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By

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ABSTRACT

HARMONIC BLOCH FUNCTIONS ON THE UPPER HALF SPACE

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It is known that a holomorphic Bloch function f on the open unit disk of the complex plane need not have radial limit at any boundary point. Nevertheless, Ullrich showed that such an f does have "boundary values" in an average sense. Furthermore, these average boundary values reflect the behavior of f inside the disk in a manner analogous to the case of bounded holomorphic functions. These results transfer easily to the upper half plane. They are also valid if we merely assume that f is harmonic; here we need only remember that a real harmonic function is Bloch if and only if its harmonic conjugate is Bloch (by Cauchy-Riemann equations). In this paper we generalize these results to harmonic Bloch functions on upper half spaces of arbitrary dimension. In higher dimensions, a new problem arises: A harmonic Bloch function may well have harmonic conjugates that are not Bloch. However, we show it is always possible to choose harmonic conjugates that are Bloch. (This choice is unique up to an additive constant.) From this we obtain several results relating the "boundary

values" of a harmonic Bloch function to its behavior inside the domain. We also obtain a number of "bounded mean oscillation" characterizations of the harmonic Bloch space as well as characterizations involving higher derivatives (one of which may be a little surprising).

To my father (posthume). To my dear mother. To my beautiful wife Saeeda Rayssi. To my wonderful children Ameur, Allem, and Walid (I have not seen him yet). To my sisters Fatma, Zohra, and Aycha. To my brothers Sidi, Ayadi, Hachmi, Chedli, Ali, Moncef, and Faycal.

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Chapter 1

Introduction

Let $D = \{z \in \mathbf{C} : \ |z| < 1\}$ denote the open unit disk of the complex plane $\mathbf{C}$. A holomorphic function f on D is said to be a *Bloch function* if

$$||f||_{\mathcal{B}} = \sup_{D} \, (1 - |z|^2)|f'(z)| < \infty.$$

The *Bloch space* $\mathcal{B}_a(D)$ is the set of all holomorphic Bloch functions on D. (The subscript a stands for "analytic"; later we will be looking at harmonic Bloch functions.) Even though $||f||_{\mathcal{B}}$ is not a norm, we will refer to $||f||_{\mathcal{B}}$ as the Bloch norm of f. The quantity $|f(0)| + ||f||_{\mathcal{B}}$ does define a norm on $\mathcal{B}_a(D)$, and equipped with this norm, $\mathcal{B}_a(D)$ is a Banach space.

Let us discuss some of the history of this subject. The basic idea of a Bloch function goes back to the following remarkable theorem of André Bloch.

Bloch's theorem: *There exists a finite positive constant b such that if g is a holomorphic function on D, normalized so that $g(0) = 0$ and $g'(0) = 1$, then there is a disk Δ contained in D on which g is one-to-one and such that $g(\Delta)$ contains a disk of radius b.*

Although the exact value of b is unknown, it is known that $.43 < b < .47$.

Before we discuss the connection between Bloch's theorem and the Bloch norm, we introduce some notation. For $a \in \mathbf{C}$ and $r > 0$, we denote the set $\{z \in \mathbf{C} : |z-a| < r\}$ by $D(a,r)$. For a holomorphic function f on D and a point z in D, let $d_f(z)$ be the supremum of all $r > 0$ for which there exists an open connected neighborhood Ω of z such that f is one-to-one on Ω and $f(\Omega) = D(f(z), r)$. (If $f'(z) = 0$, in which case f cannot be one-to-one on any neighborhood of z, then we set $d_f(z) = 0$.) Here is the connection between Bloch's theorem and the Bloch norm: For any holomorphic function f on D,

$$\sup_D d_f(z) \le ||f||_{\mathcal{B}} \le \frac{1}{b} \sup_D d_f(z).$$

This follows (easily, but not obviously) from Bloch's theorem and the invariant form of the Schwarz Lemma.

Besides this very appealing connection between the geometrically defined $d_f(z)$ and the analytically defined $||f||_{\mathcal{B}}$, a number of other interesting properties of the holomorphic Bloch space have been obtained; we discuss some of these below. The main purpose of this paper is to explore some of these properties for the *harmonic* Bloch space, especially in higher dimensions.

For the harmonic Bloch space, we take as our principal setting the open upper half-space $H = H_n$ of $\mathbf{R}^n$ defined by

$$H = \{(x,y) : x \in \mathbf{R}^{n-1},\ y > 0\}.$$

A harmonic function u on H is said to be a *Bloch function* if

$$||u||_{\mathcal{B}} = \sup\ y|\nabla(u(x,y))|\ <\ \infty,$$

where the supremum is taken over all $(x,y) \in H$, and ∇u denotes the gradient of u. In this paper the term "harmonic" means "real-valued and harmonic" unless otherwise stated. We denote the vector space of harmonic Bloch functions on H by $\mathcal{B}(H)$. As above, we call $||u||_{\mathcal{B}}$ the *Bloch norm* of u, even though it is not a norm; we obtain a norm on $\mathcal{B}(H)$ by adding $|u(0,1)|$ to $||u||_{\mathcal{B}}$. We call $\mathcal{B}(H)$ the *harmonic Bloch space* of the upper half-space. For the convenience of the reader, let us sketch a proof that, equipped with this norm, $\mathcal{B}(H)$ is a Banach space. The fact that $\mathcal{B}(H)$ is a real vector space is clear. If (u_k) is a Cauchy sequence in $\mathcal{B}(H)$, then it is easy to see that (∇u_k) is uniformly Cauchy on compact subsets of H. Therefore, because (u_k) converges at the point $(0,1)$, (u_k) must converge uniformly on compact subsets of H to some function u. The function u is harmonic, and it is easy to see that $u \in \mathcal{B}(H)$ and that (u_k) converges to u in norm.

Letting $B = B_n = \{x \in \mathbf{R}^n : |x| < 1\}$ denote the unit ball of $\mathbf{R}^n$, we could also discuss the Bloch space $\mathcal{B}(B)$ (whose definition can be guessed easily). When $n = 2$, there is no great difference between $\mathcal{B}(H)$ and $\mathcal{B}(B)$. Indeed, the map

$$\phi(z) = -\frac{iz+1}{z+i}$$

is easily seen to induce an isomorphism between these two spaces. In higher dimensions, the natural replacement for the map ϕ is an appropriate modification of the Kelvin transform (see 7.15 of [ABR]). Unfortunately, this map does not carry $\mathcal{B}(B)$ into $\mathcal{B}(H)$. Thus when $n > 2$, there is no obvious correspondence between $\mathcal{B}(B)$ and $\mathcal{B}(H)$.

So why choose H over B? The main reason is that in H, we have a transitive group of self-mappings that preserves harmonic functions—namely the family of maps generated by horizontal translations and dilations. This is a luxury that we don't find in B. We do have the rotations of B (which correspond to horizontal translations of H), but there is nothing in B resembling the dilation structure of H (which allows us to pull all of H in towards the point 0 in the boundary of H). On the other hand, the fact that H is unbounded creates a few problems of its own.

In the remainder of this introduction, we discuss some analogies between $\mathcal{B}_a(D)$ and $\mathcal{B}(H)$. The numbered theorems below are the ones whose proofs appear later in this paper.

GROWTH ESTIMATES: The next theorem, which is well-known, shows that a holomorphic Bloch function cannot grow faster than a logarithm near the boundary of D.

Theorem: *If $f \in \mathcal{B}(D)$, then*

$$|f(z)| \leq |f(0)| + ||f||_{\mathcal{B}} \log \frac{1}{1-|z|}$$

for all $z \in D$.

This easily follows from integrating f' from 0 to z. A similar type of growth estimate (requiring a little more work) is valid in $\mathcal{B}(H)$.

2.4 Theorem: *If $u \in \mathcal{B}(H)$, then*

$$|u(x,y)| \;\leq\; |u(0,1)| + ||u||_{\mathcal{B}}[1 + |\log y| + 2\log(1+|x|)]$$

for all $(x,y) \in H$.

CONFORMAL INVARIANCE: The invariant form of the Schwarz's Lemma states that if $\varphi : D \longrightarrow D$ is a holomorphic function, then

$$(1-|z|^2)|\varphi'(z)| \leq 1 - |\varphi(z)|^2$$

for all $z \in D$. Therefore, if $\varphi : D \to D$ is holomorphic, then

1.1 $$||f \circ \varphi||_{\mathcal{B}} \leq ||f||_{\mathcal{B}}$$

for all holomorphic functions f on D, and equality in 1.1 holds when φ is an automorphism of D. Thus, $\mathcal{B}_a(D)$ is invariant under composition with holomorphic self mappings of D.

Looking for analogues of conformal invariance in the upper half-space, we can easily check that the upper half-space is invariant under the maps

$$(x, y) \to (x + a, y), \quad \text{and} \quad (x, y) \to (rx, ry);$$

here $a \in \mathbf{R}^{n-1}$ and $r > 0$. For such a and r and any function u defined on H, we may define the *horizontal translation* $\tau_a u$ by

$$\tau_a u(x, y) = u(x + a, y),$$

and the *r-dilate* u_r by

$$u_r(z) = u(rz).$$

Clearly, u_r and $\tau_a u$ are harmonic on H whenever u is harmonic on H. Also, a straightforward computation shows

$$||\tau_a u||_{\mathcal{B}} = ||u||_{\mathcal{B}} = ||u_r||_{\mathcal{B}}$$

for all a and r as above. Hence, $\mathcal{B}(H)$ is invariant under horizontal translations and dilations.

We'll see later on that the dilation invariance of the Bloch norm is crucial. This is a property that the Bloch norm shares in common with the L^∞-norm and the BMO-norm. We should thus expect (and will prove later) that $\mathcal{B}(H)$ "behaves like" an L^∞-space or a BMO-space in some respects. Note that most norms do not have this property. For example, the L^p-norm is not dilation invariant if $1 \le p < \infty$.

When $n = 2$, we have more than horizontal translations and dilations. For example, the map $z \to 1/\overline{z} = z/|z|^2$ preserves H and harmonicity, and allows us to interchange the boundary points 0 and ∞. It also preserves the Bloch space (this is straightforward computation). When $n > 2$, we could hope that the Kelvin transform, which preserves H and harmonicity (see 4.4 of [ABR]), has analogous properties. Recall that the *Kelvin transform* of a function u defined on H is

$$K[u](z) = |z|^{2-n}u(\frac{z}{|z|^2})$$

for all $z \in H$. Unfortunately, the Kelvin transform does not preserve $\mathcal{B}(H)$ when $n > 2$. Indeed, letting $u \equiv 1$ we have $K[1](z) = |z|^{2-n} \notin \mathcal{B}(H)$, since $y|\nabla K[1](0,y)| = (n-2)y^{2-n}$ is unbounded. Thus, when $n > 2$, there seem to be no other self-mappings of H, other than the ones generated by the dilations and the horizontal translations, that preserve $\mathcal{B}(H)$.

HIGHER DERIVATIVES: In dealing with higher derivatives the following notation is useful. We denote the *k-th* derivative of a holomorphic function f by $f^{(k)}$. In the several variable case, we define a *multi-index* α to be an n-tuple of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$. We use $|\alpha|$ to mean $\alpha_1 + \ldots + \alpha_n$. We denote the α_k^{th} partial derivative with respect to the j^{th} coordinate variable by $D_j^{\alpha_k}$, and when $j = n$ we often write $D_y^{\alpha_k}$ instead of $D_n^{\alpha_k}$. The partial differentiation operator D^α is defined to be $D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ (D_j^0 denotes the identity operator).

The following theorem characterizes the holomorphic Bloch functions in terms of their higher derivatives. Here (and in the rest of this paper) the expression $A(f) \approx B(f)$ means that there are two positive constants c and C such that the nonnegative quantities $A(f)$, $B(f)$ satisfy the inequalities

$$cA(f) \leq B(f) \leq CA(f)$$

for all f under consideration.

Theorem: *Let f be holomorphic on D, and let m be an integer greater than* 1. *Then*

$$||f||_{\mathcal{B}} \approx \sup_{D} \ (1-|z|)^m |f^{(m)}(z)| \ + \sum_{k=1}^{m-1} |f^{(k)}(0)|.$$

A straightforward induction argument, using Cauchy's estimates and the fundamental theorem of calculus, gives the theorem.

One might think that a natural analogue in H for the above theorem would be: If u is harmonic on H and m is an integer greater than 1, then

1.2 $$||u||_{\mathcal{B}} \approx \sup_{H} \ y^m \sum_{|\alpha|=m} |D^\alpha u(x,y)| + \sum_{|\alpha|<m} |D^\alpha u(0,1)|.$$

Unfortunately, this doesn't work. Indeed, the harmonic function $u(x,y) = y$ is an easy counterexample for any integer $m > 1$. In fact, if $m > 1$, then 1.2 fails for any nonconstant harmonic polynomial of degree less than m. The reason is that, unlike a holomorphic polynomial (which is in $\mathcal{B}_a(D)$), a nonconstant harmonic polynomial is never in $\mathcal{B}(H)$. Thus, the harmonic polynomials should somehow be taken into consideration in the statement of the analogue of the above theorem in H. We do this in the following way: Let $\mathcal{P}_m$ denote the set of all harmonic polynomials of degree less or equal to m and define

$$||u+\mathcal{P}_m|| = \inf_{p\in\mathcal{P}_m} ||u+p||_{\mathcal{B}}$$

for any harmonic function u on H. Note that although the above quantity could be infinite for some u, it behaves like a quotient norm in the sense that it vanishes on $\mathcal{P}_m$. We are now able to state the analogue we are after in $\mathcal{B}(H)$.

2.11 Theorem: *If u is a harmonic function on H and m is a positive integer, then*

$$||u+\mathcal{P}_{m-1}|| \approx \sup_{H} \ y^m \sum_{|\alpha|=m} |D^\alpha u|.$$

If we know ahead of time that u is a Bloch function, then we obtain the following equivalence.

2.16 Theorem: *If $u \in \mathcal{B}(H)$ and m is a positive integer, then*

$$||u||_{\mathcal{B}} \approx \sup_{H} \, y^m \sum_{|\alpha|=m} |D^\alpha u| \approx \sup_{H} \, y^m \sum_{k=1}^{n} |D_y^{m-1} D_k u|.$$

REMARKS: 1. The last quantity in 2.16 contains only first order partial derivatives with respect to the first $n-1$ variables $x_1, \ldots, x_{n-1}$, whereas the second quantity contains partial derivatives with respect to $x_1, \ldots, x_{n-1}$ of order less or equal to m. This is the surprising fact mentioned in the abstract; it seems to suggest that for a Bloch function, the partial derivatives with respect to y are more important.

2. Because $||u||_{\mathcal{B}}$ does not depend on m, we also deduce that for different integers m_1, m_2

$$\sup_{H} \, y^{m_1} \sum_{|\alpha|=m_1} |D^\alpha u| \approx \sup_{H} \, y^{m_2} \sum_{|\alpha|=m_2} |D^\alpha u|.$$

3. The condition $|\alpha| = m$ in 2.16 can be replaced by $|\alpha| \leq m$.

4. Unlike the case of $\mathcal{B}_a(D)$, neither 2.11 nor 2.16 involves evaluating functions at some fixed point in H.

BMO CONDITIONS: Here we'll see that membership in $\mathcal{B}(H)$ is equivalent to satisfying certain BMO-type conditions. The realization that the Bloch space can be viewed as a BMO-type space first occurred in [CRW]; see also [A]. For completeness, we prove one of these results for $\mathcal{B}_a(D)$. We let dA denote Lebesgue area measure on $\mathbf{C}$, normalized so that $A(D) = 1$, and we let $D_a = \{z \in D : |z - a| < (1 - |a|)/2\}$.

Theorem: *Let $1 \leq p < \infty$ and let f be a holomorphic function on D. Then*

$$||f||_{\mathcal{B}} \approx \sup\left\{\frac{1}{A(D_a)}\int_{D_a}|f(z)-f(a)|^p dA\right\}^{\frac{1}{p}},$$

where the supremum is taken over all $a \in D$.

We now prove this theorem; the proof comes from [RU1]. (We will need some of these ideas in Chapter 2). Let a be in D and z be in D_a. Fix $p \in [1, \infty)$. The inequalities

$$\begin{aligned} |f(z)-f(a)| &\leq \int_0^1 |z-a||f'(a+t(z-a))|dt \\ &\leq ||f||_{\mathcal{B}} \int_0^1 \frac{|z-a|}{1-|a+t(z-a)|}dt \\ &\leq ||f||_{\mathcal{B}} \end{aligned}$$

(the last inequality uses the fact that $|z-a| < (1-|a|)/2)$ easily show that

$$\sup_{a\in D}\left\{\frac{1}{A(D_a)}\int_{D_a}|f(z)-f(a)|^p dA\right\}^{\frac{1}{p}} \leq ||f||_{\mathcal{B}}.$$

For the other inequality, we use the fact that

1.3 $$|f'(0)| \leq \int_D |f| dA$$

for all f holomorphic on D (recall $A(D) = 1$). Now fix $a \in D$ and let $r = (1-|a|)/2$. We apply 1.3 to $g(z) = f(rz+a) - f(a)$, getting

$$\begin{aligned} r|f'(a)| &\leq \int_D |f(rz+a)-f(a)|dA(z) \\ &= \frac{1}{r^2}\int_{D_a}|f(z)-f(a)|dA(z) \\ &\leq \left\{\frac{1}{r^2}\int_{D_a}|f(z)-f(a)|^p dA(z)\right\}^{\frac{1}{p}}, \end{aligned}$$

the conclusion of the theorem follows.□

REMARK: The conclusion of this theorem is actually valid for all $p \in (0, \infty)$.

We now state a more ambitious version of the above theorem that we will prove in Chapter 2. We let $B(a,r) = \{z \in \mathbf{R}^n : |z-a| < r\}$ and $\Omega(a,r) = B(a,r) \cap H$. Given $u \in L^1(\Omega)$, we define

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\, dV.$$

Here dV denotes the Lebesgue volume measure and $|\Omega|$ denotes the volume of Ω.

2.17 Theorem: *Suppose u is a harmonic function on H that is volume integrable on every bounded subset of H. If $1 \le p < \infty$, then*

2.18
$$||u||_{\mathcal{B}} \approx \sup\left\{\frac{1}{|\Omega|}\int_\Omega |u-u_\Omega|^p dV\right\}^{\frac{1}{p}},$$

where the supremum is taken over all $\Omega = \Omega(a,r)$ with a in the closure of H.

Note that in this theorem, we don't restrict Ω to be a ball that stays away from ∂H (as we did earlier for $\mathcal{B}_a(D)$). For example, Ω can be a ball tangent to ∂H, or half a ball with center in ∂H, or part of a ball with center in H. In the context of $\mathcal{B}_a(D)$, the authors in [CRW] also allowed such general Ω's; however, our proof of 2.17 will not rely on the machinery developed in [CRW].

There is another BMO-condition that characterizes $\mathcal{B}(H)$, this one involving the Hardy-space norm. For $\mathcal{B}_a(D)$, this first appeared in [RU1]. Recall that if u is a harmonic function on the open unit ball of $\mathbf{R}^n$ and $1 \le p < \infty$, then the Hardy-space norm of u is

$$||u||_{h^p} = \sup_{0\le r<1}\left[\int_S |u(r\zeta)|^p d\sigma(\zeta)\right]^{\frac{1}{p}},$$

where S denotes the unit sphere of $\mathbf{R}^n$ and $d\sigma$ denotes the normalized surface area measure on S, so that $\sigma(S) = 1$.

2.20 Theorem: *If u is a harmonic function on H and $1 \le p < \infty$, then*

$$||u||_{\mathcal{B}} \approx \sup ||u(a + rx) - u(a)||_{h^p},$$

where the supremum is taken over all a, r such that $B(a, r) \subset H$.

In particular, Theorem 2.20 shows that if $u \in \mathcal{B}(H)$, then "u belongs to h^p of every ball in H tangent to ∂H".

HARMONIC CONJUGATES: Here we explore the notion of harmonic conjugates in higher dimensions. This subject is not as widely known as in the case $n = 2$. Specifically, we'll address the following question: Given $u \in \mathcal{B}(H)$, will its harmonic conjugates also be in $\mathcal{B}(H)$?

We first make clear what we mean by harmonic conjugates in higher dimensions. Let u be a harmonic function on H. The functions $v_1, \ldots, v_{n-1}$ on H are said to be *harmonic conjugates* of u if

1.4 $$(v_1, \ldots, v_{n-1}, u) = \nabla f$$

for some harmonic function f on H. If such v_j's exist, then they must be harmonic, being partial derivatives of a harmonic function. Also 1.4 and the condition that f is harmonic are equivalent to the "generalized Cauchy-Riemann" equations

1.5 $$D_k v_j = D_j v_k \ ; \ D_y v_j = D_j u$$

1.6 $$\sum_{j=1}^{n-1} D_j v_j + D_y u = 0.$$

Indeed, if 1.4 holds, then 1.5 will hold since f is twice continuously differentiable on H; also 1.6 easily follows from the fact that f is harmonic. Conversely, if 1.5 and 1.6 hold, then the differential form

$$\sum_{j=1}^{n-1} v_j \, dx_j + u \, dy$$

is closed. Therefore, since H is simply connected, 1.4 holds (see Theorem 8.4 of [Fl]) for some function f, which is harmonic by 1.6.

The terminology "harmonic conjugates" comes from the case $n = 2$: By the Cauchy-Riemann equations, 1.5 and 1.6 hold if and only if $u + iv_1$ is holomorphic on H, as easily checked. Here of course the harmonic conjugate exists and is unique up to an additive constant. Thus, 1.4 is a natural generalization of the case $n = 2$. However, showing the existence of the harmonic conjugates in the case $n > 2$ requires more work. There is also a greater degree of non-uniqueness of harmonic conjugates when $n > 2$, as we show below. For now, we just state the following:

3.5 Theorem: *If u is harmonic on H, then harmonic conjugates of u exist.*

We need more than just the existence of harmonic conjugates. We now return to our basic question: If $u \in \mathcal{B}(H)$, need the harmonic conjugates $v_1, \ldots, v_{n-1}$ belong to $\mathcal{B}(H)$? In the case $n = 2$ it is easy to see via the Cauchy-Riemann equations that if u is Bloch, then so is any of its conjugates. In the case $n > 2$ it is not true that all the conjugates of a Bloch function are Bloch. For example, letting $u \equiv 0$, then $v_1(x_1, x_2, y) = x_1$ and $v_2(x_1, x_2, y) = -x_2$ are harmonic conjugates for u that are not Bloch (here $f(x_1, x_2, y) = (x_1^2 - x_2^2)/2$). On the other hand, $v_1(x_1, x_2, y) = a$ and $v_2(x_1, x_2, y) = b$ (where a, b are constants) are also conjugates for u (here $f(x_1, x_2, y) = ax_1 + bx_2$) and v_1, v_2 are this time Bloch. The next theorem shows that the above example is typical: Harmonic conjugates of a Bloch function may always be chosen to lie in the Bloch space.

3.12 Theorem: *Let $u \in \mathcal{B}(H)$. Then there exist unique harmonic conjugates $v_1, \ldots, v_{n-1}$ of u on H such that $v_j \in \mathcal{B}(H)$ and $v_j(0,1) = 0$ for each j. Moreover, there exists a constant M, depending only on n, such that $||v_j||_{\mathcal{B}} \leq M||u||_{\mathcal{B}}$ for each j.*

This theorem will be very useful in studying the boundary behavior of the harmonic Bloch functions on H.

BOUNDARY BEHAVIOR OF BLOCH FUNCTIONS: There are several well-known results in the literature concerning the behavior of harmonic functions at an individual boundary point. Perhaps the best-known of these is due to Fatou. Let μ be a complex Borel measure on $\mathbf{R}$, and let u denote the Poisson integral of μ so that u is well defined and harmonic in the upper half-plane. Fatou showed that

1.7 $$\lim_{h \to 0+} \frac{1}{2h} \int_{-h}^{h} d\mu = L$$

implies

1.8 $$\lim_{y \to 0+} u(iy) = L,$$

where $L \in \mathbf{C}$ (see [Fa]).

In general 1.8 does not imply 1.7, as Loomis showed in [L]. But in the same paper Loomis proved that 1.8 implies 1.7 if the measure μ is assumed to be positive. It follows that 1.8 implies 1.7 if $\mu = f dx$, where $f \in L^\infty(\mathbf{R})$ and dx is the Lebesgue measure on $\mathbf{R}$. However, 1.8 need not imply 1.7 if $f \in L^p(\mathbf{R})$ and $p < \infty$; in fact there is an $f \in \bigcap_{p<\infty} L^p(\mathbf{R})$ for which 1.8 does not imply 1.7. (We recall some classical terminology: The implication 1.7 $\Rightarrow$ 1.8, which is true under a wide array of conditions, is called the "abelian" direction. The direction 1.8 $\Rightarrow$ 1.7 is called the "tauberian" direction; it holds only when a certain condition—a "tauberian" condition—is added. Thus positivity is the tauberian condition for the Loomis result.)

W.Rudin generalized the Loomis result for positive measures to higher dimensions; he also showed that this result fails when $L = +\infty$ (see [R1]). W.Ramey and D.Ullrich gave a different proof of Rudin's result and extended it to the case $d\mu = f dx$, where $f \in BMO$. They also showed that in this latter case, 1.8 still implies 1.7 if f is real-valued and $L = \pm\infty$ (see [RU2]).

The techniques used by Ullrich and Ramey in [RU2]—most importantly, dilation invariance and normal families—are available for the class of Bloch functions. Thus, it seems natural to ask about the relation between 1.7 and 1.8 for Bloch functions. But we run into an immediate difficulty: If u is a Bloch function, then u need not be the Poisson integral of a measure μ on ∂H. Indeed, if every Bloch function were a Poisson integral, then every Bloch function would have almost everywhere boundary limits by the Fatou Theorem. But it is well known that there are Bloch functions that fail to have almost everywhere boundary limits. For example, on D,

$$f(z) = \sum_{k=0}^{\infty} z^{2^k}$$

is such a function (see [ACP]). (After a conformal mapping, we obtain a similar example for H.) Thus, in discussing the relation between 1.7 and 1.8 for Bloch functions, we need something on the boundary in place of the measure μ. One possible way to arrive at a "boundary function" for a Bloch function is through the theory of distributions:

4.1 Theorem: *If u is a Bloch function in the upper-half space, then*

$$\lim_{y \to 0} \int_{\mathbf{R}^{n-1}} u(x,y)\varphi(x)dx$$

exists for every smooth function φ with compact support. Moreover, this limit defines a distribution on the space of smooth test functions with compact support.

Of course when u is the Poisson integral of a measure μ as above, then the distribution of Theorem 4.1 is just the measure μ. However, for our problem we need to go beyond the theory of distributions, because in 1.7 we are integrating μ not against a smooth test function but against the characteristic function of an interval. This is what Ullrich did in [U]. More precisely, given a holomorphic function f on D, $0 < r < 1$ and $0 < t \le \pi$, set

$$A_t^r f = \frac{1}{2t}\int_{-t}^{t} f(re^{i\theta})d\theta.$$

Ullrich showed that if f is a holomorphic Bloch function on D, then

$$\lim_{r\to 1} A_t^r f = A_t f$$

exists. Moreover, this limit defines a bounded linear functional on $\mathcal{B}_a(D)$. Finally, Ullrich obtained the equivalence of 1.7 and 1.8 for Bloch functions. In other words, for $f \in \mathcal{B}_a(D)$,

$$\lim_{r\to 1} f(r) = L \quad \text{if and only if} \quad \lim_{t\to 0} A_t f = L,$$

where L is any complex number. Because $L^\infty \subset BMO \subset Bloch$, the equivalences between 1.7 and 1.8 for L^∞ and BMO mentioned earlier follow as special cases.

We now discuss analogues of Ullrich's results for harmonic Bloch functions in the upper half-space in any dimension. Let $u \in \mathcal{B}(H)$ and let Ω be a bounded measurable subset of $\mathbf{R}^{n-1}$. We then define

$$I_\Omega^h u = \int_\Omega u(x,h)dx$$

for $h > 0$. We first show that the integrals $I_\Omega^h u$ have good limiting behavior as $h \to 0$ if Ω is nice enough.

4.2 Theorem: *Suppose $u \in \mathcal{B}(H)$, Ω is a bounded open set with C^1-boundary, and $0 \in \Omega$. Then*

$$I_\Omega u = \lim_{h \to 0} I_\Omega^h u$$

exists and defines a bounded linear functional on $\mathcal{B}(H)$. Moreover, there is a constant C, depending only on n, such that

$$||I_\Omega|| \leq C(1 + |\partial\Omega| d + |\Omega|)(1 + |\log d|),$$

where d equals the diameter of Ω.

REMARKS: The operator norm $||I_\Omega||$ is, of course, computed relative to the norm $||u||_{\mathcal{B}} + |u(0,1)|$ of $\mathcal{B}(H)$. By $|\partial\Omega|$ we mean the surface area measure of $\partial\Omega$.

The proof of Theorem 4.2 uses the divergence theorem, as we'll see in Chapter 4. Thus, the condition that $\partial\Omega$ is C^1 could easily be replaced by the weaker condition that $\partial\Omega$ is piecewise C^1. This suggests a natural question: What is the largest class of Ω's for which Theorem 4.2 will hold? We have not been able to settle this question.

To discuss the equivalence between 1.7 and 1.8 for $\mathcal{B}(H)$, we specialize to the case where Ω is a ball centerd at the origin. Let us define the averages

$$A_r u = \lim_{h \to 0} \frac{1}{|B_r|} \int_{B_r} u(x,h)\,dx,$$

where B_r is the ball in $\mathbf{R}^{n-1}$ centered at the origin of radius r. The proof of the above theorem will give us the estimate

$$||A_r u|| \leq C(1 + |\log r|)(||u||_{\mathcal{B}} + |u(0,1)|)$$

for all $r > 0$. This estimate will be instrumental in proving the equivalence of 1.7 and 1.8 for harmonic Bloch functions on H, which is what the next theorem asserts.

4.14 Theorem: *If $u \in \mathcal{B}(H)$ and $L \in [-\infty, +\infty]$, then*

$$\lim_{y \to 0} u(0, y) = L \quad \textit{if and only if} \quad \lim_{r \to 0} A_r u = L.$$

Chapter 2

Basic Properties of Harmonic Bloch Functions

The Cauchy-Riemann equations show that a real-valued harmonic function on D is Bloch if and only if it is the real part of a holomorphic Bloch function on D. Hence, most properties of real-valued harmonic functions on D can be obtained by studying the corresponding properties of holomorphic Bloch functions on D. Thus, in a way, the harmonic Bloch space on D, which we denote $\mathcal{B}(D)$, is as well-known as $\mathcal{B}_a(D)$.

In this chapter we focus on the less-known harmonic Bloch space $\mathcal{B}(H)$, where

$$H = \{z = (x, y) : x \in \mathbf{R}^{n-1},\ y > 0\}.$$

A real-valued harmonic function on H is said to be a *Bloch function* if

$$\sup\ y|\nabla(u(x,y))|\ <\ \infty,$$

where the supremum is taken over all $(x, y) \in H$, and ∇u denotes the gradient of u.

The simplest Bloch functions on H are the real-valued bounded harmonic functions: For if u is harmonic and bounded on H, then by Cauchy's estimates, there is a constant C such that

$$|\nabla u(z)| \leq \frac{C}{d(z, \partial H)} = \frac{C}{y}$$

for all $z = (x, y) \in H$ (see 2.6 of [ABR]). As is well-known, the class of bounded harmonic functions on H is exactly the set of Poisson integrals of bounded measurable functions on $\mathbf{R}^{n-1}$ (see 7.14(b) of [ABR]). Thus u is bounded and harmonic on H if and only if $u = P[f]$, where $f \in L^\infty(\mathbf{R}^{n-1})$ and

2.1 $$P[f](x,y) = \int_{\mathbf{R}^{n-1}} \frac{c_n\, y\, f(t)}{(|x-t|^2 + y^2)^{\frac{n}{2}}} dt.$$

Here c_n is chosen so that

$$c_n \int_{\mathbf{R}^{n-1}} \frac{dt}{(|t|^2+1)^{\frac{n}{2}}} = 1.$$

We can easily check (for later purposes) that 2.1 defines a harmonic function on H whenever f is a measurable function on $\mathbf{R}^{n-1}$ such that

2.2 $$\int_{\mathbf{R}^{n-1}} \frac{|f(t)|}{(|t|^2+1)^{\frac{n}{2}}} dt < \infty.$$

Examples of unbounded Bloch functions on H are the functions

$$u(x,y) \;=\; \log(x_k^2 + y^2) \qquad k = 1, \ldots, n-1,$$

as easily checked. These particular functions are, respectively, the Poisson integrals of the functions $2\log|t_k| \in BMO(\mathbf{R}^{n-1})$. In fact, $P[f] \in \mathcal{B}(H)$ whenever $f \in BMO(\mathbf{R}^{n-1})$. To see this, let $f \in BMO$. Then f satisfies 2.2 (see [FS]). Moreover, setting $u = P[f]$, we know by [FS] that $y|\nabla u(x,y)|^2 dV$ is a Carleson measure on H. This means that there is a positive constant A such that

2.3 $$\int_{C_{a,h}} y|\nabla u(x,y)|^2 dV \le A h^{n-1}$$

for all $a \in \mathbf{R}^{n-1}$ and $h > 0$. Here $C_{a,h}$ is the cylinder $\{|x-a| < h\} \times (0,h)$. Now fix $z = (x,y) \in H$, and let $B_z = B(z, y/2)$. Because $|\nabla u|^2$ is subharmonic, we have

$$\begin{aligned} y^2|\nabla u(z)|^2 &\le y^2\left\{\frac{1}{|B_z|}\int_{B_z} |\nabla u(s,t)|^2 ds dt\right\} \\ &\le \frac{2y}{|B_z|}\int_{B_z} t|\nabla u(s,t)|^2 ds dt. \end{aligned}$$

Note that $|B_z| = \alpha y^n$ (where α is a constant depending only on n). Also note that B_z is contained in $C_{x,3y/2}$. Enlarging the domain of integration from B_z to $C_{x,3y/2}$, we see that inequality 2.3 implies that $y|\nabla u(x,y)|$ is bounded on H; i.e, $u \in \mathcal{B}(H)$ as desired. (I thank Dr. Frazier for pointing out this simple way of seeing that $P[f] \in \mathcal{B}(H)$ whenever $f \in BMO$.)

However, not every harmonic Bloch function is the Poisson integral of a BMO function. Indeed, it is well-known that the holomorphic Bloch function on D

$$g(z) = \sum_{k=1} z^{2^k}$$

has finite boundary values nowhere on ∂D. Hence, by Fatou's theorem, g cannot be the Poisson integral of any $f \in L^1(\partial D)$. (After a conformal mapping we obtain a similar example for H.)

Recall in Chapter 1, we introduced the horizontal translations and dilations of a function defined on H, and asked the reader to check that the Bloch norm is invariant under these transformations. We will also need the vertical translations: If u is a function on H, then its *vertical translate* $\tau_h u$ is the function on H defined by

$$\tau_h u(x,y) = u(x, y+h)$$

for $h > 0$. If $u \in \mathcal{B}(H)$, then $\tau_h u$ is also in $\mathcal{B}(H)$, and $||\tau_h u||_{\mathcal{B}} \leq ||u||_{\mathcal{B}}$. Indeed, the harmonicity of $\tau_h u$ is clear, and

$$y\,|\nabla \tau_h u(x,y)| = y\,|\nabla u(x, y+h)| \leq \frac{y||u||_{\mathcal{B}}}{y+h} \leq ||u||_{\mathcal{B}}$$

for all $(x,y) \in H$.

We'll soon see that $\tau_h u$ equals $P[u(\cdot,h)]$, which will be important for our work later. But in order for $P[u(\cdot,h)]$ to make sense, we need a growth estimate on u. That's the object of the next theorem, which shows that a Bloch function can grow no faster than $|\log y|$ near ∂H and ∞.

2.4 Theorem: *If $u \in \mathcal{B}(H)$, then*

$$|u(x,y)| \;\leq\; |u(0,1)| + ||u||_{\mathcal{B}}[1 + |\log y| + 2\log(1+|x|)].$$

PROOF: Fix $(x,y) \in H$. We have that

$$\begin{aligned} |u(x,y) \;-\; u(x,1)| &= |\int_1^y D_n u(x,s)ds| \\ &\leq |\int_1^y \frac{||u||_{\mathcal{B}}}{s}\,ds| \\ &= ||u||_{\mathcal{B}}|\log y|. \end{aligned}$$

Hence,

$$|u(x,y)| \leq |u(x,1)| \;+\; ||u||_{\mathcal{B}}|\log y|.$$

To get an estimate on $|u(x,1)|$, we let $r = 1 + |x|$ and use the triangle inequality

$$|u(x,1) - u(0,1)| \leq |u_r(\frac{x}{r},\frac{1}{r}) - u_r(\frac{x}{r},1)| + |u_r(\frac{x}{r},1) \;-\; u_r(0,1)| + |u(0,r) \;-\; u(0,1)|.$$

Denote I, II, III respectively the three terms on the right hand side of the inequality. We then have

$$\begin{aligned} I &= |\int_{\frac{1}{r}}^1 D_y u_r(\frac{x}{r},s)\,ds| \\ &\leq \int_{\frac{1}{r}}^1 \frac{||u_r||_{\mathcal{B}}}{s}\,ds \;=\; ||u_r||_{\mathcal{B}}\,\log r = ||u||_{\mathcal{B}}\,\log r; \\ II &= |u(x,r) \;-\; u(0,r)| \\ &\leq \sup_{t\in[(x,r),(0,r)]} |\nabla u(t)|\,|x| \;\leq\; \frac{||u||_{\mathcal{B}}}{r}|x| \;\leq\; ||u||_{\mathcal{B}}; \\ III &= |\int_1^r D_y u(0,t)\,dt| \\ &\leq ||u||_{\mathcal{B}}\log r. \end{aligned}$$

(Note that the dilation invariance of the Bloch norm was used in I.) The conclusion of the theorem follows.$\square$

We are now able to prove an important property of the Bloch functions in the upper half-space: The vertical translates of a Bloch function are the Poisson integrals of their boundary values. More precisely, we have

2.5 Theorem: *If $u \in \mathcal{B}(H)$, then*

$$\tau_h u = P_H[u(\cdot, h)]$$

for every $h > 0$.

For the proof of this theorem, we need the following three lemmas.

2.6 Lemma: *If f is continuous on $\mathbf{R}^{n-1}$ and 2.2 holds, then $P[f]$ can be extended continuously to the closure of H, with boundary values f.*

PROOF: This is a simple variation on the proof that the Dirichlet problem for H with bounded continuous boundary data is solvable.□

2.7 Lemma: *If u is harmonic on $\mathbf{R}^n$ and*

$$|u(x)| \leq A(1 + |x|^p) \quad (x \in \mathbf{R}^n)$$

for some constant A, then u is a polynomial of degree $\leq p$.

PROOF: Fix $x \in \mathbf{R}^n$ and let α be a multi-index. By Cauchy's estimates, there is a positive constant C_α such that

$$|D^\alpha u(x)| \leq \frac{C_\alpha A(1 + r^p)}{r^{|\alpha|}}$$

for all $r > |x|$. Letting r go to infinity gives us $D^\alpha u(x) = 0$ for all α such that $|\alpha| > p$. Thus, u is a polynomial of degree $\leq p$.□

2.8 Lemma: *If f is a measurable function on $\mathbf{R}^{n-1}$ and there exists $p \in (0,1)$ such that*

$$|f(t)| \leq A + B|t|^p$$

for some constants A and B, then there exist constants C and D such that

$$|P[f](x,y)| \leq C + D|(x,y)|^p \qquad ((x,y) \in H).$$

For the proof of this lemma we need the polar coordinate formula for integration on $\mathbf{R}^n$: If g is a Borel measurable, integrable function on $\mathbf{R}^n$, then

2.9
$$\frac{1}{n|B|}\int_{\mathbf{R}^n} g\,dV = \int_0^\infty r^{n-1}\int_S g(r\zeta)d\sigma(\zeta)dr.$$

PROOF OF LEMMA 2.8: Let f be as in the statement of the lemma. The fact that f satisfies the condition 2.2 (and hence $P[f]$ is well defined on H) follows easily by using 2.9. We also have

$$\begin{aligned} |P[f](x,y)| &\leq \int_{\mathbf{R}^{n-1}} \frac{c_n\,(A + B\,|t|^p\,y)}{(y^2\,+\,|t-x|^2)^{\frac{n}{2}}}dt \\ &= A + B\,c_n\int_{\mathbf{R}^{n-1}} \frac{y\;|t|^p}{(y^2 + |t-x|^2)^{\frac{n}{2}}} \end{aligned}$$

for all $(x,y) \in H$. We split the last integral into two parts, by integrating over $\{|t| < 2|x|\}$ and over $\{|t| \geq 2|x|\}$. In the first part, we replace $|t|^p$ in the numerator by $|2x|^p$ and integrate over all of $\mathbf{R}^{n-1}$. We then see that the first part is bounded by a constant times $|x|^p$. In the second part, we use the estimate

$$y^2 + |t-x|^2 \geq y^2 + \frac{|t|^2}{4}$$

(valid for the domain of integration), then integrate over all of $\mathbf{R}^{n-1}$ using the change of variables $t = ys$. We then see that the second part is bounded by a constant times y^p. The conclusion of the lemma follows.□

PROOF OF THEOREM 2.5: Let $u \in \mathcal{B}(H)$ and fix $h > 0$. We easily check using the growth estimate 2.4 and the formula 2.9 that $u(t, h)$ satisfies the condition 2.2. Define on H the harmonic functions

$$v = P[u(\cdot, h)], \ w = \tau_h u - v.$$

By Lemma 2.6, v extends to be continuous on the closure of H with boundary values $u(\cdot, h)$. Hence, w extends to be continuous on the closure of H with boundary values identically 0. We now extend w to all of $\mathbf{R}^n$ by setting $w(x, -y) = -w(x, y)$. By the Schwarz Reflection Principle, w is harmonic on $\mathbf{R}^n$ (see 4.9 of [ABR]). We now show that $w \equiv 0$ on $\mathbf{R}^n$.

We have by the growth estimate 2.4 that

$$\begin{aligned} |u(t,h)| &\leq |u(0,1)| + ||u||_{\mathcal{B}}[1 + |\log h| + 2\log(1 + |t|)] \\ &\leq A + B|t|^{\frac{1}{2}} \end{aligned}$$

for all $t \in \mathbf{R}^{n-1}$ (Here we choose the exponent 1/2 for no particular reason; any positive exponent less than 1 will do.) Therefore, by Lemma 2.8

$$|v(x,y)| \leq A + B|(x,y)|^{\frac{1}{2}}$$

for all $(x, y) \in H$. Also, by the growth estimate 2.4,

$$\begin{aligned} |\tau_h u(x,y)| &\leq |u(0,1)| + ||u||_{\mathcal{B}}[1 + |\log(y + h)| + 2\log(1 + |x|)] \\ &\leq A + B|(x,y)|^{\frac{1}{2}} \end{aligned}$$

for all $(x, y) \in H$. Hence,

$$|w(x,y)| \leq |\tau_h u(x,y)| + |v(x,y)| \leq A + B|(x,y)|^{\frac{1}{2}}$$

for all $(x, y) \in H$. Therefore w is a polynomial of degree 0 (by Lemma 2.7), hence $w \equiv 0$ on $\mathbf{R}^n$ (since $w \equiv 0$ on ∂H). Thus $\tau_h u = P[u(\cdot, h)]$ on H as desired.□

In the remainder of this chapter, we characterize the Bloch functions on H in terms of their higher derivatives, and in terms of two bounded mean oscillation conditions. Here is first a simple characterization of the Bloch functions in terms of the last coordinate variable y. It shows that the constant functions are the only Bloch functions in H that don't depend on y.

2.10 Lemma: *If $u \in \mathcal{B}(H)$ doesn't depend on y, then u is constant.*

PROOF: Our hypothesis implies that there exists a smooth function f defined on $\mathbf{R}^{n-1}$ such that $u(x,y) = f(x)$ for all $(x,y) \in H$. Therefore

$$\begin{aligned} |\nabla f(x)| &= |\nabla u(x,y)| \\ &\leq \frac{||u||_{\mathcal{B}}}{y} \end{aligned}$$

for all $(x,y) \in H$. Letting $y \to \infty$, we see that $\nabla f(x) = 0$ for all $x \in \mathbf{R}^{n-1}$. Thus (since H is connected), f is constant, and therefore so is u.□

Before we come to our characterization of the Bloch functions in terms of their higher derivatives, recall from chapter 1 that $\mathcal{P}_m$ denotes the set of all harmonic polynomials of degree less or equal to m and that for any harmonic function u on H we define

$$||u + \mathcal{P}_m|| = \inf_{p \in \mathcal{P}_m} ||u + p||_{\mathcal{B}}.$$

2.11 Theorem: *If u is a harmonic function on H and m is a positive integer, then*

$$||u + \mathcal{P}_{m-1}|| \approx \sup_{H} \; y^m \sum_{|\alpha|=m} |D^\alpha u|.$$

To prove the theorem we need the following two calculus-type lemmas.

2.12 Lemma: *If a smooth function f defined on $(0,\infty)$ satisfies*

$$x^{k+1}|f^{(k)}(x)| \leq M$$

for some integer $k \geq 1$, then there exists a polynomial p_{k-1} of degree $\leq k-1$ such that

$$x|f(x) - p_{k-1}(x)| \leq \frac{M}{k!} \qquad (x \in (0\,,\infty)).$$

PROOF: By induction on k. Let $k = 1$. We show first that $f(\infty) = \lim_{x\to\infty} f(x)$ exists. For $t \leq x$ we have

$$\begin{aligned} |f(x) - f(t)| &\leq \int_t^x |f'| \\ &\leq M\,|\frac{1}{x} - \frac{1}{t}| \\ &\leq \frac{2M}{t} \end{aligned}$$

This shows that $f(\infty)$ exists. Because

$$|f(x) - f(\infty)| \leq \int_x^\infty |f'| \leq \frac{M}{x},$$

we are done in the case $k = 1$ (take $p_0 = f(\infty)$).

Now suppose the lemma is true for k and $x^{k+2}|f^{(k+1)}(x)| \leq M$. As above, this implies that $f^{(k)}(\infty)$ exists. Furthermore,

$$|f^{(k)}(x) - f^{(k)}(\infty)| \leq \int_x^\infty |f^{(k+1)}| \leq \frac{M}{(k+1)x^{k+1}},$$

so that

$$x^{k+1}|f^{(k)}(x) - f^{(k)}(\infty)| \leq \frac{M}{k+1}.$$

Hence

$$x^{k+1}|(f(x) - \frac{f^{(k)}(\infty)\,x^k}{k!})^{(k)}| \leq \frac{M}{k+1}.$$

Thus (by induction hypothesis), there is a polynomial p_{k-1} such that

$$|f(x) - \frac{f^{(k)}(\infty)\, x^k}{k!} - p_{k-1}| \leq \frac{M}{(k+1)k!}.$$

Taking $p_k(x) = x^k f^{(k)}(\infty)/k! + p_{k-1}$ shows that the lemma is true for $k+1$. Therefore, the lemma is true for all positive integers k.□

2.13 Lemma: *If u is a smooth function on H such that*

$$M = \sup_H \, y^m \sum_{|\alpha|=m} |D^\alpha u| < \infty$$

for some integer $m \geq 2$, then there is a polynomial p_{m-1} of degree less or equal to $m-1$ and a positive constant C, depending only on m and n, such that

$$\sup_H \, y^2 \sum_{|\alpha|=2}^{n} |D^\alpha (u - p_{m-1})| \leq CM.$$

PROOF: By induction on m. The case $m = 2$ is clear (any linear polynomial will do). Now suppose the lemma is true for m and that

$$\sup_H \, y^{m+1} \sum_{|\alpha|=m+1} |D^\alpha u| = M < \infty.$$

Fix $x \in \mathbf{R}^{n-1}$ and let β be a multi-index such that $|\beta| = m$. Define $f(y) = D^\beta u(x,y)$ for $y > 0$. We easily see that

$$y^{m+1}|f'(y)| \leq M$$

for all $y > 0$. Therefore, because M is finite, $f(\infty) = L$ exists (see the proof of Lemma 2.11). Furthermore, L is independent of x. Indeed, the inequality

$$|D^\beta u(x,y) - D^\beta u(0,y)| \leq \sup_{z\in[(x,y),\ (0,y)]} |D^\alpha u(z)|\ |x| \leq \frac{M|x|}{y^{m+1}}$$

for all $(x,y) \in H$, shows that

$$\begin{aligned} L = f(\infty) &= \lim_{y\to\infty} D^\beta u(x,y) \\ &= \lim_{y\to\infty} D^\beta u(0,y). \end{aligned}$$

It also follows from the fundamental theorem of calculus that

$$y^m|f(y) - L| = y^m\,|D^\beta u(x,y) - L| \leq \frac{M}{m} \leq M$$

for all $(x, y) \in H$. Hence

$$y^m|D^\beta(u - q_\beta)(x,y))| \leq M$$

for all $(x, y) \in H$, where q_β is a polynomial of degree less or equal to m such that $D^\beta q_\beta = L$. Thus,

$$\sup_H\; y^m\,|D^\beta(u - q_\beta)| \leq M.$$

Letting $q = \sum_{|\beta|=m} q_\beta$ and C equal the number of possible multi-indices β such that $|\beta| = m$, we obtain

$$\sup_H\; y^m \sum_{|\beta|=m} |D^\beta(u - q)| \leq CM.$$

Hence (by induction hypothesis), there is a polynomial p_{m-1} such that

$$\sup_H\; y^2 \sum_{|\alpha|=2}^{n} |D^\alpha(u - q - p_{m-1})| \leq CM.$$

Taking $p_m = q - p_{m-1}$ shows that the Lemma is true for $m + 1$. Thus the Lemma is true for all integers $m \geq 2$.□

If we add to the hypotheses of the above lemma the assumption that u is harmonic, we obtain

2.14 Corollary: *Let u and p_{m-1} be as in Lemma* 2.13. *If u is harmonic on H, then p_{m-1} is harmonic on H.*

PROOF: Because u is harmonic, we obtain (from Lemma 2.13)

$$\sup_H\; y^2|\Delta(u - p_{m-1})| = \sup_H\; y^2|\Delta(p_{m-1})| \leq CM.$$

Here Δ denotes the Laplacian: $\Delta = D_1^2 + \cdots + D_n^2$. Because $\Delta(p_{m-1})$ is a polynomial, the above inequality is possible only when $\Delta(p_{m-1}) = 0$.□

PROOF OF THEOREM 2.11: The case $m = 1$ is clear, so we assume that $m > 1$. By Cauchy's Estimates, there is a positive constant C such that

$$\begin{aligned}\sup_H y^m \sum_{|\alpha|=m} |D^\alpha u| &= \sup_H y^m \sum_{|\alpha|=m} |D^\alpha(u+p)| \\ &\le C\|u+p\|_{\mathcal{B}}\end{aligned}$$

for all harmonic functions u on H and all $p \in \mathcal{P}_{m-1}$. Therefore,

$$\sup_H y^m \sum_{|\alpha|=m} |D^\alpha u| \le C\|u + \mathcal{P}_{m-1}\|.$$

For the other inequality, fix u as in the statement of the theorem and assume that

$$\sup_H y^m \sum_{|\alpha|=m} |D^\alpha u| = M < \infty.$$

We'll show that there is a polynomial $q \in \mathcal{P}_{m-1}$ such that $\|u - q\|_{\mathcal{B}}$ is less than or equal to a constant multiple of M. To do that, we fix $x \in \mathbf{R}^{n-1}$ and $j \in \{1, \ldots, n\}$. Define $f(y) = D_j(u - p_{m-1})(x, y)$ for $y > 0$; p_{m-1} is as in 2.13. Then

$$y^2|f'(y)| = y^2|D_y D_j(u - p_{m-1})(x,y)| \le CM$$

for all $y > 0$. Hence, $f(\infty) = L_j$ exists. Moreover, L_j is independent of x (as in the proof of 2.13). Furthermore, by the fundamental theorem of calculus, we obtain

2.15 $$y|D_j(u-p_{m-1})(x,y)-L_j| = y|D_j(u(x,y)-p_{m-1}(x,y)-L_jx_j)| \le CM$$

for all $(x, y) \in H$. Now take

$$q(x,y) = p_{m-1}(x,y) - \sum_{j=1}^{n} L_j x_j.$$

Because u is harmonic, we obtain by 2.14 that p_{m-1} is harmonic. Thus q is harmonic. Therefore, $q \in \mathcal{P}_{m-1}$. Also, using 2.15, we easily obtain that $||u-q||_{\mathcal{B}} \leq nCM$. Thus,

$$||u + \mathcal{P}_{m-1}|| \leq n\,C \sup_{H}\; y^m \sum_{|\alpha|=m} |D^\alpha u(x,y)|$$

for all harmonic function u on H for which

$$\sup_{H}\; y^m \sum_{|\alpha|=m} |D^\alpha u| < \infty.$$

For the rest of the harmonic functions u on H, we may take the same constant nC as above so that

$$||u + \mathcal{P}_{m-1}|| \leq nC \sup_{H}\; y^m \sum_{|\alpha|=m} |D^\alpha u(x,y)|$$

for all harmonic functions u on H.□

If we know ahead of time that $u \in \mathcal{B}(H)$, then we can say more.

2.16 Theorem: *Let u be a Bloch function on H and let m be a positive integer. Then*

$$||u||_{\mathcal{B}} \approx \sup_{H}\; y^m \sum_{|\alpha|=m} |D^\alpha u(x,y)| \approx \sup_{H}\; y^m \sum_{i=1}^{n} |D_y^{m-1} D_i u(x,y)|.$$

PROOF: Denote the three quantities of the theorem from left to right I, II, III. We'll prove: $I \approx II$, and $I \approx III$. We start by proving $I \approx II$. The fact that II is less or equal to a constant multiple of I follows from Cauchy's Estimates. For the other inequality, we apply Theorem 2.11. Fix $\epsilon > 0$. Because $||u + \mathcal{P}_{m-1}|| \leq ||u||_{\mathcal{B}} < \infty$, there is a harmonic polynomial p_ϵ of degree less or equal to $m-1$ such that

$$\begin{aligned} ||u + p_\epsilon||_{\mathcal{B}} &\leq \epsilon + ||u + \mathcal{P}_{m-1}|| \\ &\leq \epsilon + C \sup_{H}\; y^m \sum_{|\alpha|=m} |D^\alpha u(x,y)| < \infty \quad \text{(by Theorem 2.11).} \end{aligned}$$

Hence, since $||p_\epsilon||_{\mathcal{B}} \leq ||u + p_\epsilon||_{\mathcal{B}} + ||u||_{\mathcal{B}}$, we get p_ϵ is Bloch. Because $\mathcal{B}(H)$ does not contain any nonconstant polynomial, p_ϵ must be constant. Thus, $||u + p_\epsilon||_{\mathcal{B}} = ||u||_{\mathcal{B}}$. Therefore,

$$||u||_{\mathcal{B}} \leq \epsilon + C \sup_H y^m \sum_{|\alpha|=m} |D^\alpha u(x,y)|.$$

Letting ϵ go to 0 gives the desired inequality.

The fact that III is less or equal to a constant multiple of I is again an easy consequence of Cauchy's Estimates. The other inequality follows by an easy induction on m and by using the fundamental theorem of calculus.□

We now look at two BMO-type conditions, each of which characterizes the Bloch functions on H. Recall that $\Omega(a,r) = \{|z-a| < r\} \cap H$. Here is the first one:

2.17 Theorem: *Suppose u is a harmonic function on H that is volume integrable on every bounded subset of H. If $1 \leq p < \infty$, then*

2.18
$$||u||_{\mathcal{B}} \approx \sup \left\{ \frac{1}{|\Omega|} \int_\Omega |u - u_\Omega|^p dV \right\}^{\frac{1}{p}},$$

where the supremum is taken over all $\Omega = \Omega(a,r)$ with a in the closure of H.

PROOF: Let u and p be as in the statement of the theorem. For $a \in H$, put $B_a = B(a, a_n/2)$. Noting that $u(a) = u_{B_a}$, for u harmonic on H, we have

2.19
$$||u||_{\mathcal{B}} \approx \sup \left\{ \frac{1}{|B_a|} \int_{B_a} |u - u(a)|^p dV \right\}^{\frac{1}{p}},$$

where the supremum is taken over all $a \in H$. The proof of 2.19 is almost the same as the one given on page 9 of Chapter 1; we leave the easy changes to the reader. The equivalence 2.19 shows that $||u||_{\mathcal{B}}$ is less than or equal to a constant times the right side of 2.18.

For the other direction in 2.18, dilation and translation invariance will be crucial. After an appropriate dilation, as well as vertical and horizontal translations, we may assume that $\Omega = \Omega(a,r)$ touches ∂H, that $r = 1$, and that a lies on the y-axis. The triangle inequality gives

$$\left\{\frac{1}{|\Omega|}\int_\Omega |u - u_\Omega|^p dV\right\}^{\frac{1}{p}} \leq \left\{\frac{1}{|\Omega|}\int_\Omega |u - u(0,1)|^p dV\right\}^{\frac{1}{p}} + |u_\Omega - u(0,1)|.$$

Writing $u_\Omega - u(0,1)$ as $(1/|\Omega|)\int_\Omega (u - u(0,1))dV$, and applying Jensen's inequality, we see that the above is less than or equal to

$$2\left\{\frac{1}{|\Omega|}\int_\Omega |u - u(0,1)|^p dV\right\}^{\frac{1}{p}}.$$

Now with our assumptions on Ω, we have $\Omega \subset C$, where $C = \{|x| < 1\} \times (0,2)$. We also have $|\Omega| \geq |B_n(0,1)|/2$. Thus the last expression displayed is less than or equal to a constant (depending only on n and p) times

$$\left\{\int_C |u - u(0,1)|^p dV\right\}^{\frac{1}{p}}.$$

The growth estimate 2.4 and an easy integration now finishes the proof.□.

The second BMO-type condition involves the Hardy-space norms $||\cdot||_{h^p}$ on B (see page 10 of Chapter 1).

2.20 Theorem: *If u is a harmonic function on H and $1 \leq p < \infty$, then*

$$||u||_{\mathcal{B}} \approx \sup ||u(a + rx) - u(a)||_{h^p},$$

where the supremum is taken over all a, r such that $B(a,r) \subset H$.

For the proof of the theorem, we need the following lemma, which can be proved by using A.7 of [ABR] and then switching to polar coordinates.

2.21 Lemma: *If $p \in (0,\infty)$, then*

$$\int_S \left|\frac{\log(1+\zeta_n)}{\zeta_n}\right|^p d\sigma(\zeta) < \infty.$$

PROOF OF THEOREM 2.20: Fix $p \in [1,\infty)$ and $B(a,r) \subset H$. Note that since $B(a,r) \subset H$, the radius r of the ball $B(a,r)$ cannot exceed a_n, the n^{th} coordinate of a. It is clear that the function $u(a+rx) - u(a)$ is well defined and harmonic on B. Now fix $s \in [0,1)$ and $\zeta \in S$. We have

$$\begin{aligned} |u(a+rs\zeta) - u(a)| &= \left|\int_0^1 \nabla u(a+tsr\zeta)\cdot rs\zeta dt\right| \\ &\leq \int_0^1 |\nabla u(a+tsr\zeta)||rs\zeta|dt \\ &\leq \int_0^1 \frac{r||u||_{\mathcal{B}}}{a_n + rs\zeta_n t}dt \\ &\leq \int_0^1 \frac{||u||_{\mathcal{B}}}{1+s\zeta_n t}dt. \end{aligned}$$

In the last inequality we used the fact that $r \leq a_n$. If $\zeta_n \geq 0$, then obviously the last integral is less than or equal to $||u||_{\mathcal{B}}$. If $\zeta_n > 0$, the last integrand is dominated by $||u||_{\mathcal{B}}/(1+\zeta_n t)$, which implies that

$$|u(a+rs\zeta) - u(a)| \leq ||u||_{\mathcal{B}} \left|\frac{\log(1+\zeta_n)}{\zeta_n}\right|.$$

We now have an estimate independent of $s \in [0,1)$. The definition of the h^p-norm and Lemma 2.21 now show that there is a constant C, depending only on n and p, such that

$$||u(a+rx) - u(a)||_{h^p} \leq C||u||_{\mathcal{B}}.$$

For the other inequality, note that if $v \in h^p(B)$, then

2.22 $$|\nabla v(0)| \leq C||v||_{h^p},$$

where C is a positive constant depending only on n and p. (This follows from the Poisson integral representation formula of v.) Now fix $a = (x,y)$ in H and apply 2.22 to $v(z) = u(a+yz) - u(a)$. We obtain, since $|\nabla v(0)| = y|\nabla u(a)|$,

$$y|\nabla u(a)| \leq C||u(a+yz) - u(a)||_{h^p}.$$

Taking the sup on both sides of the above inequality yields the desired inequality.□

Chapter 3

Conjugate Harmonic Functions

THE UPPER HALF PLANE: If u is a real-valued harmonic function on H_2, then a harmonic conjugate of u is any real-valued function v on H_2 such that $u + iv$ is holomorphic on H_2. It is well known that harmonic conjugates exist and are unique up to additive constants. Hence, there is only one such v as above that satisfies $v(0, 1) = 0$. It is immediate from the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

that if u is a Bloch function, then so is v. Moreover, u and v have the same Bloch norm.

THE UPPER HALF SPACE: Recall our definition from Chapter 1: Given a harmonic function u on H, the functions $v_1, \ldots, v_{n-1}$ are said to be harmonic conjugates of u if

3.1 $$(v_1, \ldots, v_{n-1}, u) = \nabla f$$

for some harmonic function f on H. The functions $v_1, \ldots, v_{n-1}$ are automatically harmonic, since they are partial derivatives of a harmonic function. Also, 3.1 and the condition that f is harmonic are equivalent to the "generalized Cauchy-Riemann equations"

3.2 $$D_k v_j = D_j v_k \ ; \ D_y v_j = D_j u$$

3.3 $$\sum_{j=1}^{n-1} D_j v_j + D_y u = 0.$$

When $n = 2$, by the Cauchy-Riemann equations, 3.2 and 3.3 hold if and only if $u + iv$ is holomorphic on H. Thus, 3.1 is a natural generalization of the case $n = 2$. However, unlike the case $n = 2$, there is a greater degree of non-uniqueness of the v_j's, as we'll see shortly.

EXISTENCE OF HARMONIC CONJUGATES: Perhaps the best-known result concerning harmonic conjugates on H is the following: If

$$u(x,y) = P[f](x,y) = c_n \int_{\mathbf{R}^{n-1}} \frac{y\, f(t)}{(|x-t|^2 + y^2)^{\frac{n}{2}}} dt,$$

where $f \in L^2(\mathbf{R}^{n-1})$, then the functions v_j given by

3.4 $$v_j(x,y) = c_n \int_{\mathbf{R}^{n-1}} \frac{(x_j - t_j)\, f(t)}{(|x-t|^2 + y^2)^{\frac{n}{2}}} dt$$

are harmonic conjugates of u. Furthermore, each v_j is the Poisson integral of an L^2 function on $\mathbf{R}^{n-1}$ (see page 78 of [S]).

Unfortunately, not every harmonic function u on H is such a Poisson integral as above, so we need to do something else for such a function.

3.5 Theorem: *If u is harmonic on H, then harmonic conjugates of u exist.*

PROOF: We need to find a solution for 3.2 and 3.3. The solutions for the second part of 3.2 are of the form

3.6 $$v_j(x,y) = \int_1^y D_y u(x,t)dt + \varphi_j(x),$$

where the φ_j's are any $n-1$ smooth functions on $\mathbf{R}^{n-1}$ that satisfy

3.7
$$D_k\varphi_j = D_j\varphi_k$$

for all $j,k \in \{1,2,\ldots,n-1\}$. Note that when $n=2$, 3.7 holds trivially. To find the φ_j's, we proceed as follows. First, we find $\varphi_1 = \varphi$. To that end, we substitute 3.6 into 3.3 and we differentiate the new equation with respect to x_1. We obtain

$$\Delta\varphi(x) + D_1D_yu(x,1) = 0$$

(having used the harmonicity of u and the equations $D_1D_j\varphi_j = D_jD_1\varphi_j = D_j^2\varphi$ obtained from 3.7). This is just the Poisson equation in $\mathbf{R}^{n-1}$. Because $D_1D_yu(x,1)$ is smooth on $\mathbf{R}^{n-1}$, this last equation has a smooth solution on $\mathbf{R}^{n-1}$; see pages 195 and 201 of [R2]. Thus, φ_1 exists. To obtain the rest of the φ_j's, we proceed similarly. By 3.7, we have that

$$\varphi_j(x) = \int_0^{x_1} D_j\varphi(t,x_2,\ldots,x_{n-1})dt + \psi_j(x_2,\ldots,x_{n-1})$$

for $j = 2,\ldots,n-1$, where the ψ_j's are smooth functions on $\mathbf{R}^{n-2}$ that satisfy

$$D_k\psi_j = D_j\psi_k.$$

To find the ψ_j's, we fix $\psi_2 = \psi$ and follow similar steps that led to the existence of φ_1. We repeat the same process $n-2$ times till we obtain all the v_j's. Just by construction, the v_j's obtained will satisfy 3.2 and 3.3.□

NON-UNIQUENESS OF HARMONIC CONJUGATES: Before we proceed let us look at an example. Let $u(x_1,x_2,y) = 2y$. We can easily check that $v_1(x_1,x_2,y) = 0$ and $v_2(x_1,x_2,y) = -2x_2$ are harmonic conjugates of u (here $f(x_1,x_2,y) = y^2 - x_2^2$). Slightly more complicated harmonic conjugates of u are $v_1(x_1,x_2,y) = 2x_1x_2$ and

$v_2(x_1, x_2, y) = x_1^2 - x_2^2 - 2x_2$ (here $f(x_1, x_2, y) = x_1^2 x_2 - x_2^3/3 - x_2^2 + y^2$). This example shows that when $n > 2$, harmonic conjugates for the same u may differ by more than just a constant. In fact, the proof of theorem 3.5 shows that adding any harmonic function g of $x_1, \ldots, x_{n-1}$ to v_1 will generate another set of $n-1$ harmonic conjugates, say $v_1 + g$, $w_1, \ldots$, w_{n-2}. We can again add any harmonic function h of $x_2, \ldots, x_{n-1}$ to w_1. It will generate new harmonic conjugates for u of the form $v_1 + g$, $w_1 + h$, $s_1, \ldots$, s_{n-3}, and so on. In the above example, the harmonic function added to v_1 is $2x_1x_2$. Note that in the case $n = 2$, if we add a harmonic function of x_1 (which must be of the form $ax_1 + b$) to v_1, then $v_1 + ax_1 + b$ is a harmonic conjugate of u if and only if $a = 0$, as easily checked. This is compatible with the well-known fact that when $n = 2$, the harmonic conjugate is unique up to an additive constant.

HARMONIC CONJUGATES OF BLOCH FUNCTIONS: Given $u \in \mathcal{B}(H)$, need the harmonic conjugates $v_1, \ldots$, v_{n-1} belong to $\mathcal{B}(H)$? The answer is yes when $n = 2$; see the beginning of this chapter. In higher dimensions, the answer is no as we can easily see by taking $u = 0$. Here, $u = 0$ is Bloch but the harmonic conjugates $v_1(x_1, x_2, y) = x_1$ and $v_2(x_1, x_2, y) = -x_2$ are not Bloch.

Nevertheless, there are certain properties that any conjugate of a Bloch function must have. For example, we have control over the normal derivatives of the conjugates. Indeed, since

$$D_y v_j = D_j u,$$

we have

$$y|D_y v_j(x, y)| \leq ||u||_{\mathcal{B}}$$

for all $(x, y) \in H$ and all j. We also have a uniform bound on some of the second partial derivatives of the harmonic conjugates. More precisely, there exists a positive constant C such that

3.8 $$y^2|D_yD_kv_j(x,y)| \leq C||u||_{\mathcal{B}}$$

for all $(x,y) \in H$ and all $j \in \{1,\ldots,n-1\}$ and $k \in \{1,\ldots,n\}$. Indeed,

$$\begin{aligned} y^2|D_yD_kv_j(x,y)| &= y^2|D_kD_yv_j(x,y)| \\ &\leq y^2|D_kD_ju(x,y)| \qquad \text{(by 3.2)} \\ &\leq C||u||_{\mathcal{B}} \qquad \text{(Cauchy's Estimates)} \end{aligned}$$

for all $(x,y) \in H$ and all $j \in \{1,\ldots, n-1\}$ and $k \in \{1,\ldots, n\}$.

We now begin to discuss the primary question of this chapter: Given $u \in \mathcal{B}(H)$, can we choose harmonic conjugates of u that are Bloch? Referring to the discussion above, suppose we know that for every $x \in \mathbf{R}^{n-1}$, $D_kv_j(x,y) \to 0$ as $y \to \infty$. We can then fix $x \in \mathbf{R}^{n-1}$ and define on $(0,\infty)$ the function $f(y) = D_kv_j(x,y)$. Then 3.8 implies that $y^2|f'(y)| \leq C||u||_{\mathcal{B}}$. Therefore, since $f(\infty) = 0$, we get from the fundamental theorem of calculus that

3.9 $$y|f(y)| = y|D_kv_j(x,y)| \leq C||u||_{\mathcal{B}}$$

for all $(x,y) \in H$; note that the constant C in 3.9 is the same as in 3.8 and hence is independent of u.

Summarizing: To get Bloch conjugates for a Bloch function u on H, it suffices to find harmonic conjugates $v_1,\ldots,v_{n-1}$ of u such that

3.10 $$\lim_{y\to\infty} D_kv_j(x,y) = 0,$$

for all $x \in \mathbf{R}^{n-1}$ and for all $k \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,n-1\}$. Moreover, if 3.10 is satisfied, then (by 3.9) there is a positive constant C independent of u such that $||v_j||_{\mathcal{B}} \leq C||u||_{\mathcal{B}}$.

We now show that this can be done, essentially by using 3.4.

3.11 Lemma: *For every $h > 0$, the functions*

$$v_j^h(x,y) = c_n \int_{\mathbf{R}^{n-1}} \left(\frac{x_j - t_j}{(|x-t|^2 + y^2)^{\frac{n}{2}}} + \frac{t_j}{(|t|^2+1)^{\frac{n}{2}}}\right) u(t,h)\,dt$$

are Bloch conjugates for $\tau_h u$ and there is a positive constant M, depending only on n, such that $||v_j^h||_{\mathcal{B}} \leq M||u||_{\mathcal{B}}$ for all $h > 0$ and all j.

REMARKS: The term $t_j/(|t|^2+1)^{n/2}$ has been added here to ensure that the integrands belong to $L^1(\mathbf{R}^{n-1})$, and to ensure that $v_j^h(0,1) = 0$.

PROOF: Fix $h > 0$. The existence of the above integrals is checked by getting a common denominator for the integrand. A straightforward computation shows that $v_1^h, \ldots, v_{n-1}^h$ together with $\tau_h u$ satisfy 3.2 and 3.3. Therefore, $v_1^h, \ldots, v_{n-1}^h$ are harmonic conjugates for τ_h. To show that $v_1^h, \ldots, v_{n-1}^h$ are Bloch, it suffices to show that 3.10 is satisfied. Fix $j \in \{1, \ldots, n-1\}$. After differentiation under the integral sign and making the change of variable $x - t = ys$, we get

$$\begin{aligned} |D_j v_j^h(x,y)| &\leq \frac{C}{y}\int_{\mathbf{R}^{n-1}} \frac{|u(x-ys,h)|}{(|s|^2+1)^{\frac{n}{2}}}dt + \frac{C}{y}\int_{\mathbf{R}^{n-1}} \frac{s_j^2|u(x-ys,h)|}{(|s|^2+1)^{\frac{n+2}{2}}}dt \\ &\leq \frac{C}{y}\int_{\mathbf{R}^{n-1}} \frac{|u(x-ys,h)|}{(|s|^2+1)^{\frac{n}{2}}}dt \end{aligned}$$

for all $(x,y) \in H$. Using the growth estimate 2.4 in the above inequality and the estimate

$$\log(1+|x-ys|) \leq \log(1+|x|) + \log(1+y) + \log(1+|s|)$$

for all x and s in $\mathbf{R}^{n-1}$ and all $y > 0$, we easily see that there are two positive constants α and β such that

$$|D_j v_j^h(x,y)| \leq \alpha\frac{1+\log(1+|x|)}{y} + \beta\frac{\log(1+y)}{y}$$

for all $(x,y) \in H$. Therefore, $\lim_{y\to\infty} D_j v_j^h(x,y) = 0$ for all $x \in \mathbf{R}^{n-1}$. Similarly, when $k \neq j$, $\lim_{y\to\infty} D_k v_j^h(x,y) = 0$ for all $x \in \mathbf{R}^{n-1}$. Thus, 3.10 holds for $v_1^h, \ldots, v_{n-1}^h$, which implies that $v_j^h \in \mathcal{B}(H)$. Furthermore, by 3.9 the Bloch norms of $v_1^h, \ldots, v_{n-1}^h$ are uniformly bounded by a constant multiple of $||u||_{\mathcal{B}}$ (recall that $||\tau_h u||_{\mathcal{B}} \leq ||u||_{\mathcal{B}}$).□

We are now ready to prove the main result of this chapter.

3.12 Theorem: *Let $u \in \mathcal{B}(H)$. Then there exist unique harmonic conjugates $v_1, \ldots, v_{n-1}$ of u on H such that $v_j \in \mathcal{B}(H)$ and $v_j(0,1) = 0$ for each j. Moreover, there exists a constant M, depending only on n, such that $||v_j||_{\mathcal{B}} \leq M||u||_{\mathcal{B}}$ for each j.*

PROOF: Referring to $v_1^h, \ldots, v_{n-1}^h$ of Lemma 3.11, we see that the vector-valued family of harmonic functions $(v_1^h, \ldots, v_{n-1}^h)$ is uniformly bounded on compact subsets of H. (This follows because $v_j^h(0,1) = 0$ for all $h > 0$ and all j, and since $||v_j^h||_{\mathcal{B}} \leq M||u||_{\mathcal{B}}$ for all $h > 0$ and all j.) Therefore, there is a sequence (h_k) that converges to 0 such that $(v_1^{h_k}, \ldots, v_{n-1}^{h_k})$ converges uniformly on compact subsets of H to some vector of harmonic functions $(v_1, \ldots, v_{n-1})$ (this is an easy generalization of 2.6 in [ABR]). Furthermore, the sequence of partial derivatives converges uniformly on compact subsets to the corresponding partial derivatives of $(v_1, \ldots, v_{n-1})$ (see 1.19 of [ABR]). It follows easily by letting h_k go to 0 in the generalized Cauchy-Riemann equations that $v_1, \ldots, v_{n-1}$ are harmonic conjugates of u. It is also an easy consequence of the uniform boundedness of the Bloch norms $||v_j^{h_k}||_{\mathcal{B}}$, that each v_j belongs to $\mathcal{B}(H)$ and that $||v_j||_{\mathcal{B}} \leq M||u||_{\mathcal{B}}$ for each j. Indeed,

$$\begin{aligned} y|\nabla v_j(x,y)| &= \lim_{h_k\to 0} y|\nabla v_j^{h_k}(x,y)| \\ &\leq \lim_{h_k\to 0} ||v_j^{h_k}||_{\mathcal{B}} \leq M||u||_{\mathcal{B}}, \end{aligned}$$

for all $(x, y) \in H$ and all j. The fact that $v_j(0,1) = 0$ for all j is clear since $v_j^h(0,1) = 0$ for all $h > 0$ and all j. Finally, the uniqueness of $v_1, \ldots, v_{n-1}$ follows from the fact that $D_y v_j = D_j u$ and the fact that a Bloch function on H that doesn't depend on y is necessarily a constant (see Lemma 2.10).□

The last theorem is crucial for obtaining some results regarding the boundary behavior of the harmonic Bloch functions, as we'll see in the next chapter.

Chapter 4

Boundary Behavior of Bloch Functions

Recall some of our discussion in Chapter 1: Although holomorphic Bloch functions on D need not have finite radial limits at any point on the boundary, they do have "average radial limits" over an interval on the boundary; this is what Ullrich showed in [U]. It easily follows that harmonic Bloch functions on D have this property. Moreover, it terms of these averages, Ullrich obtained a necessary and sufficient condition for the existence of a radial limit at a given boundary point. The natural analogues of these results follow for harmonic Bloch functions in the upper half-plane after a conformal change of variables. The main purpose of this chapter is to explore these ideas in higher dimensions. But first, we prove that harmonic Bloch functions on H have "boundary values" in the sense of distributions. More precisely:

4.1 Theorem: *If u is a Bloch function in the upper half-space, then*

$$\lim_{y\to 0}\int_{\mathbf{R}^{n-1}} u(x,y)\varphi(x)dx$$

exists for every smooth function φ with compact support. Moreover, this limit defines a distribution on the space of smooth test functions with compact support.

PROOF: Let φ be as in the statement of the theorem. We define on $(0,\infty)$ the function

$$f(y) = \int_{\mathbf{R}^{n-1}} u(x,y)\varphi(x)dx.$$

Differentiating under the integral sign yields

$$\begin{aligned} f''(y) &= \int_{\mathbf{R}^{n-1}} \frac{\partial^2}{\partial y^2} u(x,y)\phi(x)dx \\ &= -\int_{\mathbf{R}^{n-1}} \sum_{k=1}^{n-1} D_k^2 u(x,y)\phi(x)dx \qquad \text{(since } u \text{ is harmonic)} \\ &= -\int_{\mathbf{R}^{n-1}} u(x,y)\Delta\phi(x)dx \qquad \text{(integration by parts).} \end{aligned}$$

Therefore, using the growth estimates 2.4, we obtain

$$|f''(y)| \leq C \sup|\Delta\varphi|(1 + |\log y|)$$

for all $y \in (0,\infty)$. Hence,

$$|f'(y) - f'(1)| \leq C \sup|\Delta\varphi| \int_y^1 (1 + \log\frac{1}{s})ds \leq C \sup|\Delta\varphi|$$

for all $y \in (0,1)$. Because $u \in \mathcal{B}(H)$, it easily follows that $|f'(1)| \leq C \sup|\varphi|$. Therefore

$$|f'(y)| \leq C(\sup|\Delta\varphi| + \sup|\varphi|)$$

for all $y \in (0,1)$. Thus $\lim_{y\to 0} f(y)$ exists, and

$$|f(y) - f(1)| \leq C\,(\sup|\Delta\varphi| + \sup|\varphi|)\,|y-1|$$

for all $y \in (0,1)$. The above inequality together with the easy fact that $|f(1)| \leq C \sup|\varphi|$ will finish the proof.□

Ullrich showed that when $n = 2$, the limit in 4.1 still exists if φ is the characteristic function of an open bounded interval symmetric around the origin; see the discussion in Chapter 1. We now generalize this result by letting φ be the characteristic function of an open bounded set with smooth boundary in any dimension. For this purpose, we define

$$I_\Omega^h u = \int_\Omega u(x,h)dx$$

for any bounded measurable subset Ω of $\mathbf{R}^{n-1}$. We obtain:

4.2 Theorem: *Suppose $u \in \mathcal{B}(H)$, Ω is a bounded open set with C^1-boundary, and $0 \in \Omega$. Then*

$$I_\Omega u = \lim_{h\to 0} I_\Omega^h u$$

exists and defines a bounded linear functional on $\mathcal{B}(H)$. Moreover, there is a constant C, depending only on n, such that

$$||I_\Omega|| \le C(1 + |\partial\Omega|d + |\Omega|)(1 + |\log d|),$$

where d equals the diameter of Ω.

PROOF: Fix $u \in \mathcal{B}(H)$ and Ω as in the statement of the theorem. We will be interested in the cylinder $\mathcal{C} = \Omega \times (h,d)$ for $0 < h < d$. Because $\mathcal{C}$ has piecewise smooth boundary, we may use the divergence theorem:

$$\int_{\mathcal{C}} \operatorname{div} \mathbf{w}\, dV = \int_{\partial\mathcal{C}} \mathbf{w}\cdot\mathbf{n}\; ds.$$

Here we take $\mathbf{w} = (v_1,\ldots,v_{n-1},u)$, where $v_1,\ldots,v_{n-1}$ are any harmonic conjugates of u, and $\operatorname{div}\mathbf{w}$, the divergence of $\mathbf{w}$, is defined to be $D_1v_1 + \ldots + D_yu$. The vector $\mathbf{n} = (\eta_1,\ldots,\eta_n)$ is the outward unit normal to $\partial\mathcal{C}$, ds denotes surface area measure on $\partial\mathcal{C}$, and $\cdot$ denotes the usual Euclidean inner product. Now from the definition of harmonic conjugates, we have $\operatorname{div}\mathbf{w} = 0$ (see 3.3). Thus, because $\mathbf{n} = (0,\ldots,0,1)$ on the top of $\mathcal{C}$ and $\mathbf{n} = (0,\ldots,0,-1)$ on the bottom of $\mathcal{C}$, we get

4.3 $$I_\Omega^h u = \int_\Omega u(x,h)dx = \int_\Omega u(x,d)dx + \int_h^d \int_{\partial\Omega} \sum_{j=1}^{n-1} \eta_j(x)v_j(x,y)ds(x)dy.$$

We need to estimate the integrand in the above double integral. First, since the vector $\mathbf{n}$ has length 1, we have

$$\sum_{j=1}^{n-1} |\eta_j v_j| \le \sum_{j=1}^{n-1} |v_j|$$

on $\partial\Omega \times (h, d)$. Second,

$$\begin{aligned}
|v_j(x,y)| &\le |v_j(x,1)| + |\int_y^1 D_n v_j(x,t)dt| \\
&\le |v_j(x,1)| + |\int_y^1 \frac{||u||_{\mathcal{B}}}{t} dt| \qquad \text{(recall that } D_n v_j = D_j u) \\
&= |v_j(x,1)| + ||u||_{\mathcal{B}} |\log y|
\end{aligned}$$

for all $(x,y) \in H$ and all j. Therefore,

$$|\sum_{j=1}^{n-1} \eta_j(x) v_j(x,y)| \le (n-1)||u||_{\mathcal{B}} |\log y| + \sum_{j=1}^{n-1} |v_j(x,1)|$$

for all $(x,y) \in \partial\Omega \times (0,d)$. Because $|\log y|$ is integrable on $(0,d)$, we see that

4.4 $$\lim_{h\to 0} I_\Omega^h = I_\Omega u = \int_\Omega u(x,d)dx + \int_0^d \int_{\partial\Omega} \sum_{j=1}^{n-1} \eta_j v_j(x,y) ds(x) dy.$$

Note that up until now, $v_1, \dots, v_{n-1}$ could have been any harmonic conjugates of u. However, to show that $I_\Omega u$ is bounded, we resort to our result in 3.12: We may choose harmonic conjugates $v_1, \dots, v_{n-1}$ of u that are Bloch. Indeed, if we do that, then the growth estimate 2.4 shows that

4.5 $$|v_j(x,y)| \le M\, ||u||_{\mathcal{B}} [1 + |\log y| + 2\log(1+|x|)]$$

for all $(x,y) \in H$ and all j. Here we have used the fact that $v_j(0,1) = 0$ for all j and the estimate $||v_j||_{\mathcal{B}} \le M||u||_{\mathcal{B}}$ from 3.12. Using the estimate 4.5 and the formula 4.4, we obtain

$$\begin{aligned}
|I_\Omega u| &\le \int_\Omega [|u(0,1)| + ||u||_{\mathcal{B}}(1 + |\log d| + 2\log(1+|x|))] dx \\
&+ M\, ||u||_{\mathcal{B}} \sum_{j=1}^{n-1} \int_0^d \int_{\partial\Omega} [1 + |\log y| + 2\log(1+|x|)] ds(x) dy
\end{aligned}$$

An easy estimate shows that

$$\int_0^d |\log y| dy \leq d(1 + |\log d|).$$

This inequality, together with the obvious estimates, give the desired conclusion (we recall that the constant M depends only on n).□

With Ω as in 4.2, let us define the constant

$$C_\Omega = (|\partial\Omega| d + |\Omega|)(1 + |\log d|).$$

Thus Theorem 4.2 gives

$$||I_\Omega|| \leq C\, C_\Omega.$$

The last estimate allows us to estimate $||I_\Omega^h u||$, for any $h > 0$: Because $I_\Omega^h u = I_\Omega(\tau_h u)$, we have

4.6
$$\begin{aligned} |I_\Omega^h u| &\leq C\, C_\Omega(|\tau_h u(0,1)| + ||\tau_h u||_{\mathcal{B}}) \\ &\leq C\, C_\Omega(|u(0,1)| + ||u||_{\mathcal{B}} \log(1+h) + ||u||_{\mathcal{B}}) \\ &\leq C\, C_\Omega(1 + \log(1+h))(|u(0,1)| + ||u||_{\mathcal{B}}) \end{aligned}$$

for all $h > 0$.

The proof of Theorem 4.2 yields the following three corollaries:

4.7 Corollary: *With Ω as in Theorem 4.2, we have*

$$||I_\Omega^h - I_\Omega|| \leq C\, |\partial\Omega|(1 + \log(1+d))\,(h + h|\log h|),$$

where $0 < h < d$ and C is a constant depending only on n.

PROOF: Fix $u \in \mathcal{B}(H)$. We subtract the formula 4.4 from the formula 4.3 to get

$$|I_\Omega^h u - I_\Omega u| = |\int_0^h \int_{\partial\Omega} \sum_{j=1}^{n-1} \eta_j(x) v_j(x,y) ds(x) dy|.$$

Hence, by the estimate 4.5, we obtain (since $|\eta_j| \leq 1$)

$$\begin{aligned} |I_\Omega^h u - I_\Omega u| &\leq M\,||u||_{\mathcal{B}} \sum_{j=1}^{n-1} \int_0^h \int_{\partial\Omega} [1 + |\log y| + 2\log(1+|x|)]ds(x)dy \\ &\leq C\,|\partial\Omega|(1+\log(1+d))\,(h + h|\log h|)||u||_{\mathcal{B}}. \end{aligned}$$

The desired conclusion follows.□

Note that Corollary 4.7 implies in particular that

$$\lim_{h\to 0} ||I_\Omega^h - I_\Omega|| = 0.$$

The next result shows that the linear functional I_Ω is continuous in a stronger sense than that expressed in 4.2.

4.8 Corollary: *Let Ω be as in Theorem 4.2. If (u_k) is a sequence of Bloch functions on H such that the norms $||u_k||_{\mathcal{B}}$ are uniformly bounded and such that (u_k) converges to u uniformly on compact subsets of H, then*

$$I_\Omega u = \lim_{k\to\infty} I_\Omega u_k.$$

REMARK: The hypotheses here do not imply that $u_k \to u$ in the norm of $\mathcal{B}(H)$. For example, in $\mathcal{B}_a(D)$, the sequence $f_k(z) = z^k$ is uniformly bounded on D, hence uniformly bounded in the Bloch norm. This sequence converges to 0 uniformly on compact subsets of D, yet

$$||f_k||_{\mathcal{B}} \geq (1 - (1 - \frac{1}{k}))\,k\,(1 - \frac{1}{k})^{k-1},$$

which is on the order of $1/e$ for large k. Thus, $||f_k||_{\mathcal{B}} \not\to 0$. (After a conformal mapping we obtain a similar example for H.)

PROOF OF COROLLARY 4.8: We first observe that the function u is Bloch and that $||u||_{\mathcal{B}} \leq \sup ||u_k||_{\mathcal{B}}$ (the argument was done in the proof of 3.12). Hence, $I_\Omega u$

makes sense. Now let $h < d$. We have that

$$|I_\Omega u - I_\Omega u_k| \le |I_\Omega u - I_\Omega^h u| + |I_\Omega^h u - I_\Omega^h u_k| + |I_\Omega^h u_k - I_\Omega u_k|$$

for all k. The first term on the right of this inequality is small if h is small by 4.2. The second term converges to 0 as $k \to \infty$ by the uniform convergence of (u_k) to u on compact subsets of H. For the third term, note that our hypotheses imply that there is a positive constant A such that $|u_k(0,1)| + ||u_k||_B \le A$ for all k. Thus, using Corollary 4.7, we obtain

$$|I_\Omega^h u_k - I_\Omega u_k| \le CA|\partial\Omega|(1 + \log(1+d))(h + h|\log h|)$$

for all k, which is small independently of k if h is small. The desired conclusion follows.□

The next corollary states, roughly speaking, that for all $y > 0$, the average of u over $y\Omega$ cannot be too far away from $u(0,y)$. Again the dilation invariance of the Bloch norm will be used.

4.9 Corollary: *Let Ω be as in Theorem 4.2. Then*

$$\left|\frac{1}{|y\Omega|} I_{(y\Omega)} u - u(0,y)\right| \le \frac{C\,C_\Omega}{|\Omega|} ||u||_B$$

for all $y > 0$.

PROOF: Fix $y > 0$ and Ω as in statement of the corollary. Easy manipulations show that

$$\frac{1}{|y\Omega|} I_{(y\Omega)} u = I_\Omega u_y.$$

Hence, because the Bloch function $u_y - u(0,y)$ vanishes at the point $(0,1)$, Theorem 4.2 implies that

$$\begin{aligned}
\left|\frac{1}{|y\Omega|}I_{(y\Omega)}u - u(0,y)\right| &= \left|\frac{1}{|\Omega|}I_\Omega(u_y - u(0,y))\right| \\
&\leq \frac{C\,C_\Omega}{|\Omega|}||u_y||_{\mathcal{B}} \\
&= \frac{C\,C_\Omega}{|\Omega|}||u||_{\mathcal{B}}.\square
\end{aligned}$$

We need to discuss the particular case where $\Omega = B_r$, the open ball in $\mathbf{R}^{n-1}$ centered at the origin of radius r (B_r has obviously smooth boundary). Recall from Chapter 1 the averages

$$A_r^h u = \frac{1}{|B_r|}\int_{B_r} u(x,h)dx.$$

We deduce from Theorem 4.2 that

$$\lim_{h\to 0} A_r^h u = A_r u$$

exists for all fixed $r > 0$. Also, the linear functional $A_r u$ is bounded on $\mathcal{B}(H)$ for all $r > 0$, and since there are positive constants α_n and β_n such that $|B_r| = \alpha_n r^{n-1}$ and $|\partial B_r| = \beta_n r^{n-2}$ for all $r > 0$, we easily obtain from 4.6 (here $d = 2r$)

4.10 $$||A_r^h u|| \leq C(1+|\log r|)(1+\log(1+h))(||u||_{\mathcal{B}}+|u(0,1)|)$$

for all $r > 0$ and all $h > 0$, where C is a positive constant depending only on n. Thus, letting $h \to 0$ in 4.10, we obtain

$$||A_r u|| \leq C(1 + |\log r|)(||u||_{\mathcal{B}} + |u(0,1)|)$$

for all $r > 0$.

We now prove a lemma involving the averages $A_r u$ that we will need for the main result of this chapter. Recall that c_n is the normalizing constant for the Poisson kernel (see 2.1).

4.11 Lemma: *If $u \in \mathcal{B}(H)$, then*

$$u(0,y) = n|B_1|c_n \int_0^\infty \frac{y\, r^n}{(r^2+y^2)^{\frac{n+2}{2}}} A_r u\, dr$$

for all $y > 0$.

PROOF: Fix $u \in \mathcal{B}(H)$ and $y > 0$. Also fix $h \in (0,1)$ for the moment. Estimate 4.10 gives

4.12
$$||A_r^h u|| \le C\,(1+|\log r|)(||u||_{\mathcal{B}}+|u(0,1)|)$$

for all $r > 0$ and all $h \in (0,1)$, where C is a constant depending only on n.

From the Poisson integral representation in Theorem 2.5, we have

$$u(0,y+h) = c_n \int_{\mathbf{R}^{n-1}} \frac{y\; u(t,h)}{(|t|^2+y^2)^{\frac{n}{2}}} dt.$$

We go to polar coordinates to obtain

$$u(0,y+h) = (n-1)|B_1|c_n\, y \int_0^\infty \frac{r^{n-2}}{(r^2+y^2)^{\frac{n}{2}}} \int_S u(r\zeta,h)d\sigma(\zeta)\, dr$$

for all $y > 0$. Now

$$A_r^h u = \frac{n-1}{r^{n-1}} \int_0^r t^{n-2} \int_S u(t\zeta,h)d\sigma(\zeta)dt$$

(again by going to polar coordinates). Therefore,

$$u(0,y+h) = c_n|B_1|\, y \int_0^\infty \frac{d(r^{n-1}A_r^h u)}{dr} \frac{dr}{(r^2+y^2)^{\frac{n}{2}}}.$$

Thus, integrating by parts in the formula of $u(x,y+h)$, we obtain

$$u(0,y+h) = n|B_1|c_n\, y \int_0^\infty \frac{r^n}{(r^2+y^2)^{\frac{n+2}{2}}} A_r^h u\, dr.$$

Note that the boundary terms in the integration by parts vanish by the estimate 4.12. This estimate also shows that the last integrand is bounded by a constant times

$$\frac{(1+|\log r|)r^n}{(r^2+y^2)^{\frac{n+2}{2}}}$$

independently of $h \in (0,1)$. Because this last expression is integrable on $(0,\infty)$ for any $y > 0$, we can now let $h \to 0$ to obtain the desired result (by the Lebesgue Dominated Convergence Theorem).□

We now prove a result concerning the x-radialization of a given function on H: Given a continuous function u on H, its *x-radialization* $R[u]$ is the function on H defined by

$$R[u](x,y) = \int_S u(|x|\zeta, y)\,d\sigma(\zeta).$$

4.13 Proposition: *If u is harmonic on H, then $R[u]$ is harmonic on H.*

PROOF: We use the converse of the mean value property (see 1.20 of [ABR]). The function $R[u]$ is clearly continuous on H. Also, we can view $R[u]$ as

$$R[u](x,y) = \int_{G_n} u(T(x,y))\,dT,$$

where G_n denotes the group of orthogonal transformations on $\mathbf{R}^n$ that leave the y-axis invariant, and dT denotes the Haar measure on G_n. Now, let $z \in H$ and let $r > 0$. Then

$$\begin{aligned}
\int_S R[u](z + r\zeta)\,d\sigma(\zeta) &= \int_S\int_{G_n} u(T(z+r\zeta))\,dT\,d\sigma(\zeta) \\
&= \int_{G_n}\int_S u(Tz + rT(\zeta))\,d\sigma(\zeta)\,dT \qquad \text{(Fubini and linearity of } T) \\
&= \int_{G_n}\int_S u(Tz + r\eta)\,d\sigma(\eta)\,dT \qquad \text{(change of variables } \eta = T(\zeta)) \\
&= \int_{G_n} u(Tz)\,dt \qquad (u \text{ is harmonic}) \\
&= R[u](z).
\end{aligned}$$

Thus $R[u]$ is harmonic on H.□

We are now ready to prove our main result of this chapter: An "abelian-tauberian" theorem characterizing the existence of a radial limit at a given boundary point in terms of the functionals $A_r u$. We'll prove it at the origin, but at any other point $(a, 0)$ of ∂H, τ_{-a} will take us back to the origin.

4.14 Theorem: *If $u \in \mathcal{B}(H)$ and $L \in [-\infty, +\infty]$, then*

$$\lim_{y \to 0} u(0, y) = L \quad \textit{if and only if} \quad \lim_{r \to 0} A_r u = L.$$

PROOF: We do the proof for the case $n > 2$, and we'll indicate the necessary changes for the case $n = 2$. We first assume $L = 0$. Let us call the statement that $\lim_{r \to 0} A_r u = 0$ implies $\lim_{y \to 0} u(0, y) = 0$ the "abelian" implication; the other half of the theorm is the "tauberian" half.

As one might think, the abelian direction is rather straightforward. Indeed, setting

$$K_y(r) = n|B_1|c_n \frac{y\, r^n}{(r^2 + y^2)^{\frac{n+2}{2}}}$$

in 4.11, we see that $u(0, y)$ is the integral of $A_r u$ against the positive kernel $K_y(r)$, most of whose mass is near 0 for y small. (Loosely speaking, $K_y(r)$ is an approximate identity converging to the delta function at 0.) To deduce that $u(0, y) \to 0$ if $A_r u \to 0$ is then a standard argument and we leave it to the reader.

The tauberian half of the theorem is a normal families argument: Suppose

$$\lim_{y \to 0} u(0, y) = 0 \quad \text{but} \quad \lim_{r \to 0} A_r u \neq 0.$$

Then there exists $\epsilon > 0$ and a sequence of positive numbers $r_k \to 0$ such that $|A_{r_k} u| > \epsilon$ for all k. Now we consider the sequence of dilates $u_k(z) = u(r_k z)$. First, observe that $A_{r_k} u = A_1 u_k$ for all k (this can easily be done by an adequate change of variables in $A^h_{r_k} u$). Hence,

$$|A_1 u_k| > \epsilon \tag{4.15}$$

for all k. Second, because of the dilation invariance of the Bloch norm, (u_k) is uniformly bounded on compact subsets of H. Indeed, this follows from the following inequalities

$$\begin{aligned} |u_k(x,y)| &\le |u_k(0,1)| + ||u_k||_{\mathcal{B}}(1 + |\log y| + 2\log(1+|x|)) && \text{(estimate 2.4)} \\ &= |u(0,r_k)| + ||u||_{\mathcal{B}}(1 + |\log y| + 2\log(1+|x|)) && (||u_k||_{\mathcal{B}} = ||u||_{\mathcal{B}}) \\ &\le C + ||u||_{\mathcal{B}}(1 + |\log y| + 2\log(1+|x|)) && (\lim_{t\to 0} u(0,t) = 0) \end{aligned}$$

for all $(x,y) \in H$ and all k. Therefore, (u_k) has a subsequence, which we still call (u_k), that converges uniformly on compact subsets of H to a harmonic function v on H (see 2.6 of [ABR]).

Examining the limit function, we have (since $\lim_{t\to 0} u(0,t) = 0$)

$$\begin{aligned} v(0,y) &= \lim_{k\to\infty} u_{r_k}(0,y) \\ &= \lim_{k\to\infty} u(0, r_k y) = 0 \end{aligned}$$

for all $y > 0$. Now because the Bloch norms $||u_k||_{\mathcal{B}}$ are uniformly bounded (by $||u||_{\mathcal{B}}$), it follows from Corollary 4.8, that

$$A_1 v = \lim_{k\to\infty} A_1 u_k.$$

Hence, (by 4.15)

4.16 $$|A_1 v| > \epsilon.$$

To get a contradiction, we show that $A_1 v = 0$. To do that we first observe that since $v(0,y) = 0$ for all $y > 0$, we have $R[v](0,y) = 0$ for all $y > 0$. Also, $R[v]$ is radial in x and harmonic on H (by 4.13). This is enough to give, by Proposition 2.11 of [RU 2], $R[v] \equiv 0$ on H. Now we go to polar coordinates to obtain

$$\begin{aligned} A_1^h v &= (n-1)\int_0^1 r^{n-2}\int_S v(r\zeta,h)\,d\sigma(\zeta)\,dr \\ &= (n-1)\int_0^1 r^{n-2}R[v](r\eta,h)\,dr \qquad\qquad (\eta \text{ is any element on } S) \\ &= 0 \end{aligned}$$

for all $0 < h < 1$. Thus $A_1 v = 0$, contradicting 4.16.

The case where L is a nonzero real number follows from the case $L = 0$ by considering the function $u - L$.

The case $L = \pm\infty$ follows easily from Corollary 4.9. Indeed, if r is fixed, then letting $\Omega = B$ in 4.9, gives (since $rB = B_r$)

$$\begin{aligned} |A_r u - u(0,r)| &= |\frac{1}{|rB|}I_{(rB)}u - u(0,r)| \\ &\leq C||u||_B, \end{aligned}$$

where C is a constant depending only on n. Thus, $\lim_{r\to 0} A_r u = \infty$ if and only if $\lim_{y\to 0} u(0,y) = \infty$.

The case $n = 2$ is easier, with the following change. Instead of using Proposition 2.11 of [RU 2], we use the following elementary fact: *If u is a harmonic function on the upper half-plane and $u \equiv 0$ on the y-axis, then $u(-x,y) = -u(x,y)$.*□

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