sqrt[3]{\frac{54a^5}{125b}}$

(d) $\sqrt{\frac{9x^2+30x+25}{x^2-14x+49}}$

(e) $\sqrt[3]{\frac{(x-3)(x^2-6x+9)}{250x^3y^5z^{11}}}$

(f) $\sqrt[3]{\frac{8-27y^3}{27y^3}}$

(g) $\sqrt{\frac{(x-2)^4(2x+5)^3}{2x+5}}$

45. (a) $\frac{\sqrt{252}}{\sqrt{98}} = \frac{\sqrt{2^2 \cdot 3^2 \cdot 7}}{\sqrt{7^2 \cdot 2}} = \frac{6\sqrt{7}}{7\sqrt{2}}$

46. In part g of the last frame, we got $\frac{\sqrt{2x+5}}{\sqrt{2x+5}}$ and this equals 1 as it

(b) $\dfrac{\sqrt{6^2\cdot a^2\cdot a\cdot b^2\cdot b^2\cdot 2}}{\sqrt{x^2\cdot y^2\cdot y^2\cdot y^2}} = \dfrac{6ab^2\sqrt{2a}}{xy^3}$

(c) $\dfrac{\sqrt[3]{3^3\cdot 2a^3a^2}}{\sqrt[3]{5^3\cdot b}} = \dfrac{3a\sqrt[3]{2a^2}}{5\sqrt[3]{b}}$

(d) $\dfrac{\sqrt{(3x+5)^2}}{\sqrt{(x-7)^2}} = \dfrac{3x+5}{x-7}$

(e) $\dfrac{\sqrt[3]{(x-3)(x-3)(x-3)}}{\sqrt[3]{5^3\cdot 2x^3y^3y^2z^9z^2}} = \dfrac{x-3}{5xyz^3\sqrt[3]{2y^2z^2}}$

(f) $\dfrac{\sqrt[3]{8-27y^3}}{3y}$

$8-27y^3$ can be factored, but it is not a perfect cube. It is the difference of two cubes.

(g) $\dfrac{\sqrt{(x-2)^4(2x+5)^2(2x+5)}}{\sqrt{2x+5}}$

$= \dfrac{(x-2)^2(2x+5)\sqrt{2x+5}}{\sqrt{2x+5}} = (x-2)^2(2x+5)$

represents a number divided by itself and a number divided by itself equals one unless the number is 0.
However, in this case, we need the added condition that 2x+5 is not a negative number.

Why can't 2x+5 be a negative number?

46. The square root of a negative number is not a real number.

47. The square root of a number which is not a perfect square is an irrational number. The cube root of a number which is not a perfect cube is an irrational number. The general case is the nth root of a number which is not a perfect nth power or, in other words, the nth root of a number which cannot be factored into n identical factors is an irrational number.

At the beginning of the term, we defined a rational number as one of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

The definition of an irrational number then is _______________.

47. numbers which can't be expressed as the quotient of two integers.

48. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{2}$, $\sqrt{3/2}$ are examples of ________ numbers.

48. irrational

49. Is $\sqrt{-4}$ an irrational number? Why?

49. No. -4 is not a perfect square so it isn't a rational number.

50. If you are in doubt, refer back to frame 9.

In addition, the number under the radical sign, must be such that a real number taken as a factor the required number of times equals the number under the radical. Thus, $\sqrt{-4} = a$ only if $a \cdot a = -4$ where a is a real number would make $\sqrt{-4}$ a real number. Since there is no real number which when multiplied by itself equals -4, $\sqrt{-4}$ is not an irrational number.
Note - irrational numbers are real numbers.

Irrational numbers of the form $\sqrt{\frac{a}{b}}$ can be written as $\frac{1}{b}\sqrt{ab}$. This is called the rationalized form of an irrational number.

$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}}$ because a/b and $ab/b^2$ are equivalent fractions.

$$\sqrt{\frac{ab}{b^2}} = \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{\sqrt{ab}}{b} = \frac{1}{b}\sqrt{ab}.$$

Express $\sqrt{\frac{3}{2}}$ in rationalized form.

50. $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$ or $\frac{1}{2}\sqrt{6}$

51. Express $\sqrt{\frac{5}{8}}$ in rationalized form.

51. $\sqrt{\frac{5}{8}} = \sqrt{\frac{40}{64}} = \frac{\sqrt{40}}{\sqrt{64}} = \frac{\sqrt{40}}{8}$

52. The object in rationalizing is to write the number under the radical as an integer and this means that the denominator must be a perfect square for it to be brought outside the square root sign.

In the last problem, instead of changing $\sqrt{\frac{5}{8}}$ to $\sqrt{\frac{40}{64}}$ we could have changed it to $\sqrt{\frac{10}{16}}$. In both cases the denominators are perfect squares and so we can find an equivalent fraction where we have a positive integer under the radical.

$$\sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{\sqrt{10}}{4}$$

If both of these procedures are correct then we should have equivalent answers.
How would you determine if $\frac{\sqrt{10}}{4}$ and $\frac{\sqrt{40}}{8}$ are equivalent fractions?

52. reduce the fraction $\frac{\sqrt{40}}{8}$ or multiply both numerator and denominator of $\frac{\sqrt{10}}{4}$ by the same number to see if it equals $\sqrt{40}/8$.

53. Reduce $\frac{\sqrt{40}}{8}$.

53. $\frac{\sqrt{4 \cdot 10}}{8} = \frac{2\sqrt{10}}{8} = \frac{\sqrt{10}}{4}$

54. Thus, you can see that $\sqrt{\frac{5}{8}}$ can be

rationalized by changing either to $\frac{\sqrt{40}}{8}$ or to $\frac{\sqrt{10}}{4}$. It is usually to your advantage to change to the smallest equivalent fraction, which has a denominator which is a perfect square under the radical.

Rationalize. $\sqrt{\frac{2}{27}}$

Express your final answer in lowest terms.

54. $\sqrt{\frac{54}{27^2}} = \frac{\sqrt{54}}{\sqrt{27^2}} = \frac{\sqrt{9 \cdot 6}}{27} = \frac{3\sqrt{6}}{27} = \frac{\sqrt{6}}{9}$

or $\sqrt{\frac{6}{81}} = \frac{\sqrt{6}}{\sqrt{81}} = \frac{\sqrt{6}}{9}$

55. Either procedure given in the last answer is correct but you will notice it is quicker to use the second one. Use the procedure you understand the best.

Rationalize $\sqrt{\frac{8}{75}}$ and express your final answer in lowest terms.

55. $\sqrt{\frac{8}{25 \cdot 3}} = \frac{\sqrt{24}}{\sqrt{25 \cdot 9}} = \frac{\sqrt{4 \cdot 6}}{5 \cdot 3} = \frac{2\sqrt{6}}{15}$

56. You will notice that in this last problem, both numerator and denominator of the fraction under the radical were multiplied by 3. In order to determine what to multiply by, factor the denominator into a factor which represents a perfect power and another factor. Multiply both numerator and denominator by a number which will make this second factor a perfect power.

You will notice that then the two factors do not need to be multiplied together. It is not wrong if you do the multiplication, but it is not necessary.

To rationalize $\sqrt[3]{\frac{3}{2}}$ you can use the same procedure, but this time the denominator must be a perfect cube in order for a positive integer to be left under the radical.

Rationalize $\sqrt[3]{\frac{3}{2}}$.

56. $\sqrt[3]{\frac{12}{2 \cdot 2 \cdot 2}} = \frac{\sqrt[3]{12}}{\sqrt[3]{2 \cdot 2 \cdot 2}} = \frac{\sqrt[3]{12}}{2}$

57. Rationalize each of the following.

(a) $\sqrt{\frac{21}{108}}$ (b) $\sqrt[3]{\frac{5}{4}}$ (c) $\sqrt{\frac{3a^3}{4b^5}}$

(d) $\sqrt{\frac{28x^2y^7}{63a^2b^5}}$ (e) $\sqrt[4]{\frac{4x^5}{27}}$

57. (a) $\sqrt{\frac{21}{9\cdot4\cdot3}} = \frac{\sqrt{21\cdot3}}{\sqrt{9\cdot4\cdot9}} = \frac{3\sqrt{7}}{3\cdot2\cdot3} = \frac{\sqrt{7}}{6}$

(b) $\sqrt[3]{\frac{10}{8}} = \frac{\sqrt[3]{10}}{2}$

(c) $\frac{\sqrt{3a^3b}}{\sqrt{4b^6}} = \frac{a\sqrt{3ab}}{2b^3}$

(d) $\frac{\sqrt{28\cdot7x^2y^7b}}{\sqrt{9\cdot7\cdot7a^2b^6}} = \frac{14xy^3\sqrt{yb}}{3\cdot7ab^3} = \frac{2xy^3\sqrt{yb}}{3ab^3}$

(e) $\frac{x\sqrt[4]{12x}}{3}$

58. If we are to rationalize $\frac{\sqrt{3}}{\sqrt{5}}$ we can follow either of two procedures. We know that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ so we can multiply both numerator and denominator by a number which will give a perfect power in the denominator under the radical.

$$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}\sqrt{5}}{\sqrt{5}\sqrt{5}} =$$

58. $\frac{\sqrt{15}}{\sqrt{5\cdot5}} = \frac{\sqrt{15}}{5}$

59. Instead of this last procedure, we know that $(\frac{a}{b})^{1/n} = \frac{a^{1/n}}{b^{1/n}}$ and so $\frac{\sqrt{3}}{\sqrt{5}} = \sqrt{\frac{3}{5}}$ and this can be rationalized as we did previously.

Rationalize the following.

(a) $\frac{\sqrt{6}}{\sqrt{7}}$ (b) $\frac{\sqrt{8}}{\sqrt{32}}$ (c) $\frac{\sqrt[3]{27}}{\sqrt[3]{16}}$

(d) $\sqrt{\frac{20a^3}{7b}}$ (e) $\sqrt{\frac{2x}{x-3}}$ (f) $\frac{\sqrt{8x^5}}{\sqrt{x-y}}$

59. (a) $\frac{\sqrt{42}}{7}$ (b) $\sqrt{\frac{8}{32}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

(c) $\frac{\sqrt[3]{27}\sqrt[3]{4}}{\sqrt[3]{8\cdot2\cdot4}} = \frac{3\sqrt[3]{4}}{2\cdot2} = \frac{3\sqrt[3]{4}}{4}$

(d) $\sqrt{\frac{4\cdot5\cdot7a^3b}{7b\cdot7b}} = \frac{2a\sqrt{35ab}}{7b}$

(e) $\sqrt{\frac{2x(x-3)}{(x-3)^2}} = \frac{\sqrt{2x(x-3)}}{x-3}$

(f) $\frac{\sqrt{4\cdot2x^5}\sqrt{x-y}}{\sqrt{x-y}\sqrt{x-y}} = \frac{2x^2\sqrt{2x}\sqrt{x-y}}{x-y}$

60. What answers to the last frame represent irrational numbers?

60. all except b

61. We can say that $\sqrt{3}\cdot\sqrt{15} = \sqrt{45}$ because we know that $(ab)^{1/n} = a^{1/n}b^{1/n}$ or $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ .

Can we use this law to multiply $\sqrt[3]{4}$ by $\sqrt{5}$ ? Give a reason for your answer.

61. No, $\sqrt[3]{4} = 4^{1/3}$ and $\sqrt{5} = 5^{1/2}$. Since we have different roots, we can't use the law $(ab)^{1/n} = a^{1/n}b^{1/n}$, as in this law all quantities are raised to the same power.

62. So in order to multiply irrational numbers, we must have the same root. The number which indicates the root is called the <u>index</u> number.

In $\sqrt[3]{4}$, the 3 is known as the _____ number.

62. index

63. $\sqrt[3]{4} = 4^{1/3}$ because of the definition of fractional exponents.

$4^{1/3} = 4^{2/6}$ because 1/3 and 2/6 are equivalent fractions.

$\sqrt{5} = 5^{1/2}$ and this equals $5^{3/6}$.

So $\sqrt[3]{4} \cdot \sqrt{5} = 4^{1/3} \cdot 5^{1/2} = 4^{2/6} \cdot 5^{3/6}$.

Note that in the last quantity, the roots are the same in both cases, namely the roots are 6.

Now using the meaning of fractional exponents, $4^{2/6} \cdot 5^{3/6} = (\underline{\ \ ?\ \ })^{1/6}$

63. $(4^2)^{1/6}(5^3)^{1/6} = (4^2 \cdot 5^3)^{1/6}$

64. $(4^2 \cdot 5^3)^{1/6} = (2000)^{1/6}$

This can't be reduced as 2000 is the product of three 5's and two 4's which is the product of three 5's and four 2's and there aren't six identical factors here.

Sometimes it is possible to simplify radicals by changing the order. For example, $(9)^{1/4}$ can be written as $(3^2)^{1/4}$ which equals $3^{1/2}$. When the order is changed, that is when the index is changed, note that the base is also changed.

(a) $\sqrt[3]{2}\sqrt[4]{2}$ = ____ = ____ = ____
(b) $\sqrt[6]{27}$ = ____ = ____ = ____
(c) $\sqrt[6]{16}$ = ____ = ____ = ____
(d) $\sqrt[3]{6}\sqrt{12}$ = ____ = ____ = ____

64. (a) $2^{1/3} \cdot 2^{1/4} = 2^{4/12} \cdot 2^{3/12} = (2^4 \cdot 2^3)^{1/12} = 2^{7/12} = \sqrt[12]{128}$

65. Radicals which are similar can be added and subtracted by using the distributive law.

Note that in this case the <u>bases</u> are alike so we could have multiplied these by using the multiplication law $a^m \cdot a^n = a^{m+n}$.

(b) $(27)^{1/6} = (3^3)^{1/6} = 3^{1/2} = \sqrt{3}$
(c) $(16)^{1/6} = (2^4)^{1/6} = 2^{2/3} = \sqrt[3]{2^2}$

(d) $6^{1/3} \cdot 12^{1/2} = 6^{2/6} \cdot 12^{3/6} =$

$(6^2 \cdot 12^3)^{1/6}$. Instead of multiplying $6^2$ by $12^3$, factor these numbers to see if you have a perfect sixth power and so can reduce the radical.
$(3 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 3)^{1/6} =$

$(3^5 \cdot 2^6 \cdot 2^2)^{1/6} = 2(3^5 \cdot 2^2)^{1/6} = 2\sqrt[6]{972}$

Similar radicals have the same index number and the same number under the radical sign.

Are $3\sqrt{6}$ and $-2\sqrt{6}$ similar radicals? Why?

65. Yes because they have the same index number of 2 and the same number under the radical sign of 6.

66. Since $3\sqrt{6}$ and $-2\sqrt{6}$ are similar radicals they can be added by using the distributive law.
$3\sqrt{6} + -2\sqrt{6} = \sqrt{6}(3+-2) = -1\sqrt{6}$ or $-\sqrt{6}$.

Combine.
(a) $5\sqrt{3} + 6\sqrt{3} - 4\sqrt{3}$
(b) $x\sqrt{a} + 3x\sqrt{a} - 5x\sqrt{a}$
(c) $3\sqrt{ab} + 5\sqrt{2a} - 4\sqrt{ab} - 5\sqrt{2a}$
(d) $4\sqrt[3]{5} - 6\sqrt{5} - 7\sqrt[3]{5}$

66. (a) $7\sqrt{3}$
(b) $\sqrt{a}(x+3x-5x) = -x\sqrt{a}$
(c) $\sqrt{2a}(5-5) + \sqrt{ab}(3-4) = -1\sqrt{ab}$

(d) $\sqrt[3]{5}(4-7) - 6\sqrt{5} = -3\sqrt[3]{5} - 6\sqrt{5}$

67. Remember you can only add and subtract similar terms. You can indicate the sum or difference of terms which are not similar.

For example, $x\sqrt{3} + y\sqrt{3} = (x+y)\sqrt{3}$.

If terms <u>arenot</u> similar, it is sometimes possible to get similar terms if the radicals are reduced and rationalized.

For example, $\sqrt{2}$, $\sqrt{\frac{1}{2}}$, and $\sqrt{18}$ don't look like similar terms, but when the radicals are reduced and rationalized they will be similar terms.

Rationalize and reduce these.

67. $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{1}{2}\sqrt{2}$
$\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$
Note that these <u>are</u> similar terms.

68. <u>Add the following.</u>
$\sqrt{75} - 4\sqrt{27} + 3\sqrt{243}$

68. $\sqrt{75} = 5\sqrt{3}$, $4\sqrt{27} = 12\sqrt{3}$,

$3\sqrt{243} = 27\sqrt{3}$

$5\sqrt{3} - 12\sqrt{3} + 27\sqrt{3} = 20\sqrt{3}$

69. Perform the following operations.

(a) $\sqrt{20} - 3\sqrt{80} + 2\sqrt{45}$

(b) $\sqrt{\frac{2}{27}} + \sqrt{150} - \sqrt{\frac{3}{2}}$

(c) $\sqrt{16a^2b} + a\sqrt{25b} - \frac{\sqrt{a^4b}}{2a}$

(d) $\sqrt{50xy} - 3x\sqrt{2/3} + \frac{1}{4}\sqrt{18xy}$

(e) $\frac{\sqrt{16a}}{a} - \sqrt{\frac{9}{a}} + \frac{9}{\sqrt{a}}$

(f) $\sqrt[3]{24x^5y^3} - \sqrt[3]{128x^5y^3} - 3\sqrt[3]{2x^2}$

69. (a) $2\sqrt{5} - 12\sqrt{5} + 6\sqrt{5} = -4\sqrt{5}$

(b) $\frac{1}{9}\sqrt{6} + 5\sqrt{6} - \frac{1}{2}\sqrt{6} = \frac{83}{18}\sqrt{6}$

(c) $4a\sqrt{b} + 5a\sqrt{b} - \frac{1}{2}a\sqrt{b} = \frac{17a}{2}\sqrt{b}$

(d) $5\sqrt{2xy} - x\sqrt{6} + \frac{3}{4}\sqrt{2xy} =$

$\frac{23}{4}\sqrt{2xy} - x\sqrt{6}$

(e) $\frac{4\sqrt{a}}{a} - \frac{3\sqrt{a}}{a} + \frac{9\sqrt{a}}{a} = \frac{10\sqrt{a}}{a}$

(f) $2xy\sqrt[3]{3x^2} - 4xy\sqrt[3]{2x^2} - 3\sqrt[3]{2x^2}$

$= 2xy\sqrt[3]{3x^2} + \sqrt[3]{2x^2}\,(-4xy-3)$

70. Rationalize the following two fractions and add the results.

$\sqrt{\frac{x-2y}{x+2y}} =$

$\sqrt{\frac{x+2y}{x-2y}} =$

The sum of these two results is ________.

70. $\sqrt{\frac{x-2y}{x+2y}} = \sqrt{\frac{(x-2y)(x+2y)}{(x+2y)^2}} =$

$\frac{\sqrt{x^2-4y^2}}{x+2y}$

$\sqrt{\frac{x+2y}{x-2y}} = \sqrt{\frac{(x+2y)(x-2y)}{(x-2y)^2}} =$

$\frac{\sqrt{x^2-4y^2}}{x-2y}$

$\frac{\sqrt{x^2-4y^2}}{x+2y} + \frac{\sqrt{x^2-4y^2}}{x-2y} = \sqrt{x^2-4y^2}\left(\frac{1}{x+2y} + \frac{1}{x-2y}\right) =$

$\frac{\sqrt{x^2-4y^2}\,(2x)}{(x+2y)(x-2y)}$

71. State which of the following are true.

(a) $(a+b)^2 = a^2+b^2$

(b) $\sqrt{x^2-y^2} = x-y$

(c) $\sqrt{x^2-y^2} = \sqrt{x-y}\sqrt{x+y}$

(d) $\sqrt{3+5} = \sqrt{3}\sqrt{5}$

(e) $\sqrt[3]{x^3-27y^3} = x-3y$

71. only c is true.

Make sure you know why the others are false.

72. In parts a, b, d and e of the last frame, write true statements keeping the left side of the equation as it is given.

72. (a) $(a+b)^2 = a^2+2ab+b^2$

(b) $\sqrt{x^2-y^2} = \sqrt{x-y}\,\sqrt{x+y}$

(d) $\sqrt{3+5} = \sqrt{8} = 2\sqrt{2}$

(e) $\sqrt[3]{x^3-27y^3} = \sqrt[3]{x-3y}\,\sqrt[3]{x^2+3xy+9y^2}$

73. (a) $\frac{x-3+2(x-4)}{\sqrt{x-4}} = \frac{3x-11}{\sqrt{x-4}} = \frac{(3x-11)\sqrt{x-4}}{x-4}$

(b) $\frac{\sqrt{(x+2y)(x+3y)}}{x+3y} - \frac{\sqrt{(x+2y)(x+3y)}}{x+2y} = \frac{-y\sqrt{(x+3y)(x+2y)}}{(x+3y)(x+2y)}$

(c) $\frac{y+3-(y+2)}{\sqrt{y+3}} = \frac{\sqrt{y+3}}{y+3}$

(d) $\frac{x-4-(x-2)}{\sqrt[3]{(x-4)^2}\sqrt{x-2}} = \frac{-2\sqrt[3]{x-4}\sqrt{x-2}}{(x-4)(x-2)}$

74. $= 3\sqrt{4}+\sqrt{20}-12\sqrt{10}-4\sqrt{50} =$

$6+2\sqrt{5}-10\sqrt{10}-20\sqrt{2}$

73. Add each of the following and simplify.

(a) $\frac{x-3}{\sqrt{x-4}} + 2\sqrt{x-4}$

(b) $\sqrt{\frac{x+2y}{x+3y}} - \sqrt{\frac{x+3y}{x+2y}}$

(c) $\sqrt{y+3} - \frac{y+2}{\sqrt{y+3}}$

(d) $\frac{\sqrt[3]{x-4}}{\sqrt{x-2}} - \frac{\sqrt{x-2}}{\sqrt[3]{(x-4)^2}}$

74. In parts a, c and d of the last frame, you might have rationalized the fractions first and then have gotten the LCD. It makes no difference whether you add and then rationalize or do it in reverse order.

Since we can multiply irrational numbers and also can add them, we can multiply irrational numbers consisting of several dissimilar parts by using the distributive law.

$(\sqrt{2}-4\sqrt{5})(3\sqrt{2}+\sqrt{10})$ becomes
$\sqrt{2}(3\sqrt{2}+\sqrt{10}) - 4\sqrt{5}(3\sqrt{2}+\sqrt{10})$
by the distributive law.

Finish the multiplication being sure to reduce the radicals in your final answer.

75. Find the following products. Be sure to reduce all radicals and combine similar terms.

(a) $(3\sqrt{2}-4\sqrt{5})(\sqrt{3}+\sqrt{15})$
(b) $4\sqrt{6}(4-3\sqrt{6}+2\sqrt{12})$

(c) $(4\sqrt{2}-6\sqrt{3})(2\sqrt{2}+4\sqrt{3})$
(d) $(5\sqrt{6}-\sqrt{2})(2\sqrt{6}-3\sqrt{2})$
(e) $(3\sqrt{8}-2\sqrt{5})(5\sqrt{8}+\sqrt{5})$

(f) $(\sqrt{2}-4\sqrt{3})(\sqrt{2}+4\sqrt{3})$
(g) $(3\sqrt{a}-2\sqrt{b})(4\sqrt{a}-\sqrt{b})$
(h) $(\sqrt{x}+\sqrt{y})(\sqrt{x}+2\sqrt{xy}+\sqrt{y})$

(i) $(\sqrt{x^2}-\sqrt{y^2})(\sqrt{x^4}+\sqrt{x^2y^2}+\sqrt{y^4})$

75. (a) $3\sqrt{6}-4\sqrt{15}+3\sqrt{30}-20\sqrt{3}$

(b) $16\sqrt{6}-12\sqrt{36}+8\sqrt{72} =$
$16\sqrt{6}-72+48\sqrt{2}$

(c) $8\sqrt{4}-12\sqrt{6}+16\sqrt{6}-24\sqrt{9} =$
$16+4\sqrt{6}-72 = 4\sqrt{6} - 56$

(d) $10\sqrt{36}-2\sqrt{12}-15\sqrt{12}+3\sqrt{4} =$
$66 - 17\sqrt{12} = 66 - 34\sqrt{3}$

(e) $15\sqrt{64}-10\sqrt{40}+3\sqrt{40}-2\sqrt{25} =$
$110 - 14\sqrt{10}$

(f) $\sqrt{4}-16\sqrt{9} = 2 - 48 = -46$

(g) $12a-11\sqrt{ab}+2b$

(h) $x+2\sqrt{xy}+2x\sqrt{y}+2y\sqrt{x}$

(i) $x^3 - y^3$

76. If we are to multiply $(1-\sqrt{3})$ by $(1+\sqrt{3})$, we are multiplying the difference of two numbers by the sum of the same two numbers. In this case, the cross products will add up to 0 and so we get the square of the first term minus the square of the last term.

That is, $(1-\sqrt{3})(1+\sqrt{3}) =$
$1-\sqrt{3}+\sqrt{3}-\sqrt{9} = 1-3 = -2.$

In this product, there is no irrational number.

In other words, we can rationalize a binomial which contains square roots by multiplying by a binomial which consists of the same two terms but with one opposite sign.

To rationalize $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{5}}$

we must multiply both numerator and denominator by the same number and this number must be such that the new denominator will not be irrational or will not contain a radical.

What would you multiply both numerator and denominator by here to rationalize the denominator?

76. $\sqrt{3} + \sqrt{5}$

77. $\dfrac{\sqrt{2}(\sqrt{3}+\sqrt{5})}{(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{5})} = \dfrac{\sqrt{6}+\sqrt{10}}{-2}$

78. Multiply both numerator and denominator by $1+\sqrt{3}$ so that the new denominator will contain no radicals.

$\dfrac{(4+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \dfrac{4+5\sqrt{3}+\sqrt{9}}{1-\sqrt{9}} =$

$\dfrac{7+5\sqrt{3}}{-2}$

79. Multiply both numerator and denominator by $2\sqrt{2} + 2$ .

77. Rationalize $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{5}}$

78. Rationalize $\dfrac{4+\sqrt{3}}{1-\sqrt{3}}$

79. Rationalize $\dfrac{3-\sqrt{2}}{2\sqrt{2}-2}$

80. Fractions should always be left in their lowest terms - that is, the numerator and denominator

$$\frac{(3-\sqrt{2})(2\sqrt{2}+2)}{(2\sqrt{2}-2)(2\sqrt{2}+2)} = \frac{6+4\sqrt{2}-2\sqrt{4}}{4\sqrt{4}-4}$$

$$= \frac{2+4\sqrt{2}}{-4}$$

should have no common factors. Look at the answer to the last problem. Her both numerator and denominator have a common factor of ____.

Reduce this fraction to its lowest terms.

80. common factor of 2 or -2.

$$\frac{2+4\sqrt{2}}{-4} = \frac{1+2\sqrt{2}}{-2}$$

81. Is $-\frac{1+2\sqrt{2}}{2}$ also a correct answer? Why?

81. Yes, because two factors have have been multiplied by -1.

82. Rationalize the following fractions.

(a) $\frac{3\sqrt{2}-2\sqrt{5}}{\sqrt{6}}$

(b) $\frac{3\sqrt{5}-2\sqrt{3}}{\sqrt{2}-4\sqrt{3}}$

(c) $\frac{\sqrt{6}-3\sqrt{2}}{2\sqrt{6}+4\sqrt{2}}$

(d) $\frac{5+3\sqrt{2}}{2-5\sqrt{2}}$

(e) $\frac{x+3\sqrt{xy}+y}{\sqrt{x}-\sqrt{y}}$

82. (a) $\frac{3\sqrt{2}-2\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{6\sqrt{3}-12\sqrt{5}}{6}$

$= \sqrt{3} - 2\sqrt{5}$

(b) $\frac{(3\sqrt{5}-2\sqrt{3})(\sqrt{2}+4\sqrt{3})}{(\sqrt{2}-4\sqrt{3})(\sqrt{2}+4\sqrt{3})} =$

$\frac{3\sqrt{10}-2\sqrt{6}+12\sqrt{15}-8\sqrt{9}}{\sqrt{4}-16\sqrt{9}} =$

$\frac{3\sqrt{10}-2\sqrt{6}+12\sqrt{15}-24}{-30}$

(c) $\frac{(\sqrt{6}-3\sqrt{2})(2\sqrt{6}-4\sqrt{2})}{(2\sqrt{6}+4\sqrt{2})(2\sqrt{6}-4\sqrt{2})} =$

$\frac{2\sqrt{36}-10\sqrt{12}+12\sqrt{4}}{4\sqrt{36}-16\sqrt{4}} = \frac{36-20\sqrt{3}}{-8} = \frac{9-5\sqrt{3}}{-2}$

(d) $\frac{(5+3\sqrt{2})(2+5\sqrt{2})}{(2-5\sqrt{2})(2+5\sqrt{2})} =$

$\frac{10+31\sqrt{2}+15\sqrt{4}}{4-25\sqrt{4}} = \frac{40+31\sqrt{2}}{-46}$

83. Earlier we stated that the $\sqrt{-4}$ didn't exist because there is no real number such that when that that number is squared the result is -4. In fact, we went on to say that the nth root of a negative number is not a real number, when n is even.

What about the odd noot of a negative number? Is this a real number?

For instance, can you find a real number which is the cube root of -8?

(e) $\frac{(x+3\sqrt{xy}+y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} =$

$\frac{x\sqrt{x}+3x\sqrt{y}+y\sqrt{x}+x\sqrt{y}+3y\sqrt{x}+y\sqrt{y}}{x - y}$

83. Yes to both questions.
$\sqrt[n]{-N}$ is a real number when n is odd.
It is not a real number when n is even.

84. If we try to reduce $\sqrt{-4}$ by the same procedures we used a few frames back in reducing radicals,

we obtain, $\sqrt{-4} = \sqrt{(-1)(4)} =$

$\sqrt{-1}\,\sqrt{4} = 2\sqrt{-1}.$

Let us define $\sqrt{-1} = i$.

Then $2\sqrt{-1} = 2i$. This is an example of an imaginary number. Imaginary numbers are the even roots of negative numbers.

Express each of the following in terms of i. Reduce all radicals.
(a) $\sqrt{-8}$  (b) $\sqrt{-1}$  (c) $3\sqrt{-16}$

(d) $\sqrt{25} + \sqrt{-36}$  (e) $\sqrt{-12}$

84. (a) $2i\sqrt{2}$
(b) i
(c) 12i
(d) 5+6i
(e) $2i\sqrt{3}$

85. Note the answer to part d of the last frame. 5+6i consists of a real number plus an imaginary number.
The real number is ______.

The imaginary number is ______.

85. real part is 5

imaginary part is 6i

86. A number such as 5+6i is called a complex number.

A complex number is a number of the form a+bi where a and b are real numbers.

To add or subtract complex numbers, procedd as in any other addition or subtraction problem by combining similar terms.

For example, the sum of 3+4i and 5+3i is found by adding the real parts and adding the imaginary parts.
(3+4i)+(5+3i) = 3+5+4i+3i = 8+i(4+3) = 8+7i.

Find the sum of
(a) $3-6i$ and $-2+5i$
(b) $2-i$ and $-3-9i$

Find the difference of
(c) $4+3i$ and $7-2i$
(d) $5i-6$ and $2+8i$

86. (a) $3-6i+(-2+5i) = 1-i$
(b) $2-i+(-3-9i) = -1-10i$
(c) $4+3i-(7-2i) = -3+5i$
(d) $5i-6-(2+8i) = -8-3i$

87. In multiplication of complex numbers, we use the distributive law as we would if we were dealing with real numbers. The commutative, associative and distributive laws hold for addition and multiplication of complex numbers.

$(3+4i)(2-7i) =$

87. $6-13i-28i^2$

88. Let us consider $i^2$ to see if this can be expressed in another form.

$i = \sqrt{-1}$ , therefore $i^2 = (\sqrt{-1})^2$

$= \left[(-1)^{1/2}\right]^2 = -1.$

So $-28i^2 = -28(-1) = 28.$

Then $6-13i-28i^2 =$ ________.

88. $6-13i+28 = 34-13i$

89. The product of two complex numbers can always be expressed as a complex number.

Express each of the following as a complex number.
(a) $(2+9i)(-5-2i)$

(b) $(3-5i)^2$

(c) $(3i\sqrt{5} - 2)(2i\sqrt{5} + 3)$

(d) $(\sqrt{2} - i\sqrt{6})(3\sqrt{2} + 2i\sqrt{6})$

89. (a) $-10-49i-18i^2 = 8-49i$

(b) $(3-5i)(3-5i) = 9-30i+25i^2 = -16-30i$

(c) $6i^2\sqrt{25}+5i\sqrt{5}-6 = -30+5i\sqrt{5}-6 = -36+5i\sqrt{5}$

(d) $3\sqrt{4}-i\sqrt{12}-2i^2\sqrt{36} =$

$18 - 2i\sqrt{3}$

90. Fractions which have complex numbers in the denominator must be rationalized in the same way as fractions which have irrational numbers in the denominator.

For example, $\frac{1+3i}{3-4i}$ can be rationalized by multiplying both numerator and denominator by a number which will make the denominator a

What can $3-4i$ be multiplied by so that the result will be a real number - that is, so that the product will not contain i?

90. multiply by the conjugate which in this case is $3+4i$.

91. Rationalize $\frac{1+3i}{3-4i}$

91. $\frac{(1+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{3+13i+12i^2}{9 - 16i^2} =$

$\frac{-9+13i}{25}$

92. $\frac{-9+13i}{25}$ can be written as $-\frac{9}{25} + \frac{13i}{25}$

or as $-\frac{9}{25} + \frac{13}{25}i$ .

You will note that this result is also a complex number or is of the form a+bi where a and b are real numbers.

The real part of this number is ___.

The imaginary part is _____.

92. real part is $-9/25$

imaginary part is $13i/25$.

93. The quotient of two complex numbers can always be expressed as a complex number.

Express $\frac{5+3i}{2-4i}$ as a complex number.

93. $\frac{(5+3i)(2+4i)}{(2-4i)(2+4i)} = \frac{10+26i+12i^2}{4 - 16i^2} =$

$\frac{-2+26i}{20} = -\frac{2}{20} + \frac{26}{20}i = -\frac{1}{10} + \frac{13}{10}i$

94. Let us consider the values of $i^3$ and $i^4$. We already know the values of i and $i^2$, so we shall express $i^3$ and $i^4$ in terms of these quantities.

$i^3 = i^2 \cdot i = (-1)\ i = -i$.

$i^4 =$ _______ $=$ _______ $=$ ______.

94. $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$.

95. What is the value of $i^5$?

95. $i^5 = i^4 \cdot i = (1)\ i = i$

96. What is the value of $i^6$?

96. $i^6 = i^4 \cdot i^2 = (1)(-1) = -1$.

97. If you examine the powers of i further, you would find that all the odd powers of i are imaginary numbers.

What kind of numbers will all the even powers of i be?

97. real numbers

98. Not only will even powers of i be real numbers, but even powers of i will either equal +1 or -1.

Express as a complex number:

$i^3+4i^5-2i^6$

98. $i^3 = -i$, $i^5 = i$, $i^6 = -1$.

$i^3+4i^5-2i^6 = -i+4(i)-2(-1) = 2+3i$

99. All real numbers are complex numbers.

Consider the real number -5. It can be expressed as $-5+0\cdot i$.

Can any real number be expressed as $a+0\cdot i$?

99. Of course it can. 0 multiplied by any number is 0 regardless of whether the number is a real number or an imaginary number.

100. Are all complex numbers real numbers? Give an example to illustrate your answer.

100. No.
Any number which contains i or the even root of a negative number is not a real number.

101. Let us review the various kinds of numbers with which we have dealt.
The largest set of numbers is the set of complex numbers.

Complex numbers are composed of two kinds of numbers - the ______ numbers and the ____ numbers.

101. real and imaginary

102. The real numbers are composed of two kinds of numbers - the ______ numbers and the ___ numbers.

102. rational and irrational

103. The set of rational numbers is composed of two parts - the _____ and the _______.

103. the integers and the rational numbers which are <u>not</u> integers. The rational numbers which are not integers are sometimes referred to as fractions or nonintegers.

104. A chart of the numbers we work with would look like this.

- complex numvers -
  - real numbers -
    - rational -
      - integers
      - nonintegers
    - irrational
  - imaginary numbers

Complete the following.
(a) Complex numbers are numbers _____.

(b) Imaginary numbers are ______
____________________.

(c) Rational numbers are ______
____________________.

(d) Irrational numbers are ______
____________________.

104. (a) of the form a+bi where a and b are real numbers or are composed of a real part and an imaginary part.
(b) even roots of negative numbers
(c) Rational numbers are real numbers which can be expressed as the quotient of two integers where the denominator isnot 0.
(d) real numbers which <u>can't</u> be expressed as the quotient of two integers.

105. State whether the numbers in the following are complex numbers, real numbers or rational numbers. Use the smallest possible classification.

(a) $-2, \sqrt{16}, \sqrt[3]{-8}$, 0, 23/4

(b) $\sqrt{24}, \sqrt[5]{-32}, \sqrt{64}$

(c) $\sqrt{-4}$, 0, $\sqrt[3]{-64}$

105. Note: in order to decide on the smallest possible classification, you may have to change the form of the given number.

(a) rational
(b) real
(c) complex

106. What symbol have we come across which doesn't represent any number - that is, what symbol isn't defined?

106. A fraction with 0 in the denominator <u>doesn't</u> represent a number.

## Chapter 8 - Quadratic Equations

We will learn how to solve quadratic equations and problems which lead to quadratic equations in this chapter.

1. A quadratic equation in one variable has only one variable and contains the second power of this variable but no higher power of the variable.

   For example, $3x^2 = 3-4x$ is a quadratic equation in x.

   Is $4x^2 - 5x = 0$ a quadratic equation in one variable? Why?

Answer 1. Yes because it involves the second power of the variable but no higher power.

2. Which of the following are quadratic equations in x?

   (a) $4 - 3x^2 = 7x$

   (b) $5 + 2x^2$

   (c) $4x(x^2 - 3) = 7$

   (d) $9x^2 = -9$

   (e) $4x^2 = 0$

Answer 2. Parts a, d and e are quadratic equations in x.
Part b is not an equation as an equation needs two equal quantities.
Part c will be $4x^3-12x = 7$ when the parentheses are removed and the highest power here is 3, so this isn't a quadratic equation.

You must remove parenthese and fractions and collect terms before deciding what degree equation you have.

3. One of the theorems in algebra states that an equation where the highest power of the variable is n and n is a positive integer will have precisely n roots.

   In a quadratic equation, the highest power of the variable is ______ and so a quadratic equation has precisely ______ roots.

Answer 3. two

two

4. Every quadratic equation is of the form $ax^2 + bx + c = 0$. If the equation isn't given in this form, it can be put into this form.

   Put the equation $4 - 3x^2 = 7x$ in the above form.

4. Either $-3x^2-7x+4 = 0$ or $3x^2+7x-4 = 0$.

5. In $ax^2+bx+c = 0$, a stands for the coefficient of the squared term, b stands for the coefficient of the first powered term and c stands for the constant term or for the term which doesn't contain the variable.

In the equation $3x^2+7x-4 = 0$, what are the values of a, b and c?

5. $a = 3$, $b = 7$ and $c = -4$.

6. In the equation $6x^2-5 = 7x$, what are the values of a, b and c?

6. First, set up the equation so that all the terms are on one side of the equation equal to 0. In $6x^2-5-7x = 0$, $a = 6$, $b = -7$, and $c = -5$. In $7x-6x^2+5 = 0$, $a = -6$, $b = 7$ and $c = 5$.

7. In the equation $4x^2 = 9$, what are the values of a, b and c?

7. In $4x^2-9 = 0$, $a = 4$, $b = 0$ and $c = -9$.

In $9-4x^2 = 0$, $a = -4$, $b = 0$, and $c = 9$.

8. We will first consider quadratic equations of the form $ax^2+c = 0$ or where $b = 0$.

In equations of the form $ax^2+c = 0$, arrange the equation so that $1x^2$ is equal to some number and then take the square root of both sides of the equation.

For example, in $4x^2 = 9$, first find out what $x^2$ equals.

$x^2 =$ ________

8. $x^2 = \frac{9}{4}$

9. If you wrote $2\frac{1}{4}$, change it back to the fractional form. $2\frac{1}{4}$ is correct, but it is much more convenient to deal with fractions in taking roots.

We have $x^2 = \frac{9}{4}$. Now take the square root of both sides of the equation. There are <u>two</u> square roots of every complex number.

In this case, $x = +3/2$ and also $x = -3/2$. When you square either of these numbers you get $9/4$.

Find the roots of $3x^2 = 75$.

9. $x^2 = 25$, $x = 5$, $x = -5$

10. The roots of this equation are often written $x = \pm 5$ meaning the roots are 5 or -5.

How would you check to wee if 5 and -5 are the correct roots to the equation $3x^2 = 75$?

10. Do two substitutions in the original equation - one when $x = 5$ and the other when $x = -5$ and see if you obtain the same number on both sides of the equation.

11. Solve and check each of the following.

(a) $4x^2 - 144 = 0$
(b) $9y^2 = 27$
(c) $9x^2 = 5$
(d) $2x^2 + 32 = 0$
(e) $4x^2 + 32 = 0$
(f) $7y^2 = 2$
(g) $8y^2 + 24 = 0$

11. (a) $x = 6$, $x = -6$
(b) $y = \sqrt{3}$, $y = -\sqrt{3}$
(c) $x = \sqrt{5}/3$, $x = -\sqrt{5}/3$
(e) $x^2 = -8$, $x = \pm\sqrt{-8}$, $x = \pm 2i\sqrt{2}$
(d) $x^2 = -16$, $x = \pm\sqrt{-16}$, $x = \pm 4i$
(f) $y = \pm\sqrt{\frac{2}{7}}$ or $y = \pm\frac{\sqrt{14}}{7}$
(g) $y = \pm\sqrt{-3}$ or $y = \pm i\sqrt{3}$

12. We can only use this procedure when b = 0. That is, we can only use the above procedure when we have a term containing the variable to the second power and a constant term.

Suppose we have a quadratic equation containing all three terms or one which contains a constant term of 0. First arrange the equation so that all the terms are equal to 0.

For example, in the equation $4x^2 = 5x$, we would write _____.

12. $4x^2 - 5x = 0$ or $5x - 4x^2 = 0$

13. Then if the quadratic expression is factorable, factor it.

Thus $4x^2 - 5x = 0$ becomes _____.

13. $x(4x-5) = 0$

14. Consider this equation. $x(4x-5) = 0$

On the left side of the equation, we have the product of two numbers. On the right side of the equation, we have 0.

What do you know about two numbers if their product is 0?

14. One of the numbers or else both of the numbers equals 0. That is, if ab = 0, then a or b or both equal zero.

15. Then in $x(4x-5) = 0$, either $x = 0$ $4x-5 = 0$. This means that either $x = 0$ or $x =$ _______.

15. $x = 5/4$

16. To solve the equation $x(x-3) = 4$, we should first ________.

16. Set everything on one side of the equation equal to 0.

17. Note: we cannot set each factor in the given equation equal to some number in this case. Which numbers would we use? There are many possibilities of two numbers whose product is 4. We can only use the principle of setting each factor equal to some number when the product of the factors equals 0.

Now we have the equation $x^2-3x-4 = 0$. Finish solving.

17. $(x-4)(x+1) = 0$
$x-4 = 0 \qquad x+1 = 0$
$x = 4 \qquad x = -1$

18. How would you check to see if these are the correct roots to this equation?

18. Substitute in the original equation.
You must do two substitutions, one using $x = 4$ and the second using $x = -1$.

19. For $x = 4$, the left side of the equation becomes $4(4-3)$ which equals $4(1)$ or 4. The right side of the equation is 4 and since the two sides of the equation are the same, $x = 4$ is a correct root of the equation $x(x-3) = 4$.

For $x = -1$, the left side of the original equation becomes $-1(-1-3)$ which equals $-1(-4)$ or 4 and since this is the same as the right side of the equation, $x = -1$ is a root of the equation $x(x-3) = 4$.

Solve and check.
(a) $8x^2 - 64x = 0$
(b) $4x^2 = 10x$
(c) $2x^2 + 5x = 3$
(d) $6(x^2-1) + 5x = 0$
(e) $5x(3x+4) = x+10$
(f) $6x^2 = 11x+7$
(g) $x^2 + 2ax - 3a^2 = 0$ (Solve for x in terms of a.)

19. (a) $x = 0$, $x = 8$
(b) $x = 0$, $x = 5/2$
(c) $x = \frac{1}{2}$, $x = -3$
(d) Be sure to remove parentheses and set equal to 0 before trying to factor.

20. There are many quadratic expressions which do not factor easily, some which factor only when we use irrational numbers and some which factor only when we use complex numbers. It would not be very

$x = -3/2$, $x = 2/3$

(e) Be sure to multiply and collect similar terms and set equal to 0 before factoring.
$x = 2/5$, $x = -5/3$

(f) $x = -\frac{1}{2}$, $x = 7/3$

(g) $(x+3a)(x-a) = 0$
$x+3a = 0$ $x-a = 0$
$x = -3a$ $x = a$

convenient to solve all quadratic equations by factoring. We can use the first form we talked about to solve quadratic equations, but this form only works when the term containing x has a coefficient of 0.

If we had $(x+a)^2 = N$, we could then take the square root of both sides of the equation and solve the resulting equations for x.

For example, $(x-3)^2 = 16$ is an equation of this sort. Taking the square roots of both sides of this equation, we get ______.

20. $x - 3 = \pm 4$

or

$x - 3 = 4$ and $x - 3 = -4$

21. $x-3 = \pm 4$ is equivalent to having the two equations $x-3 = 4$ and $x-3 = -4$.

Solving these two equations for x, we get ________.

21. $x = 7$, $x = -1$

22. Our problem then is to put any quadratic equation in the form of $(x+a)^2 = N$. We can do this by the process of completing the square.

Let us examine $(x+a)^2$. This equals ________.

22. $x^2+2ax+a^2$

23. $x^2+2ax+a^2$ is sometimes referred to as a perfect square trinomial, because it is the result of squaring a binomial.

The coefficient of x in this expression is ______.

23. 2a

24. Take half of this coefficient and square the result.

What do you get?

24. $a^2$

25. Note that this result of $a^2$ is the same as the constant term in the expression $x^2+2ax+a^2$. $a^2$ is the term which is necessary to complete the square of $x^2+2ax$. The necessary constant term can always be found by taking half of the coefficient

of the first powered term and squaring the result <u>providing the coefficient of the squared term is 1</u>.

Complete the square of $x^2-6x$.

25. The coefficient of x is -6. The coefficient of $x^2$ is 1. Therefore, take half the coefficient of x which is -3 and square it getting 9.

+9 completes the square of $x^2-6x$.

26. Find the factors of $x^2-6x+9$.

26. $(x-3)(x-3)$

27. Is $x^2-6x+9$ a perfect square trinomial? What must be true in order to have a perfect square trinomial?

27. Yes, because a perfect square trinomial must factor into a binomial multiplied by itself.

28. Complete the square of $x^2-3x$ and then factor the resulting expression.

28. 9/4 completes the square.

$x^2-3x+\frac{9}{4}$

$(x-\frac{3}{2})(x-\frac{3}{2})$

29. Then to solve the equation $x^2+2x-2 = 0$ by the method of completing the square, we will first arrange the equation so that the terms containing the variable are on one side of the equation and all other terms are on the other side of the equation.

$x^2+2x-2 = 0$ becomes ________.

29. $x^2+2x = 2$

30. In order to complete the square of $x^2+2x$ we need ______.

30. +1

31. If we add a number to one side of an equation, we must add the same number to the other side in order to have equivalent equations.

$x^2+2x = 2$ becomes $x^2+2x+1 = 2+1$

or $x^2+2x+1 = 3$.

Factoring the left side of the equation, we get $(x+1)(x+1) = 3$. This can be expressed as

$(x+1)^2 = 3$. This is the form mentioned in frame 20 and so we can take the square root of both sides

of this equation.
We get ____________.

31. $x + 1 = \pm\sqrt{3}$

or
$x + 1 = \sqrt{3}$ and $x+1 = -\sqrt{3}$

32. Solving these two equations, we will get the roots of the equation $x^2+2x-2 = 0$.

The two equations are ______ and ______. Therefore, the roots of the equation are ______ and ______.

32. $x+1 = \sqrt{3}$ and $x+1 = -\sqrt{3}$

roots are $\sqrt{3} - 1$ and $\sqrt{3} + 1$

33. Check these by substituting in the original equation of $x^2+2x-2=0$.

33. $(\sqrt{3}-1)^2+2(\sqrt{3}-1)-2 =$

$3-2\sqrt{3}+1+2\sqrt{3}-2-2 = 0$ and since this is the same as the right side of the original equation, $\sqrt{3}-1$ is a root of $x^2+2x-2 = 0$.

Substitute in the original equation again using $x = -\sqrt{3}-1$.

34. To review, in solving an equation by the method of completing the square, we first arranged the equation so that all the terms containing the variable were on one side of the equation and all other terms were on the other side.
Next, we completed the square for the terms containing the variable being sure to add the same number to both sides of the equation.
We then took the square root of both sides of the equation obtaining two linear equation.
Now solve these two linear equations for the values of the variable.
Check both of these values to see if they are roots to the given equation.

Solve by completing the square:
$x^2-3x-5 = 0$.

34. $x^2-3x = 5$
$x^2-3x+9/4 = 5+9/4$ (adding the same number to both sides of the equation)
$(x-\frac{3}{2})^2 = \frac{29}{4}$
$x - \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$ (taking the square root of both sides)

$x - \frac{3}{2} = \frac{\sqrt{29}}{2}$ or $x - \frac{3}{2} = -\frac{\sqrt{29}}{2}$
$x = \frac{\sqrt{29}}{2} + \frac{3}{2}$ , $x = -\frac{\sqrt{29}}{2} + \frac{3}{2}$

Be sure and check both of these.

35. Remember we can complete the square by taking half the coefficient of the first powered term and squaring it only when the coefficient of the squared term is 1.

Suppose we had the equation $2x^2-5x-6 = 0$ and we wished to solve it by completing the square. What could we do so that the coefficient of $x^2$ becomes 1 and we have an equivalent equation to $2x^2-5x-6 = 0$?

35. Divide both sides of the equation by 2 getting
$x^2 - \frac{5}{2}x - 3 = 0.$

36. $x^2 - \frac{5}{2}x + \frac{25}{16} = 3 + \frac{25}{16}$
$(x + \frac{5}{4})^2 = \frac{71}{16}$
$x + \frac{5}{4} = \frac{\sqrt{71}}{4}$ , $x + \frac{5}{4} = -\frac{\sqrt{71}}{4}$
$x = \frac{\sqrt{71}}{4} - \frac{5}{4}$ , $x = -\frac{\sqrt{71}}{4} - \frac{5}{4}$
Check both of these in the original equation $x^2-3x-5 = 0$.

37. Make the coefficient of $x^2$ equal to 1 by dividing both sides of the equation by a.
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

38. Arrange the equation so that the terms containing x are on one side of the equation and all other terms are on the other side of the equation.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$

39. Express the left side as the square of a binomial and combine terms on the right side of the equation.
$(x + \frac{b}{2a})^2 = \frac{-4ac+b^2}{4a^2}$

40. Taking the square root of both sides of the equation, we get $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

36. Solve $x^2 - \frac{5}{2}x - 3 = 0$ by completing the square.

37. If we solve the general quadratic equation of $ax^2+bx+c = 0$ by completing the square, we will derive a formula that can be used to solve every quadratic equation.

To solve $ax^2+bx+c = 0$ by completing the square, we would first have to ________.
Do this.

38. Before we complete the square, we would have to ________.
Do this and then complete the square.

39. What has to be done next?

Do this.

40. Complete the solution.

41. In $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, the denominator of both fractions on the right hand side of the equation is the same, so we may

write $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

This represents the two roots to a quadratic equation where a is the coefficient of $x^2$, b is the coefficient of x and c is the

constant term after all the terms are put on one side of the equation equal to 0.

One root is $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

and the other root is

$x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

To solve $x^2 = 5+3x$ using the quadratic formula, we would first ________.
Do this.

41. First arrange the terms of the equation so that all terms are on one side of the equation equal to 0. Then choose the values of a, b and c for this particular equation.

$x^2-5-3x = 0$, so $a = 1$, $b = -3$ and $c = -5$.

42. It doesn't make any difference in what order the terms are arranged in choosing the values of a, b and c as long as all the terms are on one side of the equation equal to 0. This is because a is always the coefficient of the squared term, b is always the coefficient of the term to the first power, and c is always the constant term or the term which doesn't contain the variable.

Using $a = 1$, $b = -3$ and $c = -5$, solve for the roots of the equation $x^2 = 5+3x$ using the quadratic formula.

42. Quadratic formula:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So in this case,

$$x = \frac{-(-3) \pm\sqrt{9-4(1)(-5)}}{2(1)}$$

$x = \frac{3\pm\sqrt{29}}{2}$, which can be written

$x = \frac{3+\sqrt{29}}{2}$, $x = \frac{3-\sqrt{29}}{2}$

43. In this case $\sqrt{29}$ cannot be reduced. If we obtain a radical which can be reduced, we do so in order to get a fraction which is in its lowest terms.

Solve and check using the quadratic formula.

$$3x^2-5 = 4x$$

43. $3x^2-5-4x = 0$ so $a = 3$, $b = -4$ and $c = -5$.

$$x = \frac{-(-4) \pm\sqrt{16-4(3)(-5)}}{2(3)}$$

$x = \frac{4+\sqrt{76}}{6}$, $x = \frac{4-\sqrt{76}}{6}$

44. If you haven't reduced the radical and then reduced the fraction to its lowest terms, do so now.

44. $x = \frac{4 + 2\sqrt{19}}{6}$, $x = \frac{4-2\sqrt{19}}{6}$

Both of these fractions have a common factor of 2 in both numerator and denominator. Reducing fractions, we get

$x = \frac{2 \pm \sqrt{19}}{3}$ for the two roots.

Don't forget to do the check.

45. $3y^2-4y+5 = 0$ so $a = 3$, $b = -4$ and $c = 5$.

$$y = \frac{-(-4) \pm \sqrt{16-4(3)(5)}}{2(3)}$$

$$y = \frac{4 \pm \sqrt{-44}}{6}, \quad y = \frac{4 \pm 2i\sqrt{11}}{6}$$

$$y = \frac{2 \pm i\sqrt{11}}{3}$$

46. (a) $x = \frac{-1 \pm \sqrt{-3}}{2}$

or $x = (-1 \pm i\sqrt{3})/2$

(b) You can factor this.
$3x(4x-7) = 0$, so $x = 0$, $x = 7/4$.

(c) Remove parentheses and combine terms getting $3x^2+4x+1=0$. You can factor this getting $x = -1$, $x = -1/3$.

(d) Remove parentheses and combine terms, getting $x^2-7x-6 = 0$.
Solve using the formula.

$$x = \frac{7 \pm \sqrt{73}}{2}$$

(e) You can solve this by factoring or you can set the equation up as $4x^2 = 100$ and then take the square root of both sides of the equation.
Roots are 5 and -5.

45. Using the quadratic formula, solve $3y(y-1) = y-5$.

46. Solve the following equations. Use factoring where possible. Check.

(a) $x^2+x+1 = 0$

(b) $12x^2= 21x$

(c) $3x(x+2) = 2x-1$

(d) $2x(x-6) = (x-3)(x-2)$

(e) $4x^2-100 = 0$

(f) $9y^2+6y+4 = 0$

47. Be sure you do the check particularly in parts a, c and f as this will give you further work on handling irrational and complex numbers.

Some equations will involve fractions. It is usually easier to obtain an equivalent equation which contains no fractions.

Remember - If you multiply both sides of the equation by $(x-2)$, you obtain equivalent fractions <u>only</u> if you haven't multiplied by 0. In this case, then, $\underline{x \neq 2}$.

If you have an equation with several letters in it, choose the variable you wish to solve for and then determine the coefficients of that variable.

For example, if you are to solve for x in the equation $a^4x^2-3a^2x+5=0$, you need to find the values of

(f) solve using the formula

$y = \frac{-6 \pm \sqrt{-108}}{18}$. Reduce the radical and put the fraction in lowest terms.

$y = \frac{-1 \pm i\sqrt{3}}{3}$

a, b and c in order to use the quadratic formula.

In this case, $a = a^4$, b = ____ and c = ____.

47. $b = -3a^2$ and $c = 5$.

Now substitute these in the formula, getting

$x = \frac{3a^2 \pm \sqrt{9a^4 - 4(a^4)(5)}}{2(a^4)}$

You can combine and simplify as you did in the other problems.

48. Solve for x in each of the following and check.

(a) $\frac{x-3}{x-4} - \frac{x-1}{x-2} = \frac{2}{15}$

(b) $\frac{5x-8}{2x^2-13x+15} - \frac{6x+4}{6x^2-5x-6} = \frac{4x+4}{3x^2-13x-10}$

(c) $3\sqrt{5}\, x^2 + \sqrt{5} = 10x$

(d) $x^2+2xy+4y^2 = 0$

(e) $x^2 + 16y^4 = 4xy^2$

(f) $\frac{4x-1}{x+2} - \frac{x+1}{x+4} = 3$

(g) $\frac{2-x}{x-3} - \frac{3x-2}{x+3} = \frac{-6}{x^2-9}$

48. (a) LCD is $15(x-4)(x-2)$. Multiplying both sides of the equation by this gives

$15(x-3)(x-2)-15(x-1)(x-4)=2(x-4)(x-2)$.

Roots are 7 and -1.

(b) LCD is $(2x-3)(x-5)(3x+2)$. Multiplying both sides of the LCD gives

$(5x-8)(3x+2)-(6x+4)(x-5)=(4x+4)(2x-3)$.

The resulting equation must be solved using the formula.

Roots are $x = \frac{-16 \pm \sqrt{192}}{2}$.

The fraction can be reduced, so

the roots are $x = -8 \pm 4\sqrt{3}$.

(c) Solve using the quadratic formula with $a = 3\sqrt{5}$, $b = -10$ and $c = \sqrt{5}$.

Roots are $x = \frac{10 \pm \sqrt{40}}{6\sqrt{5}}$.

49. In part a of the last frame, we multiplied both sides of the equation by $15(x-4)(x-2)$. This means that 4 and 2 <u>cannot</u> be roots of the equation because our new equation is <u>not</u> equivalent to the given equation if x = 4 or if x = 2.

Why don't we obtain an equivalent when x = 4 or x = 2?

What values of x can't be roots of the equation given in part b?

of the equation given in part f?

of the equation given in part g?

Reducing the radical and the fraction, we obtain

$$x = \frac{5 \pm \sqrt{10}}{3\sqrt{5}}$$

(d) Solve using the formula, with a = 1, b = 2y and c = $4y^2$.

Roots are $x = \frac{-2y \pm \sqrt{-12y^2}}{2}$.

Reducing the radical and the fraction gives $x = -y \pm yi\sqrt{3}$.

(e) Solve using the formula with a = 1, b = $-4y^2$ and c = $16y^4$.

Roots are $x = \frac{4y^2 \pm \sqrt{-48y^4}}{2}$.

Reducing the radical and fraction gives $x = 2y^2 \pm 3iy^2\sqrt{3}$.

(f) LCD is (x+2)(x+4).

Multiplying both sides of the equation by this gives

$(4x-1)(x+4)-(x+1)(x+2)=3(x+2)(x+4)$.

When we multiply and collect terms, we do not have any squared terms, so this is a linear equation and has only one root. Root is x = -5.

(g) LCD is (x-3)(x+3).

Multiplying both sides of the equation by this gives

$(2-x)(x+3) - (3x-2)(x-3) = -6$.

Solving this, you obtain $x = -\frac{1}{2}$, x = 3. However, note that the 3 doesn't check in the original equation because it makes one of the denominators equal 0 and we can't divide by 0. Therfore, there is only one root to this equation, namely $x = -\frac{1}{2}$.

49. In part b, 3/2, 5 and -2/3 can't be roots.
In part f, -2 and -4 can't be roots.
In part g, 3 and -3 can't be roots.

50. The roots of the equation in part e of the last frame are given as

$$x = \frac{5 \pm \sqrt{10}}{3\sqrt{5}}.$$

In order to rationalize this fraction, we would have to ____________.

Do the rationalization.

50. Multiply both numerator and denominator by $\sqrt{5}$.

$$\frac{5 \pm \sqrt{10}}{3\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5} \pm \sqrt{50}}{15}$$

Reducing the radical and the fraction gives

$$\frac{\sqrt{5} \pm \sqrt{2}}{3}$$

51. Usually fractions involving radicals are rationalized as then they can be expressed as a rational number multiplied by an irrational number.

What values of a, b and c would we use to solve

$x^2+4xy+16y^2 = 16$ for x using the quadratic formula?

51. $a = 1$, $b = 4y$, $c = 16y^2-16$.

52. Using these values, solve for x

in $x^2+4xy+16y^2 = 16$ using the quadratic formula.

52. $$x = \frac{-4y \pm \sqrt{16y^2-4(1)(16y^2-16)}}{2(1)}$$

$$x = \frac{-4y \pm \sqrt{-48y^2+64}}{2}$$

53. If you haven't reduced the radical, do so now, and then reduce the fraction, if possible.

Remember, to reduce a radical you must have what kind of quantities under the radical?

53. You must have <u>factors</u> under the radical in order to reduce it.

$$x = \frac{-4y \pm \sqrt{16(-3y^2+4)}}{2}$$

$$x = \frac{-4y \pm 4\sqrt{4-3y^2}}{2}$$

$$x = -2y \pm 2\sqrt{4-3y^2}$$

54. At this point, we have solved for x in terms of y or in other words, the values of x depend on the values of y.

Let us now consider <u>irrational equations</u>.

An example of an irrational equation is $\sqrt{x-4} + 3 = 0$.

The solution of irrational equations depends on a law of exponents which involves raising a quantity to a power.

We know that $(a^{1/n})^n = a$.

Therefore, if we had $(\sqrt{x-4})^2$, this would equal ______.

54. $\left[(x-4)^{1/2}\right]^2 = x-4$

55. <u>What</u> can we do to the equation $\sqrt{x-4} + 3 = 0$ so that if we squared both sides of the equation, we would have <u>only</u> $\sqrt{x-4}$ as the quantity to be squared?

55. Add -3 to both sides of the equation getting

$\sqrt{x-4} = -3$

56. If we now square both sides of the equation $\sqrt{x-4} = -3$, we obtain __________.

56. $x-4 = 9$

57. From $x-4 = 9$, we find that $x = 13$.

When both sides of an equation are raised to the same power, we do not always get equivalent equations.

Therefore, we must always check the value or values we obtain to see if they are roots of the original equation.

Is $x = 13$ a root of $\sqrt{x-4} + 3 = 0$?

57. No.

If $x = 13$, $\sqrt{x-4} + 3$ becomes $\sqrt{13-4} + 3$ which becomes 6. This is not equal to the number on the right side of the equation.

58. Therefore, $\sqrt{x-4} + 3 = 0$ has no solution.

To solve $3\sqrt{2x-7} - 5 = -2$ for x, we would first arrange the equation so that the term containing the radical is on one side of the equation and all other terms are on the other side. Then when we raise both sides of the equation to the same power, we will no longer have an irrational equation.

Arrange the equation in this way, square both sides and finish solving.

58. $3\sqrt{2x-7} = 3$
$9(2x-7) = 9$
$x = 4$

What do you have to do now to see if this value of x is a root to the equation?
Be sure you do this.

59. Remember when you raise both sides to the same power, you do not always get equivalent equations, so you must always check the values of the variable to see if they are roots of the equation.

$x = 4$ is a root to the equation $3\sqrt{2x-7} - 5 = -2$.

Solve for x: $3 - 2\sqrt{2x+5} = 4$.

59. Arrange the equation so that the radical is on one side of the equation and all other terms are on the other side.
$-2\sqrt{2x+5} = 1$
$4(2x+5) = 1$, $x = -19/4$.
No solution as this doesn't check.

60. If you had the equation

$\sqrt[3]{2x+3} = 2$, what could you do in order to get an equation which doesn't contain a radical?

60. Raise both sides of the equation to the third power
**or**
cube both sides of the equation.

61. Solve $\sqrt[3]{2x+3} = 2$ for x.

61. cubing both sides of the equation, we get $2x+3 = 8$. Solving, we get $x = 5/2$ and this is a root as it checks in the original equation.

62. The equation $\sqrt{x-2} - \sqrt{3x-2} = -2$ has two radicals in it. However, we can use the same principle to get an equation which doesn't contain any radicals.
First arrange the equation so that one radical is alone on one side of the equation and all other terms are on the other side.
Then since we have square roots, square both sides of the equation.

Do this being sure to square the complete side of the equation.

62. $\sqrt{x-2} = -2 + \sqrt{3x-2}$
Squaring both sides of the equation
$x-2 = (-2 + \sqrt{3x-2})^2$

$x-2 = (-2 + \sqrt{3x-2})(-2 + \sqrt{3x-2})$

$x-2 = 4 - 4\sqrt{3x-2} + 3x - 2$

63. Note that the right side of the equation before it was squared was a binomial. Therefore, when it is squared, you are multiplying two binomials together.

Now we have an irrational equation which only contains one radical and we are interested in getting an equation which doesn't contain any radicals.

How can we do this?

63. By repeating the same process we used before.
Arrange the equation so that the term containing the radical is on one side of the equation and all other terms are on the other side. Then square both sides.

64. Complete the solution.

64. $-2x-4 = -4\sqrt{3x-2}$
Squaring both sides of this equation, we get
$4x^2+16x+16 = 16(3x-2)$.
This gives us the quadratic equation $4x^2 - 32x + 48 = 0$.
Solving this, we obtain $x = 6$, $x = 2$.

Only $x = 2$ checks, so $x = 2$ is the only root of the given equation.

65. After squaring both sides of the equation $-2x-4 = -4\sqrt{3x-2}$, we obtained $4x^2+16x+16 = 16(3x-2)$. We can see that when terms are combined, we will still have a squared term. This means that we have a quadratic equation to solve.

What methods do we have of solving quadratic equations?

65. factoring, completing the square, and the quadratic formula.

66. In the solution of quadratic equations by the methods of factoring and the quadratic formula, how must the equation be arranged?

66. All the terms on one side of the equation equal to 0.

67. Thus in this case, we get $4x^2-32x+48 = 0$.

In this equation, the left side of the equation has a common factor of 4. We obtain an equivalent equation when we divide both sides of an equation by a nonzero number, so we can divide both sides of this equation by 4, getting ________.

67. $x^2-8x+12 = 0$

68. We don't have to do this, but you can see that it will be easier to factor this equation or to solve it using the quadratic formula than to solve the equation given in frame 67 by either method as the coefficients are so much smaller.

In fact, we could have divided both sides of the equation by a common factor when we had the equation $-2x-4 = -4\sqrt{3x-2}$.

Don't forget to check the values of x you obtain in solving $x^2-8x+12 = 0$ in the original equation which you were given in frame 62 to solve.

What procedure would you do first in solving
$\sqrt{14+2x} + 7 - 3\sqrt{x+6} = 0$?

68. Arrange the equation so that one radical is by itself on one side of the equation and all other terms are on the other side. Then square both sides of the equation.

69. Do this procedure.

69. $\sqrt{14+2x} = 3\sqrt{x+6} - 7$
Squaring both sides of the equation, we get
$14+2x = (3\sqrt{x+6} - 7)(3\sqrt{x+6} - 7)$
$14+2x = 9(x+6) - 42\sqrt{x+6} + 49$

70. Now what would you do?

70. Repeat the same process.

71. Do this and obtain an equation which doesn't have any radical.

71. $-7x-89 = -42\sqrt{x+6}$
Squaring both sides of this equation, we get
$49x^2+1246x+7921 = 1764(x+6)$.
This becomes
$49x^2-518x-2703 = 0$

72. What method would you use to solve this quadratic equation? Why?

72. Use the quadratic formula or completing the square because it would be too difficult to find suitable factors if there are any?

73. After you obtained values of x for the equation $49x^2-518x-2703 = 0$ by using the quadratic formula, what would you have to do?

73. Check these values in the original equation to see if they were roots to the given equation.

74. You must check both roots as you may have a case where there aren't any roots as well as the cases where there is one root or where there are two roots.

We will not solve this quadratic equation as we are primarily interested in the procedure by which irrational equations are solved. If you need the practice in using the quadratic formula, you can solve this equation.

Solve the following.
(a) $\sqrt{x-1} - x + 3 = 0$

(b) $\sqrt{3x-5} - \sqrt{x-3} = 2$

(c) $\sqrt{2x^2-5x-7} - 2\sqrt{x+1} = 0$

(d) $\sqrt{x-1} - \sqrt{7x+1} + \sqrt{2x+6} = 0$

74. (a) $\sqrt{x-1} = x - 3$
$x-1 = x^2-6x+9$
$x^2-7x+10 = 0$
The only root is $x = 5$.

(b) $\sqrt{3x-5} = 2 + \sqrt{x-3}$
$3x-5 = 4 + 2\sqrt{x-3} + x-3$
$2x-6 = 2\sqrt{x-3}$
Dividing both sides of the equation by 2, we get
$x-3 = \sqrt{x-3}$.
$x^2-6x+9 = x-3$
$x^2-7x+12 = 0$
Solution is $x = 3$.

75. Set us the equations needed to solve the following problems. Complete the solutions.
(a) The product of two numbers is 24. Twice the sum of the two numbers less three times one of the numbers equals two more than the other number. Find the two numbers.

(b) Two numbers differ by 4. The square of one of the numbers is 10 less than the other number. Find the two numbers.

(c) $\sqrt{2x^2-5x-7} = 2\sqrt{x+1}$

$2x^2-5x-7 = 4(x+1)$
$2x^2-9x-11 = 0$
Roots are -1 and 11/2 as both check in the original equation.

(d) $\sqrt{x-1} = \sqrt{7x+1} - \sqrt{2x+6}$
$x-1 = (\sqrt{7x+1} - \sqrt{2x+6})(\sqrt{7x+1} - \sqrt{2x+6})$
$x-1 = 7x+1 - 2\sqrt{(7x+1)(2x+6)} + 2x+6$
$-8x-8 = -2\sqrt{(7x+1)(2x+6)}$
$64x^2+128x+64 = 4(14x^2+44x+6)$
Roots are 5 and 1 as both check in the original equation.

75. (a) Let x = one number
let y = second number
$xy = 24$
$2(x+y) - 3x = y+2$

numbers are 6 and 4 or are -6 and -4.

(b) Let n = one number
Then n-4 = other number

$n^2 = n-4 + 10$

numbers are 3 and -1 or are -2 and -6.

(c) Let w = width
Then 2w-3 = length
$w(2w-3) = 104$

dimensions are 8 and 13.

Note that we do not use the negative value here as a negative dimension has no meaning.

(d) Let w = width
Then w+3 = length

$w(w+3) - 4 = (w-1)(w+4)$

dimensions are 9 and 12.

Again we do not use the negative value as a negative dimension has no meaning.

(c) The area of a rectangle is 104 sq. in. The length is 3" less than twice the width. Find the dimensions.

(d) The length of a certain rectangle is 3 more than the width. If the length is increased by 1 and the width decreased by 1, the area is decreased by 4 sq. units. Find the dimensions.

Test A

1. $34.8 \div .0004 =$ ______

2. $15\frac{5}{6} \div 23\frac{3}{4} =$ ______

3. Division is the inverse of ______.

4. Given the numerals $-6, \frac{1}{4}, 4, \frac{25}{4}, \sqrt{2}, \sqrt{5(4+1)}, .34, \pi$.

   (a) the integers are ________.
   (b) the fractions are _______.
   (c) the rational numbers are ________.
   (d) the irrational numbers are ______.
   (e) the real numbers are _________.

5. If $a \neq 0$ and $b \neq 0$, then $\frac{0}{-3a^2b} =$ ______.

6. $\frac{4xy^3}{0} =$ ___

7. The <u>sum</u> of the absolute values of -5, 2, -3 and -6 is ______.

8. The <u>absolute value</u> of the sum of -2, -3, 15 and -12 is ______.

9. Add: $4x-3y+8-10c$ and $-x-5y+4c$.

10. Subtract $-3p+5m-2q+4r$ from $2p-4m-6q+4r$.

11. $(2a - 5b)^2 =$ ____________

12. $(6x+5y)(3x-4y) =$ _________

13. $\frac{-72a^7b^4}{8a^4b^6} =$ ______

14. $\frac{-36x^2y^3 + 9xy^2}{9xy^2} =$ ________

15. Divide $36x-2x^2+8x^4-10$ by $2x^2-3x+5$. Show work.

16. $[2p - 5(3r-q)][2p + 5(3r-q)] =$ ________

17. In $4x^3+5c$
    (a) 3 is called an _______
    (b) 4 is called an _______
    (c) $4x^3$ is called a _______

18. $|3 - 4| =$ _________

19. $|3| - |4| =$ _______

20. If a, b and c represent real numbers, state the law or property of real numbers illustrated in each of the following.

(a) $b(a+c) = (a+c)b$ ____________

(b) $c + (b+a) = (c+b) + a$ __________

(c) $a(b+c) = ab + ac$ ____________

(d) $b \cdot 0 = 0$ __________

(e) $c \cdot 1 = c$ __________

(f) $0 + a = a$ _________

Test B

1. Factor the following.
   (a) $32ab - 8b^2$
   (b) $ab + 6c + 3a + 2bc$
   (c) $16x^4 - 81y^4$
   (d) $25a^2 - 49b^2$
   (e) $(4x - 3y)^2 - 25$
   (f) $16x^2 - 40x + 25$
   (g) $4x^2 + 29xy - 24y^2$
   (h) $54y^4 - 250y$
   (i) $x^2 - 13x - 12$

2. Perform the indicated operations and simplify. Show work.
   (a) $\frac{4y}{3x} + \frac{3x}{4y} - \frac{8y^2 + 6x2}{8xy}$
   (b) $\frac{5}{a - 2} - \frac{3 + a}{2 - a}$
   (c) $\frac{2a-6}{a^2-9} - \frac{2}{a + 3}$
   (d) $\frac{(2a - 5)^2 - 2(2a + 1)}{(2a - 5) - 4}$
   (e) $\frac{(a - 2)(a + 2)}{ab - 3a + 6b - 18} \div \frac{(a^2 - 4)(b + 3)}{ab(a^2 - 9)}$
   (f) $\frac{4x - 16}{2 - \frac{12}{x + 2}}$

3. Solve the following equations for x. Show work.
   (a) $\frac{1}{4} = \frac{4}{3x} - \frac{5}{6x} + \frac{1}{4x}$
   (b) $\frac{4x - 1}{4x + 1} = \frac{2x + 1}{2x - 1}$

4. (a) If you can buy b books for c cents, represent the cost of one book. Represent the cost of r books.

   (b) George's age now is 2a + 1 years. Mary's age now is twice as much as George's age was 3 years ago. Represent Mary's age now.

   (c) A car travels d miles in h hours. Represent how far it travels

in x hours.

5. (a) Write as a single fraction.

$$\frac{\frac{\frac{1}{6}}{\frac{1}{3}}}{\frac{1}{4}}$$

(b) Is this any different from

$$\frac{\frac{1}{6}}{\frac{\frac{1}{3}}{\frac{1}{4}}}\ ?$$

## Test C

Show work on <u>all</u> problems.

1. Solve by substitution: $\begin{cases} 2x - y = 5 \\ x + 3y = 6 \end{cases}$

2. Solve by addition or subtraction: $\begin{cases} \frac{x + 2y}{2} - \frac{2x + y}{6} = \frac{1}{6} \\ 2x + 3y = 2 \end{cases}$

3. Evaluate: (a) $(-8)^{2/3}$

   (b) $(-8)^{-2/3}$

   (c) $4^{-1/2} + 4^0 + 4^2$

4. Perform the indicated operations and simplify. Express all results with positive exponents.

   (a) $(-2x^3y)(xy)^2$

   (b) $\frac{x^8}{y^3} \div x^2y$

   (c) $(-3x^{1/2}y^{-3})(2x^{1/3}y^{1/2})$

   (d) $(81x^{-3}y^{1/2})^{-1/2}$

   (e) $\frac{x^{-2} + 2x^{-1}}{(1 + 2x)^{-1}}$

5. Perform the indicated operations and simplify.

   (a) $2\sqrt{3}(3\sqrt{3} - 5 + 5\sqrt{18})$

   (b) $(2\sqrt{3} - 4\sqrt{6})(3\sqrt{3} - 2\sqrt{6})$

   (c) $\sqrt{150} - \sqrt{\frac{3}{2}} + \sqrt{54} - \sqrt{27}$

   (d) $\frac{\sqrt{3} + \sqrt{6}}{2\sqrt{3} - \sqrt{6}}$

   (e) $\sqrt{\frac{x}{x - y}}$

6. <u>Set up only</u>.

   In a certain rectangle the length is 2 feet less than 5 times the width. The number of square feet in the area is 7 more than the number of feet in the perimeter. Find the dimensions.

## Test D

1. Solve: $x^2 = x + 6.$

2. Solve: $2x^2 - 6x + 3 = 0.$

3. Solve: $\sqrt{3x+7} - x = 1$

4. Solve: $2x + \frac{21}{x + 4} = 9$

5. Factor: (a) $x^3 + 6x^2 - 7x$

   (b) $16x^4 - y^4$

   (c) $ax + bx - 2a - 2b$

6. $(2a - 3b)^2 =$ _____

7. Express in the form a+bi where a and b are real numbers. $\frac{2 - 3i}{5 + i}$

8. Simplify: $(\frac{x^2}{y^2} - 4)(\frac{y^3}{x + 2y}) \div (\frac{x}{y} - 2)$

9. Express as a single fraction: $3 - \frac{x + 2}{3x}$

10. Express with positive exponents. Simplify.

    (a) $(x^{-2} + y^{-2})^{-1}$

    (b) $(9^{-1/2}a^4b^6)^{-1}$

11. (a) $\frac{0}{152} =$

    (b) $\frac{1}{0} =$

12. Simplify: $(3\sqrt{3} + 4\sqrt{8})(2\sqrt{3} - 3\sqrt{8}).$

13. Each of r couples has 2 children and each of s other couples has 3 children. How many children do the r+s couples have?

14. The difference between the square of a positive integer and 6 times the integer is 16. Find the integer.