

ON PROPERTIES OF THE APPROXIMATE
PEANO DERIVATIVE

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
BRUCE SCOTT BABCOCK
1973

This is to certify that the
thesis entitled

ON PROPERTIES OF THE APPROXIMATE
PEANO DERIVATIVE

presented by

Bruce Scott Babcock

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Clifford E. Weil

Major professor

Date May 23, 1973

MICHIGAN STATE UNIVERSITY LIBRARIES



3 1293 00651 8405

L

522 (A 199)



ABSTRACT
ON PROPERTIES OF THE APPROXIMATE
PEANO DERIVATIVE

By

Bruce Scott Babcock

The ordinary derivative has been studied extensively and many properties of it have been discovered. Although an ordinary derivative need not be continuous, it does possess certain properties worth investigating. The following four properties, defined here for an arbitrary function g , have been shown to hold for an ordinary derivative:

1. g is in the first class of Baire.
2. g has the Darboux property.
3. g has the Denjoy property.
4. g has the Zahorski property.

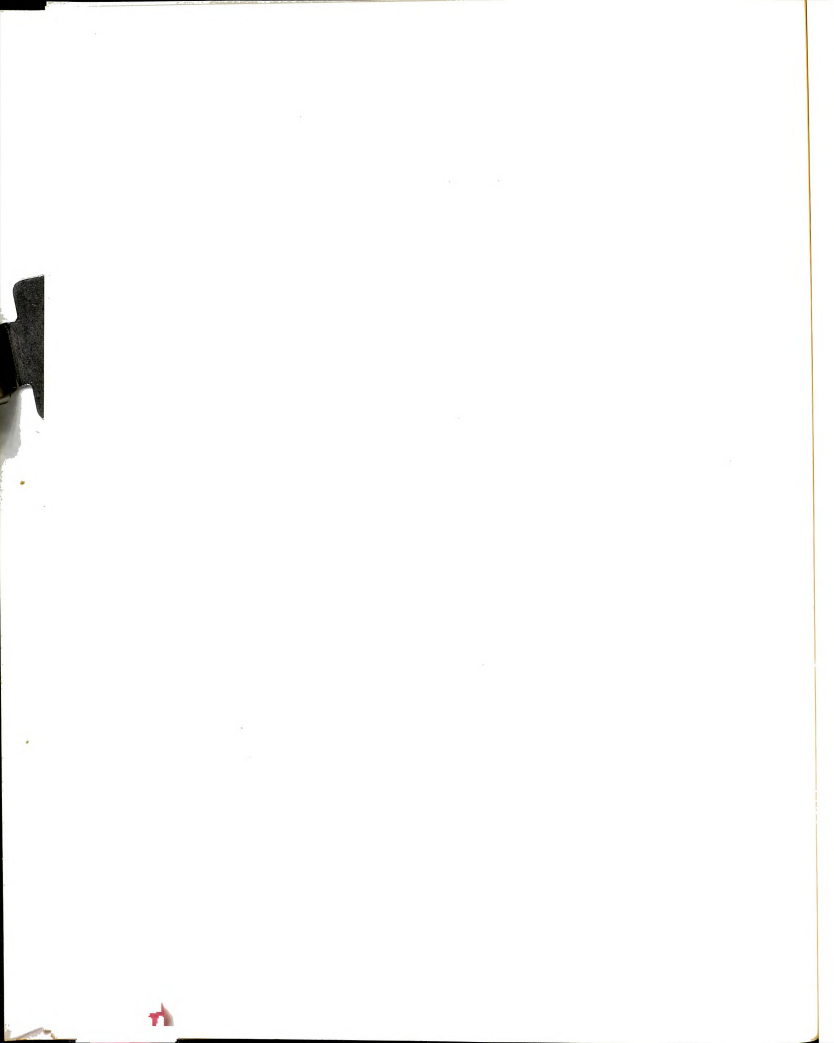
C. E. Weil has recently introduced a new property which he calls property Z. He has shown that property Z is stronger than the Zahorski property in the class of functions having the Darboux property. In addition he has shown that an ordinary derivative has property Z.

H. W. Oliver showed more generally that if a function f has a k^{th} Peano derivative f_k then f_k has properties 1, 2, and 3 listed above. C. E. Weil showed, furthermore, that f_k also has property 4 and property Z.

A. P. Calderón and A. Zygmund have generalized the k^{th} Peano derivative by means of the k^{th} L_p derivative, where $0 < p \leq \infty$. M. J. Evans has recently shown that if a function f has a k^{th} L_p derivative $f_{k,p}$, where $1 \leq p \leq \infty$, then $f_{k,p}$ has the four properties listed above.

The notion of k^{th} L_p differentiation, where $0 < p \leq \infty$, is contained in one that is more general. It is called k^{th} approximate Peano differentiation. M. J. Evans was the first to investigate this type of differentiation and has further shown that if a function f has a k^{th} approximate Peano derivative $f_{(k)}$ then $f_{(k)}$ has property 1 given above.

In this paper we first examine the concepts discussed above. We then proceed to prove our main result that when a function f has a k^{th} approximate Peano derivative $f_{(k)}$ and if $f_{(k)}$ is bounded above or below on an interval then $f_{(k)} = f^{(k)}$, the ordinary k^{th} derivative. From our main result the properties 2 and 3 given above are then easily shown to hold for a k^{th} approximate Peano derivative from known theorems. As our final result we prove that a k^{th} approximate Peano derivative has property 4 by verifying that a k^{th} approximate Peano derivative satisfies the stronger property Z.



ON PROPERTIES OF THE APPROXIMATE
PEANO DERIVATIVE

By

Bruce Scott Babcock

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

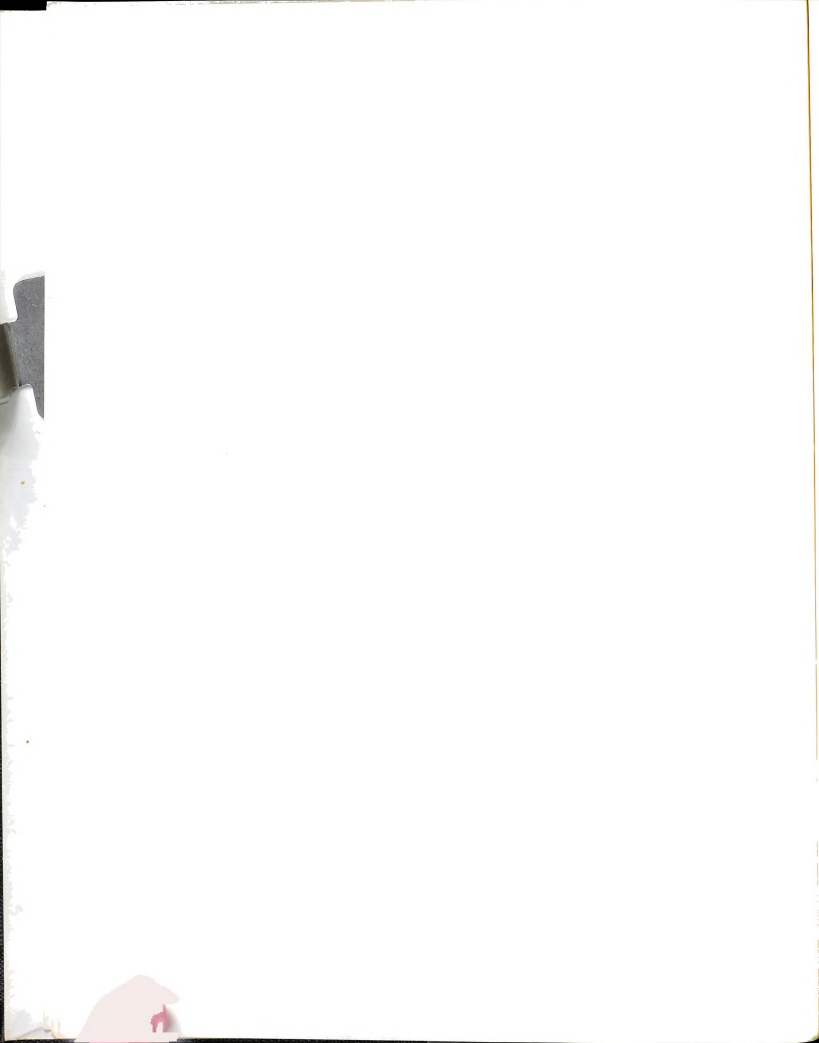
Department of Mathematics

1973



G85012

To JoAnn
and
Mom and Dad



ACKNOWLEDGEMENTS

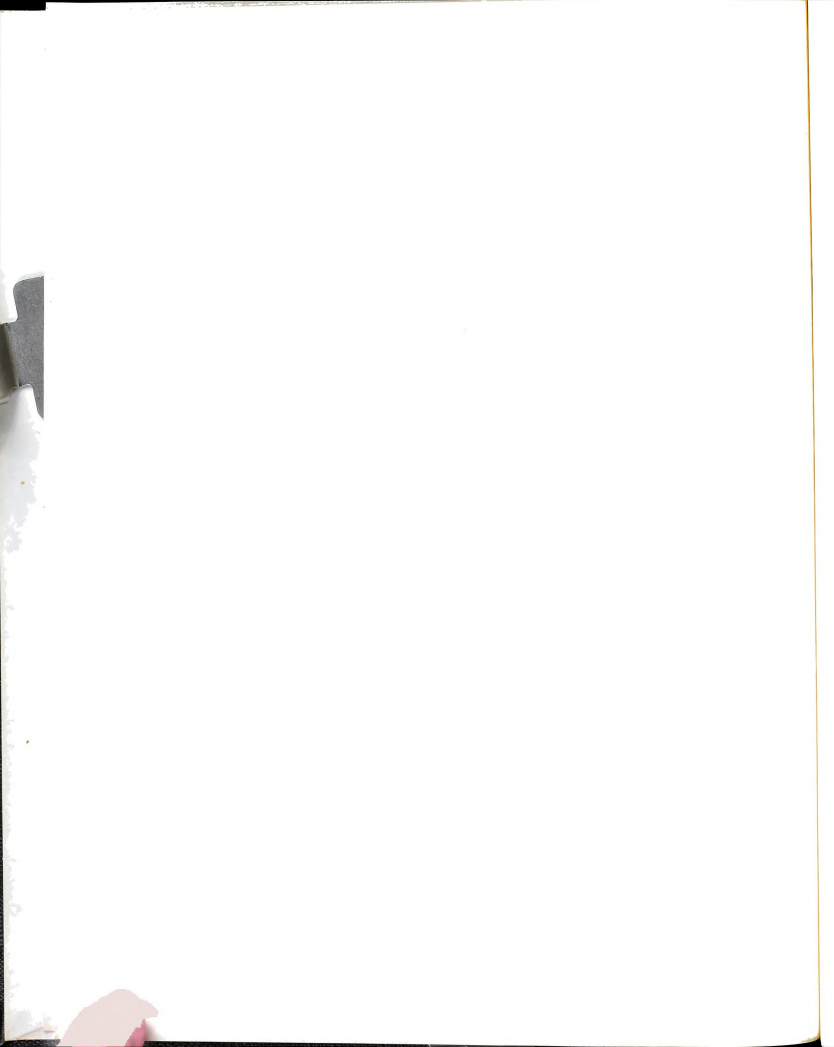
I would like to take this opportunity to express my deepest appreciation to my major Professor C. E. Weil for his helpful guidance, for the interest he has displayed, and especially his patience in the preparation of this thesis.

I am also deeply appreciative to my wife, JoAnn, for her encouragement, moral support, and also her faith in me.



TABLE OF CONTENTS

Chapter I.	Introduction and Definitions.	1
Chapter II.	Examples.	12
Chapter III.	A Preliminary Result.	18
Chapter IV.	The Major Theorem	29
Chapter V.	Property Z for Approximate Peano Derivatives	47
BIBLIOGRAPHY.		59



CHAPTER I

INTRODUCTION AND DEFINITIONS

All of the functions in this paper are assumed to be real-valued, measurable functions defined on a nondegenerate closed interval $I = [a, b]$, unless specified otherwise. Also whenever we write $x+h$ ($x \in I$) our assumption is that $x+h \in I$. Furthermore, the notation $E\text{-}\lim_{y \rightarrow x}$ will denote that the limit is computed only for those values of y in E . Finally, if E is a measurable set then we denote the measure of E by either $m(E)$ or $|E|$.

Since the turn of the century the ordinary derivative has been studied extensively and many properties of it have been discovered. Although an ordinary derivative need not be continuous, it does possess certain properties worth investigating. The simplest of these properties is that every ordinary derivative is a function of Baire class one.

Definition 1.1. A function f , defined on I , is said to be a function of Baire class one on I if there is a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions, each defined on I , such that

$$\lim_{n \rightarrow \infty} f_n(x) = f(x),$$

for each x in I .



Perhaps the best known property for an ordinary derivative is the Darboux property or intermediate value property.

Definition 1.2. A function f , defined on I , is said to have the Darboux property if whenever x_1 and x_2 are distinct points of I and y is a number between $f(x_1)$ and $f(x_2)$, there is an x_3 between x_1 and x_2 such that $f(x_3) = y$.

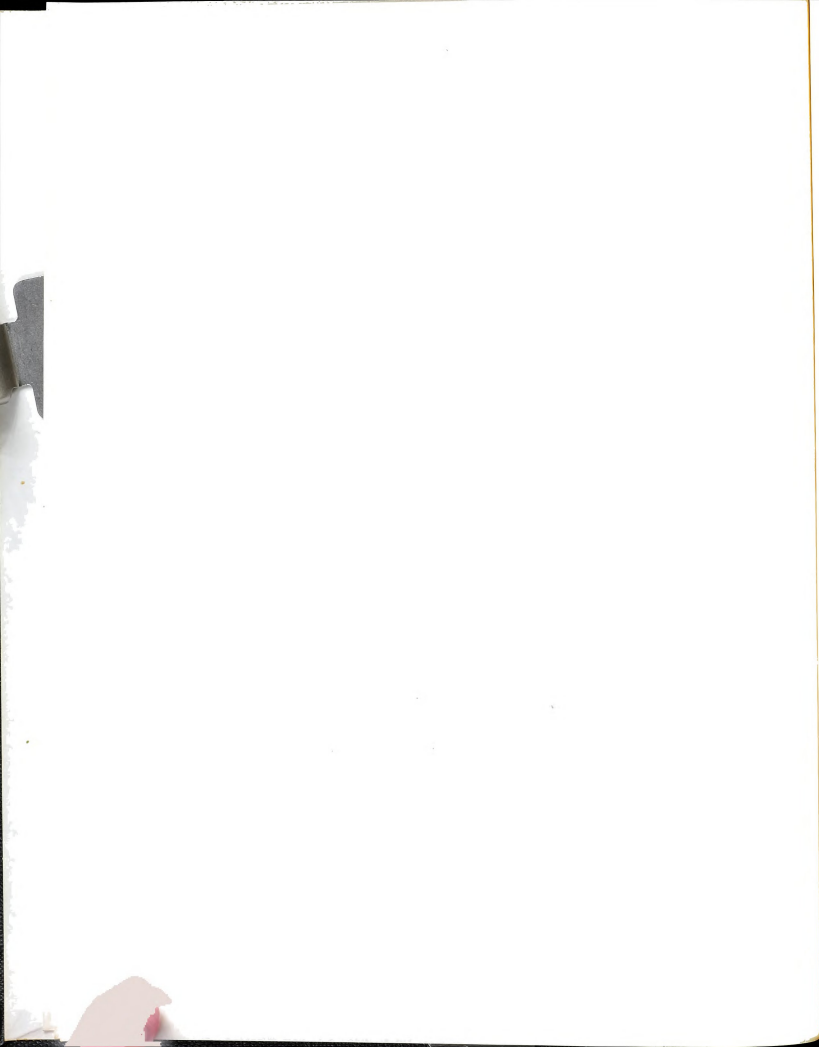
In 1916, A. Denjoy [2] proved that an ordinary derivative has what we shall call the Denjoy property.

Definition 1.3. A function f , defined on I , is said to have the Denjoy property on I if for every open interval (α, β) , $f^{-1}((\alpha, \beta))$ either is empty or has positive measure.

The Denjoy property was further strengthened by Z. Zahorski in 1950 [12]. We call this property the Zahorski property and define it in the following manner.

Definition 1.4. A sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$ is said to converge to a point x if x is not in the union of the I_n 's and if every neighborhood of x contains all but a finite number of the intervals I_n .

Definition 1.5. A function f , defined on I , is said to have the Zahorski property on I if for every open



interval (α, β) , x in $f^{-1}((\alpha, \beta))$ and $\{I_n\}_{n=1}^{\infty}$, a sequence of closed subintervals of I converging to x with

$$m(f^{-1}((\alpha, \beta)) \cap I_n) = 0$$

for every n , implies

$$\lim_{n \rightarrow \infty} \frac{m(I_n)}{\text{dist.}(x, I_n)} = 0,$$

where $\text{dist.}(x, I_n) = \inf\{|x-y| : y \in I_n\}$.

Zahorski showed that an ordinary derivative also possesses the above property.

Recently C. E. Weil [11] has introduced a new property, property Z, which is stronger than the Zahorski property.

Definition 1.6. A function f , defined on I , has property Z on I if for each $x \in I$, each $\epsilon > 0$ and each sequence $\{I_n\}_{n=1}^{\infty}$ of closed subintervals of I converging to x such that for each n , $f(y) \geq f(x)$ on I_n or $f(y) \leq f(x)$ on I_n ,

$$\lim_{n \rightarrow \infty} \frac{m\{y \in I_n : |f(y) - f(x)| \geq \epsilon\}}{m(I_n) + \text{dist.}(x, I_n)} = 0.$$

In addition Weil has shown that an ordinary derivative has the stronger property Z.

These five properties, however, do not classify derivatives. Indeed, they are possessed by more general types of derivatives.



Suppose a function f , defined on I , possesses an ordinary derivative $f'(x)$ at a point $x \in I$. Then

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x),$$

or equivalently,

$$f(x+h) - f(x) - hf'(x) = o(h)$$

as $h \rightarrow 0$.

This last equation motivates a more general first order derivative in terms of the L_∞ -norm. In order to understand this definition, recall that if g is a function defined on an interval J then $\|g\|_{\infty, J}$ is defined by

$$\|g\|_{\infty, J} = \operatorname{ess\,sup}_{t \in J} |g(t)|$$

where $\operatorname{ess\,sup}_{t \in J} |g(t)| = \inf\{M : m\{t \in J : |g(t)| > M\} = 0\}$.

Definition 1.7. A function f , defined on I , is said to have a first L_∞ derivative at a point $x \in I$ if there exists a number $f_{1, \infty}(x)$ such that if

$$g(t) = f(x+t) - f(x) - tf_{1, \infty}(x)$$

then

$$\|g\|_{\infty, \langle 0, h \rangle} = o(h),$$

where $\langle 0, h \rangle = [0, h]$ if $h > 0$ and $\langle 0, h \rangle = [h, 0]$ if $h < 0$.

The number $f_{1, \infty}(x)$ is called the first L_∞ derivative of f at x .



Replacing the L_∞ -norm by the L_p -norm where $0 < p < \infty$, suitably normalized so that the function identically 1 has L_p -norm 1, gives the following definition.

Definition 1.8. A function f , defined on I , is said to have a first L_p derivative at a point $x \in I$, $0 < p < \infty$, if there exists a number $f_{1,p}(x)$ such that

$$\left(\frac{1}{h} \int_0^h |f(x+t) - f(x) - t f_{1,p}(x)|^p dt \right)^{1/p} = o(h)$$

as $h \rightarrow 0$. The number $f_{1,p}(x)$ is called the first L_p derivative of f at x .

For its application to Fourier analysis it suffices to consider $p \geq 1$, but for establishing properties of the derivatives that arise, $0 < p < 1$ may also be considered.

All of these methods for the first order differentiation are contained in one that is more general. It is called approximate differentiation and is defined in the following manner.

Definition 1.9. Let E be a measurable set and let x be a real number. Define

$$d(x, E) = \lim_{h \rightarrow 0} \frac{m(E \cap [x-h, x+h])}{2h},$$

$$d_+(x, E) = \lim_{h \rightarrow 0^+} \frac{m(E \cap [x, x+h])}{h},$$

and



$$d_-(x, E) = \lim_{h \rightarrow 0^+} \frac{m(E \cap [x-h, x])}{h}.$$

If $d(x, E) = 1$ then x is called a point of density of E ; if $d(x, E) = 0$ then x is called a point of dispersion of E . If $d_+(x, E) = 1$ then x is called a right-hand point of density of E ; if $d_+(x, E) = 0$ then x is called a right-hand point of dispersion of E . Similarly, if $d_-(x, E) = 1$ (0) then x is called a left-hand point of density (dispersion) of E .

Definition 1.10. A function f , defined on $I = [a, b]$, is said to have an approximate derivative at a point $x \in I$ if there exists a number $f'_{ap}(x)$ and a measurable set E having 0 as a point of density such that

$$E\text{-}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'_{ap}(x).$$

We call $f'_{ap}(x)$ the approximate derivative of f at x .

Remark. In Definition 1.10 if $x = a$ ($x = b$) then the expression, there exists a measurable set E having 0 as a point of density, is understood to mean that $E \subseteq [0, \infty)$ ($E \subseteq (-\infty, x]$) and 0 is a right-hand (left-hand) point of density of E .

This same convention will be adopted in Definition 1.13

Since approximate differentiation is more general than ordinary differentiation and first L_p differentiation, $0 < p \leq \infty$, every property possessed by the approximate derivatives is also possessed by the ordinary derivatives and the first L_p derivatives, $0 < p \leq \infty$.



In 1960, C. Goffman and C. J. Neugebauer [4] gave concise proofs of the facts that every approximate derivative is a function of Baire class one and possesses the Darboux property. These facts were first established by G. Tolstoff [8]. The Denjoy property for an approximate derivative was established by S. Marcus [5] and C. E. Weil [10]. Weil, in addition, verified the Zahorski property in [10] and property Z in [11] for approximate derivatives. Another important property of an approximate derivative, a proof of which can be found in the paper of Goffman and Neugebauer, is that if an approximate derivative is bounded above or below on an interval, then it is an ordinary derivative on that interval.

Even more can be said for k^{th} order differentiation. If a function f has a k^{th} ordinary derivative at a point x , then by Taylor's theorem

$$f(x+h) - f(x) - hf'(x) - \dots - \frac{h^k}{k!} f^{(k)}(x) = o(h^k)$$

as $h \rightarrow 0$. In many instances it was only the existence of such a polynomial that was ever needed, even though the function was assumed to have k derivatives at x . This led to the introduction of the following so called k^{th} Peano derivative.

Definition 1.11. A function f , defined on I , is said to have a k^{th} Peano derivative at a point $x \in I$,



$k = 1, 2, \dots$, if there exist numbers $f_1(x), f_2(x), \dots, f_k(x)$, such that

$$f(x+h) - f(x) - hf_1(x) - \dots - \frac{h^k}{k!} f_k(x) = o(h^k)$$

as $h \rightarrow 0$. The number $f_k(x)$ is called the k^{th} Peano derivative of f at x .

That every k^{th} Peano derivative has the Darboux property, is a function of Baire class one, and possesses the Denjoy property was first proved by H. W. Oliver [7]. Weil in [10] also gave an independent proof of the Denjoy property together with a proof of the Zahorski property; and in [11], he gave a proof of property Z for the k^{th} Peano derivative. Oliver also showed in his paper that if a k^{th} Peano derivative is bounded above or below on an interval, then it is an ordinary k^{th} derivative. Recently, S. Verblunsky [9] showed how to prove this last property from the definition without using the Darboux property or other properties as Oliver had done.

Proceeding in the same fashion, as in the case of the first ordinary derivative, the k^{th} Peano derivative may be generalized by means of the L_p -norm, $0 < p \leq \infty$. We thus have the following definition.

Definition 1.12. Let f be a function defined on I . Let $x \in I$ and let k be a positive integer. If there exist numbers $f_{1,\infty}(x), f_{2,\infty}(x), \dots, f_{k,\infty}(x)$ such that



$$\operatorname{ess\,sup}_{t \in \langle 0, h \rangle} |f(x+t) - f(x) - tf_{1,\infty}(x) - \dots - \frac{t^k}{k!} f_{k,\infty}(x)| = o(h^k)$$

then f is said to have a k^{th} L_∞ derivative at x . The number $f_{k,\infty}(x)$ is called the k^{th} L_∞ derivative of f at x .

If there exist numbers $f_{1,p}(x)$, $f_{2,p}(x)$, ..., $f_{k,p}(x)$, where $0 < p < \infty$, such that

$$\left(\frac{1}{h} \int_0^h |f(x+t) - f(x) - tf_{1,p}(x) - \dots - \frac{t^k}{k!} f_{k,p}(x)|^p dt \right)^{1/p} = o(h^k)$$

then f is said to have a k^{th} L_p derivative at x . The number $f_{k,p}(x)$ is called the k^{th} L_p derivative of f at x .

This concept was first introduced by A. P. Calderón and A. Zygmund [1] but their interest was only in the case $p \geq 1$.

In a manner analogous to the way the approximate derivative was introduced, the k^{th} approximate Peano derivative may be defined. This type of differentiation is more general than k^{th} Peano differentiation and k^{th} L_p differentiation where $0 < p \leq \infty$.

Definition 1.13. A function f , defined on I , is said to have a k^{th} approximate Peano derivative at a point $x \in I$, $k = 1, 2, \dots$, if there exist numbers $f_{(0)}(x) = f(x)$, $f_{(1)}(x)$, ..., $f_{(k)}(x)$, and a measurable set E having 0 as a point of density such that

$$f(x+h) - f(x) - hf_{(1)}(x) - \dots - \frac{h^k}{k!} f_{(k)}(x) = o(h^k)$$



as $h \rightarrow 0$ and $h \in E$. The number $f_{(k)}(x)$ is called the k^{th} approximate Peano derivative of f at x .

This latter generalized derivative was first studied by M. J. Evans [3] where he showed that every such derivative is a function of Baire class one. He was also able to establish the other properties (Darboux, Denjoy, and Zahorski) but only for the k^{th} L_p derivatives with $p \geq 1$. Property Z for the k^{th} L_p derivatives can be established in the same way as Evans established the Zahorski property for these derivatives.

In Chapter II we prove that k^{th} approximate Peano differentiation is a true generalization by giving an example of a function having a k^{th} approximate Peano derivative at 0 but no k^{th} L_p derivative for $0 < p \leq \infty$, at 0. We also show, given two real numbers p, q , with $0 < p < q \leq \infty$, how to construct a function having a k^{th} L_p derivative at 0 but no k^{th} L_q derivative at 0.

In Chapter III we prove that if 0 is a point of density of a measurable set E then there exists a sequence of positive real numbers $\{\lambda_n\}_{n=1}^{\infty}$, strictly increasing to 1 (strictly decreasing to 1), so that 0 is a point of density of the set $\cap_{n=1}^{\infty} \lambda_n E$, where $\lambda_n E = \{\lambda_n x : x \in E\}$, $n = 1, 2, \dots$. This result will play a key role in Chapter IV where we prove our major theorem that when a k^{th} approximate Peano derivative is bounded above or below on an interval, it is an



ordinary k^{th} derivative. The Darboux and Denjoy properties follow easily then from known theorems.

In Chapter V, our final chapter, we give a proof of property Z for the k^{th} approximate Peano derivatives. The weaker Zahorski property then follows for the k^{th} approximate Peano derivatives.



CHAPTER II

EXAMPLES

Let f be a function, defined on I , possessing a k^{th} approximate Peano derivative at a point $x \in I$. Then there exist numbers $f_{(1)}(x), f_{(2)}(x), \dots, f_{(k)}(x)$, and a measurable set E having 0 as a point of density such that

$$(2.1) \quad f(x+h) - f(x) - hf_{(1)}(x) - \dots - \frac{h^k}{k!} f_{(k)}(x) = o(h^k)$$

as $h \rightarrow 0$ and $h \in E$.

The numbers $f_{(1)}(x), f_{(2)}(x), \dots, f_{(k)}(x)$, can easily be shown to be unique and for each $n, n = 1, 2, \dots, k$, (2.1) can be rewritten as

$$f(x+h) - f(x) - hf_{(1)}(x) - \dots - \frac{h^n}{n!} f_{(n)}(x) = o(h^n).$$

Thus f has an n^{th} approximate Peano derivative $f_{(n)}(x)$ at x for $n = 1, 2, \dots, (k-1)$, and $f_{(1)}(x) = f'_{\text{ap}}(x)$, the first approximate derivative. Moreover, if f has a k^{th} Peano derivative at x then $f_1(x) = f'(x)$, the ordinary first derivative. Notice that if f has an ordinary k^{th} derivative $f^{(k)}(x)$, at x , then Taylor's theorem shows that $f_{(k)}(x)$ exists and equals $f^{(k)}(x)$. However, as we shall show, $f_{(k)}(x)$ may exist at a point x without $f_{k,p}(x)$ existing for any $p, 0 < p \leq \infty$, at x . In [3], M. J. Evans has shown that



if a function f , defined on I , possesses a k^{th} L_p derivative $f_{k,p}(x)$ at a point $x \in I$, where $0 < p \leq \infty$, then f has a k^{th} approximate Peano derivative $f_{(k)}(x)$ at x and furthermore, $f_{k,p}(x) = f_{(k)}(x)$.

Let k be a positive integer and let p be a positive real number. We now show how to construct a function which has a k^{th} L_p derivative at 0 but not a k^{th} L_q derivative at 0, where $p < q \leq \infty$.

Example 2.2. Let k be a positive integer and let p be a positive real number. Suppose q is a real number such that $p < q < \infty$. Set $c = 1 + k + \frac{1}{p}$ and $M = \frac{cp}{q-p}$. Let

$$I_n = \left[\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{2^{Mqn}} \right],$$

$n = 1, 2, \dots$, and $E = I - \bigcup_{n=1}^{\infty} I_n$, where $I = [0, 1]$. Let us first show $d_+(0, E) = 1$. Let $0 < h < 1$, and choose the positive integer N so that

$$\frac{1}{2^N} \leq h < \frac{1}{2^{N-1}}.$$

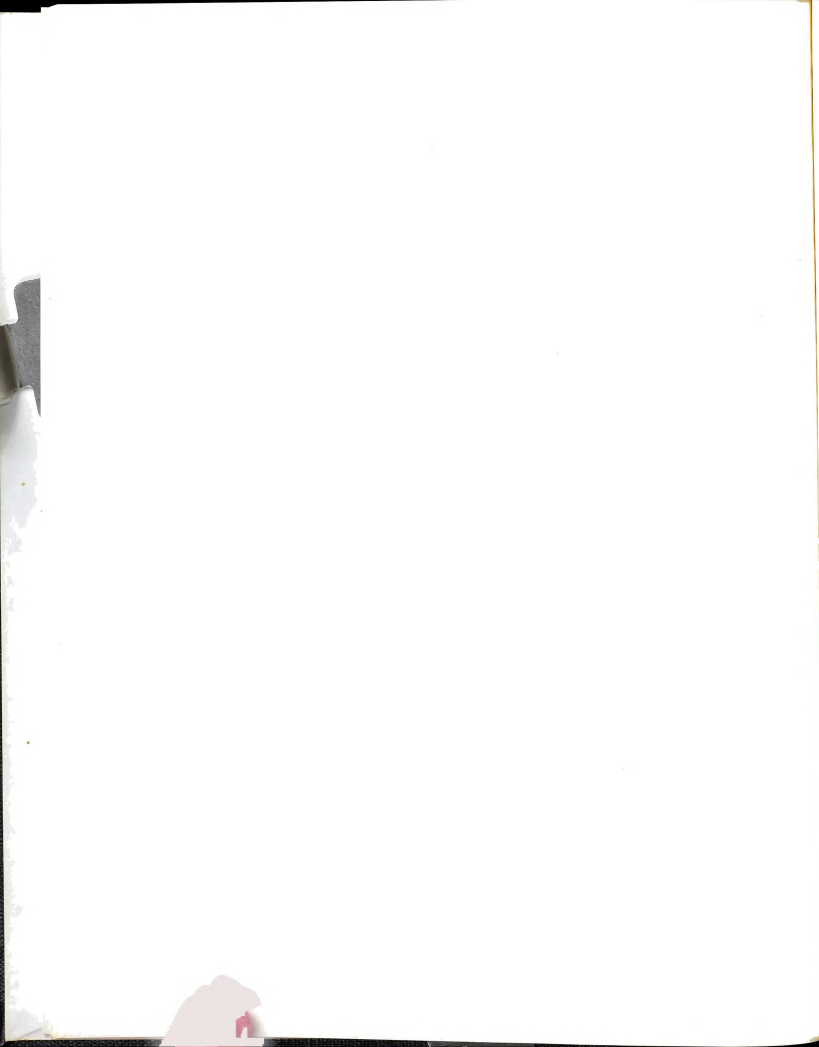
Now it can be easily shown that

$$1 - \frac{1}{2^{N(Mq-1)}(2^{Mq-1})} \leq \frac{|E \cap [0, h]|}{h} \leq 1.$$

As $h \rightarrow 0^+$, $N \rightarrow \infty$, and since $Mq-1 > 0$ it follows that

$$\lim_{h \rightarrow 0^+} \frac{|E \cap [0, h]|}{h} = 1.$$

Thus, $d_+(0, E) = 1$.



Define a function f on I as follows:

$$f(x) = \begin{cases} 2^{Mn} & , \text{ if } x \in I_n, \\ 0 & , \text{ if } x \in E. \end{cases}$$

Since

$$f(h) = o(h^k)$$

as $h \rightarrow 0^+$, $h \in E$, f has a k^{th} approximate Peano derivative $f_{(k)}(0)$ at 0. Furthermore, $f_{(n)}(0) = 0$, for $n = 0, 1, \dots, k$.

If f has a k^{th} L_r derivative at 0, $f_{k,r}(0)$, where $0 < r \leq \infty$, then as was mentioned in the beginning of this chapter $f_{k,r}(0) = f_{(k)}(0) = 0$. We first show that $f_{k,p}(0)$ exists then we show that $f_{k,q}(0)$ does not exist. To show $f_{k,p}(0) = 0$ it suffices to show

$$\lim_{h \rightarrow 0^+} \frac{1}{h^k} \left(\frac{1}{h} \int_0^h |f(t)|^p dt \right)^{1/p} = 0.$$

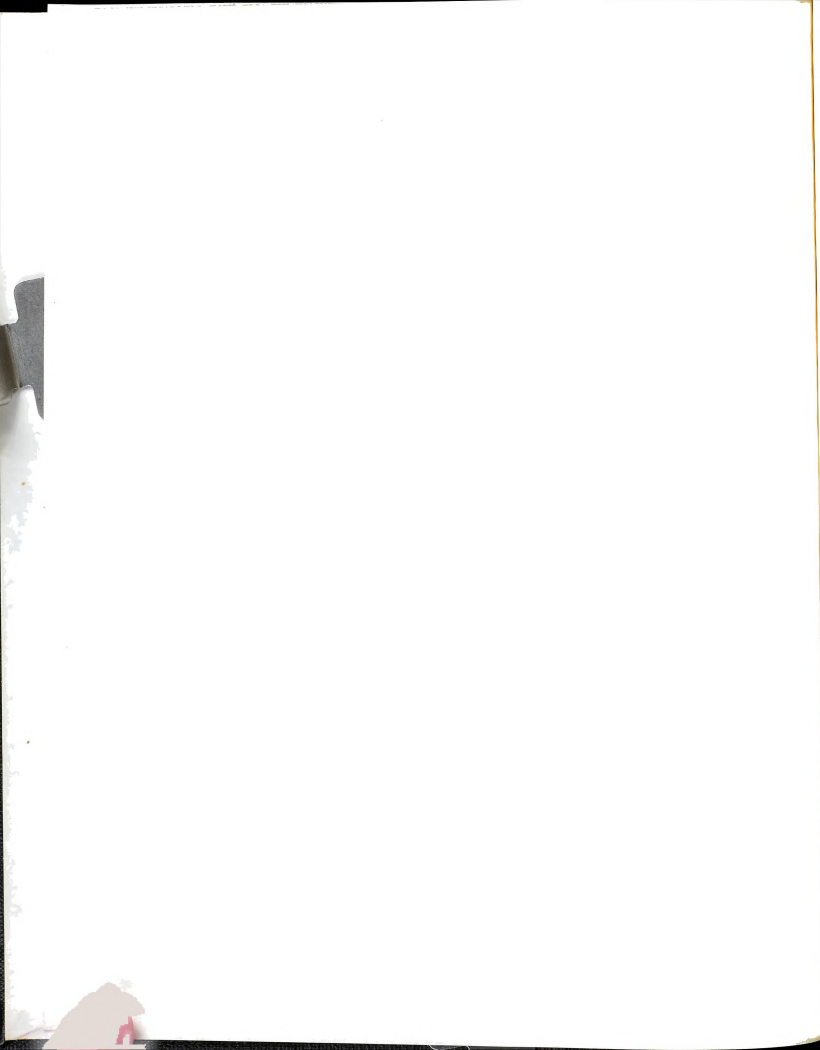
Let $0 < h < 1$, and choose the nonnegative integer N so that

$$\frac{1}{2^{N+1}} \leq h < \frac{1}{2^N}.$$

For notational convenience set $a_m = \frac{1}{2^m} + \frac{1}{2^{Mqm}}$, $m = 1, 2, \dots$.

Then

$$\begin{aligned} \frac{1}{h^k} \left(\frac{1}{h} \int_0^h |f(t)|^p dt \right)^{1/p} &\leq 2^{k(N+1)} \left(2^{N+1} \int_0^{a_N} |f(t)|^p dt \right)^{1/p} \\ &\leq 2^{k(N+1)} \left(2^{N+1} \sum_{n=N}^{\infty} \int_{I_n} |f(t)|^p dt \right)^{1/p} \\ &\leq 2^{k(N+1)} \left(2^{N+1} \sum_{n=N}^{\infty} \frac{2^{Mpn}}{2^{Mqn}} \right)^{1/p} \end{aligned}$$



$$\begin{aligned}
&\leq 2^{k(N+1)} \left(2^{N+1} \sum_{n=N}^{\infty} \frac{1}{2^{c p n}} \right)^{1/p} \\
&\leq 2^{k(N+1)} \left(2^{N+1} \left(\frac{2^{c p}}{2^{c p N} (2^{c p - 1})} \right) \right)^{1/p} \\
&\leq \frac{2^{c + p - 1 + k}}{(2^{c p - 1})^{1/p}} \cdot \frac{1}{2^{N(c - k - p - 1)}} \\
&\leq \frac{2^{c + p - 1 + k}}{(2^{c p - 1})^{1/p}} \cdot \frac{1}{2^N}.
\end{aligned}$$

As $h \rightarrow 0^+$, $N \rightarrow \infty$, and $\frac{1}{2^N} \rightarrow 0$. Thus $f_{k,p}(0) = 0$.

We now show that $f_{k,q}(0)$ does not exist. Assume to the contrary that $f_{k,q}(0)$ exists. Then $f_{n,q}(0) = 0$, for $n = 0, 1, \dots, k$. Thus,

$$(2.3) \quad \lim_{h \rightarrow 0^+} \frac{1}{h} k \left(\frac{1}{h} \int_0^h |f(t)|^q dt \right)^{1/q} = 0.$$

However, if we let $0 < h < 1$, and choose the nonnegative integer N so that

$$\frac{1}{2^{N+1}} \leq h < \frac{1}{2^N}$$

then

$$\begin{aligned}
\frac{1}{h} k \left(\frac{1}{h} \int_0^h |f(t)|^q dt \right)^{1/q} &\geq 2^{Nk} \left(2^N \int_0^{2^{N+2}} |f(t)|^q dt \right)^{1/q} \\
&\geq 2^{Nk} \left(2^N \sum_{n=N+2}^{\infty} \int_{I_n} |f(t)|^q dt \right)^{1/q} \\
&\geq 2^{Nk} \left(2^N \sum_{n=N+2}^{\infty} \int_{I_n} 2^{Mqn} dt \right)^{1/q}
\end{aligned}$$



$$\geq 2^{Nk} \left(2^{\sum_{n=N+2}^{\infty} \frac{Mqn}{2^{Mqn}}} \right)^{1/q}$$

$$= \infty.$$

This contradicts (2.3) and thus $f_{k,q}(0)$ does not exist.

Remark. From the nonexistence of $f_{k,q}(0)$ we further have the nonexistence of $f_k(0)$ and $f_{k,\infty}(0)$.

Thus, $k^{\text{th}} L_p$ differentiation is more general than $k^{\text{th}} L_q$ differentiation, where $0 < p < q \leq \infty$, and k^{th} Peano differentiation.

Example 2.4. Let k be a positive integer. Here we construct a function having a k^{th} approximate Peano derivative at 0 but no $k^{\text{th}} L_p$ derivative at 0, for $0 < p \leq \infty$, and no k^{th} Peano derivative at 0.

Let

$$I_n = \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{2^n} \right],$$

$n = 2, 3, \dots$. Define a function f on $I = [0, 1]$ as follows:

$$f(x) = \begin{cases} 2^{n^2}, & \text{if } x \in I_n, \\ 0 & \text{if } x \in E, \end{cases}$$

where $E = I - \bigcup_{n=2}^{\infty} I_n$.

Again it can be shown that $d_+(0, E) = 1$ and that $f_{(n)}(0) = 0$, for $n = 0, 1, \dots, k$. Let p be a real number such that $0 < p < \infty$. Let $0 < h < 1$ and choose N a positive integer so that

$$\frac{1}{N+1} \leq h < \frac{1}{N}.$$



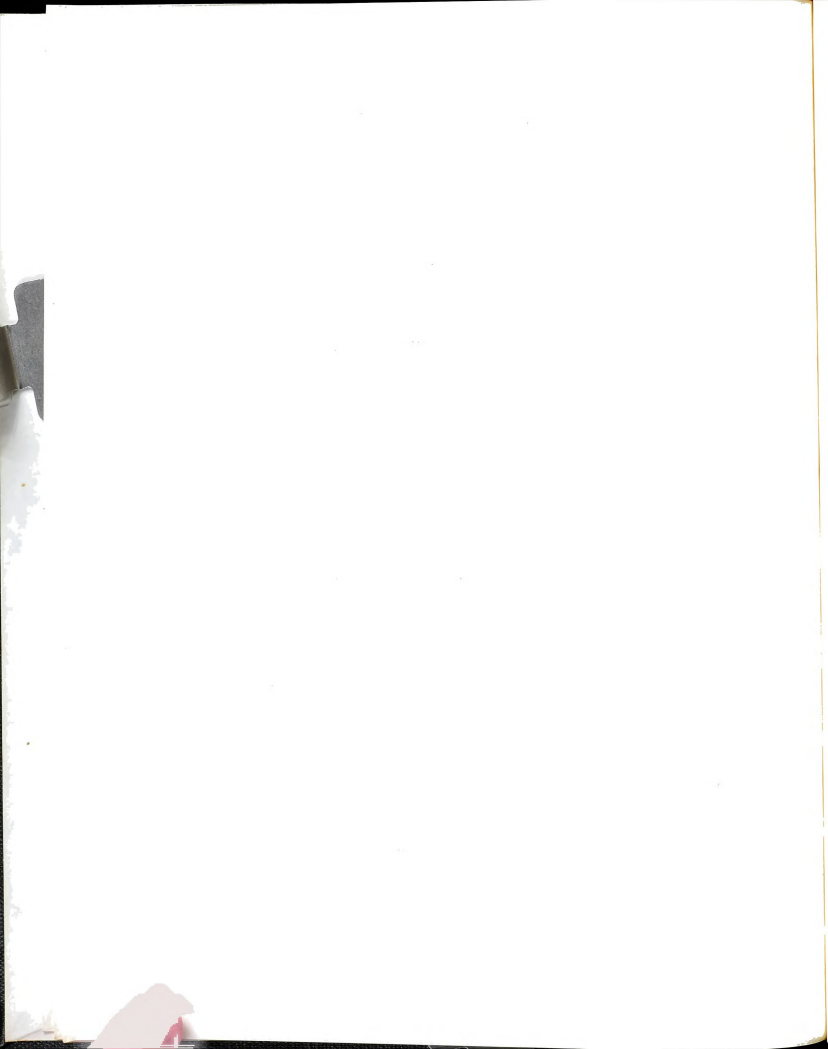
For notational convenience set $b_m = \frac{1}{m+2} + \frac{1}{2^{m+2}}$, $m = 1, 2, \dots$.
Then

$$\begin{aligned}
 \left(\frac{1}{h} k \left(\frac{1}{h} \int_0^h |f(t)|^p dt \right) \right)^{1/p} &\geq N^k \left(N \int_0^{b_N} |f(t)|^p dt \right)^{1/p} \\
 &\geq N^k \left(N \sum_{n=N+2}^{\infty} \int_{I_n} |f(t)|^p dt \right)^{1/p} \\
 &\geq N^k \left(N \sum_{n=N+2}^{\infty} \int_{I_n} 2^{pn^2} \right)^{1/p} \\
 &\geq N^k \left(N \sum_{n=N+2}^{\infty} \left(\frac{1}{2^n} \cdot 2^{pn^2} \right) \right)^{1/p} \\
 &\geq N^k \left(N \sum_{n=N+2}^{\infty} 2^{pn^2-n} \right) \\
 &= \infty.
 \end{aligned}$$

Thus, $f_{k,p}(0) \neq 0$. Hence $f_{k,p}(0)$ does not exist. Thus for each p , $0 < p < \infty$, $f_{k,p}(0)$ does not exist. From the non-existence of $f_{k,p}(0)$ we further have that $f_{k,\infty}(0)$ and $f_k(0)$ do not exist.

Remarks. 1. This last example shows that indeed the notion of k^{th} approximate Peano differentiation is a true generalization of k^{th} L_p differentiation for $0 < p \leq \infty$, and k^{th} Peano differentiation.

2. In the examples given it would be possible to construct the function f so that it is infinitely differentiable on $(0,1]$.



CHAPTER III

A PRELIMINARY RESULT

In general, if one has a sequence of measurable sets, say $\{E_n\}_{n=1}^{\infty}$, such that $d_+(0, E_n) = 0$ for $n = 1, 2, \dots$, then it is not necessarily true that $d_+(0, \bigcup_{n=1}^{\infty} E_n) = 0$. For example, if one takes $E_n = [\frac{1}{n}, 1]$ then $d_+(0, E_n) = 0$ for $n = 1, 2, \dots$, yet $d_+(0, \bigcup_{n=1}^{\infty} E_n) = 1$.

Suppose E is a measurable set such that $d_+(0, E) = 0$. Furthermore, let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of positive real numbers, with the α_n 's strictly increasing to 1 and the β_n 's strictly decreasing to 1. (It will be shown in Lemma 3.3 that if λ is a positive real number then $d_+(0, \lambda E) = 0$ where $\lambda E = \{\lambda x : x \in E\}$.) Is it necessarily true that

$$d_+(0, \bigcup_{n=1}^{\infty} \alpha_n E) = 0 \text{ or } d_+(0, \bigcup_{n=1}^{\infty} \beta_n E) = 0?$$

In this chapter we will give an example showing that this need not be true. We then prove a theorem, which will play a key role in the next chapter, showing that it is possible to choose sequences for which it is true. Before constructing the example a few lemmas are required.



Lemma 3.1. Let E be a measurable set and let λ be a positive real number. Then

$$|\lambda E| = \lambda |E|.$$

Lemma 3.2. Let E and F be subsets of the real numbers and let λ be a real number, $\lambda \neq 0$. Then

$$\lambda E \cap F = \lambda(E \cap \frac{1}{\lambda} F).$$

Proof. Let $x \in \lambda E \cap F$. Then $x \in F$ and $x = \lambda e$, where $e \in E$. Hence $e = \frac{1}{\lambda} x \in \frac{1}{\lambda} F$ and $e \in E \cap \frac{1}{\lambda} F$. Therefore, $x = \lambda e \in \lambda(E \cap \frac{1}{\lambda} F)$. Since all the above arguments are reversible the proof is complete.

Lemma 3.3. Let E be a measurable set and let λ be a positive real number. If $d_+(0, E) = 0$ then $d_+(0, \lambda E) = 0$.

Proof. By Lemma 3.2 and Lemma 3.1 we have

$$\frac{|\lambda E \cap [0, h]|}{h} = \frac{|\lambda(E \cap [0, \frac{1}{\lambda} h])|}{h} = \frac{|E \cap [0, \frac{1}{\lambda} h]|}{\frac{1}{\lambda} h} \rightarrow 0$$

as $h \rightarrow 0^+$. Therefore $d_+(0, \lambda E) = 0$.

Lemma 3.4. Let E be a measurable set. Then a necessary and sufficient condition for $d_+(0, E) = 0$ is that

$$\lim_{m \rightarrow \infty} \frac{|E \cap [0, \frac{1}{m}]|}{\frac{1}{m}} = 0.$$

Proof. The necessity of the condition is obvious. Thus, to establish the sufficiency, let $\epsilon > 0$ be given. Choose N , a positive integer, such that for every $m \geq N$

$$\frac{|E \cap [0, \frac{1}{m}]|}{\frac{1}{m}} < \frac{\epsilon}{2}.$$

Set $\delta = \frac{1}{N}$ and let $0 < h < \delta$. Choose $n \geq N$ such that

$$\frac{1}{n+1} \leq h < \frac{1}{n}.$$

Then

$$\begin{aligned} \frac{|E \cap [0, h]|}{h} &\leq \frac{|E \cap [0, \frac{1}{n}]|}{\frac{1}{n+1}} \\ &\leq \frac{|E \cap [0, \frac{1}{n}]|}{\frac{1}{n}} \cdot \frac{1}{\frac{1}{n+1}} \\ &\leq 2 \frac{|E \cap [0, \frac{1}{n}]|}{\frac{1}{n}} \\ &< \epsilon. \end{aligned}$$

Hence $d_+(0, E) = 0$.

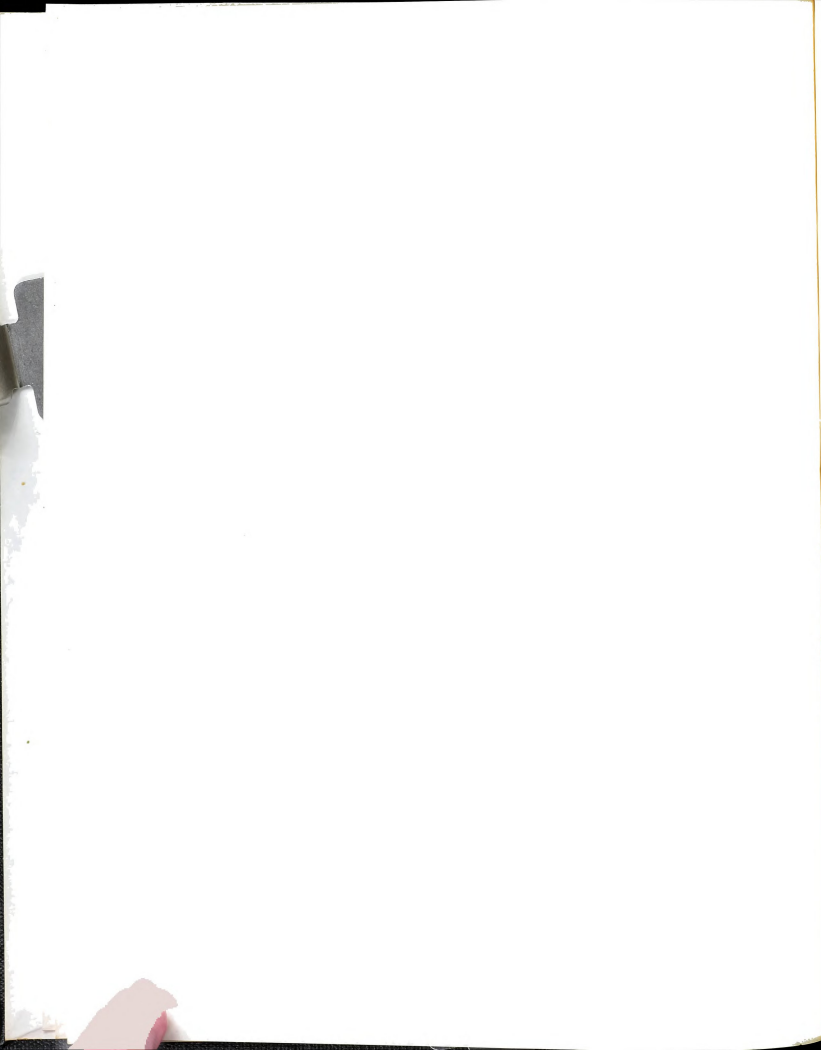
Example 3.5. We now construct a measurable set E and a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers, strictly increasing to 1, such that $d_+(0, E) = 0$ and

$$d_+(0, \cup_{n=1}^{\infty} \alpha_n E) = 1.$$

Let $E = \cup_{n=7}^{\infty} ([\frac{1}{n}, \frac{1}{n} + \frac{1}{2^n}])$. We first show that

$d_+(0, E) = 0$. Now for $m \geq 7$

$$\begin{aligned} |E \cap [0, \frac{1}{m}]| &= |(\cup_{n=7}^{\infty} [\frac{1}{n}, \frac{1}{n} + \frac{1}{2^n}]) \cap [0, \frac{1}{m}]| \\ &= \sum_{n=m}^{\infty} \frac{1}{2^{n+1}} \\ &= \frac{1}{2^m}. \end{aligned}$$



Therefore

$$\lim_{m \rightarrow \infty} \frac{|E \cap [0, \frac{1}{m}]|}{\frac{1}{m}} = \lim_{m \rightarrow \infty} \frac{m}{2^m} = 0.$$

Thus by Lemma 3.4, $d_+(0, E) = 0$.

Define for $n \geq 7$

$$\alpha_{n1} = \frac{n}{n+1} + \frac{n}{2^{n+1}}.$$

Then the sequence $\{\alpha_{n1}\}_{n=7}^{\infty}$ is strictly increasing to 1.

For n fixed define for $j = 2, 3, \dots$,

$$\alpha_{nj} = \alpha_{n1} \left(1 + \frac{n}{2^n}\right)^{j-1}.$$

Note there are only a finite number of j 's for which

$$\alpha_{nj} \left(\frac{1}{n} + \frac{1}{2^n}\right) < \frac{1}{n}.$$

Let k_n be the positive integer so that for $1 \leq j \leq k_n$

$$\alpha_{nj} \left(\frac{1}{n} + \frac{1}{2^n}\right) < \frac{1}{n}$$

and for $j = k_n + 1$

$$\alpha_{nj} \left(\frac{1}{n} + \frac{1}{2^n}\right) \geq \frac{1}{n}.$$

Observe that for each n

$$\left[\frac{1}{n+1} + \frac{1}{2^{n+1}}, \frac{1}{n}\right] \subseteq \bigcup_{j=1}^{k_n} \alpha_{nj} \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{2^n}\right].$$

Now consider the sequence

$$(3.6) \quad \alpha_{71}, \alpha_{72}, \dots, \alpha_{7k_7}, \alpha_{81}, \alpha_{82}, \dots, \alpha_{8k_8}, \dots$$

It is obvious that we can relabel the sequence (3.6), say $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$, in such a way that the α_n 's are strictly



increasing to 1. Now it is easy to see that

$$(0, \frac{1}{7}] \subseteq \bigcup_{n=1}^{\infty} \alpha_n E.$$

Therefore $d_+(0, \bigcup_{n=1}^{\infty} \alpha_n E) = 1$.

A similar example can be constructed for a sequence of positive real numbers strictly decreasing to 1.

In order to prove the theorem mentioned at the beginning of this chapter we will need a few technical lemmas.

Lemma 3.7. Let E be a subset of the real numbers and let λ be a positive real number. Then $(\lambda E)^C = \lambda(E^C)$, where E^C is the complement of E .

Proof. $x \in (\lambda E)^C \iff \frac{x}{\lambda} \in E^C \iff x \in \lambda(E^C)$.

Lemma 3.8. Let E and F be two measurable sets.

(i) If $d_+(0, E) = 0$ and $d_+(0, F) = 0$ then

$$d_+(0, E \cup F) = 0.$$

(ii) $d_+(0, E) = 0$ if and only if $d_+(0, E^C) = 1$.

Proof. Follows easily from Definition 1.9.

Lemma 3.9. Let E be a measurable set and let Δ be a real number, $\Delta > 0$. If $d_+(0, E) = 0$ then there exists a sequence of positive integers $\{a_n\}_{n=1}^{\infty}$, increasing to ∞ , such that for every $\lambda \geq \Delta$

$$\frac{|\lambda E \cap [0, h]|}{h} < \frac{1}{n^2}$$

whenever $0 < h < \frac{1}{a_n}$.



Proof. Let $\epsilon > 0$ be given. Since $d_+(0, E) = 0$ there exists a $\delta > 0$ such that

$$\frac{|E \cap [0, h]|}{h} < \epsilon$$

whenever $0 < h < \delta$. For $0 < h < \Delta\delta$ and $\lambda \geq \Delta$

$$\frac{|\lambda E \cap [0, h]|}{h} = \frac{|E \cap [0, \frac{1}{\lambda}h]|}{\frac{1}{\lambda}h} < \epsilon$$

since $\frac{1}{\lambda}h \leq \frac{h}{\Delta} < \delta$. The proof of the lemma may now be completed by letting ϵ progress through the numbers $1/n^2$ and choosing a_n so that $1/a_n$ is smaller than the corresponding $\Delta\delta$ and also the a_n 's increase to ∞ .

Lemma 3.10. Let E be a set of finite measure and let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$|\lambda E - E| < \epsilon$$

whenever $|1 - \lambda| < \delta$.

Proof. If $|E| = 0$ then the result is obvious. Thus assume $|E| > 0$. First assume $E = (a, b)$. Then it is easy to see that there exists a $\delta > 0$ such that

$$|\lambda(a, b) - (a, b)| < \epsilon$$

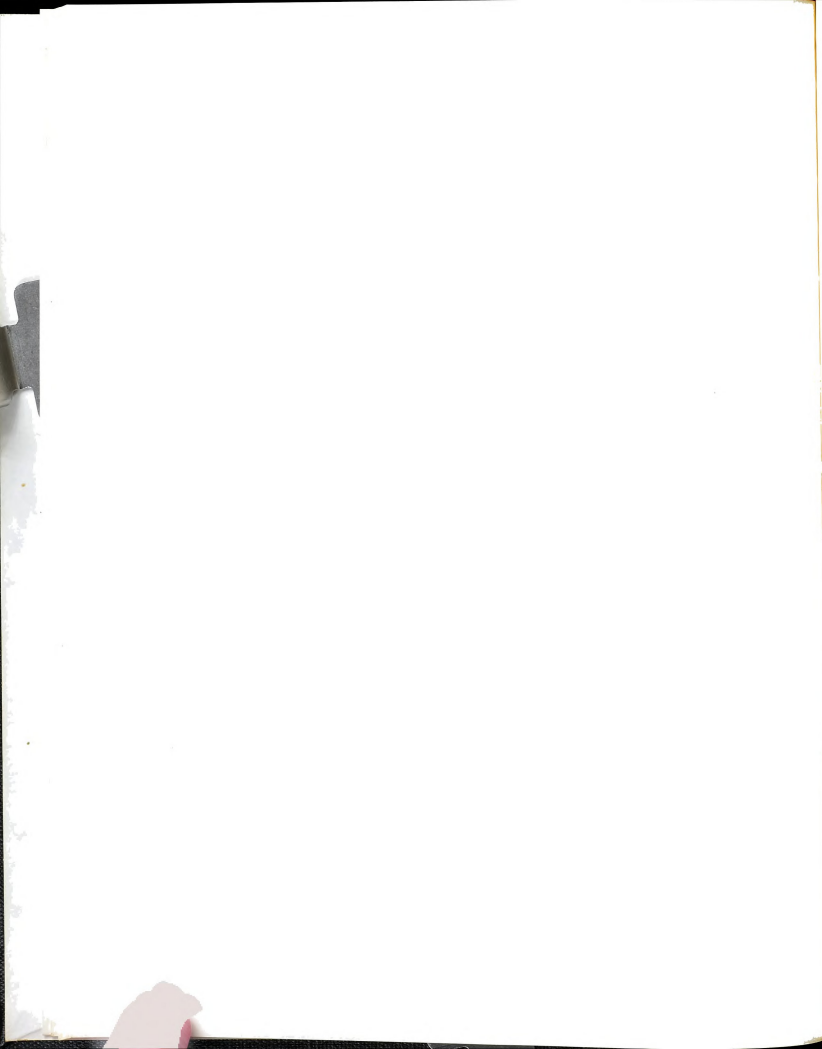
whenever $|1 - \lambda| < \delta$. Assuming $E = \bigcup_{n=1}^N (a_n, b_n)$, then again it is obvious that there exists a $\delta > 0$ such that

$$|\lambda E - E| < \epsilon$$

whenever $|1 - \lambda| < \delta$.

Now assume $E = \bigcup_{n=1}^{\infty} (a_n, b_n)$. Choose N such that

$$|\bigcup_{n=N+1}^{\infty} (a_n, b_n)| < \frac{\epsilon}{4}.$$



Setting $F = \bigcup_{n=1}^N (a_n, b_n)$ and $H = \bigcup_{n=N+1}^{\infty} (a_n, b_n)$ we have
 $E = F \cup H$. Choose $0 < \delta < 1$ so that for all λ , $|1-\lambda| < \delta$,

$$|\lambda F - F| < \frac{\varepsilon}{2}.$$

Since $\lambda E - E \subseteq \lambda F - F = (\lambda F \cup \lambda H) - F \subseteq (\lambda F - F) \cup \lambda H$, and $0 < \lambda < 2$

$$|\lambda E - E| \leq |\lambda F - F| + |\lambda H| < \frac{\varepsilon}{2} + \lambda |H| < \varepsilon$$

whenever $|1-\lambda| < \delta$.

Finally, assume E is a set of finite measure and let G be an open set such that $E \subseteq G$ and $|G - E| < \frac{\varepsilon}{2}$. Choose δ , $0 < \delta < 1$, so that for all λ , $|1-\lambda| < \delta$,

$$|\lambda G - G| < \frac{\varepsilon}{2}.$$

Since $(\lambda E - E) \subseteq \lambda G - E \subseteq (\lambda G - G) \cup (G - E)$

$$|\lambda E - E| \leq |\lambda G - G| + |G - E| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|1-\lambda| < \delta$.

Lemma 3.11. Let E be a set of finite measure. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $\lim_{n \rightarrow \infty} a_n = \infty$. Then there exist two sequences of positive real numbers $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$, with the α_n 's strictly increasing to 1 and the β_n 's strictly decreasing to 1, such that for each n , $n = 1, 2, \dots$,

$$|(\alpha_n E - E) \cap [0, \frac{1}{m}]| < \frac{1}{m 2^n} \quad \text{and} \quad |(\beta_n E - E) \cap [0, \frac{1}{m}]| < \frac{1}{m 2^n}$$

whenever $1 \leq m \leq a_n$.

Proof. We first show the existence of the sequence $\{\alpha_n\}_{n=1}^{\infty}$. By Lemma 3.10 choose α_1 , $0 < \alpha_1 < 1$, such that



$$|\alpha_1 E - E| < \frac{1}{a_1^2}.$$

Then for each m , $1 \leq m \leq a_1$,

$$|(\alpha_1 E - E) \cap [0, \frac{1}{m}]| \leq |\alpha_1 E - E| < \frac{1}{a_1^2} \leq \frac{1}{m^2}.$$

By Lemma 3.10 choose α_2 such that $\max\{\alpha_1, 1 - \frac{1}{2}\} < \alpha_2 < 1$ and

$$|\alpha_2 E - E| < \frac{1}{a_2^2}.$$

Then for each m , $1 \leq m \leq a_2$,

$$|(\alpha_2 E - E) \cap [0, \frac{1}{m}]| \leq |\alpha_2 E - E| < \frac{1}{a_2^2} \leq \frac{1}{m^2}.$$

Inductively define α_n as follows: By Lemma 3.10 choose α_n such that $\max\{\alpha_{n-1}, 1 - \frac{1}{n}\} < \alpha_n < 1$ and

$$|\alpha_n E - E| < \frac{1}{a_n^2}.$$

Then for each m , $1 \leq m \leq a_n$,

$$|(\alpha_n E - E) \cap [0, \frac{1}{m}]| \leq |\alpha_n E - E| < \frac{1}{a_n^2} \leq \frac{1}{m^2}.$$

The existence of the sequence $\{\alpha_n\}_{n=1}^\infty$ thus follows by induction.

A proof similar to the one given above can be given to show the existence of the sequence $\{\beta_n\}_{n=1}^\infty$.

Theorem 3.12. Let E be a measurable set and let $d_+(0, E) = 0$. Then there exist two sequences of positive real numbers $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$, with the α_n 's strictly increasing to 1 and the β_n 's strictly decreasing to 1, such that

$$d_+(0, \bigcup_{n=1}^\infty \alpha_n E) = 0 \quad \text{and} \quad d_+(0, \bigcup_{n=1}^\infty \beta_n E) = 0.$$



Proof. We first prove the existence of the sequence $\{\alpha_n\}_{n=1}^\infty$. Set $F = E \cap J$, where $J = [0, 1]$; then $d_+(0, F) = 0$. By Lemma 3.9, there exists a sequence of positive integers $\{a_n\}_{n=1}^\infty$, where the a_n 's increase to ∞ , such that for each λ , $\frac{1}{2} \leq \lambda < 1$

$$(3.13) \quad \frac{|\lambda F \cap [0, h]|}{h} < \frac{1}{n^2}$$

whenever $0 < h < \frac{1}{a_n}$. By Lemma 3.11 there exists a sequence of positive numbers $\{\alpha_n\}_{n=1}^\infty$, where we may assume that for each n , $\frac{1}{2} \leq \alpha_n < 1$, strictly increasing to 1 and corresponding to the sequence $\{a_n\}_{n=1}^\infty$, such that for each n

$$(3.14) \quad |(\alpha_n F - F) \cap [0, \frac{1}{m}]| < \frac{1}{m^2 n}$$

whenever $1 \leq m \leq a_n$. Given an $\varepsilon > 0$, choose a positive integer k so that

$$\frac{1}{k} + \frac{1}{2^k} < \varepsilon.$$

Set $\delta = \frac{1}{a_k}$ and let $0 < \frac{1}{m} < \delta$. Choose j , $j \geq k$, so that

$$\frac{1}{a_{j+1}} \leq \frac{1}{m} < \frac{1}{a_j}.$$

Now

$$\begin{aligned} |[\cup_{n=1}^\infty (\alpha_n F - F)] \cap [0, \frac{1}{m}]| &\leq \sum_{n=1}^j |(\alpha_n F - F) \cap [0, \frac{1}{m}]| \\ &\quad + \sum_{n=j+1}^\infty |(\alpha_n F - F) \cap [0, \frac{1}{m}]|. \end{aligned}$$

Moreover by (3.13)

$$\sum_{n=1}^j |(\alpha_n F - F) \cap [0, \frac{1}{m}]| \leq \frac{1}{m} \sum_{n=1}^j \frac{|\alpha_n F \cap [0, \frac{1}{m}]|}{\frac{1}{m}} < \frac{j}{mj^2} = \frac{1}{mj}$$



and by (3.14)

$$\sum_{n=j+1}^{\infty} |(\alpha_n^{F-F}) \cap [0, \frac{1}{m}]| < \sum_{n=j+1}^{\infty} \frac{1}{m2^n} = \frac{1}{m2^j}.$$

Therefore,

$$\frac{|[\bigcup_{n=1}^{\infty} (\alpha_n^{F-F})] \cap [0, \frac{1}{m}]|}{\frac{1}{m}} < \frac{1}{j} + \frac{1}{2^j} \leq \frac{1}{k} + \frac{1}{2^k} < \varepsilon.$$

Thus,

$$\lim_{m \rightarrow \infty} \frac{|[\bigcup_{n=1}^{\infty} (\alpha_n^{F-F})] \cap [0, \frac{1}{m}]|}{\frac{1}{m}} = 0.$$

Therefore by Lemma 3.4, $d_+(0, \bigcup_{n=1}^{\infty} (\alpha_n^{F-F})) = 0$. Furthermore, since $F \cap [\bigcup_{n=1}^{\infty} \alpha_n F] \subseteq F$ we have

$$d_+(0, F \cap [\bigcup_{n=1}^{\infty} \alpha_n F]) = 0.$$

Therefore, by Lemma 3.8(i),

$$d_+(0, \bigcup_{n=1}^{\infty} \alpha_n F) = 0.$$

Now $\alpha_n E \cap \alpha_1 J \subseteq \alpha_n E \cap \alpha_n J$, for $n = 1, 2, \dots$; and since

$$\begin{aligned} (\bigcup_{n=1}^{\infty} \alpha_n E) \cap [0, \alpha_1] &= \bigcup_{n=1}^{\infty} (\alpha_n E \cap \alpha_1 J) \\ &\subseteq \bigcup_{n=1}^{\infty} (\alpha_n E \cap \alpha_n J) \\ &\subseteq \bigcup_{n=1}^{\infty} \alpha_n F \end{aligned}$$

it follows that $d_+(0, \bigcup_{n=1}^{\infty} \alpha_n E) = 0$.

The proof for the existence of the β_n 's is analogous except we take $J = [0, \frac{1}{2})$, and we choose the β_n 's so that $1 < \beta_n \leq \frac{3}{2}$ for each n . This completes the proof of the theorem.

By Lemma 3.7 and Lemma 3.8(ii), Theorem 3.12 can be stated in the following form which will play a key role in the next chapter.

Theorem 3.15. Let E be a measurable set. If $d_+(0, E) = 1$ then there exist two sequences of positive real numbers $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$, with the α_n 's strictly increasing to 1 and the β_n 's strictly decreasing to 1, such that

$$d_+(0, \bigcap_{n=1}^{\infty} \alpha_n E) = d_+(0, \bigcap_{n=1}^{\infty} \beta_n E) = 1.$$



CHAPTER IV

THE MAJOR THEOREM

In this chapter we deduce the fundamental result stated in the following theorem.

Theorem. Suppose f possesses a k^{th} approximate Peano derivative $f_{(k)}$ everywhere on the interval $[a,b]$.

- (i) If $f_{(k)} > 0$ on $[a,b]$, then $f_{(k-1)}$ is continuous and increasing on $[a,b]$.
- (ii) If $f_{(k)}$ is bounded either above or below on $[a,b]$, then $f_{(k)} = f^{(k)}$ on $[a,b]$.

The proof of this theorem will require some additional definitions and lemmas.

Lemma 4.1. Assume f to have a k^{th} approximate Peano derivative $f_{(k)}$ for each point in $[a,b]$ and that $f_{(1)}$ is increasing in $[a,b]$. If $k > 2$ furthermore assume

$$f_{(2)}(a) = f_{(3)}(a) = \dots = f_{(k-1)}(a) = 0.$$

Then $(f_{(1)})_{(k-1)}(a) = f_{(k)}(a)$, that is, there exists a measurable set $E \subseteq [0,1]$ having 0 as a point of right-hand density such that

$$E\text{-}\lim_{h \rightarrow 0} \frac{f_{(1)}(a+h) - f_{(1)}(a)}{h^{k-1}} = \frac{f_{(k)}(a)}{(k-1)!}.$$

Proof. By subtracting from f a multiple of x , we may assume that $f_{(1)}(a) = 0$. By hypothesis there exists a measurable set $F \subseteq [0,1]$ having 0 as a point of right-hand density and such that

$$(4.2) \quad F\text{-}\lim_{h \rightarrow 0} \frac{1}{h^k} \{f(a+h) - f(a) - Ah^k\} = 0,$$

where $A = f_{(k)}(a)/k!$.

By Theorem 3.15 there exist two sequences of positive real numbers $\{\theta_m^*\}_{m=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ such that

$$\lim_{m \rightarrow \infty} \theta_m^* = \lim_{n \rightarrow \infty} \theta_n = 0$$

and

$$d_+(0, \cap_{m=1}^\infty (1 - \theta_m^*)F) = d_+(0, \cap_{n=1}^\infty (1 + \theta_n)F) = 1.$$

Let $E = F \cap [\cap_{m=1}^\infty (1 - \theta_m^*)F] \cap [\cap_{n=1}^\infty (1 + \theta_n)F]$. By Lemma 3.8 $d_+(0, E) = 1$. To complete the proof of the lemma we show

$$E\text{-}\lim_{h \rightarrow 0} \frac{f_{(1)}(a+h)}{h^{k-1}} = Ak = \frac{f_{(k)}(a)}{(k-1)!}.$$

Let $\varepsilon > 0$ be given. Choose θ_n and θ_m^* such that if

$$\alpha = \frac{\theta_n}{1 + \theta_n}, \quad \beta = \frac{\theta_m^*}{1 - \theta_m^*}$$

then

$$A \sum_{j=2}^k (-1)^{j+1} \binom{k}{j} \alpha^{j-1} > -\frac{\varepsilon}{2} \quad \text{and} \quad A \sum_{j=2}^k \binom{k}{j} \beta^{j-1} < \frac{\varepsilon}{2}.$$

Set

$$\varepsilon' = \min \left\{ \frac{\varepsilon \beta}{2[(1+\beta)^{k+1}]}, \frac{\varepsilon \alpha}{2[(1-\alpha)^{k+1}]} \right\}.$$

By (4.2) there exists a $\delta' > 0$ such that



$$|f(a+h)-f(a)-Ah^k| < \epsilon' h^k \text{ whenever } 0 < h < \delta', h \in F.$$

If t_1 and t_2 are values of h such that $0 < t_1 < t_2 < \delta'$ and $t_1, t_2 \in F$ then

$$|[f(a+t_2)-f(a+t_1)]-A(t_2^k-t_1^k)| < \epsilon' (t_2^k+t_1^k).$$

Hence

$$A \frac{(t_2^k-t_1^k)}{t_2-t_1} - \epsilon' \frac{(t_2^k+t_1^k)}{t_2-t_1} < \frac{f(a+t_2)-f(a+t_1)}{t_2-t_1} < A \frac{(t_2^k-t_1^k)}{t_2-t_1} + \epsilon' \frac{(t_2^k+t_1^k)}{t_2-t_1}.$$

Since $f_{(1)}$ is increasing on $[a, b]$ and $f_{(1)} = f'_{ap}$ we have

$f_{(1)} = f'$ on $[a, b]$ (see [4]) and hence

$$f_{(1)}(a+t_1) \leq \frac{f(a+t_2)-f(a+t_1)}{t_2-t_1} \leq f_{(1)}(a+t_2).$$

Thus, whenever $0 < t_1 < t_2 < \delta'$ and $t_1, t_2 \in F$

$$(4.3) \quad f_{(1)}(a+t_1) < A \frac{(t_2^k-t_1^k)}{t_2-t_1} + \epsilon' \frac{(t_2^k+t_1^k)}{t_2-t_1}$$

and

$$(4.4) \quad f_{(1)}(a+t_2) > A \frac{(t_2^k-t_1^k)}{t_2-t_1} - \epsilon' \frac{(t_2^k+t_1^k)}{t_2-t_1}.$$

Set $\delta = \min\{\delta'/(1+\beta), \delta'(1-\alpha)\}$ and let $h \in E$ such that $0 < h < \delta$. Since $h \in (1-\theta_m^*)F$, there exists a $t_2 \in F$ such that $h = (1-\theta_m^*)t_2$. Hence

$$t_2 = (1 + \frac{\theta_m^*}{1-\theta_m^*})h = (1+\beta)h$$

and $h < t_2 < \delta'$. Thus from (4.3) we have



$$\begin{aligned}
(4.5) \quad \frac{f_{(1)}(a+h)}{h^{k-1}} &< A \frac{[h^k(1+\beta)^k - h^k]}{\beta h^k} + \epsilon' \frac{[h^k(1+\beta)^k + h^k]}{\beta h^k} \\
&< A \frac{[(1+\beta)^k - 1]}{\beta} + \epsilon' \frac{[(1+\beta)^k + 1]}{\beta} \\
&< Ak + A \sum_{j=2}^k \binom{k}{j} \beta^{j-1} + \frac{\epsilon}{2} \\
&< Ak + \epsilon.
\end{aligned}$$

Moreover, since $h \in (1+\theta_n)F$, there exists a $t_1 \in F$ such that $h = (1+\theta_n)t_1$. Hence

$$t_1 = \left(1 - \frac{\theta_n}{1+\theta_n}\right)h = (1-\alpha)h$$

and $t_1 < h < \delta'$. Thus from (4.4) we have

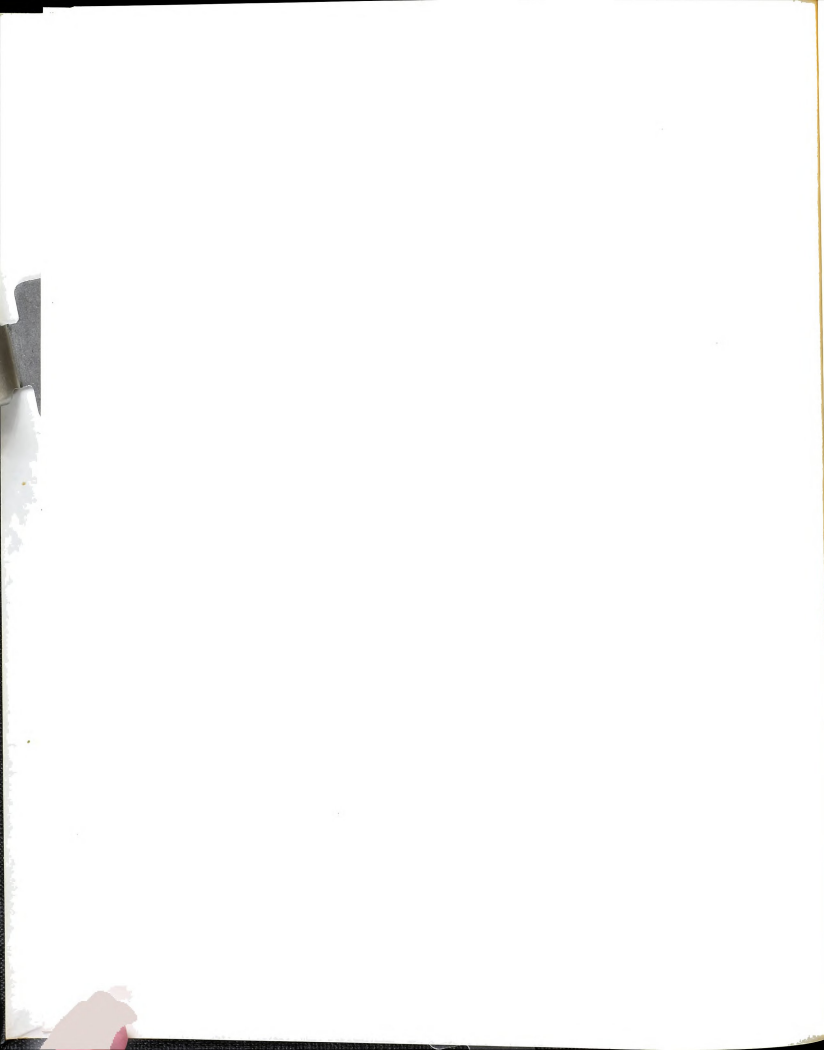
$$\begin{aligned}
(4.6) \quad \frac{f_{(1)}(a+h)}{h^{k-1}} &> A \frac{[h^k - h^k(1-\alpha)^k]}{\alpha h^k} - \epsilon' \frac{[h^k + h^k(1-\alpha)^k]}{\alpha h^k} \\
&> A \frac{[1 - (1-\alpha)^k]}{\alpha} - \epsilon' \frac{[1 + (1-\alpha)^k]}{\alpha} \\
&> Ak + A \sum_{j=2}^k (-1)^{j+1} \binom{k}{j} \alpha^{j-1} - \frac{\epsilon}{2} \\
&> Ak - \epsilon.
\end{aligned}$$

Thus from (4.5) and (4.6) we have

$$Ak - \epsilon < \frac{f_{(1)}(a+h)}{h^{k-1}} < Ak + \epsilon$$

whenever $0 < h < \delta$ and $h \in E$. Hence

$$E\text{-}\lim_{h \rightarrow 0} \frac{f_{(1)}(a+h)}{h^{k-1}} = Ak = \frac{f_{(k)}(a)}{(k-1)!}.$$



Corollary 4.7. Assume f to have a k^{th} approximate Peano derivative $f_{(k)}$ for each $x \in [a, b]$, and that $f_{(1)}$ is increasing on $[a, b]$. If $k > 2$ furthermore assume

$$f_{(2)}(b) = f_{(3)}(b) = \dots = f_{(k-1)}(b) = 0.$$

Then $(f_{(1)})_{(k-1)}(b) = f_{(k)}(b)$.

Proof. Define a function g on $[-b, -a]$ as follows:

$$g(x) = f(-x) \text{ for each } x \in [-b, -a].$$

Then $g_{(k)}(x)$ exists for each $x \in [-b, -a]$ and

$$g_{(n)}(x) = (-1)^n f_{(n)}(-x)$$

for $n = 0, 1, \dots, k$, where $f_{(0)}(-x) = f(-x)$. Now it is easily shown that $g_{(1)}$ is increasing on $[-b, -a]$. Also, if $n > 2$ then

$$g_{(2)}(-b) = g_{(3)}(-b) = \dots = g_{(k-1)}(-b) = 0.$$

By Lemma 4.1 there exists a measurable set $E \subseteq [0, 1]$ such that 0 is a point of right-hand density of E and

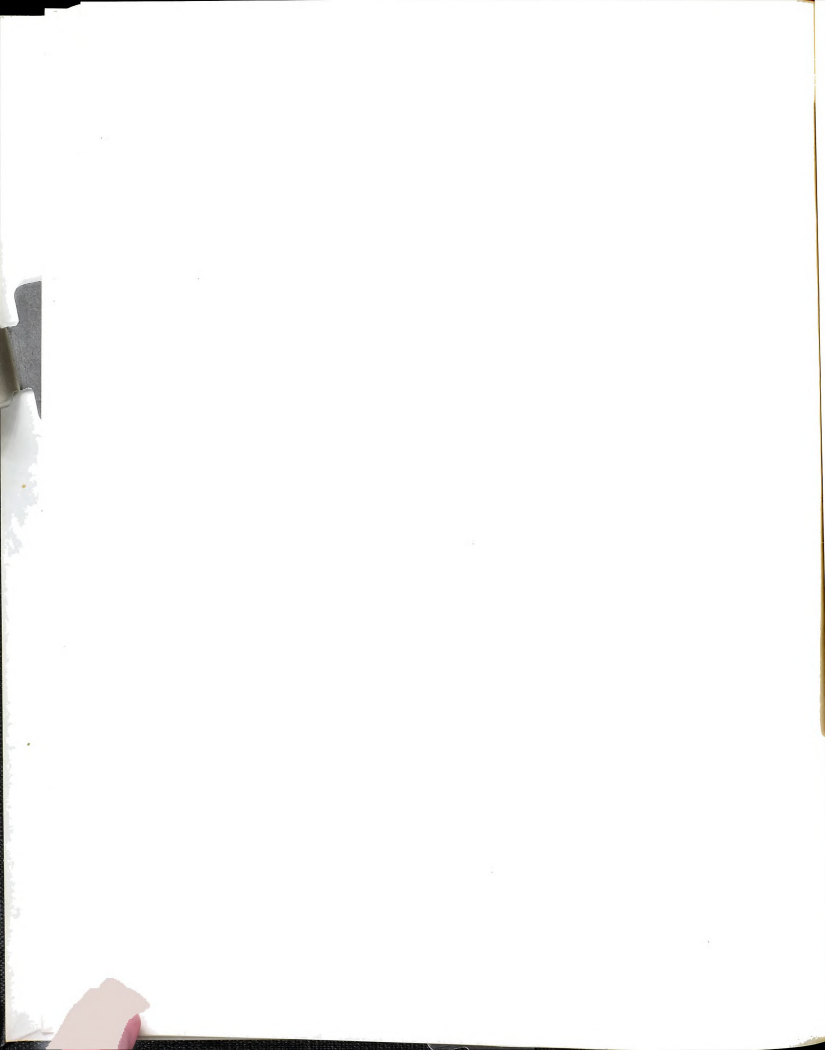
$$E\text{-}\lim_{h \rightarrow 0} \frac{g_{(1)}(-b+h) - g_{(1)}(-b)}{h^{k-1}} = \frac{g_{(k)}(-b)}{(k-1)!}.$$

Hence

$$E\text{-}\lim_{h \rightarrow 0} \frac{f_{(1)}(b-h) - f_{(1)}(b)}{(-h)^{k-1}} = \frac{f_{(k)}(b)}{(k-1)!},$$

that is, $(f_{(1)})_{(k-1)}(b) = f_{(k)}(b)$.

Corollary 4.8. Assume f to have a second approximate Peano derivative $f_{(2)}$ for each point in $[a, b]$, and that $f_{(1)}$ is increasing on $[a, b]$. Then, $(f_{(1)})_{(1)}(x) = f_{(2)}(x)$ for each $x \in [a, b]$.



Proof. Follows immediately from Lemma 4.1 and Corollary 4.7.

Lemma 4.9. Suppose f has $(k-1)$ derivatives at the point x , then for each sufficiently small non-zero h , there is a θ , $0 < \theta < 1$, depending on h such that

$$\frac{(k-2)!}{h^{k-2}} \{ f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) \} = f^{(k-2)}(x+\theta h) - f^{(k-2)}(x) - \theta h f^{(k-1)}(x)$$

where $f^{(0)}(x) = f(x)$.

Proof. Let

$$(4.10) \quad g(t) = f(x+t) - \sum_{n=0}^{k-1} \frac{t^n}{n!} f^{(n)}(x).$$

Then g is $(k-2)$ times differentiable around 0 and

$$(4.11) \quad g^{(j)}(t) = f^{(j)}(x+t) - \sum_{n=0}^{k-j-1} \frac{t^n}{n!} f^{(n+j)}(x)$$

for $j = 0, 1, \dots, (k-2)$. By the extended mean value theorem for each sufficiently small h there exists a θ , $0 < \theta < 1$, depending on h so that

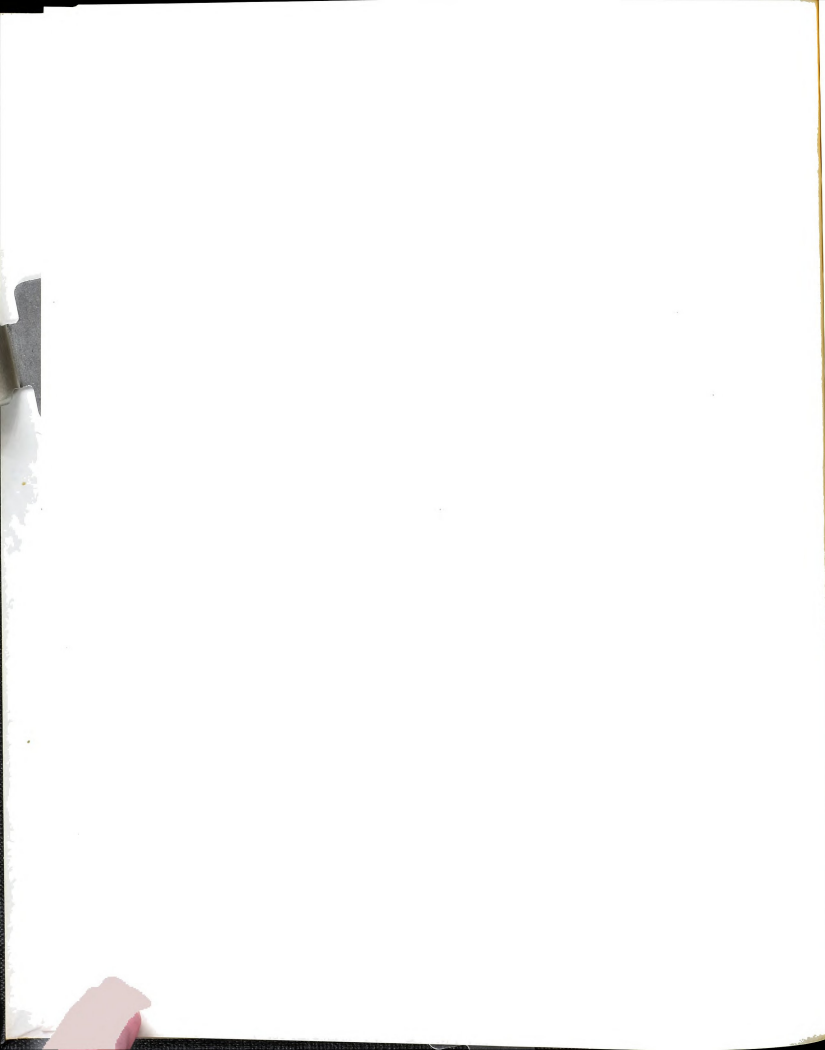
$$g(h) = \sum_{n=0}^{k-3} \frac{h^n}{n!} g^{(n)}(0) + \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h)$$

where $g^{(0)}(0) = g(0)$. By (4.11) it follows that $g^{(j)}(0) = 0$ for $j = 0, 1, \dots, (k-3)$; hence

$$(4.12) \quad g(h) = \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h).$$

Thus, by replacing the left-hand side of (4.10) by (4.12) we have

$$f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x) = \frac{h^{k-2}}{(k-2)!} g^{(k-2)}(\theta h).$$



If $h \neq 0$ then this last equation together with (4.11) yields

$$\begin{aligned}
 & \frac{(k-2)!}{h^{k-2}} \{f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f^{(n)}(x)\} \\
 &= g^{(k-2)}(\theta h) \\
 &= f^{(k-2)}(x+\theta h) - \sum_{n=0}^1 \frac{(\theta h)^n}{n!} f^{(k+n-2)}(x) \\
 &= f^{(k-2)}(x+\theta h) - f^{(k-2)}(x) - \theta h f^{(k-1)}(x).
 \end{aligned}$$

Definition 4.13. A function f defined on an interval is said to be convex if for every pair of points P_1, P_2 on the curve $y = f(x)$ the points of the arc P_1P_2 are below, or on, the chord P_1P_2 .

The following lemma is due to S. Verblunsky [9].

Lemma 4.14. Let f have a finite derivative at each point of (a,b) . Suppose that, for each $x_0 \in (a,b)$ there are, in every neighborhood of $(x_0, f(x_0))$, points of the graph of f above the line $y = f(x_0) + f'(x_0)(x - x_0)$. Then f is convex in (a,b) .

Proof. If possible suppose that there are points c, d , $a < c < d < b$, such that the arc $y = f(x)$ ($c \leq x \leq d$) has points above the chord joining $(c, f(c))$ and $(d, f(d))$. Let

$$\kappa = \frac{f(d) - f(c)}{d - c}.$$

Now the function $f(x) - f(c) - \kappa(x - c)$ is continuous and so it will attain its maximum at some point γ in $[c, d]$. By our assumption $c < \gamma < d$. Let



$$\mu = \frac{f(\gamma) - f(c)}{\gamma - c}.$$

Then $\mu > \kappa$. Since $(f(x) - f(c))/(x - c)$ is a continuous function on $[\gamma, d]$ it will attain all the values between μ and κ somewhere between γ and d . Let τ be such that $\kappa < \tau < \mu$. Then there exists a ω , $\gamma < \omega < d$, such that

$$\tau = \frac{f(\omega) - f(c)}{\omega - c}.$$

Now the function

$$(4.15) \quad g(x) = f(x) - f(c) - \tau(x - c)$$

is continuous and so it will attain a maximum at some point ξ in $[c, \omega]$. Since $\mu > \tau$, we conclude $c < \xi < \omega$. Also from (4.15), $\tau = f'(\xi)$.

Now choose $\delta > 0$ such that for each $x \in (\xi - \delta, \xi + \delta)$, $g(x) \leq g(\xi)$. This implies using (4.15) that

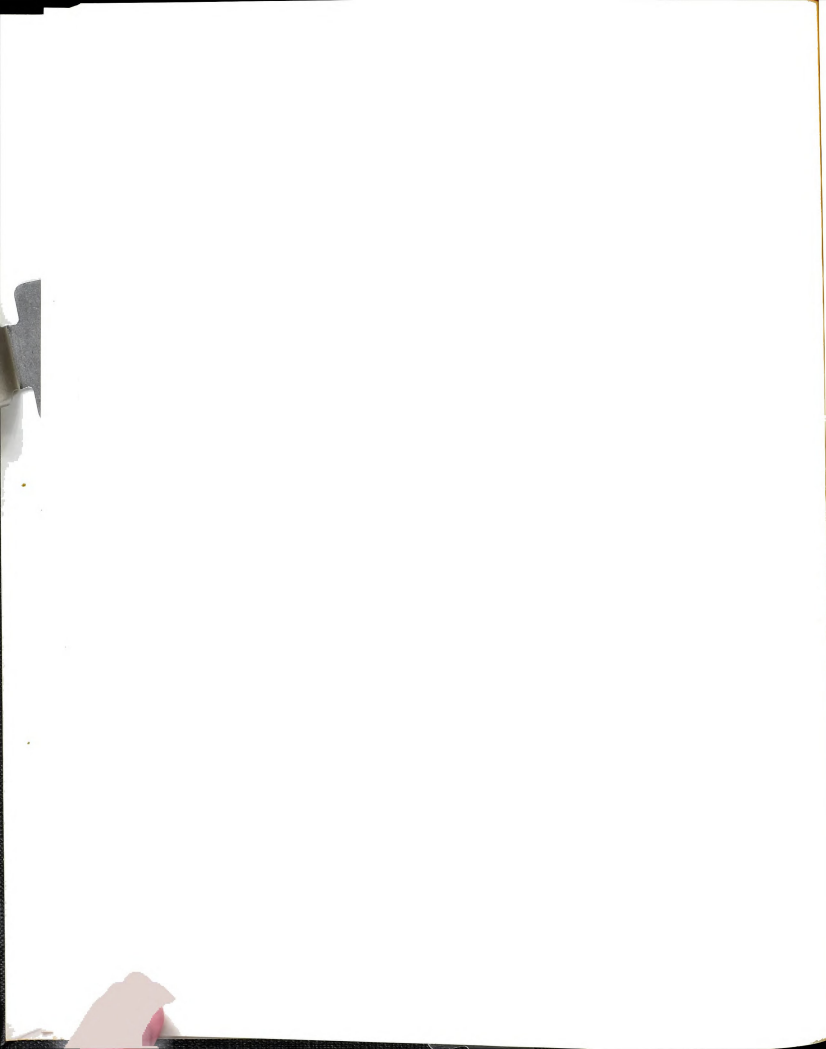
$$f(x) \leq f(\xi) + f'(\xi)(x - \xi)$$

for $x \in (\xi - \delta, \xi + \delta)$. Hence the line $y = f(\xi) + f'(\xi)(x - \xi)$ has the property that there exists a neighborhood of the point $(\xi, f(\xi))$ such that no point of the graph of f is above the line. This, however, contradicts our hypothesis.

Definition 4.16. Let f be a function defined in a neighborhood of x . Then define

$$\bar{D}_2 f(x) = \limsup_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

$\bar{D}_2 f(x)$ is called the upper symmetric second derivative of f at x .



Remark. It can easily be shown that if $f''(x)$ exists then $\bar{D}_2 f(x) = f''(x)$. However, the upper symmetric second derivative may exist without the second derivative existing.

A proof of the following lemma can be found in [13].

Lemma 4.17. A necessary and sufficient condition for a continuous function f to be convex in (a,b) is that $\bar{D}_2 f(x) \geq 0$ for each x in (a,b) .

Lemma 4.18. Suppose f has a second approximate Peano derivative $f_{(2)}$ at a point $x \in (a,b)$. Then there exists a measurable set $E \subseteq [0,1]$ such that 0 is a point of right-hand density of E and

$$E\text{-}\lim_{h \rightarrow 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f_{(2)}(x).$$

Proof. By hypothesis there exists a measurable set F having 0 as a point of density and such that

$$F\text{-}\lim_{h \rightarrow 0} \frac{1}{h^2} \{f(x+h)-f(x)-hf_{(1)}(x)-\frac{h^2}{2}f_{(2)}(x)\} = 0.$$

Set $F_1 = F \cap [0,1]$. Then 0 is a point of right-hand density of F_1 . Also, 0 is a point of left-hand density of $F \cap [-1,0]$. Setting $F_2 = \{h : -h \in F \cap [-1,0]\}$, then 0 is a right-hand point of density of F_2 . It follows from Lemma 3.8 that 0 is a point of right-hand density of $E = F_1 \cap F_2$.

Let $\varepsilon > 0$ be given. Then there exists a $0 < \delta < 1$ such that



$$\left| \frac{f(x+h)-f(x)-hf_{(1)}(x)}{h^2} - \frac{f_{(2)}(x)}{2} \right| < \frac{\varepsilon}{2}$$

whenever $h \in F$ and $|h| < \delta$. Let $h \in E$ so that $0 < h < \delta$.

Then $h \in F_1$; thus

$$\left| \frac{f(x+h)-f(x)-hf_{(1)}(x)}{h^2} - \frac{f_{(2)}(x)}{2} \right| < \frac{\varepsilon}{2},$$

and $h \in F_2$, that is, $-h \in F$; thus

$$\left| \frac{f(x-h)-f(x)+hf_{(1)}(x)}{h^2} - \frac{f_{(2)}(x)}{2} \right| < \frac{\varepsilon}{2}.$$

Hence for $0 < h < \delta$, $h \in E$

$$\begin{aligned} & \left| \frac{f(x+h)-f(x-h)-2f(x)}{h^2} - f_{(2)}(x) \right| \\ & \leq \left| \frac{f(x+h)-f(x)-hf_{(1)}(x)}{h^2} - \frac{f_{(2)}(x)}{2} \right| \\ & \quad + \left| \frac{f(x-h)-f(x)+hf_{(1)}(x)}{h^2} - \frac{f_{(2)}(x)}{2} \right| \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus

$$\lim_{h \rightarrow 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f_{(2)}(x).$$

Corollary 4.19. Suppose f has a second approximate Peano derivative $f_{(2)}$ at each point in (a,b) , and $f_{(2)} \geq 0$ on (a,b) . Then $\overline{D}_2 f(x) \geq 0$ for each $x \in (a,b)$.

Proof. Follows immediately from Lemma 4.18.



In what follows we shall use without specific reference several well known results. We list these results here without proof.

Let g be a function defined on an interval J and let g have an ordinary derivative g' on J . If g is convex on J then g' is increasing on J .

Let g be a function defined on $[a,b]$. If g is monotone on (a,b) and has the Darboux property on $[a,b]$ then g is monotone on $[a,b]$.

Let g be a function defined on an interval J . If g is monotone on J and has the Darboux property on J , then g is continuous on J .

Let g be a function of Baire class one on $[a,b]$. Then every non-empty closed set F , contained in $[a,b]$, contains points of continuity of g relative to F .

Let g be a function defined on an interval J and assume g'_{ap} exists at each point in J . Then the following are true (see [4]):

- 1) g'_{ap} is a function of Baire class one on J ,
- 2) g'_{ap} has the Darboux property on J ,
- 3) if g'_{ap} is bounded above or below on J then

$$g'_{ap} = g' \text{ on } J.$$



Lemma 4.20. Let f be a function satisfying the following two conditions on $[a,b]$:

- (i) $f'_{ap}(x)$ exists for each x in $[a,b]$,
- (ii) $\overline{D}_2 f(x) \geq 0$ for each x in (a,b) .

Then f'_{ap} is continuous and increasing on $[a,b]$.

Proof. Let G be the set of all points x in $[a,b]$ with the property that there is a neighborhood of x on which f'_{ap} is bounded. Then G is an open set. Let $(c,d) \subseteq G$. Then a simple compactness argument shows f'_{ap} is bounded on $[c',d']$, where $c < c' < d' < d$. Hence $f'_{ap} = f'$ on $[c',d']$. Therefore it follows that $f'_{ap} = f'$ on (c,d) . Since f is continuous on (c,d) and $\overline{D}_2 f(x) \geq 0$ for each $x \in (c,d)$, f is convex on (c,d) by Lemma 4.17. Hence f'_{ap} is increasing on (c,d) . Moreover since f'_{ap} has the Darboux property on $[c,d]$ it follows that f'_{ap} is continuous and increasing on $[c,d]$. In particular, f'_{ap} is continuous and increasing in the closure of each component of G .

To complete the proof of the lemma we show $G = [a,b]$. Let $H = [a,b] - G$. From above H is a closed set having no isolated points. Suppose H is non-empty. Then H is a perfect set. Since f'_{ap} is a function of Baire class one on $[a,b]$ there exists an $x_0 \in H$ such that f'_{ap} is continuous at x_0 relative to H . Hence there exist numbers $M \geq 0$ and $\delta > 0$ so that $|f'_{ap}(x)| \leq M$ for each $x \in [x_0 - \delta, x_0 + \delta] \cap H$. Let

$$a' = \min\{x : x \in [x_0 - \delta, x_0] \cap H\},$$

$$b' = \max\{x : x \in [x_0, x_0 + \delta] \cap H\}.$$



Note that since H is perfect $a', b' \in H$ and $a' < b'$. Also, if $x \in [a', b'] \cap H$ then $|f'_{ap}(x)| \leq M$. Let $x \in (a', b') - H$. Then there exists a component of G , say (α, β) , where $\alpha, \beta \in H$, such that $x \in (\alpha, \beta) \subseteq (a', b')$. From the first part of the proof f'_{ap} is increasing on $[\alpha, \beta]$. Hence

$$-M \leq f'_{ap}(\alpha) \leq f'_{ap}(x) \leq f'_{ap}(\beta) \leq M.$$

Thus for each $x \in (a', b')$, $|f'_{ap}(x)| \leq M$ and so $(a', b') \subseteq G$.

First assume $x_0 \in (a', b')$. Then from above $x_0 \in G$, which contradicts x_0 being contained in H . Secondly, assume $x_0 = a'$. Then $(x_0 - \delta, x_0) \subseteq G$ and there exists a number M' , $M' \geq 0$, so that f'_{ap} is bounded by M' on $[x_0 - \delta, x_0]$. In the last paragraph we showed f'_{ap} was bounded by M on (x_0, b') . Thus f'_{ap} is bounded by the $\max\{M, M'\}$ on $(x_0 - \delta, b')$, and again $x_0 \in G$ which is a contradiction. In a similar fashion a contradiction is obtained, if $x_0 = b'$. Thus H must be empty. Therefore $G = [a, b]$ and the proof of the lemma is complete.

Theorem 4.21. Suppose f possesses a k^{th} approximate Peano derivative $f_{(k)}$ everywhere on an interval $[a, b]$.

(i) If $f_{(k)} > 0$ at each point in $[a, b]$, then $f_{(k-1)}$ is continuous and increasing on $[a, b]$.

(ii) If $f_{(k)}$ is bounded either above or below on $[a, b]$, then $f_{(k)} = f^{(k)}$ on $[a, b]$.

Proof. Consider first the case $k = 1$. Recall that $f_{(1)} = f'_{ap}$. Thus, if $f_{(1)} > 0$ on $[a, b]$ then $f_{(1)} = f'$ on $[a, b]$. Thus, $f_{(0)} = f$ is continuous and increasing on $[a, b]$. Moreover, if $f_{(1)}$ is bounded either above or below on $[a, b]$



then $f_{(1)} = f'$ on $[a,b]$. Thus the theorem holds when $k = 1$.

Secondly, consider $k = 2$. By Corollary 4.19 and Lemma 4.20 the proof of (i) is immediate. Turning to case (ii), it is no loss of generality to assume $f_{(2)} > 0$ on $[a,b]$. From (i) it follows that $f_{(1)}$ is increasing on $[a,b]$; hence $f_{(1)} = f'$ on $[a,b]$. By Corollary 4.8, $(f')_{(1)} = f_{(2)}$ on $[a,b]$. Moreover by assumption $(f')_{(1)} > 0$ on $[a,b]$; hence $(f')_{(1)} = (f')' = f_{(2)}$. Therefore $f_{(2)} = f_{(2)}$ on $[a,b]$.

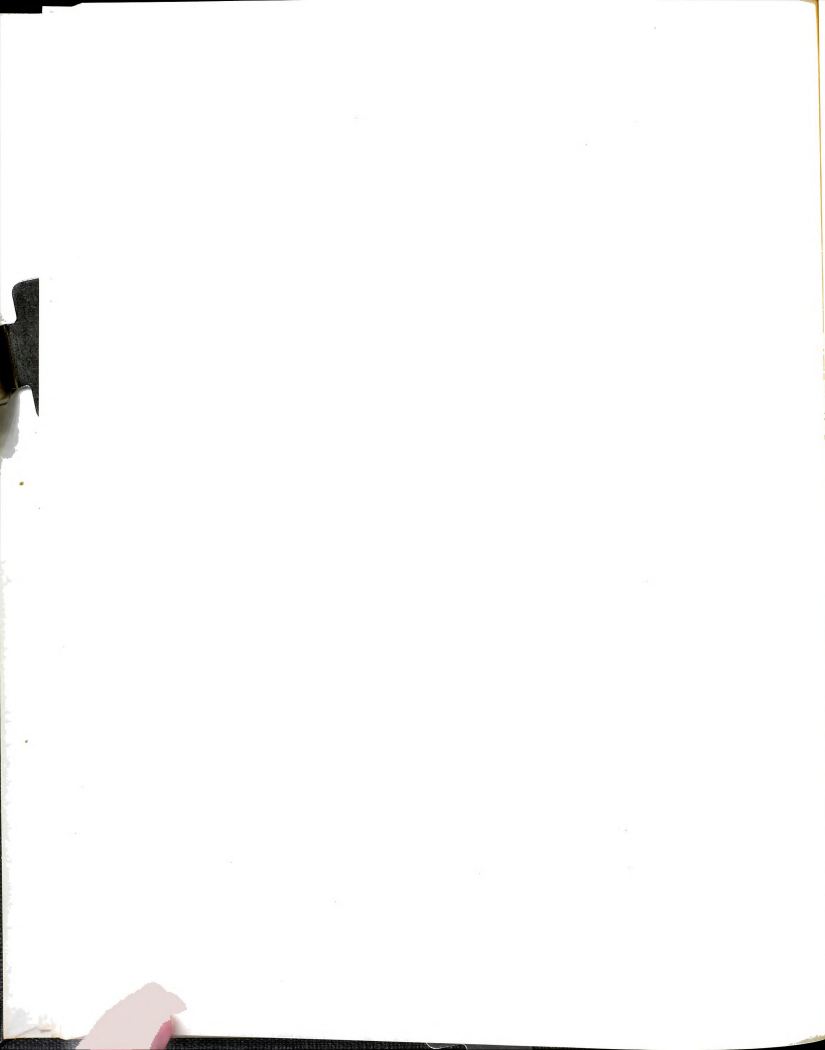
We may now assume that $k > 2$, and we can complete the proof by induction. We therefore assume the following:

If f possesses a $(k-1)^{\text{th}}$ approximate Peano derivative everywhere on an interval $[a,b]$, then for $1 \leq n \leq (k-1)$

(i) if $f_{(n)} > 0$ on $[a,b]$, then $f_{(n-1)}$ is continuous and increasing in $[a,b]$,

(ii) if $f_{(n)}$ is bounded either above or below on $[a,b]$, then $f_{(n)} = f_{(n)}$ on $[a,b]$.

Let $k > 2$ and assume $f_{(k)} > 0$ at each point in $[a,b]$. Let G be the set of all points x of $[a,b]$ with the property that there is a neighborhood of x on which $f_{(k-1)}$ is bounded. Obviously G is open. Let $(c,d) \subseteq G$. If $c < \alpha < \beta < d$, then a simple compactness argument shows $f_{(k-1)}$ is bounded on $[\alpha,\beta]$. By (ii) of the induction hypothesis, $f_{(k-1)} = f_{(k-1)}$ on $[\alpha,\beta]$ and therefore $f_{(k-2)} = f_{(k-2)}$ on $[\alpha,\beta]$. Moreover these relations hold on (c,d) . Thus $f_{(k-1)} = f'_{(k-2)}$ on (c,d) and $f_{(k-2)}$ is continuous on (c,d) . If $x \in (c,d)$ then there exists a measurable set E such that 0 is a point of



density of E and

$$f(x+h) = \sum_{n=0}^{k-1} \frac{h^n}{n!} f_{(n)}(x) + \frac{h^k}{k!} (f_{(k)}(x) + \epsilon(x, h))$$

where $E\text{-}\lim_{h \rightarrow 0} \epsilon(x, h) = 0$. From Lemma 4.9 for each sufficiently small non-zero $h \in E$ there is a θ , depending on h , between 0 and 1 such that

$$\begin{aligned} \frac{(k-2)!}{h^{k-2}} \{f(x+h) - \sum_{n=0}^{k-1} \frac{h^n}{n!} f_{(n)}(x)\} &= f_{(k-2)}(x+\theta h) - f_{(k-2)}(x) \\ &\quad - \theta h f_{(k-1)}(x). \end{aligned}$$

Hence

$$\begin{aligned} \frac{(k-2)!}{h^{k-2}} \left\{ \frac{h^k}{k!} (f_{(k)}(x) + \epsilon(x, h)) \right\} &= f_{(k-2)}(x+\theta h) - f_{(k-2)}(x) \\ &\quad - \theta h f_{(k-1)}(x). \end{aligned}$$

Thus

$$\begin{aligned} f_{(k-2)}(x+\theta h) &= f_{(k-2)}(x) + \theta h f_{(k-1)}(x) + \frac{h^2}{k(k-1)} \{f_{(k)}(x) + \epsilon(x, h)\} \\ &> f_{(k-2)}(x) + \theta h f_{(k-1)}(x) \end{aligned}$$

for all sufficiently small non-zero $h \in E$. Thus, it follows by Lemma 4.14 that $f_{(k-2)}$ is convex on (c, d) ; hence $f_{(k-1)}$ is increasing on (c, d) . Choose γ between c and d . Then $f_{(k-1)}$ is bounded below on $[\gamma, d]$. Applying (ii) of the induction hypothesis to the function $f_{(k-1)}$ on the interval $[\gamma, d]$, it follows that $f_{(k-1)} = f^{(k-1)}$ on $[\gamma, d]$. Now since $f_{(k-1)}$ is increasing on (γ, d) and has the Darboux property on $[\gamma, d]$ we have that $f_{(k-1)}$ is continuous and increasing on $[\gamma, d]$. Similarly, since $f_{(k-1)}$ is bounded above on $[c, \gamma]$, we deduce that $f_{(k-1)}$ is continuous and increasing on $[c, \gamma]$.



Thus it follows that $f_{(k-1)}$ is continuous and increasing on $[c,d]$. In particular, $f_{(k-1)}$ is increasing and continuous in the closure of each component of G .

To complete the proof of (i) we show $G = [a,b]$. Let $H = [a,b] - G$. From above H is a closed set having no isolated points. Suppose H is non-empty. Then H is a perfect set. Since $f_{(k-1)}$ is a function of Baire class one on $[a,b]$ (see [3]), the same type of argument given in the proof of Lemma 4.20 shows H is empty. Hence $G = [a,b]$ and the proof of (i) is complete.

Consider, finally, (ii) for $k > 2$. It is no loss of generality to suppose that $f_{(k)} > 0$ on $[a,b]$. By (i), $f_{(k-1)}$ is increasing on $[a,b]$ and by (ii) of the induction hypothesis $f_{(k-1)} = f^{(k-1)}$ on $[a,b]$. Thus it follows that $f_{(1)} = f'$ on $[a,b]$. We shall prove that $(f_{(1)})_{(k-1)} = f_{(k)}$ on $[a,b]$. It will then follow by the induction hypothesis (ii) applied to $f_{(1)}$ that in $[a,b]$, $f_{(k)} = (f_{(1)})_{(k-1)} = (f_{(1)})^{(k-1)} = f^{(k)}$.

It suffices to prove that in $[a,b)$ the $(k-1)^{\text{th}}$ approximate Peano derivative of $f_{(1)}$ on the right, equals $f_{(k)}$. For, applying Corollary 4.7, it will follow that in $(a,b]$ the $(k-1)^{\text{th}}$ approximate Peano derivative of $f_{(1)}$ on the left equals $f_{(k)}$. Without altering $f_{(k)}$ by adding to f a suitable polynomial of degree less than n , we may assume that $f_{(j)}(a) = 0$, for $j = 2, 3, \dots, (k-1)$. Note, since $f^{(k-1)}(a) = 0$ and $f^{(k-1)}$ is increasing on $[a,b]$, $f^{(k-1)} \geq 0$ on $[a,b]$. Now for each h , $0 < h < (b-a)$, there exists by the extended mean value theorem a number ξ , $a < \xi < a+h$



such that

$$f^{(2)}(a+h) = \frac{h^{k-3}}{(k-3)!} f^{(k-1)}(\xi).$$

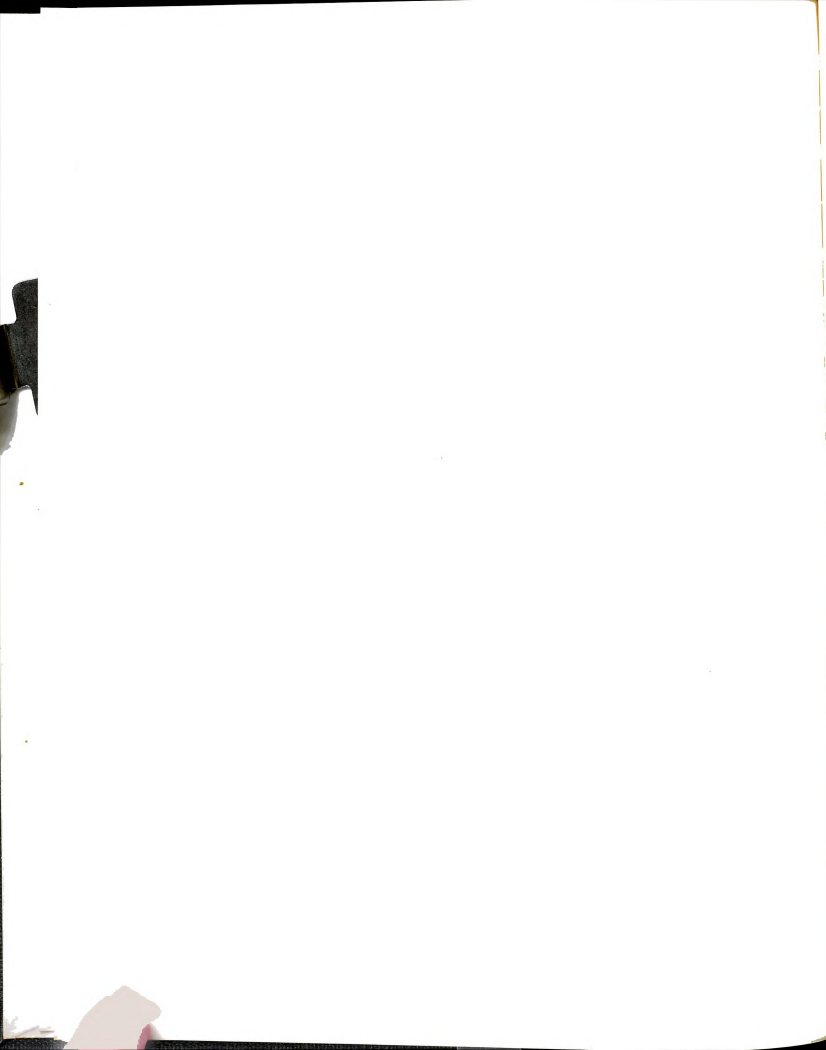
Hence $f^{(2)} \geq 0$ in (a,b) . Thus $f_{(1)}$ is increasing on $[a,b]$. By Lemma 4.1, $(f_{(1)})_{(k-1)}(a) = f_{(k)}(a)$. Since a may be replaced throughout by any $\alpha \in [a,b)$ the proof of the theorem is complete.

In [6], C. J. Neugebauer proved that if g is a function of Baire class one on an interval J , then g has the Darboux property on J if and only if for each real number λ , the sets $E^\lambda = \{x : g(x) \geq \lambda\}$ and $E_\lambda = \{x : g(x) \leq \lambda\}$ have closed connected components.

We thus have the following corollary to the last theorem.

Corollary 4.22. If f possesses a k^{th} approximate Peano derivative at each point of an interval $[a,b]$, then $f_{(k)}$ has the Darboux property on $[a,b]$.

Proof. Since $f_{(k)}$ is of Baire class one on $[a,b]$ (see [3]), in order to show $f_{(k)}$ has the Darboux property we need only show that the connected components of the sets $E^\lambda = \{x : f_{(k)}(x) \geq \lambda\}$ and $E_\lambda = \{x : f_{(k)}(x) \leq \lambda\}$ are closed for every real number λ . So suppose $f_{(k)}(x) \geq \lambda$ for all x in the interval (α, β) . We must show that $f_{(k)}(\alpha) \geq \lambda$ and $f_{(k)}(\beta) \geq \lambda$. Now since $f_{(k)}$ is bounded below on (α, β) , $f_{(k)}$ is bounded below on $[\alpha, \beta]$. Thus by Theorem 4.21, $f_{(k)} = f^{(k)}$ on $[\alpha, \beta]$. Since $f^{(k)}$ has the Darboux property on $[\alpha, \beta]$,



$f^{(k)}(\alpha) \geq \lambda$ and $f^{(k)}(\beta) \geq \lambda$. Hence, $f_{(k)}(\alpha) \geq \lambda$ and also $f_{(k)}(\beta) \geq \lambda$. Thus the connected components of E^λ are closed. Similarly the connected components of E_λ are closed. Thus, $f_{(k)}$ has the Darboux property on $[a,b]$.

In [10], C. E. Weil proved that a function g of Baire class one has the Denjoy property on an interval J if, for every subinterval L of J on which g is bounded either above or below, g restricted to L has the Denjoy property.

Using this result along with the facts that a k^{th} approximate Peano derivative is a function of Baire class one and that an ordinary k^{th} derivative has the Denjoy property, we also have the following corollary to the last theorem.

Corollary 4.23. If f possesses a k^{th} approximate Peano derivative at each point of an interval $[a,b]$, then $f_{(k)}$ has the Denjoy property on $[a,b]$.



CHAPTER V

PROPERTY Z FOR APPROXIMATE PEANO DERIVATIVES

In this our final chapter we prove property Z for the k^{th} approximate Peano derivatives. To prove that every k^{th} approximate Peano derivative has property Z we first prove a lemma, which is a modification of a lemma proved by C. E. Weil [11]. Before proving the lemma some facts are established that will be needed.

Let A be a measurable set and $[c,d]$ a closed interval. Define a function on $[c,d]$ as follows:

$$F(x) = m(A \cap [c,x]) = \int_c^x \chi(s)ds$$

where χ denotes the characteristic function of A . Then F is absolutely continuous and $F'(x) = \chi(x)$ a.e. on $[c,d]$. Moreover, almost everywhere

$$\frac{d}{dx} F^{j+1} = (j+1)F^j \chi \leq (j+1)F^j.$$

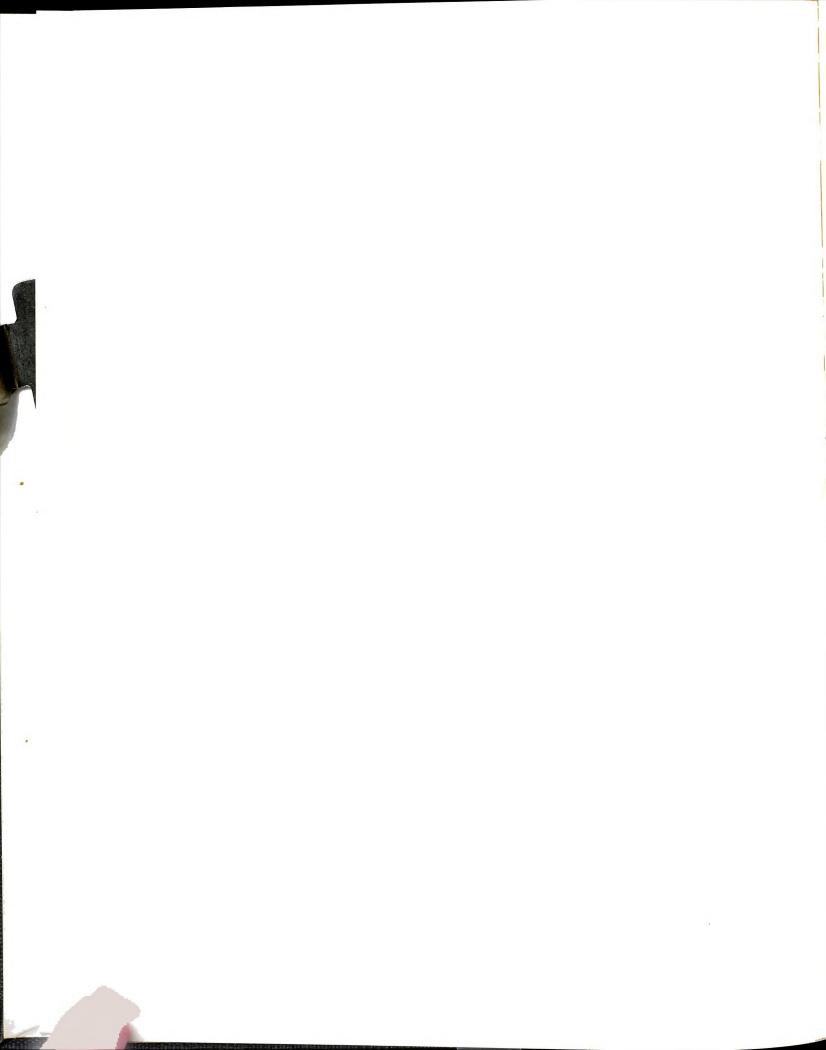
Consequently,

$$\int_c^d (m(A \cap [c,s]))^j ds \geq \frac{1}{(j+1)} (m(A \cap [c,d]))^{j+1}.$$

Likewise if we define on $[c,d]$

$$G(x) = m(A \cap [x,d]) = \int_x^d \chi(s)ds,$$

then G is absolutely continuous and $G'(x) = -\chi(x)$ a.e. on $[c,d]$. In addition almost everywhere



$$\frac{d}{dx} G^{j+1} = (j+1)G^j(-x) \geq -(j+1)G^j.$$

Therefore,

$$\int_c^d -(m(A \cap [s,d]))^j ds \leq \frac{-1}{(j+1)} (m(A \cap [c,d]))^{j+1}.$$

Lemma 5.1. Suppose f is a function whose k^{th} derivative exists and is nonnegative on the interval $[a,b]$, and let

$$A = \{x \in [a,b] : f^{(k)}(x) \geq \epsilon\}$$

where ϵ is a fixed positive number. Then there exists a partition $\{a = t_0 < t_1 < \dots < t_\ell = b\}$ of the interval $[a,b]$ with $\ell \leq 2^k$ and such that for each $i = 1, 2, \dots, \ell$, with $x, y \in [t_{i-1}, t_i]$ and $x \leq y$

$$|f(y) - f(x)| \geq \left(\frac{\epsilon}{k!}\right) (m(A \cap [x,y]))^k.$$

Proof. It will be shown by induction that for each integer $j = 1, 2, \dots, k$, there is a partition of $[a,b]$

$$\{a = t_{0,j} < t_{1,j} < \dots < t_{\ell(j),j} = b\}$$

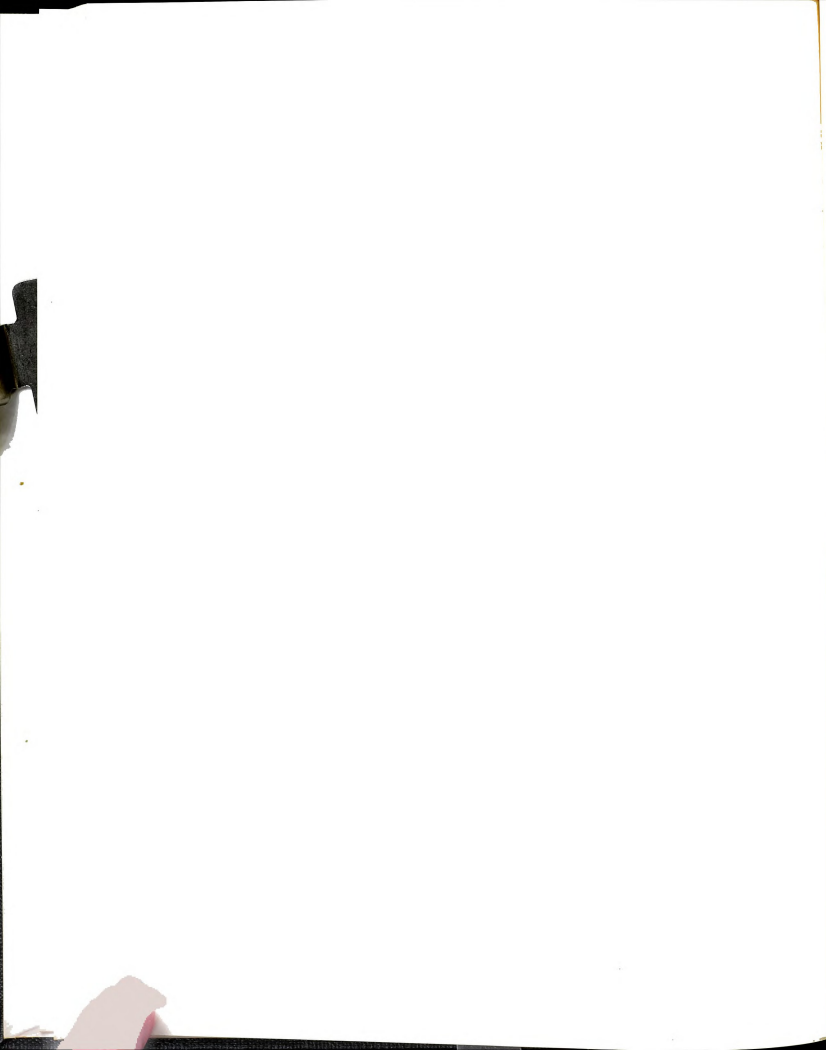
with $\ell(j) \leq 2^j$ and such that for each $i = 1, 2, \dots, \ell(j)$, one of the following holds for each x, y in $[t_{i-1,j}, t_{i,j}]$ with $x \leq y$:

$$1(j): f^{(k-j)}(y) - f^{(k-j)}(x) \geq \left(\frac{\epsilon}{j!}\right) (m(A \cap [x,y]))^j$$

$$\text{and } f^{(k-j)}(x) \geq 0.$$

$$2(j): f^{(k-j)}(y) - f^{(k-j)}(x) \geq \left(\frac{\epsilon}{j!}\right) (m(A \cap [x,y]))^j$$

$$\text{and } f^{(k-j)}(y) \leq 0.$$



$$3(j): f^{(k-j)}(x) - f^{(k-j)}(y) \geq \left(\frac{\varepsilon}{j!}\right)(m(A \cap [x,y]))^j$$

$$\text{and } f^{(k-j)}(x) \leq 0.$$

$$4(j): f^{(k-j)}(x) - f^{(k-j)}(y) \geq \left(\frac{\varepsilon}{j!}\right)(m(A \cap [x,y]))^j$$

$$\text{and } f^{(k-j)}(y) \geq 0.$$

The desired partition is then the one corresponding to $j = k$ and the desired inequality is obtained by taking absolute values, where, of course $f^{(0)} = f$.

The induction assertion is first proved for $j = 1$. Since $f^{(k)}(x) \geq 0$ for all $x \in [a,b]$, $f^{(k-1)}$ is nondecreasing and continuous on $[a,b]$. Let $|f^{(k-1)}|$ attain its minimum value on $[a,b]$ at $t \in [a,b]$. There are three cases to consider: $t = a$, $t = b$, and $a < t < b$.

If $t = a$, then since $f^{(k-1)}$ is nondecreasing on $[a,b]$, $f^{(k-1)}(a) \geq 0$ and for each $x, y \in [a,b]$ with $x \leq y$, $f^{(k-1)}(x) \geq 0$ and

$$f^{(k-1)}(y) - f^{(k-1)}(x) = \int_x^y f^{(k)}(s) ds$$

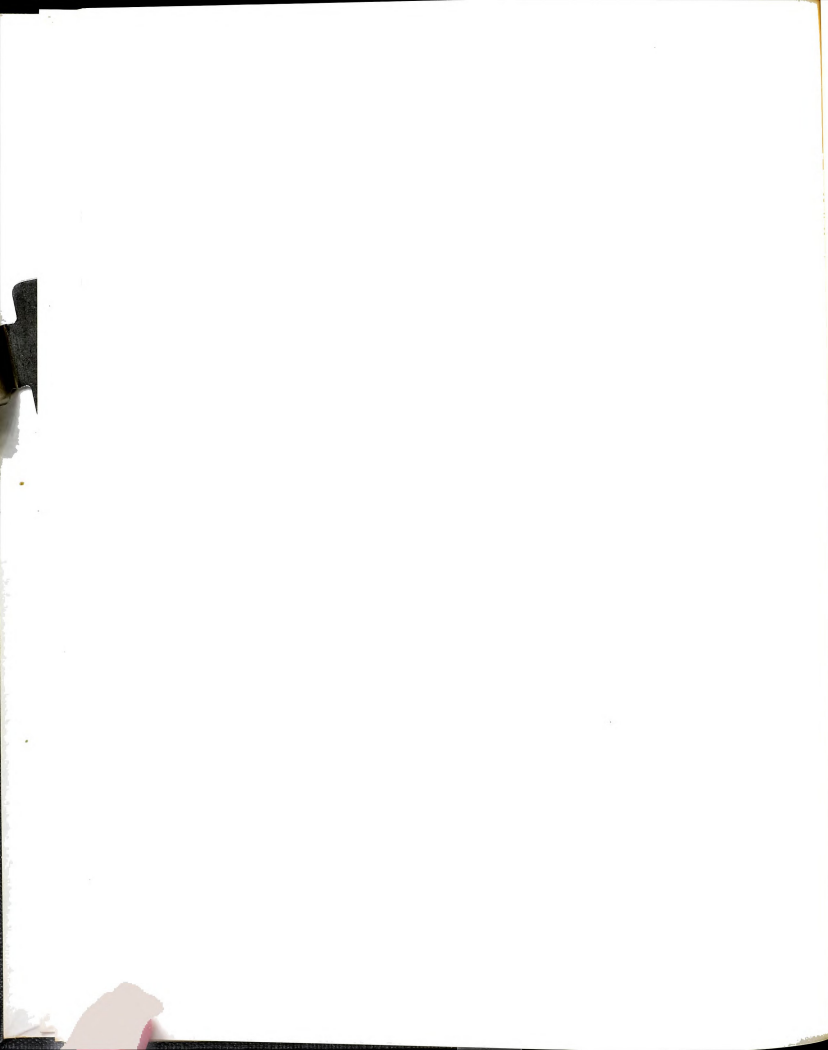
$$\geq \varepsilon m(A \cap [x,y]).$$

In this case 1(1) holds for each $x, y \in [a,b]$ with $x \leq y$ if we take $t_0 = a$ and $t_1 = b$.

If $t = b$, then since $f^{(k-1)}$ is nondecreasing on $[a,b]$, $f^{(k-1)}(b) \leq 0$ and for each $x, y \in [a,b]$ with $x \leq y$, $f^{(k-1)}(y) \leq 0$ and

$$f^{(k-1)}(y) - f^{(k-1)}(x) = \int_x^y f^{(k)}(s) ds$$

$$\geq \varepsilon m(A \cap [x,y]).$$



In this case 2(1) holds for each $x, y \in [a, b]$ with $x \leq y$ if we take $t_0 = a$ and $t_1 = b$.

If $a < t < b$, then since $f^{(k-1)}$ is nondecreasing on $[a, b]$, $f^{(k-1)}(t) = 0$. Arguments like the two just presented show that 2(1) holds on $[a, t]$ and 1(1) holds on $[t, b]$. In this case take $t_0 = a$, $t_1 = t$, and $t_2 = b$. The induction assertion is established for $j = 1$. (Note that only cases 1 and 2 occurred. In the induction step it will be seen that 3 and 4 arise from 2.)

Assume the assertion true for j , and let i be an integer between 1 and $\ell(j)$. As a notational convenience denote the interval $[t_{i-1,j}, t_{i,j}]$ by $[c, d]$ and the function $f^{(k-(j+1))}$ by g so that $g' = f^{(k-j)}$. It must be shown that if 1(j), 2(j), 3(j), or 4(j) holds on $[c, d]$, then $[c, d]$ can be divided into no more than two subintervals so that on each subinterval 1($j+1$), 2($j+1$), 3($j+1$), or 4($j+1$) holds.

First assume that 1(j) holds for each $x, y \in [c, d]$ with $x \leq y$. Then

$$\begin{aligned} g'(y) &\geq \left(\frac{\varepsilon}{j!}\right)(m(A \cap [x, y]))^j + g'(x) \\ &\geq \left(\frac{\varepsilon}{j!}\right)(m(A \cap [x, y]))^j \\ &\geq 0 \end{aligned}$$

since $g'(x) \geq 0$. Hence g is nondecreasing and continuous on $[c, d]$. Let $|g|$ attain its minimum value on $[c, d]$ at $t \in [c, d]$. There are as before three cases to consider: $t = c$, $t = d$, and $c < t < d$.



If $t = c$, then $g(c) \geq 0$ and for each $x, y \in [c, d]$ with $x \leq y$, $g(x) \geq 0$ and

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\geq \int_x^y \left(\frac{\epsilon}{j!} \right) (m(A \cap [x, s]))^j ds \\ &\geq \left(\frac{\epsilon}{(j+1)!} \right) (m(A \cap [x, y]))^{j+1}. \end{aligned}$$

(The last step was made by using one of the facts established previous to the lemma.) Thus, $1(j+1)$ holds for each x, y in $[c, d]$ with $x \leq y$.

If $t = d$, then $g(d) \leq 0$ and for each $x, y \in [c, d]$ with $x \leq y$, $g(y) \leq 0$ and

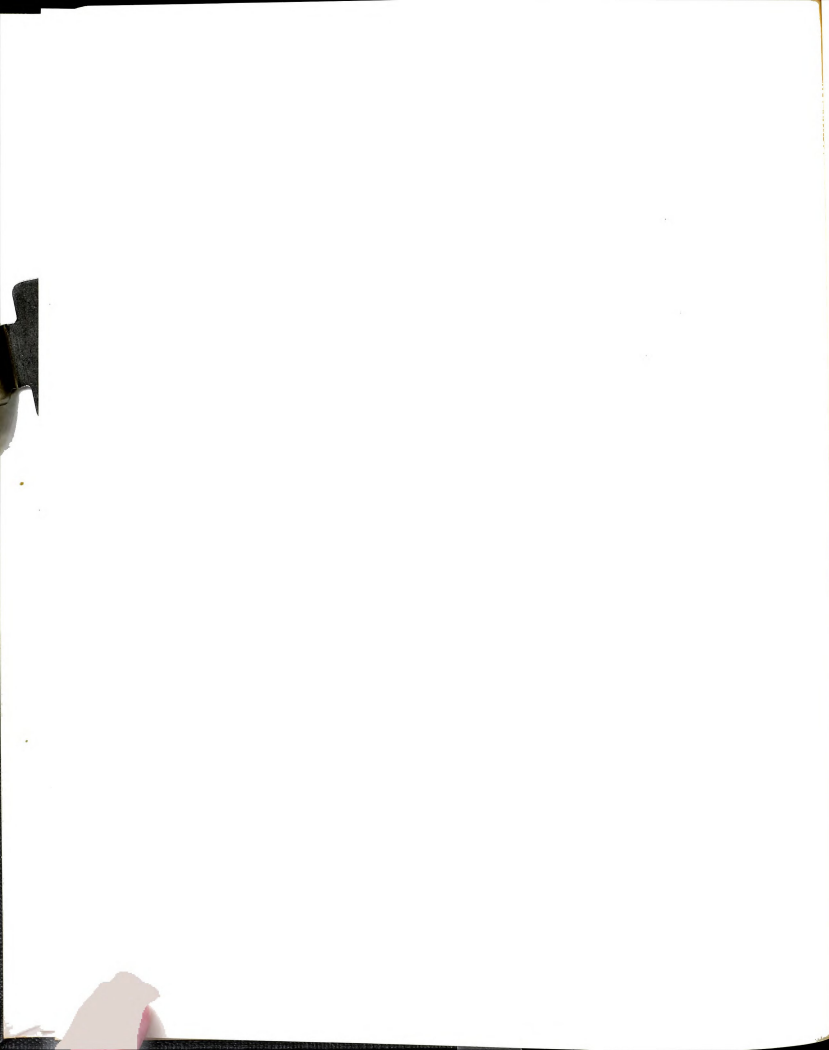
$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\geq \int_x^y \left(\frac{\epsilon}{j!} \right) (m(A \cap [x, s]))^j ds \\ &\geq \left(\frac{\epsilon}{(j+1)!} \right) (m(A \cap [x, y]))^{j+1}. \end{aligned}$$

Thus, $2(j+1)$ holds for each x, y in $[c, d]$ with $x \leq y$.

If $c < t < d$, then $g(t) = 0$ and it can be shown by arguments like the two just given that $2(j+1)$ holds for each $x, y \in [c, t]$ with $x \leq y$, and that $1(j+1)$ holds for each $x, y \in [t, d]$ with $x \leq y$.

Second, assume that $2(j)$ holds for all $x, y \in [c, d]$ with $x \leq y$. Then

$$\begin{aligned} g'(x) &\leq \left(\frac{-\epsilon}{j!} \right) (m(A \cap [x, y]))^j + g'(y) \\ &\leq \left(\frac{-\epsilon}{j!} \right) (m(A \cap [x, y]))^j \\ &\leq 0 \end{aligned}$$



since $g'(y) \leq 0$. Thus, g is nonincreasing and continuous on $[c,d]$. Let $|g|$ attain its minimum value on $[c,d]$ at t . There are three cases.

If $t = c$, then $g(c) \leq 0$ and for each $x, y \in [c,d]$ with $x \leq y$, $g(x) \leq 0$ and

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\leq \int_x^y \left(\frac{-\epsilon}{j!} \right) (m(A \cap [s,y]))^j ds \\ &\leq \left(\frac{-\epsilon}{(j+1)!} \right) (m(A \cap [x,y]))^{j+1}. \end{aligned}$$

Thus,

$$g(x) - g(y) \geq \left(\frac{\epsilon}{(j+1)!} \right) (m(A \cap [x,y]))^{j+1}.$$

Therefore, 3(j+1) holds for each x, y in $[c,d]$ with $x \leq y$.

If $t = d$, then $g(d) \geq 0$ and for each $x, y \in [c,d]$ with $x \leq y$, $g(y) \geq 0$ and

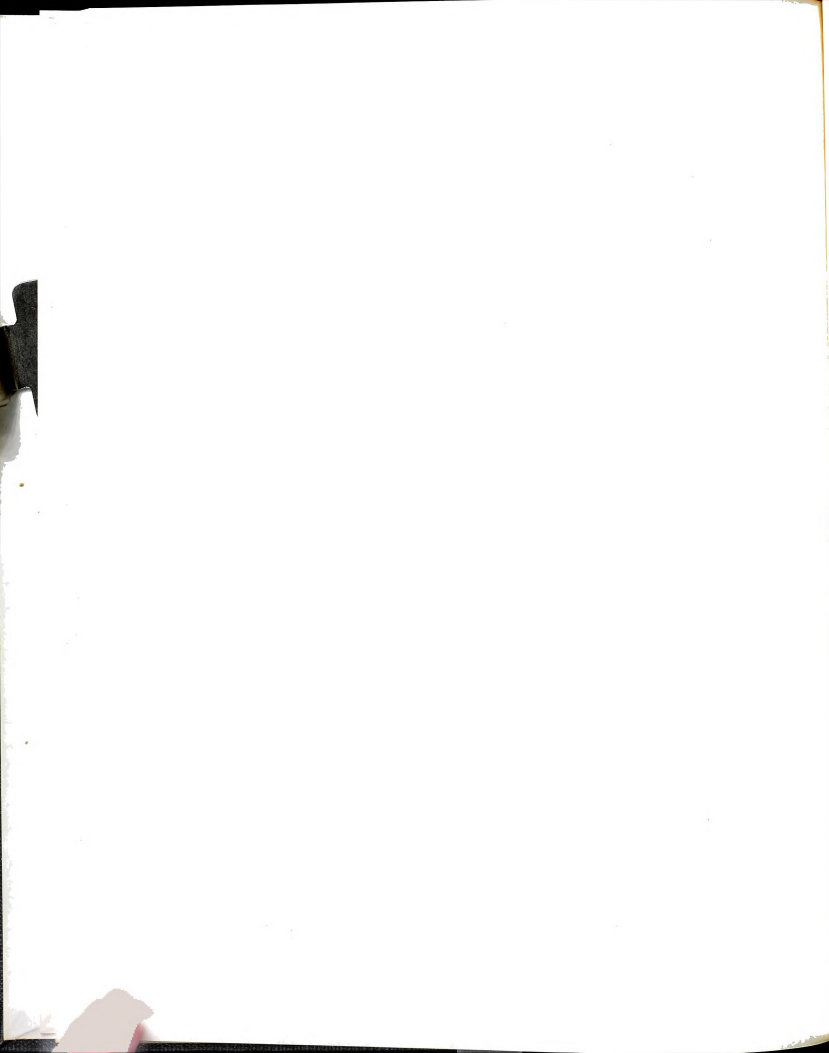
$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\leq \int_x^y \left(\frac{-\epsilon}{j!} \right) (m(A \cap [s,y]))^j ds \\ &\leq \left(\frac{-\epsilon}{(j+1)!} \right) (m(A \cap [x,y]))^{j+1}. \end{aligned}$$

Thus,

$$g(x) - g(y) \geq \left(\frac{\epsilon}{(j+1)!} \right) (m(A \cap [x,y]))^{j+1}.$$

Therefore, 4(j+1) holds for each x, y in $[c,d]$ with $x \leq y$.

If $c < t < d$, then $g(t) = 0$ and using the same reasoning as above it can be established that 4(j+1) holds for all $x, y \in [c,t]$ with $x \leq y$, while 3(j+1) holds for all $x, y \in [t,d]$ with $x \leq y$. (It should be observed that cases 3 and 4 arise from 2 and thus are essential.)



Third assume that 3(j) holds for all $x, y \in [c, d]$ with $x \leq y$. Then

$$\begin{aligned} g'(y) &\leq \left(\frac{-\varepsilon}{j!}\right)(m(A \cap [x, y]))^j + g'(x) \\ &\leq \left(\frac{-\varepsilon}{j!}\right)(m(A \cap [x, y]))^j \\ &\leq 0 \end{aligned}$$

since $g'(x) \leq 0$. Hence g is nonincreasing and continuous on $[c, d]$. Let $|g|$ attain its minimum value on $[c, d]$ at t . If $t = c$, then $g(c) \leq 0$ and for all $x, y \in [c, d]$ with $x \leq y$, $g(x) \leq 0$ and

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\leq \int_x^y \left(\frac{-\varepsilon}{j!}\right)(m(A \cap [x, s]))^j ds \\ &\leq \left(\frac{-\varepsilon}{(j+1)!}\right)(m(A \cap [x, y]))^{j+1}. \end{aligned}$$

Thus,

$$g(x) - g(y) \geq \left(\frac{\varepsilon}{(j+1)!}\right)(m(A \cap [x, y]))^{j+1}.$$

Therefore, 3(j+1) holds for each $x, y \in [c, d]$ with $x \leq y$.

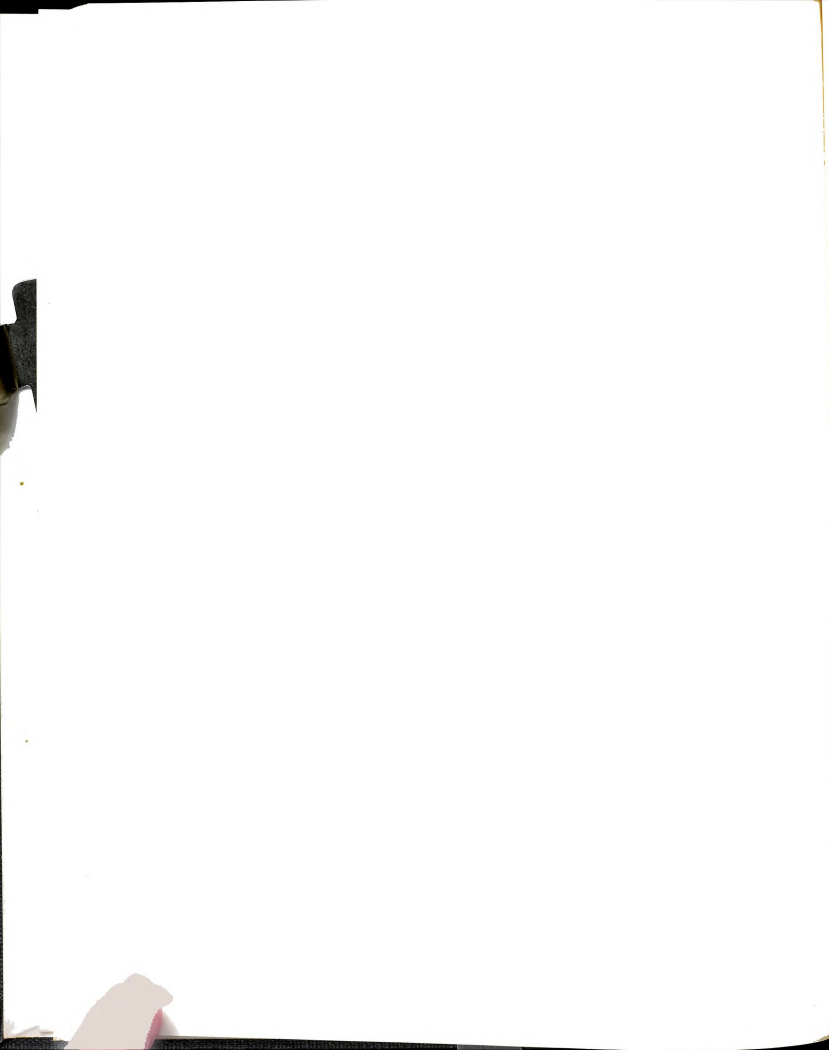
If $t = d$, then $g(d) \geq 0$ and for all $x, y \in [c, d]$ with $x \leq y$, $g(y) \geq 0$ and

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\leq \int_x^y \left(\frac{-\varepsilon}{j!}\right)(m(A \cap [x, s]))^j ds \\ &\leq \left(\frac{-\varepsilon}{(j+1)!}\right)(m(A \cap [x, y]))^{j+1}. \end{aligned}$$

Thus,

$$g(x) - g(y) \geq \left(\frac{\varepsilon}{(j+1)!}\right)(m(A \cap [x, y]))^{j+1}.$$

Therefore, 4(j+1) holds for each $x, y \in [c, d]$ with $x \leq y$.



If $c < t < d$, then $g(t) = 0$. By using the same reasoning as above it can be established that $4(j+1)$ holds on $[c,t]$ and $3(j+1)$ on $[t,d]$.

Finally, assume $4(j)$ holds for all x,y in $[c,d]$ with $x \leq y$. Then since $g'(y) \geq 0$ it follows that $g'(x) \geq 0$. Hence g is nondecreasing and continuous on $[c,d]$. Now let $|g|$ attain its minimum value on $[c,d]$ at t . If $t = c$, then $g(c) \geq 0$ and for all $x,y \in [c,d]$ with $x \leq y$, $g(x) \geq 0$ and

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\geq \int_x^y \left(\frac{\epsilon}{j!}\right) (m(A \cap [s,y]))^j ds \\ &\geq \left(\frac{\epsilon}{(j+1)!}\right) (m(A \cap [x,y]))^{j+1}. \end{aligned}$$

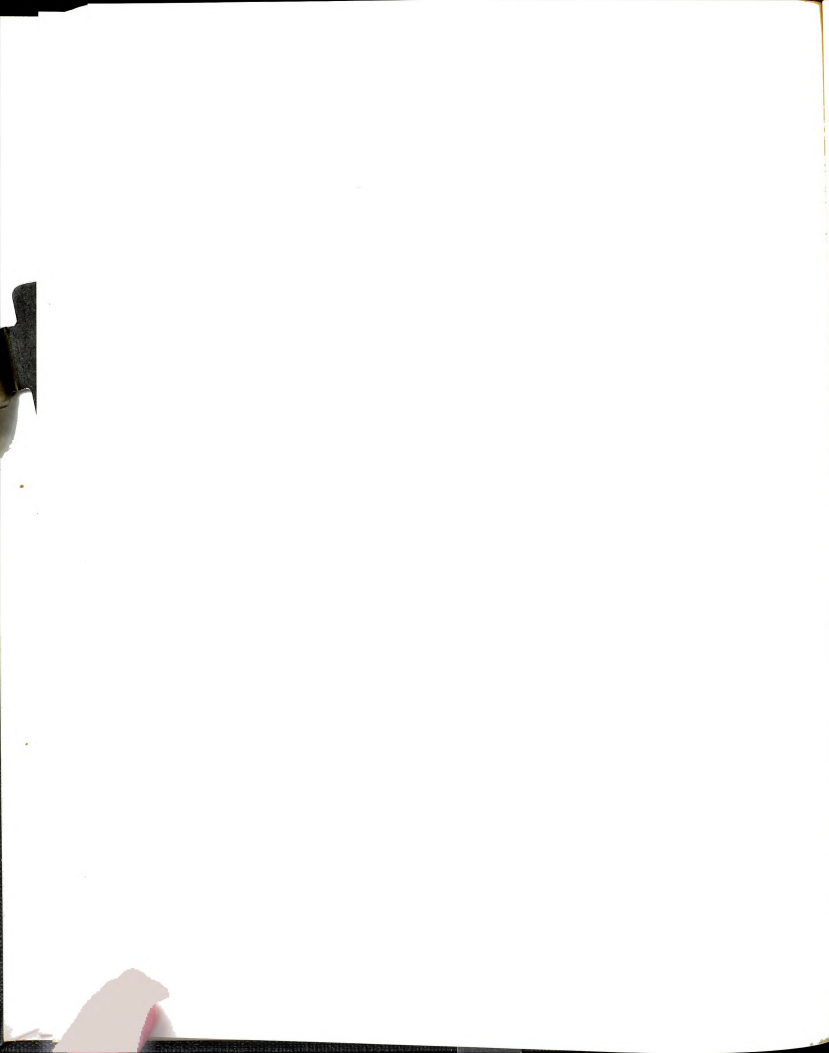
Thus, $1(j+1)$ holds for each $x,y \in [c,d]$ with $x \leq y$.

If $t = d$, then $g(d) \leq 0$ and for all $x,y \in [c,d]$ $x \leq y$, $g(y) \leq 0$ and

$$\begin{aligned} g(y) - g(x) &= \int_x^y g'(s) ds \\ &\geq \int_x^y \left(\frac{\epsilon}{j!}\right) (m(A \cap [s,y]))^j ds \\ &\geq \left(\frac{\epsilon}{(j+1)!}\right) (m(A \cap [x,y]))^{j+1}. \end{aligned}$$

Thus, $2(j+1)$ holds for each $x,y \in [c,d]$ with $x \leq y$.

If $c < t < d$, then $g(t) = 0$, and proceeding as has already been demonstrated it can be established that $2(j+1)$ holds on $[c,t]$ and $1(j+1)$ on $[t,d]$. This completes the proof of the lemma.



Theorem 5.2. If f has a k^{th} approximate Peano derivative $f_{(k)}$ everywhere on $[a,b]$ then $f_{(k)}$ has property Z on $[a,b]$.

Proof. Let x be contained in $[a,b]$ and $\epsilon > 0$. It suffices to show that if given an $\eta > 0$ there exists a $\delta > 0$ such that if the closed interval $[\alpha,\beta]$ is contained in $(x-\delta, x+\delta) \cap [a,b]$, $x \notin [\alpha,\beta]$ and $f_{(k)}(y) \geq f_{(k)}(x)$ for each $y \in [\alpha,\beta]$ or $f_{(k)}(y) \leq f_{(k)}(x)$ for each $y \in [\alpha,\beta]$ then

$$(5.3) \quad \frac{m\{y \in [\alpha,\beta] : |f_{(k)}(y) - f_{(k)}(x)| \geq \epsilon\}}{(\beta - \alpha) + \text{dist.}(x, [\alpha,\beta])} < \eta.$$

Let $\eta > 0$ be given and set

$$g(y) = f(y) - \sum_{n=0}^k \frac{(y-x)^n}{n!} f_{(n)}(x).$$

Then $g_{(k)}(y)$ exists for each $y \in [a,b]$ and furthermore

$$g_{(k)}(y) = f_{(k)}(y) - f_{(k)}(x).$$

From the existence of $f_{(k)}$, there exists a $\delta > 0$ and a measurable set $E \subseteq [a,b]$ such that x is a point of density of E , and so that

$$(5.4) \quad |g(y)| \leq \frac{\epsilon(\eta/2)^k}{k! \cdot 2^{k(k+1)}} \cdot |y-x|^k$$

for $|y-x| < \delta$ and $y \in E$,

$$(5.5) \quad m(J \cap E^c) \leq m(J) \cdot \frac{\eta}{2}$$

for J an interval contained in $(x-\delta, x+\delta) \cap [a,b]$ and $x \in J$, where $E^c = [a,b] - E$.

Let $[\alpha,\beta]$ be a closed interval contained in $(x-\delta, x+\delta) \cap [a,b]$ such that $x \notin [\alpha,\beta]$. First assume that



$f_{(k)}(y) \geq f_{(k)}(x)$ for each $y \in [\alpha, \beta]$. By Theorem 4.21, $f_{(k)} = f^{(k)}$ on $[\alpha, \beta]$. Applying Lemma 5.1 to the function g , which satisfies

$$g^{(k)}(y) = f_{(k)}(y) - f_{(k)}(x)$$

for each $y \in [\alpha, \beta]$, there exists a partition of $[\alpha, \beta]$

$$\{\alpha = t_0 < t_1 < \dots < t_\ell = \beta\}$$

with $\ell \leq 2^k$ such that for each $i = 1, 2, \dots, \ell$, and each $s, w \in [t_{i-1}, t_i]$ with $s \leq w$

$$(5.6) \quad |g(w) - g(s)| \geq \frac{\varepsilon}{k!} (m(A \cap [s, w]))^k$$

where $A = \{y \in [\alpha, \beta] : |g^{(k)}(y)| = |f_{(k)}(y) - f_{(k)}(x)| \geq \varepsilon\}$.

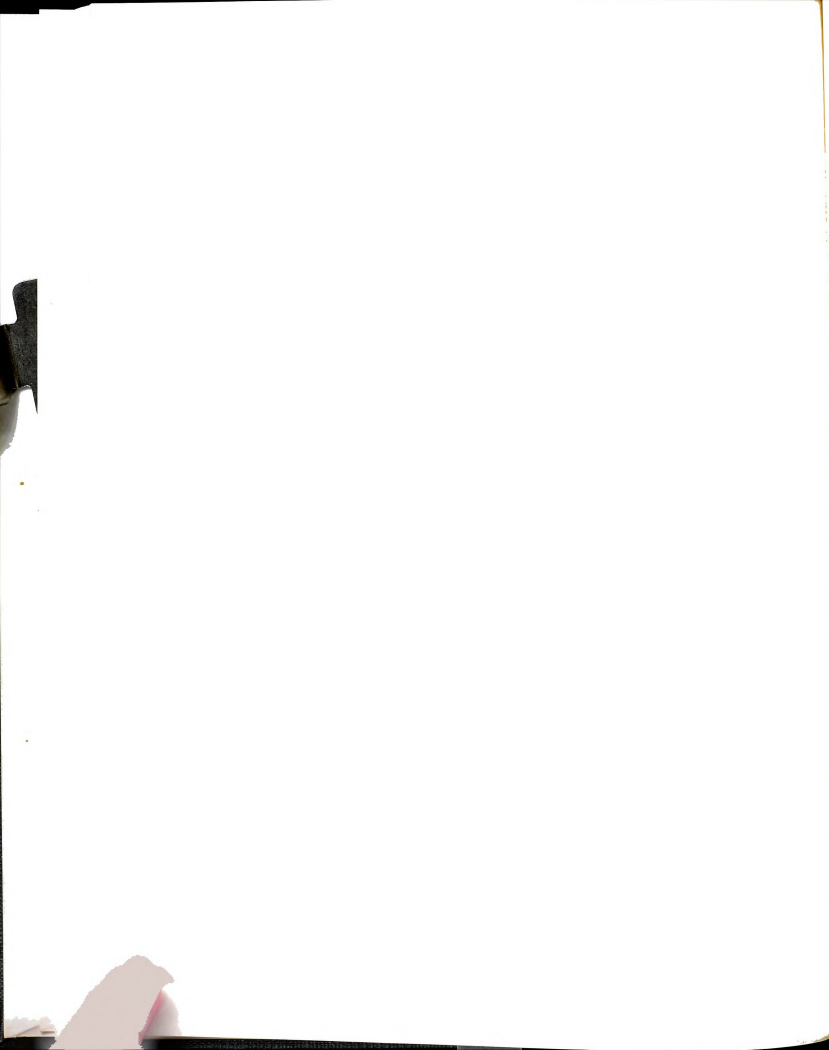
If $f_{(k)}(y) \leq f_{(k)}(x)$ for each $y \in [\alpha, \beta]$, then consider $-g$ and apply Lemma 5.1 to obtain precisely the same inequality (5.6).

We first obtain an estimate for $m(A \cap E)$. For this purpose assume $[t_{i-1}, t_i] \cap E \neq \emptyset$. Let

$$t_{i-1} \leq t'_i \leq t''_i \leq t_i$$

with $t'_i, t''_i \in E$. Then by (5.6) and (5.4)

$$\begin{aligned} m(A \cap [t'_i, t''_i]) &\leq \left(\frac{k!}{\varepsilon}\right)^{1/k} |g(t''_i) - g(t'_i)|^{1/k} \\ &\leq \left(\frac{k!}{\varepsilon}\right)^{1/k} (|g(t''_i)|^{1/k} + |g(t'_i)|^{1/k}) \\ &\leq \left(\frac{k!}{\varepsilon}\right)^{1/k} \left(\varepsilon \left(\frac{\eta}{2}\right)^k / k! \cdot 2^{k(k+1)}\right)^{1/k} (|t''_i - x| + |t'_i - x|) \\ &\leq \frac{\eta}{2 \cdot 2^k} [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)] \\ &\leq \frac{\eta}{2\ell} [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$



If

$$s'_i = \inf\{t'_i : t'_i \in [t_{i-1}, t_i] \cap E\}$$

and

$$s''_i = \sup\{t''_i : t''_i \in [t_{i-1}, t_i] \cap E\}$$

then it follows from the above inequality that

$$\begin{aligned} m(A \cap E \cap [t_{i-1}, t_i]) &= m(A \cap E \cap [s'_i, s''_i]) \\ &\leq m(A \cap [s'_i, s''_i]) \\ &\leq \frac{\eta}{2\ell} [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$

Clearly the same estimate holds if $[t_{i-1}, t_i] \cap E = \emptyset$. Hence

$$\begin{aligned} (5.7) \quad m(A \cap E) &= m(A \cap E \cap (\cup_{i=1}^{\ell} [t_{i-1}, t_i])) \\ &= \sum_{i=1}^{\ell} m(A \cap E \cap [t_{i-1}, t_i]) \\ &\leq \sum_{i=1}^{\ell} \frac{\eta}{2\ell} [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)] \\ &\leq \frac{\eta}{2} [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)]. \end{aligned}$$

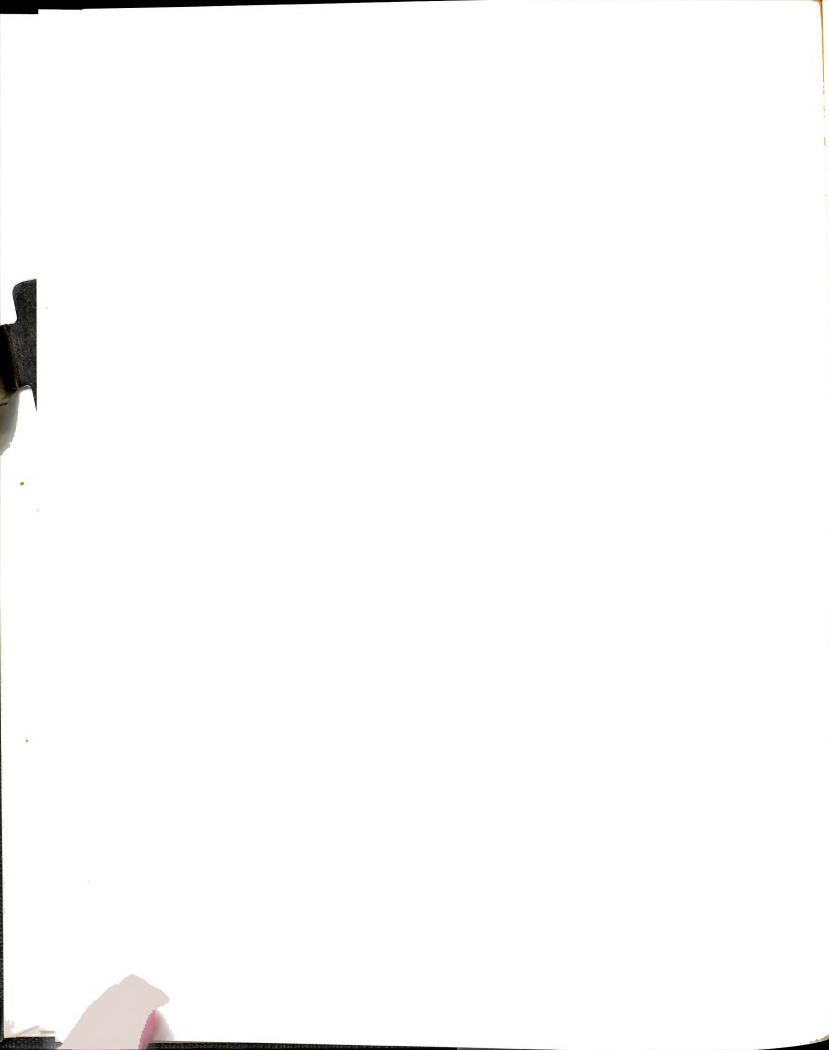
Secondly, we obtain an estimate of $m(A \cap E^c)$. Let J be the smallest closed interval in $[a, b]$ containing both x and $[\alpha, \beta]$. Using (5.5) we have the following estimate

$$(5.8) \quad m(A \cap E^c) \leq m(J \cap E^c) \leq m(J) \cdot \frac{\eta}{2}.$$

Note that $m(J) = \text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)$.

Therefore by (5.7) and (5.8)

$$\begin{aligned} m(A) &= m(A \cap E) + m(A \cap E^c) \\ &\leq m(A \cap E) + m(J \cap E^c) \\ &\leq [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)] \cdot \frac{\eta}{2} + m(J) \cdot \frac{\eta}{2} \\ &= [\text{dist.}(x, [\alpha, \beta]) + (\beta - \alpha)] \cdot \eta, \end{aligned}$$

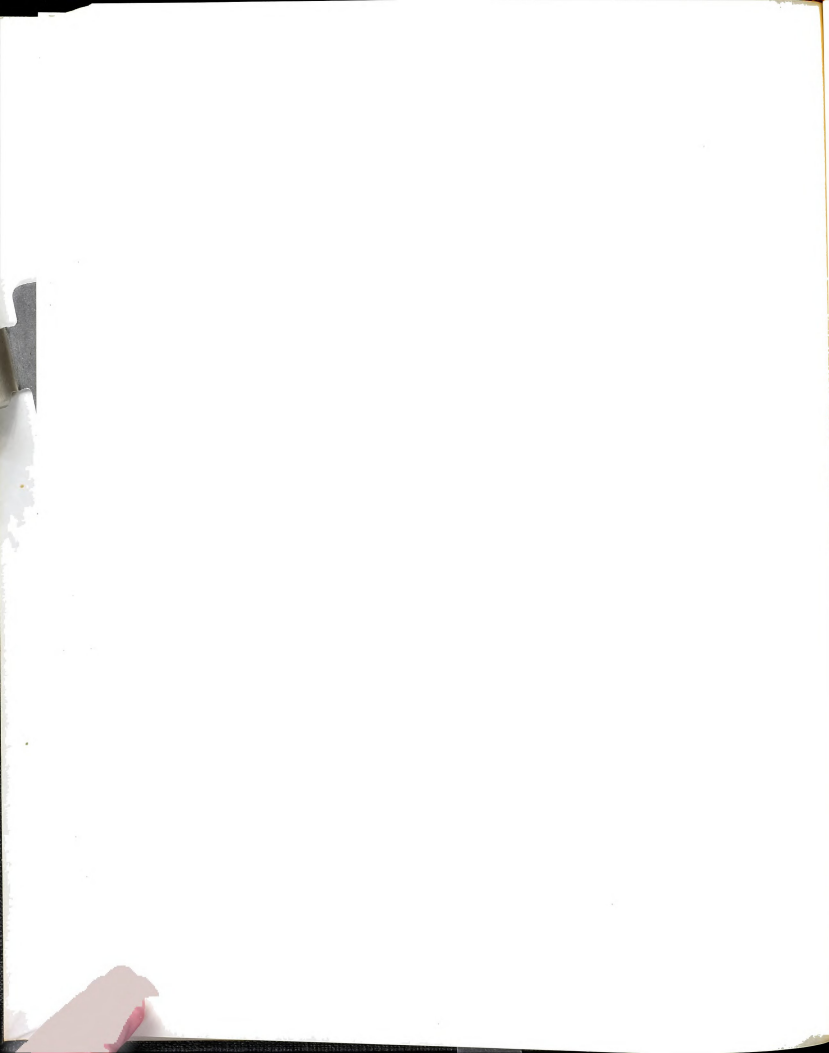


and (5.3) holds. Thus $f_{(k)}$ has property Z on $[a,b]$ and the proof is complete.

As was mentioned in the introduction, C. E. Weil introduced property Z in [11]. He further showed in [11] that if a function g has the Darboux property and property Z on an interval J then g has the Zahorski property on J (an example of a function having the Darboux property and the Zahorski property but not property Z can also be found in [11]). Thus in the class of functions having the Darboux property, property Z is strictly stronger than the Zahorski property.

Thus, by Corollary 4.22 and the previous paragraph, we have the following corollary to the last theorem.

Corollary 5.9. If f possesses a k^{th} approximate Peano derivative $f_{(k)}$ everywhere on $[a,b]$, then $f_{(k)}$ has the Zahorski property on $[a,b]$.

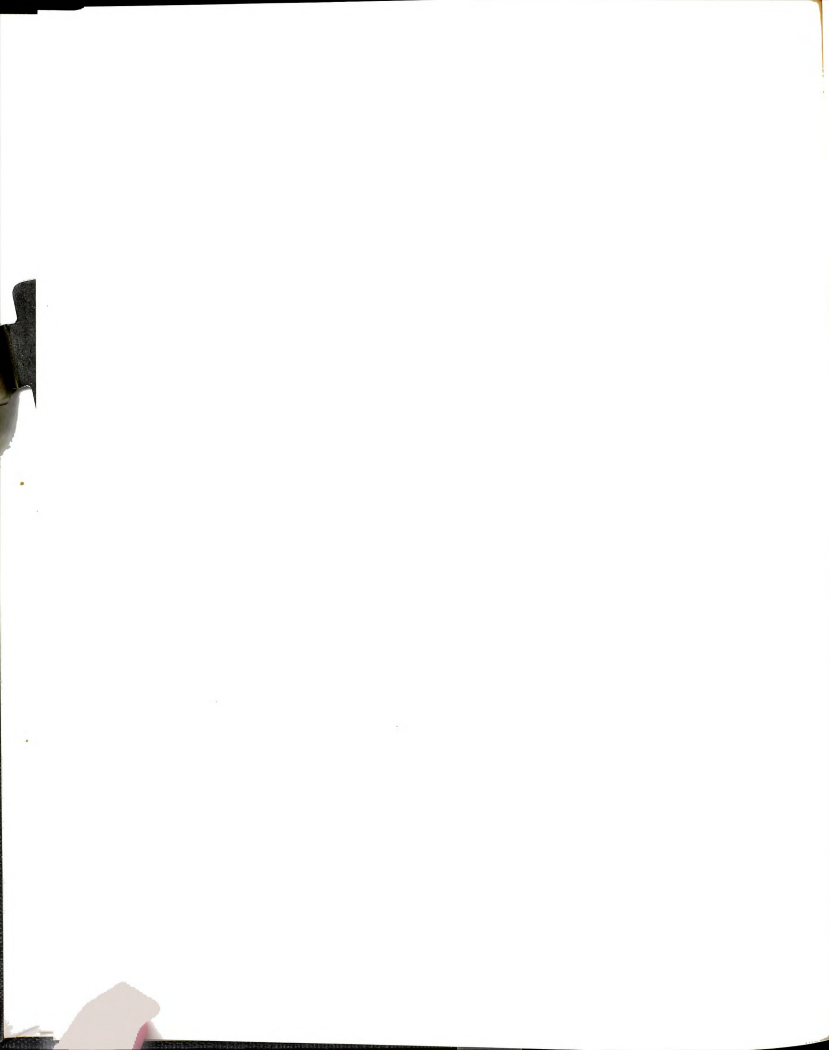


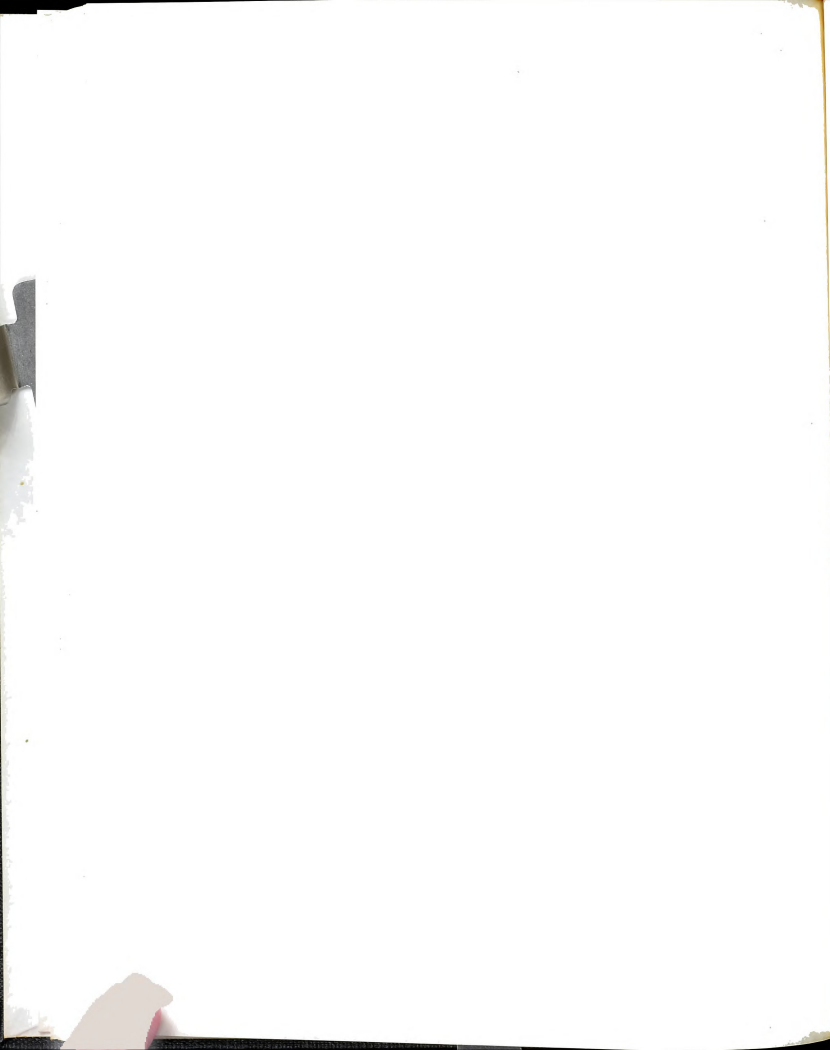
BIBLIOGRAPHY

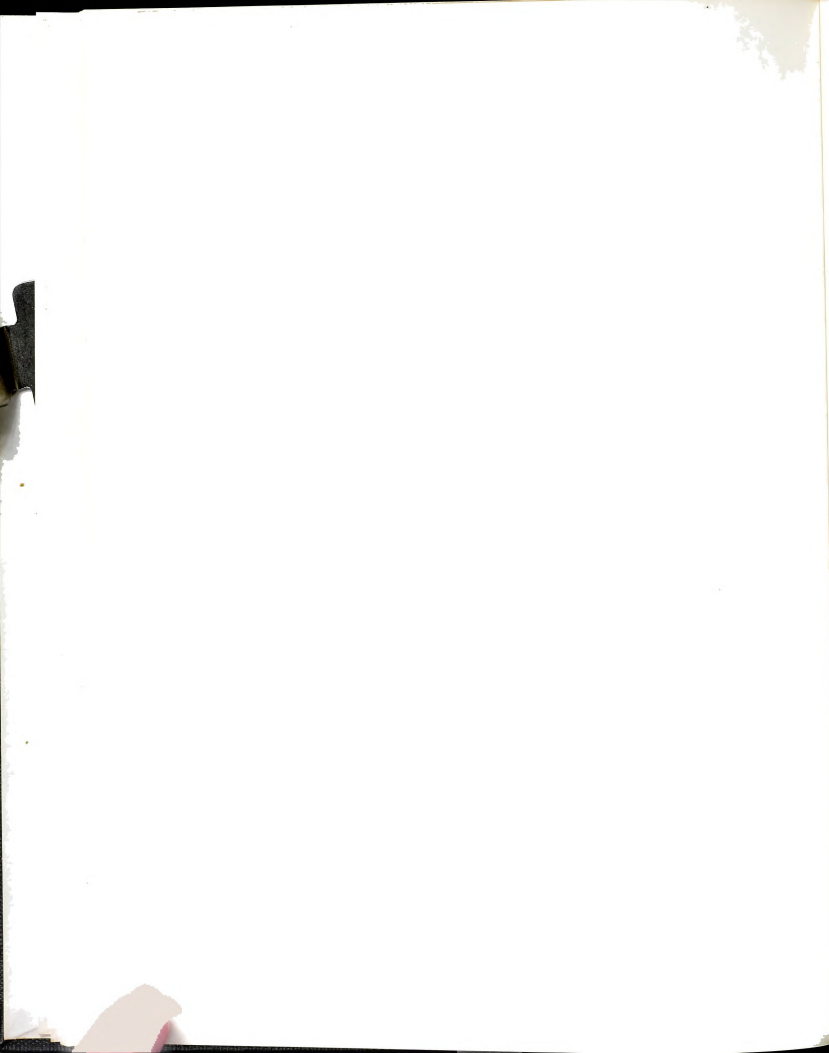
56

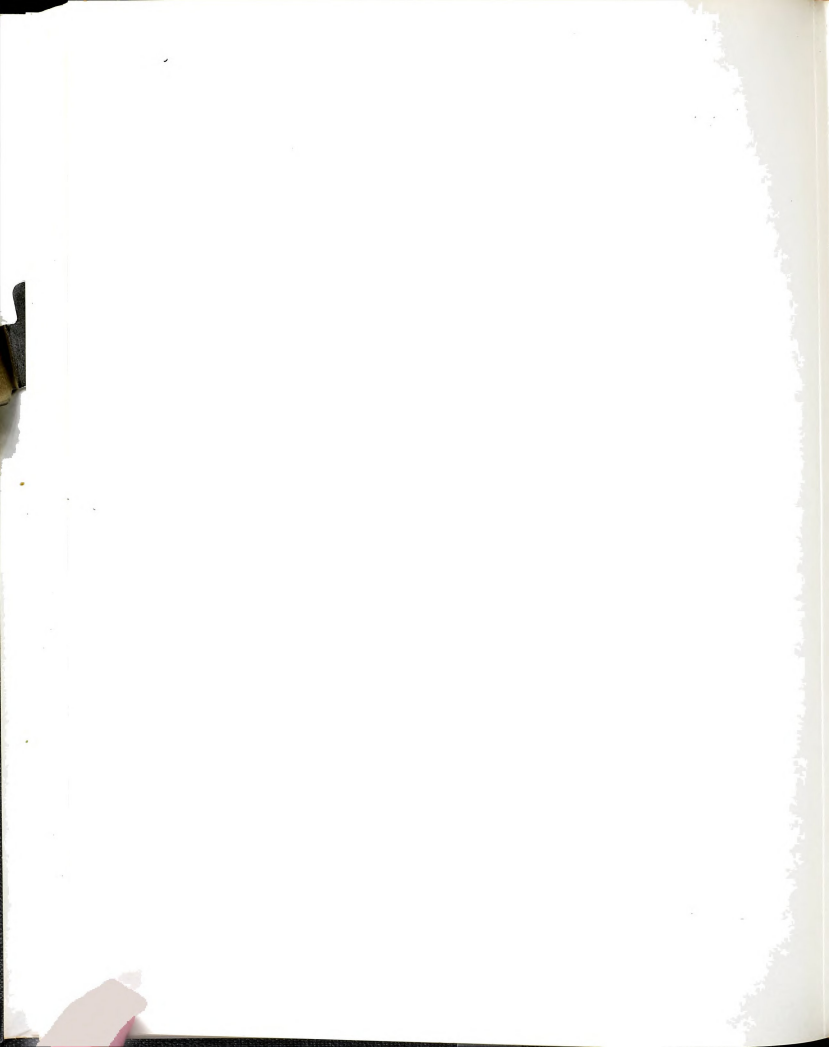
BIBLIOGRAPHY

1. A. P. Calderón and A. Zygmund, "Local properties of solutions of elliptic partial differential equations", Studia Math., 20 (1961), pp. 171-225.
2. A. Denjoy, "Sur une propriété des fonctions dérivées exactes", L'Enseignement Mathématique, 18 (1916), pp. 320-328.
3. M. J. Evans, " L_p derivatives and approximate Peano derivatives", Trans. Amer. Math. Soc., 165 (1972), pp. 381-388.
4. C. Goffman and C. J. Neugebauer, "On approximate derivatives", Proc. Amer. Math. Soc., 11 (1960), pp. 962-966.
5. S. Marcus, "On a theorem of Denjoy and on approximate derivatives", Monatsh. Math., 66 (1962), pp. 435-440.
6. C. J. Neugebauer, "Darboux functions of Baire class one and derivatives", Proc. Amer. Math. Soc., 13 (1962), pp. 838-843.
7. H. W. Oliver, "The exact Peano derivative", Trans. Amer. Math. Soc., 76 (1954), pp. 444-456.
8. G. Tolstoff, "Sur la dérivée approximative exacte", Rec. Math. (Mat. Sbornik) N.S. 4 (1938), pp. 499-504.
9. S. Verblunsky, "On the Peano derivatives", Proc. London Math. Soc., (3) 33 (1971), pp. 313-324.
10. C. E. Weil, "On properties of derivatives", Trans. Amer. Math. Soc., 114 (1965), pp. 363-376.
11. _____, "A property for certain derivatives", to appear in Indiana J. Math.
12. Z. Zahorski, "Sur la première dérivée", Trans. Amer. Math. Soc., 69 (1950), pp. 1-54.
13. A. Zygmund, Trigonometric Series, 2nd edn., Cambridge, 1959.









MICHIGAN STATE UNIV. LIBRARIES



31293006518405