BROADBAND ANALYSIS OF RADIATING, RECEIVING AND SCATTERING CHARACTERISTICS OF MICROSTRIP ANTENNAS AND ARRAYS

By

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ABSTRACT

BROADBAND ANALYSIS OF RADIATING, RECEIVING AND SCATTERING CHARACTERISTICS OF MICROSTRIP ANTENNAS AND ARRAYS

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Considered are a pin fed circular patch, an array of pin fed circular patches and an array of microstrip dipoles, all mounted on a dielectric above a ground plane. In each case, the solution is obtained via an inverse Fourier transform dyadic Green function Galerkin's method formulation. The solution is potentially exact, including higher-order resonances. An attachment current distribution provides for continuity at the feed pin-patch junction.

In the scattering case, the feed pin is attached to both the patch and the ground plane, or, for the dipole array, the dipole has no load impedance. The structure is illuminated by a plane wave.

In the transmitting case, a voltage source is inserted between the feed pin and the ground plane, or at the center of the dipole to drive the structure.

In the receiving case, an arbitrary load impedance replaces the voltage source and a plane wave again serves to drive the structure. This case is found as a combination of the first two cases, with the voltage in the transmitting case chosen so as to be equal to the voltage across the load impedance due to the current flowing through it.

For the single antenna, the electric field dyadic Green function components relating current and electric field components parallel to the ground plane on the dielectric-cover interface and normal to the ground plane within the dielectric region are obtained.

Current on the patch surface is expanded as a summation over a set of smooth continuous currents. The feed pin current distribution is constant over the feed pin surface. An additional patch current distribution flows across the patch surface radially away from the patch-feed pin junction such that current flows continuously from the feed pin onto the patch with no discontinuous charge build-up.

Current amplitudes are obtained via Galerkin's method. The current distributions are chosen so that all spatial integrations arising from Galerkin's method, including those involving the patch component of the feed pin current, can be performed in closed form. Spectral integration is performed via real-line integration.

For the array, the patches are identical and arranged in an infinite and periodic rectangular array. The inverse Fourier transforms of the single patch dyadic Green function components are converted to infinite summations, avoiding spectral integration.

For the microstrip dipole array, the dipoles are arranged in an infinite rectangular array. Only current flowing along the length of the dipole is assumed.

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I. INTRODUCTION.

1. Introduction.

While the transmitting properties of patches and patch arrays have been the object of much study [1-7], the scattering and receiving modes have had relatively little attention [8,9]. Further, the majority of the analyses have concentrated on a narrow frequency band near the fundamental resonance, neglecting off-resonance behavior, and the effects of higher order resonances.

This dissertation presents a theoretical analysis of the receiving characteristics of a single microstrip patch antenna and of a patch antenna array of infinite extent consisting of a circular patch(es) with offset feed pin(s). With an assumed plane wave excitation, the power delivered to a load impedance connected to the feedpin(s) is determined as a function of incidence angle and array parameters, and has been numerically calculated over a frequency range of over three to one.

Since the receiving mode may be viewed as a superposition of transmitting and scattering cases, an accurate analysis of the array acting as both scatterer and transmitter is needed over a wide bandwidth. A potentially exact approach is therefore undertaken, using electric field dyadic Green functions for horizontal current sources above a grounded dielectric slab, and for vertical current sources within the slab. This approach allows the coupling between the feed pins and the patches to be included explicitly and fully, an effect only recently receiving attention of researchers.

The Green's functions are determined via a two dimensional spatial Fourier transform over the coordinates transverse to the grounded dielectric slab. Coupled integral equations are developed for the current induced on the patch(es) and feed pin(s) both by an applied load voltage for the antenna/array acting as a transmitter, and by the incident plane wave for the antenna/array acting as a scatterer. Solutions are undertaken using Galerkin's method, and are used to calculate the input impedance of the transmitting antenna/array, and the current on the feed pin(s) of the scattering antenna/array. From these, the power delivered to the load of the receiving antenna/array is determined.

Mathematical expansion of surface current on a circular patch is rigorously determined through a taylor series expansion. The current distribution basis functions are expressed in terms of products of sinusoids (varying azimuthally) and Tchebychef polynomials (varying radially). This choice of basis functions is prudent in that it forms a complete set, and in that the spatial integrals arising from the application of Galerkin's method can be evaluated in closed form [5]. An additional term is included to account for the divergent nature of the patch current near the feed pin junction. Such a "singular" current has been used before [10] in patch current modelling, although of a different form; its purpose is to provide for continuity of current at the feed pin junction, and also to accelerate the convergence of the surface current expansion in the vicinity of the feed pin.

Two different singular current distributions are introduced here. The first distribution, used in the case of an infinite array, requires some numerical integration of the Galerkin's method integrations. Current flows radially from the feed pin junction to the edge of the patch, falling to zero there. A second singular current distribution used for the isolated patch is smooth and continuous, although non-zero over only part of the patch. The current flows radially away from the feed pin, falling to zero at a constant radius from the feed pin, generally not at the edge. The advantage of this second distribution is that all Galerkin's method integrations may be evaluated in closed form, in terms of simple Bessel functions

For the case of the infinite array, the spectral integrations from the Fourier transform are converted into an infinite double summation over discrete values of the transform variables. Each term in the summation represents a single Flouquet mode.

For the case of the single patch antenna, the two dimensional inverse Fourier transform is manipulated into polar form, and the azimuthal spectral variable is integrated out to obtain a single spectral integration from zero to infinity, which is performed using real-line integration. Large argument asymptotic forms of this integration are isolated and performed analytically, leaving the remaining integration to be easily performed numerically to the limit of infinity.

This dissertation also presents a theoretical analysis of the receiving characteristics of a microstrip dipole array of infinite extent. With an assumed plane wave excitation, the power delivered to load impedances centered on the dipoles is determined as a function of incidence angle and array parameters, and has been numerically calculated over a frequency range of 3 to 1.

As in the case of the infinite patch array, a potentially exact approach is undertaken, using electric field dyadic Green functions for horizontal current sources above a grounded dielectric slab.

The Green functions used in the infinite patch array are conscripted for use here. Coupled integral equations are developed for the current induced on the dipoles both by an applied load voltage for the array acting as a transmitter, and by the incident plane wave for the array acting as a scatterer. Solutions are undertaken using Galerkin's method, and are used to calculate the input impedance of the transmitting array, and the current at the dipole center of the scattering array. From these, the power delivered to the load of the receiving array is determined.

Mathematical expansion of the surface currents on the strip dipoles is accomplished using piecewise sinusoidal basis functions, with no variation across the width of the dipoles.

II. CIRCULAR PATCH ANTENNA.

2. Problem Description.

2.1 Geometry.

The geometry of the circular microstrip patch antenna to be analyzed is depicted in Figure 1. A dielectric substrate of permitivity ε_2 , permeability μ_o and thickness d, (region 2), is located between the $z = 0$ and $z = -d$ planes. The dielectric is mounted on a conducting ground plane at $z = -d$, and is covered by a material with constitutive parameters ε_1 and μ_o (region 1). The patch has radius b and is connected to a feed pin of radius a centered at a point $\vec{r}_o = x_o\hat{x} + y_o\hat{y}$ running from the patch to a load impedance Z_L at the ground plane. The patch is located at the dielectric-cover interface in the $z = 0$ plane, with center located at the origin, and is assumed to be perfectly conducting and infinitely thin.

Illumination of the structure is taken to be through an incident plane wave of frequency ω at an arbitrary incidence angle. The plane wave is expressed in terms of a coordinate system x'', y'', z'' rotated with respect to the coordinate system of the patch, as shown in Figure 2, with ϕ_i the angle between the x and x'' axes, and θ_i the angle between the wave vector $\vec{k}^i$ and the z axis. The z and z'' axes coincide. The factors

$$u = \sin(\theta_i)\cos(\phi_i) \tag{1}$$

and

$$v = \sin(\theta_i)\sin(\phi_i) \tag{2}$$

are the direction cosines for the wave vector $\vec{k}^i$ with the $-x$ and $-y$ axes respectively.

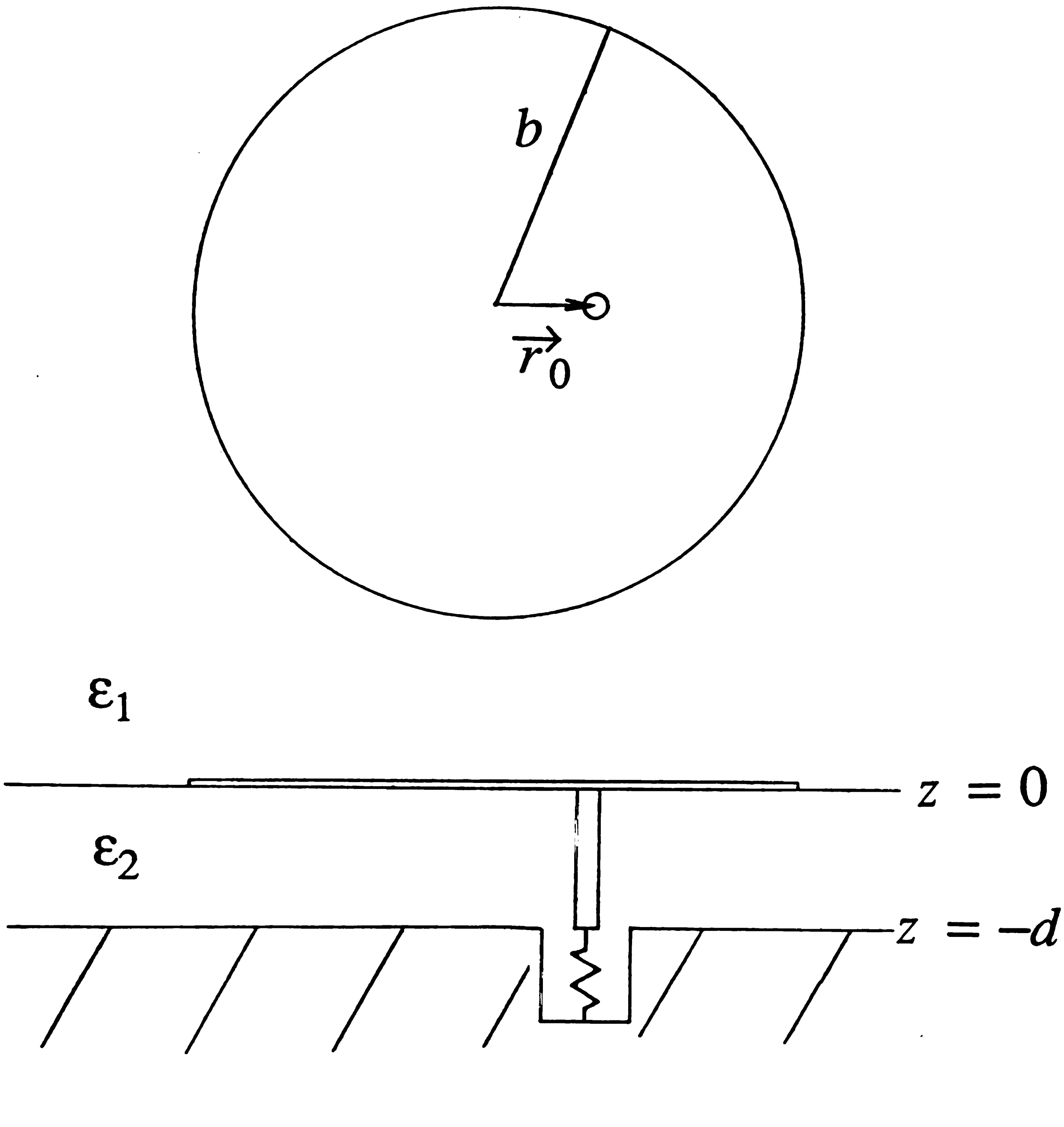
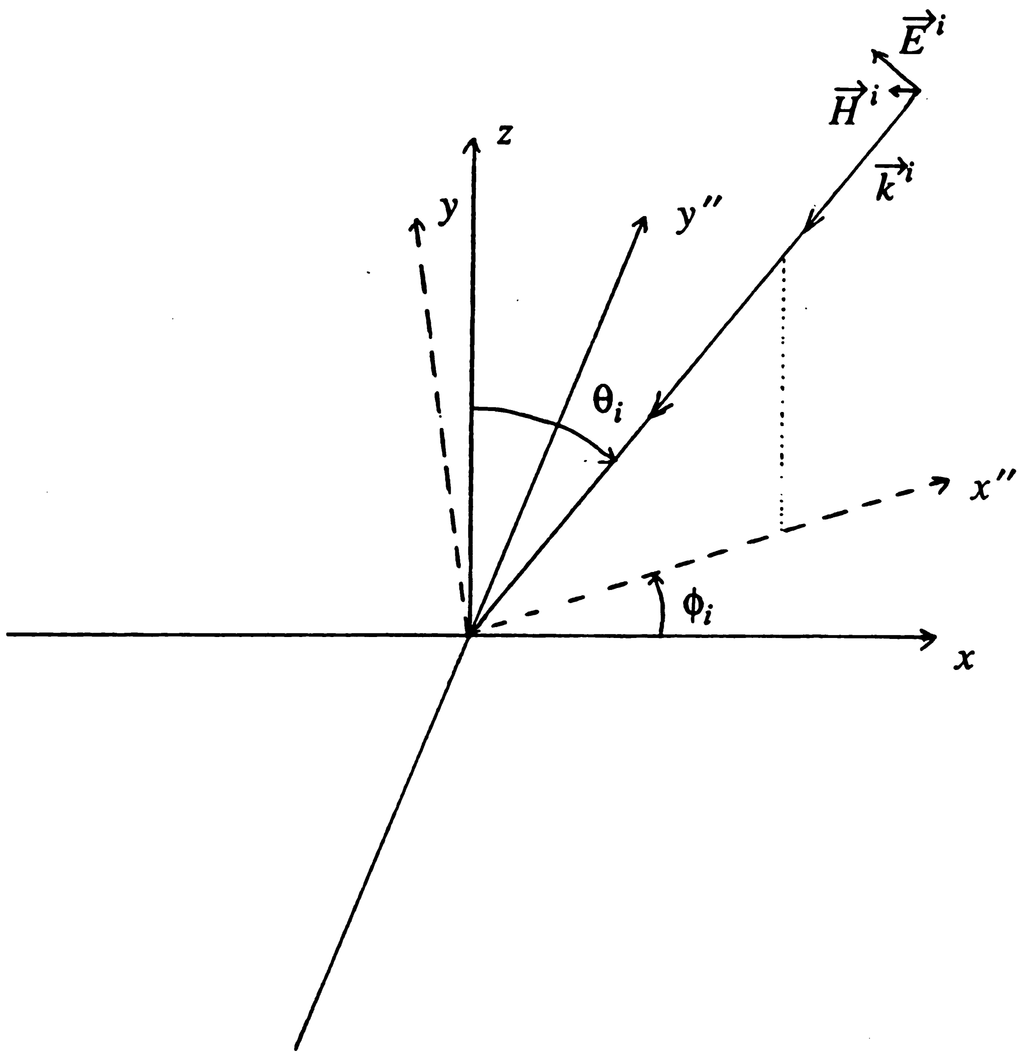


Figure 1. Patch with feed pin.



$$x'' = x\ \cos(\phi_i) + y''\ \sin(\phi_i)$$

$$y'' = -x\ \sin(\phi_i) + y''\ \cos(\phi_i)$$

$$z'' = z$$

Figure 2. Excitation field coordinate system.

2.2 Problem Decomposition.

Using the principle of superposition, the antenna acting as a receiver can be decomposed into scattering and transmitting cases. In the scattering case, the load impedance is replaced with a short-circuit to the ground plane, the antenna is illuminated by an incident plane wave, and the feed pin current, I_1, is determined. For the transmitting case, no illuminating plane wave is present. Instead, a driving voltage V is applied at the base of the feed pin in place of the load impedance, representing the voltage drop which would exist across the load in the receiving case due to current flowing on the feed pin. The resulting current on the feed pin, I_2, due to V is found, and the input impedance is then calculated from

$$Z_{in} = \frac{V}{I_2}. \tag{1}$$

Since V is the voltage drop across the load impedance Z_L due to the total current $I = I_1 + I_2$,

$$V = -IZ_L = -(I_1+I_2)Z_L \tag{2}$$

But $I_2 = \frac{V}{Z_{in}}$, so the voltage V can be obtained as

$$V = -I_1\, Z_L - V\,\frac{Z_L}{Z_{in}} \tag{3}$$

or

$$V = \frac{-I_1\, Z_L}{1 + Z_L/Z_{in}} = \frac{-I_1 Z_{in}\, Z_L}{Z_{in}+Z_L} \tag{4}$$

The total current I is then found as

$$I = I_1 + \frac{1}{Z_{in}}\left[\frac{-I_1 Z_{in}\, Z_L}{Z_{in}+Z_L}\right] \tag{5}$$

or

$$I = \frac{I_1 Z_{in}}{Z_{in}+Z_L} \tag{6}$$

The power received by the structure is then

$$P_L = \frac{1}{2} I^2 R_L = \frac{1}{2} |I_1|^2 \left| \frac{Z_{in}}{Z_{in}+Z_L} \right|^2 R_L \tag{7}$$

where R_L is the real part of Z_L. For the currents on the patch surface,

$$\vec{K}_{patch} = \vec{K}_1 + \vec{K}_2 \tag{8}$$

where $\vec{K}_{patch}$ is the total patch current in the receiving case, $\vec{K}_1$ is the patch current in the scattering case, and $\vec{K}_2$ is the patch current in the transmitting case.

To determine the current in the post and the power delivered to the load, the scattering case is solved for I_1, and the transmitting case is solved to obtain Z_{in}. Other quantities of the general receiving case can also be found using superposition of the scattering and transmitting cases.

3. Scattering Case.

3.1 Excitation Field.

Consider first the case where the patch antenna with feed pin short-circuited to the ground plane is illuminated by an incident plane wave. The excitation field is that field generated by the incident plane wave in the presence of the ground plane and dielectric coating only. The incident plane wave electric field can be written as

$$\vec{E}^{\,i}(\vec{r}\,) = E^i \,\hat{\gamma}^i e^{-j\vec{k}^i\cdot\vec{r}} \tag{1}$$

where $\hat{\gamma}^i$ describes the incident polarization and $\vec{k}^i$ is the wave vector of the incident plane wave. By applying boundary conditions at the conducting plane and at the dielectric-cover region interface, the total electric field in the dielectric and cover regions can be found. This is done in chapter 12, and the results are given here. They are, first for region 2,

$$\vec{E}_2^{\,e}(\vec{r}\,) = \vec{E}_{2\perp}(\vec{r}) + \vec{E}_{2\,||}(\vec{r}\,) \qquad -d \le z \le 0$$

where the subscripts $||$ and $\perp$ stand for polarization (of the incident plane wave) parallel and perpendicular to the plane of incidence, the $(x''-z)$ plane. The two components of the field are

$$\vec{E}_{2\,||}(\vec{r}\,) = 2T_{||}\, E^i_{||}\, e^{jk_2(x''\sin\theta_r + d\cos\theta_r)} \Big[\hat{z}\sin\theta_r\cos(k_2(z+d)\cos\theta_r) - \hat{x}''j\cos\theta_r\sin(k_2(z+d)\cos\theta_r)\Big]$$

$$-d \le z \le 0 \tag{3}$$

and

$$\vec{E}_{2\perp}(\vec{r}\,) = -2jT_{\perp}\, E^i_{\perp}\, e^{jk_2(x''\sin\theta_r + d\cos\theta_r)}\hat{y}''\sin(k_2(z+d)\cos\theta_r) \qquad -d \le z \le 0 \tag{4}$$

with

$$E^i_{||} = E^i\Big[\gamma_z^{i\,2} + (\gamma_x^i\cos\phi_i + \gamma_y^i\sin\phi_i)^2\Big]^{\frac{1}{2}} \tag{5}$$

$$E_{\perp}^{i} = E^{i}\left[\gamma_y^i\cos\phi_i - \gamma_x^i\sin\phi_i\right] \tag{6}$$

and

$$T_{||} = \frac{j\eta_2\cos\theta_i e^{-jk_2 d\cos\theta_r}}{\eta_2\cos\theta_r\sin(k_2 d\cos\theta_r) - j\eta_1\cos\theta_i\cos(k_2 d\cos\theta_r)} \tag{7}$$

$$T_{\perp} = \frac{j\eta_2\cos\theta_i e^{-jk_2 d\cos\theta_r}}{\eta_2\cos\theta_i\sin(k_2 d\cos\theta_r) - j\eta_1\cos\theta_r\cos(k_2 d\cos\theta_r)} \tag{8}$$

For region 1, the fields are

$$\vec{E}_1^{\,e}(\vec{r}) = \vec{E}_{1\perp}(\vec{r}) + \vec{E}_{1\,||}(\vec{r}) \qquad z \geq 0 \tag{9}$$

where

$$\vec{E}_{1\,||}(\vec{r}) = 2E_{||}^{i}\, e^{j(\frac{1}{2}\phi_{||}+k_1 x''\sin\theta_i)}\left[\hat{z}j\sin(\frac{1}{2}\phi_{||}-k_1 z\cos\theta_i)\sin\theta_i + \hat{x}''\cos(\frac{1}{2}\phi_{||}-k_1 z\cos\theta_i)\cos\theta_i\right] \qquad z \geq 0 \tag{10}$$

and

$$\vec{E}_{1\perp}(\vec{r}) = 2E_{\perp}^{i}\, e^{j(\frac{1}{2}\phi_{\perp}+k_1 x''\sin\theta_i)}\hat{y}'\cos(\frac{1}{2}\phi_{\perp}-k_1 z\cos\theta_i) \qquad z \geq 0 \tag{11}$$

with

$$\Gamma_{||} = \frac{\eta_2\cos\theta_r\sin(k_2 d\cos\theta_r) + j\eta_1\cos\theta_i\cos(k_2 d\cos\theta_r)}{\eta_2\cos\theta_r\sin(k_2 d\cos{+}\theta_r) - j\eta_1\cos\theta_i\cos(k_2 d\cos\theta_r)} \tag{12}$$

$$\Gamma_{\perp} = \frac{\eta_2\cos\theta_i\sin(k_2 d\cos\theta_r) + j\eta_1\cos\theta_r\cos(k_2 d\cos\theta_r)}{\eta_2\cos\theta_i\sin(k_2 d\cos\theta_r) - j\eta_1\cos\theta_r\cos(k_2 d\cos\theta_r)} \tag{13}$$

In the above, the following relations and definitions hold.

$$k_1 = \omega\sqrt{\mu_o\varepsilon_1} \tag{14a}$$

$$k_2 = \omega\sqrt{\mu_o\varepsilon_2} \tag{14b}$$

$$x'' = x\cos\phi_i + y\sin\phi_i \tag{14c}$$

$$y'' = -x\sin\phi_i + y\cos\phi_i \tag{14d}$$

$$\hat{x}'' = \hat{x}\cos\phi_i + \hat{y}\sin\phi_i \tag{14e}$$

$$\hat{y}'' = -\hat{x}\sin\phi_i + \hat{y}\cos\phi_i \tag{14f}$$

$$\phi_i = \tan^{-1}\left[\frac{k_y^i}{k_x^i}\right] \tag{14g}$$

$$k^i = k_1 \tag{14h}$$

$$\theta_i = \cos^{-1}\left[\frac{k_z^i}{k^i}\right] \tag{14i}$$

$$\phi_{||} = -j\log_e(\Gamma_{||}) \quad or \quad \Gamma_{||} = e^{j\phi_{||}} \tag{14j}$$

$$\phi_{\perp} = -j\log_e(\Gamma_{\perp}) \quad or \quad \Gamma_{\perp} = e^{j\phi_{\perp}} \tag{14k}$$

$$\eta_1 = \sqrt{\frac{\mu_o}{\varepsilon_1}} \tag{14l}$$

$$\eta_2 = \sqrt{\frac{\mu_o}{\varepsilon_2}} \tag{14m}$$

$$\theta_r = \cos^{-1}\left(\frac{k_1}{k_2}\sin\theta_i\right) \tag{14n}$$

3.2 GREEN FUNCTIONS FOR SCATTERED FIELD.

The electric field supported by the induced surface currents on the patch antenna and feed pin is determined using a Fourier transform Green function approach. The electric field is written as an integral of the product of a dyadic Green function and the induced currents. The field $\vec{E}(\vec{r})$ in region a due to a surface current $\vec{K}(\vec{r}\,')$ in region b is

$$\vec{E}(\vec{r}) = \iint_s \overleftrightarrow{g}^{a,b}(\vec{r} \mid \vec{r}\,') \cdot \vec{K}(\vec{r}\,')\, ds' \tag{1}$$

where $\overleftrightarrow{g}^{a,b}(\vec{r} \mid \vec{r}\,')$ is the dyadic Green function for electric field in region a due to currents in region b and where s is the surface in region b where the induced currents $\vec{K}(\vec{r}\,')$ flow. Here, a and b can be either 1 or 2 representing regions 1 and 2.

The tangential electric field over the patch generated by currents induced on the patch is needed. The Green functions relating horizontal components of electric field on the dielectric-cover interface to the same components of current have been used by previous researchers, [2], [11], in various forms. They have been derived in chapter 13, and are given in section 13.11 as

$$g_{\alpha\beta}^{11} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} g_{\alpha\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad \alpha = x,y \qquad \beta = x,y \tag{2}$$

with

$$g_{xx}(\vec{k}) = \frac{1}{j\omega\epsilon_1} \frac{(k_1^2\epsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\epsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{3}$$

$$g_{xy}(\vec{k}) = g_{yx}(\vec{k}) = \frac{1}{j\omega\epsilon_1} \frac{-k_x k_y \left[p_1 + p_2 \tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\epsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{4}$$

$$g_{yy}(\vec{k}) = \frac{1}{j\omega\epsilon_1} \frac{(k_1^2\epsilon_r - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\epsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{5}$$

In the above,

$$\vec{k} = k_x\hat{x} + k_y\hat{y} \tag{6a}$$

$$d^2k = dk_x\, dk_y \tag{6b}$$

$$p_1^2 = k_x^2+k_y^2-k_1^2 \tag{6c}$$

$$p_2^2 = k_x^2+k_y^2-k_2^2 \tag{6d}$$

$$k_1 = \omega\sqrt{\mu_o\,\varepsilon_1} \tag{6e}$$

$$k_2 = \omega\sqrt{\mu_o\,\varepsilon_2} \tag{6f}$$

$$\varepsilon_r = \frac{\varepsilon_2}{\varepsilon_1} \tag{6g}$$

and

$$T_e = \sinh(p_2d)+\frac{p_2}{p_1}\cosh(p_2d) \tag{7}$$

$$T_m = \varepsilon_r\cosh(p_2d)+\frac{p_2}{p_1}\sinh(p_2d) \tag{8}$$

The tangential electric field along the length of the feed pin generated by the currents induced on the patches, as well as the tangential fields over the patch surface generated by the feed pin current are also needed. The Green functions relating the vertical component of electric field in the dielectric region to horizontal currents on the dielectric-cover interface, and relating the horizontal component of electric field on the dielectric-cover interface to vertical currents in the dielectric region are, from section 13.11,

$$g_{z\beta}^{21} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} g_{z\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z+d)) \qquad \beta=x,y \tag{9}$$

$$g_{\alpha z}^{12} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} g_{\alpha z}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z'+d)) \qquad \alpha = x,y \tag{10}$$

with

$$g_{zx}(\vec{k}) = \frac{1}{j\omega\epsilon_1} e^{-p_1 z'} \frac{jk_x}{T_m} \tag{11}$$

$$g_{zy}(\vec{k}) = \frac{1}{j\omega\epsilon_1} e^{-p_1 z'} \frac{jk_y}{T_m} \tag{12}$$

$$g_{xz}(\vec{k}) = \frac{1}{j\omega\epsilon_1} e^{-p_1 z} \frac{-jk_x}{T_m} \tag{13}$$

$$g_{yz}(\vec{k}) = \frac{1}{j\omega\epsilon_1} e^{-p_1 z} \frac{-jk_y}{T_m} \tag{14}$$

Finally, the tangential electric field along the feed pin due to the current flowing in the feed pin is needed. The Green function dyad for vertical electric field in region 2 due to vertical current in region 2 is, from section 13.11,

$$g_{zz}^{2,2}(\vec{r}\,|\,\vec{r}\,') = \frac{1}{(2\pi)^2}\iint \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\epsilon_2} \times$$

$$\left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1}\,\frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>})-\epsilon_r p_1 \sinh(p_2 z^{>}))}{T_m}\right] d^2k \quad -d \le z,z' \le 0 \tag{15}$$

where $\delta(x)$ is the Dirac delta function, and

$$z^{>} = \max(z,z') \tag{16a}$$

$$z^{<} = \min(z,z') \tag{16b}$$

The divergent nature of the above Green function integrals is overcome by the additional spatial integrations introduced by Galerkin's method. When these integrations are performed prior to spectral integration, the resulting spectral integrals are convergent. Equivalent results can be obtained using potential Green functions, which give convergent spectral integrals prior to application of Galerkin's method [12].

3.3 DERIVATION OF COUPLED INTEGRAL EQUATIONS.

The induced current on the patch and feed pin surfaces must satisfy a system of two coupled integral equations constructed by employing the boundary conditions that the tangential electric field over the patch and feed pin surfaces must be zero. The scattered electric field on the patch surface results from current on both the patch surface s and feed pin surface p, and is given by

$$\vec{E}_1^s(x,y,z{=}0) = \iint_s \overline{\overline{G}}^{1,1}(x,y,z{=}0 \mid x',y',z'{=}0) \cdot \vec{K}(x',y')dx'dy'$$

$$+ \iint_{s_p} \overline{\overline{G}}^{1,2}(x,y,z{=}0 \mid r_p{=}a,\phi_p,z_p)\cdot\vec{K}_z(z_p)adz_p\,d\phi_p \tag{1}$$

where the feed pin current $\vec{K}_z$ is assumed independent of ϕ_p. The integration variables ϕ_p and z_p are from the feed pin centered coordinate system shown in Figure 47, and described in section 14.2. The scattered field in region 2 due to current on the patch and feed pin is

$$\vec{E}_2^s(x,y,z) = \iint_s \overline{\overline{G}}^{2,1}(x,y,z \mid x',y',z'{=}0)\cdot\vec{K}(x',y')dx'dy'$$

$$+ \iint_{s_p} \overline{\overline{G}}^{2,2}(x,y,z \mid r_p{=}a,\phi_p,z_p)\cdot\vec{K}_z(z_p)adz_p\,d\phi_p \tag{2}$$

The boundary condition on the tangential electric field at the patch surface yields one integral equation

$$\hat{z} \times \left[\vec{E}_1^s(x,y,z{=}0) + \vec{E}_1^e(x,y,z{=}0)\right] = 0 \tag{3}$$

where $\vec{E}_1^e$ is the total excitation field in the cover region due to the incident plane wave. This equation must hold for all points on the patch surface. The boundary condition on the tangential electric field at the feed pin surface yields the second integral equation. This boundary condition is enforced along the surface of the feed pin, giving the second integral equation,

$$\hat{z} \cdot \left[\vec{E}_2^s(\vec{r}_0+\vec{a}) + \vec{E}_2^e(\vec{r}_0+\vec{a}) \right] = 0 \qquad -d \le z \le 0 \qquad 0 \le \phi_p \le 2\pi \tag{4}$$

where $\vec{E}_2^e$ is the total excitation field in region 2 due to the incident plane wave, and where $\vec{a}(\phi_p)$ is a vector from the center of the feed pin-patch junction to the pin surface at the junction.

3.4 General Matrix Formulation.

The induced current on the patch and feed pin surfaces is to be expressed as a summation over a set of basis function current distributions. The solution for the unknown amplitudes of the chosen current distributions is to be formulated as a matrix equation. This will be done using Galerkin's method--the boundary conditions on the surface of the patches and feed pins will be forced to hold in an integral sense where the same set of current distributions will be used as testing functions. The boundary condition used is the physical constraint that the electric field tangential to the surface of the patches and feed pins be zero.

Let $\vec{E}^i$ be the electric field due to the incident plane wave and let $\vec{E}^s$ be the scattered electric field due to induced current on the conducting surface of the patch and feed pin. Then

$$\vec{E}^s = \iint_s \overline{\overline{\gamma}} \cdot \vec{K}\, ds' \tag{1}$$

Here, s is the conducting surface of the antenna structure, $\vec{K}$ is the surface current on s and $\overline{\overline{\gamma}}$ is the dyadic Green function. The boundary condition on s requires that

$$\hat{t} \cdot (\vec{E}^i + \vec{E}^s) = 0 \tag{2}$$

for all arbitrary unit vectors $\hat{t}$ tangential to the surface s.

Now let

$$\vec{K} = \sum_{j=1}^{\infty} C_j\, \vec{K}_j \tag{3}$$

where $\{\vec{K}_j\}$ is an appropriate set of independant and complete current distributions, and where C_j is the corresponding amplitude of $\vec{K}_j$ in $\vec{K}$. Then

$$\vec{E}^s = \iint_s \overline{\overline{\gamma}} \cdot \left[\sum_{j=1}^{\infty} C_j\, \vec{K}_j\right] ds' = C_j \sum_{j=1}^{\infty} \iint_s \overline{\overline{\gamma}} \cdot \vec{K}_j\, ds' \tag{4}$$

By choosing $\{\vec{K}_j\}$ appropriately, the summation can be truncated after a finite number of terms and the current obtained will approximate $\vec{K}$. For a finite summation, it is no longer possible to require $\hat{t} \cdot (\vec{E}^i + \vec{E}^s) = 0$ at every point on s. Instead, Galerkin's method is used. For $i = 1, 2, 3, \ldots, N$, where N is the number of current distributions used, require

$$\iint_s \vec{K}_i \cdot \vec{E}^s \, ds = -\iint_s \vec{K}_i \cdot \vec{E}^i \, ds \, . \tag{5}$$

Substituting the expression for $\vec{E}^s$ using the truncated summation,

$$\iint_s ds \; \vec{K}_i \cdot \left[\sum_{j=1}^{N} C_j \iint_s \overline{\overline{\Gamma}} \cdot \vec{K}_j \, ds' \right] = -\iint_s ds \vec{K}_i \cdot \vec{E}^i \qquad i = 1, 2, 3, \ldots, N \tag{6}$$

or,

$$\sum_{j=1}^{N} C_j \iint_s ds \; \vec{K}_i \cdot \iint_s \overline{\overline{\Gamma}} \cdot \vec{K}_j \, ds' = -\iint_s ds \; \vec{K}_i \cdot \vec{E}^i \qquad i = 1, 2, 3, \ldots, N \tag{7}$$

Making the definitions

$$Z_{ij} = \iint_s ds \; \vec{K}_i \cdot \iint_s ds' \; \overline{\overline{\Gamma}} \cdot \vec{K}_j \tag{8}$$

$$V_i = -\iint_s ds \; \vec{K}_i \cdot \vec{E}^i \tag{9}$$

equation (4) becomes

$$\sum_{j=1}^{N} C_j \, Z_{ij} = V_i \qquad i = 1, 2, 3, \ldots, N \tag{10}$$

This can be written in matrix form as

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \tag{11}$$

This is the general form of the equation used to obtain the current amplitudes $C_1, C_2, \ldots, C_N$ and hence the approximate solution for $\vec{K}$ via (3).

3.5 Current Distributions.

The surface current on the patch is represented using a set of current distributions based on a two dimensional taylor series expansion expressed in terms of Tchebychef polynomials radially and complex exponentials azimuthally, along with a singular current distribution used to model the curent flowing from the feed pin junction onto the patch surface. A single basis function is used to model the current on the feed pin. On the patch surface,

$$\vec{K} = \sum_{m=0}^{M} \sum_{l=-L}^{L} \left[C_{rlm} \vec{K}_{rlm} + C_{\phi lm} \vec{K}_{\phi lm} \right] + C_s \vec{K}_s \tag{1}$$

while on the feed pin,

$$\vec{K} = C_z \vec{K}_z \tag{2}$$

The Tchebychef current distributions are either purely radial or purely azimuthal. The radial variation of the current distributions are modified by an appropriate factor to account for the known edge behavior of the two components of current. These current distributions are, from 14.1(47) and 14.1(48),

$$\vec{K}_{rlm} = \hat{r}\, K_{rlm} = \hat{r}\, T_m(r/b)\, e^{jl\phi} \sqrt{1 - \frac{r^2}{b^2}} \qquad l+m \text{ odd} \tag{3}$$

$$\vec{K}_{\phi lm} = \hat{\phi}\, K_{\phi lm} = \hat{\phi}\, T_m(r/b)\, e^{jl\phi} \frac{1}{\sqrt{1 - \frac{r^2}{b^2}}} \qquad l+m \text{ odd} \tag{4}$$

The condition that $l+m$ be odd is implicated by enforcing the patch current, its divergence, and all further derivatives to be continuous at the center of the patch.

The current on the feed pin is modeled as flowing in the $\hat{z}$ direction uniformly over the surface of the feed pin.

$$\vec{K}_z = 1\, \hat{z} \tag{5}$$

The singular patch current distribution, $\vec{K}_s$, is designed to model the current flowing from the feed pin junction onto the surface of the patch. Its amplitude is forced to exactly account for the current flowing along the feed pin to the junction, so that no charge build-up at the patch-feed pin junction occurs. This current distribution is chosen to flow radially away from the junction. Thus,

$$\vec{K}_s = \hat{r}_p \, f(r_p) \tag{6}$$

where $\hat{r}_p$ is a unit vector directed radially away from the junction and r_p is the radial distance from the center of the junction to a point on the patch surface. From 14.2(12), $f(r_p)$ is constructed as

$$f(r_p) = A \, \frac{a}{r_p} + B \, \frac{r_p}{a} + C \left[\frac{r_p}{a} \right]^3 + D \left[\frac{r_p}{a} \right]^5 \qquad a \le r_p \le R \tag{7}$$

where A, B, C and D are constants determined in section 14.2, and where $R = b - r_0$ is the minimum distance from the center of the feed pin junction to the edge of the patch. Outside the region from a to R, $f(r_p)$ is zero.

3.6 Specific Matrix Formulation.

The current distributions used to model the patch current are the Tchebychef distributions $\vec{K}_{rlm}$ and $\vec{K}_{\phi lm}$ from equations 3.5(3) and 3.5(4), along with $\vec{K}_s$ from 3.5(6). The feed pin current is modeled by $\vec{K}_z$ from 3.5(5). The current amplitudes C_j become the corresponding current amplitudes for the Tchebychef current distributions along with a common amplitude for $\vec{K}_z$ and $\vec{K}_s$. The approximate equation for $\vec{K}$ then becomes

$$\vec{K} = \sum_{m=0}^{M} \sum_{l=-L}^{L} \left[C_{rlm} \vec{K}_{rlm} + C_{\phi lm} \vec{K}_{\phi lm} \right] + C_z \left[\vec{K}_z + \vec{K}_s \right] \tag{1}$$

where $\vec{K}_z$ flows over the feed pin surface, and all other current distributions flow on the patch surface.

The matrix equation 3.4(11) becomes

$$\begin{bmatrix} Z_{r'l'm'}^{rlm} & Z_{\phi'l'm'}^{rlm} & Z_{z'}^{rlm} \\ Z_{r'l'm'}^{\phi lm} & Z_{\phi'l'm'}^{\phi lm} & Z_{z'}^{\phi lm} \\ Z_{r'l'm'}^{z} & Z_{\phi'l'm'}^{z} & Z_{z'}^{z} \end{bmatrix} \cdot \begin{bmatrix} C_{r'l'm'} \\ C_{\phi'l'm'} \\ C_{z'} \end{bmatrix} = \begin{bmatrix} V_{rlm} \\ V_{\phi lm} \\ V_z \end{bmatrix} \tag{2}$$

Letting γ represent either r or ϕ and letting δ' represent either r' or ϕ', (2) can be written more compactly as

$$\begin{bmatrix} Z_{\delta'l'm'}^{\gamma lm} & Z_{z'}^{\gamma lm} \\ Z_{\delta'l'm'}^{z} & Z_{z'}^{z} \end{bmatrix} \cdot \begin{bmatrix} C_{\delta'l'm'} \\ C_{z'} \end{bmatrix} = \begin{bmatrix} V_{\gamma lm} \\ V_z \end{bmatrix} \tag{3}$$

The matrix elements $Z_{\delta'l'm'}^{\gamma lm}$ are

$$Z_{\delta'l'm'}^{\gamma lm} = \iint_s ds\, \vec{K}_{\gamma lm} \cdot \iint_s ds'\, \bar{\bar{g}}^{1,1} \cdot \vec{K}_{\delta'l'm'} \tag{4}$$

where s is the surface of the patches.

The surface current distributions, which are purely radially or azimuthally directed, must be separated into $\hat{x}$ and $\hat{y}$ components, since $\overleftrightarrow{g}$ is represented using rectangular components. Written component-wise, $\overleftrightarrow{g}$ is

$$\overleftrightarrow{g} = \begin{bmatrix} g_{xx}^{1,1} & g_{xy}^{1,1} & g_{xz}^{1,2} \\ g_{yx}^{1,1} & g_{yy}^{1,1} & g_{yz}^{1,2} \\ g_{zx}^{2,1} & g_{zy}^{2,1} & g_{zz}^{2,2} \end{bmatrix} \tag{5}$$

where the superscripts on the components are those relevent to the case being considered. The current distributions are decomposed as

$$\begin{aligned} \vec{K}_{\gamma lm} &= (\vec{K}_{\gamma lm} \cdot \hat{x})\,\hat{x} + (\vec{K}_{\gamma lm} \cdot \hat{y})\,\hat{y} \\ &= K_{\gamma lmx}\hat{x} + K_{\gamma lmy}\hat{y} \\ &= \sum_{\alpha=x,y} K_{\gamma lm\alpha}\,\hat{\alpha} \end{aligned} \tag{6}$$

and similarly,

$$\vec{K}_{\delta l'm'} = \sum_{\beta=x,y} K_{\delta l'm'\beta}\hat{\beta} \tag{7}$$

Using (6) and (7), and carrying out the scalar products, (4) becomes

$$Z_{\delta l'm'}^{\gamma lm} = \sum_{\substack{\alpha=x,y \\ \beta=x,y}} \iint_s ds\, K_{\gamma lm\alpha} \iint_s ds'\, g_{\alpha\beta}^{1,1}\, K_{\delta l'm'\beta} \tag{8}$$

In (8), the two superscripted "1"s on the Green functions indicate that the field and source integrals respectively are carried out in region 1, above the dielectric interface.

The matrix elements $Z_s^{\gamma lm}$ are

$$Z_s^{\gamma lm} = \iint_s ds\, \vec{K}_{\gamma lm} \cdot \left[\iint_{s_p} ds_p'\, \overleftrightarrow{g}^{1,2} \cdot \vec{K}_s + \iint_s ds'\, \overleftrightarrow{g}^{1,1} \cdot \vec{K}_s \right] \tag{9}$$

where s_p is the feed pin surface. The feed pin associated patch current distribution, $\vec{K}_s$ is separated into rectangular components as

$$\vec{K}_s = \sum_{\beta=x,y} K_{s\beta}\hat{\beta} \tag{10}$$

The testing function $\vec{K}_z$ has only a $\hat{z}$ component, and is

$$\vec{K}_z = \hat{z} \tag{11}$$

so (9) becomes

$$Z_z^{\gamma lm} = \sum_{\alpha=x,y} \iint_s ds\ K_{\gamma lm\alpha} \cdot \left[\iint_{s_p} ds_p{}'\ g_{\alpha z'}^{1,2} + \sum_{\beta'=x',y'} \iint_s ds'\ g_{\alpha\beta'}^{1,1}\ K_{s\beta'} \right] \tag{12}$$

In the above equations, $K_{s\beta'}$ is the component in the $\beta = \hat{x}$ or $\hat{y}$ direction of the singular current distribution associated with the feed pin. The elements $Z_{\delta' l' m'}^{z}$ are

$$Z_{\delta' l' m'}^{z} = \iint_{s_p} ds_p\ \vec{K}_z \cdot \iint_s ds'\ \overleftrightarrow{g}^{2,1} \cdot \vec{K}_{\delta' l' m'} \tag{13}$$

Equation (13) becomes

$$Z_{\delta' l' m'}^{z} = \sum_{\beta=x,y} \iint_{s_p} ds_p\ 1 \iint_s ds'\ g_{z\beta}^{21}\ K_{\delta' l' m \beta} \tag{14}$$

The matrix element $Z_{z'}^{z}$ is

$$Z_{z'}^{z} = \iint_{s_p} ds_p\ \vec{K}_z \cdot \left[\iint_{s_p} ds_p{}'\ \overleftrightarrow{g}^{2,2} \cdot \vec{K}_z + \iint_s ds'\ \overleftrightarrow{g}^{2,1} \cdot \vec{K}_s \right] \tag{15}$$

This can be written as

$$Z_{z'}^{z} = \iint_{s_p} ds_p\ dz \left[\iint_{s_p} ds_p{}'\ g_{zz'}^{2,2} + \sum_{\beta'=x',y'} \iint_s ds'\ g_{z\beta'}^{2,1}\ K_{s\beta'} \right] \tag{16}$$

From 3.4(9), the elements $V_{\gamma lm}$ are

$$V_{\gamma lm} = -\iint_s ds\ \vec{K}_{\gamma lm} \cdot \vec{E}^{\,1} \tag{17}$$

$$V_z = -\int_{s_p} ds_p\ \hat{z} \cdot \vec{E}^{\,2} \tag{18}$$

The patch testing functions $\vec{K}_{\gamma lm}$ in (17) and (18) don't need to be separated into $\hat{x}$ and $\hat{y}$ components. The impressed electric fields $\vec{E}^{\,1}$ and $\vec{E}^{\,2}$ are the fields in regions 1 and 2 respectively generated by the incident plane wave, $\vec{E}^{i}$. This field is separated into two components, $\vec{E}_{||}^{\,i}$ and $\vec{E}_{\perp}^{\,i}$, parallel and perpendicular to the plane of incidence, as shown in chapter 14. The fields $\vec{E}^{\,1}$ and $\vec{E}^{\,2}$ are the also separated into two components, and (17) and (18) become

$$V_{\gamma lm} = -\iint_s ds\ \vec{K}_{\gamma lm} \cdot \vec{E}_{||}^{\,1} - \iint_s ds\ \vec{K}_{\gamma lm} \cdot \vec{E}_{\perp}^{\,1} \tag{19}$$

$$V_z = -\iint_{s_p} ds_p\ \hat{z} \cdot \vec{E}_{||}^{\,2} - \iint_{s_p} ds_p\ \hat{z} \cdot \vec{E}_{\perp}^{\,2} \tag{20}$$

By setting first one then the other of $\vec{E}_{\perp}^{\,i}$ and $\vec{E}_{||}^{\,i}$ to be non-zero, the two cases of incident polarization may be solved separately. They may then be appropriately combined to give the solution for any arbitrary polarization.

The matrix elements can be manipulated into a less complicated form. Substituting 3.2(2) through 3.2(5) into (8), the matrix element becomes

$$Z_{\delta l'm'}^{\gamma lm} = \sum_{\substack{\alpha=x,y\\ \beta=x,y}} \iint_s ds\ K_{\gamma lm\alpha} \iint_s ds' \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k\ g_{\alpha\beta}(\vec{k}\,)\ e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\ K_{\delta l'm'\beta} \tag{21}$$

Rearranging the order of integration, this becomes

$$Z_{\delta l'm'}^{\gamma lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\substack{\alpha=x,y\\ \beta=x,y}} \left\{ \left[\iint_s ds\ K_{\gamma lm\alpha}\ e^{j\vec{k}\cdot\vec{r}} \right] \left[\iint_s ds'\ K_{\delta l'm'\beta}\ e^{-j\vec{k}\cdot\vec{r}\,'} \right] g_{\alpha\beta}(\vec{k}) \right\} \tag{22}$$

Define

$$\iint_s ds\ K_{rlmx}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{rlmx}^{\pm}\ (\vec{k}\,) \tag{23}$$

$$\iint_s ds\ K_{rlmy}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{rlmy}^{\pm}\ (\vec{k}\,) \tag{24}$$

$$\iint_s ds\ K_{\phi lmx}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{\phi lmx}^{\pm}\ (\vec{k}\,) \tag{25}$$

$$\iint_s ds\ K_{rlmy}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{\phi lmy}^{\pm}\ (\vec{k}\,) \tag{26}$$

Using (23) through (26), (22) becomes

$$Z_{\delta' l' m'}^{\gamma l m} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\substack{\alpha = x, y \\ \beta = x, y}} I_{\gamma l m \alpha}^{+} (\vec{k}) \, I_{\delta' l' m' \beta}^{-} (\vec{k}) \, g_{\alpha\beta}(\vec{k}) \tag{27}$$

Substituting 3.2(10) and 3.2(2) through 3.2(5) into (12),

$$Z_s^{\gamma l m} = \sum_{\alpha = x, y} \iint_s ds \, K_{\gamma l m \alpha} \left[\iint_{s_p} ds'_p \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; g_{\alpha z}(\vec{k}) \, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(P_2(z'+d)) \right.$$

$$\left. + \sum_{\beta = x, y} \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \iint_s ds' \, g_{\alpha\beta}(\vec{k}) \, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} K_{s\beta} \right] \tag{28}$$

In (28), both integrations over d^2k, and the exponential factor contained within them, are identical. They can thus be combined into a single integration, and (28) can be rearranged as

$$Z_s^{\gamma l m} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\alpha = x, y} \left\{ \left[\iint_s ds \, K_{\gamma l m \alpha} \, e^{j\vec{k}\cdot\vec{r}} \right] \times \right.$$

$$\left. \times \left[g_{\alpha z}(\vec{k}) \int_{-d}^{0}\int_{0}^{2\pi} a\, d\phi'_p \, dz' \, e^{-j\vec{k}\cdot\vec{r}'} \cosh(p_2(z'+d)) + \sum_{\beta = x, y} g_{\alpha\beta}(\vec{k}) \iint_s ds' \, e^{-j\vec{k}\cdot\vec{r}'} K_{s\beta} \right] \right\} \tag{29}$$

This leaves several integrals to be evaluated. The first, involving $K_{\gamma l m \alpha}$ is defined in (23) through (26). The last, involving $K_{s\beta'}$ is defined in a similar manner as

$$\iint_s ds \; e^{\pm j\vec{k}\cdot\vec{r}} \, K_{s\beta'} \equiv I_{s\beta'}^{\pm} \qquad \beta = x, y \tag{30}$$

The remaining integral can be evaluated immediately. For $\vec{r}\,'$ on the surface of the feed pin,

$$\vec{r}\,' = \vec{r}_o + \vec{a} + z\,\hat{z} \tag{31}$$

where $\vec{a}$ is a vector from $\vec{r}_o$ to the feed pin surface where it meets the patch. Since $\vec{k}$ has no $\hat{z}$ component, the $\hat{z}$ component of $\vec{r}\,'$ doesn't contribute. Thus,

$$\int_{-d}^{0}\int_{0}^{2\pi} a\, d\phi'_p \, dz' \, e^{\pm j\vec{k}\cdot\vec{r}} \cosh(P_2(z'+d)) = e^{\pm j\vec{k}\cdot\vec{r}_o} \int_{-d}^{0} dz' \cosh(p_2(z'+d)) \int_{0}^{2\pi} d\phi'_p \; a \; e^{\pm j\vec{k}\cdot\vec{a}'} \tag{32}$$

The integration over z is found as

$$\int_{-d}^{0} dz'\cosh(p_2(z'+d)) = \frac{\sinh(p_2 d)}{p_2} \tag{33}$$

and the integration over ϕ_p is obtained using

$$\int_0^{2\pi} d\phi'_p \ a \ e^{\pm j\vec{k}\cdot\vec{r}\,'} = a \int_0^{2\pi} d\phi'_p \ e^{\pm jka\cos(\phi_p - \theta)} = 2\pi a \ J_0(ka) \tag{34}$$

Thus

$$\int_{-d}^{0}\int_0^{2\pi} ds' \ e^{\pm j\vec{k}\cdot\vec{r}\,'} \cosh(P_2(z'+d)) = e^{\pm j\vec{k}\cdot\vec{r}_o} \frac{\sinh(p_2 d)}{p_2} \ 2\pi a \ J_0(ka) \tag{35}$$

Using (23) through (26), (30) and (35), (29) becomes

$$Z_z^{\gamma lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\alpha=x,y} I^{+}_{\gamma lm\alpha} \left[g_{\alpha z}(\vec{k}) \frac{\sinh(p_2 d)}{p_2} \ 2\pi a \ J_0(ka) + \sum_{\beta=x,y} g_{\alpha\beta}(\vec{k}) \ I_{z\beta}^{-} \right] e^{-j\vec{k}\cdot\vec{r}_0} \tag{36}$$

Substituting 3.2(9) into (14),

$$Z^{z}_{\delta' l' m'} = \sum_{\beta=x,y} \iint_{s_p} ds_p \ dz \ 1 \iint_{s} ds' \ K_{\delta' l' m' \beta} \times$$

$$\times \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \ g_{z\beta}(\vec{k}) \ e^{-j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \cosh(P_2(z+d)) \tag{37}$$

Equation (37) can then be rearranged to become

$$Z^{z}_{\delta' l' m'} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \ e^{j\vec{k}\cdot\vec{r}_o} \left[\int_{-d}^{0}\int_0^{2\pi} a\,d\phi_p \ dz \ \cosh(P_2(z+d)) \ e^{j\vec{k}\cdot\vec{a}} \right] \times$$

$$\times \sum_{\beta=x,y} g_{z\beta}(\vec{k}) \iint_{s} ds' \ K_{\delta' l' m' \beta} \ e^{-j\vec{k}\cdot\vec{r}\,'} \tag{38}$$

The integral over the pin surface is given in (35), and the integral over the patch surface is given in (23) through (26). Thus, (38) becomes

$$Z^{z}_{\delta' l' m'} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \ 2\pi a J_0(ka) \frac{\sinh(P_2 d)}{P_2} \sum_{\beta=x,y} g_{z\beta}(\vec{k}) \ I^{-}_{\delta' l' m' \beta} (\vec{k}) \ e^{j\vec{k}\cdot\vec{r}_0} \tag{39}$$

Using 3.2(15) and 3.2(9), (16) becomes

$$Z^{s}_{z'} = \iint_{s_p} ds_p \left\{ \sum_{\beta'=x',y'} \iint_{s} ds' \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; g_{z\beta'}(\vec{k}) \, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z+d)) \, K_{s\beta'} \right.$$

$$+ \int \int_{s_p} ds'_p \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left[\frac{1}{j\omega\epsilon_2}\right] e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \, .$$

$$\left. \left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1} \frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>}) - \overline{\epsilon} p_1 \sinh(p_2 z^{>}))}{T_m} \right] \right\} \tag{40}$$

Rearranging, this becomes

$$Z^{s}_{z'} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left\{ \sum_{\beta=x,y} g_{z\beta}(\vec{k}) \times \right.$$

$$\times \int_0^{2\pi} \int_{-d}^{0} a\,d\phi_p \; dz \; e^{j\vec{k}\cdot\vec{r}} \cosh(p_2(z+d)) \iint_{s} ds' \, e^{-j\vec{k}\cdot\vec{r}'} K_{s\beta} +$$

$$+ \frac{1}{j\omega\epsilon_2} \int_0^{2\pi} a\,d\phi_p \; e^{j\vec{k}\cdot\vec{a}} \int_0^{2\pi} e^{-j\vec{k}\cdot\vec{a}'} a\,d\phi'_p \times$$

$$\left. \times \int_{-d}^{0} \int_{-d}^{0} dz \; dz' \left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1} \frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>}) - \overline{\epsilon} p_1 \sinh(p_2 z^{>}))}{T_m} \right] \right\} \tag{41}$$

Using (35), (30) and (34), (41) becomes

$$Z^{s}_{z'} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left\{ \left[e^{j\vec{k}\cdot\vec{r}_0} \sum_{\beta=x,y} \frac{2\pi a}{p_2} \sinh(p_2 d) \, g_{z\beta} \, J_0(ka) \, I_{s\beta} \right] + \frac{(2\pi a)^2 J_0^2(ka)}{j\omega\epsilon_2} I_z \right\} \tag{42}$$

where

$$I_z = \int_{-d}^{0} \int_{-d}^{0} dz \; dz' \left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1} \frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>}) - \overline{\epsilon} p_1 \sinh(p_2 z^{>}))}{T_m} \right] \tag{43}$$

When (43) is evaluated in chapter 16, I_z is found to be (16(65))

$$I_z = d \, \frac{k_2^2}{p_2^2} - \frac{\overline{\epsilon}\, k^2}{p_2^2 \, T_m \, P_2} \sinh(p_2 d) \tag{44}$$

Thus, (42) becomes

$$Z_{z'}^{z} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left\{ \sum_{\beta'=x',y'} \frac{2\pi a}{p_2} \sinh(p_2 d)\, g_{z\beta'}\, J_0(ka)\, I_{z\beta'}\, e^{j\vec{k}\cdot\vec{r}_0} \right.$$

$$\left. + (2\pi a)^2 \frac{J_o^2(ka)}{j\omega\varepsilon_2} \left[d\, \frac{k_2^2}{p_2^2} - \frac{\bar{\varepsilon}\, k^2}{p_2^2\, T_m\, p_2} \sinh(p_2 d) \right] \right\} \quad (45)$$

The matrix elements for the incident field, (19) and (20) are also evaluated in chapter 16. For parallel incidence, these are found to be

$$V_{rlm\,||} = \bar{E}_{||}^{1}\, e^{jl\phi_i}\, \pi\, j^{l+1} \left[B_{l+1,m}^{-}(k_1 \sin(\theta_i)) + B_{l-1,m}^{-}(k_1 \sin(\theta_i)) \right] \quad (46)$$

$$V_{\phi lm\,||} = \bar{E}_{||}^{1}\, e^{jl\phi_i}\, \pi\, j^{l} \left[A_{l+1,m}^{-}(k_1 \sin(\theta_i)) + A_{l-1,m}^{-}(k_1 \sin(\theta_i)) \right] \quad (47)$$

$$V_z = -4\pi a\, T_{||}\, E_{||}^{i}\, e^{jk_2 \bar{x}\sin(\theta_r) + d\cos(\theta_r)} \frac{\sin(\theta_r)}{k_2 \cos(\theta_r)} \quad (48)$$

where $\bar{E}_{||}^{1}$ and $E_{||}^{i}$ are defined in chapter 12, and where $A_{l,m}^{\pm}$ and $B_{l,m}^{\pm}$ are obtained in section 16 as

$$A_{l,m}^{\pm} = \frac{\pi b^2}{4} \left[J_{\frac{|l|+m-1}{2}}\left(\frac{kb}{2}\right) J_{\frac{|l|-m+1}{2}}\left(\frac{kb}{2}\right) + J_{\frac{|l|+m+1}{2}}\left(\frac{kb}{2}\right) J_{\frac{|l|-m-1}{2}}\left(\frac{kb}{2}\right) \right] \cdot \left\{ \begin{array}{ll} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{array} \right\} \quad (49)$$

$$B_{l,m}^{\pm} = \frac{1}{4} \left[2 A_{l,m}^{\pm} - A_{l,|m-2|}^{\pm} - A_{l,m+2}^{\pm} \right] = \frac{1}{4} \left[2 A_{l,m}^{\pm} - A_{l,m-2}^{\pm} - A_{l,m+2}^{\pm} \right] \quad (50)$$

and are evaluated there. For the case of perpendicular incidence, the source matrix elements are

$$V_{rlm\,\perp} = -\bar{E}_{\perp}^{1}\, e^{jl\phi_i}\, \pi\, j^{l} \left[B_{l+1,m}^{-}(k_1 \sin(\theta_i)) + B_{l-1,m}^{-}(k_1 \sin(\theta_i) \right] \quad (51)$$

$$V_{\phi lm\,\perp} = -\bar{E}_{\perp}^{1}\, e^{jl\phi_i}\, \pi\, j^{l+1} \left[A_{l+1,m}^{-}(k_1 \sin(\theta_i)) - A_{l-1,m}^{-}(k_1 \sin(\theta_i)) \right] \quad (52)$$

$$V_z = 0 \quad (53)$$

where $\bar{E}_1^1$ is defined in chapter 12

4. Transmission Case.

The transmission case is arrived at in a manner analogous to that of the scattering case. The antenna is driven by a voltage at the base of the feed pin which represents the voltage drop across the load generated by current flowing on the feed pin when the antenna is acting as a receiver. To find the currents due to this applied voltage, a slice generator with one volt is assumed to be connected at the base of the feed pin at $z = -d$. This gives a tangential electric field along the feed pin

$$\vec{E}^{\,t} = \hat{z}\ \delta(z+d) \tag{1}$$

Since there is no plane wave excitation in this case, the feed pin voltage is the only excitation present. The integral equations to be satisfied are identical to those for the scattering case, with zero excitation field on the patch, and $\vec{E}_2^{\,e}(z)$ on the post replaced with $\vec{E}^{\,t}$. Thus, from 3.3(3) and 3.3(4)

$$\hat{z} \times \vec{E}_1^{\,s}(x,y,z{=}0)\Big|_s = 0 \tag{2}$$

and

$$\hat{z} \cdot \left[\vec{E}_2^{\,s}(\vec{r}_0+\vec{a},z) + \vec{E}^{\,t}\right] = 0 \qquad -d \le z \le 0 \tag{3}$$

Through the application of Galerkin's method as in the scattering case, these equations are replaced by the set

$$\iint_s \vec{K}_{lm\alpha}(r,\phi) \cdot \vec{E}_1^{\,s}(r,\phi,z{=}0)\, r\,dr\,d\phi = 0 \tag{4}$$

$$l = -L,\ldots,-1,0,1,\ldots,L$$

$$m = 0,1,\ldots,M$$

$$\alpha = r,\phi$$

and

$$\int_{-d}^{0}\int_{0}^{2\pi} \hat{z} \cdot \vec{E}_2^{\,s}(x_o+a\cos\phi,y_0+a\sin\phi,z)dz\ ad\phi_p = -\int_{-d}^{0}\int_{0}^{2\pi} \hat{z} \cdot \vec{E}^{\,t}dz\ ad\phi_p = -2\pi a\ V \tag{5}$$

Finally, this is written in matrix form in the same manner as for the scattering case. In particular, the left hand side of 3.6(2) is unchanged and the right hand side is replaced with either 0 or $-2\pi a$ as appropriate, so

$$\begin{bmatrix} Z^{rlm}_{r'l'm'} & Z^{rlm}_{\phi'l'm'} & Z^{rlm}_{z'} \\ Z^{\phi lm}_{r'l'm'} & Z^{\phi lm}_{\phi'l'm'} & Z^{\phi lm}_{z'} \\ Z^{z}_{r'l'm'} & Z^{z}_{\phi'l'm'} & Z^{z}_{z'} \end{bmatrix} \cdot \begin{bmatrix} C_{r'l'm'} \\ C_{\phi'l'm'} \\ C_{z'} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2\pi a \end{bmatrix} \tag{6}$$

Thus, once the scattering case has been solved, the same matrix is used to solve the transmission case, with only a simple substitution in the driving terms of the matrix equation. The current on the feed pin is then solved for and the input impedance obtained through 2.2(1). The input impedance obtained in this manner is more accurate than can be obtained using the induced EMF method. If (6) is rearranged into the form commonly used in the induced EMF method, the term which is solved to obtain Z_{in} is found to give $Z_{in} - Z^{z}_{z'}$. The input impedance obtained is thus off by the self-impedance of the feed pin.

III. INFINITE ARRAY OF PATCH ANTENNAS.

5. Problem Description.

The geometry of the infinite array of circular microstrip patch antennas to be analyzed is depicted in figure 3. A dielectric substrate of permitivity ε_2 , permeability μ_o and thickness d, (region 2), is located between the $z = 0$ and $z = -d$ planes, as in the case for a single patch antenna. Again, the dielectric is mounted on a conducting ground plane at $z = -d$, and is covered by a material with constitutive parameters ε_1 and μ_o (region 1). Each patch element has radius b and is connected to a load impedance Z_L through a feed pin of radius a centered at a point $\vec{r}_o = x_o\hat{x} + y_o\hat{y}$ in the local patch coordinate system. The patches are located at the dielectric-cover interface in the $z = 0$ plane, and are assumed to be perfectly conducting and infinitely thin. They are spaced distances d_x and d_y apart in the x and y directions respectively, their centers located at $\vec{r}_{pq} = p\ d_x\ \hat{x} + q\ d_y\ \hat{y}$ in the global coordinate system of the array, with p and q integers.

Illumination of the structure is again taken to be through an incident plane wave of frequency ω at an arbitrary incidence angle. The plane wave is expressed in terms of a coordinate system x'', y'', z'' rotated with respect to the global coordinate system of the array, as shown in Figure 2, with ϕ_i the angle between the x and x'' axes, and θ_i the angle between the wave vector $\vec{k}^i$ and the z axis. The z and z'' axes coincide. The factors

$$u = \sin(\theta_i)\cos(\phi_i) \tag{1}$$

and

$$v = \sin(\theta_i)\sin(\phi_i) \tag{2}$$

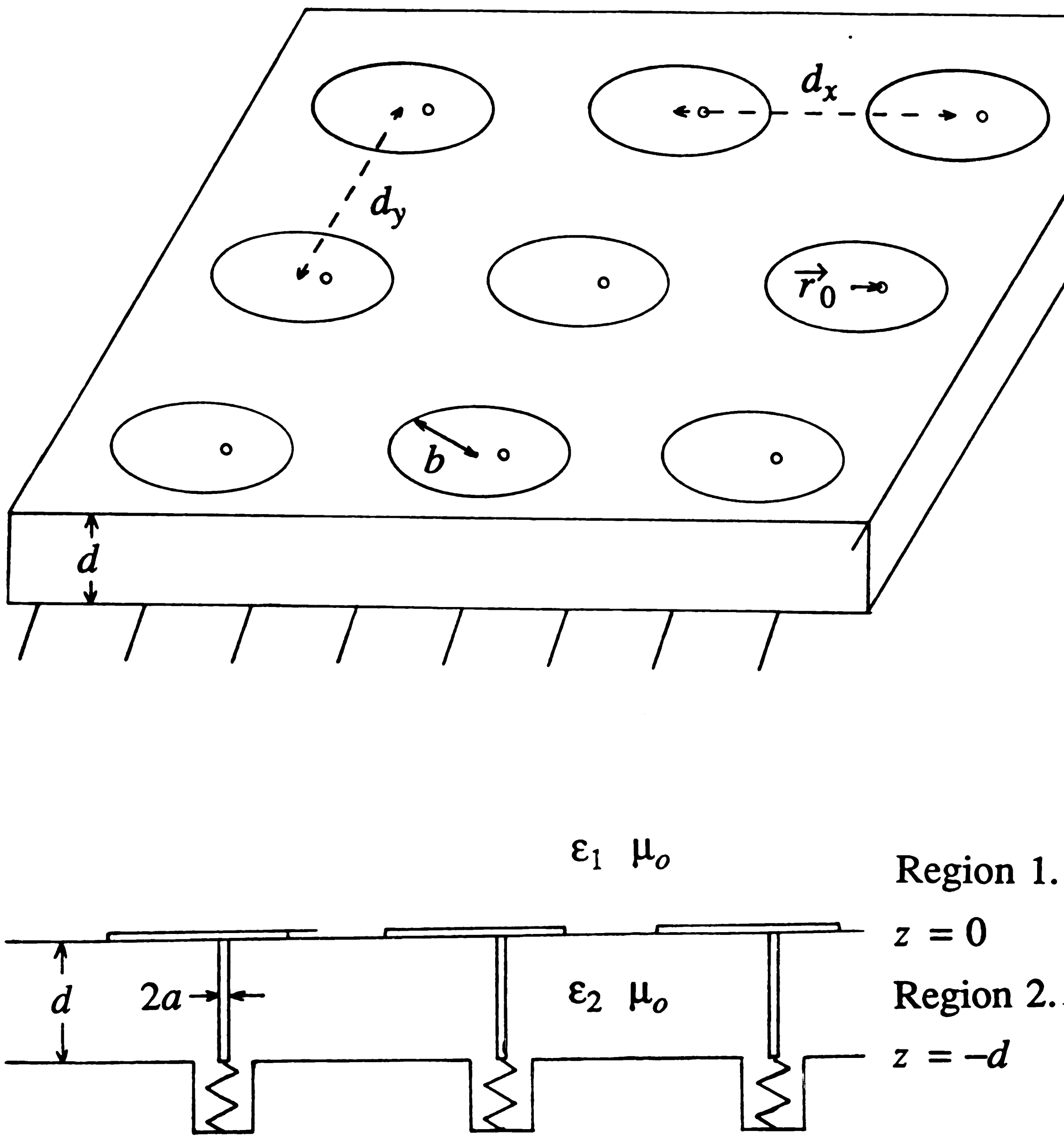


Figure 3. Geometry of an infinite rectangular array of patch antennas.

are the same as defined for the single patch antenna.

The decomposition of the receiving case into scattering and receiving cases is accomplished in the same manner as described in section 2.2 for the isolated patch.

6. Scattering and Transmission Cases.

For the case of an array of patch antennas, the excitation field is the same as that found for the isolated patch, for both the scattering and transmitting cases.

6.1 Green's Functions for Infinite Array.

The various Green function components for an isolated current element are obtained in chapter 13. The corresponding Green functions for an infinite array are obtained in section 13.12, expressed as a summation of Flouquet modes. For the infinite array, the Green functions for horizontal electric field at the dielectric interface due to horizontal currents at the interface are

$$g^{11}_{\alpha\beta'}(\vec{r} \mid \vec{r}\,') = \sum_{p,q=-\infty}^{\infty} g^{\infty}_{\alpha\beta'}(\vec{k}\,)\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad \alpha = x,y \qquad \beta' = x',y' \tag{1}$$

where

$$g^{\infty}_{xx}(\vec{k}\,) = \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{2}$$

$$g^{\infty}_{xy}(\vec{k}\,) = g^{\infty}_{yx}(\vec{k}\,) = \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{-k_x k_y \left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{3}$$

$$g^{\infty}_{yy}(\vec{k}\,) = \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{(k_1^2\varepsilon_r - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{4}$$

In the above,

$$\vec{k} = k_x\hat{x} + k_y\hat{y} = (k_1 u + p\frac{2\pi}{d_x})\,\hat{x} + (k_1 v + q\frac{2\pi}{d_y})\,\hat{y} \tag{5}$$

where u and v are given in equations 5(1) and 5(2). All other definitions needed are given in section 3.2.

The Green functions relating the vertical component of electric field in the dielectric to the horizontal currents on the dielectric-cover interface, and relating horizontal component of electric field on the dielectric-cover interface to vertical currents in the dielectric for the infinite structure are, from section 13.12,

$$g^{12}_{\alpha z'}(\vec{r} \mid \vec{r}\,') = \sum_{p,q=-\infty}^{\infty} g^{\infty}_{\alpha z'}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \cosh(p_2(z'+d)) \qquad \alpha = x,y \tag{6}$$

where

$$g^{\infty}_{xz'}(\vec{k}) \equiv \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{-jk_x}{T_m} \tag{7}$$

$$g^{\infty}_{yz'}(\vec{k}) \equiv \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{-jk_y}{T_m} \tag{8}$$

and

$$g^{21}_{z\beta'}(\vec{r} \mid \vec{r}\,') = \sum_{p,q=-\infty}^{\infty} g^{\infty}_{z\beta'}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \cosh(p_2(z+d)) \qquad \beta' = x',y' \tag{9}$$

where

$$g^{\infty}_{zx'}(\vec{k}) \equiv \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{jk_x}{T_m} \tag{10}$$

$$g^{\infty}_{zy'}(\vec{k}) \equiv \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{jk_y}{T_m} \tag{11}$$

Finally, the tangential electric field along the feed pin due to the current flowing in the feed pin is needed. The Green function component for vertical electric field in region 2 due to vertical current in region 2 for an infinite array is obtained in section 13.12 and is found to be,

$$g^{22}_{zz}(\vec{r} \mid \vec{r}\,') = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\epsilon_2} \Bigg\{ -\delta(z - z') + \left[\frac{k_2^2 + p_2^2}{p_2}\right] \frac{\cosh(p_2(z^< + d))}{T_m} \left[\frac{p_2}{p_1}\cosh(p_2 z^>) - \epsilon_r \sinh(p_2 z^>)\right] \Bigg\} d^2k \tag{12}$$

where $\delta(x)$ is the Dirac delta function, and

$$z^{>} = \max(z, z') \tag{13}$$

$$z^{<} = \min(z, z') \tag{14}$$

The spatial integrations introduced by Galerkin's method are performed prior to spectral summation.

6.2 Current Distributions.

For the infinite array, the Tchebychef current distributions used in the case of the single patch antenna are again used.

$$\vec{K}_{rlm} = \hat{r}\, K_{rlm} = \hat{r}\, T_m(r/b)\, e^{jl\phi} \sqrt{1 - \frac{r^2}{b^2}} \qquad l+m \text{ odd} \tag{1}$$

$$\vec{K}_{\phi lm} = \hat{\phi}\, K_{\phi lm} = \hat{\phi}\, T_m(r/b)\, e^{jl\phi} \frac{1}{\sqrt{1 - \frac{r^2}{b^2}}} \qquad l+m \text{ odd} \tag{2}$$

The current on the feed pin is also the same.

$$\vec{K}_z = 1\,\hat{z} \tag{3}$$

A singular current distribution is again used to model the current flowing from the feed pin onto the surface of the patch, but a different distribution is used here. For the infinite array, from a development in section 15.3,

$$\vec{K}_s(\vec{r}_p) = \frac{a\,\hat{r}_p}{r_p} \frac{\cos(\frac{\pi r_p}{2R_e})}{\cos(\frac{\pi a}{2R_e})} \tag{4}$$

where $R_e = R_e(\phi_p)$ is the distance from the center of the feed pin-patch junction to the edge of the patch. This distribution has the disadvantage of requiring numerical integration in one of the Galerkin's method integrations. It has the advantage of flowing over the entire patch surface. For a feed pin located very near the edge of the patch, this distribution would seem to be a more accurate model of the current no the patch than the other singular current, and fewer of the Tchebychef current distributions would be needed to accurately model the patch current.

6.3 Specific Matrix Formulation.

For the case of the infinite array of patch antennas, the current distributions used to model the patch and feed pin current are $\vec{K}_{rlm}$ and $\vec{K}_{\phi lm}$ from equations 7.2(1) and 7.2(2), along with $\vec{K}_s$ from 7.2(4) and $\vec{K}_z$ from 7.2(3). Only $\vec{K}_s$ differs from the current distributions used for the single patch antenna. The current amplitudes C_j are again the current amplitudes for the current distributions. The equation for $\vec{K}$ is

$$\vec{K} = \sum_{m=0}^{M} \sum_{l=-L}^{L} \left[C_{rlm}\, \vec{K}_{rlm} + C_{\phi lm}\, \vec{K}_{\phi lm}\right] + C_z \left[\vec{K}_s + \vec{K}_z\right] \tag{1}$$

where, as for the single patch antenna, $\vec{K}_z$ flows over the feed pin surface and all other current distributions flow over the patch surface.

The matrix equation becomes

$$\begin{bmatrix} Z_{r'l'm'}^{rlm} & Z_{\phi'l'm'}^{rlm} & Z_{z'}^{rlm} \\ Z_{r'l'm'}^{\phi lm} & Z_{\phi'l'm'}^{\phi lm} & Z_{z'}^{\phi lm} \\ Z_{r'l'm'}^{z} & Z_{\phi'l'm'}^{z} & Z_{z'}^{z} \end{bmatrix} \cdot \begin{bmatrix} C_{r'l'm'} \\ C_{\phi'l'm'} \\ C_{z'} \end{bmatrix} = \begin{bmatrix} V_{rlm} \\ V_{\phi lm} \\ V_z \end{bmatrix} \tag{2}$$

Letting γ represent either r or ϕ and letting δ' represent either r' or ϕ', (2) can be written more compactly as

$$\begin{bmatrix} Z_{\delta'l'm'}^{\gamma lm} & Z_{z'}^{\gamma lm} \\ Z_{\delta'l'm'}^{z} & Z_{z'}^{z} \end{bmatrix} \cdot \begin{bmatrix} C_{\delta'l'm'} \\ C_{z'} \end{bmatrix} = \begin{bmatrix} V_{\gamma lm} \\ V_z \end{bmatrix} \tag{3}$$

The matrix elements $Z_{\delta'l'm'}^{\gamma lm}$ are

$$Z_{\delta'l'm'}^{\gamma lm} = \iint_s ds\, \vec{K}_{\gamma lm} \cdot \iint_s ds'\, \bar{\bar{g}}^{1,1} \cdot \vec{K}_{\delta'l'm'} \tag{4}$$

where s is the surface of the patches.

The surface current distributions, which are purely radially or azimuthally directed, must be separated into $\hat{x}$ and $\hat{y}$ components, since $\bar{\bar{g}}$ is represented using rectangular components. Written component-wise, $\bar{\bar{g}}$ is

$$\overleftrightarrow{g} = \begin{bmatrix} g_{xx}^{1,1} & g_{xy}^{1,1} & g_{xz}^{1,2} \\ g_{yx}^{1,1} & g_{yy}^{1,1} & g_{yz}^{1,2} \\ g_{zx}^{2,1} & g_{zy}^{2,1} & g_{zz}^{2,2} \end{bmatrix} \tag{5}$$

where the superscripts on the components are those relevent to the case being considered. The two superscripts on the Green function dyadic components indicate whether the field and source integrals respectively are carried out in region 1, above the dielectric interface, or in region 2, within the dielectric. The current distributions are decomposed as

$$\begin{aligned} \vec{K}_{\gamma lm} &= (\vec{K}_{\gamma lm} \cdot \hat{x})\,\hat{x} + (\vec{K}_{\gamma lm} \cdot \hat{y})\,\hat{y} \\ &= K_{\gamma lmx}\hat{x} + K_{\gamma lmy}\hat{y} \\ &= \sum_{\alpha=x,y} K_{\gamma lm\alpha}\,\hat{\alpha} \end{aligned} \tag{6}$$

and similarly,

$$\vec{K}_{\delta l'm'} = \sum_{\beta'=x',y'} K_{\delta l'm'\beta'}\hat{\beta}' \tag{7}$$

Using (6) and (7), and carrying out the scalar products, (4) becomes

$$Z_{\delta l'm'}^{\gamma lm} = \sum_{\substack{\alpha=x,y \\ \beta'=x',y'}} \iint_s ds\, K_{\gamma lm\alpha} \iint_s ds'\, g_{\alpha\beta'}^{1,1}\, K_{\delta l'm'\beta'} \tag{8}$$

The matrix elements $Z_s^{\gamma lm}$ are

$$Z_s^{\gamma lm} = \iint_s ds\, \vec{K}_{\gamma lm} \cdot \left[\iint_{s_p} dz'\, a d\phi'_p\, \overleftrightarrow{g}^{1,2} \cdot \vec{K}_s + \iint_s ds'\, \overleftrightarrow{g}^{1,1} \cdot \vec{K}_s \right] \tag{9}$$

where s_p is the feed pin surface. The feed pin associated patch current distribution, $\vec{K}_s$ is separated into rectangular components as

$$\vec{K}_s = \sum_{\beta'=x',y'} K_{s\beta'}\hat{\beta}' \tag{10}$$

The testing function $\vec{K}_z$ has only a $\hat{z}$ component, and is

$$\vec{K}_z = 1\,\hat{z} \tag{11}$$

so (9) becomes

$$Z_z^{\gamma lm} = \sum_{\alpha=x,y} \iint_s ds\, K_{\gamma lm\alpha} \cdot \left[\iint_{s_p} dz'\, ad\phi'_p\, g_{\alpha z'}^{1,2} \frac{1}{2\pi a} + \sum_{\beta'=x',y'} \iint_s ds'\, g_{\alpha\beta'}^{1,1} K_{s\beta'} \right] \tag{12}$$

In the above equations, $K_{s\beta'}$ is the component in the $\hat{\beta}' = \hat{x}'$ or $\hat{y}'$ direction of the singular current distribution associated with the feed pin. The elements $Z_{\delta' l' m'}^z$ are

$$Z_{\delta' l' m'}^z = 2\pi a \int_{-d}^{0} dz\, \vec{K}_z \cdot \iint_s ds'\, \overline{\overline{g}}^{2,1} \cdot \vec{K}_{\delta' l' m'} \tag{13}$$

Testing is done along a line on the surface of the feed pin, hence the single testing integral. The factor $2\pi a$ in (13) is solely for dimensional consistancy with the rest of the formulation. Equation (13) becomes

$$Z_{\delta' l' m'}^z = 2\pi a \sum_{\beta'=x',y'} \int_{-d}^{0} dz \iint_s ds'\, g_{z\beta'}^{21}\, K_{\delta' l' m' \beta'} \tag{14}$$

The matrix element $Z_{z'}^z$ is

$$Z_{z'}^z = 2\pi a \int_{-d}^{0} dz\, \vec{K}_z \cdot \left[\iint_{s_p} dz' d\phi_p\, \overline{\overline{g}}^{2,2} \cdot \vec{K}_z + \iint_s ds'\, \overline{\overline{g}}^{2,1} \cdot \vec{K}_s \right] \tag{15}$$

This can be written as

$$Z_{z'}^z = 2\pi a \int_{-d}^{0} dz \left[\iint_{s_p} dz'\, ad\phi'_p\, g_{zz'}^{2,2} + \sum_{\beta'=x',y'} \iint_s ds'\, g_{z\beta'}^{2,1}\, K_{s\beta'} \right] \tag{16}$$

From 3.4(9), the elements $V_{\gamma lm}$ are

$$V_{\gamma lm} = - \iint_s ds\, \vec{K}_{\gamma lm} \cdot \vec{E}^{\,1} \tag{17}$$

and the element V_z

$$V_z = - 2\pi a \int_{-d}^{0} dz\, \hat{z} \cdot \vec{E}^{\,2} \tag{18}$$

The patch testing functions $\vec{K}_{\gamma lm}$ here don't need to be separated into $\hat{x}$ and $\hat{y}$ components. The impressed electric fields $\vec{E}^1$ and $\vec{E}^2$ are the fields in regions 1 and 2 respectively generated by the incident plane wave, $\vec{E}^i$. This field is separated into two components, $\vec{E}_{||}^i$ and $\vec{E}_{\perp}^i$, parallel and perpendicular to the plane of incidence, as shown in chapter 13. The fields $\vec{E}^1$ and $\vec{E}^2$ are the also separated into two components, and (17) and (18) become

$$V_{\gamma lm} = - \iint_s ds\ \vec{K}_{\gamma lm} \cdot \vec{E}_{||}^1 - \iint_s ds\ \vec{K}_{\gamma lm} \cdot \vec{E}_{\perp}^1 \tag{19}$$

$$V_z = - 2\pi a \int_{-d}^{0} dz\ \hat{z} \cdot \vec{E}_{||}^2 - 2\pi a \int_{-d}^{0} dz\ \hat{z} \cdot \vec{E}_{\perp}^2 \tag{20}$$

The matrix elements can be manipulated into a less complicated form. Substituting 7.1(1) through 7.1(4) into (8), the matrix element becomes

$$Z_{\delta l'm'}^{\gamma lm} = \sum_{\substack{\alpha = x,y \\ \beta = x,y}} \iint_s ds\ K_{\gamma lm\alpha} \iint_s ds' \sum_{p,q=-\infty}^{\infty} \bar{\bar{g}}_{\alpha\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} K_{\delta l'm'\beta} \tag{21}$$

Rearranging the order of integration and summation, this becomes

$$Z_{\delta l'm'}^{\gamma lm} = \sum_{p,q=-\infty}^{\infty} \sum_{\substack{\alpha = x,y \\ \beta = x,y}} \left\{ \left[\iint_s ds\ K_{\gamma lm\alpha}\, e^{j\vec{k}\cdot\vec{r}} \right] \left[\iint_s ds'\ K_{\delta l'm'\beta}\, e^{-j\vec{k}\cdot\vec{r}'} \right] \bar{\bar{g}}_{\alpha\beta}(\vec{k}) \right\} \tag{22}$$

Define

$$\iint_s ds\ K_{rlmx}\, e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{rlmx}^{\pm}(\vec{k}) \tag{23}$$

$$\iint_s ds\ K_{rlmy}\, e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{rlmy}^{\pm}(\vec{k}) \tag{24}$$

$$\iint_s ds\ K_{\phi lmx}\, e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{\phi lmx}^{\pm}(\vec{k}) \tag{25}$$

$$\iint_s ds\ K_{rlmy}\, e^{\pm j\vec{k}\cdot\vec{r}} \equiv I_{\phi lmy}^{\pm}(\vec{k}) \tag{26}$$

Using (23) through (26), (22) becomes

$$Z_{\delta l'm'}^{\gamma lm} = \sum_{p,q=-\infty}^{\infty} \sum_{\substack{\alpha=x,y \\ \beta=x,y}} I_{\gamma lm\alpha}^{+}(\vec{k})\, I_{\bar{\delta l'm'}\beta}^{-}(\vec{k})\, \bar{\bar{g}}_{\alpha\beta}(\vec{k}) \tag{27}$$

Substituting 7.1(12) and 7.1(13) into (14),

$$Z_{\delta l'm'}^{z} = 2\pi a \sum_{\beta=x,y} \int_{-d}^{0} dz \; 1 \iint_{s} ds' \, K_{\delta l'm'\beta} \sum_{p,q=-\infty}^{\infty} \bar{\bar{g}}_{z\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z+d)) \tag{28}$$

In the above equation, $\vec{r}$ is constant since the z integral takes place along a single line on the feed pin at $\vec{r} = \vec{r}_o + \vec{a}_o$, where $\vec{r}_o$ is the position vector of the center of the feed pin junction at the patch, and $\vec{a}_o$ is the vector in the plane of the patches from the feed pin center to the point on the circumference where the electric field along the feed pin is forced to zero. Equation (3.7.28) can then be rearranged to become

$$Z_{\delta l'm'}^{z} = 2\pi a \sum_{p,q=-\infty}^{\infty} e^{j\vec{k}\cdot(\vec{r}_o+\vec{a}_o)} \left[\int_{-d}^{0} dz \, \cosh(p_2(z+d))\right] \sum_{\beta=x,y} \bar{\bar{g}}_{z\beta}(\vec{k}) \iint_{s} ds' \, K_{\delta l'm'\beta}\, e^{-j\vec{k}\cdot\vec{r}'} \tag{29}$$

The integral over z is evaluated immediately as,

$$\int_{-d}^{0} dz \, \cosh(p_2(z+d)) = \frac{1}{p_2}(\sinh(p_2 d) - \sinh(0)) = \frac{1}{p_2}\sinh(P_2 d) \tag{30}$$

and the second integral, involving $K_{s\beta}$ is defined the same as for the isolated patch, the only difference being the current distribution $\vec{K}_s$.

$$\iint_{s} ds' \, e^{\pm j\vec{k}\cdot\vec{r}} K_{s\beta} \equiv I_{s\beta}^{\pm} \qquad \beta=x,y \tag{31}$$

Using (30), (31) and (23) through (26), (29) becomes

$$Z_{\delta l'm'}^{z} = 2\pi a \sum_{p,q=-\infty}^{\infty} e^{j\vec{k}\cdot\vec{a}_o} \frac{\sinh(p_2 d)}{p_2} \sum_{\beta=x,y} \bar{\bar{g}}_{z\beta}(\vec{k})\, I_{\bar{\delta l'm'}\beta}^{-}(\vec{k}) \tag{32}$$

Substituting 7.1(1) through 7.1(4) and 7.1(9) and 7.1(10) into (12),

$$Z_{z}^{\gamma lm} = \sum_{\alpha=x,y} \iint_{s} dz' \, a\, d\phi'_p \, K_{\gamma lm\alpha} \cdot \left[\iint_{s_p} ds' \sum_{p,q=-\infty}^{\infty} \bar{\bar{g}}_{\alpha z}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z'+d)) + \right.$$

$$\left. + \sum_{\beta=x,y} \iint_{s} ds' \sum_{p,q=-\infty}^{\infty} \bar{\bar{g}}_{\alpha\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} K_{s\beta}\right] \tag{33}$$

In (33), both summations over p and q, and the exponential factor contained within them, are identical. They can thus be combined into a single summation, and (33) can be rearranged as

$$Z_z^{\gamma lm} = \sum_{p,q=-\infty}^{\infty} \sum_{\alpha=x,y} \left\{ \left[\iint_s ds\ K_{\gamma lm\alpha}\ e^{j\vec{k}\cdot\vec{r}} \right] \times \right.$$

$$\left. \times \left[g_{\alpha z}(\vec{k}) \iint_{s_p} dz'\ a d\phi'_p\ e^{j\vec{k}\cdot\vec{r}'} \frac{\cosh(p_2(z'+d))}{2\pi a} + \sum_{\beta'=x',y'} g_{\alpha\beta'}(\vec{k}) \iint_s ds\ e^{j\vec{k}\cdot\vec{r}'} K_{s\beta'} \right] \right\} \quad (34)$$

This leaves three integrals to be evaluated. The first, involving $K_{\gamma lm\alpha}$ is defined in (23) through (26). The last has been defined in equation (31) and the remaining integral can be evaluated immediately. For $\vec{r}'$ on the surface of the feed pin,

$$\vec{r}' = \vec{r}_o + \vec{a} + z\ \hat{z} \quad (35)$$

where $\vec{a}$ is a vector from $\vec{r}_o$ to the surface of the feed pin at the patch-feed pin junction. The magnitude of $\vec{a}$ is simply a. Thus,

$$\iint_{s_p} ds'\ e^{-j\vec{k}\cdot\vec{r}} \cosh(p_2(z'+d)) = e^{-j\vec{k}\cdot\vec{r}_o} \int_{-d}^{0} dz' \cosh(p_2(z'+d)) \int_0^{2\pi} d\phi'\ a\ e^{-j\vec{k}\cdot\vec{a}}$$

$$= 2\pi a\ e^{-j\vec{k}\cdot\vec{r}'_o} \left(\frac{\sinh(p_2 d)}{p_2}\right) J_0(ka) \quad (36)$$

where the integral over z' is the same as done in (7.3.30), and the integral over ϕ_p has been evaluated to J_0, the ordinary Bessel function of zero order, using

$$\int_0^{2\pi} e^{-j\vec{k}\cdot\vec{a}}\ d\phi_p = \int_0^{2\pi} e^{-jka\cos(\phi_p - \theta)}\ d\phi_p = 2\pi\ J_o(ka) \quad (37)$$

Using (23) through (26), (31) and (36), (34) becomes

$$Z_z^{\gamma lm} = \sum_{p,q=-\infty}^{\infty} \sum_{\alpha=x,y} I^{+}_{\gamma lm\alpha} \left[g_{\alpha z}^{-}(\vec{k}) \left(\frac{\sinh(p_2 d)}{p_2}\right) 2\pi a\ J_0(ka) + \sum_{\beta=x,y} g_{\alpha\beta}^{-}(\vec{k})\ I_{s\beta}^{-} \right] \quad (38)$$

Using 7.1(12), 7.1(13) and 7.1(14), (16) becomes

$$Z_z^s = 2\pi a \int_{-d}^{0} dz \left\{ \sum_{\beta=x,y} \iint_s ds' \sum_{p,q=-\infty}^{\infty} g_{z\bar{\beta}'}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z+d))\, K_{s\beta'} \right.$$

$$+ \int_0^{2\pi}\int_{-d}^{0} dz'\, a\, d\phi'_p \sum_{p,q=-\infty}^{\infty} \left[\frac{1}{j\omega\varepsilon_2 d_x d_y}\right] e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \times$$

$$\left. \times \left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1} \, \frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>}) - \bar{\varepsilon} p_1 \sinh(p_2 z^{>}))}{T_m} \right] \right\} \qquad (39)$$

Rearranging, this becomes

$$Z_z^s = 2\pi a \sum_{p,q=-\infty}^{\infty} \left\{ \sum_{\beta=x,y} g_{z\bar{\beta}}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}_o+\vec{a}_o)} \int_{-d}^{0} \cosh(p_2(z+d))\, dz \iint_s ds'\, e^{-j\vec{k}\cdot\vec{r}'} K_{s\beta} + \right.$$

$$+ a\, \frac{e^{j\vec{k}\cdot(\vec{r}_0+\vec{a}_0-\vec{r}_0)}}{j\omega\varepsilon_2 d_x d_y} \int_0^{2\pi} e^{-j\vec{k}\cdot\vec{a}}\, d\phi_p \int_{-d}^{0}\int_{-d}^{0} dz\, dz' \times$$

$$\left. \times \left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1} \, \frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>}) - \bar{\varepsilon} p_1 \sinh(p_2 z^{>}))}{T_m} \right] \right\} \qquad (40)$$

Using (30), (31) and (37), (40) becomes

$$Z_z^s = 2\pi a \sum_{p,q=-\infty}^{\infty} \left\{ \left[\sum_{\beta=x,y} \frac{\sinh(p_2 d)}{p_2}\, g_{z\bar{\beta}}(\vec{k})\, e^{j\vec{k}\cdot\vec{a}_o}\, e^{j\vec{k}\cdot\vec{r}_o}\, I_{s\bar{\beta}} \right] + \left[\frac{e^{j\vec{k}\cdot\vec{a}_o}}{j\omega\varepsilon_2 d_x d_y} J_o(ka)\, I_z \right] \right\} \qquad (41)$$

where

$$I_z = \int_{-d}^{0}\int_{-d}^{0} dz\, dz' \left[-\delta(z-z') + \frac{k_2^2+p_2^2}{p_2 p_1} \, \frac{\cosh(p_2(z^{<}+d))(p_2\cosh(p_2 z^{>}) - \bar{\varepsilon} p_1 \sinh(p_2 z^{>}))}{T_m} \right] \qquad (42)$$

Equation (42) has been evaluated in chapter 17 to be

$$I_z = d\, \frac{k_2^2}{p_2^2} - \frac{\bar{\varepsilon}\, k^2}{p_2^2\, T_m\, P_2} \sinh(p_2 d) \qquad (43)$$

Thus, (41) becomes

$$Z_z^s = 2\pi a \sum_{p,q=-\infty}^{\infty} \left\{ \sum_{\beta=x,y} \frac{\sinh(p_2 d)}{p_2}\, g_{z\bar{\beta}}(\vec{k})\, e^{j\vec{k}\cdot\vec{a}_o}\, e^{j\vec{k}\cdot\vec{r}_o}\, I_{s\bar{\beta}} \right.$$

$$\left. + \frac{e^{j\vec{k}\cdot\vec{a}_o}}{j\omega\varepsilon_2 d_x d_y} J_o(ka) \left[d\, \frac{k_2^2}{p_2^2} - \frac{\bar{\varepsilon}\, k^2}{p_2^2\, T_m\, p_2} \sinh(p_2 d) \right] \right\} \qquad (44)$$

The matrix elements for the incident field, (19) and (20) are the same as given in section 3.6 for the case of the single patch. The difference in the singular current distribution used in each case doesn't affect the source elements. Copying those here,

$$V_{rlm\,||} = \bar{E}^{1}_{||}\; e^{jl\phi_i}\; \pi\; j^{l+1} \left[A^{-}_{l+1,m}(k_1 \sin(\theta_i)) + A^{-}_{l-1,m}(k_1\sin(\theta_i))\right] \tag{45}$$

$$V_{\phi lm\,||} = \bar{E}^{1}_{||}\; e^{jl\phi_i}\; \pi\; j^{l} \left[A^{-}_{l+1,m}(k_1 \sin(\theta_i)) + A^{-}_{l-1,m}(k_1\sin(\theta_i))\right] \tag{46}$$

$$V_z = -4\pi a\; T_{||}\; E^{i}_{||}\; e^{jk_2\bar{x}\sin(\theta_r) + d\cos(\theta_r)}\; \frac{\sin(\theta_r)}{k_2\cos(\theta_r)} \tag{47}$$

where $A^{\pm}_{l,m}$ and $B^{\pm}_{l,m}$ are obtained in section 17 as

$$A^{\pm}_{l,m} = \frac{\pi b^2}{4}\left[J_{\frac{|l|+m-1}{2}}\left(\frac{kb}{2}\right) J_{\frac{|l|-m+1}{2}}\left(\frac{kb}{2}\right) + J_{\frac{|l|+m+1}{2}}\left(\frac{kb}{2}\right) J_{\frac{|l|-m-1}{2}}\left(\frac{kb}{2}\right)\right]$$

$$\cdot\begin{cases} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l<0 \end{cases} \tag{48}$$

$$B^{\pm}_{l,m} = \frac{1}{4}\left[2\,A^{\pm}_{l,m} - A^{\pm}_{l,|m-2|} - A^{\pm}_{l,m+2}\right] = \frac{1}{4}\left[2\,A^{\pm}_{l,m} - A^{\pm}_{l,m-2} - A^{\pm}_{l,m+2}\right] \tag{49}$$

and are evaluated there. For the case of perpendicular incidence, the source matrix elements are

$$V_{rlm\,\perp} = -\,\bar{E}^{1}_{\perp}\; e^{jl\phi_i}\; \pi\; j^{l} \left[B^{-}_{l+1,m}(k_1 \sin(\theta_i)) + B^{-}_{l-1,m}(k_1\sin(\theta_i)\right] \tag{50}$$

$$V_{\phi lm\,\perp} = -\,\bar{E}^{1}_{\perp}\; e^{jl\phi_i}\; \pi\; j^{l+1} \left[A^{-}_{l+1,m}(k_1 \sin(\theta_i)) - A^{-}_{l-1,m}(k_1\sin(\theta_i))\right] \tag{51}$$

$$V_z = 0 \tag{52}$$

6.4 Convergence of Feed pin-Feed pin Matrix Element.

Equation 6.3(44) is rewritten here.

$$Z_z^z = 2\pi a \sum_{p,q=-\infty}^{\infty} \left\{ \sum_{\beta=x,y} \frac{\sinh(p_2 d)}{p_2} g_{z\beta}^{\infty}(\vec{k})\, e^{j\vec{k}\cdot\vec{a}_o}\, e^{j\vec{k}\cdot\vec{r}_o}\, I_{s\beta}^{-} \right.$$

$$\left. + \frac{e^{j\vec{k}\cdot\vec{a}_o}}{j\omega\varepsilon_2 d_x d_y} J_o(ka) \left[d\, \frac{k_2^2}{p_2^2} - \frac{\bar{\varepsilon}\, k^2}{p_2^2\, T_m\, p_2} \sinh(p_2 d) \right] \right. \tag{1}$$

Looking at the second term in equation (1) as $k \to \infty$, it would seem that the summation wouldn't converge. This is seen most easily by considering that

$$\frac{1}{d_x\, d_y} \sum_{p,q=-\infty}^{\infty}$$

crudely approximates an integration of the form

$$\iint d^2k = \int_0^{\infty}\int_0^{2\pi} k\; dk\; d\theta$$

Performing the integral over $d\theta$, the exponential $e^{j\vec{k}\cdot\vec{a}_o}$ becomes $J_0(ka)$. The integrand is then asymptotically proportional to $J_0^2(ka)$, and the integral diverges. The solution to this difficulty lies in the term $I_{s\beta'}^{-}$ which implicitly is of the form close enough to $J_0(ka)$ capable of canceling this divergent term. To see this more clearly requires some investigation.

From 6.1(10) and 6.1(11)

$$g_{zx'}^{\infty}(\vec{k}) = \frac{1}{j\omega\varepsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{jk_x}{T_m} \tag{2}$$

$$g_{zy'}^{\infty}(\vec{k}) = \frac{1}{j\omega\varepsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{jk_y}{T_m} \tag{3}$$

and from 14.3(13)

$$I_{s\beta'}^{-} = e^{j\vec{k}\cdot\vec{r}_0} \int_0^{2\pi} d\phi_p\; \hat{r}_p \cdot \hat{\beta}'\; \frac{1}{\cos\left[\dfrac{\pi a}{2R_E}\right]}\, I_R^{-} \tag{4}$$

The dot product above becomes for the two cases of β

$$\hat{r}_p \cdot \hat{x} = \cos(\phi_p) \qquad \hat{r}_p \cdot \hat{y} = \sin(\phi_p) \tag{5a,b}$$

A Bessel identity [16] gives,

$$J_0(ka) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{jka\cos(\phi-\theta)} \tag{6}$$

Using equations (2) through (6), using $k_2^2 = k^2 - p_2^2$ and changing the dummy variable ϕ_p in (4) and (5) to ϕ, (1) becomes

$$Z_z^z = \sum_{p,q=-\infty}^{\infty} \frac{2\pi a \sinh(p_2 d)\, e^{j\vec{k}\cdot\vec{a}_o}}{j\omega\epsilon_1 d_x d_y \, p_2 \, T_m} \int_0^{2\pi} d\phi \left[\frac{jk(\cos\phi\cos\theta + \sin\phi\sin\theta)\, I_R^-}{\cos\left[\frac{\pi a}{2R_e}\right]} - \frac{1}{2\pi} e^{jka\cos(\phi-\theta)} \right] +$$

$$+ \sum_{p,q=-\infty}^{\infty} \frac{2\pi a \, e^{j\vec{k}\cdot\vec{a}_o} k_2^2}{j\omega\epsilon_2 d_x d_y \, p_2} \left[d - \frac{\epsilon_r \sinh(p_2 d)}{T_m \, p_2} \right] J_0(ka) \tag{7}$$

The last summation in (7) is convergent. Using $\cos\phi\cos\theta + \sin\phi\sin\theta = \cos(\phi-\theta)$, the integral in the first summation becomes

$$\int_0^{2\pi} d\phi \left[\frac{jk\cos(\phi-\theta)\, I_R^-}{\cos\left[\frac{\pi a}{2R_e}\right]} - \frac{e^{jka\cos(\phi-\theta)}}{2\pi} \right] \tag{8}$$

From 15.3(12), with $\phi_p \to \phi$

$$I_R^- = \frac{-jR_e}{2\pi(\pi - 2R_e k\cos(\phi-\theta))} \left[e^{j(\frac{\pi}{2} - kR_e\cos(\phi-\theta))} - e^{j(\frac{\pi a}{2R_e} - ka\cos(\phi-\theta))} \right]$$

$$- \frac{jR_e}{2\pi(\pi + 2R_e k\cos(\phi-\theta))} \left[e^{-j(\frac{\pi}{2} + kR_e\cos(\phi-\theta))} - e^{-j(\frac{\pi a}{2R_e} + ka\cos(\phi-\theta))} \right] \tag{9}$$

Since the dependance on $\phi-\theta$ is cyclic, and since the integration covers a complete cycle, let $\phi-\theta \to \phi$ in (8) and (9). Bringing both terms in (8) over a common denominator, (8) becomes

$$\int_0^{2\pi} d\phi \frac{kR_e\cos(\phi)\left[je^{-jkR_e\cos\phi} - e^{j\frac{\pi a}{2R_e}}e^{-jka\cos\phi}\right] - (\pi/2 - kR_e\cos\phi)\, e^{-jka\cos\phi}\cos\left[\frac{\pi a}{2R_e}\right]}{2\pi\,(\pi - 2R_e k\cos\phi)\cos\left[\frac{\pi a}{2R_e}\right]} +$$

$$+ \int_0^{2\pi} d\phi \frac{-kR_e\cos(\phi)\left[-je^{-jkR_e\cos\phi} - e^{-j\frac{\pi a}{2R_e}}e^{-jka\cos\phi}\right] - (\pi/2 + kR_e\cos\phi)\, e^{-jka\cos\phi}\cos\left[\frac{\pi a}{2R_e}\right]}{2\pi\,(\pi + 2R_e k\cos\phi)\cos\left[\frac{\pi a}{2R_e}\right]} \quad (10)$$

Using Euler's identity,

$$e^{\pm j\frac{\pi a}{2R_e}} = \cos\left[\frac{\pi a}{2R_e}\right] \pm j\sin\left[\frac{\pi a}{2R_e}\right] \quad (11)$$

in (10), the cosine portion of $e^{\pm j\frac{\pi a}{2R_e}}$ cancels.

$$\int_0^{2\pi} d\phi \frac{kR_e\cos(\phi)\left[je^{-jkR_e\cos\phi} - j\sin\left[\frac{\pi a}{2R_e}\right]e^{-jka\cos\phi}\right] - \frac{\pi}{2}\,e^{-jka\cos\phi}\cos\left[\frac{\pi a}{2R_e}\right]}{2\pi\,(\pi - 2R_e k\cos\phi\,)\cos\left[\frac{\pi a}{2R_e}\right]} +$$

$$+ \int_0^{2\pi} d\phi \frac{kR_e\cos(\phi)\left[je^{-jkR_e\cos\phi} - j\sin\left[\frac{\pi a}{2R_e}\right]e^{-jka\cos\phi}\right] - \frac{\pi}{2}\,e^{-jka\cos\phi}\cos\left[\frac{\pi a}{2R_e}\right]}{2\pi\,(\pi + 2R_e k\cos\phi\,)\cos\left[\frac{\pi a}{2R_e}\right]} \quad (12)$$

Putting each integral in (12) over a common denominator, and combining the two integrands together, (12) becomes

$$\int_0^{2\pi} d\phi \frac{2\pi\, kR_e\cos(\phi)\left[je^{-jkR_e\cos\phi} - j\sin\left[\frac{\pi a}{2R_e}\right]e^{-jka\cos\phi}\right] - \pi^2\,e^{-jka\cos\phi}\cos\left[\frac{\pi a}{2R_e}\right]}{2\pi\,(\pi^2 - 4k^2R_e^2\cos^2\phi\,)\cos\left[\frac{\pi a}{2R_e}\right]} \quad (13)$$

or

$$\int_0^{2\pi} d\phi \left[\frac{jkR_e\cos(\phi)\,e^{-jkR_e\cos\phi}}{(\pi^2 - 4k^2R_e^2\cos^2\phi\,)\cos\left[\frac{\pi a}{2R_e}\right]} + \right. \quad (14)$$

$$+ \frac{jkR_e \cos(\phi)\, e^{-jka\cos\phi} \left[\frac{\pi}{2} \cos\left[\frac{\pi a}{2R_e}\right] - \sin\left[\frac{\pi a}{2R_e}\right] \right]}{2\pi\, (\pi^2 - 4k^2 R_e^2 \cos^2\phi\,) \cos\left[\frac{\pi a}{2R_e}\right]} \Bigg]$$

Looking at (14) as $k \to \infty$, the term with $e^{-jka\cos\phi}$ will integrate to give something similar to $\frac{1}{k} J_0(ka)$. This will multiply coherently with $e^{j\vec{k}\cdot\vec{a}_0}$ in the Floquet summation, but the factor of $\frac{1}{k}$ will ensure convergence. The other term integrates to give something similar to $\frac{1}{k} J_0(k\ \ R_e\ \ )$ as $k \to \infty$, where R_e is some value in the range of $R_e(\phi)$, and so converges also. Thus, the summation in equation (7) and hence (1) converges, so the matrix element exists.

7. NUMERICAL RESULTS.

Galerkin's method solutions to the integral equations of both the scattering and transmitting cases have been implemented on a Cray X-MP/48 supercomputer at Pittsburgh Supercomputing Center. The matrix elements are computed by evaluating the summation in 6.3(27), 6.3(32), 6.3(38) and 6.3(44) over a range of $\vec{k}_{pq}$ sufficiently large to ensure convergence. Filling the Z-matrix for 6.2(44) takes by far the most computation time. Total C.P.U. time for each frequency is about 150 seconds, over 140 of which are spent filling the Z-matrix. Once it is filled, the scattering case is solved with electric field polarization both parallel and perpendicular to the plane of incidence. The current I_1 and the patch currents $\vec{K}_1$ are obtained for both of these cases. The transmission case is then solved and the input impedance and $\vec{K}_2$ are obtained. For an assumed load impedance, the total currents on both the feed pin and patch surface are computed using 6.3(3).

Numerical results are obtained for input impedance, feed pin current, and power received over the frequency range 3 to 10 GHz, for both cases of electric field polarization. The dimensions and parameters of the antenna array are chosen as follows.

$b = 1.0\ cm.$

$d = 0.15875\ cm.$

$\varepsilon_1 = \varepsilon_o$

$\varepsilon_2 = 2.5\varepsilon_o$

$a = 0.05\ cm.$

$d_x = d_y = 3.0\ cm.$

$\vec{r}_o = x_o\ \hat{x} + y_o\ \hat{y} = 0.3cm\ \hat{x}$

Nearly normal incidence is chosen, with $\theta_i = 0.0001$ and $\phi_i = 0$ ($\theta_i = 0$ can't be chosen due to indeterminant forms in the solution). For parallel polarization, the electric field is parallel to the patch diameter containing the feed pin center. The incident field strength is taken to be 1 V/m. For matching the electric field to zero on the feed pin, the line along $\phi_p = \frac{\pi}{4}$ is chosen. In 6.3(44), the term $e^{j\vec{k}\cdot\vec{a}_0}$ interferes coherently with $J_0(ka)$ in those terms in the summation over p and q for which $|\vec{k}\cdot\vec{a}_0| \approx ka$. Choosing $phis_p = \frac{\pi}{4}$ puts these terms on the diagonal of the square region of $\vec{k}$ space covered by p and q both ranging over $\pm M$. A range on the expansion functions of $l = -5$ to 5, $m = 0$ to 5, and on p and q of -25 to 25 has been found to give adequate convergence of the matrix elements over the frequency range covered. For the feed pin-feed pin matrix element, this range is increased to ±70 Results are found for two cases of load impedance, $Z_L = 50\,\Omega$ and $Z_L = Z_{in}^*$, the complex conjugate of Z_{in} (Z_L matched).

The scattered feed pin current, input resistance and input reactance are calculated by solving 6.3(2) for each of the scatering and transmitting cases and using 2.2(4). For the case of parallel polarization, the current magnitude on the feed pin for the scattering case, $|I_1|$, is shown in Figure 4. The input resistance and reactance from solving the transmission case are shown in Figures 5 and 6 respectively. The feed pin current for the receiving case is found via 2.2(6) and is shown in Figure 7 for $Z_L = 50\,\Omega$. The power received for both $Z_L = 50\,\Omega$ and $Z_L = Z_{in}^*$ is found using 2.2(7) and is shown in Figure 8.

The lowest order resonance appears between 5 and 6 GHz as a peak in the feed pin current and input resistance. The resonance peaks for the total feed pin current and for the input resistance both occur near 5.2 GHz, and the input reactance passes through zero near these points. For $Z_L = 50\,\Omega$, the power absorbed also peaks at 5.2 GHz. This resonance behavior is very similar to that reported for a single circular patch [10]. For Z_L matched, the power received is nearly constant for frequencies below about 9.7 GHz. The current flow on the patch for the dominant mode, near 5.2 GHz, is shown in Figure 9 for $Z_L = 50\ \Omega$, and is similar to that expected from the simple cavity model for a single element [13]. Current is flowing from the patch onto the feed pin. The simple cavity model gives 5.56 GHz as the resonant frequency of the lowest order mode, slightly higher than that obtained here. It is found that resonances occur consistantly at lower frequencies than given by the cavity model for other resonances also, and that the difference becomes more pronounced at the higher order resonances. Due to mutual interaction between the elements and the presence of the feed pin, the simple cavity model is not expected to give more than qualitative results. For the scattering case, however, the frequency for peak feed pin current is extremely close, occurring between 5.5 and 5.6 GHz

Higher order resonance behavior can be observed above 8 GHz. The first incidence of this occurs around 8.4 GHz, where the feed pin current goes through a null followed by a sharp peak. The input reactance becomes very large in magnitude, and changes sign, at the frequency where the null in the feed pin current occurs. This type of behavior is well known as an anti-resonance in the study of various antenna structures. At 8.44 GHz, the feed pin current and power received for $Z_L = 50\ \Omega$ are near maximum. At this frequency, current flow on the patch surface is as shown in Figure 10 for $Z_L = 50\ \Omega$, with current flowing out of the feed pin. From the cavity model, current flow such as that shown in Figure 10 is expected at the second resonance, though the cavity model gives 9.22 GHz as the resonant frequency of the second mode.

At 9.45 GHz, the feed pin current goes through another null. The input reactance again becomes very large in magnitude and changes sign. This anti-resonance is less sharp than the one at 8.44 GHz, both in the current null and in the approach of the reactance towards infinity. For both of these cases, the input resistance remains almost constant in spite of the large fluctuations of the input reactance.

Between 9.6 and 9.9 GHz, the feed pin current and power absorbed reach maximums while the current distribution on the patch changes form. At 9.75 GHz, the scattered feed pin current rises to a large peak, greater even than that of the principle resonance. The patch current is shown in Figure 11, and again, current is flowing out of the feed pin. Its overall magnitude is smaller than the current at either of the two lower resonances. The patch current distribution in Figure 11 can be separated into two main components--a purely radial component which is expected for the third resonance from the cavity model at 11.6 GHz, and a smaller component similar to the form shown in Figure 9.

At 9.85 GHz, the input resistance encounters a jump in magnitude. The current distribution on the patch surface is shown in Figure 12, with current flowing out of the feed pin. This distribution is a combination of current similar to that shown in Figure 9 with a current distribution with 3-fold rotational symmetry. The cavity model predicts a current distribution with this 3-fold symmetry for the fourth resonance, at 12.7 GHz.

Between 9.75 GHz and 9.85 GHz, the numerical solution breaks down. For the dimensions we used, the lowest order TM_o surface wave mode has a wave length of 3 cm (the spacing between the feed pins) at 9.8 GHz. As noted in [14], the zeros of T_m correspond to surface wave poles. When the Poisson summation is carried out near this frequency, $T_m \approx 0$ for the term with $p = q = 0$. Since T_m occurs in the denominator of the Green functions, the solution becomes numerically unstable.

To understand physically why an instability results at this frequency, consider the current on each feed pin and patch as a source for a surface wave. For frequencies away from 9.8 GHz, the surface waves from each element in the array add out of phase and tend to cancel. At 9.8 GHz, these surface waves add in phase along the array axes, and would radiate strongly in these directions for a finite array. Since the analysis is for an infinite array, the level of this radiation would be unbounded, so no steady state solution exists.

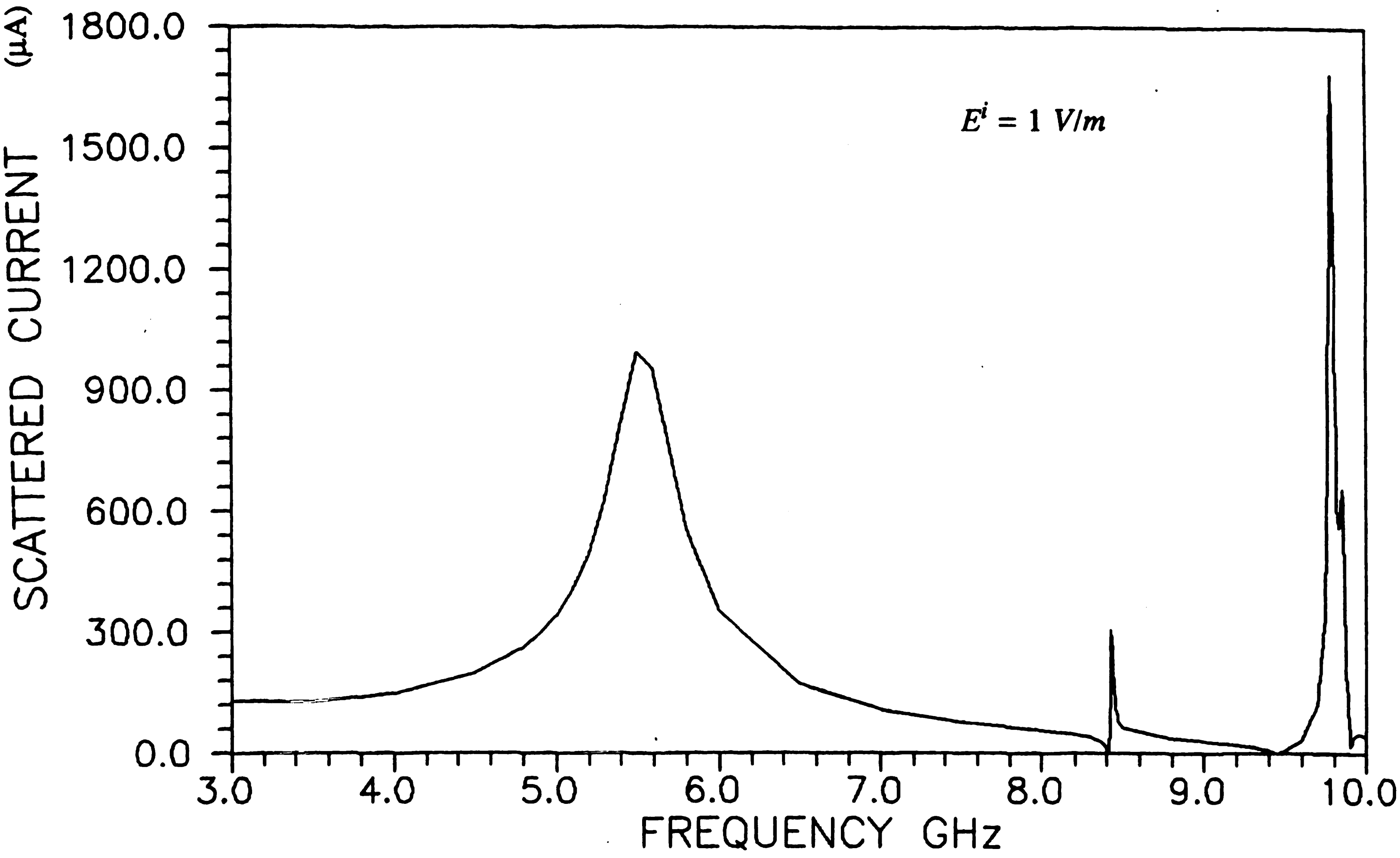


Figure 4. Scattered current magnitude for parallel electric field polarization.

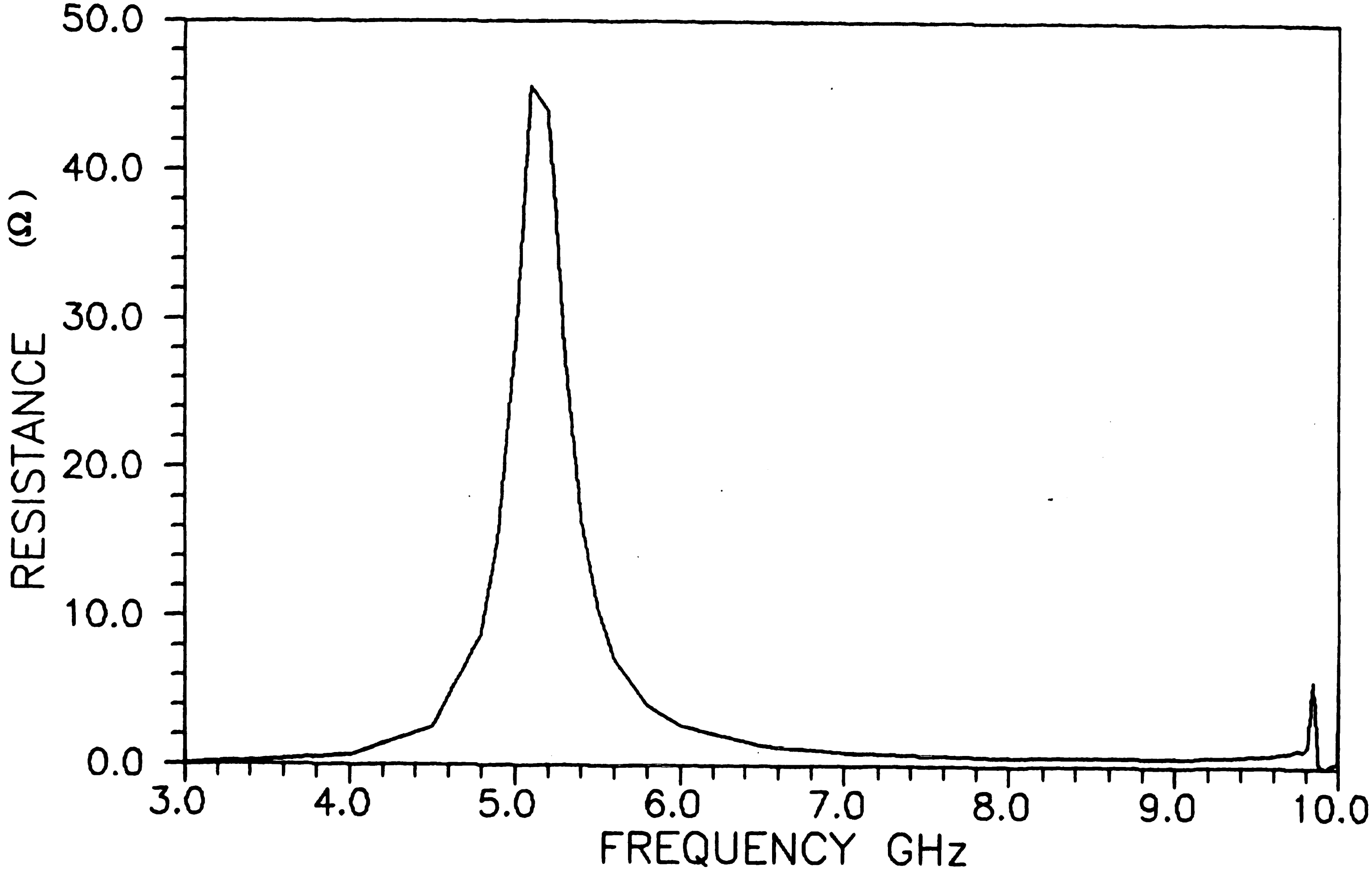


Figure 5. Input resistance for each element of array.

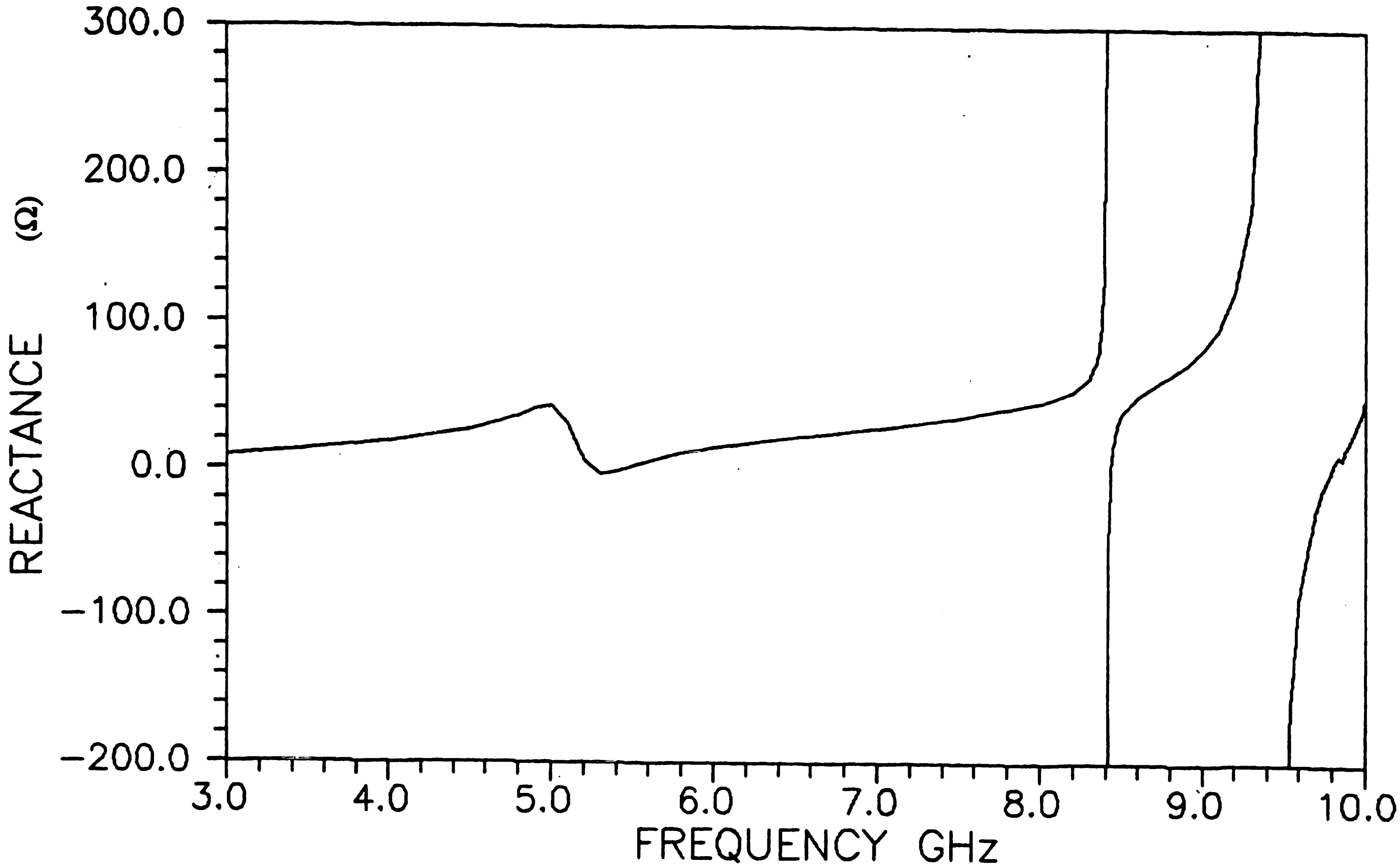


Figure 6. Input reactance for each element of array.

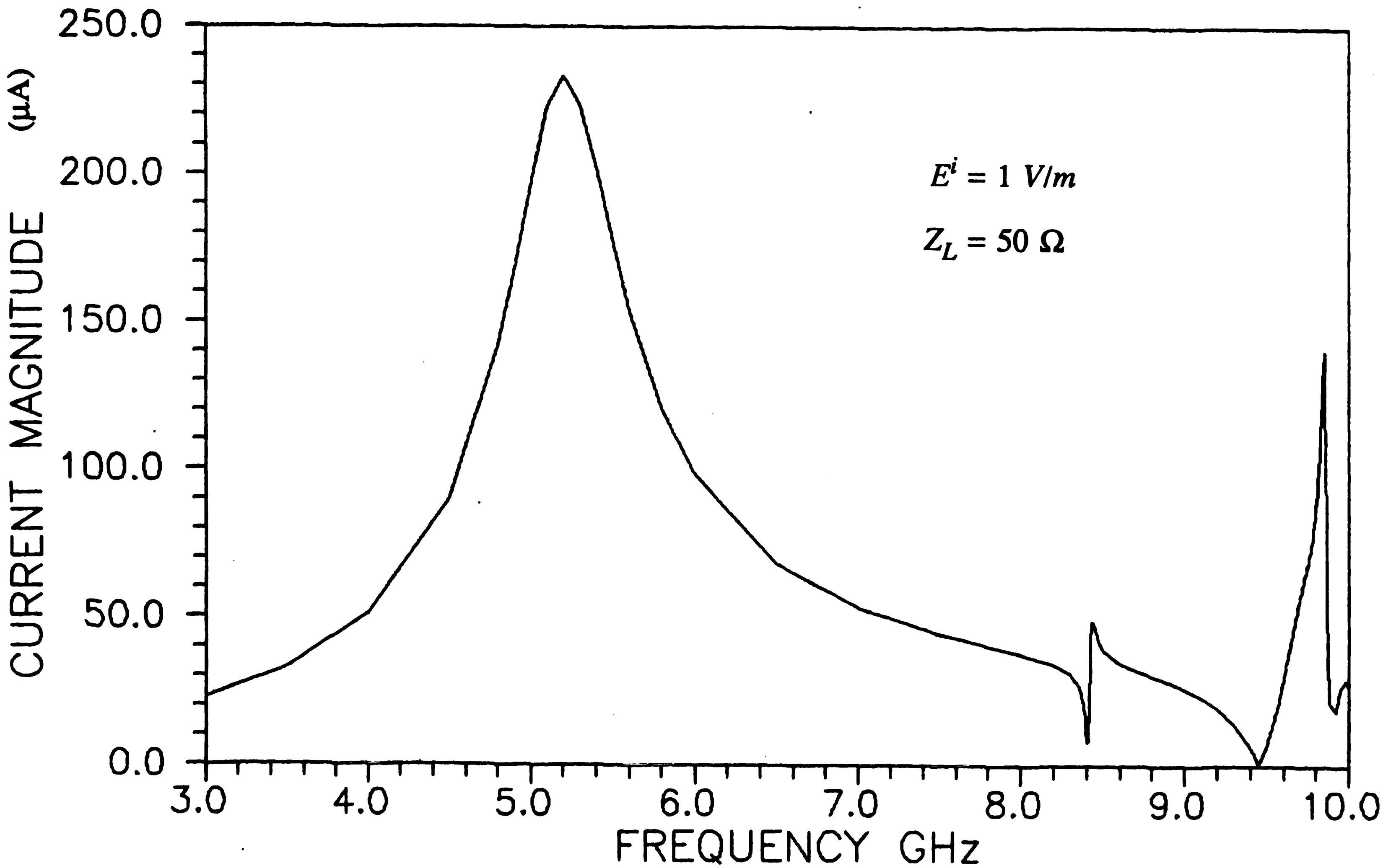


Figure 7. Total feed pin current magnitude for parallel electric filed polarization.

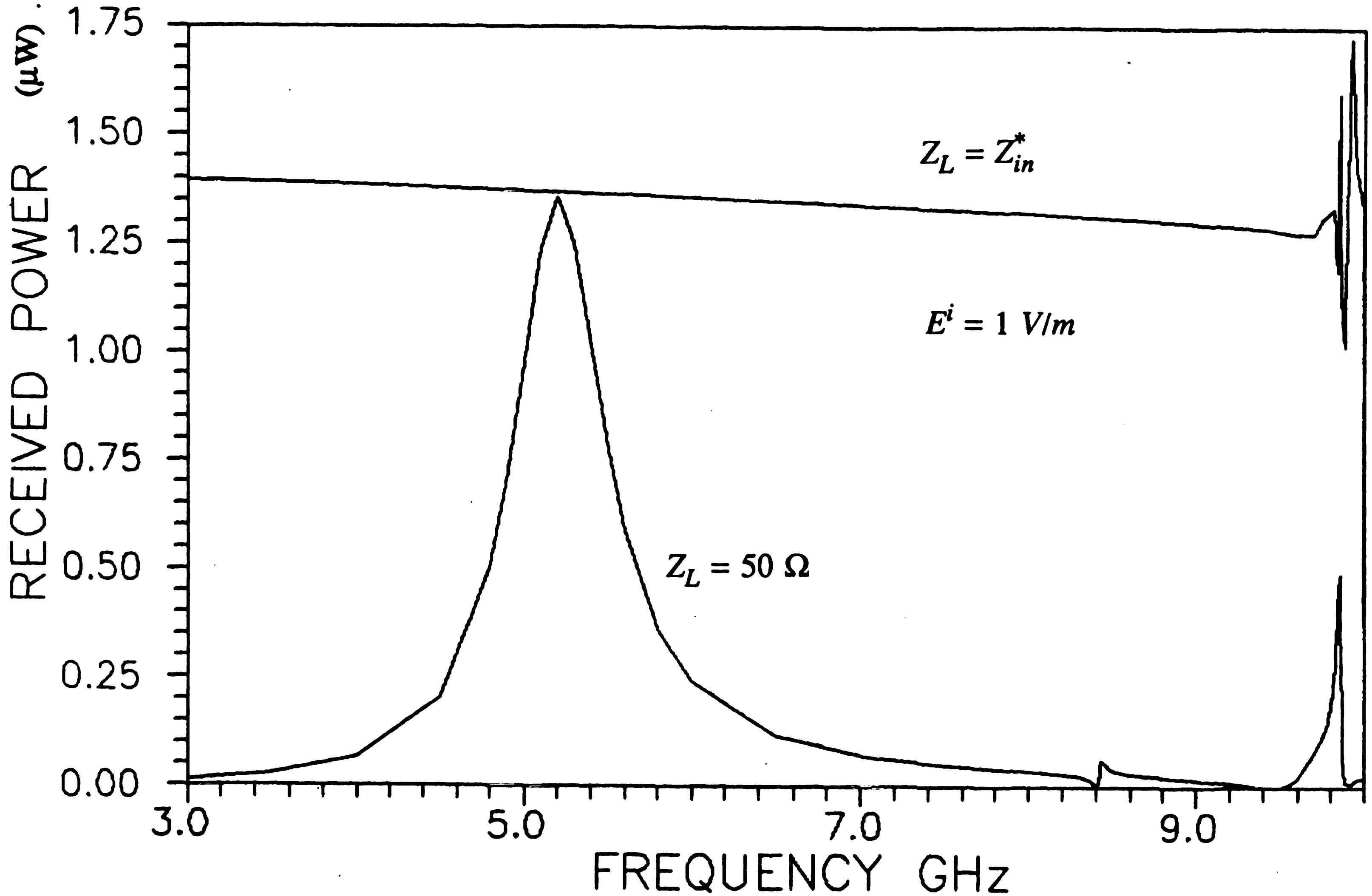


Figure 8. Received power for parallel electric field polarization.

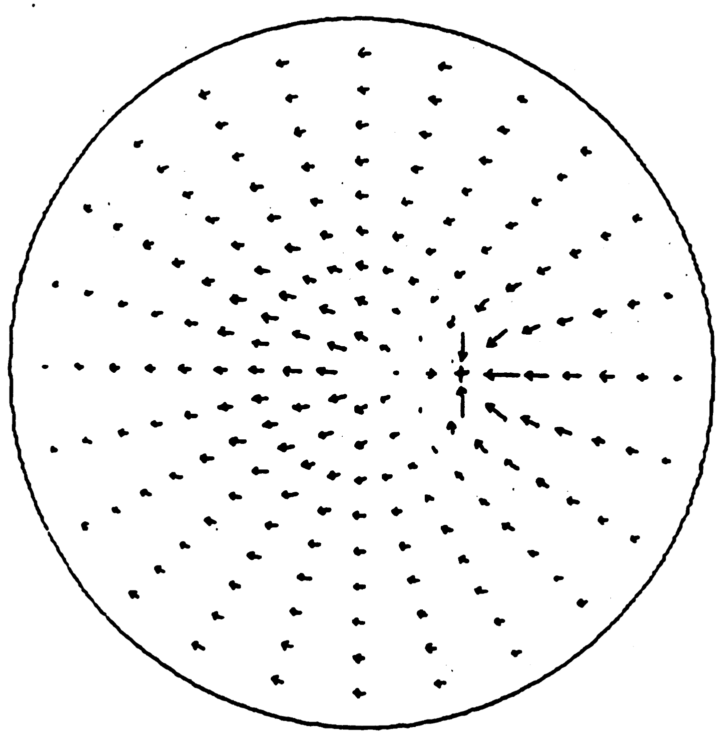


Figure 9. Current distribution at 5.2 GHz.

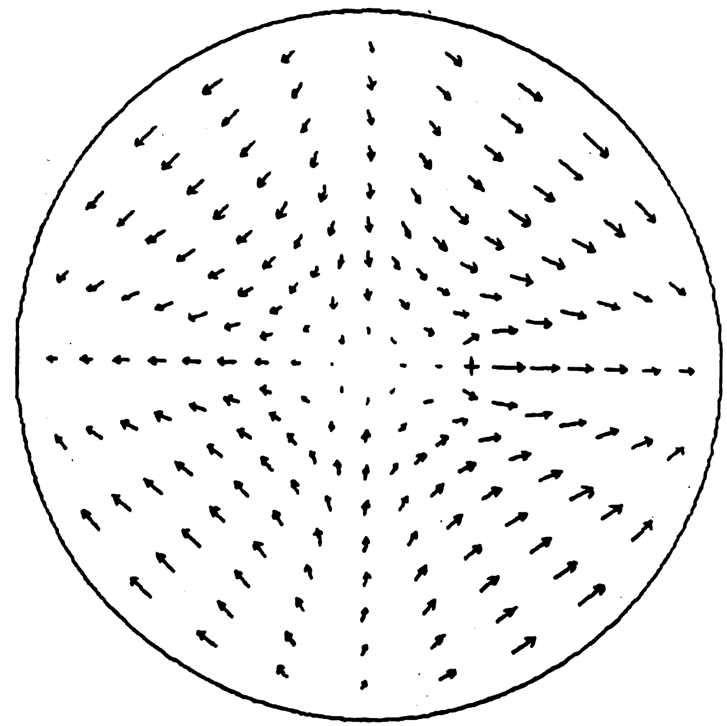


Figure 10. Current distribution at 8.44 GHz.

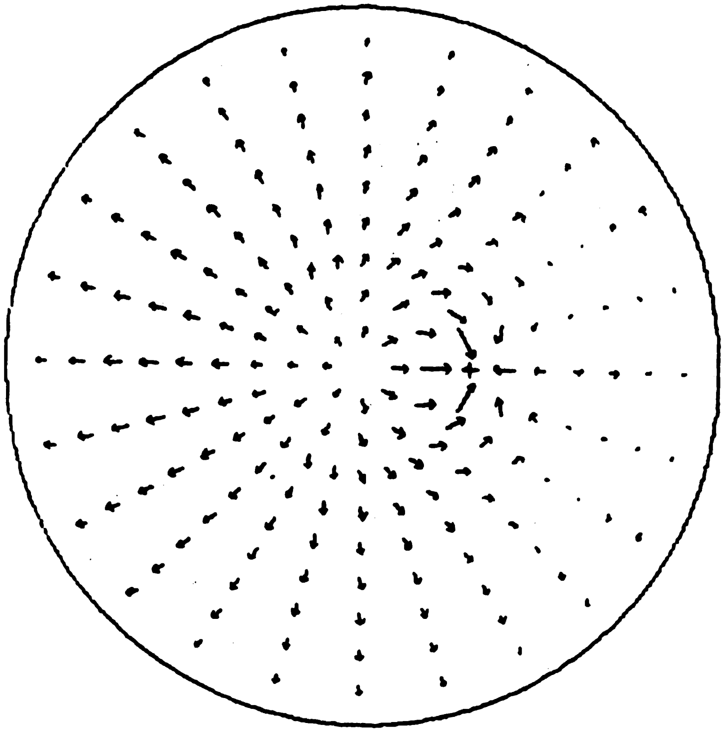


Figure 11. Current distribution at 9.75 GHz.

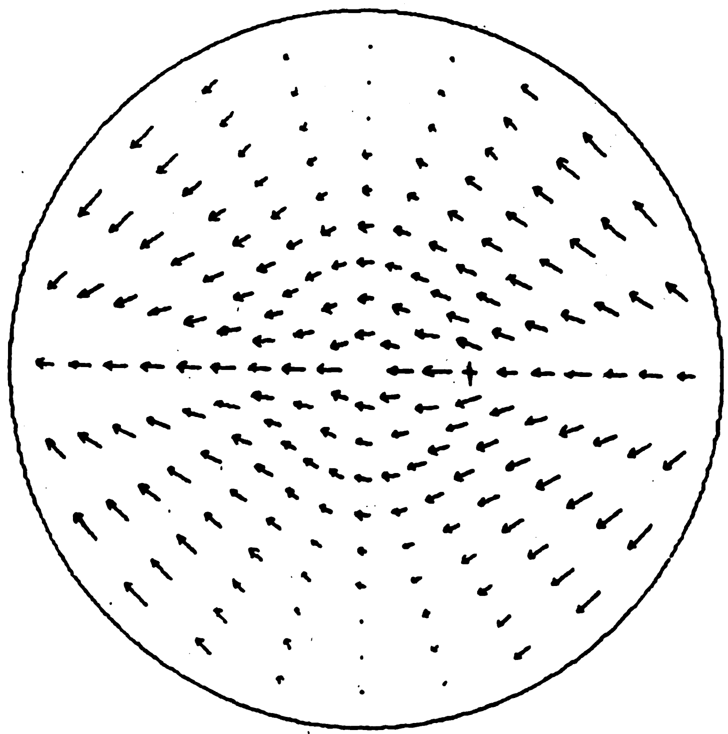


Figure 12. Current distribution at 9.85 GHz.

For the case of perpendicular polarization, the electric field is perpendicular to the diameter containing the feed pin. The feed pin current for the scattering case is shown in Figure 13, and the feed pin current for the receiving case and with $Z_L = 50\,\Omega$ is shown in Figure 14. The input impedance is the same as for parallel polarization, shown in Figures 5 and 6. The power received for a load impedance of $50\,\Omega$ is shown in Figure 15.

The feed pin current frequency responses for both scattered and receiving cases are similar to those obtained for parallel polarization, but at a greatly reduced magnitude. The magnitudes of the feed pin currents near the first resonance are smaller in proportion to the current over the rest of the frequency band covered than for the other polarization case. The major difference beyond this occurs between the third and fourth resonances, where the feed pin current dips in magnitude for both the scattered and receiving feed pin currents. This occurs right around 9.8 GHz, where the program becomes numerically unstable. The received power for Z_L equal to $50\,\Omega$ follows the same trends as the feed pin currents. The power received is much smaller in magnitude for perpendicular polarization than for parallel polarization, with a larger drop in proportion to the rest of the frequencies covered occuring near the first resonance. A dip in power received also occurs near 9.8 GHz.

The resonance behavior predicted here is qualitatively consistent with published data. Changes in substrate thickness and dielectric constant give shifts in the position of the dominant resonance frequency similar to trends described in [8]. In addition, the higher order resonance behavior seen in Figure 8 has been observed in experiments performed at Boeing Advanced Systems Company.

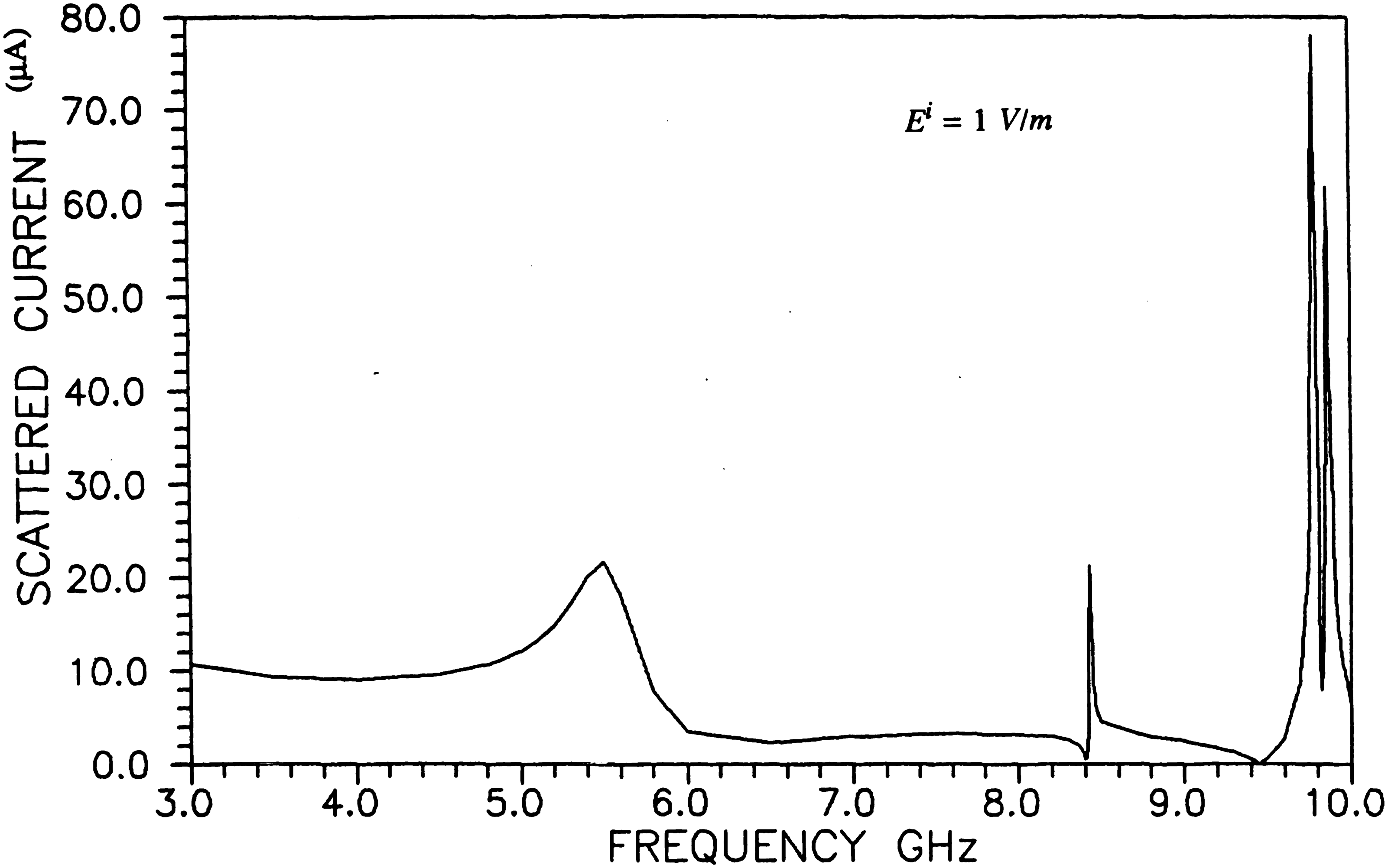


Figure 13. Scattered current magnitude for perpendicular electric field polarization.

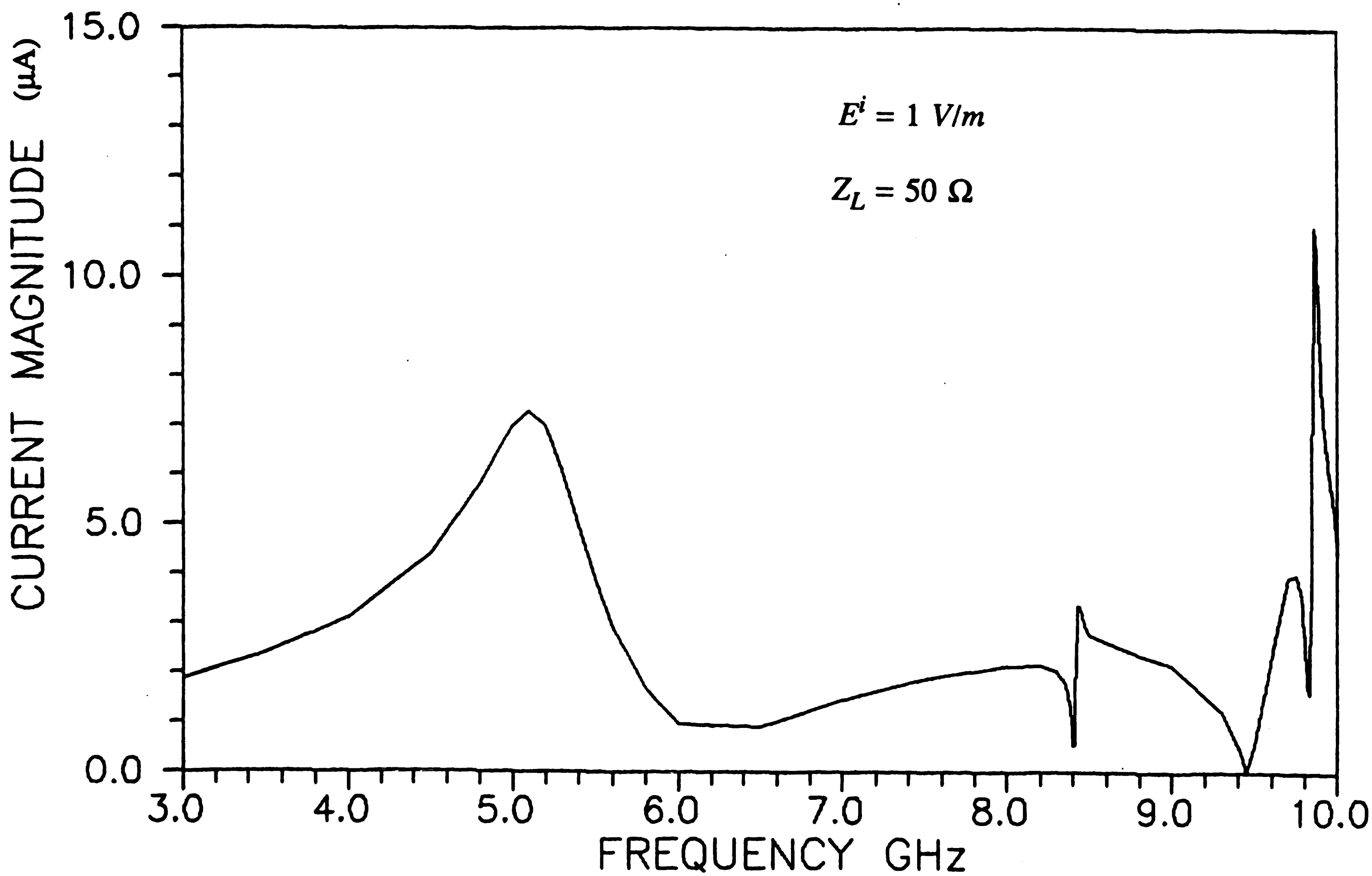


Figure 14. Total feed pin current magnitude for perpendicular electric field polarization.

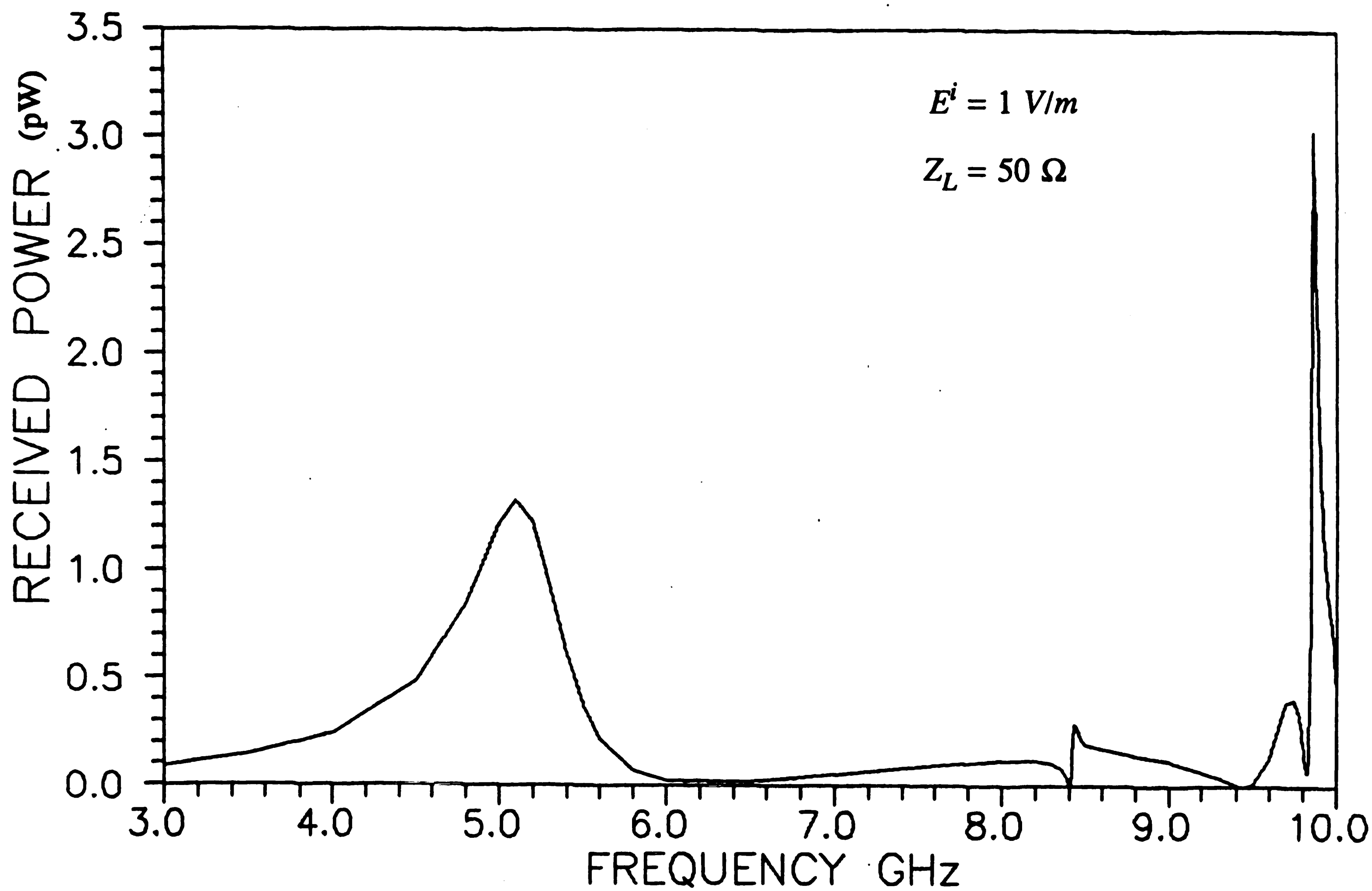


Figure 15. Received power for perpendicular electric field polarization: $Z_L = 50\Omega$.

IV. MICROSTRIP DIPOLE ANTENNA ARRAY

8. Problem Description.

8.1 Geometry.

The geometry of the infinite array of microstrip dipole patch antennas to be analyzed is depicted in figure 16. A dielectric substrate of permitivity ε_2, permeability μ_o and thickness d, (region 2), is located between the $z = 0$ and $z = -d$ planes. The dielectric is mounted on a conducting ground plane at $z = -d$, and is covered by a material with constitutive parameters ε_1 and μ_o (region 1). Each element has half-length b and half-width a. A load impedance Z_L is located at the dipole center. The microstrip dipoles are located at the dielectric-cover interface in the $z = 0$ plane, and are assumed to be perfectly conducting and infinitely thin. They are spaced distances d_x and d_y apart in the x and y directions respectively, their centers located at $\vec{r}_{pq} = p\, d_x\, \hat{x} + q\, d_y\, \hat{y}$ in the global coordinate system of the array, with p and q integers.

Illumination of the structure is taken to be through an incident plane wave of frequency ω at an arbitrary incidence angle. The plane wave is expressed in terms of a coordinate system x'', y'', z'' rotated with respect to the global coordinate system of the array, as shown in Figure 2, with ϕ_i the angle between the x and x'' axes, and θ_i the angle between the wave vector $\vec{k}^i$ and the z axis. The z and z'' axes coincide. The factors

$$u = \sin(\theta_i)\cos(\phi_i) \tag{1}$$

and

$$v = \sin(\theta_i)\sin(\phi_i) \tag{2}$$

are the direction cosines for the wave vector $\vec{k}^i$ with the $-x$ and $-y$ axes respectively.

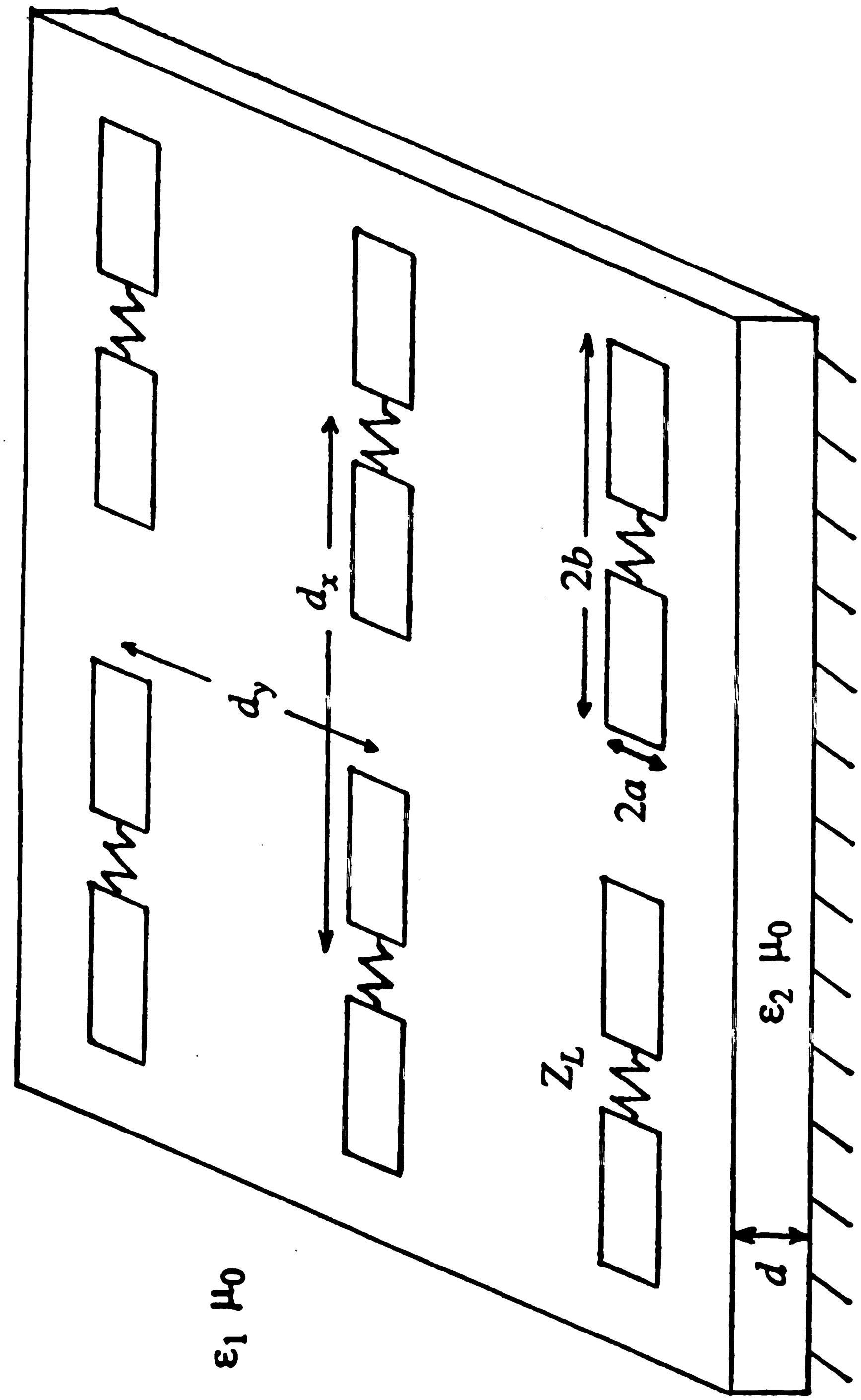


Figure 16. Geometry of an infinite array of strip dipoles.

8.2 Problem Decomposition.

Using the principle of superposition, the array acting as a receiver can be decomposed into scattering and transmission cases, as in Figure 17. In the scattering case, the load impedance for each element of the array is short-circuited at the dipole center, the array is illuminated by an incident plane wave, and the current I_1 at the dipole center is determined. For the transmission case, no illuminating plane wave is present. Instead, a driving voltage V is applied at the center of the feed pin, representing the voltage drop which would exist across the load in the receiving case due to current flowing through the load impedance. The resulting current at the dipole center, I_2, due to V is found, and the input impedance is then calculated from

$$Z_{in} = \frac{V}{I_2}. \tag{1}$$

Following the same development as section 2.2, the total current I is found to be

$$I = \left[\frac{Z_{in}}{Z_{in}+Z_L}\right] I_1 \tag{2}$$

and the voltage V across the load impedance in the receiving case is

$$V = \left[\frac{-I_1 Z_{in}}{Z_{in}+Z_L}\right] Z_L \tag{3}$$

The power received by the structure is then

$$P_L = \frac{1}{2} I^2 R_L = \frac{1}{2} |I_1|^2 \left|\frac{Z_{in}}{Z_{in}+Z_L}\right|^2 R_L \tag{4}$$

where R_L is the real part of Z_L.

To determine the current at the dipole center and the power delivered to the load, the scattering case is solved for I_1, and the transmission case is solved to obtain Z_{in}. Other quantities of the general receiving case can also be found using superposition of the scattering and transmitting cases.

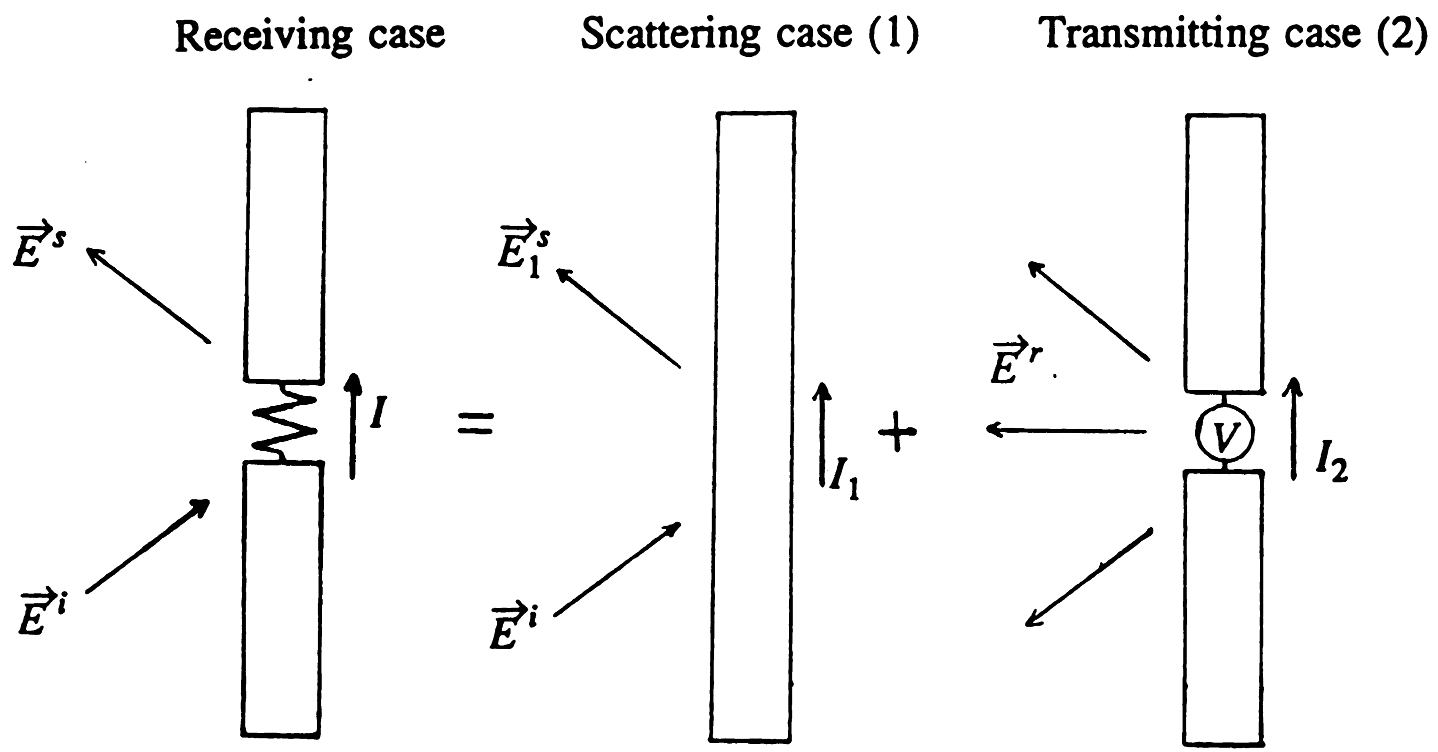


Figure 17. Decomposition of receiving case into transmitting and scattering cases.

9. Scattering Case.

9.1 Excitation Field.

Consider first the case where the short circuited dipole array is illuminated by an incident plane wave. The total excitation field is the field generated by the incident plane wave in the presence of the ground plane and dielectric coating. The incident plane wave electric field can be written as

$$\vec{E}^{\,i}(\vec{r}) = E^i\,\hat{\gamma}\,e^{-j\vec{k}^i\cdot\vec{r}} \tag{1}$$

where $\hat{\gamma}$ describes the incident polarization. By applying boundary conditions at the conducting plane and at the dielectric-cover region interface, the total electric field in the dielectric and cover regions can be found. This is done in chapter 12, and the results are given here. They are

$$\vec{E}_1^{\,e}(\vec{r}) = \vec{E}_{1\perp}(\vec{r}) + \vec{E}_{1\,||}(\vec{r}) \qquad z \geq 0 \tag{2}$$

where

$$\vec{E}_{1\,||}(\vec{r}) = 2\,E^i_{||}\,e^{j(\frac{1}{2}\phi_{||}+k_1x''\sin\theta_i)} \times$$

$$\times\left[\hat{z}j\sin(\frac{1}{2}\phi_{||}-k_1z\cos\theta_i)\sin\theta_i + \hat{x}''\cos(\frac{1}{2}\phi_{||}-k_1z\cos\theta_i)\cos\theta_i\right] \qquad z \geq 0 \tag{3}$$

and

$$\vec{E}_{1\perp}(\vec{r}) = 2\,E^i_{\perp}\,e^{j(\frac{1}{2}\phi_{\perp}+k_1x''\sin\theta_i)}\,\hat{y}''\cos(\frac{1}{2}\phi_{\perp}-k_1z\cos\theta_i) \qquad z \geq 0 \tag{4}$$

with

$$\Gamma_{||} = \frac{\eta_2\cos\theta_r\sin(k_2d\cos\theta_r) + j\eta_1\cos\theta_i\cos(k_2d\cos\theta_r)}{\eta_2\cos\theta_r\sin(k_2d\cos{+}\theta_r) - j\eta_1\cos\theta_i\cos(k_2d\cos\theta_r)} \tag{5}$$

$$\Gamma_{\perp} = \frac{\eta_2\cos\theta_i\sin(k_2d\cos\theta_r) + j\eta_1\cos\theta_r\cos(k_2d\cos\theta_r)}{\eta_2\cos\theta_i\sin(k_2d\cos\theta_r) - j\eta_1\cos\theta_r\cos(k_2d\cos\theta_r)} \tag{6}$$

In the above, the following relations and definitions hold.

$$k_1 = \omega\sqrt{\mu_o \varepsilon_1} \tag{7a}$$

$$k_2 = \omega\sqrt{\mu_o \varepsilon_2} \tag{7b}$$

$$x'' = x\cos\phi_i + y\sin\phi_i \tag{7c}$$

$$y'' = -x\sin\phi_i + y\cos\phi_i \tag{7d}$$

$$\hat{x}'' = \hat{x}\cos\phi_i + \hat{y}\sin\phi_i \tag{7e}$$

$$\hat{y}'' = -\hat{x}\sin\phi_i + \hat{y}\cos\phi_i \tag{7f}$$

$$\phi_i = \tan^{-1}\left[\frac{k_y^i}{k_x^i}\right] \tag{7g}$$

$$k^i = k_1 \tag{7h}$$

$$\theta_i = \cos^{-1}\left[\frac{k_z^i}{k^i}\right] \tag{7i}$$

$$\phi_{||} = -j\ \log_e(\Gamma_{||}) \quad or \quad \Gamma_{||} = e^{j\phi_{||}} \tag{7j}$$

$$\phi_{\perp} = -j\ \log_e(\Gamma_{\perp}) \quad or \quad \Gamma_{\perp} = e^{j\phi_{\perp}} \tag{7k}$$

$$\eta_1 = \sqrt{\frac{\mu_o}{\varepsilon_1}} \tag{7l}$$

$$\eta_2 = \sqrt{\frac{\mu_o}{\varepsilon_2}} \tag{7m}$$

$$\theta_r = \cos^{-1}\left(\frac{k_1}{k_2}\sin\theta_i\right) \tag{7n}$$

9.2 Green Functions for Scattered Field.

The electric field supported by the induced surface currents on the dipoles is determined using a Fourier transform Green function approach. The electric field is written as an integral of the product of a Green function dyadic and the induced currents. The field $\vec{E}(\vec{r})$ due to an infinite array of dipoles, where the central element has surface current $\vec{K}(\vec{r}\,')$ is

$$\vec{E}(\vec{r}) = \iint_s \overline{\overline{g}}(\vec{r} \mid \vec{r}\,') \cdot \vec{K}(\vec{r}\,')\, ds' \tag{1}$$

Here $\overline{\overline{g}}(\vec{r} \mid \vec{r}\,')$ is the infinite array Green dyadic for electric field at the interface due to currents on the interface and s is the surface of the central patch element. The currents of the other dipoles are identical to the center dipole to within a phase factor. The phase factor is due to the progression of the phase of the incident plane wave at each dipole location.

The tangential electric field over the dipole surfaces, generated by currents induced on the dipoles, is to be set to zero. The Green functions relating horizontal components of electric field on the dielectric-cover interface to horizontal components of current for an infinite array used for the patch array are used here as well. They are

$$g^{11}_{\alpha\beta}(\vec{r} \mid \vec{r}\,') = \sum_{p,q=-\infty}^{\infty} g_{\alpha\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad \alpha,\beta = x,y \tag{2}$$

where

$$g_{xx}^{\infty}(\vec{k}) = \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{(k_1^2\overline{\varepsilon} - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\overline{\varepsilon}p_1 + p_2\tanh(p_2 d)\right]} \tag{3}$$

$$g_{xy}^{\infty}(\vec{k}) = g_{yx}^{\infty}(\vec{k}) = \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{-k_x k_y \left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\overline{\varepsilon}p_1 + p_2\tanh(p_2 d)\right]} \tag{4}$$

$$g_{yy}^{=}(\vec{k}) = \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{(k_1^2\bar{\varepsilon} - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\bar{\varepsilon}p_1 + p_2\tanh(p_2 d)\right]} \tag{5}$$

In the above,

$$\vec{k} = k_x\hat{x} + k_y\hat{y} = (k_1 u + p\frac{2\pi}{d_x})\hat{x} + (k_1 v + q\frac{2\pi}{d_y})\hat{y} \tag{6a}$$

where u and v are given in equations (9.1.1) and (9.1.2),

$$d^2k = dk_x\, dk_y \tag{6b}$$

$$p_1^2 = k_x^2 + k_y^2 - k_1^2 \tag{6c}$$

$$p_2^2 = k_x^2 + k_y^2 - k_2^2 \tag{6d}$$

$$k_1 = \omega\sqrt{\mu_o\varepsilon_1} \tag{6e}$$

$$k_2 = \omega\sqrt{\mu_o\varepsilon_2} \tag{6f}$$

$$\bar{\varepsilon} = \frac{\varepsilon_2}{\varepsilon_1} \tag{6g}$$

and

$$T_e = \sinh(p_2 d) + \frac{p_2}{p_1}\cosh(p_2 d) \tag{7}$$

$$T_m = \bar{\varepsilon}\cosh(p_2 d) + \frac{p_2}{p_1}\sinh(p_2 d) \tag{8}$$

The divergent nature of the above summations is overcome by the spatial integrations introduced by Galerkin's method. When these integrals are performed prior to spectral summation, the resulting summations are all convergent.

Only x-directed currents are assumed, and only the x component of the electric field will be tested in the application of Galerkin's method. Thus, the only component of the dyadic Green function needed is the x–x component, 9.2(3).

9.3 Derivation of Coupled Integral Equations.

The induced currents on the dipole surfaces must satisfy an integral equation constructed by employing the boundary conditions that the tangential electric field over the dipole surfaces must be zero. The scattered electric field on the dipole surface resulting from currents on the dipole surface s is given by

$$\vec{E}_1^{\,s}(x,y,z{=}0) = \iint_s \overleftrightarrow{\mathcal{G}}^{1,1}(x,y,z{=}0 \mid x',y',z'{=}0) \cdot \vec{K}(x',y')\, dx'dy' \tag{1}$$

The boundary condition on the tangential electric field at the strip dipole surface yields an integral equation

$$\hat{z} \times \left[\vec{E}_1^{\,s}(x,y,z{=}0) + \vec{E}_1^{\,e}(x,y,z{=}0)\right] = 0 \tag{2}$$

where $\vec{E}_1^{\,e}$ is the total excitation field in the cover region due to the incident plane wave. This equation must hold at all points on the dipole surface.

9.4 Matrix Formulation.

The induced current on the strip dipole surfaces is to be expressed as a summation over a set of current distributions. The solution for the unknown amplitudes of the chosen current distributions is to be formulated as a matrix equation. This will be done using Galerkin's method--the boundary conditions on the surface of the dipoles will be forced to hold in an integral sense where the same set of current distributions will be used as testing functions. The boundary condition used is the physical constraint that the electric field tangential to the dipole surface be zero.

Let $\vec{E}^i$ be the electric field due to the incident plane wave and let $\vec{E}^s$ be the scattered electric field due to induced currents on the conducting surfaces of the strip dipoles. Then

$$\vec{E}^s = \iint_s \bar{\bar{\gamma}} \cdot \vec{K}\, ds' \tag{1}$$

Here, s is the conducting surface of the antenna structure, $\vec{K}$ is the surface current on s and $\bar{\bar{\gamma}}$ is the dyadic Green function. The boundary condition on s requires that

$$\hat{t} \cdot (\vec{E}^i + \vec{E}^s) = 0 \tag{2}$$

for all unit vectors, $\hat{t}$, tangential to the surface s.

Now let

$$\vec{K} = \sum_{n=1}^{\infty} C_n \vec{K}_n \tag{3}$$

where $\{\vec{K}_n\}$ is an appropriate set of independent and complete current distributions in $\vec{K}$, and where C_n is the corresponding amplitude of $\vec{K}_n$ in $\vec{K}$. Then

$$\vec{E}^s = \iint_s \bar{\bar{\gamma}} \cdot \left[\sum_{n=1}^{\infty} C_n \vec{K}_n\right] ds' \tag{4}$$

or, bringing the summation through the integral,

$$\vec{E}^s = \sum_{n=1}^{\infty} C_n \iint_s \bar{\bar{\gamma}} \cdot \vec{K}_n\, ds' \tag{5}$$

Obviously, the summation over an infinite set of current distributions is an impossibility. By choosing $\{\vec{K}_n\}$ appropriatly, the summation can be truncated at some suitably small value, N, and the current obtained will be approximately $\vec{K}$. That is,

$$\sum_{n=1}^{N} C_n \vec{K}_n \approx \vec{K} \tag{6}$$

Using the above summation to obtain $\vec{E}^s$, it is no longer possible to require $\hat{t} \cdot (\vec{E}^i + \vec{E}^s) = 0$ at every point on s. Instead, Galerkin's method is used. For $m = 1, 2, 3, \ldots, N$ require

$$\iint_s \vec{K}_m \cdot \vec{E}^s \, ds = -\iint_s \vec{K}_m \cdot \vec{E}^i \, ds \,. \tag{7}$$

Substituting the expression for $\vec{E}^s$ using the truncated summation,

$$\iint_s ds\vec{K}_m \cdot \left[\sum_{n=1}^{N} C_n \iint_s \overleftrightarrow{\gamma} \cdot \vec{K}_n \, ds' \right] = -\iint_s ds\vec{K}_m \cdot \vec{E}^i \tag{8}$$

$$m = 1, 2, 3, \ldots, N$$

or,

$$\sum_{n=1}^{N} C_n \iint_s ds\vec{K}_m \cdot \iint_s \overleftrightarrow{\gamma} \cdot \vec{K}_n \, ds' = -\iint_s ds\vec{K}_m \cdot \vec{E}^i \tag{9}$$

$$m = 1, 2, 3, \ldots, N$$

Making the definitions

$$Z_{mn} = \iint_s ds\vec{K}_m \cdot \iint_s ds' \overleftrightarrow{\gamma} \cdot \vec{K}_n \tag{10}$$

$$V_m = -\iint_s ds\vec{K}_m \cdot \vec{E}^i \tag{11}$$

equation (4) becomes

$$\sum_{n=1}^{N} C_n Z_{mn} = V_m \qquad m = 1, 2, 3, \ldots, N \tag{12}$$

This can be written in matrix form as

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \tag{13}$$

This is the equation used to obtain the current amplitudes $C_1, C_2, ..., C_N$ and hence the approximate solution for $\vec{K}$ via (6). The current decomposition in (6) is constructed as a piecewise sinusoidal series with N pieces along the length of the dipole, and with no variation across the width. The component current distributions are

$$\vec{K}_n = K_n\,\hat{x} = \hat{x}\,\frac{\sin[(\bar{b} - |x_n|)\,k_0]}{\sin(\bar{b}k_0)} \qquad b_{n-1} < x < b_{n+1} \quad -a < y < a \tag{14}$$

where

$$\bar{b} = \frac{2b}{N+1} \tag{15}$$

$$b_n = \bar{b}\left[n - \frac{N+1}{2}\right] \tag{16}$$

and where

$$x_n = x - b_n \tag{17}$$

From (10), using

$$\hat{x}\cdot\overline{\overline{g}}\cdot\hat{x} = \sum_{p,q=-\infty}^{\infty} g_{xx}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \tag{18}$$

where $g_{xx}^{\infty}(\vec{k})$ is given in equation 10.2(3), the matrix element Z_{mn} becomes

$$Z_{mn} = \int_{-a}^{a} dy \int_{-b}^{b} dx\, K_m \int_{-a}^{a} dy' \int_{-b}^{b} dx'\, K_n \sum_{p,q=-\infty}^{\infty} g_{xx}^{\infty}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \tag{19}$$

Rearranging, and using (14), this becomes

$$Z_{mn} = \sum_{p,q=-\infty}^{\infty} g_{xx}^{\infty}(\vec{k}) \left[\frac{1}{\sin(\bar{b}k_0)} \int_{-a}^{a} dy\, e^{jk_y y} \int_{b_{m-1}}^{b_{m+1}} dx\, \sin[(\bar{b} - |x_m|)\,k_0]\, e^{jk_x x} \right]$$

$$\left[\frac{1}{\sin(\bar{b}k_0)} \int_{-a}^{a} dy' \, e^{-jk_y y'} \int_{b_{n-1}}^{b_{n+1}} dx' \sin[(\bar{b} - |x_n'|) k_0] \, e^{-jk_x x'} \right] \tag{20}$$

where $x_n' = x' - b_n$. This can be written compactly as

$$Z_{mn} = \sum_{p,q=-\infty}^{\infty} g_{xx}^{=} (\vec{k}) \, I_m^+(\vec{k}) \, I_n^-(\vec{k}) \tag{21}$$

where

$$I_n^{\pm}(\vec{k}) \equiv \frac{1}{\sin(\bar{b}k_0)} \int_{-a}^{a} dy \, e^{\pm jk_y y} \int_{b_{n-1}}^{b_{n+1}} dx \, \sin[(\bar{b} - |x_n|) k_0] \, e^{\pm jk_x x} \tag{22}$$

or

$$I_n^{\pm}(\vec{k}) = \frac{1}{\sin(\bar{b}k_0)} \int_{-a}^{a} dy \, e^{\pm jk_y y} \left[\int_{-\bar{b}}^{0} dx_n \, \sin[(\bar{b} + x_n) k_0] \, e^{\pm jk_x x_n} e^{\pm jk_x b_n} + \int_{0}^{\bar{b}} dx_n \, \sin[(\bar{b} - x_n) k_0] \, e^{\pm jk_x x_n} e^{\pm jk_x b_n} \right] \tag{23}$$

Evaluating the integral over y, and using

$$\sin(z) = \frac{-j}{2} \left[e^{jz} - e^{-jz} \right] \tag{24}$$

in the integral over x, equation (23) becomes

$$I_n^{\pm}(\vec{k}) = \frac{1}{\pm jk_y \sin(\bar{b}k_0)} \left[e^{\pm jk_y a} - e^{-(\pm jk_y a)} \right] e^{\pm jk_x b_n} \times$$

$$\times \left\{ \int_{-\bar{b}}^{0} dx_n \, \frac{-j}{2} \left[e^{j(\bar{b} + x_n) k_0} e^{\pm jk_x x_n} - e^{-j(\bar{b} + x_n) k_0} e^{\pm jk_x x_n} \right] + \right.$$

$$\left. + \int_{0}^{\bar{b}} dx_n \, \frac{-j}{2} \left[e^{j(\bar{b} - x_n) k_0} e^{\pm jk_x x_n} - e^{-j(\bar{b} - x_n) k_0} e^{\pm jk_x x_n} \right] \right\} \tag{25}$$

using (24) again, and evaluating the integral over x, (25) becomes

$$I_n^{\pm}(\vec{k}\,) = \frac{2j\ \sin(\pm k_y a)}{\pm jk_y\ \sin(\bar{b}k_0)}\ e^{\pm jk_x\, b_n}\ \frac{-j}{2} \times$$

$$\times \left\{ \frac{e^{jk_0\bar{b}}}{j(k_0 \pm k_x)}\ e^{j(k_0 \pm k_x)x_n} - \frac{e^{-jk_0\bar{b}}}{j(-k_0 \pm k_x)}\ e^{j(-k_0 \pm k_x)\,x_n}\ \Bigg|_{-\bar{b}}^{0} \right.$$

$$\left. + \frac{e^{jk_0\bar{b}}}{j(-k_0 \pm k_x)}\ e^{j(-k_0 \pm k_x)x_n} - \frac{e^{-jk_0\bar{b}}}{j(k_0 \pm k_x)}\ e^{j(k_0 \pm k_x)\,x_n}\ \Bigg|_{0}^{\bar{b}} \right\} \tag{26}$$

Evaluating at the limits, this becomes

$$I_n^{\pm}(\vec{k}\,) = \frac{-\sin(k_y a)}{k_y\ \sin(\bar{b}k_0)}\ e^{\pm jk_x\, b_n} \times$$

$$\times \left\{ \frac{e^{jk_0\bar{b}}}{(k_0 \pm k_x)} \left[1 - e^{-jk_0\bar{b}}\ e^{-(\pm jk_x\bar{b})}\right] - \frac{e^{-jk_0\bar{b}}}{(-k_0 \pm k_x)} \left[1 - e^{jk_0\bar{b}}\ e^{-(\pm jk_x\bar{b})}\right] + \right.$$

$$\left. + \frac{e^{jk_0\bar{b}}}{(-k_0 \pm k_x)} \left[e^{-jk_0\bar{b}}\ e^{\pm jk_x\bar{b}} - 1\right] - \frac{e^{-jk_0\bar{b}}}{(k_0 \pm k_x)} \left[e^{jk_0\bar{b}}\ e^{\pm jk_x\bar{b}} - 1\right] \right\} \tag{27}$$

Putting the terms over a common denominator gives

$$I_n^{\pm}\ (\vec{k}) = \frac{-\sin(k_y a)}{k_y\ \sin(k_0\bar{b})}\ e^{\pm jk_x b_n} \times$$

$$\times \left\{ \frac{(-k_0 \pm k_x)}{(-k_0^2\ \pm k_x^2)} \left[e^{jk_0\bar{b}} - e^{-(\pm k_x\bar{b})} - e^{\pm jk_x\bar{b}} + e^{-jk_0\bar{b}}\right] + \right.$$

$$\left. + \frac{(k_0 \pm k_x)}{(-k_0^2\ \pm k_x^2)} \left[-e^{-jk_0\bar{b}} + e^{-(\pm k_x\bar{b})} + e^{\pm jk_x\bar{b}} - e^{jk_0\bar{b}}\right] \right\} \tag{28}$$

Using

$$\cos(z) = \frac{1}{2}\left[e^{jz} + e^{-jz}\right] \tag{29}$$

leads to

$$I_n^{\pm}(\vec{k}\,) = \frac{\sin(k_y a)}{k_y\ \sin(k_0\bar{b})}\ e^{\pm jk_x\, b_n} \left[\frac{(-k_0 \pm k_x)}{(-k_0^2\ + k_x^2)} \left[2\cos(k_0\bar{b}) - 2\cos(\pm\, k_x\bar{b})\right] - \right.$$

$$- \frac{(k_0 \pm k_x)}{(-k_0^2 + k_x^2)} \left[2\cos(k_0\bar{b}) - 2\cos(\pm k_x \bar{b}) \right] \Bigg] \tag{30}$$

collecting terms, equation (30) becomes

$$I_n^{\pm}(\vec{k}) = \frac{4\sin(k_y a)}{k_y \sin(k_0 \bar{b})} e^{\pm j k_x b_n} \frac{-k_0}{k_x^2 - k_0^2} \left[\cos(k_x \bar{b}) - \cos(k_0 \bar{b}) \right] \tag{31}$$

Notice that the $I_n^{\pm}$ differ only by a phase factor $e^{\pm j k_x b_n}$. With the definition

$$I^0(\vec{k}) = \frac{4\sin(k_y a)}{k_y \sin(k_0 \bar{b})} \left[\frac{-k_0}{k_x^2 - k_0^2} \left[\cos(k_x \bar{b}) - \cos(k_0 \bar{b}) \right] \right] \tag{32}$$

equation (31) can be written as

$$I_n^{\pm}(\vec{k}) = I^0(\vec{k})\, e^{\pm j k_x b_n} \tag{33}$$

With this, equation (21) can be written as

$$Z_{mn} = \sum_{p,q=-\infty}^{\infty} g_{xx}^{\infty}(\vec{k})\, I^{0^2} e^{+jk_x b_m} e^{-jk_x b_n} \tag{34}$$

or, using the definition of b_n, equation (16),

$$Z_{mn} = \sum_{p,q=-\infty}^{\infty} g_{xx}^{\infty}(\vec{k})\, I^{0^2} e^{+jk_x \bar{b}(m-n)} \tag{35}$$

From (35), it can easily be seen that

$$Z_{m+1,n+1} = Z_{m,n} \tag{36}$$

Once the first row and first column of the Z-matrix are filled, the rest of the elements correspond to one of these, and can be easily filled in.

The excitation matrix elements (11) now must be evaluated. Since the $\vec{K}_n$ only have an x component, only the x component of the excitation field is needed. For the scattering case for parallel incidence, from chapter 12, equation 12(45) is specialized to $z = 0$ to become

$$\vec{E}_{||x}^{1} = \hat{x}'' \cdot \hat{x}\; 2 E_{||}^{i}\, e^{j(\frac{1}{2}\Phi_{||} + k_1 x'' \sin(\theta_i))} \cos(\tfrac{1}{2}\Phi_{||}) \cos(\theta_i) \tag{37}$$

From 12(4),

$$\hat{x}'' \cdot \hat{x} = \cos(\phi_i) \tag{38}$$

and with the definition

$$E_1 = 2\, E^i_{||}\, e^{j\frac{1}{2}\Phi_{||}} \cos(\frac{1}{2}\Phi_{||}) \cos(\theta_i) \cos(\phi_i) \tag{39}$$

equation (37) becomes

$$\vec{E}^{\,1}_{||x} = E_1\, e^{jk_1 x'' \sin(\theta_i)} \tag{40}$$

using 12(2) and u and v from equations 8.1(1) and 8.1(2), this becomes

$$\vec{E}^{\,1}_{||x} = E_1\, e^{jk_1 vy}\, e^{jk_1 ux} \tag{41}$$

With this and equation (14), (11) becomes

$$V_n = -E_1 \int_{-a}^{a} e^{jk_1 vy}\, dy \int_{b_{n-1}}^{b_{n+1}} \frac{\sin[(\bar{b} - |x_n|)\, k_0]}{\sin(k_0 \bar{b})}\, e^{jk_1 ux}\, dx \tag{42}$$

or

$$V_n = -E_1 \int_{-a}^{a} e^{jk_1 vy}\, dy \int_{\bar{b}}^{-\bar{b}} \frac{\sin[(\bar{b} - |x_n|)\, k_0]}{\sin(k_0 \bar{b})}\, e^{jk_1 ux_n}\, e^{jk_1 ub_n}\, dx_n \tag{43}$$

comparing this to equation (22), it can be seen that

$$V_n = -E_1\, I_n^+(k_1 u\, \hat{x} + k_1 v\, \hat{y}) \tag{44}$$

For the scattering case of perpendicular polarization, the x component of $\vec{E}_\perp^{\,1}$ is, for $z = 0$ from equation 12(75),

$$\vec{E}^{\,1}_{\perp x} = \hat{y}'' \cdot \hat{x}\; 2\, E^i_\perp\, e^{j(\frac{1}{2}\Phi_\perp + k_1 x'' \sin(\theta_i))} \cos(\frac{1}{2}\Phi_\perp) \tag{45}$$

From 12(5),

$$\hat{y}'' \cdot \hat{x} = -\sin(\phi_i) \tag{46}$$

Making the definition

$$E_2 = -\sin(\phi_i)\, 2\, E_\perp^i\, e^{j\frac{1}{2}\Phi_\perp} \cos(\frac{1}{2}\Phi_\perp) \tag{47}$$

equation (45) becomes

$$\vec{E}_{\perp x}^{\,1} = E_2\, e^{jk_1 x'' \sin(\theta_i)} \tag{48}$$

using 13(3) and u and v from equations 9.1(1) and 9.1(2), this becomes

$$\vec{E}_{\perp x}^{\,1} = E_2\, e^{jk_1 vy}\, e^{jk_1 ux} \tag{49}$$

With this and equation (14), (11) becomes

$$V_n = -E_2 \int_{-a}^{a} e^{jk_1 vy}\, dy \int_{b_{n-1}}^{b_{n+1}} \frac{\sin[(\bar{b} - |x_n|)\, k_0]}{\sin(k_0 \bar{b})}\, e^{jk_1 ux}\, dx \tag{50}$$

or

$$V_n = -E_2 \int_{-a}^{a} e^{jk_1 vy}\, dy \int_{\bar{b}}^{-\bar{b}} \frac{\sin[(\bar{b} - |x_n|)\, k_0]}{\sin(k_0 \bar{b})}\, e^{jk_1 ux_n}\, e^{jk_1 ub_n}\, dx_n \tag{51}$$

comparing this to equation (22), it can be seen that

$$V_n = -E_2\, I_n^+(k_1 u\, \hat{x} + k_1 v\, \hat{y}) \tag{52}$$

10. Transmission Case.

The transmission case is arrived at in a manner analogous to that of the scattering case. The array is driven by a voltage at the center of the dipole which represents the voltage drop across the load generated by current flowing at the dipole center when the array is acting as a receiver. To find the currents on the array due to this applied voltage, a slice gap generator with amplitude one volt is assumed to be located at the center of each dipole. This gives a tangential electric field along the central dipole

$$\vec{E}^{t} = \hat{x}\ \delta(x). \tag{1}$$

At each off-center dipole, the excitation field carries a phase factor due to the incident wave.

Since there is no plane wave excitation in this case, this voltage is the only excitation present. The integral equation to be satisfied is identical to that of the scattering case, with zero excitation field on the dipoles, except at their centers. Thus, from 9.3(2)

$$\hat{z} \times \vec{E}_1^{s}(x,y,z{=}0) = -\hat{z}\ \mathrm{x}\ \hat{x}\ \delta(x) \qquad\qquad |y| < a. \tag{2}$$

For this case, the integral in (9.4.11) is trivially evalutated to obtain

$$V_n = \begin{cases} -2a & n = n_c \equiv (N+1)/2 \\ 0 & \text{else} \end{cases} \tag{3}$$

This is written in matrix form in the same manner as for the scattering case. In particular, the left hand side of 9.4(13) is unchanged and the right hand side is replaced with either 0 or $(-2a)$ as appropriate, so

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ \vdots & \vdots & & \vdots \\ Z_{n_c1} & Z_{n_c2} & \cdots & Z_{n_cN} \\ \vdots & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ \vdots \\ C_{n_c} \\ \vdots \\ C_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -2a \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

Thus, once the scattering case has been solved, the same matrix is used to solve the transmission case, with only an easy substitution in the driving terms of the matrix equation. The current at the dipole center is then solved for and the input impedance obtained through

$$Z_{in} = \frac{1}{I_2} = \frac{1}{2a\ C_{n_c}} \tag{5}$$

11. NUMERICAL RESULTS.

To test the accuracy of the solution, the results were compared to those obtained in [4]. The reflection coefficient was obtained for two cases as a function of incidence angle. The reflection coefficient is defined by

$$R = \sqrt{1 - \frac{P_r}{P_i}} \tag{1}$$

where P_r is the power received by each element of the structure, and P_i is the power in the incident plane wave landing each unit cell of the array.

The results are shown in Figures 18 and 19. The parameters and dimensions of the array are:

$2b = 0.39\ \lambda$, $2a = .002\ \lambda$

$d = 0.19\ \lambda$

$\varepsilon_1 = \varepsilon_0$, $\varepsilon_2 = 2.55\ \varepsilon_0$

$d_x = 0.5\ \lambda$, $d_y = 0.5\ \lambda$ in Figure 4.

$d_x = 0.5155\ \lambda$, $d_y = 0.5\ \lambda$ in Figure 5.

Figures 18a and 19a are the results obtained by Pozar, while Figures 18b and 19b are the results obtained here for the E-plane. In both Figures 18 and 19, the load impedance is matched for normal incidence, $Z_l = 75\ \Omega$ in Figures 18, while $Z_l = 73 + j\ 4\ \Omega$ in Figures 19. Note that the curves match perfectly in Figures 18, and in Figures 19 for θ_i less than 70^o. For θ_i greater than 70^o, a grating lobe exists in Figure 19. In Figure 18a, the reflection coeficient is defined for transmission from the array, and power transmitted into the grating lobe is included. In Figure 19b, however, it is defined for the case of reception of power from a plane wave, and power scattered into the grating lobe is not included. The difference between plots 19a and 19b represents power scattered into the grating lobe.

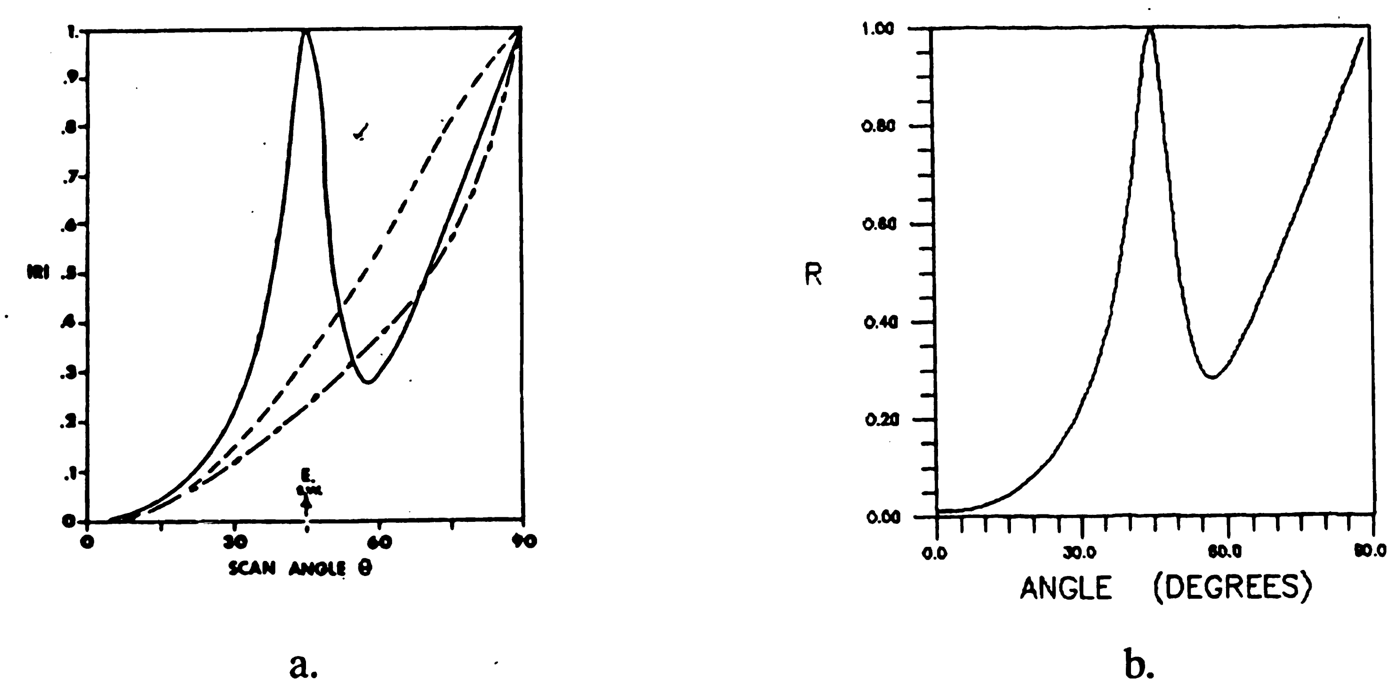
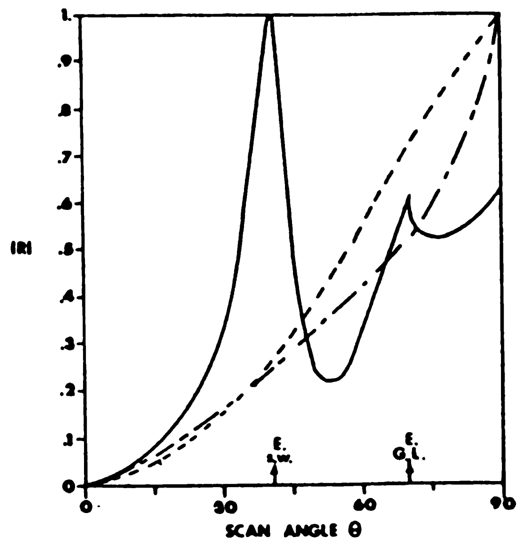
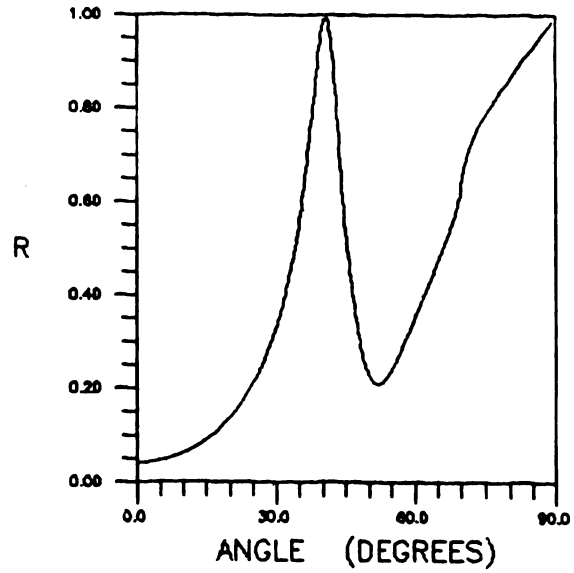


Figure 18. Reflection coeficient. a. Pozar et al. b. Chen et al.



a.



b.

Figure 19. Reflection coeficient. a. Pozar et al. b. Chen et al.

The Galerkin's method solution was implemented on an I.B.M. P.C. compatible personal computer. The number of piecewise sinusoidal basis functions used is 13. The summation variables p and q were allowed to vary from +50 to −50. The time per run at each frequency selected is about 5 min. The dimensions and other parameters were chosen as

$\varepsilon_1 = \varepsilon_0$ $\qquad\qquad$ $\varepsilon_2 = 2.5\ \varepsilon_0$

$d = 0.15875$ cm. = 1 / 16 in.

$d_x = d_y = 1.27$ cm.

$b = 0.508$ cm. $\qquad\qquad$ $a = 0.0254$ cm.

$|\vec{E}^{\,i}| = 1$ V/m.

$\theta = 0.0001^{\circ}$ $\qquad\qquad$ $\phi = 0.0^{\circ}$

Numerical results for input impedance and scattered current, along with power received for several load impedances, for a frequency range of 8 to 24 GHz are obtained. In Figure 20., the input resistance for each element of the array is given. It peaks at 17 GHz. and has a dip at 21.4 GHz.

In Figure 21., the input reactance is given. It goes through zero near 9.8 GHz, and shows a standard dipole anti-resonance curve in the vicinity of 17 GHz, where the resistance peaked, and a small blip near 21.4 GHz, where the resistance dipped.

In Figure 22., the scattered current magnitude is given. It has a large peak at 9.8 Ghz, and a dip at 21.4 GHz. At 21.4 GHz, the guided wavelength of the lowest order surface wave mode is equal to the dipole spacing. Near this frequency, coupling to surface waves becomes strong, leading to the observed frequency response in this and the two previous figures.

For the $p = q = 0$ term in the Flouquet mode summation, and for nearly normal incidence, $k_y \approx 0$ and $k_x \approx 0$. For this term, $\varepsilon_r\, p_1 + p_2 \tanh(p_2 d)$ or equivilently T_m

becomes zero near 21.4 GHz, and this term of $g_{xx}^{-}(\vec{k})$ would seem to diverge. However, since k_x is very small, the numerator also becomes nearly zero. Thus, only very near the exact location of the zero of T_m will the term become large. By way of comparison with the patch array, g_{xy}, g_{xx} etc. don't have such a pole-cancelling term in the numerator, thus leading to divergent behavior in a larger vicintity of the pole.

With the input impedance and the scattered current, the total power received can now be obtained for any load impedance. In Figure 23, the received power is shown as a function of frequency for a load impedance of 50 Ω. The maximum power received, near 9.8 GHz, is about 65 % of the incident power. A sharp dip in received power is located at 21.4 GHz.

In Figure 24, the received power is shown for a load impedance of 100 Ω. The maximum power received in this case occurs near 11 GHz, where about half the incident power is received.

In Figure 25, the received power is shown for the load impedance set equal to the input impedance. There are two peaks, located at 9.8 and 17 GHz, the two locations where the input reactance goes to zero. The power received at these two peaks is nearly 100 % of the incident power.

In Figure 26, the power received is shown for the load impedance set equal to the complimentary impedance of the load impedance. If the input impedance Z_{in} was for an array of dipoles in free space, the complimentary impedance, Z_c, would be that due to an array of slot dipoles with the same dimensions;

$$Z_c = Z_0^2 \,/\, (4Z_{in})$$

where Z_0 is the impedance of free space. For this case, the power received again has two peaks, near 13 and 23.5 GHz. These corresond to the locations where the magnitude of Z_{in} is equal to Z_0. The power received at these peaks is about 100 % of the incident power.

In Figure 27, the received power for the load impedance matched to the input impedance is given. For this case, 100 % of the incident power is absorbed for frequencies below about 23.5 GHz. This frequency corresponds to the frequency where the dipole spacing is equal to a wavelength in the cover region, the cut-off frequency for the appearance of grating lobes.

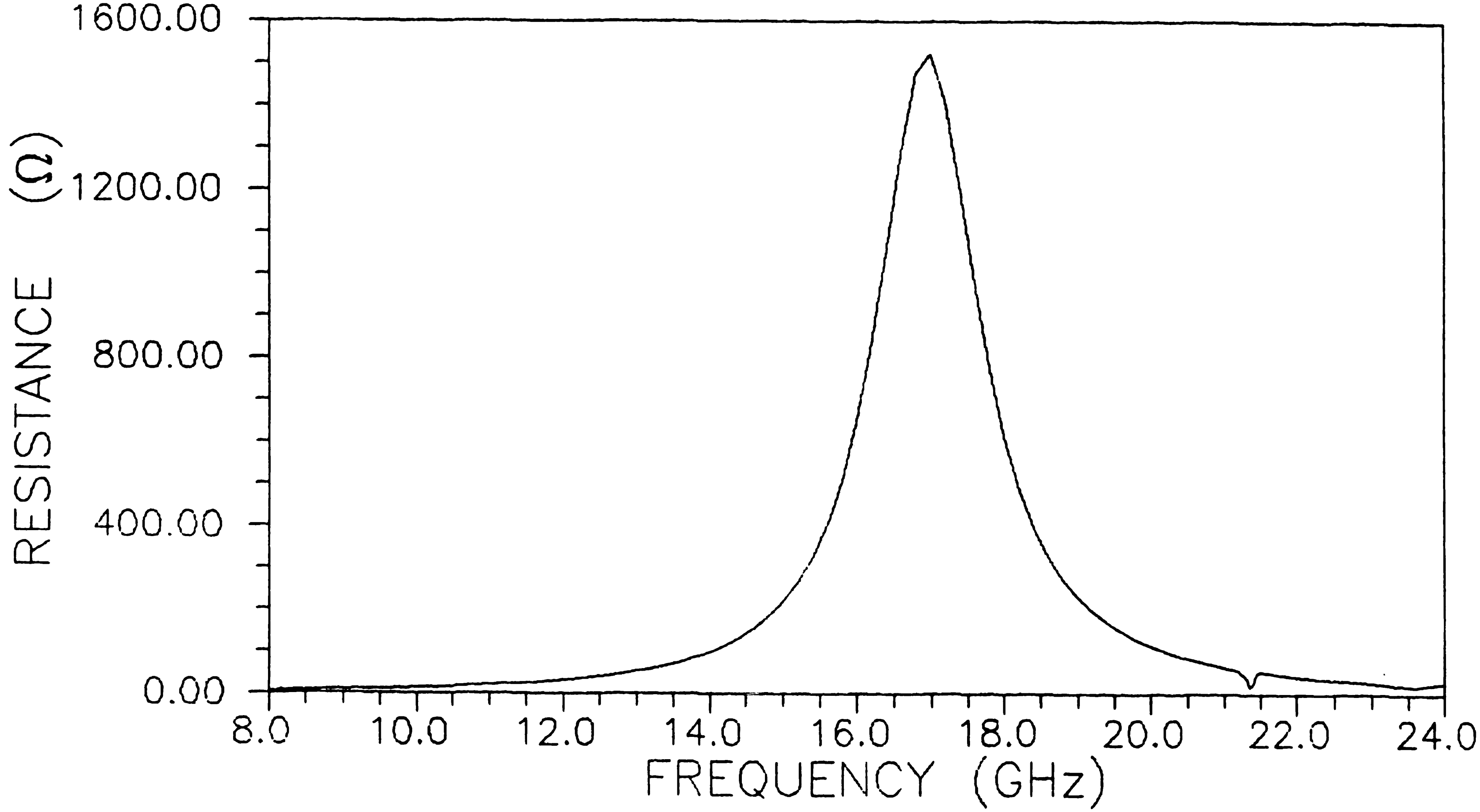


Figure 20. Input resistance for each element of array.

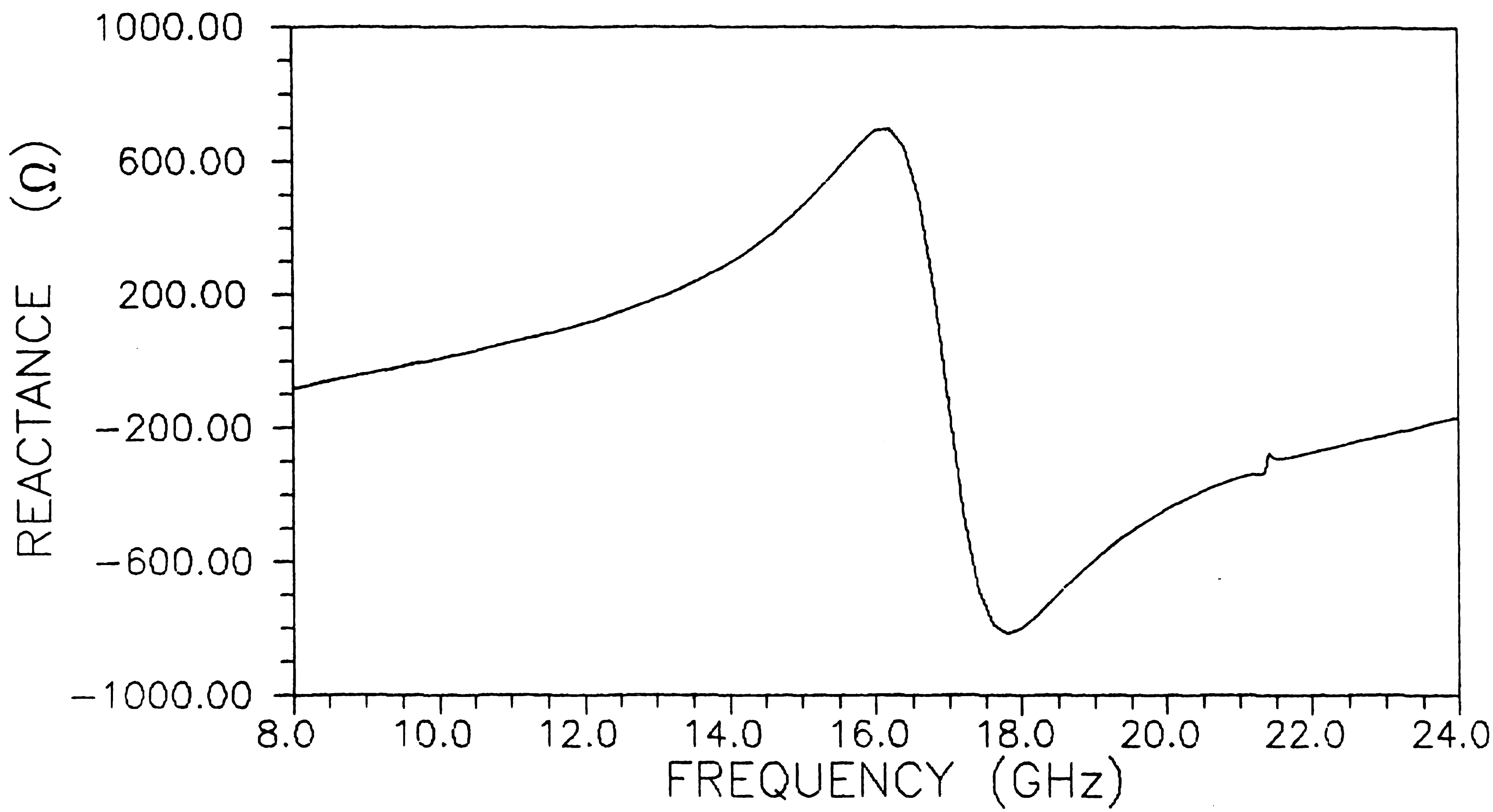


Figure 21. Input reactance for each element of array.

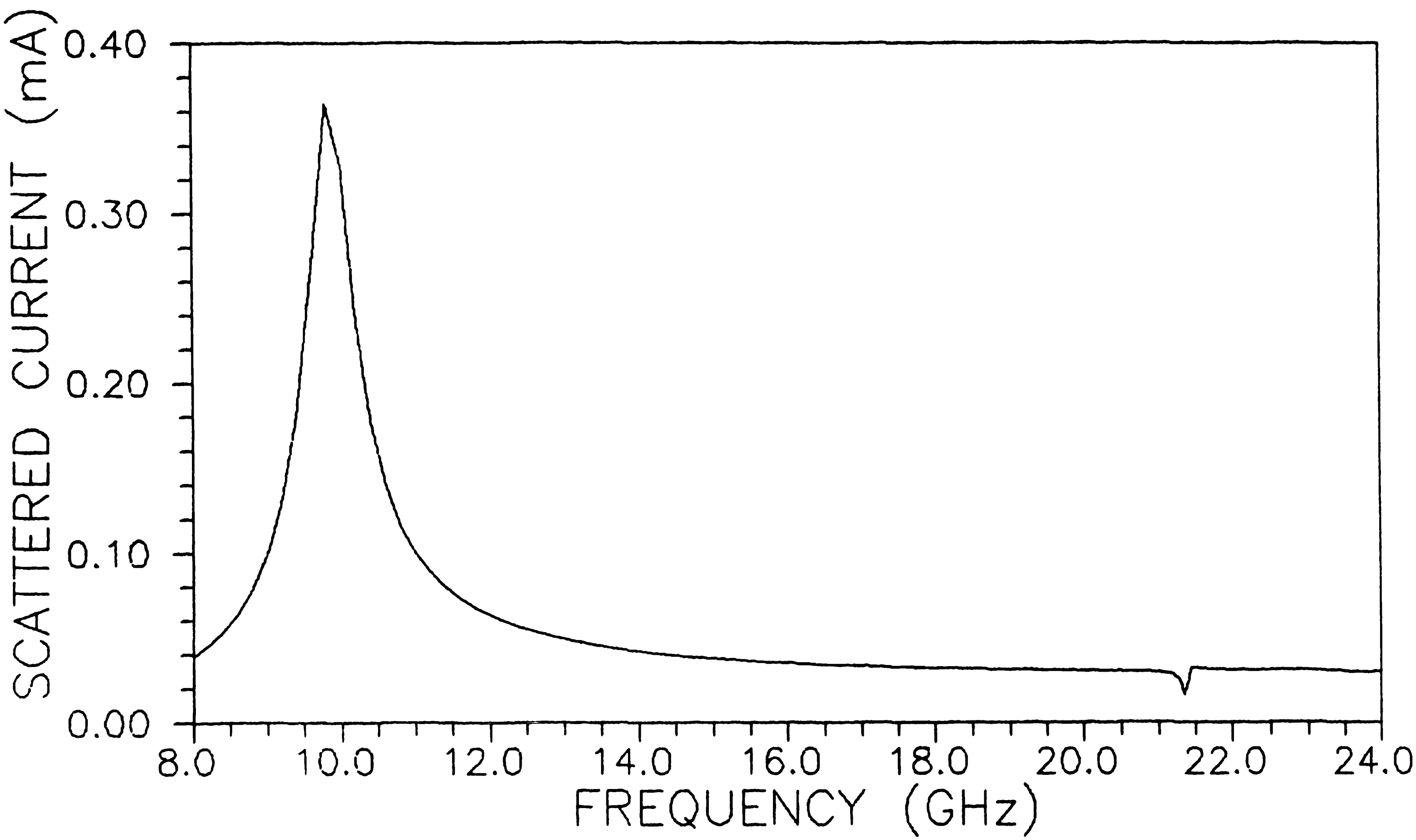


Figure 22. Scattered current magnitude at center of dipole.

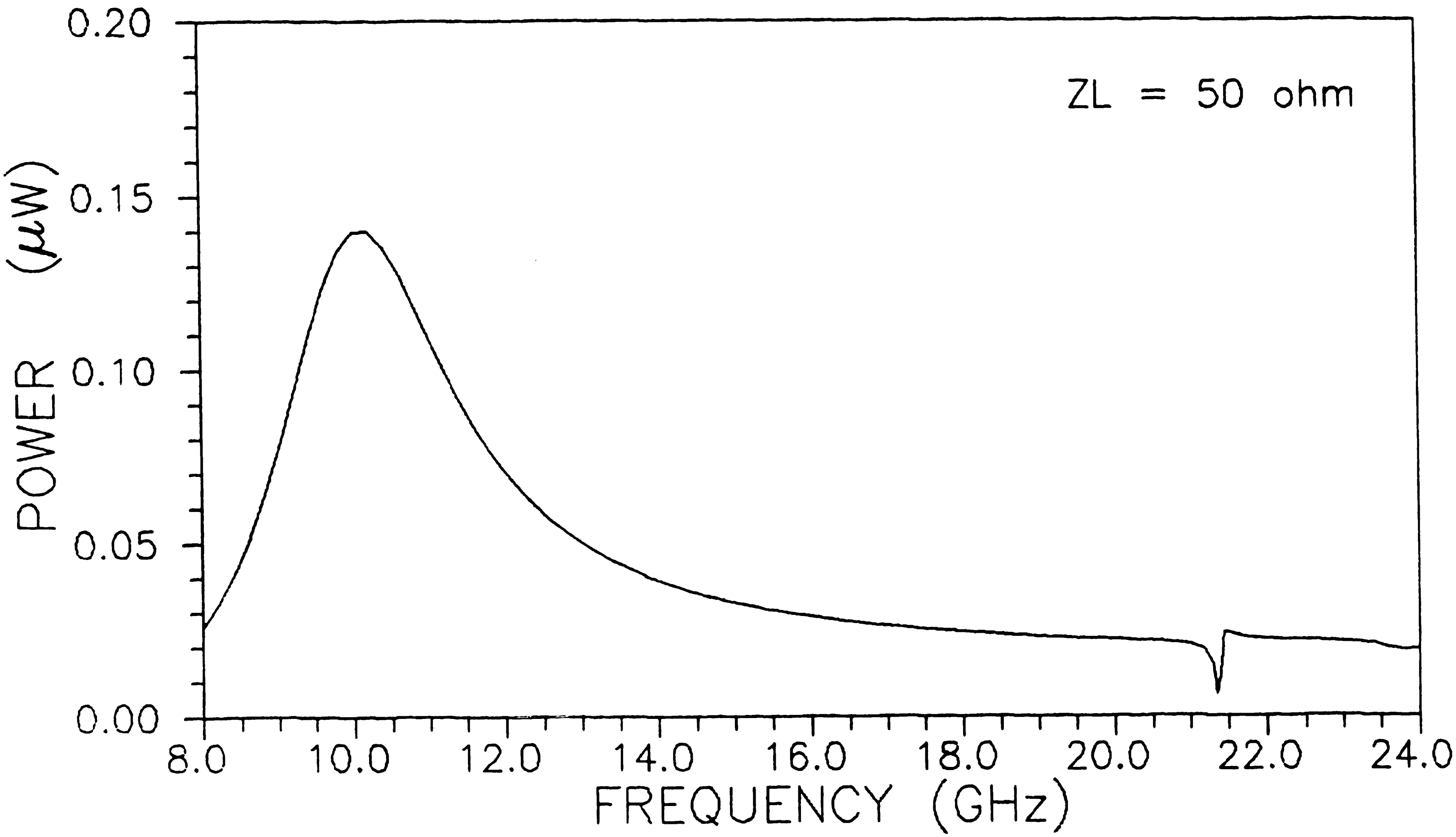


Figure 23. Received power for $Z_L = 50\ \Omega$.

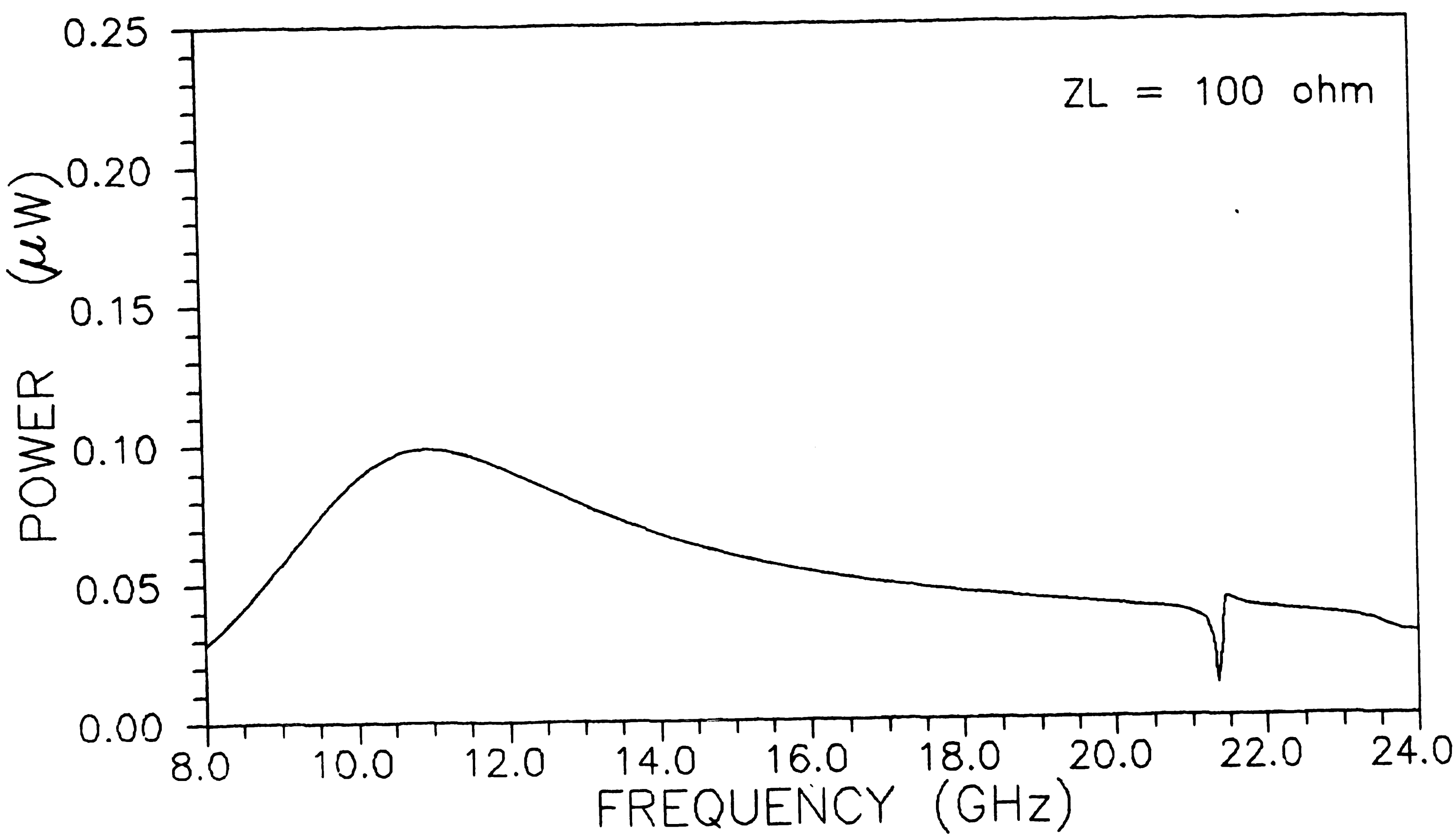


Figure 24. Received power for $Z_L = 100\ \Omega$.

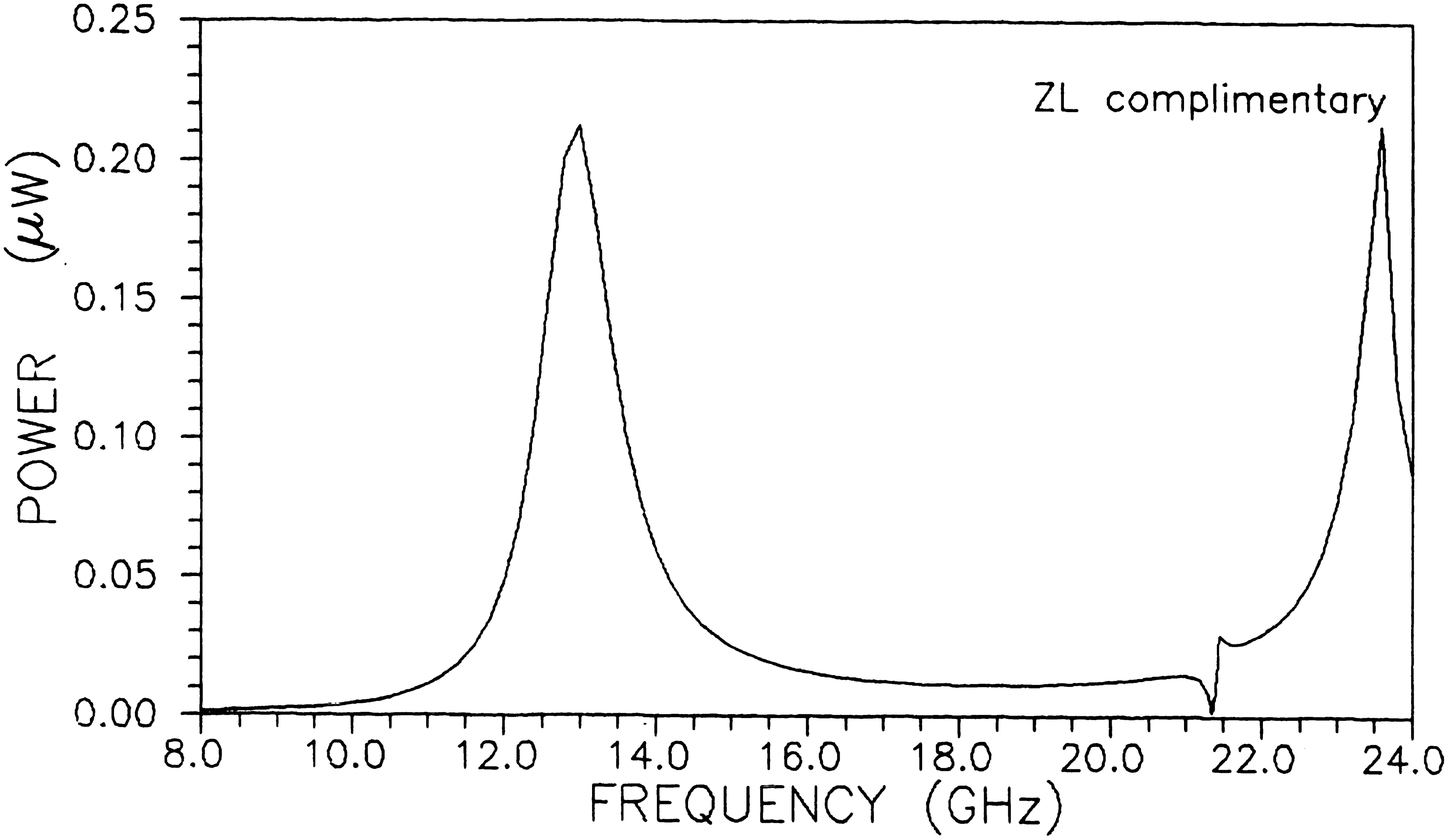


Figure 25. Received power for $Z_L = Z_{in}$.

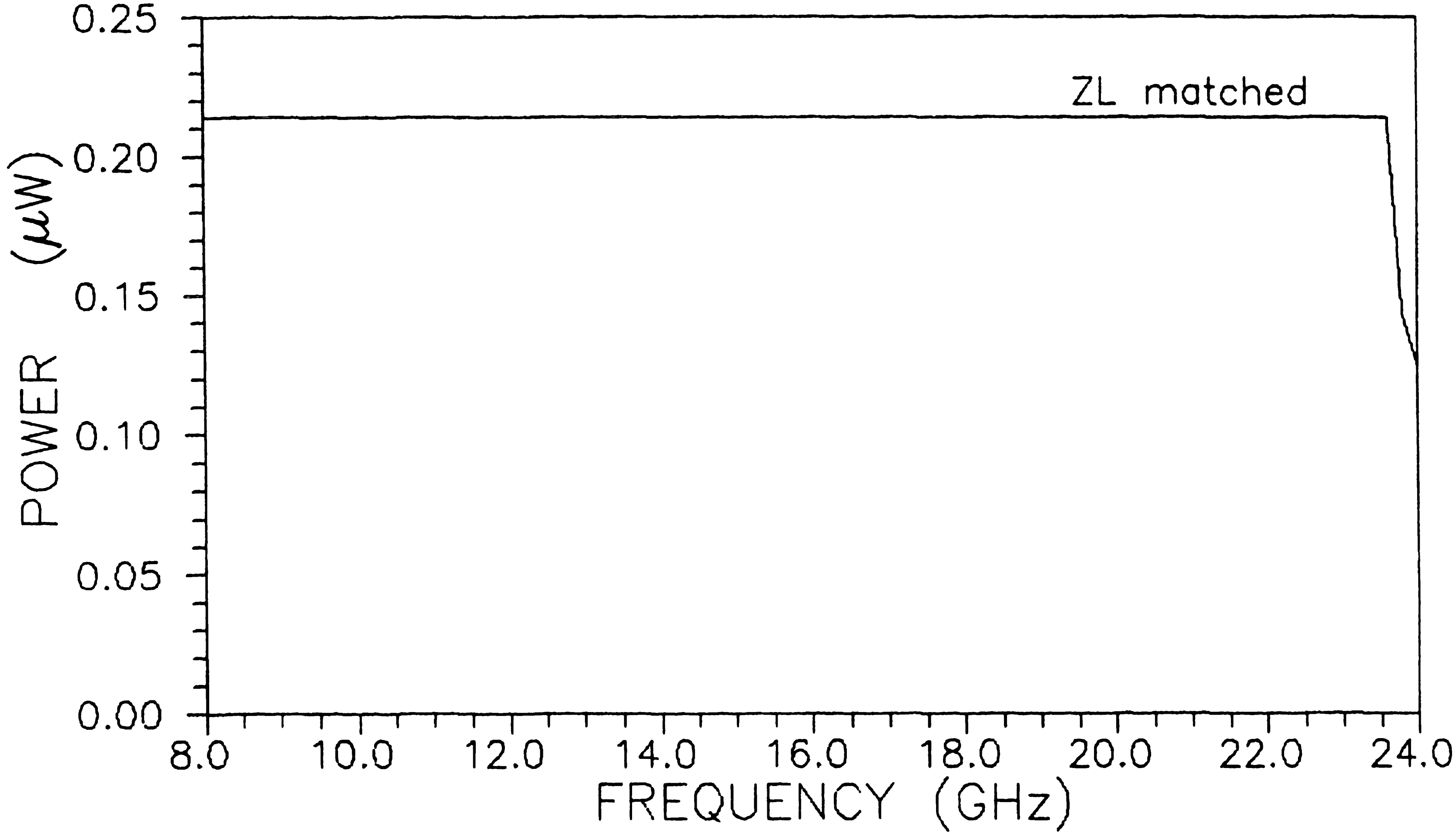


Figure 26. Received power for Z_L complimentary to Z_{in}.

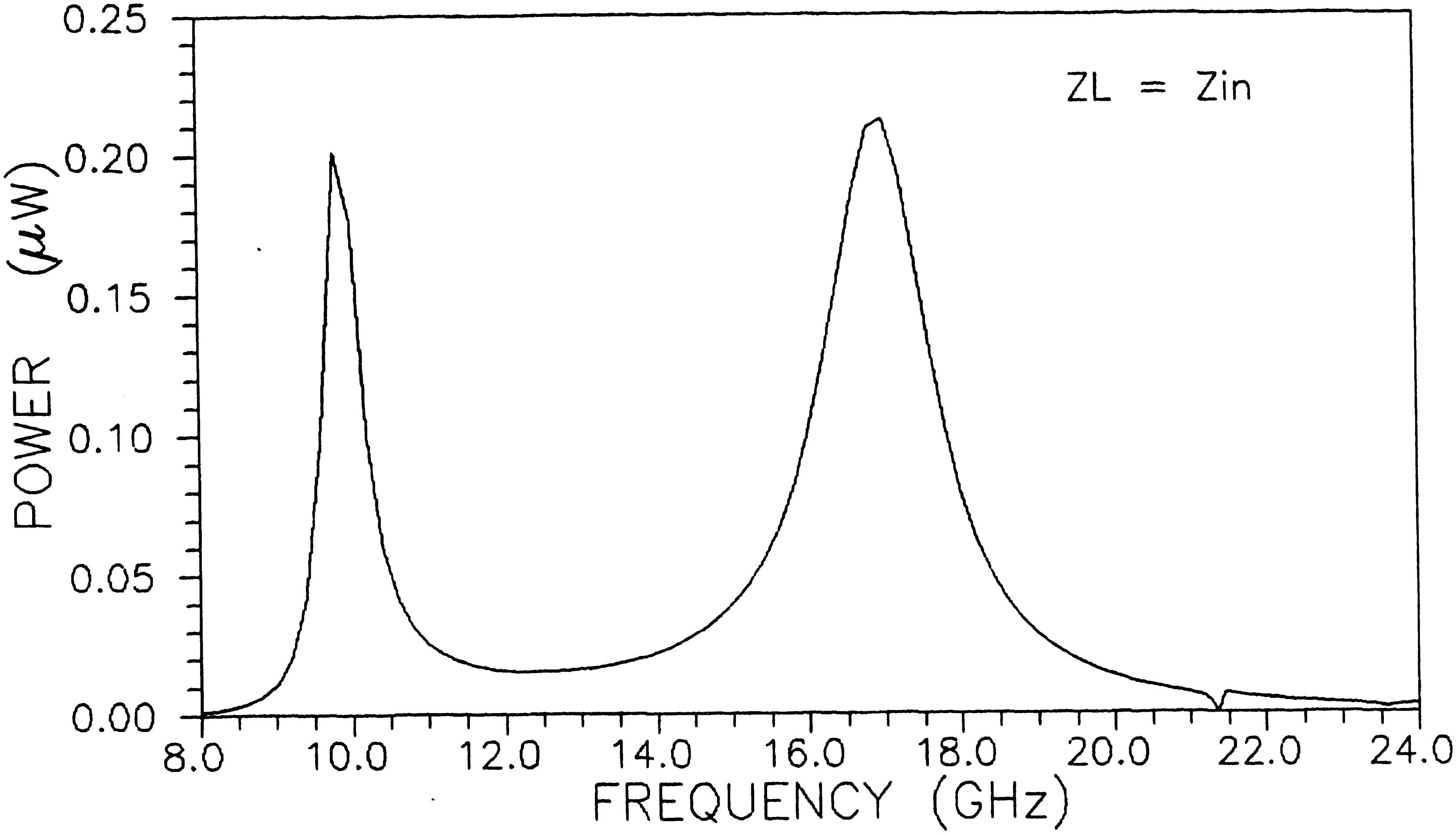


Figure 27. Received power for $Z_L = Z_{in}^*$.

In the remaining figures, numerical results for input impedance and scattered current, along with power received for several load impedances, are given as the incidence angle, θ_i, is swept from 0 to 89 degrees in increments of 1 degree. The first set of figures is for frequencies of 17 and 18 GHz, near the resistance peak.

Figure 28 gives the input resistance of each element of the array. The elements are driven with a phase progression corresponding to the phase progression of the incident wave in the receiving case; hence the dependance on incidence angle. The input resistance is nearly constant with respect to angle, except in the vicinity of the blind spot, [4], where the resistance peaks and then drops to zero as the angle is varied. The blind spot occurs when the rate of phase progression of the incident wave along the array corresponds to the rate of phase progression of a surface wave mode, so that the fields produced by each strip dipole add in phase, and energy is coupled into the mode. Note that as the frequency increases, the angular location of the blind spot approaches zero degrees, occuring at zero degrees at a frequency of 21.4 GHz (see Figure 18).

Similarly, Figure 29 gives the input reactance as a function of angle. The input reactance is also nearly constant as a function of angle, except in the vicinity of the blind spot. Near this angle, the reactance dips then peaks as a function of angle. At 18 GHz, the dip is barely noticeable.

In Figure 30, the scattered current magnitude is given as a function of angle. Near the blind spot, the scattered current magnitude dips, and then peaks sharply as incidence angle is increased. Note that the angular location of the blind spots for the scattering case correspond to those of the input impedances obtained from the transmission case, as they should.

Figures 31 through 35 give received power as a function of incidence angle θ_i for the same cases of load impedance as Figures 21 through 25. In Figure 15, the received power for a load impedance of 50 Ω is given. For both frequencies, the received power experiences a sharp drop, followed by a peak as the incidence angle is increased through the blind spot region. The received power absorbed at zero degrees is about 10 % of the total incident power. At the peaks, the power absorbed is between 25 and 30 %.

Figure 32 gives the received power as a function of frequency for a load impedance of 100 Ω. The shape of the curves is similar to that of the preceding case, with about twice the power received.

In Figure 33, the received power is given for the load impedance set equal to the input impedance. At 17 GHz, where the received power is nearly 100 % at normal incidence, the received power experiences a drop near the blind spot in an otherwise smooth curve. At 18 GHz, the plot looks similar to that of the previous figure, a dip followed by a peak.

In Figure 34, the received power for a load impedance set equal to the complimentary impedance of the array is shown. For both frequencies shown, the received power drops to zero at the blind spot, followed by a sharp peak in power absorbed.

Finally, in Figure 35, the received power is given for a load impedance matched to the input impedance. In each case, the power received is about 100 % until the incidence angle increases to between 20 or 25 degrees. The angle at which the drop occurs varies with frequency, decreasing as the frequency increases. The drop is due to phase progression of the excitation of the dipoles becoming large enough that the radiation into the cover region of each individual dipole can add in phase, so that power can be scattered into the cover region in the form of grating lobes.

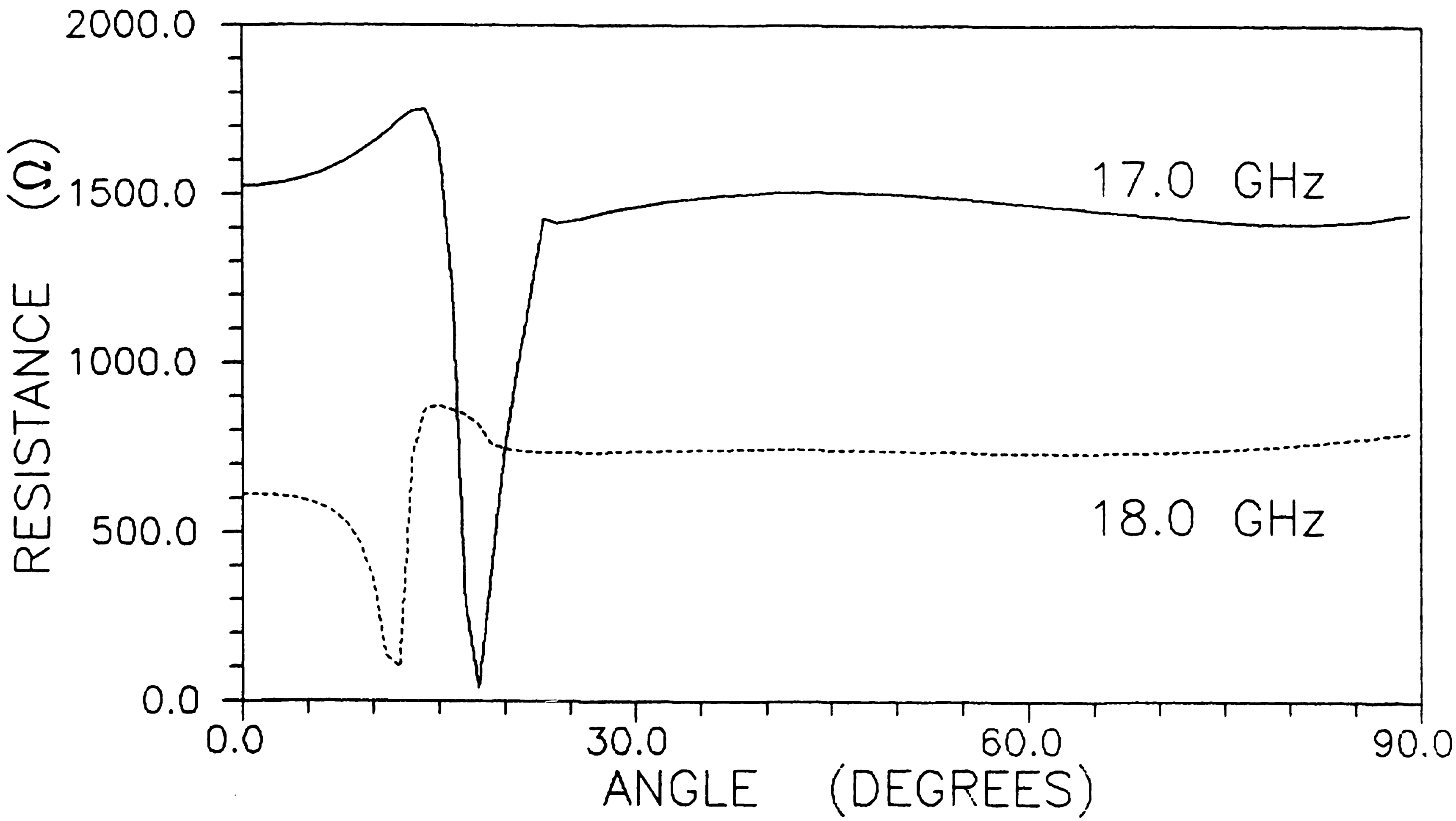


Figure 28. Input resistance for each element of array at 17.0 and 18.0 GHz.

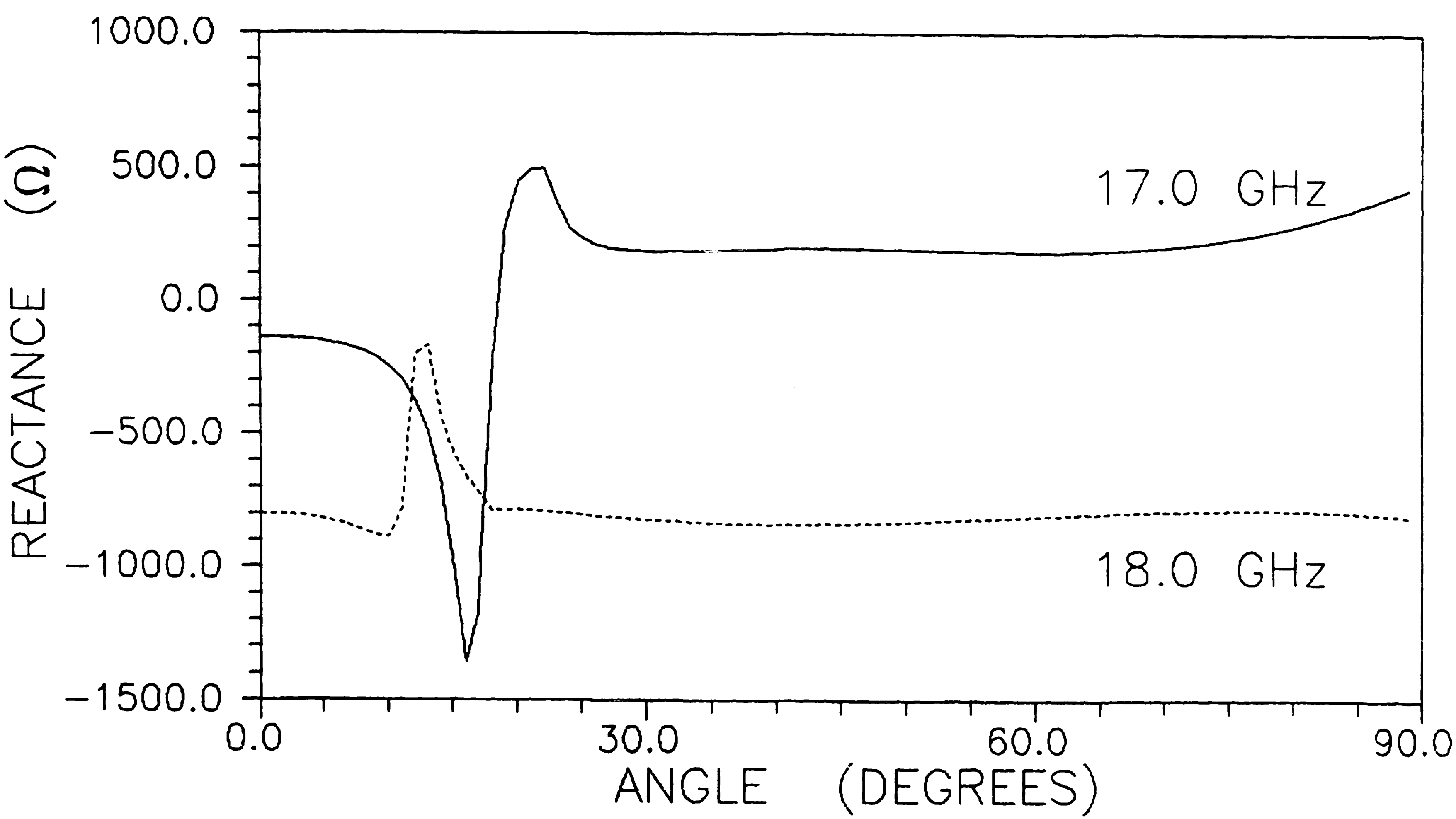


Figure 29. Input reactance for each element of array at 17.0 and 18.0 GHz.

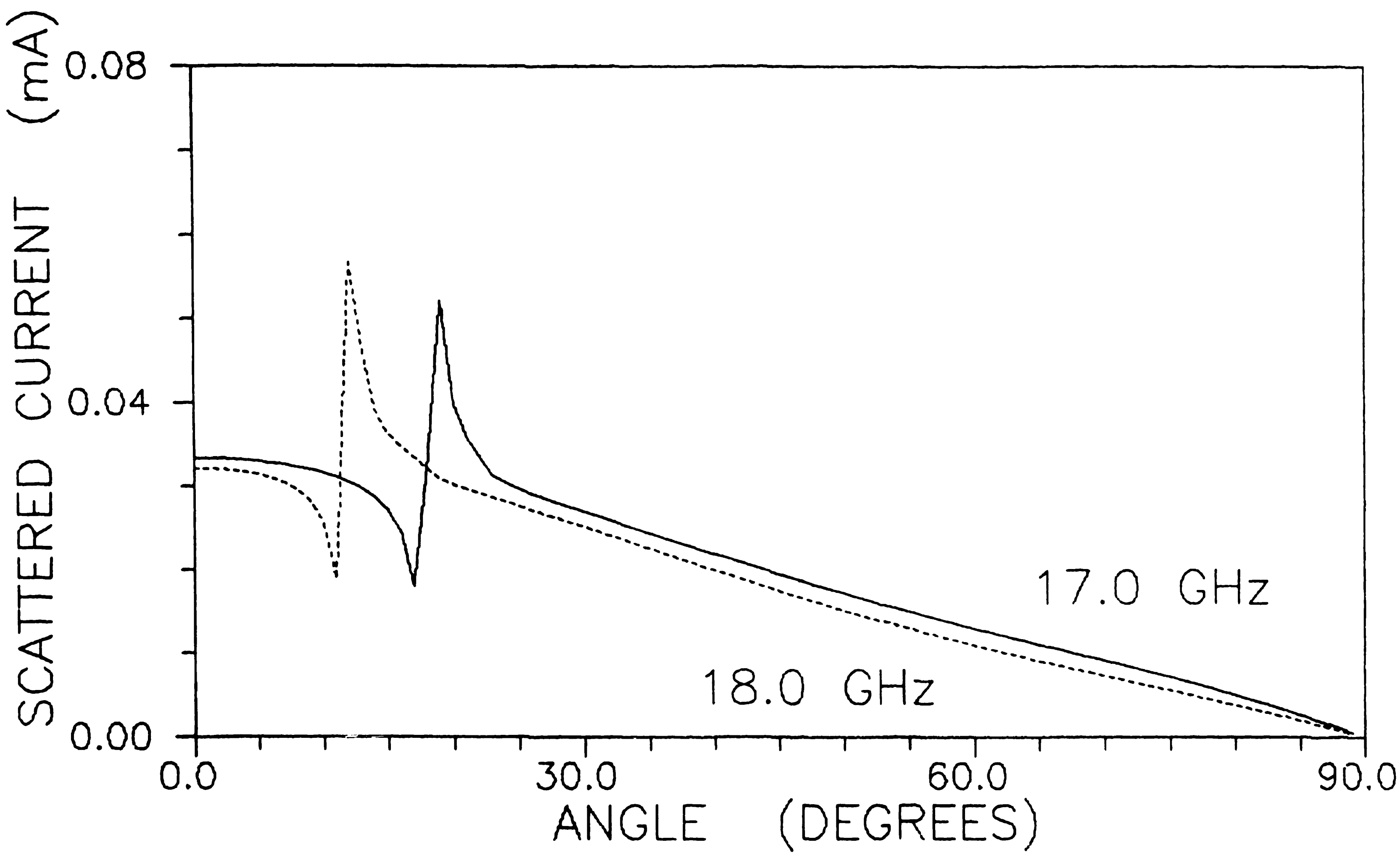


Figure 30. Scattered current magnitude at center of dipole at 17.0 and 18.0 GHz.

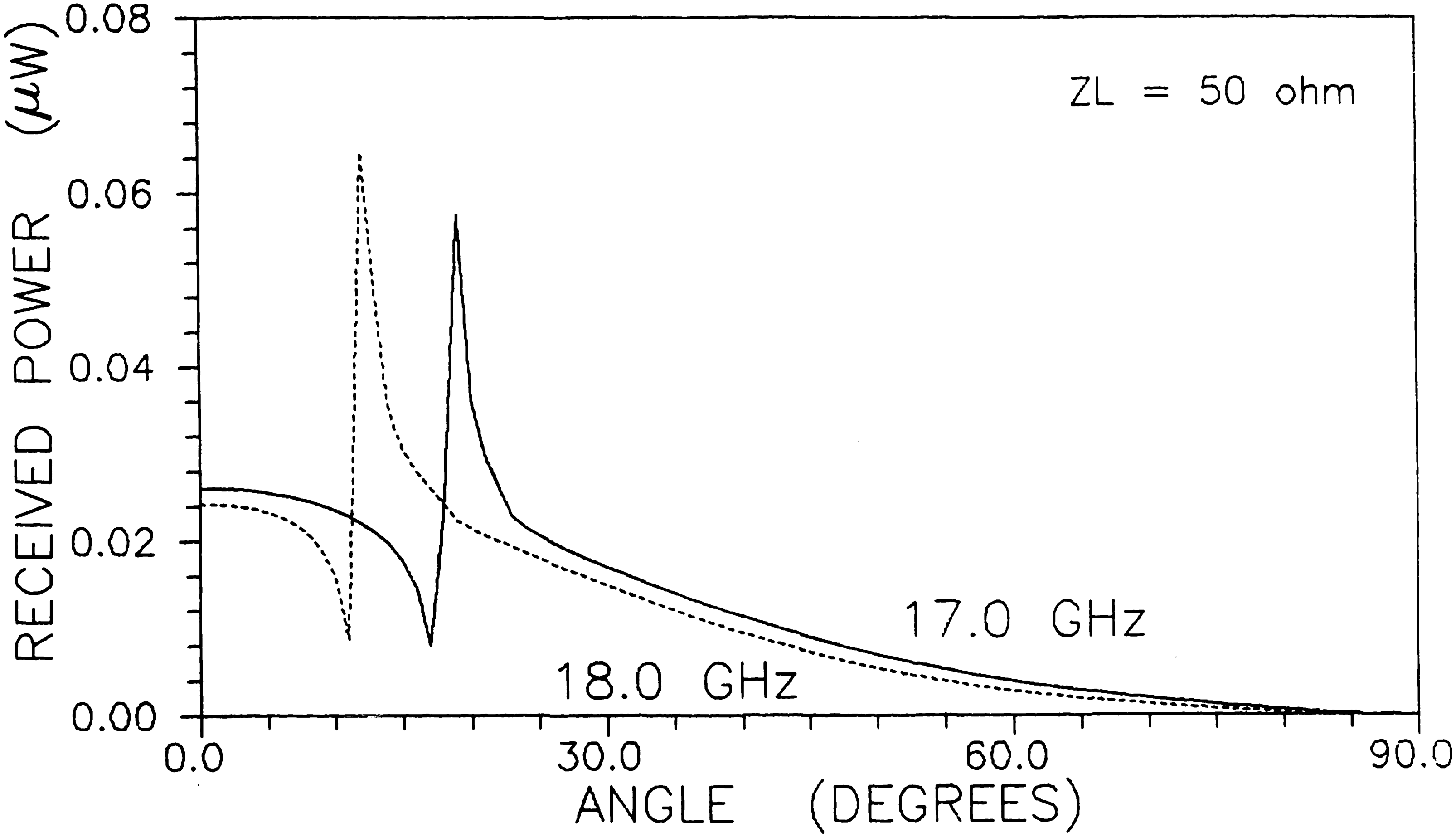


Figure 31. Received power for $Z_L = 50\ \Omega$ at 17.0 and 18.0 GHz.

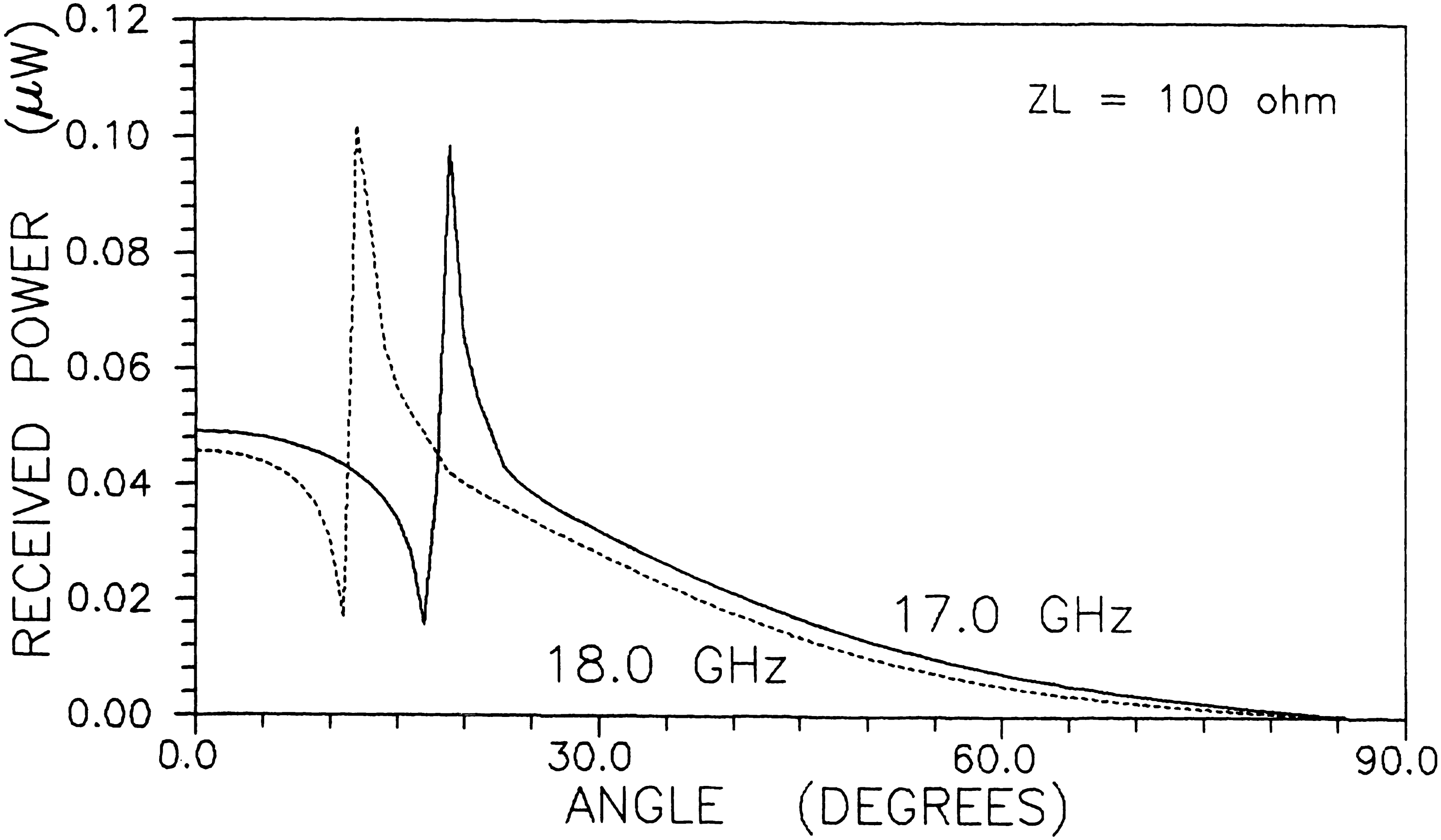


Figure 32. Received power for Z_L = 100 Ω at 17.0 and 18.0 GHz.

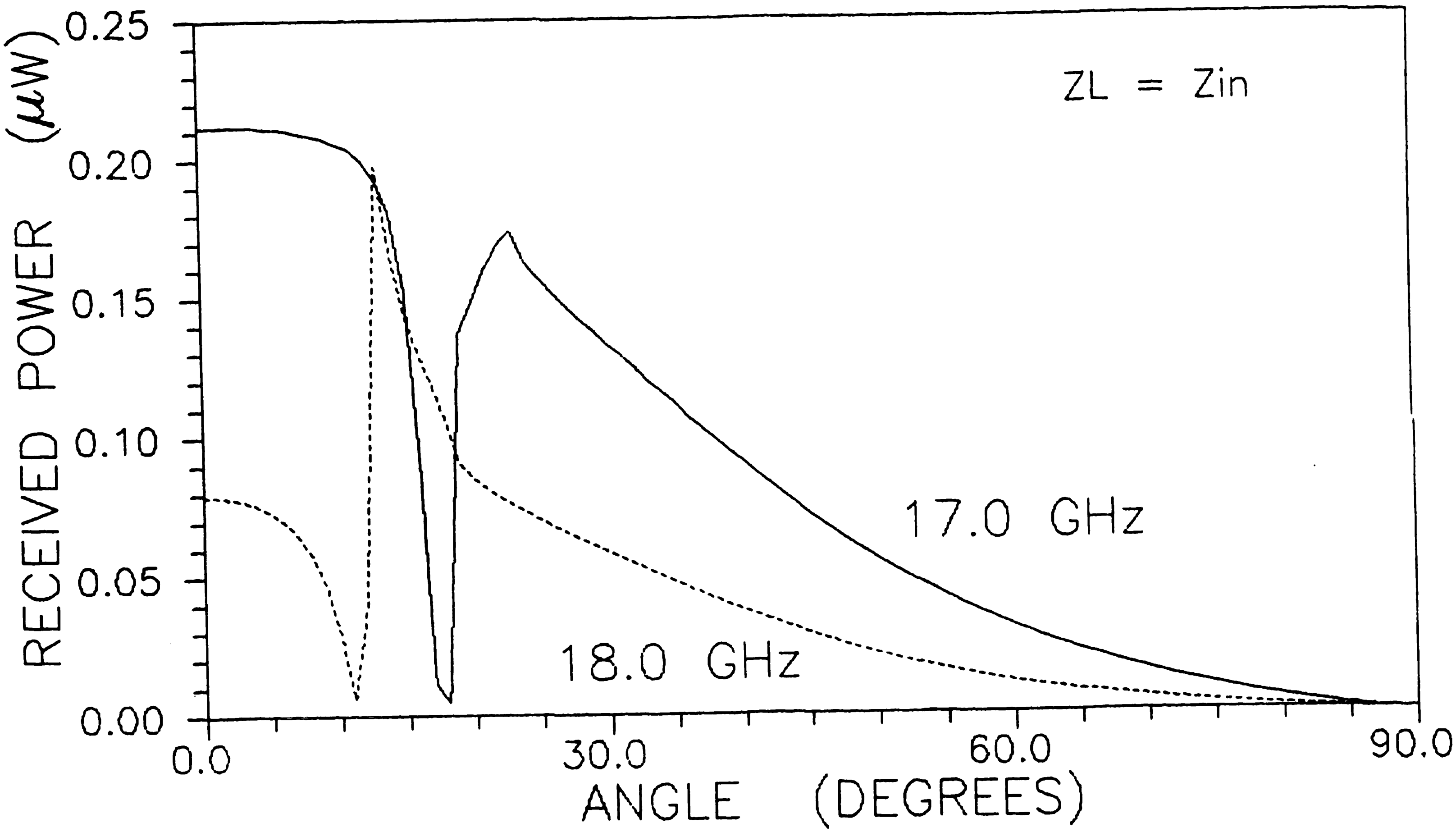


Figure 33. Received power for $Z_L = Z_{in}$ at 17.0 and 18.0 GHz.

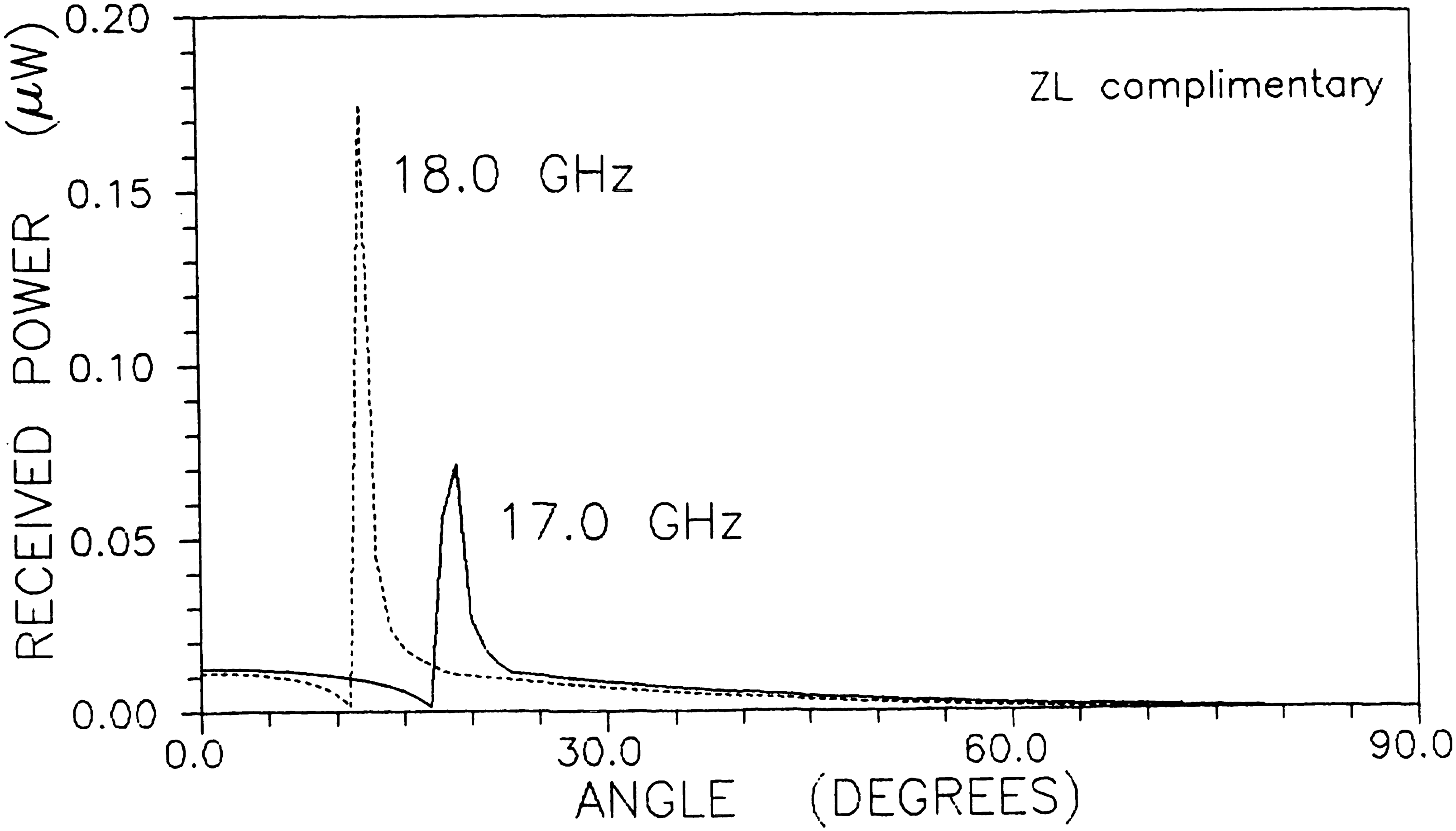


Figure 34. Received power for Z_L complimentary to Z_{in} at 17.0 and 18.0 GHz.

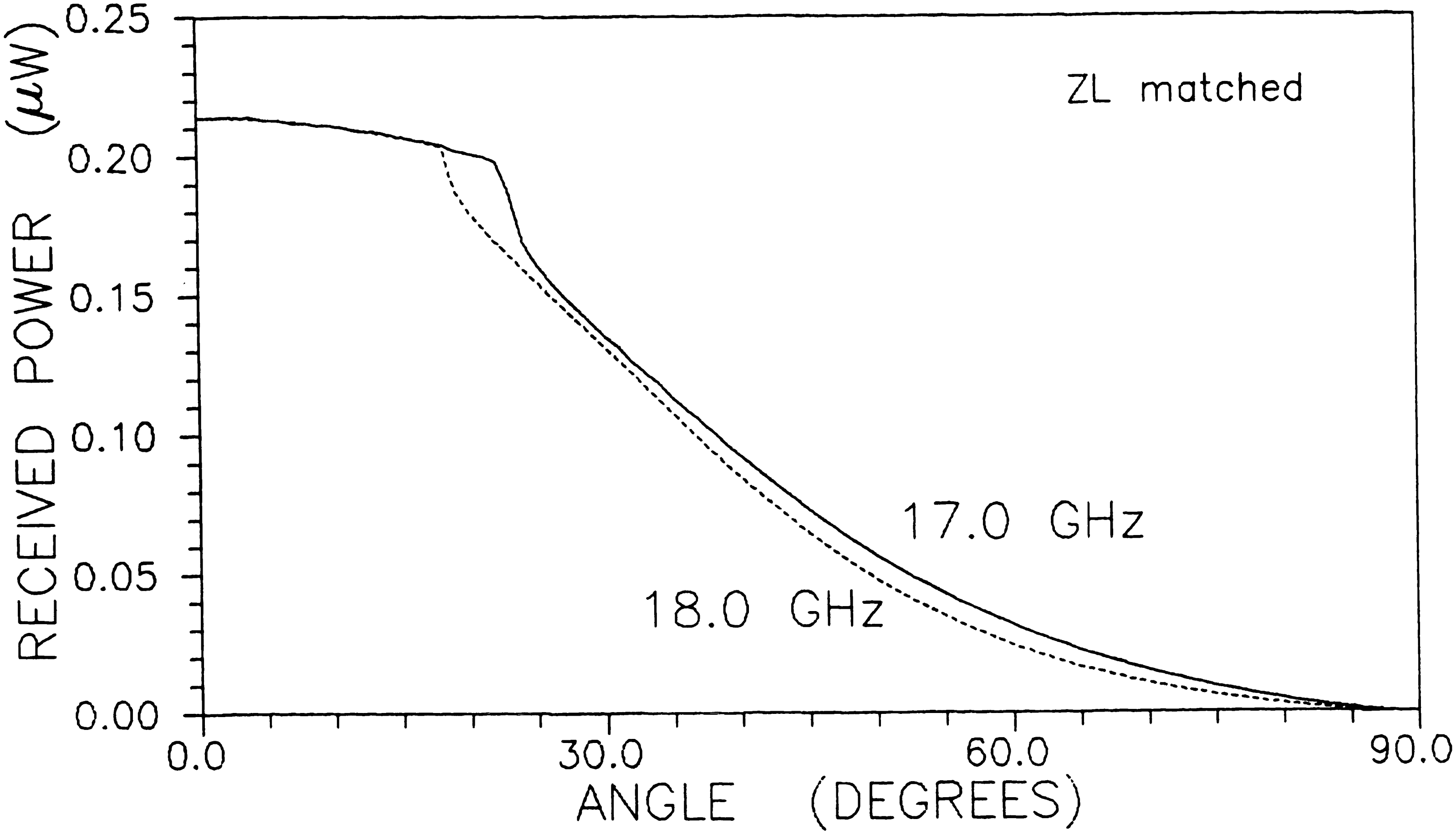


Figure 35. Received power for $Z_L = Z_{in}^*$ at 17.0 and 18.0 GHz.

The final set of figures gives the angular dependance for input impedance, scattered current and received power for a frequency of 9.8 GHz, where the scattered current experiences a peak, and the input impedance is nearly pure real.

Figure 36 shows the input resistance. It varies smoothly from about 12 Ω down to about 8 Ω at 80°, then drops more quickly, approaching 0 as θ_i approaches 90°.

In Figure 37, the reactance is shown. It varies from -3 Ω to -21 Ω as θ_i varies from 0 to 89°.

In Figure 38, the scattered current is given. It drops towards 0 in a smooth arc as θ_i varies from 0 to 90°.

In Figures 39 through 43, the received power is shown for the various cases of load impedance. Note that the units of power in Figure 42 are nW, not μW. The power received is so much below that of the other cases because the received power near 13 GHz has not yet peaked and is still very small. In all five Figures, maximum received power occurs at $\theta_i = 0°$, dropping smoothly towards 0 as θ_i approaches 90°.

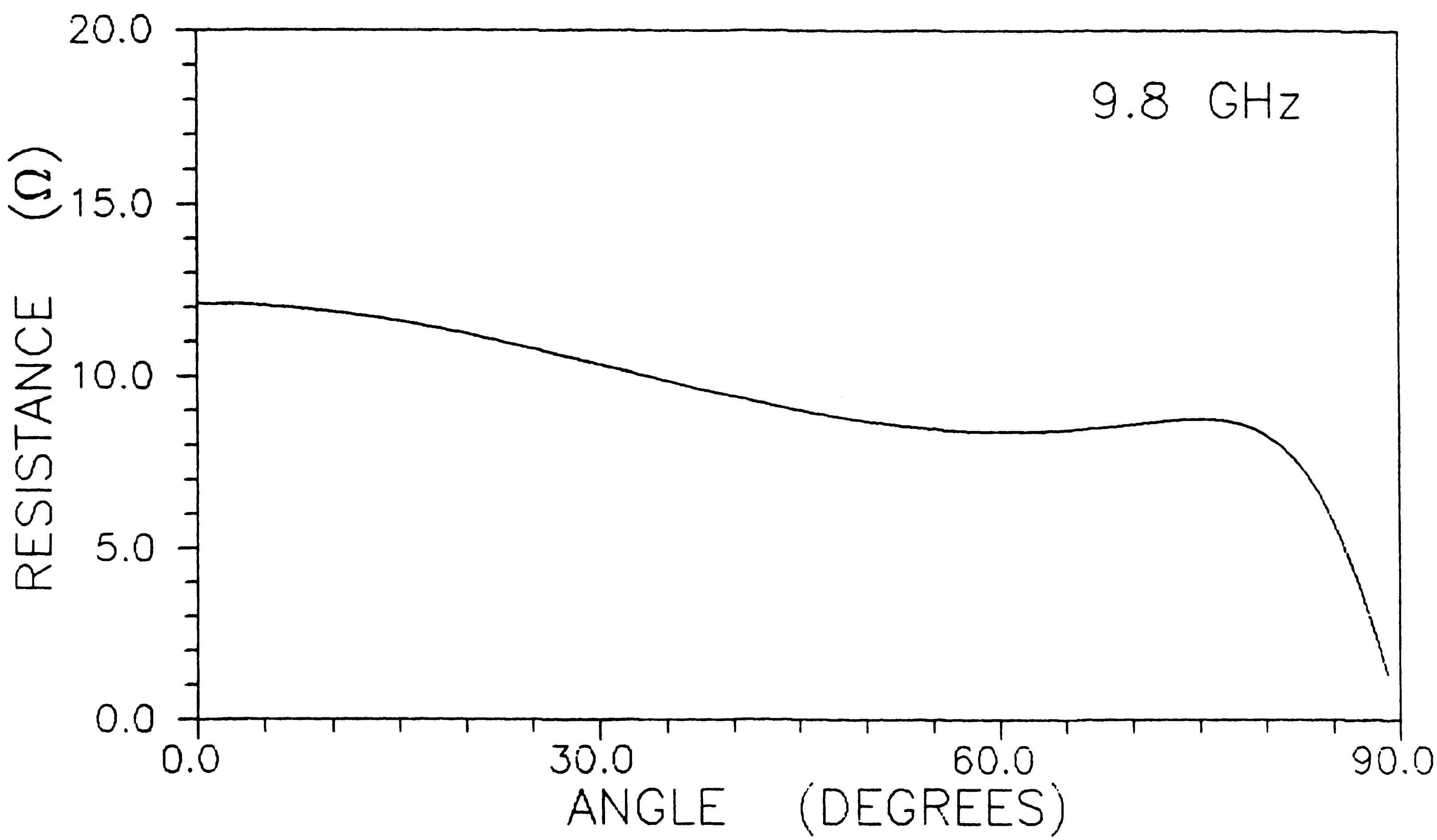


Figure 36. Input resistance for each element of array at 9.8 GHz.

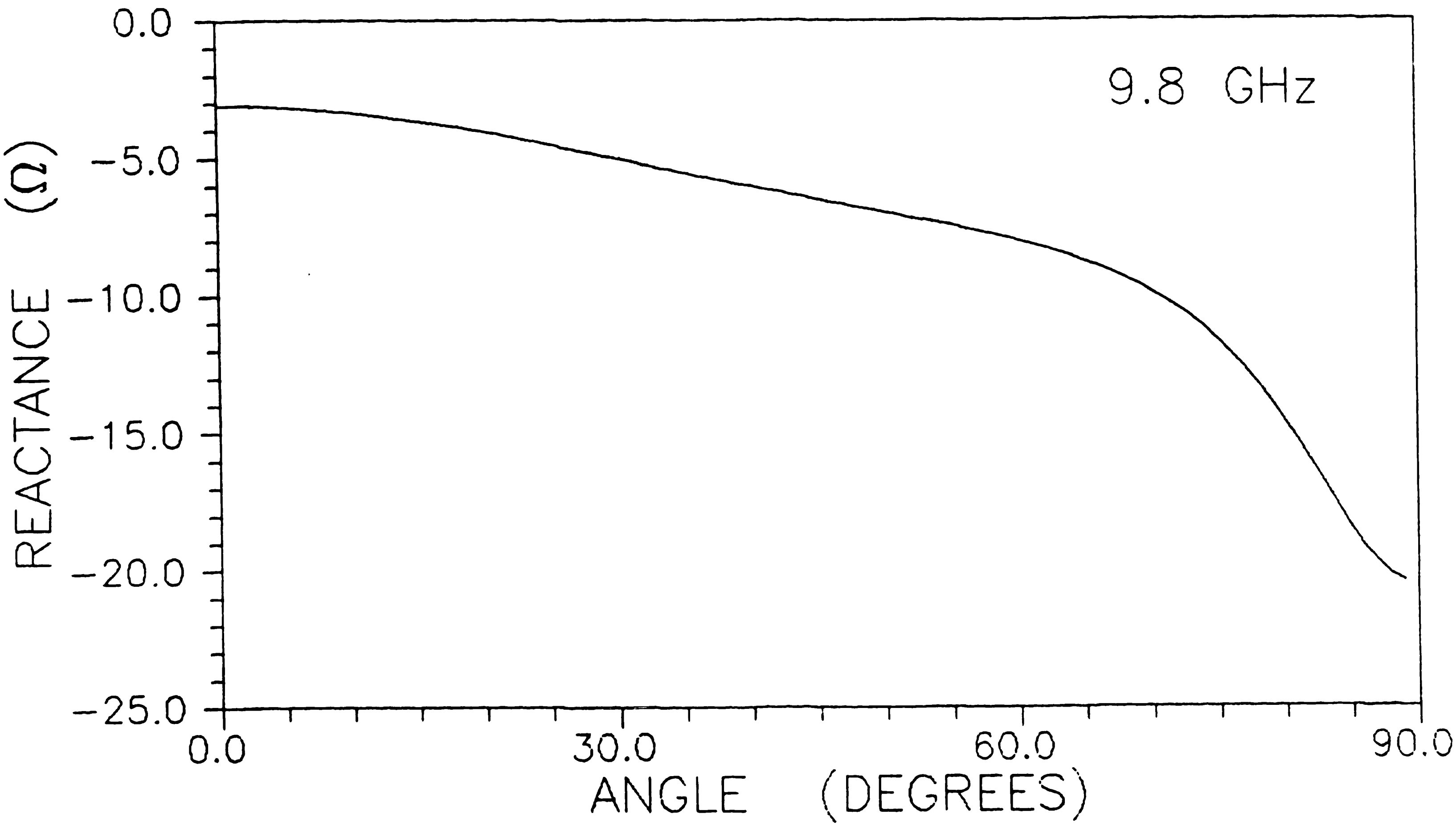


Figure 37. Input reactance for each element of array at 9.8 GHz.

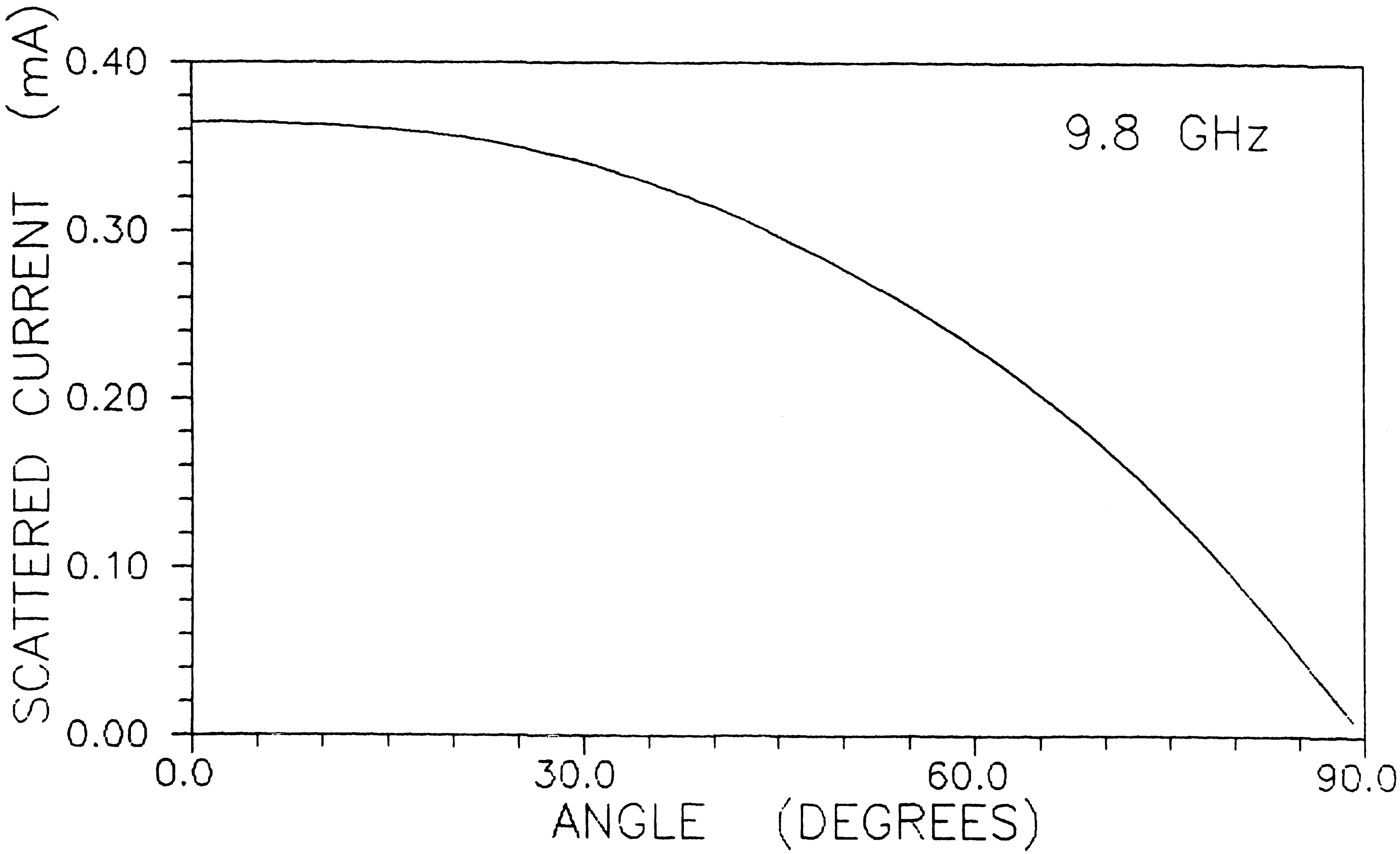


Figure 38. Scattered current magnitude at center of dipole at 9.8 GHz.

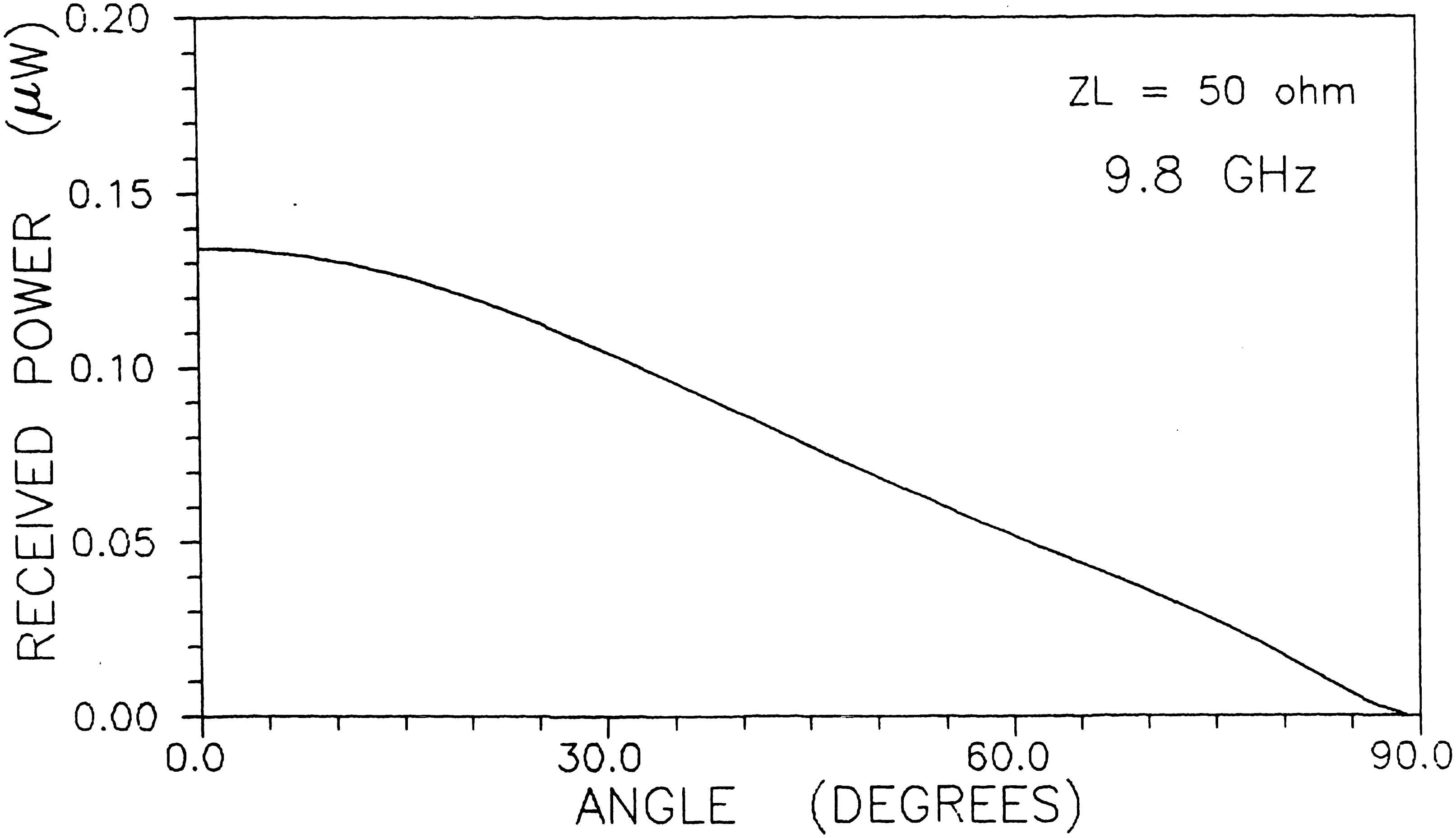


Figure 39. Received power for $Z_L = 50\ \Omega$ at 9.8 GHz.

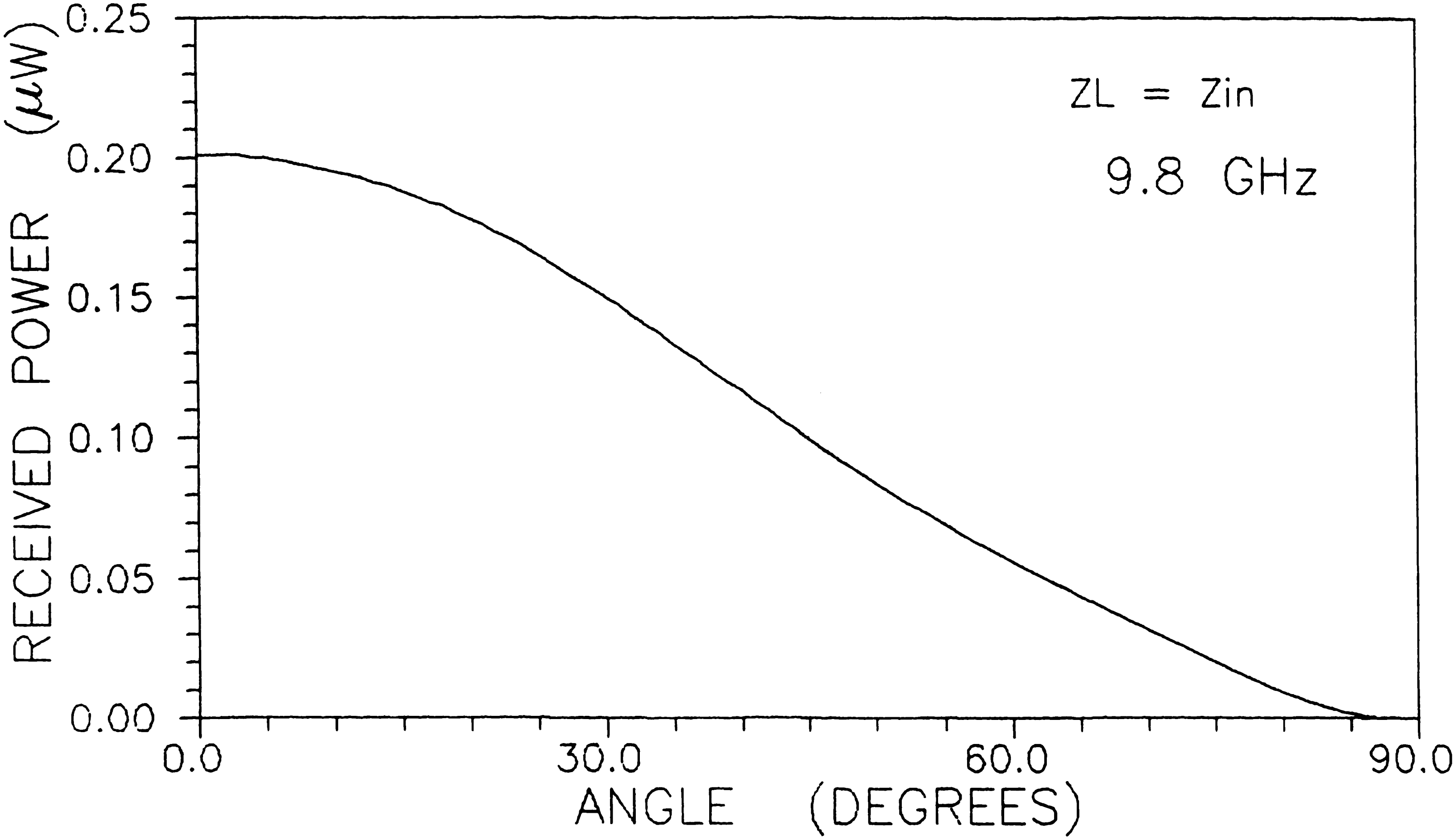


Figure 40. Received power for Z_L = 100 Ω at 9.8 GHz.

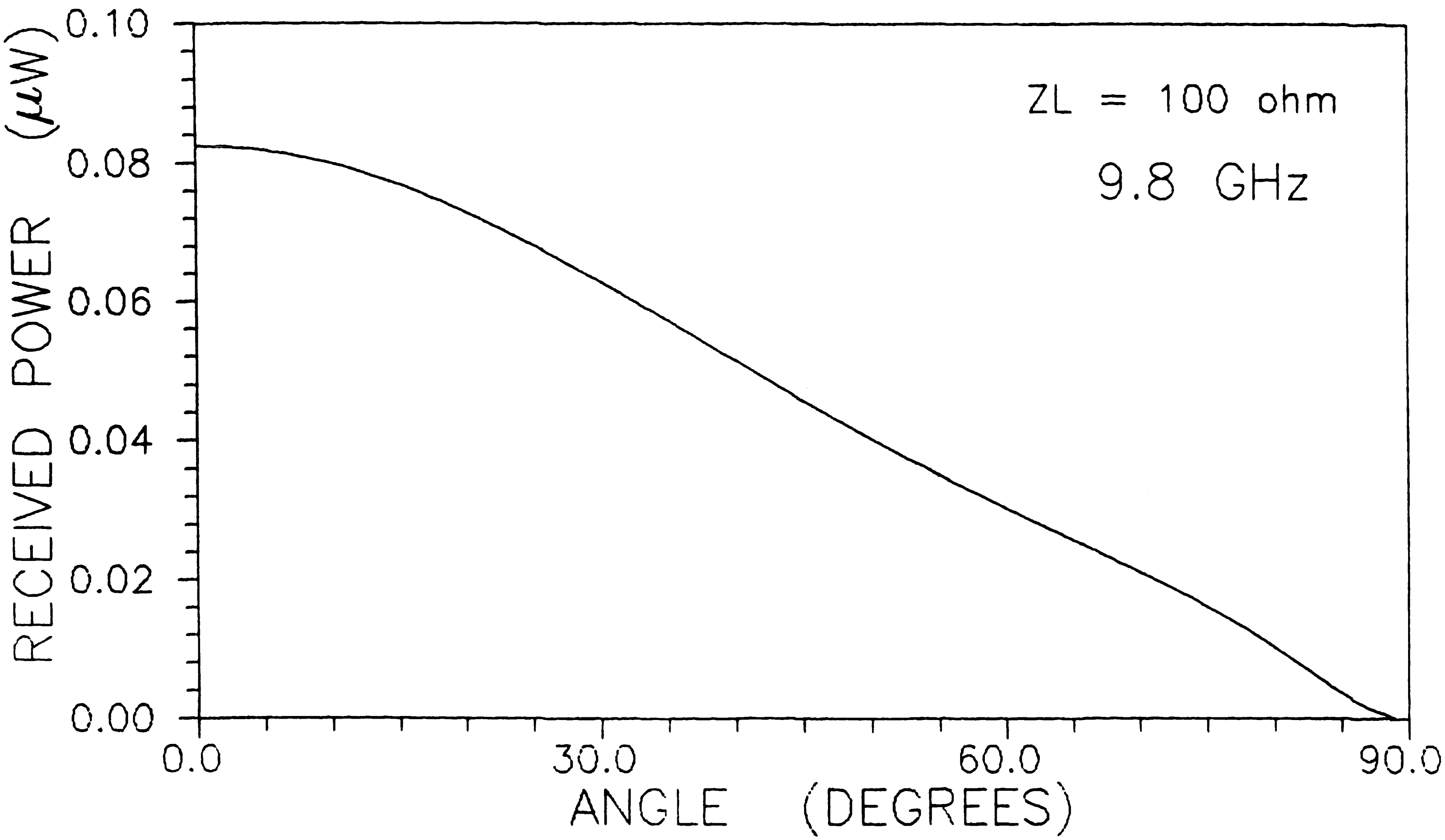


Figure 41. Received power for $Z_L = Z_{in}$ at 9.8 GHz.

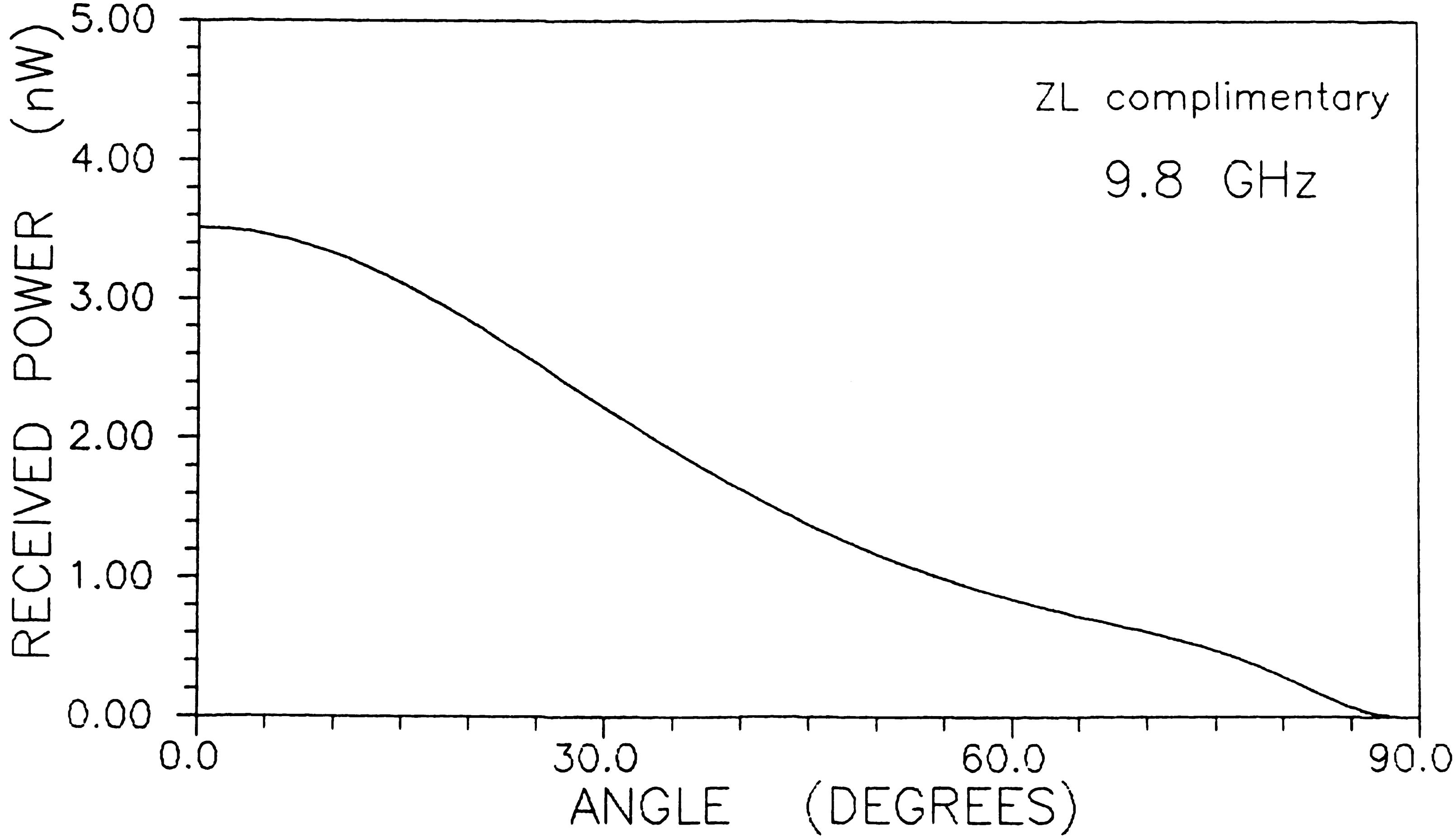


Figure 42. Received power for Z_L complimentary to Z_{in} at 9.8 GHz.

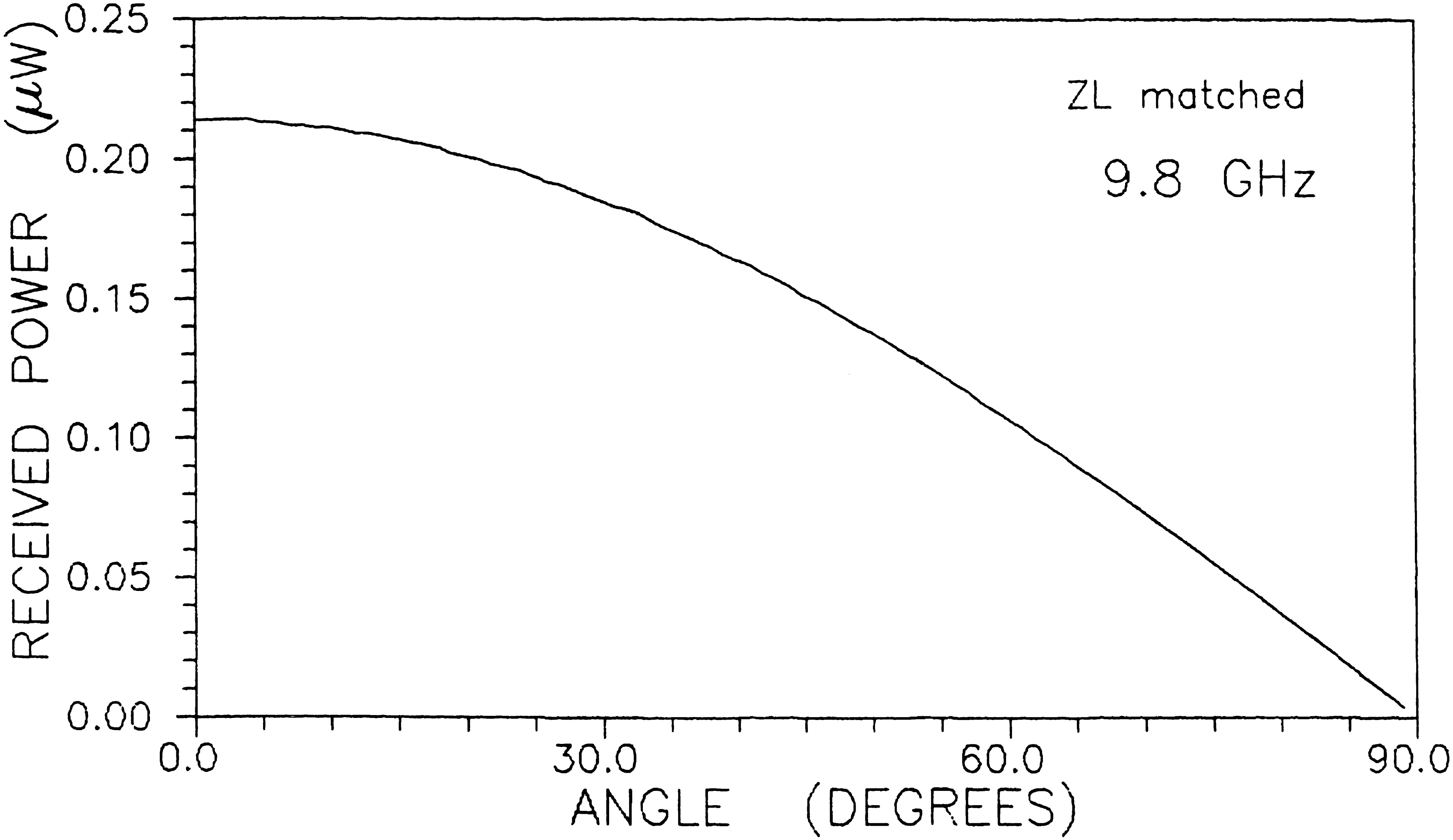


Figure 43. Received power for $Z_L = Z_{in}^*$ at 9.8 GHz.

V. EXCITATION FIELD, GREEN FUNCTION AND CURRENT DERIVATIONS.

12. Plane Wave Reflected by a Coated Conductor

The excitation field for the antenna or array is the field due to a plane wave reflected by a coated conductor. The total field (incident plane plus scattered) in the presence of the coated conductor supports the scattering elements.

Consider a plane wave illuminating a coated conductor as shown in Figures 1, 3 and 16, but without the antenna or array elements present. The form of the illuminating electric field is

$$\vec{E}^{\,i}(\vec{r}\,) = E^i \hat{\gamma}\, e^{-j\vec{k}^i \cdot \vec{r}} \tag{1}$$

where $\hat{\gamma}$ describes the incident plane wave polarization and $\vec{k}^i$ describes the incident direction of propagation. By applying the boundary conditions at the conducting plane and at the interface between the coating and the surround, the total electric field can be found.

The case of arbitrary incidence can be handled most easily by superposing waves parallel to and perpendicular to the plane of incidence.

Case 1: $\vec{E}$ parallel to plane of incidence.

The component of $\vec{E}^{\,i}$ parallel to the plane of incidence is the projection of $\vec{E}^{\,i}$ onto the x'–z plane. As shown in Figure 2 the x' and y' axes are rotated through an angle ϕ_i from the x and y axes, yielding a coordinate transformation

$$x'' = x\, \cos(\phi_i) + y\, \sin(\phi_i) \tag{2}$$

$$y'' = y\, \cos(\phi_i) - x\, \sin(\phi_i) \tag{3}$$

$$\hat{x}'' = \hat{x}\, \cos(\phi_i) + \hat{y}\, \sin(\phi_i) \tag{4}$$

$$\hat{y}'' = \hat{y}\, \cos(\phi_i) - \hat{x}\, \sin(\phi_i) \tag{5}$$

The projection of $\vec{E}^{\,i}$ onto the x'–z plane is, in vector form

SCATTERING CASE

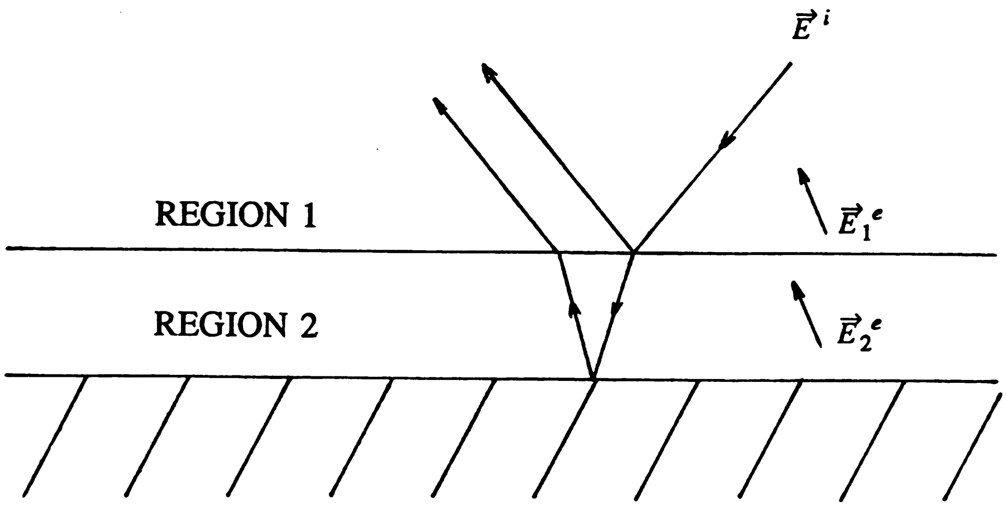


Figure 44. Incident plane wave: parallel polarization.

$$\vec{E}_{||}^{\,i} = (\vec{E}^i \cdot \hat{z})\,\hat{z} + (\vec{E}^i \cdot \hat{x}'')\,\hat{x}'' \tag{6}$$

Substituting (1) gives

$$\vec{E}_{||}^{\,i} = E^i \left[(\hat{\gamma}^i \cdot \hat{z})\,\hat{z} + (\hat{\gamma}^i \cdot \hat{x}'')\,\hat{x}''\right] e^{-j\vec{k}^i \cdot \vec{r}} \tag{7}$$

The magnitude of $\vec{E}_{||}^{\,i}$ is then

$$E_{||}^{\,i} = E^i \left[(\hat{\gamma}^i \cdot \hat{z})^2 + (\hat{\gamma}^i \cdot \hat{x}'')^2\right]^{\frac{1}{2}} \tag{8}$$

Using (4) then yields

$$E_{||}^{\,i} = E^i \left[\gamma_z^{i2} + (\gamma_x^i \cos(\phi_i) + \gamma_y^i \sin(\phi_i))^2\right]^{\frac{1}{2}} \tag{9}$$

Figure 44 shows a plane wave incident on a coated conductor, polarized parallel to the plane of incidence. The direction of propagation makes an angle θ_i with the z-axis. A reflected wave is also present in region 1, due to the reflection from the dielectric interface and from the conductor. Its direction of propagation also makes an angle of θ_i with the z-axis. Refracted and reflected waves are present in region 2, the coating, due to refraction at the dielectric interface and reflection from the conductor. The direction of propagation of these waves makes an angle θ_r with the z-axis.

The amplitudes of the reflected and refracted waves can be determined in terms of the incident wave by applying the boundary conditions at the dielectric interface and at the conductor. Refering to Figure 44, the incident, reflected and refracted waves can be written in vector form as

$$\vec{E}^{\,i}_{||} = \left[-\hat{z}\, E^{i}_{||}\, \sin(\theta_i) + \hat{x}''\, E^{i}_{||}\, \cos(\theta_i)\right] e^{-j\vec{k}^{i}\cdot\vec{r}} \tag{10}$$

$$\vec{E}^{\,r}_{||} = \left[+\hat{z}\, E^{r}_{||}\, \sin(\theta_i) + \hat{x}''\, E^{r}_{||}\, \cos(\theta_i)\right] e^{-j\vec{k}^{r}\cdot\vec{r}} \tag{11}$$

$$\vec{E}^{\,-}_{||} = \left[-\hat{z}\, E^{-}_{||}\, \sin(\theta_r) + \hat{x}''\, E^{-}_{||}\, \cos(\theta_r)\right] e^{-j\vec{k}^{-}\cdot\vec{r}} \tag{12}$$

$$\vec{E}^{\,+}_{||} = \left[+\hat{z}\, E^{+}_{||}\, \sin(\theta_r) + \hat{x}''\, E^{+}_{||}\, \cos(\theta_r)\right] e^{-j\vec{k}^{+}\cdot\vec{r}} \tag{13}$$

$$\vec{H}^{\,i}_{||} = -\hat{y}''\, \frac{E^{i}_{||}}{\eta_1}\, e^{-j\vec{k}^{i}\cdot\vec{r}} \tag{14}$$

$$\vec{H}^{\,r}_{||} = +\hat{y}''\, \frac{E^{r}_{||}}{\eta_1}\, e^{-j\vec{k}^{r}\cdot\vec{r}} \tag{15}$$

$$\vec{H}^{\,-}_{||} = -\hat{y}''\, \frac{E^{-}_{||}}{\eta_2}\, e^{-j\vec{k}^{-}\cdot\vec{r}} \tag{16}$$

$$\vec{H}^{\,+}_{||} = +\hat{y}''\, \frac{E^{+}_{||}}{\eta_2}\, e^{-j\vec{k}^{+}\cdot\vec{r}} \tag{17}$$

where

$$\vec{k}^{i} = (-\hat{x}''\sin(\theta_i) - \hat{z}\cos(\theta_i))k_1 \tag{18}$$

$$\vec{k}^{r} = (-\hat{x}''\sin(\theta_i) + \hat{z}\cos(\theta_i))k_1 \tag{19}$$

$$\vec{k}^{-} = (-\hat{x}''\sin(\theta_r) - \hat{z}\cos(\theta_r))k_2 \tag{20}$$

$$\vec{k}^{+} = (-\hat{x}''\sin(\theta_r) + \hat{z}\cos(\theta_r))k_2 \tag{21}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \tag{22a,b}$$

$$k_1 = \omega\sqrt{\mu_1 \varepsilon_1} \qquad k_2 = \omega\sqrt{\mu_2 \varepsilon_2} \tag{23a,b}$$

Here, (14) through (17) have been determined from (10) through (13) and (18) through (21) using the relationship for plane waves,

$$\vec{H} = \frac{\vec{k} \times \vec{E}}{\eta} \tag{24}$$

The field amplitudes $E^r_{||}$, $E^-_{||}$, and $E^+_{||}$ can be determined in terms of $E^i_{||}$ by applying the following boundary conditions.

a) Tangential $\vec{E} = 0$ at $z = -d$

$$E^-_{||x''}(z=-d) + E^+_{||x''}(z=-d) = 0 \tag{25}$$

or, using (12), (13), (20), and (21)

$$E^-_{||} \cos(\theta_r)\, e^{jk_2 x'' \sin(\theta_r)}\, e^{-jk_2 d \cos(\theta_r)} + E^+_{||} \cos(\theta_r)\, e^{jk_2 x'' \sin(\theta_r)}\, e^{jk_2 d \cos(\theta_r)} = 0 \tag{26}$$

which reduces to

$$E^-_{||} = -E^+_{||}\, e^{+2jk_2 d \cos(\theta_r)} \tag{27}$$

b) Tangential $\vec{E}$ continuous at $z = 0$

$$E^i_{||x''}(z=0) + E^r_{||x''}(z=0) - E^-_{||x''}(z=0) - E^+_{||x''}(z=0) = 0 \tag{28}$$

or, using (10) through (13) and (18) to (21),

$$E^i_{||} \cos(\theta_i)\, e^{jk_1 x'' \sin(\theta_i)} + E^r_{||} \cos(\theta_i)\, e^{jk_1 x'' \sin(\theta_i)}$$
$$- E^-_{||} \cos(\theta_r)\, e^{jk_2 x'' \sin(\theta_r)} - E^+_{||} \cos(\theta_r)\, e^{jk_2 x'' \sin(\theta_r)} = 0 \tag{29}$$

For this equation to hold for all x', the phase terms must be equal, giving

$$k_1 \sin(\theta_i) = k_2 \sin(\theta_r) \tag{30}$$

which is Snell's law. Then, (29) reduces to

$$E^i_{||} \cos(\theta_i) + E^r_{||} \cos(\theta_i) - E^-_{||} \cos(\theta_r) - E^+_{||} \cos(\theta_r) = 0 \tag{31}$$

c) Tangential $\vec{H}$ continuous at $z = 0$

$$H^{i}_{||y''}(z{=}0) + H^{r}_{||y''}(z{=}0) - H^{-}_{||y''}(z{=}0) - H^{+}_{||y''}(z{=}0) = 0 \tag{32}$$

Using equations (14) through (17) and (18) to (21) again gives (30), and hence

$$-\frac{E^{i}_{||}}{\eta_1} + \frac{E^{r}_{||}}{\eta_1} + \frac{E^{-}_{||}}{\eta_2} - \frac{E^{+}_{||}}{\eta_2} = 0 \tag{33}$$

Equations (27), (31), and (33) represent a system of three equations in the unknown field amplitudes $E^{r}_{||}$, $E^{+}_{||}$, and $E^{-}_{||}$. To solve this system, first substitute (13.27) into (13.31), giving

$$E^{i}_{||} + E^{r}_{||} = E^{+}_{||}\,\frac{\cos(\theta_r)}{\cos(\theta_i)}\left[1 - e^{2jk_2 d\cos(\theta_r)}\right] \tag{34}$$

which eliminates $E^{-}_{||}$. Next, substitute (27) into (33) to give

$$E^{i}_{||} - E^{r}_{||} = -E^{+}_{||}\,\frac{\eta_1}{\eta_2}\left[1 + e^{2jk_2 d\cos(\theta_r)}\right] \tag{35}$$

Adding (34) and (35) will then eliminate $E^{r}_{||}$, giving

$$E^{i}_{||} = -E^{+}_{||}\,e^{jk_2 d\cos(\theta_r)}\left[\frac{\eta_1}{\eta_2}\cos(k_2 d\cos(\theta_r)) + j\,\frac{\cos(\theta_r)}{\cos(\theta_i)}\sin(k_2 d\cos(\theta_r))\right] \tag{36}$$

As a convenient notation, let

$$E^{+}_{||} = T_{||}E^{i}_{||} \tag{37}$$

where $T_{||}$ is the transmission coefficient into the coating

$$T_{||} = \frac{j\eta_2\cos(\theta_i)\,e^{-jk_2 d\cos(\theta_r)}}{\eta_2\cos(\theta_r)\,\sin(k_2 d\cos(\theta_r)) - j\eta_1\cos(\theta_i)\cos(k_2 d\cos(\theta_r))} \tag{38}$$

Substituting (37) into (35) gives $E^{r}_{||}$

$$E^{r}_{||} = \Gamma_{||}\,E^{i}_{||} \tag{39}$$

where

$$\Gamma_{||} = \frac{\eta_2\cos(\theta_r)\,\sin(k_2 d\cos(\theta_r)) + j\eta_1\cos(\theta_i)\cos(k_2 d\cos(\theta_r))}{\eta_2\cos(\theta_r)\,\sin(k_2 d\cos(\theta_r)) - j\eta_1\cos(\theta_i)\cos(k_2 d\cos(\theta_r))} \tag{40}$$

is the reflection coefficient from the coating. Note that if k_2 is real, the magnitude of the reflection coeficient is

$$| \Gamma_{||} | = 1 \tag{41}$$

This suggests that $\Gamma_{||}$ be written as

$$\Gamma_{||} = 1\, e^{j\Phi_{||}} \tag{42}$$

If k_2 is complex, (42) can still be used, but $\Phi_{||}$ is then also complex.

With the field amplitudes now known, the total electric field in each region can be determined. The field in the surround (region 1) is

$$\vec{E}^{\,1}_{||} = \vec{E}^{\,i}_{||} + \vec{E}^{\,r}_{||} \tag{43}$$

Substituting (10) and (11) and using (39) gives

$$\vec{E}^{\,1}_{||} = \hat{z}\, E^i_{||}\, e^{jk_1 x'' \sin(\theta_i)} \left[\Gamma_{||}\, e^{-jk_1 z \cos(\theta_i)} - e^{jk_1 z \cos(\theta_i)}\right] \sin(\theta_i)$$

$$+ \hat{x}''\, E^i_{||}\, e^{jk_1 x'' \sin(\theta_i)} \left[\Gamma_{||}\, e^{-jk_1 z \cos(\theta_i)} + e^{jk_1 z \cos(\theta_i)}\right] \cos(\theta_i) \tag{44}$$

Now, using (42) allows this to be written as

$$\vec{E}^{\,1}_{||} = \hat{z}\, 2j\, E^i_{||}\, e^{j(\frac{1}{2}\Phi_{||} + k_1 x'' \sin(\theta_i))} \sin(\tfrac{1}{2}\Phi_{||} - k_1 z \cos(\theta_i)) \sin(\theta_i)$$

$$+ \hat{x}''\, 2\, E^i_{||}\, e^{j(\frac{1}{2}\Phi_{||} + k_1 x'' \sin(\theta_i))} \cos(\tfrac{1}{2}\Phi_{||} - k_1 z \cos(\theta_i)) \cos(\theta_i) \tag{45}$$

which shows $\vec{E}^{\,1}_{||}$ to be a travelling wave in the x'' direction and a standing wave in the z direction.

The field in the coating (region 2) is

$$\vec{E}^{\,2}_{||} = \vec{E}^{\,-}_{||} + \vec{E}^{\,+}_{||} \tag{46}$$

Substituting (12) and (13) and using (27) and (37) gives

$$\vec{E}^{\,2}_{||} = \hat{z}\, 2\, T_{||}\, E^i_{||}\, e^{jk_2(x'' \sin(\theta_r) + d \cos(\theta_r))} \sin(\theta_r) \cos(k_2(z + d) \cos(\theta_r))$$

$$- \hat{x}''\, 2j\, T_{||}\, E^i_{||}\, e^{jk_2(x'' \sin(\theta_r) + d \cos(\theta_r))} \cos(\theta_r) \sin(k_2(z + d) \cos(\theta_r)) \tag{47}$$

which is, again, a travelling wave in the x'' direction and a standing wave in the z direction.

SCATTERING CASE

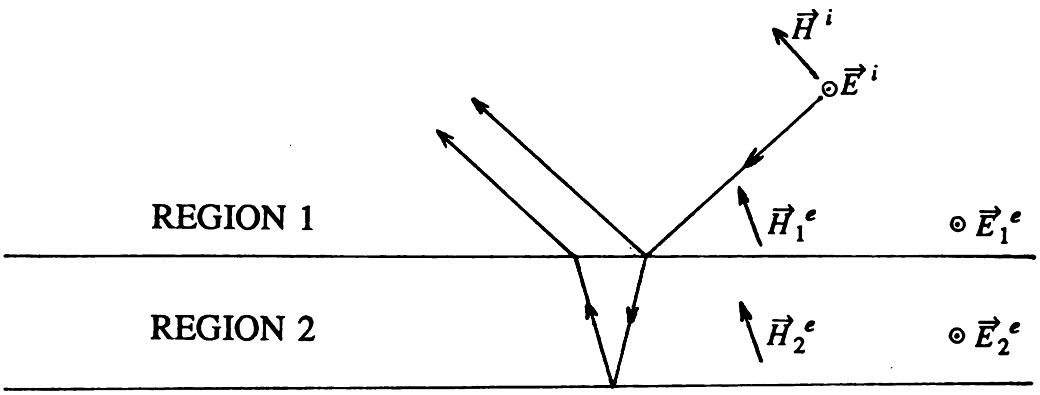


Figure 45. Incident plane wave: perpendicular polarization.

Case 2: $\vec{E}$ perpendicular to plane of incidence

The component of $\vec{E}^i$ perpendicular to the plane of incidence is merely the y'' component, as seen in figure 45. Thus,

$$\vec{E}_{\perp}^{i} = (\vec{E}^{i} \cdot \hat{y}'')\, \hat{y}'' \tag{48}$$

Substituting (13.1) yields

$$\vec{E}_{\perp}^{i} = E^{i}\, \hat{y}''(\hat{\gamma}^{i} \cdot \hat{y}'')\, e^{-j\vec{k}^{i} \cdot \vec{r}} \tag{49}$$

The magnitude of $\vec{E}_{\perp}^{i}$ is thus

$$E_{\perp}^{i} = E^{i}\, [\hat{\gamma}^{i} \cdot \hat{y}''] \tag{50}$$

which, upon using (5) becomes

$$E_{\perp}^{i} = E^{i}\, [\gamma_y^i \cos(\phi_i) - \gamma_x^i \sin(\phi_i)] \tag{51}$$

Figure 45 shows a plane wave incident on a coated conductor, polarized perpendicular to the plane of incidence. As with the parallel case, there is a reflected wave in region 1 and reflected and transmitted waves in region 2. Referring to Figure 45, these waves can be written in vector form as

$$\vec{E}_\perp^{\,i} = \hat{y}''\, E_\perp^i \; e^{-j\vec{k}^i \cdot \vec{r}} \tag{52}$$

$$\vec{E}_\perp^{\,r} = \hat{y}''\, E_\perp^r \; e^{-j\vec{k}^r \cdot \vec{r}} \tag{53}$$

$$\vec{E}_\perp^{\,-} = \hat{y}''\, E_\perp^- \; e^{-j\vec{k}^- \cdot \vec{r}} \tag{54}$$

$$\vec{E}_\perp^{\,+} = \hat{y}''\, E_\perp^+ \; e^{-j\vec{k}^+ \cdot \vec{r}} \tag{55}$$

$$\vec{H}_\perp^{\,i} = \left[-\hat{z}\, \frac{E_\perp^i}{\eta_1} \sin(\theta_i) + \hat{x}''\, \frac{E_\perp^i}{\eta_1} \cos(\theta_i) \right] e^{-j\vec{k}^i \cdot \vec{r}} \tag{56}$$

$$\vec{H}_\perp^{\,r} = \left[-\hat{z}\, \frac{E_\perp^r}{\eta_1} \sin(\theta_i) - \hat{x}''\, \frac{E_\perp^r}{\eta_1} \cos(\theta_i) \right] e^{-j\vec{k}^r \cdot \vec{r}} \tag{57}$$

$$\vec{H}_\perp^{\,-} = \left[-\hat{z}\, \frac{E_\perp^-}{\eta_2} \sin(\theta_r) + \hat{x}''\, \frac{E_\perp^-}{\eta_2} \cos(\theta_r) \right] e^{-j\vec{k}^- \cdot \vec{r}} \tag{58}$$

$$\vec{H}_\perp^{\,+} = \left[-\hat{z}\, \frac{E_\perp^+}{\eta_2} \sin(\theta_r) - \hat{x}''\, \frac{E_\perp^+}{\eta_2} \cos(\theta_r) \right] e^{-j\vec{k}^+ \cdot \vec{r}} \tag{59}$$

where the $\vec{k}$ vectors are again given by equations (18) through (21).

The wave amplitudes are determined by applying the following boundary conditions.

a) Tangential $\vec{E} = 0$ at $z = -d$

$$E_{\perp y''}^-(z=-d) + E_{\perp y''}^+(z=-d) = 0 \tag{60}$$

or, using (54), (55), (20), and (21)

$$E_\perp^- \, e^{jk_2 x'' \sin(\theta_r)} \, e^{-jk_2 d \cos(\theta_r)} + E_\perp^+ \, e^{jk_2 x'' \sin(\theta_r)} \, e^{jk_2 d \cos(\theta_r)} = 0 \tag{61}$$

which reduces to

$$E_\perp^- = -E_\perp^+ \, e^{+2jk_2 d \cos(\theta_r)} \tag{62}$$

b) Tangential $\vec{E}$ continuous at $z = 0$

$$E^i_{\perp y''}(z{=}0) + E^r_{\perp y''}(z{=}0) - E^-_{\perp y''}(z{=}0) - E^+_{\perp y''}(z{=}0) = 0 \tag{63}$$

Using (52) through (55) and (18) through (21) gives Snell's law again, along with

$$E^i_\perp + E^r_\perp - E^-_\perp - E^+_\perp = 0 \tag{64}$$

c) Tangential $\vec{H}$ continuous at $z = 0$

$$H^i_{\perp x''}(z{=}0) + H^r_{\perp x''}(z{=}0) - H^-_{\perp x''}(z{=}0) - H^+_{\perp x''}(z{=}0) = 0 \tag{65}$$

Using (56) through (59) and (18) to (21) again gives (30), and hence

$$\frac{E^i_\perp}{\eta_1}\cos(\theta_i) - \frac{E^r_\perp}{\eta_1}\cos(\theta_i) - \frac{E^-_\perp}{\eta_2}\cos(\theta_r) + \frac{E^+_\perp}{\eta_2}\cos(\theta_r) = 0 \tag{66}$$

To solve for $E^r_\perp$, $E^-_\perp$ and $E^+_\perp$ in terms of $E^i_\perp$, first substitute (62) into (64) and (67), giving

$$E^i_\perp + E^r_\perp = E^+_\perp \left[1 - e^{2jk_2 d\cos(\theta_r)}\right] \tag{67}$$

$$E^i_\perp - E^r_\perp = -E^+_\perp \frac{\cos(\theta_r)}{\cos(\theta_i)} \frac{\eta_1}{\eta_2} \left[1 + e^{2jk_2 d\cos(\theta_r)}\right] \tag{68}$$

Now, these two equations are of the same form as (6.34) and (6.35). Thus, using the replacements

$$\frac{\cos(\theta_r)}{\cos(\theta_i)} \rightarrow 1 \qquad\qquad \frac{\eta_1}{\eta_2} \rightarrow \frac{\eta_1}{\eta_2}\frac{\cos(\theta_r)}{\cos(\theta_i)} \tag{69a,b}$$

gives

$$E^r_\perp = \Gamma_\perp E^i_\perp \tag{70}$$

$$E^+_\perp = T_\perp E^i_\perp \tag{71}$$

where, following equations (40) and (38),

$$\Gamma_\perp = \frac{\eta_2\cos(\theta_i)\sin(k_2 d\,\cos(\theta_r)) + j\eta_1\cos(\theta_r)\cos(k_2 d\,\cos(\theta_r))}{\eta_2\cos(\theta_i)\sin(k_2 d\,\cos(\theta_r)) - j\eta_1\cos(\theta_r)\cos(k_2 d\,\cos(\theta_r))} \tag{72}$$

$$T_\perp = \frac{j\eta_2\cos(\theta_i)\,e^{-jk_2 d\cos(\theta_r)}}{\eta_2\cos(\theta_i)\sin(k_2 d\,\cos(\theta_r)) - j\eta_1\cos(\theta_r)\cos(k_2 d\,\cos(\theta_r))} \tag{73}$$

The electric field in region 1 is now found to be

$$\vec{E}_{\perp}^{\,1} = \vec{E}_{\perp}^{\,i} + \vec{E}_{\perp}^{\,r} \tag{74}$$

which, upon substitution of (52), (53), (18), and (19) gives

$$\vec{E}_{\perp}^{\,1} = \hat{y}''\, 2\, E_{\perp}^{i}\, e^{j(\frac{1}{2}\Phi_{\perp} + k_1 x'' \sin(\theta_i))} \cos(\frac{1}{2}\Phi_{\perp} - k_1 z\, \cos(\theta_i)) \tag{75}$$

where $\Phi_{\perp}$ is determined from

$$\Gamma_{\perp} = 1\, e^{j\Phi_{\perp}} \tag{76}$$

Lastly, the electric field in region 2 is found from

$$\vec{E}_{\perp}^{\,2} = \vec{E}_{\perp}^{\,+} + \vec{E}_{\perp}^{\,-} \tag{77}$$

which, using (54), (55), (20), and (21) gives

$$\vec{E}_{\perp}^{\,2} = -\hat{y}''\, 2j\, T_{\perp}\, E_{\perp}^{i}\, e^{jk_2(x''\sin(\theta_r) + d\cos(\theta_r))} \sin(k_2(z + d)\cos(\theta_r)) \tag{78}$$

An interesting special case is normal incidence. Letting $\theta_i = \theta_r = 0$ gives

$$\vec{E}_{||}^{\,1} = \hat{x}''\, 2\, E_{||}^{i}\, e^{j\frac{1}{2}\Phi_{||}} \cos(\frac{1}{2}\Phi_{||} - k_1 z) \tag{79}$$

$$\vec{E}_{||}^{\,2} = -\hat{x}''\, 2j\, E_{||}^{i}\, T_{||}\, e^{jk_2 d} \sin(k_2(z + d)) \tag{80}$$

$$\vec{E}_{\perp}^{\,1} = \hat{y}''\, 2\, E_{\perp}^{i}\, e^{j\frac{1}{2}\Phi_{\perp}} \cos(\frac{1}{2}\Phi_{\perp} - k_1 z) \tag{81}$$

$$\vec{E}_{\perp}^{\,2} = -\hat{y}''\, 2j\, E_{\perp}^{i}\, T_{\perp}\, e^{jk_2 d} \sin(k_2(z + d)) \tag{82}$$

where

$$T_{||} = T_{\perp} = \frac{j\eta_2\, e^{-jk_2 d}}{\eta_2 \sin(k_2 d) - j\,\eta_1 \cos(k_2 d)} \tag{83}$$

$$\Gamma_{||} = \Gamma_{\perp} = \frac{\eta_2 \sin(k_2 d) + j\,\eta_1 \cos(k_2 d)}{\eta_2 \sin(k_2 d) - j\,\eta_1 \cos(k_2 d)} \tag{84}$$

13. Derivation of Green Functions for sources in the presence of a grounded dielectric slab.

13.1 Preliminaries.

Consider a grounded dielectric slab of thickness d as shown in figures 1, 3 and 16, but without the patches, feed pins or dipoles present. The region above the slab (region 1, $z \geq 0$) is assumed to have the constitutive parameters ε_1 and μ_1, while the slab (region 2, $-d < z < 0$) has parameters ε_2 and μ_2.

If a current source $\vec{J}$ is placed in region 1, an electric field will be maintained in both regions 1 and 2. The field in region 1 can be decomposed into a primary wave $\vec{E}_1^p$ and a reflected wave $\vec{E}_1^r$. The primary wave is equivalent to the field produced by the current source in unbounded space filled with the material composing region 1, while $\vec{E}_1^r$ represents the reflection of the primary wave from the dielectric interface. The field in region 2 can be decomposed into a wave $\vec{E}_1^-$ propagating in the $-z$ direction due to transmittance of the primary wave at the interface, and a wave $\vec{E}_1^+$ propagating in the $+z$ direction resulting from reflection at the conductor.

If a current source is placed in region 2, an electric field will also be maintained in both regions 1 and 2. The field in region 2 is now decomposed into a primary wave $\vec{E}_2^p$, a wave $\vec{E}_2^+$ resulting from reflection at the conductor, and a wave $\vec{E}_2^-$ due to reflection at the dielectric interface. The field in region 1 now consists merely of a wave $\vec{E}_1^t$ representing transmittance through the interface of the primary and upward travelling reflected waves.

13.2 Representation of Field Quantities using Hertzian Potentials.

Each of the electric fields given above, and their associated magnetic fields, can be represented in terms of electric Hertzian potentials $\vec{\Pi}$ as

$$\vec{E} = k^2 \vec{\Pi} + \nabla(\nabla \cdot \vec{\Pi}) \tag{1}$$

$$\vec{H} = j\omega\epsilon \nabla \times \vec{\Pi} \tag{2}$$

where k is the propagation constant of the medium,

$$k^2 = \omega^2 \mu \epsilon \tag{3}$$

and the potentials satisfy the inhomogeneous wave equation

$$\nabla^2 \vec{\Pi} + k^2 \vec{\Pi} = -\frac{\vec{J}}{j\omega\epsilon} \tag{4}$$

In component form, equations (1) and (2) can be written as

$$E_x = k^2 \Pi_x + \frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}) \tag{5}$$

$$E_y = k^2 \Pi_y + \frac{\partial}{\partial y}(\nabla \cdot \vec{\Pi}) \tag{6}$$

$$E_z = k^2 \Pi_z + \frac{\partial}{\partial z}(\nabla \cdot \vec{\Pi}) \tag{7}$$

$$H_x = j\omega\epsilon \left[\frac{\partial \Pi_z}{\partial y} - \frac{\partial \Pi_y}{\partial z} \right] \tag{8}$$

$$H_y = j\omega\epsilon \left[\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x} \right] \tag{9}$$

$$H_z = j\omega\epsilon \left[\frac{\partial \Pi_y}{\partial x} - \frac{\partial \Pi_x}{\partial y} \right] \tag{10}$$

13.3 Boundary Conditions on Hertzian Potentials

To determine the Hertzian potentials which describe the fields produced by current sources in the presence of the grounded slab, it is necessary to employ the boundary conditions at the dielectric interface and at the conductor. The boundary conditions on the potentials may be deduced by examining the boundary conditions on the fields. Letting $\vec{E}_1$ represent the total field in region 1 and $\vec{E}_2$ represent the total field in region 2, the boundary conditions are

a) $$E_{1x}(z=0^+) = E_{2x}(z=0^-) \tag{1}$$

or $$k_1^2\Pi_{1x} + \frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_1) = k_2^2\Pi_{2x} + \frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_2) \qquad z = 0 \tag{2}$$

b) $$E_{1y}(z=0^+) = E_{2y}(z=0^-) \tag{3}$$

or $$k_1^2\Pi_{1y} + \frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_1) = k_2^2\Pi_{2y} + \frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_2) \qquad z = 0 \tag{4}$$

c) $$H_{1x}(z=0^+) = H_{2x}(z=0^-) \tag{5}$$

or $$\varepsilon_1\left[\frac{\partial\Pi_{1z}}{\partial y} - \frac{\partial\Pi_{1y}}{\partial z}\right] = \varepsilon_2\left[\frac{\partial\Pi_{2z}}{\partial y} - \frac{\partial\Pi_{2y}}{\partial z}\right] \qquad z = 0 \tag{6}$$

d) $$H_{1y}(z=0^+) = E_{2y}(z=0^-) \tag{7}$$

or $$\varepsilon_1\left[\frac{\partial\Pi_{1x}}{\partial z} - \frac{\partial\Pi_{1z}}{\partial x}\right] = \varepsilon_2\left[\frac{\partial\Pi_{2x}}{\partial z} - \frac{\partial\Pi_{2z}}{\partial x}\right] \qquad z = 0 \tag{8}$$

e) $$E_{2x}(z=-d) = 0 \tag{9}$$

or $$k_2^2\Pi_{2x} + \frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_2) = 0 \qquad z = -d \tag{10}$$

f) $$E_{2y}(z=-d) = 0 \tag{11}$$

or $$k_2^2\Pi_{2y} + \frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_2) = 0 \qquad z = -d \tag{12}$$

Each rectangular component of current gives rise to one or two components of Hertzian potential. It is convenient to examine the current components individully, and catalog the boundary conditions on the potentials which result from each component.

1. Horizontally directed source: $\vec{J} = \hat{x}\, J_x$

The fields maintained by a current distribution directed along x can be described completely by a Hertzian potential with x and z components but no y component

$$\vec{\Pi} = \hat{x}\, \Pi_x + \hat{z}\, \Pi_z \tag{13}$$

Both x and z components of potential are needed to satisfy the boundary conditions on the fields. Any other combination of components results in a contradiction between boundary conditions at the dielectric interface.

The boundary conditions on $\vec{\Pi}$ for an x-directed current source are found as specializations of (1-12) with $\Pi_y = 0$. Equation (4) gives

$$\frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_1) = \frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_2) \qquad z = 0 \tag{14}$$

The spatial invariance with respect to y of the grounded slab guide allows this equation to be simplified to

$$\nabla\cdot\vec{\Pi}_1 = \nabla\cdot\vec{\Pi}_2 \qquad z = 0 \tag{15}$$

A mathematical justification for this step is easily obtained using the Fourier transform representations of the potentials, which is done later on. Using (15), equation (12) reduces to

$$k_1^2\Pi_{1x} = k_2^2\Pi_{2x} \qquad z = 0 \tag{16}$$

or

$$\Pi_{1x} = \varepsilon_r\, \Pi_{2x} \qquad z = 0 \tag{17}$$

where

$$\varepsilon_r \equiv \frac{\varepsilon_2}{\varepsilon_1} \tag{18}$$

Now, rewriting (15) gives (remembering $\Pi_y = 0$),

$$\left[\frac{\partial \Pi_{1z}}{\partial z} - \frac{\partial \Pi_{2z}}{\partial z}\right] = -\left[\frac{\partial \Pi_{1x}}{\partial x} - \frac{\partial \Pi_{2x}}{\partial x}\right] \qquad z = 0 \tag{19}$$

Using (17) then yields

$$\left[\frac{\partial \Pi_{1z}}{\partial z} - \frac{\partial \Pi_{2z}}{\partial z}\right] = -(\varepsilon_r - 1)\frac{\partial \Pi_{2x}}{\partial x} \qquad z = 0 \tag{20}$$

Next, equation (6) reduces to

$$\varepsilon_1 \frac{\partial \Pi_{1z}}{\partial y} = \varepsilon_2 \frac{\partial \Pi_{2z}}{\partial y} \qquad z = 0 \tag{21}$$

or

$$\Pi_{1z} = \varepsilon_r \, \Pi_{2z} \qquad z = 0 \tag{22}$$

where the same argument used to obtain (15) has been used. Continuing, (8) can be written as

$$\left[\varepsilon_1 \frac{\partial \Pi_{1x}}{\partial z} - \varepsilon_2 \frac{\partial \Pi_{2x}}{\partial z}\right] = \left[\varepsilon_1 \frac{\partial \Pi_{1z}}{\partial x} - \varepsilon_2 \frac{\partial \Pi_{2z}}{\partial x}\right] \qquad z = 0 \tag{23}$$

which, with the help of (14.3.22), reduces to

$$\frac{\partial \Pi_{1x}}{\partial z} = \varepsilon_r \frac{\partial \Pi_{2x}}{\partial z} \qquad z = 0 \tag{24}$$

Note that this equation does not reduce to $\Pi_{1x} = \varepsilon_r \, \Pi_{\Pi_{2x}}$, since the variation of the geometry of the slab in the z-direction does not allow the derivatives to be removed.

Next, equation (12) gives

$$\frac{\partial}{\partial y}\left[\frac{\partial \Pi_{2x}}{\partial x} + \frac{\partial \Pi_{2z}}{\partial z}\right] = 0 \qquad z = -d \tag{25}$$

or,

$$\frac{\partial \Pi_{2x}}{\partial x} + \frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \tag{26}$$

while (10) yields

$$k_2^2\, \Pi_{2x} + \frac{\partial}{\partial x}\left[\frac{\partial \Pi_{2x}}{\partial x} + \frac{\partial \Pi_{2z}}{\partial z}\right] = 0 \qquad z = -d \tag{27}$$

Using (26) in (27) merely gives

$$\Pi_{2x} = 0 \qquad z = -d \tag{28}$$

and using (28) back in (26) gives

$$\frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \tag{29}$$

In summary, then, the boundary conditions on the Hertzian potentials for an x-directed current source are

a) $$\Pi_{1x} = \varepsilon_r\, \Pi_{2x} \qquad z = 0 \tag{17}$$

b) $$\left[\frac{\partial \Pi_{1z}}{\partial z} - \frac{\partial \Pi_{2z}}{\partial z}\right] = -(\varepsilon_r - 1)\frac{\partial \Pi_{2x}}{\partial x} \qquad z = 0 \tag{20}$$

c) $$\Pi_{1z} = \varepsilon_r\, \Pi_{2z} \qquad z = 0 \tag{22}$$

d) $$\frac{\partial \Pi_{1x}}{\partial z} = \varepsilon_r\, \frac{\partial \Pi_{2x}}{\partial z} \qquad z = 0 \tag{24}$$

e) $$\Pi_{2x} = 0 \qquad z = -d \tag{28}$$

f) $$\frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \tag{29}$$

2. Horizontally directed source: $\vec{J} = \hat{y}\, J_y$

For a current source directed along y, the electric and magnetic fields can be completely described by a Hertzian potential with y and z components and with no x component.

$$\vec{\Pi} = \hat{y}\, \Pi_y + \hat{z}\, \Pi_z \tag{30}$$

The boundary conditions on $\vec{\Pi}$ for a y-directed current source are found as specializations of (1) through (12) with $\Pi_x = 0$. Beginning with (2) gives

$$\frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_1) = \frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_2) \qquad z = 0 \qquad (31)$$

or

$$\nabla\cdot\vec{\Pi}_1 = \nabla\cdot\vec{\Pi}_2 \qquad z = 0 \qquad (32)$$

Using this in (14.3.4) then gives

$$k_1^2\Pi_{1y} = k_2^2\Pi_{2y} \qquad z = 0 \qquad (33)$$

or

$$\Pi_{1y} = \varepsilon_r\, \Pi_{2y} \qquad z = 0 \qquad (34)$$

Now, rewriting (32) gives

$$\left[\frac{\partial\Pi_{1z}}{\partial z} - \frac{\partial\Pi_{2z}}{\partial z}\right] = -\left[\frac{\partial\Pi_{1y}}{\partial y} - \frac{\partial\Pi_{2y}}{\partial y}\right] \qquad z = 0 \qquad (35)$$

and (34) then simplifies this to

$$\left[\frac{\partial\Pi_{1z}}{\partial z} - \frac{\partial\Pi_{2z}}{\partial z}\right] = -(\varepsilon_r - 1)\frac{\partial\Pi_{2y}}{\partial y} \qquad z = 0 \qquad (36)$$

Next, equation (8) reduces to

$$\varepsilon_1\frac{\partial\Pi_{1z}}{\partial x} = \varepsilon_2\frac{\partial\Pi_{2z}}{\partial x} \qquad z = 0 \qquad (37)$$

or

$$\Pi_{1z} = \varepsilon_r\, \Pi_{2z} \qquad z = 0 \qquad (38)$$

Continuing, (6) can be written as

$$\left[\varepsilon_1\frac{\partial\Pi_{1y}}{\partial z} - \varepsilon_2\frac{\partial\Pi_{2y}}{\partial z}\right] = \left[\varepsilon_1\frac{\partial\Pi_{1z}}{\partial y} - \varepsilon_2\frac{\partial\Pi_{2z}}{\partial y}\right] \qquad z = 0 \qquad (39)$$

which, using (38), reduces to

$$\frac{\partial \Pi_{1y}}{\partial z} = \varepsilon_r \frac{\partial \Pi_{2y}}{\partial z} \qquad z = 0 \qquad (40)$$

Next, equation (10) gives

$$\frac{\partial}{\partial x}\left[\frac{\partial \Pi_{2y}}{\partial y} + \frac{\partial \Pi_{2z}}{\partial z}\right] = 0 \qquad z = -d \qquad (41)$$

or

$$\frac{\partial \Pi_{2y}}{\partial y} + \frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \qquad (42)$$

while (12) yields

$$k_2^2\, \Pi_{2y} + \frac{\partial}{\partial y}\left[\frac{\partial \Pi_{2y}}{\partial y} + \frac{\partial \Pi_{2z}}{\partial z}\right] = 0 \qquad z = -d \qquad (43)$$

Using (42) in (43) gives

$$\Pi_{2y} = 0 \qquad z = -d \qquad (44)$$

and using (44) back in (42) gives

$$\frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \qquad (45)$$

In summary, the boundary conditions on the Hertzian potentials for a y-directed current source are

a) $$\Pi_{1y} = \frac{\varepsilon_2}{\varepsilon_1}\, \Pi_{2y} = \varepsilon_r\, \Pi_{2y} \qquad z = 0 \qquad (34)$$

b) $$\left[\frac{\partial \Pi_{1z}}{\partial z} - \frac{\partial \Pi_{2z}}{\partial z}\right] = -\,(\varepsilon_r - 1)\frac{\partial \Pi_{2y}}{\partial y} \qquad z = 0 \qquad (36)$$

c) $$\Pi_{1z} = \varepsilon_r\, \Pi_{2z} \qquad z = 0 \qquad (38)$$

d) $$\frac{\partial \Pi_{1y}}{\partial z} = \varepsilon_r \frac{\partial \Pi_{2y}}{\partial z} \qquad z = 0 \qquad (40)$$

e) $$\Pi_{2y} = 0 \qquad z = -d \qquad (44)$$

f) $$\frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \qquad (45)$$

3. Vertically directed source: $\vec{J} = \hat{z}\, J_z$

For a current source directed along z, the electric and magnetic fields can be completely described by a Hertzian potential with a z component, but with no x or y component.

$$\vec{\Pi} = \hat{z}\, \Pi_z \qquad (46)$$

The boundary conditions for a z-directed current source are found as specializations of (1) through (12) with $\Pi_x = \Pi_y = 0$. Beginning with (4) gives

$$\frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_1) = \frac{\partial}{\partial y}(\nabla\cdot\vec{\Pi}_2) \qquad z = 0 \qquad (47)$$

or

$$\nabla\cdot\vec{\Pi}_1 = \nabla\cdot\vec{\Pi}_2 \qquad z = 0 \qquad (48)$$

which reduces to

$$\frac{\partial \Pi_{1z}}{\partial z} = \frac{\partial \Pi_{2z}}{\partial z} \qquad z = 0 \qquad (49)$$

Next, (2) gives

$$\frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_1) = \frac{\partial}{\partial x}(\nabla\cdot\vec{\Pi}_2) \qquad z = 0 \qquad (50)$$

which again reduces to (49).

Continuing, (6) reduces to

$$\varepsilon_1 \frac{\partial \Pi_{1z}}{\partial y} = \varepsilon_2 \frac{\partial \Pi_{2z}}{\partial y} \qquad z = 0 \qquad (51)$$

$$\Pi_{1z} = \varepsilon_r\, \Pi_{2z} \qquad z = 0 \qquad (52)$$

Continuing, (8) gives

$$\varepsilon_1 \frac{\partial \Pi_{1z}}{\partial x} = \varepsilon_2 \frac{\partial \Pi_{2z}}{\partial x} \qquad z = 0 \qquad (53)$$

which again yields (52). Finally, (10) gives

$$\frac{\partial}{\partial x}(\nabla \cdot \vec{\Pi}_2) = 0 \qquad z = -d \qquad (54)$$

or,

$$\frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \qquad (55)$$

while (12) reduces to

$$\frac{\partial}{\partial y}(\nabla \cdot \vec{\Pi}_2) = 0 \qquad z = -d \qquad (56)$$

which again yields (54).

In summary, the boundary conditions on $\vec{\Pi}$ for a z-directed source are

a) $$\frac{\partial \Pi_{1z}}{\partial z} = \frac{\partial \Pi_{2z}}{\partial z} \qquad z = 0 \qquad (49)$$

b) $$\Pi_{1z} = \varepsilon_r \Pi_{2z} \qquad z = 0 \qquad (52)$$

c) $$\frac{\partial \Pi_{2z}}{\partial z} = 0 \qquad z = -d \qquad (55)$$

13.4 Fourier-Integral Representation of Hertz Potentials

Because of the invariance of the grounded slab geometry along the x and y directions, it is prudent to represent the Hertzian potentials as a linear superposition of plane waves propagating along these directions. The two-dimensional spatial Fourier transform of the α'th scalar component of $\vec{\Pi}$ is defined through

$$\tilde{\Pi}_\alpha(k_x, k_y, z) = \iint_{-\infty}^{\infty} \Pi_\alpha(x, y, z)\, e^{-j(k_x x + k_y y)}\, dx\, dy \tag{1}$$

where α represents x, y, or z, and

$$\vec{k} = k_x \hat{x} + k_y \hat{y} \tag{2}$$

is the two-dimensional spatial frequency, or transform variable. The definition of inverse transform then states

$$\Pi_\alpha(x, y, z) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{\Pi}_\alpha(k_x, k_y, z)\, e^{+j(k_x x + k_y y)}\, dk_x\, dk_y \tag{3}$$

Using the relations

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{4}$$

$$d^2k = dk_x\, dk_y \tag{5}$$

$$d^2r = dx\, dy \tag{6}$$

allows (1) and (3) to be written as

$$\tilde{\Pi}_\alpha(\vec{k}, z) = \iint_{-\infty}^{\infty} \Pi_\alpha(\vec{r})\, e^{-j\vec{k}\cdot\vec{r}}\, d^2r \tag{7}$$

$$\Pi_\alpha(\vec{r}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{\Pi}_\alpha(\vec{k}, z)\, e^{j\vec{k}\cdot\vec{r}}\, d^2k \tag{8}$$

It is readily seen from (8) that each component of the Hertzian potential can be represented as a continuous sum of plane waves propagating perpendicular to the z-axis, with amplitudes given by the Fourier spectrum of the potential.

The two dimensional Fourier-integral representation of the potentials provides a simple approach for solving the wave equation 13.2(4). With the representation given in (8), the wave equation reduces from a partial differential equation to an ordinary differential equation with simple solutions. By retaining the variation of the potentials along the z-direction in the transform representation, the boundary conditions on $\vec{\Pi}$ at $z = 0$, and $z = -d$ can be explicitly enforced.

13.5 Solution for the Scattered Hertzian Potentials

The electric fields, $\vec{E}_1^{\,r}$, $\vec{E}_1^{\,-}$, $\vec{E}_1^{\,+}$, $\vec{E}_2^{\,-}$, and $\vec{E}_2^{\,+}$ represent the transmittance or reflection of the primary fields maintained by the current source distributions in regions 1 and 2. Letting $\vec{\Pi}_1^r$, $\vec{\Pi}_1^-$, $\vec{\Pi}_1^+$, $\vec{\Pi}_2^-$, and $\vec{\Pi}_2^+$ be the Hertzian potential representations of these fields, respectively, and letting $\vec{\Pi}^s$ represent any of these "scattered" potentials, the wave equation 13.2(4) reduces to the homogeneous form

$$\nabla^2 \vec{\Pi}^s + k^2 \vec{\Pi}^s = 0 \tag{1}$$

Thus, each component of the scattered Hertzian potentials must obey the scalar Helmholtz equation

$$(\nabla^2 + k^2)\, \Pi_\alpha^s = 0 \tag{2}$$

where α represents x, y, or z.

The scalar Helmholtz equation can be solved quite readily by using the Fourier-integral representation of the scattered potentials.. Substituting 13.4(8) into (2) and expanding the Laplacian operator gives

$$\left[k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{\Pi}_\alpha^s (\vec{k}, z)\, e^{j\vec{k}\cdot\vec{r}}\, d^2k = 0 \tag{3}$$

Bringing the derivitives inside the inversion integral yields

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left(k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \tilde{\Pi}_\alpha^s (\vec{k}, z)\, e^{j\vec{k}\cdot\vec{r}}\, d^2k = 0 \tag{4}$$

Taking the derivitives, and remembering that $\vec{k}\cdot\vec{r}$ is not a function of z, gives

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left\{ \frac{\partial^2 \tilde{\Pi}_\alpha^s (\vec{k}, z)}{\partial z^2} - (k_x^2 + k_y^2 - k^2)\, \tilde{\Pi}_\alpha^s (\vec{k}, z) \right\} e^{j\vec{k}\cdot\vec{r}}\, d^2k = 0 \tag{5}$$

This integral merely represents the inverse transform of the function. If the inverse transform of the function is identically zero, the function being transformed must also be zero. Thus,

$$\left[\frac{\partial^2}{\partial z^2} - p^2\right] \tilde{\Pi}_\alpha^s(\vec{k}, z) = 0 \tag{6}$$

where

$$p(\vec{k}) \equiv \sqrt{k_x^2 + k_y^2 - k^2} \tag{7}$$

Equation (6) is a second order ordinary differential equation for the transform domain representation of the scattered Hertzian potential. It has obvious solutions

$$\tilde{\Pi}_\alpha^s(\vec{k}, z) = W_\alpha^s \, e^{\pm pz} \tag{8}$$

Potentials corresponding to waves traveling in the $+z$ direction will assume the negative sign in the exponential while potentials corresponding to waves travelling in the $-z$ direction will assume the positive sign in the expontial.

Using the transform solution (8), the scattered Hertzian potentials are recovered from the inverse transform (8) as

$$\Pi_\alpha^s(\vec{r}) = \iint_{-\infty}^{\infty} \frac{W_\alpha^s(\vec{k})}{(2\pi)^2} \, e^{j\vec{k}\cdot\vec{r}} \, e^{\pm p(\vec{k})z} \, d^2k \tag{9}$$

13.6 Solution for the Primary Hertzian Potentials

The electric fields $\vec{E}_1^p$ and $\vec{E}_2^p$ are the primary fields maintained by the current sources in regions 1 and 2, respectively. Letting $\vec{\Pi}_1^p$ and $\vec{\Pi}_2^p$ be the Hertzian potential representations of these fields, and letting $\vec{\Pi}^p$ represent either of these primary potentials, the wave equation 13.2(4) requires

$$\nabla^2 \vec{\Pi}^p + k^2 \vec{\Pi}^p = -\frac{\vec{J}}{j\omega\varepsilon} \tag{1}$$

This equation is most readily solved using a Green function technique.

From standard potential theory, the Green function for the primary Hertzian potential in unbounded space, $G^p(\vec{r} \mid \vec{r}\,')$, must satisfy

$$(\nabla^2 + k^2)\, G^p(\vec{r} \mid \vec{r}\,') = -\delta(\vec{r} - \vec{r}\,') \tag{2}$$

That is, G^p represents the potential due to a point source excitation. Since the medium is unbounded, the Green function depends only on the vector difference between the source and field point position vectors

$$G^p(\vec{r} \mid \vec{r}\,') = G^p(\vec{r} - \vec{r}\,') \tag{3}$$

Thus, for convenience, the condition $\vec{r}\,' = 0$ can be taken. At the end of the solution, the point charge position will be returned to an arbitrary point. With this, (2) becomes

$$(\nabla^2 + k^2)\, G^p(\vec{r}\,) = -\delta(\vec{r}\,) \tag{4}$$

For compatibility with the geometry of the grounded slab, the Fourier transform of the Green function is utilized

$$\tilde{G}^p(\vec{k}, z) = \iint_{-\infty}^{\infty} G^p(\vec{r}\,)\, e^{-j\vec{k}\cdot\vec{r}}\, d^2r \tag{5}$$

$$G^p(\vec{r}\,) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}^p(\vec{k}, z)\, e^{j\vec{k}\cdot\vec{r}}\, d^2k \tag{6}$$

Using (6), equation (4) becomes

$$(\nabla^2 + k^2)\, \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{G}^p(\vec{k}, z)\, e^{j\vec{k}\cdot\vec{r}}\, d^2k = -\delta(\vec{r}\,) \tag{7}$$

Next, the Fourier-integral representation of the two dimensional delta function

$$\delta(x - x')\,\delta(y - y') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{j\vec{k}\cdot(\vec{r} - \vec{r}\,')}\, d^2k \tag{8}$$

is employed to give

$$\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) = -\frac{\delta(z)}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{j\vec{k}\cdot\vec{r}}\, d^2k \tag{9}$$

Substituting this into (7) and taking the derivitives yields

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left\{ \left[\frac{\partial^2}{\partial z^2} - (k_x^2 + k_y^2 - k^2) \right] \tilde{G}^p\,(\vec{k}, z) + \delta(z) \right\} e^{j\vec{k}\cdot\vec{r}}\, d^2k = 0 \tag{10}$$

Setting the function being inverted equal to zero results in

$$\left[\frac{\partial^2}{\partial z^2} - p^2(\,\vec{k}\,) \right] \tilde{G}^p\,(\vec{k}, z) = -\delta(z) \tag{11}$$

This is an inhomogenous second order ordinary differential equaton which can be solved by transforming the z-dependance of $\tilde{G}^p(\vec{k}, z)$ as

$$\tilde{\tilde{G}}^p(\vec{k}, Z) = \int_{-\infty}^{\infty} \tilde{G}^p(\vec{k}, z)\, e^{-jZz}\, dz \tag{12}$$

$$\tilde{G}^p(\vec{k}, z) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \tilde{\tilde{G}}^p(\vec{k}, Z)\, e^{jZz}\, dZ \tag{13}$$

Substituting (13) into (11), along with the Fourier-integral relationship of $\delta(z)$

$$\delta(z) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{jZz}\, dZ \tag{14}$$

gives

$$\left[\frac{\partial^2}{\partial z^2} - p^2(\vec{k}\,) \right] \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \tilde{\tilde{G}}^p(\vec{k}, Z)\, e^{jZz}\, dZ = -\frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{jZz}\, dZ \tag{15}$$

Combining results in

$$\frac{1}{(2\pi)}\int_{-\infty}^{\infty}\left\{\left[\frac{\partial^2}{\partial z^2}-p^2(\vec{k})\right]\tilde{\tilde{G}}^P(\vec{k},Z)\,e^{jZz}+e^{jZz}\right\}dZ=0 \tag{16}$$

Taking the derivitives then gives

$$\frac{1}{(2\pi)}\int_{-\infty}^{\infty}\left\{\left[Z^2+p^2(\vec{k})\right]\tilde{\tilde{G}}^P(\vec{k},Z)-1\right\}e^{jZz}\,dZ=0 \tag{17}$$

Setting the function being inverse transformed to zero yields

$$\tilde{\tilde{G}}^P(\vec{k},Z)=\frac{1}{Z^2+p^2} \tag{18}$$

The desired transform representation of the Green function can now be recovered by inverting (18) as

$$\tilde{G}^P(\vec{k},z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{jZz}}{Z^2+p^2}\,dZ \tag{19}$$

The inversion integral involved in equation (19) is most easily computed using contour integration. Rewriting (19) as

$$\tilde{G}^P(\vec{k},z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{jZz}}{(Z+jp)(Z-jp)}\,dZ \tag{20}$$

reveals that the integrand has poles at $Z=\pm jp$. To remove ambiguity, the sign on p is chosen such that

$$\mathrm{Re}\,\{p\}>0 \qquad\qquad \mathrm{Im}\{p\}>0 \tag{21}$$

The positions of the resulting poles in the complex Z-plane are shown in figure 47. If the real line integration given by (20) is closed in the complex plane along an infinite semicircular contour, C^{∞}, as shown in figure 47, then Cauchy's residue theorem can be used. For $z>0$, the line integral is closed in the upper half plane to give

$$\int_{-\infty}^{\infty}\frac{e^{jZz}}{Z^2+p^2}\,dZ+\int_{C_{\infty}^{+}}\frac{e^{jZz}}{Z^2+p^2}\,dZ=2\pi j\sum\text{Residues} \tag{22}$$

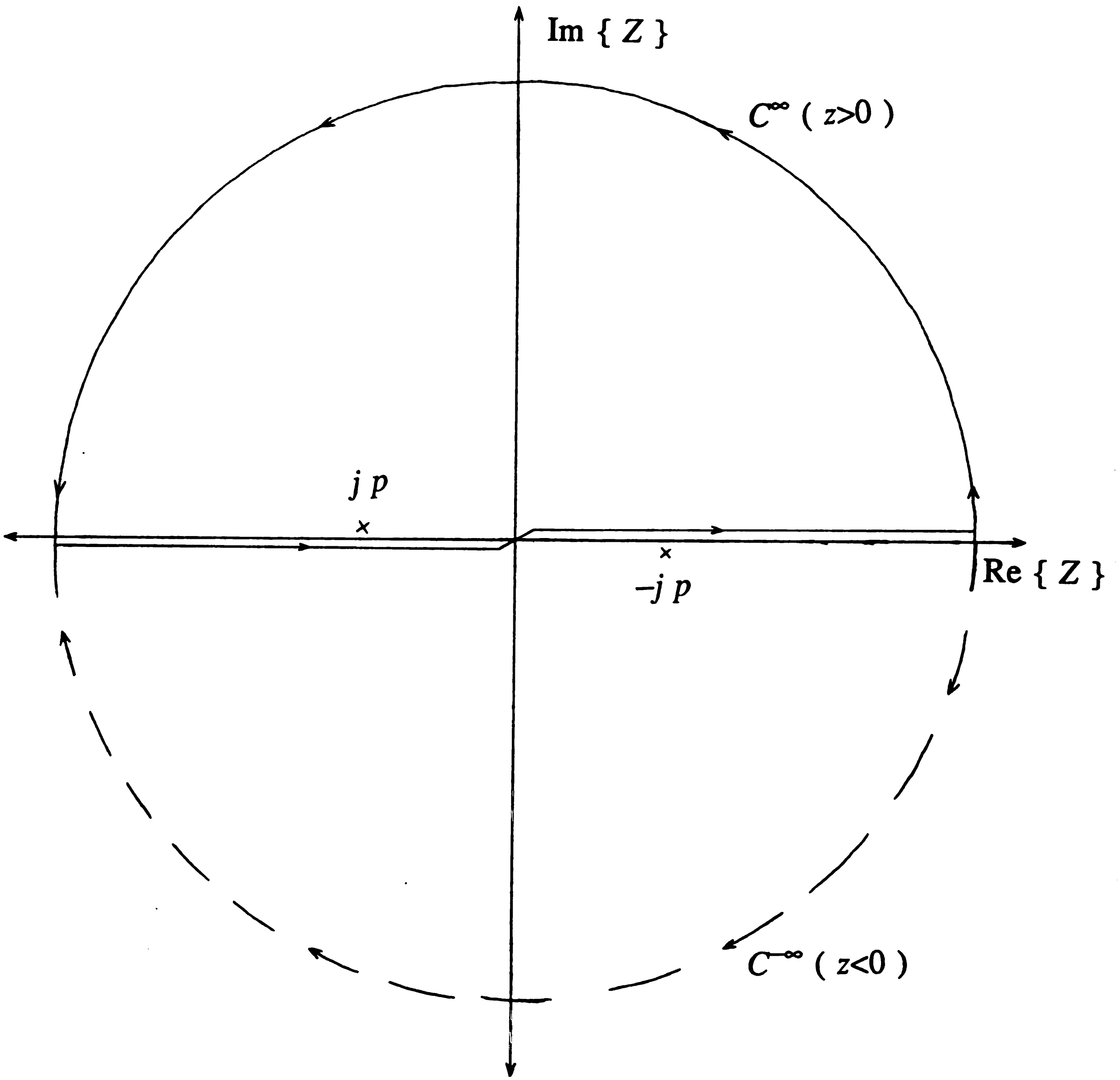


Figure 46. Complex Z-plane.

by Cauchy's residue theorem. The contribution from the second term is zero, since the numerator is exponentially decaying for Im $\{Z\} > 0$, and since the denominator is decaying as Z^{-2}. Thus, (22) becomes

$$\int_{-\infty}^{\infty} \frac{e^{jZz}}{Z^2+p^2}\, dZ = 2\pi j \left. \frac{e^{jZz}}{Z+jp} \right|_{Z=jp} = \pi \frac{e^{-pz}}{p} \tag{23}$$

Thus, the Green's function becomes

$$\tilde{G}^p(\vec{k},z) = \frac{e^{-pz}}{2p} \qquad z > 0 \tag{24}$$

For $z < 0$, the line integral is closed in the lower half plane along dashed contour $C^{-\infty}$ to give

$$\int_{-\infty}^{\infty} \frac{e^{jZz}}{Z^2+p^2}\, dZ + \int_{C_{\infty}^{-}} \frac{e^{jZz}}{Z^2+p^2}\, dZ = -2\pi j \sum \text{Residues} \tag{25}$$

where the negative sign on the right hand side is due to the reversed direction of the path of integration. Again, the contribution from the second term is zero, since the numerator is exponentially decaying for Im $\{Z\} < 0$, and since the denominator is decaying as Z^{-2}. Thus, (25) becomes

$$\int_{-\infty}^{\infty} \frac{e^{jZz}}{Z^2+p^2}\, dZ = -2\pi j \left. \frac{e^{jZz}}{Z-jp} \right|_{Z=-jp} = \pi \frac{e^{+pz}}{p} \tag{26}$$

and the Green function is

$$\tilde{G}^p(\vec{k},z) = \frac{e^{+pz}}{2p} \qquad z < 0 \tag{27}$$

Equations (24) and (27) can be combined to give

$$\tilde{G}^p(\vec{k},z) = \frac{e^{-p|z|}}{2p} \qquad -\infty < z < \infty \tag{28}$$

Note that with the restriction on the sign of Re $\{p\}$ given by (14.6.21), the Green's function decays as $|z| \to \infty$, as is required.

The final step in obtaining the Green function is to perform the inverse Fourier transform of (28) to give

$$G^p(\vec{r}\,) = \iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot\vec{r}}\, e^{-p(\vec{k}\,)|z|}}{2(2\pi)^2\, p(\vec{k}\,)}\, d^2k \tag{29}$$

and to return the source point to an arbitrary position determined by $\vec{r}\,'$, resulting in

$$G^p(\vec{r} \mid \vec{r}\,') = \iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p(\vec{k}\,)|z-z'|}}{2(2\pi)^2\, p(\vec{k}\,)}\, d^2k \tag{30}$$

Lastly, the Hertzian potential for the primary wave can be obtained through the use of (30) and Green's theorem as

$$\vec{\Pi}^p(\vec{r}) = \int_v \frac{\vec{J}(\vec{r}\,')}{j\omega\epsilon}\, G^p(\vec{r} \mid \vec{r}\,')dv' \tag{31}$$

Note that since the primary waves are viewed as radiating into unbounded space, (31) can also be regarded as a superposition of point source responses.

13.7 Green Function for the Hertzian Potential Maintained by a Horizontal Source in Region 1.

A current source placed in region 1 will maintain electric and magnetic fields in both regions 1 and 2. In region 1, the field will be composed of a primary contribution and a contribution due to reflection from the dielectric interface. Thus, the total potential in region 1 is

$$\vec{\Pi}_1 = \vec{\Pi}_1^{\,p} + \vec{\Pi}_1^{\,r} \qquad z > 0 \tag{1}$$

In region 2, the field will be made up of a wave transmitted by the interface, propagating in the $-z$ direction, plus a wave reflected from the conductor, propagating in the $+z$ direction. Thus, the total potential in region 2 is

$$\vec{\Pi}_2 = \vec{\Pi}_2^{\,+} + \vec{\Pi}_2^{\,-} \qquad -d < z < 0 \tag{2}$$

The total potentials in each region can be determined by employing the appropriate boundary conditions from section 13.3. This is most easily done by considering the effects of each component of the horizontal current distribution separately.

1. $\vec{J}_1 = \hat{x}\, J_{1x}$

For an x-directed current, there will be only x and z components of the corresponding scattered Hertzian potentials, and, as seen from 13.6(31), only an x component of primary Hertzian Potential. The required boundary conditions are employed as follows.

a) Employ 13.3(17)

$$\Pi_{1x}^{p} + \Pi_{1x}^{r} = \varepsilon_r (\Pi_{2x}^{+} + \Pi_{2x}^{-}) \qquad \text{at } z = 0 \tag{3}$$

Substituting 13.5(9) and 13.6(31) gives

$$\iint\limits_{-\infty}^{\infty} d^2k \int\limits_{v} \frac{J_{1x}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p_1 z'}}{2(2\pi)^2 p_1}\, dv' + \iint\limits_{-\infty}^{\infty} \frac{W_{\alpha}^{r}(\vec{k})}{(2\pi)^2}\, e^{j\vec{k}\cdot\vec{r}}\, d^2k$$

$$= \varepsilon_r \iint_{-\infty}^{\infty} \frac{W_{2z}^{+}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} d^2k + \varepsilon_r \iint_{-\infty}^{\infty} \frac{W_{2z}^{-}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} d^2k \tag{4}$$

Since $z = z'$ when employing the boundary conditions, the condition

$$|z-z'| = z'-z \tag{5}$$

has been used to obtain the first term of (4). Grouping terms now gives

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left\{ \int_v \frac{J_{1z}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{-j\vec{k}\cdot\vec{r}\,'} e^{-p_1 z'}}{2p_1} dv' + W_{1z}^{r} - \varepsilon_r \left[W_{2z}^{+} + W_{2z}^{-} \right] \right\} e^{j\vec{k}\cdot\vec{r}} = 0 \tag{6}$$

Setting the function being inverted in (6) to zero yields

$$-W_{1z}^{r} + \varepsilon_r \left[W_{2z}^{+} + W_{2z}^{-} \right] = V_{1z} \tag{7}$$

where

$$\vec{V}_1(\vec{k}) \equiv \int_v \frac{\vec{J}_1(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{-j\vec{k}\cdot\vec{r}\,'} e^{-p_1 z'}}{2p_1} dv' \tag{8}$$

b) Employ 13.3(21)

$$\Pi_{1z}^{p} + \Pi_{1z}^{r} = \varepsilon_r(\Pi_{2z}^{+} + \Pi_{2z}^{-}) \qquad \text{at } z = 0 \tag{9}$$

A procedure identical to that used to derive (7) yields

$$-W_{1z}^{r} + \varepsilon_r \left[W_{2z}^{+} + W_{2z}^{-} \right] = V_{1z} = 0 \tag{10}$$

Note that V_z is zero since there is no z component of current (or, as a result, no z component of primary potential.)

c) Employ 13.3(23)

$$\frac{\partial}{\partial z} (\Pi_{1x}^{p} + \Pi_{1x}^{r}) = \varepsilon_r(\Pi_{2x}^{+} + \Pi_{2x}^{-}) \qquad \text{at } z = 0 \tag{11}$$

Substituting gives

$$\frac{\partial}{\partial z} \left\{ \iint_{-\infty}^{\infty} d^2k \int_v \frac{J_{1x}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} e^{-p_1(z'-z)}}{2(2\pi)^2 p_1} dv' + \iint_{-\infty}^{\infty} \frac{W_{1x}^{r}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_1 z} d^2k \right\}$$

$$= \varepsilon_r \frac{\partial}{\partial z} \left\{ \iint_{-\infty}^{\infty} \frac{W_{2x}^{+}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_2 z} d^2k + \varepsilon \iint_{-\infty}^{\infty} \frac{W_{2x}^{-}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{p_2 z} d^2k \right\} \quad \text{at } z = 0 \qquad (12)$$

Performing the indicated differentiation inside the integrals and then setting $z = 0$ results in

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left\{ V_{1x}(\vec{k}\,) - W_{1x}^{r} - \varepsilon_r \frac{p_2(\vec{k}\,)}{p_1(\vec{k})} \left[W_{2x}^{-} - W_{2x}^{+} \right] \right\} p_1(\vec{k}\,)\, e^{j\vec{k}\cdot\vec{r}} = 0 \qquad (13)$$

Lastly, setting the function being inverted to zero yields

$$W_{1x}^{r} + \varepsilon_r \frac{p_2}{p_1} \left[W_{2x}^{-} - W_{2x}^{+} \right] = V_{1x} \qquad (14)$$

d) Employ 13.3(19)

$$\frac{\partial}{\partial z} (\Pi_{1z}^{f} + \Pi_{1z}^{r} - \Pi_{12}^{+} - \Pi_{2z}^{-}) = -(\varepsilon_r - 1) \frac{\partial}{\partial x} (\Pi_{2x}^{+} + \Pi_{2x}^{-}) \qquad \text{at } z = 0 \qquad (15)$$

Substituting gives, with $\Pi_{1z}^{f} = 0$ for an x-directed current source,

$$\frac{\partial}{\partial z} \left\{ \iint_{-\infty}^{\infty} \frac{W_{1z}^{r}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_1 z} d^2k - \iint_{-\infty}^{\infty} \frac{W_{2z}^{+}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_2 z} d^2k \right.$$

$$\left. - \iint_{-\infty}^{\infty} \frac{W_{2z}^{-}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{p_2 z} d^2k \right\}$$

$$= (\varepsilon_r - 1) \frac{\partial}{\partial x} \left\{ \iint_{-\infty}^{\infty} \frac{W_{2x}^{+}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_2 z} d^2k + \iint_{-\infty}^{\infty} \frac{W_{2x}^{-}(\vec{k}\,)}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{p_2 z} d^2k \right\} \quad \text{at } z = 0 \qquad (16)$$

or, taking the derivative and setting z to zero,

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \left\{ -p_1 W_{1z}^{r} - p_2 \left[W_{2z}^{-} - W_{2z}^{+} \right] + (\varepsilon_r - 1) jk_x \left[W_{2x}^{-} + W_{2x}^{+} \right] \right\} e^{j\vec{k}\cdot\vec{r}} = 0 \qquad (17)$$

Setting the transform to zero then yields

$$W_{1z}^{r} + \frac{p_2}{p_1} \left[W_{2z}^{-} - W_{2z}^{+} \right] = (\varepsilon_r - 1) \frac{jk_x}{p_1} \left[W_{2x}^{-} + W_{2x}^{+} \right] \qquad (18)$$

e) Employ 13.3(27)

$$\Pi_{2x}^{+} + \Pi_{2x}^{-} = 0 \qquad \text{at } z = -d \qquad (19)$$

Substituting

$$\iint_{-\infty}^{\infty} \frac{W_{2x}^{+}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_2 z} d^2k + \iint_{-\infty}^{\infty} \frac{W_{2x}^{-}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{p_2 z} d^2k = 0 \qquad \text{at } z = -d \qquad (20)$$

Combining gives

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left[W_{2x}^{+}(\vec{k})\, e^{p_2 d} + W_{2x}^{-}(\vec{k})\, e^{-p_2 d}\right] e^{j\vec{k}\cdot\vec{r}} d^2k = 0 \qquad (21)$$

Thus

$$W_{2x}^{+}(\vec{k})\, e^{p_2 d} + W_{2x}^{-}(\vec{k})\, e^{-p_2 d} = 0 \qquad (22)$$

f) Employ (13.3.28)

$$\frac{\partial}{\partial z}(\Pi_{2x}^{+} + \Pi_{2x}^{-}) = 0 \qquad \text{at } z = -d \qquad (23)$$

Substituting gives

$$\frac{\partial}{\partial z} \iint_{-\infty}^{\infty} \frac{W_{2x}^{+}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_2 z} d^2k + \frac{\partial}{\partial z} \iint_{-\infty}^{\infty} \frac{W_{2x}^{-}(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{p_2 z} d^2k = 0 \qquad \text{at } z = -d \qquad (24)$$

Performing the derivitives and combining gives

$$\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \left[W_{2x}^{-}(\vec{k})\, e^{-p_2 d} - W_{2x}^{+}(\vec{k})\, e^{p_2 d}\right] p_2\, e^{j\vec{k}\cdot\vec{r}} d^2k = 0 \qquad (25)$$

Setting the function being transformed equal to zero yields

$$W_{2x}^{-}(\vec{k})\, e^{-p_2 d} - W_{2x}^{+}(\vec{k})\, e^{p_2 d} = 0 \qquad (26)$$

In summary, then, the boundary conditions for an x-directed source current yield

a) $-W_{1x}^{r} + \varepsilon_r \left[W_{2x}^{+} + W_{2x}^{-}\right] = V_{1x}$ (7)

b) $-W_{1z}^{r} + \varepsilon_r \left[W_{2z}^{+} + W_{2z}^{-}\right] = V_{1z} = 0$ (10)

c) $W_{1x}^{r} + \varepsilon_r \dfrac{p_2}{p_1} \left[W_{2x}^{-} - W_{2x}^{+}\right] = V_{1x}$ (14)

d) $W^r_{1z} + \frac{p_2}{p_1}\left[W^-_{2z} - W^+_{2z}\right] = (\varepsilon_r - 1)\frac{jk_x}{p_1}\left[W^-_{2x} + W^+_{2x}\right]$ (18)

e) $W^+_{2x}(\vec{k}\,)\, e^{p_2 d} + W^-_{2x}(\vec{k}\,)\, e^{-p_2 d} = 0$ (22)

f) $W^-_{2z}(\vec{k}\,)\, e^{-p_2 d} - W^+_{2z}(\vec{k}\,)\, e^{p_2 d} = 0$ (26)

The equations labeled (a) through (f) above represent a system of six equations in the six unknown transform amplitudes W^r_{1x}, W^r_{1z}, W^+_{2x}, W^-_{2x}, W^+_{2z}, and W^-_{2z}. These amplitudes can be solved for as follows.

Solving (a) for W^r_{1x} and substituting into (c), and solving (b) for W^r_{1z} and substituting into (d) gives the reduced set of equations

c') $\varepsilon_r \left[1 + \frac{p_2}{p_1}\right] W^-_{2x} + \varepsilon_r \left[1 - \frac{p_2}{p_1}\right] W^+_{2x} = 2V_{1x}$ (27)

d') $\varepsilon_r \left[W^+_{2z} + W^-_{2z}\right] + \frac{p_2}{p_1}\left[W^-_{2z} - W^+_{2z}\right] = (\varepsilon_r - 1)\frac{jk_x}{p_1}\left[W^-_{2x} + W^+_{2x}\right]$ (28)

e) $W^+_{2x}(\vec{k}\,)\, e^{p_2 d} + W^-_{2x}(\vec{k}\,)\, e^{-p_2 d} = 0$ (22)

f) $W^-_{2z}(\vec{k}\,)\, e^{-p_2 d} - W^+_{2z}(\vec{k}\,)\, e^{p_2 d} = 0$ (26)

Now, solving (7.7.22) for W^-_{2x} gives

$$W^-_{2x}(\vec{k}\,) = -W^+_{2x}(\vec{k}\,)\, e^{2p_2 d} \quad (29)$$

Substituting this into (27) yields

$$W^+_{2x}\left\{\varepsilon_r \left[1 - \frac{p_2}{p_1}\right] - \varepsilon_r \left[1 + \frac{p_2}{p_1}\right] e^{2p_2 d}\right\} = 2V_{1x} \quad (30)$$

or,

$$W^+_{2x}\left\{\varepsilon_r\, e^{p_2 d}(e^{-p_2 d} - e^{p_2 d}) - \varepsilon_r \frac{p_2}{p_1}\, e^{p_2 d}(e^{-p_2 d} + e^{p_2 d})\right\} = 2V_{1x} \quad (31)$$

Using the definitions of sinh and cosh,

$$\sinh(x) = \frac{1}{2}\left[e^{x} - e^{-x}\right] \qquad \cosh(x) = \frac{1}{2}\left[e^{x} + e^{-x}\right] \quad (32a,b)$$

equation (31) becomes

$$W_{2x}^{+}\left\{\varepsilon_r\, e^{p_2 d}\sinh(p_2 d) - \varepsilon_r \frac{p_2}{p_1}\, e^{p_2 d}\cosh(p_2 d)\right\} = V_{1x} \tag{33}$$

Now, defining

$$T_e \equiv \sinh(p_2 d) + \frac{p_2}{p_1}\cosh(p_2 d) \tag{34}$$

allows W_{2x}^{+} to be solved for from (33) as

$$W_{2x}^{+} = -\frac{\varepsilon_r^{-1} V_{1x}\, e^{-p_2 d}}{T_e} \tag{35}$$

Also, using (29) in (35) yields

$$W_{2x}^{-} = +\frac{\varepsilon_r^{-1} V_{1x}\, e^{+p_2 d}}{T_e} \tag{36}$$

Note that the zeroes of T_e correspond to the eigenfrequencies of the TE surface wave modes of the grounded slab. Next, (26) is solved for W_{2x}^{-}

$$W_{2x}^{-} = W_{2x}^{+}\, e^{2p_2 d} \tag{37}$$

Substituting (37), (35), and (36) into (28) then yields

$$W_{2x}^{+}\left\{\left[\varepsilon_r - \frac{p_2}{p_1}\right] + \left[\varepsilon_r + \frac{p_2}{p_1}\right] e^{2p_2 d}\right\} = (\varepsilon_r - 1)\frac{jk_x}{p_1}\left\{\varepsilon_r^{-1} 2V_{1x}\,\frac{\sinh(p_2 d)}{T_e}\right\} \tag{38}$$

Rearranging this equation gives

$$W_{2x}^{+}\left\{\varepsilon_r\left[e^{p_2 d} + e^{-p_2 d}\right] + \frac{p_2}{p_1}\left[e^{p_2 d} - e^{-p_2 d}\right]\right\} e^{p_2 d} = (\varepsilon_r - 1)\frac{jk_x}{p_1}\left\{\varepsilon_r^{-1} 2V_{1x}\,\frac{\sinh(p_2 d)}{T_e}\right\} \tag{39}$$

Using equations (34) and defining

$$T_m \equiv \varepsilon_r \cosh(p_2 d) + \frac{p_2}{p_1}\sinh(p_2 d) \tag{40}$$

allows (39) to be solved as

$$W_{2z}^{+} = - V_{1x}\, e^{-p_2 d}\, \frac{(\varepsilon_r^{-1} - 1)\dfrac{jk_x}{p_1} \sinh(p_2 d)}{T_e\, T_m} \tag{41}$$

Using (37) in (41) also gives

$$W_{2z}^{-} = - V_{1x}\, e^{p_2 d}\, \frac{(\varepsilon_r^{-1} - 1)\dfrac{jk_x}{p_1} \sinh(p_2 d)}{T_e\, T_m} \tag{42}$$

Note that the zeroes of T_m correspond to the eigenfrequencies of the TM surface wave modes of the grounded slab.

Now, equations (35) and (36) can be substituted back into (7), yielding

$$W_{1x}^{r} = \frac{V_{1x}}{T_e} \left[e^{p_2 d} - e^{-p_2 d} \right] - V_{1x} \tag{43}$$

or, using (32a)

$$W_{1x}^{r} = -V_{1x} \left[1 - 2\frac{\sinh(p_2 d)}{T_e} \right] \tag{44}$$

Similarly, equation (10) gives

$$W_{1z}^{r} = -\varepsilon_r\, \frac{(\varepsilon_r^{-1} - 1)\, \dfrac{jk_x}{p_1}}{T_e\, T_m} \sinh(p_2 d) \left[e^{-p_2 d} + e^{p_2 d} \right] V_{1x} \tag{45}$$

or,

$$W_{1z}^{r} = 2V_{1x}\, \frac{(\varepsilon_r - 1)\, \dfrac{jk_x}{p_1} \sinh(p_2 d) \cosh(p_2 d)}{T_e\, T_m} \tag{46}$$

The total potentials in region 1 due to an x-directed current source can now be calculated with the help of 13.3(13), 13.5(9), 13.6(31), and (1) as

$$\Pi_{1z}(\vec{r}) = \Pi_{1z}^{r}(\vec{r})$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} 2V_{1x}(\vec{k}) \frac{(\varepsilon_r - 1)\dfrac{jk_x}{p_1} \sinh(p_2 d) \cosh(p_2 d)}{T_e\, T_m}\, e^{j\vec{k}\cdot\vec{r}}\, e^{-p_1 z}\, d^2k \tag{47}$$

and

$$\Pi_{1z}(\vec{r}) = \Pi_{1z}^{p}(\vec{r}) + \Pi_{1z}^{r}(\vec{r})$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \int_{v} \frac{J_z(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} e^{-p_1|z-z'|}}{2p_1} dv'$$

$$- \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} V_{1z}(\vec{k}) \left[1 - 2\frac{\sinh(p_2 d)}{T_e}\right] e^{j\vec{k}\cdot\vec{r}} e^{-p_1 z} d^2k \tag{48}$$

By substituting (8), equation (47) can also be written as

$$\Pi_{1z}(\vec{r}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \int_{v} \frac{J_{1z}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} e^{-p_1(z+z')}}{p_1} dv' \times$$

$$\times \left[\frac{(\varepsilon_r - 1)\frac{jk_x}{p_1} \sinh(p_2 d)\cosh(p_2 d)}{T_e\, T_m}\right] \tag{49}$$

while (48) gives

$$\Pi_{1z}(\vec{r}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \int_{v} \frac{J_x^1(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{2p_1} \times$$

$$\times \left\{e^{-p_1|z-z'|} - \left[1 - 2\,\frac{\sinh(p_2 d)}{T_e}\right] e^{-p_1(z+z')}\right\} dv' \tag{50}$$

Similarly, the potentials in region 2 are found using (2) to be

$$\Pi_{2x}(\vec{r}) = \Pi_{2x}^{+}(\vec{r}) + \Pi_{2x}^{-}(\vec{r})$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{j\vec{k}\cdot\vec{r}} \frac{V_{1x}}{T_e \varepsilon_r} \left\{-e^{-p_2 d} e^{-p_2 z} + e^{p_2 d} e^{p_2 z}\right\} d^2k$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \int_{v} \frac{J_{1x}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} e^{-p_1 z'}}{p_1\, T_e\, \varepsilon_r} dv' \sinh(p_2(d+z)) \tag{51}$$

$$\Pi_{2z}(\vec{r}) = \Pi_{2z}^{+}(\vec{r}) + \Pi_{2z}^{-}(\vec{r})$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{j\vec{k}\cdot\vec{r}} \left[-V_{1x} \frac{(\varepsilon_r^{-1} - 1)\frac{jk_x}{p_1}\sinh(p_2 d)}{T_e\, T_m}\right] \left[e^{-p_2 d} e^{-p_2 z} + e^{p_2 d} e^{p_2 z}\right] d^2k$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; \frac{(\varepsilon_r^{-1} - 1)\frac{jk_x}{p_1}\sinh(p_2 d)\cosh(p_2(d+z))}{T_e \, T_m} \times$$

$$\times \int_v \frac{J_{1x}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{-e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} e^{-p_1 z'}}{p_1} dv' \qquad (52)$$

2. $\vec{J}_1 = \hat{y}\, J_{1y}$

For a y-directed current, there will be only y and z components of the corresponding scattered Hertzian potentials, and, as seen from (31), only a y component of primary Hertzian Potential. The required boundary conditions are employed in the same manner as for an x-directed source.

a) Employ 13.3(33)

$$\Pi_{1y}^p + \Pi_{1y}^r = \varepsilon_r(\Pi_{2y}^+ + \Pi_{2y}^-) \qquad \text{at } z = 0 \qquad (53)$$

Proceeding in a manner analogous to that used to obtain (7) gives

$$-W_{1y}^r + \varepsilon_r \left[W_{2y}^+ + W_{2y}^-\right] = V_{1y} \qquad (54)$$

where $\vec{V}_1$ has been defined in (8).

b) Employ 13.3(37)

$$\Pi_{1z}^p + \Pi_{1z}^r = \varepsilon_r(\Pi_{2z}^+ + \Pi_{2z}^-) \qquad \text{at } z = 0 \qquad (55)$$

This gives

$$-W_{1z}^r + \varepsilon_r \left[W_{2z}^+ + W_{2z}^-\right] = V_{1z} = 0 \qquad (56)$$

c) Employ 13.3(39)

$$\frac{\partial}{\partial z}(\Pi_{1y}^p + \Pi_{1y}^r) = \varepsilon_r(\Pi_{2y}^+ + \Pi_{2y}^-) \qquad \text{at } z = 0 \qquad (57)$$

Proceeding as was done to obtain (14) leads to

$$W_{1y}^r + \varepsilon_r \frac{p_2}{p_1}\left[W_{2y}^- - W_{2y}^+\right] = V_{1y} \tag{58}$$

d) Employ 13.3(35)

$$\frac{\partial}{\partial z}(\Pi_{1z}^p + \Pi_{1z}^r - \Pi_{2z}^+ - \Pi_{2z}^-) = -(\varepsilon_r - 1)\frac{\partial}{\partial y}(\Pi_{2y}^+ + \Pi_{2y}^-) \qquad \text{at } z = 0 \tag{59}$$

This results in, similar to (18)

$$W_{1z}^r + \frac{p_2}{p_1}\left[W_{2z}^- - W_{2z}^+\right] = (\varepsilon_r - 1)\frac{jk_y}{p_1}\left[W_{2y}^- + W_{2y}^+\right] \tag{60}$$

e) Employ 13.3(43)

$$\Pi_{2y}^+ + \Pi_{2y}^- = 0 \qquad \text{at } z = -d \tag{61}$$

or, as in (22)

$$W_{2y}^+ e^{p_2 d} + W_{2y}^- e^{-p_2 d} = 0 \tag{62}$$

f) Employ 13.3(44)

$$\frac{\partial}{\partial z}(\Pi_{2z}^+ + \Pi_{2z}^-) = 0 \qquad \text{at } z = -d \tag{63}$$

This yields

$$W_{2z}^- e^{-p_2 d} - W_{2z}^+ e^{p_2 d} = 0 \tag{64}$$

In summary, then, the boundary conditions for a y-directed source current yield

a) $$-W_{1y}^r + \varepsilon_r\left[W_{2y}^+ + W_{2y}^-\right] = V_{1y} \tag{54}$$

b) $$-W_{1z}^r + \varepsilon_r\left[W_{2z}^+ + W_{2z}^-\right] = V_{1z} = 0 \tag{56}$$

c) $$W_{1y}^r + \varepsilon_r \frac{p_2}{p_1}\left[W_{2y}^- - W_{2y}^+\right] = V_{1y} \tag{58}$$

d) $$W_{1z}^r + \frac{p_2}{p_1}\left[W_{2z}^- - W_{2z}^+\right] = (\varepsilon_r - 1)\frac{jk_y}{p_1}\left[W_{2y}^- + W_{2y}^+\right] \tag{60}$$

e) $$W_{2y}^{+}\, e^{p_2 d} + W_{2y}^{-}\, e^{-p_2 d} = 0 \tag{62}$$

f) $$W_{2z}^{-}\, e^{-p_2 d} - W_{2z}^{+}\, e^{p_2 d} = 0 \tag{64}$$

The solution of the above equations for W_{1y}^{r}, W_{1z}^{r}, W_{2y}^{+}, W_{2y}^{-}, W_{2z}^{+}, and W_{2z}^{-} is identical to that obtained for an x-directed current source, with the obvious substitutions

$$W_x \to W_y \qquad V_x \to V_y \qquad jk_x \to jk_y \tag{65}$$

Thus,

$$W_{2y}^{+} = -\frac{\varepsilon_r^{-1} V_{1y}\, e^{-p_2 d}}{T_e} \tag{66}$$

$$W_{2y}^{-} = +\frac{\varepsilon_r^{-1} V_{1y}\, e^{+p_2 d}}{T_e} \tag{67}$$

$$W_{2z}^{+} = -V_{1y}\, e^{-p_2 d}\, \frac{(\varepsilon_r^{-1} - 1)\dfrac{jk_y}{p_1} \sinh(p_2 d)}{T_e\, T_m} \tag{68}$$

$$W_{2z}^{-} = -V_{1y}\, e^{p_2 d}\, \frac{(\varepsilon_r^{-1} - 1)\dfrac{jk_y}{p_1} \sinh(p_2 d)}{T_e\, T_m} \tag{69}$$

$$W_{1y}^{r} = -V_{1y} \left[1 - 2\frac{\sinh(p_2 d)}{T_e} \right] \tag{70}$$

$$W_{1z}^{r} = 2V_{1y}\, \frac{(\varepsilon_r - 1)\, \dfrac{jk_y}{p_1} \sinh(p_2 d) \cosh(p_2 d)}{T_e\, T_m} \tag{71}$$

The total potentials in region 1 due to a y-directed current source can now be calculated as

$$\Pi_{1z}(\vec{r}) = \Pi_{1z}^{r}(\vec{r})$$

$$= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \int_{v} \frac{J_{1y}(\vec{r}\,')}{j\omega\varepsilon_1}\, \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p_1(z+z')}}{p_1}\, dv'\, \frac{(\varepsilon_r - 1)\dfrac{jk_y}{p_1} \sinh(p_2 d) \cosh(p_2 d)}{T_e\, T_m} \tag{72}$$

and

$$\Pi_{1y}(\vec{r}) = \Pi^p_{1y}(\vec{r}) + \Pi^r_{1y}(\vec{r})$$

$$= \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} d^2k \int_v \frac{J_y^1(\vec{r}\,')}{j\omega\epsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{2p_1}\left\{e^{-p_1|z-z'|} - \left[1 - 2\frac{\sinh(p_2d)}{T_e}\right]e^{-p_1(z+z')}\right\}dv' \quad (73)$$

Similarly, the potentials in region 2 are found using (2) to be

$$\Pi_{2y}(\vec{r}) = \Pi^+_{2y}(\vec{r}) + \Pi^-_{2y}(\vec{r})$$

$$= \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} d^2k \int_v \frac{J_{1y}(\vec{r}\,')}{j\omega\epsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,e^{-p_1z'}}{p_1}\, dv'\, \epsilon_r^{-1}\frac{\sinh(p_2(d+z))}{T_e} \quad (74)$$

$$\Pi_{2z}(\vec{r}) = \Pi^+_{2z}(\vec{r}) + \Pi^-_{2z}(\vec{r})$$

$$= \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} d^2k \frac{(\epsilon_r^{-1}-1)\frac{jk_y}{p_1}\sinh(p_2d)\cosh(p_2(d+z))}{T_e\,T_m}\int_v \frac{J_{1y}(\vec{r}\,')}{j\omega\epsilon_1}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,e^{-p_1z'}}{p_1}\,dv' \quad (75)$$

It becomes prudent at this point to write the potentials maintained in regions 1 and 2 by a horizontal current source in region 1 in terms of a dyadic green function. The following notation will be used:

$$\vec{\Pi}_a(\vec{r}) = \int_v \overleftrightarrow{G}^{a,b}(\vec{r}\mid\vec{r}\,')\cdot\vec{J}_b(\vec{r}\,')\,dv' \quad (76)$$

Here $\vec{\Pi}_a(\vec{r})$ is the potential in region a maintained by a current source in region b, where a and b represent either 1 or 2, designating regions 1 and 2, respectively.

With this notation, the potentials maintained by a horizontal current source in region 1 can be represented using

$$G^{1,1}_{xx}(\vec{r}\mid\vec{r}\,') = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\epsilon_1 2p_1}\left\{e^{-p_1|z-z'|} - \left[1 - 2\frac{\sinh(p_2d)}{T_e}\right]e^{-p_1(z+z')}\right\} \quad (77)$$

$$G^{1,1}_{yx}(\vec{r}\mid\vec{r}\,') = 0 \quad (78)$$

$$G^{1,1}_{zx}(\vec{r}\mid\vec{r}\,') = \frac{1}{(2\pi)^2}\iint\limits_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,e^{-p_1(z+z')}}{j\omega\epsilon_1 p_1}\left[\frac{(\epsilon_r-1)\frac{jk_x}{p_1}\sinh(p_2d)\cosh(p_2d)}{T_e\,T_m}\right] \quad (79)$$

$$G_{xy}^{1,1}(\vec{r} \mid \vec{r}\,') = 0 \tag{80}$$

$$G_{yy}^{1,1}(\vec{r} \mid \vec{r}\,') = G_{xx}^{1,1}(\vec{r} \mid \vec{r}\,') \tag{81}$$

$$G_{zy}^{1,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p_1(z+z')}}{j\omega\varepsilon_1 p_1} \; \frac{(\varepsilon_r - 1)\dfrac{jk_y}{p_1} \sinh(p_2 d)\cosh(p_2 d)}{T_e\, T_m} \tag{82}$$

$$G_{xx}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p_1 z'}}{j\omega\varepsilon_1 p_1} \; \frac{\sinh(p_2(d+z))}{\varepsilon_r\, T_e} \tag{83}$$

$$G_{zx}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; \frac{-e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p_1 z'}}{j\omega\varepsilon_1 p_1} \; \frac{(\varepsilon_r^{-1} - 1)\dfrac{jk_x}{p_1}\sinh(p_2 d)\cosh(p_2(d+z))}{T_e\, T_m} \tag{84}$$

$$G_{yx}^{2,1}(\vec{r} \mid \vec{r}\,') = 0 \tag{85}$$

$$G_{xy}^{2,1}(\vec{r} \mid \vec{r}\,') = 0 \tag{86}$$

$$G_{yy}^{2,1}(\vec{r} \mid \vec{r}\,') = G_{xx}^{2,1}(\vec{r} \mid \vec{r}\,') \tag{87}$$

$$G_{zy}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; \frac{-e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, e^{-p_1 z'}}{j\omega\varepsilon_1 p_1} \; \frac{(\varepsilon_r^{-1} - 1)\dfrac{jk_y}{p_1}\sinh(p_2 d)\cosh(p_2(d+z))}{T_e\, T_m} \tag{88}$$

13.8 Green Function for the Electric Field Maintained by a Horizontal Source in Region 1.

The electric field maintained by a horizontal source in region 1 can also be written in terms of a dyadic Green function. The notation used is

$$\vec{E}_a(\vec{r}\,) = \int_v \overleftrightarrow{\mathcal{G}}^{a,b}(\vec{r} \mid \vec{r}\,') \cdot \vec{J}_b(\vec{r}\,')\, dv' \tag{1}$$

Here $\vec{E}_a$ is the electric field in region a maintained by a current source in region b, where a and b represent either 1 or 2.

The electric field is calculated from the Hertzian potentials by using 14.2(1). Thus, the fields in region a are given by

$$\vec{E}_a = k_a^2 \vec{\Pi}_a + \nabla(\nabla \cdot \vec{\Pi}_a) \tag{2}$$

Substituting 13.7(76) gives

$$\vec{E}_a(\vec{r}\,) = \int_v \left\{ (k_a^2 + \nabla\nabla \cdot\,) \overleftrightarrow{G}^{a,b}(\vec{r} \mid \vec{r}\,') \right\} \cdot \vec{J}_b(\vec{r}\,')\, dv' \tag{3}$$

and thus it is possible to identify

$$\overleftrightarrow{\mathcal{G}}^{a,b}(\vec{r} \mid \vec{r}\,') = (k_a^2 + \nabla\nabla \cdot\,) \overleftrightarrow{G}^{a,b}(\vec{r} \mid \vec{r}\,') \tag{4}$$

as the Green function for the electric field. Using the transform notation introduced in 13.4(8), equation (4) can also be written as

$$\overleftrightarrow{\mathcal{G}}^{a,b}(\vec{r} \mid \vec{r}\,') = (k_a^2 + \nabla\nabla \cdot\,) \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \overleftrightarrow{\tilde{G}}^{a,b}(z \mid \vec{r}\,', \vec{k}\,)\, e^{j\vec{k} \cdot \vec{r}}\, d^2k \tag{5}$$

where

$$\overleftrightarrow{\tilde{G}}^{a,b}(z \mid \vec{r}\,', \vec{k}\,) = \iint_{-\infty}^{\infty} \overleftrightarrow{G}^{a,b}(\vec{r} \mid \vec{r}\,')\, e^{-j\vec{k} \cdot \vec{r}}\, d^2r \tag{6}$$

is the Fourier transform of $\overleftrightarrow{G}^{a,b}(\vec{r} \mid \vec{r}\,')$.

At this point further simplification is difficult unless certain assumptions are made about the convergence properties of the spectral integral. If the integral in (5) converges rapidly enough, the derivitives can be brought inside to operate on the integrand, giving

$$\overleftrightarrow{g}^{a,b}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \overleftrightarrow{\tilde{g}}^{a,b}(z \mid \vec{r}\,', \vec{k}) \, e^{j\vec{k}\cdot\vec{r}} \, d^2k \tag{7}$$

where

$$\overleftrightarrow{\tilde{g}}^{a,b}(z \mid \vec{r}\,', \vec{k}) = (k_a^2 + \nabla\nabla \cdot)\overleftrightarrow{\tilde{G}}^{a,b}(z \mid \vec{r}\,', \vec{k}) \tag{8}$$

is immediately identified as the Fourier transform of the electric field dyadic Green function. Expanding out the operations indicated in (8) gives

$$\nabla(\nabla \cdot \overleftrightarrow{\tilde{G}}) = \hat{x}\,\frac{\partial}{\partial x}(\nabla \cdot \overleftrightarrow{\tilde{G}}) + \hat{y}\,\frac{\partial}{\partial y}(\nabla \cdot \overleftrightarrow{\tilde{G}}) + \hat{z}\,\frac{\partial}{\partial z}(\nabla \cdot \overleftrightarrow{\tilde{G}}) \tag{9}$$

where

$$\begin{aligned} \nabla \cdot \overleftrightarrow{\tilde{G}} = {} & \frac{\partial}{\partial x}[\tilde{G}_{xx}\,\hat{x} + \tilde{G}_{xy}\,\hat{y} + \tilde{G}_{xz}\,\hat{z}] \\ & + \frac{\partial}{\partial y}[\tilde{G}_{yx}\,\hat{x} + \tilde{G}_{yy}\,\hat{y} + \tilde{G}_{yz}\,\hat{z}] \\ & + \frac{\partial}{\partial z}[\tilde{G}_{zx}\,\hat{x} + \tilde{G}_{zy}\,\hat{y} + \tilde{G}_{zz}\,\hat{z}] \end{aligned} \tag{10}$$

For current applications, only the horizontal components of the electric field in region 1 are required. Using (9) and (10) in (8), the appropriate components of the electric field dyadic Green function will be found from

$$\tilde{g}_{xx}^{1,1}(z \mid \vec{r}\,', \vec{k}) = \frac{\partial^2}{\partial x^2}\tilde{G}_{xx}^{1,1} + \frac{\partial^2}{\partial x \partial z}\tilde{G}_{zx}^{1,1} + k_1^2\tilde{G}_{xx}^{1,1} \tag{11}$$

$$\tilde{g}_{xy}^{1,1}(z \mid \vec{r}\,', \vec{k}) = \frac{\partial^2}{\partial x \partial y}\tilde{G}_{yy}^{1,1} + \frac{\partial^2}{\partial x \partial z}\tilde{G}_{zy}^{1,1} \tag{12}$$

$$\tilde{g}_{yx}^{1,1}(z \mid \vec{r}\,', \vec{k}) = \frac{\partial^2}{\partial y \partial x}\tilde{G}_{xx}^{1,1} + \frac{\partial^2}{\partial y \partial z}\tilde{G}_{zx}^{1,1} \tag{13}$$

$$\tilde{g}_{yy}^{1,1}(z \mid \vec{r}\,', \vec{k}\,) = \frac{\partial^2}{\partial y^2}\tilde{G}_{yy}^{1,1} + \frac{\partial^2}{\partial y \partial z}\tilde{G}_{zy}^{1,1} + k_1^2\tilde{G}_{yy}^{1,1} \tag{14}$$

In deriving the above equations, it is remembered that $\tilde{G}_{yx}^{1,1} = \tilde{G}_{xy}^{1,1} = 0$, and that there is no vertical component of current in region 1. Using equations 13.7(77) through 13.7(81) in (11) to (14) then gives

$$\tilde{g}_{xx}^{1,1}(z \mid \vec{r}\,', \vec{k}\,) = (k_1^2 - k_x^2)\tilde{G}_{xx}^{1,1} - jk_x p_1 \tilde{G}_{zx}^{1,1} \tag{15}$$

$$\tilde{g}_{xy}^{1,1}(z \mid \vec{r}\,', \vec{k}\,) = -k_x k_y \tilde{G}_{yy}^{1,1} - jk_x p_1 \tilde{G}_{zy}^{1,1} \tag{16}$$

$$\tilde{g}_{yx}^{1,1}(z \mid \vec{r}\,', \vec{k}\,) = -k_x k_y \tilde{G}_{xx}^{1,1} - jk_y p_1 \tilde{G}_{zx}^{1,1} \tag{17}$$

$$\tilde{g}_{yy}^{1,1}(z \mid \vec{r}\,', \vec{k}\,) = (k_1^2 - k_y^2)\tilde{G}_{yy}^{1,1} - jk_y p_1 \tilde{G}_{zy}^{1,1} \tag{18}$$

Similarly, only the vertical component of the electric field in region 2 is needed. This component will be formed from

$$\tilde{g}_{zx}^{2,1}(z \mid \vec{r}\,', \vec{k}\,) = \frac{\partial^2}{\partial x \partial z}\tilde{G}_{xx}^{2,1} + \frac{\partial^2}{\partial z^2}\tilde{G}_{zx}^{2,1} + k_2^2\tilde{G}_{zx}^{2,1} \tag{19}$$

$$\tilde{g}_{zy}^{2,1}(z \mid \vec{r}\,', \vec{k}\,) = \frac{\partial^2}{\partial y \partial z}\tilde{G}_{yy}^{2,1} + \frac{\partial^2}{\partial z^2}\tilde{G}_{zy}^{2,1} + k_2^2\tilde{G}_{zy}^{2,1} \tag{20}$$

where $\tilde{G}_{yx}^{2,1} = \tilde{G}_{xy}^{2,1} = 0$ has been used. Using equations 13.7(82) through 13.7(88) in (19) and (20) gives

$$\tilde{g}_{zx}^{2,1}(z \mid \vec{r}\,', \vec{k}\,) = (k_2^2 + p_2^2)\tilde{G}_{zx}^{2,1} + jk_x p_2 \coth(p_2(d+z))\tilde{G}_{xx}^{2,1} \tag{21}$$

$$\tilde{g}_{zy}^{2,1}(z \mid \vec{r}\,', \vec{k}\,) = (k_2^2 + p_2^2)\tilde{G}_{zy}^{2,1} + jk_y p_2 \coth(p_2(d+z))\tilde{G}_{yy}^{2,1} \tag{22}$$

Explicit formulas can be calculated for (15) to (18) by substituting equations 13.7(77) through 13.7(81). However, to employ the boundary conditions on tangential electric field at the surface of a patch element, knowledge of the field quantities is only required at $z = 0$, and only for a horizontal source residing entirely at $z' = 0$. The proper specializations of (15) through (18) are thus found to be

$$\tilde{g}_{xx}^{1,1}(z{=}0 \mid z'{=}0, x', y', \vec{k}\,) =$$

$$= \frac{e^{-j\vec{k}\cdot\vec{r}'}}{j\omega\epsilon_1 p_1}\left\{(k_1^2 - jk_x^2)\left[\frac{\sinh(p_2 d)}{T_e}\right] - k_x p_1\left[\frac{(\epsilon_r - 1)\frac{jk_x}{p_1}\sinh(p_2 d)\cosh(p_2 d)}{T_e T_m}\right]\right\} \quad (23)$$

$\tilde{g}_{xy}^{1,1}(z=0 \mid z'=0, x', y', \vec{k}) =$

$$= \frac{e^{-j\vec{k}\cdot\vec{r}'}}{j\omega\epsilon_1 p_1}\left\{-k_x k_y\left[\frac{\sinh(p_2 d)}{T_e}\right] - jk_x p_1\left[\frac{(\epsilon_r - 1)\frac{jk_y}{p_1}\sinh(p_2 d)\cosh(p_2 d)}{T_e T_m}\right]\right\} \quad (24)$$

$\tilde{g}_{yx}^{1,1}(z=0 \mid z'=0, x', y', \vec{k}) =$

$$= \frac{e^{-j\vec{k}\cdot\vec{r}'}}{j\omega\epsilon_1 p_1}\left\{-k_x k_y\left[\frac{\sinh(p_2 d)}{T_e}\right] - jk_y p_1\left[\frac{(\epsilon_r - 1)\frac{jk_x}{p_1}\sinh(p_2 d)\cosh(p_2 d)}{T_e T_m}\right]\right\} \quad (25)$$

$\tilde{g}_{yy}^{1,1}(z=0 \mid z'=0, x', y', \vec{k}) =$

$$= \frac{e^{-j\vec{k}\cdot\vec{r}'}}{j\omega\epsilon_1 p_1}\left\{(k_1^2 - jk_y^2)\left[\frac{\sinh(p_2 d)}{T_e}\right] - k_y p_1\left[\frac{(\epsilon_r - 1)\frac{jk_y}{p_1}\sinh(p_2 d)\cosh(p_2 d)}{T_e T_m}\right]\right\} \quad (26)$$

The relationships (23) through (26) can be simplified as follows. Let

$$F = (k_1^2 - k_x^2)\left[\frac{\sinh(p_2 d)}{T_e}\right] + \frac{(\epsilon_r - 1)k_x^2 \sinh(p_2 d)\cosh(p_2 d)}{T_e T_m} \quad (27)$$

Putting both terms over a common denominator and substituting equation 14.7(40) for T_m gives

$$F = \frac{(k_1^2 - k_x^2)\sinh(p_2 d)\left[\epsilon_r \cosh(p_2 d) + \frac{p_2}{p_1}\sinh(p_2 d)\right] + (\epsilon_r - 1)k_x^2 \sinh(p_2 d)\cosh(p_2 d)}{T_e\, T_m} \quad (28)$$

which reduces to

$$F = \frac{(k_1^2 \epsilon_r - k_x^2)\sinh(p_2 d)\cosh(p_2 d) + (k_1^2 - k_x^2)\frac{p_2}{p_1}\sinh^2(p_2 d)}{T_e\, T_m} \quad (29)$$

Dividing through by $\sinh(p_2 d)\cosh(p_2 d)$ gives

$$F = \frac{(k_1^2\varepsilon_r - k_x^2) + (k_1^2 - k_x^2)\dfrac{p_2}{p_1}\tanh(p_2 d)}{\left[\sinh(p_2 d)\cosh(p_2 d)\right]^{-1} T_e\, T_m} \tag{30}$$

Finally, substituting for T_e from 13.7(34) and for T_m from 13.7(40) yields

$$F = p_1 \frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{31}$$

Next, let

$$G = \frac{-k_x k_y \sinh(p_2 d)}{T_e} + \frac{(\varepsilon_r - 1)k_x k_y \sinh(p_2 d)\cosh(p_2 d)}{T_e T_m} \tag{32}$$

Putting both terms over a common denominator and substituting for T_m yields

$$G = \frac{-k_x k_y \sinh(p_2 d)\left[\varepsilon_r \cosh(p_2 d) + \dfrac{p_2}{p_1}\sinh(p_2 d)\right] + (\varepsilon_r - 1)k_x k_y \sinh(p_2 d)\cosh(p_2 d)}{T_e\, T_m} \tag{33}$$

Simplifying gives

$$G = \frac{-k_x k_y \left[\dfrac{p_2}{p_1}\sinh^2(p_2 d) + \sinh(p_2 d)\cosh(p_2 d)\right]}{T_e\, T_m} \tag{34}$$

which, upon substitution for T_e and T_m, reduces to

$$G = p_1 \frac{-k_x k_y \left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{35}$$

Using (31) in (23) and an analogous formula in (26), and using (35) in (24) and (25), results in, with the help of (7), the electric field dyadic Green function components are

$$g_{xx}^{1,1}(z=0,x,y \mid z'=0,x',y',\vec{k}) = \frac{1}{(2\pi)^2}\int\!\!\int_{-\infty}^{\infty} \frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}\, d^2k \tag{36}$$

$$g_{xy}^{1,1}(z{=}0,x,y \mid z'{=}0,x',y',\vec{k}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}\times$$

$$\times\frac{-k_x k_y\left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}d^2k \tag{37}$$

$$g_{yx}^{1,1}(z{=}0,x,y \mid z'{=}0,x',y',\vec{k}) = g_{xy}^{1,1}(z{=}0,x,y \mid z'{=}0,x',y',\vec{k}) \tag{38}$$

$$g_{yy}^{1,1}(z{=}0,x,y \mid z'{=}0,x',y',\vec{k}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}\times$$

$$\times\frac{(k_1^2\varepsilon_r - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}d^2k \tag{39}$$

Next, explicit formulas can be obtained for (21) and (22) by substituting equations 13.7(82) through 13.7(88). This gives

$$\tilde{g}_{zx}^{2,1}(z \mid \vec{r}',k) = \frac{e^{-j\vec{k}\cdot\vec{r}'}}{j\omega\varepsilon_1 p_1}\left\{(k_2^2 + p_2^2)\,e^{-p_1 z'}\left[-\frac{(\varepsilon_r^{-1}-1)\frac{jk_x}{p_1}\sinh(p_2 d)\cosh(p_2(d+z))}{T_e T_m}\right]+\right.$$

$$\left. + jk_x p_2\, e^{-p_1 z'}\left[\frac{\cosh(p_2(d+z))}{\varepsilon_r\, T_e}\right]\right\} \tag{40}$$

$$\tilde{g}_{zy}^{2,1}(z \mid \vec{r}',k) = \frac{e^{-j\vec{k}\cdot\vec{r}'}}{j\omega\varepsilon_1 p_1}\left\{(k_2^2 + p_2^2)\,e^{-p_1 z'}\left[-\frac{(\varepsilon_r^{-1}-1)\frac{jk_y}{p_1}\sinh(p_2 d)\cosh(p_2(d+z))}{T_e T_m}\right]+\right.$$

$$\left. + jk_y p_2\, e^{-p_1 z'}\left[\frac{\cosh(p_2(d+z))}{\varepsilon_r\, T_e}\right]\right\} \tag{41}$$

To simplify these, let

$$F = -(k_2^2 + p_2^2)\frac{\frac{1}{p_1}(\varepsilon_r^{-1}-1)\sinh(p_2 d)}{T_e\, T_m} + p_2\varepsilon_r^{-1}\frac{1}{T_e} \tag{42}$$

Putting these two terms over a common denominator, and substituting for T_m gives

$$F = \frac{-(k_2^2 + p_2^2)\frac{1}{p_1}(\varepsilon_r^{-1}-1)\sinh(p_2 d) + p_2\varepsilon_r^{-1}\left[\varepsilon_r\cosh(p_2 d) + \frac{p_2}{p_1}\sinh(p_2 d)\right]}{T_e\, T_m} \tag{43}$$

which simplifies to

$$F = \frac{\frac{k_2^2}{p_1}(\varepsilon_r^{-1}-1)\sinh(p_2 d) + p_2\cosh(p_2 d) + \frac{p_2^2}{p_1}\sinh(p_2 d)}{T_e\, T_m} \tag{44}$$

Here, note that

$$k_2^2\ \varepsilon_r^{-1} = k_1^2 \tag{45}$$

so

$$-k_2^2\ \varepsilon_r^{-1} + k_2^2 + p_2^2 = -k_1^2 + k_2^2 + p_2^2 \tag{46}$$

But, from 14.5(7)

$$k_2^2 + p_2^2 = k_x^2 + k_y^2 = k_1^2 + p_1^2 \tag{47}$$

so

$$-k_2^2\ \varepsilon_r^{-1} + k_2^2 + p_2^2 = p_1^2 \tag{48}$$

Using (48) in (44) gives

$$F = \frac{p_1\sinh(p_2 d) + p_2\cosh(p_2 d)}{T_e\, T_m} \tag{49}$$

which, using 13.7(34) for T_e, reduces to

$$F = \frac{p_1}{T_m} \tag{50}$$

Now, substituting (50) into (40) gives

$$\tilde{g}_{zx}^{2,1}(z \mid \vec{r}\,', k) = \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{j\omega\varepsilon_1}\ e^{-p_1 z'}\frac{jk_x\cosh(p_2(d+z))}{T_m} \tag{51}$$

A similar set of steps gives

$$\tilde{g}_{zy}^{2,1}(z \mid \vec{r}\,',k) = \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{j\omega\epsilon_1}\, e^{-p_1 z'} \frac{jk_y \cosh(p_2(d+z))}{T_m} \tag{52}$$

Thus, the electric field dyadic Green function components are

$$g_{zx}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\epsilon_1}\, e^{-p_1 z'} \frac{jk_x \cosh(p_2(d+z))}{T_m} d^2k \tag{53}$$

$$g_{zy}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\epsilon_1}\, e^{-p_1 z'} \frac{jk_y \cosh(p_2(d+z))}{T_m} d^2k \tag{54}$$

13.9 Green Function for the Hertzian Potential Maintained by a Vertical Source in Region 2.

A current source placed in region 2 will maintain electric and magnetic fields in both regions 1 and 2. In region 2 the field will be composed of a primary contribution, a contribution due to reflection from the conductor, and a contribution due to reflection from the dielectric interface. Thus, the total potential in region 2 can be written as

$$\vec{\Pi}_2 = \vec{\Pi}_2^p + \vec{\Pi}_2^+ + \vec{\Pi}_2^- \qquad -d < z < 0 \tag{1}$$

In region 1, the field will be composed merely of a wave transmitted by the interface. Thus, the potential in region 1 is

$$\vec{\Pi}_1 = \vec{\Pi}_1^t \qquad z > 0 \tag{2}$$

The total potentials in each region can be determined by employing the appropriate boundary conditions from section 13.3.

For a z-directed current, there will be only a z component of both primary and scattered Hertzian potentials. The required boundary conditions are employed as follows.

a) Employ 13.3(48)

$$\frac{\partial}{\partial z}\Pi_{1z}^t = \frac{\partial}{\partial z}(\Pi_{2z}^p + \Pi_{2z}^+ + \Pi_{2z}^-) \qquad \text{at } z = 0 \tag{3}$$

Substituting 13.5(9) and 13.6(31) gives

$$\frac{\partial}{\partial z}\left\{\iint_{-\infty}^{\infty} \frac{W_{1z}^t(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_1 z} d^2k\right\} = \frac{\partial}{\partial z}\left\{\iint_{-\infty}^{\infty} d^2k \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\epsilon_2} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} e^{-p_2(z-z')}}{2(2\pi)^2 p_2} dv'\right.$$

$$\left. + \iint_{-\infty}^{\infty} \frac{W_{2z}^+(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{-p_2 z} d^2k + \iint_{-\infty}^{\infty} \frac{W_{2z}^-(\vec{k})}{(2\pi)^2} e^{j\vec{k}\cdot\vec{r}} e^{p_2 z} d^2k\right\} \quad \text{at } z = 0 \tag{4}$$

Since $z > z'$ when applying the boundary conditions, the condition

$$|z - z'| = z - z' \tag{5}$$

has been used to obtain (4). Differentiating, and grouping terms gives, at $z = 0$

$$\iint_{-\infty}^{\infty}\left\{-\frac{p_1}{p_2}W_{1z}^{t} + (W_{2z}^{+} - W_{2z}^{-}) + \int_{v}\frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2}\frac{e^{-j\vec{k}\cdot\vec{r}\,'}e^{p_2 z'}}{2p_2}dv'\right\}p_2\, e^{j\vec{k}\cdot\vec{r}}\frac{d^2k}{(2\pi)^2} = 0 \tag{6}$$

Setting the quantity being transformed in (6) to zero then yields

$$\frac{p_1}{p_2}W_{1z}^{t} - (W_{2z}^{+} - W_{2z}^{-}) = V_{2z} \tag{7}$$

where

$$\vec{V}_2(\vec{k}\,) = \int_{v}\frac{\vec{J}_{2z}(\vec{r}\,')}{j\omega\varepsilon_2}\frac{e^{-j\vec{k}\cdot\vec{r}\,'}e^{p_2 z'}}{2p_2}dv' \tag{8}$$

b) Employ 13.3(51)

$$\Pi_{1z}^{t} = \overline{\varepsilon}(\Pi_{2z}^{p} + \Pi_{2z}^{+} + \Pi_{2z}^{-}) \qquad \text{at } z = 0 \tag{9}$$

Substituting gives

$$\iint_{-\infty}^{\infty}\frac{W_{1z}^{t}(\vec{k}\,)}{(2\pi)^2}\, e^{j\vec{k}\cdot\vec{r}}\, e^{-p_1 z}\, d^2k = \varepsilon_r\left\{\iint_{-\infty}^{\infty} d^2k\int_{v}\frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}e^{-p_2(z-z')}}{2(2\pi)^2 p_2}dv'\right.$$

$$\left. + \iint_{-\infty}^{\infty}\frac{W_{2z}^{+}(\vec{k}\,)}{(2\pi)^2}\, e^{j\vec{k}\cdot\vec{r}}\, e^{-p_2 z}d^2k + \iint_{-\infty}^{\infty}\frac{W_{2z}^{-}(\vec{k}\,)}{(2\pi)^2}\, e^{j\vec{k}\cdot\vec{r}}\, e^{p_2 z}d^2k\right\} \quad \text{at } z = 0 \tag{10}$$

Simplifying this yields

$$\varepsilon_r^{-1}W_{1z}^{t} - (W_{2z}^{+} + W_{2z}^{-}) = V_{2z} \tag{11}$$

c) Employ 13.3(54)

$$\frac{\partial}{\partial z}(\Pi_{2z}^{p} + \Pi_{2z}^{+} + \Pi_{2z}^{-}) = 0 \qquad \text{at } z = -d \tag{12}$$

Substituting gives

$$\frac{\partial}{\partial z}\left\{\iint_{-\infty}^{\infty} d^2k\int_{v}\frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}e^{-p_2(z'-z)}}{2(2\pi)^2 p_2}dv'\right.$$

$$\left. + \iint_{-\infty}^{\infty}\frac{W_{2z}^{+}(\vec{k}\,)}{(2\pi)^2}\, e^{j\vec{k}\cdot\vec{r}}\, e^{-p_2 z}d^2k + \iint_{-\infty}^{\infty}\frac{W_{2z}^{-}(\vec{k}\,)}{(2\pi)^2}\, e^{j\vec{k}\cdot\vec{r}}\, e^{p_2 z}d^2k\right\} = 0 \quad \text{at } z = -d \tag{13}$$

Performing the indicated derivitives then gives, at $z = -d$

$$\iint_{-\infty}^{\infty}\left\{W_{2z}^{+}\, e^{p_2 d} - W_{2z}^{-}\, e^{-p_2 d} - \int\frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2}\frac{e^{-j\vec{k}\cdot\vec{r}\,'}e^{-p_2(z'+d)}}{2p_2}dv'\right\}p_2\, e^{j\vec{k}\cdot\vec{r}}\frac{d^2k}{(2\pi)^2} = 0 \tag{14}$$

Setting the function being transformed in (14) to zero yields

$$W_{2z}^{+}\, e^{p_2 d} - W_{2z}^{-}\, e^{-p_2 d} = e^{-p_2 d}\, \bar{V}_{2z} \tag{15}$$

where

$$\vec{V}_2(\vec{k}) \equiv \int_v \frac{\vec{J}_{2z}(\vec{r}\,')}{j\omega\varepsilon_2} \frac{e^{-j\vec{k}\cdot\vec{r}\,'} e^{-p_2 z'}}{2p_2} dv' \tag{16}$$

In summary, then, the boundary conditions for a z-directed source current yield

$$\frac{p_1}{p_2} W_{1z}^{t} - (W_{2z}^{+} - W_{2z}^{-}) = V_{2z} \tag{7}$$

$$\varepsilon_r^{-1} W_{1z}^{t} - (W_{2z}^{+} + W_{2z}^{-}) = V_{2z} \tag{11}$$

$$W_{2z}^{+}\, e^{p_2 d} - W_{2z}^{-}\, e^{-p_2 d} = e^{-p_2 d}\, \bar{V}_{2z} \tag{15}$$

The above system of equations can be easily solved as follows. Equation (15) can be rearranged to yield

$$W_{2z}^{+} = (W_{2z}^{-} + \bar{V}_{2z})\, e^{-2p_2 d} \tag{17}$$

which can then be substituted into (7) to give

$$\frac{p_1}{p_2} W_{1z}^{t} - W_{2z}^{-}\, e^{-2p_2 d} - \bar{V}_{2z}\, e^{-2p_2 d} + W_{2z}^{-} = V_{2z} \tag{18}$$

or

$$\frac{p_1}{p_2} W_{1z}^{t} - W_{2z}^{-}\, (e^{-2p_2 d} - 1) = U_{2z} \tag{19}$$

where

$$\vec{U}_2 \equiv \vec{V}_2 + \vec{\bar{V}}_2\, e^{-2p_2 d} \tag{20}$$

Equation (17) can also be substituted into (11) to give

$$\varepsilon_r^{-1} W_{1z}^{t} - W_{2z}^{-}\, e^{-2p_2 d} - \bar{V}_{2z}\, e^{-2p_2 d} - W_{2z}^{-} = V_{2z} \tag{21}$$

or

$$\varepsilon_r^{-1} W_{1z}^t - W_{2z}^- (e^{-2p_2 d} + 1) = U_{2z} \tag{22}$$

which then yields

$$W_{2z}^- = \frac{\varepsilon_r^{-1} W_{1z}^t - U_{2z}}{e^{-2p_2 d} + 1} \tag{23}$$

Substituting (23) into (19) gives

$$\frac{p_1}{p_2} W_{1z}^t - (\varepsilon_r^{-1} W_{1z}^t - U_{2z}) \frac{e^{-2p_2 d} - 1}{e^{-2p_2 d} + 1} = U_{2z} \tag{24}$$

Factoring out W_{1z}^t and using 13.7(32a,b) results in

$$W_{1z}^t \left[\frac{p_1}{p_2} + \varepsilon_r^{-1} \tanh(p_2 d) \right] = U_2 \left[1 + \tanh(p_2 d) \right] \tag{25}$$

Solving this equation gives

$$W_{1z}^t = U_{2z} \frac{1 + \tanh(p_2 d)}{\dfrac{p_1}{p_2} + \varepsilon_r^{-1} \tanh(p_2 d)} \tag{26}$$

The numerator of (26) can be rewritten using

$$1 + \tanh(p_2 d) = \frac{e^{p_2 d}}{\cosh p_2 d} \tag{27}$$

and, from (20)

$$U_{2z} = V_{2z} + \bar{V}_{2z} e^{-2p_2 d} \tag{28}$$

First, substituting (8) and (16) into (28) gives

$$U_{2z} = e^{-p_2 d} \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2} \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{2p_2} \left[e^{p_2(z' + d)} + e^{-p_2(z' + d)} \right] dv' \tag{29}$$

which simplifies to

$$U_{2z} = e^{-p_2 d} \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2} \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{p_2} \cosh(p_2(z' + d)) dv' \tag{30}$$

Now, substituting (30) and (27) into (26) gives

$$W_{1z}^{t} = \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2} e^{-j\vec{k}\cdot\vec{r}\,'} \frac{\cosh(p_2(z'+d))}{p_1\cosh(p_2 d) + \varepsilon_r^{-1} p_2 \sinh(p_2 d)} dv' \tag{31}$$

or

$$W_{1z}^{t} = \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{p_1} \frac{\cosh(p_2(z'+d))}{T_m} dv' \tag{32}$$

Next, solutions for W_{2z}^{-} and W_{2z}^{+} can be obtained quite easily. Subtracting (11) from (7) gives

$$2W_{2z}^{-} = W_{1z}^{t} \left[\varepsilon_r^{-1} - \frac{p_1}{p_2}\right] \tag{33}$$

which upon substitution of (32) yields

$$W_{2z}^{-} = \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_1} \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{2p_1} \left[\frac{\varepsilon_r^{-1} - \frac{p_1}{p_2}}{T_m}\right] \cosh(p_2(z'+d)) dv' \tag{34}$$

or

$$W_{2z}^{-} = \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2} \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{2p_2} \left[\frac{\frac{p_2}{p_1} - \varepsilon_r}{T_m}\right] \cosh(p_2(z'+d)) dv' \tag{35}$$

Now, substituting (35) and (16) into (17) gives

$$W_{2z}^{+} = \int_v \frac{J_{2z}(\vec{r}\,')}{j\omega\varepsilon_2} e^{-2p_2 d} \frac{e^{-j\vec{k}\cdot\vec{r}\,'}}{2p_2} \left\{\left[\frac{\frac{p_2}{p_1} - \varepsilon_r}{T_m}\right] \cosh(p_2(z'+d)) + e^{-p_2 z'}\right\} dv' \tag{36}$$

With the transform amplitudes W_{1z}^{t}, W_{2z}^{+}, and W_{2z}^{-} determined, it is now possible to formulate expressions for the total Hertzian potential maintained by a vertical current source in region 2.

The potential in region 1 is easily determined by using (2). Substituting (32) into 13.5(9) gives

$$\Pi_{1z}(\vec{r}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \int_v \frac{J_{2z}(\vec{r}')}{j\omega\epsilon_1} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-p_1 z}}{p_1} \frac{\cosh(p_2(z'+d))}{T_m} dv' \tag{37}$$

The dyadic Green function representation for the potential in region 1 is then easily identified from 13.7(76) as

$$G_{zz}^{1,2}(\vec{r} \mid \vec{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \; \frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')} e^{-p_1 z}}{j\omega\epsilon_1\, p_1} \frac{\cosh(p_2(z'+d))}{T_m} \tag{38}$$

The potential in region 2 is determined by using (1). Using 13.6(31) and substituting (35) and (36) into 13.5(9) gives

$$\Pi_{2z}(\vec{r}') = \iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot\vec{r}}}{(2\pi)^2} d^2k \int_v \frac{J_{2z}(\vec{r}')}{j\omega\epsilon_2} \frac{e^{-j\vec{k}\cdot\vec{r}'}}{2p_2} \left\{ e^{-p_2|z-z'|} \right.$$

$$+ \left[\frac{\frac{p_2}{p_1} - \epsilon_r}{T_m} \cosh(p_2(z'+d)) + \right.$$

$$\left. \left. + e^{-p_2 z'} \right] e^{-p_2 z} e^{-2p_2 d} + \frac{\frac{p_2}{p_1} - \epsilon_r}{T_m} \cosh(p_2(z'+d))\, e^{p_2 d} \right\} dv' \tag{39}$$

This can be written more succinctly as

$$\Pi_{2z}(\vec{r}') = \iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot\vec{r}}}{(2\pi)^2} d^2k \int_v \frac{J_{2z}(\vec{r}')}{j\omega\epsilon_2} \frac{e^{-j\vec{k}\cdot\vec{r}'}}{2p_2} \left\{ F \right\} dv' \tag{40}$$

where F can be simplified by considering the following two cases.

a) $z > z'$

For the case $z > z'$, the absolute value in (39) can be replaced by

$$|z - z'| = z - z' \tag{41}$$

Thus, F becomes

$$F = e^{-p_2 z} e^{p_2 z'} + e^{-p_2 z'} e^{-p_2 z} e^{-2p_2 d} + \frac{\frac{p_2}{p_1} - \epsilon_r}{T_m} \cosh(p_2(z'+d)) \left[e^{-p_2 z} e^{-2p_2 d} + e^{p_2 z} \right] \tag{42}$$

Combining the exponentials in the first two terms and using 13.7(32b) gives

$$F = \cosh(p_2(z' + d))\, e^{-p_2 d} \left\{ 2e^{-p_2 z} + \frac{\frac{p_2}{p_1} - \varepsilon_r}{T_m} \left[e^{-p_2 z}\, e^{-p_2 d} + e^{p_2 z}\, e^{p_2 d} \right] \right\} \tag{43}$$

Putting both terms of (43) over a common denominator, and expanding T_m in terms of exponentials then gives

$$F = \cosh(p_2(z' + d))\, \frac{e^{-p_2 d}}{T_m} \left\{ \varepsilon_r\, e^{-p_2 z}\, e^{p_2 d} + \varepsilon_r\, e^{-p_2 z}\, e^{-p_2 d} + \frac{p_2}{p_1} e^{-p_2 z}\, e^{p_2 d} - \frac{p_2}{p_1} e^{-p_2 z}\, e^{-p_2 d} \right.$$

$$\left. + \frac{p_2}{p_1} e^{-p_2 z}\, e^{-p_2 d} + \frac{p_2}{p_1} e^{p_2 z}\, e^{p_2 d} - \varepsilon_r\, e^{-p_2 z}\, e^{-p_2 d} - \varepsilon_r\, e^{p_2 z}\, e^{p_2 d} \right\} \tag{44}$$

Canceling, and factoring out common terms results in

$$F = \cosh(p_2(z' + d))\, \frac{e^{-p_2 d}}{T_m} \left\{ \frac{p_2}{p_1} e^{p_2 d} \left(e^{p_2 z} + e^{-p_2 z}\right) - \varepsilon_r\, e^{p_2 d} \left(e^{p_2 z} - e^{-p_2 z}\right) \right\}$$

$$= 2\frac{\cosh(p_2(z' + d))}{T_m} \left[\frac{p_2}{p_1} \cosh(p_2 z) - \varepsilon_r \sinh(p_2 z) \right] \tag{45}$$

b) $z < z'$

For the case $z < z'$, the absolute value in (39) can be replaced by

$$|\, z - z' \,| = z' - z \tag{46}$$

with this, F becomes

$$F = e^{-p_2 z'}\, e^{p_2 z} + e^{-p_2 z'}\, e^{-p_2 z}\, e^{-2p_2 d} + \frac{\frac{p_2}{p_1} - \varepsilon_r}{T_m} \cosh(p_2(z' + d)) \left[e^{-p_2 z}\, e^{-2p_2 d} + e^{p_2 z} \right] \tag{47}$$

combining the exponentials gives

$$F = 2\cosh(p_2(z + d))\, e^{-p_2 d}\, e^{-p_2 z'} + 2\cosh(p_2(z' + d) \left(\frac{\frac{p_2}{p_1} - \varepsilon_r}{T_m} \right) e^{-p_2 d} \cosh(p_2(z + d)) \tag{48}$$

or,

$$F = \cosh(p_2(z+d))\, e^{-p_2 d} \left\{ 2e^{-p_2 z'} + \frac{\frac{p_2}{p_1} - \varepsilon_r}{T_m} \left[e^{-p_2 z'} e^{-p_2 d} + e^{p_2 z'} e^{p_2 d} \right] \right\} \tag{49}$$

Note that (49) is identical to (43), with the roles of z and z' reversed. Thus, by analogy, (49) can be written using (45) as

$$F = 2\frac{\cosh(p_2(z+d))}{T_m} \left[\frac{p_2}{p_1}\cosh(p_2 z') - \varepsilon_r \sinh(p_2 z') \right] \tag{50}$$

Finally, the dyadic Green function representation for the potential in region 2 is identified from 13.7(76) and (40) as

$$G_{zz}^{2,2}(\vec{r} \mid \vec{r}\,') = \begin{cases} \dfrac{1}{(2\pi)^2}\displaystyle\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_2 p_2} \frac{\cosh(p_2(z+d))}{T_m} \left[\frac{p_2}{p_1}\cosh(p_2 z') - \varepsilon_r \sinh(p_2 z') \right] & z < z' \\ \dfrac{1}{(2\pi)^2}\displaystyle\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_2 p_2} \frac{\cosh(p_2(z'+d))}{T_m} \left[\frac{p_2}{p_1}\cosh(p_2 z) - \varepsilon_r \sinh(p_2 z) \right] & z > z' \end{cases} \tag{51}$$

13.10 Green Function for the Electric Field Maintained by a Vertical Source in Region 2.

The electric field dyadic Green function can be obtained from the Hertzian potential dyadic Green function exactly as was done in Section 14.8.

In region 1 only the horizontal components of field are of immediate interest. Since there is only a z-component of potential in region 1, the appropriate components of the dyadic greens function spectrum are determined from 13.8(8), 13.8(9) and 13.8(10) as

$$\tilde{g}_{xz}^{1,2}(z \mid \vec{r}\,', \vec{k}) = \frac{\partial^2 \tilde{G}_{zz}^{1,2}}{\partial x \, \partial z} \tag{1}$$

$$\tilde{g}_{yz}^{1,2}(z \mid \vec{r}\,', \vec{k}) = \frac{\partial^2 \tilde{G}_{zz}^{1,2}}{\partial y \, \partial z} \tag{2}$$

The relationships can be calculated explicitly with the help of 13.9(38). Taking the derivatives gives

$$\tilde{g}_{xz}^{1,2}(z \mid \vec{r}\,', \vec{k}) = -jk_x p_1 \tilde{G}_{zz}^{1,2} \tag{3}$$

$$\tilde{g}_{yz}^{1,2}(z \mid \vec{r}\,', \vec{k}) = -jk_y p_1 \tilde{G}_{zz}^{1,2} \tag{4}$$

Substituting from 13.9(38) and performing the Fourier inversion then gives

$$g_{xz}^{1,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k} \cdot (\vec{r} - \vec{r}\,')}}{j\omega\varepsilon_1} (-jk_x) e^{-p_1 z} \frac{\cosh(p_2(z'+d))}{T_m} \tag{5}$$

$$g_{yz}^{1,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k} \cdot (\vec{r} - \vec{r}\,')}}{j\omega\varepsilon_1} (-jk_y) e^{-p_1 z} \frac{\cosh(p_2(z'+d))}{T_m} \tag{6}$$

In region 2, only the vertical component of field is of immediate interest. Since there is only a z-component of potential in region 2, the vertical component of field is given via 13.2(7) as

$$E_{2z}(\vec{r}) = k_2^2 \Pi_{2z}(\vec{r}) + \frac{\partial^2 \Pi_{2z}(\vec{r})}{\partial z^2} \tag{7}$$

The derivatives indicated in (7) must be handled with special care, since the definition of $G_{zz}^{2,2}$ depends on the value of $z - z'$. First, use 13.7(76) to write

$$\Pi_{2z}(\vec{r}) = \int_v G_{zz}^{2,2}(\vec{r} \mid \vec{r}\,')J_{2z}(\vec{r}\,')dv' \tag{8}$$

Then,

$$\frac{\partial^2 \Pi_{2z}}{\partial z^2} = \frac{\partial^2}{\partial z^2} \int_v G_{zz}^{2,2}(\vec{r} \mid \vec{r}\,')J_{2z}(\vec{r}\,')dv' \tag{9}$$

Obviously, the derivatives cannot be brought blindly inside the integral. This is even more apparent when (9) is written using 13.9(51) as

$$\frac{\partial^2 \Pi_{2z}}{\partial z^2} = \frac{\partial^2}{\partial z^2} \iint_{x'y'} \left[\int_{z'=-d}^{z} G^{>}(\vec{r} \mid \vec{r}\,')J_{2z}(\vec{r}\,')\, dz' \right] dx'\, dy'$$
$$+ \frac{\partial^2}{\partial z^2} \iint_{x'y'} \left[\int_{z'=z}^{0} G^{<}(\vec{r} \mid \vec{r}\,')J_{2z}(\vec{r}\,')\, dz' \right] dx'\, dy' \tag{10}$$

where

$$G_{zz}^{2,2}(\vec{r} \mid \vec{r}\,') = \begin{cases} G^{>}(\vec{r} \mid \vec{r}\,') & z > z' \\ G^{<}(\vec{r} \mid \vec{r}\,') & z < z' \end{cases} \tag{11}$$

Since the limits of integration in (10) are variable, the differentiation must be performed using Leibnitz' rule.

It is possible to bring the derivatives indicated in (9) inside the integral, if $G_{zz}^{2,2}$ is represented properly. Let

$$G_{zz}^{2,2}(\vec{r} \mid \vec{r}\,') = U(z' - z)G^{<}(\vec{r} \mid \vec{r}\,') + U(z - z')G^{>}(\vec{r} \mid \vec{r}\,') \tag{12}$$

where $U(x)$ is the unit step function

$$U(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{13}$$

With this representation, derivatives on Π_{2z} can be safely brought inside the integral. Thus, the first derivative on Π_{2z} is

$$\frac{\partial \Pi_{2z}}{\partial z} = \int_v \frac{\partial G_{zz}^{2,2}}{\partial z} J_{2z}(\vec{r}\,')dv' \tag{14}$$

The first derivative of (12) is found using the product rule to be

$$\frac{\partial G_{zz}^{2,2}(\vec{r} \mid \vec{r}\,')}{\partial z} = U(z' - z)\frac{\partial G^{<}(\vec{r} \mid \vec{r}\,')}{\partial z} + G^{<}(\vec{r} \mid \vec{r}\,')\frac{\partial U(z' - z)}{\partial z}$$

$$+ U(z - z')\frac{\partial G^{>}(\vec{r} \mid \vec{r}\,')}{\partial z} + G^{>}(\vec{r} \mid \vec{r}\,')\frac{\partial U(z - z')}{\partial z} \tag{15}$$

Now, the derivative of the step function can be written as

$$\frac{\partial U(x)}{\partial x} = \delta(x) \tag{16}$$

where $\delta(x)$ is the Dirac δ-function. Thus,

$$\frac{\partial U(z' - z)}{\partial z} = \frac{\partial U(z' - z)}{\partial(z' - z)}\frac{\partial(z' - z)}{\partial z} = -\delta(z' - z) \tag{17}$$

$$\frac{\partial U(z - z')}{\partial z} = \frac{\partial U(z - z')}{\partial(z - z')}\frac{\partial(z - z')}{\partial z} = -\delta(z - z') \tag{18}$$

Substituting (15), (17) and (18) into (14) then gives

$$\frac{\partial \Pi_{2z}(\vec{r})}{\partial z} = \int_v J_{2z}(\vec{r}\,')\left\{U(z' - z)\frac{\partial G^{<}(\vec{r} \mid \vec{r}\,')}{\partial z} - \delta(z' - z)G^{<}(\vec{r} \mid \vec{r}\,')\right.$$

$$\left. + U(z - z')\frac{\partial G^{>}(\vec{r} \mid \vec{r}\,')}{\partial z} + \delta(z - z')G^{>}(\vec{r} \mid \vec{r}\,')\right\}dv' \tag{19}$$

This can also be written as

$$\frac{\partial \Pi_{2z}(\vec{r})}{\partial z} = \int_{x'}\int_{y'} dx'dy' \int_{z'} J_{2z}(\vec{r}\,')\left\{- \delta(z' - z)G^{<}(\vec{r} \mid \vec{r}\,') + \delta(z - z')G^{>}(\vec{r} \mid \vec{r}\,')\right\}dv'$$

$$+ \int_v J_{2z}(\vec{r}\,')\left\{U(z' - z)\frac{\partial G^{<}(\vec{r} \mid \vec{r}\,')}{\partial z} + U(z - z')\frac{\partial G^{>}(\vec{r} \mid \vec{r}\,')}{\partial z}\right\}dv' \tag{20}$$

The first term of (20) is of particular interest. Let

$$I(x', y' \mid \vec{r}) = \int_{z'} J_{2z}(\vec{r}\,')\left\{- \delta(z' - z)G^{<}(\vec{r} \mid \vec{r}\,') + \delta(z - z')G^{>}(\vec{r} \mid \vec{r}\,')\right\}dz' \tag{21}$$

and consider the following two cases.

a) Integration over z' does not pass through the observation point z.

In this case, the integral property of the δ-function gives

$$\int_{z'} J_{2z}(\vec{r}\,')\left\{-\delta(z'-z)G^{<}(\vec{r}\mid\vec{r}\,')\right\}dz' = \int_{z'} J_{2z}(\vec{r}\,')\left\{\delta(z-z')G^{>}(\vec{r}\mid\vec{r}\,')\right\}dz' = 0 \qquad (22)$$

Thus,

$$I(x', y' \mid \vec{r}) = 0 \qquad (23)$$

b) Integration over z' passes through observation point z.

In this case, the integral property of the δ-function gives

$$I(x', y' \mid \vec{r}) = J_{2z}(z)\left[-G^{<}(x', y', z \mid x, y, z) + G^{>}(x, y, z \mid x, y, z)\right] \qquad (24)$$

But, 13.9(51) shows that

$$G^{<}(x, y, z \mid x', y', z' = z) = G^{>}(x, y, z \mid x', y', z' = z) \qquad (25)$$

and thus

$$I(x', y' \mid \vec{r}) = 0 \qquad (26)$$

Thus, the first term of (20) vanishes, and so

$$\frac{\partial \Pi_{2z}(\vec{r})}{\partial z} = \int_{v} J_{2z}(\vec{r}\,')\left\{U(z'-z)\frac{\partial G^{<}(\vec{r}\mid\vec{r}\,')}{\partial z} + U(z-z')\frac{\partial G^{>}(\vec{r}\mid\vec{r}\,')}{\partial z}\right\}dv' \qquad (27)$$

The second derivative of Π_{2z} can be calculated in a similar manner to above. Differentiating (27) gives

$$\frac{\partial^2 \Pi_{2z}(\vec{r})}{\partial z^2} = \int_{v} J_{2z}(\vec{r}\,')\left\{-\delta(z'-z)\frac{\partial G^{<}(\vec{r}\mid\vec{r}\,')}{\partial z} + U(z'-z)\frac{\partial^2 G^{<}(\vec{r}\mid\vec{r}\,')}{\partial z^2}\right.$$

$$\left. + \delta(z-z')\frac{\partial G^{>}(\vec{r}\mid\vec{r}\,')}{\partial z} + U(z-z')\frac{\partial^2 G^{>}(\vec{r}\mid\vec{r}\,')}{\partial z^2}\right\}dv' \qquad (28)$$

Substituting (28) into (7) and using (12) gives

$$E_{2z}(\vec{r}) = \int_{v} J_{2z}(\vec{r}\,')\left\{-\delta(z'-z)\frac{\partial G^{<}(\vec{r}\mid\vec{r}\,')}{\partial z} + U(z'-z)\left[k_2^2 + \frac{\partial^2}{\partial z^2}\right]G^{<}(\vec{r}\mid\vec{r}\,')\right.$$

$$\left. + \delta(z-z')\frac{\partial G^{>}(\vec{r}\mid\vec{r}\,')}{\partial z} + U(z-z')\left[k_2^2 + \frac{\partial^2}{\partial z^2}\right]G^{>}(\vec{r}\mid\vec{r}\,')\right\}dv' \qquad (29)$$

Note that the first and third terms of (29) would not appear if the derivatives were brought inside the integral of equation (9) without using (12).

From (29), the dyadic Green function component for the z component of field in region 2 produced by a vertical current source in region 2 is easily identified using 13.8(1) as

$$g_{zz}^{2,2} = -\delta(z'-z)\frac{\partial G^{<}(\vec{r}\mid\vec{r}')}{\partial z} + U(z'-z)\left[k_2^2 + \frac{\partial^2}{\partial z^2}\right]G^{<}(\vec{r}\mid\vec{r}')$$

$$+\delta(z-z')\frac{\partial G^{>}(\vec{r}\mid\vec{r}')}{\partial z} + U(z-z')\left[k_2^2 + \frac{\partial^2}{\partial z^2}\right]G^{>}(\vec{r}\mid\vec{r}') \qquad (30)$$

Equation (30) can be written in a more explicit form, using 13.9(51), assuming the derivitives can be passed through the inversion integral. In this case, the required derivatives become

$$\frac{\partial G^{<}(\vec{r}|\vec{r}')}{\partial z} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_2}\frac{\sinh(p_2(z+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2z') - \varepsilon_r\sinh(p_2z')\right] \qquad (31)$$

$$\frac{\partial^2 G^{<}(\vec{r}|\vec{r}')}{\partial z^2} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_2}p_2\frac{\cosh(p_2(z+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2z') - \varepsilon_r\sinh(p_2z')\right] \qquad (32)$$

$$\frac{\partial G^{>}(\vec{r}|\vec{r}')}{\partial z} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_2}\frac{\cosh(p_2(z'+d))}{T_m}\left[\frac{p_2}{p_1}\sinh(p_2z) - \varepsilon_r\cosh(p_2z)\right] \qquad (33)$$

$$\frac{\partial^2 G^{>}(\vec{r}|\vec{r}')}{\partial z^2} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_2}p_2\frac{\cosh(p_2(z'+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2z) - \varepsilon_r\sinh(p_2z)\right] \qquad (34)$$

with these, equation (30) becomes

$$g_{zz}^{2,2}(\vec{r}|\vec{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_2}\left\{\delta(z-z')\frac{\cosh(p_2(z'+d))}{T_m}\left[\frac{p_2}{p_1}\sinh(p_2z) - \varepsilon r\cosh(p_2z)\right]\right.$$

$$-\delta(z'-z)\frac{\sinh(p_2(z+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2z') - \varepsilon_r\sinh(p_2z')\right]$$

$$+U(z-z')\left[\frac{k_2^2+p_2^2}{p_2}\right]\frac{\cosh(p_2(z'+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2z) - \varepsilon_r\sinh(p_2z)\right]$$

$$\left.+U(z'-z)\left[\frac{k_2^2+p_2^2}{p_2}\right]\frac{\cosh(p_2(z+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2z') - \varepsilon_r\sinh(p_2z')\right]\right\}d^2k \qquad (35)$$

The above expression can be further simplified by using

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y) \tag{36}$$

and

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y) \tag{37}$$

Since $\delta(x) = \delta(-x)$, the coefficients multiplying the δ-function can be combined and written as

$$\frac{\cosh(p_2(z'+d))}{T_m}\left[\frac{p_2}{p_1}\sinh(p_2 z) - \varepsilon_r \cosh(p_2 z)\right] - \frac{\sinh(p_2(z+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2 z') - \varepsilon_r \sinh(p_2 z')\right]$$

$$= \frac{1}{T_m}\frac{p_2}{p_1}\Big[[\cosh(p_2 z')\cosh(p_2 d) + \sinh(p_2 z')\sinh(p_2 d)]\sinh(p_2 z)$$

$$- [\sinh(p_2 z)\cosh(p_2 d) + \cosh(p_2 z)\sinh(p_2 d)]\cosh(p_2 z')\Big]$$

$$- \frac{\varepsilon_r}{T_m}\Big[[\cosh(p_2 z')\cosh(p_2 d) + \sinh(p_2 z')\sinh(p_2 d)]\cosh(p_2 z)$$

$$- [\sinh(p_2 z)\cosh(p_2 d) + \cosh(p_2 z)\sinh(p_2 d)]\sinh(p_2 z')\Big]$$

$$= \Bigg\{\frac{1}{T_m}\frac{p_2}{p_1}\Big[\sinh(p_2 z')\sinh(p_2 z) - \cosh(p_2 z)\cosh(p_2 z')\Big]\sinh(p_2 d)$$

$$- \frac{\varepsilon_r}{T_m}\Big[\cosh(p_2 z')\cosh(p_2 z) - \sinh(p_2 z')\sinh(p_2 z)\Big]\cosh(p_2 d)\Bigg\} \tag{38}$$

Using (36) and (37) again, this can be written as

$$= \left\{\frac{-1}{T_m}\frac{p_2}{p_1}\cosh(z'-z)\sinh(p_2 d) - \frac{\varepsilon_r}{T_m}\cosh(z'-z)\cosh(p_2 d)\right\} \tag{39}$$

Since the above multiplies $\delta(z-z')$, set z' equal to z and get

$$= \left\{\frac{-1}{T_m}\frac{p_2}{p_1}\sinh(p_2 d) - \frac{\varepsilon_r}{T_m}\cosh(p_2 d)\right\} \tag{40}$$

From the definition of T_m, equation 13.7(40), this becomes simply

$$= -1 \tag{41}$$

Using the above result and making the definitions

$$z^{>} = \max(z, z') \tag{42}$$

$$z^{<} = \min(z, z') \tag{43}$$

equation (35) may then be written as

$$g_{zz}^{2,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2}\int\!\!\int_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_2}\left\{-\delta(z - z') + \left[\frac{k_2^2 + p_2^2}{p_2}\right]\frac{\cosh(p_2(z^{<}+d))}{T_m}\left[\frac{p_2}{p_1}\cosh(p_2 z^{>}) - \varepsilon_r \sinh(p_2 z^{>})\right]\right\} d^2k \tag{44}$$

13.11 Electric Field Green Function Summary.

From the development of previous sections, the Green functions for the electric field which will be needed are

$$g_{xx}^{1,1}(z=0,x,y\mid z'=0,x',y',\vec{k}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}\times$$

$$\times\frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}d^2k \quad (1)$$

$$g_{xy}^{1,1}(z=0,x,y\mid z'=0,x',y',\vec{k}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}\times$$

$$\times\frac{-k_x k_y\left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}d^2k \quad (2)$$

$$g_{yx}^{1,1}(z=0,x,y\mid z'=0,x',y',\vec{k}) = g_{xy}^{1,1}(z=0,x,y\mid z'=0,x',y',\vec{k}) \quad (3)$$

$$g_{yy}^{1,1}(z=0,x,y\mid z'=0,x',y',\vec{k}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}\times$$

$$\times\frac{(k_1^2\varepsilon_r - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}d^2k \quad (4)$$

$$g_{zx}^{2,1}(\vec{r}\mid\vec{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}e^{-p_1 z'}\frac{jk_x\cosh(p_2(d+z))}{T_m}d^2k \quad (5)$$

$$g_{zy}^{2,1}(\vec{r}\mid\vec{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}e^{-p_1 z'}\frac{jk_y\cosh(p_2(d+z))}{T_m}d^2k \quad (6)$$

$$g_{xz}^{1,2}(\vec{r}\mid\vec{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}d^2k\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}(-jk_x)e^{-p_1 z}\frac{\cosh(p_2(z'+d))}{T_m} \quad (7)$$

$$g_{yz}^{1,2}(\vec{r}\mid\vec{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}d^2k\frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}')}}{j\omega\varepsilon_1}(-jk_y)e^{-p_1 z}\frac{\cosh(p_2(z'+d))}{T_m} \quad (8)$$

$$g_{zz}^{2,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} d^2k \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_2}\left\{ -\delta(z - z') + \right.$$

$$\left. + \left[\frac{k_2^2 + p_2^2}{p_2}\right] \frac{\cosh(p_2(z^< + d))}{T_m} \left[\frac{p_2}{p_1}\cosh(p_2 z^>) - \varepsilon_r \sinh(p_2 z^>)\right]\right\} d^2k \quad (9)$$

Making the following definitions,

$$g_{xx}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \quad (10)$$

$$g_{xy}(\vec{k}) \equiv g_{yx}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{-k_x k_y \left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \quad (11)$$

$$g_{yy}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{(k_1^2\varepsilon_r - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \quad (12)$$

$$g_{zx}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} e^{-p_1 z'} \frac{jk_x}{T_m} \quad (13)$$

$$g_{zy}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} e^{-p_1 z'} \frac{jk_y}{T_m} \quad (14)$$

$$g_{xz}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} e^{-p_1 z} \frac{-jk_x}{T_m} \quad (15)$$

$$g_{yz}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} e^{-p_1 z} \frac{-jk_y}{T_m} \quad (16)$$

the Green functions involving horizontal components are written compactly as

$$g_{\alpha\beta}^{11} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} g_{\alpha\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad \alpha = x,y \qquad \beta = x,y \quad (17)$$

$$g_{z\beta}^{21} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} g_{z\beta}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \cosh(p_2(z+d)) \qquad \beta = x,y \quad (18)$$

$$g_{\alpha z}^{12} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} g_{\alpha z}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')} \cosh(p_2(z'+d)) \qquad \alpha = x,y \quad (19)$$

13.12 Green Functions for an Infinite Antenna Array.

The Geometry of an infinite rectangular array of circular patches on a coated ground plane is shown in figure 3. This is to be driven by an infinite plane wave of arbitrary incidence angle. The incident field at each patch is then identical to the incident field at the patch located at the origin to within a phase factor. For the p,qth patch, the phase factor is

$$e^{jk_1(\sin(\theta_i)\cos(\phi_i)\,x_p + \sin(\theta_i)\sin(\phi_i)\,y_q)} = e^{jk_1(ux_p + vy_q)} \tag{1}$$

where

$$u = \sin(\theta_i)\cos(\phi_i) \qquad v = \sin(\theta_i)\sin(\phi_i) \tag{2a,b}$$

and where, for a patch spacing of d_x in the x-direction and d_y in the y-direction,

$$x_p = p\ d_x \qquad y_q = q\ d_y \tag{3a,b}$$

For the center, (0,0), element, each component of both the electric field and Hertzian potential Green Function dyadics can be written as

$$g(\vec{r} \mid \vec{r}\,') = \iint_{-\infty}^{\infty} f(k_x, k_y, z, z')\, e^{jk_x(x-x')}\, e^{jk_y(y-y')}\, dk_x\, dk_y \tag{4}$$

Each point $\vec{r}\,'_{p,q}$ on the p,qth patch corresponds to a point on the (0,0) patch such that

$$\vec{r}\,'_{p,q} = p\ d_x\ \hat{x} + q\ d_y\ \hat{y} + \vec{r}\,'_{0,0} \tag{5}$$

or,

$$x'_{p,q} = p\ d_x + x'_{0,0} \qquad y'_{p,q} = q\ d_y + y'_{0,0} \tag{6a,b}$$

The phase of a current element at $\vec{r}\,'_{p,q}$ differs from that of a current element at $\vec{r}\,'_{0,0}$ by $e^{jk_1(u\,p\,d_x + v\,q\,d_y)}$. The Green function for field on the (0,0) patch due to current on the p,qth patch is then

$$g(\vec{r} \mid \vec{r}\,'_{p,q}) = e^{jk_1(u\,p\,d_x + v\,q\,d_y)} \iint_{-\infty}^{\infty} f(k_x, k_y, z, z')\, e^{jk_x(x - p\,d_x - x')}\, e^{jk_y(y - q\,d_y - y')}\, dk_x dk_y \quad (7)$$

where $x' = x'_{oo}$, $y' = y'_{oo}$. The total Green function due to all the elements is then

$$g_t(\vec{r} \mid \vec{r}\,') = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} e^{jk_1(u\,p\,d_x + v\,q\,d_y)}$$

$$\iint_{-\infty}^{\infty} f(k_x, k_y, z, z')\, e^{jk_x(x - p\,d_x - x')}\, e^{jk_y(y - q\,d_y - y')}\, dk_x dk_y \quad (8)$$

or,

$$g_t(\vec{r} \mid \vec{r}\,') = \sum_{q=-\infty}^{\infty} e^{jk_1 v\,q\,d_y} \int_{-\infty}^{\infty} dk_y\, e^{jk_y(y - q\,d_y - y')}$$

$$\left\{ \sum_{p=-\infty}^{\infty} e^{jk_1 u\,p\,d_x} \int_{-\infty}^{\infty} dk_x\, f(k_x, k_y, z, z')\, e^{jk_x(x - p\,d_x - x')} \right\} \quad (9)$$

Since the choice of the (0,0) patch is arbitrary, the above Green function will hold for any $\vec{r}$ and $\vec{r}\,'$.

Each summation and corresponding integration for both x and y can be converted into solely a summation using the method of Poisson summation [14]. Consider first the bracketted term. Define

$$S_x \equiv \sum_{p=-\infty}^{\infty} e^{jk_1 u\,p\,d_x} \int_{-\infty}^{\infty} dk_x\, f(k_x, k_y, z, z')\, e^{jk_x(x - p\,d_x - x')}$$

$$= \sum_{p=-\infty}^{\infty} e^{jk_1 u\,p\,d_x} \int_{-\infty}^{\infty} dk_x\, \tilde{f}_x(k_x, k_y, z, z')\, e^{-jk_x\,p\,d_x} \quad (10)$$

where

$$\tilde{f}_x(k_x, k_y, z, z') \equiv f(k_x, k_y, z, z')\, e^{jk_x(x - x')} \quad (11)$$

Recalling standard Fourier transform theory,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad (12)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \tag{13}$$

Making the correspondance $k_x \rightarrow t$ and $p\ d_x \rightarrow \omega$,

$$\bar{F}_x(p\ d_x, k_y, z, z') = \int_{-\infty}^{\infty} dk_x\, \bar{f}_x(k_x, k_y, z, z')\, e^{-jk_x p\ d_x} \tag{14}$$

where

$$\bar{F}_x(\omega, k_y, z, z') = \int_{-\infty}^{\infty} \bar{f}_x(t, k_y, z, z')\, e^{-j\omega t}\, dt \tag{15}$$

Then

$$S_x = \sum_{p=-\infty}^{\infty} e^{jk_1 u\, p\, d_x}\, \bar{F}_x(p\ d_x, k_y, z, z') \tag{16}$$

The method of Poisson summation is now applied. For any function $f(t)$ and its transform $F(\omega)$, [15],

$$\sum_{p=-\infty}^{\infty} f(t + p\ T) = \frac{1}{T} \sum_{p=-\infty}^{\infty} e^{jp\ \omega_o t}\, F(p\ \omega_o) \tag{17}$$

where

$$T = \frac{2\pi}{\omega_o} \tag{18}$$

making the correspondance $d_x \rightarrow \omega_o$ and $k_1\ u \rightarrow t$, equation (16) becomes

$$S_x = \frac{2\pi}{d_x} \sum_{p=-\infty}^{\infty} \bar{f}_x(k_1\ u + p\ \frac{2\pi}{d_x}, k_y, z, z') \tag{19}$$

From (11), this becomes

$$S_x = \frac{2\pi}{d_x} \sum_{p=-\infty}^{\infty} f(k_1\ u + p\ \frac{2\pi}{d_x}, k_y, z, z')\, e^{j(k_1 u + p \frac{2\pi}{d_x})(x - x')} \tag{20}$$

Using (20), equation (9) then becomes

$$g_t(\vec{r} \mid \vec{r}\,') = \sum_{q=-\infty}^{\infty} e^{jk_1 v\, q\, d_y} \int_{-\infty}^{\infty} dk_y\, e^{jk_y(y - q\, d_y - y')}$$

$$\left\{\frac{2\pi}{d_x} \sum_{p=-\infty}^{\infty} f(k_1 u + p \frac{2\pi}{d_x}, k_y, z, z') \, e^{j(k_1 u + p \frac{2\pi}{d_x})(x - x')}\right\} \tag{21}$$

Bringing the final summation through the integration and first summation, this becomes

$$g_t(\vec{r} \mid \vec{r}\,') = \frac{2\pi}{d_x} \sum_{p=-\infty}^{\infty} e^{j(k_1 u + p \frac{2\pi}{d_x})(x - x')}$$

$$\sum_{q=-\infty}^{\infty} e^{jk_1 v q d_y} \int_{-\infty}^{\infty} dk_y \, e^{jk_y(y - q d_y - y')} f(k_1 u + p \frac{2\pi}{d_x}, k_y, z, z') \tag{22}$$

Now, let

$$S_y = \sum_{q=-\infty}^{\infty} e^{jk_1 v q d_y} \int_{-\infty}^{\infty} dk_y \, f(k_1 u + p \frac{2\pi}{d_x}, k_y, z, z') \, e^{jk_y(y - q d_y - y')}$$

$$= \sum_{q=-\infty}^{\infty} e^{jk_1 v q d_y} \int_{-\infty}^{\infty} dk_y \, \bar{f}_y(k_1 u + p \frac{2\pi}{d_x}, k_y, z, z') \, e^{-jk_y q d_y} \tag{23}$$

where

$$\bar{f}_y(k_1 u + p \frac{2\pi}{d_x}, k_y, z, z') \equiv f(k_1 u + p \frac{2\pi}{d_x}, k_y, z, z') \, e^{jk_y(y - y')} \tag{24}$$

Equation (23) is now in the same form as (10), and following a development similar to that for S_x yields an equation which corresponds to equation (20),

$$S_y = \frac{2\pi}{d_y} \sum_{q=-\infty}^{\infty} f(k_1 u + p \frac{2\pi}{d_x}, k_1 v + q \frac{2\pi}{d_y}, z, z') \, e^{j(k_1 v + p \frac{2\pi}{d_y})(y - y')} \tag{25}$$

Substituting this back into equation (22) yields

$$g_t(\vec{r} \mid \vec{r}\,') = \frac{2\pi}{d_x} \left[\sum_{p=-\infty}^{\infty} e^{j(k_1 u + p \frac{2\pi}{d_x})(x - x')} \right] \times$$

$$\times \frac{2\pi}{d_y} \sum_{q=-\infty}^{\infty} f(k_1 u + p \frac{2\pi}{d_x}, k_1 v + q \frac{2\pi}{d_y}, z, z') \, e^{j(k_1 v + p \frac{2\pi}{d_y})(y - y')} \tag{26}$$

Rearranging the summations gives

$$g_t(\vec{r} \mid \vec{r}\,') = \frac{(2\pi)^2}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty}$$

$$f\left(k_1 u + p\frac{2\pi}{d_x},\, k_1 v + q\frac{2\pi}{d_y},\, z,\, z'\right) e^{j(k_1 u + p\frac{2\pi}{d_x})(x - x')}\, e^{j(k_1 v + p\frac{2\pi}{d_y})(y - y')} \tag{27}$$

If a single element Green dyadic component is of the form of equation (4), the corresponding Green dyadic component for an infinite array driven by a plane wave is of the form of equation (27). By inspection, a simple prescription for obtaining the infinite array Green function from the single element Green function is

$$\iint_{-\infty}^{\infty} dk_x\, dk_y \;\longrightarrow\; \frac{(2\pi)^2}{d_x\, d_y} \sum_{p,q=-\infty}^{\infty} \tag{28}$$

and

$$k_x \longrightarrow k_1 u + p\frac{2\pi}{d_x} \qquad\qquad k_y \longrightarrow k_1 u + p\frac{2\pi}{d_y} \tag{29a,b}$$

From the summary of the previous section, the Green functions for the electric field which will be needed are

$$g_{xx}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}\, d^2k \tag{30}$$

$$g_{xy}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{-k_x k_y \left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]}\, d^2k \tag{31}$$

$$g_{yx}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) = g_{xy}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) \tag{32}$$

$$g_{yy}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \frac{e^{-j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{(k_1^2 \varepsilon_r - k_y^2) p_1 + (k_1^2 - k_y^2) p_2 \tanh(p_2 d)}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \, d^2k \tag{33}$$

$$g_{zx}^{2,1}(\vec r \mid \vec r\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_1} \, e^{-p_1 z'} \frac{jk_x \cosh(p_2(d+z))}{T_m} d^2k \tag{34}$$

$$g_{zy}^{2,1}(\vec r \mid \vec r\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_1} \, e^{-p_1 z'} \frac{jk_y \cosh(p_2(d+z))}{T_m} d^2k \tag{35}$$

$$g_{xz}^{1,2}(\vec r \mid \vec r\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} d^2k \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_1} (-jk_x) e^{-p_1 z} \frac{\cosh(p_2(z'+d))}{T_m} \tag{36}$$

$$g_{yz}^{1,2}(\vec r \mid \vec r\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} d^2k \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_1} (-jk_y) e^{-p_1 z} \frac{\cosh(p_2(z'+d))}{T_m} \tag{37}$$

$$g_{zz}^{2,2}(\vec r \mid \vec r\,') = \frac{1}{(2\pi)^2} \int\!\!\int_{-\infty}^{\infty} d^2k \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_2} \Bigg\{ -\delta(z - z')$$

$$+ \left[\frac{k_2^2 + p_2^2}{p_2}\right] \frac{\cosh(p_2(z^< + d))}{T_m} \left[\frac{p_2}{p_1} \cosh(p_2 z^>) - \varepsilon_r \sinh(p_2 z^>)\right] \Bigg\} d^2k \tag{38}$$

where

$$z^> = \max(z, z') \qquad z^< = \min(z, z') \tag{39a,b}$$

Letting

$$\vec k = (k_1 u + p \frac{2\pi}{d_x}) \hat x + (k_1 v + q \frac{2\pi}{d_y}) \hat y \tag{40}$$

and using (28), the above Green functions become, for an infinite array,

$$g_{xx}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec k) = \frac{1}{d_x d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{(k_1^2 \varepsilon_r - k_x^2) p_1 + (k_1^2 - k_x^2) p_2 \tanh(p_2 d)}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \tag{41}$$

$$g_{xy}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec k) = \frac{1}{d_x d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec k \cdot (\vec r - \vec r\,')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{-k_x k_y \left[p_1 + p_2 \tanh(p_2 d)\right]}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \tag{42}$$

$$g_{yx}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) = g_{xy}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) \tag{43}$$

$$g_{yy}^{1,1}(z=0, x, y \mid z'=0, x', y', \vec{k}) = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1} \times$$

$$\times \frac{(k_1^2 \varepsilon_r - k_y^2) p_1 + (k_1^2 - k_y^2) p_2 \tanh(p_2 d)}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \tag{44}$$

$$g_{zx}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1}\, e^{-p_1 z'} \frac{jk_x \cosh(p_2(d+z))}{T_m} \tag{45}$$

$$g_{zy}^{2,1}(\vec{r} \mid \vec{r}\,') = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1}\, e^{-p_1 z'} \frac{jk_y \cosh(p_2(d+z))}{T_m} \tag{46}$$

$$g_{xz}^{1,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1} (-jk_x) e^{-p_1 z} \frac{\cosh(p_2(z'+d))}{T_m} \tag{47}$$

$$g_{yz}^{1,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_1} (-jk_y) e^{-p_1 z} \frac{\cosh(p_2(z'+d))}{T_m} \tag{48}$$

$$g_{zz}^{2,2}(\vec{r} \mid \vec{r}\,') = \frac{1}{d_x\, d_y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{e^{j\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{j\omega\varepsilon_2} \Bigg\{ -\delta(z - z') +$$

$$+ \left[\frac{k_2^2 + p_2^2}{p_2}\right] \frac{\cosh(p_2(z^< + d))}{T_m} \left[\frac{p_2}{p_1}\cosh(p_2 z^>) - \varepsilon_r \sinh(p_2 z^>)\right] \Bigg\} d^2k \tag{49}$$

Making the following definitions

$$g_{xx}^{\infty}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{(k_1^2 \varepsilon_r - k_x^2) p_1 + (k_1^2 - k_x^2) p_2 \tanh(p_2 d)}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \tag{50}$$

$$g_{xy}^{\infty}(\vec{k}) \equiv g_{yx}^{\infty}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{-k_x k_y \left[p_1 + p_2 \tanh(p_2 d)\right]}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \tag{51}$$

$$g_{yy}^{\infty}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1 d_x d_y} \frac{(k_1^2 \varepsilon_r - k_y^2) p_1 + (k_1^2 - k_y^2) p_2 \tanh(p_2 d)}{\left[p_1 + p_2 \coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2 \tanh(p_2 d)\right]} \tag{52}$$

$$g_{zx}^{\infty}(\vec{k}) = \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{jk_x}{T_m} \tag{53}$$

$$g_{zy}^{\infty}(\vec{k}) = \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{jk_y}{T_m} \tag{54}$$

$$g_{xz}^{\infty}(\vec{k}) = \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{-jk_x}{T_m} \tag{55}$$

$$g_{yz}^{\infty}(\vec{k}) = \frac{1}{j\omega\epsilon_1 d_x d_y}\, e^{-p_1 z'} \frac{-jk_y}{T_m} \tag{56}$$

the infinite array Green functions involving horizontal components are written compactly as

$$g_{\alpha\beta}^{11} = \sum_{p,q=-\infty}^{\infty} g_{\alpha\beta}^{\infty}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \qquad \alpha = x,y \qquad \beta = x,y \tag{57}$$

$$g_{z\beta}^{21} = \sum_{p,q=-\infty}^{\infty} g_{z\beta}^{\infty}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z+d)) \qquad \beta = x,y \tag{58}$$

$$g_{\alpha z}^{12} = \sum_{p,q=-\infty}^{\infty} g_{\alpha z}^{\infty}(\vec{k})\, e^{j\vec{k}\cdot(\vec{r}-\vec{r}')} \cosh(p_2(z'+d)) \qquad \alpha = x,y \tag{59}$$

14. Current Distribution Derivations.

14.1 Patch Current Distributions.

An arbitrary surface current on the patch is to be modeled as a summation over a set of current distributions which are independant and which form a complete set. The current distributions used are based upon a two dimensional Taylor expansion of two orthogonal current components.

A scalar function $\psi(x,y)$, continuous and with all derivitives continuous over the region of interest, can be represented as a power series in x and y. Let

$$A^{o} \equiv \psi(0,0) \tag{1a}$$

$$A^{x} \equiv \left.\frac{\partial \psi}{\partial x}\right|_{x=y=0} \qquad A^{y} \equiv \left.\frac{\partial \psi}{\partial y}\right|_{x=y=0} \tag{1b,c}$$

$$A^{x^2} \equiv \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{x=y=0} \qquad A^{y^2} \equiv \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{x=y=0} \tag{1d,e}$$

$$A^{xy} \equiv \left.\frac{\partial^2 \psi}{\partial x \partial y}\right|_{x=y=0} \tag{1f}$$

Since $\psi(x,y)$ and all of its derivatives are continuous,

$$\left.\frac{\partial^2 \psi}{\partial x \partial y}\right|_{x=y=0} = \left.\frac{\partial^2 \psi}{\partial y \partial x}\right|_{x=y=0} \tag{2}$$

In general then,

$$A^{x^n y^p} \equiv \left.\frac{\partial^{n+p} \psi}{\partial x^n \partial y^p}\right|_{x=y=0} \tag{3}$$

where the order of differentiation with respect to x and y makes no difference, again because $\psi(x,y)$ and all its derivitives are continuous.

The power series expansion of $\psi(x,y)$ is then

$$\psi(x,y) = A^{o} + A^{x}x + A^{y}y + A^{x^2}x^2 + A^{xy}xy + A^{y^2}y^2 + ... + A^{x^n y^p}x^n y^p + ... \tag{4}$$

Converting this to polar form using

$$x = r\cos(\phi) \tag{5a}$$

and

$$y = r\sin(\phi), \tag{5b}$$

$\psi(x,y)$ becomes

$$\psi(r,\phi) = A^{o} + A^{x} r\cos(\phi) + A^{y} r\sin(\phi) + A^{x^2} r^2\cos^2(\phi) + A^{xy} r^2\sin(\phi)\cos(\phi) + A^{y^2} r^2\sin^2(\phi)$$

$$+ \ldots + A^{x^n y^p} r^{n+p}\cos^n(\phi)\sin^p(\phi) + \ldots \tag{6}$$

$\psi(r,\phi)$ is to be represented as a summation over terms of the form

$$B^{ml}\, r^m\, e^{jl\phi} \tag{7}$$

where m is a non-negative integer and l is any integer. B^{ml} is the amplitude coefficient and is in general complex. Then

$$\psi(r,\phi) = \sum_{m=0}^{\infty} \sum_{l=-\infty}^{\infty} B^{ml} r^m e^{jl\phi} \tag{8}$$

Many of the terms in (8) are not needed to represent $\psi(r,\phi)$, and so the corresponding coefficients are zero. By comparing the two expressions for $\psi(r,\phi)$, (6) and (8), the coefficients B^{ml} which are zero can be determined.

Since the various powers of r within each representation of $\psi(r,\phi)$ are independent radial functions, equating expressions (6) and (8) yields independent equations for each power of r. Thus,

$$A^{o} = \sum_{l=-\infty}^{\infty} B^{0l}\, e^{jl\phi} \tag{9}$$

$$A^{x} r\cos(\phi) + A^{y} r\sin(\phi) = \sum_{l=-\infty}^{\infty} B^{1l}\, r\, e^{jl\phi} \tag{10}$$

$$A^{x^2} r^2\cos^2(\phi) + A^{xy} r^2\sin(\phi)\cos(\phi) + A^{y^2}\sin^2(\phi) = \sum_{l=-\infty}^{\infty} B^{2l}\, r^2\, e^{jl\phi} \tag{11}$$

and in general,

$$\sum_{n=0}^{m} A^{x^n y^{m-n}} \cos^n(\phi) \sin^{m-n}(\phi) r^m = \sum_{l=-\infty}^{\infty} B^{ml}\, r^m\, e^{jl\phi} \tag{12}$$

Further simplifications are possible. In each of (9) through (12), the summation over l needn't go from minus to plus infinity. By inspection, (9) can be reduced to

$$B^{0,0} = A^0 \;; \qquad B^{0,l} = 0 \quad l \neq 0 \tag{13}$$

following from the orthogonality of the exponential terms. The left hand side of (10) can be rewritten as

$$\frac{A^x - jA^y}{2}\,(\cos(\phi) + j\sin(\phi))r + \frac{A^x + jA^y}{2}\,(\cos(\phi) - j\sin(\phi))r \tag{14}$$

Using Euler's identity, (14) becomes

$$\frac{A^x - jA^y}{2}\, e^{j\phi} r + \frac{A^x + jA^y}{2} e^{-j\phi} r = \sum_{l=-\infty}^{\infty} B^{1l}\, r\, e^{jl\phi} \tag{15}$$

By comparing exponential terms,

$$B^{1,1} = \frac{A^x - jA^y}{2} \qquad\qquad B^{1,-1} = \frac{A^x + jA^y}{2} \tag{16a,b}$$

$$B^{1,l} = 0 \quad l \neq \pm 1 \tag{16c}$$

Continuing with (3.5.11), the sin and cos products may be written as

$$\cos^2(\phi) = \frac{\cos(2\phi)+1}{2} \tag{17a}$$

$$\cos(\phi)\sin(\phi) = \frac{\sin(2\phi)}{2} \tag{17b}$$

$$\sin^2(\phi) = \frac{1-\cos(2\phi)}{2} \tag{17c}$$

By inspection, when these are written as a sum of exponentials, $e^{jl\phi}$, l must be 0, 2, or -2. Thus

$$B^{2,l} \begin{Bmatrix} \neq 0 & l = 0, \pm 2 \\ = 0 & \text{else} \end{Bmatrix} \tag{18}$$

In general, the coefficients which are non-zero are

$$B^{m,l} \begin{cases} \neq 0 & l = -m, -m+2, \ldots, m-2, m \\ = 0 & \text{else} \end{cases} \tag{19}$$

$\psi(r,\phi)$ then becomes

$$\psi(r,\phi) = \sum_{m=0}^{\infty} \sum_{l=-m}^{m} B^{m,l}\, r^m\, e^{jl\phi} \qquad l+m \text{ even} \tag{20}$$

Let $\vec{K}(r,\phi)$ be the current which is to be modeled on the patch. This can be represented using two scalar components, $K_x(r,\phi)$ and $K_y(r,\phi)$,

$$\vec{K}(r,\phi) = K_x(r,\phi)\hat{x} + K_y(r,\phi)\hat{y}. \tag{21}$$

The current components $K_x(r,\phi)$ and $K_y(r,\phi)$ will both be continuous over the surface of the patch. Both can then be represented as a sum of the form of equation (20). It is desired to obtain $\vec{K}(r,\phi)$ in the form

$$\vec{K}(r,\phi) = K_r(r,\phi)\hat{r} + K_\phi(r,\phi)\hat{\phi}. \tag{22}$$

The two components above, however, can't be represented using (20) since they are both discontinuous at the origin due to the discontinuity there of $\hat{r}$ and $\hat{\phi}$ in the polar coordinate system. Thus, $K_r(r,\phi)$ and $K_\phi(r,\phi)$ must be obtained via rectangular components. Conversion from rectangular to polar components is made easier through the use of a third set of current components. Define

$$\hat{p} \equiv \hat{x} + j\hat{y} \quad ; \qquad \hat{q} \equiv \hat{x} - j\hat{y} \tag{23a,b}$$

then

$$\vec{K}(r,\phi) = K_p(r,\phi)\hat{p} + K_q(r,\phi)\hat{q} \tag{24}$$

where

$$K_p(r,\phi) = \frac{K_x(r,\phi) - jK_y(r,\phi)}{2} \tag{25}$$

$$K_q(r,\phi) = \frac{K_x(r,\phi) + jK_y(r,\phi)}{2} \tag{26}$$

The components $K_p(r,\phi)$ and $K_q(r,\phi)$ are continuous and can be represented using (20). Doing this, writing the summation in long form,

$$K_p(r,\phi) = P^{0,0} + P^{1,1}\, re^{j\phi} + P^{1,-1}\, re^{-j\phi} + P^{0,2}\, r^2 + P^{2,2}\, r^2e^{e^{2j\phi}} + P^{2,-2}\, r^2e^{-2j\phi} + \cdots \quad (27)$$

$$K_q(r,\phi) = Q^{0,0} + Q^{1,1}\, re^{j\phi} + Q^{1,-1}\, re^{-j\phi} + Q^{0,2}\, r^2 + Q^{2,2}\, r^2e^{e^{2j\phi}} + Q^{2,-2}\, r^2e^{-2j\phi} + \cdots \quad (28)$$

where $P^{m,l}$ and $Q^{m,l}$ are the amplitude coeficients corresponding to $B^{m,l}$ in (20). Clearly, the current components in (22) can be found as

$$K_r(r,\phi) = \vec{K}(r,\phi)\cdot\hat{r} = K_p(r,\phi)\,\hat{p}\cdot\hat{r} + K_q(r,\phi)\,\hat{q}\cdot\hat{r} \quad (29)$$

$$K_\phi(r,\phi) = \vec{K}(r,\phi)\cdot\hat{\phi} = K_p(r,\phi)\,\hat{p}\cdot\hat{\phi} + K_q(r,\phi)\,\hat{q}\cdot\hat{\phi} \quad (30)$$

From the definition of $\hat{p}$ and $\hat{q}$, equations (23a,b),

$$\hat{p}\cdot\hat{r} = \cos(\phi) + j\sin(\phi) = e^{j\phi} \quad (31a)$$

$$\hat{p}\cdot\hat{\phi} = -\sin(\phi) + j\cos(\phi) = je^{j\phi} \quad (31b)$$

$$\hat{p}\cdot\hat{r} = \cos(\phi) - j\sin(\phi) = e^{-j\phi} \quad (31c)$$

$$\hat{p}\cdot\hat{r} = -\sin(\phi) - j\cos(\phi) = -je^{-j\phi} \quad (31d)$$

The simple exponentials obtained above are the sole reason for using K_p and K_q rather than K_x and K_y, which give $\sin(\phi)$ and $\cos(\phi)$. Substituting equations (31) into (29) and (30),

$$K_r(r,\phi) = K_p(r,\phi)\, e^{j\phi} + K_q(r,\phi)\, e^{-j\phi} \quad (32)$$

$$K_\phi(r,\phi) = jK_p(r,\phi)\, e^{j\phi} - jK_q(r,\phi)\, e^{-j\phi} \quad (33)$$

Substituting the expressions for K_p and K_q into the above,

$$\begin{aligned} K_r(r,\phi) = {} & P^{0,0}\, e^{j\phi} + Q^{0,0}\, e^{-j\phi} + P^{1,1}\, r\, e^{2j\phi} + P^{1,-1}\, r + Q^{1,1}\, r + Q^{1,-1}\, r\, e^{-2j\phi} \\ & + P^{2,0}\, r^2\, e^{j\phi} + P^{2,2}\, r^2\, e^{3j\phi} + P^{2,-2}\, r^2\, e^{-j\phi} \\ & + Q^{2,0}\, r^2\, e^{-j\phi} + Q^{2,2}\, r^2\, e^{j\phi} + Q^{2,-2}\, r^2\, e^{-3j\phi} + \cdots \end{aligned} \quad (34)$$

$$K_\phi(r,\phi) = jP^{0,0}\, e^{j\phi} - jQ^{0,0}\, e^{-j\phi} + jP^{1,1}\, r\, e^{2j\phi} + jP^{1,-1}\, r - jQ^{1,1}\, r - jQ^{1,-1}\, r\, e^{-2j\phi}$$

$$+ jP^{2,0}\, r^2\, e^{j\phi} + jP^{2,2}\, r^2\, e^{3j\phi} + jP^{2,-2}\, r^2\, e^{-j\phi}$$

$$- jQ^{2,0}\, r^2\, e^{-j\phi} - jQ^{2,2}\, r^2\, e^{j\phi} - jQ^{2,-2}\, r^2\, e^{-3j\phi} + \cdots \tag{35}$$

Combining like terms and renaming coeficients,

$$K_r(r,\phi) = R^{0,1}\, e^{j\phi} + R^{0,-1}\, e^{-j\phi} + R^{1,0}\, r + R^{1,2}\, r\, e^{2j\phi} + R^{1,-2}\, r\, e^{-2j\phi}$$

$$+ R^{2,1}\, r^2\, e^{j\phi} + R^{2,-1}\, r^2\, e^{-j\phi} + R^{2,3}\, r^2\, e^{3j\phi} + R^{2,-3}\, r^2\, e^{-3j\phi} + \cdots \tag{36}$$

$$K_\phi(r,\phi) = \Phi^{0,1}\, e^{j\phi} + \Phi^{0,-1}\, e^{-j\phi} + \Phi^{1,0}\, r + \Phi^{1,2}\, r\, e^{2j\phi} + \Phi^{1,-2}\, r\, e^{-2j\phi}$$

$$+ \Phi^{2,1}\, r^2\, e^{j\phi} + \Phi^{2,-1}\, r^2\, e^{-j\phi} + \Phi^{2,3}\, r^2\, e^{3j\phi} + \Phi^{2,-3}\, r^2\, e^{-3j\phi} + \cdots \tag{37}$$

Again, $R^{m,l}$ and $\Phi^{m,l}$ are amplitude coefficients for the various terms in (36) and (37). By comparison of (36) and (37) with (34) and (35),

$$\Phi^{0,1} = jR^{0,1} \qquad \Phi^{0,-1} = -jR^{0,-1} \tag{38a}$$

$$\Phi^{1,2} = jR^{1,2} \qquad \Phi^{1,-2} = -jR^{1,-2} \tag{38b}$$

and generally,

$$\Phi^{n,n+1} = jR^{n,n+1} \qquad \Phi^{n,-n-1} = -jR^{n,-n-1} \tag{38c}$$

From the expansions of K_r and K_ϕ, two facts may be observed. First, for odd powers of r, the azimuthal variation is even, and for even powers of r, the azimuthal variation is odd. That is, both are a summation of terms of the form,

$$r^m\, e^{jl\phi} \tag{39}$$

where $m+l$ is odd. The second observation is that, for a given value of m,

$$|l| - 1 \le m \tag{40}$$

The currents $K_r(r,\phi)$ and $K_\phi(r,\phi)$ then become

$$K_r(r,\phi) = \sum_{m=0}^{\infty} \sum_{l=-m-1}^{m+1} R^{m,l}\, r^m\, e^{jl\phi} \tag{41}$$

$$K_\phi(r,\phi) = \sum_{m=0}^{\infty} \sum_{l=-m-1}^{m+1} \Phi^{m,l}\, r^m\, e^{jl\phi} \tag{42}$$

with $R^{m,l} = 0$ and $\Phi^{m,l} = 0$ if $m+l$ is even.

It is desirable to modify equations (41) and (42) to a better form for the problem at hand. In particular, the radial variation will not be modeled as a power series. First, in place of r^m, the Tchebychef polynomial $T_m(r/b)$, where b is the patch radius, is used. This has a leading term of r^m, but also has terms of r^{m-2}, and r^{m-4} down to a term with r^1 or a constant term, depending on whether m is odd or even. This modification is made so that, after a second modification is made, integrals involving the currents which will be obtained later can be easily evaluated in closed form.

The second modification is a multiplicative factor which forces the known edge conditions, $J_r = 0$ and J_ϕ singular at the edge of the patch. The multiplicative edge factors are,

$$\sqrt{1 - \frac{r^2}{b^2}} \tag{43}$$

for the radial current, and

$$\frac{1}{\sqrt{1 - \frac{r^2}{b^2}}} \tag{44}$$

for the azimuthal currents.

Since the Tchebychef polynomials are even or odd about the origin, and since both multiplicative factors are even about the origin, their product is even or odd about the origin as well. When $K_r(r,\phi)$ and $K_\phi(r,\phi)$ were expressed using powers of r, an even power of r was associated with an odd variation in ϕ. This property is preserved, so that (41) and (42) are replaced with

$$K_r(r,\phi) = \sum_{m=0}^{\infty} \sum_{\substack{l=-\infty \\ m+l \text{ odd}}}^{\infty} R^{m,l}\, T_m(r/b)\, e^{jl\phi} \sqrt{1 - \frac{r^2}{b^2}} \qquad m+l \text{ odd} \tag{45}$$

$$K_\phi(r,\phi) = \sum_{m=0}^{\infty} \sum_{\substack{l=-\infty \\ m+l \text{ odd}}}^{\infty} \Phi^{m,l}\, T_m(r/b)\, e^{jl\phi} \frac{1}{\sqrt{1 - \frac{r^2}{b^2}}} \qquad m+l \text{ odd} \tag{46}$$

Since every polynomial $T_m(r/b)$ contains either a constant term, or a term which varies as a single power of r, the restriction $|l| - 1 \le m$ is no longer valid.

As a further step, the identification is made

$$\vec{K}_{rlm} = \hat{r}\, K_{rlm} = T_m(r/b)\, e^{jl\phi} \sqrt{1 - \frac{r^2}{b^2}}\, \hat{r} \tag{47}$$

$$\vec{K}_{\phi lm} = \hat{\phi}\, K_{\phi lm} = T_m(r/b)\, e^{jl\phi} \frac{1}{\sqrt{1 - \frac{r^2}{b^2}}}\, \hat{\phi} \tag{48}$$

$$C_{rlm} = R^{m,l} \tag{49}$$

$$C_{\phi lm} = \Phi^{m,l} \tag{50}$$

The expression for the total current $\vec{K}(r,\phi)$ on the surface of the patch is then

$$\vec{K}(r,\phi) = \sum_{m=o}^{\infty} \sum_{l=-\infty}^{\infty} \left[C_{rlm}\, \vec{K}_{rlm} + C_{\phi lm}\, \vec{K}_{\phi lm} \right\} \tag{51}$$

where the coefficients C_{rlm} and $C_{\phi lm}$ are to be determined.

14.2 Feed Pin and Singular Current Distributions for Single Patch Antenna.

The current on the feed pin is assumed to be constant over the surface and flowing in the z direction, so the surface current distribution describing this current is chosen as

$$\vec{K}_z = 1\,\hat{z} \qquad\qquad -d \le z \le 0 \tag{1}$$

The assumed current on the feed pin leads to a build-up of charge at the patch-feed pin junction. Physically, the current flowing on the patch from the feed pin will be continuous, with little charge build-up. Mathematically, it is necessary to have a patch surface current distribution which cancels the current discontinuity caused by the feed pin current in order to insure convergent integrals. The singular patch current associated with the feed pin is written as $\vec{K}_s$.

It is convenient to use a feed pin-centered coordinate system to describe the "singular" patch current distribution. This is shown in figure 47. The feed pin is centered at $\vec{r}_o$ relative to the patch coordinate system, which has its origin at the center of the patch. A point located at $\vec{r}$ is located at

$$\vec{r}_p = \vec{r} - \vec{r}_o = r_p\,\hat{r}_p \tag{2}$$

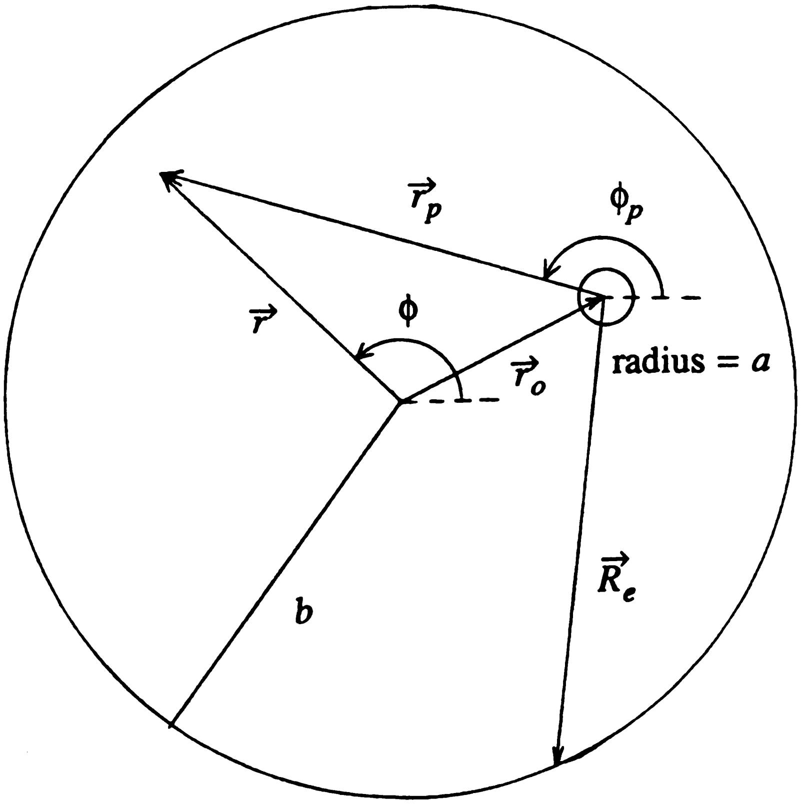
in the feed pin system. It is desired that this current distribution yield evaluable integrals upon the application of Galerkin's method. The integral in question from equation 3.6(30) is

$$\vec{I}_s^{\pm} = \int_0^{2\pi} d\phi_p \int_a^{r_e} dr_p\; r_p\; \vec{K}_s(\vec{r}_p)\; e^{\pm j\vec{k}\cdot(\vec{r}_p + \vec{r}_0)} \tag{3}$$

Let

$$R \equiv \min[\,R_e(\phi_p)\,] = b - |\vec{r}_0| \tag{4}$$

and assume current flowing radially away from the patch-feed pin junction, so that



$$\vec{r}_p = \vec{r} - \vec{r}_o = r_p\,\hat{r}_p \qquad \vec{R}_e = b\hat{r} - \vec{r}_o = R_e\,\hat{r}_p$$

Figure 47. Feed pin-centered coordinate system.

$$\vec{K}_s(\vec{r}_p) = \hat{r}_p\, f(|\vec{r}_p|) = \hat{r}_p\, f(r_p) \tag{5}$$

Since $f(r_p)$ is not a function of ϕ_p, it must be enforced that

$$f(r_p) \equiv 0 \qquad\qquad r_p \geq R \tag{6}$$

Equation (2) can then be written as

$$\vec{I}_s^{\pm} = e^{\pm j\vec{k}\cdot\vec{r}_0} \int_0^{2\pi} d\phi_p\, \hat{r}_p \int_a^R dr_p\, r_p\, f(r_p)\, e^{\pm jkr_p \cos(\phi_p - \theta)} \tag{7}$$

It will be required that $\hat{r}_p\, f(r_p)$ satisfy four conditions, namely that

$$f(a) = 1 \tag{8}$$

$$f(R) = 0 \tag{9}$$

$$\nabla \cdot \hat{r}_p\, f(r_p)\Big|_{r_p = a} = 0 \tag{10}$$

$$\nabla \cdot \hat{r}_p\, f(r_p)\Big|_{r_p = R} = 0 \tag{11}$$

The function $f(r_p)$ will be constructed as a power series in r_p. The dominant behavior in the vicinity of the junction is expected to be $\approx \frac{1}{r_p}$, so the lowest power of r_p used will be -1. There are four conditions to be satisfied, and only odd powers of r_p will lead to closed form evaluable integrals, so construct $f(r_p)$ as

$$f(r_p) = A\,\frac{a}{r_p} + B\,\frac{r_p}{a} + C\left[\frac{r_p}{a}\right]^3 + D\left[\frac{r_p}{a}\right]^5 \tag{12}$$

where A, B, C, D are constants to be determined.

From equation (8),

$$A + B + C + D = 1 \tag{13}$$

and from equation (9),

$$A\,\frac{a}{R} + B\,\frac{R}{a} + C\left[\frac{R}{a}\right]^3 + D\left[\frac{r}{a}\right]^5 = 0. \tag{14}$$

Taking the divergence of $\hat{r} f(r_p)$ gives

$$\nabla \cdot (\hat{r} f(r_p)) = \frac{1}{r_p} \frac{d}{dr_p} (r_p f(r_p)) = \frac{f(r_p)}{r_p} + f'(r_p) \tag{15}$$

Using (9) and (15), (10) gives

$$f'(a) = \frac{-1}{a} \tag{16}$$

while (11) gives

$$f'(R) = 0 \tag{17}$$

From equation (12),

$$f'(r_p) = \frac{-A\ a}{r_p^2} + \frac{B}{a} + \frac{3C\ r_p^2}{a^3} + \frac{5D\ r_p^4}{a^5} \tag{18}$$

so equations (16) and (17) become

$$-A + B + 3C + 5D = -1 \tag{19}$$

$$-A \left[\frac{a}{R}\right]^2 + B + 3C \left[\frac{r}{a}\right]^2 + 5D \left[\frac{r}{a}\right]^4 = 0. \tag{20}$$

Solving (20) for A,

$$A = 1 + B + 3C + 5D \tag{21}$$

Adding (13) to (19) and solving for B gives

$$B = -2C - 3D \tag{22}$$

so (21) becomes

$$A = 1 + C + 2D \tag{23}$$

Using (22) and (23) in (14) and (20) gives

$$(1 + C + 2D) \frac{a}{R} - (2C + 3D) \frac{R}{a} + C \left[\frac{R}{a}\right]^3 + D \left[\frac{R}{a}\right]^5 = 0. \tag{24}$$

and

$$-(1 + C + 2D)\left[\frac{a}{R}\right]^2 - (2C + 3D) + 3C\left[\frac{r}{a}\right]^2 + 5D\left[\frac{r}{a}\right]^4 = 0. \tag{25}$$

Collecting terms, (24) and (25) become

$$\left[1 - 2\left[\frac{R}{a}\right]^2 + \left[\frac{R}{a}\right]^4\right]C + \left[2 - 3\left[\frac{R}{a}\right]^2 + \left[\frac{R}{a}\right]^6\right]D = -1 \tag{26}$$

and

$$\left[-1 - 2\left[\frac{R}{a}\right]^2 + 3\left[\frac{R}{a}\right]^4\right]C + \left[-2 - 3\left[\frac{R}{a}\right]^4 + 5\left[\frac{R}{a}\right]^6\right]D = 1 \tag{27}$$

Adding equations (26) and (27) gives

$$\left[-4\left[\frac{R}{a}\right]^2 + 4\left[\frac{R}{a}\right]^4\right]C + \left[-6\left[\frac{R}{a}\right]^4 + 6\left[\frac{R}{a}\right]^6\right]D = 0 \tag{28}$$

or

$$C = \frac{-3}{2}\left[\frac{R}{a}\right]^2 D \tag{29}$$

Using (29) in equation (26) gives

$$\left[-3\left[\frac{R}{a}\right]^2 + 6\left[\frac{R}{a}\right]^4 - 3\left[\frac{R}{a}\right]^6 + 4 - 6\left[\frac{R}{a}\right]^2 + 2\left[\frac{R}{a}\right]^6\right]D = -1 \tag{30}$$

or

$$D = \frac{1}{\left[\frac{R}{a}\right]^6 - 6\left[\frac{R}{a}\right]^4 + 9\left[\frac{R}{a}\right]^2 - 4} \tag{31}$$

Using (31) in (29),

$$C = \frac{\frac{-3}{2}\left[\frac{R}{a}\right]^2}{\left[\frac{R}{a}\right]^6 - 6\left[\frac{R}{a}\right]^4 + 9\left[\frac{R}{a}\right]^2 - 4} \tag{32}$$

Equation (22) then gives

$$B = \frac{3\left[\frac{R}{a}\right]^2 - 3}{\left[\frac{R}{a}\right]^6 - 6\left[\frac{R}{a}\right]^4 + 9\left[\frac{R}{a}\right]^2 - 4} \tag{33}$$

and (23) gives

$$A = 1 + \frac{2 - \frac{3}{2}\left[\frac{R}{a}\right]^2}{\left[\frac{R}{a}\right]^6 - 6\left[\frac{R}{a}\right]^4 + 9\left[\frac{R}{a}\right]^2 - 4} \tag{34}$$

The four coefficients A, B, C, D are now known. Substituting equation (3.106) into equation (6), $\vec{I}_s^{\pm}$ becomes

$$\vec{I}_s^{\pm} = e^{\pm j\vec{k}\cdot\vec{r}_0}\, d\phi_p\, \hat{r}_p \int_a^R dr_p\, a \left[A + B\left[\frac{r_p}{a}\right]^2 + C\left[\frac{r_p}{a}\right]^4 + D\left[\frac{r_p}{a}\right]^6\right] e^{\pm jkr_p \cos(\phi_p - \theta)} \tag{35}$$

or, interchanging the order of integration,

$$\vec{I}_s^{\pm} = e^{\pm j\vec{k}\cdot\vec{r}_0} \int_a^R dr_p \left[A + B\left[\frac{r_p}{a}\right]^2 + C\left[\frac{r_p}{a}\right]^4 + D\left[\frac{r_p}{a}\right]^6\right] \int_0^{2\pi} a\, d\phi_p\, \hat{r}_p\, e^{\pm jkr_p \cos(\phi_p - \theta)} \tag{36}$$

The unit vector $\hat{r}_p$ can be written as

$$\hat{r}_p = \hat{x}\cos(\phi_p) + \hat{y}\sin(\phi_p) \tag{37}$$

or

$$\hat{r}_p = \hat{x}\left[\cos(\phi_p-\theta)\cos(\theta) + \sin(\phi_p-\theta)\sin(\theta)\right] + \tag{38}$$

$$+\ \hat{y}\left[\cos(\phi_p-\theta)\sin(\theta) - \sin(\phi_p-\theta)\cos(\theta)\right] \tag{39}$$

The terms with $\sin(\phi_p - \theta)$ give zero after integration [16]. Using

$$\int_0^{2\pi} e^{\pm iz\cos(\theta)}\cos(n\theta)\, d\theta = 2\pi(\pm i)^n J_n(z) \tag{40}$$

the terms with $\cos(\phi_p - \theta)$ integrate to give

$$\vec{I}_s^{\pm} = e^{\pm j\vec{k}\cdot\vec{r}_0}\,(\,\hat{x}\cos\theta + \hat{y}\sin\theta\,) \times$$

$$\times \int_a^R dr_p \left[A + B \left[\frac{r_p}{a}\right]^2 + C \left[\frac{r_p}{a}\right]^4 + D \left[\frac{r_p}{a}\right]^6 \right] (\pm j 2\pi a)\, J_1(kr_p) \tag{41}$$

The integrations over r_p of even powers of r_p times $J_1(kr_p)$ now must be obtained. Two useful formulae are, from [16] ((11.3.20) and (11.3.21)),

$$\int_a^b J_1(t)\, dt = -J_0(b) + J_0(a) \tag{42}$$

and

$$\int_a^b t^2 J_1(t)\, dt = b^2 J_2(b) - a^2 J_2(a) \tag{43}$$

Using (42), the term in (41) with coefficient A becomes

$$A \int_a^R J_1(kr_p)\, dr_p = \frac{A\ a}{ka} \left[J_0(ka) - J_0(kR) \right] \tag{44}$$

For the feed pin-patch terms, it is useful to eliminate J_0 in favor of J_1 and J_2 with

$$J_0(t) = \frac{2}{t} J_1(t) - J_2(t) \tag{45}$$

so (44) can also be written as

$$A \int_a^R J_1(kr_p)\, dr_p = \frac{A\ a}{ka} \left[\frac{2}{ka} J_1(ka) - J_2(ka) - \frac{2}{kR} J_1(kR) + J_2(kR) \right] \tag{46}$$

Using (3), the term with coefficient B becomes

$$B \int_a^R \frac{r_p^2}{a^2} J_1(kr_p)\, dr_p = \frac{B\ a}{(ka)^3} \left[k^2 R^2 J_2(kR) - k^2 a^2 J_2(ka) \right] \tag{47}$$

For the feed pin-feed pin term, it is desired to have this in terms of J_0 and J_1. Using (45),

$$B \int_a^R \frac{r_p^2}{a^2} J_1(kr_p)\, dr_p = \frac{B\ a}{(ka)^3} \left[2kR\ J_1(kR) - k^2 R^2 J_0(kR) - 2ka\ J_1(ka) + (ka)^2 J_0(ka) \right] \tag{48}$$

The terms with coefficients C and D will now be obtained via integration by parts.

$$\int_a^b t^4 J_1(t)\,dt = -t^4 J_0(t)\Big|_a^b + \int_a^b 4t^3 J_0(t)\,dt \tag{49}$$

or

$$\int_a^b t^4 J_1(t)\,dt = -t^4 J_0(t)\Big|_a^b + 4\int_a^b t^2 \Big[t\, J_0(t) \Big]\,dt \tag{50}$$

Another useful identity for Bessel functions from [16] (Eq. (9.1.30)) gives

$$t\, J_0(t) = \frac{d}{dt}\Big[t\, J_1(t) \Big] \tag{51}$$

so equation (50) becomes

$$\int_a^b t^4 J_1(t)\,dt = \Big[-t^4 J_0(t) + 4t^2 \,(\, t\, J_1(t)\,) \Big]_a^b - 8\int_a^b t^2 J_1(t)\,dt \tag{52}$$

Using (43) then gives

$$\int_a^b t^4 J_1(t)\,dt = \Big[-t^4 J_0(t) + 4t^3 J_1(t) - 8t^2 J_2(t) \Big]_a^b \tag{53}$$

or, in terms of J_1 and J_2

$$\int_a^b t^4 J_1(t)\,dt = \Big[2t^3 J_1(t) + (t^4 - 8t^2)\, J_2(t) \Big]_a^b \tag{54}$$

In terms of J_0 and J_1, this becomes

$$\int_a^b t^4 J_1(t)\,dt = \Big[(-t^4 + 8t^2)\, J_0(t) + (4t^3 - 16t)\, J_1(t) \Big]_a^b \tag{55}$$

Following essentially the same path,

$$\int_a^b t^6 J_1(t)\,dt = -t^6 J_0(t)\Big|_a^b + 6\int_a^b t^5 J_0(t)\,dt \tag{56}$$

$$\int_a^b t^6 J_1(t)\,dt = \Big[-t^6 J_0(t) + 6t^4 \,(\, t\, J_1(t)\,) \Big]_a^b - 24\int_a^b t^4 J_1(t)\,dt \tag{57}$$

Now equation (55) can be used to obtain

$$\int_a^b t^6 J_1(t)\, dt = \Big[-t^6 J_0(t) + 6t^5 J_1(t) + (24t^4 - 192t^2) J_0(t) - (96t^3 - 384t) J_1(t) \Big]_a^b \quad (58)$$

or, using (45)

$$\int_a^b t^6 J_1(t)\, dt = \Big[(4t^5 - 48t^3) J_1(t) + (t^6 - 24t^4 + 192\, t^2) J_2(t) \Big]_a^b \quad (59)$$

Using (54), the term with coefficient C becomes

$$C \int_a^R \frac{r_p^4}{a^4} J_1(kr_p)\, dr_p = \frac{C\,a}{(ka)^5} \Big[2(kt)^3 J_1(kt) + ((kt)^4 - 8(kt)^2) J_2(kt) \Big]_a^R \quad (60)$$

or, using (55)

$$C \int_a^R \frac{r_p^4}{a^4} J_1(kr_p)\, dr_p = \frac{C\,a}{(ka)^5} \Big[(-(kt)^4 + 8kt^2) J_0k(t) + (4(kt)^3 - 16kt) J_1(kt) \Big]_a^R \quad (61)$$

Using (58), the term with coefficient D becomes

$$D \int_a^R \frac{r_p^6}{a^6} J_1(kr_p)\, dr_p = \frac{D\,a}{(ka)^7} \Big[(-(kt)^6 + 24(kt)^4 - 192(kt)^2) J_0(kt) +$$

$$+ (6(kt)^5 - 96(kt)^3 + 384(kt)) \Big]_a^R \quad (62)$$

or, using (59)

$$D \int_a^R \frac{r_p^6}{a^6} J_1(kr_p)\, dr_p = \frac{D\,a}{(ka)^7} \Big[(4(kt)^5 - 48(kt)^3) J_1(kt) + ((kt)^6 - 24(kt)^4 + 192(kt)^2) J_2(kt) \Big]_a^R \quad (63)$$

Equations (46), (47), (60) and (63) give all the integrations needed. Equation (41) can then be written as

$$\vec{I}_s^{\pm} = \pm j 2\pi a^2 e^{\pm j\vec{k}\cdot\vec{r}_0} (\hat{x} \cos\theta + \hat{y} \sin\theta) \Big[f_1(kt) J_1(kt) + f_2(kt) J_2(kt) \Big]_a^R \quad (64)$$

where, using (46), (47), (60) and (63),

$$f_1(kt) = -\frac{A}{ka} \frac{2}{(kt)} + 2\frac{C}{(ka)^5} (kt)^3 + \frac{D}{(ka)^7} (4(kt)^5 - 48(kt)^3) \quad (65)$$

$$f_2(kt) = \frac{A}{ka} + \frac{B}{(ka)^3} (kt)^2 + \frac{C}{(ka)^5} ((kt)^4 - 8(kt)^2) + \frac{D}{(ka)^7} ((kt)^6 - 24(kt)^4 + 192(kt)^2) \quad (66)$$

Equivalently, (41) can be written as

$$\vec{I}_s^{\pm} = \pm j 2\pi a^2 e^{\pm j\vec{k}\cdot\vec{r}_0} (\hat{x} \cos\theta + \hat{y} \sin\theta) \Big[\bar{f}_0(kt) J_0(kt) + \bar{f}_1(kt) J_1(kt) \Big]_a^R \tag{67}$$

where, using (44), (48), (61) and (62),

$$\bar{f}_0(kt) = -\frac{A}{ka} - \frac{B}{(ka)^3}(kt)^2 - \frac{C}{ka^5}((kt)^4 - 8(kt)^2) - \frac{D}{(ka)^7}((kt)^6 - 24(kt)^4 + 192(kt)^2) \tag{68}$$

$$\bar{f}_1(kt) = \frac{B}{(ka)^3} 2kt + \frac{C}{(ka)^5} (4(kt)^3 - 16kt) + \frac{D}{(ka)^7} (6(kt)^5 - 96(kt)^3 + 384kt) \tag{69}$$

14.3 Feed Pin And Singular Current Distribution for Patch Array.

The current on the feed pin is assumed to be constant over the surface and flowing in the z direction, so the surface current distribution describing this current is chosen as

$$\vec{K}_z = 1\,\hat{z} \qquad\qquad -d \le z \le 0 \tag{1}$$

The assumed current on the feed pin leads to a build-up of charge at the patch-feed pin junction. Physically, the current flowing on the patch from the feed pin will be continuous, with little charge build-up. A current distribution term is developed which has a near singularity at the feed pin junction that matches the known approximate current flow in the vicinity of the junction.

It is convenient to use a feed pin-centered coordinate system to describe the singular patch current distribution, denoted $\vec{K}_s$. This is the same as shown in figure 3 in section 3.6. The feed pin is centered at $\vec{r}_o$ relative to the local patch coordinate system, which has its origin at the center of the patch. A point located at $\vec{r}$ is located at

$$\vec{r}_p = \vec{r} - \vec{r}_o = r_p\,\hat{r}_p \tag{2}$$

in the feed pin system. In particular, the edge of the patch, at $|\vec{r}\,| = b$, is located at

$$\vec{R}_e = b\hat{r} - \vec{r}_o = R_e\,\hat{r}_p \tag{3}$$

From (3), $R_e(\phi_p)$ may be determined via

$$b^2 = |\vec{R}_e + \vec{r}_o\,|^2$$

$$= (R_e\cos(\phi_p) + x_o)^2 + (R_e\sin(\phi_p) + y_o)^2$$

$$= R_e^2\cos^2(\phi_p) + 2R_e x_o\cos(\phi_p) + x_o^2 + R_e^2\sin^2(\phi_p) + 2R_e y_o\sin(\phi_p) + y_o^2 \tag{4}$$

or

$$R_e^2 + 2R_e(x_o\cos(\phi_p) + y_o\sin(\phi_p)) - b^2 + x_o^2 + y_o^2 = 0 \tag{5}$$

so

$$R_e(\phi_p) = -(x_o\cos(\phi_p) + y_o\sin(\phi_p))$$

$$+ \sqrt{(x_o\cos(\phi_p) + y_o\sin(\phi_p))^2 + b^2 - x_o^2 - y_o^2} \quad (6)$$

where the + sign is chosen on the radical since $R_e > 0$.

Five properties are desired in the choice of $\vec{K}_s(\vec{r}_p)$:

1) $|\vec{K}_s(r_p = a)| = |\vec{K}_s|$

2) For r_p small, $\vec{K}_s(\vec{r}_p) \approx \frac{\hat{r}_p}{r_p}$

3) For $\vec{r}_p = \vec{R}_e$, $\vec{K}_s(\vec{r}_p) \cdot \hat{r} = 0$

4) Continuous and smooth for $a < |\vec{r}_p| < R_e$

5) Integrable in closed form over patch.

The first property simply provides that current flowing on the feed pin surface flows entirely onto the patch. The second property guarantees that $\nabla \cdot \vec{K}_s(\vec{r}_p) \approx 0$ for small r_p, which, along with the first property, ensures no charge build-up near the feed pin junction. This form of current is expected physically. The third property enforces the condition that normal current flow is zero at the edge. The fourth is demanded by Maxwell's equations, which are continous and continuously differentiable to all orders.

For the fifth property, the integral in question is, from Eq. 3.6(30),

$$I_{s\beta} = \int_0^{2\pi} d\phi_p \int_a^{R_e} dr_p \; r_p \; \hat{\beta} \cdot \vec{K}_s(\vec{r}_p) \, e^{-j\vec{k} \cdot \vec{r}'} \qquad \beta = x, y \quad (7)$$

where the exponential term comes from the Green function. Closed form integration is desired since it lessens the computation time needed.

The choice of $\vec{K}_s(\vec{r}_p)$ used for the infinite array is

$$\vec{K}_s(\vec{r}_p) = \frac{a \; \hat{r}_p}{r_p} \frac{\cos(\frac{\pi r_p}{2R_e})}{\cos(\frac{\pi a}{2R_e})} \quad (8)$$

Using $ds = r_p\, dr_p\, d\phi_p$ and $\vec{r}\,' = \vec{r}_p + \vec{r}_0$, (7) becomes

$$I_{s\bar{\beta}} = \int_0^{2\pi} d\phi_p\ \hat{r}_p \cdot \beta \frac{1}{\cos\left[\frac{\pi a}{2R_e}\right]} e^{j\vec{k}\cdot\vec{r}_0} \int_a^{R_e} \frac{1}{2\pi} \cos\left[\frac{\pi r_p}{2R_E}\right] e^{-j\vec{k}\cdot\vec{r}_p}\, dr_p \tag{9}$$

Now, define

$$I_{\bar{R}} = \int_a^{R_e} dr_p\ \frac{1}{2\pi} \cos\left[\frac{\pi r_p}{2R_E}\right] e^{-jkr\cos(\phi_p - \theta)}$$

$$= \int_a^{R_e} dr_p\ \frac{1}{4\pi} \left[e^{j\frac{\pi r_p}{2R_e}} + e^{j\frac{\pi r_p}{2R_e}} \right] e^{-jkr\cos(\phi_p - \theta)}$$

$$= \int_a^{R_e} dr_p\ \frac{1}{4\pi} \left[e^{jr\left[-\frac{\pi}{2R_e} - k\cos(\phi_p - \theta)\right]} + e^{jr\left[\frac{\pi}{2R_e} - k\cos(\phi_p - \theta)\right]} \right] \tag{10}$$

where

$$\theta = \tan^{-1}\left[\frac{k_y}{k_x}\right] \tag{11}$$

This is evaluated to obtain

$$I_{\bar{R}} = \frac{-jR_e}{2\pi(\pi - 2R_e k\cos(\phi_p - \theta))} \left[e^{j(\frac{\pi}{2} - kR_e\cos(\phi_p - \theta))} - e^{j(\frac{\pi a}{2R_e} - ka\cos(\phi_p - \theta))} \right]$$

$$- \frac{jR_e}{2\pi(\pi + 2R_e k\cos(\phi_p - \theta))} \left[e^{-j(\frac{\pi}{2} + kR_e\cos(\phi_p - \theta))} - e^{-j(\frac{\pi a}{2R_e} + ka\cos(\phi_p - \theta))} \right] \tag{12}$$

with (12), (7) becomes

$$I_{s\bar{\beta}'} = e^{j\vec{k}\cdot\vec{r}_0} \int_0^{2\pi} d\phi_p\ \hat{r}_p \cdot \hat{\beta}' \frac{1}{\cos\left[\frac{\pi a}{2R_E}\right]} I_{\bar{R}} \tag{13}$$

Since R_e is a complicated function of ϕ_p, if $\vec{K}_s$ to go to zero only at the edge of the patch, the ϕ_p integral will likely be difficult or impossible to do in closed form. Property 5) is thus relaxed to performing the azimuthal integral to be obtained numerically.

For most of the range of ϕ_p, (13) can easily be computed numerically. At $\frac{\pi}{2R_e} = \pm k\cos(\phi_p - \theta)$, I_R^- becomes indeterminate. Near these values of ϕ_p, make the definition

$$z \equiv \frac{\pi b}{2R_e} \pm kb\cos(\phi_p - \theta) \tag{14}$$

Using the minus sign above, the first term of I_R^- becomes

$$\frac{1}{z}\left[e^{jR_e z/b} - e^{jz}\right] \qquad z \text{ small} \tag{15}$$

$$\approx \frac{1}{z}\left[1 + \frac{jR_e z}{b} - \frac{(R_e z)}{2b^2} + ... - 1 - jz + \frac{z^2}{2} + ...\right] \tag{16}$$

$$\approx \left[j(\frac{R_e}{b} - 1) - \frac{z}{2}(\frac{R_e^2}{b^2} - 1) + ...\right] \tag{17}$$

If the plus sign in (14) is taken, the second term from (13) takes on the form shown in (15) through (17) as well. Thus (13) is well behaved for all values of ϕ_p.

VI. MATRIX ELEMENT EVALUATIONS

15. Single Patch Matrix Elements.

15.1 Patch-Patch Matrix Elements.

The matrix elements relating the current distributions on the patch surface to the tangential electric fields on the patch surface are given in 3.6(27) as

$$Z_{\delta l'm'}^{\gamma lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\substack{\alpha=x,y \\ \beta=x,y}} I_{\gamma lm\alpha}^{+}(\vec{k})\, I_{\bar{\delta} l'm'\beta}(\vec{k})\, g_{\alpha\beta}(\vec{k}) \tag{1}$$

where $g_{\alpha\beta}(\vec{k})$ are given by

$$g_{xx}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{(k_1^2\varepsilon_r - k_x^2)p_1 + (k_1^2 - k_x^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{2}$$

$$g_{xy}(\vec{k}) \equiv g_{yx}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{-k_x k_y \left[p_1 + p_2\tanh(p_2 d)\right]}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{3}$$

$$g_{yy}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{(k_1^2\varepsilon_r - k_y^2)p_1 + (k_1^2 - k_y^2)p_2\tanh(p_2 d)}{\left[p_1 + p_2\coth(p_2 d)\right]\left[\varepsilon_r p_1 + p_2\tanh(p_2 d)\right]} \tag{4}$$

Expanding the sum over α and β then yields

$$\begin{aligned} Z_{\delta l'm'}^{\gamma lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k\, & \frac{1}{j\omega\varepsilon_1\,(p_1 + p_2\coth(p_2 d))\,(\varepsilon_r p_1 + p_2\tanh(p_2 d))} \\ \times \Big\{ & \left[(k_1^2\varepsilon_r - k_x^2)\,p_1 + (k_1^2 - k_x^2)\,p_2\tanh(p_2 d)\right] I_{\gamma lmx}^{+}(\vec{k})\, I_{\bar{\delta} l'm'x}(\vec{k}) \\ + & \left[(k_1^2\varepsilon_r - k_y^2)\,p_1 + (k_1^2 - k_y^2)\,p_2\tanh(p_2 d)\right] I_{\gamma lmy}^{+}(\vec{k})\, I_{\bar{\delta} l'm'y}(\vec{k}) \\ - & k_x k_y (p_1 + p_2\tanh(p_2 d)) \left[I_{\gamma lmx}^{+}(\vec{k})\, I_{\bar{\delta} l'm'y}(\vec{k}) + I_{\gamma lmy}^{+}(\vec{k})\, I_{\bar{\delta} l'm'x}(\vec{k})\right] \Big\} \end{aligned} \tag{5}$$

or

$$Z^{\gamma lm}_{\delta l'm'} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} d^2k \; \frac{1}{j\omega\varepsilon_1 \, (p_1 + p_2 \coth(p_2 d)) \, (\varepsilon_r \, p_1 + p_2 \tanh(p_2 d))}$$

$$\times \Bigg\{ \Big[p_1 + p_2 \tanh(p_2 d)\Big] \Big[-k_x^2 \, I^+_{\gamma lmx}(\vec{k}\,) \, I^-_{\delta l'm'x}(\vec{k}\,) - k_y^2 \, I^+_{\gamma lmy}(\vec{k}\,) \, I^-_{\delta l'm'y}(\vec{k}\,) -$$

$$- k_x k_y \, I^+_{\gamma lmx}(\vec{k}\,) \, I^-_{\delta l'm'y}(\vec{k}\,) + k_x k_y \, I^+_{\gamma lmy}(\vec{k}\,) \, I^-_{\delta l'm'x}(\vec{k}\,) \Big]$$

$$- \; k_1^2 \, (\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)) \Big[I^+_{\gamma lmx}(\vec{k}\,) \, I^-_{\delta l'm'x}(\vec{k}\,) + I^+_{\gamma lmy}(\vec{k}\,) \, I^-_{\delta l'm'y}(\vec{k}\,) \Big] \Bigg\} \qquad (6)$$

Equation (6) will now be expanded for each case of δ and γ.

CASE I : $\gamma = r$, $\delta = r$.

expanding $I^{\pm}_{rlmx}$, and $I^{\pm}_{rlmy}$ using equations 16.1(15) and 16.1(20)

$$I^{\pm}_{rlmx} = \pi \, j^l \, e^{jl\theta} \Big[j \, e^{j\theta} \, B^{\pm}_{l+1,m} - j \, e^{-j\theta} \, B^{\pm}_{l-1,m} \Big] \qquad (7)$$

$$I^{\pm}_{rlmy} = \pi \, j^l \, e^{jl\theta} \Big[e^{j\theta} \, B^{\pm}_{l+1,m} + e^{-j\theta} \, B^{\pm}_{l-1,m} \Big] \qquad (8)$$

The four possible products of (7) and (8) are

$$I^+_{rlmx} \, I^-_{rl'm'x} = \pi^2 \, j^{l+l'} \, e^{j(l+l')\theta} \Big\{ -e^{2j\theta} \, B^+_{l+1,m} \, B^-_{l'+1,m'} - e^{-2j\theta} \, B^+_{l-1,m} \, B^-_{l'-1,m'}$$

$$+ B^+_{l+1,m} \, B^-_{l'-1,m'} + B^+_{l-1,m} \, B^-_{l'+1,m'} \Big\} \qquad (9)$$

$$I^+_{rlmy} \, I^-_{rl'm'y} = \pi^2 \, j^{l+l'} \, e^{j(l+l')\theta} \Big\{ e^{2j\theta} \, B^+_{l+1,m} \, B^-_{l'+1,m'} + e^{-2j\theta} \, B^+_{l-1,m} \, B^-_{l'-1,m'}$$

$$+ B^+_{l+1,m} \, B^-_{l'-1,m'} + B^+_{l-1,m} \, B^-_{l'+1,m'} \Big\} \qquad (10)$$

$$I^+_{rlmx} \, I^-_{rl'm'y} = \pi^2 \, j^{l+l'} \, e^{j(l+l')\theta} \Big\{ je^{2j\theta} \, B^+_{l+1,m} \, B^-_{l'+1,m'} - je^{-2j\theta} \, B^+_{l-1,m} \, B^-_{l'-1,m'}$$

$$+ j \, B^+_{l+1,m} \, B^-_{l'-1,m'} - j \, B^+_{l-1,m} \, B^-_{l'+1,m'} \Big\} \qquad (11)$$

$$I^{+}_{rlmy}\, I^{-}_{rl'm'x} = \pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ je^{2j\theta}\, B^{+}_{l+1,m}\, B^{-}_{l'+1,m'} - je^{-2j\theta}\, B^{+}_{l-1,m}\, B^{-}_{l'-1,m'} \right.$$

$$\left. - j\, B^{+}_{l+1,m}\, B^{-}_{l'-1,m'} + j\, B^{+}_{l-1,m}\, B^{-}_{l'+1,m'} \right\} \quad (12)$$

Using (11) and (12),

$$I^{+}_{rlmx}\, I^{-}_{rl'm'y} + I^{+}_{rlmy}\, I^{-}_{rl'm'x} = \pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ 2je^{2j\theta}\, B^{+}_{l+1,m}\, B^{-}_{l'+1,m'} - 2je^{-2j\theta}\, B^{+}_{l-1,m}\, B^{-}_{l'-1,m'} \right\} \quad (13)$$

The terms in (6) with coefficient ($p_1 + p_2 \tanh(p_2 d)$) are

$$-k_x^2\, I^{+}_{rlmx}(\vec{k})\, I^{-}_{rl'm'x}(\vec{k}) - k_y^2\, I^{+}_{rlmy}(\vec{k})\, I^{-}_{rl'm'y}(\vec{k}) -$$

$$- k_x k_y \left[I^{+}_{rlmx}(\vec{k})\, I^{-}_{rl'm'y}(\vec{k}) + I^{+}_{rlmy}(\vec{k})\, I^{-}_{rl'm'x}(\vec{k}) \right] \quad (14)$$

Using equations (9), (10) and (13), these terms become

$$\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ (k_x^2 - k_y^2 - 2jk_x k_y)\, e^{2j\theta}\, B^{+}_{l+1,m}\, B^{-}_{l'+1,m'} + \right.$$

$$+ (k_x^2 - k_y^2 + 2jk_x k_y)\, e^{-2j\theta}\, B^{+}_{l-1,m}\, B^{-}_{l'-1,m'} -$$

$$\left. - (k_x^2 + k_y^2)\, (B^{+}_{l+1,m}\, B^{-}_{l'-1,m'} + B^{+}_{l-1,m}\, B^{-}_{l'+1,m'}) \right\} \quad (15)$$

Noting that

$$k_x = k\, \cos(\theta) \quad (16)$$

and

$$k_x = k\, \cos(\theta) \quad (17)$$

the following relations hold:

$$k_x^2 - k_y^2 + 2jk_x k_y = (k_x + jk_y)^2 = k^2\, e^{2j\theta} \quad (18)$$

$$k_x^2 - k_y^2 - 2jk_x k_y = (k_x - jk_y)^2 = k^2\, e^{-2j\theta} \quad (19)$$

$$k_x^2 + k_y^2 = k^2 \tag{20}$$

The terms in (14) then become

$$\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}\, k^2 \left\{ B^+_{l+1,m}\, B^-_{l'+1,m'} + B^+_{l-1,m}\, B^-_{l'-1,m'} - (\, B^+_{l+1,m}\, B^-_{l'-1,m'} + B^+_{l-1,m}\, B^-_{l'+1,m'}\,) \right\} \tag{21}$$

The terms in (6) with coefficients k_1^2 ($\varepsilon_r\, p_1 + p_2 \tanh(p_2 d)$) are

$$I^+_{rlmx}\, I^-_{rl'm'x} + I^+_{rlmy}\, I^-_{rl'm'y} \tag{22}$$

Using equations (9) and (10), this becomes

$$\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}\, 2 \left[B^+_{l+1,m}\, B^-_{l'-1,m'} + B^+_{l-1,m}\, B^-_{l'+1,m'} \right] \tag{23}$$

Using equations (21) and (23), (6) then becomes, for $\gamma = r$ and $\delta = r$,

$$Z^{rlm}_{rl'm'} = \frac{1}{4\pi^2} \iint d^2k\, \frac{1}{j\omega\varepsilon_1}\, \frac{\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}}{(p_1 + p_2 \coth(p_2\, d))}$$

$$\times \left\{ k_1^2\; 2 \left[B_{l+1,m}\, B_{l'-1,m'} + B_{l-1,m}\, B_{l'+1,m'}\,) \right] + \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)}\, k^2 \right.$$

$$\left. \times \left[B_{l+1,m}\, B_{l'+1,m'} + B_{l-1,m}\, B_{l'-1,m'} - (B_{l+1,m}\, B_{l'-1,m'} + B_{l-1,m}\, B_{l'+1,m'}\,) \right] \right\} \tag{24}$$

Performing the integral with respect to θ in equation (23), the factor $e^{j(l+l')\theta}$ integrates to zero unless $l = -l'$, in which case a factor of 2π is obtained.

$$Z^{rlm}_{rl'm'} = \frac{1}{2\pi} \int_0^{\infty} dk\, \frac{2k}{j\omega\varepsilon_1}\, \frac{\pi^2\, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2\, d))}$$

$$\times \left\{ \left[k_1^2 - \frac{k^2}{2}\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \right] \left[B_{l+1,m}\, B_{-l-1,m'} + B_{l-1,m}\, B_{-l+1,m'} \right] \right.$$

$$\left. + \frac{k^2}{2}\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \left[B_{l+1,m}\, B_{-l+1,m'} + B_{l-1,m}\, B_{-l-1,m'}\,) \right] \right\} \tag{25}$$

CASE II : $\gamma = \phi$, $\delta = \phi$.

expanding $I^{\pm}_{\phi lmx,y}$ using equations 16.1(25) and 16.1(31)

$$I^{\pm}_{\phi lmx} = -\pi\, j^{l}\, e^{jl\theta} \left[e^{j\theta} A^{\pm}_{l+1,m} + e^{-j\theta} A^{\pm}_{l-1,m} \right] \tag{26}$$

$$I^{\pm}_{\phi lmy} = \pi\, j^{l}\, e^{jl\theta} \left[j\, e^{j\theta} A^{\pm}_{l+1,m} - j\, e^{-j\theta} A^{\pm}_{l-1,m} \right] . \tag{27}$$

The four possible products of (26) and (27) are

$$I^{+}_{\phi lmx}\, I^{-}_{\phi l'm'y} = \pi^{2}\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ -je^{2j\theta} A^{+}_{l+1,m} A^{-}_{l'+1,m'} + je^{-2j\theta} A^{+}_{l-1,m} A^{-}_{l'-1,m'} + \right.$$

$$\left. + j\, A^{+}_{l+1,m} A^{-}_{l'-1,m'} - j\, A^{+}_{l-1,m} A^{-}_{l'+1,m'} \right\} \tag{28}$$

$$I^{+}_{\phi lmx}\, I^{-}_{\phi l'm'x} = \pi^{2}\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ e^{2j\theta} A^{+}_{l+1,m} A^{-}_{l'+1,m'} + e^{-2j\theta} A^{+}_{l-1,m} A^{-}_{l'-1,m'} \right.$$

$$\left. + A^{+}_{l+1,m} A^{-}_{l'-1,m'} + A^{+}_{l-1,m} A^{-}_{l'+1,m'} \right\} \tag{29}$$

$$I^{+}_{\phi lmy}\, I^{-}_{\phi l'm'y} = \pi^{2}\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ -e^{2j\theta} A^{+}_{l+1,m} A^{-}_{l'+1,m'} - e^{-2j\theta} A^{+}_{l-1,m} A^{-}_{l'-1,m'} \right.$$

$$\left. + A^{+}_{l+1,m} A^{-}_{l'-1,m'} + A^{+}_{l-1,m} A^{-}_{l'+1,m'} \right\} \tag{30}$$

$$I^{+}_{\phi lmy}\, I^{-}_{\phi l'm'x} = \pi^{2}\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ -je^{2j\theta} A^{+}_{l+1,m} A^{-}_{l'+1,m'} + je^{-2j\theta} A^{+}_{l-1,m} A^{-}_{l'-1,m'} \right.$$

$$\left. - j\, A^{+}_{l+1,m} A^{-}_{l'-1,m'} + j\, A^{+}_{l-1,m} A^{-}_{l'+1,m'} \right\} \tag{31}$$

Using (28) and (31),

$$I^{+}_{\phi lmx}\, I^{-}_{\phi l'm'y} + I^{+}_{\phi lmy}\, I^{-}_{\phi l'm'x} = -2\pi^{2}\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ je^{2j\theta} A^{+}_{l+1,m} A^{-}_{l'+1,m'} - je^{-2j\theta} A^{+}_{l-1,m} A^{-}_{l'-1,m'} \right\} \tag{32}$$

The terms is (6) with coefficient ($p_1 + p_2 \tanh(p_2 d)$) are

$$-k_x^{2}\, I^{+}_{\phi lmx}(\vec{k}\,)\, I^{-}_{\phi l'm'x}(\vec{k}\,) - k_y^{2}\, I^{+}_{\phi lmy}(\vec{k}\,)\, I^{-}_{\phi l'm'y}(\vec{k}\,) -$$

$$- k_x k_y \left[I^+_{\phi lmx}(\vec{k}\,)\, I^-_{\phi l'm'y}(\vec{k}\,) + I^+_{\phi lmy}(\vec{k}\,)\, I^-_{\phi l'm'x}(\vec{k}\,) \right] \qquad (33)$$

Using equations (29), (30) and (33), these terms become

$$-\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \Big\{ (k_x^2 - k_y^2 - 2jk_x k_y\,)\, e^{2j\theta}\, A^+_{l+1,m}\, A^-_{l'+1,m'} +$$

$$+ (k_x^2 - k_y^2 + 2jk_x k_y^2\,)\, e^{-2j\theta}\, A^+_{l-1,m}\, A^-_{l'-1,m'} +$$

$$+ (k_x^2 + k_y\,)\,(\, A^+_{l+1,m}\, A^-_{l'-1,m'} + A^+_{l-1,m}\, A^-_{l'+1,m'}\,) \Big\} \qquad (34)$$

Using (18), (19) and (20), the terms in (34) become

$$-\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}\, k^2 \Big\{ A^+_{l+1,m}\, A^-_{l'+1,m'} + A^+_{l-1,m}\, A^-_{l'-1,m'} + (\, A^+_{l+1,m}\, A^-_{l'-1,m'} + A^+_{l-1,m}\, A^-_{l'+1,m'}\,) \Big\} \qquad (35)$$

The terms with coefficients $k_1^2\,(\,\varepsilon_r\, p_1 + p_2 \tanh(p_2 d)\,)$ are

$$I^+_{\phi lmx}\, I^-_{\phi l'm'x} + I^+_{\phi lmy}\, I^-_{\phi l'm'x} \qquad (36)$$

Using equations (29) and (30), this becomes

$$\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}\, 2 \left[A^+_{l+1,m}\, A^-_{l'-1,m'} + A^+_{l-1,m}\, A^-_{l'+1,m'} \right] \qquad (37)$$

Using equations (35) and (37), (6) becomes, for $\gamma = \phi$ and $\delta = \phi$,

$$Z^{\phi lm}_{\phi l'm'} = \frac{1}{4\pi^2} \iint d^2k\, \frac{1}{j\omega\varepsilon_1}\, \frac{\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}}{(p_1 + p_2 \coth(p_2\, d))}$$

$$\times \Big\{ k_1^2\; 2 \left[A_{l+1,m}\, A_{l'-1,m'} + A_{l-1,m}\, A_{l'+1,m'}\,) \right] - \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)}\, k^2$$

$$\times \left[A_{l+1,m}\, A_{l'+1,m'} + A_{l-1,m}\, A_{l'-1,m'} + (A_{l+1,m}\, A_{l'-1,m'} + A_{l-1,m}\, A_{l'+1,m'}\,) \right] \Big\} \qquad (38)$$

Performing the integral with respect to θ in equation (37) gives the factor $2\pi\, \delta_{l,-l'}$. Thus,

$$Z^{\phi lm}_{\phi l'm'} = \frac{1}{2\pi} \int_0^\infty dk\, \frac{2k}{j\omega\varepsilon_1}\, \frac{\pi^2\, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2\, d))}$$

$$\times \left\{ \left[k_1^2 - \frac{k^2}{2} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \right] \left[A_{l+1,m}\, A_{-l-1,m'} + A_{l-1,m}\, A_{-l+1,m'} \right] \right.$$

$$\left. - \frac{k^2}{2} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \left[A_{l+1,m}\, A_{-l+1,m'} + A_{l-1,m}\, A_{-l-1,m'} \right] \right\} \qquad (39)$$

CASE III : $\gamma = r$, $\delta = \phi$.

Using equations (7), (8), (26) and (27),

$$I^{+}_{rlmx}\, I^{-}_{\phi l'm'y} = \pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ -e^{2j\theta}\, B^{+}_{l+1,m}\, A^{-}_{l'+1,m'} - e^{-2j\theta}\, B^{+}_{l-1,m}\, A^{-}_{l'-1,m'} \right.$$

$$\left. + B^{+}_{l+1,m}\, A^{-}_{l'-1,m'} + B^{+}_{l-1,m}\, A^{-}_{l'+1,m'} \right\} \qquad (40)$$

$$I^{+}_{rlmx}\, I^{-}_{\phi l'm'x} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ je^{2j\theta}\, B^{+}_{l+1,m}\, A^{-}_{l'+1,m'} - je^{-2j\theta}\, B^{+}_{l-1,m}\, A^{-}_{l'-1,m'} \right.$$

$$\left. + j\, B^{+}_{l+1,m}\, A^{-}_{l'-1,m'} - j\, B^{+}_{l-1,m}\, A^{-}_{l'+1,m'} \right\} \qquad (41)$$

$$I^{+}_{rlmy}\, I^{-}_{\phi l'm'y} = \pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ je^{2j\theta}\, B^{+}_{l+1,m}\, A^{-}_{l'+1,m'} - je^{-2j\theta}\, B^{+}_{l-1,m}\, A^{-}_{l'-1,m'} \right.$$

$$\left. - j\, B^{+}_{l+1,m}\, A^{-}_{l'-1,m'} + j\, B^{+}_{l-1,m}\, A^{-}_{l'+1,m'} \right\} \qquad (42)$$

$$I^{+}_{rlmy}\, I^{-}_{\phi l'm'x} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ e^{2j\theta}\, B^{+}_{l+1,m}\, A^{-}_{l'+1,m'} + e^{-2j\theta}\, B^{+}_{l-1,m}\, A^{-}_{l'-1,m'} \right.$$

$$\left. + B^{+}_{l+1,m}\, A^{-}_{l'-1,m'} + B^{+}_{l-1,m}\, A^{-}_{l'+1,m'} \right\} \qquad (43)$$

Using (40) and (43),

$$I^{+}_{rlmx}\, I^{-}_{\phi l'm'y} + I^{+}_{rlmy}\, I^{-}_{\phi l'm'x} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}\, 2 \left\{ e^{2j\theta}\, B^{+}_{l+1,m}\, A^{-}_{l'+1,m'} + e^{-2j\theta}\, B^{+}_{l-1,m}\, A^{-}_{l'-1,m'} \right\} \qquad (44)$$

The terms is (6) with coefficient ($p_1 + p_2 \tanh(p_2 d)$) are

$$-k_x^2\, I^{+}_{rlmx}(\vec{k}\,)\, I^{-}_{\phi l'm'x}(\vec{k}\,) - k_y^2\, I^{+}_{rlmy}(\vec{k}\,)\, I^{-}_{\phi l'm'y}(\vec{k}\,) -$$

$$- k_x k_y \left[I^{+}_{rlmx}(\vec{k}\,) \, I^{-}_{\phi l'm'y}(\vec{k}\,) + I^{+}_{rlmy}(\vec{k}\,) \, I^{-}_{\phi l'm'x}(\vec{k}\,) \right] \tag{45}$$

Using equations (41), (42) and (44), the terms in (45) become

$$\pi^2 \, j^{l+l'} \, e^{j(l+l')\theta} \Big\{ (k_x^2 - k_y^2 - 2jk_x k_y \,) \, j \, e^{2j\theta} \, B^{+}_{l+1,m} \, A^{-}_{l'+1,m'} \, -$$

$$- (k_x^2 - k_y^2 + 2jk_x k_y \,) \, j \, e^{-2j\theta} \, B^{+}_{l-1,m} \, A^{-}_{l'-1,m'} \, +$$

$$+ j \, (k_x^2 + k_y^2 \,) \, (\, B^{+}_{l+1,m} \, A^{-}_{l'-1,m'} - B^{+}_{l-1,m} \, A^{-}_{l'+1,m'} \,) \Big\} \tag{46}$$

Using (18), (19) and (20), the terms in (46) become

$$\pi^2 \, j^{l+l'} \, e^{j(l+l')\theta} \, k^2 \, j \, \Big\{ B^{+}_{l+1,m} \, A^{-}_{l'+1,m'} - B^{+}_{l-1,m} \, A^{-}_{l'-1,m'} + (\, B^{+}_{l+1,m} \, A^{-}_{l'-1,m'} - B^{+}_{l-1,m} \, A^{-}_{l'+1,m'} \,) \Big\} \tag{47}$$

The terms in (6) with coefficients $k_1^2 \; (\, \varepsilon_r \, p_1 + p_2 \tanh(p_2 d) \,)$ are

$$I^{+}_{rlmx} \, I^{-}_{\phi l'm'x} + I^{+}_{rlmy} \, I^{-}_{\phi l'm'x} \tag{48}$$

Using equations (41) and (42), this becomes

$$-\pi^2 \, j^{l+l'} \, e^{j(l+l')\theta} \, 2 \, j \, \left[B^{+}_{l+1,m} \, A^{-}_{l'-1,m'} - B^{+}_{l-1,m} \, A^{-}_{l'+1,m'} \right] \tag{49}$$

Using equations (47) and (49), (6) becomes, for $\gamma = r$ and $\delta = \phi$,

$$Z^{\phi lm}_{\phi l'm'} = \frac{1}{4\pi^2} \iint d^2k \; \frac{1}{j\omega\varepsilon_1} \; \frac{\pi^2 \, j^{l+l'} \, e^{j(l+l')\theta}}{(p_1 + p_2 \coth(p_2 \, d))}$$

$$\times \Big\{ -j \, k_1^2 \, 2 \, \left[B_{l+1,m} \, A_{l'-1,m'} - B_{l-1,m} \, A_{l'+1,m'} \,) \right] + j \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 \, d)} \, k^2$$

$$\times \left[B_{l+1,m} \, A_{l'+1,m'} - B_{l-1,m} \, A_{l'-1,m'} + (B_{l+1,m} \, A_{l'-1,m'} - B_{l-1,m} \, A_{l'+1,m'} \,) \right] \Big\} \tag{50}$$

Performing the integral with respect to θ in equation (50) yieldss the factor $2\pi \, \delta_{l,-l'}$. Thus,

$$Z^{\phi lm}_{rl'm'} = \frac{1}{2\pi} \int_0^{\infty} dk \; \frac{2k}{j\omega\varepsilon_1} \; \frac{\pi^2 \, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2 \, d))}$$

$$\times \left\{ -j \left[k_1^2 - \frac{k^2}{2} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \right] \left[B_{l+1,m}\, A_{-l-1,m'} - B_{l-1,m}\, A_{-l+1,m'} \right] \right.$$

$$\left. + \frac{jk^2}{2} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \left[B_{l+1,m}\, A_{-l+1,m'} - B_{l-1,m}\, A_{-l-1,m'} \right] \right\} \tag{51}$$

CASE IV : $\gamma = \phi,\ \delta = r$.

Using equations (7), (8), (26) and (27),

$$I^+_{\phi lmx}\, I^-_{rl'm'y} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ e^{2j\theta} A^+_{l+1,m}\, B^-_{l'+1,m'} + e^{-2j\theta} A^+_{l-1,m}\, B^-_{l'-1,m'} + A^+_{l+1,m}\, B^-_{l'-1,m'} + A^+_{l-1,m}\, B^-_{l'+1,m'} \right\} \tag{52}$$

$$I^+_{\phi lmx}\, I^-_{rl'm'x} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ je^{2j\theta} A^+_{l+1,m}\, B^-_{l'+1,m'} - j\, e^{-2j\theta} A^+_{l-1,m}\, B^-_{l'-1,m'} - j\, A^+_{l+1,m}\, B^-_{l'-1,m'} + j\, A^+_{l-1,m}\, B^-_{l'+1,m'} \right\} \tag{53}$$

$$I^+_{\phi lmy}\, I^-_{rl'm'y} = \pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ je^{2j\theta} A^+_{l+1,m}\, B^-_{l'+1,m'} - je^{-2j\theta} A^+_{l-1,m}\, B^-_{l'-1,m'} + j\, A^+_{l+1,m}\, B^-_{l'-1,m'} - j\, A^+_{l-1,m}\, B^-_{l'+1,m'} \right\} \tag{54}$$

$$I^+_{\phi lmy}\, I^-_{rl'm'x} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta} \left\{ e^{2j\theta} A^+_{l+1,m}\, B^-_{l'+1,m'} + e^{-2j\theta} A^+_{l-1,m}\, B^-_{l'-1,m'} - A^+_{l+1,m}\, B^-_{l'-1,m'} - A^+_{l-1,m}\, B^-_{l'+1,m'} \right\} \tag{55}$$

Using (52) and (55),

$$I^+_{\phi lmx}\, I^-_{rl'm'y} + I^+_{\phi lmy}\, I^-_{rl'm'x} = -\pi^2\, j^{l+l'}\, e^{j(l+l')\theta}\, 2 \left\{ e^{2j\theta} A^+_{l+1,m}\, B^-_{l'+1,m'} + e^{-2j\theta} A^+_{l-1,m}\, B^-_{l'-1,m'} \right\} \tag{56}$$

The terms is (6) with coefficient $(\, p_1 + p_2 \tanh(p_2 d)\,)$ are

$$-k_x^2\, I^+_{\phi lmx}(\vec{k}\,)\, I^-_{rl'm'x}(\vec{k}\,) - k_y^2\, I^+_{\phi lmy}(\vec{k}\,)\, I^-_{rl'm'y}(\vec{k}\,) -$$

$$- k_x k_y \left[I^+_{\phi lmx}(\vec{k}\,)\, I^-_{rl'm'y}(\vec{k}\,) + I^+_{\phi lmy}(\vec{k}\,)\, I^-_{rl'm'x}(\vec{k}\,) \right] \tag{57}$$

Using equations (53), (54) and (56), these terms become

$$\pi^2 j^{l+l'} e^{j(l+l')\theta} \Big\{ (k_x^2 - k_y^2 - 2jk_xk_y)\, j\, e^{2j\theta} A^+_{l+1,m} B^-_{l'+1,m'} -$$

$$- (k_x^2 - k_y^2 + 2jk_xk_y)\, j\, e^{-2j\theta} A^+_{l-1,m} B^-_{l'-1,m'} -$$

$$- j\, (k_x^2 + k_y^2)\, (A^+_{l+1,m} B^-_{l'-1,m'} - A^+_{l-1,m} B^-_{l'+1,m'}) \Big\} \tag{58}$$

Using (18), (19) and (20), the terms in (58) become

$$\pi^2 j^{l+l'} e^{j(l+l')\theta} k^2 j \Big\{ A^+_{l+1,m} B^-_{l'+1,m'} - A^+_{l-1,m} B^-_{l'-1,m'} - (A^+_{l+1,m} B^-_{l'-1,m'} - A^+_{l-1,m} B^-_{l'+1,m'}) \Big\} \tag{59}$$

The terms with coefficients $k_1^2\ (\varepsilon_r\, p_1 + p_2 \tanh(p_2 d))$ are

$$I^+_{\phi lmx}\, I^-_{rl'm'x} + I^+_{\phi lmy}\, I^-_{rl'm'x} \tag{60}$$

Using equations (53) and (55), this becomes

$$\pi^2 j^{l+l'} e^{j(l+l')\theta}\, 2\, j \left[A^+_{l+1,m} B^-_{l'-1,m'} - A^+_{l-1,m} B^-_{l'+1,m'} \right] \tag{61}$$

Using equations (59) and (61), (6) becomes, for $\gamma = \phi$ and $\delta = r$,

$$Z^{\phi lm}_{\phi l'm'} = \frac{1}{4\pi^2} \iint d^2k\, \frac{1}{j\omega\varepsilon_1}\, \frac{\pi^2 j^{l+l'} e^{j(l+l')\theta}}{(p_1 + p_2 \coth(p_2 d))}$$

$$\times \Big\{ j\, k_1^2\, 2 \left[A_{l+1,m} B_{l'-1,m'} - A_{l-1,m} B_{l'+1,m'}) \right] + j\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2 d)}\, k^2$$

$$\times \left[A_{l+1,m} B_{l'+1,m'} - A_{l-1,m} B_{l'-1,m'} - (A_{l+1,m} B_{l'-1,m'} - A_{l-1,m} B_{l'+1,m'}) \right] \Big\} \tag{62}$$

Performing the integral with respect to θ in equation (62) givess the factor $2\pi\, \delta_{l,-l'}$.

Thus,

$$Z^{rlm}_{\phi l'm'} = \frac{1}{2\pi} \int_0^\infty dk\, \frac{2k}{j\omega\varepsilon_1}\, \frac{\pi^2\, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2 d))}$$

$$\times \left\{ j \left[k_1^2 - \frac{k^2}{2} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \right] \left[A_{l+1,m}\, B_{-l-1,m'} - A_{l-1,m}\, B_{-l+1,m'} \right] \right.$$

$$\left. + j\, \frac{k^2}{2} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2\, d)} \left[A_{l+1,m}\, B_{-l+1,m'} - A_{l-1,m}\, B_{-l-1,m'} \right] \right\} \tag{63}$$

15.2 Patch-feed pin Matrix Elements.

The matrix elements relating current distributions on the patch surface to tangential electric fields on the feed-pin surface, from Eq. 3.6(39), are given by

$$Z^{z}_{\delta l'm'} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \; 2\pi a \; J_0(ka) \, e^{j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2} \sum_{\beta=x,y} g_{z\beta}(\vec{k}) \, I_{\bar{\delta} l' m' \beta} \; (\vec{k}) \tag{1}$$

with, from section 13.11,

$$g_{zx}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{jk_x}{T_m} = \frac{k_x}{\omega\varepsilon_1} \frac{1}{T_m} \tag{2}$$

and

$$g_{zy}(\vec{k}) \equiv \frac{1}{j\omega\varepsilon_1} \frac{jk_y}{T_m} = \frac{k_y}{\omega\varepsilon_1} \frac{1}{T_m} \tag{3}$$

CASE V: $\delta = r$

$$Z^{z}_{r'l'm'} = \frac{a}{(2\pi)\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k \; J_0(ka) \, e^{j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2 \, T_m} \left[k_x \, I_{\bar{r} l' m' x} \, (\vec{k}) + k_y \, I_{\bar{r} l' m' y} \, (\vec{k}) \right] \tag{4}$$

Using 15.1(7) and 15.1(8), this becomes

$$Z^{z}_{r'l'm'} = \frac{a\pi \, j^{l'}}{(2\pi)\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k \; J_0(ka) \, e^{j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2 \, T_m} \, e^{jl'\theta} \times$$

$$\times \left[B_{\bar{l'}+1,m'} \, e^{j\theta} \, (jk_x + k_y) + B_{\bar{l'}-1,m'} \, e^{-j\theta} \, (-jk_x + k_y) \right] \tag{5}$$

Using equations 16.1(15) and 16.1(16),

$$jk_x + k_y = j \; (k_x - jk_y) = jk \; e^{-j\theta} \tag{6}$$

$$-jk_x + k_y = -j \; (k_x + jk_y) = -jk \; e^{j\theta} \tag{7}$$

so (5) becomes

$$Z^{z}_{r'l'm'} = \frac{a\pi \, j^{l'}}{(2\pi)\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k \; J_0(ka) \, e^{jkr_0\cos(\theta-\theta_0)} \frac{\sinh(p_2 d)}{p_2 \, T_m} \, e^{jl'\theta} \, jk \left[B_{\bar{l'}+1,m'} - B_{\bar{l'}-1,m'} \right] \tag{8}$$

Rearranging the integration order,

$$Z^{z}_{r\,l'm'} = \frac{ja\pi\, j^{l'}}{(2\pi)\,\omega\varepsilon_1} \int_0^{\infty} dk\; k^2\, J_0(ka)\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[B_{\bar{l'}+1,m'} - B_{\bar{l'}-1,m'} \right] \int_0^{2\pi} d\theta\; e^{jkr_0\cos(\theta-\theta_0)}\, e^{jl'\theta} \quad (9)$$

A useful identity gives [16]

$$\int_0^{2\pi} d\theta\; e^{\pm jz\cos(\theta-\phi)}\, e^{jn\theta} = 2\pi\, (\pm j)^n\, e^{jn\phi}\, J_n\,(z) \quad (10)$$

Using the identity (10), (9) becomes

$$Z^{z}_{r\,l'm'} = \frac{ja\pi\, (-1)^{l'} e^{jl'\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk\; k^2\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[B_{\bar{l'}+1,m'} - B_{\bar{l'}-1,m'} \right] J_{l'}(kr_0)\, J_0(ka) \quad (11)$$

CASE VI: $\delta = \phi$

$$Z^{z}_{\phi\,l'm'} = \frac{1}{(2\pi)^2\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k\; 2\pi a\; J_0(ka)\, e^{j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2\, T_m} \left[k_x\, I_{\bar{\phi} l'm'x}\,(\vec{k}) + k_y\, I_{\bar{\phi} l'm'y}\,(\vec{k}) \right] \quad (12)$$

Using 16.1(25) and 16.1(26), this becomes

$$Z^{z}_{\phi\,l'm'} = \frac{a\pi\, j^{l'}}{(2\pi)\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k\; J_0(ka)\, e^{j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2\, T_m}\, e^{jl'\theta} \times$$

$$\times \left[A_{\bar{l'}+1,m'}\, e^{j\theta}\, (-k_x + jk_y) + A_{\bar{l'}-1,m'}\, e^{-j\theta}\, (-k_x - jk_y) \right] \quad (13)$$

Using equations (6) and (7), equation (13) becomes

$$Z^{z}_{\phi\,l'm'} = \frac{a\pi\, j^{l'}}{(2\pi)\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k\; J_0(ka)\, e^{jkr_0\cos(\theta-\theta_0)}\, \frac{\sinh(p_2 d)}{p_2\, T_m}\, e^{jl'\theta}\, (-k) \left[A_{\bar{l'}+1,m'} + A_{\bar{l'}-1,m'} \right] \quad (14)$$

Rearranging the integration order to separate out the θ-dependance,

$$Z^{z}_{\phi\,l'm'} = \frac{-a\pi\, j^{l'}}{(2\pi)\,\omega\varepsilon_1} \int_0^{\infty} dk\; k^2\, J_0(ka)\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[A_{\bar{l'}+1,m'} + A_{\bar{l'}-1,m'} \right] \int_0^{2\pi} d\theta\; e^{jkr_0\cos(\theta-\theta_0)}\, e^{jl'\theta} \quad (15)$$

Using the identity (10), (14) becomes

$$Z^{z}_{\phi\,l'm'} = \frac{-a\pi\, (-1)^{l'} e^{jl'\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk\; k^2\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[A_{\bar{l'}+1,m'} + A_{\bar{l'}-1,m'} \right] J_{l'}(kr_0)\, J_0(ka) \quad (16)$$

15.3 Feed pin-patch Matrix Elements.

The matrix elements relating the current distribution on the feed-pin surface along with the associated singular patch current distribution to tangential electric fields on the patch surface are given in 3.6(36) as

$$Z_{t}^{\gamma lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\alpha=x,y} I_{\gamma lm\alpha}^{+} \left[g_{\alpha z}(\vec{k}) \frac{\sinh(p_2 d)}{p_2} 2\pi a\ J_0(ka) + \sum_{\beta=x,y} g_{\alpha\beta}(\vec{k})\, I_{s\beta}^{-} \right] e^{-j\vec{k}\cdot\vec{r}_0} \tag{1}$$

$$Z_{t}^{\gamma lm} = Z_{s'}^{\gamma lm} + Z_{p'}^{\gamma lm} \tag{2}$$

where p' refers to the feed-pin current contribution, and s' refers to the associated singular patch current contribution

CASE VII: $\gamma = r$

Looking first at the contribution due to currents on the feed-pin,

$$Z_{p'}^{rlm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k\ 2\pi a\ J_0(ka)\, e^{-j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2} \sum_{\alpha=x,y} g_{\alpha z}(\vec{k})\, I_{\gamma lm\alpha}^{+}(\vec{k}) \tag{3}$$

with

$$g_{xz}(\vec{k}) \equiv \frac{-k_x}{\omega\epsilon_1 T_m} \tag{4}$$

and

$$g_{yz}(\vec{k}) \equiv \frac{-k_y}{\omega\epsilon_1 T_m} \tag{5}$$

Expanding the sum over α,

$$Z_{p'}^{rlm} = \frac{1}{(2\pi)^2\, \omega\epsilon_1} \iint_{-\infty}^{\infty} d^2k\ 2\pi a\ J_0(ka)\, e^{-j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2 T_m} \left[-k_x I_{rlmx}^{+}(\vec{k}) - k_y I_{rlmy}^{+}(\vec{k}) \right] \tag{6}$$

Using 15.1(7) and 15.1(8), this becomes

$$Z_{p'}^{rlm} = \frac{a\pi j^l}{(2\pi)\, \omega\epsilon_1} \iint_{-\infty}^{\infty} d^2k\ J_0(ka)\, e^{-j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2 T_m} e^{jl\theta} \times$$

$$\times \left[-B^{+}_{l+1,m}\, e^{j\theta}\,(jk_x + k_y) - B^{+}_{l-1,m}\, e^{-j\theta}\,(-jk_x + k_y) \right] \tag{7}$$

Using equations 15.2(6) and 15.2(7), (6) becomes

$$Z_{p'}^{rlm} = \frac{-a\pi\, j^l}{(2\pi\ \omega\varepsilon_1} \iint\limits_{-\infty}^{\infty} d^2k\ J_0(ka)\, e^{-jkr_0\cos(\theta-\theta_0)}\, \frac{\sinh(p_2 d)}{p_2\, T_m}\, e^{jl\theta}\, jk \left[B^{+}_{l+1,m'} - B^{+}_{l-1,m'} \right] \tag{8}$$

Rearranging the integration order,

$$Z_{p'}^{rlm} = \frac{-ja\pi\, j^l}{(2\pi)\ \omega\varepsilon_1} \int\limits_0^{\infty} dk\ k^2\, J_0(ka)\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[B^{+}_{l+1,m} - B^{+}_{l-1,m} \right] \int\limits_0^{2\pi} d\theta\ e^{-jkr_0\cos(\theta-\theta_0)}\, e^{jl\theta} \tag{9}$$

Using the Bessel identity 14.2(10), then gives

$$Z_{p'}^{rlm} = \frac{-ja\pi\, e^{jl\theta_0}}{\omega\varepsilon_1} \int\limits_0^{\infty} dk\ k^2\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[B^{+}_{l+1,m} - B^{+}_{l-1,m} \right] J_l(kr_0)\, J_0(ka) \tag{10}$$

Looking at the associated singular current component,

$$Z_{s'}^{rlm} = \frac{1}{(2\pi)^2} \iint\limits_{-\infty}^{\infty} d^2k \sum_{\substack{\alpha=x,y \\ \beta=x,y}} I^{+}_{rlm\alpha}\,(\vec{k})\, I^{-}_{s\beta}\,(\vec{k})\, g_{\alpha\beta}(\vec{k}) \tag{11}$$

with $g_{\alpha\beta}(\vec{k})$ given in expressions 15.1(2) through 15.1(4).

Expanding equation (11) in the same manner as Eq. 15.1(1),

$$Z_{s'}^{rlm} = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} d^2k\ \frac{1}{j\,\omega\varepsilon_1\,(p_1 + p_2\coth(p_2 d))\,(\varepsilon_r\, p_1 + p_2\tanh(p_2 d))}$$

$$\times \left\{ \left[p_1 + p_2\tanh(p_2 d) \right] \left[-k_x^2\, I^{+}_{rlmx}(\vec{k})\, I^{-}_{sx}(\vec{k}) - k_y^2\, I^{+}_{rlmy}(\vec{k})\, I^{-}_{sy}(\vec{k}) - \right. \right.$$

$$\left. - k_x k_y\, I^{+}_{rlmx}(\vec{k})\, I^{-}_{sy}(\vec{k}) + k_x k_y\, I^{+}_{rlmy}(\vec{k})\, I^{-}_{sx}(\vec{k}) \right]$$

$$\left. - k_1^2\, (\varepsilon_r\, p_1 + p_2\tanh(p_2 d)) \left[I^{+}_{rlmx}(\vec{k})\, I^{-}_{sx}(\vec{k}) + I^{+}_{rlmy}(\vec{k})\, I^{-}_{sy}(\vec{k}) \right] \right\} \tag{12}$$

From equation 14.2(64),

$$I^{-}_{sx}(\vec{k}) = -j\,2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \frac{k_x}{k} \left[J_1(kt)\, f_1\,(kt) + J_2(kt)\, f_2\,(kt) \right]_a^R \tag{13}$$

$$I_{sy}^{-}(\vec{k}) = -j2\pi a^2 e^{-j\vec{k}\cdot\vec{r}_0} \frac{k_y}{k} \left[J_1(kt) f_1(kt) + J_2(kt) f_2(kt) \right]_a^R \tag{14}$$

where R is given by

$$R \equiv b - r_0 \tag{15}$$

The terms in (12) with coefficient ($p_1 + p_2 \tanh(p_2 d)$) are

$$-k_x^2 I_{rlmx}^{+}(\vec{k}) I_{sx}^{-}(\vec{k}) - k_y^2 I_{rlmy}^{+}(\vec{k}) I_{sy}^{-}(\vec{k}) - k_x k_y \left[I_{rlmx}^{+}(\vec{k}) I_{sy}^{-}(\vec{k}) + I_{rlmy}^{+}(\vec{k}) I_{sx}^{-}(\vec{k}) \right] \tag{16}$$

Using equations 15.1(6) and (13),

$$I_{rlmx}^{+}(\vec{k}) I_{sx}^{-}(\vec{k}) = -j2\pi a^2 e^{-j\vec{k}\cdot\vec{r}_0} \pi j^l e^{jl\theta} \frac{k_x}{k} \sum_{i=1,2} J_i(kt) f_i(kt)_a^R \times$$

$$\times \left[je^{j\theta} B_{l+1,m}^{+} - je^{-j\theta} B_{l-1,m}^{+} \right] \tag{17}$$

and using 15.1(7) and (14)

$$I_{rlmy}^{+}(\vec{k}) I_{sy}^{-}(\vec{k}) = -j2\pi a^2 e^{-j\vec{k}\cdot\vec{r}_0} \pi j^l e^{jl\theta} \frac{k_y}{k} \sum_{i=1,2} J_i(kt) P_i(kt)_a^R \left[e^{j\theta} B_{l+1,m}^{+} + e^{-j\theta} B_{l-1,m}^{+} \right] \tag{18}$$

Using 15.1(6), 15.1(7), (13) and (14),

$$I_{rlmx}^{+}(\vec{k}) I_{sy}^{-}(\vec{k}) + I_{rlmy}^{+}(\vec{k}) I_{sx}^{-}(\vec{k}) = 2\pi a^2 e^{-j\vec{k}\cdot\vec{r}_0)} \pi j^l e^{jl\theta} \sum_{i=1,2} J_i(kt) f_i(kt)_a^R$$

$$\left\{ \frac{-jk_y}{k} \left[je^{j\theta} B_{l+1,m}^{+} - je^{-j\theta} B_{l-1,m}^{+} \right] - \frac{jk_x}{k} \left[e^{j\theta} B_{l+1,m}^{+} + e^{-j\theta} B_{l-1,m}^{+} \right] \right\} \tag{19}$$

Rearranging (19) gives

$$I_{rlmx}^{+}(\vec{k}) I_{sy}^{-}(\vec{k}) + I_{rlmy}^{+}(\vec{k}) I_{sx}^{-}(\vec{k}) = 2\pi a^2 e^{-j\vec{k}\cdot\vec{r}_0} \pi j^l e^{jl\theta} \sum_{i=1,2} J_i(kt) f_i(kt)_a^R$$

$$\frac{1}{k} \left\{ (k_y - jk_x) e^{j\theta} B_{l+1,m}^{+} - (k_y + jk_x) e^{-j\theta} B_{l-1,m}^{+} \right\} \tag{20}$$

Using (17), (18) and (20), (16) becomes

$$2\pi a^2 e^{-j\vec{k}\cdot\vec{r}_0} \pi j^l e^{jl\theta} \sum_{i=1,2} J_i(kt) f_i(kt)_a^R$$

$$\left\{ \frac{-k_x}{k}\, k_x^2 \left[e^{j\theta} B_{l+1,m}^{+} - e^{-j\theta} B_{l-1,m}^{+} \right] + \frac{jk_y}{k}\, k_y^2 \left[e^{j\theta} B_{l+1,m}^{+} + e^{-j\theta} B_{l-1,m}^{+} \right] \right.$$

$$\left. - \frac{k_x\, k_y}{k} \left[(k_y - jk_x)\, e^{j\theta} B_{l+1,m}^{+} + (k_y + jk_x)\, e^{-j\theta} B_{l-1,m}^{+} \right] \right\} \tag{21}$$

Rearranging (21),

$$-2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt) \Big|_a^R$$

$$\left\{ \left[k_x^2\, \frac{(k_x - jk_y)}{k} + k_y^2 \frac{(k_x - jk_y)}{k} \right] e^{j\theta} B_{l+1,m}^{+} \right.$$

$$\left. - \left[k_x^2\, \frac{(k_x + jk_y)}{k} + k_y^2 \frac{(k_x + jk_y)}{k} \right] e^{-j\theta} B_{l-1,m}^{+} \right\} \tag{22}$$

Using 15.2(6) and 15.2(7),

$$-2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt) \Big|_a^R \left\{ k^2 B_{l+1,m}^{+} - k^2 B_{l-1,m}^{+} \right\} \tag{23}$$

The terms in (12) with coefficients $k_1^2\ (\varepsilon_r P_1 + p_2 \tanh(p_2 d))$ are

$$I_{rlmx}^{+}(\vec{k}\,)\, I_{sx}^{-}(\vec{k}\,) + I_{rlmy}^{+}(\vec{k}\,)\, I_{sy}^{-}(\vec{k}\,) \tag{24}$$

Using (17) and (18), this becomes

$$I_{rlmx}^{+}(\vec{k}\,)\, I_{sx}^{-}(\vec{k}\,) + I_{rlmy}^{+}(\vec{k}\,)\, I_{sy}^{-}(\vec{k}\,) = -2\pi j a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt) \Big|_a^R$$

$$\left\{ \frac{(k_y + jk_x)}{k}\, e^{j\theta} B_{l+1,m}^{+} + \frac{(k_y - jk_x)}{k}\, e^{-j\theta} B_{l-1,m}^{+} \right\} \tag{25}$$

or, using 15.2(6) and 15.2(7),

$$I_{rlmx}^{+}(\vec{k}\,)\, I_{sx}^{-}(\vec{k}\,) + I_{rlmy}^{+}(\vec{k}\,)\, I_{sy}^{-}(\vec{k}\,) =$$

$$= 2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt) \Big|_a^R \left\{ B_{l+1,m}^{+} - B_{l-1,m}^{+} \right\} \tag{26}$$

Using (23) and (26), (12) becomes

$$Z_{s'}^{rlm} = \frac{-1}{(2\pi)} \iint_{-\infty}^{\infty} d^2k \; \frac{1}{j\;\omega\;\varepsilon_1} \; \frac{a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\; e^{jl\theta}}{(p_1 + p_2 \coth(p_2\, d))}$$

$$\times \left[\sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \right] \left[k^2 \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ B_{l+1,m}^{+} - B_{l-1,m}^{+} \right\} \tag{27}$$

separating out the θ-dependance,

$$Z_{s'}^{rlm} = \frac{-1}{(2\pi)} \int_0^{\infty} k\;dk \; \frac{1}{j\;\omega\;\varepsilon_1} \; \frac{a^2\,\pi}{(p_1 + p_2 \coth(p_2\, d))} \left[\sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \right]$$

$$\times \left[k^2 \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ B_{l+1,m}^{+} - B_{l-1,m}^{+} \right\} \int_0^{2\pi} d\theta\; j^l\; e^{-j\vec{k}\cdot\vec{r}_0}\, e^{jl\theta} \tag{28}$$

Using 15.2(10), this becomes

$$Z_{s'}^{rlm} = \frac{-1}{(2\pi)} \int_0^{\infty} k\;dk \; \frac{1}{j\;\omega\;\varepsilon_1} \; \frac{a^2\,\pi}{(p_1 + p_2 \coth(p_2\, d))} \left[\sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \right] \times$$

$$\times \left[k^2 \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ B_{l+1,m}^{+} - B_{l-1,m}^{+} \right\} 2\pi\, e^{jl\theta_0} J_l(kr_0) \tag{29}$$

CASE VIII: $\gamma = \phi$

Again, looking first at the contribution due to currents on the feed-pin,

$$Z_{p}^{\phi lm} = \frac{1}{(2\pi)^2\, \omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k \;\; 2\pi a J_0(ka)\; e^{j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2} \sum_{\beta=x,y} g_{z\beta}(\vec{k})\, I_{\phi l'm\beta}^{-}\;(\vec{k}) \tag{30}$$

or, with (4) and (5),

$$Z_{p}^{\phi lm} = \frac{a}{(2\pi)\;\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k \;\; J_0(ka)\; e^{-j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2\, T_m} \left[-k_x\; I_{rlmx}^{+}\;(\vec{k}) - k_y\; I_{\phi lmy}^{+}\;(\vec{k}) \right] \tag{31}$$

Using 15.1(25) and 15.1(26), this becomes

$$Z_{p}^{\phi lm} = \frac{a\pi\; j^l}{(2\pi)\;\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k \;\; J_0(ka)\; e^{-j\vec{k}\cdot\vec{r}_0} \frac{\sinh(p_2 d)}{p_2\, T_m}\; e^{jl\theta} \times$$

$$\times \left[A_{l+1,m}^{+}\; e^{j\theta}\,(k_x - jk_y) + A_{l-1,m}^{-}\; e^{-j\theta}\,(k_x + jk_y) \right] \tag{32}$$

Using equations 15.2(6) and 15.2(7), (32) becomes

$$Z_{p}^{\phi lm} = \frac{a\pi\, j^{l}}{(2\pi)\,\omega\varepsilon_1} \iint_{-\infty}^{\infty} d^2k\; J_0(ka)\, e^{-jkr_0\cos(\theta-\theta_0)}\, \frac{\sinh(p_2 d)}{p_2\, T_m}\, e^{jl\theta}\, k \left[A^{+}_{l+1,m'} + A^{+}_{l-1,m'} \right] \tag{33}$$

Rearranging the integration order,

$$Z_{p}^{\phi lm} = \frac{a\pi\, j^{l}}{(2\pi)\,\omega\varepsilon_1} \int_0^{\infty} dk\; k^2\, J_0(ka)\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[A^{+}_{l+1,m} + A^{+}_{l-1,m} \right] \int_0^{2\pi} d\theta\; e^{-jkr_0\cos(\theta-\theta_0)}\, e^{jl\theta} \tag{34}$$

Using the identity 15.2(10), then gives

$$Z_{p}^{\phi lm} = \frac{a\pi\, e^{jl\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk\; k^2\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[A^{+}_{l+1,m} + A^{+}_{l-1,m} \right] J_l(kr_0)\, J_0(ka) \tag{35}$$

Looking at the associated singular current component,

$$Z_{s}^{\phi lm} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \sum_{\substack{\alpha=x,y\\ \beta=x,y}} I^{+}_{\gamma lm\alpha}(\vec{k})\, I^{-}_{s\beta}(\vec{k})\, g_{\alpha\beta}(\vec{k}) \tag{36}$$

with $g_{\alpha\beta}(\vec{k})$ given in expressions (2) through (4).

Expanding equation (36),

$$\begin{aligned} Z_{sl'm'}^{\phi lm} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} d^2k\; & \frac{1}{j\omega\varepsilon_1\,(p_1 + p_2\coth(p_2 d))\,(\varepsilon_r\, p_1 + p_2\tanh(p_2 d))} \\ \times \Bigg\{ & \Big[p_1 + p_2\tanh(p_2 d)\Big] \Big[-k_x^2\, I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sx}(\vec{k}) - k_y^2\, I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sy}(\vec{k}) - \\ & - k_x k_y\, I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sy}(\vec{k}) + k_x k_y\, I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sx}(\vec{k}) \Big] \\ & - k_1^2\, (\varepsilon_r\, p_1 + p_2\tanh(p_2 d)) \Big[I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sx}(\vec{k}) + I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sy}(\vec{k}) \Big] \Bigg\} \end{aligned} \tag{37}$$

The terms in 15.1(6) with coefficient $(p_1 + p_2\tanh(p_2 d))$ are

$$\begin{aligned} & -k_x^2\, I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sx}(\vec{k}) - k_y^2\, I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sy}(\vec{k}) - \\ & \qquad - k_x k_y \Big[I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sy}(\vec{k}) + I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sx}(\vec{k}) \Big] \end{aligned} \tag{38}$$

Using equations 15.1(6) and (7),

$$I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sx}(\vec{k}) = 2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \frac{k_x}{k} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \left[e^{j\theta} A^{+}_{l+1,m} - e^{-j\theta} A^{+}_{l-1,m} \right] \tag{39}$$

and using 15.1(7) and (8)

$$I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sy}(\vec{k}) = -j2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \frac{k_y}{k} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \left[e^{j\theta} A^{+}_{l+1,m} + e^{-j\theta} A^{+}_{l-1,m} \right] \tag{40}$$

Using 15.1(25), 15.1(26), (13) and (14),

$$I^{+}_{\phi lmx}(\vec{k})\, I^{-}_{sy}(\vec{k}) + I^{+}_{\phi lmy}(\vec{k})\, I^{-}_{sx}(\vec{k}) = -j2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R$$

$$\frac{1}{k} \left\{ (-k_y + jk_x)\, e^{j\theta} A^{+}_{l+1,m} + (-k_y - jk_x)\, e^{-j\theta} A^{+}_{l-1,m} \right\} \tag{41}$$

Using (39), (40) and (41), (38) becomes

$$j2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R$$

$$\left\{ \frac{-k_x}{k}\, k_x^2 \left[e^{j\theta} A^{+}_{l+1,m} + e^{-j\theta} A^{+}_{l-1,m} \right] + \frac{jk_y}{k}\, k_y^2 \left[e^{j\theta} A^{+}_{l+1,m} - e^{-j\theta} A^{+}_{l-1,m} \right] \right.$$

$$\left. - \frac{k_x\, k_y}{k} \left[(k_y - jk_x)\, e^{j\theta} A^{+}_{l+1,m} - (k_y + jk_x)\, e^{-j\theta} A^{+}_{l-1,m} \right] \right\} \tag{42}$$

Rearranging (42) gives

$$-2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R$$

$$\left\{ \left[k_x^2\, \frac{(-k_x + jk_y)}{k} + k_y^2 \frac{(-k_x + jk_y)}{k} \right] e^{j\theta} A^{+}_{l+1,m} \right.$$

$$\left. - \left[k_x^2\, \frac{(k_x + jk_y)}{k} + k_y^2 \frac{(k_x + jk_y)}{k} \right] e^{-j\theta} A^{+}_{l-1,m} \right\} \tag{43}$$

Using (41) and (42), (43) becomes

$$-j2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \left[k^2 A^{+}_{l+1,m} + k^2 A^{+}_{l-1,m} \right] \tag{44}$$

The terms in (37) with coefficients $k_1^2\ (\varepsilon_r P_1 + p_2 \tanh(p_2 d))$ are

$$I^{+}_{\phi lmx}(\vec{k}\,)\, I^{-}_{sx}(\vec{k}\,) + I^{+}_{\phi lmy}(\vec{k}\,)\, I^{-}_{sy}(\vec{k}\,) \tag{45}$$

Using (13) and (14), this becomes

$$I^{+}_{\phi lmx}(\vec{k}\,)\, I^{-}_{sx}(\vec{k}\,) + I^{+}_{\phi lmy}(\vec{k}\,)\, I^{-}_{sy}(\vec{k}\,) = j2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R$$

$$\left\{ \frac{(k_x - jk_x}{k}\, e^{j\theta}\, A^{+}_{l+1,m} + \frac{(k_x + jk_x)}{k}\, e^{-j\theta}\, A^{+}_{l-1,m} \right\} \tag{46}$$

or, using 15.2(6) and 15.2(7),

$$I^{+}_{\phi lmx}(\vec{k}\,)\, I^{-}_{sx}(\vec{k}\,) + I^{+}_{\phi lmy}(\vec{k}\,)\, I^{-}_{sy}(\vec{k}\,) =$$

$$= j2\pi a^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta} \sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \left[A^{+}_{l+1,m} + A^{+}_{l-1,m} \right] \tag{47}$$

Using (44) and (47), (37) becomes

$$Z_s^{\phi lm} = \frac{-1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k\, \frac{1}{2\pi j\ \omega\, \varepsilon_1}\, \frac{ja^2\, e^{-j\vec{k}\cdot\vec{r}_0}\, \pi j^l\, e^{jl\theta}}{(p_1 + p_2 \coth(p_2\, d))}$$

$$\times \left[\sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \right] \left[k^2\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left[A^{+}_{l+1,m} + A^{+}_{l-1,m} \right] \tag{48}$$

Separating out the θ-dependance,

$$Z_s^{\phi lm} = \frac{-1}{(2\pi)} \int_0^{\infty} k\ dk\, \frac{1}{\omega\, \varepsilon_1}\, \frac{a^2\, \pi}{(p_1 + p_2 \coth(p_2\, d))} \left[\sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \right]$$

$$\times \left[k^2\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ A^{+}_{l+1,m} + A^{+}_{l-1,m} \right\} \int_0^{2\pi} d\theta\, j^l\, e^{-j\vec{k}\cdot\vec{r}_0}\, e^{jl\theta} \tag{49}$$

Using 15.2(10), this becomes

$$Z_s^{\phi lm} = \int_0^{\infty} k\ dk\, \frac{-1}{\omega\, \varepsilon_1}\, \frac{a^2\, \pi}{(p_1 + p_2 \coth(p_2\, d))} \left[\sum_{i=1,2} J_i(kt)\, f_i(kt)\Big|_a^R \right] \times$$

$$\times \left[k^2\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ A^{+}_{l+1,m} + A^{+}_{l-1,m} \right\} e^{jl\theta_0} J_l(kr_0) \tag{50}$$

15.4. Feed pin-feed pin matrix elements

The matrix element relating the current distribution on the feed pin along with the associated singular patch current distribution to tangential electric field on the feed pin surface is given by

$$Z^{z}_{s'} = Z^{z}_{p'} + Z^{z}_{s'} \tag{1}$$

where s' refers to the component on the patch surface, and p' to the component on the feed pin.

Looking first at the component due to the associated patch current,

$$Z^{z}_{s'} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d^2k \, \frac{\sinh(p_2 d)}{p_2} \, 2\pi a \, J_0(ka) \sum_{\beta=x,y} g_{zx} \, \tilde{I}_{s\beta} \, e^{j\vec{k}\cdot\vec{r}_0} \tag{2}$$

From equation 14.2(67),

$$\tilde{I}_{sx}(\vec{k}) = -j\,2\pi a^2 \, e^{-j\vec{k}\cdot\vec{r}_0} \, \frac{k_x}{k} \left[J_0(kt)\, \tilde{f}_0(kt) + J_1(kt)\, \tilde{f}_1(kt) \right]_a^R \tag{3}$$

$$\tilde{I}_{sy}(\vec{k}) = -j\,2\pi a^2 \, e^{-j\vec{k}\cdot\vec{r}_0} \, \frac{k_y}{k} \left[J_0(kt)\, \tilde{f}_0(kt) + J_1(kt)\, \tilde{f}_1(kt) \right]_a^R \tag{4}$$

Using 15.2(2), 15.2(3), (3) and (4) equation (2) becomes

$$Z^{z}_{s'} = \frac{1}{(2\pi)} \iint_{-\infty}^{\infty} d^2k \, \frac{a}{\omega\varepsilon_1} \, \frac{\sinh(p_2 d)}{p_2\, T_m} \, J_0(ka)\, e^{j\vec{k}\cdot\vec{r}_0}$$

$$\times (-2\pi j a^2)\, e^{-j\vec{k}\cdot\vec{r}_0} \sum_{i=1}^{2} \tilde{f}_i(kt)\, J_i(kt) \Big|_a^R \, \frac{(k_x^2 + k_y^2)}{k} \tag{5}$$

Integration over θ gives a factor of 2π.

$$Z^{z}_{s'} = \int_0^{\infty} dk \, k \, \frac{2\pi a^2}{j\omega\varepsilon_1} \, \frac{\sinh(p_2 d)}{p_2\, T_m} \, J_0(ka)\, ka \sum_{i=0}^{1} \tilde{f}_i(kt)\, J_i(kt) \Big|_a^R \tag{6}$$

From 3.6(45),

$$Z_{p'}^{s} = \iint\limits_{-\infty}^{\infty} d^2k \, \frac{a^2}{j\omega\varepsilon_2} J_0^2(ka) \left[d\frac{k_2^2}{p_2^2} - \frac{\varepsilon_r \, k^2}{p_2^2 \, T_m} \, \frac{\sinh(p_2 d)}{p_2} \right] \tag{7}$$

The θ dependance is integrated out to give

$$Z_{p'}^{s} = \int\limits_{0}^{\infty} k \, dk \, \frac{2\pi k a^2}{j\omega\varepsilon_2} J_0^2(ka) \left[d\frac{k_2^2}{p_2^2} - \frac{\varepsilon_r \, k^2}{p_2^2 \, T_m} \, \frac{\sinh(p_2 d)}{p_2} \right] \tag{8}$$

Both $Z_{p'}^{s}$ and $Z_{i'}^{s}$ have a nonconverging $(J_0(ka))^2$ term, but taken together, the terms cancel one another. To show this explicitly,

$$Z_{i'}^{s} = 2\pi a^2 \int\limits_{0}^{\infty} dk \, \frac{k \, J_0(ka)}{j\omega\varepsilon_1} \left\{ \frac{\sinh(p_2 d)}{p_2 T_m} \, ka \sum_{i=0}^{1} \bar{f}_i(kt) \, J_i(kt) \Big|_{a}^{R} + \right.$$

$$\left. + J_0(ka) \, \frac{k_2^2 d}{\varepsilon_r p_2^2} - \frac{k^2}{p_2^2 \, T_m} \, \frac{\sinh(p_2 d)}{p_2} J_0(ka) \right\} \tag{9}$$

or

$$Z_{i'}^{s} = 2\pi a^2 \int\limits_{0}^{\infty} dk \, \frac{k}{j\omega\varepsilon_1} \left\{ \frac{J_0^2(ka) \, k_2^2 d}{\varepsilon_r \, p_2^2} + ka \, \frac{\sinh(p_2 d)}{p_2 T_m} \times \right.$$

$$\left. \times \left[-J_0^2(ka) \left[\bar{f}_0(ka) + \frac{k}{a \, p_2^2} \right] + \bar{f}_0(kR) \, J_0(ka) \, J_0(kR) + J_0(ka) \, \bar{f}_1(kt) \, J_1(kt) \Big|_{a}^{R} \right] \right\} \tag{10}$$

For the integral in (10) to exist, $\bar{f}_0(ka) + \frac{k}{a \, p_2^2}$ must approach zero faster than $\frac{1}{ka}$.

Examining this term using 14.2(68)

$$\bar{f}_0(ka) + \frac{k}{a \, p_2^2} = \frac{-1}{ka} \left[A + B + C + D - \frac{8C}{(ka)^2} - \frac{24D}{(ka)^2} + \frac{192D}{(ka)^4} - \frac{k^2}{p_2^2} \right] \tag{11}$$

But $k^2 = p_2^2 + k_2^2$ and $A + B + C + D = 1$, so (11) becomes

$$\bar{f}_0(ka) + \frac{k}{a \, p_2^2} = \frac{-1}{ka} \left[1 - \frac{8C}{(ka)^2} - \frac{24D}{(ka)^2} + \frac{192D}{(ka)^4} - \frac{k_2^2}{p_2^2} - 1 \right] \tag{12}$$

or

$$\bar{f}_0(ka) + \frac{k}{a\, p_2^2} = \frac{1}{ka} \left[\frac{8C}{(ka)^2} + \frac{24D}{(ka)^2} - \frac{192D}{(ka)^4} + \frac{k_2^2}{p_2^2} \right] \tag{13}$$

The right hand side of (13) approaches zero like $\frac{1}{(ka)^3}$, so the integral in (10) exists. Equation (10) then becomes

$$Z_{i'}^{i} = 2\pi a^2 \int_0^{\infty} dk \, \frac{k}{j\omega\varepsilon_1} \left\{ J_0^2(ka) \left[\frac{k_2^2 d}{\varepsilon_r\, p_2^2} - \frac{ka \sinh(p_2 d)}{p_2 T_m} \times \right.\right.$$

$$\left. \times \left[\frac{8C + 24D}{ka^3} - \frac{192D}{ka^5} + \frac{k_2^2}{p_2^2\, ka} \right] \right] +$$

$$\left. + \frac{ka \sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \left[\bar{f}_0(kR)\, J_0(kR) - \bar{f}_1(ka)\, J_1(ka) + \bar{f}_1(kR)\, J_1(kR) \right] \right\} \tag{14}$$

The expressions for $\bar{f}_i(kt)$ from section 14.2 are

$$\bar{f}_0(kt) = -\frac{A}{ka} - \frac{B}{ka^3}(kt)^2 - \frac{C}{ka^5}\left((kt)^4 - 8(kt)^2 \right) - \frac{D}{(ka)^7}\left((kt)^6 - 24(kt)^4 + 192(kt)^2 \right) \tag{15}$$

$$\bar{f}_1(kt) = \frac{B}{ka^3} 2kt + \frac{C}{ka^5}\left(4(kt)^3 - 16kt \right) + \frac{D}{(ka)^7}\left(6(kt)^5 - 96(kt)^3 + 384kt \right) \tag{16}$$

Using (15) and (16), the matrix element $Z_{i'}^{i}$ is found to be

$$Z_{i'}^{i} = 2\pi a \int_0^{\infty} dk \, \frac{ka}{j\omega\varepsilon_1} \left\{ J_0^2(ka) \left[\frac{k_2^2 d}{\varepsilon_r\, p_2^2} - \frac{ka \sinh(p_2 d)}{p_2 T_m} \left[\frac{8C + 24D}{(ka)^3} - \frac{192D}{(ka)^5} + \frac{k_2^2}{p_2^2\, ka} \right] \right] + \right.$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_0(kR) \left[- \left[A + B\frac{R^2}{a^2} + C\frac{R^4}{a^4} + D\frac{R^6}{a^6} \right] + \right.$$

$$\left. + \frac{8}{(ka)^2}\left[C\frac{R^2}{a^2} + 3D\frac{R^4}{a^4} \right] - \frac{192D}{(ka)^4}\frac{R^2}{a^2} \right] -$$

$$- \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_1(ka) \left[\frac{1}{ka}\left[2B + 4C + 6D \right] - \frac{16}{(ka)^3}\left(C + 6D \right) + \frac{384D}{(ka)^5} \right] +$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_1(kR) \left[\frac{1}{ka}\left[2B\frac{R}{a} + 4C\frac{R^3}{a^3} + 6D\frac{R^5}{a^5} \right] - \right.$$

$$\left.\left. - \frac{16}{(ka)^3}\left[C\frac{R}{a} + 6D\frac{R^3}{a^3} \right] + \frac{384D}{(ka)^5}\frac{R}{a} \right] \right\} \tag{17}$$

15.5. Matrix element summary.

The matrix elements obtained in the previous sections are here summarized

$$Z^{rlm}_{rl'm'} = \frac{1}{2\pi}\int_0^\infty dk\ \frac{2k}{j\omega\varepsilon_1}\ \frac{\pi^2\,\delta_{l,-l'}}{(p_1 + p_2\coth(p_2 d))}$$

$$\times\left\{\left[k_1^2 - \frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\right]\left[B_{l+1,m}\, B_{-l-1,m'} + B_{l-1,m}\, B_{-l+1,m'}\right]\right.$$

$$\left. + \frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\left[B_{l+1,m}\, B_{-l+1,m'} + B_{l-1,m}\, B_{-l-1,m'})\right]\right\} \qquad (1)$$

$$Z^{\phi lm}_{\phi l'm'} = \frac{1}{2\pi}\int_0^\infty dk\ \frac{2k}{j\omega\varepsilon_1}\ \frac{\pi^2\,\delta_{l,-l'}}{(p_1 + p_2\coth(p_2 d))}$$

$$\times\left\{\left[k_1^2 - \frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\right]\left[A_{l+1,m}\, A_{-l-1,m'} + A_{l-1,m}\, A_{-l+1,m'}\right]\right.$$

$$\left. - \frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\left[A_{l+1,m}\, A_{-l+1,m'} + A_{l-1,m}\, A_{-l-1,m'}\right]\right\} \qquad (2)$$

$$Z^{\phi lm}_{rl'm'} = \frac{1}{2\pi}\int_0^\infty dk\ \frac{2k}{j\omega\varepsilon_1}\ \frac{\pi^2\,\delta_{l,-l'}}{(p_1 + p_2\coth(p_2 d))}$$

$$\times\left\{-j\left[k_1^2 - \frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\right]\left[B_{l+1,m}\, A_{-l-1,m'} - B_{l-1,m}\, A_{-l+1,m'}\right]\right.$$

$$\left. + \frac{jk^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\left[B_{l+1,m}\, A_{-l+1,m'} - B_{l-1,m}\, A_{-l-1,m'}\right]\right\} \qquad (3)$$

$$Z^{rlm}_{\phi l'm'} = \frac{1}{2\pi}\int_0^\infty dk\ \frac{2k}{j\omega\varepsilon_1}\ \frac{\pi^2\,\delta_{l,-l'}}{(p_1 + p_2\coth(p_2 d))}$$

$$\times\left\{j\left[k_1^2 - \frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\right]\left[A_{l+1,m}\, B_{-l-1,m'} - A_{l-1,m}\, B_{-l+1,m'}\right]\right.$$

$$\left. + j\,\frac{k^2}{2}\ \frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2\tanh(p_2 d)}\left[A_{l+1,m}\, B_{-l+1,m'} - A_{l-1,m}\, B_{-l-1,m'}\right]\right\} \qquad (4)$$

15.5. Matrix element summary.

The matrix elements obtained in the previous sections are here summarized

$$Z_{rl'm'}^{rlm} = \frac{1}{2\pi} \int_0^\infty dk \; \frac{2k}{j\omega\varepsilon_1} \; \frac{\pi^2 \, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2 d))}$$

$$\times \Bigg\{ \Bigg[k_1^2 - \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Bigg] \Big[B_{l+1,m} \, B_{-l-1,m'} + B_{l-1,m} \, B_{-l+1,m'} \Big]$$

$$+ \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Big[B_{l+1,m} \, B_{-l+1,m'} + B_{l-1,m} \, B_{-l-1,m'}) \Big] \Bigg\} \tag{1}$$

$$Z_{\phi l'm'}^{\phi lm} = \frac{1}{2\pi} \int_0^\infty dk \; \frac{2k}{j\omega\varepsilon_1} \; \frac{\pi^2 \, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2 d))}$$

$$\times \Bigg\{ \Bigg[k_1^2 - \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Bigg] \Big[A_{l+1,m} \, A_{-l-1,m'} + A_{l-1,m} \, A_{-l+1,m'} \Big]$$

$$- \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Big[A_{l+1,m} \, A_{-l+1,m'} + A_{l-1,m} \, A_{-l-1,m'} \Big] \Bigg\} \tag{2}$$

$$Z_{rl'm'}^{\phi lm} = \frac{1}{2\pi} \int_0^\infty dk \; \frac{2k}{j\omega\varepsilon_1} \; \frac{\pi^2 \, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2 d))}$$

$$\times \Bigg\{ -j \Bigg[k_1^2 - \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Bigg] \Big[B_{l+1,m} \, A_{-l-1,m'} - B_{l-1,m} \, A_{-l+1,m'} \Big]$$

$$+ \frac{jk^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Big[B_{l+1,m} \, A_{-l+1,m'} - B_{l-1,m} \, A_{-l-1,m'} \Big] \Bigg\} \tag{3}$$

$$Z_{\phi l'm'}^{rlm} = \frac{1}{2\pi} \int_0^\infty dk \; \frac{2k}{j\omega\varepsilon_1} \; \frac{\pi^2 \, \delta_{l,-l'}}{(p_1 + p_2 \coth(p_2 d))}$$

$$\times \Bigg\{ j \Bigg[k_1^2 - \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Bigg] \Big[A_{l+1,m} \, B_{-l-1,m'} - A_{l-1,m} \, B_{-l+1,m'} \Big]$$

$$+ j \, \frac{k^2}{2} \, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r \, p_1 + p_2 \tanh(p_2 d)} \Big[A_{l+1,m} \, B_{-l+1,m'} - A_{l-1,m} \, B_{-l-1,m'} \Big] \Bigg\} \tag{4}$$

$$Z^{z}_{r'l'm'} = \frac{ja\pi\,(-1)^{l'} e^{jl'\theta_0}}{\omega\varepsilon_1} \int_0^\infty dk\ k^2\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[B^{-}_{l'+1,m'} - B^{-}_{l'-1,m'} \right] J_{l'}(kr_0)\, J_0(ka) \qquad (5)$$

$$Z^{z}_{\phi'l'm'} = \frac{-a\pi\,(-1)^{l'} e^{jl'\theta_0}}{\omega\varepsilon_1} \int_0^\infty dk\ k^2\, \frac{\sinh(p_2 d)}{p_2\, T_m} \left[A^{-}_{l'+1,m'} + A^{-}_{l'-1,m'} \right] J_{l'}(kr_0)\, J_0(ka) \qquad (6)$$

$$Z^{rlm}_{s^{z'}} = \frac{-1}{(2\pi)} \int_0^\infty k\ dk\ \frac{1}{j\ \omega\ \varepsilon_1}\ \frac{a^2\,\pi}{(p_1 + p_2 \coth(p_2\, d))} \left[\sum_{i=1,2} J_i(kt)\, f_i(kt) \Big|_a^R \right] \times$$

$$\times \left[k^2\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ B^{+}_{l+1,m} - B^{+}_{l-1,m} \right\} 2\pi\, e^{jl\theta_0} J_l(kr_0) \qquad (7)$$

$$Z^{\phi lm}_{s^{z'}} = \int_0^\infty k\ dk\ \frac{-1}{\omega\ \varepsilon_1}\ \frac{a^2\,\pi}{(p_1 + p_2 \coth(p_2\, d))} \left[\sum_{i=1,2} J_i(kt)\, f_i(kt) \Big|_a^R \right] \times$$

$$\times \left[k^2\, \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r p_1 + p_2 \tanh(p_2 d)} - k_1^2 \right] \left\{ A^{+}_{l+1,m} + A^{+}_{l-1,m} \right\} e^{jl\theta_0} J_l(kr_0) \qquad (8)$$

$$Z^{z}_{z'} = 2\pi a \int_0^\infty dk\ \frac{ka}{j\omega\varepsilon_1} \left\{ J_0^2(ka) \left[\frac{k_2^2 d}{\varepsilon_r\, p_2^2} - \frac{ka\ \sinh(p_2 d)}{p_2 T_m} \left[\frac{8C + 24D}{(ka)^3} - \frac{192D}{(ka)^5} + \frac{k_2^2}{p_2^2\ ka} \right] \right] + \right.$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m}\, J_0(kR) \left[- \left[A + B\, \frac{R^2}{a^2} + C\, \frac{R^4}{a^4} + D\, \frac{R^6}{a^6} \right] + \right.$$

$$\left. + \frac{8}{(ka)^2} \left[C\, \frac{R^2}{a^2} + 3D\, \frac{R^4}{a^4} \right] - \frac{192D}{(ka)^4}\, \frac{R^2}{a^2} \right] -$$

$$- \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m}\, J_1(ka) \left[\frac{1}{ka} \left[2B + 4C + 6D \right] - \frac{16}{(ka)^3}\,(C + 6D) + \frac{384D}{(ka)^5} \right] +$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m}\, J_1(kR) \left[\frac{1}{ka} \left[2B\, \frac{R}{a} + 4C\, \frac{R^3}{a^3} + 6D\, \frac{R^5}{a^5} \right] - \right.$$

$$\left.\left. - \frac{16}{(ka)^3} \left[C\, \frac{R}{a} + 6D\, \frac{R^3}{a^3} \right] + \frac{384D}{(ka)^5}\, \frac{R}{a} \right] \right\} \qquad (9)$$

16. Matrix Element Integral Evaluation

The integrals to be evaluated are given in equations 3.6(23) through 3.6(26), 3.6(30) and 3.6(43). Rewriting these here,

$$\iint_s ds\ K_{rlmx}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I^{\pm}_{rlmx}\ (\vec{k}\ ) \qquad (1)$$

$$\iint_s ds\ K_{rlmy}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I^{\pm}_{rlmy}\ (\vec{k}\ ) \qquad (2)$$

$$\iint_s ds\ K_{\phi lmx}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I^{\pm}_{\phi lmx}\ (\vec{k}\ ) \qquad (3)$$

$$\iint_s ds\ K_{rlmy}\ e^{\pm j\vec{k}\cdot\vec{r}} \equiv I^{\pm}_{\phi lmy}\ (\vec{k}\ ) \qquad (4)$$

$$\iint_s ds\ e^{\pm j\vec{k}\cdot\vec{r}}\ K_{s\beta'} \equiv I^{\pm}_{s\beta'} \qquad \beta = x, y \qquad (5)$$

$$I_z = \int_{-d}^{0}\int_{-d}^{0} dz\ dz' \left[-\delta(z-z') + \frac{k_2^2 + p_2^2}{p_2 p_1}\ \frac{\cosh(p_2(z^< + d))(p_2\cosh(p_2 z^>) - \varepsilon_r p_1 \sinh(p_2 z^>))}{T_m}\right] \qquad (6)$$

The integrals in (1) through (4) now will be evaluated, beginning with (1). Let

$$ds = r\ dr\ d\phi \qquad (7)$$

$$K_{rlmx} = T_m(r/b)\ e^{jl\phi}\ (\hat{r}\cdot\hat{x})\sqrt{1+\frac{r^2}{b^2}} \qquad (8)$$

$$\hat{r}\cdot\hat{x} = \cos(\phi) = \frac{1}{2}\left[e^{j\phi} + e^{-j\phi}\right] \qquad (9)$$

So, (1) becomes

$$I^{\pm}_{rlmx} = \int_0^b dr\ T_m(r/b)\ r\sqrt{1-\frac{r^2}{b^2}}\int_0^{2\pi} d\phi\ e^{\pm j\vec{k}\cdot\vec{r}}\ \frac{1}{2}\left[e^{j(l+1)\phi} + e^{j(l-1)\phi}\right] \qquad (10)$$

Let

$$e^{\pm j\vec{k}\cdot\vec{r}} = e^{\pm jkr\cos(\phi-\theta)} \qquad (11)$$

where

$$\theta = \tan^{-1}\left[\frac{k_x}{k_y}\right] \tag{12}$$

An identity that will be used frequently [16] is

$$\int_0^{2\pi} e^{jz\cos(\phi-\theta)}\, e^{jn\phi}\, d\phi = e^{jn\theta}\, 2\,\pi\, j^n\, J_n(z) \tag{13}$$

Thus

$$I^{\pm}_{rlmx} = \int_0^b dr\ T_m(r/b)\, r\, \sqrt{1-\frac{r^2}{b^2}} \int_0^{2\pi} d\phi\ e^{\pm jkr\cos(\phi-\theta)}\, \frac{1}{2}\left[e^{j(l+1)\phi} + e^{j(l-1)\phi}\right]$$

$$= \int_0^b dr\ T_m(r/b)\, r\, \sqrt{1-\frac{r^2}{b^2}} \left[\pi\, j^{(l+1)}\, e^{j(l+1)\theta}\, J_{l+1}(\pm kr) + \pi\, j^{(l-1)}\, e^{j(l-1)\theta}\, J_{(l-1)}(\pm kr)\right] \tag{14}$$

Rearranging,

$$I^{\pm}_{rlmx} = \pi\, j^{l+1}\, e^{j(l+1)\theta}\, B^{\pm}_{l+1,m} + \pi\, j^{l-1}\, e^{j(l-1)\theta}\, B^{\pm}_{l-1,m} \tag{15}$$

where

$$B^{\pm}_{l,m} = \int_0^b dr\ T_m(r/b)\, r\, \sqrt{1-\frac{r^2}{b^2}}\, J_l(\pm kr) \tag{16}$$

Proceeding similarly for (2),

$$K_{rlmy} = T_m(r/b)\, e^{jl\phi}\, (\hat{r}\cdot\hat{y})\sqrt{1+\frac{r^2}{b^2}} \tag{17}$$

$$\hat{r}\cdot\hat{y} = \sin(\phi) = \frac{-j}{2}\left[e^{j\phi} - e^{-j\phi}\right] \tag{18}$$

So, (2) becomes

$$I^{\pm}_{rlmy} = \int_0^b dr\ T_m(r/b)\, r\, \sqrt{1-\frac{r^2}{b^2}} \int_0^{2\pi} d\phi\ e^{\pm jkr\cos(\phi-\theta)}\, \frac{-j}{2}\left[e^{j(l+1)\phi} - e^{j(l-1)\phi}\right]$$

$$= \int_0^b dr\ T_m(r/b)\, r\, \sqrt{1-\frac{r^2}{b^2}} \left[\pi\, j^l\, e^{j(l+1)\theta}\, J_{l+1}(\pm kr) + \pi\, j^l\, e^{j(l-1)\theta}\, J_{(l-1)}(\pm kr)\right] \tag{19}$$

or

$$I^{\pm}_{rlmy} = \pi\, j^l\, e^{j(l+1)\theta} B^{\pm}_{l+1,m} + \pi\, j^l\, e^{j(l-1)\theta} B^{\pm}_{l-1,m} \tag{20}$$

Continuing for (3)

$$K_{\phi lmx} = T_m(r/b)\, e^{jl\phi}\, (\hat{\phi} \cdot \hat{x}) \frac{1}{\sqrt{1+\frac{r^2}{b^2}}} \tag{21}$$

$$\hat{\phi} \cdot \hat{x} = -\sin(\phi) = \frac{j}{2}\left[e^{j\phi} - e^{-j\phi}\right] \tag{22}$$

So, (3) becomes

$$I^{\pm}_{\phi lmx} = \int_0^b dr\ T_m(r/b) \frac{r}{\sqrt{1-\frac{r^2}{b^2}}} \int_0^{2\pi} d\phi\, e^{\pm jkr\cos(\phi-\theta)} \frac{j}{2}\left[e^{j(l+1)\phi} - e^{j(l-1)\phi}\right] \tag{23}$$

Using (13), (23) becomes

$$I^{\pm}_{\phi lmx} = \int_0^b dr\ T_m(r/b) \frac{r}{\sqrt{1-\frac{r^2}{b^2}}} \frac{2\pi j}{2}\left[j^{l+1}\, e^{j(l+1)\theta} J_{l+1}(\pm kr) - j^{l-1}\, e^{j(l-1)\theta} J_{(l-1)}(\pm kr)\right] \tag{24}$$

or

$$I^{\pm}_{\phi lmx} = -\pi\, j^l\, e^{j(l+1)\theta} A^{\pm}_{l+1,m} - \pi\, j^l\, e^{j(l-1)\theta} A^{\pm}_{l-1,m} \tag{25}$$

where

$$A^{\pm}_{l,m} = \int_0^b dr\ T_m(r/b) \frac{r}{\sqrt{1-\frac{r^2}{b^2}}} J_l(\pm kr) \tag{26}$$

Finally, for (4)

$$K_{\phi lmy} = T_m(r/b)\, e^{jl\phi}\, (\hat{\phi} \cdot \hat{y}) \frac{1}{\sqrt{1+\frac{r^2}{b^2}}} \tag{27}$$

$$\hat{\phi} \cdot \hat{y} = \cos(\phi) = \frac{1}{2}\left[e^{j\phi} + e^{-j\phi}\right] \tag{28}$$

So, (4) becomes

$$I^{\pm}_{\phi lmy} = \int_0^b dr\ T_m(r/b)\ \frac{r}{\sqrt{1-\frac{r^2}{b^2}}} \int_0^{2\pi} d\phi\ e^{\pm jkr\cos(\phi-\theta)}\ \frac{1}{2}\left[e^{j(l+1)\phi} + e^{j(l-1)\phi}\right] \tag{29}$$

Using (13), (29) becomes

$$I^{\pm}_{\phi lmy} = \int_0^b dr\ T_m(r/b)\ \frac{r}{\sqrt{1-\frac{r^2}{b^2}}}\ \frac{2\pi}{2}\left[j^{l+1}\ e^{j(l+1)\theta}\ J_{l+1}(\pm kr) + j^{l-1}\ e^{j(l-1)\theta}\ J_{(l-1)}(\pm kr)\right] \tag{30}$$

or

$$I^{\pm}_{\phi lmy} = \pi\ j^{l+1}\ e^{j(l+1)\theta}\ A^{\pm}_{l+1,m} + \pi\ j^{l-1}\ e^{j(l-1)\theta}\ A^{\pm}_{l-1,m} \tag{31}$$

The integrals in (26) and (16) will now be evaluated. Let $z = r/b$ and perform a change of variable on the integral in (26), giving

$$A^{\pm}_{l,m} = b^2 \int_0^1 dz\ T_m(z)\ \frac{z}{\sqrt{1-z^2}}\ J_l(\pm kbz) \tag{32}$$

where $T_m(z)$ is the Tchebychef polynomial of order m.

The recurrence relationship for Tchebychef polynomials is [16]

$$2\ z\ T_m(z) = T_{m-1}(z) + T_{m+1}(z) \tag{33}$$

If m is 0, $m-1$ is negative and Tchebychef polynomials aren't normally defined for negative orders. It is desired to use the formula for $m = 0$, so (33) will be considered as defining $T_m(z)$ for m negative. When this is done, it is found that

$$T_m(z) = T_{|m|}(z) \qquad \text{all integer } m \tag{34}$$

Thus, (33) may be used for all m if (34) is applied afterwards. Equation (32) then becomes

$$A^{\pm}_{l,m} = \frac{b^2}{2} \int_0^1 dz\ \left[T_{|m-1|}(z) + T_{m+1}(z)\right] \frac{1}{\sqrt{1-z^2}}\ J_l(\pm kbz) \tag{35}$$

A standard integral from [17] is

$$\int_0^1 T_m(x)\,\frac{1}{\sqrt{1-x^2}}\,J_l(xy)\,dx = \frac{\pi}{2}\,J_{\frac{l+m}{2}}\!\left(\frac{y}{2}\right)J_{\frac{l-m}{2}}\!\left(\frac{y}{2}\right) \qquad (36)$$

$$l > -m-1$$

Since the Tchebyshev polynomials in (35) have a non-negative order, $l \geq 0$ satisfies the restriction on (36). For $l < 0$, use

$$J_l(\pm kbz) = (-1)^l\,J_{-l}(\pm kbz) \qquad l < 0. \qquad (37)$$

Using, in addition, the relation [16]

$$J_l(-kbz) = (-1)^l\,J_l(kbz) \qquad (38)$$

the Bessel function in (37) can be written as

$$J_l(\pm kbz) = J_{|l|}(kbz)\cdot\begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{Bmatrix} \qquad (39)$$

Using (36) and (34) in (35),

$$A^{\pm}_{l,m} = \frac{\pi b^2}{4}\left[J_{\frac{|l|+|m-1|}{2}}\left[\frac{kb}{2}\right]J_{\frac{|l|-|m-1|}{2}}\left[\frac{kb}{2}\right] + J_{\frac{|l|+m+1}{2}}\left[\frac{kb}{2}\right]J_{\frac{|l|-m-1}{2}}\left[\frac{kb}{2}\right]\right]\times$$

$$\times\begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{Bmatrix} \qquad (40)$$

By inspection, the absolute values on the $m-1$ subscripts may be dropped, since the bracketed term as a whole will not be affected. Thus, equation (40) becomes

$$A^{\pm}_{l,m} = \frac{\pi b^2}{4}\left[J_{\frac{|l|+m-1}{2}}\left[\frac{kb}{2}\right]J_{\frac{|l|-m+1}{2}}\left[\frac{kb}{2}\right] + J_{\frac{|l|+m+1}{2}}\left[\frac{kb}{2}\right]J_{\frac{|l|-m-1}{2}}\left[\frac{kb}{2}\right]\right]$$

$$\cdot\begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{Bmatrix} \qquad (41)$$

Proceeding similarly for (16), let $z = r/b$ and perform a change of variable to obtain

$$B_{l,m}^{\pm} = b^2 \int_0^1 dz \; T_m(z)\, z\, \sqrt{1-z^2}\, J_l(\pm kbz) \tag{42}$$

or

$$B_{l,m}^{\pm} = b^2 \int_0^1 dz \; T_m(z)\, (z - z^3)\, \frac{1}{\sqrt{1-z^2}}\, J_l(\pm kbz) \tag{43}$$

Using the Tchebychef recursion relationship repeatedly,

$$8\, z^3 T_m(z) = 4\, z^2 T_{m-1}(z) + 4\, z^2\, T_{m+1}(z) \tag{44a}$$

$$= 2\, z\, T_{m-2}(z) + 4\, z\, T_m(z) + 2\, T_{m+2}(z) \tag{44b}$$

$$= T_{m-3}(z) + 3\, Tm-1(z) + 3\, T_{m+1}(z) + T_{m+3}(z) \tag{44c}$$

With (34),

$$z^3 T_m(z) = \frac{1}{8}\, (\, T_{|m-3|}(z) + 3\, T_{|m-1|}(z) + 3\, T_{m+1}(z) + T_{m+3}(z)\,) \tag{45}$$

so, using (33),

$$(z - z^3)\, T_m(z) = \frac{1}{8}\, (\, -T_{|m-3|}(z) + T_{|m-1|}(z) + T_{m+1}(z) - T_{m+3}(z)\,) \tag{46}$$

Equation (43) is then, using (39) and (46),

$$B_{l,m}^{\pm} = b^2 \int_0^1 dz \left[\frac{1}{8}\, (\, -T_{|m-3|}(z) + T_{|m-1|}(z) + T_{m+1}(z) - T_{m+3}(z)\,) \right] \frac{1}{\sqrt{1-z^2}}\, J_{|l|}(kbz) \cdot \begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l<0 \end{Bmatrix} \tag{47}$$

Reworking the bracketed term,

$$B_{l,m}^{\pm} = \frac{b^2}{2} \int_0^1 dz\, \frac{1}{4} \Big[-T_{|m-3|}(z) - T\,|m-1|(z) + 2(\, T_{|m-1|}(z) + T\,|m+1|(z)\,) - - T_{m+1}(z) - T_{m+3}(z) \Big] \frac{1}{\sqrt{1-z^2}}\, J_{|l|}(kbz) \cdot \begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l<0 \end{Bmatrix} \tag{48}$$

or

$$B^{\pm}_{l,m} = \frac{b^2}{2}\int_0^1 dz \, \frac{-1}{4}\left[T_{|m-3|}(z) + T_{|m-1|}(z)\right]\frac{1}{\sqrt{1-z^2}} J_{|l|}(kbz)\begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ -(\pm 1)^l & l<0 \end{Bmatrix} +$$

$$+\frac{b^2}{2}\int_0^1 dz \, \frac{2}{4}\left[T_{|m-1|}(z) + T_{|m+1|}(z)\right]\frac{1}{\sqrt{1-z^2}} J_{|l|}(kbz)\begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ -(\pm 1)^l & l<0 \end{Bmatrix} +$$

$$+\frac{b^2}{2}\int_0^1 dz \, \frac{-1}{4}\left[T_{|m+1|}(z) + T_{|m+3|}(z)\right]\frac{1}{\sqrt{1-z^2}} J_{|l|}(kbz)\begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ -(\pm 1)^l & l<0 \end{Bmatrix} \quad (49)$$

Then, using (35),

$$B^{\pm}_{l,m} = \frac{1}{4}\left[2A^{\pm}_{l,m} - A^{\pm}_{l,|m-2|} - A^{\pm}_{l,m+2}\right] = \frac{1}{4}\left[2A^{\pm}_{l,m} - A^{\pm}_{l,m-2} - A^{\pm}_{l,m+2}\right] \quad (50)$$

Equations (1) through (4) are now fully evaluated. Rewriting (5),

$$e^{-j\vec{k}\cdot\vec{r}_o} I_{s\beta'} = \iint_s ds' \, e^{j\vec{k}\cdot\vec{r}} K_{s\beta'} \qquad \beta' = x', y' \quad (51)$$

where, from 14.2(6),

$$K_{s\beta'} = \hat{r}_p \, f\,(r_p) \quad (52)$$

Using $ds = r_p \, dr_p \, d\phi_p$ and $\vec{r}_p = \vec{r}\,' - \vec{r}_o$, (5) becomes

$$I^{-}_{s\beta'} = \int_0^{2\pi} d\phi_p \, \hat{r}_p \cdot \hat{\beta}' \int_a^R f\,(r_p)\, e^{-j\vec{k}\cdot\vec{r}_p} \, r_p \, dr_p \quad (53a)$$

This is evaluated in section 14.2 with the result in equations 14.2(64) and 14.2(67) for the case of a single patch.

For the case of the infinite array, (51) is evaluated in section 14.3 to obtain

$$I^{-}_{s\beta'} = e^{j\vec{k}\cdot\vec{r}_0}\int_0^{2\pi} d\phi_p \, \hat{r}_p \cdot \hat{\beta}' \, \frac{1}{\cos\left[\dfrac{\pi a}{2R_E}\right]} I^{-}_R \quad (53b)$$

with I^{-}_R given in Equation 13.3(12).

The final integral to be evaluated is (6). Rewriting this,

$$I_z = \int_{-d}^{0}\int_{-d}^{0} dz\, dz' \left[-\delta(z-z') + \frac{k_2^2 + p_2^2}{p_2 p_1} \frac{\cosh(p_2(z^< + d))(p_2\cosh(p_2 z^>) - \varepsilon_r p_1 \sinh(p_2 z^>))}{T_m} \right] \tag{6}$$

Remembering that

$$z^> = \max(z, z') \tag{54a}$$

$$z^< = \min(z, z') \tag{54b}$$

the above integral can be separated according to whether z or z' is larger. Equation (6) then becomes

$$I_z = \int_{-d}^{0} dz \left\{ \int_{-d}^{0} -\delta(z - z')dz' + \frac{k_2^2 + p_2^2}{p_2 p_1 T_m} \left[p_2\cosh(p_2 z) - \varepsilon_r p_1 \sinh(p_2 z) \right] \int_{-d}^{z} \cosh(p_2(z' + d))dz' \right.$$

$$\left. + \cosh(p_2(z + d)) \int_{z}^{0} (p_2 \cosh(p_2 z') - \varepsilon_r p_1 \sinh(p_2 z'))dz' \right\} \tag{55}$$

The hyperbolic integrals in (55) are evaluated as

$$\int_{-d}^{z} \cosh(p_2(z' + d))\, dz' = \frac{1}{p_2} \left[\sinh(p_2(z + d)) - \sinh(0) \right] = \frac{\sinh(p_2(z + d))}{p_2} \tag{56}$$

$$\int_{z}^{0} \sinh(p_2 z)\, dz' = \frac{1}{p_2} \left[\cosh(0) - \cosh(p_2 z) \right] = \frac{1}{p_2} \left[1 - \cosh(p_2 z) \right] \tag{57}$$

$$\int_{z}^{0} \cosh(p_2 z')\, dz' = \frac{1}{p_2} \left[\sinh(0) - \sinh(p_2 z) \right] = \frac{\sinh(p_2 z)}{p_2} \tag{58}$$

Using (56) through (58) and the δ-function, equation (55) becomes

$$I_z = \int_{-d}^{0} dz \left\{ -1 + \frac{k_2^2 + p_2^2}{p_2^2\, p_1 T_m} \left[p_2 \cosh(p_2 z) - \varepsilon_r p_1 \sinh(p_2 z) \right] \sinh(p_2(z + d)) \right.$$

$$\left. + \frac{k_2^2 + p_2^2}{p_2^2\, p_1 T_m} \left[-p_2 \sinh(p_2 z) + \varepsilon_r p_1 \cosh(p_2 z) - \varepsilon_r \right] \cosh(p_2(z + d)) \right\} \tag{59}$$

Using 13.10(36) and 13.10(37), this becomes

$$I_z = -d + \frac{k_2^2 + p_2^2}{p_2^2\, p_1 T_m} \int_{-d}^{0}$$

$$\Big\{ \left[p_2\cosh(p_2 z) - \varepsilon_r p_1 \sinh(p_2 z)\right]\left[\sinh(p_2 z)\cosh(p_2 d) + \sinh(p_2 d)\cosh(p_2 z)\right] +$$

$$+ \left[\cosh(p_2 z)\cosh(p_2 d) + \sinh(p_2 d)\sinh(p_2 z)\right]\left[-p_2 \sinh(p_2 z) + \varepsilon_r p_1 \cosh(p_2 z) - \varepsilon_r p_1\right]\Big\} dz \quad (60)$$

With cancellation of several terms, this becomes

$$I_z = -d + \frac{k_2^2 + p_2^2}{p_2^2\, p_1\, T_m} \int_{-d}^{0} \Big\{ p_2 \left[\cosh^2(p_2 z)\sinh(p_2 d) - \sinh^2(p_2 z)\sinh(p_2 d)\right]$$

$$+ \varepsilon_r p_1 \left[\cosh^2(p_2 z)\cosh(p_2 d) - \sinh^2(p_2 z)\cosh(p_2 d)\right]$$

$$- \varepsilon_r p_1 \left[\cosh(p_2 z)\cosh(p_2 d) + \sinh(p_2 d)\sinh(p_2 z)\right]\Big\} dz \quad (61)$$

Using $\cosh^2(x) - \sinh^2(x) = 1$, and 13.10(37) again gives

$$I_z = -d + \frac{k_2^2 + p_2^2}{p_2^2\, p_1\, T_m} \int_{-d}^{0} \Big\{ p_2 \sinh(p_2 d) + \varepsilon_r p_1 \cosh(p_2 d) - \varepsilon_r p_1 \cosh(p_2(z + d)) \Big\} \quad (62)$$

Using the definiion of T_m, equation (14.7.40), this is written as

$$I_z = -d + \frac{k_2^2 + p_2^2}{p_2^2} \int_{-d}^{0} \left[1 - \frac{\varepsilon_r}{T_m} \cosh(p_2(z + d))\right] \quad (63)$$

Performing the integral, with the use of 3.6(33) gives

$$I_z = d \left[\frac{k_2^2 + p_2^2}{p_2^2} - 1\right] - \frac{\varepsilon_r\, k^2}{p_2^2\, T_m\, p_2} \sinh(p_2 d) \quad (64)$$

or

$$I_z = d\, \frac{k_2^2}{p_2^2} - \frac{\varepsilon_r\, k^2}{p_2^2\, T_m\, p_2} \sinh(p_2 d) \quad (65)$$

The matrix elements for the incident field, equations 3.6(19) and 3.6(20), will now also be evaluated. Rewriting these,

$$V_{\gamma lm} = - \iint_s ds\, \vec{K}_{\gamma lm} \cdot \vec{E}_{||}^{\,1} - \iint_s ds\, \vec{K}_{\gamma lm} \cdot \vec{E}_{\perp}^{\,1} \quad (66)$$

$$V_z = -\int_{-d}^{0} dz\ 2\pi a K_z\ \hat{z} \cdot \vec{E}_{||}^{2} - \int_{-d}^{0} dz\ 2\pi a K_z\ \hat{z} \cdot \vec{E}_{\perp}^{2} \tag{67}$$

CASE 1 $\vec{E}^{i}$ parallel to plane of incidence

For this case, equation (66) reduces to

$$V_{\gamma lm} = V_{\gamma lm\ ||} = -\iint_{s} ds\ \vec{K}_{\gamma lm} \cdot \vec{E}_{||}^{1} \tag{68}$$

For $\gamma = r$, this becomes

$$V_{rlm\ ||} = -\iint_{s} ds\ \vec{K}_{rlm} \cdot \hat{x}''\ E_{||x''}^{1} \tag{69}$$

since $\vec{E}_{||}^{1}$ has no y'' component. Let

$$E_{||x''}^{1} = \bar{E}_{||}^{1}\ e^{jk_1 x'' \sin(\theta_i)} \tag{70}$$

where, from 13(45),

$$\bar{E}_{||}^{1} = 2\, E_{||}^{i}\ e^{j1/2\Phi_{||}} \cos(1/2\Phi_{||} - k_1 z\ \cos(\theta_i))\cos(\theta_i) \tag{71}$$

Now,

$$x''\ \sin(\theta_i) = x\ \sin(\theta_i)\cos(\phi_i) + y \sin(\theta_i)\sin(\phi_i)$$

$$= \sin(\theta_i)\ \left[r\cos(\phi)\cos(\phi_i) + r\ \sin(\phi)\sin(\phi_i) \right]$$

$$= r\ \sin(\theta_i)\cos(\phi - \phi_i) \tag{72}$$

so (71) becomes

$$E_{||x''}^{1} = \bar{E}_{||}^{1}\ e^{jk_1 r\ \sin(\theta_i)\cos(\phi - \phi_i)} \tag{73}$$

From 13(4),

$$\hat{r} \cdot \hat{x}'' = \cos(\phi)\cos(\phi_i) + \sin(\phi)\sin(\phi_i) = \cos(\phi - \phi_i) \tag{74}$$

Equation (69) then becomes

$$V_{rlm\ ||} = \iint_s \bar{E}^1_{||}\ e^{jk_1 r \sin(\theta_i)\cos(\phi-\phi_i)}\ e^{jl\phi}\cos(\phi-\phi_i)\ T_m(\ r/b\ )\sqrt{1-(\ r/b\ )^2}\ rdrd\phi \tag{75}$$

Using Euler's identity, and the identities

$$2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b) \qquad 2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b) \tag{76a,b}$$

equation (75) becomes

$$V_{rlm\ ||} = \frac{\bar{E}^1_{||}}{2}\int_0^b\int_0^{2\pi} rdrd\phi\ e^{jl\phi_i}\ e^{jk_1 r\sin(\theta_i)\cos(\phi-\phi_i)}\ T_m(\ r/b\ )\sqrt{1-(\ r/b\ )^2}$$

$$\left[\cos((l+1)(\phi-\phi_i)) + \cos((l-1)(\phi-\phi_i)) + j\ \sin((l+1)(\phi-\phi_i)) + j\ \sin((l-1)(\phi-\phi_i))\right] \tag{77}$$

Using the Bessel identities [16]

$$\int_0^{2\pi} e^{i\ z\cos(\theta)}\cos(n\theta)\ d\theta = 2\pi\ i^n\ J_n(\ z\ ) \tag{78a}$$

$$\int_0^{2\pi} e^{i\ z\cos(\theta)}\sin(n\theta)\ d\theta = 0 \tag{78b}$$

equation (77) becomes

$$V_{rlm\ ||} = \bar{E}^1_{||}\ e^{jl\phi_i}\ \pi\ j^{l+1}\int_0^b \left[J_{l+1}(k_1 r\ \sin(\theta_i)) - J_{l-1}(k_1 r\ \sin(\theta_i))\right]T_m(r/b)\sqrt{1-(r/b)^2}\ rdr \tag{79}$$

Using (16), this becomes

$$V_{rlm\ ||} = \bar{E}^1_{||}\ e^{jl\phi_i}\ \pi\ j^{l+1}\left[B^-_{l+1,m}(k_1\ \sin(\theta_i)) - B^-_{l-1,m}(k_1\sin(\theta_i))\right] \tag{80}$$

For $\gamma = \phi$, equation (68) becomes

$$V_{\phi lm\ ||} = -\iint_s ds\ \vec{K}_{\phi lm}\cdot\hat{x}''\ E^1_{||} \tag{81}$$

$$\hat{\phi}\cdot\hat{x}'' = \cos(\phi)\sin(\phi') - \sin(\phi)\cos(\phi') = -\sin(\phi-\phi') \tag{82}$$

Using (73), (82) and 15.1(44), equation (81) becomes

$$V_{\phi lm\ ||} = \iint_s \bar{E}^1_{||}\ e^{jk_1 r\sin(\theta_i)\cos(\phi-\phi_i)}\ e^{jl\phi}\sin(\phi-\phi_i)\ T_m(\ r/b\ )\frac{1}{\sqrt{1-(r/b)^2}}\ rdrd\phi \tag{83}$$

Using Euler's identity, (78b) and

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b) \tag{84}$$

equation (83) becomes

$$V_{\phi lm\,||} = \frac{\bar{E}^{1}_{||}}{2} \int_0^b \int_0^{2\pi} r dr d\phi \; e^{jl\phi_i} \; e^{jk_1 r \sin(\theta_i)\cos(\phi-\phi_i)} \; T_m(r/b) \frac{1}{\sqrt{1-(r/b)^2}}$$

$$\Big[-j\cos((l+1)(\phi-\phi_i)) + j\cos((l-1)(\phi-\phi_i)) + \sin((l+1)(\phi-\phi_i)) - \sin((l-1)(\phi-\phi_i))\Big] \tag{85}$$

Using (78a) and (78b), this becomes

$$V_{\phi lm\,||} = \bar{E}^{1}_{||} \; e^{jl\phi_i} \pi \, j^l \int_0^b \Big[J_{l+1}(k_1 r\sin(\theta_i)) + J_{l-1}(k_1 r\sin(\theta_i))\Big] T_m(r/b) \frac{1}{\sqrt{1-(r/b)^2}} \, r dr \tag{86}$$

Using (32), this becomes

$$V_{\phi lm\,||} = \bar{E}^{1}_{||} \; e^{jl\phi_i} \; \pi \, j^{l+1} \Big[A^{-}_{l+1,m}(k_1\sin(\theta_i)) + A^{-}_{l-1,m}(k_1\sin(\theta_i))\Big] \tag{87}$$

For $\vec{E}^{\,i}$ parallel to the plane of incidence, (68) reduces to

$$V_{z\,||} = -\int_{-d}^{0} dz \; 2\pi a K_z \hat{z} \cdot \vec{E}^{\,2}_{||} \tag{88}$$

From 13(47),

$$E^{2}_{||z} = 2\,T_{||}\,E^{i}_{||} \; e^{jk_2 x''\sin(\theta_r) + d\cos(\theta_r)} \sin(\theta_r)\cos(k_2(z+d)\cos(\theta_r)) \tag{89}$$

Equation (88) then becomes

$$V_{z\,||} = -2\pi a \; 2\,T_{||}\,E^{i}_{||} \; e^{jk_2 x''\sin(\theta_r) + d\cos(\theta_r)} \sin(\theta_r) \int_{-d}^{0} dz \, \cos(k_2(z+d)\cos(\theta_r)) \tag{90}$$

Evaluating the integral, this becomes

$$V_{z\,||} = -4\pi a \; T_{||}\,E^{i}_{||} \; e^{jk_2 x''\sin(\theta_r) + d\cos(\theta_r)} \frac{\sin(\theta_r)}{k_2\cos(\theta_r)} \tag{91}$$

Case 2 $\quad \vec{E}^{\,i}$ perpendicular to plane of incidence

For this case, equation (66) reduces to

$$V_{\gamma lm} = V_{\gamma lm\,\perp} = -\iint_s ds\ \vec{K}_{\gamma lm} \cdot \vec{E}_\perp^{\,1} \tag{92}$$

For $\gamma = r$, this becomes

$$V_{rlm\,\perp} = -\iint_s ds\ \vec{K}_{rlm} \cdot \hat{y}''\ E_{\perp y''}^{1} \tag{93}$$

since $\vec{E}_\perp^{\,1}$ has no component in the x'' direction. Let

$$E_{\perp y''}^{1} = \bar{E}_\perp^{1}\ e^{jk_1 x'' \sin(\theta_i)} \tag{94}$$

where, from 12(75),

$$\bar{E}_\perp^{1} = 2\, E_\perp^{i}\ e^{j1/2\Phi_{||}} \cos(1/2\Phi_{||} - k_1 z\ \cos(\theta_i)) \tag{95}$$

Using (72), equation (94) becomes

$$E_{\perp y''}^{1} = \bar{E}_\perp^{1}\ e^{jk_1 r\ \sin(\theta_i) \cos(\phi - \phi_i)} \tag{96}$$

Using, from equation 13(5)

$$\hat{r} \cdot \hat{y}'' = \sin(\phi)\ \cos(\phi_i) - \cos(\phi)\ \sin(\phi_i) = \sin(\phi - \phi_i) \tag{97}$$

equation (93) then becomes

$$V_{rlm\,\perp} = -\iint_s \bar{E}_{\perp y}^{1}\ e^{jk_1 r\ \sin(\theta_i) \cos(\phi - \phi_i)}\ e^{jl\phi} \sin(\phi - \phi_i)\ T_m(\ r/b\ )\ \sqrt{1 - (\ r/b\ )^2}\ rdrd\phi \tag{98}$$

Using Euler's identity, (76b), and (86), this becomes

$$V_{rlm\,\perp} = \frac{\bar{E}_{\perp y''}^{1}}{2} \int_0^b \int_0^{2\pi} rdrd\phi\ j\ e^{jl\phi_i}\ e^{jk_1 r\ \sin(\theta_i) \cos(\phi - \phi_i)}\ T_m(\ r/b\ )\ \sqrt{1 - (\ r/b\ )^2}$$

$$\Big[\cos((l+1)\ (\phi - \phi_i)) + \cos((l-1)\ (\phi - \phi_i)) + j\ \sin((l+1)\ (\phi - \phi_i)) - j\ \sin((l-1)\ (\phi - \phi_i))\Big] \tag{99}$$

Using the Bessel identities (78a) and (78b), this becomes

$$V_{rlm\,\perp} = -\bar{E}_\perp^{1} e^{jl\phi_i}\ \pi\ j^{l+1} \int_0^b \Big[J_{l+1}(k_1 r\ \sin(\theta_i)) + J_{l-1}(k_1 r\ \sin(\theta_i))\Big] T_m(r/b)\ \sqrt{1 - (\ r/b\ )^2} rdr \tag{100}$$

Using (16), this becomes

$$V_{rlm\perp} = -\bar{E}_{\perp}^{1}\, e^{jl\phi_i}\, \pi\, j^{l} \left[B_{l+1,m}^{-}(k_1 \sin(\theta_i)) + B_{l-1,m}^{-}(k_1 \sin(\theta_i))\right] \tag{101}$$

For $\gamma = \phi$, equation (66) becomes

$$V_{\phi lm\perp} = -\iint_s ds\, \vec{K}_{\phi lm} \cdot \hat{y}''\, E_{\perp}^{1} \tag{102}$$

$$\hat{\phi} \cdot \hat{y}'' = \cos(\phi)\cos(\phi') + \sin(\phi)\sin(\phi') = \cos(\phi - \phi') \tag{103}$$

Using (73), (103) and 14.1(44), equation (102) becomes

$$V_{\phi lm\perp} = -\iint_s \bar{E}_{\perp}^{1}\, e^{jk_1 r \sin(\theta_i)\cos(\phi - \phi_i)}\, e^{jl\phi} \cos(\phi - \phi_i)\, T_m(r/b)\, \frac{1}{\sqrt{1-(r/b)^2}}\, r dr d\phi \tag{104}$$

Using Euler's identity, and (78a,b) equation (104) becomes

$$V_{\phi lm\perp} = -\frac{\bar{E}_{\perp}^{1}}{2} \int_0^b \int_0^{2\pi} r dr d\phi\, e^{jl\phi_i}\, e^{jk_1 r \sin(\theta_i)\cos(\phi - \phi_i)}\, T_m(r/b)\, \frac{1}{\sqrt{1-(r/b)^2}}$$

$$\left[\cos((l+1)(\phi - \phi_i)) + \cos((l-1)(\phi - \phi_i)) + j\sin((l+1)(\phi - \phi_i)) - j\sin((l-1)(\phi - \phi_i))\right] \tag{105}$$

Using (78a) and (78b), this becomes

$$V_{\phi lm\perp} = -\bar{E}_{\perp}^{1} e^{jl\phi_i}\, \pi\, j^{l+1} \int_0^b \left[J_{l+1}(k_1 r \sin(\theta_i)) - J_{l-1}(k_1 r \sin(\theta_i))\right] T_m(r/b)\, \frac{1}{\sqrt{1-(r/b)^2}}\, r dr \tag{106}$$

Using (26), this becomes

$$V_{\phi lm\perp} = -\bar{E}_{\perp}^{1}\, e^{jl\phi_i}\, \pi\, j^{l+1} \left[A_{l+1,m}^{-}(k_1 \sin(\theta_i)) - A_{l-1,m}^{-}(k_1 \sin(\theta_i))\right] \tag{107}$$

For perpendicular incidence, equation (68) reduces to

$$V_{z\perp} = -\int_{-d}^{0} dz\; 2\pi a K_z \hat{z} \cdot \vec{E}_{\perp}^{2} \tag{108}$$

From 12(82), $\vec{E}_{\perp}^{2}$ has no component in the z direction, so

$$V_{z\perp} = 0 \tag{109}$$

Below is a summary of the integrals evaluated.

$$I_{rlmx}^{\pm} = \pi\, j^{l+1}\, e^{j(l+1)\theta}\, B_{l+1,m}^{\pm} + \pi\, j^{l-1}\, e^{j(l-1)\theta}\, B_{l-1,m}^{\pm} \tag{15}$$

$$I^{\pm}_{rlmy} = \pi\, j^{l}\, e^{j(l+1)\theta}\, B^{\pm}_{l+1,m} + \pi\, j^{l}\, e^{j(l-1)\theta}\, B^{\pm}_{l-1,m} \tag{20}$$

$$I^{\pm}_{\phi lmx} = -\pi\, j^{l}\, e^{j(l+1)\theta}\, A^{\pm}_{l+1,m} - \pi\, j^{l}\, e^{j(l-1)\theta}\, A^{\pm}_{l-1,m} \tag{25}$$

$$I^{\pm}_{\phi lmy} = \pi\, j^{l+1}\, e^{j(l+1)\theta}\, A^{\pm}_{l+1,m} + \pi\, j^{l-1}\, e^{j(l-1)\theta}\, A^{\pm}_{l-1,m} \tag{31}$$

$$A^{\pm}_{l,m} = \frac{\pi b^2}{4}\left[J_{\frac{|l|+m-1}{2}}\left[\frac{kb}{2}\right] J_{\frac{|l|-m+1}{2}}\left[\frac{kb}{2}\right] + J_{\frac{|l|+m+1}{2}}\left[\frac{kb}{2}\right] J_{\frac{|l|-m-1}{2}}\left[\frac{kb}{2}\right]\right]$$

$$\cdot \begin{Bmatrix} (\pm 1)^{l} & l \geq 0 \\ (-\pm 1)^{l} & l < 0 \end{Bmatrix} \tag{41}$$

$$B^{\pm}_{l,m} = \frac{1}{4}\left[2\, A^{\pm}_{l,m} - A^{\pm}_{l,|m-2|} - A^{\pm}_{l,m+2}\right] = \frac{1}{4}\left[2\, A^{\pm}_{l,m} - A^{\pm}_{l,m-2} - A^{\pm}_{l,m+2}\right] \tag{50}$$

$$I_z = d\, \frac{k_2^2}{p_2^2} - \frac{\varepsilon_r\, k^2}{p_2^2\, T_m\, p_2} \sinh(p_2 d) \tag{65}$$

$$V_{rlm\,||} = \bar{E}^{1}_{||}\, e^{jl\phi_i}\, \pi\, j^{l+1}\left[B^{-}_{l+1,m}(k_1 \sin(\theta_i)) - B^{-}_{l-1,m}(k_1 \sin(\theta_i))\right] \tag{80}$$

$$V_{\phi lm\,||} = \bar{E}^{1}_{||}\, e^{jl\phi_i}\, \pi\, j^{l+1}\left[A^{-}_{l+1,m}(k_1 \sin(\theta_i)) + A^{-}_{l-1,m}(k_1 \sin(\theta_i))\right] \tag{87}$$

$$V_{z\,||} = -4\pi a\; T_{||}\, E^{i}_{||}\; e^{jk_2 x'' \sin(\theta_r) + d\cos(\theta_r)}\, \frac{\sin(\theta_r)}{k_2 \cos(\theta_r)} \tag{91}$$

$$V_{rlm\,\perp} = -\,\bar{E}^{1}_{\perp}\, e^{jl\phi_i}\, \pi\, j^{l}\left[B^{-}_{l+1,m}(k_1 \sin(\theta_i)) + B^{-}_{l-1,m}(k_1 \sin(\theta_i))\right] \tag{101}$$

$$V_{\phi lm\,\perp} = -\,\bar{E}^{1}_{\perp}\, e^{jl\phi_i}\, \pi\, j^{l+1}\left[A^{-}_{l+1,m}(k_1 \sin(\theta_i)) - A^{-}_{l-1,m}(k_1 \sin(\theta_i))\right] \tag{107}$$

$$V_{z\perp} = 0 \tag{109}$$

VII. PROGRAMMING DETAILS FOR SINGLE PATCH

17. Programming details for single patch.

17.1 Expansion of $A_{l,m}^{\pm}$ and $B_{l,m}^{\pm}$.

The matrix elements of section 15.1 are obtained in terms of products of $A_{l,m}$ and $B_{l,m}$. These products are here expressed in terms of quadrupole products of Bessel functions. Also, $B_{l,m}$ is obtained in terms of $A_{l,m}$. From equation 16(41),

$$A_{l,m}^{\pm} = \frac{\pi b^2}{4} \left[J_{\frac{|l|+m-1}{2}}\left[\frac{kb}{2}\right] J_{\frac{|l|-m+1}{2}}\left[\frac{kb}{2}\right] + J_{\frac{|l|+m+1}{2}}\left[\frac{kb}{2}\right] J_{\frac{|l|-m-1}{2}}\left[\frac{kb}{2}\right] \right] \cdot \begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{Bmatrix} \quad (1)$$

In practice, $A_{l,m}^{\pm}$ is needed only where $m+l$ is even. This implies the Bessel function orders are all half-odd-integer. Consequently, the use of spherical Bessel functions is indicated.

$$J_{n+1/2}(z) = \sqrt{\frac{2z}{\pi}}\, j_n(z) \quad (2)$$

where $j_n(z)$ is the spherical Bessel function of order n [16]

Rewriting $A_{l,m}^{\pm}$ then gives

$$A_{l,m}^{\pm} = \frac{b^2}{2} \left[\frac{kb}{2}\, j_{\frac{|l|+m-2}{2}}\left[\frac{kb}{2}\right] j_{\frac{|l|-m}{2}}\left[\frac{kb}{2}\right] + \frac{kb}{2}\, j_{\frac{|l|+m}{2}}\left[\frac{kb}{2}\right] j_{\frac{|l|-m-2}{2}}\left[\frac{kb}{2}\right] \right] \cdot \begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{Bmatrix} \quad (3)$$

The products needed are in the forms

$$A_{l+1,m}^{+} A_{-l-1,m'}^{-} \pm A_{l-1,m}^{+} A_{-l+1,m'}^{-} \quad (4)$$

and

$$A^{+}_{l+1,m}\,A^{-}_{-l+1,m'} \pm A^{+}_{l-1,m}\,A^{-}_{-l-1,m'} \tag{5}$$

Equation (4) is expanded using equation (3) to give

$$A^{+}_{l+1,m}\,A^{-}_{-l-1,m'} \pm A^{+}_{l-1,m}\,A^{-}_{-l+1,m'} =$$

$$= \frac{b^4}{4}\Bigg\{\Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$+ \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'-2}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$+ \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$+ \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'-2}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$\pm \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$\pm \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'-2}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$\pm \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'}{2}}\left(\frac{kb}{2}\right)\Bigg]$$

$$\pm \Bigg[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'-2}{2}}\left(\frac{kb}{2}\right)\Bigg]\Bigg\} \tag{6}$$

Equation (5) is expanded using equation (3) to give

$$A^{+}_{l+1,m}\,A^{-}_{-l+1,m'} \pm A^{+}_{l-1,m}\,A^{-}_{-l-1,m'} =$$

$$\begin{aligned}
= \frac{b^4}{4} \Bigg\{ & \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'}{2}}\left(\frac{kb}{2}\right) \right] \\
& + \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'-2}{2}}\left(\frac{kb}{2}\right) \right] \\
& + \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'}{2}}\left(\frac{kb}{2}\right) \right] \\
& + \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l+1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m'-2}{2}}\left(\frac{kb}{2}\right) \right] \\
& \pm \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'}{2}}\left(\frac{kb}{2}\right) \right] \\
& \pm \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'-2}{2}}\left(\frac{kb}{2}\right) \right] \\
& \pm \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'}{2}}\left(\frac{kb}{2}\right) \right] \\
& \pm \left[\left[\frac{kb}{2}\right]^2 j_{\frac{|l-1|+m}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l-1|-m-2}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|+m'}{2}}\left(\frac{kb}{2}\right) j_{\frac{|l+1|-m'-2}{2}}\left(\frac{kb}{2}\right) \right] \Bigg\}
\end{aligned} \tag{7}$$

Expressions (4) and (5) are used directly in the patch-patch matrix elements relating radial electric fields to radial surface currents. For patch-patch matrix elements relating azimuthal electric fields to azimuthal surface currents, the products needed involve $B^{\pm}_{l,m}$. These can be expressed in terms of products of $A^{\pm}_{l,m}$ already obtained.

$$B^{+}_{l+1,m}\,B^{-}_{-l-1,m'} \pm B^{+}_{l-1,m}\,B^{-}_{-l+1,m'} \tag{8}$$

$$B^{+}_{l+1,m}\,B^{-}_{-l+1,m'} \pm B^{+}_{l-1,m}\,B^{-}_{-l-1,m'} \tag{9}$$

From equation 17.( ),

$$B^{\pm}_{l,m} = \frac{1}{4}\left[2\,A^{\pm}_{l,m} - A^{\pm}_{l,|m-2|} - A^{\pm}_{l,m+2}\right] \tag{10}$$

Using (10), equation (8) is found in terms of $A^{\pm}_{l,m}$ as

$$\begin{aligned}
B^{+}_{l+1,m}\,B^{-}_{-l-1,m'} + B^{+}_{l-1,m}\,B^{-}_{-l+1,m'} &= \frac{1}{4}\left[A^{+}_{l+1,m}\,A^{-}_{-l-1,m'} + A^{+}_{l-1,m}\,A^{-}_{-l+1,m'}\right] + \\
&+ \frac{1}{16}\left[A^{+}_{l+1,m+2}\,A^{-}_{-l-1,m'+2} + A^{+}_{l+1,m+2}\,A^{-}_{-l-1,m'+2} + A^{+}_{l+1,m-2}\,A^{-}_{-l-1,m'+2} + A^{+}_{l+1,m-2}\,A^{-}_{-l-1,m'-2}\right] + \\
&+ \frac{1}{16}\left[A^{+}_{l-1,m+2}\,A^{-}_{-l+1,m'+2} + A^{+}_{l-1,m+2}\,A^{-}_{-l+1,m'+2} + A^{+}_{l-1,m-2}\,A^{-}_{-l+1,m'+2} + A^{+}_{l-1,m-2}\,A^{-}_{-l+1,m'-2}\right] + \\
&+ \frac{-1}{8}\left[A^{+}_{l+1,m-2}\,A^{-}_{-l-1,m'} + A^{+}_{l+1,m+2}\,A^{-}_{-l-1,m'} + A^{+}_{l+1,m}\,A^{-}_{-l-1,m'-2} + A^{+}_{l+1,m}\,A^{-}_{-l-1,m'+2}\right] + \\
&+ \frac{-1}{8}\left[A^{+}_{l-1,m-2}\,A^{-}_{-l+1,m'} + A^{+}_{l-1,m+2}\,A^{-}_{-l+1,m'} + A^{+}_{l-1,m}\,A^{-}_{-l+1,m'-2} + A^{+}_{l-1,m}\,A^{-}_{-l+1,m'+2}\right]
\end{aligned} \tag{11}$$

Using (10), equation (9) is found in terms of $A^{\pm}_{l,m}$ as

$$\begin{aligned}
B^{+}_{l+1,m}\,B^{-}_{-l+1,m'} + B^{+}_{l-1,m}\,B^{-}_{-l-1,m'} &= \frac{1}{4}\left[A^{+}_{l+1,m}\,A^{-}_{-l+1,m'} + A^{+}_{l-1,m}\,A^{-}_{-l-1,m'}\right] + \\
&+ \frac{1}{16}\left[A^{+}_{l+1,m+2}\,A^{-}_{-l+1,m'+2} + A^{+}_{l+1,m+2}\,A^{-}_{-l+1,m'+2} + A^{+}_{l+1,m-2}\,A^{-}_{-l+1,m'+2} + A^{+}_{l+1,m-2}\,A^{-}_{-l+1,m'-2}\right] + \\
&+ \frac{1}{16}\left[A^{+}_{l-1,m+2}\,A^{-}_{-l-1,m'+2} + A^{+}_{l-1,m+2}\,A^{-}_{-l-1,m'+2} + A^{+}_{l-1,m-2}\,A^{-}_{-l-1,m'+2} + A^{+}_{l-1,m-2}\,A^{-}_{-l-1,m'-2}\right] + \\
&+ \frac{-1}{8}\left[A^{+}_{l+1,m-2}\,A^{-}_{-l+1,m'} + A^{+}_{l+1,m+2}\,A^{-}_{-l+1,m'} + A^{+}_{l+1,m}\,A^{-}_{-l+1,m'-2} + A^{+}_{l+1,m}\,A^{-}_{-l+1,m'+2}\right] + \\
&+ \frac{-1}{8}\left[A^{+}_{l-1,m-2}\,A^{-}_{-l-1,m'} + A^{+}_{l-1,m+2}\,A^{-}_{-l-1,m'} + A^{+}_{l-1,m}\,A^{-}_{-l-1,m'-2} + A^{+}_{l-1,m}\,A^{-}_{-l-1,m'+2}\right]
\end{aligned} \tag{12}$$

In addition, the patch-feed pin and feed pin-patch involve cross products of $A^{\pm}_{l,m}$ and $B^{\pm}_{l,m}$. The terms needed are

$$B^+_{l+1,m}\, A^-_{-l-1,m'} \pm B^+_{l-1,m}\, A^-_{-l+1,m'} \tag{13}$$

$$B^+_{l+1,m}\, A^-_{-l+1,m'} \pm B^+_{l-1,m}\, A^-_{-l-1,m'} \tag{14}$$

$$A^+_{l+1,m}\, B^-_{-l-1,m'} \pm A^+_{l-1,m}\, B^-_{-l+1,m'} \tag{15}$$

$$A^+_{l+1,m}\, B^-_{-l+1,m'} \pm A^+_{l-1,m}\, B^-_{-l-1,m'} \tag{16}$$

These are expanded using (15.1.10) to give

$$B^+_{l+1,m}\, A^-_{-l-1,m'} - B^+_{l-1,m}\, A^-_{-l+1,m'} = \frac{1}{2}\left[A^+_{l+1,m}\, A^-_{-l-1,m'} - A^+_{l-1,m}\, A^-_{-l+1,m'}\right] +$$

$$+ \frac{-1}{4}\left[A^+_{l+1,m+2}\, A^-_{-l-1,m'} + A^+_{l+1,m-2}\, A^-_{-l-1,m'} - A^+_{l-1,m+2}\, A^-_{-l+1,m'} - A^+_{l-1,m-2}\, A^-_{-l+1,m'}\right] \tag{17}$$

$$B^+_{l+1,m}\, A^-_{-l+1,m'} - B^+_{l-1,m}\, A^-_{-l-1,m'} = \frac{1}{2}\left[A^+_{l+1,m}\, A^-_{-l+1,m'} - A^+_{l-1,m}\, A^-_{-l-1,m'}\right] +$$

$$+ \frac{-1}{4}\left[A^+_{l+1,m+2}\, A^-_{-l+1,m'} + A^+_{l+1,m-2}\, A^-_{-l+1,m'} - A^+_{l-1,m+2}\, A^-_{-l-1,m'} - A^+_{l-1,m-2}\, A^-_{-l-1,m'}\right] \tag{18}$$

$$A^+_{l+1,m}\, B^-_{-l-1,m'} - A^+_{l-1,m}\, B^-_{-l+1,m'} = \frac{1}{2}\left[A^+_{l+1,m}\, A^-_{-l-1,m'} - A^+_{l-1,m}\, A^-_{-l+1,m'}\right] +$$

$$+ \frac{-1}{4}\left[A^+_{l+1,m}\, A^-_{-l-1,m'+2} + A^+_{l+1,m}\, A^-_{-l-1,m'-2} - A^+_{l-1,m}\, A^-_{-l+1,m'+2} - A^+_{l-1,m}\, A^-_{-l+1,m'-2}\right] \tag{19}$$

$$A^+_{l+1,m}\, B^-_{-l+1,m'} - A^+_{l-1,m}\, B^-_{-l-1,m'} = \frac{1}{2}\left[A^+_{l+1,m}\, A^-_{-l+1,m'} - A^+_{l-1,m}\, A^-_{-l-1,m'}\right] +$$

$$+ \frac{-1}{4}\left[A^+_{l+1,m}\, A^-_{-l+1,m'+2} + A^+_{l+1,m}\, A^-_{-l+1,m'-2} - A^+_{l-1,m}\, A^-_{-l-1,m'+2} - A^+_{l-1,m}\, A^-_{-l-1,m'-2}\right] \tag{20}$$

The expansions in (11), (12) and (17) through (20) are all sums of terms of the forms of (4) and (5).

For the case of $l = 0$, the expressions (17) through (20) reduce to

$$B^+_{1,m}\, A^-_{-1,m'} - B^+_{-1,m}\, A^-_{1,m'} = \frac{1}{2}\left[A^+_{1,m}\, A^-_{-1,m'} - A^+_{-1,m}\, A^-_{+1,m'}\right] +$$

$$+ \frac{-1}{4}\left[A^+_{1,m+2}\, A^-_{-1,m'} + A^+_{1,m-2}\, A^-_{-1,m'} - A^+_{-1,m+2}\, A^-_{1,m'} - A^+_{-1,m-2}\, A^-_{1,m'}\right] \tag{21}$$

$$B^+_{1,m}\, A^-_{1,m'} - B^+_{-1,m}\, A^-_{-1,m'} = \frac{1}{2}\left[A^+_{1,m}\, A^-_{1,m'} - A^+_{-1,m}\, A^-_{-1,m'}\right] +$$

$$+ \frac{-1}{4} \left[A^+_{1,m+2} A^-_{1,m'} + A^+_{1,m-2} A^-_{1,m'} - A^+_{-1,m+2} A^-_{-1,m'} - A^+_{-1,m-2} A^-_{-1,m'} \right] \tag{22}$$

$$A^+_{1,m} B^-_{-1,m'} - A^+_{-1,m} B^-_{1,m'} = \frac{1}{2} \left[A^+_{1,m} A^-_{-1,m'} - A^+_{-1,m} A^-_{1,m'} \right] +$$

$$+ \frac{-1}{4} \left[A^+_{1,m} A^-_{-1,m'+2} + A^+_{1,m} A^-_{-1,m'-2} - A^+_{-1,m} A^-_{1,m'+2} - A^+_{-1,m} A^-_{1,m'-2} \right] \tag{23}$$

$$A^+_{1,m} B^-_{1,m'} - A^+_{-1,m} B^-_{-1,m'} = \frac{1}{2} \left[A^+_{1,m} A^-_{1,m'} - A^+_{-1,m} A^-_{-1,m'} \right] +$$

$$+ \frac{-1}{4} \left[A^+_{1,m} A^-_{1,m'+2} + A^+_{1,m} A^-_{1,m'-2} - A^+_{-1,m} A^-_{-1,m'+2} - A^+_{-1,m} A^-_{-1,m'-2} \right] \tag{24}$$

By inspection of (1) for $l = \pm 1$, it can be seen that

$$A^-_{-1,m} = A^+_{1,m} \tag{25}$$

and that

$$A^+_{-1,m} = A^-_{1,m} = -A^+_{1,m} \tag{26}$$

The right hand sides of equations (21) through (24) thus reduce to zero.

The double and quadruple products of spherical Bessel functions needed for equations 14.1(6) and 14.1(7) will be obtained as a sum of polynomials and polynomials times simple trigonometric functions.

From [16] (equations (10.1.8) and (10.1.9), Abramowitz and Stegun, page 437),

$$j_n(z) = z^{-1} \left[P(n+1/2,\, z) \sin(z - n\pi/2) + Q(n+1/2,\, z) \cos(z - n\pi/2) \right] \tag{27}$$

$$j_{-(n+1)}(z) = z^{-1} \left[P(n+1/2,\, z) \cos(z + n\pi/2) - Q(n+1/2,\, z) \sin(z + n\pi/2) \right] \tag{28}$$

where

$$P(n+1/2,\, z) = \sum_{0}^{[n/2]} (-1)^k \frac{(n+2k)!}{(2k)!\,(n-2k)!} (2z)^{-2k} \tag{29}$$

$$Q(n+1/2,\, z) = \sum_{0}^{[(n-1)/2]} (-1)^k \frac{(n+2k+1)!}{(2k+1)!\,(n-2k-1)!} (2z)^{-2k-1} \tag{30}$$

From equation (28),

$$j_{-n}(z) = z^{-1}\left[P(n-1/2,\, z)\cos(z+(n-1)\pi/2) - Q(n-1/2,\, z)\sin(z+(n-1)\pi/2)\right] \tag{31}$$

Using

$$\cos(z+(n-1)\pi/2) = \sin(z+n\pi/2) \tag{32}$$

and

$$\sin(z+(n-1)\pi/2) = -\cos(z+n\pi/2) \tag{33}$$

and letting $-n \to n$,

$$j_n(z) = z^{-1}\left[P(|n+1|/2,\, z)\sin(z-n\pi/2) + Q(|n+1|/2,\, z)\cos(z-n\pi/2)\right] \tag{34}$$

Equation (34) holds for all integer n.

Using (34) then gives

$$\begin{aligned} z\, j_n(z)\, j_m(z) = z^{-1} \Big[& P(|n+1|/2,\, z)\, P(|m+1/2|,\, z)\sin(z-n\pi/2)\sin(z-m\pi/2) + \\ & + Q(|n+1|/2,\, z)\, Q(|m+1|/2,\, z)\cos(z-n\pi/2)\cos(z-m\pi/2) \\ & + P(|n+1|/2,\, z)\, Q(|m+1|/2,\, z)\sin(z-n\pi/2)\cos(z-m\pi/2) + \\ & + Q(|n+1|/2,\, z)\, P(|m+1|/2,\, z)\cos(z-n\pi/2)\sin(z-m\pi/2) \Big] \end{aligned} \tag{35}$$

Using

$$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y) \tag{36a}$$

$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y) \tag{36b}$$

$$2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y) \tag{36c}$$

equation (35) becomes

$$\begin{aligned} z\, j_n(z)\, j_m(z) = (2z)^{-1} \Big[& P(|n+1|/2,\, z)\, P(|m+1|/2,\, z)\left(\cos((n-m)\pi/2) - \cos(2z-(m+n)\pi/2)\right) + \\ & + Q(|n+1|/2,\, z)\, Q(|m+1|/2,\, z)\left(\cos((n-m)\pi/2) + \cos(2z-(m+n)\pi/2)\right) \\ & + P(|n+1|/2,\, z)\, Q(|m+1|/2,\, z)\left(\sin((n-m)\pi/2) + \sin(2z-(m+n)\pi/2)\right) + \\ & + Q(|n+1|/2, z)\, P(|m+1|/2, z)\left(-\sin((n-m)\pi/2) + \sin(2z-(m+n)\pi/2)\right) \Big] \end{aligned} \tag{37}$$

Using

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y) \tag{38a}$$

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y) \tag{38b}$$

to obtain

$$\cos(2z-(m+n)\pi/2) = \cos((n+m)\pi/2)\cos(2z) + \sin((n+m)\pi/2)\sin(2z) \tag{39}$$

$$\sin(2z-(m+n)\pi/2) = \cos((n+m)\pi/2)\sin(2z) - \sin((n+m)\pi/2)\cos(2z) \tag{40}$$

equation (37) can be rewritten as

$$\begin{aligned} z\, j_n(z)\, j_m(z) = (2z)^{-1} \times & \\ \times \Big\{ & \Big[P(|n+1|/2, z)\, P(|m+1|/2, z) + Q(|n+1|/2, z)\, Q(|m+1|/2, z)\Big] \cos((n-m)\pi/2) + \\ + & \Big[P(|n+1|/2, z)\, Q(|m+1|/2, z) - Q(|n+1|/2, z)\, P(|m+1|/2, z)\Big] \sin((n-m)\pi/2) + \\ - & \cos(2z) \Big[\Big[P(|n+1|/2,z)\, P(|m+1|/2,z) - Q(|n+1|/2,z)\, Q(|m+1|/2,z)\Big] \cos((n+m)\pi/2) + \\ + & \Big[P(|n+1|/2,z)\, Q(|m+1|/2,z) + Q(|n+1|/2,z)\, P(|m+1|/2,z)\Big] \sin((n+m)\pi/2)\Big] \\ - & \sin(2z) \Big[\Big[P(|n+1|/2,z)\, P(|m+1|/2,z) - Q(|n+1|/2,z)\, Q(|m+1|/2,z)\Big] \sin((n+m)\pi/2) + \\ - & \Big[P(|n+1|/2,z)\, Q(|m+1|/2,z) + Q(|n+1|/2,z)\, P(|m+1|/2,z)\Big] \cos((n+m)\pi/2)\Big]\Big\} \end{aligned} \tag{41}$$

Making the definitions

$$D^{\pm}_{n,m}(z) = (2z)^{-1} \Big[P(|n+1|/2, z)\, P(|m+1|/2, z) \pm Q(|n+1|/2, z)\, Q(|m+1|/2, z)\Big] \tag{42}$$

and

$$E^{\pm}_{n,m}(z) = (2z)^{-1} \Big[P(|n+1|/2, z)\, Q(|m+1|/2, z) \pm Q(|n+1|/2, z)\, P(|m+1|/2, z)\Big] \tag{43}$$

equation (35) becomes

$$z\ j_n(z)\ j_m(z) = D^+_{n,m}(z) \cos((n-m)\pi/2) - E^-_{n,m}(z) \sin((n-m)\pi/2) +$$

$$+ \cos(2z) \left[-D^-_{n,m}(z) \cos((n+m)\pi/2) - E^+_{n,m}(z) \sin((n+m)\pi/2) \right]$$

$$+ \sin(2z) \left[-D^-_{n,m}(z) \sin((n+m)\pi/2) + E^+_{n,m}(z) \cos((n+m)\pi/2) \right] \tag{44}$$

Equation (44) is sufficient for $A_{l,m}$ and $B_{l,m}$, but not for products of them. Using equation (44), and making the definitions

$$c(n) \equiv \cos(\frac{n\pi}{2}) \tag{45a}$$

and

$$s(n) \equiv \sin(\frac{n\pi}{2}) \tag{45b}$$

the product of four spherical Bessel function is found as

$$z^2\, j_n(z)\, j_m(z)\, j_{n'}(z)\, j_{m'}(z) =$$

$$= D^+_{n,m}(z)\, D^+_{n',m'}(z)\, c(n-m)\, c(n'-m') + E^-_{n,m}(z)\, E^-_{n',m'}(z)\, s(n-m)\, s(n'-m')$$

$$- D^+_{n,m}(z)\, E^-_{n',m'}(z)\, c(n-m)\, s(n'-m') - E^-_{n,m}(z)\, D^-_{n',m'}(z)\, s(n-m)\, c(n'-m')$$

$$+ \cos(2z) \Big[-D^+_{n,m}(z)\, D^-_{n',m'}(z)\, c(n-m)\, c(n'+m') + E^-_{n,m}(z)\, E^+_{n',m'}(z)\, s(n-m)\, s(n'+m')$$

$$- D^+_{n,m}(z)\, E^+_{n',m'}(z)\, c(n-m)\, s(n'+m') + E^-_{n,m}(z)\, D^-_{n',m'}(z)\, s(n-m)\, c(n'+m')$$

$$- D^-_{n,m}(z)\, D^+_{n',m'}(z)\, c(n+m)\, c(n'-m') + E^+_{n,m}(z)\, E^-_{n',m'}(z)\, s(n+m)\, s(n'-m')$$

$$+ D^-_{n,m}(z)\, E^-_{n',m'}(z)\, c(n+m)\, s(n'-m') - E^+_{n,m}(z)\, D^+_{n',m'}(z)\, s(n+m)\, c(n'-m') \Big]$$

$$+ \sin(2z) \Big[-D^+_{n,m}(z)\, D^-_{n',m'}(z)\, c(n-m)\, s(n'+m') - E^-_{n,m}(z)\, E^+_{n',m'}(z)\, s(n-m)\, c(n'+m')$$

$$+ D^+_{n,m}(z)\, E^+_{n',m'}(z)\, c(n-m)\, c(n'+m') + E^-_{n,m}(z)\, D^-_{n',m'}(z)\, s(n-m)\, s(n'+m')$$

$$- D^-_{n,m}(z)\, D^+_{n',m'}(z)\, s(n+m)\, c(n'-m') - E^+_{n,m}(z)\, E^-_{n',m'}(z)\, c(n+m)\, s(n'-m')$$

$$+ D^-_{n,m}(z)\, E^-_{n',m'}(z)\, s(n+m)\, s(n'-m') + E^+_{n,m}(z)\, D^+_{n',m'}(z)\, c(n+m)\, c(n'-m') \Big]$$

$$+ \frac{\sin(4z)}{2} \Big[D^-_{n,m}(z)\, D^-_{n',m'}(z)\, c(n+m)\, s(n'+m') - E^+_{n,m}(z)\, E^+_{n',m'}(z)\, s(n+m)\, c(n'+m')$$

$$- D^-_{n,m}(z)\, E^+_{n',m'}(z)\, c(n+m)\, c(n'+m') + E^+_{n,m}(z)\, D^-_{n',m'}(z)\, s(n+m)\, s(n'+m')$$

$$+ D^-_{n,m}(z)\, D^-_{n',m'}(z)\, s(n+m)\, c(n'+m') - E^+_{n,m}(z)\, E^+_{n',m'}(z)\, c(n+m)\, s(n'+m')$$

$$+ D^-_{n,m}(z)\, E^+_{n',m'}(z)\, s(n+m)\, s(n'+m') - E^+_{n,m}(z)\, D^-_{n',m'}(z)\, c(n+m)\, c(n'+m') \Big]$$

$$+ \frac{1+\cos(4z)}{2} \Big[D^-_{n,m}(z)\, D^-_{n',m'}(z)\, c(n+m)\, c(n'+m') + E^+_{n,m}(z)\, E^+_{n',m'}(z)\, s(n+m)\, s(n'+m')$$

$$+ D^-_{n,m}(z)\, E^+_{n',m'}(z)\, c(n+m)\, s(n'+m') + E^+_{n,m}(z)\, D^-_{n',m'}(z)\, s(n+m)\, c(n'+m') \Big]$$

$$+ \frac{1-\cos(4z)}{2} \Big[D^-_{n,m}(z)\, D^-_{n',m'}(z)\, s(n+m)\, s(n'+m') + E^+_{n,m}(z)\, E^+_{n',m'}(z)\, c(n+m)\, c(n'+m')$$

$$- D^-_{n,m}(z)\, E^+_{n',m'}(z)\, s(n+m)\, c(n'+m') - E^+_{n,m}(z)\, D^-_{n',m'}(z)\, c(n+m)\, s(n'+m') \Big] \tag{46}$$

17.2 Asymptotic Forms for Spectral Integrands.

In this section, the asymptotic forms for evaluating the matrix elements are obtained. The matrix elements relating electric field over the patch surface to Tcheby-chef surface current distributions on the patch surface are

$$Z^{rlm}_{rl'm'} = \frac{1}{2\pi}\int_0^\infty dk\,\frac{2k}{j\omega\varepsilon_1}\,\frac{\pi^2\,\delta_{l,-l'}}{(p_1+p_2\coth(p_2 d))}\times$$

$$\times\left\{\left[k_1^2-\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\right]\left[B_{l+1,m}\,B_{-l-1,m'}+B_{l-1,m}\,B_{-l+1,m'}\right]+\right.$$

$$\left.+\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\left[B_{l+1,m}\,B_{-l+1,m'}+B_{l-1,m}\,B_{-l-1,m'}\right]\right\} \tag{1}$$

$$Z^{\phi lm}_{\phi l'm'} = \frac{1}{2\pi}\int_0^\infty dk\,\frac{2k}{j\,\omega\,\varepsilon_1}\,\frac{\pi^2\,\delta_{l,-l'}}{(p_1+p_2\coth(p_2 d))}\times$$

$$\times\left\{\left[k_1^2-\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\right]\left[A_{l+1,m}\,A_{-l-1,m'}+A_{l-1,m}\,A_{-l+1,m'}\right]+\right.$$

$$\left.-\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\left[A_{l+1,m}\,A_{-l+1,m'}+A_{l-1,m}\,A_{-l-1,m'}\right]\right\} \tag{2}$$

$$Z^{\phi lm}_{rl'm'} = \frac{1}{2\pi}\int_0^\infty dk\,\frac{2k}{j\,\omega\,\varepsilon_1}\,\frac{\pi^2\,\delta_{l,-l'}}{(p_1+p_2\coth(p_2 d))}\times$$

$$\times\left\{-j\left[k_1^2-\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\right]\left[B_{l+1,m}\,A_{-l-1,m'}-B_{l-1,m}\,A_{-l+1,m'}\right]+\right.$$

$$\left.+\frac{jk^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\left[B_{l+1,m}\,A_{-l+1,m'}-B_{l-1,m}\,A_{-l-1,m'}\right]\right\} \tag{3}$$

$$Z^{rlm}_{\phi l'm'} = \frac{1}{2\pi}\int_0^\infty dk\,\frac{2k}{j\,\omega\,\varepsilon_1}\,\frac{\pi^2\,\delta_{l,-l'}}{(p_1+p_2\coth(p_2 d))}\times$$

$$\times\left\{j\left[k_1^2-\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\right]\left[A_{l+1,m}\,B_{-l-1,m'}-A_{l-1,m}\,B_{-l+1,m'}\right]+\right.$$

$$\left.+j\,\frac{k^2}{2}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2 d)}\left[A_{l+1,m}\,B_{-l+1,m'}-A_{l-1,m}\,B_{-l-1,m'}\right]\right\} \tag{4}$$

The products of A and B are expressed as products of four spherical Bessel functions of argument $\frac{kb}{2}$ times $\left[\frac{kb}{2}\right]^2$ in section 17.1. For kb less than some value μ, the product of four spherical Bessel functions is expressed as a power series. For kb greater than μ, the product of four spherical Bessel functions of a single argument z times that argument squared is that obtained in 17.1(39). The product is obtained in terms of a polynomial in $2z$ times each of the following: 1, $\sin(2z)$, $\cos(2z)$, $\sin(4z)$ or $\cos(4z)$. Since a polynomial is simply a power series, the products of A and B are expressable as a sum over terms, each of which is a power series in kb times one of 1, $\sin(kb)$, $\cos(kb)$, $\sin(2kb)$ or $\cos(2kb)$. Labeling these various terms as $P_j(kb)$ where each $P_j(kb)$ is a power of kb times 1 or one of the sin or cos factors, each of (1) through (4) can be written as

$$Z_{\gamma l'm'}^{\delta lm} = \frac{\pi\delta_{l,-l'}}{j\omega\varepsilon_1}\int_{\mu}^{\infty} dk \left[\frac{k\,k_1^2}{(p_1+p_2\coth(p_2\,d))}\sum_j c_j\,P_j(kb) + \right.$$

$$\left. + \frac{k^2}{2}\,\frac{1}{(p_1+p_2\coth(p_2\,d))}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2\,d)}\sum_j c_j\,P_j(kb)\right] +$$

$$+\int_0^{\mu} dk\,\left[\,\cdots\,\right] \tag{5}$$

where the integration from 0 to μ will be dealt with in later sections. Equation (5) can be written as

$$Z_{\gamma l'm'}^{\delta lm} = \frac{\pi\delta_{l,-l'}}{j\omega\varepsilon_1}\left[\sum_j c_j\int_{\mu}^{\infty} dk\;g(k)\,P_j(kb) + \sum_j c_j\int_{\mu}^{\infty} dk\;h(k)\,P_j(kb)\right] + \int_0^{\mu} dk\,\left[\,\cdots\,\right] \tag{6}$$

where

$$g(k) = \frac{k^3}{2}\,\frac{1}{(p_1+p_2\coth(p_2\,d))}\,\frac{p_1+p_2\tanh(p_2 d)}{\varepsilon_r\,p_1+p_2\tanh(p_2\,d)} \tag{7}$$

and

$$h(k) = \frac{k\,k_1^2}{(p_1+p_2\coth(p_2\,d))} \tag{8}$$

Each of the integrations of $P_j(kb)$ times $g(k)$ and $h(k)$ in Eq. (6) are performed individually and stored. Once all necessary integrations have been performed, the matrix elements are assembled using Eq. (6). The asymptotic portions of the integrations are subtracted off, performed analytically, then added to the portions obtained via numerical integration. The separation of the asymptotic portion of the integrands is done by finding the bahavior of $h(k)$ and $g(k)$ as k becomes large.

Define

$$g_1(k) \equiv k^2 \lim_{k\to\infty} \frac{g(k)}{k^2} \tag{9}$$

or

$$g_1(k) = k^2 \frac{k}{2} \frac{2}{\varepsilon_r + 1} \frac{1}{2k} = \frac{k^2}{2(\varepsilon_r + 1)} \tag{10}$$

Then let

$$g_1^-(k) \equiv g(k) - g_1(k) \tag{11}$$

or

$$g_1^-(k) = \frac{k^2}{2}\left[\frac{k}{(p_1 + p_2 \coth(p_2 d))} \frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2 d)} - \frac{1}{(\varepsilon_r + 1)}\right] \tag{12}$$

Now let

$$g_2(k) \equiv \lim_{k\to\infty} g_1^-(k) \tag{13}$$

Taking the limit by using

$$p_{1,2} \approx k\left(1 - \frac{k_{1,2}^2}{2k^2}\right) \tag{14}$$

yields

$$g_2(k) = \frac{k^2}{2}\left[\frac{k}{2k - \frac{1}{2k}(k_1^2 + k_2^2)}\left[\frac{2k - \frac{(k_1^2 + k_2^2)}{2k}}{(\varepsilon_r + 1)k - \frac{(\varepsilon_r k_1^2 + k_2^2)}{2k}} - \frac{1}{(\varepsilon_r + 1)}\right]\right] \tag{15}$$

or

$$g_2(k) = \frac{k^2}{2}\left[\frac{1}{(\varepsilon_r + 1) - \varepsilon_r \frac{k_1^2}{k^2}} - \frac{1}{(\varepsilon_r + 1)}\right] \tag{16}$$

Putting the right hand side of Eq. (16) over a common denominator gives

$$g_2(k) = \frac{k^2}{2}\,\frac{(\varepsilon_r + 1) - (\varepsilon_r + 1) + \frac{k_1^2}{k^2}\varepsilon_r}{(\varepsilon_r + 1)((\varepsilon_r + 1) - \varepsilon_r \frac{k_1^2}{k^2})} \tag{17}$$

or

$$g_2(k) = \frac{\varepsilon_r}{(\varepsilon_r + 1)^2} \tag{18}$$

Now define

$$g_2^-(k) \equiv g(k) - g_2(k) - g_1(k) \tag{19}$$

or

$$g_2^-(k) = \frac{1}{2}\left[\frac{k^3}{(p_1 + p_2 \coth(p_2 d))}\,\frac{p_1 + p_2 \tanh(p_2 d)}{\varepsilon_r\, p_1 + p_2 \tanh(p_2 d)} - \frac{k^2}{(\varepsilon_r + 1)} - \frac{k_1^2\, \varepsilon_r}{(\varepsilon_r + 1)^2}\right] \tag{20}$$

Doing the same for $h(k)$, define

$$h_1(k) \equiv k^2 \lim_{k \to \infty} \frac{h(k)}{k^2} \tag{21}$$

or

$$h_1(k) = 0 \tag{22}$$

so

$$h_1^-(k) \equiv h(k) - h_1(k) = h(k) \tag{23}$$

Continuing,

$$h_2(k) \equiv \lim_{k \to \infty} h_1^-(k) \tag{24}$$

Taking the limit gives

$$h_2(k) = \frac{k_1^2}{2} \tag{25}$$

Lastly,

$$h_2^-(k) = \frac{k\ k_1^2}{(p_1 + p_2 \coth(p_2\, d))} - \frac{k_1^2}{2} \tag{25}$$

The integrals involving $g(k)$ are

$$\int_\mu^\infty dk\ \ g(k)\, P_j(kb) \tag{26}$$

The integral in (26) is performed in by separating the asymptotic portions of $g(k)$ and evaluating these pieces analytically. The factor P_j can be written as

$$P_j = \frac{\text{csp}(mkb)}{(kb)^n} \tag{27}$$

where

$$\text{csp}(mkb) \text{ represents } \begin{Bmatrix} \cos(mkb) \\ \sin(mkb) \\ 1 \end{Bmatrix} \tag{28}$$

where m is either 1 or 2.

The separation of the asympotic portions depends on n, and on whether csp is a trigonometric function or simply 1. The cases where csp is trigonometric will be hand- led first, followed by csp = 1. First, for n=2 the integral is broken up as follows.

$$\int_\mu^\infty dk\ \ g(k)\, \frac{\text{csp}(mkb)}{kb^2} = \int_\mu^\infty dk\ \ g_2(k)\, \frac{\text{csp}(mkb)}{kb^2} + \int_\mu^\infty dk\ \ g_1(k)\, \frac{\text{csp}(mkb)}{kb^2} + $$

$$+ \int_\mu^\infty dk\ \ g_2^-(k)\, \frac{\text{csp}(mkb)}{kb^2} \tag{29}$$

In Eq. (29), the integral involving $g_2^-(k)$ is handled numerically, while the other two are handled analytically. For n=3 or 4 the integral is broken up as

$$\int_{\mu}^{\infty} dk \; g(k) \, \frac{\mathrm{csp}(mkb)}{kb^n} = \int_{\mu}^{\infty} dk \; g_1(k) \, \frac{\mathrm{csp}(mkb)}{kb^n} + \int_{\mu}^{\infty} dk \; g_1^-(k) \, \frac{\mathrm{csp}(mkb)}{kb^n} \tag{30}$$

with the integral involving $g_1^-(k)$ handled numerically, while the one involving $g_1(k)$ is handled analytically. For n greater than 4, $g(k)$ is used and the integration is performed numerically.

The asymptotic integrations needed for (29) and (30) are

$$\int_{\mu}^{\infty} dz \, \frac{\sin(mz)}{z^n} \tag{31}$$

and

$$\int_{\mu}^{\infty} dz \, \frac{\cos(mz)}{z^n} \tag{32}$$

for $n = 0, 1, 2$ and for each of m equal to 1 and 2. For n equal to 0, the integration must be performed with the understanding that the integrand contains an infinitessimal decay factor, due to small but finite loss in the dielectric. With this understanding, (31) and (32) become, for n=0,

$$\int_{\mu}^{\infty} dz \; \sin(mz) = \int_{0}^{\infty} dz \; \sin(mz) - \int_{0}^{\mu} dz \; \sin(mz) \tag{33}$$

or

$$\int_{\mu}^{\infty} dz \; \sin(mz) = \frac{1}{m} \left[1 - \cos(\mu m) \right] \tag{34}$$

and

$$\int_{\mu}^{\infty} dz \; \cos(mz) = \int_{0}^{\infty} dz \; \cos(mz) - \int_{0}^{\mu} dz \; \cos(mz) \tag{35}$$

$$\int_{\mu}^{\infty} dz \; \cos(mz) = \frac{1}{m} \left[0 + \sin(\mu m) \right] = \frac{\sin(\mu m)}{m} \tag{36}$$

For $n = 1$ and 2, (31) and (32) are obtained in terms of sin and cos integrals [16]. For n=1, using Eqs. (5.2.26) and (5.2.27) in [ ], (31) and (32) are found to be

$$\int_{\mu}^{\infty} dz \, \frac{\sin(mz)}{z} \equiv - \mathrm{Si}\,(\mu m) \tag{37}$$

and

$$\int_{\mu}^{\infty} dz \, \frac{\cos(mz)}{z} \equiv - \mathrm{Ci}\,(\mu m) \tag{38}$$

Finally, for n=2, (31) and (32) become, using Eqs. (4.3.120) and (4.3.124) from [16],

$$\int_{\mu}^{\infty} dz \, \frac{\sin(mz)}{z^2} = m \left[\frac{\sin(\mu m)}{\mu m} - \mathrm{Ci}\,(\mu m) \right] \tag{39}$$

and

$$\int_{\mu}^{\infty} dz \, \frac{\cos(mz)}{z^2} = m \left[\frac{\cos(\mu m)}{\mu m} - \mathrm{Si}\,(\mu m) + \frac{\pi}{2} \right] \tag{40}$$

For csp = 1, the integration is broken up as follows. For n=3, the integration is not needed since all terms involving this case cancel exactly. For n=4, use

$$\int_{\mu}^{\infty} dk \, \frac{g(k)}{kb^2} = \int_{\mu}^{\infty} dk \, \frac{g_2(k)}{kb^4} + \int_{\mu}^{\infty} dk \, \frac{g_1(k)}{kb^4} + \int_{\mu}^{\infty} dk \, \frac{g_2^-(k)}{kb^4} \tag{41}$$

In Eq. (41), the integral involving $g_2^-(k)$ is handled numerically, while the other two are handled analytically. For n=5 or 6 the integral is broken up as

$$\int_{\mu}^{\infty} dk \, \frac{g(k)}{kb^n} = \int_{\mu}^{\infty} dk \, \frac{g_1(k)}{kb^n} + \int_{\mu}^{\infty} dk \, \frac{g_1^-(k)}{kb^n} \tag{42}$$

with the integral involving $g_1^-(k)$ handled numerically, while the one involving $g_1(k)$ handled analytically. For n greater than 6, $g(k)$ is used and the integration is performed numerically.

The integration needed in (41) and (42) for the asymptotic integrations is

$$\int_{\mu}^{\infty} dz \, \frac{1}{z^n} = \frac{1}{n-1} \, \frac{1}{\mu^{n-1}} \tag{43}$$

for $n = 2, 3$ and 4.

The integrations involving $h(k)$ are handled in the same manner as $g(k)$, but with $h_1(k)$ equal to zero

The matrix elements relating Tchebychef current distributions on the patch to tangential electric field along the feed pin are

$$Z^{z}_{rl'm'} = \frac{ja\pi \, (-1)^{l'} e^{jl'\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk \; k^2 \, \frac{\sinh(p_2 d)}{p_2 \, T_m} \left[B^{-}_{l'+1,m'} - B^{-}_{l'-1,m'} \right] J_{l'}(kr_0) \, J_0(ka) \tag{44}$$

$$Z^{z}_{\phi l'm'} = \frac{-a\pi \, (-1)^{l'} e^{jl'\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk \; k^2 \, \frac{\sinh(p_2 d)}{p_2 \, T_m} \left[A^{-}_{l'+1,m'} + A^{-}_{l'-1,m'} \right] J_{l'}(kr_0) \, J_0(ka) \tag{45}$$

The components of the matrix elements relating current on the feed pin to electric field tangential to the patch surface are

$$Z^{rlm}_{p} = \frac{-ja\pi \, e^{jl\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk \; k^2 \, \frac{\sinh(p_2 d)}{p_2 \, T_m} \left[B^{+}_{l+1,m} - B^{+}_{l-1,m} \right] J_l(kr_0) \, J_0(ka) \tag{46}$$

$$Z^{\phi lm}_{p} = \frac{a\pi \, e^{jl\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk \; k^2 \, \frac{\sinh(p_2 d)}{p_2 \, T_m} \left[A^{+}_{l+1,m} + A^{+}_{l-1,m} \right] J_l(kr_0) \, J_0(ka) \tag{47}$$

Equations (44) through (47) may be written in the form

$$Z = \frac{a\pi \, e^{jl\theta_0}}{\omega\varepsilon_1} \int_{\mu}^{\infty} dk \; f(k) \, J_l(kr_0) \, J_0(ka) \sum_j C_j \, P_j + \int_0^{\mu} dk \left[\; \cdots \; \right] \tag{48}$$

where

$$f(k) = \frac{p_1 \, k^2}{p_2 \, (\varepsilon_r \, p_1 \coth(p_2 d) + p_2)} \tag{49}$$

and where

$$P_j = \frac{\mathrm{csp}(kb)}{(kb)^n} \tag{50}$$

with $\mathrm{csp}(x)$ defined in (28). Again, the integration from 0 to μ will be handled in a later section.

Substituting for P_j, the integration from μ to infinity in (48) may be expressed as a summation over integrals of the form

$$\int_{\mu}^{\infty} dk\, f(k)\, J_l(kr_0)\, J_0(ka)\, \frac{\text{csp}(kb)}{(kb)^n} \tag{51}$$

Define

$$f_1(k) \equiv k \lim_{k\to\infty} \frac{f(k)}{k} \tag{52}$$

or

$$f_1(k) = \frac{k}{(\varepsilon_r + 1)} \tag{53}$$

Then let

$$f_1^-(k) \equiv f(k) - f_1(k) \tag{54}$$

or

$$f_1^-(k) = \frac{p_1 k^2}{p_2\,(\varepsilon_r\, p_1 \coth(p_2 d) + p_2)} - \frac{k}{(\varepsilon_r + 1)} \tag{55}$$

Now define

$$f_2(k) \equiv \frac{1}{k} \lim_{k\to\infty} k\, f_1^-(k) \tag{56}$$

Taking the limit by using (14) yields

$$f_2(k) = \frac{1}{k} \lim_{k\to\infty} \left[\frac{k(1 - \frac{k_1^2}{k^2})}{k(1 - \frac{k_2^2}{k^2})} \frac{k^3}{k(1 - \frac{k_2^2}{k^2}) + \varepsilon_r k(1 - \frac{k_1^2}{k^2})} - \frac{k^2}{(1 + \varepsilon_r)} \right] \tag{57}$$

or, working to first order only in $1/k^2$,

$$f_2(k) = \frac{1}{k} \lim_{k\to\infty} \left[(1 - \frac{k_1^2}{k^2} + \frac{k_2^2}{k^2}) \frac{k^2}{(\varepsilon_r + 1) - \frac{k_2^2}{k^2} - \varepsilon_r \frac{k_1^2}{k^2}} - \frac{k^2}{(1 + \varepsilon_r)} \right] \tag{58}$$

or

$$f_2(k) = \frac{1}{k} \lim_{k \to \infty} \left[\frac{k^2}{(\varepsilon_r + 1)} + \left[\frac{-k_1^2}{k^2} + \frac{k_2^2}{k^2} \right] \frac{k^2}{(\varepsilon_r + 1)} + \frac{k^2}{(\varepsilon_r + 1)} \frac{k_2^2 + \varepsilon_r k_1^2}{k^2 (\varepsilon_r + 1)} - \frac{k^2}{(\varepsilon_r + 1)} \right] \tag{59}$$

or

$$f_2(k) = \frac{1}{k} \lim_{k \to \infty} \left[\frac{k_2^2 - k_1^2}{\varepsilon_r + 1} + \frac{k_2^2 + \varepsilon_r k_1^2}{(\varepsilon_r + 1)^2} \right] \tag{60}$$

Using

$$\varepsilon_r \, k_1^2 = k_2^2 \tag{61}$$

and taking the limit,

$$f_2(k) = \frac{1}{k} \left[k_1^2 \, \frac{(\varepsilon_r - 1)}{(\varepsilon_r + 1)} - \frac{2\varepsilon_r k_1^2}{(\varepsilon_r + 1)^2} \right] \tag{62}$$

or

$$f_2(k) = \frac{k_1^2}{k} \left[\frac{(\varepsilon_r - 1)}{(\varepsilon_r + 1)} - \frac{2\varepsilon_r}{(\varepsilon_r + 1)^2} \right] \tag{63}$$

Again, the asymptotic portions of the integration in (51) is separated and handled analytically, with the remainder evaluated numerically. For n=1, (51) becomes

$$\int_\mu^\infty dk \; f(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)} = \int_\mu^\infty dk \; f_1(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)} + \int_\mu^\infty dk \; f_2(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)} + \int_\mu^\infty dk \; f_2^{-}(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)} \tag{64}$$

For n = 2 or 3, (51) becomes

$$\int_\mu^\infty dk \; f(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)^n} = \int_\mu^\infty dk \; f_1(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)^n} + \int_\mu^\infty dk \; f_1^{-}(k) \, J_l(kr_0) \, J_0(ka) \, \frac{\mathrm{csp}(kb)}{(kb)^n} \tag{65}$$

The integrals in (64) and (65) involving $f_1(k)$ and $f_2(k)$ are performed in closed form, while those involving $f_1^{-(k)}$ and $f_2^{-(k)}$ are handled numerically. For n greater than 3, (51) is integrated numerically.

The integrations needed to evaluate the asymptotic integrations in (64) and (65) are of the form

$$\int_{\mu}^{\infty} dx\ J_l(ax)\, J_m(bx)\, \mathrm{csp}(cx)\, (cx)^{-n} \tag{66}$$

where m is zero for (64) and (65), but will be left general for later use.

For csp(x) trigonometric,

$$\mathrm{csp}(x) = J_\rho(x) \left[\frac{\pi x}{2}\right]^{\frac{1}{2}} \tag{67}$$

where $\rho = \pm 1/2$. Using (67), (66) can be written as

$$\sqrt{\frac{\pi}{2}}\, c^{-n} \int_{\mu}^{\infty} dx\ J_l(ax)\, J_m(bx)\, J_\rho(cx)\, x^{-n+1/2} \tag{68}$$

Letting $\lambda = -n + 3/2$, (68) becomes

$$\sqrt{\frac{\pi}{2}}\, c^{-n} \left[\int_0^{\infty} dx\ J_l(ax)\, J_0(bx)\, J_\rho(cx)\, x^{\lambda-1} - \int_0^{\mu} dx\ J_l(ax)\, J_m(bx)\, J_\rho(cx)\, x^{\lambda-1}\right] \tag{69}$$

The integration in (69) from 0 to μ can be done numerically. The integration from 0 to infinity is found in [19].

$$\int_0^{\infty} J_l(ax)\, J_0(bx)\, J_\rho(cx)\, x^{\lambda-1}\, dx = \frac{2^{\lambda-1}\, a^l\, b^m\, \Gamma((\lambda + l + m + \rho)/2)}{c^{\lambda+l+m}\, \Gamma(l+1)\, \Gamma(m+1)\, \Gamma(1-(\lambda + l + m - \rho)/2)} \times$$

$$\times F_4\left[\frac{1}{2}(\lambda + l + m - \rho),\ \frac{1}{2}(\lambda + l + m + \rho);\ l+1,\ m+1;\ \frac{a^2}{c^2},\ \frac{b^2}{c^2}\right] \tag{70}$$

where

$$\mathrm{Re}\,(\lambda + l + m + \rho) > 0\ ; \quad \mathrm{Re}\,(\lambda) < \frac{5}{2}\ ; \quad c > a + b \tag{71a,b,c}$$

and where [20]

$$F_4\left[\alpha, \beta; \gamma, \gamma'; x, y\right] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\beta)_{m+n}}{(\gamma)_m\,(\gamma')_n\,m!\,n!}\,x^m\,y^n \tag{72}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{73}$$

For csp(x) = 1, (66) becomes

$$\int_{\mu}^{\infty} dx\ J_l(ax)\,J_m(bx)\,(cx)^{-n} = c^{-n}\left[\int_0^{\infty} dx\ J_l(ax)\,J_m(bx)\,x^{-n} + \int_0^{\mu} dx\ J_l(ax)\,J_m(bx)\,x^{-n}\right] \tag{74}$$

The integral from 0 to μ may be performed analytically, while the one from 0 to infinity is found in [21],

$$\int_0^{\infty} dx\ J_l(ax)\,J_m(bx)\,x^{-n} = \frac{a^l\,\Gamma\left[\dfrac{l+m-n+1}{2}\right]}{2^n\,b^{l-n+1}\,\Gamma\left[\dfrac{-l+m+n+1}{2}\right]\,l!} \times$$

$$\times F\left[\frac{l+m-n+1}{2}, \frac{l-m-n+1}{2}; l+1; \frac{a^2}{b^2}\right] \tag{75}$$

where [16]

$$F\left[\alpha, \beta; \gamma; x\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n\,n!}\,x^n \tag{76}$$

The asymptotic portions of the components of the matrix elements relating the singular current distribution to tangential electric field at the patch surface will now be separated. Those matrix components are

$$Z_s^{rlm} = \frac{ja\pi\,e^{jl\theta_0}}{\omega\epsilon_1}\int_0^{\infty} dk\ \frac{k}{p_1 + p_2\coth(p_2 d)}\left[\frac{p_1 + p_2\tanh(p_2 d)}{\epsilon_r p_1 + p_2\tanh(p_2 d)} - k_1^2\right] \times$$

$$\times\left[B_{l+1,m}^{+} - B_{l-1,m}^{+}\right] J_l(kr_0)\,a\sum_{i=1}^{2}\left[f_i(kR)\,J_i(kR) - f_i(ka)\,J_i(ka)\right] \tag{77}$$

$$Z_s^{\phi lm} = \frac{-a\pi\, e^{jl\theta_0}}{\omega\varepsilon_1} \int_0^{\infty} dk \; \frac{k}{p_1 + p_2\coth(p_2 d)} \left[\frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r p_1 + p_2\tanh(p_2 d)} - k_1^2 \right] \times$$

$$\times \left[A^{+}_{l+1,m} + A^{+}_{l-1,m} \right] J_l(kr_0)\, a \sum_{i=1}^{2} \left[f_i(kR)\, J_i(kR) - f_i(ka)\, J_i(ka) \right] \qquad (78)$$

Equations (77) and (78) may be written in the form

$$Z_s^{\gamma lm} = \frac{a\pi\, e^{jl\theta_0}}{\omega\varepsilon_1} \int_{\mu}^{\infty} dk \;\, s(k)\, J_l(kr_0)\, a \sum_{i=1}^{2} \left[f_i(kR)\, J_i(kR) - f_i(ka)\, J_i(ka) \right] \sum_j C_j\, P_j(kb) +$$

$$+ \int_0^{\mu} dk \,\left[\; \cdots \;\right] \qquad (79)$$

where

$$s(k) \equiv \frac{k}{p_1 + p_2\coth(p_2 d)} \left[\frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r p_1 + p_2\tanh(p_2 d)} - k_1^2 \right] \qquad (80)$$

Make the definition

$$s_1(k) \equiv k^2 \lim_{k \to \infty} \frac{1}{k^2}\, s(k) \qquad (81)$$

or

$$s_1(k) \equiv \frac{k^2}{\varepsilon_r + 1} \qquad (82)$$

then let

$$s_1^-(k) \equiv s(k) - s_1(k) \qquad (83)$$

$$s_1^-(k) \equiv \frac{k}{p_1 + p_2\coth(p_2 d)} \left[\frac{p_1 + p_2\tanh(p_2 d)}{\varepsilon_r p_1 + p_2\tanh(p_2 d)} - k_1^2 \right] - \frac{k^2}{\varepsilon_r + 1} \qquad (84)$$

Now define

$$s_2(k) \equiv \lim_{k \to \infty} s_1^-(k) \qquad (85)$$

Using Equation (14) and (85) in (84),

$$s_2(k) = \lim_{k\to\infty} \frac{k}{k(1-\frac{k_1^2}{k^2}) + k(1-\frac{k_2^2}{k^2})} \left[\frac{k(1-\frac{k_1^2}{k^2}) + k(1-\frac{k_2^2}{k^2})}{\varepsilon_r k(1-\frac{k_1^2}{k^2}) + k(1-\frac{k_2^2}{k^2})} - k_1^2 \right] - \frac{k^2}{\varepsilon_r + 1} \tag{86}$$

Working to order $1/k^2$, and using $k_1 \varepsilon_r = k_2^2$, (86) becomes

$$s_2(k) = \lim_{k\to\infty} \frac{1}{4} \left[2 + \frac{k_1^2}{k^2} + \frac{k_2^2}{k^2} \right] \left[\frac{k^2(2 - \frac{k_1^2}{k^2} - \frac{k_2^2}{k^2})}{(\varepsilon_r + 1) - 2\frac{k_2^2}{k^2}} - k_1^2 \right] - \frac{k^2}{\varepsilon_r + 1} \tag{87}$$

or

$$s_2(k) = \lim_{k\to\infty} \frac{1}{4} \left[2 + \frac{k_1^2}{k^2} + \frac{k_2^2}{k^2} \right] \left[\frac{k^2}{\varepsilon_r + 1}(2 - \frac{k_1^2}{k^2} - \frac{k_2^2}{k^2} + \frac{4}{(\varepsilon_r + 1)} \frac{k_2^2}{k^2}) - k_1^2 \right] - \frac{k^2}{\varepsilon_r + 1} \tag{88}$$

$$s_2(k) = \lim_{k\to\infty} \frac{k^2}{\varepsilon_r + 1} + \frac{1}{2(\varepsilon_r + 1)} \left[k_1^2 + k_2^2 - k_1^2 - k_2^2 + \frac{4}{\varepsilon_r + 1} k_2^2 \right] - \frac{k_1^2}{2} - \frac{k^2}{\varepsilon_r + 1} \tag{89}$$

$$s_2(k) = \lim_{k\to\infty} \frac{1}{2(\varepsilon_r + 1)} \left[\frac{4}{\varepsilon_r + 1} k_2^2 \right] - \frac{k_1^2}{2} \tag{90}$$

or, taking the limit

$$s_2(k) = \frac{2k_2^2}{(\varepsilon_r + 1)^2} - \frac{k_1^2}{2} \tag{91}$$

Finally, define

$$s_2^{-(k)} = s(k) - s_1(k) - s_2(k) \tag{92}$$

Since the functions in (79), $f_i(x)$, are polynomials, when $\mathrm{csp}(kb)$ is substituted for $P_j(kb)$, the result is expressable in terms of integrals of the form

$$\int_{\mu}^{\infty} dk\ s(k)\, J_l(kr_0)\, J_i(kt)\, \mathrm{csp}(kb)\, (kb)^n \tag{93}$$

Substituting for $s(k)$ using either (83) or (92), the integrations involving $s_1(k)$ and $s_2(k)$ are of the form of (66). The integrations involving $s_1^-(k)$ and $s_2^-(k)$ are performed numerically.

The final matrix element relates the electric field on the feed pin due to the feed pin and singular patch current distributions. This matrix element is, from 16.4(13),

$$Z^{s}_{s'} = 2\pi a \int_0^\infty dk \, \frac{ka}{j\omega\varepsilon_1} \left\{ J_0^2(ka) \left[\frac{k_2^2 d}{\varepsilon_r \, p_2^2} - \frac{\sinh(p_2 d)}{p_2 T_m} \left[\frac{8C + 24D}{ka^2} - \frac{192D}{ka^4} + \frac{k_2^2}{p_2^2} \right] \right] + \right.$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_0(kR) \left[- \left[A + B \frac{R^2}{a^2} + C \frac{R^4}{a^4} + D \frac{R^6}{a^6} \right] + \right.$$

$$\left. + \frac{8}{ka^2} \left[C \frac{R^2}{a^2} + 3D \frac{R^4}{a^4} \right] - \frac{192D}{ka^4} \frac{R^2}{a^2} \right] -$$

$$- \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_1(ka) \left[\frac{1}{ka} \left[2B + 4C + 6D \right] - \frac{16}{ka^3} (C + 6D) + \frac{384D}{ka^5} \right] +$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_1(kR) \left[\frac{1}{ka} \left[2B \frac{R}{a} + 4C \frac{R^3}{a^3} + 6D \frac{R^5}{a^5} \right] - \right.$$

$$\left. \left. - \frac{16}{ka^3} \left[C \frac{R}{a} + 6D \frac{R^3}{a^3} \right] + \frac{384D}{ka^5} \frac{R}{a} \right] \right\} \qquad (94)$$

The dominant term in the integrand is the term involving $J_0(ka)\, J_0(kR)$. Using

$$\frac{\sinh(p_2 d)}{p_2 T_m} = \frac{\sinh(p_2 d)}{p_2 \left[\varepsilon \cosh(P_2 d) + \frac{P_2}{P_1} \sinh(P_2 d) \right]} = \frac{1}{p_2 \left[\varepsilon \coth(P_2 d) + \frac{P_2}{P_1} \right]} \qquad (95)$$

this term becomes

$$\frac{-J_0(ka)}{p_2 \left[\varepsilon_r \coth(P_2 d) + \frac{p_2}{p_1} \right]} J_0(kR) \left[A + B \frac{R^2}{a^2} + C \frac{R^4}{a^4} + D \frac{R^6}{a^6} \right] \qquad (96)$$

Taking the limit as k approaches infinity in p_1, p_2 and $\coth(p_2 d)$, the integrand becomes

$$\frac{-J_0(ka)\, J_0(kR)}{(\varepsilon_r + 1)} \left[A + B \frac{R^2}{a^2} + C \frac{R^4}{a^4} + D \frac{R^6}{a^6} \right] \qquad (97)$$

With the asymptotic portion separated out, (94) becomes

$$Z^{s}_{s'} = 2\pi a \int_0^\infty dk \, \frac{ka}{j\omega\varepsilon_1} \left\{ J_0^2(ka) \left[\frac{k_2^2 d}{\varepsilon_r \, p_2^2} - \frac{\sinh(p_2 d)}{p_2 T_m} \left[\frac{8C + 24D}{ka^2} - \frac{192D}{ka^4} + \frac{k_2^2}{p_2^2} \right] \right] - \right.$$

$$- \left[\frac{\sinh(p_2 d)}{p_2 T_m} - \frac{1}{k(\varepsilon_r + 1)} \right] J_0(ka)\, J_0(kR) \left[A + B \frac{R^2}{a^2} + C \frac{R^4}{a^4} + D \frac{R^6}{a^6} \right] +$$

$$+ \frac{\sinh(p_2 d)}{p_2 T_m} J_0(ka)\, J_0(kR) \left[\frac{8}{ka^2} \left[C \frac{R^2}{a^2} + 3D \frac{R^4}{a^4} \right] - \frac{192D}{ka^4} \frac{R^2}{a^2} \right] -$$

$$- \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_1(ka) \left[\frac{1}{ka} \left[2B + 4C + 6D \right] - \frac{16}{ka^3} (C + 6D) + \frac{384D}{ka^5} \right] +$$

$$+ \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} J_1(kR) \left[\frac{1}{ka} \left[2B \frac{R}{a} + 4C \frac{R^3}{a^3} + 6D \frac{R^5}{a^5} \right] - \right.$$

$$\left. - \frac{16}{ka^3} \left[C \frac{R}{a} + 6D \frac{R^3}{a^3} \right] + \frac{384D}{ka^5} \frac{R}{a} \right] \Bigg\} -$$

$$- \frac{2\pi a^2}{j\omega\varepsilon_1} \int_0^\infty dk \frac{J_0(ka)\, J_0(kR)}{(\varepsilon_r + 1)} \left[A + B \frac{R^2}{a^2} + C \frac{R^4}{a^4} + D \frac{R^6}{a^6} \right] \tag{98}$$

The first integral in (98) is handled numerically and the second is performed using (75).

17.3 Products of Bessel Functions for Argument Approaching Zero.

In this section, the necessary products of $A_{l,m}^{\pm}$ and $B_{l,m}^{\pm}$ are obtained in a form useful for small arguments. From 17.1(3)

$$A_{l,m}^{\pm} = \frac{b^2}{2}\left[\frac{kb}{2}\, j_{\frac{|l|+m-2}{2}}\left[\frac{kb}{2}\right] j_{\frac{|l|-m}{2}}\left[\frac{kb}{2}\right] + \frac{kb}{2}\, j_{\frac{|l|+m}{2}}\left[\frac{kb}{2}\right] j_{\frac{|l|-m-2}{2}}\left[\frac{kb}{2}\right]\right] \times$$

$$\times \begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l < 0 \end{Bmatrix} \tag{1}$$

As the argument $kb/2$ approaches zero, it is efficient to express the spherical Bessel functions above in a power series [16].

$$j_n(z/2) = \frac{(z/2)^n}{1\cdot 3\cdot 5 \cdots (2n+1)}\left\{1 - \frac{z^2/8}{(2n+3)} + \frac{(z^2/8)^2}{2!\,(2n+3)(2n+5)} - \cdots\right\} \qquad n \geq 0 \tag{2}$$

and

$$j_n(z/2) = (-1)^n\, \frac{(-1)\cdot 1\cdot 3 \cdots (-2n-3)}{(z/2)^{-n}}\left\{1 - \frac{z^2/8}{(2n+3)} + \frac{(z^2/8)^2}{2!\,(2n+3)(2n+5)} - \cdots\right\} \qquad n < 0 \tag{3}$$

Equations (2) and (3) may be approximated, for $z/2$ small, as

$$j_n(z/2) \approx (z/2)^n\, c_n \left[1 - (z/2)^2 d_n + (z/2)^4 e_n\right] \tag{4}$$

where

$$c_n = \begin{Bmatrix} \dfrac{1}{1\cdot 3\cdot 5 \cdots (2n+1)} & n \geq 0 \\ (-1)^n\,(-1)\cdot 1\cdot 3\cdot 5 \cdots (-2n-3) & n < 0 \end{Bmatrix} \tag{5}$$

$$d_n = \frac{1}{2(3+2n)} \tag{6}$$

$$e_n = \frac{1}{8(3+2n)(5+2n)} \tag{7}$$

Using equations (4) through (7), (1) can be written as

$$A_{l,m}^{\pm} = \frac{b^2}{2}(kb/2)^{|l|} \cdot \begin{Bmatrix} (\pm 1)^l & l \geq 0 \\ (-\pm 1)^l & l<0 \end{Bmatrix}$$

$$\times \Bigg\{ c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} \Bigg[1 - \frac{(kb)^2}{4} (d_{\frac{|l|+m-2}{2}} + d_{\frac{|l|-m}{2}}) +$$

$$+ \frac{(kb)^4}{16} (e_{\frac{|l|+m-2}{2}} + e_{\frac{|l|-m}{2}} + d_{\frac{|l|+m-2}{2}} d_{\frac{|l|-m}{2}}) \Bigg] +$$

$$+ c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} \Bigg[1 - \frac{(kb)^2}{4} (d_{\frac{|l|+m}{2}} + d_{\frac{|l|-m-2}{2}}) +$$

$$+ \frac{(kb)^4}{16} (e_{\frac{|l|+m}{2}} + e_{\frac{|l|-m-2}{2}} + d_{\frac{|l|+m}{2}} d_{\frac{|l|-m-2}{2}}) \Bigg] \Bigg\} \tag{8}$$

Using equation (8),

$$A_{l,m}^{+} A_{l',m'}^{-} = \frac{b^4}{4}(kb/2)^{|l|+|l'|} \cdot \begin{Bmatrix} 1 & l \geq 0 \\ (-1)^l & l<0 \end{Bmatrix} \cdot \begin{Bmatrix} (-1)^{l'} & l \geq 0 \\ 1 & l<0 \end{Bmatrix} \times$$

$$\times \left[a_1 - \frac{(kb)^2}{4} a_2 + \frac{(kb)^4}{16} a_3 \right] \tag{9}$$

where

$$a_1 \equiv \left[c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} + c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} \right] \cdot \left[c_{\frac{|l'|+m-2}{2}} c_{\frac{|l'|-m}{2}} + c_{\frac{|l'|+m}{2}} c_{\frac{|l'|-m-2}{2}} \right] \tag{10}$$

$$a_2 \equiv \left[c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} (d_{\frac{|l|+m-2}{2}} + d_{\frac{|l|-m}{2}}) + c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} (d_{\frac{|l|+m}{2}} + d_{\frac{|l|-m-2}{2}}) \right] \times$$

$$\times \left[c_{\frac{|l'|+m-2}{2}} c_{\frac{|l'|-m}{2}} + c_{\frac{|l'|+m}{2}} c_{\frac{|l'|-m-2}{2}} \right] +$$

$$+ \left[c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} + c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} \right] \times$$

$$\times \left[c_{\frac{|l'|+m-2}{2}} c_{\frac{|l'|-m}{2}} (d_{\frac{|l'|+m-2}{2}} + d_{\frac{|l'|-m}{2}}) + c_{\frac{|l'|+m}{2}} c_{\frac{|l'|-m-2}{2}} (d_{\frac{|l'|+m}{2}} + d_{\frac{|l'|-m-2}{2}}) \right] \tag{11}$$

$$a_3 \equiv \left[c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} \left(d_{\frac{|l|+m-2}{2}} + d_{\frac{|l|-m}{2}} \right) + c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} \left(d_{\frac{|l|+m}{2}} + d_{\frac{|l|-m-2}{2}} \right) \right] \times$$

$$\times \left[c_{\frac{|l'|+m-2}{2}} c_{\frac{|l'|-m}{2}} \left(d_{\frac{|l'|+m-2}{2}} + d_{\frac{|l'|-m}{2}} \right) + c_{\frac{|l'|+m}{2}} c_{\frac{|l'|-m-2}{2}} \left(d_{\frac{|l'|+m}{2}} + d_{\frac{|l'|-m-2}{2}} \right) \right] +$$

$$+ \left[c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} \left(e_{\frac{|l|+m-2}{2}} + e_{\frac{|l|-m}{2}} + d_{\frac{|l|+m-2}{2}} d_{\frac{|l|-m}{2}} \right) + \right.$$

$$\left. + c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} \left(e_{\frac{|l|+m}{2}} + e_{\frac{|l|-m-2}{2}} + d_{\frac{|l|+m}{2}} d_{\frac{|l|-m-2}{2}} \right) \right] \times$$

$$\times \left[c_{\frac{|l'|+m-2}{2}} c_{\frac{|l'|-m}{2}} + c_{\frac{|l'|+m}{2}} c_{\frac{|l'|-m-2}{2}} \right] +$$

$$+ \left[c_{\frac{|l'|+m-2}{2}} c_{\frac{|l'|-m}{2}} \left(e_{\frac{|l'|+m-2}{2}} + e_{\frac{|l'|-m}{2}} + d_{\frac{|l'|+m-2}{2}} d_{\frac{|l'|-m}{2}} \right) + \right.$$

$$\left. + c_{\frac{|l'|+m}{2}} c_{\frac{|l'|-m-2}{2}} \left(e_{\frac{|l'|+m}{2}} + e_{\frac{|l'|-m-2}{2}} + d_{\frac{|l'|+m}{2}} d_{\frac{|l'|-m-2}{2}} \right) \right] +$$

$$\times \left[c_{\frac{|l|+m-2}{2}} c_{\frac{|l|-m}{2}} + c_{\frac{|l|+m}{2}} c_{\frac{|l|-m-2}{2}} \right] \qquad (12)$$

Equation (9) along with equations (10) through (12) is used to obtain the necessary products of $A_{l,m}$ from equations 17.1(4) and 17.1(5). The products involving $B_{l,m}$ are then obtained from the products of $A_{l,m}$ folowing the development of section 17.1.

17.4 Asymptotic Forms for Spectral Integrands as Argument Approaches Zero.

From the form of $A_{l,m}$ from the last section, the components of the integrand of the matrix elements of 17.2(1-4), and 17.2(44-47) are all well behaved as k approaches zero. The numerical integrations from 0 to μ in 17.2(5) and 17.2(48) may thus be carried out easily. From equations 17.2(77), 17.2(78) and 17.2(80), the integration from 0 to μ in 17.2(79) may be written as

$$Z_s^{\gamma lm} = \frac{-a\pi\, e^{jl\theta_0}}{\omega\epsilon_1} \int_0^{\mu} dk\ s(k) \left[A_{l+1,m}^{\pm} + A_{l-1,m}^{\pm} \right] J_l(kr_0)\, a \sum_{i=1}^{2} \left[f_i(kR)\, J_i(kR) - f_i(ka)\, J_i(ka) \right] \quad (1)$$

As k approaches 0, $s(k)$ is proportional to k, and

$$\left[A_{l+1,m}^{\pm} + A_{l-1,m}^{\pm} \right] \approx (kb)^{l\pm1} \quad (2)$$

so as k approaches zero, the integrand in (1) becomes proportional to

$$k\ (kb)^{|l\pm1|}\, J_l(kr_0) \sum_{i=1}^{2} \left[f_i(kR)\, J_i(kR) - f_i(ka)\, J_i(ka) \right] \quad (3)$$

In the worst case for integration to the limit $k = 0$, $l = 0$ or ±1. For these cases,

$$(kb)^{|l\pm1|}\, J_l(kr_0) \approx k \quad (4)$$

so (3) becomes

$$k^2 \sum_{i=1}^{2} \left[f_i(kR)\, J_i(kR) - f_i(ka)\, J_i(ka) \right] \quad (5)$$

For (5) to be integrable to 0,

$$\sum_{i=1}^{2} \left[f_i(kR)\, J_i(kR) - f_i(ka)\, J_i(ka) \right] \quad (6)$$

must diverge to infinity no quicker than $1/k^2$ as k approaches 0. Letting t be either a

or R, look at the term

$$f_1(kt)\, J_1(kt) + f_2(kt)\, J_2(kt) \tag{7}$$

From 14.2(55) and 14.2(56)

$$f_1(kt) = -\frac{A}{ka}\,\frac{2}{(kt)} + 2\,\frac{C}{ka^5}\,(kt)^3 + \frac{D}{(ka)^7}\,(\,4(kt)^5 - 48(kt)^3\,) \tag{8}$$

$$f_2(kt) = \frac{A}{ka} + \frac{B}{ka^3}\,(kt)^2 + \frac{C}{ka^5}\,(\,(kt)^4 - 8(kt)^2\,) + \frac{D}{(ka)^7}\,(\,4(kt)^6 - 24(kt)^4 + 192(kt)^2\,) \tag{9}$$

Since

$$J_l(kt) \quad \frac{(kt)^l}{2^l\, l!} \tag{10}$$

for k approaching zero, by inspection the terms involving coefficients A, B and C are sufficiently well-behaved, as are portions of the terms involving D. The portions of $f_1(kt)$ and $f_2(kt)$ that must be examined more carefully are

$$f_1(kt) \approx -\frac{D}{(ka)^7}\,48(kt)^3 \tag{11}$$

$$f_2(kt) \approx \frac{D}{(ka)^7}\,192(kt)^2 \tag{12}$$

Using (10) through (12), (7) becomes, for k approaching zero

$$-\frac{D}{(ka)^7}\,48(kt)^3\,\frac{kt}{2} + \frac{D}{(ka)^7}\,192(kt)^2\,\frac{(kt)^2}{8} = 48\frac{D}{(ka)^7}\left[4\frac{(kt)^4}{8} - \frac{(kt)^4}{2}\right] = 0 \tag{13}$$

Thus the nonconverging term cancels, so the integral in (1) exists.

The final matrix element integration to check is that shown in equation 17.2(98). Rewriting this here,

$$Z^{z}_{z'} = 2\pi a \int_0^\infty dk\,\frac{ka}{j\omega\varepsilon_1}\left\{J_0^2(ka)\left[\frac{k_2^2 d}{\varepsilon_r\, p_2^2} - \frac{\sinh(p_2 d)}{p_2 T_m}\left[\frac{8C + 24D}{ka^2} - \frac{192D}{ka^4} + \frac{k_2^2}{p_2^2}\right]\right] + \right.$$

$$- \left[\frac{\sinh(p_2 d)}{p_2 T_m} - \frac{1}{k(\varepsilon_r + 1)}\right] J_0(ka)\, J_0(kR)\left[A + B\,\frac{R^2}{a^2} + C\,\frac{R^4}{a^4} + D\,\frac{R^6}{a^6}\right] +$$

$$+\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_0(kR)\left[\frac{8}{ka^2}\left[C\,\frac{R^2}{a^2}+3D\,\frac{R^4}{a^4}\right]-\frac{192D}{ka^4}\,\frac{R^2}{a^2}\right]-$$

$$-\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_1(ka)\left[\frac{1}{ka}\left[2B+4C+6D\right]-\frac{16}{ka^3}\,(C+6D)+\frac{384D}{ka^5}\right]+$$

$$+\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_1(kR)\left[\frac{1}{ka}\left[2B\,\frac{R}{a}+4C\,\frac{R^3}{a^3}+6D\,\frac{R^5}{a^5}\right]-\right.$$

$$\left.-\frac{16}{ka^3}\left[C\,\frac{R}{a}+6D\,\frac{R^3}{a^3}\right]+\frac{384D}{ka^5}\,\frac{R}{a}\right]\Bigg\} \qquad (14)$$

Separating the terms in (14) which pose no difficulty at the lower integration limit from those which do, (14) becomes

$$Z^{z}_{z'}=2\pi a\int_0^\infty dk\,\frac{ka}{j\omega\varepsilon_1}\left\{J_0^2(ka)\left[\frac{k_2^2d}{\varepsilon_r\,p_2^2}-\frac{\sinh(p_2d)}{p_2T_m}\,\frac{k_2^2}{p_2^2}\right]+\right.$$

$$-\left[\frac{\sinh(p_2d)}{p_2T_m}-\frac{1}{k(\varepsilon_r+1)}\right]J_0(ka)\,J_0(kR)\left[A+B\,\frac{R^2}{a^2}+C\,\frac{R^4}{a^4}+D\,\frac{R^6}{a^6}\right]+$$

$$-\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_1(ka)\left[\frac{1}{ka}\left[2B+4C+6D\right]\right]+$$

$$\left.+\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_1(kR)\,\frac{1}{ka}\left[2B\,\frac{R}{a}+4C\,\frac{R^3}{a^3}+6D\,\frac{R^5}{a^5}\right]\right\}$$

$$+2\pi a\int_0^\infty dk\,\frac{ka}{j\omega\varepsilon_1}\left\{J_0^2(ka)\left[-\frac{\sinh(p_2d)}{p_2T_m}\left[\frac{8C+24D}{ka^2}-\frac{192D}{ka^4}\right]\right]+\right.$$

$$+\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_0(kR)\left[\frac{8}{ka^2}\left[C\,\frac{R^2}{a^2}+3D\,\frac{R^4}{a^4}\right]-\frac{192D}{ka^4}\,\frac{R^2}{a^2}\right]-$$

$$-\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_1(ka)\left[-\frac{16}{ka^3}\,(C+6D)+\frac{384D}{ka^5}\right]+$$

$$\left.+\frac{\sinh(p_2d)\,J_0(ka)}{p_2T_m}\,J_1(kR)\left[-\frac{16}{ka^3}\left[C\,\frac{R}{a}+6D\,\frac{R^3}{a^3}\right]+\frac{384D}{ka^5}\,\frac{R}{a}\right]\right\}+ \qquad (15)$$

The second integration in (15) will be manipulated to make explicit the convergence at the lower limit. Rearrange the second integration to

$$2\pi a \int_0^\infty dk \, \frac{ka}{j\omega\varepsilon_1} \, \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \left\{ - J_0(ka) \left[\frac{8C + 24D}{(ka)^2} - \frac{192D}{(ka)^4} \right] + \right.$$

$$+ J_0(kR) \left[\frac{8}{(ka)^2} \left[C \, \frac{R^2}{a^2} + 3D \, \frac{R^4}{a^4} \right] - \frac{192D}{(ka)^4} \, \frac{R^2}{a^2} \right] -$$

$$+ J_1(ka) \left[\frac{16}{ka^3} \, (C + 3D + 3D) - \frac{384D}{ka^5} \right] -$$

$$\left. - J_1(kR) \left[\frac{16}{(ka)^3} \left[C \, \frac{R}{a} + 3D \, \frac{R^3}{a^3} + 3D \, \frac{R^3}{a^3} \right] - \frac{384D}{(ka)^5} \, \frac{R}{a} \right] \right\} + \qquad (16)$$

A Bessel identity gives [16]

$$J_{l+1}(z) = \frac{2l}{z} \, J_l(z) - J_{l-1}(z) \qquad (17)$$

Using (17) with l=1, (16) becomes

$$2\pi a \int_0^\infty dk \, \frac{ka}{j\omega\varepsilon_1} \, \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \left\{ J_2(ka) \left[\frac{8C + 24D}{ka^2} - \frac{192D}{(ka)^4} \right] + J_1(ka) \, \frac{48D}{(ka)^3} - \right.$$

$$\left. - J_2(kR) \left[\frac{8}{(ka)^2} \left[C \, \frac{R^2}{a^2} + 3D \, \frac{R^4}{a^4} \right] - \frac{192D}{(ka)^4} \, \frac{R^2}{a^2} \right] - J_1(kR) \, \frac{48D}{(ka)^3} \, \frac{R^3}{a^3} \right\} \qquad (18)$$

Using (17) with l=2, (18) becomes

$$2\pi a \int_0^\infty dk \, \frac{ka}{j\omega\varepsilon_1} \, \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \left\{ J_2(ka) \, \frac{8C + 24D}{ka^2} - J_3(ka) \, \frac{48D}{(ka)^3} - \right.$$

$$\left. - J_2(kR) \, \frac{8}{(ka)^2} \left[C \, \frac{R^2}{a^2} + 3D \, \frac{R^4}{a^4} \right] + J_3(kR) \, \frac{48D}{(ka)^3} \, \frac{R^3}{a^3} \right\} \qquad (19)$$

By inspections using (10), equation (19) may be integrated to the limit 0 without difficulty.

Using (19) in place of the second integral in (15) and combining with the first integral gives

$$Z^{s}_{s'} = 2\pi a \int_0^{\infty} dk \, \frac{ka}{j\omega\varepsilon_1} \left\{ J_0^2(ka) \left[\frac{k_2^2 d}{\varepsilon_r \, p_2^2} - \frac{\sinh(p_2 d)}{p_2 T_m} \, \frac{k_2^2}{p_2^2} \right] + \right.$$

$$- \left[\frac{\sinh(p_2 d)}{p_2 T_m} - \frac{1}{k(\varepsilon_r + 1)} \right] J_0(ka)\, J_0(kR) \left[A + B \, \frac{R^2}{a^2} + C \, \frac{R^4}{a^4} + D \, \frac{R^6}{a^6} \right] +$$

$$- \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \, J_1(ka) \left[\frac{1}{ka} \left[2B + 4C + 6D \right] \right] +$$

$$\left. + \frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \, J_1(kR) \, \frac{1}{ka} \left[2B \, \frac{R}{a} + 4C \, \frac{R^3}{a^3} + 6D \, \frac{R^5}{a^5} \right] \right\}$$

$$\frac{\sinh(p_2 d)\, J_0(ka)}{p_2 T_m} \left\{ J_2(ka) \, \frac{8C + 24D}{ka^2} - J_3(ka) \, \frac{48D}{(ka)^3} - \right.$$

$$\left. - J_2(kR) \, \frac{8}{(ka)^2} \left[C \, \frac{R^2}{a^2} + 3D \, \frac{R^4}{a^4} \right] + J_3(kR) \, \frac{48D}{(ka)^3} \, \frac{R^3}{a^3} \right\} \qquad (20)$$

Equation (20) is in the form used for integration to the lower limit of 0, replacing the form in equation 17.2(98).

VIII CONCLUSION.

A potentially exact solution was obtained for each structure, allowing off resonance and higher order resonance behavior to be obtained accurately. In the case of the patch antenna and patch array, a set of current distributions capable of modelling an arbitrary surface current on a circular patch has been developed. Also, two additional current distributions were developed, each of which models the surface current diverging from the feed pin onto the patch surface. One of these was used in the infinite array and the other was used for the single patch.

For the case of the single patch, the Sommerfeld integrations were performed via real line integration. For all but one of the matrix elements, the integrand involved a sum of many terms which were products of two or four spherical Bessel functions. These products can be expressed most simply in terms of a power series plus power series multiplied by sin and cos. Rather than integrating each matix element separately--which would involve summing many products of Bessel functions, each of which is a summation of power series--all the terms needed for the power series for some matrix element are integrated. The matrix elements are then pieced together from the various integrations.

In addition, the asymptotic forms of the integrations for the matrix elements are identified, separated and performed analytically.

For the infinite arrays, the Sommerfeld integrals are converted into a doubly infinite summation, each term of which represents a Flouquet mode. A sufficient number of Flouquet modes are summed to ensure convergence. The range of terms necessary for convergence in the patch array is greater for the matrix elements involving the feed pin.

The current distributions on the patch surcace are obtained at the primary resonance, and at three higher resonances. They match well qualitatively with the expected resonance currents. The resonances come in the expected order also, but at lower frequencies than the simplest models predict. The third and fourth resonances occur very close together.

The received power is obtained for a constant load impedance, and for a matched load impedance. For the load impedance matched, the power received is flat until the frequency is high enough to excite surface waves with wavelength equal to the patch spacing. At this frequency, the received power varies wildly, and it is believed the solution is inaccurate here.

For the array of patches, the solution was obtained for a frequency range of over 3 to 1, from below the primary resonance to above the fourth resonance. The development is for an arbitrary plane wave incidence angle, but only normal incidence is investigated. Difficulty was encountered in the vicinity of the frequency where the lowest order surface wave has a spatial period that matches the spacing of the patches. Approaching this frequency, the central term in the Flouquet summation approaches the location of the surface wave pole and diverges.

For the dipole array, the solution is again over a 3 to 1 frequency band for near normal incidence. The the lowest order surface wave is encountered, but doesn't seem to damage the solution. The component of the Green function dyad used contains a term which cancels the pole for normal incidence. The effect of the surface wave pole is thus smaller at nearly normal incidence.

Solutions for the dipole array as a function of incidence angle, varying from normal to nearly grazing, are also obtained at several frequencies. At the frequencies where the TM_0 surface wave pole is implicated, this occurs away from normal incidence, so the effect is greater on the array properties.

LIST OF REFERENCES

[1] Pozar, D. M. and D. H. Schaubert, "Comparison of architectures for monolithic phased array antennas," Microwave journal, pp. 93-104, March 1986.

[2] Pozar, D. M. and D. H. Schaubert, "Analysis of an infinite array of rectangular microstrip patches with idealized probe feeds," IEEE Transactions on Antennas and Propagation, vol. AP-32, no. 10, pp. 1101-1107, October 1984.

[3] Pozar, D. M., "Finite phased arrays of rectangular microstrip patches," IEEE Transactions on Antennas and Propagation, vol. AP-34, no. 5, pp. 658-665, May 1986.

[4] Lo, Y. T., D. Solomon, and W. F. Richards, "Theory and experiment on microstrip antennas," IEEE Transactions on Antennas and Propagation, vol. AP-27, March 1979.

[5] Araki, K. and T. Itoh, "Hankel transform domain analysis of open circular microstrip radiating structures," IEEE Transactions on Antennas and Propagation, vol. AP-29, no. 1, pp. 84-89, January 1981.

[6] Yano, S. and A. Ishimaru, "A theoretical study of the input impedance of a circular microstrip dish antenna," IEEE Transactions on Antennas and Propagation, vol. AP-29, no. 1, pp. 77-83, January 1981.

[7] Chew, W. C. and J. A. Kong, "Resonances of nonaxial symmetric modes in circular microstrip disk antenna," Journal of Mathematical Physics, vol. 21, no. 10, pp. 2590-2598, October 1980.

[8] Liu, C-C, et. al., "Plane wave reflection from microstrip-patch arrays -- theory and experiment," IEEE Transactions on Antennas and Propagation, vol. AP-33, no. 4, pp. 426-435, April 1985.

[9] Chen, C-C, "Scattering by a two-dimensional periodic array of conducting plates," IEEE Transactions on Antennas and Propagation, vol. AP-18, no. 5, pp. 660-665, September 1970.

[10] Chew, W. C. and J. A. Kong, "Analysis of a circular microstrip disk antenna with a thick dielectric substrate," IEEE Transactions on Antennas and Propagation, vol. AP-29, no. 1, pp. 68-76, January 1981.

[11] Uzunoglu, N. K., N. G. Alexopoulos, and J. G. Fikioris, "Radiation properties of microstrip dipoles," IEEE Transactions on Antennas and Propagation, vol. AP, no. 6, pp. 853-858, November 1979.

[12] Alexopoulos, N. G. and I. E. Rana, "Mutual impedance computation between printed dipoles," IEEE Transactions on Antennas and Propagation, vol. AP-29, no. 1, pp. 106-111, January 1981

[13] Long, S. A., L. C. Shen, and P. B. Morel, "Theory of the circular-disk printed-circuit antenna," Proceedings in the IEE, vol. 125, no. 10., October 1978.

[14] Pozar, D. M., and D. H. Schaubert, "Scan blindness in infinite phased arrays of printed dipoles," IEEE Transactions on Antennas and Propagation, vol. AP-32, no. 6, pp. 602-610, June 1984.

[15] Papoulis, A., The Fourier Integral and its Applications, Mcgraw-Hill, New York, 1962, p. 47.

[16] Abramowitz, M. A., and Stegun, I. A., Handbook of Mathematical Functions, Dover Publications, New York, 1970.

[17] Bateman, H., Tables of Integral Transforms, Vol 2., Mcgraw-Hill, New York, 1954, p. 42.

[19] Bailey, W. N., "Some infinite integrals involving Bessel Functions," Proceedings London Mathematical Society (2), Vol. 40, 1936, p. 45.

[20] Bailey, W. N., "A reducible case of the fourth type of Appell's hypergeometric functions of two variables," Quarterly Journal of Mathematics (Oxford), 4, (1933), p. 305.

[21] Gradshteyn, I. S., Ryzhik, I. M., Table of Integrals, Series, and Products, Academic Press, Florida, 1980.