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RADIATIVE DECAYS IN THE BETHE-SALPETER EQUATION WITH A STATIC KERNEL AND HARMONIC OSCILLATOR POTENTIAL

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ABSTRACT

RADIATIVE DECAYS IN THE BETHE-SALPETER EQUATION
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Radiative decay widths are calculated for the radiative decay processes observed experimentally in the charmonium system. The model uses a Bethe-Salpeter equation with a static kernel and harmonic oscillator potentials to model the $c\text{-}\bar{c}$ system. Each decay width is calculated for 21 different choices of the c-quark mass. The potential used was a linear combination of a vector coupled and a scalar coupled harmonic oscillator potential. The quark mass and the scalar to vector coupling ratio were determined by trying to fit simultaneously the $\psi'(3685) - \psi(3095)$ mass difference, the $\psi(3095) \rightarrow e^+ + e^-$ decay width and the ^3P_J mass splittings. A single choice of the quark mass and scalar to vector coupling ratio could not simultaneously fit all these constraints. The "best fit" to these constraints occurred when the quark mass was 5.5 and the scalar to vector coupling ratio parameter was -0.16. The decay width calculations are shown graphically for values of the quark mass from 0.00 to 16 Gev. The decay widths were calculated two different ways: (i) using the matrix elements of the quark momentum; (ii) using the matrix elements of the quark position. Most of the published calculations use method (ii). The widths computed by methods (i) and (ii) are quite different for all masses and all transitions implying that the usual method (ii) gives incorrect results, and the fits with experimental data are fortuitous.

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Chapter 1

Introduction

With the discovery of the π-meson in 1947 a certain stage of "explanation" of atomic structure was reached. The electron, proton and neutron were the fundamental building blocks of nature. The photon was the instrument that was responsible for transmission of the electromagnetic forces from one charged particle to another charged particle. The π-meson did the analogous thing for the nuclear forces found to be acting between protons and neutrons. The neutrino was needed to conserve angular momentum in beta decay of nuclei. The only oddity was the μ-meson (or muon) whose existence escaped reason. Certain other particles, like the positron, were known or strongly suspected to exist on the basis of a symmetry of the Dirac equation, which was believed to describe the behavior of massive spin 1/2 particles.

The discovery of several additional particles ruined this nicely reasoned structure. These strange particles came in two varieties - called mesons (particles with masses between the π-meson and the proton) and hyperons (masses greater than the proton's mass) which eventually decayed into a proton or a neutron. These particles are now called $K^{\pm}$, K^{o} and $\bar{K}^{o}$ mesons, and Λ^{0}, Σ^{o}, $\Sigma^{\pm}$, Ξ^{o} and Ξ^{-} for the hyperons. If one looks at the eight particles p, n, Λ^{0}, Σ^{o}, $\Sigma^{\pm}$, Ξ^{o}, Ξ^{-} one finds that many of their properties are identical e.g. spin, parity and their interaction strength with nuclei. In an attempt to explain this similarity of properties Gell-Mann and Ne'eman (independently) developed the SU(3) or unitary symmetry model of elementary particles in 1961.

The group SU(3) has an eight dimensional representation which accomodated these eight heavy particles. This group also has representations of dimensions 3, 6, 10, 27, ... At this point the question arises, do there exist other, still to be found particles that could be described by these other allowed representations. The 10-dimensional representation was found to be an acceptable representation; it described certain resonances and predicted a new S = -3 heavy particle, the Ω^-. This Ω^- was subsequently found in an experiment conducted at Brookhaven in the 1960's. So far no particles have been detected directly that do not belong to the representations of dimensions 1, 8, or 10. Fairbanks, et al, [43] have reported results that seem to imply the existence of free quarks. These experiments still await verification by other experimental groups. This peculiar behavior has puzzled particle physicists for a long time. If one ascribed to the fundamental, or three-dimensional, representation, hypothetical particles called quarks, then all of the observed "particles" corresponded to representations of SU(3) obtained by taking the product representations $[SU(3)]_3 \otimes [\overline{SU(3)}]_3 = I \oplus [SU(3)]_8$ or $[SU(3)]_3 \otimes [SU(3)]_3 \otimes [SU(3)]_3 = 1 \oplus [SU(3)]_8 \oplus [SU(3)]_{10} \oplus [\overline{SU(3)}]_8$. These hypothetical quark particles have the following (unusual) properties.

TABLE 1

QUARK PROPERTIES

	spin	charge	baryon number	other quantum numbers
up	1/2	2/3	1/3	none
down	1/2	-1/3	1/3	none
strange	1/2	-1/3	1/3	S = -1
charm	1/2	2/3	1/3	c = +1
bottom	1/2	-1/3	1/3	b = +1
top	1/2	2/3	1/3	t = +1

This scheme could be looked upon as indicating the existence of three "elementary particles" belonging to the three-dimensional representation of SU(3), and that the observed particles, e.g. π-mesons, protons, etc., were bound states of these quarks. The results from high energy electron- proton scattering also seem to indicate the existence of a point-like substructure for the proton. The point-like substructure had the same properties that were being ascribed to the SU(3) quarks. One should note here that in the 1960's only the first three quarks in TABLE 1 were believed "to exist." The only essential difference between then and now is the replacement of SU(3) by SU(6) as the underlying symmetry group.

Eventually, as things usually do, this simple quark picture became somewhat more complicated. If quarks obeyed the usual Pauli spin-statistics theorem, then the quark model of elementary particles caused problems. The Ω^- and the π + p (1238) resonance were viewed as being composed of three quarks called (s,s,s) and (u,u,u) respectively, and all three quarks were to have zero orbital angular momentum. This condition violated the Pauli spin-statistics theorem, which stated that no two spin 1/2 particles could have the same set of quantum numbers. In the case of the Ω^- and π + p (1238) resonance, all three quarks did have the same set of quantum numbers. Another problem for the quark model surfaced in the analysis of high energy $e^+ + e^-$ annihilation experiments. The ratio of $e^+ + e^- \rightarrow$ hadrons to that for $e^+ + e^- \rightarrow \mu^+ + \mu^-$ is predictable on the basis of the quark model [44]. On the basis of the simple model above, the predicted and observed ratios differ by a factor of three. The predicted decay rate of $\pi^0 \rightarrow 2\gamma$ is also off by this same factor when one uses this simple quark model. In an effort to eliminate these problems, it was then hypothesized that each of the quarks u, d, s (and also subsequently discovered or inferred quarks like c, b, t) come in three varieties, e.g. red, green, and blue. The quark model now predicts that Ω^- is the bound state of three S quarks, one red, one green, and one blue. Since these S quarks are now distinguishable (this new quantity called the "color" of the quark is the new quantum number), the spin-statistic problem goes away immediately. The additional multiplicity of quarks also eliminates the problem of the factor of three in the π^0 decay rate and the $e^+ + e^-$ annihilation experiments mentioned above. The picture or model that emerges for those elementary particles that participate in strong

interactions is that they are composed of quarks or quark-antiquark combinations. Other considerations of nature seem to restrict this same set of elementary particles to be made up of either three quarks or of quark-antiquark pairs. No deviation from these conditions seems to exist. The particles to be subsequently considered here are a special class of heavy mesons and are composed of a quark and its associated antiquark.

The simplest thing to do is to assume a certain potential energy function for the $q-\bar{q}$ interaction, insert this into the Schroedinger wave equation and solve for the energy levels of this $q-\bar{q}$ system. This is what was done in [29] and [30]. The assumed potential was $V(r) = \alpha r + \beta/r$. There were reasons for this choice, aside from giving good agreement with the observed energy levels for the $c-\bar{c}$ system. The αr term arises from the requirement that the quarks are permanently bound or confined, i.e. free quarks do not exist. As the separation increases between two quarks, the potential energy also increases. The only way the two initial quarks can have arbitrarily large separations is to produce a new $q-\bar{q}$ pair, one of which escapes with the first quark and the other stays behind with the remaining quark. The β/r term comes from the known coulomb interaction occurring at short distances between the quarks. This treatment is completely non-relativistic in character. Since the discovery of the $\Psi(3095)$ meson and other members of the charmonium system, a number of other potential functions have been used to describe these new particles. A recent comparison of the predictions of several of these model calculations with experimental findings is given in [41]. Because of the close agreement between these non—relativistic calculations and experiments, one assumes that the

motion of the quarks in these charmonium states is non-relativistic. Most of these calculations have problems predicting the splittings of the various P states (1^3P_0, 1^3P_1, and 1^3P_2) as well as the relative decay rates from the 2^3S_1 state to these afore-mentioned P states. The radiative decay rates of these triplet P states to 1^3S_1 are for the most part not well enough determined to be useful in making theoretical comparisons between various model potentials.

In the case of the hydrogen atom the non-relativistic Schroedinger equation with a pure coulomb potential is unable to reproduce the fine structure observed. Initially additional terms were incorporated into the potential and treated perturbatively to produce the observed fine structure. With the development of the Dirac equation, which treated the electron's motion relativistically, the fine structure of the hydrogen atom was correctly produced, the major defect of the Dirac equation being the failure to account for the ($2S_{1/2}$ - $2P_{1/2}$) splitting (Lamb effect) discovered about twenty years later. To account for this, one needed a relativistic quantum field theory of electrons and the electromagnetic field. This construction was first attempted in the late 1920's and early 1930's. These initial attempts were able to handle some things, e.g. the calculation of the Klein-Nishina formula for compton scattering. The Klein-Nishina formula, from a relativistic quantum field theory viewpoint, represented the effect of the leading term in a perturbative expansion. When one tried to compute the radiative corrections to this result, certain divergent integrals arose. It was unclear how to deal with these divergences in the theory. In the late 1940's, Feynman, Dyson, Schwinger, and others developed a covariant renormalization scheme which

enabled them to calculate these radiative correction terms. This method is described in several papers in the collection [23].

Shortly after the introduction of the Dirac equation, G. Breit [16] and others attempted to generalize this equation to describe the interaction of two or more relativistic particles, the immediate application being to the fine structure of the helium atom. The initial attempts presented problems. (See discussion in [17]). These problems were only finally resolved with the developments in relativistic quantum field theory in the late 1940's. The initial paper making use of these methods is by Bethe and Salpeter [18]. Most of the work done on the Bethe-Salpeter equation is directed to applications involving radiative corrections to the hydrogen atom or positronium. A recent survey on the applications to positronium is given by Stroscio [22]. The starting point for our calculations makes use of equations 2.1 to 2.10 and equations 4.20 to 4.25 of [26]. The derivation of the Bethe-Salpeter equation from quantum field theory is given in appendix C of [22] and will not be reproduced here. The derivation given in [22] as well as the original one given by Bethe and Salpeter in [18] is based upon Feynman's approach to quantum field theory described in papers 21 and 22 of reference [23].

The method of reduction of the general form of the Bethe-Salpeter equation as given by

$$H(\vec{p})\phi(\vec{p}) + \phi(\vec{p})H(-\vec{p}) - \lambda_n \phi(\vec{p}) = - u_{++}(\vec{p},\vec{p}')\phi(\vec{p}') \qquad (1)$$

to get the radial equation(s) needed to determine the eigenvalues λ_n is done in several steps. The function $\phi(\vec{p})$ can be expanded as follows.

$$\phi(\vec{p}) = S(\vec{p})C + \gamma_\mu V_\mu(\vec{p})C + \sigma_{\mu\nu}T_{\mu\nu}(\vec{p})C + \gamma_5\gamma_\mu A_\mu(\vec{p})C + \gamma_5 P(\vec{p})C \quad (2)$$

The functions $S(\vec{p})$, $V_\mu(\vec{p})$, ..., $P(\vec{p})$ are not all independent of one another. Relativistic kinematics imposes certain restrictions, and these relations are derived in Appendix I.1. It is seen there that the function $\phi(\vec{p})$ splits up into a singlet and a triplet case. We choose as the independent function for the singlet case $P(\vec{p})$ and for the triplet case $\vec{V}(\vec{p})$. The next step is to insert these singlet and triplet wavefunctions back into equation (1) and carry out the necessary Dirac algebra to get equations 4.3, 4.8, and 4.19 of [26]. The program Schoonschip was used to perform this reduction. The program to do this for the singlet wavefunction is shown in Appendix I.2. Similar but more complicated computer programs are needed for the triplet cases and are not illustrated here. The wavefunctions for the singlet and triplet states of the scalar potential are related to those of the vector potential by

$$\phi_s(\vec{p}) = \beta\phi_v(\vec{p})\beta \quad (3)$$

One consequence of (3) is that by making the substitutions $\vec{p}' \to -\vec{p}'$ and $\phi(\vec{p}') \to -\phi(\vec{p}')$ in equations 4.20, 4.21, and 4.22 of [26] one could convert the results for the "coulomb potential" given there to the case of a scalar potential. For the singlet case this reduction was explicitly carried out and checked by Schoonschip. For the triplet cases a complete reduction was not done, but this substitution rule was used to obtain the equations analogous to those of 4.21 and 4.22 of [26] for the scalar potential.

The remainder of Appendix I gives the details of other reductions needed to put the Bethe-Salpeter equations for the singlet

and triplet states into a form suitable for numerical solution. Appendix II gives the details of the calculations of formulae for the leptonic and radiative decay rates of various $q-\bar{q}$ systems. The results of the numerical calculations of these decay rates will be discussed in chapter 3.

The point of the work described below is to calculate properties of the $q-\bar{q}$ system, with an admixture of a vector and scalar harmonic coupling, and the comparison of these calculations with the experimental results on the charmonium system. The spectrum of the $q-\bar{q}$ system using a purely vector coupling was carried out in [31] and it was found that the level structure in the $^3(J\pm1)_J$ system was not the same as that found for this spin assignment in charmonium. In [31] the level scheme is 1^3S_1, 1^3D_1, 2^3S_1, 2^3D_1, etc. In charmonium the appropriate level structure is 1^3S_1, 2^3S_1, 1^3D_1, etc. The addition of the second, scalar harmonic, coupling here was an attempt to obtain the correct ordering of these states. This attempt was not successful, as will be seen below. This scalar potential gives us another parameter to vary to see if one can obtain better agreement with either decay rates or the fine structure of the P-states within this model. Initially attempts were made to incorporate a 1/r type of term in the potential. This attempt produced problems described in chapter 3. We hope to look further into these problems with a view to incorporating this potential form into our work. It is possible, based upon work using the linear plus coulomb type potential described in [29], that the use of a 1/r potential would correct the above level structure problem.

Chapter 2

Outline of the Calculational Scheme

The Bethe-Salpeter equation is a relativistic two body wave equation which represents a generalization of the Dirac equation. The Dirac equation is a linear partial differential equation. The Bethe-Salpeter equation is an integro-differential equation in general. For special choices of the potential function, the Bethe-Salpeter equation can be reduced to a partial differential equation. One of these special potentials is the instantaneous harmonic oscillator potential. This potential was picked for this work for precisely this reason.

Once we have this differential equation form for the Bethe-Salpeter equation, next arises the problem of obtaining solutions of this equation. One of the tried and true methods for solving partial differential equations is to use the method of separation of variables to reduce the problem to a set of ordinary differential equations. We follow this time honored method. The next problem is the solution of the resulting ordinary differential equations. If we use spherical coordinates in our separation of variables technique, then the two equations involving the polar angles are solvable and yield the well known spherical harmonic functions as solutions. The radial equation which results from this separation procedure appears not to be a well known equation whose solution is easily recognized. In such cases one time honored method of solution says to express the (radial) function as a power series in terms of the independent variable, substitute this series into the differential equation, collect terms of the same power

in the independent variable and set the coefficients of these terms equal to zero. This method yields a recurrence relation for the coefficients of the assumed power series solution. In the workable cases one has a two term recurrence relation. Three or more term recurrence relations have historically been very hard to deal with. One wants to be able to express the nth coefficient of the power series as a function of n and the first term in the recurrence relation. To be able to solve the recurrence relation when it involves three or more terms is in general very difficult. It is only within the last couple of years that the 3-term recurrence relation arising from the non-relativistic Schroedinger equation with a linear potential has been solved using non-standard techniques. In the radial equation resulting from our Bethe-Salpeter equation, an eleven term recurrence relation results — so much for this time honored method.

An alternate time honored method of obtaining approximate solutions of differential equations is to recast them as a variational calculus problem. This is the procedure to be used here on the radial equation to obtain an approximate solution to the overall Bethe-Salpeter equation. One of the basic types of problems in the calculus of variations is:

Given a function $f(x,y(x),y'(x))$, find that function $y(x)$ such that the integral

$$I = \int f(x,y(x),y'(x))dx$$

is a minimum, subject to the constraint condition that

$$\int p(x)[y(x)]^2 dx = 1.$$

For particular choices of $\mathfrak{L}(x,y(x),y'(x))$ and $p(x)$, this problem requires, as a necessary condition for its solution, that $y(x)$ obey a linear second order ordinary differential equation. Since any second order ordinary differential equation can be cast into a self-adjoint or Sturm-Liouville form, one restricts oneself to those choices of $\mathfrak{L}(x,y(x),y'(x))$ which give rise to such equations.

The reduction of the ordinary differential equations obtained by the introduction of spherical coordinates is given in appendix I.3. The further reduction to the Sturm-Liouville form and the obtaining of the associated Lagrangian is described in appendix I.4. We note here that the Lagrangian associated with the differential equation

$$d/dx(p(x)\ df/dx) + (q(x) + \lambda\ r(x))\ f(x) = 0$$

is

$$\mathfrak{L} = (1/2)p(x)[df/dx]^2 - (1/2)q(x)[f(x)]^2$$

with the subsidiary condition

$$(1/2)\int r(x)[f(x)]^2 dx = 1.$$

That this Lagrangian leads to the given differential equation is easily checked by inserting $\mathfrak{L}$ into the Euler-Lagrange equations. $(d/dx(\partial\mathfrak{L}/\partial(df/dx)) - \partial\mathfrak{L}/\partial f = 0)$

The method we use to attack this variational problem is called the direct method. We do not rely on the associated differential equation beyond using it to obtain the Lagrangian for the variational problem. This method says to select a set of functions $\{A_i(x)\}_{i=1}^{\infty}$ and to

write the function f(x) that minimizes the integral $\int f(x,y(x),y'(x))dx$, subject to the constraint $(1/2)\int r(x)f^2(x)dx = 1$, as

$$f(x) = a_0A_0(x) + a_1A_1(x) + \ldots + a_nA_n(x) + \ldots$$

The usefulness of such a method depends upon how fast the above series for f(x) converges to the correct (or exact) solution. The choice made here was to set

$$A_n(x) = (x^kAi(x)x^n)/(a + x^{k+1})$$

The ideas behind making such a choice are the following.

(i) We wanted to approximate as nearly as possible the correct asymptotic behavior ($x \to \infty$).

(ii) As $x \to 0$, the exact solution must go to zero as x^k where k is the orbital angular momentum of the two particle system.

(iii) The correct asymptotic behavior ($x \to \infty$) is given by $(x^kAi(x))/(a + bx^{k+1})$ and this is also compatible with the choice (ii). It is shown in Appendix I.6 that Ai(x)/x is the appropriate solution in the $x \to \infty$ limit. The third term is one that should not affect either of these limiting behaviors. This means that f(x) should reduce to one as $x \to 0$ and as $x \to \infty$. Several such terms were tried, none successfully. The one finally chosen was just x^n. If one recalls that the correct solution is a linear combination of functions of the form $A_n(x)$, then one sees that the $x \to 0$ limit is still valid provided that $a_0 \neq 0$. The $x \to \infty$ limit however is not maintained for any choice of $a_i \neq 0$, $i > 1$. The $x \to \infty$ limit is still dominated by the Airy function since as $x \to \infty$ it goes like $\exp[-2x^{3/2}/3]$.

The choice of x^n for the third part of our trial functions was

tested for the case in which the quark mass m = 0. In this case the known solution is just a "displaced Airy function," i.e. $f(x) = Ai(x-x_0)$, times a normalization constant, where x_0 is one of the zeros of the Airy function. By choosing b = 1 and a = 2.385, the correct function was obtained to five decimal places over the interval from $x_o = 0$ to $x \approx 38$. This corresponds to a range of values in f(x) from $f(x) \approx 1.00$ to $f(x) \approx 10^{-64}$. We next discuss the principle steps in the calculational scheme.

(A) The Calculation of the Airy function and its Derivative

The Airy function and its first derivative are related to Bessel functions of fractional order through the following identities:

$$Ai(x) = (1/\pi)(x/3)^{1/2}K_{1/3}(z)$$

$$= (1/3)\sqrt{x}\,\{I_{-1/3}(z) - I_{1/3}(z)\}$$

$$Ai'(x) = (-x/\pi)(1/\sqrt{3})K_{2/3}(z)$$

$$= (-x/3)\{I_{-2/3}(z) - I_{2/3}(z)\}$$

where $z = 2x^{3/2}/3$ and $x > 0$.

The determination of Ai(x) and Ai'(x) is done by making use of series expansions of $I_{\pm 1/3}(x)$, $I_{\pm 2/3}(x)$, and $K_{1/3}(x)$ and $K_{2/3}(x)$. The expressions used are the following:

$$x^{-\nu}I_{\nu}(x) = \sum_{n=0}^{\infty} A_n(\nu)T_{2n}(x/8)$$

where $\nu = \pm 1/3$ and $-8 \leq x \leq +8$. The series for $\nu = \pm 1/3$ was terminated

at 17 terms. The constants $A_n(\nu)$ are taken from [32]. The quantities $T_{2n}(y)$ are Chebyshev polynomials. (See below.)

The expansion for $x \geq 5$ makes use of the following approximation:

$$K_{1/3}(x) = \sqrt{(\pi/2x)}\ e^{-x} \sum_{n=0}^{\infty} C_n T_n^*(5/x), \quad x \geq 5$$

The values of the constants C_n are also from [32]. The quantities $T_n^*(y)$ are again Chebyshev polynomials. (See below.)

The expansions for $I_{\pm 2/3}(x)$ and $K_{2/3}(x)$ have the same form as those for the case of $I_{\pm 1/3}(x)$ and $K_{1/3}(x)$, and the relevant constants for these expansions appear again in [32].

The Chebyshev polynomials which appear in the above series expansions are evaluated by the use of the following recurrence relations:

$$T_0(x) = 1$$

$$T_1(x) = xT_0(x) = x$$

$$T_2(x) = (2x^2-1)T_0(x) = 1(2x^2-1)$$

$$T_3(x) = 2(2x^2-1)T_1(x) - T_1(x) = 4x^3-3x$$

or in general

$$T_{2n+2}(x) = 2(2x^2-1)T_{2n}(x) - T_{2n-2}(x), \quad n>0$$

$$T_{2n+3}(x) = 2(2x^2-1)T_{2n+1}(x) - T_{2n-1}(x), \quad n>0$$

Note that $T_n^*(x) = T_n(2x-1)$ so that

$$T_{n+1}^*(x) = 2(2x-1)T_n^*(x) - T_{n-1}^*(x), \quad n \geq 1$$

The source for these recurrence relations is [32].

(B) The Transformation of the Trial Wavefunctions

The next step is the transformation of the set of trial functions $\{Ai(x)x^k x^n/(a + bx^{k+1})\}_{n=0}^{9}$ to a second set of orthonormal functions. These new orthonormal functions are orthogonal with respect to a weight function x^2. The important quantity here is the matrix C_{mn} that transforms the functions $\{Ai(x)x^k x^n/(a+bx^{k+1})\}_{n=0}^{9}$ to the orthonormal set $\phi_m(x)$. The determination of these matrices C_{mn} is done separately for each value of k and is stored in the program for future reference. These matrices were used when some of the wavefunctions were plotted for values of x which differed from those used in the integration algorithm. One should note here that the value of x=0, when used in the $^1S_0$ functions, yields zero, but this is incorrect. This is due to the fact that the above expansions for Ai(x) fail at x=0. To eliminate this problem the value of Ai(x), for x=0, is explicitly specified in the main body of the program.

(C) The Integration Algorithm

In the calculation of the matrix C_{mn} mentioned in (B), several integrations were needed. The method of integration that was used throughout the program was one of the gaussian quadrature algorithms. Three different gaussian methods were tried. The first was a standard

32 point Gauss-Laguerre quadrature. The second was a truncated 68 point Gauss-Laguerre quadrature. The third one tried was a 28 point guassian quadrature over the interval from x = 0 to x = 20. In the Gauss-Laguerre routines, the values of x are prescribed, and in the standard 32 point case several values of x were sufficiently large to cause the value of Ai(x) to be less than 10^{-300}, which the machine sets equal to zero. The truncated Gauss- Laguerre routine used only the first 32 points of the 68 points available. The x-value of the last point used in this routine was approximately x = 38.094, and the value of the Airy function in this region of the x-axis is about 10^{-68}. The truncated Gauss-Laguerre also gives the best fit to the computed eigenvalues when compared to the known results for the $^1S_0$ case. For all cases, the lowest eigenvalue is reasonably accurate, but for the 2 $^1S_0$, 3 $^1S_0$, etc., the truncated 68 point routine proved to be the best.

(D) Reduction of the Differential Equation(s) to Finite Dimensional Matrix Problems

The final step in the solution of the differential equations is to write f(x), the solution of the differential equation, as a linear combination of the orthonormal functions $\phi_n(x)$ obtained in (B). This formal series

$$f(x) = \sum_{n=0}^{9} a_n \phi_n(x)$$

was substituted into the Lagrangian giving rise to the original differential equation. Note that for terms like $[f(x)]^2$ in the Lagrangian, we substitute in $\sum_{m,n} a_m a_n \phi_m(x)\phi_n(x)$. We then integrate over x. The resulting terms

are of the form $\int a_m a_n g_{mn}(x)dx$, a_m and a_n being constants independent of x, and where $g_{mn}(x)$ is a product of the form

$$q(x)\phi_m(x)\phi_n(x) + r(x)\phi_m'(x)\phi_n'(x)$$

and where q(x) and r(x) are the coefficients of $[f(x)]^2$ and $(df/dx)^2$ in the appropriate Lagrangian function. If we set $M_{mn} = \int g_{mn}(x)dx$ and view M_{mn} now as just a matrix, we have the typical linear algebra problem of finding the relative minimums of the quadratic form

$$\sum_{m,n} a_m M_{mn} a_n$$

subject to the constraint that the vector $\vec{a}$ have unit length. The minimum value(s) of this bilinear form occurs when the vector $\vec{a} = (a_0, a_1, a_2, \ldots, a_9)$ is an eigenvector of the matrix M_{mn}, the minimum being associated with the lowest eigenvalue λ_1, the next lowest relative minimum being associated with the next lowest eigenvalue λ_2, etc. The sought after approximate solution to the original differential equation is then given by

$$f(x) = \sum_{i=0}^{9} a_i \phi_i(x)$$

where the a_i's are the components of the eigenvector associated with the eigenvalue λ_1, λ_2, etc. This approach is expected to produce good results only for the lowest few eigenvalues and eigenfunctions for the solution of the original differential equations. To show that this expectation is verified, TABLE 2 lists the eigenvalues for the $^1S_0$ case. The known eigenvalues for this equation are the zeros of the Airy function. The zeros of the Airy function are "well known" and are also listed in TABLE 2 for purposes of comparison. One can see from TABLE 2

that the lowest two eigenvalues are indeed very accurately calculated, but as we go to the larger eigenvalues the agreement becomes progressively worse, and in the last several cases, one can safely say there is no agreement whatsoever. Also shown in TABLE 2 are the corresponding results for including only 6 or 8 terms in the expansion for f(x). In TABLES 3 and 4 we compare the known solutions for the lowest three eigenvalues of the $^1S_0$ equation. The case of the lowest or ground state solution is indeed very good, that of the next lowest one not quite so good. In principle one can increase the accuracy of the calculations by increasing the number of orthonormal functions used. In practice this presents two different types of problems: the first involves the amount of storage needed to do the calculations; the second is how bad the round off errors become in the process of constructing the orthonormal functions $\phi_n(x)$, or what is the same thing, the matrix C_{mn}. To get some idea of the round off error problem, the program calculates numerically the quantity

$$M_{nm} = \int_0^\infty x^2 \phi_n(x) \phi_m(x) dx$$

This matrix, if there is no round off error, should be just the unit matrix. In practice it is not of course. Some examples of what M_{nm} comes out to be in a typical run are shown in TABLES 5,6,7, and 8.

From these examples we see that the round off errors differ by a few parts in 10^{-19} from the exact value of zero. One should also point out that the initial data used to start the calculation had an accuracy of one part in 10^{-29}. So the resulting round off error due to machine computations is amplified by a factor of 10^{10}.

TABLE 2

COMPARISON OF THE COMPUTED AND EXACT VALUES
OF THE ZEROS OF THE AIRY FUNCTION

	COMPUTED VALUE	EXACT VALUE
	NEXP=6	
N= 1	.2338107412E+01	.233810741 E+01
N= 2	.4088312211E+01	.408794944 E+01
N= 3	.5611404421E+01	.552055983 E+01
N= 4	.8096949395E+01	.678670809 E+01
N= 5	.1543180161E+02	.794413359 E+01
N= 6	.5566569603E+02	.902265085 E+01
	NEXP=8	
N= 1	.2338107411E+01	.233810741 E+01
N= 2	.4087949656E+01	.408794944 E+01
N= 3	.5521632290E+01	.552055983 E+01
N= 4	.6883352602E+01	.678670809 E+01
N= 5	.8962481837E+01	.794413359 E+01
N= 6	.1360923309E+02	.902265085 E+01
N= 7	.2712791080E+02	.1004017434E+02
N= 8	.1004682793E+03	.1100852430E+02
	NEXP=10	
N= 1	.2338107410E+01	.233810741 E+01
N= 2	.4087949444E+01	.408794944 E+01
N= 3	.5520562028E+01	.552055983 E+01
N= 4	.6788694482E+01	.678670809 E+01
N= 5	.8042637663E+01	.794413359 E+01
N= 6	.9863035191E+01	.902265085 E+01
N= 7	.1333017852E+02	.1004017434E+02
N= 8	.2095336528E+02	.1100852430E+02
N= 9	.4282500750E+02	.1193601556E+02
N=10	.1610089334E+03	.1282877675E+02

TABLE 3

CHECK ON THE FIRST THREE SINGLET S ZERO WAVEFUNCTIONS

X BETWEEN .211 AND 38.0

X(I)	N=1	N=2	N=3
.211069E-01	-.999827E+00	.999704E+00	.997956E+00
	.999827E+00	.999697E+00	.999591E+00
.111223E+00	-.995301E+00	.991706E+00	.988945E+00
	.995301E+00	.991707E+00	.988771E+00
.273399E+00	-.972800E+00	.951497E+00	.934374E+00
	.972800E+00	.951498E+00	.934266E+00
.507755E+00	-.912816E+00	.843205E+00	.788431E+00
	.912816E+00	.843205E+00	.788570E+00
.814421E+00	-.799778E+00	.639957E+00	.523295E+00
	.799778E+00	.639958E+00	.523181E+00
.119356E+01	-.634660E+00	.354204E+00	.178495E+00
	.634660E+00	.354204E+00	.178549E+00
.164537E+01	-.441984E+00	.551956E-01	-.116453E+00
	.441984E+00	.551956E-01	-.116456E+00
.217010E+01	-.261690E+00	-.158693E+00	-.219319E+00
	.261690E+00	-.158693E+00	-.219349E+00
.276803E+01	-.127633E+00	-.228837E+00	-.112739E+00
	.127633E+00	-.228837E+00	-.112690E+00
.343948E+01	-.496930E-01	-.182127E+00	.591506E-01
	.496930E-01	-.182127E+00	.591014E-01
.418481E+01	-.149694E-01	-.981915E-01	.139638E+00
	.149694E-01	-.981916E-01	.139667E+00
.500445E+01	-.338155E-02	-.370865E-01	.110621E+00
	.338155E-02	-.370865E-01	.110624E+00
.589883E+01	-.555171E-03	-.977252E-02	.509622E-01
	.555171E-03	-.977240E-02	.509223E-01
.686846E+01	-.641862E-04	-.176512E-02	.147425E-01
	.641862E-04	-.176522E-02	.147830E-01
.791388E+01	-.506221E-05	-.213317E-03	.268755E-02
	.506221E-05	-.213457E-03	.273990E-02
.903570E+01	-.263715E-06	-.167705E-04	.302775E-03

TABLE 3(CONT'D)

X(I)	N=1	N=2	N=3
.102346E+02	-.878275E-08	-.831496E-06	.205044E-04
	.878258E-08	-.838389E-06	.233031E-04
.115112E+02	-.180880E-09	-.251494E-07	.809859E-06
	.180869E-09	-.256429E-07	.102245E-05
.128663E+02	-.222694E-11	-.448093E-09	.180869E-07
	.222655E-11	-.465977E-09	.263552E-07
.143007E+02	-.158329E-13	-.453591E-11	.221354E-09
	.158266E-13	-.486603E-11	.387046E-09
.158153E+02	-.627448E-16	-.251373E-13	.143778E-11
	.626952E-16	-.282128E-13	.313479E-11
.174111E+02	-.133659E-18	-.734396E-16	.479642E-14
	.133469E-18	-.876490E-16	.135328E-13
.190890E+02	-.147441E-21	-.108850E-18	.794358E-17
	.147098E-21	-.140650E-18	.300500E-16
.208501E+02	-.810499E-25	-.787113E-22	.630493E-20
	.807638E-25	-.112245E-21	.330744E-19
.226957E+02	-.213384E-28	-.266822E-25	.231190E-23
	.212323E-28	-.428369E-25	.173628E-22
.246269E+02	-.258229E-32	-.407018E-29	.376933E-27
	.256545E-32	-.750718E-29	.417700E-26
.266451E+02	-.137651E-36	-.267872E-33	.262552E-31
	.136559E-36	-.579290E-33	.441760E-30
.287517E+02	-.309212E-41	-.728245E-38	.749358E-36
	.306472E-41	-.188422E-37	.196707E-34
.309482E+02	-.279528E-46	-.781792E-43	.838930E-41
	.277056E-46	-.246885E-42	.352555E-39
.332363E+02	-.969233E-52	-.316246E-48	.351942E-46
	.962143E-52	-.124303E-47	.242684E-44
.356176E+02	-.122602E-57	-.459064E-54	.527375E-52
	.122157E-57	-.228932E-53	.610937E-50
.380942E+02	-.536897E-64	-.227238E-60	.268432E-58
	.538470E-64	-.146493E-59	.534384E-56

COMPUTED SOLUTION IS ON SAME LINE AS X(I) VALUES

TABLE 4

CHECK ON THE FIRST THREE SINGLET S ZERO WAVEFUNCTIONS

X BETWEEN ZERO AND 7.75

X(I)	N=1	N=2	N=3
0.	-.100000E+01	.100001E+01	.997442E+00
.250000E+00	-.977106E+00	.959227E+00	.944917E+00
	.977106E+00	.959227E+00	.944736E+00
.500000E+00	-.915237E+00	.847583E+00	.794269E+00
	.915237E+00	.847583E+00	.794414E+00
.750000E+00	-.825840E+00	.686489E+00	.582872E+00
	.825840E+00	.686489E+00	.582778E+00
.100000E+01	-.720513E+00	.500483E+00	.349834E+00
	.720513E+00	.500483E+00	.349780E+00
.125000E+01	-.609593E+00	.312709E+00	.132405E+00
	.609593E+00	.312709E+00	.132476E+00
.150000E+01	-.501320E+00	.141730E+00	-.416246E-01
	.501320E+00	.141730E+00	-.415743E-01
.175000E+01	-.401525E+00	-.789459E-04	-.157424E+00
	.401525E+00	-.788153E-04	-.157461E+00
.200000E+01	-.313722E+00	-.106706E+00	-.212720E+00
	.313722E+00	-.106705E+00	-.212780E+00
.225000E+01	-.239453E+00	-.177627E+00	-.215059E+00
	.239453E+00	-.177627E+00	-.215068E+00
.250000E+01	-.178756E+00	-.216346E+00	-.178057E+00
	.178756E+00	-.216346E+00	-.178011E+00
.275000E+01	-.130654E+00	-.228775E+00	-.117592E+00
	.130654E+00	-.228775E+00	-.117541E+00
.300000E+01	-.935876E-01	-.221772E+00	-.486397E-01
	.935876E-01	-.221772E+00	-.486283E-01
.325000E+01	-.657508E-01	-.202015E+00	.168483E-01
	.657508E-01	-.202015E+00	.168149E-01
.350000E+01	-.453415E-01	-.175279E+00	.709170E-01
	.453415E-01	-.175279E+00	.708673E-01
.375000E+01	-.307110E-01	-.146077E+00	.109603E+00
	.307110E-01	-.146077E+00	.109572E+00

TABLE 4 (CONT'D)

X(I)	N=1	N=2	N=3
.400000E+01	-.204436E-01	-.117589E+00	.132262E+00
	.204436E-01	-.117589E+00	.132267E+00
.425000E+01	-.133823E-01	-.917967E-01	.140577E+00
	.133823E-01	-.917969E-01	.140612E+00
.450000E+01	-.861847E-02	-.697032E-01	.137535E+00
	.861847E-02	-.697033E-01	.137580E+00
.475000E+01	-.546340E-02	-.515996E-01	.126567E+00
	.546340E-02	-.515997E-01	.126597E+00
.500000E+01	-.341048E-02	-.373086E-01	.110924E+00
	.341048E-02	-.373086E-01	.110927E+00
.525000E+01	-.209730E-02	-.263872E-01	.933180E-01
	.209730E-02	-.263872E-01	.932932E-01
.550000E+01	-.127105E-02	-.182788E-01	.757680E-01
	.127105E-02	-.182786E-01	.757255E-01
.575000E+01	-.759404E-03	-.124145E-01	.596039E-01
	.759404E-03	-.124144E-01	.595586E-01
.600000E+01	-.447436E-03	-.827453E-02	.455610E-01
	.447436E-03	-.827442E-02	.455272E-01
.625000E+01	-.260058E-03	-.541667E-02	.339171E-01
	.260058E-03	-.541661E-02	.339039E-01
.650000E+01	-.149148E-03	-.348499E-02	.246333E-01
	.149148E-03	-.348500E-02	.246439E-01
.675000E+01	-.844280E-04	-.220506E-02	.174797E-01
	.844280E-04	-.220514E-02	.175119E-01
.700000E+01	-.471837E-04	-.137288E-02	.121332E-01
	.471837E-04	-.137301E-02	.121812E-01
.725000E+01	-.260398E-04	-.841504E-03	.824675E-02
	.260398E-04	-.841655E-03	.830365E-02
.750000E+01	-.141946E-04	-.508026E-03	.549336E-02
	.141946E-04	-.508184E-03	.555253E-02
.775000E+01	-.764442E-05	-.302205E-03	.358897E-02
	.764442E-05	-.302355E-03	.364520E-02

COMPUTED SOLUTION IS ON SAME LINE AS X(I) VALUES

TABLE 5

ORTHOGONALITY CHECK, CASE L = 0

.100E+01	-.178E-27	.211E-27	.760E-27	-.108E-25	.545E-25	-.202E-24	.396E-24	.904E-24	-.151E-22
-.178E-27	.100E+01	.424E-27	-.168E-26	.490E-26	-.143E-25	.403E-25	-.385E-24	.317E-23	-.226E-22
.211E-27	.424E-27	.100E+01	.210E-25	-.676E-25	.187E-24	-.468E-24	.106E-23	-.241E-23	.258E-23
.760E-27	-.168E-26	.210E-25	.100E+01	.298E-24	-.801E-24	.211E-23	-.601E-23	.186E-22	-.607E-22
-.108E-25	.490E-26	-.676E-25	.298E-24	.100E+01	.271E-23	-.791E-23	.243E-22	-.766E-22	.237E-21
.545E-25	-.143E-25	.187E-24	-.801E-24	.271E-23	.100E+01	.335E-22	-.113E-21	.356E-21	-.106E-20
-.202E-24	.403E-25	-.468E-24	.211E-23	-.791E-23	.335E-22	.100E+01	.504E-21	-.162E-20	.473E-20
.396E-24	-.385E-24	.106E-23	-.601E-23	.243E-22	-.113E-21	.504E-21	.100E+01	.640E-20	-.191E-19
.904E-24	.317E-23	-.241E-23	.186E-22	-.766E-22	.356E-21	-.162E-20	.640E-20	.100E+01	.761E-19
-.151E-22	-.226E-22	.258E-23	-.607E-22	.237E-21	-.106E-20	.473E-20	-.191E-19	.761E-19	.100E+01

TABLE 6

ORTHOGONALITY CHECK, CASE L = 1

.100E+01	-.541E-28	-.685E-27	.466E-26	-.180E-25	.359E-25	-.552E-26	-.885E-24	.843E-23	-.501E-22
-.541E-28	.100E+01	.786E-27	.314E-26	-.305E-25	.125E-24	-.357E-24	.939E-25	.691E-23	-.533E-22
-.685E-27	.786E-27	.100E+01	-.457E-26	.590E-25	-.227E-24	.465E-24	-.197E-24	-.819E-24	-.613E-23
.466E-26	.314E-26	-.457E-26	.100E+01	.524E-25	-.423E-24	.298E-23	-.146E-22	.562E-22	-.187E-21
-.180E-25	-.305E-25	.590E-25	.524E-25	.100E+01	.383E-23	-.167E-22	.627E-22	-.206E-21	.613E-21
.359E-25	.125E-24	-.227E-24	-.423E-24	.383E-23	.100E+01	.505E-22	-.152E-21	.452E-21	-.138E-20
-.552E-26	-.357E-24	.465E-24	.298E-23	-.167E-22	.505E-22	.100E+01	.100E-21	.212E-21	-.124E-20
-.885E-24	.939E-25	-.197E-24	-.146E-22	.627E-22	-.152E-21	.100E-21	.100E+01	-.977E-20	.417E-19
.843E-23	.691E-23	-.819E-24	.562E-22	-.206E-21	.452E-21	.212E-21	-.977E-20	.100E+01	-.293E-18
-.501E-22	-.533E-22	-.613E-23	-.187E-21	.613E-21	-.138E-20	-.124E-20	.417E-19	-.293E-18	.100E+01

TABLE 7

ORTHOGONALITY CHECK, CASE L = 2

.100E+01	-.232E-27	.109E-26	-.604E-26	.271E-25	-.532E-25	.260E-26	-.212E-24	.683E-23	-.554E-22
-.232E-27	.100E+01	-.120E-26	.582E-26	-.292E-25	.120E-24	-.269E-24	-.343E-24	.999E-23	-.687E-22
.109E-26	-.120E-26	.100E+01	.312E-25	-.141E-24	.589E-24	-.220E-23	.737E-23	-.211E-22	.641E-22
-.604E-26	.582E-26	.312E-25	.100E+01	.174E-23	-.727E-23	.254E-22	-.796E-22	.243E-21	-.773E-21
.271E-25	-.292E-25	-.141E-24	.174E-23	.100E+01	.410E-22	-.140E-21	.431E-21	-.129E-20	.402E-20
-.532E-25	.120E-24	.589E-24	-.727E-23	.410E-22	.100E+01	.566E-21	-.171E-20	.496E-20	-.150E-19
.260E-26	-.269E-24	-.220E-23	.254E-22	-.140E-21	.566E-21	.100E+01	.539E-20	-.145E-19	.397E-19
-.212E-24	-.343E-24	.737E-23	-.796E-22	.431E-21	-.171E-20	.539E-20	.100E+01	.306E-19	-.630E-19
.683E-23	.999E-23	-.211E-22	.243E-21	-.129E-20	.496E-20	-.145E-19	.306E-19	.100E+01	-.382E-20
-.554E-22	-.687E-22	.641E-22	-.773E-21	.402E-20	-.150E-19	.397E-19	-.630E-19	-.382E-20	.100E+01

TABLE 8

ORTHOGONALITY CHECK, CASE L = 3

.100E+01	.187E-27	-.241E-27	-.825E-28	.623E-26	.708E-26	-.504E-25	.118E-23	-.132E-22	.104E-21
.187E-27	.100E+01	.848E-26	-.252E-25	.569E-25	-.113E-24	.498E-24	-.215E-23	.187E-23	.509E-22
-.241E-27	.848E-26	.100E+01	.103E-24	-.185E-24	.142E-24	.749E-24	-.506E-23	.167E-22	-.248E-22
-.825E-28	-.252E-25	.103E-24	.100E+01	.879E-24	-.238E-23	.568E-23	-.969E-23	.276E-23	.809E-22
.623E-26	.569E-25	-.185E-24	.879E-24	.100E+01	.209E-22	-.716E-22	.186E-21	-.353E-21	.528E-21
.708E-26	-.113E-24	.142E-24	-.238E-23	.209E-22	.100E+01	.377E-21	-.101E-20	.200E-20	-.336E-20
-.504E-25	.498E-24	.749E-24	.568E-23	-.716E-22	.377E-21	.100E+01	.367E-20	-.718E-20	.133E-19
.118E-23	-.215E-23	-.506E-23	-.969E-23	.186E-21	-.101E-20	.367E-20	.100E+01	.160E-19	-.245E-19
-.132E-22	.187E-23	.167E-22	.276E-23	-.353E-21	.200E-20	-.718E-20	.160E-19	.100E+01	-.861E-19
.104E-21	.509E-22	-.248E-22	.809E-22	.528E-21	-.336E-20	.133E-19	-.245E-19	-.861E-19	.100E+01

(E) Scaling and Fitting Parameters

In the calculations used to obtain the starting differential equations, the mass and the variable x were scaled by the factor $\sqrt[3]{k/2}$. (See Appendix I.6) In the initial equations we have three undetermined parameters k, RAT, and m. The scaling transformation eliminates the explicit dependence of the quantity k from the differential equations. To determine the transition widths for the radiative decays one needs to know k, m, and RAT, i.e. the decay widths for a given mass depend upon the values of k and RAT. We have chosen to determine k and m by imposing the following two constraints:

(1) That the difference between the two lowest eigenvalues of the $^3S_1$ equation be equal to .587(Gev).

(2) The second constraint also involves the $^3S_1$ equation. The lowest eigenfunction of this state is supposed to describe the $\psi(3095)$ meson. This particle has, as one of its decay modes, $\psi(3095) \rightarrow e^+ + e^-$. A derivation of the formula giving this decay width is presented in Appendix II.1. Using this expression, which also depends upon m, k, and RAT (the last two implicitly), we determine a decay width. This decay width depends upon k^3.

To determine the last parameter RAT, which is the ratio of the scalar to the vector potential coupling strength, we make use of a spectral constraint. The initial hope was to vary RAT so as to produce a level structure in the J=1 mixed state of the form 1S, 2S, 1D. It was not possible to produce this type of ordering and still obtain reasonable fits to the $2\ ^3S_1 - 1\ ^3S_1$ mass difference and the $\psi(3095) \rightarrow e^+ + e^-$ decay width. To determine RAT the splittings between the various p-states were used. The two particular choices used were the

$^3P_2 - {}^3P_0$ and $^3P_2 - {}^3P_1$ mass differences. One reason for choosing the p-states for this fit was to obtain a constraint that was independent of the ones used to determine m and k. A simultaneous fit to both of these mass differences was not possible. In addition there is another possible constraint. The $1^3P_2 - 1^3S_1$ mass difference could not be satisfied with either of the above mass differences from the above mentioned p-states.

(F) Radiative Decay Width Equations

The derivations of the equations of the radiative decay widths are given in appendices II.2, II.3, II.4, and II.5. In appendix II.2 the current matrix element is derived. In appendix II.3 the functions describing the spin states of the $q\text{-}\bar{q}$ systems discussed below are obtained. Appendix II.5 gives the Schoonschip computer program for evaluating the current matrix element for the transition $2^3S_1 \rightarrow 1^3P_0 + \gamma$. Certain polarization constants needed to evaluate the transition widths, but omitted from the Schoonschip program(s) are given in TABLE 9. The output of the Schoonschip current matrix element calculations is shown in TABLE 10. The final expressions used in the calculation of the radiative decay widths are: (i) for the electric dipole cases

$$\Gamma(B' \rightarrow B+\gamma) = 8(m_B/m_{B'})\alpha|\vec{p}||C_e|^2R^2(p)P$$

and (ii) for the magnetic dipole cases

$$\Gamma(B' \rightarrow B+\gamma) = 8(m_B/m_{B'})\alpha|\vec{p}|^3|C_m|^2R^2(p)P.$$

The quantities P are the spin sums and are given in TABLE 11. The constants C_e and C_m are given in TABLES 12 and 13 respectively. These equations along with the final steps of their derivations are given in appendix II.4.

TABLE 9

POLARIZATION PROGRAM CORRECTIONS

State	Correct Polarization Quantity	Programmed Quantity	Needed Factor
$^3S_1$	$(1/\sqrt{4\pi})\hat{\xi}$	$\hat{\xi}$	$1/\sqrt{4\pi}$
$^3D_1$	$-3/(2\sqrt{2\pi})[(\hat{\epsilon}\cdot\hat{p})\hat{p} - \hat{\epsilon}/3]$	$(\hat{\epsilon}\cdot\hat{p})\hat{p} - \hat{\epsilon}/3$	$-3/(2\sqrt{2\pi})$
$^3P_0$	$\hat{p}/\sqrt{4\pi}$	$\hat{p}$	$1/\sqrt{4\pi}$
$^3P_1$	$-\frac{1}{2}\sqrt{3/(2\pi)}(\hat{\xi}\times\hat{p})$	$\hat{\xi}\times\hat{p}$	$-\frac{1}{2}\sqrt{3/(2\pi)}$
$^3P_2$	$\frac{1}{2}\sqrt{3/\pi}\ E\cdot\hat{p}$	$E\cdot\hat{p}$	$\frac{1}{2}\sqrt{3/\pi}$
$^1S_0$	$1/\sqrt{4\pi}$	1	$1/\sqrt{4\pi}$

TABLE 10

SCHOONSCHIP EVALUATION OF CURRENT MATRIX ELEMENTS

Transition	Schoonschip Result	Omitted Factor
$^3S_1 \to {}^3P_0$	$-\frac{8}{3} i f_i f_f \frac{p}{E}[p_0 + \frac{1}{2}E_T - E]\hat{\xi}\cdot\hat{e}$	$1/12$
$^3S_1 \to {}^3P_1$	$-\frac{8}{3} f_i f_f \frac{p}{E}[p_0 + \frac{1}{2}E_T - E]\hat{\xi}'\cdot(\hat{e}\times\hat{\xi})$	$-\sqrt{6}/24$
$^3S_1 \to {}^3P_2$	$\frac{8}{3} i \frac{p}{E} f_i f_f [p_0 + \frac{1}{2}E_T - E]\hat{e}\cdot E\cdot\hat{\xi}$	$\sqrt{3}/12$
$^3D_1 \to {}^3P_0$	$\frac{16i}{3} f_i f_f \frac{p}{E}[p_0 + \frac{1}{2}E_T - E]\hat{\xi}\cdot\hat{e}$	$-\sqrt{2}/8$
$^3D_1 \to {}^3P_1$	$\frac{8}{3} f_i f_f \frac{p}{E}[p_0 + \frac{1}{2}E_T - E]\hat{\xi}\cdot(\hat{e}\times\hat{\xi}')$	$\sqrt{3}/8$
$^3D_1 \to {}^3P_2$	$-\frac{8}{5} i f_i f_f \frac{p}{E}[p_0 + \frac{1}{2}E_T - E]\hat{\xi}\cdot E\cdot\hat{e}$	$-\sqrt{6}/8$
$^3S_1 \to {}^1S_0$	$\frac{2}{3} i f_i f_f ((E+2m)/E^2)[p_0 + \frac{1}{2}E_T - E]\hat{k}\cdot(\hat{\xi}\times\hat{e})$	$1/12$
$^3D_1 \to {}^1S_0$	$\frac{4}{9} i f_i f_f ((E-m)/E^2)[p_0 + \frac{1}{2}E_T - E]\hat{k}\cdot(\hat{\xi}\times\hat{e})$	$-\sqrt{2}/8$

TABLE 11

SPIN AVERAGE RESULTS

$^3S_1 \to {}^3P_0$	2/3	$^3S_1 \to {}^3P_1$	4/3	$^3S_1 \to {}^3P_2$	10/9	$^3S_1 \to {}^1S_0$	2/3
$^3P_0 \to {}^3S_1$	2	$^3P_1 \to {}^3S_1$	4/3	$^3P_2 \to {}^3S_1$	2/3	$^1S_0 \to {}^3S_1$	2

TABLE 12

ELECTRIC DIPOLE CONSTANTS

$^3S_1 \to {}^3P_0$	$4\pi/9$	$^3D_1 \to {}^3P_0$	$4\pi\sqrt{2}/3$
$^3S_1 \to {}^3P_1$	$-2\pi i\sqrt{6}/9$	$^3D_1 \to {}^3P_1$	$-2\pi i\sqrt{3}/3$
$^3S_1 \to {}^3P_2$	$4\pi\sqrt{3}/9$	$^3D_1 \to {}^3P_2$	$-2\pi\sqrt{6}/5$

TABLE 13

MAGNETIC DIPOLE CONSTANTS

$^3S_1 \to {}^1S_0$	$\pi/9$
$^3D_1 \to {}^1S_0$	$2\sqrt{2}\pi/9$

Chapter 3

Calculational Results and Conclusions

(A) Calculational Results

The calculational results of this work will be presented in a series of graphs. In most cases the quantity being varied is the quark mass and usually the variation in mass is from zero to 20, in dimensionless units. The conversion to energy units is made by requiring the calculated splitting in the $2^3S_1 - 1^3S_1$ to be equal to the $\psi'(3685) - \psi(3095)$ splitting. This is one of the quantities plotted in the graphs presented below, and since this calculated splitting varies with the mass parameter, so does this conversion factor. To date most of the published calculations relating to the charmonium system give only the results for the parameters that result in the best fit to the system. I have tried to show how the different calculated quantities vary as a function of the parameters in this model. My hope is that by presenting the results in this form this will give some feeling as to how sensitive the calculated results are to the fitting parameters.

It was pointed out in Chapter 2 that one of the programming checks was to see if the computed results were consistent with the known behavior of these equations in various limiting cases. The case in which RAT=0 and m→0 is known to yield the Schroedinger equation with a linear potential. In this case, systems with the same total angular momentum, J, will have the same eigenvalue, or binding energy, for this mass. In the case in which the mass becomes very large, our equations go over into the Schroedinger equation for the spherical harmonic oscillator. In this case we know that the difference in the binding

Figure 1. Graphs of the Binding Energies of the Ground States of the $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3S_1$, $^3D_1$, $^3P_2$, $^3F_2$ configurations of the $q\bar{q}$ system for RAT = 0 as a function of the quark mass.

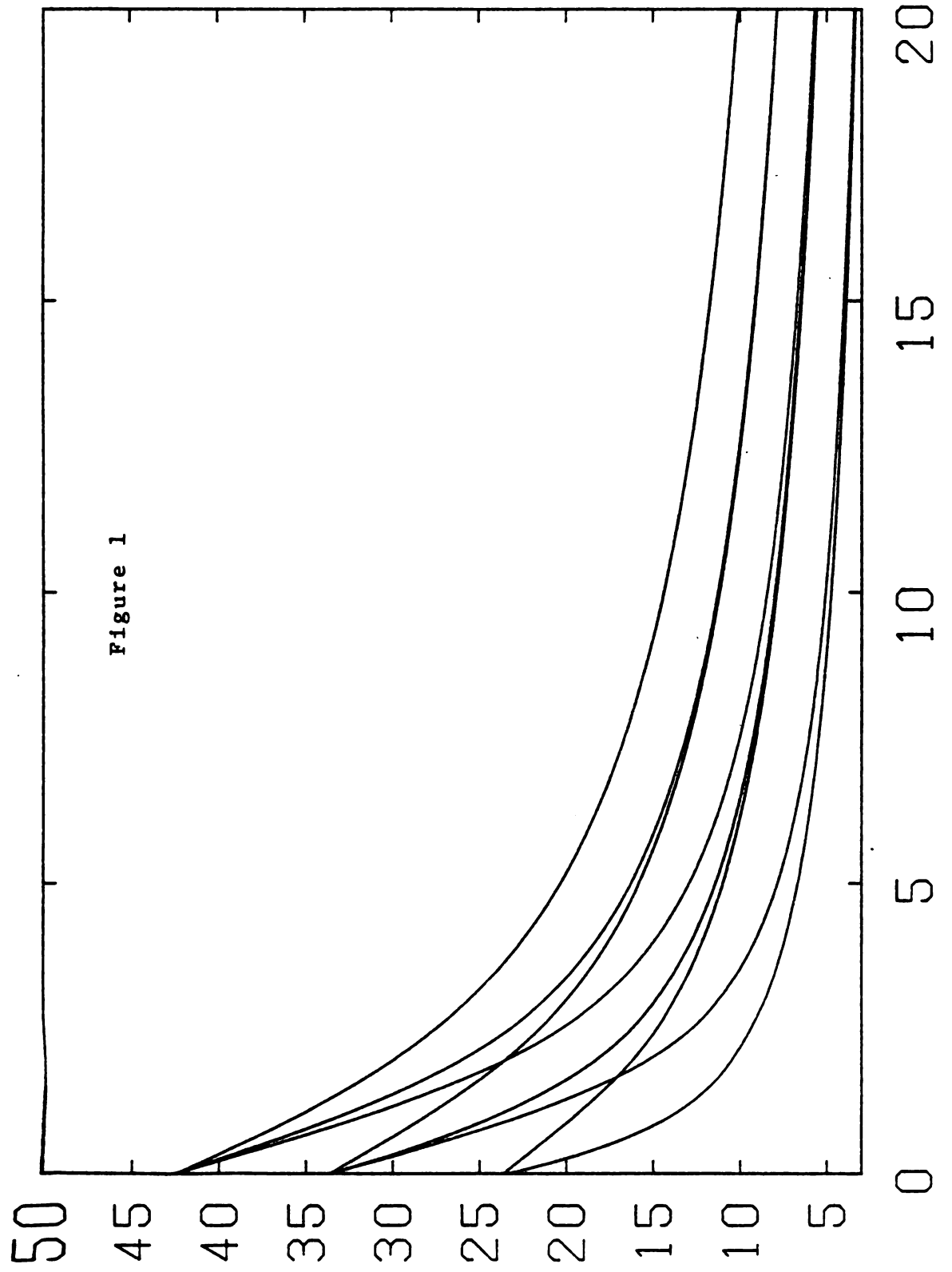


Figure 1

Figure 2. Graphs of the Binding Energies of the Ground State and First Excited State of the $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3S_1$, $^3D_1$, $^3P_2$, $^3F_2$ configurations of the $q\bar{q}$ system for RAT = 0 as a function of the quark mass.

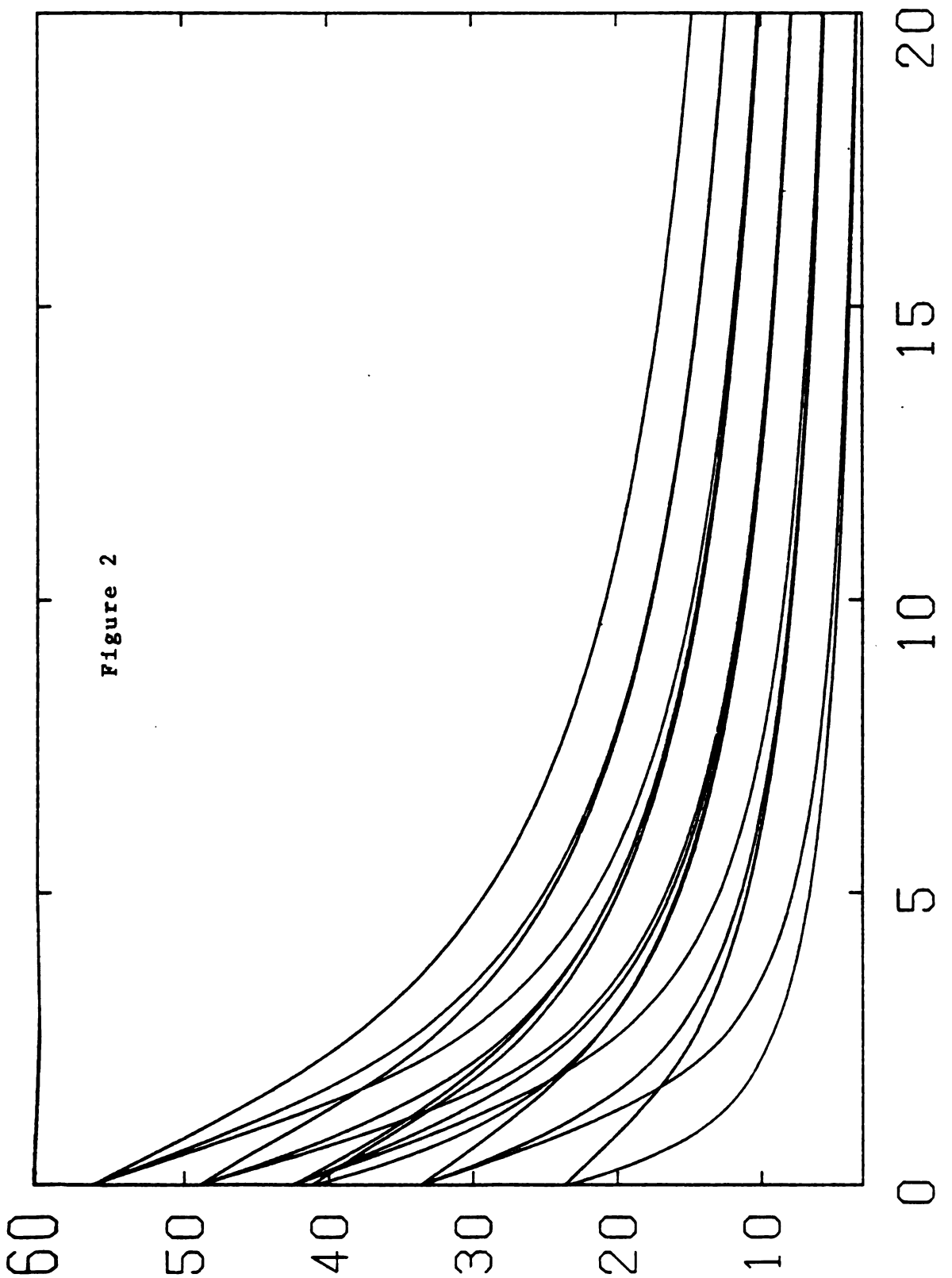


Figure 2

Figure 3. Graphs of the Binding Energies of the First and Second Excited States of the $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$; $^3S_1$, $^3D_1$, $^3P_2$, $^3F_2$ configurations of the $q\bar{q}$ system for RAT = 0 as a function of the quark mass.

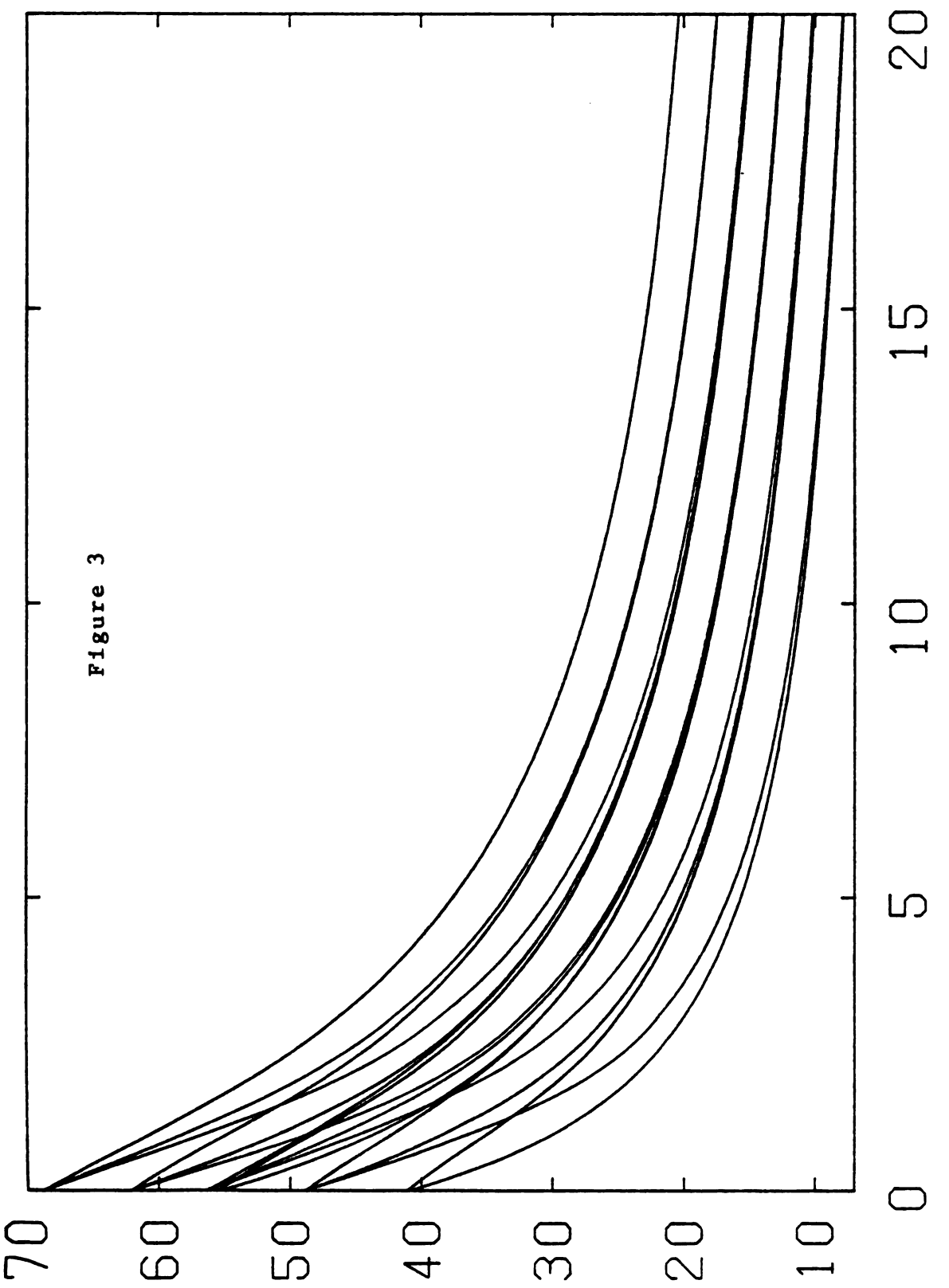


Figure 3

Figure 4. Graphs of the Binding Energies of the Ground State and First Excited State of the $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3S_1$, $^3D_1$, $^3P_2$, $^3F_2$ Configurations of the $q\bar{q}$ system for RAT = +0.33 as a function of the quark mass

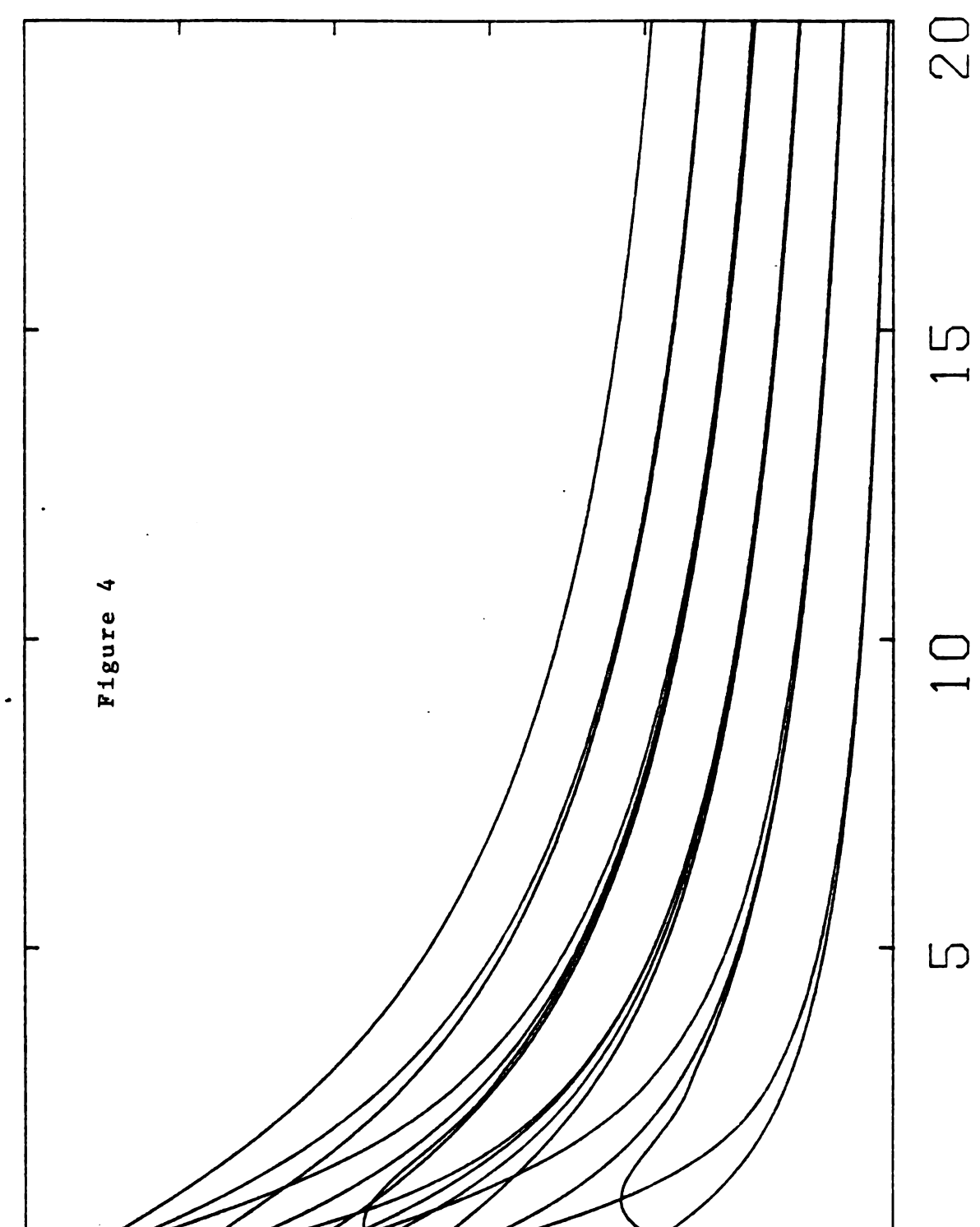


Figure 4

energy between the (n+2)nd level and the nth level is just twice the difference in binding energy between the (n+1)st and nth levels. This large mass harmonic oscillator character is independent of the value of RAT used here. That these limiting behaviors are observed can be seen by looking at figures 1, 2, 3, and 4. In figure 1, we show a plot of the binding energy of the eight states 1^1S_0, 1^1P_1, 1^1D_2, 1^3P_0, 1^3S_1, 1^3D_1, 1^3P_2, and 1^3F_2 as a function of the quark mass parameter. In figure 2 is shown a similar plot for the ground state and first excited state of these same configurations. In figure 3 is shown a similar plot for the first and second excited states of these same configurations. For these first three figures RAT=0. In figure 4, the same states as are plotted in figure 2 are graphed, but now for RAT=0.33. In these graphs one can see that the desired limiting behavior is found. At m=0 in figures 1, 2, and 3, states with the same value of J have the same binding energy, and as one goes to m=20 one sees the convergence of the binding energies for the states with the same value of the orbital angular momentum. The equal spacing can also be seen between the levels with L, the orbital angular momentum, values differing by one unit. Figures 2 and 3 were included to show a further degeneracy known to occur for the harmonic oscillator, namely the equality of the binding energies for the 2S and 1D states and also for the 3S and 2D states and a similar behavior for the 2P-1F and 3P-2F states. Figure 4 is included to show a similar behavior for large masses in a case in which RAT $\neq$ 0. Similar results also occur for values of RAT between $\pm$0.50 but are not shown here. In figure 5, for RAT=0, we have plotted several different functions that were used in the determination of the parameters RAT, k, and the quark mass of this model. The curve in figure 5 labeled 1 is

Figure 5. Determination of Potential Parameters, RAT = 0.00

Curve 1. k = the observed ψ'- ψ mass difference (0.587 Gev) divided by the mass difference computed by the program (in dimensionless units)

Curve 2. The difference in the computed masses for the 1^3P_2 - 1^3S_1 states

Curve 3. k = the cube root of A/B where

A = observed decay width for $\Gamma(\psi(3095) \rightarrow \mu^+ + \mu^-)$

B = computed decay width for $\Gamma(\psi(3095) \rightarrow \mu^+ + \mu^-)$

Curve 4. k = the same as for curve 3 except that $\psi(3095)$ is replaced by $\psi'(3685)$

Curve 5. The ratio (C-D)/C where

C = $(^3P_2, ^3P_1)$ observed mass difference

D = $(^3P_2, ^3P_1)$ computed mass difference

Curve 6. The same as for curve 5 but for the $^3P_2$ and $^3P_0$ states

The vertical scale for curves 1, 2, 3, and 4 is Gev.

The vertical scale for curves 5 and 6 is in dimensionless units.

The horizontal scale is the dimensionless mass number (used in the calculation).

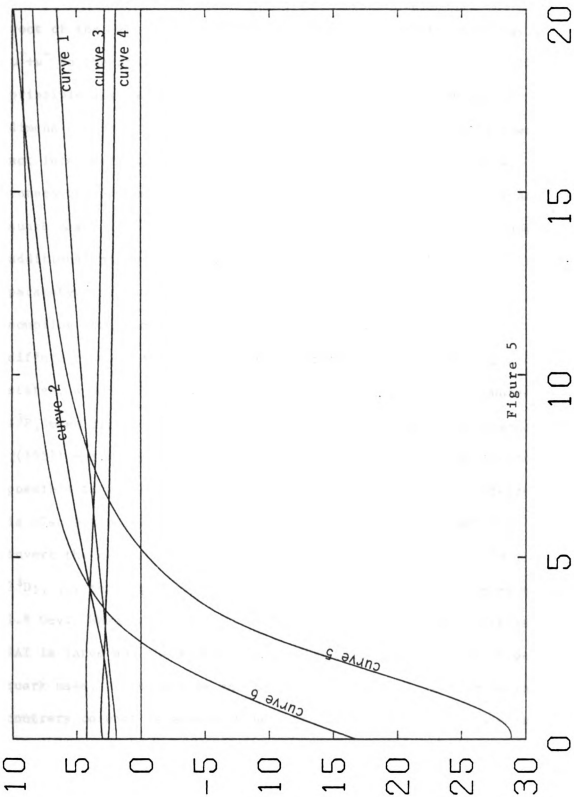

10
5
0
-5
-10
-15
-20
-25
-30
0
5
10
15
20
curve 1
curve 2
curve 3
curve 4
curve 5
curve 6

Figure 5

the ratio $0.587/(\lambda_2-\lambda_1)$ where λ_2 and λ_1 are the computed masses of the 2^3S_1 and 1^3S_1 states in this model. The curve labeled 3 is the cube root of the ratio of the observed leptonic decay rate of the $\psi(3095) \rightarrow \mu^+ + \mu^-$ to the computed leptonic decay rate. Curves 1,3, and 4 all in principle give a value for k, the conversion factor between my dimensionless units and energy (Gev) units. Curves 3 and 4 however do not intersect, so only curves 1 and 3 are used to determine k. The intersection point of these two curves (1 and 3) determines k and the quark mass. The other curves in figure 5 can be used to fit any additional parameters that remain in the potential. The only other parameter currently available is the ratio of the scalar to vector coupling strength, RAT. By requiring that the $1^3P_2 - 1^3P_0$ mass difference come out to be the value observed for the $\chi(3551)$ and $\chi(3414)$ states, a value RAT(1) is determined. By requiring the computed $1^3P_2 - 1^3P_1$ mass difference to come out to the observed mass difference between $\chi(3551) - \chi(3507)$, a different value RAT(2) is found. It is not possible to obtain a fit for the $\chi(3551) - \psi(3095)$ mass difference. It is also not possible to obtain a satisfactory value of RAT that will invert the calculated level structure, i.e. go from 1^3S_1, 1^3D_1, 2^3S_1, 2^3D_1, ... to 1^3S_1, 2^3S_1, 1^3D_2, ... and still have a quark mass of about 1.8 Gev. When trial runs to obtain this result are made, the value of RAT is large and the k,m determinations produce very large values of the quark mass. This implies an extremely relativistic configuration, contrary to what is generally believed to be the case for the charmonium system. From the discussion in [29] it seems that their β/r potential is mainly responsible for the correct level structure 1^3S_1, 2^3S_1, 1^3D_1, ... It was hoped to include this potential form in our model. The

problem preventing this usage to this point is an inability to perform numerically certain singular integrals. These singular integrals arise from the need to use a momentum representation for the β/r potential. When this is done, integrals with terms like $\ln(p-p')$ are encountered. Attempts were made to check out the integration routine for the coulomb potential in momentum space by computing numerically the expectation values of $1/r$, $1/r^2$, etc. whose values are known [17]. The results were inconsistent with those given in [17]. One therefore concluded that this method of dealing with singular integrals could not be trusted and the attempt was temporarily dropped. I had hoped to look into this matter again to see if this potential form can be included. I would like to look at this problem again to see if these difficulties can be overcome and an α/r form can be included in the potential structure.

In figures 6, 7, and 8 are shown graphs of the wavefunctions for the first three singlet S states, singlet P_1 states, and singlet D_2 states. In [35] a procedure for the solution of the Schroedinger equation with a linear potential is given. The method developed in [35] can be used for values of L, the orbital angular momentum, different from zero. Reference [33] discusses these wavefunctions, and graphs of some of these wavefunctions are presented. Comparison of figures 7 and 8 with figures 2 and 3 of [33] shows that there is good agreement between our m=0, RAT=0 wavefunctions and their results.

In the system of constraints used here to determine m, k, and RAT, the values of m determined were found to fall within the interval $3.0 \leq m \leq 7.0$. Figures 9, 10, 11, and 12 show most of the same quantities that were plotted in figure 5 for this restricted mass range and for values of RAT=0.00, -0.16, -0.275, and -0.33. In figure 5,

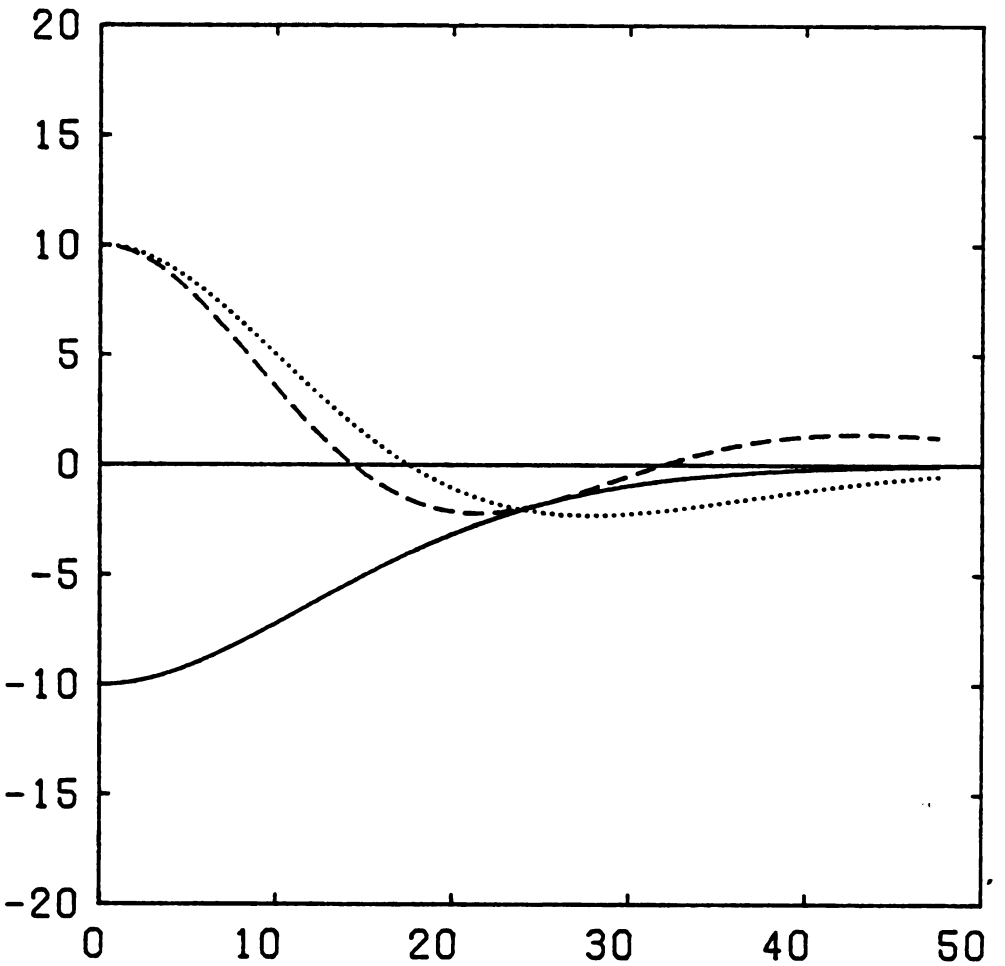


Figure 6. Plot of the Wavefunctions of the first three states of the $^1S_0$ system, for m = RAT = 0.

_____ 1^1S_0 ..... 2^1S_0 ----- 3^1S_0

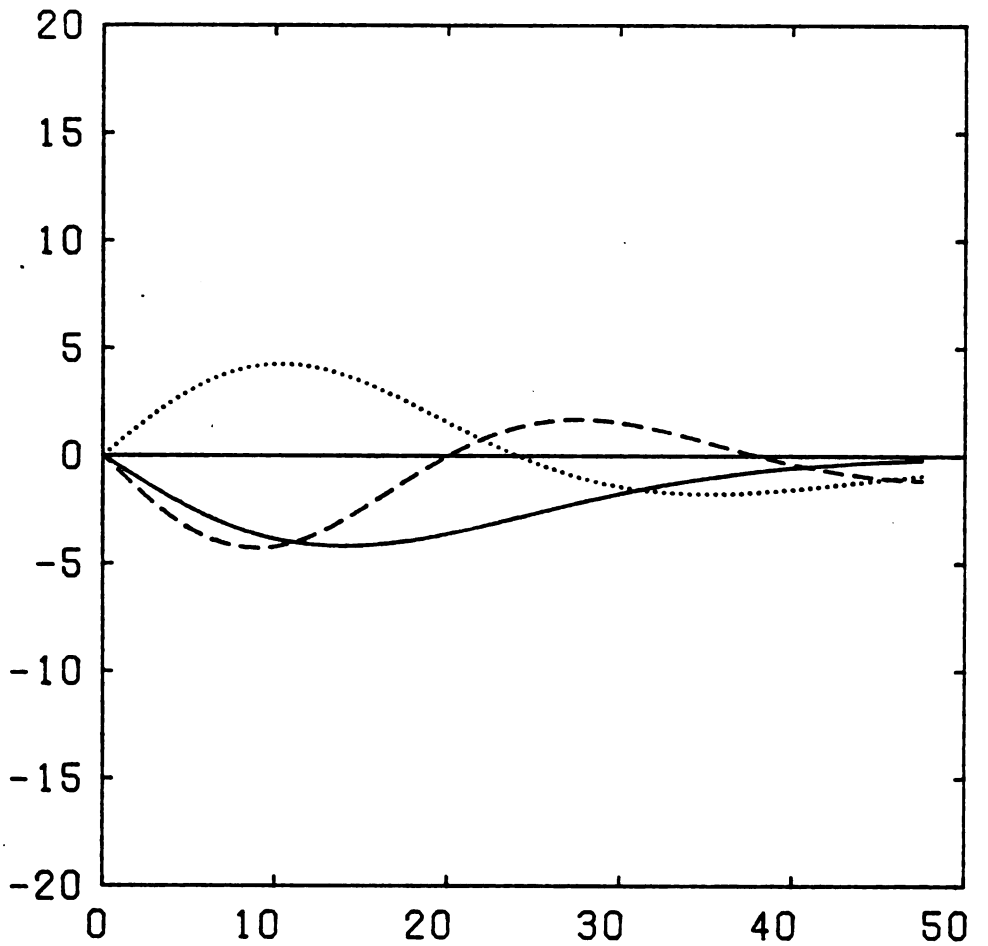


Figure 7. Plot of the Wavefunctions of the first three states of the $^1P_1$ system, for m = RAT = 0.

_____ 1^1P_1 ..... 2^1P_1 ----- 3^1P_1

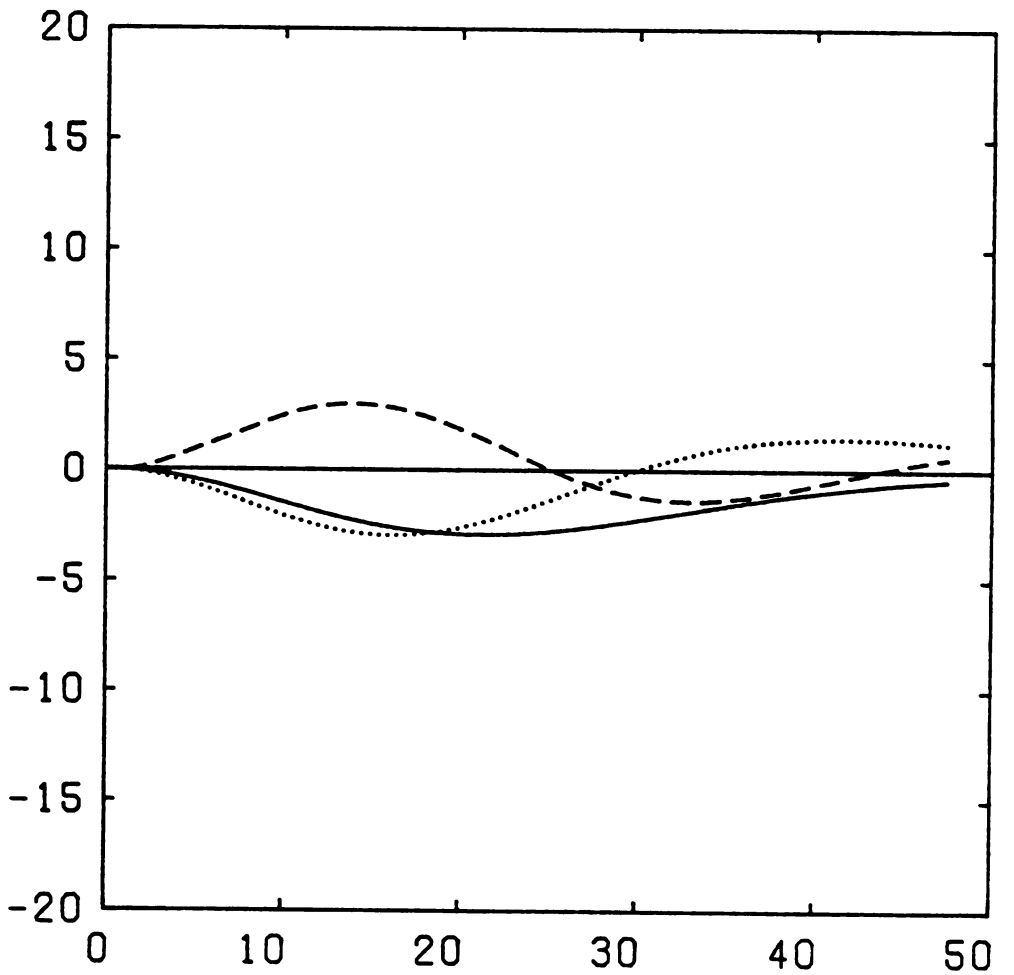


Figure 8. Plot of the Wavefunctions of the first three states of the $^1D_2$ system, for m = RAT = 0.

_____ 1^1D_2 ..... 2^1D_2 ----- 3^1D_2

Figure 9. Determination of Potential Parameters, RAT = 0.00

_____ k = the observed ψ'- ψ mass difference (0.587 Gev) divided by the mass difference computed by the program (in dimensionless units)

····· The difference in the computed masses for the 1^3P_2 - 1^3S_1 states

----- k = the cube root of A/B where

A = upper limit of observed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

B = computed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

-·-·- k = the same as for ----- curve but with A = the lower limit

-·-·-·- The ratio (C-D)/C where

C = ($^3P_2$, $^3P_1$) observed mass difference

D = ($^3P_2$, $^3P_1$) computed mass difference

-·-·-·-·- The same as for -·-·-·- curve but for the $^3P_0$ and $^3P_2$ states

The vertical scale for _____, ·····, -----, and -·-·- is Gev.

The vertical scale for -·-·-·- and -·-·-·-·- is in dimensionless units.

The horizontal scale is in dimensionless mass units.

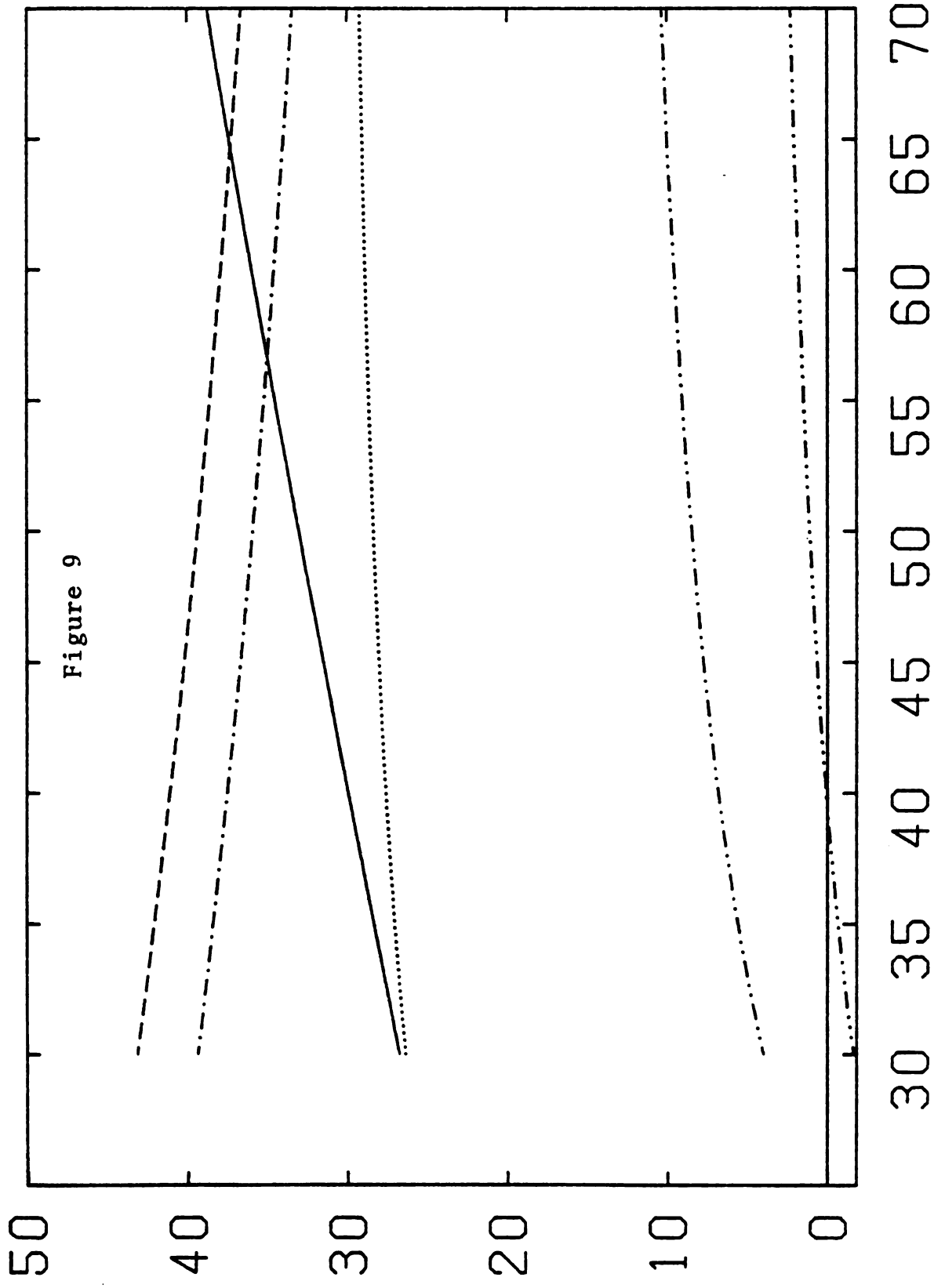


Figure 9

Figure 10. Determination of Potential Parameters, RAT = -0.16

_____ k = the observed ψ' - ψ mass difference (0.587 Gev) divided by the mass difference computed by the program (in dimensionless units)

····· The difference in the computed masses for the 1^3P_2 - 1^3S_1 states

----- k = the cube root of A/B where

A = upper limit of observed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

B = computed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

-·-·- k = the same as for ----- curve but with A = the lower limit

-·-·-·- The ratio (C-D)/C where

C = ($^3P_2$, $^3P_1$) observed mass difference

D = ($^3P_2$, $^3P_1$) computed mass difference

-·-·-·-·- The same as for -·-·-·- curve but for the $^3P_0$ and $^3P_2$ states

The vertical scale for _____, ·····, -----, and -·-·- is Gev.

The vertical scale for -·-·-·- and -·-·-·-·- is in dimensionless units.

The horizontal scale is in dimensionless mass units.

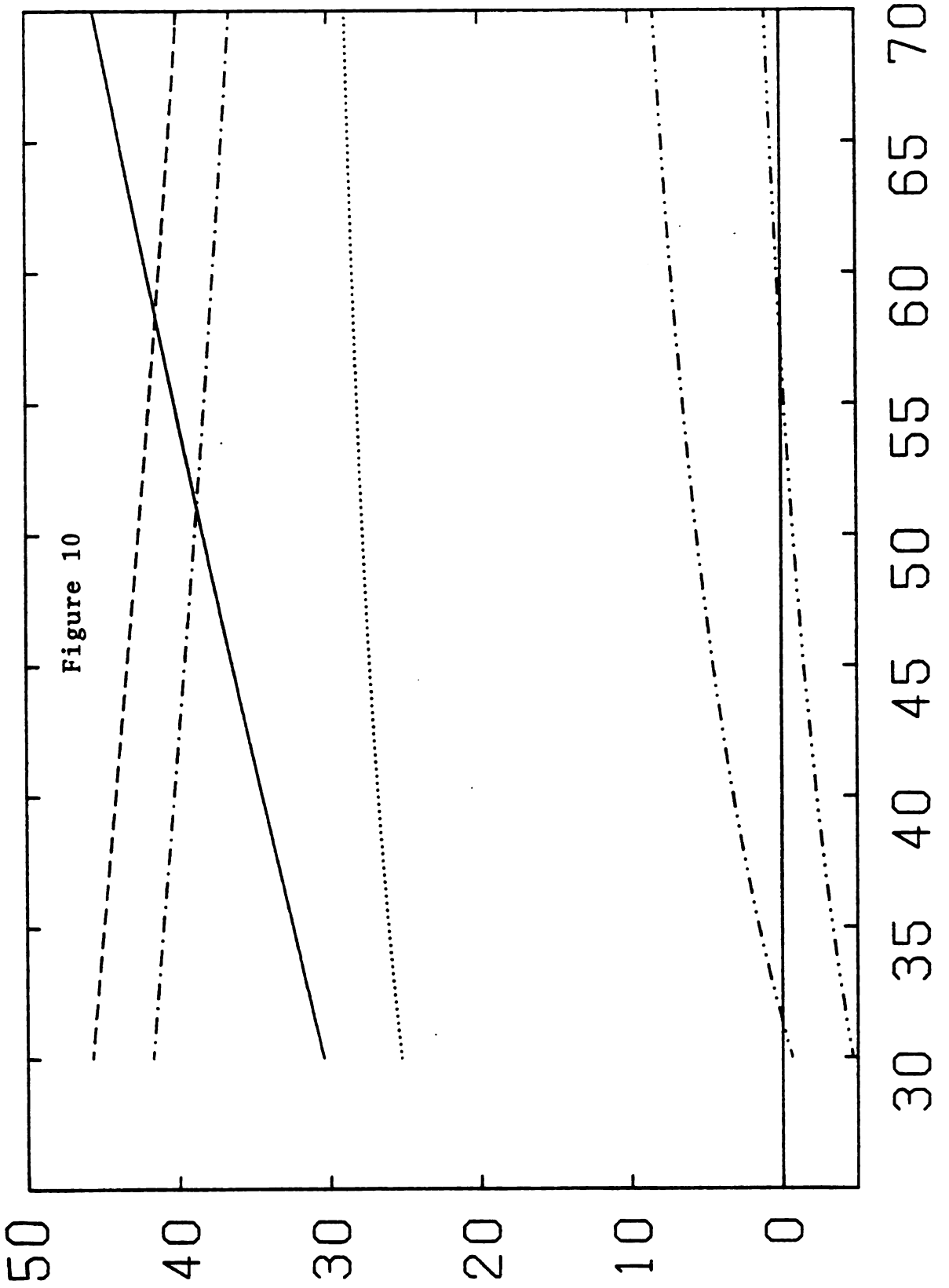


Figure 10

Figure 11. Determination of Potential Parameters, RAT = -0.275

_____ k = the observed ψ' - ψ mass difference (0.587 Gev) divided by the mass difference computed by the program (in dimensionless units)

····· The difference in the computed masses for the 1^3P_2 - 1^3S_1 states

----- k = the cube root of A/B where

A = upper limit of observed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

B = computed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

-·-·- k = the same as for ----- curve but with A = the lower limit

-·-·-·- The ratio (C-D)/C where

C = ($^3P_2$, $^3P_1$) observed mass difference

D = ($^3P_2$, $^3P_1$) computed mass difference

-·-·-·-·- The same as for -·-·-·- curve but for the $^3P_0$ and $^3P_2$ states

The vertical scale for _____, ·····, -----, and -·-·- is Gev.

The vertical scale for -·-·-·- and -·-·-·-·- is in dimensionless units.

The horizontal scale is in dimensionless mass units.

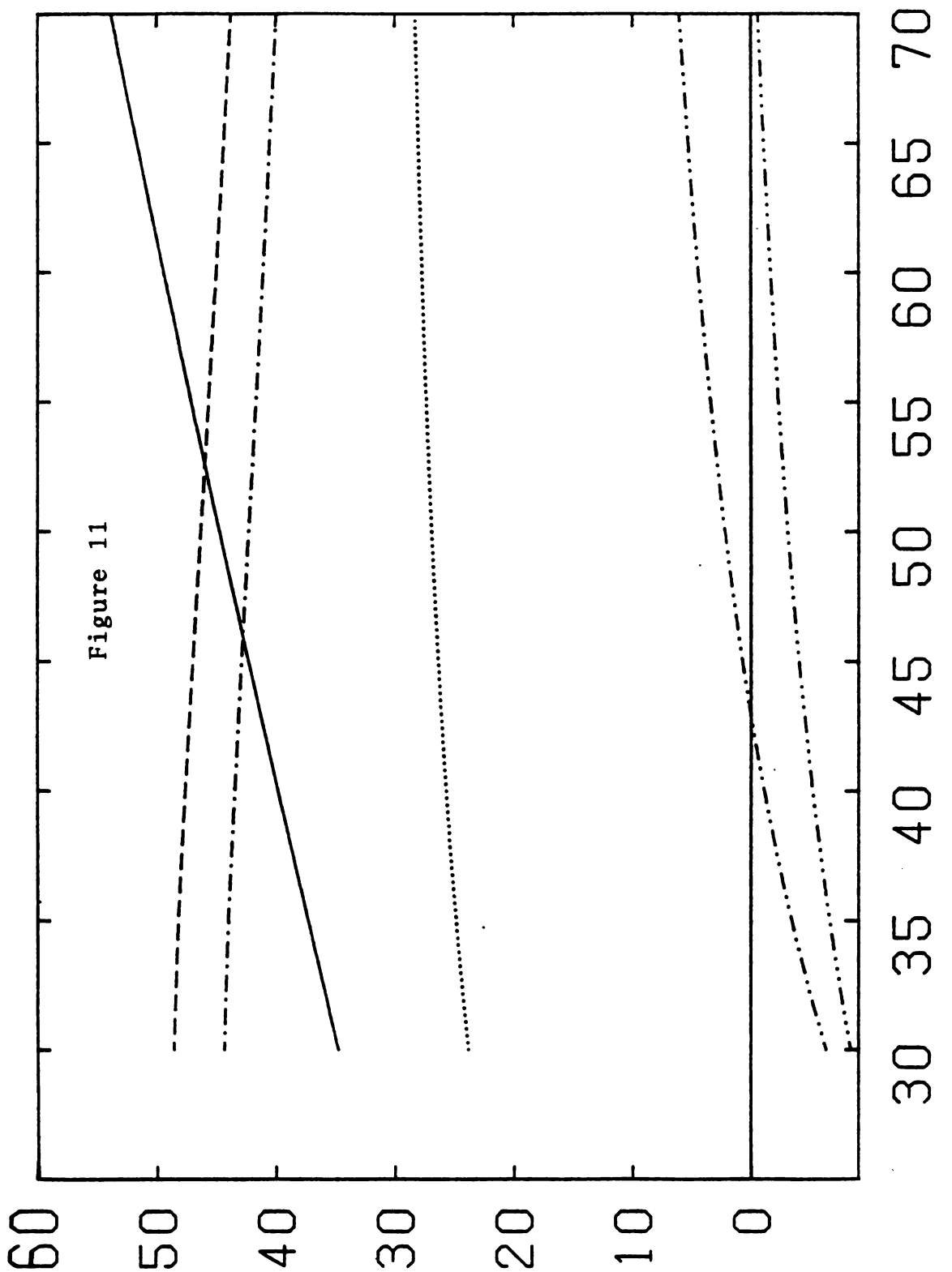


Figure 11

Figure 12. Determination of Potential Parameters, RAT = -0.33

_____ k = the observed ψ'- ψ mass difference (0.587 Gev) divided by the mass difference computed by the program (in dimensionless units)

····· The difference in the computed masses for the 1^3P_2 - 1^3S_1 states

----- k = the cube root of A/B where

A = upper limit of observed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

B = computed decay width for $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$

-·-·- k = the same as for ----- curve but with A = the lower limit

-·-·-·- The ratio (C-D)/C where

C = ($^3P_2$, $^3P_1$) observed mass difference

D = ($^3P_2$, $^3P_1$) computed mass difference

-·-·-·-·- The same as for -·-·-·- curve but for the $^3P_0$ and $^3P_2$ states

The vertical scale for _____, ·····, -----, and -·-·- is Gev.

The vertical scale for -·-·-·- and -·-·-·-·- is in dimensionless units.

The horizontal scale is in dimensionless mass units.

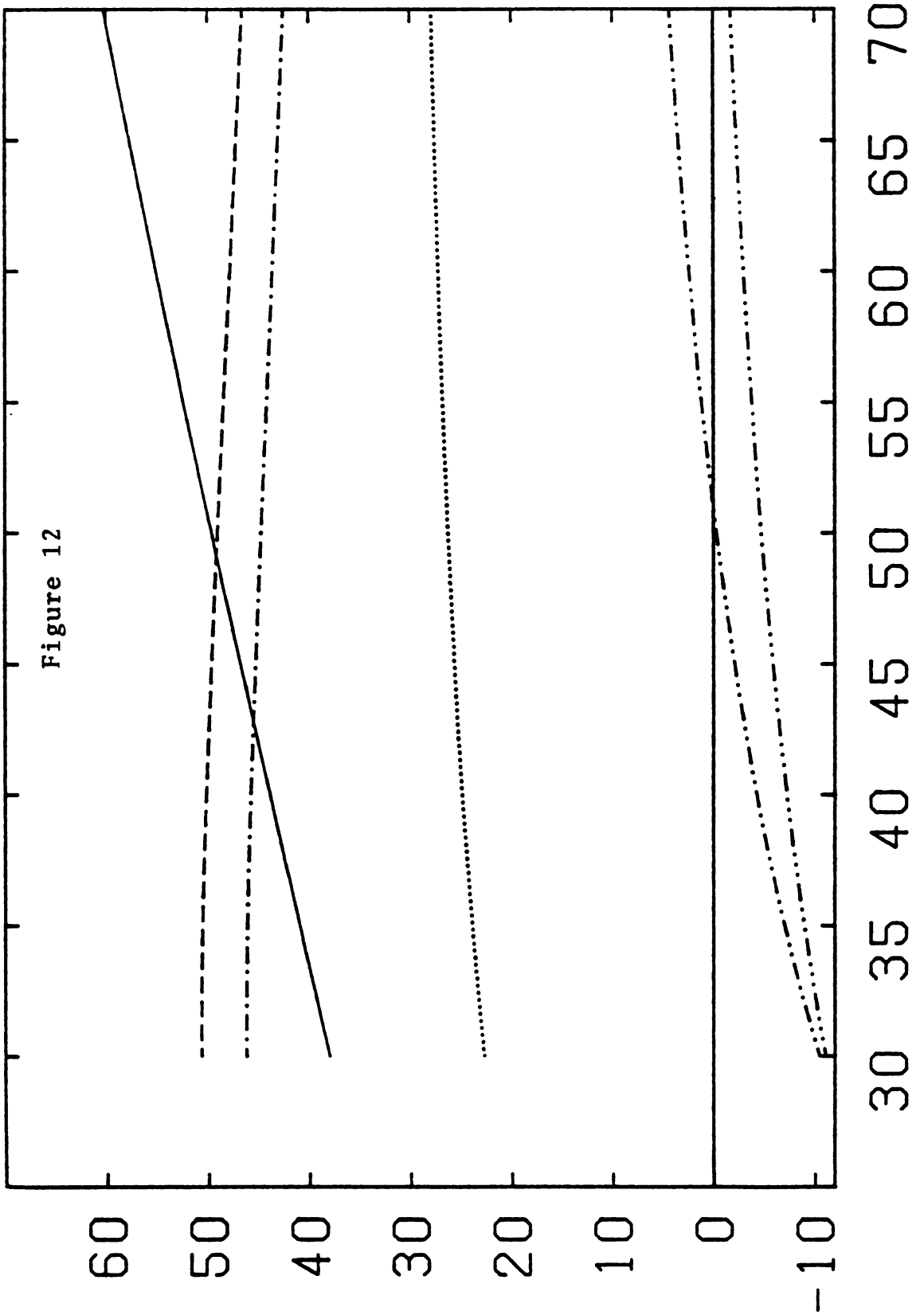


Figure 12

curves 3 and 4 give the values of k as determined by the ψ and ψ' leptonic decay widths. Note that for the ψ leptonic decay the value of 4.8 kev was used in figure 5. In figures 9-12 the experimental electronic decay width limits of 3.8 and 5.0 were used to determine the range of quark masses and k values permitted by this experimental uncertainty in the value of $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$. One should observe that as RAT goes from 0 to -0.33, the quark mass ranges determined by these plots decrease from about 6.5 to about 4.0 in my dimensionless units.

The calculated masses of several of the charmonium states are given in TABLE 14 for quark masses between 3.0 and 7.0, and RAT = 0.00, -0.16, and -0.33. Note that in some cases the $^3P_0$ and $^3P_1$ masses are below the 1^3S_1 state.

The expressions used in this work to calculate the radiative decay widths are given in appendix II.4. In appendices II.2 and II.3 the current matrix element and certain polarization quantities are calculated. These polarization quantities were used in the Schoonschip program that was used to reduce the current matrix element and to calculate the radiative decay widths. An example of one of these Schoonschip programs is given in appendix II.5, namely that used for the transition $2^3S_1 \rightarrow 1^3P_0$. The Schoonschip programs performing the reductions for the other transitions are similar.

The radiative decay widths were calculated by three alternative methods. All methods yielded approximately the same results. In the simplest case (case 1) the program made use of the input parameters m, RAT, the $\psi'(3685) - \psi(3095)$ mass difference, and $\psi(3095)$'s mass to calculate the masses of the various other charmonium states. These calculated masses were then used in the calculation of the matrix

TABLE 14

VARIATION OF THE CHARMONIUM MASSES AS A FUNCTION OF THE QUARK MASS PARAMETER AND THE SCALAR VECTOR COUPLING CONSTANT RATIO

STATE	RAT=0.00	RAT=-0.16	RAT=-0.33	OBSERVED
		M=3		
1(1S0)	2.947	2.861	2.690	2.980
1(3P0)	3.233	3.142	2.952	3.415
1(1P1)	3.298	3.229	3.098	NOT SEEN
1(3P1)	3.307	3.249	3.132	3.507
1(1D2)	3.647	3.622	3.569	NOT SEEN
1(3D2)	3.658	3.631	3.595	NOT SEEN
1(3P2)	3.472	3.494	3.546	3.551
2(1S0)	3.574	3.512	3.376	3.592
		M=4		
1(1S0)	2.991	2.929	2.807	2.980
1(3P0)	3.279	3.209	3.067	3.415
1(1P1)	3.318	3.269	3.169	NOT SEEN
1(3P1)	3.329	3.285	3.195	3.507
1(1D2)	3.654	3.632	3.585	NOT SEEN
1(3D2)	3.664	3.645	3.607	NOT SEEN
1(3P2)	3.450	3.467	3.506	3.551
2(1S0)	3.602	3.553	3.450	3.592
		M=5		
1(1S0)	3.017	2.970	2.807	2.980
1(3P0)	3.305	3.250	3.139	3.415
1(1P1)	3.333	3.294	3.215	NOT SEEN
1(3P1)	3.342	3.307	3.236	3.507
1(1D2)	3.659	3.639	3.598	NOT SEEN
1(3D2)	3.668	3.650	3.616	NOT SEEN
1(3P2)	3.435	3.449	3.478	3.551
2(1S0)	3.619	3.580	3.498	3.592

TABLE 14(CONT'D)

STATE	RAT=0.00	RAT=-0.16	RAT=-0.33	OBSERVED
		M=6		
1(1S0)	3.034	2.996	2.921	2.980
1(3P0)	3.323	3.277	3.187	3.415
1(1P1)	3.344	3.311	3.247	NOT SEEN
1(3P1)	3.351	3.322	3.264	3.507
1(1D2)	3.662	3.645	3.609	NOT SEEN
1(3D2)	3.669	3.654	3.625	NOT SEEN
1(3P2)	3.425	3.436	3.459	3.551
2(1S0)	3.630	3.598	3.531	3.592
		M=7		
1(1S0)	3.047	3.014	2.953	2.980
1(3P0)	3.335	3.297	3.221	3.415
1(1P1)	3.351	3.324	3.271	NOT SEEN
1(3P1)	3.358	3.333	3.284	3.507
1(1D2)	3.665	3.650	3.619	NOT SEEN
1(3D2)	3.671	3.658	3.632	NOT SEEN
1(3P2)	3.418	3.427	3.446	3.551
2(1S0)	3.639	3.612	3.555	3.592

element and phase space factors for the radiative decay widths. A second method made use of the calculated masses in the matrix element for the radiative decays but used the physical masses for the phase space factor (case 2). In the last case (case 3) one made use of the physical masses to calculate the matrix element and phase space factors. The transition widths were calculated by these three different methods for RAT = -0.16 and RAT = -0.33. For RAT = -0.16, case 1 yielded results quite different from cases 2 and 3, which generally produced quite similar results. In case 1 some of the transition widths came out negative. The reason for this is that for some values of the quark mass the 1^3P_1 and 1^3P_0 levels fall below that of the 1^3S_1 level. When these values are used in the phase space factor of the expression for the transition width the negative value results. The results to be displayed graphically below are all computed by using the physical masses in the phase space factor but not in the radial matrix element integral (i.e. case 2). The results of the radiative decay width calculations are again shown graphically as a function of the quark mass as this mass varies from 0 to 20 in dimensionless units.

Figures 13, 14, and 15 show the results of the $2^3S_1 \rightarrow 1^3P_0 + \gamma$, $2^3S_1 \rightarrow 1^3P_1 + \gamma$, and $2^3S_1 \rightarrow 1^3P_2 + \gamma$. The two horizontal lines at 1.07×10^{-5} and at 2.15×10^{-5} (Gev) correspond to the lower limit given in the particle data compilation [40], and to the value given in [41] for the $\psi'(3685) \rightarrow {}^3P_0 + \gamma$ decay width, respectively. In figure 13 the value of RAT = -0.16 was used while in figures 14 and 15 this value was set at -.275 and at -0.33 respectively. The calculations were also done at RAT = -0.16 and at RAT = -0.33 for cases 1 and 3 and the results differed from those of case 2 in some instances. For RAT = -0.275 only

Figure 13. Transition Widths for $2^3S_1 \rightarrow {}^3P + \gamma$ states, RAT = -0.16

_____ $2^3S_1 \rightarrow {}^3P_0 + \gamma$ decay width as a function of the quark mass m

····· $2^3S_1 \rightarrow {}^3P_1 + \gamma$ decay width as a function of the quark mass m

----- $2^3S_1 \rightarrow {}^3P_2 + \gamma$ decay width as a function of the quark mass m

-·-·- Upper limit of $2^3S_1 \rightarrow {}^3P_0 + \gamma$ experimentally determined decay width as given in [41] = 21.4 kev

-·-·-·- Lower limit of $2^3S_1 \rightarrow {}^3P_0 + \gamma$ given in particle data table [40] = 10.7 kev

The vertical scale is in units of 10 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

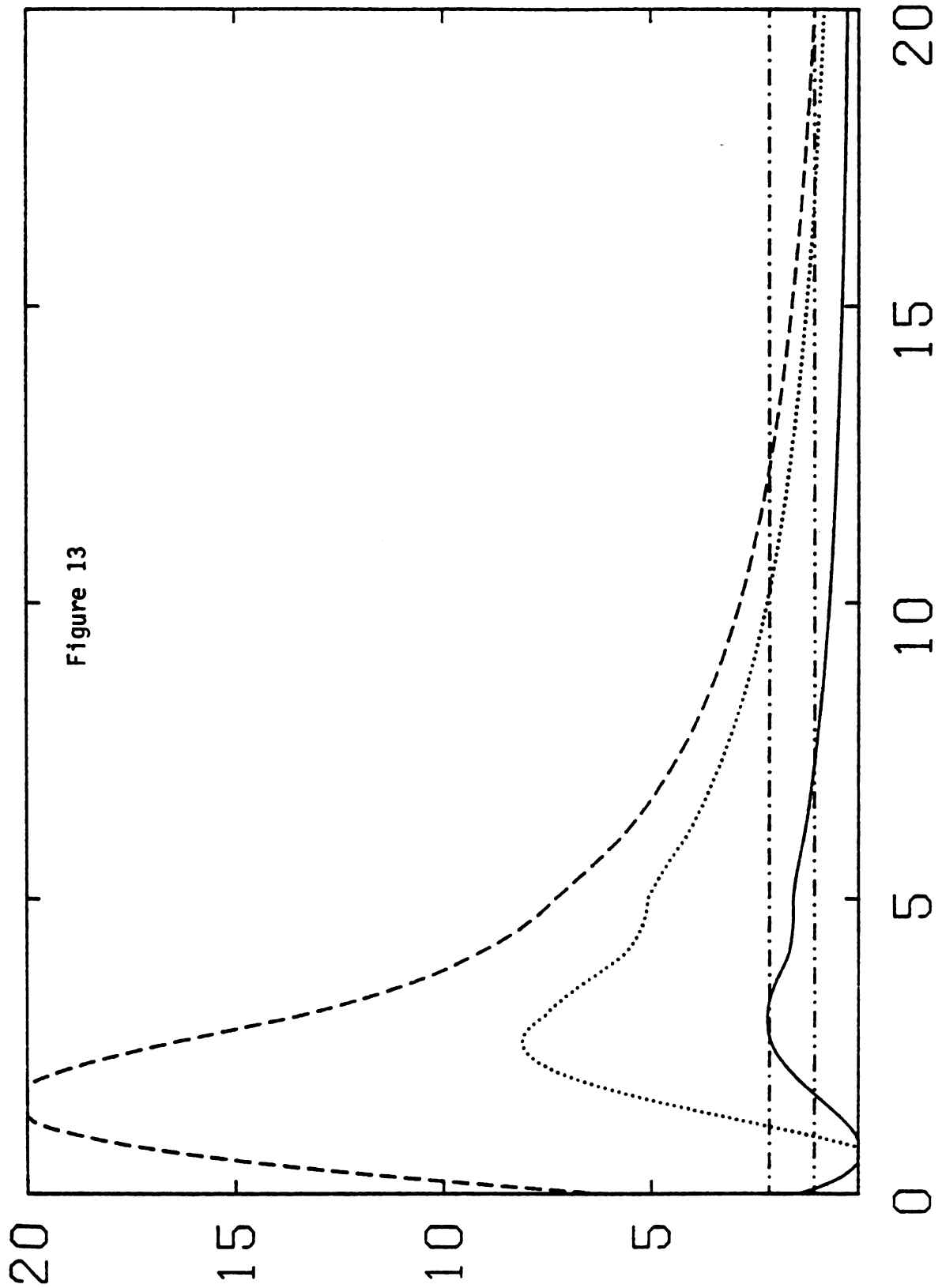


Figure 13

Figure 14. Transition Widths for $2^3S_1 \rightarrow {}^3P + \gamma$ states, RAT = -0.275

_____ $2^3S_1 \rightarrow {}^3P_0 + \gamma$ decay width as a function of the quark mass m

····· $2^3S_1 \rightarrow {}^3P_1 + \gamma$ decay width as a function of the quark mass m

----- $2^3S_1 \rightarrow {}^3P_2 + \gamma$ decay width as a function of the quark mass m

-·-·- Upper limit of $2^3S_1 \rightarrow {}^3P_0 + \gamma$ experimentally determined decay width as given in [41] = 21.4 kev

-·-·-·- Lower limit of $2^3S_1 \rightarrow {}^3P_0 + \gamma$ given in particle data table [40] = 10.7 kev

The vertical scale is in units of 10 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

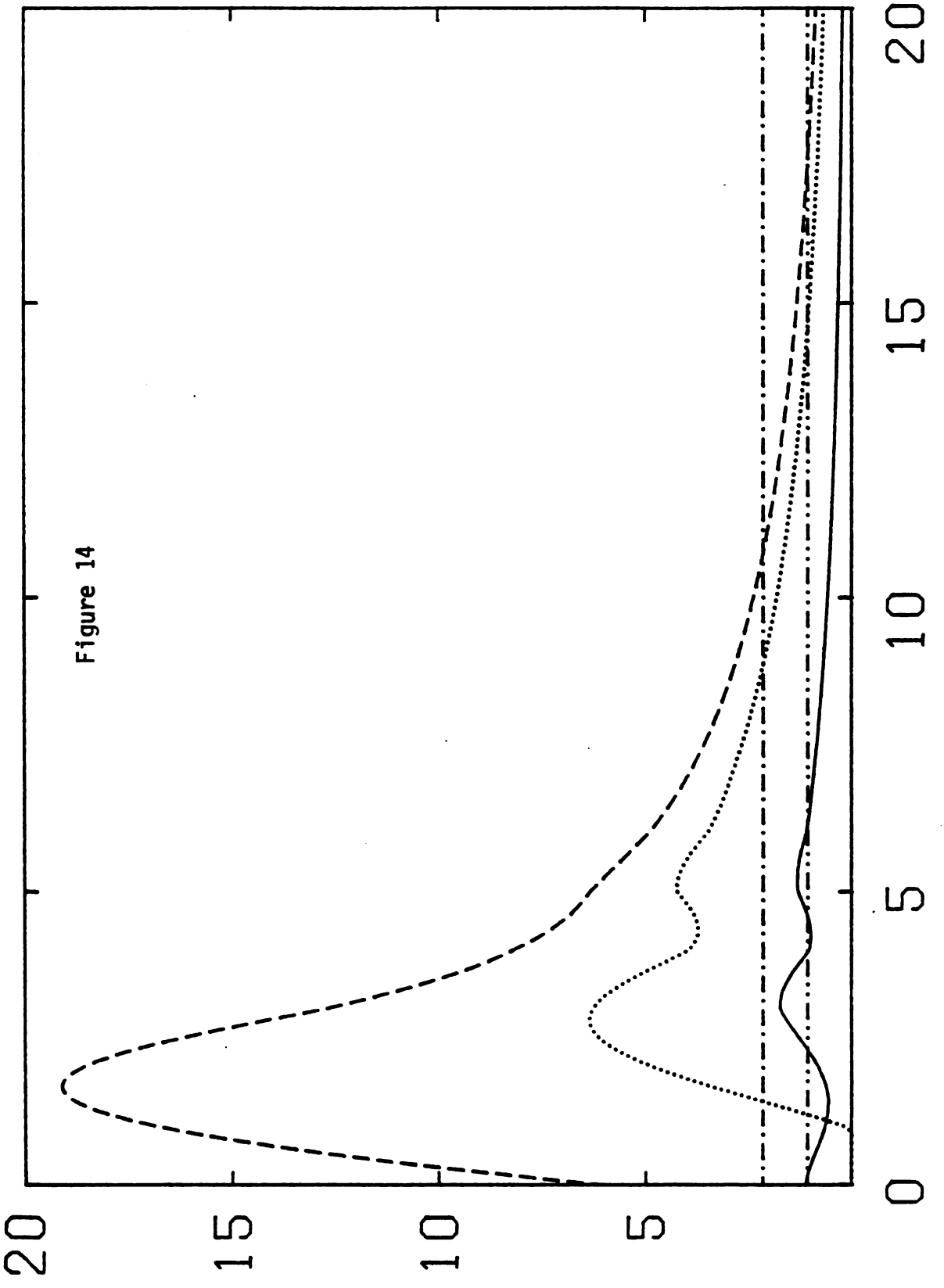


Figure 14

Figure 15. Transition Widths for $2^3S_1 \rightarrow {}^3P + \gamma$ states, RAT = -0.33

_____ $2^3S_1 \rightarrow {}^3P_0 + \gamma$ decay width as a function of the quark mass m

••••• $2^3S_1 \rightarrow {}^3P_1 + \gamma$ decay width as a function of the quark mass m

----- $2^3S_1 \rightarrow {}^3P_2 + \gamma$ decay width as a function of the quark mass m

-•-•- Upper limit of $2^3S_1 \rightarrow {}^3P_0 + \gamma$ experimentally determined decay width as given in [41] = 21.4 kev

-••••••- Lower limit of $2^3S_1 \rightarrow {}^3P_0 + \gamma$ given in particle data table [40] = 10.7 kev

The vertical scale is in units of 10 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

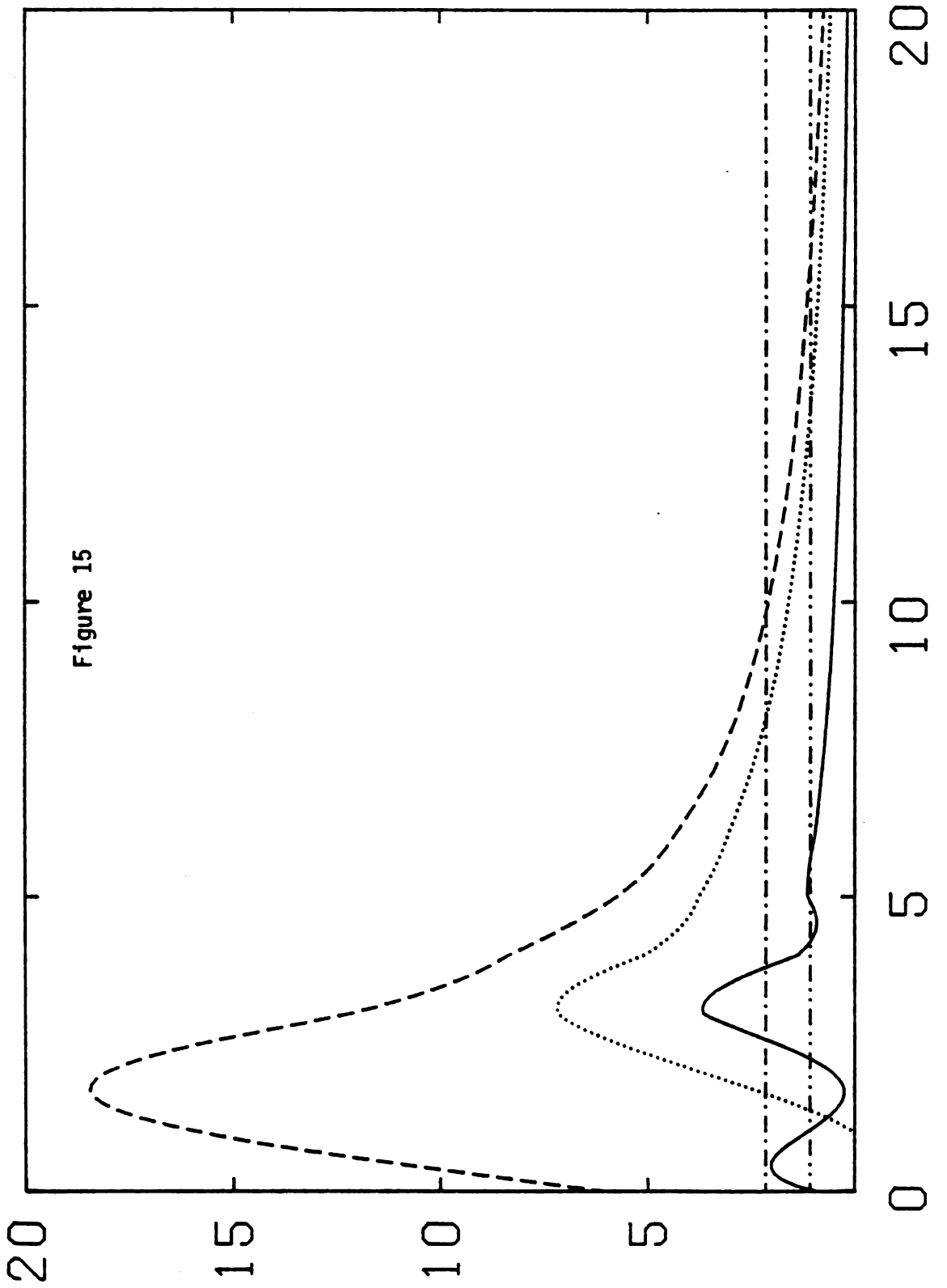


Figure 15

case 2 was done. For RAT = -0.16, the biggest difference between cases 1, 2, and 3 occurs in the $\psi'(3685) \rightarrow {}^3P_1 + \gamma$ decay width, the low mass peak being about twice as big in cases 2 and 3. For cases 2 and 3 the difference in the calculated results for the decays leading to the ${}^3P_1$ and ${}^3P_2$ states is small. The decay width for the $\psi'(3685) \rightarrow {}^3P_0 + \gamma$ process in case 1 is also larger than that computed by case 2, the peak being about 30 kev higher in case 1, and for $m \approx 5$ the decay width is about 30 kev. This decay width, for case 3, is about 10 kev higher than that computed for case 2 in the low mass region ($m \approx 5$) where the fit is desired. For RAT = -0.33 the calculations in cases 2 and 3 produce identical results for the ${}^3P_2$ process. The case 3 results for the other two decay widths are different from those obtained from case 2. The ${}^3P_1$ width in case 2 is higher for $m < 5$, but for $m > 5$ cases 2 and 3 give the same results. In the ${}^3P_0$ case the oscillations present for $m < 5$ are more pronounced in case 2 than in case 3. For $m > 5$ case 3 yields a larger decay width than does case 2. At $m = 5$ the difference in the decay widths leading to the ${}^3P_0$ state calculated by means of cases 2 and 3 is about 5 kev, and as m increases this difference drops off to a couple of kev at $m = 20$. For case 1 the computed decay widths are much larger, the ${}^3P_1$ peak being about 2.5 times that of case 2. In case 1 the oscillatory behavior present for $m < 5$ in the ${}^3P_0$ process is much larger in amplitude, the peak being at 125 kev instead of the 75 kev value of case 2. The $\psi'(3685) \rightarrow {}^3P_2 + \gamma$ decay widths are also larger in case 1, being for $m \approx 5$ about 80-90 kev compared to the case 2 value of about 50 kev.

Figures 16, 17, and 18 show the results of the decay width calculations for the transitions $1{}^3P_1 \rightarrow 1{}^3S_1 + \gamma$ and $1{}^3P_2 \rightarrow 1{}^3S_1 + \gamma$ for

Figure 16. Transition Widths for 1^3P_2 and $1^3P_1 \rightarrow 1^3S_1 + \gamma$ state, RAT = -0.16

_____ $1^3P_1 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

••••• $1^3P_2 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

----- This curve corresponds to a decay width of 100 kev

-•-•- This curve corresponds to a decay width of 300 kev

The vertical scale is in units of 100 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

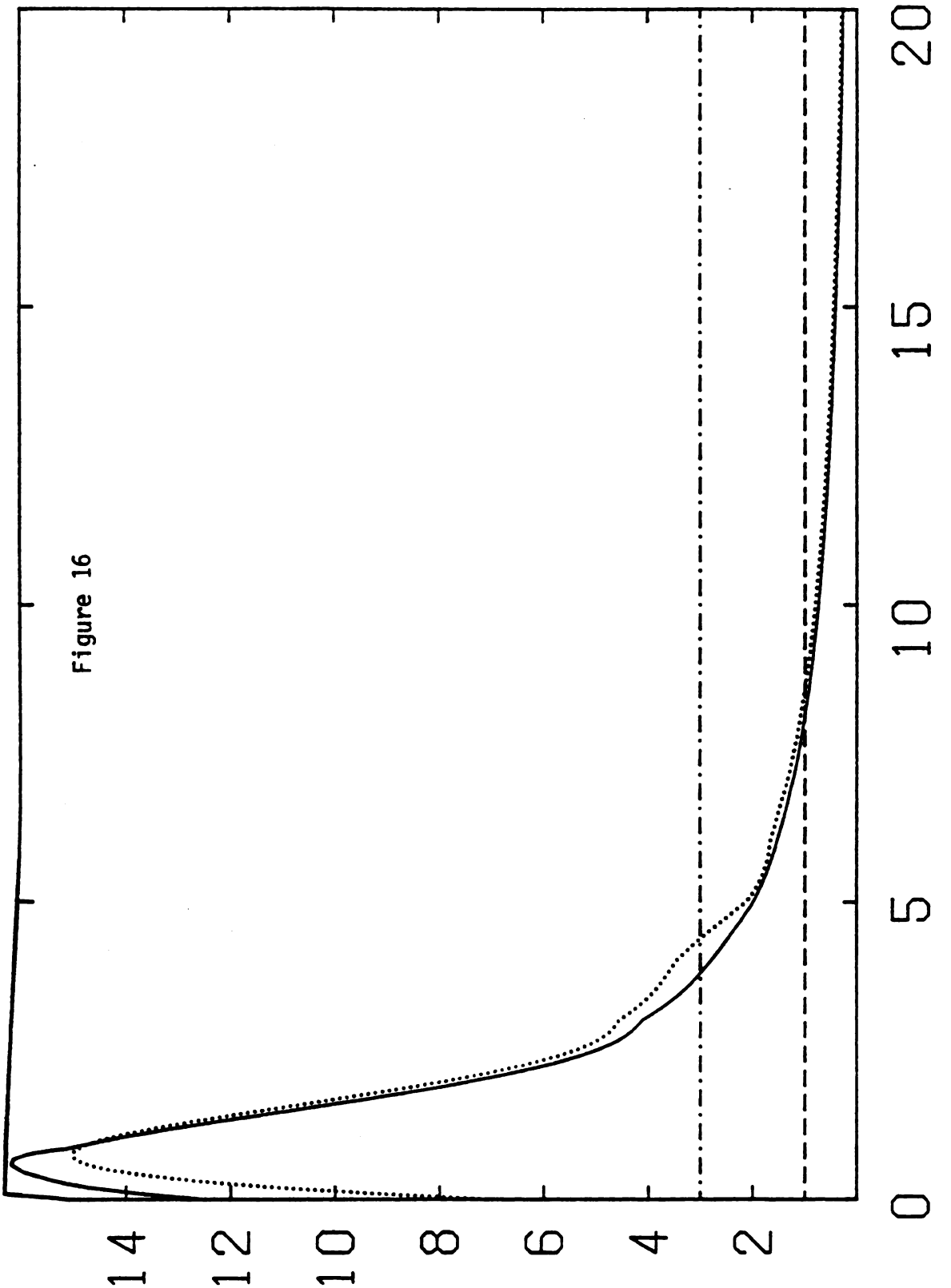


Figure 16

Figure 17. Transition Widths for 1^3P_2 and $1^3P_1 \rightarrow 1^3S_1 + \gamma$ state, RAT = -0.275

_____ $1^3P_1 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

••••• $1^3P_2 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

----- This curve corresponds to a decay width of 100 kev

-•-•- This curve corresponds to a decay width of 300 kev

The vertical scale is in units of 100 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

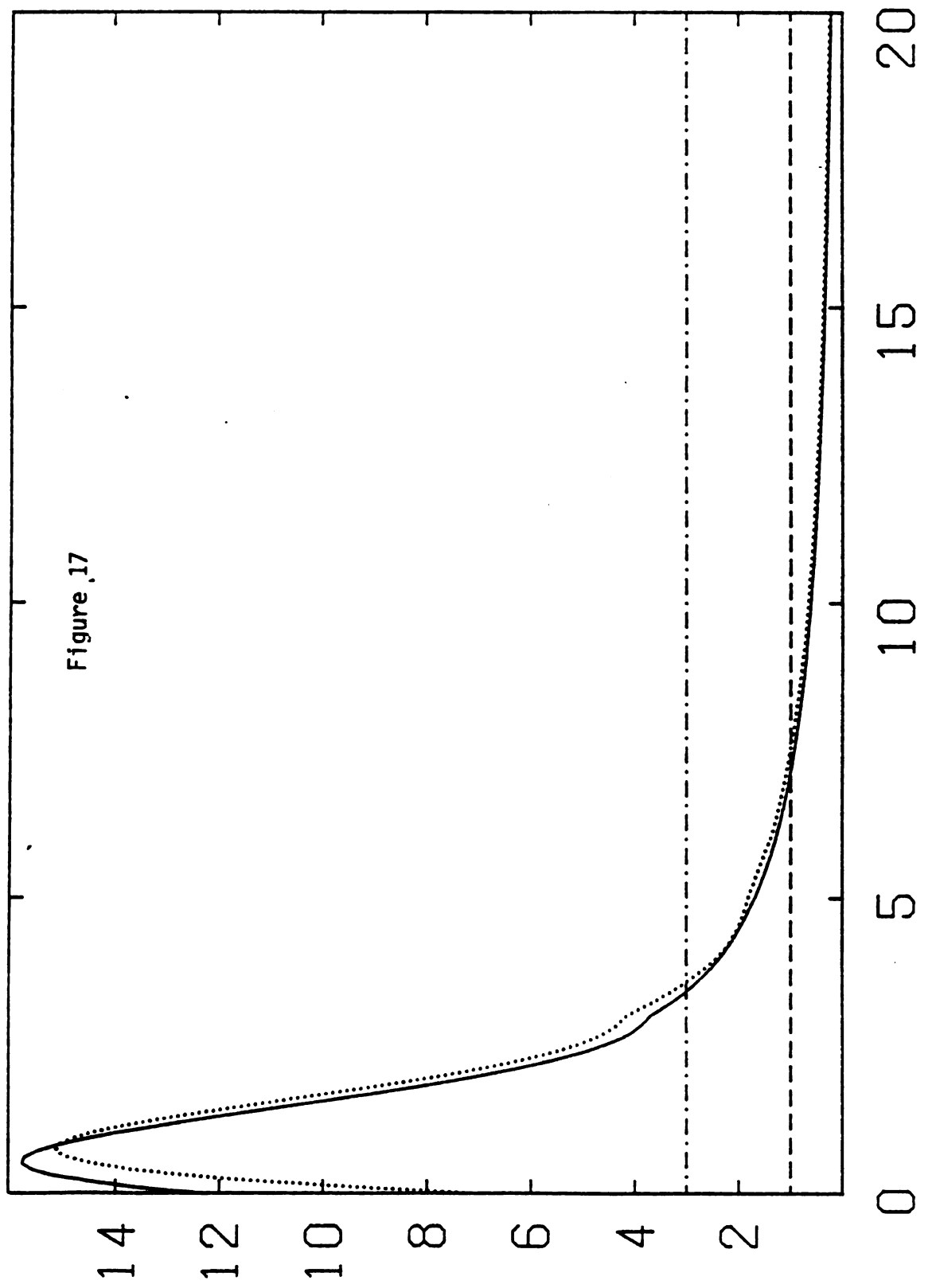


Figure 17

Figure 18. Transition Widths for 1^3P_2 and $1^3P_1 \rightarrow 1^3S_1 + \gamma$ state, RAT = $^{-0.33}$

_____ $1^3P_1 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

••••• $1^3P_2 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

----- This curve corresponds to a decay width of 100 kev

-•-•- This curve corresponds to a decay width of 300 kev

The vertical scale is in units of 100 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

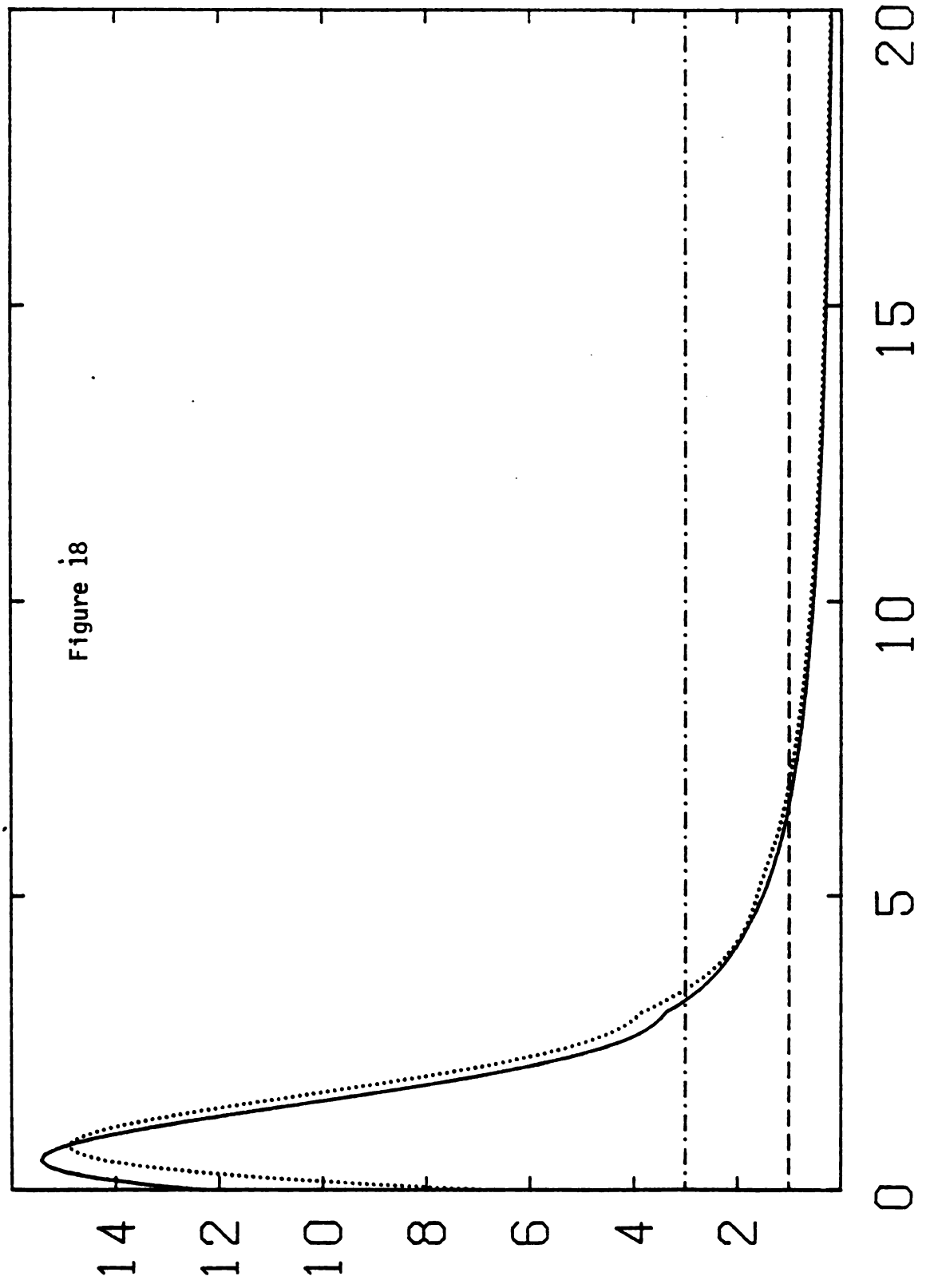


Figure 18

values of RAT = -0.16, -0.275, and -0.33. The horizontal lines in these graphs correspond to 10.0×10^{-5} and 30.0×10^{-5} (Gev) respectively. The decay widths for the transitions $1^3P_2 \rightarrow 1^3S_1 + \gamma$ and $1^3P_1 \rightarrow 1^3S_1 + \gamma$ are essentially undetermined experimentally. The results quoted in [41] are $4.9 \pm 3.3 \times 10^{-4}$ Gev and $< 7 \times 10^{-4}$ Gev respectively. The horizontal lines in figures 16-18 therefore serve only as a guide to the order of magnitude of the calculations and their variation with the quark mass. The decay width computations for the $^3P_2 \rightarrow \psi(3095) + \gamma$ and $^3P_1 \rightarrow \psi(3095) + \gamma$ were also done for cases 1 and 3 with RAT = -0.16 and -0.33. As in the $\psi' \rightarrow {}^3P_J + \gamma$ cases, the results of the three different cases did not always agree. For RAT = -0.16 the results of the case 3 calculations produced larger decay widths than did those of case 1, for $m < 5$, and approximately the same results for $m > 5$. In case 1 there is no characteristic low mass hump in the $^3P_1$ decay. For large masses ($m \rightarrow 20$), the $^3P_2$ and $^3P_1$ decay widths are about equal in each case. There are only slight differences in the results calculated in cases 2 and 3. The differences occur mainly in the region $3 < m < 5$ and amount to no more than 10-20 kev out of the computed values of 300 kev for this mass range. For RAT = -0.33 and $m < 5$, case 1 gives larger values for the $^3P_2 \rightarrow \psi(3095) + \gamma$ decay width than does case 3. For $m > 5$, the decay widths are not appreciably different. For $m < 5$, the $^3P_1$ decay width in case 1 is negative because of the inversion of the $^3P_1$ and 1^3S_1 levels. For case 1 the decay width of $^3P_1 \rightarrow 1^3S_1 + \gamma$ is always less than 100 kev. The difference in the computed decay width in cases 2 and 3 is negligible.

Figures 19, 20, and 21 show the results of the calculation for $1^3P_0 \rightarrow 1^3S_1 + \gamma$, for RAT = -0.16, -0.275, and -0.33. The horizontal

Figure 19. Transition Width for $1^3P_0 \rightarrow 1^3S_1 + \gamma$, RAT = -0.16

_____ Reference line = 100 kev

····· $1^3P_0 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

The vertical scale is in units of 100 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

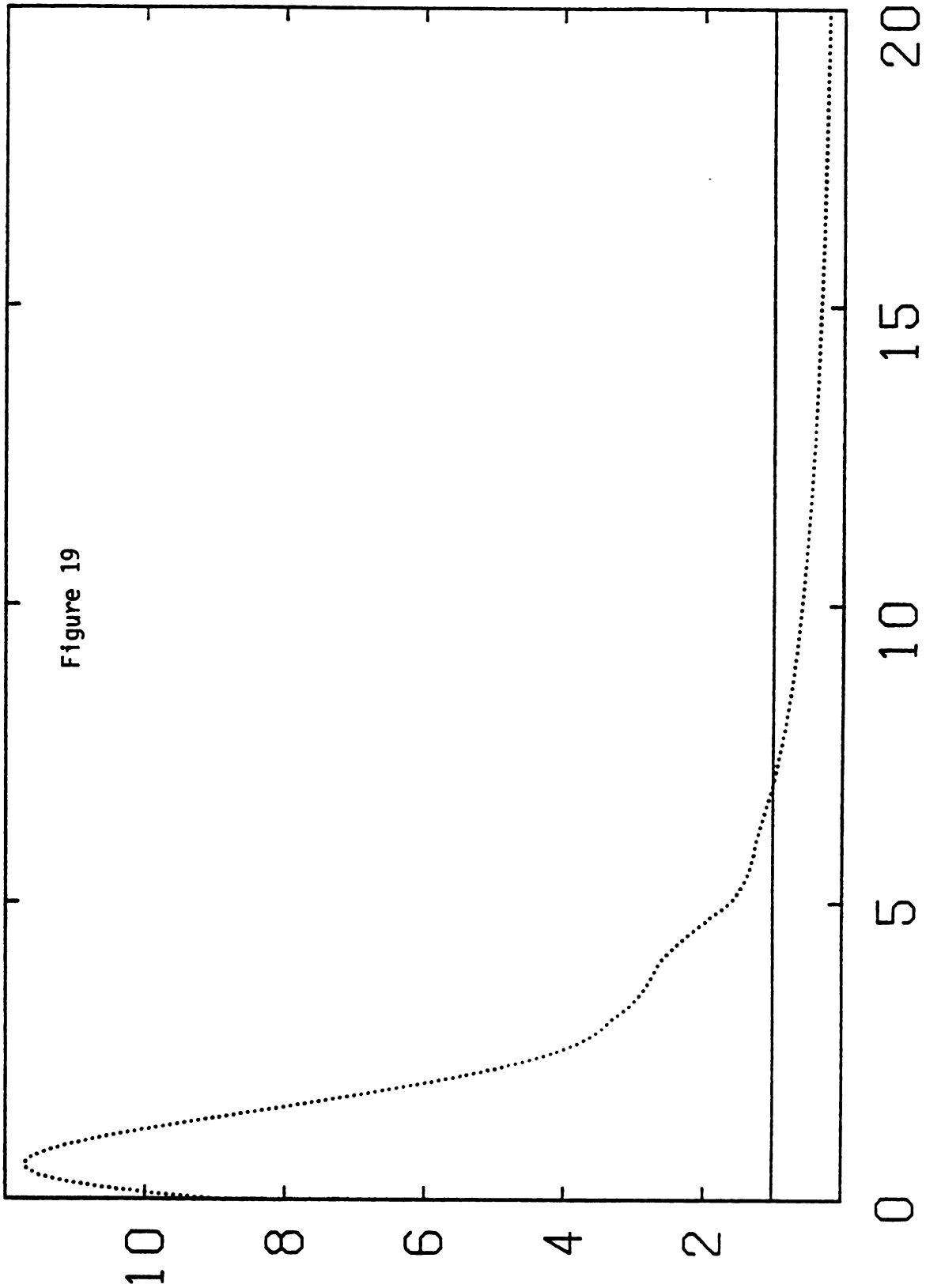


Figure 19

Figure 20. Transition Width for $1^3P_0 \rightarrow 1^3S_1 + \gamma$, RAT = -0.275

_____ Reference line = 100 kev

····· $1^3P_0 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

The vertical scale is in units of 100 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

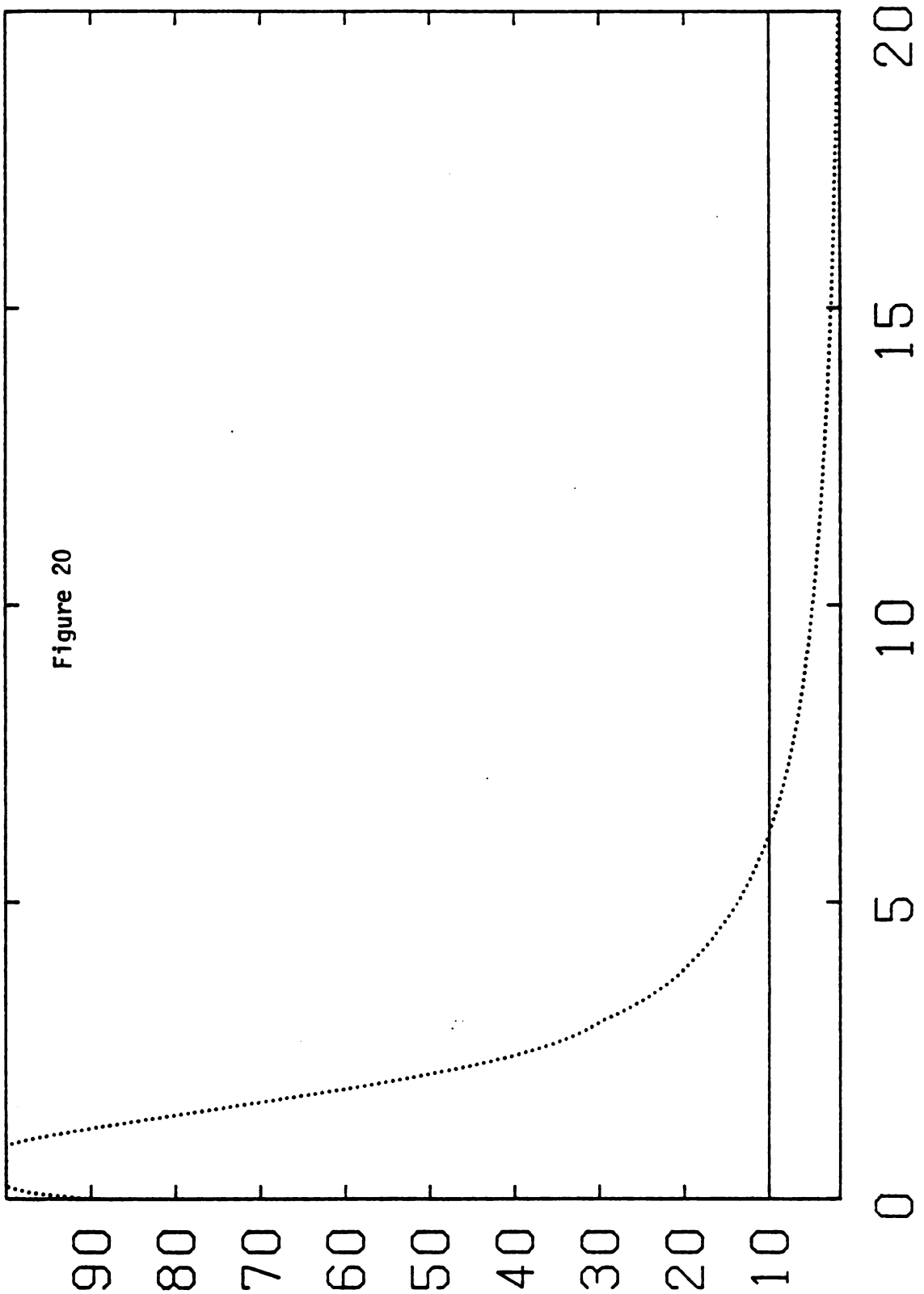


Figure 20

Figure 21. Transition Width for $1^3P_0 \rightarrow 1^3S_1 + \gamma$, RAT = -0.33

_____ Reference line = 100 kev

····· $1^3P_0 \rightarrow 1^3S_1 + \gamma$ decay width as a function of the quark mass m

The vertical scale is in units of 100 kev.

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

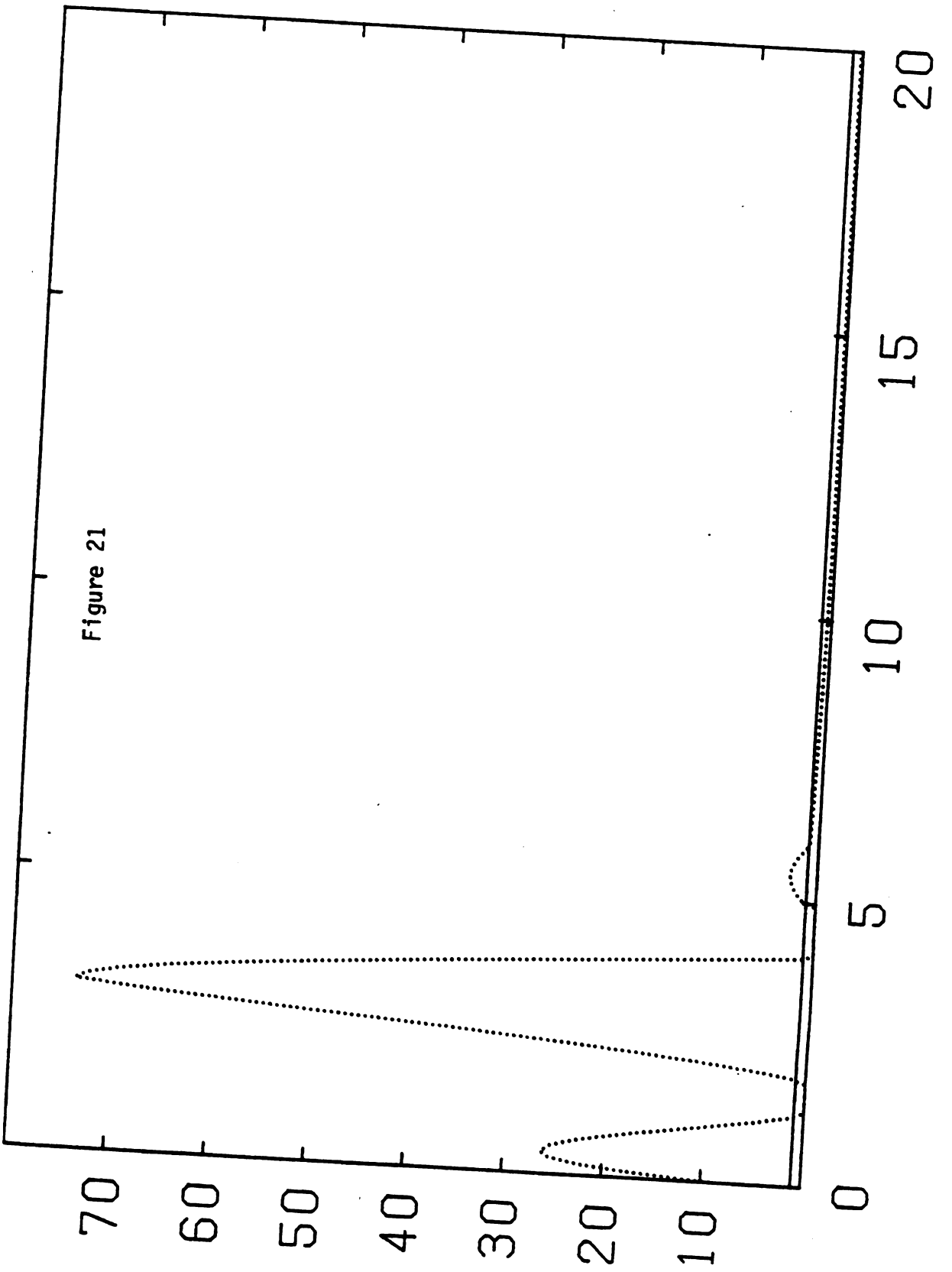


Figure 21

line here corresponds to a decay width of 10.0×10^{-5} Gev. The experimentally determined width given in [41] is $9.7 \pm 3.8 \times 10^{-5}$ Gev. In the $^3P_0 \rightarrow 1^3S_1 + \gamma$ decay the computations were done at RAT = -0.16 and at -0.33 for all three cases. For RAT = -0.16, cases 1 and 3 produce quite different results. The main effect comes from the fact that the $^3P_0$ level lies below the 1^3S_1 level for low masses. (See Table 14.) This results in the appearance of unphysical (negative) values for the decay widths for $m < 5$. The low mass hump seen in most of the decay plots is not present in the case 1 results. The graphs of the case 2 and 3 computations are very similar, the only difference being a slight bump in case 2 at $m \approx 4.5$ of about 30-50 kev. For RAT = -0.33 the calculations produce quite different results for $m < 5$. The results of case 3 look very much like the curves for $^3P_2 \rightarrow \psi + \gamma$ in the previous plots. Case 1 for $m < 4.5$ is negative and large. The decay width for $m \approx 5$ is about 100 kev and for $m > 6$ is small (<30 kev). The results of case 2 are shown in figure 21. The behaviors in cases 1 and 2 for $m < 4.5$ are almost mirror images of each other. For case 3 the computed decay widths at $m = 4.0$ and $m = 5.0$ are 166 kev and 120 kev respectively.

In figures 22, 23, and 24 are shown the computed decay widths for the transitions $2^3S_1 \rightarrow 1^1S_0 + \gamma$, $1^3S_1 \rightarrow 1^1S_0 + \gamma$, and $2^3S_1 \rightarrow 2^1S_0 + \gamma$. The horizontal lines in these figures are the experimental limits quoted in [41] for the decay $\psi'(3685) \rightarrow \eta_c'(3592) + \gamma$, namely processes are still not determined very precisely, but their values, even within their given errors, still fall within the .43 to 2.795 range. The graphs in figure 22 were computed with a value of RAT = -0.16; the plots in figures 23 and 24 were done with values of RAT = -0.275 and -0.33

Figure 22. Magnetic Dipole Transition Widths, RAT = -0.16

_____ $\log_{10}$ of $2^3S_1 \rightarrow 1^1S_0 + \gamma$ decay width as a function of the quark mass m

••••• $\log_{10}$ of $1^3S_1 \rightarrow 1^1S_0 + \gamma$ decay width as a function of the quark mass m

----- $\log_{10}$ of $2^3S_1 \rightarrow 2^1S_0 + \gamma$ decay width as a function of the quark mass m

-•-•-• $\log_{10}$ (4.300×10^{-7})

-••••••- $\log_{10}$ (2.795×10^{-6})

The vertical scale is the $\log_{10}$ (computed decay width in Gev).

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

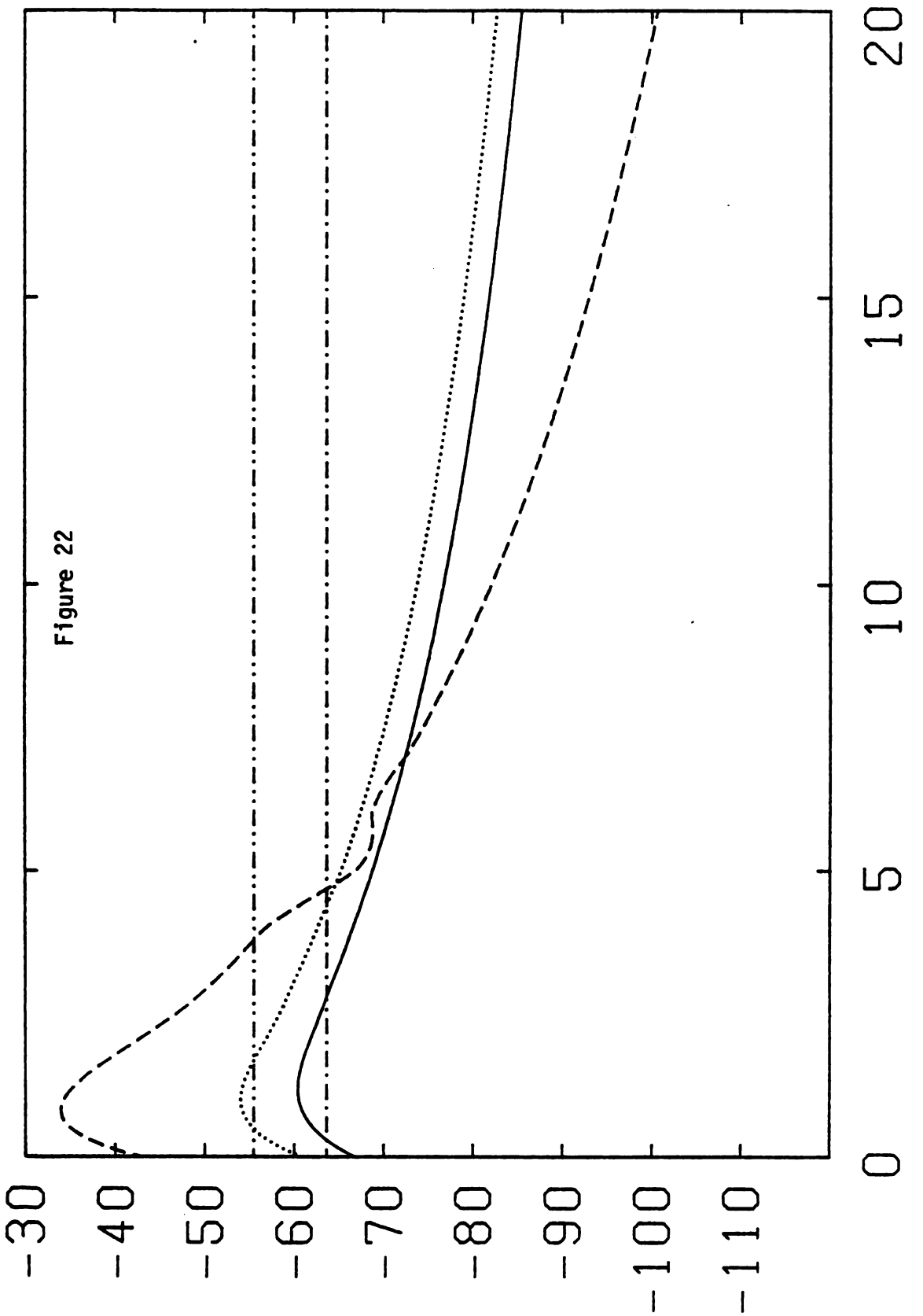


Figure 22

Figure 23. Magnetic Dipole Transition Widths, RAT = -0.275

_____ $\log_{10}$ of $2^3S_1 \rightarrow 1^1S_0 + \gamma$ decay width as a function of the quark mass m

••••• $\log_{10}$ of $1^3S_1 \rightarrow 1^1S_0 + \gamma$ decay width as a function of the quark mass m

----- $\log_{10}$ of $2^3S_1 \rightarrow 2^1S_0 + \gamma$ decay width as a function of the quark mass m

-•-•- $\log_{10}$ (4.300×10^{-7})

-••••••- $\log_{10}$ (2.795×10^{-6})

The vertical scale is the $\log_{10}$ (computed decay width in Gev).

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

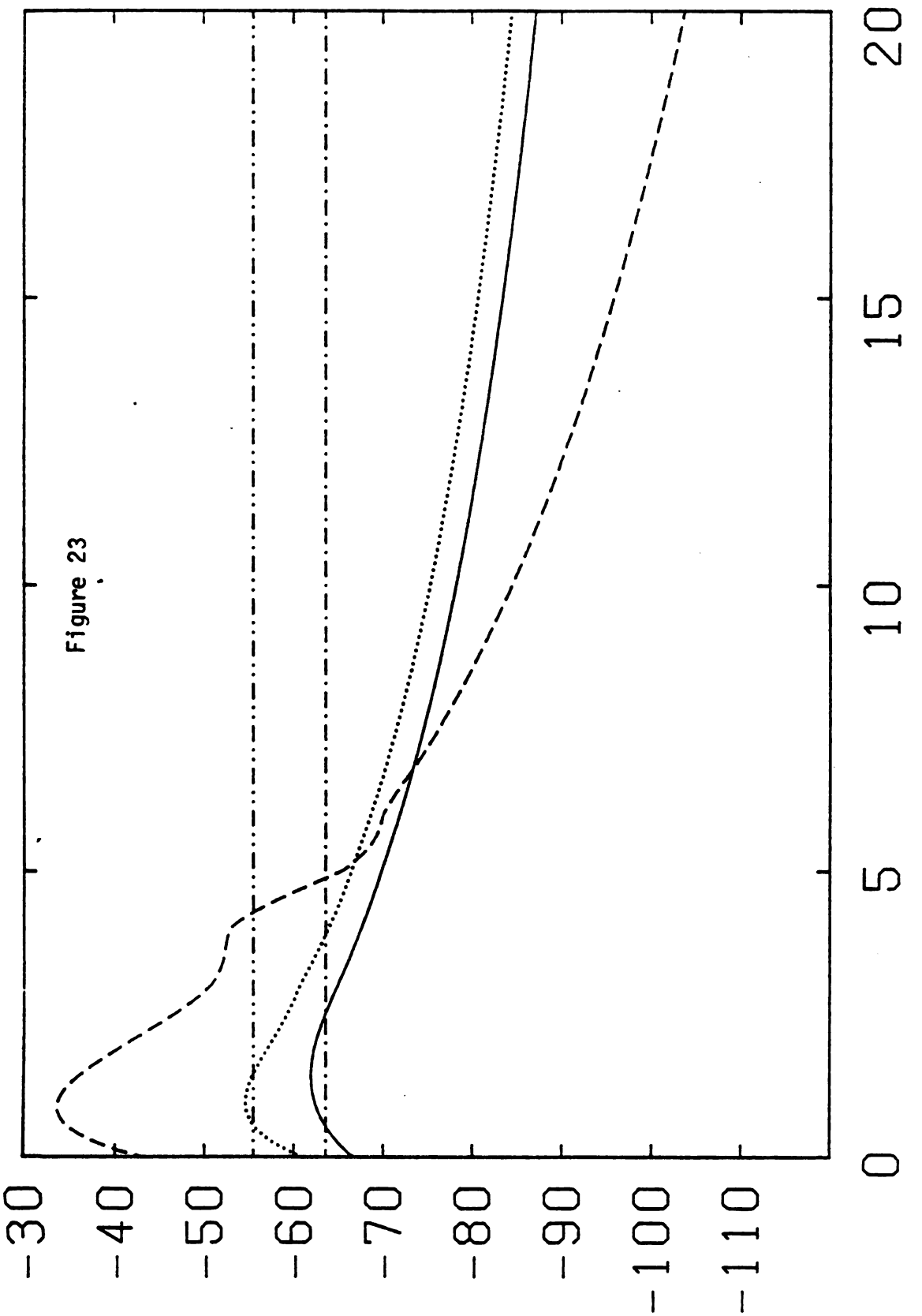
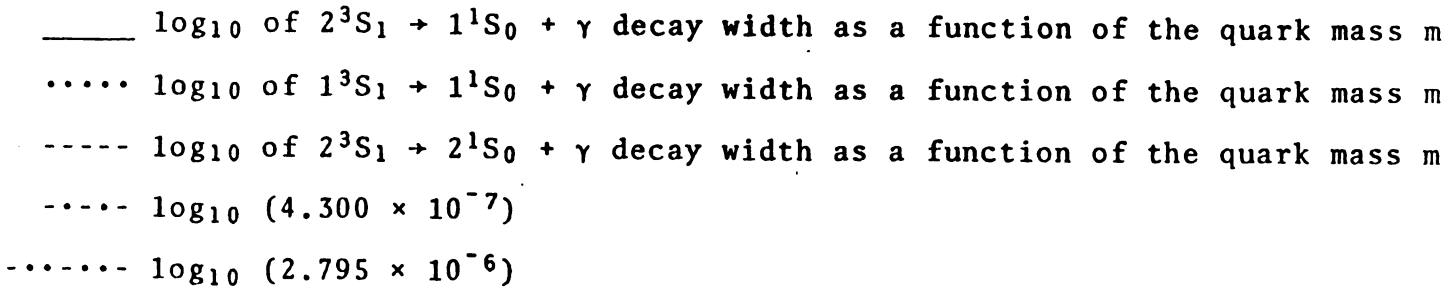


Figure 23

Figure 24. Magnetic Dipole Transition Widths, RAT = -0.33



The vertical scale is the $\log_{10}$ (computed decay width in Gev).

The horizontal scale is the dimensionless quark mass, m=0.00 to m=20.00.

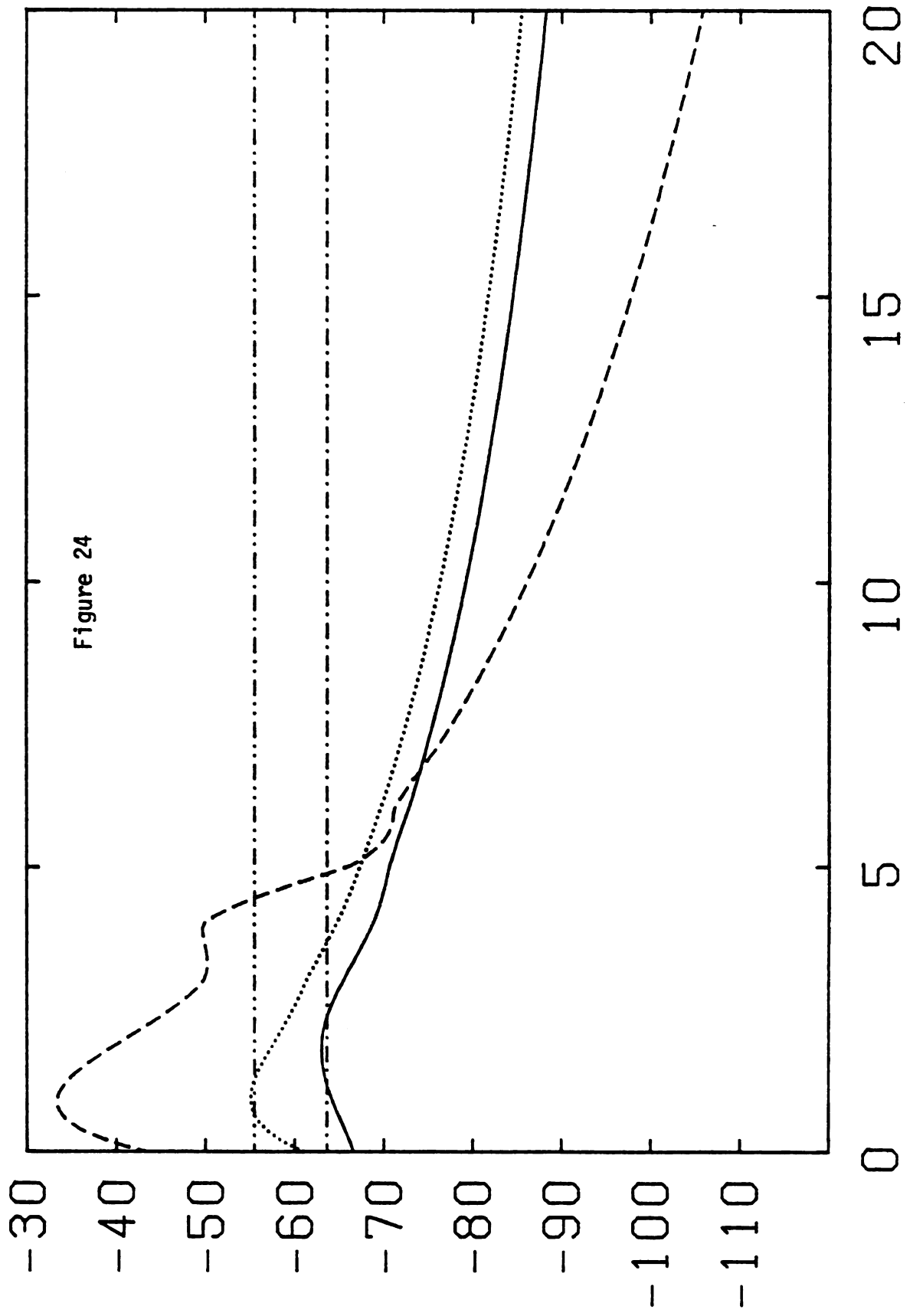


Figure 24

respectively. In these figures (22 to 24) the $\log_{10}$ of the decay width is plotted rather than the decay width itself. The reason for this is that the values of these decay widths, as a function of the quark mass, vary over a few orders of magnitude and the variation of these widths with the quark mass could not be shown clearly using only the decay widths proper. For RAT = -0.16 the case 1 decay width calculations are always larger than those obtained in case 3, sometimes by 2 or 3 in the logarithm of the decay width. In the region of fitting, the differences however are less than one in the logarithm of the decay widths. The results of the case 2 and 3 calculations produce no noticable differences for RAT = -0.16. For RAT = -0.33, cases 2 and 3 also produce almost identical results, the only difference being in the $\psi' \rightarrow \eta_c + \gamma$ for $m \simeq 4$, and giving in case 3 about 0.2 (in the logarithm) larger a value. The case 1 decay widths are larger for $m < 5$ and generally smaller for $m > 8$.

From figures 22, 23, and 24, it appears that the magnetic dipole widths do have a dependence upon RAT. To get a better feeling as to the actual variation of these magnetic dipole transition widths, we give the actual values of these transition widths for the quark masses 3.0 to 7.0 in TABLE 15 for RAT = -0.16, -0.275, and -0.33. For an explanation of the notation used in this table see TABLE 16.

Additional information on these radiative decay processes can be found in Tables 17 to 21. Tables 17 to 20 give the values of the radial integrals for RAT = 0.00, -0.16, -0.275, and -0.33. The key to the organization of these radial integral tables is shown in TABLE 16. TABLE 21, for RAT = 0.00 and m = 5.0 shows the way the radial integral is built up. It is seen from TABLE 21 that the main contribution comes

TABLE 15

VARIATION OF MAGNETIC DIPOLE WIDTHS WITH RAT(GEV)

PROCESS	RAT=-0.16	RAT=-0.275	RAT=-0.33	OBSERVED
		M=3		
RSSZ	.8570E-05	.8247E-05	.1041E-04	.645E-06
RATEC	.1043E-05	.8399E-06	.7371E-06	.756E-06
RATEA	.3871E-06	.3184E-06	.2850E-06	.43 TO 2.8E-06
		M=4		
RSSZ	.2081E-05	.5100E-05	.1060E-04	.645E-06
RATEC	.5491E-06	.3989E-06	.3371E-06	.756E-06
RATEA	.2210E-06	.1731E-06	.1322E-06	.43 TO 2.8E-06
		M=5		
RSSZ	.2043E-06	.2879E-06	.2507E-06	.645E-06
RATEC	.2594E-06	.2249E-06	.1854E-06	.756E-06
RATEA	.1336E-06	.1034E-06	.8672E-07	.43 TO 2.8E-06
		M=6		
RSSZ	.1369E-06	.9899E-07	.7772E-07	.645E-06
RATEC	.1824E-06	.1344E-06	.1094E-06	.756E-06
RATEA	.8508E-07	.6427E-07	.5307E-07	.43 TO 2.8E-06
		M=7		
RSSZ	.5557E-07	.3682E-07	.2761E-07	.645E-06
RATEC	.1182E-06	.8576E-07	.6906E-07	.756E-06
RATEA	.5682E-07	.4218E-07	.3434E-07	.43 TO 2.8E-06

THE ERRORS IN RSSZ AND RATEC ARE ABOUT
25 AND 40 PER CENT RESPECTIVELY

TABLE 16

RADIAL INTEGRALS KEY

quark mass (m)	RSPZ	RPZS	RATEA
quark mass (m)	RSPO	RPOS	RSSZ
quark mass (m)	RSPT	RPTS	RATEC
quark mass (m+1)	RSPZ	RPZS	RATEA

etc.

where the corresponding processes are identified by

RSPZ	$\psi'(3685) \rightarrow \chi(3414) + \gamma$
RSPO	$\psi'(3685) \rightarrow \chi(3507) + \gamma$
RSPT	$\psi'(3685) \rightarrow \chi(3551) + \gamma$
RPZS	$\chi\ (3414) \rightarrow \psi(3095) + \gamma$
RPOS	$\chi\ (3507) \rightarrow \psi(3095) + \gamma$
RPTS	$\chi\ (3551) \rightarrow \psi(3095) + \gamma$
RSSZ	$\psi'(3685) \rightarrow \eta_c(2984) + \gamma$
RATEC	$\psi\ (3095) \rightarrow \eta_c(2984) + \gamma$
RATEA	$\psi'(3685) \rightarrow \eta_c'(3592) + \gamma$

TABLE 17

THE RADIAL INTEGRALS AS A FUNCTION OF THE QUARK MASS FOR RAT=0.00

0.	.217319E+00	-.747014E+00	-.102470E+00
0.	-.150658E-02	-.796518E+00	.935898E-01
0.	-.225797E+00	-.575166E+00	-.148867E+00
.100000E+01	.467113E-01	.824063E+00	.242934E+00
.100000E+01	-.113364E+00	.852695E+00	.287832E+00
.100000E+01	-.397167E+00	.821269E+00	.378575E+00
.200000E+01	-.224919E+00	.604400E+00	.266567E+00
.200000E+01	-.273660E+00	.608421E+00	.145168E+00
.200000E+01	-.404432E+00	.604089E+00	.352293E+00
.300000E+01	.268602E+00	-.458781E+00	-.228200E+00
.300000E+01	.284168E+00	-.457804E+00	.621725E-01
.300000E+01	.337271E+00	-.462235E+00	.277449E+00
.400000E+01	.237641E+00	-.406179E+00	.192300E+00
.400000E+01	.247163E+00	-.378131E+00	-.341702E-01
.400000E+01	.280406E+00	-.402359E+00	-.224539E+00
.500000E+01	.228151E+00	-.318388E+00	.164943E+00
.500000E+01	.232434E+00	-.317823E+00	-.118132E-01
.500000E+01	-.247249E+00	.315047E+00	-.182945E+00
.600000E+01	-.209056E+00	-.279730E+00	-.143602E+00
.600000E+01	-.211284E+00	-.279550E+00	.105477E-01
.600000E+01	.219548E+00	.278809E+00	-.155771E+00
.700000E+01	-.191764E+00	-.250550E+00	-.126702E+00
.700000E+01	-.193045E+00	-.250463E+00	.725529E-02
.700000E+01	.198048E+00	.250136E+00	-.135372E+00
.800000E+01	-.176848E+00	.227644E+00	-.113175E+00
.800000E+01	-.177641E+00	.227596E+00	.517015E-02
.800000E+01	.180858E+00	-.227422E+00	.119594E+00
.900000E+01	-.164087E+00	-.209112E+00	-.102154E+00
.900000E+01	-.164605E+00	-.209082E+00	.382091E-02
.900000E+01	.166772E+00	.208980E+00	-.107056E+00

TABLE 17 (CONT')

.100000E+02	-.153126E+00	.193762E+00	-.930260E-01
.100000E+02	-.153479E+00	.193742E+00	.291216E-02
.100000E+02	-.154994E+00	.193677E+00	.968664E-01
.110000E+02	-.143639E+00	-.180808E+00	-.853557E-01
.110000E+02	-.143888E+00	-.180794E+00	.227723E-02
.110000E+02	.144980E+00	.180751E+00	-.884285E-01
.120000E+02	-.135359E+00	-.169709E+00	-.788272E-01
.120000E+02	-.135540E+00	-.169699E+00	.181939E-02
.120000E+02	.136348E+00	.169669E+00	-.813302E-01
.130000E+02	-.128075E+00	.160077E+00	-.732080E-01
.130000E+02	-.128209E+00	.160070E+00	.148024E-02
.130000E+02	-.128820E+00	.160048E+00	.752780E-01
.140000E+02	-.121617E+00	-.151630E+00	-.683234E-01
.140000E+02	-.121720E+00	-.151624E+00	.122316E-02
.140000E+02	-.122190E+00	-.151608E+00	-.700580E-01
.150000E+02	-.115853E+00	-.144152E+00	-.640403E-01
.150000E+02	-.115932E+00	-.144148E+00	.102437E-02
.150000E+02	-.116301E+00	-.144135E+00	-.655106E-01
.160000E+02	-.110674E+00	-.137480E+00	-.602553E-01
.160000E+02	-.110736E+00	-.137476E+00	.867959E-03
.160000E+02	-.111030E+00	-.137467E+00	-.615143E-01
.170000E+02	-.105995E+00	-.131484E+00	-.568873E-01
.170000E+02	-.106045E+00	-.131482E+00	.743013E-03
.170000E+02	-.106281E+00	-.131474E+00	-.579750E-01
.180000E+02	-.101745E+00	.126064E+00	-.538718E-01
.180000E+02	-.101785E+00	.126062E+00	.641852E-03
.180000E+02	-.101978E+00	.126057E+00	.548189E-01
.190000E+02	-.978668E-01	.121137E+00	-.511565E-01
.190000E+02	-.978997E-01	.121136E+00	.558965E-03
.190000E+02	.980583E-01	-.121131E+00	.519872E-01
.200000E+02	-.943119E-01	-.116636E+00	-.486993E-01
.200000E+02	-.943391E-01	-.116635E+00	.490320E-03
.200000E+02	.944709E-01	.116631E+00	-.494325E-01

TABLE 18

THE RADIAL INTEGRALS AS A FUNCTION OF THE QUARK MASS FOR RAT=-0.16

0.	-.314656E+00	.905730E+00	.121704E+00
0.	.984360E-03	.100000E+01	-.119327E+00
0.	.713235E+00	.186857E+01	.187547E+00
.100000E+01	.698416E-01	-.841996E+00	.269781E+00
.100000E+01	.198053E+00	-.847571E+00	.253649E+00
.100000E+01	-.424731E+00	.809197E+00	.393462E+00
.200000E+01	.289642E+00	.618097E+00	-.258072E+00
.200000E+01	.323079E+00	.617968E+00	-.117468E+00
.200000E+01	-.414009E+00	-.609615E+00	.344420E+00
.300000E+01	-.305935E+00	.482617E+00	.218984E+00
.300000E+01	-.315706E+00	.482904E+00	.539162E-01
.300000E+01	-.351846E+00	.482382E+00	.269863E+00
.400000E+01	-.286690E+00	.403025E+00	-.186035E+00
.400000E+01	-.289127E+00	.403384E+00	.271822E-01
.400000E+01	.303700E+00	-.403937E+00	.217333E+00
.500000E+01	-.253805E+00	.340072E+00	-.160092E+00
.500000E+01	-.254696E+00	.337297E+00	-.377494E-01
.500000E+01	.263312E+00	-.312365E+00	.180083E+00
.600000E+01	-.228462E+00	.302819E+00	-.139905E+00
.600000E+01	-.229570E+00	.302556E+00	.998030E-02
.600000E+01	-.235375E+00	.301617E+00	.153816E+00
.700000E+01	-.209550E+00	.272436E+00	-.123947E+00
.700000E+01	-.210113E+00	.272331E+00	.739016E-02
.700000E+01	.213595E+00	-.271961E+00	.133967E+00
.800000E+01	-.193298E+00	.248212E+00	-.111071E+00
.800000E+01	-.193611E+00	.248159E+00	.541835E-02
.800000E+01	.195843E+00	-.247971E+00	.118551E+00
.900000E+01	-.179427E+00	-.228436E+00	-.100511E+00
.900000E+01	-.179613E+00	-.228405E+00	.408369E-02
.900000E+01	-.181113E+00	-.228298E+00	-.106259E+00

TABLE 18(CONT'D)

.100000E+02	-.167523E+00	-.211956E+00	-.917180E-01
.100000E+02	-.167638E+00	-.211937E+00	.316016E-02
.100000E+02	-.168686E+00	-.211870E+00	-.962422E-01
.110000E+02	-.157220E+00	.197987E+00	-.842960E-01
.110000E+02	-.157295E+00	.197975E+00	.250176E-02
.110000E+02	-.158050E+00	.197931E+00	.879295E-01
.120000E+02	-.148226E+00	.185980E+00	-.779557E-01
.120000E+02	-.148276E+00	.185971E+00	.201923E-02
.120000E+02	-.148834E+00	.185941E+00	.809240E-01
.130000E+02	-.140308E+00	.175534E+00	-.724818E-01
.130000E+02	-.140343E+00	.175528E+00	.165695E-02
.130000E+02	.140765E+00	-.175506E+00	.749425E-01
.140000E+02	-.133284E+00	-.166354E+00	-.677114E-01
.140000E+02	-.133309E+00	-.166349E+00	.137918E-02
.140000E+02	-.133634E+00	-.166333E+00	-.697772E-01
.150000E+02	-.127009E+00	-.158214E+00	-.635194E-01
.150000E+02	-.127027E+00	-.158211E+00	.116227E-02
.150000E+02	-.127281E+00	-.158199E+00	-.652728E-01
.160000E+02	-.121368E+00	-.150942E+00	-.598081E-01
.160000E+02	-.121381E+00	-.150940E+00	.990150E-03
.160000E+02	.121583E+00	.150931E+00	-.613108E-01
.170000E+02	-.116265E+00	-.144401E+00	-.565005E-01
.170000E+02	-.116275E+00	-.144399E+00	.851619E-03
.170000E+02	.116438E+00	.144392E+00	-.577993E-01
.180000E+02	-.111626E+00	-.138482E+00	-.535350E-01
.180000E+02	-.111634E+00	-.138480E+00	.738715E-03
.180000E+02	-.111766E+00	-.138475E+00	-.546661E-01
.190000E+02	-.107388E+00	-.133097E+00	-.508616E-01
.190000E+02	-.107394E+00	-.133095E+00	.645661E-03
.190000E+02	-.107503E+00	-.133091E+00	-.518533E-01
.200000E+02	-.103499E+00	-.128174E+00	-.484398E-01
.200000E+02	-.103503E+00	-.128173E+00	.568189E-03
.200000E+02	-.103593E+00	-.128169E+00	-.493144E-01

TABLE 19

THE RADIAL INTEGRALS AS A FUNCTION OF

THE QUARK MASS FOR RAT=-0.275

0.	.189974E+00	.754984E+00	-.104223E+00
0.	-.240058E-02	.791531E+00	.909452E-01
0.	-.226665E+00	.571706E+00	.147512E+00
.100000E+01	.145047E+00	.782606E+00	-.200503E+00
.100000E+01	-.428813E-01	.845562E+00	.311263E+00
.100000E+01	-.375850E+00	.823551E+00	.358250E+00
.200000E+01	-.162740E+00	.582081E+00	-.271109E+00
.200000E+01	.225274E+00	-.592731E+00	-.175095E+00
.200000E+01	.395272E+00	-.597849E+00	-.358450E+00
.300000E+01	-.238835E+00	-.431842E+00	-.238024E+00
.300000E+01	-.258811E+00	-.433901E+00	.701394E-01
.300000E+01	.326959E+00	.441314E+00	-.286382E+00
.400000E+01	-.184018E+00	.346402E+00	-.197583E+00
.400000E+01	-.205073E+00	.345171E+00	-.621033E-01
.400000E+01	-.262320E+00	.335028E+00	-.222228E+00
.500000E+01	-.207322E+00	.292947E+00	.169822E+00
.500000E+01	-.212864E+00	.292821E+00	-.164110E-01
.500000E+01	.230626E+00	-.291347E+00	-.185544E+00
.600000E+01	-.190202E+00	.255893E+00	-.147056E+00
.600000E+01	.193149E+00	-.255827E+00	-.105668E-01
.600000E+01	.202977E+00	-.255343E+00	-.157548E+00
.700000E+01	.174421E+00	-.228414E+00	.129252E+00
.700000E+01	.176150E+00	-.228372E+00	.699797E-02
.700000E+01	-.182067E+00	.228147E+00	.136640E+00
.800000E+01	-.160761E+00	.207064E+00	.115107E+00
.800000E+01	-.161845E+00	.207036E+00	.487081E-02
.800000E+01	-.165633E+00	.206912E+00	.120531E+00
.900000E+01	.149063E+00	.189907E+00	.103653E+00
.900000E+01	.149779E+00	.189888E+00	-.353584E-02
.900000E+01	-.152320E+00	-.189812E+00	.107770E+00

TABLE 19(CONT'D)

.100000E+02	-.139016E+00	-.175765E+00	.942141E-01
.100000E+02	-.139508E+00	-.175751E+00	.265637E-02
.100000E+02	-.141279E+00	-.175701E+00	-.974237E-01
.110000E+02	-.130326E+00	.163872E+00	.863144E-01
.110000E+02	-.130677E+00	.163862E+00	.205264E-02
.110000E+02	.131950E+00	-.163828E+00	.888730E-01
.120000E+02	-.122749E+00	-.153710E+00	.796132E-01
.120000E+02	-.123006E+00	-.153702E+00	.162366E-02
.120000E+02	.123945E+00	.153678E+00	-.816912E-01
.130000E+02	-.116089E+00	-.144909E+00	-.738613E-01
.130000E+02	-.116281E+00	-.144903E+00	.130983E-02
.130000E+02	.116991E+00	.144886E+00	-.755758E-01
.140000E+02	-.110191E+00	.137203E+00	-.688731E-01
.140000E+02	-.110338E+00	.137199E+00	.107448E-02
.140000E+02	.110884E+00	-.137185E+00	.703071E-01
.150000E+02	-.104930E+00	.130391E+00	-.645077E-01
.150000E+02	-.105044E+00	.130388E+00	.894180E-03
.150000E+02	.105471E+00	-.130377E+00	.657213E-01
.160000E+02	-.100207E+00	-.124320E+00	-.606566E-01
.160000E+02	-.100298E+00	-.124317E+00	.753482E-03
.160000E+02	-.100637E+00	-.124309E+00	-.616943E-01
.170000E+02	-.959435E-01	-.118869E+00	-.572348E-01
.170000E+02	-.960160E-01	-.118867E+00	.641897E-03
.170000E+02	-.962892E-01	-.118860E+00	-.581303E-01
.180000E+02	-.920736E-01	.113945E+00	-.541749E-01
.180000E+02	-.921324E-01	.113944E+00	.552132E-03
.180000E+02	-.923548E-01	.113938E+00	.549540E-01
.190000E+02	-.885441E-01	-.109472E+00	-.514228E-01
.190000E+02	-.885923E-01	-.109471E+00	.479003E-03
.190000E+02	-.887753E-01	-.109466E+00	-.521055E-01
.200000E+02	-.853108E-01	-.105388E+00	-.489346E-01
.200000E+02	-.853508E-01	-.105387E+00	.418752E-03
.200000E+02	-.855029E-01	-.105383E+00	-.495369E-01

TABLE 20

THE RADIAL INTEGRALS AS A FUNCTION OF THE QUARK MASS FOR RAT=-0.33

0.	-.177196E+00	-.757340E+00	.104912E+00
0.	.293672E-02	-.788885E+00	-.896735E-01
0.	-.227065E+00	.569838E+00	-.146809E+00
.100000E+01	-.196307E+00	-.743145E+00	.166208E+00
.100000E+01	-.490867E-02	.834391E+00	.320354E+00
.100000E+01	.363868E+00	-.817294E+00	.341977E+00
.200000E+01	.129844E+00	.567324E+00	-.271818E+00
.200000E+01	-.196436E+00	-.579536E+00	-.195668E+00
.200000E+01	.389602E+00	.593293E+00	.361503E+00
.300000E+01	-.346331E+00	-.215341E+01	-.247057E+00
.300000E+01	.276173E+00	.415476E+00	.864422E-01
.300000E+01	-.320458E+00	-.425050E+00	-.294315E+00
.400000E+01	-.213941E+00	-.327997E+00	-.190705E+00
.400000E+01	.232601E+00	.328508E+00	.988935E-01
.400000E+01	-.265352E+00	-.319799E+00	.225568E+00
.500000E+01	.194519E+00	.276078E+00	.172731E+00
.500000E+01	.200548E+00	.276179E+00	.170045E-01
.500000E+01	.219480E+00	.275286E+00	-.187074E+00
.600000E+01	.178440E+00	.240454E+00	-.149125E+00
.600000E+01	.181666E+00	.240468E+00	.104493E-01
.600000E+01	.192076E+00	.240165E+00	-.158589E+00
.700000E+01	.163560E+00	-.214235E+00	-.130768E+00
.700000E+01	.165456E+00	-.214228E+00	.678930E-02
.700000E+01	-.171687E+00	.214083E+00	.137379E+00
.800000E+01	.150665E+00	.193969E+00	-.116249E+00
.800000E+01	.151856E+00	.193958E+00	-.466144E-02
.800000E+01	.155825E+00	.193875E+00	.121075E+00
.900000E+01	-.139624E+00	-.177742E+00	-.104534E+00
.900000E+01	-.140412E+00	-.177732E+00	.334741E-02
.900000E+01	.143066E+00	.177679E+00	-.108183E+00

TABLE 20 (CONT'D)

.100000E+02	-.130149E+00	-.164400E+00	-.949095E-01
.100000E+02	-.130691E+00	-.164392E+00	.249253E-02
.100000E+02	.132536E+00	.164356E+00	-.977456E-01
.110000E+02	.121961E+00	-.153202E+00	-.868738E-01
.110000E+02	.122348E+00	-.153196E+00	.191173E-02
.110000E+02	.123671E+00	-.153170E+00	.891294E-01
.120000E+02	-.114828E+00	.143647E+00	-.800706E-01
.120000E+02	-.115111E+00	.143642E+00	-.150263E-02
.120000E+02	-.116087E+00	.143623E+00	-.818992E-01
.130000E+02	-.108564E+00	.135383E+00	-.742407E-01
.130000E+02	-.108776E+00	.135379E+00	.120561E-02
.130000E+02	-.109511E+00	.135364E+00	.757472E-01
.140000E+02	-.103019E+00	.128153E+00	-.691917E-01
.140000E+02	-.103181E+00	.128149E+00	.984323E-03
.140000E+02	-.103747E+00	.128138E+00	.704502E-01
.150000E+02	-.980772E-01	.121766E+00	-.647783E-01
.150000E+02	-.982036E-01	.121763E+00	.815782E-03
.150000E+02	.986458E-01	-.121754E+00	.658423E-01
.160000E+02	.936438E-01	.116077E+00	.608887E-01
.160000E+02	.937438E-01	.116074E+00	-.684932E-03
.160000E+02	.940948E-01	.116068E+00	.617977E-01
.170000E+02	-.896431E-01	-.110972E+00	-.574355E-01
.170000E+02	-.897233E-01	-.110970E+00	.581631E-03
.170000E+02	-.900056E-01	-.110965E+00	-.582194E-01
.180000E+02	-.860135E-01	-.106363E+00	-.543499E-01
.180000E+02	-.860786E-01	-.106361E+00	.498870E-03
.180000E+02	-.863083E-01	-.106357E+00	-.550314E-01
.190000E+02	-.827046E-01	-.102177E+00	-.515764E-01
.190000E+02	-.827580E-01	-.102176E+00	.431695E-03
.190000E+02	-.829470E-01	-.102172E+00	-.521734E-01
.200000E+02	-.796745E-01	.983567E-01	-.490703E-01
.200000E+02	-.797188E-01	.983556E-01	.376532E-03
.200000E+02	.798757E-01	-.983524E-01	.495966E-01

TABLE 21

ACCUMULATION OF THE RADIALINTEGRALS

1	0.	0.	0.
1	0.	0.	0.
1	0.	0.	0.
2	-.146373E-08	-.123103E-08	.695466E-05
2	-.146184E-08	-.120817E-08	-.479859E-05
2	.140991E-08	.111657E-08	-.444891E-05
3	-.259785E-05	-.219372E-05	.449301E-03
3	-.259450E-05	-.215313E-05	-.311564E-03
3	.250269E-05	.199048E-05	-.289903E-03
4	-.142608E-03	-.123063E-03	.429800E-02
4	-.142413E-03	-.120818E-03	-.305538E-02
4	.137428E-03	.111824E-03	-.289486E-02
5	-.203201E-02	-.186883E-02	.182293E-01
5	-.202888E-02	-.183602E-02	-.138511E-01
5	.196013E-02	.170479E-02	-.138052E-01
6	-.124057E-01	-.132231E-01	.428070E-01
6	-.123856E-01	-.130097E-01	-.371321E-01
6	.119992E-01	.121578E-01	-.413986E-01
7	-.363156E-01	-.531043E-01	.593118E-01
7	-.362701E-01	-.523713E-01	-.624716E-01
7	.353171E-01	.494519E-01	-.865457E-01
8	-.431407E-01	-.133232E+00	.594584E-01
8	-.431024E-01	-.131822E+00	-.663341E-01
8	.420757E-01	.126193E+00	-.134284E+00
9	.197459E-01	-.226364E+00	.764343E-01
9	.201235E-01	-.224777E+00	-.456313E-01
9	-.217585E-01	.218342E+00	-.166148E+00
10	.126663E+00	-.288749E+00	.119040E+00
10	.128354E+00	-.287584E+00	-.238495E-01
10	-.134085E+00	.282634E+00	-.179224E+00

TABLE 21 (CONT'D)

11	.199517E+00	-.312611E+00	.151778E+00
11	.202785E+00	-.311863E+00	-.142276E-01
11	-.213966E+00	.308422E+00	-.182442E+00
12	.223716E+00	-.317744E+00	.162939E+00
12	.227787E+00	-.317149E+00	-.120792E-01
12	-.241834E+00	.314268E+00	-.182905E+00
13	.227791E+00	-.318349E+00	.164786E+00
13	.232051E+00	-.317781E+00	-.118287E-01
13	-.246784E+00	.314997E+00	-.182943E+00
14	.228137E+00	-.318387E+00	.164936E+00
14	.232418E+00	-.317821E+00	-.118137E-01
14	-.247229E+00	.315046E+00	-.182945E+00
15	.228151E+00	-.318388E+00	.164942E+00
15	.232434E+00	-.317823E+00	-.118132E-01
15	-.247248E+00	.315047E+00	-.182945E+00
16	.228151E+00	-.318388E+00	.164943E+00
16	.232434E+00	-.317823E+00	-.118132E-01
16	-.247249E+00	.315047E+00	-.182945E+00
17	.228151E+00	-.318388E+00	.164943E+00
17	.232434E+00	-.317823E+00	-.118132E-01
17	-.247249E+00	.315047E+00	-.182945E+00
18	.228151E+00	-.318388E+00	.164943E+00
18	.232434E+00	-.317823E+00	-.118132E-01
18	-.247249E+00	.315047E+00	-.182945E+00
19	.228151E+00	-.318388E+00	.164943E+00
19	.232434E+00	-.317823E+00	-.118132E-01
19	-.247249E+00	.315047E+00	-.182945E+00

TABLE 21 (CONT'D)

20	.228151E+00	-.318388E+00	.164943E+00
20	.232434E+00	-.317823E+00	-.118132E-01
20	-.247249E+00	.315047E+00	-.182945E+00
21	.228151E+00	-.318388E+00	.164943E+00
21	.232434E+00	-.317823E+00	-.118132E-01
21	-.247249E+00	.315047E+00	-.182945E+00
22	.228151E+00	-.318388E+00	.164943E+00
22	.232434E+00	-.317823E+00	-.118132E-01
22	-.247249E+00	.315047E+00	-.182945E+00
23	.228151E+00	-.318388E+00	.164943E+00
23	.232434E+00	-.317823E+00	-.118132E-01
23	-.247249E+00	.315047E+00	-.182945E+00
24	.228151E+00	-.318388E+00	.164943E+00
24	.232434E+00	-.317823E+00	-.118132E-01
24	-.247249E+00	.315047E+00	-.182945E+00
25	.228151E+00	-.318388E+00	.164943E+00
25	.232434E+00	-.317823E+00	-.118132E-01
25	-.247249E+00	.315047E+00	-.182945E+00
26	.228151E+00	-.318388E+00	.164943E+00
26	.232434E+00	-.317823E+00	-.118132E-01
26	-.247249E+00	.315047E+00	-.182945E+00
27	.228151E+00	-.318388E+00	.164943E+00
27	.232434E+00	-.317823E+00	-.118132E-01
27	-.247249E+00	.315047E+00	-.182945E+00
28	.228151E+00	-.318388E+00	.164943E+00
28	.232434E+00	-.317823E+00	-.118132E-01
28	-.247249E+00	.315047E+00	-.182945E+00
29	.228151E+00	-.318388E+00	.164943E+00
29	.232434E+00	-.317823E+00	-.118132E-01
29	-.247249E+00	.315047E+00	-.182945E+00

TABLE 21 (CONT'D)

30	.228151E+00	-.318388E+00	.164943E+00
30	.232434E+00	-.317823E+00	-.118132E-01
30	-.247249E+00	.315047E+00	-.182945E+00
31	.228151E+00	-.318388E+00	.164943E+00
31	.232434E+00	-.317823E+00	-.118132E-01
31	-.247249E+00	.315047E+00	-.182945E+00
32	.228151E+00	-.318388E+00	.164943E+00
32	.232434E+00	-.317823E+00	-.118132E-01
32	-.247249E+00	.315047E+00	-.182945E+00

from the eighth to the fifteenth terms. This same region is where the major contributions for the other cases also occur (e.g. RAT = -0.16, case 1 or case 2).

Tables 22 and 23 give the results of the 2γ and 2 gluon decay widths for some charmonium states. The results were computed by expressions similar to those used in the leptonic decay widths reported in Appendix II.1. The expressions used in these calculations can be found in [46].

The computations of the radiative decay widths reported in [30] make use of a substitution that is common in discussions of atomic spectra, namely the replacement of $\langle f|\vec{p}|i\rangle$ by $\mu\langle f|[\vec{x},H]|i\rangle = \mu(\lambda_i - \lambda_f)\langle f|\vec{x}|i\rangle$. In [30] $\lambda_i - \lambda_f$ is replaced by $m_i - m_f$. With this substitution the expression for the radiative decay width is

$$\Gamma(m_i \rightarrow m_f + \gamma) = (2/3)^2 e_q^2\ \alpha(m_i^2 m_f/4)(1-(m_f^2/m_i^2))^3 |R/\Delta\lambda|^2 P \qquad (A)$$

where $\Delta\lambda$ is $\lambda_2 - \lambda_1$, i.e. the difference in the computed masses and R is the radial integral tabulated in tables 17 to 20, and P is the polarization sum factor. These calculations were done using the subroutine Cheat listed in Appendix III, along with the rest of the computer program which is also included in Appendix III. The results of these calculations are tabulated in tables 25, 27, and 29, while tables 26, 28, and 30 give the ratio of the calculations shown in figures 13 to 21 to those calculated by (A). TABLE 24 gives the key to the meaning of the various columns in tables 25 to 30. The quantity listed in the first column of tables 25 to 30 is not the quark mass but the quark mass plus 1.0. It is clear from these tables that the two ways of doing the calculation give quite different results.

In order to determine the quark mass in Gev, one needs to know a certain conversion factor. This factor as stated above changes with the quark mass. For the dimensionless quark mass from 3 to 7, one can see how this factor varies with the quark mass from Figures 9 to 12. The average value for this factor is between 0.40 and 0.50, depending upon RAT and m. For RAT = -0.16 we have for m = 4.0, 5.0, 6.0, and 20.0, the values 0.32510, 0.38849, 0.42379, and 0.78668 respectively. A simple calculation shows that the "predicted" quark mass from the model falls in the range of 1.5 to 2.0 Gev, which is what almost all charmonium model calculations give.

TABLE 22

TWO-GAMMA DECAY WIDTHS FOR SOME CHARMONIUM STATES (KEV)

MASS	1 (1S0)	2 (1S0)	1 (3P0)	1 (3P2)
		RAT=-0.16		
3.0	3.714	5.504	5.476	3.885
4.0	5.840	7.914	11.229	6.539
5.0	8.785	11.226	21.596	11.393
6.0	12.615	15.481	38.723	19.374
7.0	17.381	20.717	65.119	31.600
		RAT=-0.275		
3.0	4.314	6.698	7.076	6.006
4.0	6.928	9.713	15.033	9.660
5.0	10.610	13.940	29.803	16.659
6.0	15.482	19.455	54.891	28.476
7.0	21.622	26.306	94.403	46.923
		RAT=-0.33		
3.0	4.758	7.625	8.375	8.037
4.0	7.756	11.121	18.255	12.304
5.0	12.028	16.100	36.975	21.422
6.0	17.749	22.675	69.384	36.787
7.0	25.025	30.904	121.212	61.099

TABLE 23

TWO-GLUON DECAY WIDTHS FOR SOME CHARMONIUM STATES (MEV)

MASS	1 (1S0)	2 (1S0)	1 (3P0)	1 (3P2)
	RAT=-0.16			
3.0	2.543	3.767	7.497	2.659
4.0	3.998	5.418	15.374	4.476
5.0	6.013	7.684	29.566	7.799
6.0	8.636	10.598	53.014	13.262
7.0	11.898	14.181	89.152	21.631
	RAT=-0.275			
3.0	2.953	4.585	9.687	4.111
4.0	4.743	6.649	20.581	6.613
5.0	7.263	9.543	40.802	11.404
6.0	10.598	13.318	75.149	19.493
7.0	14.801	18.008	129.243	32.120
	RAT=-0.33			
3.0	3.257	5.219	11.466	5.501
4.0	5.309	7.613	24.992	8.559
5.0	8.233	11.021	50.621	14.664
6.0	12.150	15.522	94.991	25.182
7.0	17.130	21.155	165.947	41.824
	OBSERVED VALUE OF THE DECAY RATE			
	12.4.	< 8.0	16.3	1.8

THE ERRORS ARE ABOUT 25 PERCENT IN THE EXPERIMENTIAL VALUES

TABLE 24

KEY TO TABLES 25 TO 30

Tables 25, 27, and 29 give the following electric dipole transition widths as calculated by the subroutine "cheat." The meaning of the columns is:

column 1 The value of the quark mass in dimensionless units + 1.0

column 2 $2^3S_1 \rightarrow 1^3P_0 + \gamma$ (RXSPZ in cheat)

column 3 $2^3S_1 \rightarrow 1^3P_1 + \gamma$ (RXSPO in cheat)

column 4 $2^3S_1 \rightarrow 1^3P_2 + \gamma$ (RXSPT in cheat)

column 5 $1^3P_0 \rightarrow 1^3S_1 + \gamma$ (RXPZS in cheat)

column 6 $1^3P_1 \rightarrow 1^3S_1 + \gamma$ (RXPOS in cheat)

column 7 $1^3P_2 \rightarrow 1^3S_1 + \gamma$ (RXPTS in cheat)

In tables 26, 28, and 30, the columns 2 through 7 give the ratios of the decay widths calculated in the main body of the program to those given by using the subroutine "cheat."

column 1 The value of the quark mass in dimensionless units + 1.0

column 2 The ratio RSPZ/RXSPO

column 3 The ratio RSPO/RXSPO

column 4 The ratio RSPT/RXSPT

column 5 The ratio RPZS/RXPZS

column 6 The ratio RPOS/RXPOS

column 7 The ratio RPTS/RXPTS

where RSPZ,...,RPTS are the transition widths for the corresponding decay processes as determined in Orthog and as shown in figures 13 to 21.

TABLE 25. CHEAT TRANSITION WIDTHS, RAT = -0.16

1	.186253E-05	.125870E-09	.595878E-05	.058344E-03	.131224E+03	.368644E-03
2	.102574E-06	.769515E-06	.258384E-04	.182628E-02	.321385E+00	.608443E-03
3	.367240E-05	.588767E-05	.440055E-04	.993786E-02	.843483E-02	.262514E-03
4	.783429E-05	.826792E-05	.226611E-04	.140354E-01	.174198E-02	.174542E-03
5	.799675E-05	.741261E-05	.119702E-04	.186343E-02	.787641E-03	.152167E-03
6	.883490E-05	.735501E-05	.788489E-05	.619902E-03	.445748E-03	.103349E-03
7	.845261E-05	.660245E-05	.559600E-05	.345942E-03	.300127E-03	.870411E-04
8	.784452E-05	.586570E-05	.424235E-05	.226827E-03	.219103E-03	.738505E-04
9	.719721E-05	.521215E-05	.336503E-05	.163222E-03	.168919E-03	.634956E-04
10	.657967E-05	.464955E-05	.275763E-05	.124730E-03	.135376E-03	.552546E-04
11	.601481E-05	.416898E-05	.231590E-05	.993911E-04	.111671E-03	.485931E-04
12	.550757E-05	.375828E-05	.198233E-05	.816740E-04	.941915E-04	.431303E-04
13	.505569E-05	.340594E-05	.172283E-05	.687110E-04	.808669E-04	.385923E-04
14	.465423E-05	.310211E-05	.151606E-05	.588857E-04	.704331E-04	.347787E-04
15	.429752E-05	.283859E-05	.134803E-05	.512254E-04	.620809E-04	.315405E-04
16	.398009E-05	.260871E-05	.120921E-05	.451138E-04	.552704E-04	.287654E-04
17	.369691E-05	.240704E-05	.109290E-05	.401432E-04	.496296E-04	.263672E-04
18	.344355E-05	.222915E-05	.994258E-06	.360346E-04	.448945E-04	.242790E-04
19	.321615E-05	.207144E-05	.909723E-06	.325911E-04	.408731E-04	.224483E-04
20	.301141E-05	.193094E-05	.836603E-06	.296703E-04	.374228E-04	.208334E-04
21	.282646E-05	.180519E-05	.772838E-06	.271668E-04	.344359E-04	.194006E-04

TABLE 26. CHEAT RATIOS, RAT = -0.16

1	.771950E+01	.169182E+02	.102028E+02	.103336E+01	.954021E-05	.192434E+01
2	.647602E+01	.156684E+02	.727980E+01	.591030E+00	.446417E-02	.237713E+01
3	.419374E+01	.119336E+02	.443224E+01	.584268E-01	.865987E-01	.298093E+01
4	.280361E+01	.916317E+01	.598572E+01	.238364E-01	.237407E+00	.262499E+01
5	.214996E+01	.773191E+01	.783276E+01	.140727E+00	.358208E+00	.228145E+01
6	.179367E+01	.689142E+01	.924510E+01	.259925E+00	.447157E+00	.205943E+01
7	.157411E+01	.634339E+01	.102712E+02	.359527E+00	.513799E+00	.191510E+01
8	.142715E+01	.596062E+01	.110248E+02	.439895E+00	.564960E+00	.181678E+01
9	.132293E+01	.568015E+01	.115911E+02	.504651E+00	.605101E+00	.174673E+01
10	.124579E+01	.546722E+01	.120268E+02	.557242E+00	.637191E+00	.169489E+01
11	.118679E+01	.530101E+01	.123694E+02	.600406E+00	.663268E+00	.165533E+01
12	.114047E+01	.516834E+01	.126439E+02	.636221E+00	.684760E+00	.162436E+01
13	.110330E+01	.506046E+01	.128675E+02	.666251E+00	.702696E+00	.159958E+01
14	.107295E+01	.497136E+01	.130524E+02	.691680E+00	.717833E+00	.157940E+01
15	.104778E+01	.489678E+01	.132073E+02	.713408E+00	.730734E+00	.156271E+01
16	.102664E+01	.483362E+01	.133384E+02	.732127E+00	.741826E+00	.154872E+01
17	.100868E+01	.477959E+01	.134505E+02	.748377E+00	.751441E+00	.153686E+01
18	.993261E+00	.473294E+01	.135473E+02	.762580E+00	.759834E+00	.152670E+01
19	.979918E+00	.469236E+01	.136315E+02	.775073E+00	.767209E+00	.151792E+01
20	.968278E+00	.465679E+01	.137053E+02	.786124E+00	.773728E+00	.151027E+01
21	.958054E+00	.462543E+01	.137704E+02	.795952E+00	.779520E+00	.150354E+01

TABLE 27. CHEAT TRANSITION WIDTHS, RAT = -0.275

1	.157771E-05	.319247E-09	.601502E-05	.120076E-02	.759065E+02	.363773E-03
2	.784511E-06	.931109E-07	.213206E-04	.804181E-03	.176412E+00	.640689E-03
3	.129145E-05	.288575E-05	.592679E-04	.103934E-02	.172206E+01	.228844E-03
4	.428449E-05	.512405E-05	.297499E-04	.705317E-02	.499159E-02	.138335E-03
5	.348477E-05	.399381E-05	.130000E-04	.169577E-01	.124325E-02	.940577E-04
6	.552549E-05	.500910E-05	.796024E-05	.153487E-02	.583112E-03	.805790E-04
7	.548179E-05	.461188E-05	.533970E-05	.571013E-03	.346910E-03	.676740E-04
8	.522935E-05	.417742E-05	.390665E-05	.308125E-03	.234773E-03	.576585E-04
9	.490633E-05	.377070E-05	.302529E-05	.197656E-03	.171968E-03	.497933E-04
10	.456934E-05	.340744E-05	.243718E-05	.140032E-03	.132849E-03	.435030E-04
11	.424287E-05	.308840E-05	.202095E-05	.105773E-03	.106611E-03	.383901E-04
12	.393718E-05	.280967E-05	.171308E-05	.835361E-04	.880290E-04	.341750E-04
13	.365582E-05	.256619E-05	.147745E-05	.681649E-04	.743091E-04	.306567E-04
14	.339914E-05	.235300E-05	.129213E-05	.570247E-04	.638406E-04	.276875E-04
15	.316598E-05	.216569E-05	.114314E-05	.486481E-04	.556381E-04	.251569E-04
16	.295457E-05	.200045E-05	.102112E-05	.421615E-04	.490686E-04	.229811E-04
17	.276289E-05	.185407E-05	.919646E-06	.370161E-04	.437098E-04	.210953E-04
18	.258897E-05	.172386E-05	.834134E-06	.328523E-04	.392697E-04	.194491E-04
19	.243094E-05	.160754E-05	.761247E-06	.294255E-04	.355413E-04	.180026E-04
20	.228710E-05	.150321E-05	.698500E-06	.265643E-04	.323739E-04	.167239E-04
21	.215590E-05	.140930E-05	.644007E-06	.241453E-04	.296555E-04	.155874E-04

TABLE 28. CHEAT RATIOS, RAT = -0.275

1	.696404E+01	.169354E+02	.101852E+02	.754529E+00	.162868E-04	.192671E+01
2	.816432E+01	.185280E+02	.790075E+01	.121057E+01	.799730E-02	.227005E+01
3	.624326E+01	.164989E+02	.314349E+01	.518162E+00	.402574E-03	.334924E+01
4	.405317E+01	.122643E+02	.428490E+01	.420264E-01	.744256E-01	.301903E+01
5	.295831E+01	.987917E+01	.631185E+01	.112474E-01	.189100E+00	.255900E+01
6	.236819E+01	.848665E+01	.796763E+01	.888715E-01	.290156E+00	.225891E+01
7	.200912E+01	.758927E+01	.920064E+01	.182275E+00	.372268E+00	.206598E+01
8	.177112E+01	.696862E+01	.101181E+02	.269138E+00	.438348E+00	.193583E+01
9	.160362E+01	.651736E+01	.108134E+02	.344789E+00	.491838E+00	.184375E+01
10	.148041E+01	.617684E+01	.113518E+02	.409365E+00	.535567E+00	.177595E+01
11	.138665E+01	.591234E+01	.117772E+02	.464242E+00	.571703E+00	.172438E+01
12	.131333E+01	.570206E+01	.121193E+02	.510966E+00	.601880E+00	.168411E+01
13	.125473E+01	.553163E+01	.123991E+02	.550930E+00	.627331E+00	.165195E+01
14	.120702E+01	.539126E+01	.126309E+02	.585309E+00	.648993E+00	.162580E+01
15	.116756E+01	.527403E+01	.128255E+02	.615062E+00	.667588E+00	.160420E+01
16	.113449E+01	.517496E+01	.129906E+02	.640969E+00	.683673E+00	.158611E+01
17	.110645E+01	.509034E+01	.131321E+02	.663658E+00	.697687E+00	.157079E+01
18	.108244E+01	.501739E+01	.132543E+02	.683642E+00	.709976E+00	.155767E+01
19	.106168E+01	.495400E+01	.133608E+02	.701336E+00	.720817E+00	.154634E+01
20	.104360E+01	.489850E+01	.134542E+02	.717079E+00	.730434E+00	.153647E+01
21	.102773E+01	.484958E+01	.135366E+02	.731152E+00	.739008E+00	.152782E+01

TABLE 29. CHEAT TRANSITION WIDTHS, RAT = -0.33

1	.143779E-05	.477729E-09	.604438E-05	.140878E-02	.723381E+02	.361053E-03
2	.125431E-05	.112144E-08	.192427E-04	.508843E-03	.458462E-01	.645207E-03
3	.642235E-06	.179343E-05	.787363E-04	.448450E-03	.217618E-01	.206880E-03
4	.712099E-05	.482657E-05	.390093E-04	.331581E-01	.255189E-01	.115626E-03
5	.383504E-05	.437195E-05	.160093E-04	.204607E-01	.213859E-02	.787364E-04
6	.405819E-05	.386979E-05	.814820E-05	.589405E-02	.763128E-03	.671898E-04
7	.410678E-05	.361485E-05	.522554E-05	.101566E-02	.402291E-03	.565895E-04
8	.398044E-05	.331312E-05	.372021E-05	.427342E-03	.254202E-03	.484181E-04
9	.378415E-05	.301985E-05	.282935E-05	.242109E-03	.178175E-03	.419767E-04
10	.356396E-05	.275145E-05	.225053E-05	.159206E-03	.133499E-03	.367982E-04
11	.334147E-05	.251138E-05	.184877E-05	.114435E-03	.104763E-03	.325673E-04
12	.312691E-05	.229856E-05	.155596E-05	.872402E-04	.850434E-04	.290630E-04
13	.292491E-05	.211038E-05	.133444E-05	.693349E-04	.708377E-04	.261259E-04
14	.273723E-05	.194391E-05	.116182E-05	.568332E-04	.602106E-04	.236382E-04
15	.256414E-05	.179634E-05	.102408E-05	.477061E-04	.520172E-04	.215114E-04
16	.240513E-05	.166515E-05	.911981E-06	.408053E-04	.455429E-04	.196775E-04
17	.225934E-05	.154814E-05	.819246E-06	.354386E-04	.403214E-04	.180843E-04
18	.212576E-05	.144342E-05	.741448E-06	.311672E-04	.360369E-04	.166905E-04
19	.200333E-05	.134938E-05	.675390E-06	.277012E-04	.324693E-04	.154635E-04
20	.189102E-05	.126464E-05	.618713E-06	.248423E-04	.294605E-04	.143769E-04
21	.178788E-05	.118802E-05	.569634E-06	.224507E-04	.268948E-04	.134097E-04

TABLE 30. CHEAT RATIOS, RAT = -0.33

1	.664828E+01	.169370E+02	.101715E+02	.647135E+00	.169762E-04	.192857E+01
2	.935333E+01	.201579E+02	.820470E+01	.172512E+01	.299650E-01	.222004E+01
3	.799185E+01	.201859E+02	.229883E+01	.114079E+01	.304541E-01	.364856E+01
4	.512791E+01	.148257E+02	.313916E+01	.222290E+00	.133478E-01	.335064E+01
5	.363344E+01	.116102E+02	.524461E+01	.835746E-02	.995737E-01	.278536E+01
6	.283852E+01	.975086E+01	.704968E+01	.205544E-01	.197226E+00	.241862E+01
7	.236038E+01	.856545E+01	.841892E+01	.904838E-01	.283630E+00	.218567E+01
8	.204607E+01	.775209E+01	.944816E+01	.170712E+00	.356250E+00	.202983E+01
9	.182622E+01	.716421E+01	.102336E+02	.247007E+00	.416630E+00	.192016E+01
10	.166527E+01	.672259E+01	.108449E+02	.315410E+00	.466907E+00	.183970E+01
11	.154327E+01	.638077E+01	.113299E+02	.375406E+00	.509015E+00	.177865E+01
12	.144820E+01	.610980E+01	.117214E+02	.427632E+00	.544541E+00	.173105E+01
13	.137241E+01	.589070E+01	.120422E+02	.473043E+00	.574749E+00	.169309E+01
14	.131086E+01	.571059E+01	.123089E+02	.512604E+00	.600629E+00	.166225E+01
15	.126007E+01	.556043E+01	.125331E+02	.547190E+00	.622966E+00	.163678E+01
16	.121758E+01	.543369E+01	.127236E+02	.577552E+00	.642376E+00	.161548E+01
17	.118161E+01	.532559E+01	.128871E+02	.604325E+00	.659353E+00	.159743E+01
18	.115085E+01	.523250E+01	.130286E+02	.628041E+00	.674289E+00	.158199E+01
19	.112430E+01	.515167E+01	.131519E+02	.649141E+00	.687504E+00	.156865E+01
20	.110119E+01	.508096E+01	.132602E+02	.667994E+00	.699256E+00	.155704E+01
21	.108093E+01	.501869E+01	.133559E+02	.684910E+00	.709756E+00	.154686E+01

(B) Discussion and Conclusions

One of the original aims of the present study was to determine the level structure of the $q-\bar{q}$ system and compare the computed results with those observed for the charmonium system, the placement of the $^1S_0$, $^1P_1$, $^1D_2$, and $^3D_2$ configurations being of most interest. When we calculate the radiative decay widths of the charmonium states, we also determine directly the masses of several of the ground state energies in Gev. The position of these levels, given as a function of RAT, for various quark mass parameters, is given in TABLE 14. The computer output gives these computed masses for all m between 0 and 20, the ones given in TABLE 14 being in the range where the best fit occurs to the harmonic potential's parameters. From the results given in TABLE 14 the $^1S_0$ states come out fairly well, but the ^3P states are rather poorly determined, being off by 100 - 200 Mev. The ^3P state splittings are also incorrect. Observationally the $(^3P_1-^3P_0)$ splitting is about three times the $(^3P_2-^3P_1)$ splitting but the computed values give the $(^3P_2-^3P_1)$ splitting to be about three times the $(^3P_1-^3P_0)$ splitting. This, along with the improper ordering of the $^3S_1$ and $^3D_1$ levels, shows that there is an important contribution to the potential missing in this model.

The calculations using a coulomb potential $-\alpha/r$ give the well known ordering 1S; 2S and 1P degenerate; 3S, 2P and 1D degenerate, etc. while the same calculations for the harmonic potential βr^2 give the ordering 1S; 1P; 2S and 1D degenerate, etc. This observation implies that a linear combination of these two potentials could be obtained to produce the desired order structure for the $^3S_1$ and $^3D_1$ states. This observation strongly implies that the missing piece in the potential is of the form α/r for some finite α. Given the matrix elements of 1/r for

our trial wavefunctions, it would be an easy matter to search for a set of acceptable parameters RAT, m, k and α to fit the spectrum and possibly give a better handle on the decay widths for the charmonium system.

In the case of the transition widths the electric dipole transitions yield the correct order of magnitude. The observed widths are in several cases lower than the ones determined in this calculation. The radiative decay widths for the $2^3S_1 \rightarrow 1^3P_2 + \gamma$ and $2^3S_1 \rightarrow 1^3P_1 + \gamma$, for the values of the quark masses that result from fitting the $\psi'(3685)$, $\psi(3095)$ mass difference, $\Gamma(\psi(3095) \rightarrow e^+ + e^-)$ decay width and the triplet P splittings, are much larger than the experimental findings. The calculated decay width for the $2^3S_1 \rightarrow 1^3P_0 + \gamma$ is close to the experimental results, and in general is between the values given by the particle data group in [40] and the latest crystal ball experimental finding [41]. The systematic behavior of the radiative decay widths for $\psi' \rightarrow {}^3P_{0,1,2}$ reported in [30] decrease as a function of J. Those calculated here increase with J. The experimental findings show a small decrease with J but the experimental errors are such one could say that the decay widths are the same for J=0,1, and 2. The decay widths for the ^3P states to the 1^3S_1 state are rather poorly determined, the $1^3P_0 \rightarrow 1^3S_1 + \gamma$ being the best determined with an error of about 40 percent. For the other cases the results given in [41] give about a 66 percent error for the $1^3P_2 \rightarrow 1^3S_1 + \gamma$ and only an upper limit of less than 700 kev for the $1^3P_1 \rightarrow 1^3S_1 + \gamma$. The $1^3P_J \rightarrow \psi(3095) + \gamma$ decay widths, in the region of the quark mass determined above, give the same result for J=1 and J=2 and about 40 kev less for J=0. The experimental results could be consistant with almost anything.

The calculated magnetic dipole radiative decay widths for the transitions $2^3S_1 \rightarrow 1^1S_0 + \gamma$, $2^3S_1 \rightarrow 2^1S_0 + \gamma$, and $1^3S_1 \rightarrow 1^1S_0 + \gamma$ are given in figures 22 to 24 and in TABLE 15.

The magnetic dipole transition widths calculated here seem to be generally lower than those determined experimentally, the computed widths ranging from a third to a fifth below those found by experiment. This is independent of the value of RAT used in the calculation, the only possible exception being the $\psi'(3685) \rightarrow \eta_c(2980) + \gamma$. For this process the computed decay width is changing very rapidly in the region of masses allowed by the fit in figure 12 and could be somewhat higher (or lower) than the measured value.

In the calculation of the radiative decay widths of the charmonium system, we needed only information on the $\psi'(3685) - \psi(3095)$ mass difference and the real masses of the charmonium system. The decay width calculation was carried out for 21 different values of the quark mass. From this calculation no prediction for the decay widths can be made because no value for the quark mass is determined. To determine a choice, or range of choices, for the quark mass, we need some additional information. This additional information comes from requiring that we obtain a simultaneous fit to the $\psi'(3685) - \psi(3095)$ mass difference and the $\psi(3095) \rightarrow e^+ + e^-$ decay width. This is done in principle by varying the force constant in the harmonic oscillator potential. For each choice of RAT this gives a range of values for m, the quark mass. If we try to specify RAT, we also need an additional fit requirement. The one chosen here was to try to make the observed $^3P_2(3551) - {}^3P_1(3507)$ mass difference be equal to the respective computed mass differences. It was not possible to obtain this simultaneous fit from RAT. (See figures

9-12.) We instead chose a value of RAT which produced a $\psi'(3685) - \psi(3095)$ and $\Gamma[\psi(3095) \rightarrow e^+ + e^-)]$ fit for the quark mass lying about half way between the zero points of the observed mass differences minus the computed mass differences of the $^3P_2(3551) - {}^3P_0(3414)$ and $^3P_2(3551) - {}^3P_1(3507)$ states. (See figure 10.) One can however make use of the decay width plots in a different way. We can ask what limits can be placed upon the quark mass by the requirement that the computed decay widths be compatible with the observed widths. From the results for the $^3P_J \rightarrow \psi(3095) + \gamma$ widths, if the quark mass is much lower than m = 5.0, the radiative decay widths become very large. If the quark mass is very large, the decay widths could become too small compared to experiment to be allowed. This alternate viewpoint could be very useful in obtaining limits on the quark masses when the decay widths for the ^3P_J states become reasonably well determined. This same technique could be used with other potential theories of $q\text{-}\bar{q}$ systems when the necessary calculations are made, e.g. for the $\alpha r + \beta/r$ potentials.

From the calculated results displayed in TABLE 23, most of the widths predicted by this model are larger than those observed experimentally, the S states being less than a factor of two different and the P states being less than a factor of ten off.

The most interesting observation to be made from the last set of tables (25 to 30) is to see where the decay widths computed by Cheat are approximately equal to those determined in Orthog. The method used in the Cheat calculation is typical of the charmonoium potential model calculations. It replaces the momentum matrix element by the position matrix element in the transition width formula, which is typical of what is done in the study of atomic transition widths. This substitution is

valid in the non-relativistic limit applicable to atomic spectra studies, but from the results displayed in tables 26, 28, and 30 seems inappropriate for charmonium model studies.

In view of the disagreement between the computed and observed properties of the charmonium system, one can say that the model as currently constituted is incorrect. It is possible that the addition of one or more terms to our effective potential could produce results more in tune with nature. The first obvious choice is the form α/r. The reason for this choice, as indicated above, is to rectify the ordering of the energy level structure in the $^3(S\text{-}D)_1$ system. The changes in the wavefunctions induced by this change in the potential might be sufficient to induce an appreciable change in the transition widths given above. A comparison of the radial integral for the $\psi' \rightarrow {}^3P_J + \gamma$ decay with the 2^3S_1 wavefunctions shows that the major contribution to the radial integral comes from a range of momentum values near the node of the 2^3S_1 wavefunction. See figures 25, 26, and 27. The corresponding curves for the magnetic dipole transitions are shown in figure 28. Any changes of the wavefunction in this region could have an appreciable effect upon the values of the radiative decay widths.

Figure 25. Radial Wavefunctions and Radial Integrals for Charmonium Transitions, RAT = -0.16

_____ Normalized Radial Wavefunctions for 1^3S_1 state

••••• Normalized Radial Wavefunctions for 1^3P_0 state

----- Normalized Radial Wavefunctions for 2^3S_1 state

-•-•- 8.0 times Radial Integral Accumulation for $2^3S_1 \rightarrow 1^3P_0 + \gamma$ transition

-••••- 6.0 times Radial Integral Accumulation for $1^3P_0 \rightarrow 1^3S_1 + \gamma$ transition

The vertical scale is in dimensionless units from -2.00 to +2.00.

The horizontal scale is the dimensionless quark momentum from 0.00 to 10.00.

The mass of the quarks used is 5.50 in dimensionless units, which corresponds to 2.23 Gev.

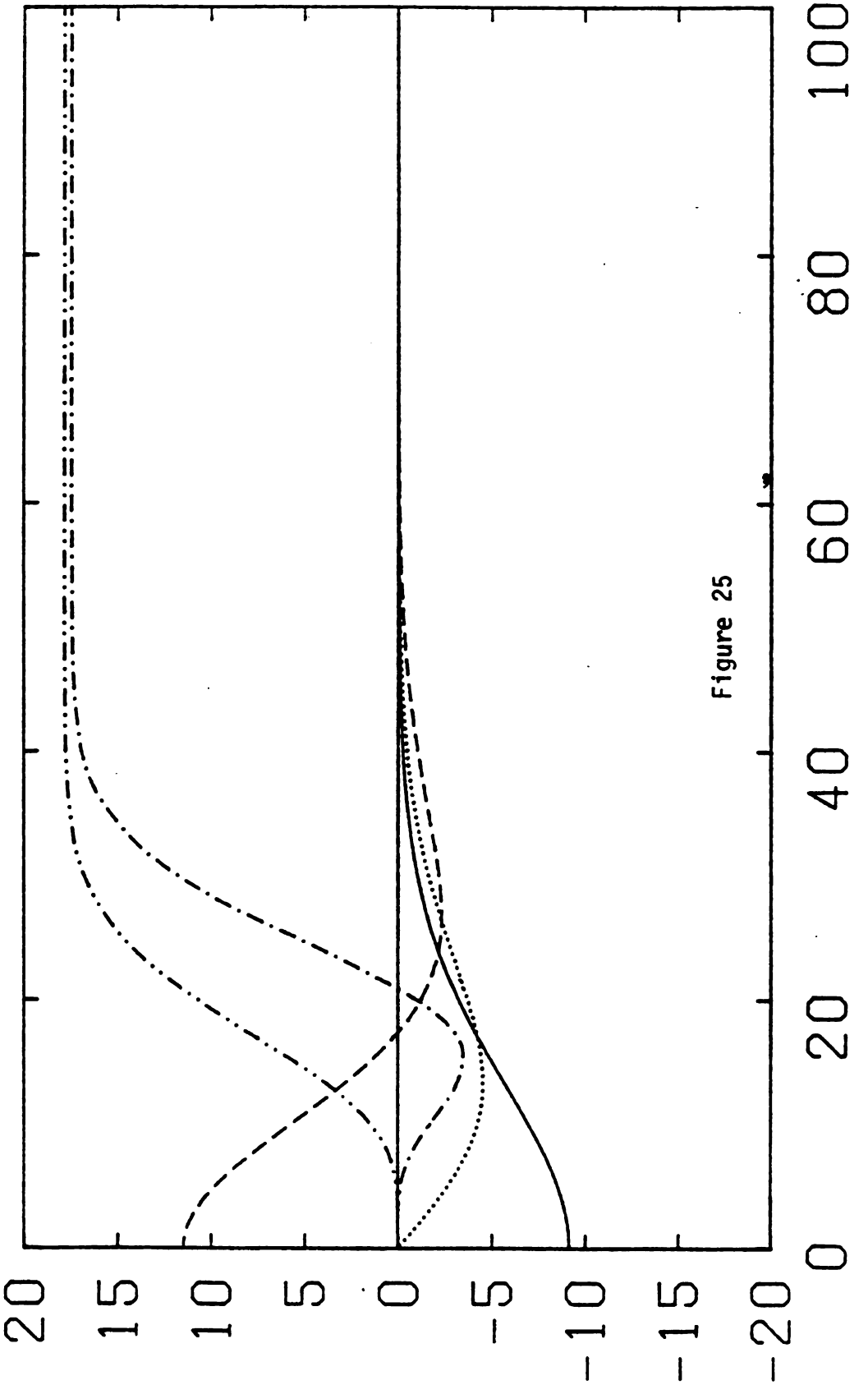


Figure 25

Figure 26. Radial Wavefunctions and Radial Integrals for Charmonium Transitions, RAT = -0.16

_____ Normalized Radial Wavefunctions for 1^3S_1 state

····· Normalized Radial Wavefunctions for 1^3P_1 state

----- Normalized Radial Wavefunctions for 2^3S_1 state

-·-·- 8.0 times Radial Integral Accumulation for $2^3S_1 \rightarrow 1^3P_1 + \gamma$ transition

-··-··- 6.0 times Radial Integral Accumulation for $1^3P_1 \rightarrow 1^3S_1 + \gamma$ transition

The vertical scale is in dimensionless units from -2.00 to +2.00.

The horizontal scale is the dimensionless quark momentum from 0.00 to 10.00.

The mass of the quarks used is 5.50 in dimensionless units, which corresponds to 2.23 Gev.

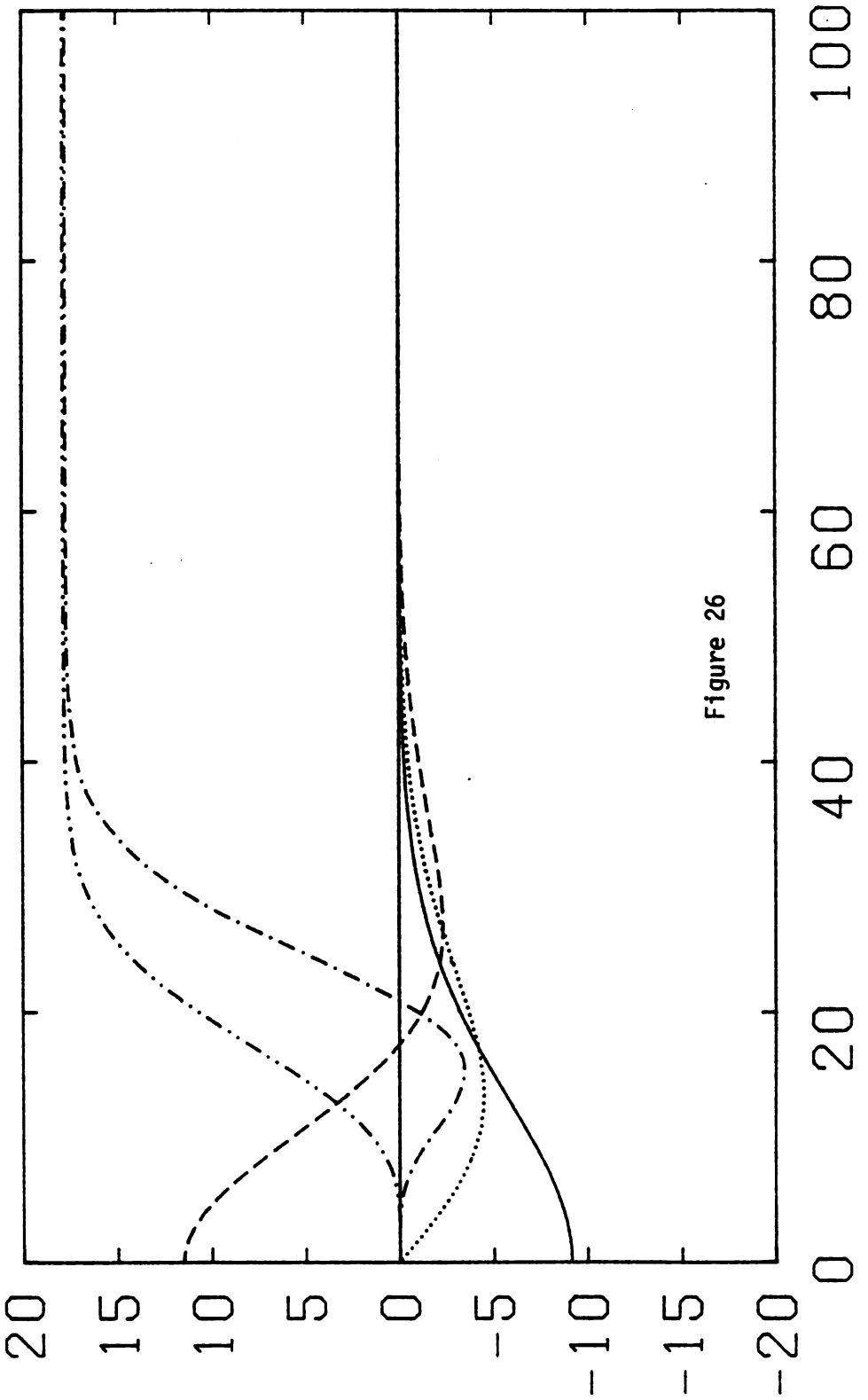


Figure 26

Figure 27. Radial Wavefunctions and Radial Integrals for Charmonium Transitions, RAT = -0.16

_____ Normalized Radial Wavefunctions for 1^3S_1 state

••••• Normalized Radial Wavefunctions for 1^3P_2 state

----- Normalized Radial Wavefunctions for 2^3S_1 state

-•-•- 8.0 times Radial Integral Accumulation for $2^3S_1 \rightarrow 1^3P_2 + \gamma$ transition

-••••- 6.0 times Radial Integral Accumulation for $1^3P_2 \rightarrow 1^3S_1 + \gamma$ transition

The vertical scale is in dimensionless units from -2.00 to +2.00.

The horizontal scale is the dimensionless quark momentum from 0.00 to 10.00.

The mass of the quarks used is 5.50 in dimensionless units, which corresponds to 2.23 Gev.

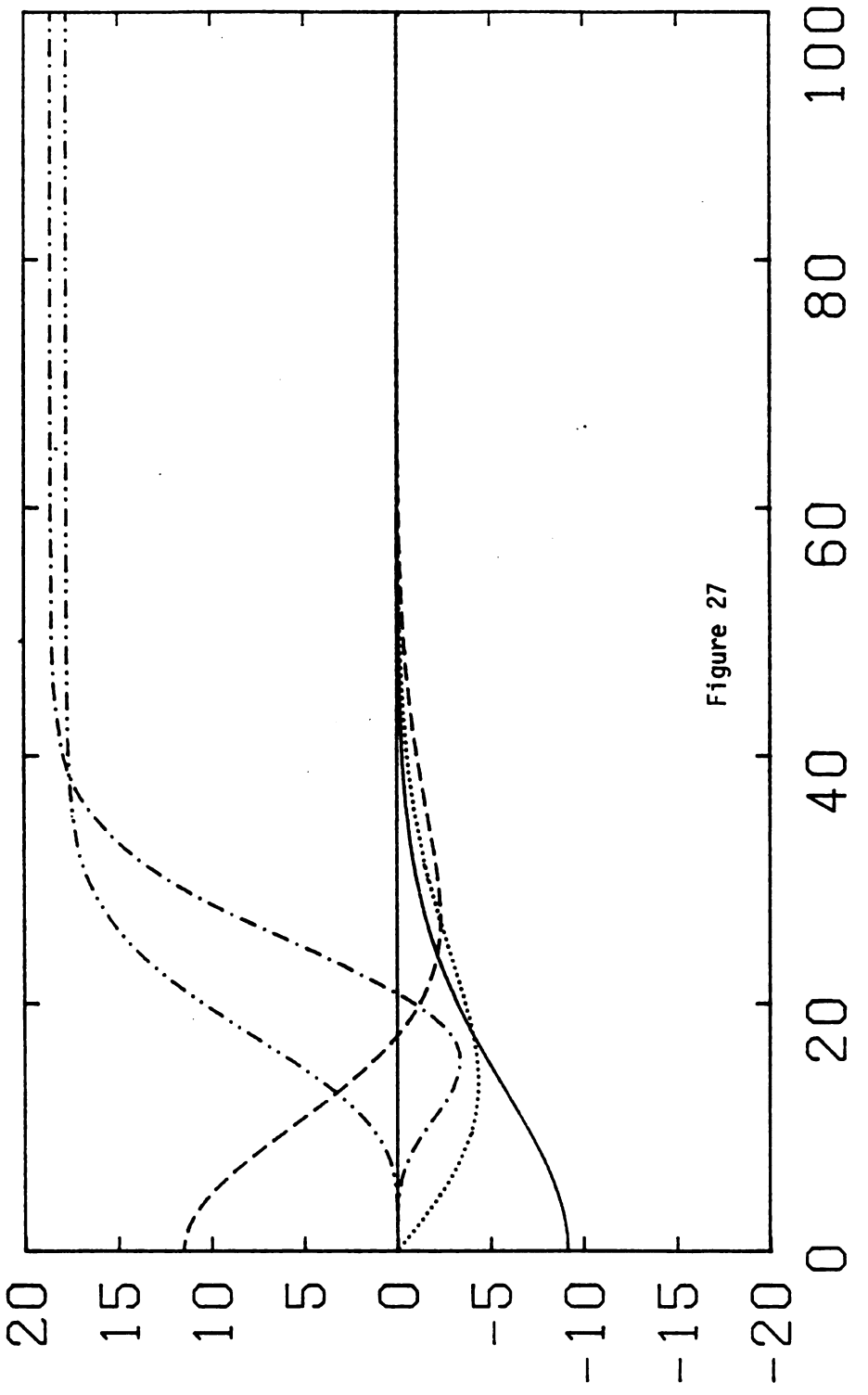


Figure 27

Figure 28. Radial Wavefunctions and Radial Integrals for Charmonium Transitions, RAT = -0.16

_____ Normalized Radial Wavefunctions for 1^3S_1 state

····· Normalized Radial Wavefunctions for 1^1S_0 state

----- Normalized Radial Wavefunctions for 2^1S_0 state

-·-·- Normalized Radial Wavefunctions for 2^3S_1 state

-·-·-·- 30.0 times Radial Integral Accumulation for $2^3S_1 \rightarrow 1^1S_0 + \gamma$ transition

---------- 10.0 times Radial Integral Accumulation for $1^3S_1 \rightarrow 1^1S_0 + \gamma$ transition

..... 10.0 times Radial Integral Accumulation for $2^3S_1 \rightarrow 2^1S_0 + \gamma$ transition

The vertical scale is in dimensionless units from -2.00 to +2.00.

The horizontal scale is the dimensionless quark momentum from 0.00 to 10.00.

The mass of the quarks used is 5.50 in dimensionless units, which corresponds to 2.23 Gev.

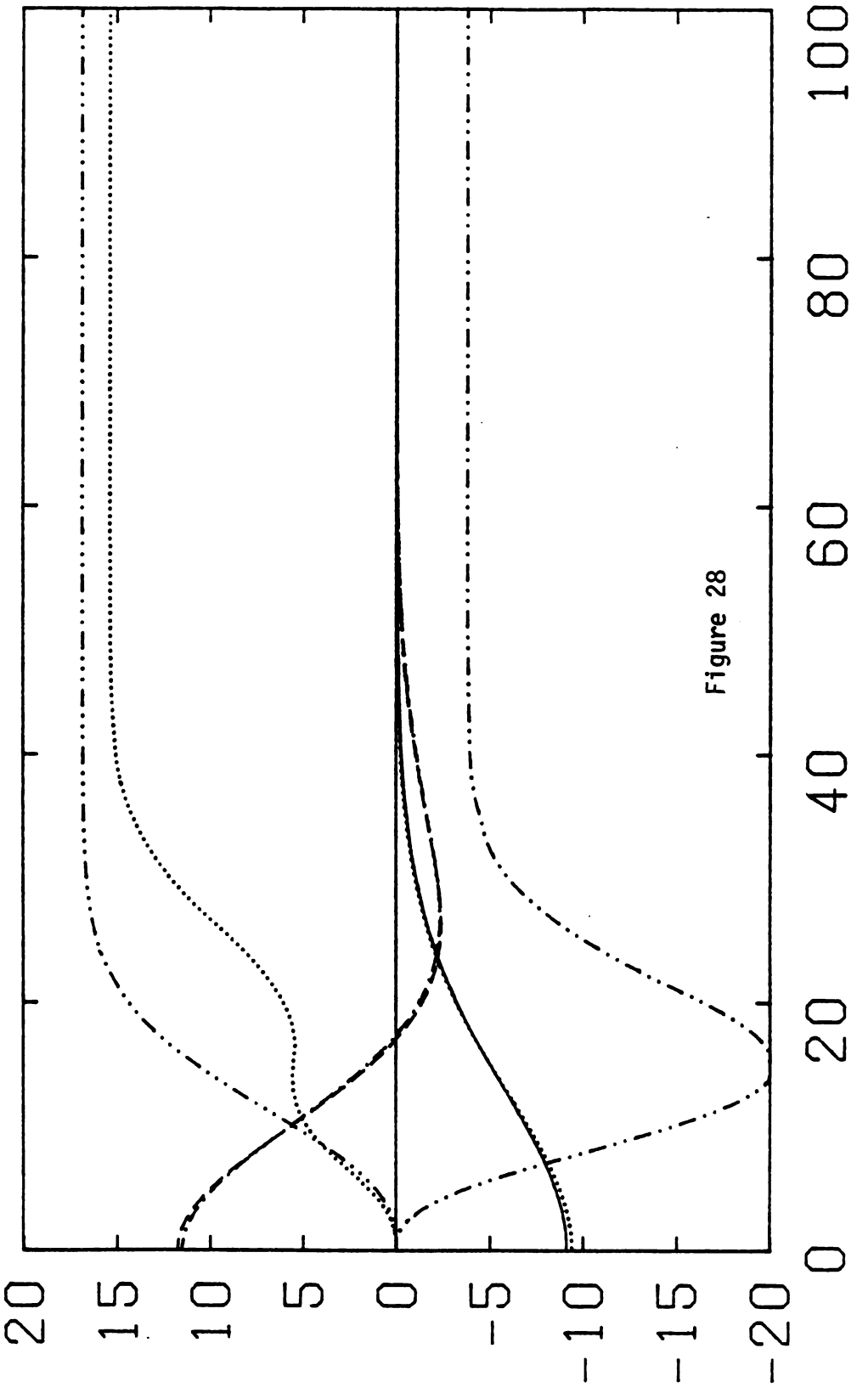


Figure 28

(C) Summary

The Bethe-Salpeter equation is a relativistic wave equation that can be used to describe bound states. This equation is in general difficult to solve. Certain simplifications are usually made in discussions of this equation. One such simplification is to replace the interaction term by some effective potential form to describe the interaction between the two particles. A review of effective potential methods and their application to the hydrogen atom is described in a review article by Grotch and Yennie [28]. The effective potential chosen in this work is a linear combination of two harmonic oscillator potentials. Using this potential between two spin 1/2 particles, the spectrum of this two particle syatem is determined. By using the coupling constants in the potential, and by shifting the overall zero point of the potential, a fit is attempted for the $c\bar{c}$ charmonium system. In this fit the ^3P states are lower than the observed states. The 1^1S_0 and 2^1S_0 states come close to the experimental values. Using the wavefunctions determined by solving, numerically, the radial Bethe-Salpeter equations, radiative decay widths are determined for all those transitions observed experimentally. The computed decay widths are within a factor of three to five of the experimentally determined decay widths. The main defects of this model are the following. (i) It predicts the wrong ordering of the $^3S_1$ and $^3D_1$ states. (ii) The $^3P_0$, $^3P_1$ and $^3P_2$ splittings are incorrect, as well as the $^3S_1$ and ^3P_J splittings. (iii) The systematic behavior of the computed and observed $\psi'(3685) \rightarrow {}^3P_J + \gamma$ decay widths are inconsistent. Experiments indicate that these decay widths are approximately the same for all three ^3P_J states, but the model here gives results that increase with J. The only

electric dipole decay widths in agreement with experiment are the $\psi'(3685) \rightarrow {}^3P_0 + \gamma$ and the ${}^3P_0 \rightarrow \psi(3095) + \gamma$, the best fit being for RAT = -0.16 and m ≅ 5.50. (iv) The magnetic dipole decay widths, when compared to the experimental values, are too low by a factor of three to five. (v) The predicted widths of the 1^1S_0, 2^1S_0, 1^3P_0, and 1^3P_2 based upon results given in [46] are too big.

On the more positive side, one of the most interesting results is the ratio of the radiative decay widths computed by $\langle f|\vec{x}|i\rangle$ and $\langle f|\vec{p}|i\rangle$ given in tables 26, 28, and 30. These ratios show the method used in some potential model charmonium calculations, e.g. [30], to be suspect. The arguments used in support of this substitution of $\langle f|\vec{x}|i\rangle$ for $\langle f|\vec{p}|i\rangle$ are qualitative and not based upon actual calculation. As far as I know this work is the first one to attempt to do both calculations for the same potential function and compare the results.

The substitution when used in this calculation changed the radiative decay widths appreciably. An example of this variation can be seen by looking at TABLE 26 for m=5.0. In this case this substitution produced lower rates for the decays into the triplet P states from the $\psi'(3685)$ level. This substitution also produced lower widths for the ${}^3P_2 \rightarrow \psi(3095) + \gamma$ decays. When this substitution was made in the ${}^3P_0 \rightarrow \psi(3095) + \gamma$ and ${}^3P_1 \rightarrow \psi(3095) + \gamma$ decay widths calculations, the widths were increased by a factor of 2 to 4. This suggests that this substitution yields very unreliable results and any agreement of these calculations with experimental data is strictly coincidental.

In terms of comparison of the calculations of others with mine on the charmonium radiative decay width, the appropriate comparisons are not with my correct calculations displayed in figures 13 to 24 but

rather with the Cheat values listed in tables 25, 27, and 29. The reason for this is that almost all calculations done to date make the replacement $\langle f|\vec{p}|i\rangle$ by $\langle f|\vec{x}|i\rangle$ as is done in the Cheat calculations. As can be seen from tables 26, 28, and 30, there can be, and usually are, appreciable differences resulting in these two methods of determining the radiative decay widths. Hence the validity of these published calculations is very questionable. A comparison of three different radiative decay width calculations with experimental results is shown in figure 29.

Another item peculiar to this work is to show how the decay widths vary as a function of the quark masses. For the electric dipole cases the decay widths are rather flat, as a function of the quark mass, for large values of the quark mass. This would indicate that when information becomes available for the higher mass analogues of the charmonium system, i.e. $b-\bar{b}$ and $t-\bar{t}$, the radiative decay widths could be compatible with experiments over a larger range in the quark masses. In the low quark mass region where the $c-\bar{c}$ system is fit, a change in quark mass of 1 to 2 units produces a large change in some of the decay widths (factors of 3 or more in some cases). This variation of the widths with quark mass can be used to place upper and lower limits to the quark masses. If the quark mass is too low, the radiative decay widths become much too large in many cases. If the quark mass is too large, the radiative decay widths will be smaller than the experimental decay widths. The most sensitive cases are the ^3P_J decays into the $\psi(3095)$ state. To be able to make use of the calculations in this way, however, the experimental uncertainties in the ^3P_J decay widths will have to be improved considerably.

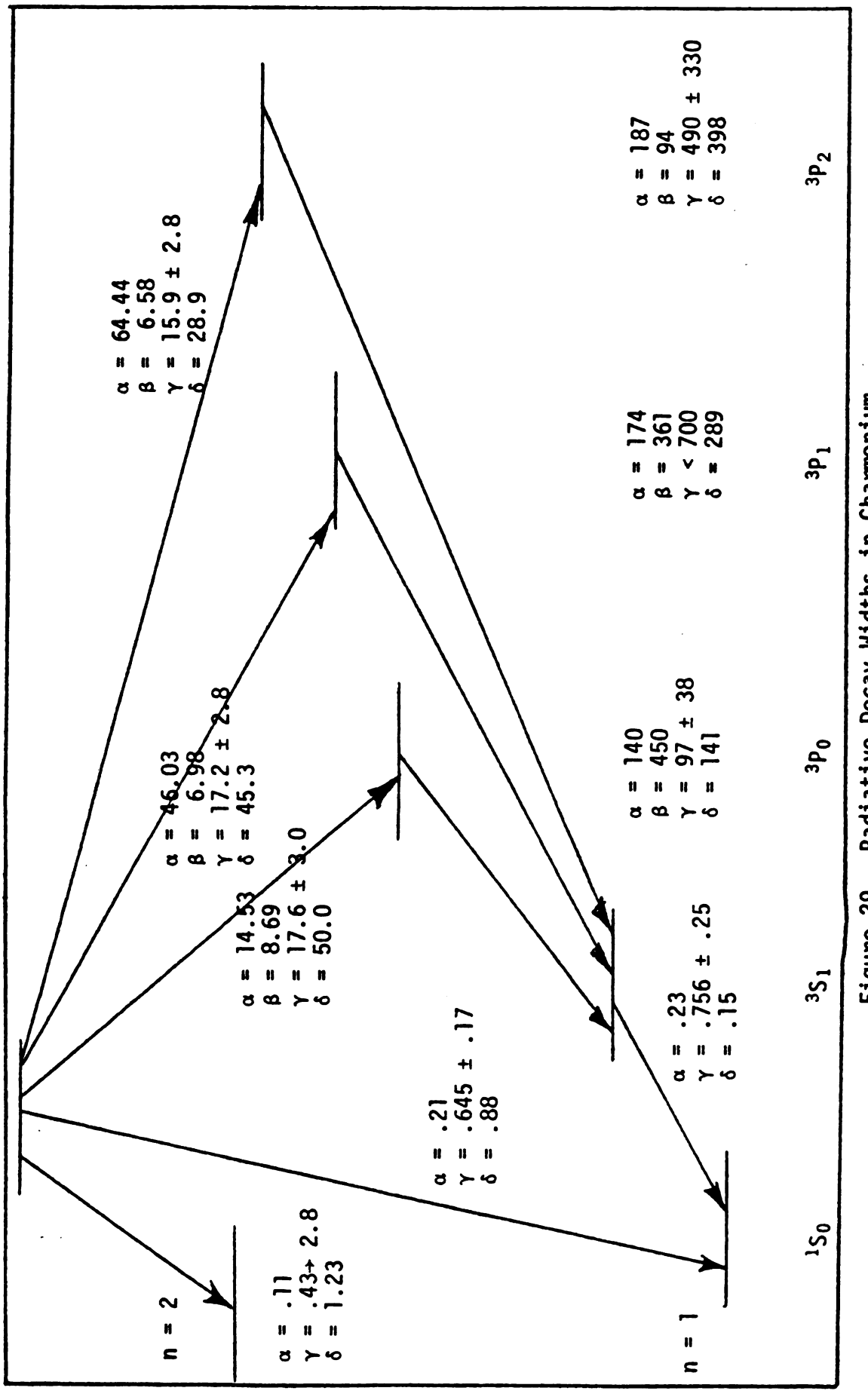


Figure 29. Radiative Decay Widths in Charmonium

α = Orthog calculation, β = Cheat calculation, γ = experimental result, δ = Cornell calculation

The inclusion of a coulomb type potential form might eliminate some or all of these problems. It is uncertain how the inclusion of an α/r potential would affect the radiative decay widths. The α/r potential would affect the wavefunctions most near the origin where the energy levels would be affected, but in this region the decay matrix element contribution is small. Therefore the inclusion of an α/r potential might remedy some of these defects.

APPENDICES

Appendix I.1

Kinematic Constraints on the Singlet and Triplet Wavefunctions

The number of known exact solutions of the Bethe-Salpeter equation is very very limited. In view of this, almost all calculations involving the Bethe-Salpeter equation start out with some truncated or approximate version of the complete equation. Two common assumptions made are:

(i) In a first approximation, retardation effects are ignored initially, and if they are required, then they are computed as perturbation effects.

(ii) One demands that the equation be correct up to order $(v/c)^2$ and be linear in the potential function describing the interaction of the two particles.

This combination of restrictions allows one to write the two-particle Bethe-Salpeter equation in the form:

$$(H(p) + p_+ v p_+)\phi = K_0\phi \qquad (1)$$

where

$$p_+ = H(2\omega + H)/8\omega^2, \qquad p_- = -H(2\omega - H)/8\omega^2 \qquad (2)$$

$$H = 2\omega p_+ - 2\omega p_-, \qquad p_0 = (4\omega^2 - H^2)/4\omega^2$$

where p_+, p_-, and p_0 are projection operators onto the eigenvalues 2ω, -2ω, and 0 of H(p). The application of p_0 and p_- to equation (1) will yield the following equations:

$$p_0[H(p) + p_+ v p_+]\phi = K_0 p_0 \phi = 0$$

So

$$p_0 H(p)\phi + p_0 p_+ v p_+ \phi = 0 \rightarrow H(p) p_0 \phi = 0 \rightarrow H^2(p)\phi/4\omega^2 = \phi$$

$$p_-[H(p) + p_+ v p_+]\phi = K_0 p_- \phi = 0 \quad \text{if } K_0 > 0$$

$$p_- H(p)\phi = H(p) p_- \phi = 0 \rightarrow p_-\phi = 0$$

$$\rightarrow (H^2(p)/8\omega^2)\phi - (2\omega H/8\omega^2)\phi = 0$$

So

$$(H^2(p)/4\omega^2)\phi = (H/2\omega)\phi = \phi \rightarrow H\phi = 2\omega\phi \qquad (3)$$

The consequences of equation (3) upon the wavefunctions can be found by expanding ϕ in terms of various tensor forms, namely

$$\phi(p) = CS(p) + \gamma_\mu V_\mu(p)C + \vec{\sigma}\cdot\vec{T}(p)C/2 + \vec{\alpha}\cdot\vec{T}'(p)C/2$$

$$+ \gamma_\mu A_\mu(p)\gamma_5 C + \gamma_5 CP(p) \qquad (4)$$

The matrix C in this decomposition represents the operation of charge conjugation.

In terms of the two sided matrix notation, equation (3) with $\phi(p)$ as given by equation (4) is:

$$H(p)\phi = (\vec{\alpha}\cdot\vec{p} + \beta m)[CS + \gamma_\mu V_\mu C + \vec{\sigma}\cdot\vec{T}C + \vec{\alpha}\cdot\vec{T}' + \gamma_\mu A_\mu \gamma_5 C + \gamma_5 CP]$$

$$+ [CS + \gamma_\mu V_\mu C + \vec{\sigma}\cdot\vec{T}C + \vec{\alpha}\cdot\vec{T}'C + \gamma_\mu A_\mu \gamma_5 C + \gamma_5 CP](-\vec{\alpha}^T\cdot\vec{p} + \beta^T m)$$

$= 2\omega\phi(p)$.

The expression $(\vec{\alpha}\cdot\vec{p} + \beta m)\phi(p) + \phi(p)(-\vec{\alpha}^T\cdot\vec{p} + \beta^T m)$ is reduced to a simpler form as follows: (i) Move the matrix C in the second term past $-\vec{\alpha}^T$ and β^T. This converts $-\vec{\alpha}^T$ into $\vec{\alpha}$ and β^T into $-\beta$. With this the second term has the form $\psi(p)(\vec{\alpha}\cdot\vec{p} - \beta m)C$ where $\psi(p) = \phi(p)C^{-1}$. (ii) We now combine the same terms in $\phi(p)$, viz., $[(\vec{\alpha}\cdot\vec{p}+\beta m)\vec{\gamma}\cdot\vec{V}(\vec{\alpha}\cdot\vec{p}-\beta m)]C$ and make use of the following identities:

$$(\vec{\alpha}\cdot\vec{A})(\vec{\alpha}\cdot\vec{B}) + (\vec{\alpha}\cdot\vec{B})(\vec{\alpha}\cdot\vec{A}) = 2(\vec{A}\cdot\vec{B})$$

$$(\vec{\alpha}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) + (\vec{\sigma}\cdot\vec{B})(\vec{\alpha}\cdot\vec{A}) = -2\gamma_5(\vec{A}\cdot\vec{B})$$

$$(\vec{\alpha}\cdot\vec{A})(\vec{\gamma}\cdot\vec{B}) + (\vec{\gamma}\cdot\vec{B})(\vec{\alpha}\cdot\vec{A}) = -2i\gamma_5\vec{\gamma}\cdot(\vec{A}\times\vec{B})$$

$$m\beta(\vec{\alpha}\cdot\vec{p}) - m(\vec{\alpha}\cdot\vec{p})\beta = -2i\vec{\gamma}\cdot\vec{p}\ m$$

$$\beta\,\vec{\sigma} - \vec{\sigma}\,\beta = 0$$

$$\beta\,\vec{\gamma} - \vec{\gamma}\,\beta = 2i\vec{\alpha}$$

$$\beta(\vec{\alpha}\cdot\vec{p})+(\vec{\alpha}\cdot\vec{p})\beta = 0$$

The reduction is now rather easily done and the terms with the same γ matrix structure are then collected together. The results are:

$$H(p)\phi = [(\vec{\alpha}\cdot\vec{p} + \beta m)SC + (\vec{\alpha}\cdot\vec{p} - \beta m)SC] + [i(\vec{p}\cdot\vec{V})\beta C + i\vec{\gamma}\cdot(\vec{p}\times\vec{V})\gamma_5 C$$

$$-im(\vec{\alpha}\cdot\vec{V})C + i\vec{\gamma}\cdot(\vec{p}\times\vec{V})\gamma_5 C - i\beta(\vec{p}\cdot\vec{V})C - im(\vec{\alpha}\cdot\vec{V})C]$$

$$+[-\gamma_5(\vec{p}\cdot\vec{T})C + i\vec{\alpha}\cdot(\vec{p}\times\vec{T})C - im(\vec{\gamma}\cdot\vec{T})\gamma_5 C - (\vec{p}\cdot\vec{T})\gamma_5 C$$

$$-i\vec{\alpha}\cdot(\vec{p}\times\vec{T})C + im(\vec{\gamma}\cdot\vec{T})\gamma_5 C]$$

$$+[(\vec{p}\cdot\vec{T}')C + i\vec{\sigma}\cdot(\vec{p}\times\vec{T}')C + im(\vec{\gamma}\cdot\vec{T}')C + (\vec{p}\cdot\vec{T}')C$$

$$-i\vec{\sigma}\cdot(\vec{p}\times\vec{T}')C + im(\vec{\gamma}\cdot\vec{T}')C]$$

$$+[-i\gamma_5\beta(\vec{p}\cdot\vec{A})C + i\gamma\cdot(\vec{p}\times\vec{A})C + im(\vec{\sigma}\cdot\vec{A})C + i\gamma_5\beta(\vec{p}\cdot\vec{A})C$$

$$-i\vec{\gamma}\cdot(\vec{A}\times\vec{p})C - im(\vec{\alpha}\cdot\vec{A})C]$$

$$+[-i(\vec{\gamma}\cdot\vec{p})A_4\gamma_5 C + mA_4\gamma_5 C + i(\vec{\gamma}\cdot\vec{p})A_4\gamma_5 C + mA_4\gamma_5 C]$$

$$+[-(\vec{\sigma}\cdot\vec{p})PC + m\beta\gamma_5 CP - (\vec{\sigma}\cdot\vec{p})PC + i\beta\gamma_5 PC]$$

$$= 2(\vec{\alpha}\cdot\vec{p})SC + [2i\vec{\gamma}\cdot(\vec{p}\times\vec{V})\gamma_5 C - 2im(\vec{\alpha}\cdot\vec{V})C]$$

$$+ [-2(\vec{p}\cdot\vec{T})\gamma_5 C] + [2(\vec{p}\cdot\vec{T}')C + 2im(\vec{\gamma}\cdot\vec{T}')C] + [2i\vec{\gamma}\cdot(\vec{p}\times\vec{A})C]$$

$$+[2mA_4\gamma_5 C] + [-2(\vec{\sigma}\cdot\vec{p})PC + 2m\beta\gamma_5 C]$$

$$= 2[\vec{p}\cdot\vec{T}'C + i(m\vec{T}' + (\vec{p}\times\vec{A}))\cdot\vec{\gamma}C - \vec{p}\cdot\vec{\sigma}PC + \vec{\alpha}\cdot(\vec{p}S - im\vec{V})C$$

$$+ i(\vec{p}\times\vec{V})\cdot\vec{\gamma}\gamma_5 C + mP\beta\gamma_5 C + (mA_4 - \vec{p}\cdot\vec{T})\gamma_5 C]$$

$$= 2\omega[SC + \vec{\gamma}\cdot\vec{V}\ C + \gamma_4 V_4 C + \vec{T}\cdot\vec{\sigma}\ C + \vec{\alpha}\cdot\vec{T}'C + \vec{\gamma}\cdot\vec{A}\ \gamma_5 C$$

$$+ A_4\beta\gamma_5 C + \gamma_5 PC].$$

The last equality is just $H\phi = 2\omega\phi$ (equation 3). Comparing the coefficients we see that

$$\vec{p}\cdot\vec{T}' = \omega S \qquad (5i)$$

$$im\vec{T}' + i(\vec{p}\times\vec{A}) = \omega\vec{V} \qquad (5ii)$$

$$-im\vec{V} + \vec{p}S = \omega\vec{T}' \qquad (5iii)$$

$$i(\vec{p}\times\vec{V}) = \omega\vec{A} \qquad (5iv)$$

$$-\vec{p}\ P = \omega\ \vec{T} \qquad (6i)$$

$$mA_4 - \vec{p}\cdot\vec{T} = \omega P \qquad (6ii)$$

$$mP = \omega A_4 \qquad (6iii)$$

The four equations in set (5) come from comparing the coefficients of the following γ matrices: C, $\vec{\gamma}C$, $\vec{\alpha}C$, and $\vec{\gamma}\ \gamma_5 C$ respectively. The set of equations in (6) likewise comes from comparing the coefficients of the following γ matrices: $\vec{\sigma}C$, $\gamma_5 C$, and $\beta\gamma_5 C$ respectively. In this derivation we assumed that the $\gamma_\mu V_\mu$ term has only three components, i.e. $\gamma_\mu V_\mu = \vec{\gamma}\cdot\vec{V} + \beta V_4$ with $V_4 = 0$.

From the set of equations (5) one can see that the scalar S, the axial vector $\vec{A}$, and the "pseudotensor three vector" $\vec{T}'$ can all be expressed as a function of the vector term $\vec{V}$, viz.,

$$S = -(i/m)\vec{p}\cdot\vec{V} \qquad (7i)$$

$$\vec{A} = (i/\omega)\vec{p} \times \vec{V} \qquad (7ii)$$

$$\vec{T}' = -(i/m\omega)(m^2\vec{V} + \vec{p}(\vec{p}\cdot\vec{V})) \qquad (7iii)$$

Equation (7i) comes from (5i) by substituting in (5i) for $\vec{p}\cdot\vec{T}'$ the relation found by dotting the vector $\vec{p}$ into (5ii). Equation (7iii) comes from (5ii) by substituting into (5ii) for $\vec{A}$ from (5iv).

The set of equations (6) can also be expressed in terms of a single potential function. In this case we choose P, and we are led to the following relations:

$$\vec{T} = (-\vec{p}/\omega)\ P \qquad (8i)$$

$$A_4 = (m/\omega)\ P \qquad (8ii)$$

Equation (8i) is just the same as (6i). Equation (8ii) comes from (6ii) using (6i) to eliminate the $\vec{T}$ dependence in favor of the P dependence. One can therefore write the wavefunction for the Bethe-Salpeter equation strictly in terms of the four functions P(p) and $\vec{V}(p)$:

$$^1\phi = (1/(2\sqrt{2}))[-\vec{\sigma}\cdot\vec{p}/\omega + m\beta\gamma_5/\omega + \gamma_5]PC \qquad (9i)$$

$$^3\phi = (1/(2\sqrt{2}))[-i\vec{p}\cdot\vec{V}/m + \vec{\gamma}\cdot\vec{V} - (i/m\omega)(m^2\vec{V} + \vec{p}(\vec{p}\cdot\vec{V}))\cdot\vec{\alpha}$$

$$+ (i/\omega)(\vec{p} \times \vec{V})\cdot\vec{\gamma}\,\gamma_5]C \qquad (9ii)$$

The state decribed by (9i) is the singlet one, and that described by (9ii) is a vector or triplet state. The factor $1/(2\sqrt{2})$ is a normalization convention which comes from the requirement that

$$\int d^3x\, P^*P = 1$$

and

$$\sum_{i,j=1}^{3} \int d^3x\, V_i^+(\delta_{ij} + (1/m^2)p_i p_j)V_j = 1.$$

APPENDIX I.2
SCHOONSCHIP DECOMPOSTION OF THE BETHE-SALPETER EQUATION

THE FOLLOWING SCHOONSCHIP COMPUTER PROGRAMS HAVE PERFORMED THE REDUCTION OF THE SINGLET J-J TO THE FORM GIVEN IN THE WORK OF CUNG AND OTHERS CITED IN THE BIBLIOGRAPHICAL ESSAY. THE REDUCTION IS PERFORMED IN A TWO STEP PROCESS. IN THE FIRST STEP ONE CHECKS THAT THE WAVE FUNCTION IS TRANSFORMED BY LAMBDA-PLUS(PQ) INTO ITSELF.THEN THE FOLLOWING SHORT COMPUTER PROGRAMS COMPLETE THE REDUCTION. THE FIRST PROGRAM INVOLVING LAMBDA-PLUS(PP) ACTS ON THE WAVEFUNCTION B(PQ) GIVING ONE OF THE TERMS IN THE FINAL REDUCTION. IN THE NEXT SET OF TWO COMPUTER PROGRAMS THE REDUCTION IS CARRIED OUT FOR THOSE TERMS THAT HAVE AN OVERALL COEFFICIENT OF C OR C*D RESPECTIVELY. IN EACH CASE THE REDUCTION IS DONE IN TWO PARTS. IN THE FIRST PART THE REDUCTION ALPHA*B *ALPHA IS PERFORMED AND THEN THE LAMBDA-PLUS(PP) TRANSFORMATION IS APPLIED TO THE RESULT,GIVING THE FINAL FORM FOR THIS TERM. IN THE SECOND CASE THE ALPHA-DOT-K TRANSFORMATION IS APPLIED FIRST, AND THEN THE LAMBDA-PLUS(PP) TRANSFORMATION IS APPLIED TO COMPLETE THIS PHASE OF THE REDUCTION.

NOTE THAT THE FIRST FOUR CARDS BELOW ARE COMMON TO ALL OF THE COMPUTER REDUCTION PROGRAMS LISTED BELOW.

FOR CONVENIENCE, A FACTOR OF ONE-HALF HAS BEEN OMITTED FROM ALL OF THE A AND C TERMS IN THE PROGRAMS LISTED BELOW.

```
S M,W,V,E,EE
I J,MU,NU
F A, B,C,F,H,
V PP,PQ,PK,PV=U,PW,PX=U,PY,PT=U,PU=U
Z=A*B*C
ID,A=1.0+1*G(J,4)*G(J,PQ)/V+M*G(J,4)/V
```

```
ID,C=1.0+I*G(J,4)*G(J,PQ)/V-M*G(J,4)/V
ID,B=I*G5(J)*G(J,4)*G(J,PQ)/V+M*G(J,4)*G5(J)/V+G5(J)
ID,TRICK,J
ID,G(J,PQ)*G(J,4)=-G(J,4)*G(J,PQ)
ID,G(J,PQ)*G(J,PQ)=PQDPQ
ID,G(J,PP)*G(J,4)=-G(J,4)*G(J,PP)
ID,G(J,PQ)*G(J,PP)=PQDPP+G5(J)*G(J,4)*G(J,PV)
ID,G(J,PP)*G(J,PQ)=PQDPP-G5(J)*G(J,4)*G(J,PV)
ID,FUNCT,PV(MU+)=EPF(MU,PQ,PP,4)
ID,PPDPP=W**2-M**2
ID,PQDPQ=V*V-M*M
ID,PQ(4)=0
ID,PP(4)=0.00
*END
```

```
Z=A*B*C
ID,A=1.0+I*G(J,4)*G(J,PP)/W+G(J,4)*M/W
ID,C=1.0+I*G(J,4)*G(J,PP)/W-G(J,4)*M/W
ID,B=I*G5(J)*G(J,4)*G(J,PQ)/V+M*G(J,4)*G5(J)/V+G5(J)
ID,TRICK,J
ID,G(J,PQ)*G(J,4)=-G(J,4)*G(J,PQ)
ID,G(J,PQ)*G(J,PQ)=PQDPQ
ID,G(J,PP)*G(J,4)=-G(J,4)*G(J,PP)
ID,G(J,PQ)*G(J,PP)=PQDPP+G5(J)*G(J,4)*G(J,PV)
ID,G(J,PP)*G(J,PQ)=PQDPP-G5(J)*G(J,4)*G(J,PV)
ID,FUNCT,PV(MU+)=EPF(MU,PQ,PP,4)
ID,PPDPP=W**2-M**2
ID,PQDPQ=V*V-M*M
ID,PQ(4)=0
ID,PP(4)=0.00
*END
```

IN THE NEXT TWO PROGRAMS, AN OVERALL MINUS SIGN HAS BEEN OMITTED. THE OUTPUT OF THESE PARTS GENERATES THE C TERM FOR THE BETHE-SALPETER EQUATION IN THE WORK OF CUNG ET.AL. CITED IN THE BIBLIOGRAPHY.

```
Z=A+C+F
ID,A=G(J,1)*B*G(J,1)
ID,C=G(J,2)*B*G(J,2)
```

```
ID,F=G(J,3)*B*G(J,3)
ID,B=1*G5(J)*G(J,4)*G(J,PQ)/V+M*G(J,4)*G5(J)/V+G5(J)
ID,TRICK,J
ID,G(J,PQ)*G(J,4)=-G(J,4)*G(J,PQ)
ID,PQ(4)=0.00
*END



Z=A*B*C
ID,A=1.0+1*G(J,4)*G(J,PP)/W+G(J,4)*M/W
ID,B=-3.0*G5(J)-3.0*M*G5(J)*G(J,4)/V-1*G5(J)*G(J,4)*G(J,PQ)/V
ID,C=1.0+1*G(J,4)*G(J,PP)/W-G(J,4)*M/W
ID,TRICK,J
ID,G(J,PQ)*G(J,4)=-G(J,4)*G(J,PQ)
ID,G(J,PP)*G(J,4)=-G(J,4)*G(J,PP)
ID,G(J,PQ)*G(J,PP)=PPDPQ+1*G5(J)*G(J,4)*G(J,PX)
ID,G(J,PP)*G(J,PQ)=PPDPQ-1*G5(J)*G(J,4)*G(J,PX)
ID,FUNCT,PX(MU+)=EPF(MU,PQ,PP,4)
ID,DOT PR,PX(MU+)=EPF(MU,PQ,PP,4)
ID,G(J,PP)*G(J,PP)=W**2-M**2
ID,PPDPP=W*W-M*M
ID,PQ(4)=0.00
ID,PP(4)=0.00
*END



Z=A*B*C
ID,A=1*G(J,4)*G(J,PK)
ID,C=1*G(J,4)*G(J,PK)
ID,B=1*G5(J)*G(J,4)*G(J,PQ)/V+M*G(J,4)*G5(J)/V+G5(J)
ID,TRICK,J
ID,G(J,PQ)*G(J,4)=-G(J,4)*G(J,PQ)
ID,G(J,PK)*G(J,4)=-G(J,4)*G(J,PK)
ID,G(J,PQ)*G(J,PQ)=PQDPQ
ID,PQDPQ=V**2-M**2
ID,PKDPK=1.0
ID,PK(4)=0.00
ID,PQ(4)=0.00
*END



Z=A*B*C
ID,A=1.0+1*G(J,4)*G(J,PP)/W+G(J,4)*M/W
ID,C=1.0+1*G(J,4)*G(J,PP)/W-G(J,4)*M/W
ID,B=G5(J)+M*G5(J)*G(J,4)/V-1*G5(J)*G(J,4)*G(J,PQ)/V
```

```
     +2.0*I*G5(J)*G(J,4)*G(J,PK)*PQDPK/V
ID,TRICK,J
ID,G(J,PP)*G(J,4)=-G(J,4)*G(J,PP)
ID,G(J,PK)*G(J,4)=-G(J,4)*G(J,PK)
ID,G(J,PQ)*G(J,4)=-G(J,4)*G(J,PQ)
ID,G(J,PQ)*G(J,PQ)=PQDPQ
ID,G(J,PQ)*G(J,PP)=PQDPP+I*G5(J)*G(J,4)*G(J,PV)
ID,G(J,PP)*G(J,PQ)=PQDPP-I*G5(J)*G(J,4)*G(J,PV)
ID,FUNCT,PV(MU+)=EPF(MU,PQ,PP,4)
ID,DOT PR,PV(MU+)=EPF(MU,PQ,PP,4)
ID,G(J,PP)*G(J,PK)=PPDPK+G5(J)*G(J,4)*G(J,PU)
ID,G(J,PK)*G(J,PP)=PPDPK-G5(J)*G(J,4)*G(J,PU)
ID,FUNCT,PU(MU+)=EPF(MU,PP,PK,4)
ID,PQDPQ=V**2-M**2
ID,PPDPP=W**2-M**2
ID,PKDPK=1.0
ID,PP(4)=0.00
ID,PQ(4)=0.00
ID,PK(4)=0.00
*END
```

Appendix I.3

(i) Derivation of the ^1J_J Equation for a Coupled Vector-Scalar Interaction

The basic two-body $q\bar{q}$ Bethe-Salpeter equation with both a vector and a scalar harmonic oscillator potential is

$$2(\omega-\mu)f(\vec{p}) + (v/(\omega\omega'))[(2\omega\omega'-m^2) + A(m^2 +\omega\omega'- \vec{p}\cdot\vec{p}')]f(\vec{p}') = 0 \quad (1)$$

Here $v = -(k/2)\nabla^2_{p'}\,\delta^{(3)}(\vec{p} - \vec{p}')$ and A is one half of the ratio of the scalar to vector coupling constant k. The derivation will consist of rewriting $\nabla^2_{p'}$ as $\nabla^2_r + (1/p^2)\nabla^2_{\hat{}}$ and performing the operations of the radial and angular parts of the Laplacian in this equation separately and then combining the results at the end. To simplify things, rewrite (1) as

$$2(\omega-\mu)f(\vec{p}) + (v/(\omega\omega'))[(2+A)\omega\omega'- m^2(1-A) - A\,\vec{p}\cdot\vec{p}']f(\vec{p}') = 0$$

This equation's radial part can be written as:

$$(-2/k)[2(\omega-\mu)]f + (2+A)\nabla^2_r f - (1-A)(m^2/\omega)\nabla^2_r(f/\omega)$$

$$- (Ap/\omega)\nabla^2_r(pf/\omega) \quad (2)$$

At this point we shall make use of the following identities:

$$\nabla^2_r(f/\omega) = (1/\omega)\nabla^2_r f - (2p/\omega^3)df/dp - (3m^2/\omega^5)f \quad (3)$$

$$\nabla^2_r(pf/\omega) = (p/\omega)\nabla^2_r f + (2m^2/\omega^3)df/dp + [(2m^4/p) - (m^2 p\ )]f/\omega^5 \quad (4)$$

$$\nabla^2_r(p^2 f/(\omega(\omega+m))) = \nabla^2_r f - m\nabla^2_r(f/\omega)$$

$$= \nabla_r^2 f - (m/\omega)\nabla_r^2 f + (2mp/\omega^3)df/dp + (3m^3/\omega^5)f \quad (5)$$

Inserting (3), (4), and (5) into (2) we have:

$$(-2/k)[2(\omega-\mu)]f + (2+A)\nabla_r^2 f - (1-A)(m^2/\omega)[(1/\omega)\nabla_r^2 f - (2p/\omega^3)df/dp$$

$$- (3m^2/\omega^5)f] - (Ap/\omega)[(p/\omega)\nabla_r^2 f + (2m^2/\omega^3)df/dp$$

$$+ ((2m^4/p) - (m^2p))f/\omega^5]$$

$$= (-2/k)[2(\omega-\mu)]f + \nabla_r^2 f[(2+A) - (1-A)(m^2/\omega^2) - (Ap^2/\omega^2)]$$

$$+ df/dp[(1-A)(2m^2p/\omega^4) - (2m^2pA/\omega^4)]$$

$$+ [(1-A)(3m^4/\omega^6) - (2m^4/\omega^6)A + (Am^2p^2/\omega^6]f$$

$$= (-2/k)[2(\omega-\mu)]f + \nabla_r^2 f[(2\omega^2-\alpha m^2)/\omega^2] + df/dp[2m^2p\alpha/\omega^4]$$

$$+ [(3m^4/\omega^6)\alpha - (Am^2/\omega^4)]f \quad (6)$$

where $\alpha = 1-2A$.

Now look at the terms involving the angular part of the Laplacian. In this case we have:

$$(v/(\omega\omega'))[(2+A)\omega\omega' - m^2(1-A) - A\,\vec{p}\cdot\vec{p}']f(\vec{p}') =$$

$$[(2+A)/p^2]\nabla_<^2 f - (1-A)(m^2/(\omega^2p^2))\nabla_<^2 f - (Ap^2/(\omega^2p^2))\hat{p}_i\nabla_<^2\hat{p}_i f \quad (7)$$

At this point we make use of the following identity:

$$[\nabla_{<}^2, \hat{p}_i] = -2\hat{p}_i + 2\nabla_i \tag{8}$$

With identity (8) the expression (7) becomes:

$$(1/(\omega^2p^2))[(2+A)\omega^2-m^2(1-A)]\nabla_{<}^2 f - A(p^2/(\omega^2p^2))\hat{p}_i[\hat{p}_i\nabla_{<}^2-2\hat{p}_i+ 2\nabla_i]f$$

$$= (1/(\omega^2p^2))[(2+A)\omega^2 - m^2(1-A) - Ap^2]\nabla_{<}^2 f + (2A/\omega^2)f + 0 \tag{9}$$

In (9) we have made use of the fact that $\hat{p}_i\nabla_i = 0$.

Now combining (9) with (6) we get the desired final expression for the ^1J_J equation:

$$(-2/k)[2(\omega-\mu)]f + [\nabla_r^2 f + (1/p^2)\nabla_{<}^2 f]((2\omega^2 - \alpha m^2)/\omega^2)$$

$$+ df/dp(2m^2p/\omega^4)\alpha + [(3m^4/\omega^6)\alpha + (Ap^2/\omega^4)]f = 0 \tag{10}$$

The quantity α appearing in expression (6) and equation (9) is just $1-2A$.

(ii) Derivation of ^3J_J Equation for a Coupled Vector-Scalar Interaction

The basic two-body $q\bar{q}$ Bethe-Salpeter equation with both a vector and a scalar harmonic oscillator potential is:

$$2(\omega-\mu)\vec{f}(\vec{p}) + (v/(\omega\omega'))\{[(\omega\omega'+\vec{p}\cdot\vec{p}')I + (\vec{p}\vec{p}'-\vec{p}'\vec{p})]$$

$$+ A[(m^2 + \omega\omega' - \vec{p}\cdot\vec{p}')I - (\vec{p}\vec{p}'-\vec{p}'\vec{p})\}\cdot\vec{f}(\vec{p}') = 0 \quad (1)$$

The quantity A is one half of the scalar to vector coupling strength. Collecting terms simplifies equation (1) somewhat. When this is done, the result is:

$$2(\omega-\mu)\vec{f}(\vec{p}) + (v/(\omega\omega'))\{[\omega\omega'(1+A) + \vec{p}\cdot\vec{p}'(1-A) + Am^2]I$$

$$+ (1-A)(\vec{p}\vec{p}'-\vec{p}'\vec{p})\}\cdot\vec{f}(\vec{p}') = 0. \quad (2)$$

We now concentrate on the term involving the v in equation (2). The quantity v for a harmonic oscillator potential takes the form $(-k/2)\delta^{(3)}(\vec{p}-\vec{p}')\nabla^2_{p'}$. For ease of handling we shall split the Laplacian up into its radial and angular parts and perform the reductions separately. We begin with the radial component reduction.

$$(-2/k)[2(\omega-\mu)f + [\nabla^2_r f(1+A) + (1-A)(p/\omega)\nabla^2_r(pf/\omega) +$$

$$+ (Am^2/\omega)\nabla^2_r(f/\omega)] + 0 \quad (3)$$

We now make use of the following identities in (3):

(i) $\nabla^2_r(f/\omega) = (1/\omega)\nabla^2_r f - (2p/\omega^3)df/dp - (3m^2/\omega^5)f$

$$\text{(ii)}\quad \nabla_r^2(pf/\omega) = (p/\omega)\nabla_r^2 f + (2m^2/\omega^3)df/dp + [(2m^4/p)-(m^2p)]f/\omega^5$$

With these substitutions we get:

$$(-2/k)[2(\omega-\mu)]f + (1+A)\nabla_r^2 f + (1-A)(p/\omega)[(p/\omega)\nabla_r^2 f + (2m^2/\omega^3)df/dp$$

$$+\{(2m^4/p) - (m^2p)\}f/\omega^5] + (Am^2/\omega)[(1/\omega)\nabla_r^2 f - (2p/\omega^3)df/dp$$

$$- (3m^2/\omega^5)f]$$

$$= -(2/k)[2(\omega-\mu)]f + \nabla_r^2 f[(1+A) + (1-A)(p^2/\omega^2) + A(m^2/\omega^2)]$$

$$+ df/dp[(1-A)(2m^2p/\omega^4) - A(2m^2p/\omega^4)]$$

$$+ [(1-A)((2m^4/\omega^6) - (m^2p^2/\omega^6)) - A(3m^4/\omega^6)]f \qquad (4)$$

The coefficients for $\nabla_r^2 f$, df/dp, and f can be further simplified as follows:

$\nabla_r^2 f$: $1+(p^2/\omega^2)=(2\omega^2-m^2)/\omega^2$; $A(1-p^2/\omega^2 + m^2/\omega^2)=A(\omega^2-p^2+m^2)/\omega^2$

$$= 2Am^2/\omega^2$$

df/dp: $(1-A)(2m^2p/\omega^4) - A(2m^2p/\omega^4) = (1-2A)(2m^2p/\omega^4)$

f: $(1-A)((2m^4/\omega^6)-(m^2p^2/\omega^6)) - (3m^4A/\omega^6) = (1-A)(3m^4/\omega^6$

$-m^2/\omega^4) - (3m^4A/\omega^6) = (3m^4/\omega^6)(1-2A) - (1-A)(m^2/\omega^4)$

Setting $\alpha = (1-2A)$, expression (4) can be rewritten as:

$$-(2/k)[2(\omega-\mu)]f + \nabla_r^2 f((2\omega^2-\alpha m^2)/\omega^2) + df/dp[2m^2p\alpha/\omega^4]$$

$$+ [3m^4\alpha/\omega^6 -(1-A)(m^2/\omega^4)]f \qquad (5)$$

To complete the derivation of the ${}^3J_J$ equation we need to reduce the part that comes from the angular part of the Laplacian. The expression to be reduced is, from (3),

$$(1+A)(1/p^2)\nabla_<^2 f_i + (1-A)(p^2/\omega^2)(1/p^2)\hat{p}_j\nabla_<^2\hat{p}_j f_i + (Am^2/(\omega^2p^2))\nabla_<^2 f_i$$

$$+ (1-A)(1/\omega^2)[\hat{p}_i\nabla_<^2\hat{p}_j f_j - \hat{p}_j\nabla_<^2\hat{p}_i f_j] \qquad (6)$$

The reduction of this expression is aided by the use of the following identities:

(i) $[\nabla_<^2, \hat{p}_i] = -2\hat{p}_i + 2\nabla_i$

(ii) $[\nabla_i, \hat{p}_j] = \delta_{ij} - \hat{p}_i\hat{p}_j$

Inserting these into (6) we get:

$$(1+A)(1/p^2)\nabla_<^2 f_i + (1-A)(1/\omega^2)\hat{p}_j[\hat{p}_j\nabla_<^2 f_i - 2\hat{p}_j f_i + 2\nabla_j f_i]$$

$$+(Am^2/(\omega^2p^2))\nabla_<^2 f_i + (1-A)(p^2/(\omega^2p^2))[(\nabla_<^2\hat{p}_i + 2\hat{p}_i - 2\nabla_i)\hat{p}_j f_j$$

$$-(\nabla_<^2\hat{p}_j + 2\hat{p}_j - 2\nabla_j)\hat{p}_i f_j]$$

$$= (1/p^2)\nabla_<^2 f_i[(1+A)+(p^2/\omega^2)(1-A)+(Am^2/\omega^2)] - 2(1-A)(1/\omega^2)f_i$$

$$+ (1-A)[(1/\omega^2)(-2\nabla_i\hat{p}_j f_j + 2\nabla_j\hat{p}_i f_j)]$$

$$= (1/p^2)((2\omega^2-\alpha m^2)/\omega^2)\nabla^2_< f_i - (1-A)(2/\omega^2)f_i$$

$$+ (1-A)(1/\omega^2)(2\nabla_j\hat{p}_i f_j - 2\nabla_i\hat{p}_j f_j).$$

Combining this with (5) we have the final expression for the ^3J_J system.

$$-(2/k)[2(\omega-\mu)]f_i + \nabla^2_r f_i((2\omega^2-\alpha m^2)/\omega^2) + df_i/dp(2m^2p\alpha/\omega^4)$$

$$+ [3m^4\alpha/\omega^6 - (1-A)m^2/\omega^4]f_i - (2(1-A)/\omega^2)f_i$$

$$+ (1/p^2)((2\omega^2-\alpha m^2)/\omega^2)\nabla^2_< f_i + (1-A)(1/\omega^2)(2\nabla_j\hat{p}_i f_j - 2\nabla_i\hat{p}_j f_j) = 0$$

and where

$$\nabla_j\hat{p}_i f_j - \nabla_i\hat{p}_j f_j = (\vec{L}\cdot\vec{S})f_i = (1/2)[j(j+1) - l(l+1) - s(s+1)]f_i$$

(iiia) The Radial Equation for the $^3(J\pm1)_J$ Bethe-Salpeter Equation with Vector Harmonic Potential

The starting equation is:

$$2(\omega-\mu)\vec{f}(\vec{p}) + (v/(\omega\omega'))[(\omega\omega' + \vec{p}\cdot\vec{p}')I - (\omega'/(\omega+m))\vec{p}\vec{p} - (\omega/(\omega'+m))\vec{p}'\vec{p}'$$

$$+ \vec{p}\vec{p}'(\vec{p}\cdot\vec{p}'/((\omega+m)(\omega'+m))+2) - \vec{p}'\vec{p}]\cdot\vec{f}(\vec{p}') = 0 \qquad (1)$$

The equation is an integral equation for $\vec{f}(\vec{p})$. The choice of $v(\vec{p},\vec{p}') = (-k/2)\nabla^2_{p'}\delta^{(3)}(\vec{p}-\vec{p}')$ converts this integral equation into a second order differential equation, which can then be separated in spherical

coordinates to yield a "simple" ordinary differential equation for the radial functions. The reduction makes use of the following relations:

$$\nabla_p^2 = \nabla_r^2 + (1/p^2)\nabla_<^2$$

I. Radial case

(i) $\nabla_r^2(f/\omega) = (1/\omega)\nabla_r^2 f - (2p/\omega^3)\partial f/\partial p - (3m^2/\omega^5)f$

(ii) $\nabla_r^2(pf/\omega) = (p/\omega)\nabla_r^2 f + (2m^2/\omega^3)\partial f/\partial p + [(2m^4/p)-(m^2p)]f/\omega^5$

(iii) $\nabla_r^2(p^2 f/(\omega(\omega+m))) = \nabla_r^2 f - m\nabla_r^2(f/\omega)$

$$= \nabla_r^2 f - (m/\omega)\nabla_r^2 f + (2mp/\omega^3)\partial f/\partial p + (3m^3/\omega^5)f$$

II. Angular case

(i) $[\nabla_<^2, \hat{p}_i] = -2\hat{p}_i + 2\nabla_i$

(ii) $[\nabla_i, \hat{p}_j] = \delta_{ij} - \hat{p}_i\hat{p}_j$

(iii) $[\nabla_<^2, \hat{p}_i\hat{p}_j] = -4\hat{p}_i\hat{p}_j + 2\nabla_i\hat{p}_j + 2\hat{p}_i\nabla_j$

The seven terms in equation (1) involving ∇_r^2 are given next along with reductions using the above relations:

(1) $(v/(\omega\omega'))\omega\omega' = \nabla_r^2 f$

$$(2)\ (v/(\omega\omega'))\vec{p}\cdot\vec{p}' = \hat{p}_i\hat{p}_j(p/\omega)\nabla_r^2(pf/\omega)$$

$$= \hat{p}_i\hat{p}_j(p/\omega)[(p/\omega)\nabla_r^2 f + (2m^2/\omega^3)\partial f/\partial p$$

$$+ (2m^4/p - m^2p)f/\omega^5]$$

$$= \hat{p}_i\hat{p}_i[(p^2/\omega^2)\nabla_r^2 f + (2m^2p/\omega^4)\partial f/\partial p$$

$$+ (3m^4/\omega^6 - m^2/\omega^4)f]$$

$$= (p^2/\omega^2)\nabla_r^2 f + (2m^2p/\omega^4)\partial f/\partial p$$

$$+ (3m^4/\omega^6 - m^2/\omega^4)f]$$

$$(3)\ (-v/(\omega\omega'))(\omega'/(\omega+m))\vec{p}\vec{p} = -\hat{p}_i\hat{p}_j(p^2/(\omega(\omega+m)))\nabla_r^2 f_j$$

$$(4)\ (-v/(\omega\omega'))(\omega/(\omega'+m))\vec{p}'\vec{p}' = -\hat{p}_i\hat{p}_j\nabla_r^2(p^2 f_j/(\omega(\omega+m)))$$

$$= -\hat{p}_i\hat{p}_j[\nabla_r^2 f_j - (m/\omega)\nabla_r^2 f_j + (2mp/\omega^3)\partial f_j/\partial p + (3m^3/\omega^5)f_j]$$

$$(5)\ (v/(\omega\omega'))\vec{p}\vec{p}'[\vec{p}\cdot\vec{p}'/((\omega+m)(\omega'+m))]$$

$$= \hat{p}_i\hat{p}_j(p^2/(\omega(\omega+m)))\nabla_r^2(p^2 f_j/(\omega(\omega+m)))$$

$$= \hat{p}_i\hat{p}_j(p^2/(\omega(\omega+m)))[\nabla_r^2 f_j - (m/\omega)\nabla_r^2 f_j + (2mp/\omega^3)\partial f_j/\partial p + (3m^3/\omega^5)f]$$

$$(6)+(7)\ (v/(\omega\omega'))(2\vec{p}\vec{p}' - \vec{p}'\vec{p}) = \hat{p}_i\hat{p}_j(p/\omega)\nabla_r^2(pf/\omega)$$

$$= \hat{p}_i\hat{p}_j(p/\omega)[(p/\omega)\nabla_r^2 f + (2m^2/\omega^3)\partial f/\partial p + (2m^4/p - m^2p)f/\omega^5]$$

$$= \hat{p}_i\hat{p}_j[(p^2/\omega^2)\nabla_r^2 f + (2m^2p/\omega^4)\partial f/\partial p + (3m^4/\omega^6 - m^2/\omega^4)f]$$

Now collect terms of the form $\nabla_r^2 f_j$, $\partial f_j/\partial p$, and $\hat{p}_i\hat{p}_j f_j$ and look at their coefficients. Doing this we get:

$$\hat{p}_i\hat{p}_j\nabla_r^2 f_j[\delta_{ij} + \delta_{ij}(p^2/\omega^2) - p^2/(\omega(\omega+m)) - 1 + (m/\omega) + p^2/(\omega(\omega+m))$$

$$- mp^2/(\omega^2(\omega+m)) + p^2/\omega^2]$$

$$= \hat{p}_i\hat{p}_j\nabla_r^2 f_j[((2\omega^2-m^2)/\omega^2)\delta_{ij} - (\omega-m)/\omega - 1 + (m/\omega) + (\omega-m)/\omega$$

$$- m(\omega-m)/\omega^2 + p^2/\omega^2]$$

$$= \hat{p}_i\hat{p}_j\nabla_r^2 f_j[((2\omega^2-m^2)/\omega^2)\delta_{ij} - 1 + (m/\omega) - 1 + (m/\omega) + 1 - (m/\omega)$$

$$- (m/\omega) + (m^2/\omega^2) + (p^2/\omega^2)]$$

$$= \hat{p}_i\hat{p}_j\nabla_r^2 f[((2\omega^2-m^2)/\omega^2)\delta_{ij} + 0]$$

$$= \nabla_r^2 f_j((2\omega^2-m^2)/\omega^2)\delta_{ij}$$

$$\hat{p}_i\hat{p}_j\partial f_j/\partial p[\delta_{ij}(2m^2p/\omega^4) - (2mp/\omega^3) + (p^2/(\omega(\omega+m)))(2mp/\omega^3)$$

$$+ (2m^2p/\omega^4)]$$

$$= \hat{p}_i\hat{p}_j\partial f_j/\partial p[\delta_{ij}(2m^2p/\omega^4) - (2mp/\omega^3) + (\omega-m)(2mp/\omega^4) + (2m^2p/\omega^4)]$$

$$= \hat{p}_i\hat{p}_j\partial f_j/\partial p[\delta_{ij}(2m^2p/\omega^4) - (2mp/\omega^3) + (2mp/\omega^3) - (2m^2p/\omega^4)$$

$$+ (2m^2p/\omega^4)]$$

$$= \hat{p}_i\hat{p}_j\partial f_j/\partial p[\delta_{ij}(2m^2p/\omega^4)]$$

$$= (2m^2p/\omega^4)\partial f_j/\partial p\ \delta_{ij}$$

$$\hat{p}_i\hat{p}_j\ f_j[\delta_{ij}(3m^4/\omega^6 - m^2/\omega^4) - (3m^3/\omega^5) + (3m^3/\omega^5)(p^2/(\omega(\omega+m)))$$

$$+ (3m^4/\omega^6) - (m^2/\omega^4)]$$

$$= \hat{p}_i\hat{p}_jf_j[\delta_{ij}(3m^4/\omega^6 - m^2/\omega^4) - (3m^3/\omega^5) + (\omega-m)3m^3/\omega^6$$

$$+ (3m^4/\omega^6) - (m^2/\omega^4)]$$

$$= \hat{p}_i\hat{p}_jf_j[\delta_{ij}(3m^4/\omega^6 - m^2/\omega^4) - (3m^3/\omega^5) + (3m^3/\omega^5) - (3m^4/\omega^6)$$

$$+ (3m^4/\omega^6) - (m^2/\omega^4)]$$

$$= \hat{p}_i\hat{p}_jf_j[\delta_{ij}(3m^4/\omega^6 - m^2/\omega^4) - m^2/\omega^4]$$

Now for the angular terms, we proceed in a similar manner.

(1) $(v/(\omega\omega'))(\omega\omega') = (1/p^2)\nabla^2_{\hat{p}}f$

(2) $(v/(\omega\omega'))(\vec{p}\cdot\vec{p}') = (\hat{p}_i/\omega)(\nabla^2_{\hat{p}}/p^2)(\hat{p}_i/\omega)f$

$$= (p^2/\omega^2)(1/p^2)\hat{p}_i\nabla^2_{\hat{p}}\hat{p}_if$$

$$= (1/\omega^2)\hat{p}_i[\hat{p}_i\nabla^2_{\hat{p}} - 2\hat{p}_i + 2\nabla_i]f$$

$$= (1/\omega^2)\nabla^2_{<} f - (2/\omega^2) f + 0$$

$$(3)\ -(v/(\omega\omega'))(\omega'/(\omega+m))\vec{p}\vec{p} = -(p^2/(\omega(\omega+m)))\hat{p}_i\hat{p}_j(1/p^2)\nabla^2_{<} f_j$$

$$= -(1/(\omega(\omega+m)))\hat{p}_i\hat{p}_j\nabla^2_{<} f_j$$

$$= -(1/(\omega(\omega+m)))[\nabla^2_{<}\hat{p}_i\hat{p}_j + 4\hat{p}_i\hat{p}_j - 2\nabla_i\hat{p}_j - 2\hat{p}_i\nabla_j] f_j$$

$$= -(1/(\omega(\omega+m)))[\nabla^2_{<}\hat{p}_i\hat{p}_j f_j + 4\hat{p}_i\hat{p}_j f_j - 2\nabla_i\hat{p}_j f_j - 2\hat{p}_i\nabla_j f_j]$$

$$(4)\ -(v/(\omega\omega'))(\omega/(\omega'+m))\vec{p}'\vec{p}' = -(p^2/(\omega(\omega+m)))(1/p^2)\nabla^2_{<}\hat{p}_i\hat{p}_j f_j$$

$$= -(1/(\omega(\omega+m)))\nabla^2_{<}\hat{p}_i\hat{p}_j f_j$$

$$(5)\ (v/(\omega\omega'))\vec{p}\vec{p}'(\vec{p}\cdot\vec{p}'/((\omega+m)(\omega'+m))$$

$$= p^4/(\omega^2(\omega+m)^2)\hat{p}_i\hat{p}_k(1/p^2)\nabla^2_{<}\ \hat{p}_k\hat{p}_j f_j$$

$$= p^2/(\omega^2(\omega+m)^2)\hat{p}_i\hat{p}_k\nabla^2_{<}\ \hat{p}_k\hat{p}_j f_j$$

$$= p^2/(\omega^2(\omega+m)^2)[\nabla^2_{<}\hat{p}_i\hat{p}_k + 4\hat{p}_i\hat{p}_k - 2\nabla_i\hat{p}_k - 2\hat{p}_i\nabla_k]\hat{p}_k\hat{p}_j f_j$$

$$= p^2/(\omega^2(\omega+m)^2)[\nabla^2_{<}\hat{p}_i\hat{p}_j f_j + 4\hat{p}_i\hat{p}_j f_j - 2\nabla_i\hat{p}_j f_j - 2\hat{p}_i\nabla_k\hat{p}_k\hat{p}_j f_j]$$

$$= p^2/(\omega^2(\omega+m)^2)[\nabla^2_{<}\hat{p}_i\hat{p}_j f_j + 4\hat{p}_i\hat{p}_j f_j - 2\nabla_i\hat{p}_j f_j$$

$$-2\hat{p}_i(\hat{p}_k\nabla_k + \delta_{kk} - \hat{p}_k\hat{p}_k)\hat{p}_j f_j]$$

$$= p^2/(\omega^2(\omega+m)^2)[\nabla_<^2\hat{p}_i\hat{p}_jf_j+4\hat{p}_i\hat{p}_jf_j-2\nabla_i\hat{p}_jf_j-0\ -4\hat{p}_i\hat{p}_jf_j]$$

$$= p^2/(\omega^2(\omega+m)^2)[\nabla_<^2\hat{p}_i\hat{p}_jf_j-2\nabla_i\hat{p}_jf_j]$$

(6) $2(v/(\omega\omega'))\vec{p}\vec{p}' = 2(p^2/\omega^2)\hat{p}_i\nabla_<^2(1/p^2)\hat{p}_jf_j$

$$= (2/\omega^2)[\nabla_<^2\hat{p}_i\ -2\hat{p}_i\ -2\nabla_i]\hat{p}_jf_j$$

$$= (2/\omega^2)[\nabla_<^2\hat{p}_i\hat{p}_jf_j\ +2\hat{p}_i\hat{p}_jf_j\ -2\nabla_i\hat{p}_jf_j]$$

(7) $-(v/(\omega\omega'))\vec{p}'\vec{p} = -(p^2/\omega^2)\hat{p}_j(1/p^2)\nabla_<^2\hat{p}_if_j$

$$= -(1/\omega^2)[\nabla_<^2\hat{p}_j\ +2\hat{p}_j\ -2\nabla_j]\hat{p}_if_j$$

$$= -(1/\omega^2)[\nabla_<^2\hat{p}_i\hat{p}_jf_j+2\hat{p}_i\hat{p}_jf_j\ -2\nabla_j\hat{p}_if_j]$$

Now combine terms, looking at the coefficients of $\nabla_<^2f_i$, $\nabla_i\hat{p}_jf_j$, $\nabla_j\hat{p}_if_j$, and $\hat{p}_i\hat{p}_jf_j$.

$$\nabla_<^2f_j[\delta_{ij}(1/p^2) + \delta_{ij}(1/\omega^2)] = \nabla_<^2\delta_{ij}f_j((2\omega^2-m^2)/(p^2\omega^2))$$

$$\nabla_<^2\hat{p}_i\hat{p}_jf_j[-(1/(\omega(\omega+m)))-(1/(\omega(\omega+m)))\ +p^2/(\omega^2(\omega+m)^2)+(2/\omega^2)-(1/\omega^2)]$$

$$= \nabla_<^2\hat{p}_i\hat{p}_jf_j[-(2/(\omega(\omega+m)))\ +((\omega-m)/(\omega^2(\omega+m)))\ +(1/\omega^2)]$$

$$= \nabla_<^2\hat{p}_i\hat{p}_jf_j[-(2/(\omega(\omega+m)))\ +(1/(\omega(\omega+m)))\ -(m/(\omega^2(\omega+m)))\ +(1/\omega^2)]$$

$$= \nabla_<^2\hat{p}_i\hat{p}_jf_j[-(1/(\omega(\omega+m))\ -(m/(\omega^2(\omega+m)))\ +(1/\omega^2)]$$

$$= \nabla^2_{<}\hat{\beta}_i\hat{\beta}_j f_j(1/(\omega^2(\omega+m)))[-\omega-m+\omega+m] = 0.$$

$$\nabla_i\hat{\beta}_j f_j[+(2/(\omega(\omega+m))) - (2p^2/(\omega^2(\omega+m)^2) - (4/\omega^2)]$$

$$= \nabla_i\hat{\beta}_j f_j[+(2/(\omega(\omega+m))) - 2(\omega-m)/(\omega^2(\omega+m)) - (4/\omega^2)]$$

$$= \nabla_i\hat{\beta}_j f_j[(2\omega-2\omega+2\omega-4\omega-4m)/(\omega^2(\omega+m))]$$

$$= -2\nabla_i\hat{\beta}_j f_j[(2\omega+m)/(\omega^2(\omega+m))]$$

$$\nabla_j\hat{\beta}_i f_j(2/\omega^2) + \hat{\beta}_i\nabla_j f_j(2/(\omega(\omega+m)))$$

$$= (2/\omega^2)[\hat{\beta}_i\nabla_j + \delta_{ij} - \hat{\beta}_i\hat{\beta}_j]f_j + \hat{\beta}_i\nabla_j f_j(2/(\omega(\omega+m)))$$

$$= 2\hat{\beta}_i\nabla_j f_j[(1/\omega^2) + (1/(\omega(\omega+m)))] + (2/\omega^2)\delta_{ij}f_j - (2/\omega^2)\hat{\beta}_i\hat{\beta}_j f_j$$

$$= 2\hat{\beta}_i\nabla_j f_j[(2\omega+m)/(\omega^2(\omega+m))] + (2/\omega^2)\delta_{ij}f_j - (2/\omega^2)\hat{\beta}_i\hat{\beta}_j f_j$$

Now we do the $\hat{\beta}_i\hat{\beta}_j$ terms. Note that we have obtained two additional terms of this type in the $\nabla_j\hat{\beta}_i f_j$ reduction.

$$\hat{\beta}_i\hat{\beta}_j f_j[-(2/\omega^2)\delta_{ij} - 4/(\omega(\omega+m)) + (4/\omega^2) - (2/\omega^2) + (2/\omega^2)\delta_{ij} - (2/\omega^2)]$$

$$= \hat{\beta}_i\hat{\beta}_j f_j[-(4/(\omega(\omega+m))]$$

To proceed further we look at the term:

$$2((2\omega+m)/(\omega^2(\omega+m)))(\hat{\beta}_i\nabla_j f_j - \nabla_i\hat{\beta}_j f_j)$$

This term looks like $\vec{p} \times \vec{r}$, i.e. an angular momentum or an $\vec{L}\cdot\vec{S}$ type structure. To proceed further just consider the $\hat{\beta}_i \nabla_j f_j - \nabla_i \hat{\beta}_j f_j$ part.

$$\hat{\beta}_i \nabla_j f_j - \nabla_i \hat{\beta}_j f_j = [\nabla_j \hat{\beta}_i - \delta_{ij} + \hat{\beta}_i \hat{\beta}_j] f_j - \nabla_i \hat{\beta}_j f_j$$

$$= \nabla_j \hat{\beta}_i f_j - \nabla_i \hat{\beta}_j f_j - \delta_{ij} f_j + \hat{\beta}_i \hat{\beta}_j f_j = I$$

Up to this point we have been somewhat sloppy in our notation for typing purposes. The symbol ∇_i used up to this point is as defined above to be given by $\nabla_p^2 = \nabla_r^2 + (1/p^2)(\nabla_i \cdot \nabla_i)$. Strictly speaking ∇_i here should be explicitly written as $(\nabla_<)_i$, but was written as ∇_i for convenience. To proceed further we need to distinguish between the angular quantity $(\nabla_<)_i$ and the various components of the gradient operator ∇_i. With this in mind we proceed on with the derivation of the $\vec{L}\cdot\vec{S}$ term.

$$(\nabla_<)_i = \nabla_i - \hat{\beta}_i(\vec{p}\cdot\vec{\nabla}), \quad (\nabla_<)_j = \nabla_j - \hat{\beta}_j(\vec{p}\cdot\vec{\nabla})$$

$$I = [\nabla_j - \hat{\beta}_j(\vec{p}\cdot\vec{\nabla})]\hat{\beta}_i f_j - [\nabla_i - \hat{\beta}_i(\vec{p}\cdot\vec{\nabla})]\hat{\beta}_j f_j - \delta_{ij} + \hat{\beta}_i \hat{\beta}_j f_j$$

$$= \nabla_j \hat{\beta}_i f_j - \nabla_i \hat{\beta}_j f_j - \hat{\beta}_j(\vec{p}\cdot\vec{\nabla})\hat{\beta}_i f_j + \hat{\beta}_i(\vec{p}\cdot\vec{\nabla})\hat{\beta}_j f_j - \delta_{ij} f_j + \hat{\beta}_i \hat{\beta}_j f_j$$

$$= i(x_j \hat{\beta}_i f_j - x_i \hat{\beta}_j f_j) - i[\hat{\beta}_j(\vec{p}\cdot\vec{x})\hat{\beta}_i - \hat{\beta}_i(\vec{p}\cdot\vec{x})\hat{\beta}_j] f_j - \delta_{ij} f_j + \hat{\beta}_i \hat{\beta}_j f_j$$

$$= -(\vec{L}\cdot\vec{S})_{ij} - i[(\vec{p}\cdot\vec{x})\hat{\beta}_i \hat{\beta}_j - i\hat{\beta}_i \hat{\beta}_j - (\vec{p}\cdot\vec{x})\hat{\beta}_i \hat{\beta}_j + i\hat{\beta}_i \hat{\beta}_j] f_j - \delta_{ij} f_j + \hat{\beta}_i \hat{\beta}_j f_j$$

So we end up with

$$2(2\omega+m)/(\omega^2(\omega+m))[\hat{p}_i\nabla_j f_j - \nabla_i \hat{p}_j f_j]]$$

$$= 2(2\omega+m)/(\omega^2(\omega+m))[-(\vec{L}\cdot\vec{S})_{ij} - \delta_{ij}+\hat{p}_i\hat{p}_j]f_j$$

$$=[-2(2\omega+m)/(\omega^2(\omega+m))\ (1/2)[j(j+1)-1(1+1)]+2(2\omega+m)/(\omega^2(\omega+m)]\delta_{ij}f_j$$
$$-2(2\omega+m)/(\omega^2(\omega+m))\delta_{ij}f_j +2(2\omega+m)/(\omega^2(\omega+m))\hat{p}_i\hat{p}_j f_j$$

$$=-(2\omega+m)/(\omega^2(\omega+m))[j(j+1)-1(1+1)]\delta_{ij}f_j +2(2\omega+m)/(\omega^2(\omega+m))\hat{p}_i\hat{p}_j f_j$$

Now we combine the $\hat{p}_i\hat{p}_j$ term with that obtained above and we have:

$$-4/(\omega(\omega+m)) +4/(\omega(\omega+m)) +2m/(\omega^2(\omega+m)) = 2m/(\omega^2(\omega+m))$$

This is the coefficient of the $\hat{p}_i\hat{p}_j$ term arising from the angular part of the Bethe-Salpeter equation reduction. Combining the above terms we have:

$$\nabla_r^2 f_j[(2\omega^2-m^2)/\omega^2]\delta_{ij} +(2m^2p/\omega^4)(\partial f_j/\partial p)\delta_{ij}$$

$$+\hat{p}_i\hat{p}_j[\delta_{ij}(3m^4/\omega^6 - m^2/\omega^4) - (m^2/\omega^4)] +\nabla_q^2((2\omega^2-m^2)/\omega^2p^2)\delta_{ij}f_j$$

$$-(2\omega+m)/(\omega^2(\omega+m))[j(j+1)-1(1+1)]\delta_{ij}f_j + (2m/(\omega^2(\omega+m))\hat{p}_i\hat{p}_j f_j$$

$$= (2/k)2(\omega-\mu)f_i \qquad (2)$$

This expression can be recast into a more convenient form by noting that

$$(i)\ ((2\omega^2-m^2)/(\omega^2p^2))\nabla_r^2 f_j\delta_{ij} = -((2\omega^2-m^2)/(\omega^2p^2))[1(1+1)]\delta_{ij}f_j$$

$$= -((2\omega^2-m^2)/(\omega^2p^2))[j(j+1) - j(j+1) + 1(1+1)]\delta_{ij}f_j$$

$$= \{-((2\omega^2-m^2)/(\omega^2p^2))j(j+1) + ((2\omega^2-m^2)/(\omega^2p^2)[j(j+1)$$

$$-1(1+1)]\}\delta_{ij}f_j$$

Now combine the $[j(j+1) - 1(1+1)]$ terms:

$$(ii)\ ((2\omega^2-m^2)/(\omega^2p^2)) - ((2\omega+m)/(\omega^2(\omega+m)))$$

$$= (1/\omega^2)[(\omega^2/p^2) + (\omega^2-m^2)/p^2 - \omega/(\omega+m) - (\omega+m)/(\omega+m)]$$

$$= (1/\omega^2)[(\omega^2/p^2) + 1 - \omega/(\omega+m) - 1]$$

$$= (1/\omega^2)[(\omega^2(\omega+m)-\omega p^2)/((\omega+m)p^2)]$$

$$= (1/\omega^2p^2)[(\omega^2(\omega+m) - \omega(\omega+m)(\omega-m))/(\omega+m)]$$

$$= (1/\omega^2p^2)[\omega^2 - \omega(\omega-m)] = (1/\omega^2p^2)[m\omega] = m/(\omega p^2)$$

So we have for

$$\nabla_r^2((2\omega^2-m^2)/(\omega^2p^2))f_i - ((2\omega+m)/(\omega^2(\omega+m)))[j(j+1) - 1(1+1)]f_i$$

the quantity

$$-((2\omega^2-m^2)/(\omega^2p^2))j(j+1)\delta_{ij}f_j + (m/\omega p^2)\delta_{ij}f_j[j(j+1)-1(1+1)]$$

With these last substitutions we have for the $^3(J\pm1)_J$ vector

potential equation

$$-(2/k)2(\omega-\mu)f_i(p) + ((2\omega^2-m^2)/\omega^2)\nabla_r^2 f - ((2\omega^2-m^2)/(\omega^2p^2))j(j+1)f_i$$

$$+(2m^2p/\omega^4)\partial f_i/\partial p + f_i(p)[(3m^4/\omega^6) - (m^2/\omega^4)]$$

$$-(m^2/\omega^4)\hat{p}_i\hat{p}_jf_j + (2m/(\omega^2(\omega+m)))\hat{p}_i\hat{p}_jf_j$$

$$+(m/(\omega p^2))[j(j+1) - l(l+1)]f_i = 0.$$

(iiib) The Radial Equation for the $^3(J\pm1)_J$ Bethe-Salpeter Equation with a Scalar Harmonic Potential

The starting point for the derivation is the equation

$$2(\omega-\mu)\vec{f}(\vec{p}) + A(v/(\omega\omega'))[m^2+\omega\omega'-\vec{p}\cdot\vec{p}')I - (\vec{p}\vec{p}'-\vec{p}'\vec{p})$$

$$+(1/((\omega+m)(\omega'+m))(2\vec{p}\vec{p}'(\vec{p}\cdot\vec{p}') - \vec{p}\vec{p}(p')^2-\vec{p}'\vec{p}'(p^2))]\cdot\vec{f}(\vec{p})=0 \quad (1)$$

The reduction, as with the other cases discussed, is carried out by first considering the radial part, then the angular part, and then adding up the two contributions. The radial part reduction is aided by the use of the following identities.

$$\text{(i)}\quad \nabla_r^2(f/\omega) = (1/\omega)\nabla_r^2 f - (2p/\omega^3)df/dp - (3m^2/\omega^5)f$$

$$\text{(ii)}\quad \nabla_r^2(pf/\omega) = (p/\omega)\nabla_r^2 f + (2m^2/\omega^3)df/dp + f(p)[2m^4/(\omega^5 p) - m^2 p/\omega^5]$$

$$\text{(iii)}\quad \nabla_r^2(p^2 f/((\omega+m)(\omega-m))) = \nabla_r^2 f - m\nabla_r^2(f/\omega)$$

$$= \nabla_r^2 f - (m/\omega)\nabla_r^2 f + (2mp/\omega^3)df/dp + (3m^3/\omega^5)f$$

Writing v as $-(k/2)\delta^{(3)}(\vec{p} - \vec{p}')\nabla_{p'}^2$ with $\nabla_{p'}^2 f = \nabla_r^2 f + (1/p'^2)\nabla_{\hat{p}}^2 f$ and considering for the moment only the ∇_r^2 terms, we have from (1):

$$-(2/k)[2(\omega-\mu)f(p)] + A(m^2/\omega)\nabla_r^2(f/\omega) + A\nabla_r^2 f - A(p/\omega)\nabla_r^2(pf/\omega) \qquad (2)$$

The remaining terms sum up to zero. To see this note that

$$(A/(\omega(\omega+m)))[2(\hat{p}_i\hat{p}_k\hat{p}_k\hat{p}_j p^2)\nabla_r^2(p^2 f/(\omega(\omega+m))) - \hat{p}_i\hat{p}_j\nabla_r^2(p^2 f/(\omega(\omega+m))) - p^2\hat{p}_i\hat{p}_j\nabla_r^2(p^2 f/(\omega(\omega+m)))]$$

is the contribution from the last three terms, and using $\hat{p}_k\hat{p}_k = 1$, we have the resulting zero contribution here. The terms in (2) can be reduced with the aid of (i) and (ii), giving

$$-(2/k)2(\omega-\mu)f + A(m^2/\omega)[(1/\omega)\nabla_r^2 f - (2p/\omega^3)df/dp - (3m^2/\omega^5)f]$$

$$+A\nabla_r^2 f - A(p/\omega)[(p/\omega)\nabla_r^2 f + (2m^2/\omega^3)df/dp + f(p)(2m^4/(\omega^5 p)$$

$$-m^2 p/\omega^5)] = -(2/k)2(\omega-\mu)f + A\nabla_r^2 f[m^2/\omega^2 + 1 - p^2/\omega^2]$$

$$-2A(2m^2 p/\omega^4)df/dp - 2Af(p)[3m^4/\omega^6 - m^2/(2\omega^4)] \qquad (3)$$

We now proceed with the angular part of the Laplacian.

$$A(m^2/(\omega^2 p^2))\nabla_<^2 f_i + A\nabla_<^2 f_i - A(p^2/(\omega^2 p^2))\hat{p}_j \nabla_<^2 \hat{p}_j f_i$$

$$-Ap^2/(\omega^2 p^2)\ (\hat{p}_i \nabla_<^2 \hat{p}_j f_j - \hat{p}_j \nabla_<^2 \hat{p}_i f_j)$$

$$+Ap^4/(\omega^2(\omega+m)^2 p^2)[2\hat{p}_i \hat{p}_k \nabla_<^2 \hat{p}_k \hat{p}_j f_j - \hat{p}_i \hat{p}_j \nabla_<^2 f_j - \nabla_<^2 \hat{p}_i \hat{p}_j f_j] \qquad (4)$$

The reduction of (4) is made easier by the use of the identities:

$$\text{(i)}\ [\nabla_<^2, \hat{p}_i] = -2\hat{p}_i + 2\nabla_i$$

$$\text{(ii)}\ [\nabla_i, \hat{p}_j] = \delta_{ij} - \hat{p}_i \hat{p}_j$$

$$\text{(iii)}\ [\nabla_<^2, \hat{p}_i \hat{p}_j] = -4\hat{p}_i \hat{p}_j + 2\nabla_i \hat{p}_j + 2\hat{p}_i \nabla_j$$

The first three terms in (4) yield:

$$A(m^2/(\omega^2p^2))\nabla_<^2 f_i + A(1/p^2)\nabla_<^2 f_i - A(p^2/(\omega^2p^2))\hat{p}_j[\hat{p}_j\nabla_<^2 - 2\hat{p}_j + 2\nabla_j]f_i$$

$$= A\nabla_<^2 f_i[m^2/(\omega^2p^2) + (1/p^2) - (p^2/(\omega^2p^2))] + 2Af_i/\omega^2.$$

The fourth and fifth terms in (4) yield:

$$-Ap^2/(\omega^2p^2)\{[\nabla_<^2\,\hat{p}_i + 2\hat{p}_i - 2\nabla_i]\hat{p}_j f_j - [\nabla_<^2\,\hat{p}_j + 2\hat{p}_j - 2\nabla_j]\hat{p}_i f_j\}$$

$$= -Ap^2/(\omega^2p^2)[-2\nabla_i\hat{p}_j f_j + 2\nabla_j\hat{p}_i f_j]$$

The last terms yield the following:

$$I = Ap^4/(\omega^2(\omega+m)^2p^2)[2\hat{p}_i\hat{p}_k(\hat{p}_k\nabla_<^2 - 2\hat{p}_k + 2\nabla_k)\hat{p}_j f_j$$

$$-(\nabla_<^2\,\hat{p}_i\hat{p}_j + 4\hat{p}_i\hat{p}_j - 2\nabla_i\hat{p}_j - 2\hat{p}_j\nabla_j + \nabla_<^2\,\hat{p}_i\hat{p}_j)f_j]$$

In this last expression we have used (i) on the first term and (iii) on the second term. Making use of (i) and (ii), I is simplified as follows:

$$I = Ap^4/(\omega^2(\omega+m)^2p^2)[2\hat{p}_i\nabla_<^2\,\hat{p}_j f_j - 4\hat{p}_i\hat{p}_j f_j - \nabla_<^2\,\hat{p}_i\hat{p}_j f_j - 4\hat{p}_i\hat{p}_j f_j$$

$$+2\nabla_i\hat{p}_j f_j + 2\hat{p}_i\nabla_j f_j - \nabla_<^2\,\hat{p}_i\hat{p}_j f_j]$$

$$= Ap^4/(\omega^2(\omega+m)^2p^2)[2(\nabla_<^2\,\hat{p}_i\hat{p}_j f_j - 2\nabla_i\hat{p}_j f_j) - 4\hat{p}_i\hat{p}_j f_j - 2\nabla_<^2\hat{p}_i\hat{p}_j f_j$$

$$-4\hat{p}_i\hat{p}_j f_j + 2\nabla_i\hat{p}_j f_j + 2\hat{p}_i\nabla_j f_j]$$

$$= Ap^2/(\omega^2(\omega+m)^2)[-4\nabla_i\hat{p}_j f_j - 4\hat{p}_i\hat{p}_j f_j + 2\nabla_i\hat{p}_j f_j + 2\hat{p}_i\nabla_j f_j]$$

$$= Ap^2/(2\omega^2(\omega+m)^2)[-2\nabla_i\hat{p}_j f_j + 2\hat{p}_i\nabla_j f_j - 4\hat{p}_i\hat{p}_j f_j]$$

The net result of these angular reductions is

$$\nabla^2_< f_i[2Am^2/(\omega^2p^2)] + (2A/\omega^2)f_i - (Ap^2/(\omega^2p^2))[2\nabla_j\hat{p}_i f_j - 2\nabla_i\hat{p}_j f_j]$$

$$+(Ap^2/(\omega^2(\omega+m)^2)[2\hat{p}_i\nabla_j f_j - 2\nabla_i\hat{p}_j f_j - 4\hat{p}_i\hat{p}_j f_j].$$

The term $2\hat{p}_i\nabla_j f_j$ can be rewritten as $2(\nabla_j\hat{p}_j - \delta_{ij} + \hat{p}_i\hat{p}_j)f_j$, which gives

$$\nabla^2_< f_i(2Am^2/(\omega^2p^2)) + (2A/\omega^2)f_i - (Ap^2/(\omega^2p^2))[2\nabla_j\hat{p}_i f_j - 2\nabla_i\hat{p}_j f_j]$$

$$+(Ap^2/(\omega^2(\omega+m)^2)[2\nabla_j\hat{p}_i f_j - 2f_i - 2\nabla_i\hat{p}_j f_j - 2\hat{p}_i\hat{p}_j f_j] \qquad (5)$$

Combining the radial and angular parts we obtain:

$$(-2/k)2(\omega-\mu)f_i(p) = 2(Am^2/\omega^2)\nabla^2_r f_i - 2(2Am^2p/\omega^4)df/dp$$

$$-2Af_i(p)[3m^4/\omega^6 - m^2/2\omega^4] + (2Am^2/(\omega^2p^2))\nabla^2_< f_i + 2A(f_i/\omega^2)$$

$$-(A/\omega^2)[2\nabla_j\hat{p}_j f_j - 2\nabla_i\hat{p}_j f_j] + (Ap^2/(\omega^2(\omega+m)^2)[2\nabla_j\hat{p}_i f_j - 2f_i$$

$$-2\nabla_j\hat{p}_j f_j - 2\hat{p}_i\hat{p}_j f_j] \qquad (6)$$

The equation (6), or rather some of its angular terms, can still be simplified.

(i) The $\nabla^2_< f_i$ term.

As noted in connection with the vector potential equation

$$\nabla_r^2 f_j = -l(l+1)f_j = -[j(j+1) - j(j+1) + l(l+1)]f_j$$

$$= -j(j+1)f_j + [j(j+1) - l(l+1)]f_j$$

(ii) $(\nabla_j \hat{\beta}_j f_j - \nabla_i \hat{\beta}_j f_j)$ terms

$$-2[A/\omega^2 - Ap^2/(\omega^2(\omega+m)^2)][\nabla_j \hat{\beta}_i f_j - \nabla_i \hat{\beta}_j f_j]$$

$$= -2(A/\omega^2)[((\omega+m)^2 - p^2)/(\omega+m)^2](\nabla_j \hat{\beta}_i f_j - \nabla_i \hat{\beta}_j f_j)$$

$$= -2(A/\omega^2)[((\omega+m)[(\omega+m)-(\omega-m)])/((\omega+m)(\omega+m))](\nabla_j \hat{\beta}_i f_j - \nabla_i \hat{\beta}_j f_j)$$

$$= -2(A/\omega^2)(2m/(\omega+m))[\nabla_j \hat{\beta}_i f_j - \nabla_i \hat{\beta}_j f_j]$$

$$= -2(2mA/(\omega^2(\omega+m)))[-\vec{L}\cdot\vec{S}]_{ij}$$

Inserting these relations into (6) we have

$$(2/k)2(\omega-\mu)f_i(p) = 2(Am^2/\omega^2)\nabla_r^2 f_i - 2(2Am^2 p/\omega^4)df/dp$$

$$-2Af_i(p)[3m^4/\omega^6 - m^2/2\omega^4] - (2Am^2/(\omega^2 p^2))j(j+1)f_i(p)$$

$$+2(A/\omega^2)f_i - 2(2mA/(\omega^2(\omega+m)))[-\vec{L}\cdot\vec{S}]_{ij} f_j$$

$$+(2Am^2/(\omega^2 p^2))[j(j+1) - l(l+1)]f_i$$

$$-2A(\omega-m)/(\omega^2(\omega+m))[\delta_{ij} - \hat{\beta}_i \hat{\beta}_j]f_j$$

Using $(\vec{L}\cdot\vec{S})_{ij} = (j^2 - l^2 - s^2)(1/2)\delta_{ij}$ we have

$$(2/k)2(\omega-\mu)f_i(p) = 2(Am^2/\omega^2)\nabla_r^2 f_i - 2(2Am^2p/\omega^4)df_i/dp$$

$$-2Af_i(p)[3m^4/\omega^6 - m^2/2\omega^4] - (2Am^2/(\omega^2p^2))j(j+1) + (2A/\omega^2)f_i$$

$$+(2mA/(\omega^2(\omega+m)))[j(j+1) - l(l+1)]f_i$$

$$+(2Am^2/(\omega^2p^2))[j(j+1)-l(l+1)]f_i + (2A(\omega-m)/(\omega^2(\omega+m)))\hat{p}_i\hat{p}_jf_j$$

$$-(2A(\omega-m)/(\omega^2(\omega+m)))f_i + 2(2mA/(\omega^2(\omega+m)))f_i(p)$$

Combining the $[j(j+1) - l(l+1)]$ terms we have for its coefficient $2mA/(\omega p^2)$. Finally we have

$$(2/k)2(\omega-\mu)f_i = (2Am^2/\omega^2)\nabla_r^2 f_i - 2(2Am^2p/\omega^4)df_i/dp$$

$$-2Af_i(p)[3m^4/\omega^6 - m^2/2\omega^4] - (2Am^2/(\omega^2p^2))j(j+1)$$

$$+[j(j+1)-l(l+1)](2mA/(\omega p^2)) + (2A(\omega-m)/(\omega^2(\omega+m)))\hat{p}_i\hat{p}_jf_j$$

(iiic) Combining the $^3(J\pm1)_J$ Scalar and Vector Potential Forms

We now combine the terms arising from the potential in the two previous equations, i.e. the Bethe-Salpeter equation with a pure vector potential and the same equation with a pure scalar potential. We proceed directly by writing out the terms indicated.

$$(2/k)2(\omega-\mu)f_i = [(2\omega^2-m^2)/\omega^2 + 2Am^2/\omega^2]\nabla_r f_i(p)$$

$$+ [(2\omega^2-m^2)/\omega^2 - 2Am^2/(\omega^2p^2)]j(j+1)f_i(p)$$

$$+ [2m^2p/\omega^4 - 2(2Am^2/\omega^2)]df_i(p)/dp + f_i(p)[3m^4/\omega^6 - m^2/\omega^4$$

$$-2A(3m^4/\omega^6 - m^2/2\omega^4)] + [m/(\omega p^2) + 2mA/(\omega p^2)][j(j+1)$$

$$-1(1+1)]f_i(p) + [2m/(\omega^2(\omega+m))+2A(\omega-m)/(\omega^2(\omega+m))]\hat{p}_i\hat{p}_jf_j(p)$$

$$-\hat{p}_i\hat{p}_j(m^2/\omega^4) = [(2\omega^2-\alpha m^2)/\omega^2]\nabla_r f_i(p)$$

$$+ [(2\omega^2-\alpha m^2)/\omega^2]j(j+1)f_i(p) + (2m^2p\alpha/\omega^4)df_i(p)/dp$$

$$+ f_i(p)[3m^4\alpha/\omega^6 - m^2\beta/\omega^4] + (m\gamma/\omega p^2)[j(j+1)-1(1+1)]f_i(p)$$

$$+ \hat{p}_i\hat{p}_jf_j[2m/(\omega^2(\omega+m)) + 2A(\omega-m)/(\omega^2(\omega+m))]$$

$$- T_{ij}(m^2/\omega^4)$$

In this last equation we have defined $\alpha = 1-2A$, $\beta = (3-2A)/2$, $\gamma = 1 + 2A$, $\delta = 1 - A$, and

$$T_{ij} = \hat{p}_i\hat{p}_j \text{ - diagonal part}$$

$$= [(1/2)I - \sqrt{j(j+1)}/(2j+1)\ \sigma_x - (1/(2(2j+1)))\ \sigma_z] - (1/2)I$$

With this the final form is the following:

$$(2/k)2(\omega-\mu)f_i(p)=((2\omega^2-\alpha m^2)/\omega^2)\nabla_r^2 f_i(p) + ((2\omega^2-\alpha m^2)/\omega^2)j(j+1)f_i(p)$$

$$+ (2m^2p\alpha/\omega^4)df_i(p)/dp + [3m^4\alpha/\omega^6 - m^2\beta/\omega^4]f_i(p)$$

$$+ (m\gamma/(\omega p^2))[j(j+1)-1(1+1)]f_i(p)$$

$$+[2m\delta/(\omega^2(\omega+m)) +2A/(\omega(\omega+m))][(1/2)I -(\sqrt{j(j+1)}/(2j+1))\sigma_x$$

$$- (1/(2(2j+1)))\sigma_z]_{ij} f_j(p)$$

$$+ [(\sqrt{j(j+1)}/(2j+1))\sigma_x + (1/(2(2j+1)))\sigma_z]_{ij}f_j(p)(m^2/\omega^4)$$

Appendix I.4

Integrating Factors and the Sturm-Liouville Form

(a) The Integrating Factor or Multiplier

The arguments of the functions f and f_i below depend only on the magnitude of the momentum p.

From Appendix I.3 the radial equations for the three cases to be considered are:

$$(-2/k)2(\omega-\mu)f + [\nabla_r^2 f - (1/p^2)j(j+1)]((2\omega^2-\alpha m^2)/\omega^2)$$

$$+ (2m^2p\alpha/\omega^4)df/dp + [3m^4\alpha/\omega^6 + Ap^2/\omega^2]f = 0 \qquad (1)$$

$$(-2/k)2(\omega-\mu)f_i + [\nabla_r^2 f_i - (1/p^2)j(j+1)]((2\omega^2-\alpha m^2)/\omega^2)$$

$$+ (2m^2p\alpha/\omega^4)df_i/dp + [3m^4\alpha/\omega^6 - (1-A)(m^2/\omega^4)]f_i = 0 \qquad (2)$$

$$(-2/k)2(\omega-\mu)f_i + [\nabla_r^2 f_i - (1/p^2)j(j+1)]((2\omega^2-\alpha m^2)/\omega^2)$$

$$+ (2m^2p\alpha/\omega^4)df_i/dp + [3m^4\alpha/\omega^6 - m^2\beta/\omega^4 + (m\gamma/(\omega p^2))(j(j+1)$$

$$- 1(1+1))] + [2m\delta/(\omega^2(\omega+m)) + 2A/(\omega(\omega+m))][(1/2)\delta_{ij}$$

$$- (\sqrt{j(j+1)}/(2j+1))\sigma_x - (1/(2(2j+1)))\sigma_z]_{ij}f_j$$

$$+ (m^2/\omega^4)[(\sqrt{j(j+1)}/(2j+1))\sigma_x + (1/(2(2j+1)))\sigma_z]_{ij}f_j = 0 \qquad (3)$$

The Sturm-Liouville (or self-adjoint) form of a second order ordinary differential equation $a(x)y'' + b(x)y' + c(x)y = 0$ is

$d/dx[p(x)y'] + q(x)y = 0$. To transform any second order ordinary differential equation into its associated Sturm-Liouville form one must determine a multiplier function $s(x)$ such that $a(x)s(x) = p(x)$ and $b(x)s(x) = dp(x)/dx$. Since the multiplier is determined by the coefficients of the first and second derivative terms alone, all three cases listed above will have the same multiplier $s(x)$. To proceed further we first rewrite the ${}^1J_J$ equation using:

$$\nabla_r^2 f = (1/p^2)(d/dp\ p^2\ d/dp)f = d^2f/dp^2 + (2/p)df/dp$$

With this the ${}^1J_J$ equation takes the form:

$$(-2/k)2(\omega-\mu)f + ((2\omega^2-\alpha m^2)/\omega^2)(d^2f/dp^2 + (2/p)df/dp)$$

$$+ (2m^2p\alpha/\omega^4)df/dp + [3m^4\alpha/\omega^6 + Ap^2/\omega^2]f = 0$$

The coefficients of the second and first derivative terms are easily seen to be:

$$a(p) = \text{coefficient of } d^2f/dp^2 = ((2\omega^2-\alpha m^2)/\omega^2) \tag{4}$$

$$b(p) = \text{coefficient of } df/dp = ((2\omega^2-\alpha m^2)/\omega^2)(2/p)+(2m^2p\alpha/\omega^4) \tag{5}$$

To determine $s(p)$ we make use of its defining relation:

$$d/dp[a(p)s(p)] = b(p)s(p)$$

whose solution is given by

$$s(p) = (1/a(p))\ \exp(\int(b(p)/a(p))dp$$

Set

$$q(p) = (b(p)/a(p)) = (p^2+m^2)/(2p^2+\tilde{m}^2)[((2p^2+\tilde{m}^2)/(p^2+m^2))(2/p)$$

$$+ (2m^2\alpha p)/(p^2+m^2)^2] = (2/p) + (2m^2\alpha p)/((2p^2+\tilde{m}^2)(p^2+m^2))$$

First note that $\int(2/p)dp = \ln(p^2)$ and that we have set $\tilde{m}^2 = (2 - \alpha)m^2$. The second integral is a little harder and we do this by means of a partial fraction expansion. Set $p^2 = y$, so that the second integral now reads

$$m^2\alpha\int(1/((2y+\tilde{m}^2)(y+m^2)))dy$$

Expanding the integrand

$$A/(2y+\tilde{m}^2) + B/(y+m^2) = 1/((2y+\tilde{m}^2)(y+m^2))$$

yields the relation

$$A(y+m^2) + B(2y+\tilde{m}^2) = 1$$

so that

$$y(A+2B) = 0, \quad m^2A + \tilde{m}^2B = 1$$

From the first of these relations we see that $A = -2B$ and from the second $-2m^2B + \tilde{m}^2B = 1$, so that

$$B = -(1/(2m^2-\tilde{m}^2)) = -(1/\alpha m^2),$$

and

$$A = +(2/\alpha m^2)$$

We can now write

$$\int(1/((y+m^2)(2y+\tilde{m}^2)))dy = \int[(2/\alpha m^2)(1/(2y+\tilde{m}^2) - (1/\alpha m^2)(1/(y+m^2))]dy$$

$$= (2/\alpha m^2)[(1/2)\ln(2y+\tilde{m}^2)] -(1/\alpha m^2)\ln(y+m^2)$$

$$= (1/\alpha m^2)[\ln((2y+\tilde{m}^2)/(y+m^2))]$$

Combining things we see that

$$s(p) = [(2\omega^2-\alpha m^2)/\omega^2]^{-1}\exp[\ln(p^2) + \ln((2p^2+\tilde{m}^2)/(p^2+m^2))]$$

$$= [(2\omega^2-\alpha m^2)/\omega^2]^{-1}\exp[\ln(p^2(2\omega^2-\alpha m^2)/\omega^2)]$$

$$= p^2$$

(b) The Sturm-Liouville Form and the Lagrangian

The Sturm-Liouville forms of the ${}^1J_J$, ${}^3J_J$, and ${}^3(J\pm1)_J$ equations are:

$$-(2p^2/k)2(\omega-\mu)f + d/dp[p^2((2\omega^2-\alpha m^2)/\omega^2)df/dp]$$

$$+ [3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4]f - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1)f = 0 \quad (1)$$

$$-(2p^2/k)2(\omega-\mu)f_i + d/dp[p^2((2\omega^2-\alpha m^2)/\omega^2)df_i/dp]$$

$$- ((2\omega^2-\alpha m^2)/\omega^2)j(j+1)f_i + [3m^4\alpha p^2/\omega^6 - (1-A)m^2p^2/\omega^4]f_i = 0 \quad (2)$$

$$-(2p^2/k)2(\omega-\mu)f_i + d/dp[p^2((2\omega^2-\alpha m^2)/\omega^2)df_i/dp]$$

$$-[(2\omega^2-\alpha m^2)/\omega^2]j(j+1)f_i + [3m^4\alpha p^2/\omega^6 - m^2p^2\beta/\omega^4$$

$$+(m\gamma/\omega)(j(j+1)-1(1+1))]f_i + [(2mp^2\delta/(\omega^2(\omega+m)))$$

$$+ (2Ap^2/(\omega(\omega+m)))][(1/2)\delta_{ij} - ((\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$-(1/(2(2j+1)))(\sigma_z)_{ij}]f_j + (m^2p^2/\omega^4)[((\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$+ (1/(2(2j+1)))(\sigma_z)_{ij}]f_j = 0 \qquad (3)$$

These equations follow from the corresponding equations in appendix I.4(a), namely equations (1),(2), and (3) there, using the substitutions for the first and second derivative terms as given by (4) and (5) in the same appendix multiplied by p^2, the integrating factor found in appendix I.4(a).

Appendix I.5

Limiting Cases of the Radial Equations and the Lagrangians for the Radial Equations

We shall look at the four limiting cases: (i) $p \to \infty$, (ii) $p \to 0$, (iii) $m \to \infty$, and (iv) $m \to 0$.

(i) $p \to \infty$

To look at $p \to \infty$, we first write out the coefficients of the first and second derivatives of our equations. Note that these will be the same for all three equations.

$$d/dp[p^2((2\omega^2-\alpha m^2)/\omega^2)df/dp] = [p^2(2\omega^2-\alpha m^2)/\omega^2]d^2f/dp^2$$

$$+ d/dp[p^2(2\omega^2-\alpha m^2)/\omega^2]df/dp$$

$$d/dp[p^2(2\omega^2-\alpha m^2)/\omega^2] = 2p(2\omega^2-\alpha m^2)/\omega^2 + (p^2/\omega^2)4p$$

$$- (p^2(2\omega^2-\alpha m^2)/\omega^4)2p$$

The coefficient of d^2f/dp^2 goes like $2p^2$ as $p \to \infty$. The coefficient of df/dp has a factor of $4p$ for its leading term as $p \to \infty$.

To complete the discussion one needs to look at the coefficients of the undifferentiated terms. In this case the three equations have different coefficients and each needs to be looked at separately.

$$^1J_J: [3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - 4p^2\omega/k$$

$$+ (2p^2/k)E]f \to -(2p^2/k)(2\omega-E)f$$

$$^3J_J: [3m^4\alpha p^2/\omega^6 - (1-A)m^2p^2/\omega^4 - 4p^2\omega/k + 2p^2E/k$$

$$- ((2\omega^2-\alpha m^2)/\omega^2)j(j+1)]f + (-2p^2/k)(2\omega-E)f$$

$^3(J\pm1)_J$: In this case one can also see by counting powers of p that the leading behavior is given by the same factor as in the ^1J_J and the ^3J_J cases. In these three cases it is appropriate to replace ω by p for large p and for finite $m \ll p$. In each case then the equation appropriate to the limit $p \to \infty$ is

$$2p^2\, d^2f/dp^2 + 4p\, df/dp - (2p^2/k)(2p-E)f = 0$$

or

$$d^2f/dp^2 + (2/p)df/dp - (1/k)(2p-E)f = 0 \qquad (1)$$

If we set

$$f(p) = u(p)/p \qquad (2)$$

then

$$df/dp = -(1/p^2)u(p) + (1/p)du(p)/dp \qquad (3)$$

$$d^2f/dp^2 = (2/p^3)u(p) - (1/p^2)du/dp - (1/p^2)du/dp + (1/p)d^2u/dp^2$$

$$= (1/p)d^2u/dp^2 - (2/p)du/dp + (2/p^3)u(p) \qquad (4)$$

Inserting equations (2), (3), and (4) into equation (1), we have

$$(1/p)d^2u/dp^2 - (2/p)du/dp + (2/p^3)u(p) + (2/p)[(1/p)du/dp$$

$$- (1/p^2)u] - (1/k)(2p-E)u(p)/p = 0$$

or

$$d^2u/dp^2 - ((2p/k) - (E/k))u(p) = 0 \qquad (5)$$

(ii) Case of $p \to 0$, for finite $m > 0$.

Look at the coefficients of the first and second derivative terms first, which from part (i) are:

$$p^2(2\omega^2-\alpha m^2)/\omega^2 \qquad \text{(second derivative)}$$

and

$$2p((2\omega^2-\alpha m^2)/\omega^2) + (p^2/\omega^2)4p - (p^2(2\omega^2-\alpha m^2)/\omega^4)2p$$

(first derivative)

As $p \to 0$, these coefficients become

$$p^2((2m^2-\alpha m^2)/m^2)$$

and

$$2p((2m^2-\alpha m^2)/m^2) + (4p^3/m^2) - 2p^3(2m^2-\alpha m^2)/m^4$$

respectively. Keeping only the leading term in p for these two cases, we have that

$p^2(2-\alpha)$ = coefficient of second derivative term

$2p(2-\alpha)$ = coefficient of first derivative term

For the ^1J_J equation the coefficient of f(p) is

$$3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - 4p^2\omega/k + 2p^2E/k$$

For small p this goes over into $[-(2-\alpha)j(j+1)]$.

For the ^3J_J case we have

$$[3m^4\alpha p^2/\omega^6 - (1-A)m^2p^2/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - (2p^2/k)2\omega$$

$$+ 2p^2E/k] \rightarrow -(2-\alpha)j(j+1).$$

For the $^3(J\pm1)_J$ case we have

$$[3m^4\alpha p^2/\omega^6 - m^2p^2\beta/\omega^4 + (m\gamma/\omega)[j(j+1)-1(1+1)]$$

$$-((2\omega^2-\alpha m^2)/\omega^2)j(j+1)]f_i - [2mp^2\delta/(\omega^2(\omega+m))$$

$$+ Ap^2/(\omega(\omega+m))][(1/2)\delta_{ij} - (\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$- (1/(2(2j+1)))(\sigma_z)_{ij}]f_j$$

$$- [(m^2p^2/\omega^4)\{(\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}+(1/(2(2j+1)))(\sigma_z)_{ij}]f_j$$

$$\rightarrow [\gamma[j(j+1)-1(1+1)] - (2-\alpha)j(j+1)]f_i.$$

Now recall that $\alpha = 1-2A$ and $\gamma = 1+2A$; so we have for the

$^3(J\pm1)_J$ case:

$$[(1+2A)j(j+1) - (1+2A)1(1+1) - (2-1+2A)j(j+1)]$$

$$= [j(j+1)\{1+2A-2+1-2A\} - (1+2A)1(1+1)]$$

$$= [j(j+1)\{0\} - (1+2A)1(1+1)] = -(1+2A)1(1+1)$$

^3J_J and ^1J_J cases:

$$-(2-\alpha)j(j+1) = -(2-1+2A)1(1+1) = -(1+2A)[1(1+1)]$$

since $j = 1$ in the ^3J_J and ^1J_J cases. Therefore all three equations have the same functional form for $p \to 0$ and for m finite and $m > 0$, namely

$$d^2f/dp^2 + (2/p)df/dp - (1(1+1)/p^2)f = 0 \qquad (6)$$

(iii) Case of $m \to \infty$

Proceeding as in the previous cases we start with the coefficients of the first and second derivatives of f.

$$\text{coefficient of } d^2f/dp^2 = p^2((2\omega^2-\alpha m^2)/\omega^2) + (2-\alpha)p^2$$

$$\text{coefficient of } df/dp = 2p((2\omega^2-\alpha m^2)/\omega^2) + 4p^3/\omega^2$$

$$- 2p^3(2\omega^2-\alpha m^2)/\omega^4 + 2p(2-\alpha) + 4p^3/m^2 - 2p^3(2-\alpha)/m^2$$

$$= 2p(2-\alpha) + 2p^3\alpha/m^2.$$

The coefficient of the undifferentiated term is

$$[3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - (2p^2/k)2\omega$$

$$+ (2p^2/k)E] + [3\alpha p^2/m^2 + Ap^4/m^4 - (2-\alpha)j(j+1)$$

$$- (2p^2/k)(2m + (p^2/m) + \ldots -E)]$$

where we have made the approximation $\omega \simeq m$ and in $(2p^2/k)2\omega$ we have made use of the expansion $\omega = \sqrt{m^2+p^2} \simeq m + (p^2/2m) + \ldots$

Here we have set $E = 2m + \lambda$. Collecting terms we now have

$$(2-\alpha)p^2d^2f/dp^2 + 2p(2-\alpha)df/dp + [-(2-\alpha)j(j+1) - (2p^2/k)(p^2/m)$$

$$+ (2p^2/k)\lambda]f = 0$$

or

$$d^2f/dp^2 + (2/p)df/dp + [2\lambda/(k(2-\alpha)) - j(j+1)/p^2$$

$$- 2p^2/(mk(2-\alpha)]f(p) = 0 \qquad (7)$$

Comparing this last equation with that for the radial equation of the three dimensional harmonic oscillator such as is found in [38], we see that they are of the same form if the force constant is taken to be $k(2-\alpha) = k(1+2A)$.

In a like manner one proceeds to show that the ${}^3J_J$ and ${}^3(J\pm1)_J$ equations give rise to this same limiting form.

(iv) Case of $m \to 0$

Proceeding as in the previous three cases we look at the coefficients of the first and second derivatives of f.

coefficient of d^2f/dp^2: $p^2((2\omega^2-\alpha m^2)/\omega^2) + 2p^2$

coefficient of df/dp: $2p((2\omega^2-\alpha m^2)/\omega^2) - (4p^3/\omega^2)$

$$- p^2((2\omega^2-\alpha m^2)/\omega^4)2p + 4p - 4p - 4p = 4p$$

coefficient of undifferentiated term:

$$[3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - (2p^2/k)2\omega$$

$$+ (2p^2/k)E] + 0 + A - 2j(j+1) - (2p^2/k)(2p-E)$$

For the ^1J_J equation in the limit $m \to 0$ we have the form:

$$2p^2d^2f/dp^2 + 4pdf/dp + [A - 2j(j+1) - 2p^2(2p-E)/k]f = 0$$

or

$$d^2f/dp^2 + (2/p)df/dp + [A/(2p^2) - j(j+1)/p^2 - (2p-E)/k]f = 0 \quad (8)$$

In this case the other two equations do not reduce to the same limiting form. For the ^3J_J equation, the coefficient of the undifferentiated term is

$$-2j(j+1) - (2p^2/k)(2p-E).$$

For the $^3(J\pm1)_J$ case this same coefficient is:

$$[-2j(j+1)]f_i + A[(1/2)\delta_{ij} - (\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$-(1/(2(2j+1)))(\sigma_z)_{ij}]f_j - (2p^2/k)(2p-E)f_i$$

In this limit these equations become

$$^3J_J: \quad d^2f/dp^2 + (2/p)df/dp + [-j(j+1)/p^2 - (2p-E)/k]f = 0 \qquad (9)$$

$$^3(J\pm1)_J: \quad d^2f_i/dp^2 + (2/p)df_i/dp + [-(j(j+1)/p^2)\delta_{ij}$$

$$+(A/2p^2)\{(1/2)\delta_{ij} - (\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$- (1/(2(2j+1)))(\sigma_z)_{ij}\} - ((2p-E)/k)\delta_{ij}]f_j = 0 \qquad (10)$$

Note if A = 0 then all three equations in the m = 0 limit have the same form again.

(v) Lagrangians of the Radial Equations

The general form of these, or any, Sturm-Liouville equation is:

$$d/dx[p(x)df/dx] + [q(x) + \lambda r(x)]f = 0.$$

It is well known that such equations arise as the Euler-Lagrange equations associated with the Isoperimetric Problem in the calculus of variations. The statement of this problem is to find a function f(x) such that the following integral

$$I = (1/2)\int_A^B [p(x)(df/dx)^2 - q(x)f^2]dx \equiv \int_A^B \mathfrak{f}(x,y,y')dx \qquad (11)$$

is an extremum, subject to the further constraint that

$$(1/2)\int_A^B r(x)f^2\,dx = 1. \qquad (12)$$

The limits A and B may be finite or infinite, p(x) must be differentiable for x in the region $A \leq x \leq B$, and q(x) and r(x) should be continuous in the same region.

The associated Lagrangians for the equations are respectively

$$\mathfrak{L}_1 = (1/2)[p^2((2\omega^2-\alpha m^2)/\omega^2)(df/dp)^2 - (1/2)[3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4$$

$$- ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - (2p^2/k)2\omega]f^2$$

$$\mathfrak{L}_2 = (1/2)[p^2((2\omega^2-\alpha m^2)/\omega^2)](df/dp)^2 - (1/2)[3m^4\alpha p^2/\omega^6$$

$$-(1-A)m^2p^2/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1) - (2p^2/k)2\omega]f^2$$

$$\mathfrak{L}_3 = (1/2)[p^2((2\omega^2-\alpha m^2)/\omega^2)](df_i/dp)^2 - (1/2)[3m^4\alpha p^2/\omega^6 - m^2p^2\beta/\omega^4$$

$$+ (m\gamma/\omega)(j(j+1)-1(1+1)) - (2p^2/k)2\omega - ((2\omega^2-\alpha m^2)/\omega^2)j(j+1)]f^2$$

$$- (1/2)[(2mp^2\delta/(\omega^2(\omega+m))) + Ap^2/(\omega(\omega+m))][(1/2)\delta_{ij}$$

$$- (\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij} - (1/(2(2j+1)))(\sigma_z)_{ij}]f_j^2$$

$$- (1/2)\{(m^2p^2/\omega^4)[(\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$+ (1/(2(2j+1)))(\sigma_z)_{ij}]\}f_j^2$$

where $\alpha = 1-2A$, $\beta = (3-2A)/2$, $\gamma = 1+2A$, $\delta = 1-A$, and the common constraint condition is

$$(2E/k)\int_A^B p^2[f_i(p)]^2\, dp = 1.$$

Appendix I.6

Solutions of Limiting Forms of the Radial Bethe-Salpeter Equations and the Choice of the Variational Trial Wave Functions

We first obtain solutions of the limiting forms of the equations derived in appendix I.5(i) and (ii). We first discuss the case of $p \to \infty$. In this case the equations have the limiting form [equation (5) of appendix I.5(i)]:

$$d^2u/dp^2 - (2/k)pu + (E/k)u = 0 \qquad (1)$$

If we set $y = \alpha(p-E/2)$, equation (1) becomes

$$d^2u/dy^2 - yu = 0 \qquad (2)$$

for the choice $\alpha = \sqrt[3]{(2/k)}$. The solution of equation (2) is the Airy function.

If $p \to 0$, then the limiting forms of these equations again all take on the same form, namely equation (6) of appendix I.5(ii):

$$d^2f/dp^2 + (2/p)df/dp - [\ell(\ell+1)/p^2]f = 0 \qquad (3)$$

The general solution of equation (3) is

$$f(p) = ap^{\ell} + bp^{-(\ell+1)}$$

We impose the requirement that $f(p) \to 0$ as p^{ℓ} which requires that $b \equiv 0$.

The idea now is to make use of these two limiting forms of the Bethe-Salpeter equations to obtain "good" approximate solutions to these equations. One method is to write the approximate solutions in the form:

$$f(p) = [p^{\ell}]B(p)[Ai(\sqrt[3]{(2/k)}\ p)/p] \qquad (4)$$

This particular form does not preserve the above limiting forms however. This particular problem can be easily taken care of by modifying equation (4) to read:

$$f(p) = [p^{\ell}]B(p)[Ai(\sqrt[3]{(2/k)}\ p)](1/(a + p^{\ell+1})) \qquad (5)$$

with $a > 0$.

The form of the Airy function is somewhat awkward to deal with for the following reasons. The constant k which appears in the argument is one of the parameters that is to be determined by comparing the eigenvalue spectrum and decay rates calculated by these equations to those observed experimentally. A second reason is that Ai(x) for $x \simeq 35$ is very small ($\simeq 10^{-300}$). The expected value of k from previous calculations is about .3. The numerical integration routine to be used is a Gauss-Laguerre algorithm whose abscissa range will pass this limit. The more points used in the numerical integration, the better the energy eigenvalues will be, especially if those points are near $x = 0$. This factor of $\sqrt[3]{(2/k)}$ can be eliminated from the argument of the Airy function by making the following change of scale.

$$p = \sqrt[3]{(k/2)}\ x \qquad m = \sqrt[3]{(k/2)}\ \kappa \qquad (6)$$

Inserting (6) into the term

$$d/dp[p^2((2\omega^2-\alpha m^2)/\omega^2)df/dp]$$

we have

$$d/dx[x^2((2\omega^2-\alpha\kappa^2)/\omega^2)df/dx]$$

where $\omega^2 = x^2 + \kappa^2$ now. In the coefficient of the f(p) terms we have

$$^1J_J: [3m^4\alpha p^2/\omega^6 + Ap^4/\omega^4 - ((2\omega^2-\alpha m^2)/\omega^2))j(j+1)$$

$$- (2p^2/k)(2\omega-E)]f(p)$$

$$+ [3\kappa^4\alpha x^2/\omega^6 + Ax^4/\omega^4 - ((2\omega^2-\alpha\kappa^2)/\omega^2))j(j+1)$$

$$- 2x^2\omega - \sqrt[3]{(2/k)}\, Ex^2]f(x)$$

The ^3J_J and the $^3(J\pm1)_J$ equations behave in a similar manner, i.e. with the substitutions $p \to x$ and $m \to \kappa$, all of the terms in the coefficient remain of the same form with the exception of the $(2p^2/k)(2\omega-E)$ term. This term, as in the ^1J_J case, becomes

$$2x^2\omega - \sqrt[3]{(2/k)}\, x^2E$$

In terms of the variables x and κ the variational problem now reads: Find those functions f(x) which render the following integrals an extremum

$$\int_0^\infty f_1(x,f(x),f'(x))dx \tag{7}$$

subject to the constraint

$$\sqrt[3]{(2/k)}\, E \int_0^\infty x^2f^2(x)dx = 1 \tag{8}$$

The functions $f_1(x,f(x),f'(x))$ are

$$\mathcal{L}_1(x,f(x),f'(x)) = (1/2)(df/dx)^2[x^2(2\omega^2-\alpha m^2)/\omega^2]$$

$$- (1/2)[3\kappa^4\alpha x^2/\omega^6 + Ax^4/\omega^4 - ((2\omega^2-\alpha\kappa^2)/\omega^2)j(j+1)$$

$$- 2x^2\omega]f^2 \qquad (9)$$

$$\mathcal{L}_2(x,f(x),f'(x)) = (1/2)(df/dx)^2[x^2(2\omega^2-\alpha m^2)/\omega^2]$$

$$- (1/2)[3\kappa^4\alpha x^2/\omega^6 - (1-A)(\kappa^2x^2/\omega^4) - ((2\omega^2-\alpha\kappa^2)/\omega^2)j(j+1)$$

$$- 2x^2\omega]f^2 \qquad (10)$$

$$\mathcal{L}_3(x,f(x),f'(x)) = (1/2)[x^2(2\omega^2-\alpha m^2)/\omega^2](df_i/dx)^2$$

$$- (1/2)[3\kappa^4\alpha x^2/\omega^6 - \kappa^2x^2\beta/\omega^4 + (\kappa\gamma/\omega)[j(j+1)-\ell(\ell+1)]$$

$$- ((2\omega^2-\alpha\kappa^2)/\omega^2)j(j+1) - 2x^2\omega]f_i^2(x)$$

$$- (1/2)[2\kappa x^2\delta/(\omega^2(\omega+x)) + Ax^2/(\omega(\omega+x))]\ [(1/2)\delta_{ij}$$

$$- (\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij} - (1/(2(2j+1)))(\sigma_z)_{ij}](f_j(x))^2$$

$$- (1/2)\{(\kappa^2x^2/\omega^4)[(\sqrt{j(j+1)}/(2j+1))(\sigma_x)_{ij}$$

$$+ (1/(2(2j+1)))(\sigma_z)_{ij}](f_j(x))^2 \qquad (11)$$

In terms of the variables x and κ the trial form of the function f(p) given in equation (5) reads:

$$f(x) = x^{\ell}B(x)Ai(x)(1/(a+x^{\ell+1})) \qquad (12)$$

The function B(x) ideally should go to unity as $x \to 0$ and as $x \to \infty$. Of the functions tried, none was found that obeyed these restrictions and yielded satisfactory results. In view of this, the function B(x) was chosen to be a polynomial

$$B(x) = a_0 + a_1x + a_2x^2 + \ldots + a_9x^9.$$

Details concerning this choice of B(x) and its calculational results are discussed above in Chapter II of the thesis.

Appendix II.1

The transition rate for $^3\psi_1(3095)\to e^+ + e^-$

We start from the S-matrix element for the process $^3\psi_1(3095)\to e^+ + e^-$,

$$\langle e^+e^-|S|{}^3\psi_1(3095)\rangle = \langle e^+e^-|T\exp(-i\int H_I(t)dt)|{}^3\psi_1(3095)\rangle,$$

where T denotes the time ordered product, and

$$H_I(t) = -\int d^3x j_\mu(\vec{x},t)A_\mu(\vec{x},t)$$

$$j_\mu(x) = j^l_\mu(\vec{x},t) + j^h_\mu(\vec{x},t) =$$

$$j^h_\mu(\vec{x},t) + ie{:}\bar{\psi}(x)\gamma_\mu\psi(x){:}.$$

With these definitions, we have for $\int H_I(t)dt$ the form

$$-\int H_I(t)dt = \int d^4x\left(j^h_\mu(x)A_\mu(x) + j^l_\mu(x)A_\mu(x)\right)$$

To proceed further we need to evaluate the S-matrix element. This we do by the use of perturbation theory. The standard approach here is to expand the exponential in the S-matrix element formally in a power series, so that

$$\exp(-i\int_{-\infty}^{+\infty} H_I(t)dt = 1-i\int_{-\infty}^{+\infty} H_I(t)dt + \frac{(-i)^2}{2!}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H_I(t)H_I(t')dtdt'+\ldots$$

We need to take into account the time ordering property explicitly given in the S-matrix above. Formally this means writing the S-matrix element

$$\langle e^+e^-|1+iT\int_{-\infty}^{+\infty} d^4x\left(j^h_\mu(x)A_\mu(x) + j^l_\mu(x)A_\mu(x)\right)$$

$$\frac{+(-i)^2}{2!}T\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^4xd^4y\left(j^h_\mu(x)A_\mu(x) + j^l_\mu(x)A_\mu(x)\right)\left(j^h_\nu(y)A_\nu(y)\right.$$

$$\left.+j^l_\nu(y)A_\nu(y)\right)+\ldots\ |{}^3\psi_1(3095)\rangle = \langle e^+e^-|-\tfrac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^4xd^4y\left(j^h_\mu(x)A_\mu(x)j^l_\nu(y)A_\nu(y)\right.$$

$$\left.+j^l_\mu(x)A_\mu(x)j^h_\nu(y)A_\nu(y)+\ldots\right)\ |{}^3\psi_1(3095)\rangle$$

The terms omitted from this last expression will give a zero expectation value because they do not contain the requisite number of e^+ and e^- creation operators, in the first or second order terms in this expansion, to produce the e^+e^- final state. The terms in this last expression can be factorized as follows:

$$\langle e^+e^-|S|^3\psi_1(3095)\rangle = \tfrac{1}{2}T\int d^4x d^4y \langle e^+e^-|j_\mu^1(x)|0\rangle\langle 0|A_\mu(x)A_\nu(y)|0\rangle$$

$$\langle 0|j_\nu^h(y)|^3\psi_1(3095)\rangle$$

+ a second term with x and y interchanged.

Since the only term in this product that contains two arguments, x and y, is the photon term $A_\mu(x)A_\nu(y)$, the time ordering need only be considered for this term. The quantity

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle$$

is called the photon propagator.

We now discuss each of these three terms separately. In what follows we shall use the Fourier expansion of the functions $\psi(x)$ that arise in the current matrix element. The form that we shall employ for this purpose is

$$\psi(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_{S=1}^{2} \int d^3p \ \frac{m}{E(p)} \left[b_S(p)u_S(p)\exp(-ipx)\right.$$

$$\left. + d_S^+(p)v_S(p)\exp(+ipx)\right]$$

(i) The term $\langle e^+e^-|j_\mu(x)|0\rangle = \langle e^+e^-|ie\bar{\psi}(x)\gamma_\mu\psi(x)|0\rangle = I$

The Fourier expansion of I is

$$I = \left(\frac{1}{2\pi}\right)^3 ie\int d^3p d^3p' \sum_{S,S'=1}^{2} \left[\frac{m^2}{E(p)E(p')}\langle e^+e^-|b_S^+(p)u_S(p)\gamma_\mu\right.$$

$$\left. d_{S'}^+(p')v_{S'}(p')\exp(-i(p+p')x)|0\rangle\right.,$$

plus other terms which involve one creation operator of either type and an annihilation operator or no creation operators of either type (i.e. b_s^+ or d_s^+).

Let $\langle e^+e^-| = \langle 0|b_r(q)d_{r'}(q')$, so that

$$\langle e^+e^-|b_s^+(p)d_{s'}^+(p')|0\rangle = \langle 0|b_r(q)d_{r'}(q')b_s^+(p)d_{s'}(p')|0\rangle$$

$$= \langle 0|b_s^+(p)d_{s'}(p')d_{r'}(q')b_r(q)|0\rangle + \delta^3_{rs}(p-q)\delta^3_{r's'}(p'-q').$$

Now with the integrations over p and p' we arrive at the result

$$\langle e^+e^-|j_\mu(x)|0\rangle = \left(\frac{1}{2\pi}\right)^3 ie\ \frac{m^2}{EE'}\ \bar{u}_r(q)\gamma_\mu v_{r'}(q')\ \exp\ (-i(q+q')x)$$

(ii) The photon propagator term

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \left(\frac{1}{2\pi}\right)^4 \int d^4k\ \frac{\exp(ik(x-y))}{k^2-i\epsilon}$$

The expression on the right hand side of the photon propagator term (ii) is obtained in the following way. We expand $A_\mu(x)$ in a Fourier expansion using

$$A_\mu(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sum_\lambda \int \frac{d^3k}{\sqrt{2\omega}}\ \left(\epsilon_\mu^\lambda(\vec{k})a_\lambda(\vec{k})\exp(ikx) + \epsilon_\mu^\lambda\ a_\lambda{}^+(\vec{k})\exp(-ikx)\right)$$

$$A_\mu(x)A_\nu(y) = \left(\frac{1}{2\pi}\right)^3 \sum_{\lambda,\alpha} \iint \frac{d^3k}{\sqrt{2\omega}}\ \frac{d^3k'}{\sqrt{2\omega'}}\ \left(\epsilon_\mu^\lambda(\vec{k})a_\lambda(\vec{k})\exp(ikx)\right.$$

$$\left.+\epsilon_\mu^\lambda(\vec{k})a_\lambda{}^+(\vec{k})\exp(-ikx)\right)\left(\epsilon_\nu^\alpha(\vec{k}')a_\alpha(\vec{k}')\exp(ik'y) + \epsilon_\nu^\alpha(\vec{k}')a_\alpha^+(\vec{k}')\exp(-ik'y)\right)$$

$$= \left(\frac{1}{2\pi}\right)^3 \sum_{\lambda,\alpha} \int \frac{d^3kd^3k'}{\sqrt{4\omega\omega'}}\ \left(\epsilon_\mu^\lambda(\vec{k})a_\lambda(\vec{k})\,\epsilon_\nu^\alpha(\vec{k}')a_\alpha(\vec{k}')\exp(ikx - ik'y)\right.$$

$$\left. + \epsilon_\mu^\lambda(\vec{k})a_\lambda^+(\vec{k})\,\epsilon_\nu^\alpha(\vec{k}')a_\alpha(\vec{k}')\exp(-ikx + ik'y)\right)$$

$$TA_\mu(x)A_\nu(y) = \theta(x_o-y_o)A_\mu(x)A_\nu(y) + \theta(y_o-x_o)A_\nu(y)A_\mu(x)$$

$$= \left(\frac{1}{2\pi}\right)^3 \sum_{\lambda,\alpha} \int \frac{d^3k d^3k'}{\sqrt{4\omega\omega'}} \left[\varepsilon_\mu^\lambda(\vec{k})a_\lambda(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')a_\alpha(\vec{k}')\exp(ikx-ik'y)\theta(x_o-y_o)\right.$$

$$\left. + \varepsilon_\mu^\lambda(\vec{k})a_\lambda^+(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')a_\alpha(\vec{k}')\exp(-ikx + ik'y)\theta(y_o-x_o)\right]$$

The vacuum expectation value of the time ordered product is zero, except for the two terms explicitly written out above. In this case we have

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \left(\frac{1}{2\pi}\right)^3 \sum_{\alpha,\lambda} \iint \frac{d^3k d^3k'}{\sqrt{4\omega\omega'}} \left(\varepsilon_\mu^\lambda(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')\delta^3_{\alpha,\lambda}(\vec{k}-\vec{k}')\right.$$

$$\exp(ikx - ik'y)\theta(x_o-y_o) + \varepsilon_\mu^\lambda(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')\delta^3_{\alpha,\lambda}(\vec{k}-\vec{k}')$$

$$\left.\exp(-ikx + ik'y)\theta(y_o-x_o)\right)$$

$$= \left(\frac{1}{2\pi}\right)^3 \sum_{\alpha,\lambda} \int \frac{d^3k}{\sqrt{4\omega^2}} \sum_{\alpha,\lambda}\left[\varepsilon_\mu^\lambda(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')\exp(ik(x-y))\theta(x_o-y_o)\right.$$

$$\left. + \varepsilon_\mu^\lambda(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')\exp(-ik(x-y))\theta(y_o-x_o)\right]$$

$$= \left(\frac{1}{2\pi}\right)^3 \sum_\alpha \int \frac{d^3k}{2\omega}\left[\varepsilon_\mu^\alpha(\vec{k})\varepsilon_\nu^\alpha(\vec{k}')\exp(ik(x-y))\right] = \left(\frac{1}{2\pi}\right)^3 \int \frac{d^3k}{2\omega}\,\delta_{\mu\nu}\exp(-ik(x-y))$$

We now note that $\int_{-\infty}^{+\infty} dk_o \dfrac{\exp(-ik_o x_o)}{-k_o^2 + \vec{k}^2 + m^2 - i\varepsilon}$

$$= \int_c dk_o \frac{\exp(ik_o x_o)}{\vec{k}^2+m^2-k_o^2+i\varepsilon} = 2\pi i \text{ Residue } \left[f(k_o)\right]$$

$$R = \text{Residue}\left(f(k_o)\right) = \text{Residue}\ \frac{\exp(ik_o x_o)}{\vec{k}^2+m^2-k_o^2}$$

$$= \text{Residue}\ \frac{\exp(ik_o x_o)}{-(k_o-\omega)(k_o+\omega)}$$

where $\omega = \vec{k}^2 + m^2$, so that

$$R = \lim_{k_o\to\omega} (k_o-\omega) f(k_o) = \lim_{k_o\to\omega} \frac{\exp(ik_o x_o)}{k_o+\omega} = \frac{\exp(i\omega x_o)}{2\omega} \ .$$

So

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \left(\frac{1}{2\pi}\right)^4 \int d^4k\ \frac{\exp(ik(x-y))}{k^2-i\varepsilon}\ \delta_{\mu\nu} \ .$$

(iii) The Hadronic Current Term:

$$H = \langle 0|j_\mu^h(y)|^3\psi_1(3095)\rangle$$

$$H = \langle 0|j_\mu^h(y)|^3\psi_1(3095)\rangle$$

$$= \langle 0|\exp(-ip_\psi y) j_\nu^h(0) \exp(ip_\psi y)|^3\psi_1(3095)\rangle$$

$$= \langle 0|j_\nu^h(0)|^3\psi_1(3095)\rangle \exp(ip_\psi y)$$

Now combine (i), (ii), and (iii) to get

$$S \equiv \langle e^+e^-|S|\psi\rangle = \iint \left(\frac{1}{2\pi}\right)^3 \bar{e}u_r(q)\gamma_\mu v_{r'}(q')\ \ \frac{m_e^2}{EE'}\langle 0|j_\nu^h(0)|^3\psi_1(3095)\rangle$$

$$\exp(ip_\psi y)\,\delta_{\mu\nu}\exp(-i(q+q')x)\left(\frac{1}{2\pi}\right)^4 \int d^4k\ \frac{\exp(ik(x-y))}{k^2-i\varepsilon}\ dxdy$$

Note that the ½ has been omitted. We have a sum of two terms, which are equal, the second term coming from interchanging x and y. Next we perform the integrations over y, k, and x, in that order, giving us

$$S = \left(\frac{1}{2\pi}\right)^7 e\delta_{\mu\nu}\langle 0|j_\nu^h(0)|^3\psi_1(3095)\rangle \bar{u}_r(q)\gamma_\mu v_{r'}(q') \frac{m_e^2}{EE'}$$

$$\int d^4k d^4x \exp(-i(q+q')x)\frac{1}{k}\exp(ikx)\int d^4y \exp(ip_\psi y)\exp(-iky)$$

$$= \left(\frac{1}{2\pi}\right)^7 e\delta_{\mu\nu}\langle 0|j_\nu^h(0)|^3\psi_1(3095)\rangle \bar{u}_r(q)\gamma_\mu v_{r'}(q') \frac{m_e^2}{EE'}$$

$$\int d^4k d^4x \frac{\exp(-i(q+q'-k)x)}{k^2} (2\pi)^4 \delta(k-p_\psi)$$

$$= \left(\frac{1}{2\pi}\right)^3 e\delta_{\mu\nu}\langle 0|j_\mu^h(0)|^3\psi_1(3095)\rangle \bar{u}_r(q)\gamma_\mu v_{r'}(q') \frac{m_e^2}{EE'}$$

$$\int d^4x \frac{\exp(-i(q+q'-p_\psi)x}{p_\psi^2}$$

$$= \left(\frac{1}{2\pi}\right)^3 e\delta_{\mu\nu}\langle 0|j_\mu^h(0)|^3\psi_1(3095)\rangle \bar{u}_r(q)\gamma_\mu v_{r'}(q') \frac{m_e^2}{EE'}$$

$$\frac{1}{p_\psi^2}(2\pi)^4\delta(q+q'-p_\psi)$$

$$= (2\pi) e\delta_{\mu\nu}\langle 0|j_\mu^h(0)|^3\psi_1(3095)\rangle \bar{u}_r(q)\gamma_\mu v_{r'}(q') \frac{m_e^2}{EE'}$$

$$\frac{1}{p_\psi^2}\delta(q+q'-p_\psi).$$

We now want to find the quantity $|\langle e^+e^-|S|\psi\rangle|^2$. We will sum this over the possible final state polarizations and integrate over

the allowed momentum states so as to determine the total decay rate of $\psi \to e^+ + e^-$. Let $T = |\langle e^+e^-|S|^3\psi_1(3095)\rangle|^2$ suitably summed and integrated over.

$$T = \sum_{r,r'} \{(2\pi)^2 e^2 (\langle e^+e^-|j_\mu^h(0)|\psi\rangle \bar{u}_r(q)\gamma_\mu v_{r'}(q') \frac{m_e^2}{EE'} \frac{\delta^{(4)}(q+q'-p_\psi)}{p_\psi^2} (\bar{v}_{r'}(q')\gamma_\nu u_r(q) \langle 0|j_\mu^h|\psi\rangle \frac{m_e^2}{EE'} \frac{\delta^{(4)}(q+q'-p_\psi)}{p_\psi^2}$$

$$= (2\pi)^2 e^2 \frac{m_e^2}{EE'} \langle\psi|j_\mu^h(0)|0\rangle\langle 0|j_\nu^h(0)|\psi\rangle \frac{\delta^{(4)}(q+q'-p_\psi)}{p_\psi^2}$$

$$\sum_{r,r'} (\bar{u}_r(q)\gamma_\mu v_{r'}(q')\bar{v}_{r'}(q')\gamma_\nu u_r(q))$$

Look first at the term summed over r and r', and note that this double sum is equivalent to the taking of the trace of the matrix product

$$\gamma_\mu \left(\frac{-1}{2m}(-i\gamma\cdot q+m)\right)\gamma_\nu\left(\frac{-1}{2m}(\gamma\cdot q'+m)\right) .$$

Note that

$$\sum_r u_r(p)\bar{u}_r(p) = \frac{1}{2m_e}(-i\gamma\cdot p + m_e)$$

$$\sum_{r'} v_{r'}(p')\bar{v}_{r'}(p') = \frac{-1}{2m_e}(i\gamma\cdot p' + m_e) .$$

By using

$$\text{Trace}\,(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) = 4\delta_{\mu\alpha}\delta_{\nu\beta} - 4\delta_{\mu\nu}\delta_{\alpha\beta} + 4\gamma_{\mu\beta}\gamma_{\nu\alpha}$$

$$\text{Trace}\,(\gamma_\mu\gamma_\nu) = 4\delta_{\mu\nu}$$

we have

$$\sum_{r,r'} \gamma_\mu v_{r'}(q')\bar{v}_{r'}(q')\gamma_\nu u_r(q)\bar{u}_r(q)$$

$$= \text{trace}\left[\gamma_\mu\left(\frac{-1}{2m_e}\right)(i\gamma\cdot q'+m_e)\gamma_\nu\,\frac{-i\gamma\cdot q+m_e}{2m_e}\right]$$

$$= -\text{trace}\left[\gamma_\mu(i\gamma\cdot q')\gamma_\nu(-i\gamma\cdot q) + \gamma_\mu m_e\gamma_\nu m_e\right.$$

$$\left. + \text{terms with 1 or 3 } \gamma\text{'s}\right]\frac{1}{4m_e^2}$$

$$= \frac{-1}{4m_e^2}\,\text{trace}\left[\gamma_\mu(\gamma\cdot q')\gamma_\nu(\gamma\cdot q) + m_e^2\gamma_\mu\gamma_\nu\right]$$

With this, the trace part becomes

$$\frac{-1}{4m_e^2}\left[4q'_\mu q_\nu - 4\delta_{\mu\nu}q\cdot q' + 4q_\mu q'_\nu + 4m_e^2\delta_{\mu\nu}\right]$$

$$= \frac{-1}{m_e^2}\left[q'_\mu q_\nu + q_\mu q'_\nu + \delta_{\mu\nu}(m_e^2 - q\cdot q')\right].$$

Here $q\cdot q' = (\vec{q}\cdot\vec{q}' - EE')$. If we choose for our coordinate system the center of mass of the decaying particle, then $\vec{q} = -\vec{q}'$ and $E=E'$ in this case since the masses of the decay products are equal. With this we have

$$q\cdot q' = -(\vec{q}')^2 - E^2 = -(E^2 - m_e^2) - E^2 = -2E^2+m_e^2.$$

Then the trace part becomes

$$\frac{-1}{m_e^2}\left[q'_\mu q_\nu + q_\mu q'_\nu + \delta_{\mu\nu}\,(2E^2)\right].$$

To complete the reduction we pick the z-axis parallel to the spin axis of the decaying particle and note that in the center of mass system that we will use here, the 4th component of $\varepsilon_\nu^{(m)}$ is zero. Therefore $\hat{\varepsilon}_\nu = \delta_{3\nu}$, $\hat{\varepsilon}_\mu = \delta_{3\mu}$, and we have

$$\varepsilon_\nu q_\nu = q\cos\theta \qquad \varepsilon_\mu q_\mu = q\cos\theta$$

$$\varepsilon_\nu q'_\nu = -\varepsilon_\nu q_\nu = -q\cos\theta \qquad \varepsilon_\mu q'_\mu = -q\cos\theta$$

and the trace part dotted into the polarization vectors $\varepsilon_\mu\varepsilon_\nu$ becomes

$$\frac{-1}{m_e^2}\left((-q\cos\theta)(q\cos\theta) + (q\cos\theta)(-q\cos\theta) + 2E^2\right)$$

$$= \frac{-1}{m_e^2}\left(-2q^2\cos^2\theta + 2E^2\right) \approx \frac{+1}{m_e^2}\left(-2E^2(1-\cos^2\theta)\right) = \frac{-2E^2}{m_e^2}(1-\cos^2\theta)$$

where we have made the approximation $q^2 \approx E^2 >> m_e^2$ and we have neglected m_e^2 in comparison to E^2. With this T becomes

$$T = -(2\pi)^2 e^2 \frac{m_e^2}{EE'}\left(\frac{\delta^4(p_\psi - q - q')}{p_\psi^2}\right)^2 |<0|j^h(0)|\psi>|^2 \left(\frac{2E^2}{m_e^2}(1-\cos^2\theta)\right)$$

We now need to give an interpretation to the factor $\left(\delta^4(p_\psi - q - q')\right)^2$. This is done in the standard way of rewriting $\left(\delta^4(p_\psi - q - q')\right)^2$ as

$$\delta^4(p_\psi - q - q')\left(\frac{1}{2\pi}\right)^4 \int d^4x \exp(i(p_\psi - q - q')x) = \frac{\delta^4(p_\psi - q - q')}{(2\pi)^4}(V\times\tau)$$

where $V\times\tau$ is a normalization space-time.

Now we would like the quantity T/τ.

$$\frac{T}{\tau} = -(2\pi)^2 \frac{m_e^2 e^2}{E_e^2} |\langle 0|j^h(0)|\psi\rangle|^2 \frac{2E_e^2}{m_e^2} (1-\cos^2\theta) \frac{1}{p_\psi^4}$$

$$\delta^4(p_\psi - q - q')V\left(\frac{1}{2\pi}\right)^4.$$

To obtain the total decay rate we must sum over the possible sets of (unpolarized) final states available. If we choose the normalizing volume to be $(2\pi)^3$, the density of final states is given by $d^3q d^3q'$ and we have

$$d\omega = \frac{T}{\tau} = -\left(\frac{1}{2\pi}\right)^2 \frac{2e^2}{p_\psi^4} (1-\cos^2\theta)|\langle 0|j^h(0)|\psi\rangle|^2 \delta^4(p_\psi - q - q')d^3q d^3q'$$

$$= \frac{2}{\pi}\frac{\alpha}{p_\psi^4} |\langle 0|j^h(0)|\psi\rangle|^2 (1-\cos^2\theta)\delta^4(p_\psi - q - q')d^3q d^3q'$$

We now integrate over d^3q' using the delta function, and we get

$$d\omega = \frac{-2\alpha}{\pi p_\psi^4} |\langle 0|j^h(0)|\psi\rangle|^2 (1-\cos^2\theta)\delta^4(E_\psi - E - E')d^3q.$$

We next integrate over $d^3q = q^2 dq d(\cos\theta) d\phi$. The $d\phi$ integral just gives us 2π. For the other two integrals we have

$$\int_0^\pi (1-\cos^2\theta)d(\cos\theta) = \left(\cos\theta - \frac{\cos^3\theta}{3}\right)_0^\pi = (-1+\tfrac{1}{3}) - (1-\tfrac{1}{3}) = \frac{4}{3}$$

$$\int\delta(E_\psi - 2E)q^2 dq = \int\delta(E_\psi - 2E)E^2 dE = \int\delta(E_\psi - x)\frac{x^2}{4}\frac{dx}{2} = \frac{1}{8}\int\delta(E_\psi - x)x^2 dx$$

$$= \frac{x^2}{8}\Big|_{x=E_\psi} = \frac{E_\psi^2}{8}.$$

$$\text{Finally } \omega = \frac{-2\alpha}{\pi p_\psi^4} \left|\langle 0|j^h(0)|\psi\rangle\right|^2 (2\pi)(-4/3)\left(\frac{p_\psi^2}{8}\right)$$

$$= \frac{+2\alpha}{3p_\psi^2} \left|\langle 0|j^h(0)|\psi\rangle\right|^2$$

The final step is to determine $\left|\langle 0|j^h(0)|\psi\rangle\right|^2$. The matrix element $\langle 0|j^h(0)|\psi\rangle$ is related to the wavefunction of the decaying particle as follows:

$$\langle 0|j^h(0)|\psi\rangle = 3\langle 0|j^h(0)|\psi_s\rangle + 3\langle 0|j^h(0)|\psi_D\rangle$$

$$= \frac{\sqrt{3}ie_q\sqrt{2}}{(2\pi)^{3/2}} \left\{ \int d^3p \frac{f_s(p)}{\sqrt{4\pi}} - \int d^3p \frac{p^2 f_s(p)}{\sqrt{4\pi}} \frac{1}{3E(E+m)} \right\}$$

$$+ \frac{\sqrt{3}ie_q\sqrt{2}}{3\pi} \int p^4 dp \frac{f_D(p)}{E(E+m)}$$

The factor of $\sqrt{3}$ that appears in these terms is to take into account the currently accepted hypothesis that there exist three different "colored" quarks that can give rise to the same $q\bar{q}$ system.

In terms of $\psi(0)$ we have

$$\omega = \frac{2\alpha}{3m_v^2} \left|\langle 0|j^h(0)|\psi\rangle\right|^2$$

$$= \frac{2\alpha}{3m_v^2} \left\{\sqrt{3}\, i\, e_q\sqrt{2}\psi(0)\right\}^2$$

$$= \frac{2\alpha}{3m_v^2} \left(3e_q^2 2|\psi(0)|^2\right)$$

where

$$e_q^2 = (eQ)^2 = \frac{e^2}{4\pi} Q^2 (4\pi) = \frac{4e^2}{9(4\pi)} (4\pi) = \frac{4}{9}\alpha(4\pi)$$

and where $Q = \frac{2}{3}$.

Thus $\omega = \frac{2\alpha}{3m_v^2} 3 \frac{4}{9}\alpha(4\pi) 2 |\psi(0)|^2$

$$= \frac{16\alpha^2}{9m_v^2} (4\pi) |\psi(0)|^2$$

$$= \frac{64\pi\alpha^2}{9m_v^2} |\psi(0)|^2$$

Appendix II.2

Derivation of the Transition Rate for the Radiative Decays in the Bethe-Salpeter Equation

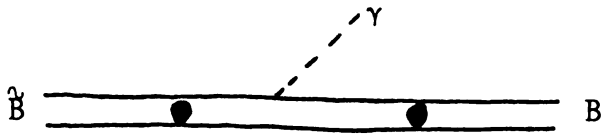
We start this derivation from the S-matrix element of the appropriate decay process, namely $\langle \gamma, B|S|\tilde{B}\rangle$, where $\tilde{B}$ is the initial state and $\gamma,\tilde{B}$ represents the final state of the transition in question. The S-matrix in the interaction representation is given by the time ordered product of $\exp[-i\int H_I(t)dt]$ where $H_I(t)$ is the interaction hamiltonian. The interaction hamiltonian is given by $-\int j_\mu(x)A_\mu(x)d^3x = -\int j_\mu^h(x)A_\mu(x)d^3x$ to the lowest order in the electromagnetic current, $A_\mu(x)$. In the second term, $j_\mu^h(x)$ stands for the hadronic component of the electromagnetic current. With these considerations one can express the S-matrix element for the desired transition in the form

$$\langle \gamma, B|T\ \exp(i\int H_I(x)d^4x)|\tilde{B}\rangle = \langle \gamma, B|[1+i\int d^4x H_I(x) + \ldots]|\tilde{B}\rangle$$

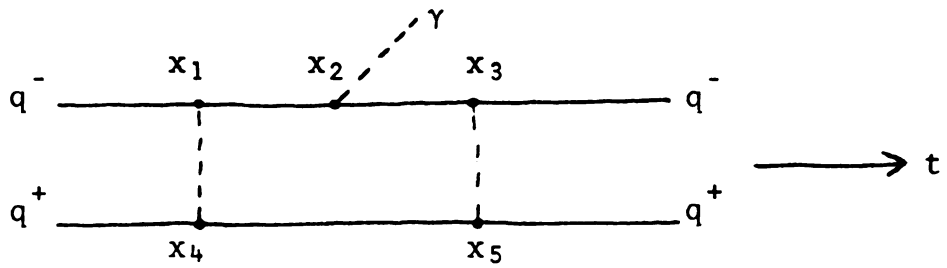
$$= i\langle \gamma, B|\int d^4x\ H_I(x)|\tilde{B}\rangle = i\int d^4x \langle \gamma|A_\mu(x)|0\rangle\ \langle B|j_\mu^h(x)|\tilde{B}\rangle$$

The expression $\langle B|j_\mu^h(x)|\tilde{B}\rangle$ is the hadronic contribution to the electromagnetic current matrix element between the states B and $\tilde{B}$.

The S-matrix element can be diagrammatically represented as



To translate this into a more useful form for calculational purposes, we assume that to lowest order this can be represented in the form



This diagram is evaluated just like the analogous one for electron-positron scattering, with the emission of a photon, in quantum electrodynamics. We start by looking at the top (or q-) line in the diagram. This contribution yields

$$I=\bar{\psi}^{-}(x_3)\gamma_{\mu}{}^{3}[i\ S_c(x_3-x_2)]\ \not{A}^{+}(x_2)[i\ S_c(x_2-x_1)]\gamma_{\nu}{}^{1}\psi^{-}(x_1)$$

and a similar contribution from the bottom line

$$II = \bar{\psi}^{+}(x_4)\ \gamma_{\alpha}{}^{4}[i\ S_c(x_4-x_5)]\ \gamma_{\beta}{}^{5}\psi^{+}(x_5)$$

The interaction part between lines I and II is given by

$$[ig_{\mu\upsilon}D_c(x_3 - x_5)][ig_{\upsilon\alpha}D_c(x_1 - x_4)].$$

(i) Conversion of $\bar{\psi}$ to ψ_c

The first step in this derivation is to convert the expression for the positively charged line into one involving the charge conjugate wavefunction, since the Bethe-Salpeter equation will be written in a form that makes use of ψ and ψ_c rather than one that makes use of ψ and $\bar{\psi}$. The charge conjugate wavefunction ψ_c is defined as

$\psi_c(x) = C\,\bar{\psi}^T(x)$. This says that $\bar{\psi} = [\psi_c]^T[C^{-1}]^T$. We will also need to know the explicit form of $\psi(x)$ expressed in terms of $\psi_c(x)$. This follows directly with a little algebra. (The second equality in the first line below is just the definition of $\bar{\psi}(x)$.)

$$\bar{\psi}(x) = [\psi_c(x)]^T[C^{-1}]^T = [\psi(x)^*]^T\beta$$

$$\psi(x)^* = \beta^T C^{-1}\psi_c(x)$$

$$\psi(x) = \beta^T[C^{-1}]^*\psi_c^*(x) = (\beta^T C^{-1})^*\psi_c^*(x)$$

Now we make use of the relation $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ in the form $C^{-1}\gamma_\mu = -\gamma_\mu^T C^{-1}$ ($\gamma_\mu = \beta$) to get

$$\psi(x) = -[C^{-1}\beta]^*\psi_c^*(x) = -C^{-1}\psi_c^{*T}(x)\,\beta^T$$

$$= -[C^{-1}]^*\bar{\psi}_c(x) = -\bar{\psi}_c(x)\,C$$

The forms that we need for the positively charged line wavefunctions are now

$$\psi_c(x) = C\bar{\psi}^T(x)$$

$$\psi(x) = +\bar{\psi}_c(x)C^{-1} = -\bar{\psi}_c(x)C$$

(ii) Conversion of II

We now make use of these relations between $\psi_c(x)$ and $\psi(x)$ in the positively charged part of the diagram representing the radiative decay of our system.

$$II = \bar{\psi}^{+}(x_4)\ \gamma_\alpha{}^4\ [i\ S_c(x_4-x_5)]\ \gamma_\beta{}^5\ \psi^{+}(x_5)$$

$$= \psi_c{}^T(x_4)\ [C^{-1}]^T\ \gamma_\alpha{}^4 [i\ S_c(x_4-x_5)]\gamma_\beta{}^5[-\bar{\psi}_c(x_5)C]$$

$$= \psi_c{}^T(x_4)\ C^*\gamma_\alpha{}^4[i\ S_c(x_4-x_5)]\ \gamma_\beta{}^5[-\bar{\psi}_c(x_5)C]$$

Now take the transpose of the quantity II.

$$II^T = -C^T\bar{\psi}_c{}^T(x_5)\ \gamma_\beta{}^{5T}[i\ S_c(x_5-x_4)]^T\ \gamma_\alpha{}^{4T}(C^*)^T\psi_c(x_4)$$

$$= -C^T\bar{\psi}_c{}^T(x_5)\ \gamma_\beta{}^{5T}[i\ S_c(x_5-x_4)]^T\gamma_\alpha{}^{4T}\ C^{-1}\psi_c(x_4)$$

$$= -\bar{\psi}_c(x_5)C\ \gamma_\beta{}^{5T}[i\ S_c(x_5-x_4)]^T\gamma_\alpha{}^{4T}\ C^{-1}\psi_c(x_4)$$

$$= -\bar{\psi}_c(x_5)C\ \gamma_\beta{}^{5T}C^{-1}C[i\ S_c(x_5-x_4)]^TC^{-1}C\ \gamma_\alpha{}^{4T}C^{-1}\psi_c(x_4)$$

$$= -\bar{\psi}_c(x_5)[-\gamma_\beta{}^5][i\ S_c(x_5-x_4)](-\gamma_\alpha{}^4)^T\ \psi_c(x_4)$$

(iii) Combining terms and the Bethe-Salpeter equation

With (i) and (ii) above the full matrix element can be written as:

$$\text{I II Interaction} = \bar{\psi}^{-}(x_3)\gamma_\mu[i\ S_c(x_3-x_2)]\ \not{A}^{+}(x_2)[i\ S_c(x_2-x_1)]$$

$$\gamma_\alpha\psi^{-}(x_1)\{-[\bar{\psi}_c(x_5)\gamma_\nu[i\ S_c(x_5-x_4)]\gamma_\beta\psi_c(x_4)]\}$$

$$ig_{\mu\nu}D_c(x_3-x_5)ig_{\alpha\beta}D_c(x_1-x_4)$$

At this point we display the matrix indices in order to keep things straight. We rewrite this matrix element in the form:

$$\text{Int} = -[\bar{\psi}^{-}(x_3)]_A[\gamma_\mu]_{AB}[i\ S_c(x_3-x_2)]_{BC}[\not{A}^{+}(x_2)]_{CD}[i\ S_c(x_2-x_1)]_{DE}$$

$$[\gamma_\alpha]_{EF}[\psi^{-}(x_1)]_F[\psi_c^{+}(x_5)]_M[\gamma_\nu]_{MN}[i\ S_c(x_5-x_4)]_{NO}[\gamma_\beta]_{OP}$$

$$[\psi_c(x_4)]_P\ ig_{\mu\nu}D_c(x_3-x_5)ig_{\alpha\beta}D_c(x_1-x_4)$$

We now collect terms that appear as "initial" or "final" state terms. The "initial" state terms are those with arguments x_1 and x_4 and adjacent matrix terms:

$$(1\text{-}4)\quad [\gamma_\alpha]_{EF}\ \psi^{-}(x_1)_F[\gamma_\beta]_{OP}[\psi_c(x_4)]_P\ ig_{\alpha\beta}D_c(x_1-x_4)$$

and in like manner the "final" state terms are:

$$(3\text{-}5)\quad [\bar{\psi}^{-}(x_3)]_A[\gamma_\mu]_{AB}\ [\bar{\psi}_c^{+}(x_5)]_M[\gamma_\nu]_{MN}\ ig_{\mu\nu}D_c(x_3-x_5)$$

(a) The (1-4) Initial State System

$$(1\text{-}4) = [\gamma_\alpha]_{EF}[\psi(x_1)]_F[\gamma_\beta]_{OP}[\psi_c(x_4)]_P[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= [\gamma_\alpha]_{EF}[\psi(x_1)]_F[\psi_c(x_4)]_P[\gamma_\beta{}^T]_{PO}[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= [\gamma_\alpha]_{EF}[\psi(x_1)]_F[\psi_c(x_4)]_P C^{-1}C[\gamma_\beta{}^T]_{PO}C^{-1}C[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= [\gamma_\alpha]_{EF}[\psi(x_1)]_F[\psi_c(x_4)]_P C^{-1}_{PQ}(C\gamma^T_\beta C^{-1})_{QR}C_{RO}[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= [\gamma_\alpha]_{EF}[\psi(x_1)]_F[\psi_c(x_4)]_P C^{-1}_{PQ}[-\gamma_\beta]_{QR}C_{RO}[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= -[\gamma_\alpha]_{EF}[\psi(x_1,x_4)]_{FP}C^{-1}{}_{PQ}[\gamma_\beta]_{QR}\ C_{RO}[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= -ig_{\alpha\beta}D_c(x_1-x_4)\{[\gamma_\alpha]_{EF}[\psi(x_1,x_4)]_{FP}\ C^{-1}{}_{PQ}\ [\gamma_\beta]_{QR}\ C_{RO}\}$$

$$= [(\not{\partial}_1+m)\ \psi(x_1,x_4)\ C^{-1}(-\not{\partial}_4+m)C]_{EO}$$

(b) The (3-5) Final State System

$$(3\text{-}5) = [\bar{\psi}(x_3)]_A[\gamma_\mu]_{AB}[\bar{\psi}_c(x_5)]_M[\gamma_\nu]_{MN}[ig_{\mu\nu}D_c(x_3-x_5)]$$

$$= [\gamma_\mu{}^T]_{BA}[\bar{\psi}(x_3)\bar{\psi}_c(x_5)]_{AM}[\gamma_\nu]_{MN}\ [ig_{\mu\nu}D_c(x_3-x_5)]$$

$$= C^{-1}C[\gamma_\mu{}^T]_{BA}C^{-1}C[\bar{\psi}(x_3)\bar{\psi}_c(x_5)]_{AM}[\gamma_\nu]_{MN}[ig_{\mu\nu}D_c(x_3-x_5)]$$

$$= C^{-1}[-\gamma_\mu]_{BA}C[\bar{\psi}(x_3)\bar{\psi}_c(x_5)]_{AM}[\gamma_\nu]_{MN}[ig_{\mu\nu}D_c(x_3-x_5)]$$

$$= C^{-1}\gamma_\mu C\bar{\psi}(x_3,x_5)\gamma_\nu[-ig_{\mu\nu}D_c(x_3-x_5)]$$

(c) An Aside on the Bethe-Salpeter Equation

We shall need to make use of the adjoint form of the Bethe-Salpeter equation and this section is devoted to obtaining such a form. We start from the Bethe-Salpeter equation in the form:

$$(\not{\partial}_1+m)\psi C^{-1}(-\not{\partial}_2+m)C = -g^2D_c(x_1-x_2)\gamma_\mu\psi C^{-1}\gamma_\mu C$$

where g^2 is a constant. We first note that

$$\bar{\psi}(x_1,x_2) = \gamma_4{}^T\psi^*(x_1,x_2)\gamma_4$$

and $C^* = C^{-1}$. We start by taking the complex conjugate of the above Bethe-Salpeter equation, giving

$$(\not{\partial}_1^* +m)\psi^*(C^{-1})^*(-\not{\partial}_2^* +m)C^* = -g^2D_c{}^*(x_1-x_2)\gamma_\mu{}^*\psi^*(C^{-1})^*\gamma_\mu{}^*C^*$$

We first consider the left hand side of this equation and express it in terms of $\bar{\psi}(x_1,x_2)$.

$$L = (\not{\partial}_1^* + m)\psi^*(C^{-1})^*(-\not{\partial}_2^* + m)C^* = (\not{\partial}_1^* + m)\psi^*C(-\not{\partial}_2^*+m)C^{-1}$$

$$= (\partial_1^* + m)\gamma_4{}^T\bar{\psi}\,\gamma_4C(-\not{\partial}_2^* + m)C^{-1}$$

$$= (\not{\partial}_1^* + m)\gamma_4{}^T\bar{\psi}(-C\gamma_4{}^T)(-\not{\partial}_2^* + m)C^{-1}$$

$$= (\not\partial_1^* + m)\gamma_4^T\bar{\psi}(-C)\ \gamma_4^T(-\not\partial_2^* + m)C^{-1}$$

Now note that

$$\gamma_4^T\gamma_\mu^*\partial_\mu = \gamma_4^T(\gamma_i^T\partial_i - \gamma_4\partial_4) = \gamma_4^T(-\gamma_i\partial_i - \gamma_4\partial_4)$$

$$= \gamma_4(-\gamma_i\partial_i - \gamma_4\partial_4) = (\gamma_i\partial_i - \gamma_4\partial_4)\gamma_4 = (-\gamma_i^T\partial_i - \gamma_4^T\partial_4)\gamma_4^T$$

where $\gamma_4 = \gamma_4^T$ and $\gamma_i = -\gamma_i^T$. So we have that, moving γ_4^T past $-\not\partial_2^*$

$$L = (\not\partial_1^* + m)\gamma_4^T\bar{\psi}(-C)(+\not\partial_2^T + m)\gamma_4^T C^{-1}$$

$$= (\not\partial_1^* + m)\gamma_4^T\bar{\psi}(-C)(+\not\partial_2^T + m)(-C^{-1}\gamma_4)$$

In a similar manner

$$\gamma_\mu^*\partial_\mu^*\gamma_4^T = (\gamma_i^T\partial_i - \gamma_4\partial_4)\gamma_4^T = (-\gamma_i\partial_i - \gamma_4\partial_4)\gamma_4$$

$$= \gamma_4(\gamma_i\partial_i - \gamma_4\partial_4) = \gamma_4(-\gamma_i^*\partial_i - \gamma_4\partial_4) = \gamma_4(-\gamma_i^T\partial_i - \gamma_4^T\partial_4)$$

$$= \gamma_4^T(\gamma_\mu^T\partial_\mu)$$

This gives, now moving the γ_4^T past $\not\partial_1^*$

$$L = \gamma_4^T(\gamma_\mu^T\partial_\mu^1 + m)\bar{\psi}\ [-C][+\not\partial_2^T + m][-C^{-1}]\gamma_4$$

$$= \gamma_4^T C^{-1}C[\gamma_\mu^T\partial_\mu^1 + m]C^{-1}C\bar{\psi}\ C[\not\partial_2^T + m]C^{-1}\gamma_4$$

$$= \gamma_4^T C^{-1}(\not\partial_1 + m)C\bar{\psi}(\not\partial_2 + m)\gamma_4$$

We next turn our attention to the right hand side of the Bethe-Salpeter equation.

$$R = -g^2 D_c^*(x_1-x_2)\gamma_\mu{}^T\psi^* \; C \; \gamma_\mu{}^T \; C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_\mu{}^T\gamma_4{}^T \; \bar{\psi} \; \gamma_4 \; C \; \gamma_\mu{}^T \; C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\{(\gamma_4\gamma_i)^T+(\gamma_4\gamma_4)^T\} \; \bar{\psi} \; [-C\gamma_4{}^T]\gamma_\mu{}^T \; C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\{(\gamma_4\gamma_i)^T+(\gamma_4\gamma_4)^T\} \; \bar{\psi} \; [-C]\{(\gamma_i\gamma_4)^T+(\gamma_4\gamma_4)^T\}C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\{-(\gamma_i\gamma_4)^T+(\gamma_4\gamma_4)^T\bar{\psi}[-C]\{-(\gamma_4\gamma_i)^T+(\gamma_4\gamma_4)^T\}C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\{-\gamma_4{}^T\gamma_i{}^T+\gamma_4{}^T\gamma_4{}^T\}\bar{\psi}[-C]\{-\gamma_i{}^T\gamma_4{}^T+\gamma_4{}^T\gamma_4{}^T\}C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_4{}^T\gamma_\mu{}^T\bar{\psi}[-C]\gamma_\mu{}^T\gamma_4{}^T C^{-1}$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_4{}^T\gamma_\mu{}^T\bar{\psi}[-C]\gamma_\mu{}^T[-C^{-1}\gamma_4]$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_4{}^T\gamma_\mu{}^T\bar{\psi}[-\gamma_\mu]\gamma_4$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_4{}^T[C^{-1}C\gamma_\mu{}^TC^{-1}C]\bar{\psi}(-\gamma_\mu)\gamma_4$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_4{}^TC^{-1}[-\gamma_\mu]C\bar{\psi}(-\gamma_\mu)\gamma_4$$

$$= -g^2 D_c^*(x_1-x_2)\gamma_4{}^TC^{-1}\gamma_\mu C \; \bar{\psi} \; \gamma_\mu\gamma_4$$

$$= -\gamma_4{}^T g^2 D_c^*(x_1-x_2)C^{-1}\gamma_\mu C \; \bar{\psi} \; \gamma_\mu\gamma_4$$

Finally L = R gives

$$C^{-1}(\not{p}_1+m)C\bar{\psi}(x_1,x_2)(-\not{p}_2+m) = -g^2 D_c^*(x_1-x_2)C^{-1}\gamma_\mu C\bar{\psi}(x_1,x_2)\gamma_\mu$$

the desired adjoint Bethe-Salpeter equation.

(d) Substitution into the Bethe-Salpeter equation

We now make use of the equation obtained in (c) to reduce the interaction term. The initial state reduction gives

$$(1\text{-}4) = [\gamma_\alpha]_{EF}[\psi^-(x_1)]_F[\psi_c(x_4)]_P[\gamma_\beta]_{OP}[ig_{\alpha\beta}D_c(x_1-x_4)]$$

$$= -i[(\not{p}_1+m)\psi(x_1,x_4)C^{-1}(-\not{p}_4+m)C]_{EO}$$

The final state reduction gives

$$(3\text{-}5) = [\bar{\psi}^-(x_3)]_A[\gamma_\mu]_{AB}[\bar{\psi}_c^+(x_5)]_M[\gamma_\nu]_{MN}[ig_{\mu\nu}D_c(x_3-x_5)]$$

$$= -i[C^{-1}(\not{p}_5+m)C\bar{\psi}(x_3,x_5)(-\not{p}_3+m)]_{NB}$$

Although it is not explicitly written down, there is an implied integration over the repeated variables in the interaction term I II Int. The above manipulations have been done on the integrand of this expression.

(iv) Final reduction of the current matrix element

Combining the various pieces from (iii) we have

$$-\text{Int} = \text{I II Int} = [C^{-1}(\not{p}_5+m)C\bar{\psi}(x_3,x_5)(-\not{p}_3+m)]_{NB}[iS_c(x_3-x_2)]_{BC}$$

$$[\not{A}^+(x_2)]_{CD}[iS_c(x_2-x_1)]_{DE}[(\not{p}_1+m)\psi(x_1,x_4)C^{-1}(-\not{p}_4+m)C]_{EO}$$

$$[iS_c(x_5-x_4)]_{NO}$$

The reduction is aided by the use of repeated integrations over the variables $x_1,\ldots,x_5$. Before we begin this process, we note two properties of $S_c(x-y)$ that will be useful in our reduction procedure. The quantity $S_c(x_2-x_1)$ has the Fourier representation

$$S_c(x_2-x_1) = (1/2\pi)^4\int d^4p\ (m-i\not{p})\exp(ip(x_2-x_1))/(p^2+m^2-i\varepsilon)$$

Also note that the application of the operator $\not{\partial} + m$ to $S_c(x_2-x_1)$ yields

$$(\not{\partial}_1+m)S_c(x_2-x_1) = (1/2\pi)^4\int d^4p(m-i\not{p})(i\not{p}+m)\exp(ip(x_2-x_1))/(p^2+m^2-i\varepsilon)$$

$$= (1/2\pi)^4\int d^4p(p^2+m^2)\exp(ip(x_2-x_1))/(p^2+m^2-i\varepsilon) = \delta^4(x_2-x_1)$$

Furthermore note that under charge conjugation we have

$$C^{-1}S_c(x-y)C = [S_c(y-x)]^T$$

In the above expression for -Int, let $\not{\partial}_1$ act to the left on $[iS_c(x_2-x_1)]_{DE}$. This gives rise to the delta function $\delta^4(x_2-x_1)$.

Likewise letting $-\not{\partial}_4$ act on $[iS_c(x_5-x_4)]_{NO}$ gives rise to a second delta function $\delta^4(x_5-x_4)$. With this -Int has the structure

$$-\text{Int} = [C^{-1}(\not{\partial}_5 +m)\bar{\psi}(x_3,x_5)[-\not{\partial}_3 +m)]_{NB}[iS_c(x_3-x_2)]_{BC}[\not{A}^+(x_2)]_{CD}$$

$$[i\delta(x_2-x_1)]_{DE}[\psi(x_1,x_4)i\delta(x_5-x_4)]_{EN}$$

Now integrate over x_1 and x_5. This gives

$$-\text{Int} = [C^{-1}(\not{\partial}_5 +m)C\bar{\psi}(x_3,x_4)(-\not{\partial}_3 +m)]_{NB}[iS_c(x_3-x_2)]_{BC}[\not{A}^+(x_2)]_{CD}$$

$$[\bar{\psi}(x_2,x_4)]_{DN}$$

Now look at the term $\bar{\psi}(x_3,x_4)(-\not\partial_3+m)iS_c(x_3-x_2)$. This equals

$$[-\partial_{3_\mu}\bar{\psi}(x_3,x_4)\gamma_\mu + m\bar{\psi}(x_3,x_4)][iS_c(x_3-x_2)] \equiv A$$

with the derivative in the initial expression acting to the left. If we now integrate by parts over x_3 we have

$$-\int[\partial_{3_\mu}\bar{\psi}(x_3,x_4)\gamma_\mu iS_c(x_3-x_4)d^4x_3$$

$$= -\int\bar{\psi}(x_3,x_4)\gamma_\mu iS_c(x_3-x_2)n_\mu d^3x_3$$

$$+\int\bar{\psi}(x_3,x_4)\gamma_\mu[\partial_{3_\mu}iS_c(x_3-x_2)]d^4x_3.$$

With this A becomes

$$A = -\int\bar{\psi}(x_3,x_4)\gamma_\mu[iS_c(x_3-x_2)]n_\mu d^3x_3$$

$$+ \int\bar{\psi}(x_3,x_4)(m+\gamma_\mu\partial_{3_\mu})[iS_c(x_3-x_2)]d^4x_3$$

The first term of this expression is zero because the function $\bar{\psi}(x_3,x_4)$ vanishes on the surface at infinity. With this -Int takes the form

$$-\text{Int} = [C^{-1}(\not\partial_4+m)C\bar{\psi}(x_3,x_4)\partial^4(x_3-x_2)]_{NC}[\not A^+(x_2)]_{CD}[\bar{\psi}(x_2,x_4)]_{DN}$$

$$= [C^{-1}(\not\partial_4+m)C\bar{\psi}(x_4,x_2)]\ \not A^+(x_2)\ \bar{\psi}(x_2,x_4)$$

(v) Final Reduction by using Center of Mass Coordinates

We now set

$$\psi(x_2,x_4) = \exp(-ikX)\ \phi(x), \quad \bar{\psi}(x_2,x_4) = \exp(+ik'X)\ \bar{\phi}(x)$$

where k is the momentum of the center of mass of the initial state of

the system and k' is the momentum of the center of mass of the final state of the system and

$$X = (x_2+x_4)/2,\ x = x_2-x_4$$

implies

$$x_2 = X + x/2,\quad x_4 = X - x/2.$$

Using the chain rule we have

$$\partial/\partial x_4 = (\partial/\partial X)(\partial X/\partial x_4) + (\partial/\partial x)(\partial x/\partial x_2) = (1/2)(\partial/\partial X) - \partial/\partial x$$

Inserting this back into -Int we have

$$[C^{-1}((1/2)\not\partial/\partial X - \not\partial/\partial x + m)C\bar{\psi}(x_2,x_4)]i\not{A}^{+}(x_2)\psi(x_2,x_4)$$

$$= [C^{-1}((1/2)\not\partial/\partial X - \not\partial/\partial x + m)C\exp(ik'X)\bar{\phi}(x)][i\not{A}^{+}(X + (1/2)x)$$

$$\exp(-ikX)\phi(x)]$$

$$= [C^{-1}(ik'/2 - \not\partial/\partial x + m)C\exp(ik'X)\bar{\phi}(x)][i\not{A}^{+}(X + (1/2)x)\exp(-ikX)\phi(x)]$$

$$= [C^{-1}(ik'/2 - \not\partial/\partial x + m)C\exp(ik'X)\bar{\phi}(x)][i\not{\epsilon}\ \exp(ik(X+(1/2)x)$$

$$\exp(-ikX)\phi(x)]$$

$$= [C^{-1}((i/2)\not{p}_f - \not\partial/\partial x + m)C\bar{\phi}(x)\ i\not{\epsilon}\ \exp(ikx/2)]\ \exp(i(p_f+k-p_i)$$

We now integrate over X. Note that $\int d^4X\ d^4x$ (integrand) = $\int d^4x_2\ d^4x_4$ (integrand), i.e. the Jacobian of the transformation $(X,x) \to (x_2,x_4)$ is

one. The result of our integration over X is just the addition of a delta function in place of the exponential term (modulo 2π's).

$$-\text{Int} = C^{-1}(i\not{p}_f/2 - \not{\partial}/\partial x + m)C\bar{\phi}(x)\; i\not{\epsilon}\; \exp(ikx/2)(2\pi)^4\delta(p_f+k-p_i)\phi(x)$$

At this point we take the trace of this expression. This was done using the 1977 version of Schoonschip we have on the CDC here. The evaluation was done by using the momentum representation of the above expression in the form

$$-\text{Int} = \phi_f((i/2)\beta E_T + i\,\vec{\gamma}\cdot\vec{p} - \beta p_0 + m)\;\phi_i\vec{\gamma}\cdot\vec{\epsilon}$$

This form is obtained by making use of the identity trace(ABC) = trace(CBA) for γ matrices. In more detail we have

$$-\text{Int} = C^{-1}[i\not{p}_f/2 - \not{\partial}/\partial x + m]C\bar{\phi}(x)i\not{\epsilon}\; \exp(ikx/2)(2\pi)^4\delta(p_f+k-p_i)\phi(x)$$

Note

$$\text{trace}[-\text{Int}] = \text{trace}\{C^{-1}[i\not{p}_f/2 - \not{\partial}/\partial x + m]C\bar{\phi}\; i\not{\epsilon}\;\phi(x)]\}(2\pi)^4\delta(p_f+k-p_i)$$

$$= (\text{constants})\;\text{trace}\{[i\not{p}_f/2 - \not{\partial}/\partial x + m]C\bar{\phi}\; i\not{\epsilon}\;\phi(x)\;C^{-1}]\}$$

$$\phi(x)C^{-1} \equiv \phi_i$$

$$\text{trace}[-\text{Int}] = (\text{constants})\;\text{trace}\{[i\not{p}_f/2 - \not{\partial}/\partial x + m]C\bar{\phi}\; i\not{\epsilon}\;\phi_i]$$

$$= (\text{constants})\;\text{trace}\{\phi_i\; i\not{\epsilon}\;\bar{\phi}\;C[i\not{p}_f/2 - \not{\partial}/\partial x + m]\}$$

$$= (\text{constants})\;\text{trace}\{\bar{\phi}C[i\not{p}_f/2 - \not{\partial}/\partial x + m]\phi_i\; i\not{\epsilon}\}$$

$$= (\text{constants})\ \text{trace}\{\phi_f[(i/2)\beta E_T + \vec{\gamma}\cdot\vec{p} - p_0\beta + m]\phi_i\ i\not{\epsilon}]$$

where $\overline{\phi}C = \phi_f$ and the operators in the square brackets act to the left. The constants in this last equation are $(2\pi)^4\,\delta(p_f + k - p_i)\,2(2e/3)$, and the delta function is the four-dimensional energy momentum delta function.

Appendix II.3

The Calculation of the Angular Momentum States of the Vector Particles Arising from $q\bar{q}$ States

In the calculation of the transition amplitude, we shall need to make use of the explicit polarization states of the particles. In this appendix we derive the required polarization states and the appropriate normalizing constants. The triplet particle states that we will be considering are $^3P_0$, $^3S_1$, $^3P_1$, $^3P_2$, and $^3D_1$.

The calculations make use of the relation

$$T_{JLM} = \sum_{\tilde{m}} C(L1J;\ M-\tilde{m},\ \tilde{m}) Y_L^{M-\tilde{m}}(\hat{r}) \vec{\xi}^{\,\tilde{m}}$$

with

$$\vec{\xi}^{\,\tilde{m}} = \begin{cases} \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} & \text{for } \tilde{m} = +1 \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \text{for } \tilde{m} = 0 \\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} & \text{for } \tilde{m} = -1 \end{cases}$$

(1) $^3S_1$. Let $J = 1$, $L = 0$.

$$T_{101} = 1/\sqrt{4\pi}\ \vec{\xi}^{+1} \qquad T_{100} = 1/\sqrt{4\pi}\ \vec{\xi}^{0} \qquad T_{10-1} = 1/\sqrt{4\pi}\ \vec{\xi}^{-1}$$

so that the appropriate polarization vector for these particles

is $1/\sqrt{4\pi}\ \vec{\xi}^{m}$.

(2i) $^3D_1$. Let J=1, L=2. M=1 case.

$$T_{JLM} = \sum_{\tilde{m}} C(L1J;\ M-\tilde{m},\ \tilde{m})\, Y_L^{M-\tilde{m}}(\hat{r})\,\vec{\xi}^{\tilde{m}}$$

$$T_{121} = \sqrt{1/10}[Y_2^0(\hat{p})\vec{\xi}^{+1}] - \sqrt{3/10}[Y_2^1(\hat{p})\vec{\xi}^{0}] + \sqrt{3/5}[Y_2^2(\hat{p})\vec{\xi}^{-1}]$$

$$= \sqrt{1/10}\left[\sqrt{5/(4\pi)}\ \tfrac{1}{2}(3\cos^2\theta - 1)\begin{pmatrix} -1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}\right]$$

$$-\sqrt{3/10}\left[-\sqrt{15/(8\pi)}\ \sin\theta\cos\theta\ e^{+i\phi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$+\sqrt{3/5}\left[\sqrt{15/(2\pi)}\ \tfrac{1}{4}\sin^2\theta\ e^{2i\phi}\begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}\right]$$

$$= -(1/\sqrt{2})(1/\sqrt{10})(1/2)\sqrt{5/(4\pi)}\ (3p_z^2 - 1)\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$+\sqrt{3/10}\ \sqrt{15/(8\pi)}\ p_z(p_x + ip_y)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$+\sqrt{3/5}\ \sqrt{15/(2\pi)}(1/4)(p_x + ip_y)^2(1/\sqrt{2})\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} (3/(8\sqrt{\pi}))(p_x^2 + 2ip_xp_y - p_y^2 - p_z^2 + 1/3) \\ (-3i/(8\sqrt{\pi}))(p_x^2 + 2ip_xp_y - p_y^2 + p_z^2 - 1/3) \\ (3/(4\sqrt{\pi}))(p_x + ip_y)p_z \end{bmatrix}$$

$$= \begin{bmatrix} (3/(8\sqrt{\pi}))(p_x^2 + 2ip_xp_y + p_x^2 - 1 + 1/3) \\ (-3i/(8\sqrt{\pi}))(2ip_xp_y - p_y^2 + 1 - p_y^2 - 1/3) \\ (3/(4\sqrt{\pi}))(p_x + ip_y)p_z \end{bmatrix}$$

$$= \begin{bmatrix} (3/(8\sqrt{\pi}))(2p_x^2 + 2ip_xp_y - 2/3) \\ (-3i/(8\sqrt{\pi}))(1 - 2p_y^2 + 2ip_xp_y - 1/3) \\ (3/(4\sqrt{\pi}))(p_x + ip_y)p_z \end{bmatrix}$$

$$= (3/(4\sqrt{\pi})) \begin{bmatrix} p_x(p_x + ip_y) - 1/3 \\ -i[1/3 + ip_y(p_x + ip_y)] \\ (p_x + ip_y)p_z \end{bmatrix}$$

<u>(2ii) $^3D_1$. J=1, L=2. M=0 case.</u>

$$T_{120} = \sqrt{3/10}\, Y_2^{-1}(\hat{p})\xi^{+1} - \sqrt{2/5}\, Y_2^0(\hat{p})\xi^0 + \sqrt{3/10}\, Y_2^1(\hat{p})\xi^{-1}$$

$$= \sqrt{3/10}\left[\sqrt{15/(8\pi)}\sin\theta\cos\theta\, e^{-i\phi} \begin{pmatrix} -1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}\right]$$

$$- \sqrt{2/5}\left[(1/2)\ \sqrt{5/(4\pi)}\ (3\cos^2\theta - 1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$+ \sqrt{3/10}\left[-\sqrt{15/(8\pi)}\ \sin\theta\ \cos\theta\ e^{+i\phi}\begin{pmatrix}1/\sqrt{2}\\-i/\sqrt{2}\\0\end{pmatrix}\right]$$

$$= (-1/\sqrt{2})\sqrt{3/10}\sqrt{15/(8\pi)}\begin{bmatrix}p_z(p_x-ip_y)\\ip_z(p_x-ip_y)\\0\end{bmatrix} - \sqrt{2/5}(1/2)\sqrt{5/(4\pi)}\begin{bmatrix}0\\0\\3p_z^2-1\end{bmatrix}$$

$$-\sqrt{3/10}\ \sqrt{15/(8\pi)}\ (1/\sqrt{2})\begin{bmatrix}p_z(p_x+ip_y)\\-ip_z(p_x+ip_y)\\0\end{bmatrix}$$

$$= -\begin{bmatrix}(3/(4\sqrt{2\pi}))p_z(p_x-ip_y) + (3/(4\sqrt{2\pi}))p_z(p_x+ip_y)\\(3i/(4\sqrt{2\pi}))p_z(p_x-ip_y) - (3i/(4\sqrt{2\pi}))p_z(p_x+ip_y)\\(3/(2\sqrt{2\pi}))(p_z^2-1/3)\end{bmatrix}$$

$$= -3/(4\sqrt{2\pi})\begin{bmatrix}2p_xp_z\\2p_yp_z\\2(p_z^2-1/3)\end{bmatrix} \qquad = -3/(2\sqrt{2\pi})\begin{bmatrix}p_xp_z\\p_yp_z\\p_z^2-1/3\end{bmatrix}$$

For a $^3D_1$ state one can write the polarization vector as $(\hat{\varepsilon}\cdot\hat{p})\hat{p} - \hat{\varepsilon}/3$ times some constant k. For a definition of $\overleftrightarrow{\varepsilon}^{\,m}$, see section 6 of appendix II.3. The point of the above calculation is to determine the appropriate constant k.

$$\hat{\varepsilon}'\cdot\hat{p} = (-1/\sqrt{2})(p_x + ip_y)$$

$$(\hat{\varepsilon}'\cdot\hat{p})\hat{p} - \hat{\varepsilon}'/3 = \begin{bmatrix} (-1/\sqrt{2})(p_x+ip_y)p_x + (1/3)(1/\sqrt{2}) \\ (-1/\sqrt{2})(p_x+ip_y)p_y + (1/3)(i/\sqrt{2}) \\ (-1/\sqrt{2})(p_x+ip_y)p_z \end{bmatrix}$$

$$= -1/\sqrt{2} \begin{bmatrix} p_x(p_x+ip_y) - 1/3 \\ p_y(p_x+ip_y) - i/3 \\ p_z(p_x+ip_y) \end{bmatrix}$$

Now comparing with the above result for m=1, we see that

$$-k/\sqrt{2} = 3/(4\sqrt{\pi}) \quad \rightarrow \quad k = -3/(2\sqrt{2\pi})$$

As a check one can carry out a similar calculation on the m=0 case and one again arrives at the same constant of proportionality, as one should.

(3) $^3P_0$ case. Here L=1, J=0.

$$T_{JLM} = \sum_{\tilde{m}} C(L1J;\ M-\tilde{m},\ \tilde{m})\ Y_L^{M-\tilde{m}}(\hat{p})\xi^{\tilde{m}}$$

$$= (1/\sqrt{3})[Y_1^{-1}(\hat{p})\xi^{+1} - Y_1^0(\hat{p})\xi^0 + Y_1^{+1}(\hat{p})\xi^{-1}]$$

$$= (1/\sqrt{3})[\sqrt{3/(8\pi)}\ \sin\theta\ (\cos\phi - i\sin\phi) \begin{pmatrix} -1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}$$

$$- \sqrt{3/(4\pi)}\ \cos\theta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$- \sqrt{3/(8\pi)}\ \sin\theta(\cos\phi + i\sin\phi) \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{pmatrix} \ ]$$

$$= [(-1/\sqrt{3})\sqrt{3/(8\pi)}\ (1/\sqrt{2}) \begin{pmatrix} p_x - ip_y \\ i(p_x - ip_y) \\ 0 \end{pmatrix} -(1/\sqrt{3})\sqrt{3/(4\pi)} \begin{pmatrix} 0 \\ 0 \\ p_z \end{pmatrix}$$

$$-(1/\sqrt{2})\sqrt{3/(8\pi)}(1/\sqrt{3}) \begin{pmatrix} p_x + ip_y \\ -i(p_x + ip_y) \\ 0 \end{pmatrix} \ ]$$

$$= -1/(4\sqrt{\pi}) \begin{pmatrix} 2p_x \\ 2p_y \\ 2p_z \end{pmatrix} \quad = -1/(2\sqrt{\pi}) \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

So for $^3P_0$ our polarization vector is $\vec{\xi}^m = \hat{p}/\sqrt{4\pi}$.

<u>(4) $^3P_1$ case. Here L=J=1.</u>

(a) m = +1 case

$$T_{JLM} = T_{11+1} = (1/\sqrt{2})[Y_1^1(\hat{p})\vec{\xi}^0 - Y_1^0(\hat{p})\vec{\xi}^1]$$

$$= (1/\sqrt{2})[\sqrt{3/(8\pi)}\ \sin\theta(\cos\phi + i\sin\phi) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$- \sqrt{3/(4\pi)}\cos\theta \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}]$$

$$= -(1/\sqrt{2})\sqrt{3/(8\pi)} \begin{pmatrix} 0 - p_z \\ 0 - ip_z \\ p_x + ip_y - 0 \end{pmatrix}$$

$$= (1/4)\sqrt{3/\pi} \begin{pmatrix} p_z \\ ip_z \\ -p_x - ip_y \end{pmatrix}$$

In this case the angular part of the wave function is just a constant times $\vec{\xi} \times \vec{p}$. To see this look at the above case.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \\ p_x & p_y & p_z \end{vmatrix} = \hat{i}(-ip_z/\sqrt{2}) + \hat{j}(p_z/\sqrt{2}) + \hat{k}(-p_y/\sqrt{2} + ip_x/\sqrt{2})$$

$$= (-i/\sqrt{2})[\hat{i}(p_z) + \hat{j}(ip_z) + \hat{k}(-p_x - ip_y)]$$

$$= (i/\sqrt{2}) \begin{pmatrix} -p_z \\ -ip_z \\ p_x + ip_y \end{pmatrix}$$

But $T_{111} = -(1/4)\sqrt{3/\pi} \begin{pmatrix} -p_z \\ -ip_z \\ p_x + ip_y \end{pmatrix}$

Therefore comparing terms, the required constant is

obtained from

$$(i/\sqrt{2})k = -(1/4)\sqrt{3/\pi} \rightarrow k = \sqrt{2}(-i)(-1/4)\sqrt{3/\pi}$$
$$= (i/(2\sqrt{2}))\sqrt{3/\pi}.$$

(5) $^3P_2$ case. Here J=2,L=1.

The polarization matrix is $\overleftrightarrow{\epsilon}\cdot\vec{p}$ for this case.

$$T_{JLM} = T_{21M} = \sum_{\tilde{m}} C(112;M-\tilde{m},\tilde{m})Y_{L=2}^{M-\tilde{m}}(\hat{p})\vec{\xi}^{\tilde{m}}$$

$$T_{212} = Y_1^1(\hat{p})\xi'$$

$$= -\sqrt{3/(8\pi)}\ \sin\theta\ (\cos\phi + i\sin\phi)\begin{pmatrix} -1/\sqrt{2} \\ -i/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= (1/\sqrt{2})\sqrt{3/(8\pi)}\begin{pmatrix} p_x+ip_y \\ i(p_x+ip_y) \\ 0 \end{pmatrix}$$

$$= (1/4)\sqrt{3/\pi}\ (p_x+ip_y)\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

The determination of the constant of proportionality k in the polarization matrix $k\overleftrightarrow{\epsilon}^m\cdot\vec{p}$ is as follows:

$$T_{212} = (k/2)\begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = (k/2)\begin{bmatrix} p_x+ip_y \\ i(p_x+ip_y) \\ 0 \end{bmatrix}$$

$$= (1/4)\sqrt{3/\pi}\begin{bmatrix} p_x+ip_y \\ i(p_x+ip_y) \\ 0 \end{bmatrix}$$

$$k/2 = (1/4)\sqrt{3/\pi} \rightarrow k = (1/2)\sqrt{3/\pi}$$

(6) Determination of $\overleftrightarrow{\varepsilon}^m$

$\overleftrightarrow{\varepsilon}^m$ is obtained from the following considerations.

$$\overleftrightarrow{\varepsilon}^m_{\mu\nu} = \sum_{\tilde{m}} C(112;M,\tilde{m})\xi_\mu^{M-\tilde{m}}\,\xi_\nu^{\tilde{m}}$$

For m=2 this just gives us $\begin{bmatrix} 1/2 & i/2 & 0 \\ i/2 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

since $\vec{\xi}' = -\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}(1/\sqrt{2})$ and the Clebsch-Gordon coefficient is one.

Appendix II.4

Radiative Decay Rate Formulae

(i) Results of the Schoonschip Evaluation of the Current Matrix Element

In the program used to evaluate the Dirac algebra associated with the current matrix elements, certain numerical factors were omitted from the main body of the program for convenience. TABLE 9 below lists those factors that depend upon the polarization state involved. In addition an overall $4\pi/8$ was omitted. The 1/8 comes from the normalization of the two body wavefunctions and the factor of 4π comes from the angular integration that was done with Schoonschip. The results of the Schoonschip evaluation of the Dirac algebra and the angular integration over the internal momentum are given in TABLE 10.

(ii) The P_0 Integration

As seen from the previous section, the Schoonschip program for the current matrix elements yields results that have the following form:

$$(\text{constants})(p/E)[p_0+(1/2)E_T-E]f_i(p)f_f(p)(\text{polarization factors})$$

For example in the $2^3S_1 \rightarrow 1^3P_0$ case we have explicitly

$$-(8/3)if_f(p)f_i(p)(p/E)[p_0+(1/2)E_T-E]\hat{\xi}\cdot\hat{e}(4\pi/32\pi)(4e/3)$$

The functions $f_i(p)$ and $f_f(p)$ are actually functions of p as well as p_0. These functions can be decomposed into the product structure

$$f_i(\vec{p},p_0) = f_i(|\vec{p}|)[1/(p_0+(1/2)E_T-E+i\varepsilon) - 1/(p_0+E-(1/2)E_T-i\varepsilon)]$$

A similar structure holds for $f_f(\vec{p},p_0)$, the only modification being that E_T is replaced by E_f. In the current matrix element we have the

products $f_i(p)f_f(p)$, so structures of the following form are encountered

$$G(p_0) = [1/(p_0+(1/2)E_T-E+i\varepsilon) - 1/(p_0-(1/2)E_T+E-i\varepsilon)]$$

$$[1/(p_0+(1/2)E_f-E+i\varepsilon) - 1/(p_0-(1/2)E_f+E-i\varepsilon)]$$

$$[p_0+(1/2)E_T-E] \qquad (1)$$

The decay width is obtained by integrating over the four momentum $(\vec{p},p_0)$. The angular integrations were relatively easy to do and were done concurrently with the current matrix element calculation in Schoonschip. What remains are the radial $|\vec{p}|$ and the p_0 integrals. The only terms involving p_0 are explicitly displayed in (1). The p_0 integral is over the real line, the $|\vec{p}|$ from 0 to ∞. For simplicity we break the integral up into four parts as follows.

$$\int dp_0 G(p_0)/(2\pi)$$

$$= \int dp_0(p_0+(1/2)E_T-E)/((p_0+(1/2)E_T-E+i\varepsilon)(p_0+(1/2)E_f-E+i\varepsilon))/(2\pi)$$

$$- \int dp_0(p_0+(1/2)E_T-E)/((p_0+(1/2)E_T-E+i\varepsilon)(p_0-(1/2)E_f+E-i\varepsilon))/(2\pi)$$

$$- \int dp_0(p_0+(1/2)E_T-E)/((p_0-(1/2)E_T+E-i\varepsilon)(p_0+(1/2)E_f-E+i\varepsilon))/(2\pi)$$

$$+ \int dp_0(p_0+(1/2)E_T-E)/((p_0-(1/2)E_T+E-i\varepsilon)(p_0-(1/2)E_f+E-i\varepsilon))/(2\pi)$$

The first two terms give just -i. The second two terms yield the following, using the method of residues.

$$\int dp_0(p_0+(1/2)E_T-E)/((p_0-(1/2)E_T+E-i\varepsilon)(p_0+(1/2)E_f-E+i\varepsilon))/(2\pi)$$

$$= +i \text{ Residue}[(p_0+(1/2)E_T-E)/(p_0+(1/2)E_f-E)]$$

$$= +i\ [(E_T/2 - E + E_T/2 - E)/(E_T/2 - E + E_f/2 - E)]$$

$$= +i\ [(E_T-2E)/((E_T+E_f)/2 - 2E)]$$

$$\int dp_0(p_0+(1/2)E_T-E)/((p_0-(1/2)E_T+E)(p_0-(1/2)E_f+E))/(2\pi)$$

$$= +i\ [2(E_T/2 - E)/((E_T-E_f)/2)$$

$$+((E_f/2 - E)+(E_T/2 - E))/((E_f-E_T)/2)]$$

$$= +i\ 2/(E_T-E_f)[E_T-2E -[(E_f+E_T)/2 - 2E]]$$

$$= +i\ 2/(E_T-E_f)[E_T-2E - E_f/2 - E_T/2 + 2E]$$

$$= +i\ 2/(E_T-E_f)[(1/2)(E_T-E_f)] = i$$

Adding together the results of the integrations we have

$$+i[-1+1-(E_T-2E)/((E_T+E_f)/2 - 2E)$$

$$= -i\ (E_T-2E)/((E_T+E_f)/2 - 2E)$$

Therefore

$$\int dp_0 G(p_0)/(2\pi) = -i(E_T-2E)/((E_T+E_f)/2 - 2E) \qquad (2)$$

(iii) The Transition Rate Formulae

The S-matrix element for the radiative decay $B' \to B + \gamma$ is given by

$$-(2\pi)^4 i[\text{current matrix element}]\ \delta^4(p_f - p_i + k)$$

The usual expression for the decay width is given in terms of a matrix element denoted by M_{fi}. (See Appendix D of [45].) If one expresses the wavefunctions $f_i(p)$ and $f_f(p)$ in the current matrix element in terms of the radial wavefunctions of the states B and B′, then our current matrix element is precisely this M-matrix element. According to [45], the decay width in terms of this M-matrix is given by

$$\Gamma(B' \to B + \gamma) = (1/2)(1/(4\pi))^2\ |\vec{p}|/M_{B'}^2\ \int d\Omega \sum_{\text{spins}} |M|^2$$

This expression however assumes a specific normalization for the states we have labeled B and B′. Sakurai's expression assumes these states are plane waves and are normalized accordingly. In the case of bound states a different normalization is conventionally used, namely

$$\int_0^\infty \Psi(p)\Psi^*(p)\, dp = 1.$$

If we use this normalization condition then the above equation for Γ reads

$$\Gamma(B' \to B + \gamma) = (8m_B/m_{B'})\,(e/4\pi)^2\ |\vec{p}|\ \int d\Omega \sum_{\text{spins}} |M|^2 \qquad (3)$$

Note here that we have factored out of our matrix M a 2e.

The sum over spins here means a sum over the final states and an average over the initial states. The quantity $|\vec{p}|$ is the magnitude of the momentum of one of the decay products in the final state. The angular integral and the spin average are easily done and in our cases we obtain the results for the spin averages listed in TABLE 11. In the case of the magnetic dipole transitions, an extra factor of k^2 arises from the

polarization term which has not been incorpowidthd into TABLE 11. The cases in which a $^3S_1$ state is replaced by a $^3D_1$ state lead to the same spin average results, e.g. $^3S_1 \to {}^3P_0$ and $^3D_1 \to {}^3P_0$ both have a spin average of 2/3.

We now carry out the calculation of the decay width for the special case of $^3S_1 \to {}^3P_0$, and just list the results for the remaining cases. We start with the M matrix element for the transition $^3S_1 \to {}^3P_0$.

$$M = (1/12)(8i/3)\int dp\ p^2\{f_i(p)f_f(p)\ (p/E)$$

$$[-i(E_T-2E)/((E_T+E_f)/2\ -2E)]\}\ \hat{\xi}\cdot\hat{e} \qquad (4)$$

$$\equiv (2/9)iR(p)(-i)\ \hat{\xi}\cdot\hat{e} = (2/9)R(p)\ \hat{\xi}\cdot\hat{e} \qquad (5)$$

where R(p) is just the p integral in (4):

$$R(p) \equiv \int dp\ p^2\{f_i(p)f_f(p)\ (p/E)$$

$$[(E_T-2E)/((E_T+E_f)/2\ -2E)]\} \qquad (6)$$

Inserting (5) into (3) we have for the decay width

$$\Gamma(B' \to B + \gamma) = (8m_B/m_{B'})\,(e/(4\pi))^2\,|\vec{p}|\,(4\pi)\,(2/9)^2\,R^2(p)\,(2/3)$$

$$= (64/243)\,\alpha\,(m_B/m_{B'})\,|\vec{p}|\,R^2(p) \qquad (7)$$

where $\alpha = e^2/(4\pi)$

Equation (7) is for the $^3S_1 \to {}^3P_0$ case. We can however write the decay width equation in the form

$$\Gamma(B' \to B+\gamma) = 8(m_B/m_{B'})(e^2/(4\pi))|\vec{p}|\,|C_e|^2R^2(p)P$$

$$= 8(m_B/m_{B'})\alpha|\vec{p}|\,|C_e|^2R^2(p)P \qquad (8)$$

In equation (8), if we specify P and C_e, which are constants coming from TABLE 11 and TABLE 12 respectively, then we can write all of the electric dipole transition widths by specifying these two constants, with the obvious understanding that the appropriate radial integrals are chosen.

For the magnetic dipole transitions the functional form changes to

$$\Gamma(B \to B+\gamma) = 8(m_B/m_{B'})e^2/(4\pi)^2|\vec{p}|^3\,|C_m|^2R^2(p)P(4\pi)$$

$$= 8(m_B/m_{B'})\alpha|\vec{p}|^3\,|C_m|^2R^2(p)P \qquad (9)$$

In this case the constants C_m are given in TABLE 13. The polarization constants P are those quantities given in TABLE 11.

```
.PS 62,80
```

APPENDIX II.5

SCHOONSCHIP REDUCTION PROGRAM

TRIPLET S ONE TO TRIPLET P ZERO

```
S EE,ES,M,MEE,COEF,P,PO
I J,J1,MU,NU,LA,MA,MM,MN,O,N,J2
V PP,TF,TI,PS,V,W,E,PT,PQ=U
F PF,PI,EP,EPS,EPP
C  THE FOLLOWING EXPRESSION NEEDS TO BE DIVIDED BY THE QUANTITY
C  (1/(4*PI)*COEF, WHERE COEF IS THE COMMON DENOMIMATOR OF THE ORIGINAL
C  CURRENT MATRIX ELEMENT I.E. (M**2*EE**4*(EE+M)**2
C    TRIPLET S ONE TO THE TRIPLET P ZERO CASE.
Z  EXPR=PF*(-.5*G(J,4)*ET+I*G(J,PP)-G(J,4)*PO+M*GI(J))*PI*G(J,E)*COEF
ID,PF=I*EE*PPDV-M*EE*G(J,V)+M**2*G(J,4)*G(J,V)+G(J,4)*G(J,PP)*PPDV
 +I*M*G(J,J1)*EPF(J1,PP,V,4)*G5(J)
ID, PI=-I*EE*PPDW+M*EE*G(J,W)+G(J,4)*G(J,W)*M**2+G(J,4)*G(J,PP)*PPDW
 +I*M*G(J,J1)*EPF(J1,PP,W,4)*G5(J)
ID,DOT PR, V(MU+)=FF*TF(MU)*ES-FF*PP(MU)*PPDTF
ID,FUNCT,  V(MU+)=FF*TF(MU)*ES-FF*PP(MU)*PPDTF
ID,DOT PR, W(NU+)=FI*TI(NU)*ES-FI*PP(NU)*PPDTI
ID,FUNCT, W(NU+)=FI*TI(NU)*ES-FI*PP(NU)*PPDTI
ID, ES=EE*EE+EE*M
ID,FUNCT,TF(LA+)=-PP(LA)/P
ID,DOT PR,TF(LA+)=-PP(LA)/P
ID,DOT PR, TI(MA+)=-PS(MA)
ID,FUNCT,TI(MA+)=-PS(MA)
*YEP
ID,TRICK,TRACE,J
ID,EDE=1
ID,PSDPS=1
ID,PPDPP=P**2
ID,PP(4)=0
ID,E(4)=0
ID,PS(4)=0
ID,PPDPS*PPDE=P**2*PSDE/3
ID,P**9=P**7*(EE*EE-M*M)
ID,P**7=P**5*(EE*EE-M*M)
ID,P**5=P**3*(EE*EE-M*M)
ID,P**3=P*(EE*EE-M*M)
ID,P**2=EE*EE-M*M
B FI,FF,PSDE,P,I,COEF,M**4
N R
*END
```

```
      program orthog(input,output,tape5,tape7,tape9)
      complex w,wa,wb,wc,wd,we,wf,wg,cx
      double precision go,g,co,d,zo,ho,yz,h,b,a,x,e,r,c,xb,xc,xd,xe,xf,
     /xg,xh,xi,ph,phd,aip,aiph,aipo,aipho,del,test,ab,am,amm,amn,aa,ge,
     /gx,ai,aid,t,u,v,td,ud,vd,wgt,sq,gg,dgg,tb,uo,vo,tdo,udo,vdo,
     /airy,dairy,ch,ts,tau,pe,sum,sim,sqx,sam,exx,sqt,sqz,yx,qe,pi,
     /sqo,tim,tum,tam,sqq
      common e(32),x(32),s(32),ai(32),yz(9),a(10,10),b(10,10)
     /,c(10,10),h(9,9),ph(32,9),sq(9),ho(9),aid(32),phd(32,9),aiph(32,9),p(32),q
     /,p(32),q(32),w(20),aip(32,9),aipo(32),aipho(32),wk(440),wa(10),wb(10),wc(1
     /10),wc(10),wd(10),we(10),wf(10),wg(20),av(10,10),bv(10,10),ca(10,
     /10),cb(10,10),ce(10,10),cd(10,10),ax(20,20),bw(10,10),bx(10,10),
     /by(10,10),bz(10,10),aipow(32),aipox(32),aipoy(32),aipoz(32),aiphow(32),
     /(32),aiphox(32),aiphoy(32),aiphoz(32),aipw(32,9),aipx(32,9),aipy(32,9)
     /2,9),xgy(32),zsp(32,10),zpp(32,10),zdp(32,10),zfp(32,10)
     /,ch(34,32),ts(20,32),yx(32),tau(32)
     /,aipz(32,9),aiphw(32,9),aiphx(32,9),aiphy(32,9),aiphz(32,9),psi(32,20),
     /,20),del(10,10),ba(55),bg(210),wba(10),zba(10,10),
     /wbc(10),zbc(10,10),wbe(10),zbe(10,10),wbf(10),zbf(10,10),wbd(10),
     /wbg(20),zbg(20,20),pbsi(32,20),pbsj(32,20),wbb(10),wbh(20),zbd(10,10),
     /10),zbb(10,10),zbh(20,20),spsi(32),dpsi(32),ppsi(32),fpsi(32),wgt(32),
     /32),gx(32),ge(32),ccs(10,10),ccp(10,10),ccd(10,10),ccf(10,10),
     /d(30),qa(32),qb(32),qc(32),qd(32),cv(10,10),
     /g(30,32),gg(30,32),dgg(30,32),go(32),isn(3),idn(3),ipn(3),ifn(3),
     /zbs(20),zbt(20),zbu(20)
      common/con/t(17),u(17),v(20),tb,uo,vo
      common/don/td(17),ud(17),vd(20),tdo,udo,vdo
      common/data/ rat,am,numb,nint,slaeq,ak,dam,cam,xval,yval,dtpso,
     /outf,owtf,oxtf,optf,drwide,dswide,rmass,dptpz,dptpo,
     /noplot,dtsos,nexp,ncomp,amo,amof,ndo,amf,amce,amcm
      common/dataa/nwave,nmat
      common/pmpar/pmes,npm
      common/wgg/wgg(21,30),rspz,rdpz,rpzs,przd,rspo,rdpo,rpos,rpod,
     /rspt,rdpt,rpts,rptd,rssz,rdsz,rszs,rszd,ratea,rateb,
     /ratec,rated
      common/pmass/pma,pmb,pmc,pme,pmf,pmg,pmh,acb,
     /abc,awa,awb,awc,awd,awe,awf,awg,awh,awk,awl
      common/matrix/cone,ctwo,cthr,cfou,cfiv,csix,csev,ceig,cnin,cten,
     /cele,ctwv,ctne,cfne,cffe,csne
      common/zst/za,zb,zc,ze,zf,zg,zaa,zab,zac,zad
      common/pqrs/apo,app,apq,apr,aps,apt,apu
      common/ctet/nchet
c if we set opt=0 then the eigenvalues of the states are printed out for each
c mass for which they are calculated.  if opt=1 then they are not printed
c the same applies to the  wave functions if out=1 or =0 resp.
c  similiar comments apply to the coupled states normalizations and
c for the e+-e- combination decay widths.(oxt in this case)
      call  datab
      rats=1.0+2.0*rat
      cats=1.0+2.0*rat
```

```
      bats=1.0-2.0*rat
      sats=1.0-rat
      opt=0
      out=0
      owt=0
      oxt=0
      mcomp=ncomp-1
      pi=4.0d+00*datan(1.00d+00)
      do 1 i=1,nint
      read 6,x(i),e(i)
    6 format(d36.29,d36.29)
      wgt(i)=e(i)*dexp(x(i))
    1 continue
      aep=0.00
      npp=nexp-1
      nbb=npp-1
      nxx=2*nexp+5
      ab=(2.0/ak)**(1.0/3.0)
      akk=2.0*((16.0/9.0)**(1.0/3.0))/ab
      abk=ab*akk
      rm=am
      tb=246.764467761021950144 76d+00
      t(1)=374.904196397761772383 56d+00
      t(2)=168.946905466551934323 42d+00
      t(3)=47.8129247970090212898 6d+00
      t(4)=9.0259418911352788118 0d+00
      t(5)=1.1985210167304122317 6d+00
      t(6)=0.1169776116931269216 6d+00
      t(7)=8.6983095974213417 6d-03
      t(8)=5.0740916172007577d-04
      t(9)=2.378736533743276d-05
      t(10)=9.1436238307318d-07
      t(11)=2.930994786363*10.0**(-8.0)
      t(12)=7.9484632582*10.0**(-10.0)
      t(13)=1.846255797*10.0**(-11.0)
      t(14)=3.7127729*10.0**(-13.0)
      t(15)=6.52493*10.0**(-15.0)
      t(16)=1.0105*10.0**(-16.0)
      t(17)=1.39*10.0**(-18.0)
      uo=65.173665488416526954 82d+00
      u(1)=95.231902704092163978 60d+00
      u(2)=39.593310343521063450 67d+00
      u(3)=10.206221859797178588 61d+00
      u(4)=1.7555258978348035815 5d+00
      u(5)=0.2135104236003738574 9d+00
      u(6)=1.9212686862831236 42d-02
      u(7)=1.3257975710634128 4d-03
      u(8)=7.220755408840446d-05
      u(9)=3.17761364290919d-06
      u(10)=1.152071846507 4*10.0**(-7.0)
```

```
u(11)=3.49793567902*10.0**(-9.0)
u(12)=9.018377386*10.0**(-11.0)
u(13)=1.99807607*10.0**(-12.0)
u(14)=3.843793*10.0**(-14.0)
u(15)=6.4790*10.0**(-16.0)
u(16)=9.65*10.0**(-18.0)
u(17)=1.3*10.0**(-19.0)
vo=0.9935364122760933892 0d+00
v(1)=-6.3144392607986313 7d-03
v(2)=1.4300958096113131d-04
v(3)=-5.78706059202472d-06
v(4)=3.2655033319976*10.0**(-7.0)
v(5)=-2.312323195077*10.0**(-8.0)
v(6)=1.93955514434*10.0**(-9.0)
v(7)=-1.8589788507*10.0**(-10.0)
v(8)=1.986842439*10.0**(-11.0)
v(9)=-2.32678966*10.0**(-12.0)
v(10)=2.9468313*10.0**(-13.0)
v(11)=-3.995293*10.0**(-14.0)
v(12)=5.75225*10.0**(-15.0)
v(13)=-8.7375*10.0**(-16.0)
v(14)=1.3927*10.0**(-16.0)
v(15)=-2.319*10.0**(-17.0)
v(16)=4.02*10.0**(-18.0)
v(17)=-7.2*10.0**(-19.0)
v(18)=1.3*10.0**(-19.0)
v(19)=-3.0*10.0**(-20.0)
v(20)=1.0*10.0**(-20.0)
tdo=469.9531279949204224581 9d+00
td(1)=726.0321650664959236718 1d+00
td(2)=340.1027233771799415599 9d+00
td(3)=100.9007585218300263176 4d+00
td(4)=19.9854228723936094622 3d+00
td(5)=2.7781011375551557522 1d+00
td(6)=0.2829128347057472595 3d+00
td(7)=2.1874833495706093 67d-02
td(8)=1.3226199574205036 5d-03
td(9)=6.408208613666998d-05
td(10)=2.53929256578414*10.0**(-6.0)
td(11)=8.372145518318*10.0**(-8.0)
td(12)=2.33062506863*10.0**(-9.0)
td(13)=5.547405288*10.0**(-11.0)
td(14)=1.14138753*10.0**(-12.0)
td(15)=2.049509*10.0**(-14.0)
td(16)=3.2389*10.0**(-16.0)
td(17)=4.54*10.0**(-18.0)
udo=32.774792403035026334 40d+00
ud(1)=46.837939074283086521 36d+00
ud(2)=18.686560256349831120 85d+00
ud(3)=4.601212707645102940 50d+00
```

```
      ud(4)=0.75671345747144813715d+00
      ud(5)=8.82417620774635346 1d-02
      ud(6)=7.63782201494049230d-03
      ud(7)=5.08553307618343 36d-04
      ud(8)=2.68011171968182 6d-05
      ud(9)=1.14414589826759*10.0**(-6.0)
      ud(10)=4.03319220640*10.0**(-8.0)
      ud(11)=1.19291942611*10.0**(-9.0)
      ud(12)=3.00138939 1*10.0**(-11.0)
      ud(13)=6.4993739*10.0**(-13.0)
      ud(14)=1.223720*10.0**(-14.0)
      ud(15)=2.0213*10.0**(-16.0)
      ud(16)=2.95*10.0**(-18.0)
      ud(17)=0.4*10.0**(-19.0)
      vdo=1.0091495380727894021 8d+00
      vd(1)=0.00897120684248359755d+00
      vd(2)=-0.00017138959826153943d+00
      vd(3)=0.00000655479254982352d+00
      vd(4)=-0.00000035951919048499d+00
      vd(5)=0.00000002502441218493d+00
      vd(6)=-0.00000000207492413355d+00
      vd(7)=0.00000000019722366561d+00
      vd(8)=-0.00000000002094647303d+00
      vd(9)=2.44093253*10.0**(-12.0)
      vd(10)=-3.0790652*10.0**(-13.0)
      vd(11)=4.160827*10.0**(-14.0)
      vd(12)=-5.97399*10.0**(-15.0)
      vd(13)=9.0528*10.0**(-16.0)
      vd(14)=-1.4400*10.0**(-16.0)
      vd(15)=2.393*10.0**(-17.0)
      vd(16)=-4.14*10.0**(-18.0)
      vd(17)=7.4*10.0**(-19.0)
      vd(18)=-1.4*10.0**(-19.0)
      vd(19)=0.3*10.0**(-19.0)
      vd(20)=-0.1*10.0**(-19.0)
      do 31 i=1,nint
   31 yx(i)=x(i)
      do 15 i=1,nint
      tau(i)=(2.0*yx(i)**1.5)/3.0
      ta=tau(i)-5.0
      if(ta)22,22,34
   22 so=1.0
      pe=tau(i)/8.0
      ch(1,i)=pe
      ch(2,i)=2.0*pe*pe-1.0
      ch(3,i)=4.0*pe*pe*pe-3.0*pe
      ch(4,i)=2.0*ch(2,i)*ch(2,i)-so
      do 23 k=2,16
      j=2*k+2
      i=2*k
```

```
      m=2*k-2
   23 ch(j,i)=2.0*ch(2,i)*ch(1,i)-ch(m,i)
      sum=tb*so
      sim=uo*so
      tum=tdo*so
      tim=udo*so
      do 26 k=1,17
      j=2*k
      tum=tum+td(k)*ch(j,i)
      tim=tim+ud(k)*ch(j,i)
      sum=sum+t(k)*ch(j,i)
   26 sim=sim+u(k)*ch(j,i)
      sqx=dsqrt(yx(i))
      ai(i)=sqx*(sum*tau(i)**(-1.0/3.0)-sim*tau(i)**(1.0/3.0))/3.0
      aid(i)=-yx(i)*(tum*tau(i)**(-2.0/3.0)-tim*tau(i)**(2.0/3.0))/3.0
      go to 15
   34 to=1.0
      qe=5.0/tau(i)
      ts(1,i)=2.0*qe-1.0
      ts(2,i)=8.0*qe*qe-8.0*qe+1.0
      do 39 k=2,19
      j=k+1
      l=k-1
   39 ts(j,i)=2.0*ts(1,i)*ts(k,i)-ts(l,i)
      sam=vo*to
      tam=vdo*to
      do 49 k=1,20
      tam=tam+vd(k)*ts(k,i)
   49 sam=sam+v(k)*ts(k,i)
      exx=dexp(-tau(i))
      sqt=dsqrt(pi/(2.0*tau(i)))
      sqz=dsqrt(yx(i)/3.0)
      ai(i)=sqz*sqt*exx*sam/pi
      sqq=dsqrt(dble(3.0))
      aid(i)=-yx(i)*sqt*exx*tam/(pi*sqq)
   15 continue
      index=0
      al=0.0
   16 bal=al+1.0
      cal=2.0*al+2.0
      dal=al-1.0
      eal=al-2.0
      alk=aep+al+1.0
      do 17 i=1,nint
      go(i)=ai(i)*ai(i)*(x(i)**cal)/((cam+x(i)**alk)*(cam+x(i)**alk))
      do 18 n=1,nxx
      gg(n,i)=x(i)**n
      dgg(n,i)=n*(x(i)**(n-1))
      g(n,i)=go(i)*gg(n,i)
   18 continue
```

```
 17 continue
    co=0.00
    do 24 i=1,nint
 24 co=co+wgt(i)*go(i)
    do 25 n=1,nxx
    d(n)=0.00
    do 21 i=1,nint
 21 d(n)=d(n)+wgt(i)*g(n,i)
 25 continue
    zo=co
    write 820,zo
820 format(1x,*zo=*,e15.8//)
    sqo=dsqrt(zo)
    do 30 k=1,npp
 30 ho(k)=d(k)/sqo
    yz(1)=d(2)-ho(1)*ho(1)
    write 825,yz(1)
825 format(1x,*yz(1)=*,e15.8//)
    sq(1)=dsqrt(yz(1))
    do 35 k=1,npp
    ka=k+1
 35 h(k,1)=d(ka)/sq(1)-ho(1)*d(k)/(sqo*sq(1))
    yz(2)=d(4)-ho(2)*ho(2)-h(2,1)*h(2,1)
    write 830,yz(2)
830 format(1x,*yz(2)=*,e15.8//)
    sq(2)=dsqrt(yz(2))
    do 40 i=1,nexp
    do 38 j=1,nexp
 38 b(i,j)=0.0
 40 continue
    b(1,1)=1.0/sqo
    b(2,2)=1.0/sq(1)
    b(3,3)=1.0/sq(2)
    do 45 i=4,nexp
 45 b(i,i)=1.0
    b(2,1)=-ho(1)/(sqo*sq(1))
    b(3,1)=-ho(2)/(sqo*sq(2))+ho(1)*h(2,1)/(sqo*sq(1)*sq(2))
    b(3,2)=-h(2,1)/(sq(1)*sq(2))
    do 50 i=1,nexp
    do 48 j=1,nexp
 48 a(i,j)=0.0
 50 continue
    do 55 i=1,nexp
 55 a(i,i)=1.0
    do 60 k=3,npp
    ka=k+1
    kb=k+2
 60 h(k,2)=b(3,1)*d(k)+b(3,2)*d(ka)+b(3,3)*d(kb)
    yz(3)=d(6)-ho(3)*ho(3)-h(3,1)*h(3,1)-h(3,2)*h(3,2)
    write 835,yz(3)
```

```
835 format(1x,*yz(3)=*,e15.8//)
    sq(3)=dsqrt(yz(3))
    a(4,4)=a(4,4)/sq(3)
    a(4,1)=-ho(3)/sq(3)
    a(4,2)=-h(3,1)/sq(3)
    a(4,3)=-h(3,2)/sq(3)
    do 700 i=1,nexp
    do 701 j=1,nexp
    c(i,j)=0.00
701 continue
700 continue
    do 980 i=1,nexp
    do 975 j=1,nexp
    do 976 k=1,nexp
    c(i,j)=c(i,j)+a(i,k)*b(k,j)
976 continue
975 continue
980 continue
    do 90 k=4,npp
    do 65 i=1,nexp
    do 63 j=1,nexp
 63 b(i,j)=c(i,j)
 65 continue
    do 70 i=1,nexp
    do 68 j=1,nexp
 68 a(i,j)=0.0
 70 continue
    do 75 i=1,nexp
 75 a(i,i)=1.0
    do 80 j=k,npp
    kk=k-1
    h(j,kk)=b(k,1)*d(j)
    do 702 m=1,nbb
    n=m+1
    mm=j+m
    h(j,kk)=h(j,kk)+b(k,n)*d(mm)
702 continue
    do 78 nn=j,npp
 78 h(j,nn)=0.0
 80 continue
    kt=2*k
    sum=0.0
    do 85 i=1,kk
 85 sum=sum+h(k,i)*h(k,i)
    yz(k)=d(kt)-ho(k)*ho(k)-sum
    test=yz(k)-1.0d-30
    if(test)998,998,999
998 yz(k)=1.0*10.0**(50.0)
999 write 840,yz(k)
840 format(1x,*yz(k)=*,e15.8//)
```

```
      sq(k)=dsqrt(yz(k))
      k1=k+1
      a(k1,1)=-ho(k)/sq(k)
      do 704 m=2,npp
      n=m-1
      a(k1,m)=-h(k,n)/sq(k)
  704 continue
      a(k1,k1)=1.0/sq(k)
      do 705 i=1,nexp
      do 706 j=1,nexp
      c(i,j)=0.00
  706 continue
  705 continue
      do 990 i=1,nexp
      do 985 j=1,nexp
      do 707 m=1,nexp
      c(i,j)=c(i,j)+a(i,m)*b(m,j)
  707 continue
  985 continue
  990 continue
   90 continue
      all=al-1.0
      if(all)710,711,712
  710 continue
      do 713 i=1,nexp
      do 714 j=1,nexp
      ccs(i,j)=c(i,j)
  714 continue
  713 continue
      go to 716
  711 continue
      do 717 i=1,nexp
      do 718 j=1,nexp
      ccp(i,j)=c(i,j)
  718 continue
  717 continue
      go to 716
  712 continue
      alt=al-3.0
      if(alt)719,720,716
  719 continue
      do 721 i=1,nexp
      do 722 j=1,nexp
      ccd(i,j)=c(i,j)
  722 continue
  721 continue
      go to 716
  720 continue
      do 723 i=1,nexp
      do 724 j=1,nexp
```

```
      ccf(i,j)=c(i,j)
  724 continue
  723 continue
  716 continue
      do 120 i=1,nint
      do 119 j=1,npp
      jj=j+1
      ph(i,j)=c(jj,1)
      phd(i,j)=0.0
      do 118 k=1,npp
      kk=k+1
      ph(i,j)=ph(i,j)+c(jj,kk)*gg(k,i)
  118 phd(i,j)=phd(i,j)+c(jj,kk)*dgg(k,i)
  119 continue
      do 110 j=1,npp
      aip(i,j)=ai(i)*ph(i,j)*x(i)**al/(cam+x(i)**alk)
      aiph(i,j)=ai(i)*phd(i,j)*(x(i)**al)/(cam+x(i)**alk)+aid(i)*ph(i,j)
     /*(x(i)**al)/(cam+x(i)**alk)+aip(i,j)*((al/x(i))-(alk*(x(i)**(alk-1.0)))
     /.0))/(cam+x(i)**alk)))
  110 continue
  120 continue
      do 122 i=1,nint
      aipo(i)=ai(i)*(x(i)**al)/(sqo*(cam+x(i)**alk))
      aipho(i)=aid(i)*(x(i)**al)/(sqo*(cam+x(i)**alk))+aipo(i)*((al/x(i))
     /)-(alk*(x(i)**(alk-1.0))/(cam+x(i)**alk)))
  122 continue
      do 850 j=1,nexp
      do 860 i=1,nexp
  860 del(i,j)=0.0
  850 continue
      do 870 i=1,nint
      xb=x(i)**2.0
  870 del(1,1)=del(1,1)+wgt(i)*xb*aipo(i)*aipo(i)
      do 865 j=2,nexp
      k=j-1
      do 875 i=1,nint
      xb=x(i)**2.0
  875 del(1,j)=del(1,j)+wgt(i)*xb*aipo(i)*aip(i,k)
  865 del(j,1)=del(1,j)
      do 890 i=2,nexp
      do 885 j=2,nexp
      ki=i-1
      kj=j-1
      do 880 k=1,nint
      xb=x(k)**2.0
  880 del(i,j)=del(i,j)+wgt(k)*xb*aip(k,ki)*aip(k,kj)
  885 continue
  890 continue
      write 893,del(1,1),del(1,2),del(1,3),del(1,4),del(1,5),del(1,6),del(1,7)
     /l(1,7),del(1,8),del(1,9),del(1,10)
```

```
893 format(1h1,1x,10e11.3/)
    do 895 i=2,10
    write 894,del(i,1),del(i,2),del(i,3),del(i,4),del(i,5),del(i,6),del(i,7),
   /l(i,7),del(i,8),del(i,9),del(i,10)
894 format(1x,10e11.3/)
895 continue
    if(dal)11,12,13
 11 continue
    do 2 i=1,nint
    aipow(i)=aipo(i)
    aiphow(i)=aipho(i)
    do 3 j=1,npp
    aipw(i,j)=aip(i,j)
  3 aiphw(i,j)=aiph(i,j)
  2 continue
    al=al+1.0
    go to 16
 12 continue
    do 4 i=1,nint
    aipox(i)=aipo(i)
    aiphox(i)=aipho(i)
    do 5 j=1,npp
    aipx(i,j)=aip(i,j)
  5 aiphx(i,j)=aiph(i,j)
  4 continue
    al=al+1.0
    go to 16
 13 if(eal)14,14,19
 14 continue
    do 7 i=1,nint
    aipoy(i)=aipo(i)
    aiphoy(i)=aipho(i)
    do 8 j=1,npp
    aipy(i,j)=aip(i,j)
  8 aiphy(i,j)=aiph(i,j)
  7 continue
    al=al+1.0
    go to 16
 19 continue
    do 9 i=1,nint
    aipoz(i)=aipo(i)
    aiphoz(i)=aipho(i)
    do 10 j=1,npp
    aipz(i,j)=aip(i,j)
 10 aiphz(i,j)=aiph(i,j)
  9 continue
920 continue
    ram=rm
    rmn=am*ab
    write 928,am,ram,rmn
```

```
928 format(1x,3e11.3//)
    11=am*amof+1.20-amo
    amm=am*am
    amn=amm*amm
    al=1.0
    bal=al+1.0
    cal=2.0*al+2.0
300 continue
    do 130 i=1,nint
    xb=x(i)**2.0
    xc=x(i)**3.0
    xe=x(i)**5.0
    rr=xb+am*am
    r=dsqrt(dble(rr))
    p(i)=wgt(i)*(2.0*xb+rats*amm)/rr
    p(i)=p(i)*xb
    xf=3.0*xb/(rr**3.0)
    xf=xf*amn*bats
    xg=al*bal*(2.0*xb+rats*amm)/rr
    xh=2.0*xb*r*slaeq
    xj=2.0*xb/(rr**2.0)
    xj=xj*amm
    xk=(al+1.0)*(bal+1.0)*(2.0*xb+rats*amm)/rr
    xl=(3.0*xb+amm)/(2.0*xb+amm)
    xm=rat*xb*xb/(rr*rr)
    xn=-2.0*rat*am/r
    xn=2.0*xn
    xo=(r-am)*rat*xb/(rr*(r+am))
    xo=xo*2.0
    xp=amm*rat/(2.0*rr*rr)
    xp=2.0*xp*xb
    xq=-6.0*rat*xb*amn/(rr**3.0)
    qa(i)=wgt(i)*(xf-xh)
    qa(i)=qa(i)+wgt(i)*xm
    qb(i)=wgt(i)*(xf-xg-xh)
    qb(i)=qb(i)+wgt(i)*xm
    qc(i)=wgt(i)*(xf-xk-xh)
    qc(i)=qc(i)+wgt(i)*xm
    s(i)=-wgt(i)*2.0*amm*xl/rr
    s(i)=s(i)+wgt(i)*(xn+xo+xp+xq)
    qd(i)=-wgt(i)*xj/2.0
    qd(i)=qd(i)*sats
    qd(i)=qd(i)-wgt(i)*xm
130 continue
    bv(1,1)=0.0
    bw(1,1)=0.0
    bx(1,1)=0.0
    by(1,1)=0.0
    bz(1,1)=0.0
    cv(1,1)=0.0
```

```
      ca(1,1)=0.0
      cb(1,1)=0.0
      cd(1,1)=0.0
      ce(1,1)=0.0
      do 124 i=1,nint
      bv(1,1)=bv(1,1)+(qa(i)+s(i))*aipox(i)*aipox(i)
      bw(1,1)=bw(1,1)+p(i)*aiphow(i)*aiphow(i)
      bx(1,1)=bx(1,1)+p(i)*aiphox(i)*aiphox(i)
      by(1,1)=by(1,1)+p(i)*aiphoy(i)*aiphoy(i)
      bz(1,1)=bz(1,1)+p(i)*aiphoz(i)*aiphoz(i)
      cv(1,1)=cv(1,1)+qa(i)*aipow(i)*aipow(i)
      ca(1,1)=ca(1,1)+qb(i)*aipox(i)*aipox(i)
      cb(1,1)=cb(1,1)+qc(i)*aipoy(i)*aipoy(i)
      cd(1,1)=cd(1,1)+(qb(i)+qd(i))*aipox(i)*aipox(i)
  124 ce(1,1)=ce(1,1)+(qc(i)+qd(i))*aipoy(i)*aipoy(i)
      do 134 j=2,nexp
      cv(1,j)=0.0
      ca(1,j)=0.0
      cb(1,j)=0.0
      cd(1,j)=0.0
      ce(1,j)=0.0
      bv(1,j)=0.0
      bw(1,j)=0.0
      bx(1,j)=0.0
      by(1,j)=0.0
      bz(1,j)=0.0
      k=j-1
      do 132 i=1,nint
      bv(1,j)=bv(1,j)+(qa(i)+s(i))*aipox(i)*aipx(i,k)
      bw(1,j)=bw(1,j)+p(i)*aiphow(i)*aiphw(i,k)
      bx(1,j)=bx(1,j)+p(i)*aiphox(i)*aiphx(i,k)
      by(1,j)=by(1,j)+p(i)*aiphoy(i)*aiphy(i,k)
      bz(1,j)=bz(1,j)+p(i)*aiphoz(i)*aiphz(i,k)
      cv(1,j)=cv(1,j)+qa(i)*aipw(i,k)*aipow(i)
      ca(1,j)=ca(1,j)+qb(i)*aipx(i,k)*aipox(i)
      cb(1,j)=cb(1,j)+qc(i)*aipy(i,k)*aipoy(i)
      cd(1,j)=cd(1,j)+(qb(i)+qd(i))*aipx(i,k)*aipox(i)
  132 ce(1,j)=ce(1,j)+(qc(i)+qd(i))*aipy(i,k)*aipoy(i)
      bv(j,1)=bv(1,j)
      bw(j,1)=bw(1,j)
      bx(j,1)=bx(1,j)
      by(j,1)=by(1,j)
      bz(j,1)=bz(1,j)
      ca(j,1)=ca(1,j)
      cb(j,1)=cb(1,j)
      cd(j,1)=cd(1,j)
      ce(j,1)=ce(1,j)
  134 cv(j,1)=cv(1,j)
      do 140 i=2,nexp
      do 138 j=2,nexp
```

```
      bv(i,j)=0.0
      bw(i,j)=0.0
      bx(i,j)=0.0
      by(i,j)=0.0
      bz(i,j)=0.0
      cv(i,j)=0.0
      ca(i,j)=0.0
      cb(i,j)=0.0
      cd(i,j)=0.0
      ce(i,j)=0.0
      ki=i-1
      kj=j-1
      do 136 k=1,nint
      bv(i,j)=bv(i,j)+(qa(k)+s(k))*aipx(k,kj)*aipx(k,ki)
      bw(i,j)=bw(i,j)+p(k)*aiphw(k,ki)*aiphw(k,kj)
      bx(i,j)=bx(i,j)+p(k)*aiphx(k,ki)*aiphx(k,kj)
      by(i,j)=by(i,j)+p(k)*aiphy(k,ki)*aiphy(k,kj)
      bz(i,j)=bz(i,j)+p(k)*aiphz(k,ki)*aiphz(k,kj)
      cv(i,j)=cv(i,j)+qa(k)*aipw(k,ki)*aipw(k,kj)
      ca(i,j)=ca(i,j)+qb(k)*aipx(k,ki)*aipx(k,kj)
      cb(i,j)=cb(i,j)+qc(k)*aipy(k,ki)*aipy(k,kj)
      cd(i,j)=cd(i,j)+(qb(k)+qd(k))*aipx(k,ki)*aipx(k,kj)
  136 ce(i,j)=ce(i,j)+(qc(k)+qd(k))*aipy(k,ki)*aipy(k,kj)
  138 continue
  140 continue
      ijob=2
      num=numb
  487 continue
      n=num
      iz=10
      ia=10
      do 301 i=1,num
      do 302 j=1,num
      av(i,j)=(bw(i,j)-cv(i,j))/2.0
      av(i,j)=av(i,j)/slaeq
  302 continue
  301 continue
      call vcvtfs(av,n,ia,ba)
      call eigrs(ba,n,ijob,wba,zba,iz,wk,ier)
      ac=(2.0/ak)**(1.0/3.0)
      ad=2.0*((16.0/9.0)**(1.0/3.0))
      cx=cmplx(ac,ad)
      do 363 i=1,num
      wa(i)=cx*wba(i)
      write 364,wba(i),wa(i),zba(i,1),zba(i,2)
  364 format(1x,5e20.10/)
  363 continue
      write 305,wk(1)
  305 format(1x,'performance index=',10e11.4/)
      do 306 i=1,num
```

```
      do 307 j=1,num
      av(i,j)=(bx(i,j)-ca(i,j))/2.0
      av(i,j)=av(i,j)/slaeq
  307 continue
  306 continue
      call vcvtfs(av,n,ia,ba)
      call eigrs(ba,n,ijob,wbb,zbb,iz,wk,ier)
      do 308 i=1,num
      wb(i)=cx*wbb(i)
      if(opt)1033,1034,1033
 1034 continue
      write 309,wbb(i),wb(i),zbb(i,1),zbb(i,2)
  309 format(1x,5e20.10/)
 1033 continue
  308 continue
      write 310,wk(1)
  310 format(1x,*performance index=*,10e11.4/)
      do 311 i=1,num
      do 312 j=1,num
      av(i,j)=(bx(i,j)-cd(i,j))/2.0
      av(i,j)=av(i,j)/slaeq
  312 continue
  311 continue
      call vcvtfs(av,n,ia,ba)
      call eigrs(ba,n,ijob,wbc,zbc,iz,wk,ier)
      do 313 i=1,num
      wc(i)=cx*wbc(i)
      if(opt)1035,1036,1035
 1036 continue
      write 314,wbc(i),wc(i)
  314 format(1x,3e20.10/)
 1035 continue
  313 continue
      write 315,wk(1)
  315 format(1x,*performance index=*,10e11.4/)
      do 316 i=1,num
      do 317 j=1,num
      av(i,j)=(by(i,j)-cb(i,j))/2.0
      av(i,j)=av(i,j)/slaeq
  317 continue
  316 continue
      call vcvtfs(av,n,ia,ba)
      call eigrs(ba,n,ijob,wbd,zbd,iz,wk,ier)
      do 318 i=1,num
      wd(i)=cx*wbd(i)
      if(opt)1037,1038,1037
 1038 continue
      write319,wbd(i),wd(i)
  319 format(1x,3e20.10/)
 1037 continue
```

```
 318 continue
     write 320,wk(1)
 320 format(1x,*performance index=*,10e11.4/)
     do 321 i=1,num
     do 322 j=1,num
     av(i,j)=(by(i,j)-ce(i,j))/2.0
     av(i,j)=av(i,j)/slaeq
 322 continue
 321 continue
     call vcvtfs(av,n,ia,ba)
     call eigrs(ba,n,ijob,wbe,zbe,iz,wk,ier)
     do 323 i=1,num
     we(i)=cx*wbe(i)
     if(opt)1039,1040,1039
1040 continue
     write 324,wbe(i),we(i)
 324 format(1x,3e20.10/)
1039 continue
 323 continue
     write 325,wk(1)
 325 format(1x,*performance index=*,10e11.4/)
     do 326 i=1,num
     do 327 j=1,num
     av(i,j)=(bx(i,j)-bv(i,j))/2.0
     av(i,j)=av(i,j)/slaeq
 327 continue
 326 continue
     call vcvtfs(av,n,ia,ba)
     call eigrs(ba,n,ijob,wbf,zbf,iz,wk,ier)
     do 911 i=1,num
     wf(i)=cx*wbf(i)
     if(opt)1041,1042,1041
1042 continue
     write 912,wbf(i),wf(i)
 912 format(1x,3e20.10/)
1041 continue
 911 continue
     write913,wk(1)
 913 format(1x,*performance index=*,10e11.4/)
     num=num+2
     ifg=num-nexp
     if(ifg)487,487,488
 488 continue
     al=1.0
     bal=al+1.0
     fal=al-1.0
     fbl=fal+1.0
     gal=al+1.0
     gbl=gal+1.0
 450 continue
```

```
      do 455 i=1,nint
      xb=x(i)**2.0
      xc=x(i)**3.0
      xe=x(i)**5.0
      rr=xb+am*am
      r=dsqrt(dble(rr))
      xf=3.0*xb/(rr**3.0)
      xf=xf*amn
      xf=xf*rats
      xg=al*bal*(2.0*xb+rats*am*am)/rr
      xh=2.0*xb*r*slaeq
      xi=3.0*amm*xb/(2.0*rr*rr)
      xi=3.0*xi-rat*xi*2.0
      xj=am*(al*bal-fal*fbl)/r
      xj=xj*cats
      xk=am*xb/(rr*(r+am))
      xl=am*(2.0*r+am)*(r-am)*xb
      xl=xl/(2.0*(2.0*al+1.0)*rr*rr*(r+am))
      xm=(am/r)*(al*bal-gal*gbl)
      xm=xm*cats
      xn=rat*(r-am)/(rr*(r+am))
      xn=xn*xb*sqrt(al*bal)/(2.0*al+1.0)
      xn=2.0*xn
      xo=rat*(r-am)/(rr*(r+am))
      xo=xo*xb
      xp=rat*(r-am)/(rr*(r+am))
      xp=xp*xb/(2.0*(2.0*al+1.0))
      xp=xp*2.0
      qa(i)=(xj-xl+xf-xg-xh-xi+xk)*wgt(i)
      qa(i)=qa(i)+wgt(i)*(xp-xo)
      qb(i)=-am*wgt(i)*sqrt(al*bal)*(2.0*r+am)*(r-am)*xb/((2.0*al+1.0)*rr
     /r*rr*(r+am))
      qb(i)=qb(i)+wgt(i)*xn
      qc(i)=(xm+xl+xf-xg-xh-xi+xk)*wgt(i)
      qc(i)=qc(i)+wgt(i)*(-xp-xo)
  455 continue
      ca(1,1)=0.0
      cb(1,1)=0.0
      ce(1,1)=0.0
      cd(1,1)=0.0
      do 460 i=1,nint
      sal=al-1.0
      if(sal)456,456,457
  456 ca(1,1)=ca(1,1)+qa(i)*aipow(i)*aipow(i)
      cb(1,1)=cb(1,1)+qb(i)*aipow(i)*aipoy(i)
      ce(1,1)=ce(1,1)+qb(i)*aipow(i)*aipoy(i)
      cd(1,1)=cd(1,1)+qc(i)*aipoy(i)*aipoy(i)
      go to 460
  457 ca(1,1)=ca(1,1)+qa(i)*aipox(i)*aipox(i)
      cb(1,1)=cb(1,1)+qb(i)*aipox(i)*aipoz(i)
```

```
      ce(1,1)=ce(1,1)+qb(i)*aipox(i)*aipoz(i)
      cd(1,1)=cd(1,1)+qc(i)*aipoz(i)*aipoz(i)
  460 continue
      do 470 j=2,nexp
      ca(1,j)=0.0
      cb(1,j)=0.0
      ce(1,j)=0.0
      cd(1,j)=0.0
      k=j-1
      do 465 i=1,nint
      if(sal)461,461,462
  461 ca(1,j)=ca(1,j)+qa(i)*aipw(i,k)*aipow(i)
      cb(1,j)=cb(1,j)+qb(i)*aipy(i,k)*aipow(i)
      ce(1,j)=ce(1,j)+qb(i)*aipw(i,k)*aipoy(i)
      cd(1,j)=cd(1,j)+qc(i)*aipy(i,k)*aipoy(i)
      go to 465
  462 ca(1,j)=ca(1,j)+qa(i)*aipx(i,k)*aipox(i)
      cb(1,j)=cb(1,j)+qb(i)*aipz(i,k)*aipox(i)
      ce(1,j)=ce(1,j)+qb(i)*aipx(i,k)*aipoz(i)
      cd(1,j)=cd(1,j)+qc(i)*aipz(i,k)*aipoz(i)
  465 continue
      ca(j,1)=ca(1,j)
      cb(j,1)=ce(1,j)
      ce(j,1)=cb(1,j)
  470 cd(j,1)=cd(1,j)
      do 485 i=2,nexp
      do 480 j=2,nexp
      ca(i,j)=0.0
      cb(i,j)=0.0
      ce(i,j)=0.0
      cd(i,j)=0.0
      ki=i-1
      kj=j-1
      do 475 k=1,nint
      if(sal)471,471,472
  471 ca(i,j)=ca(i,j)+qa(k)*aipw(k,ki)*aipw(k,kj)
      cb(i,j)=cb(i,j)+qb(k)*aipy(k,ki)*aipw(k,kj)
      ce(i,j)=ce(i,j)+qb(k)*aipw(k,ki)*aipy(k,kj)
      cd(i,j)=cd(i,j)+qc(k)*aipy(k,ki)*aipy(k,kj)
      go to 475
  472 ca(i,j)=ca(i,j)+qa(k)*aipx(k,ki)*aipx(k,kj)
      cb(i,j)=cb(i,j)+qb(k)*aipz(k,ki)*aipx(k,kj)
      ce(i,j)=ce(i,j)+qb(k)*aipx(k,ki)*aipz(k,kj)
      cd(i,j)=cd(i,j)+qc(k)*aipz(k,ki)*aipz(k,kj)
  475 continue
  480 continue
  485 continue
      num=numb
  486 continue
      do 495 i=1,num
```

```
      do 490 j=1,num
      if(sal)491,491,492
  491 ax(i,j)=(bw(i,j)-ca(i,j))/2.0
      ax(i,j)=ax(i,j)/slaeq
      go to 493
  492 continue
      ax(i,j)=(bx(i,j)-ca(i,j))/2.0
      ax(i,j)=ax(i,j)/slaeq
  493 ki=i+num
      kj=j+num
      ax(i,kj)=-cb(i,j)/2.0
      ax(i,kj)=ax(i,kj)/slaeq
      ax(ki,j)=-ce(i,j)/2.0
      ax(ki,j)=ax(ki,j)/slaeq
      if(sal)494,494,496
  494 continue
      ax(ki,kj)=(by(i,j)-cd(i,j))/2.0
      ax(ki,kj)=ax(ki,kj)/slaeq
      go to 490
  496 continue
      ax(ki,kj)=(bz(i,j)-cd(i,j))/2.0
      ax(ki,kj)=ax(ki,kj)/slaeq
  490 continue
  495 continue
      nub=2*num
      n=nub
      ia=20
      iz=20
      if(sal)925,925,926
  925 call vcvtfs(ax,n,ia,bg)
      call eigrs(bg,n,ijob,wbg,zbg,iz,wk,ier)
      write 927,ier
  927 format(1x,10x,*ier=*,i10//)
      nuc=nub-2
      do 916 i=1,nuc
      j=i+1
      k=j+1
      w(i)=cx*wbg(i)
      dw=wbg(k)-wbg(i)
      zw=0.560/dw
      dw=.587/dw
      if(opt)1043,1044,1043
 1044 continue
      write 917,wbg(i),w(i),dw,zw
  917 format(1x,5e20.10/)
 1043 continue
  916 continue
      write 921,wk(1)
  921 format(1x,*performance index=*,10e11.4/)
      num=num+2
```

```
      ifg=num-nexp
      if(ifg)486,486,801
      go to 801
  926 call vcvtfs(ax,n,ia,bg)
      call eigrs(bg,n,ijob,wbh,zbh,iz,wk,ier)
      write 915,ier
  915 format(1x,10x,*ier=*,i10//)
      do 918 i=1,nub
      wg(i)=cx*wbh(i)
      if(opt)1045,1046,1045
 1046 continue
      write 919,wbh(i),wg(i)
  919 format(1x,3e20.10/)
 1045 continue
  918 continue
      num=num+2
      ifg=num-10
      if(ifg)486,486,922
  922 continue
      go to 803
  801 if(sal)805,805,803
  805 al=2.0
      bal=al+1.0
      fal=al-1.0
      fbl=fal+1.0
      gal=al+1.0
      gbl=gal+1.0
      go to 450
  803 continue
      ik=1
      il=1
      im=1
      in=1
      mm=2*nexp
      do 937 m=1,mm
      do 938 k=1,nint
      spsi(k)=zbg(1,m)*aipow(k)
      ppsi(k)=zbh(1,m)*aipox(k)
      fpsi(k)=zbh(11,m)*aipoz(k)
      dpsi(k)=zbg(11,m)*aipoy(k)
      do 939 i=2,nexp
      j=i-1
      kk=11+j
      spsi(k)=spsi(k)+zbg(i,m)*aipw(k,j)
      ppsi(k)=ppsi(k)+zbh(i,m)*aipx(k,j)
      fpsi(k)=fpsi(k)+zbh(kk,m)*aipz(k,j)
  939 dpsi(k)=dpsi(k)+zbg(kk,m)*aipy(k,j)
  938 continue
      xsps=0.00
      xpps=0.00
```

```
      xdps=0.00
      xfps=0.00
      do 940 k=1,nint
      xb=x(k)*x(k)
      xsps=xsps+spsi(k)*spsi(k)*wgt(k)*xb
      xpps=xpps+ppsi(k)*ppsi(k)*wgt(k)*xb
      xfps=xfps+fpsi(k)*fpsi(k)*wgt(k)*xb
      xdps=xdps+dpsi(k)*dpsi(k)*wgt(k)*xb
  940 continue
      xx=xsps+xdps
      yy=xpps+xfps
      atest=0.50000-xsps
      btest=0.50000-xpps
      if(atest)2001,2002,2003
 2001 ikk=ik-4
      if(ikk)1999,3000,3000
 1999 isn(ik)=m
      ik=ik+1
      go to 3000
 2002 write 2004,wbg(m),xsps
 2004 format(5x,2e20.10//)
      go to 3000
 2003 ilk=il-4
      if(ilk)1997,3000,3000
 1997 idn(il)=m
      il=il+1
 3000 continue
      if(btest)2005,2006,2007
 2005 imk=im-4
      if(imk)1995,3001,3001
 1995 ipn(im)=m
      im=im+1
      go to 3001
 2006 write 2008,wbh(m),xpps
 2008 format(8x,2e20.10//)
      go to 3001
 2007 ink=in-4
      if(ink)1993,3001,3001
 1993 ifn(in)=m
      in=in+1
 3001 continue
      if(owt)1049,1050,1049
 1050 continue
      write 941,xsps,xdps,xx,xpps,xfps,yy
  941 format(1x,6e12.3//)
 1049 continue
  937 continue
c  psi(k,1),psi(k,2),and psi(k,3) are the three lowest singlet s-zero states
c psi(k,4),psi(k,5) and psi(k,6) are the three lowets singlet p-one states
c psi(k,7),psi(k,8), and psi(k,9) are the three lowest singlet d-two states
```

```
c psi(k,10) is the lowest triplet p-zero state
c psi(k,11) is the lowest triplet p-one state
c psi(k,12) is the lowest triplet p-two state (unmixed part of the mixes state
c  q(k) is the derivative of the lowest singlet p-one state
c  spsi and dpsi are the s and d componentss of the 3-s-1 mixed state
c  psi(i,13) and psi(i,14) are the s an d  d partss of the 2 (3-s-1) states
      if(mcomp)1047,1203,1202
 1203 continue
      xgy(1)=1.0
      do 951 i=1,nint
      j=i-1
      aj=j
      x(i)=0.25*aj
      do 952 n=2,10
      m=n-1
      xgy(n)=x(i)**m
  952 continue
      do 961 l=1,10
      zsp(i,l)=ccs(1,l)*xgy(1)
      zpp(i,l)=ccp(1,l)*xgy(1)
      zdp(i,l)=ccd(1,l)*xgy(1)
      zfp(i,l)=ccf(1,l)*xgy(1)
      do 953 m=2,10
      zsp(i,l)=zsp(i,l)+ccs(l,m)*xgy(m)
      zpp(i,l)=zpp(i,l)+ccp(l,m)*xgy(m)
      zdp(i,l)=zdp(i,l)+ccd(l,m)*xgy(m)
      zfp(i,l)=zfp(i,l)+ccf(l,m)*xgy(m)
  953 continue
  961 continue
      ai(i)=airy(x(i))
      xb=x(i)*x(i)
      xc=x(i)*xb
      xd=xb*xb
      aipow(i)=ai(i)*zsp(i,1)/(cam+x(i))
      aipox(i)=ai(i)*zpp(i,1)*x(i)/(cam+xb)
      aipoy(i)=ai(i)*zdp(i,1)*xb/(cam+xc)
      aipoz(i)=ai(i)*zfp(i,1)*xc/(cam+xd)
      do 954 k=2,10
      l=k-1
      aipw(i,l)=ai(i)*zsp(i,k)/(cam+x(i))
      aipx(i,l)=ai(i)*zpp(i,k)*x(i)/(cam+xb)
      aipy(i,l)=ai(i)*zdp(i,k)*xb/(cam+xc)
      aipz(i,l)=ai(i)*zfp(i,k)*xc/(cam+xd)
  954 continue
  951 continue
 1202 continue
      do 808 i=1,20
      zbs(i)=zbg(i,isn(2))
      zbt(i)=zbg(i,isn(1))
      zbu(i)=zbh(i,ipn(1))
```

```
      zbg(i,1)=zbt(i)
      zbg(i,3)=zbs(i)
      zbh(i,1)=zbu(i)
  808 continue
      wbt=wbg(isn(1))
      wbs=wbg(isn(2))
      wbu=wbh(ipn(1))
      wbg(1)=wbt
      wbg(3)=wbs
      wbh(1)=wbu
      do 350 k=1,nint
      psi(k,1)=zba(1,1)*aipow(k)
      psi(k,2)=zba(1,2)*aipow(k)
      psi(k,3)=zba(1,3)*aipow(k)
      psi(k,4)=zbb(1,1)*aipox(k)
      psi(k,5)=zbb(1,2)*aipox(k)
      psi(k,6)=zbb(1,3)*aipox(k)
      psi(k,7)=zbd(1,1)*aipoy(k)
      psi(k,8)=zbd(1,2)*aipoy(k)
      psi(k,9)=zbd(1,3)*aipoy(k)
      psi(k,10)=zbf(1,1)*aipox(k)
      psi(k,11)=zbc(1,1)*aipox(k)
      q(k)=zbb(1,1)*aiphox(k)
      aabb=am
      if(aabb)806,955,806
  955 continue
      if(rat)806,956,806
  806 continue
      spsi(k)=zbg(1,1)*aipow(k)
      dpsi(k)=zbg(11,1)*aipoy(k)
      psi(k,12)=zbh(1,1)*aipox(k)
      psi(k,13)=zbg(1,3)*aipow(k)
      psi(k,14)=zbg(11,3)*aipoy(k)
      go to 957
  956 spsi(k)=(zbg(1,1)+zbg(1,2))*aipow(k)
      psi(k,12)=(zbh(1,1)+zbh(1,2))*aipox(k)
      psi(k,13)=(zbg(1,3)+zbg(1,4))*aipow(k)
      psi(k,14)=zbg(11,3)*aipoy(k)
      dpsi(k)=zbg(11,1)*aipoy(k)
  957 continue
      do 933 i=2,nexp
      j=i-1
      kk=11+j
      psi(k,1)=psi(k,1)+zba(1,1)*aipw(k,j)
      psi(k,2)=psi(k,2)+zba(1,2)*aipw(k,j)
      psi(k,3)=psi(k,3)+zba(1,3)*aipw(k,j)
      psi(k,4)=psi(k,4)+zbb(1,1)*aipx(k,j)
      psi(k,5)=psi(k,5)+zbb(1,2)*aipx(k,j)
      psi(k,6)=psi(k,6)+zbb(1,3)*aipx(k,j)
      psi(k,7)=psi(k,7)+zbd(1,1)*aipy(k,j)
```

```
      psi(k,8)=psi(k,8)+zbd(i,2)*aipy(k,j)
      psi(k,9)=psi(k,9)+zbd(i,3)*aipy(k,j)
      psi(k,10)=psi(k,10)+zbf(i,1)*aipx(k,j)
      psi(k,11)=psi(k,11)+zbc(i,1)*aipx(k,j)
      q(k)=q(k)+zbb(i,1)*aiphx(k,j)
      if(aabb)807,958,807
  958 continue
      if(rat)807,959,807
  807 continue
      spsi(k)=spsi(k)+zbg(i,1)*aipw(k,j)
      dpsi(k)=dpsi(k)+zbg(kk,1)*aipy(k,j)
      psi(k,12)=psi(k,12)+zbh(i,1)*aipx(k,j)
      psi(k,13)=psi(k,13)+zbg(i,3)*aipw(k,j)
      psi(k,14)=psi(k,14)+zbg(kk,3)*aipy(k,j)
      go to 960
  959 continue
      spsi(k)=spsi(k)+(zbg(i,1)+zbg(i,2))*aipw(k,j)
      dpsi(k)=dpsi(k)+zbg(kk,1)*aipy(k,j)
      psi(k,12)=psi(k,12)+(zbh(i,1)+zbh(i,2))*aipx(k,j)
      psi(k,13)=psi(k,13)+(zbg(i,3)+zbg(i,4))*aipw(k,j)
      psi(k,14)=psi(k,14)+zbg(kk,3)*aipy(k,j)
  960 continue
  933 continue
  350 continue
      if(nwave)502,503,502
  503 continue
      acb=dtsos/(2.0*(wbg(3)-wbg(1)))
      call wavef(x,psi,nint,wgt,spsi,dpsi,am,acb,ncomp,ii)
  502 continue
      if(out)1047,1048,1047
 1048 continue
      do 930 k=1,nint
      write 934,x(k),psi(k,1),psi(k,2),psi(k,3),psi(k,4),q(k),psi(k,5)
     /,psi(k,6),psi(k,7)
  934 format(1x,1e12.3,1e15.6,7e12.3//)
  930 continue
      do 99 k=1,nint
      write 947,x(k),psi(k,8),psi(k,9),psi(k,10),spsi(k),dpsi(k)
  947 format(1x,1e12.3,1e15.6,4e12.3//)
   99 continue
      do 1150 i=1,nint
      write 1151,x(i),psi(i,11),psi(i,12),psi(i,13),psi(i,14)
 1151 format(3x,5e15.6//)
      if(ndo)841,842,841
  842 continue
      psi(i,3)=spsi(i)
      psi(i,9)=dpsi(i)
  841 continue
 1150 continue
 1047 continue
```

```
      if(mcomp)1200,1201,1200
 1201 continue
      if(noplot)1204,1205,1204
 1205 continue
      call zplott(x,psi)
 1204 continue
      go to 1113
 1200 continue
      if(mcomp)1209,1209,1210
 1210 continue
c  rspz,rspo, and rspt are the calculated decay rates from 2(3-s-1) to
c  3-p-0, 3-p-1, and 3-p-2 to 1(3-s-1)
c rpzs,rpos,rpts are the calculated radiative decay rates for the transitions
c   3-p-0, 3-p-1 and 3-p-2 to 1(3-s-1)
c  rssz is the transition from the 2(3-s-1) to the 1 (1-s-0) state
c  the expression  rdsz is for the 2(3-s-1) to 1(1-s-0) transition case
c  rszs is the transition from the 2(1-s-0) to the 1(3-s-1) stare
c   rszd is the transition from the 2(1-s-0) to the 1(3-d-1) state
c    ratea is the transition fro5 the 2(3-s-1) the  2(1-s-0) state
c  rateb is the transition from the 2(3-d-1) to the 2(1-s-0) state
c  ratec is the transition frombthe 1(3-s-1) to the 1(1-s-0) state
c  rated is the transition from the 1(3-d-1) to the 1(1-s-0) state
      rspz=0.00
      rdpz=0.00
      rspo=0.00
      rdpo=0.00
      rspt=0.00
      rdpt=0.00
      rpzs=0.00
      rpzd=0.00
      rpos=0.00
      rpod=0.00
      rpts=0.00
      rptd=0.00
      rssz=0.00
      rdsz=0.00
      rszs=0.00
      rszd=0.00
      ratea =0.00
      rateb=0.00
      ratec=0.00
      rated=0.00
      wba(1)=wba(1)*amf
      wba(2)=wba(2)*amf
      wbc(1)=wbc(1)*amf
      wbf(1)=wbf(1)*amf
      wbg(1)=wbg(1)*amf
      wbg(3)=wbg(3)*amf
      wbh(1)=wbh(1)*amf
      acb=dtsos/(wbg(3)-wbg(1))
```

```
      abc=xval/acb-wbg(1)
      fcfed=acb*amce
      fcfmd=acb*amcm
      wba(1)=wba(1)+abc
      wba(2)=wba(2)+abc
      wbc(1)=wbc(1)+abc
      wbg(1)=wbg(1)+abc
      wbg(3)=wbg(3)+abc
      wbh(1)=wbh(1)+abc
      wbf(1)=wbf(1)+abc
      write 95,wba(1),wba(2),wbc(1),wbg(1),wbg(3),wbh(1),wbf(1)
   95 format(1x,7e12.5//)
      amg=am*acb
      write 96,amg,abc,acb
   96 format(3x,3e12.5//)
      prspz=2.0/3.0
      prspo=4.0/3.0
      prszd=2.0/3.0
      prspt=10.0/9.0
      prpzs=2.0
      prpos=4.0/3.0
      prpts=2.0/3.0
      prszs=2.0
      prssz=2.0/3.0
      cone=1.0
      ctwo=1.0
      cthr=1.0
      cfou=1.0
      cfiv=1.0
      csix=1.0
      csev=cthr
      ceig=csix
      cnin=ctwo
      cten=cfiv
      cele=cone
      ctwv=cfou
      ctne=1.0
      cfne=1.0
      cffe=ctne
      csne=cfne
      pmg=wbg(1)
      pmh=wbg(3)
      apo=wba(1)
      app=wba(2)
      apq=wbc(1)
      apr=wbf(1)
      aps=wbg(1)
      apt=wbg(3)
      apu=wbh(1)
      if(npm)103,103,104
```

```
104 call phmass
    wba(1)=pma
    wba(2)=pmb
    wbc(1)=pmc
    wbf(1)=pme
    wbh(1)=pmf
103 continue
    awaa=wbg(3)+wbf(1)
    awab=wbg(3)-wbf(1)
    awa=awaa*awab/wbg(3)
    awba=wbf(1)+wbg(1)
    awbb=wbf(1)-wbg(1)
    awb=awba*awbb/wbf(1)
    awca=wbg(3)+wbc(1)
    awcb=wbg(3)-wbc(1)
    awc=awca*awcb/wbg(3)
    awda=wbc(1)+wbg(1)
    awdb=wbc(1)-wbg(1)
    awd=awda*awdb/wbc(1)
    awea=wbg(3)+wbh(1)
    aweb=wbg(3)-wbh(1)
    awe=awea*aweb/wbg(3)
    awfa=wbh(1)+wbg(1)
    awfb=wbh(1)-wbg(1)
    awf=awfa*awfb/wbh(1)
    awga=wbg(1)+wba(1)
    awgb=wbg(1)-wba(1)
    awg=awga*awgb/wbg(1)
    awha=wbg(3)+wba(2)
    awhb=wbg(3)-wba(2)
    awh=awha*awhb/wbg(3)
    awka=wba(2)+wbg(1)
    awkb=wba(2)-wbg(1)
    awk=awka*awkb/wba(2)
    awla=wbg(3)+wba(1)
    awlb=wbg(3)-wba(1)
    awl=awla*awlb/wbg(3)
    awa=awa/2.0
    awb=awb/2.0
    awc=awc/2.0
    awd=awd/2.0
    awe=awe/2.0
    awf=awf/2.0
    awg=awg/2.0
    awh=awh/2.0
    awk=awk/2.0
    awl=awl/2.0
    do 91 i=1,nint
    if(nmat)500,501,500
501 continue
```

```
      call element(x,i)
  500 continue
      xb=x(i)*x(i)
      xc=xb*x(i)
      amm=am*am
      r=dsqrt(xb+amm)
      rr=xb+amm
      bot=(awaa+awa)/2.0-2.0*r
      btt=(awba+awb)/2.0-2.0*r
      cot=(awca+awc)/2.0-2.0*r
      ctt=(awda+awd)/2.0-2.0*r
      dot=(awea+awe)/2.0-2.0*r
      dtt=(awfa+awf)/2.0-2.0*r
      eot=(awga+awg)/2.0-2.0*r
      ett=(awha+awh)/2.0-2.0*r
      fot=(awka+awk)/2.0-2.0*r
      got=(awla+awl)/2.0-2.0*r
      rspz=rspz+wgt(i)*xc*psi(i,13)*psi(i,10)*cone*(wbg(3)-2.0*r)/(r*bot)
     /)
      rdpz=rdpz+wgt(i)*xc*psi(i,14)*psi(i,10)*cfou*(wbg(3)-2.0*r)/(r*bot)
     /)
      rpzs=rpzs+wgt(i)*xc*psi(i,10)*spsi(i)*cele*(wbf(1)-2.0*r)/(r*btt)
      rpzd=rpzd+wgt(i)*xc*psi(i,10)*dpsi(i)*ctwv*(wbf(1)-2.0*r)/(r*btt)
      rspo=rspo+wgt(i)*xc*psi(i,13)*psi(i,11)*ctwo*(wbg(3)-2.0*r)/(r*cot)
     /)
      rdpo=rdpo+wgt(i)*xc*psi(i,14)*psi(i,11)*cfiv*(wbg(3)-2.0*r)/(r*cot)
     /)
      rpos=rpos+wgt(i)*xc*psi(i,11)*spsi(i)*cnin*(wbc(1)-2.0*r)/(r*ctt)
      rpod=rpod+wgt(i)*xc*psi(i,11)*dpsi(i)*(wbc(1)-2.0*r)*cten/(r*ctt)
      rspt=rspt+wgt(i)*xc*psi(i,13)*psi(i,12)*cthr*(wbg(3)-2.0*r)/(r*dot)
     /)
      rdpt=rdpt+wgt(i)*xc*psi(i,14)*psi(i,12)*csix*(wbg(3)-2.0*r)/(r*dot)
     /)
      rpts=rpts+wgt(i)*xc*psi(i,12)*spsi(i)*csev*(wbh(1)-2.0*r)/(r*dtt)
      rptd=rptd+wgt(i)*xc*psi(i,12)*dpsi(i)*ceig*(wbh(1)-2.0*r)/(r*dtt)
      rssz=rssz+wgt(i)*xb*psi(i,1)*psi(i,13)*(r+2.0*am)/3.0*(wbg(3)-2.0*r)
     /r)*ctne/(rr*got)
      rdsz=rdsz+wgt(i)*xb*psi(i,1)*psi(i,14)*(r-am)/(9.0*rr)*(wbg(3)-2.0*r
     /*r)*cfne/(rr*got)
      rszs=rszs+wgt(i)*xb*psi(i,2)*spsi(i)*(r+2.0*am)/3.0*(wba(2)-2.0*r)*cffe
     /*cffe/(rr*fot)
      rszd=rszd+wgt(i)*xb*psi(i,2)*dpsi(i)*(r-am)/(9.0*rr)*(wba(2)-2.0*r)
     /)*csne/(rr*fot)
      ratea=ratea+wgt(i)*xb*psi(i,13)*psi(i,2)*(r+2.0*am)/3.0*(wbg(3)-2.0
     /0*r)*ctne/(rr*ett)
      rateb=rateb+wgt(i)*xb*psi(i,14)*psi(i,2)*(r-am)/(9.0*rr)*(wbg(3)-2.0*r
     /.0*r)*cfne/(rr*ett)
      ratec=ratec+wgt(i)*xb*psi(i,1)*spsi(i)*(r+2.0*am)/3.0*(wbg(1)-2.0*r)
     /r)*ctne/(rr*eot)
      rated=rated+wgt(i)*xb*psi(i,1)*dpsi(i)*(r-am)/(9.0*rr)*(wbg(1)-2.0*r)
```

```
   /*r)*cfne/(rr*eot)
 91 continue
    if(nchet)600,601,600
601 call cheat
600 continue
    rspz=rspz*rspz*4.0*pi/81.0
    rspo=rspo*rspo*2.0*pi/27.0
    rspt=rspt*rspt*4.0*pi/27.0
    rdpz=rdpz*rdpz*8.0*pi/9.0
    rdpo=rdpo*rdpo*2.0*pi/6.0
    rdpt=rdpt*rdpt*6.0*pi/25.0
    rpzs=rpzs*rpzs*4.0*pi/81.0
    rpos=rpos*rpos*2.0*pi/27.0
    rpts=rpts*rpts*4.0*pi/27.0
    rpzd=rpzd*rpzd*8.0*pi/9.0
    rpod=rpod*rpod*pi/3.0
    rptd=rptd*rptd*6.0*pi/25.0
    ratea=ratea*ratea*pi/36.0
    ratec=ratec*ratec*pi/36.0
    rszs=rszs*rszs*pi/36.0
    rssz=rssz*rssz*pi/36.0
    rateb=rateb*rateb*2.0*pi
    rated=rated*rated*2.0*pi
    rszd=rszd*rszd*2.0*pi
    rdsz=rdsz*rdsz*2.0*pi
    if(npm)105,106,106
105  call phmass
    wba(1)=pma
    wba(2)=pmb
    wbc(1)=pmc
    wbf(1)=pme
    wbh(1)=pmf
106 continue
    za=4.0*wbg(3)*wbf(1)
    zb=4.0*wbg(3)*wbc(1)
    zc=4.0*wbg(3)*wbh(1)
    ze=4.0*wbf(1)*wbg(1)
    zf=4.0*wbc(1)*wbg(1)
    zg=4.0*wbh(1)*wbg(1)
    zaa=4.0*wbg(3)*wba(1)
    zab=4.0*wba(2)*wbg(1)
    zac=4.0*wbg(3)*wba(2)
    zad=4.0*wbg(1)*wba(1)
    rspz=rspz*za
    rdpz=rdpz*za
    rspo=rspo*zb
    rdpo=rdpo*zb
    rspt=rspt*zc
    rdpt=rdpt*zc
    rpzs=rpzs*ze
```

```
rpzd=rpzd*ze
rpos=rpos*zf
rpod=rpod*zf
rpts=rpts*zg
rptd=rptd*zg
rssz=rssz*zaa
rdsz=rdsz*zaa
rszs=rszs*zab
rszd=rszd*zab
ratea=ratea*zac
rateb=rateb*zac
ratec=ratec*zad
rated=rated*zad
rspz=prspz*rspz*awa/(2.0*wbg(3)*wbg(3))
rspo=prspo*rspo*awc/(2.0*wbg(3)*wbg(3))
rspt=prspt*rspt*awe/(2.0*wbg(3)*wbg(3))
rpzs=prpzs*rpzs*awb/(2.0*wbf(1)*wbf(1))
rpos=prpos*rpos*awd/(2.0*wbc(1)*wbc(1))
rpts=prpts*rpts*awf/(2.0*wbh(1)*wbh(1))
rdpz=prspz*rdpz*awa/(2.0*wbg(3)*wbg(3))
rdpo=prspo*rdpo*awc/(2.0*wbg(3)*wbg(3))
rdpt=prspt*rdpt*awe/(2.0*wbg(3)*wbg(3))
rpzd=prpzs*rpzd*awb/(2.0*wbf(1)*wbf(1))
rpod=prpos*rpod*awd/(2.0*wbc(1)*wbc(1))
rptd=prpts*rptd*awf/(2.0*wbh(1)*wbh(1))
rszs=prszs*rszs*awk**3.0/(2.0*wba(2)*wba(2))
rszd=prszd*rszd*awk**3.0/(2.0*wba(2)*wba(2))
rssz=prssz*rssz*awl**3.0/(2.0*wbg(3)*wbg(3))
rdsz=prssz*rdsz*awl**3.0/(2.0*wbg(3)*wbg(3))
ratea=prssz*ratea*awh**3.0/(2.0*wbg(3)*wbg(3))
rateb=prssz*rateb*awh**3.0/(2.0*wbg(3)*wbg(3))
ratec=prssz*ratec*awg**3.0/(2.0*wbg(1)*wbg(1))
rated=prssz*rated*awg**3.0/(2.0*wbg(1)*wbg(1))
rspz=rspz*fcfed
rspo=rspo*fcfed
rspt=rspt*fcfed
rpzs=rpzs*fcfed
rpos=rpos*fcfed
rpts=rpts*fcfed
rdpz=rdpz*fcfed
rdpo=rdpo*fcfed
rdpt=rdpt*fcfed
rpzd=rpzd*fcfed
rpod=rpod*fcfed
rptd=rptd*fcfed
rszs=rszs*fcfmd
rszd=rszd*fcfmd
rssz=rssz*fcfmd
rdsz=rdsz*fcfmd
ratea=ratea*fcfmd
```

```
      rateb=rateb*fcfmd
      ratec=ratec*fcfmd
      rated=rated*fcfmd
      write 92,am,rspz,rspo,rspt,rpzs,rpos,rpts
   92 format(1x,7e12.5//)
      write 93,am,rdpz,rdpo,rdpt,rpzd,rpod,rptd
   93 format(1x,7e12.5//)
      write 94,rssz,rdsz,rszs,rszd
   94 format(7x,4e12.5//)
      write 100,ratea,rateb,ratec,rated
  100 format(7x,4e12.5//)
      call setwgg(ii)
      wba(1)=wba(1)*acb
      wba(2)=wba(2)*acb
      wbc(1)=wbc(1)*acb
      wbf(1)=wbf(1)*acb
      wbg(1)=wbg(1)*acb
      wbg(3)=wbg(3)*acb
      wbh(1)=wbh(1)*acb
      wbb(1)=(amf*wbb(1)+abc)*acb
      wbd(1)=(amf*wbd(1)+abc)*acb
      wbe(1)=(amf*wbe(1)+abc)*acb
      write 101,wba(1),wba(2),wbc(1),wbf(1),wbg(1),wbg(3),wbh(1)
  101 format(1x,7e12.5//)
      write 102,wbb(1),wbd(1),wbe(1),wbs,wbt
  102 format(4x,5e12.5//)
      rm=rm+dam
      am=rm
      opt=optf
      oxt=oxtf
      owt=owtf
      out=outf
      dmass=rm-rmass
      if(dmass)920,920,1206
 1206 continue
      if(noplot)1207,1208,1207
c  the plots of the wavefunctions are as a function of x, the plots
c  of the transition rates are as a function of the mass.
c  all of the above calculations are implicitly a function of ak(or m)and rat.
c   for the plotting of the transitions . to do this need to
c midify the plotting program calling sequence as follows
c  on the first graph  i want the first  five  of the following
c expressions, next  the following three, then the next four,
c and lastly the two remainning ones.
 1208 call yplott(wgg,amo,amof)
 1207 continue
      go to 1113
 1209 continue
c  from here back delete for calculation of b-s functions on plotter
      xsps=0.00
```

```
      xdps=0.00
      do 935 k=1,nint
      xb=x(k)*x(k)
      xsps=xsps+spsi(k)*spsi(k)*wgt(k)*xb
      xdps=xdps+dpsi(k)*dpsi(k)*wgt(k)*xb
  935 continue
      xx=xdps+xsps
      write 936,xsps,xdps,xx
  936 format(1x,3e12.3//)
      do 942 j=1,mm
      do 943 i=1,nint
      pbsi(i,j)=zbg(1,j)*aipow(i)
      pbsj(i,j)=zbg(11,j)*aipoy(i)
      do 944 k=2,10
      kk=k+10
      kl=k-1
      pbsi(i,j)=pbsi(i,j)+zbg(k,j)*aipw(i,kl)
      pbsj(i,j)=pbsj(i,j)+zbg(kk,j)*aipy(i,kl)
  944 continue
  943 continue
      son=0.00
      stw=0.00
      don=0.00
      do 945 i=1,nint
      xb=x(i)*x(i)
      rr=xb+amm
      r=dsqrt(dble(rr))
      son=son+wgt(i)*xb*pbsi(i,j)
      stw=stw+wgt(i)*xb*xb*pbsi(i,j)/(3.0*r*(r+am))
      don=don+wgt(i)*xb*xb*pbsj(i,j)/(r+(r+am))
  945 continue
      don=don*sqrt(2.0)/3.0
      alpha=1.0/137.0
      width=(4.0*alpha)/(3.0*xval)
      width=width*width*2.0/pi
      width=width*((son-stw+don)**2.0)
      rwide=drwide/width
      rwide=(rwide)**(1.0/3.0)
      swide=width*xval/yval
      swide=dswide/swide
      swide=swide**(1.0/3.0)
      mmn=j-1
      if(mmn)97,98,97
   98 continue
c  didth is hoestlers value of psi(0) for the 3-s-1 ground state.
c  sidth  is hoestlers value of kappa obtained from the psi mass differences
      write 1053,wbg(1),width,son,stw,don,rwide,swide
 1053 format(1x,7e15.6//)
      acb=dtsos/(wbg(isn(2))-wbg(isn(1)))
      sidth=acb/ad
```

```
      csts=4.0/(3.0*pi*sqrt(2.0))
      didth=csts*(son-stw)
      sdith=csts*son
      write 948,wbg(2),didth,sidth,sdith
  948 format(4x,4e15.6//)
   97 continue
      mmn=j-isn(1)
      mmo=j-isn(2)
      if(mmn)2100,2101,2100
 2101 wgg(ii,27)=rwide
      wgg(ii,28)=swide
 2100 continue
      if(mmo)2102,2103,2102
 2103 wgg(ii,28)=swide
 2102 continue
      if(oxt)1051,1052,1051
 1052 continue
      write 946,wbg(j),width,son,stw,don,rwide,swide
  946 format(1x,7e15.6//)
 1051 continue
  942 continue
      do 1111 i=1,3
      write 1110,wbg(isn(i)),wbg(idn(i)),wbh(ipn(i)),wbh(ifn(i))
 1110 format(5x,4e20.10//)
 1111 continue
      if(mcomp)1114,1113,1113
 1114 continue
      wgg(ii,1)=wba(1)-am
      wgg(ii,2)=wbb(1)-am
      wgg(ii,3)=wbd(1)-am
      wgg(ii,4)=wbf(1)-am
      wgg(ii,5)=wbg(isn(1))-am
      wgg(ii,6)=wbg(idn(1))-am
      wgg(ii,7)=wbh(ipn(1))-am
      wgg(ii,8)=wbh(ifn(1))-am
      wgg(ii,9)=wba(2)-am
      wgg(ii,10)=wbb(2)-am
      wgg(ii,11)=wbd(2)-am
      wgg(ii,12)=wbf(2)-am
      wgg(ii,13)=wbg(isn(2))-am
      wgg(ii,14)=wbg(idn(2))-am
      wgg(ii,15)=wbh(ipn(2))-am
      wgg(ii,16)=wbh(ifn(2))-am
      wgg(ii,17)=wba(3)-am
      wgg(ii,18)=wbb(3)-am
      wgg(ii,19)=wbd(3)-am
      wgg(ii,20)=wbf(3)-am
      wgg(ii,21)=wbg(isn(3))-am
      wgg(ii,22)=wbg(idn(3))-am
      wgg(ii,23)=wbh(ipn(3))-am
```

```
      wgg(ii,24)=wbh(ifn(3))-am
      wgg(ii,25)=dtsos/(wbg(isn(2))-wbg(isn(1)))
      wgg(ii,25)=wgg(ii,25)/2.0
      wgg(ii,26)=dtpso/(wbh(ipn(1))-wbg(isn(1)))
      wgg(ii,26)=wgg(ii,26)/2.0
      wgg(ii,29)=wbh(ipn(1))-wbf(1)
      wgg(ii,29)=wgg(ii,29)/2.0
      wgg(ii,29)=dptpz-acb*wgg(ii,29)
      wgg(ii,29)=wgg(ii,29)/dptpz
      wgg(ii,30)=wbh(ipn(1))-wbc(1)
      wgg(ii,30)=wgg(ii,30)/2.0
      wgg(ii,30)=dptpo-acb*wgg(ii,30)
      wgg(ii,30)=wgg(ii,30)/dptpo
      rm=rm+dam
      am=rm
      opt=optf
      oxt=oxtf
      owt=owtf
      out=outf
      dmas=rm-rmass
      if(dmas)920,920,931
  931 continue
      do 1100 i=1,21
      write 1101,i,wgg(i,25),wgg(i,26),wgg(i,27),wgg(i,28),wgg(i,29),wgg(i,30)
     /(i,30)
 1101 format(1x,113,6e15.6//)
 1100 continue
      if(noplot)1113,1112,1113
 1112 continue
      call xplott(wgg,amo,amof)
 1113 continue
      end
c    the rest of this deck is the xplott routine for plotting masses and leptons
      subroutine xplott(wgg,amo,amof)
      dimension wgg(21,30)
            dimension ibuf(257)
      dimension  xt(30),fxt(30,16), title(50)
      dimension worka(200),workb(200), workc(200),workd(200),sizval(6)
          dimension y(200)          ,x(200)
      common /kurve/ icurve(6)
c.....icurve=kurve+1
c     kurve = 0 for solid line, 1 for dotted, 2 for dashed, (3,4,5) for
c         dash with (1,2,3) dots
      data title /21h b s wavefunctions   /
      data nd,nt,nf,isize,iann,iex,nord,ipoint,icmin,iplot/200,100,2,
     1 3,1,0,1,0,0,200/
      nord=4
            icmin=1
      nd=30
      nt=21
```

```
      if(nd .ne. 30)  stop 741
            nloc=257
      sizval(1)=10.00
      sizval(2)=7.00
            call plots(ibuf,nloc,5)
            call plimit (150)
      hi=0.6
      advance=5.0
      call letter (20,hi,80.0,0.0,0.0,  7h becker)
      call plot(advance,0.0,-3)
      call letter( 20,hi,90.0,0.0,0.0,title)
      call plot(advance,0.0,-3)
      do 508 j=1,21
      aj=j
      xt(j)=(aj+amo-1.0)/amof
  508 continue
  201    continue
      nn=4
      do 126 kk=1,nn
      go to (12,13,14,15),kk
   12 continue
      isize=0
      nf=8
      do 500 i=1,8
      do 501 j=1,21
  501 fxt(j,i)=wgg(j,i)
  500 continue
      go to 35
   13 continue
      nf=16
      isize=0
      do 502 i=1,16
      do 503 j=1,21
  503 fxt(j,i)=wgg(j,i)
  502 continue
      go to 35
   14 continue
      nf=16
      isize=0
      do 504 i=1,16
      ii=i+8
      do 505 j=1,21
  505 fxt(j,i)=wgg(j,ii)
  504 continue
          go to 35
   15 continue
      nf=6
      isize=0
      do 506 i=1,6
      ii=i+24
```

```
      do 507 j=1,21
  507 fxt(j,i)=wgg(j,ii)
  506 continue
   35 continue
      call plotb (xt,fxt,nd,nt,nf,isize,iann,iex,nord,ipoint,icmin,
     1 iplot,sizval,worka,workb,workc,workd)
  126 continue
      call plot (0.0,0.0,999)
      return
      end
      subroutine yplott(wgg,amo,amof)
      dimension wgg(21,30)
            dimension ibuf(257)
      dimension  xt(30),fxt(30,16), title(50)
      dimension worka(200),workb(200), workc(200),workd(200),sizval(6)
          dimension y(200)           ,x(200)
      common /kurve/ icurve(6)
c.....icurve=kurve+1
c     kurve = 0 for solid line, 1 for dotted, 2 for dashed, (3,4,5) for
c         dash with (1,2,3) dots
      data title /21h b s wavefunctions   /
      data nd,nt,nf,isize,iann,iex,nord,ipoint,icmin,iplot/200,100,2,
     1 3,1,0,1,0,0,200/
      nord=4
            icmin=1
      nd=30
      nt=21
      if(nd .ne. 30)  stop 741
           nloc=257
      sizval(1)=10.00
      sizval(2)=7.00
           call plots(ibuf,nloc,5)
           call plimit (150)
      hi=0.6
      advance=5.0
      call letter (20,hi,80.0,0.0,0.0,  7h becker)
      call plot(advance,0.0,-3)
      call letter( 20,hi,90.0,0.0,0.0,title)
      call plot(advance,0.0,-3)
      do 508 j=1,21
      aj=j
      xt(j)=(aj+amo-1.0)/amof
  508 continue
  201    continue
      nn=4
      do 126 kk=1,nn
      go to (12,13,14,15),kk
   12 continue
      isize=0
      nf=5
```

```
      do 500 i=1,5
      do 501 j=1,21
  501 fxt(j,i)=wgg(j,i)
  500 continue
      go to 35
   13 continue
      nf=5
      isize=0
      do 502 i=1,5
      ii=i+5
      do 503 j=1,21
  503 fxt(j,i)=wgg(j,ii)
  502 continue
      go to 35
   14 continue
      nf=4
      isize=0
      do 504 i=1,4
      ii=i+10
      do 505 j=1,21
  505 fxt(j,i)=wgg(j,ii)
  504 continue
          go to 35
   15 continue
      nf=2
      isize=0
      do 506 i=1,2
      ii=i+14
      do 507 j=1,21
  507 fxt(j,i)=wgg(j,ii)
  506 continue
   35 continue
      call plotb (xt,fxt,nd,nt,nf,isize,iann,iex,nord,ipoint,icmin,
     1 iplot,sizval,worka,workb,workc,workd)
  126 continue
      call plot (0.0,0.0,999)
      return
      end

      subroutine zplott(x,psi)
      double precision x
      dimension x(32),psi(32,20)
            dimension ibuf(257)
      dimension xt(32),fxt(32,20), title(50)
      dimension worka(200),workb(200), workc(200),workd(200),sizval(6)
      common /kurve/ icurve(6)
      data title /21h b s wavefunctions   /
      data nd,nt,nf,isize,iann,iex,nord,ipoint,icmin,iplot/200,100,2,
     1 3,1,0,1,0,0,200/
            nloc=257
```

```
      call plots(ibuf,nloc,5)
      call plimit (150)
    advance=5.0   .
    hi=0.6
      sizval(2)=6.0
      sizval(1)=6.0
    nt=28
    nd=32
    if(nd.ne.32)  stop 741
       icmin=1
    call letter (20,hi,80.0,0.0,0.0,  7h becker)
    call plot(advance,0.0,-3)
    call letter( 20,hi,90.0,0.0,0.0,title)
    call plot(advance,0.0,-3)
    nt=20
     do 105 i=1,nt
105  xt(i)=x(i)
    nord=4
    nn=4
    kkk=0-1
    do 126 kk=1,nn
    kkk=kkk+1
    go to (12,12,12,12),kk
12  nf=3
    sizval(3)=0.00
    sizval(4)=5.00
    sizval(5)=-2.00
    sizval(6)=+2.00
    do 241 if=1,nf
       do 241 i=1,nt
    ifx=if+nf*kkk
241 fxt(i,if)=psi(i,ifx)
    go to 35
 13 nt=32
      sizval(5)=-0.2    $ sizval(6)=0.1
      do 210 i=1,nt
       fxt(i,3)=ddav(i)
210      fxt(i,1)= ds(i)
     do 211 i=1,nt
211      fxt(i,2)= ds(i)
       nf=3
       go to 35
14     nf=4
    sizval(3)=0.00
    sizval(4)=5.00
    sizval(5)=-2.00
    sizval(6)=+2.00
      nt=50
       xt(1)=sizval(3)
       do 319 i=2,nt
```

```
319      xt(i)=xt(i-1)+dxt
          do 231 i=1,nt
          fxt(i,4)=gz1
         fxt(i,1)=gx1
        fxt(i,2)=gy1
          fxt(i,3)=fff*w*0.5/gapo/amax
231       continue
          go to 35
15        nf=2
         sizval(5)=-25.0
         sizval(6)=50.0
         do 239 i=1,nt
         write(6,221) xt(i),fxt(i,1),fxt(i,2)
 221 format(1x,f10.5,2e15.5)
239         continue
35    continue
      call plotb (xt,fxt,nd,nt,nf,isize,iann,iex,nord,ipoint,icmin,
     1 iplot,sizval,worka,workb,workc,workd)
126   continue
      call plot (0.0,0.0,999)
      return
      end

      double precision function airy(yx)
      implicit double precision (a-h,p-z)
      dimension  ch(34),ts(20)
      common/con/t(17),u(17),v(20),tb,uo,vo
      if(yx)10,11,10
   10 continue
      tau=(2.0*yx**1.5)/3.0
      ta=tau-5.0
      if(ta)22,22,34
   22 so=1.0
      pe=tau/8.0
      ch(1)=pe
      ch(2)=2.0*pe*pe-1.0
      ch(3)=4.0*pe*pe*pe-3.0*pe
      ch(4)=2.0*ch(2)*ch(2)-so
      do 23 k=2,16
      j=2*k+2
      l=2*k
      m=2*k-2
   23 ch(j)=2.0*ch(2)*ch(l)-ch(m)
      sum=tb*so
      sim=uo*so
      do 26 k=1,17
      j=2*k
      sum=sum+t(k)*ch(j)
      sim=sim+u(k)*ch(j)
   26 continue
```

```
      sqx=dsqrt(yx)
      airy=sqx*(sum*tau**(-1.0/3.0)-sim*tau**(1.0/3.0))/3.0
      go to 15
   34 to=1.0
      qe=5.0/tau
      ts(1)=2.0*qe-1.0
      ts(2)=8.0*qe*qe-8.0*qe+1.0
      do 39 k=2,19
      j=k+1
      l=k-1
   39 ts(j)=2.0*ts(1)*ts(k)-ts(l)
      sam=vo*to
      do 49 k=1,20
      sam=sam+v(k)*ts(k)
   49 continue
      exx=dexp(-tau)
      pi=4.0d+00*datan(1.00d+00)
      sqt=dsqrt(pi/(2.0*tau))
      sqz=dsqrt(yx/3.0)
      airy=sqz*sqt*exx*sam/pi
   15 continue
      go to 12
   11 continue
      airy=0.355028053887817d+00
   12 continue
      return
      end
      double precision function dairy(yx)
      implicit double precision (a-h,p-z)
      common/don/td(17),ud(17),vd(20),tdo,udo,vdo
      dimension ch(34),ts(20)
      if(yx)10,11,10
   10 continue
      tau=(2.0*yx**1.5)/3.0
      ta=tau-5.0
      if(ta)22,22,34
   22 so=1.0
      pe=tau/8.0
      ch(1)=pe
      ch(2)=2.0*pe*pe-1.0
      ch(3)=4.0*pe*pe*pe-3.0*pe
      ch(4)=2.0*ch(2)*ch(2)-so
      do 23 k=2,16
      j=2*k+2
      l=2*k
      m=2*k-2
   23 ch(j)=2.0*ch(2)*ch(1)-ch(m)
      tum=tdo*so
      tim=udo*so
      do 26 k=1,17
```

```
      j=2*k
      tum=tum+td(k)*ch(j)
      tim=tim+ud(k)*ch(j)
   26 continue
      dairy=-yx*(tum*tau**(-2.0/3.0)-tim*tau**(2.0/3.0))/3.0
      go to 15
   34 to=1.0
      qe=5.0/tau
      ts(1)=2.0*qe-1.0
      ts(2)=8.0*qe*qe-8.0*qe+1.0
      do 39 k=2,19
      j=k+1
      l=k-1
   39 ts(j)=2.0*ts(1)*ts(k)-ts(l)
      tam=vdo*to
      do 49 k=1,20
      tam=tam+vd(k)*ts(k)
   49 continue
      exx=dexp(-tau)
      pi=4.0d+00*datan(1.00d+00)
      sqt=dsqrt(pi/(2.0*tau))
      sqq=dsqrt(3.0d+00)
      dairy=-yx*sqt*exx*tam/(pi*sqq)
   15 continue
      go to 12
   11 continue
      dairy=0.258819403792807d+00
   12 continue
      return
      end
```

```
      SUBROUTINE  DATAB
      COMMON/PMPAR/PMES,NPM
      COMMON/KURVE/ICURVE(6)
      COMMON/DATA/ RAT,AM,NUMB,NINT,SLAEQ,AK,DAM,CAM,XVAL,YVAL,DTPSO,
     /OUTF,OWTF,OXTF,OPTF,DRWIDE,DSWIDE,RMASS,DPTPZ,DPTPO,
     /NOPLOT,DTSOS,NEXP,NCOMP,AMO,AMOF,NDO,AMF,AMCE,AMCM
      COMMON/DATAA/NWAVE,NMAT
      DOUBLE PRECISION AM
CC                                                                    CC
CC                                                                    CC
CC                                                                    CC
CC    RAT IS TWICE THE RELATIVE SCALAR/VECTOR COUPLING
C  TO GET THE PURE VECTOR CASE BACK SET RAT = 0.00
C  TO TRY TO GET THE PURE SCALAR CASE SET RAT TO BE A LARGE POSITIVE
CC     NUMBER SUCH AS 1000000, AND SLAEQ IS A NUMBER TWICE AS LARGE E.G. 2000000
C  IF NCOMP=0,THEN THE PROGRAM ONLY CALCULATES THE EIGENVALUES AND IF
C   NOPLOT IS EQUAL TO 0 IT WILL PLOT THESE VALUES AS A FUNCTION OF
CC    MASS FROM M=0 TO M=20. IF NOPLOT IS DIFFERENT FROM 0 IT WILL NOT
C  PLOT THE GRAPH.
CCXXXXXXXXXXX   ADDITIONAL NOTE ON THE CASE OF NCOMP=0. IF AMO IS NOT
CCXXXXXXXXXXX  EQUAL TO ZERO THEN  ONE NEEDS TO LOOK AT TWO OTHER ITEMS
CCXXXXXXXXXXX    NAMELY THE  VALUE OF AMO AND AMOF. IN THE PROGRAM THESE
CCXXXXXXXXXXX    QUANTITIES ARE USED TO  SCAN  SMALLER INTERVALS (OR
CCXXXXXXXXXXX    LARGER ONES DEPENDING ON THESE FACTORS) INTERVALS OF
CCXXXXXXXXXXX    THIS MASS SCALE. NOTE HOWEVER THERE ARE STILL ONLY
CCXXXXXXXXXXX     21  DIFFERENT VALUES OF THE MASS USED IN THE CALCULATION....
C.............NOTE IN THE CASE WHERE AM IS NOT EQUAL TO ZERO.............
C.................AND RMASS IS NOT EQUAL TO 20 THE QUANTITIES SHOULD .........
C.................BE LOOKED AT . IN PARTICULAR THE CASE WHEN DAM IS
C................,NOT ONE THEN AMO IS EQUAL TO AM*AMOF,WHERE AMOF IS
C.................SUCH AS TO MAKE THE PRODUCT DAM*AMOF EQUAL TO ONE..........
C  IFNCOMP=1,THEN THE WAVEFUNCTIONS ARE COMPUTED BUT ONLY AT THE VALUE
CC    OF THE MASS WHICH APPEARS ON THE DATA CARD, AND ONLY FOR THIS VALUE.
C  IF NCOMP=2, THEN ONLY THE TRANSITION RATES ARE CALCULATED.
C  IF NOPLOT=0 IN EITHER OF THESE LAST TWO CASES THEN GRAPHS ARE PLOTTED,
C   OTHERWISE NO GRAPHS ARE MADE.
CC                                                                    CC
CC                                                                    CC
CC
CC.................NOTE WHEN DOING RUNS FOR THE B QUARKS ONE MUST.............
CC.................MAKE THE FOLLOWING CHANGES(1)THE LEPTONIC
CC.................DECAY WIDTHS  MUST BE MULTIPLIED BY A FACTOR ..........
CC.................OF FOUR TO BALANCE THE DIFFERENCE IN THE QUARK.......
CC.................CHARGES TO GET THE CORRECT FITTING MASS FROM THE
CC.................NCOMP =0 SUBPLOT GRAPHS. (2)IN THE NCOMP = 2
CC.................RUNS ONE MUST BALANCE THE QUARK CHARGES BY THE
CC.................INCLUSION OF AN ADDITIONAL FACTOR OF 1/4 IN
CC.................THE AMCE AND AMCM TERMS. THE WAVE FUNCTION PART
```

```
CC.................IS UNEFFECTED BY THE CHANGES FROM C TO B QUARKS.
CC
CC
CC                              GRAPHS FOR NCOMP=0                                  CC
CC     FOR THESE PLOTS WE HAVE THE FOLLOWING ARRANGEMENTS,
CC      1)  A PLOT OF THE LOWEST EIGENVALUE AS A FUNCTION OF THE INPUT MASS    CCC
CC           FOR THE CASES  SINGLET -S-ZERO, SINGLET-P-ONE, AND SINGLET -D-TWO CC
CC           THE TRIPLET -P-ZERO, AND THE S AND D COMPONENTS OF THE MIXED       CC
CC           SPIN ONE  TRIPLET, ALONG WITH THE CORRESPONDING SET OF THE         CC
CC           MIXED SPIN TWO CASES, I.E. THE P AND F COMPONENTS OF THE           CC
CC           ORBITAL ANGULAR MOMENTUM.
CC       2)  THE SAME ARRANGEMENT AS ABOVE EXCEPT THAT THE LOWEST TWO STATES   CCC
CC           ARE PLOTTED AS A FUNCTION OF THE MASSES.                           CCC
CC      3)   IN THE THIRD CASE THE SECOND AND THIRD LOWEST EIGENVALUES ARE      CC
CC           PLOTTED FOR THE SAME SET OF STATES AS IS SPECIFIED IN CASE         CC
CC           ONE ABOVE.                                                         CC
CC      4)  ON THE LAST PLOT ARE GRAPHED THE LEPTONIC DECAY WIDTHS OF          CCC
CC           THE LOWEST TWO MIXED TRIPLET SPIN ONE STATES,(PSI AND PSI")        CCC
CC           ALONG WITH THE FRACTIONAL DIFFERENCE  BETWEEN THE COMPUTED         CCC
CC           AND THE OBSERVED TRIPLET STATES.  THAT IS THE FOLLOWING            CC
CC           DIFFERENCES ARE PLOTTED                                           CCC
CC                A)(3-P-2 MINUS 3-P-0)OBSERVED  MINUS (3-P-2 MINUS 3-P-0)   CCCC
CC                      CALCULATED DIVIDED BY  THE OBSERVED DIFFERENCE.       CCCC
CC                B) ON THIS SAME PLOT IS SHOWN THE SIMILIAR CASE FOR THE      CC
CC                       (3-P-2 MINUS 3-P-1) STATES.                          CCC
CC                                                                            CC
CC                                                                            CC
CC                                                                            CC
CC                              GRAPHS FOR NCOMP=1                            CC
CC           IN THE PLOTTING OF THE WAVEFUNCTIONS, ON THE THE FIRST GRAPH      CC
CC        WE HAVE FOR THE MASS VALUE GIVEN ON THE DATAB  SUBROUTINE,           CC
CC      IF NDO=1.....THEN WE HAVE ON THE FIRST GRAPH THE PLOTS FOR THE
CC                    FIRST THREE SINGLET-S-ZERO WAVEFUNCTIONS. ON THE
CC                    SECOND GRAPH WE HAVE PLOTTED THE FIRST THREE SINGLET-P
CC                    -ONE STATES.  ON THE THIRD GRAPH
CC                    WE HAVE PLOTTED THE FIRST THREE SINGLET-D-TWO STATES.
CC                    ON THE FOURTH GRAPH WE HAVE PLOTTED THE GROUND STATES
CC                    OF THE TRIPLET - P-ZERO, THE TRIPLET-P-ONE, AND
CC                    THE TRIPLET-P-TWO WAVEFUNCTIONS.
CC
CC       IF NDO=0....THEN ON THE FIRST SET OF GRAPHS THE THIRD SINGLET-S-
CC                    ZERO STATE IS REPLACED BY THE LOWEST TRIPLET-S-ONE
CC                    STATE, AND ON THE THIRD GRAPH THE THIRD SINGLET-D-TWO
CC                    STATE IS REPLACED BY THE TRIPLET-D-ONE COMPONENT OF
CC                    THE LOWEST TRIPLET-S-ONE STATE.....................
CC
CC                      WAVE FUNCTION PRINT OUT                              CCC
CC      ON THE FIRST  SET  THE COLUMNS ARE X(I), NEXT THE FIRST THREE STATES OF CC
```

```
CC       THE SINGLET S ZERO STATES, THE FIRST OF THE SINGLET P ONE STATES,
CC     THEN THE DERIVATIVE OF THE FIRST SINGLET P ONE STATE(NOT PLOTTED)
CC     THEN COMES THE SECOND AND THIRD  SINGLET P ONE STATES AND THE FIRST OF
CC         THE SINGLET D TWO STATES.
CC     THE SECOND SET OF DATA IS X(I), THE SECOND AND THIRD OF THE SINGLET D-TWO
CC      STATES,  THE LOWEST TRIPLET P- ZERO STATES, AND THEN THE S AND D WAVE
CC     COMPONENTS OT THE MIXED TRIPLET ONE STATS(NOT PLOTTED)
CC    THE THIRD SET OF DATA ARE FOR THE FOLLOWING STATES(ALL NOT PLOTTED). X(I),
CC      THE FIRST TRIPLET P-ONE STATE, THE P COMPONENT OT THE MIXED TRIPLET SPIN
CC      SPIN TWO STATE AND FINALLY THE S AND D COMPONENTS OF THE SECOND MIXED
CC   TRIPLET SPIN ONE CASE.
CC                                                                          CC
CC                                                                          CC
CC                              GRAPHS FOR NCOMP=2                          CC
CC  FOR THE  GRAPHS SEE THE IDENTIFICATIONS LISTED UNDER THE
CC    SUBROUTINE SETWGG(II).
CC                                                                          CC
CC.............NOTE IF NWAVE =0 THEN PROGRAM CALLS WAVEF.....................
CC                                                                          CC
CC
      NPM=-1
      NMAT=1
      NWAVE=1
      AM=3.0
      DAM=0.20
      AMO=15.0
      AMOF=5.0
      RMASS=7.00
      NCOMP=0
      RAT=-0.33
      XVAL=3.095
      YVAL=3.095
      DSWIDE=5.0E-06
      DRWIDE=3.8E-06
      DTSOS=0.587
      AMCE=1.00
      AMCM=1.00
      ICURVE(1)=1
      ICURVE(2)=2
      ICURVE(3)=3
      ICURVE(4)=4
      ICURVE(5)=5
      ICURVE(6)=6
      NDO=1
      NUMB=10
      NINT=32
      SLAEQ=1.0
      AK=0.30
```

```
      CAM=2.3850
      DTPSO=0.454
      DPTPZ=0.136
      NOPLOT=0
      DPTPO=0.044
      NEXP=10
      OUTF=1
      OPTF=1
      OWTF=1
      OXTF=1
      AMF=2.00
      PMES=0
      END
      SUBROUTINE PHMASS
      COMMON/PMPAR/PMES,NPM
      COMMON/PMASS/PMA,PMB,PMC,PME,PMF,PMG,PMH,ACB,
     /ABC,AWA,AWB,AWC,AWD,AWE,AWF,AWG,AWH,AWK,AWL
CC    IF NPM=-1  THEN WE USE THE MASSES GENERATED BY THE PROGRAM TO
CC               CALCULATE  THE MATRIX ELEMENTS, BUT WE USE THE REAL
CC               MASSES IN THE PHASE SPACE FACTOR FOR THE  THE TRANSITION
CC               RATES.
CC    IF NPM= 0  THEN WE USE THE MASSES GENERATED BY THE PROGRAM
CC               THROUGHOUT THE CALCULATION.
CC    IF NPM=+1  THEN WE USE THE REAL MASSES THROUGHOUT THE PROGRAM,
CC               MATRIX ELEMENTS AND THE PHASE SPACE FACTOR
      PMA=2.980/ACB
      PMB=3.580/ACB
      PME=3.414/ACB
      PMC=3.507/ACB
      PMF=3.551/ACB
      IF(NPM)3,1,1
    1 GO TO 2
    3 CONTINUE
      AWA=(PMH*PMH-PMC*PMC)/(2.0*PMH)
      AWB=(PME*PME-PMG*PMG)/(2.0*PME)
      AWC=(PMH*PMH-PMC*PMC)/(2.0*PMH)
      AWD=(PMC*PMC-PMG*PMG)/(2.0*PMC)
      AWE=(PMH*PMH-PMF*PMF)/(2.0*PMH)
      AWF=(PMF*PMF-PMG*PMG)/(2.0*PMF)
      AWG=(PMG*PMG-PMA*PMA)/(2.0*PMG)
      AWH=(PMH*PMH-PMB*PMB)/(2.0*PMH)
      AWK=(PMB*PMB-PMG*PMG)/(2.0*PMB)
      AWL=(PMH*PMH-PMA*PMA)/(2.0*PMH)
    2 CONTINUE
      END
      SUBROUTINE SETWGG(II)
      COMMON/WGG/WGG(21,30),RSPZ,RDPZ,RPZS,PRZD,RSPO,RDPO,RPOS,RPOD,
     /RSPT,RDPT,RPTS,RPTD,RSSZ,RDSZ,RSZS,RSZD,RATEA,RATEB,
```

```
      /RATEC,RATED
CC                                                                      CC
CC                                                                      CC
CC                                                                      CC
CC                                                                      CC
CC   THIS ROUTINE PICKS OUT THE  PARTICULAR DECAY RATES THAT ARE TO BE  CC
CC PLOTTED ON GRAPHS THAT ARE CALLED BY THE YPLOTT SUBROUTINE.          CC
CC                                                                      CC
CC     ON THESE  GRAPHS  WE HAVE THE FOLLOWING DISTRIBUTION,
CC       ON THE FIRST GRAPH WE HAVE PLOTTED THE FIRST FIVE WGG,S LISTED  C
CC     BELOW.     ON THE SECOND GRAPH WE HAVE THE NEXT THREE WGG,S LISTED CC
CC     ON THE THIRD GRAPH WE HAVE PLOTTED THE NEXT FOUR WGG"S LISTED    CC
CC    BELOW.  ON THE FOURTH AND LAST GRAPH WE HAVE THE LAST TWO WGG"S LISTED CC
CC                                                                      CC
CC                                                                      CC
CC       RSPZ  IS THE 2(TRIPLET S ONE) TO THE 1(TRIPLET P ZERO) CASE.
CC       RSPO IS THE 2(TRIPLET S ONE) TO 1(TRIPLET P ONE) TRANSITION.
CC       RSPT IS THE 2(TRIPLET S ONE ) TO 1(TRIPLET P TWO) TRANSITION.
CC       RPZS, RPOS AND RPTS ARE THE CORRESPONDING ONES FROM
CC                  THE ABOVE P-STATES TO THE 1(TRIPLET S ONE).
CC       THE SIMILIAR EXPRESSIONS WITH S REPLACED BY A D MEANS
CC             THE CORRESPONDING TRANSITIONS TO OR FROM THE
CC             ONE OR TWO (TRIPLET SPIN ONE) CASE WITH L=2.
CC       RSSZ IS THE TRANSITION FROM THE 2(3-S-1) TO THE1(1-S-O) STATE.
CC       RSZS IS THE TRANSITION FROM 2(1-S-O) TO 1(3-S-1) STATE.
CC       RATEA IS THE TRANSITION FROM 2(3-S-1) TO 2(1-S-O) STATE.
CC       RATEC IS THE TRANSITION FROM THE 1(3-S-1) TO THE 1(1-S-O) CASE.
CC       THE FOLLOWING ARE THE CORRESPONDING  TRANSITIONS TO OR FROM THE
CC           CORRESPONDING TRIPLET -D - ONE STETES,
CC             RDSZ,RSZD,RATEB,RATED.
CC                                                                      CC
CC                                                                      CC
      WGG(II,1)=RSPZ
      WGG(II,2)=RSPO
      WGG(II,3)=RSPT
      WGG(II,4)=0.0215*10**(-3.0)
      WGG(II,5)=0.0107*10**(-3.0)
      IF(RATEA)1,1,2
    1 RATEA=0.40*10**(-10.0)
    2 CONTINUE
      WGG(II,6)=ALOG10(RATEA)
      IF(RATEC)3,3,4
    3 RATEC=10**(-11.0)
    4 CONTINUE
      WGG(II,7)=ALOG10(RATEC)
      IF(RSZS)5,5,6
    5 RSZS=0.40*10**(-10.0)
    6 CONTINUE
```

```
      WGG(II,8)=ALOG10(RSZS)
      WGG(II,9)=RPOS
      WGG(II,10)=RPTS
      WGG(II,11)=0.050*10**(-3.0)
      WGG(II,12)=0.075*10**(-3.0)
      WGG(II,13)=0.075*10**(-3.0)
      WGG(II,14)=RPZS
      END
      SUBROUTINE WAVEF(X,PSI,NINT)
      DOUBLE PRECISION X
      DIMENSION X(32),PSI(32,20)
CC
CC...........   PROGRAM  WAVEF CAN BE USED TO PUNCH OUTPUT INTO ..........
CC.............CARDS OF THE WAVEFUNCTIONS COMPUTED BY ORTHOG..........
CC.............IN PRINCIPLE WAVEF COULD BE USED TO GET OUTPUT OF ..........
CC.............OTHER MATERIAL IN ORTHOG COMPUTED UP TO THIS POINT.....
CC.............NAMELY UP TO ORTHOG CARD NUMBER1121..........
CC
      DO 2 I=1,NINT
      WRITE(5,1)X(I),PSI(I,1),PSI(I,2),PSI(I,3)
    1 FORMAT(12X,4E15.6)
    2 CONTINUE
      END
      SUBROUTINE ELEMENT(X,I)
      COMMON/WGG/WGG(21,30),RSPZ,RDPZ,RPZS,PRZD,RSPO,RDPO,RPOS,RPOD,
     /RSPT,RDPT,RPTS,RPTD,RSSZ,RDSZ,RSZS,RSZD,RATEA,RATEB,
     /RATEC,RATED
      DIMENSION X(32)
      COMMON/MATRIX/CONE,CTWO,CTHR,CFOU,CFIV,CSIX,CSEV,CEIG,CNIN,CTEN,
     /CELE,CTWV,CTNE,CFNE,CFFE,CSNE
CC
CC
CC.............IF NMAT=0 THEN ONE CAN MODIFY THE MATRIX ELEMENTS OF THE.........
CC.............CALCULATION BY MEANS OF THE FACTORS CONE ETCTO BE..............
CC..............ANY FUNCTION OF THE INTERNAL MOMENTUM X(I) AND ............
CC.............OTHER CONSTANTS WHICH CAN BE INSERTED EXTERNALLY.......
CCC............ IT CAN ALSO BE USED TO PRINT OUT THE VALUE OF THE.......
CC.............MATRIX ELEMENT ITSELF,ALTHOUGH THIS IS DONE ITERATIVELY..........
CC.............. SO THAT ONE GETS EVERY TERM IN THE INTEGRATION ...
CC.............ROUTINE. THIS IS DONE TO CHECK THE INTEGRATION....
CC.............PROCEDURE AND OTHER INTERMEDIATE RESULTS..........
CC
CC
      MMAT=-1
      IF(MMAT)1,2,3
    2 CONTINUE
      CONE=1.0
      CTWO=1.0
```

```
      CTHR=1.0
      CFOU=1.0
      CFIV=1.0
      CSIX=1.0
      CSEV=1.0
      CEIG=1.0
      CNIN=1.0
      CTEN=1.0
      CELE=1.0
      CTWV=1.0
      CTNE=1.0
      CFNE=1.0
      CFFE=1.0
      CSNE=1.0
    3 CONTINUE
    1 CONTINUE
      WRITE 5,I,RSPZ,RATEA,RATEC
    5 FORMAT(3X,1I4,3E15.6/)
      END
```

```
0.211068726530635220070323832260-01 0.530370973397610549801512510900-01
0.111223048437012445800224704500+00 0.112845824655176078078574120540+00
0.273398752901179114558053494790+00 0.150824523158723628358897930330+00
0.507755460397669376483711798400+00 0.162791336311942126686867401880+00
0.814421367610832899479577355230+00 0.151856410604663669874682374250+00
0.119355999096479203214245400920+01 0.125936258232099794354463029790+00
0.164537329739714440128498508070+01 0.941989305849645317932854044650-01
0.217010279385679998723230691770+01 0.640788141334410798623791911660-01
0.276803037643665158180773817280+01 0.398456458245284030685826464290-01
0.343947921984755245461076725350+01 0.227241364420953881461517140200-01
0.418481477448765574457013799240+01 0.119122354893055369147185204210-01
0.500444589556563127113463386670+01 0.574831064380665700668678347 10-02
0.589882611848984319108532199300+01 0.255593490701437950942561655930-02
0.686845509250623008878184109940+01 0.104781228211460596729318007370-02
0.791388018477499758323865543380+01 0.396170004817089372461265053360-03
0.903569826821440482480788295970+01 0.138162062901931209084723222940-03
0.102345577080578834289312531440+02 0.444397608439257243945378045120-04
0.115111605640346479275649208380+02 0.131804665685672437551106817960-04
0.128662650273698293415573189330+02 0.360336936646654482712842944050-05
0.143006881144291036883931172920+02 0.907604328516954032332156631220-06
0.158153086418496730822978750990+02 0.210492690623815202669713209190-06
0.174110705114662393112032857620+02 0.449187564805032992153788588890-07
0.190889863373352472421705057990+02 0.881297288802648284367715818930-08
0.208501414517786565045772338060+02 0.158829738991572736262990022400-08
0.226956983327259390311675900090+02 0.262677884829491417229989497340-09
0.246269015008710592335891521570+02 0.398218573048395934725754789270-10
0.266450829424519168135856399210+02 0.552719967002581891049829453950-11
0.287516681220135784453828954020+02 0.701467442223044113857069612790-12
0.309481826595921531140185879040+02 0.812855624829291335492854268200-13
```

```
                0.3323625975867251280200248230 3D+02 0.858730494786093232525006027 52D-14
                0.3561764848542608042824918733 0D+02 0.825692287568621135842381550 64D-15
                0.3809422301661817388315711466 4D+02 0.721307609095154070014546003 29D-16
               .FLUSHFIGS .ENDPAGE
RMAIMO NTE DBFRMN&F LMNI TLNM^_
```

Bibliographical Essay

The discussion in chapter 1 is only a general outline of the development of the main facts of elementary particle physics and is geared to motivating the reason for this work. In the past few years several elementary books on elementary particles have been published that discuss the material in chapter 1 in much more detail, e.g.

[1] Polkinghore, J.C., "The Particle Play," W.H. Freeman, San Francisco, Calif. (1979).

[2] Trefil, J., "From Atoms to Quarks: An Introduction to the Strange World of Particle Physics," Scribners, New York (1982).

[3] Segre, E., "From X-rays to Quarks" W.H. Freeman, San Francisco, Calif. (1980).

The book by Segre [3] deals extensively with the early development of nuclear physics, quantum mechanics and particle physics. The most recent developments are not treated as extensively as the earlier pre-world war II material. The other two books deal with the recent developments more extensively and are easy to read. The book [2] by Trefil contains somewhat more detail than that of Polkinghore [1] and is somewhat more recent. More theoretical surveys of many of the topics discussed in these books and in chapter 1 are to be found in the following books by Roman, Sakurai, Gell-Mann and Ne'eman, and Lichtenberg.

[4] Roman, Paul, "Theory of Elementary Particles," revised edition, Interscience, New York (1961).

[5] Sakurai, J.J., "Invariance Principles in Elementary Particle Physics," Princeton University Press, Princeton, New Jersey (1964).

[6] Gell-Mann, M. and Y. Ne'eman, "The Eight-Fold way," W.A. Benjamin, New York (1964).

[7] Lichtenberg, D.B., "Unitary Symmetry and Elementary Particles," second edition, Academic Press, New York (1978).

The book by Roman was written before the development of unitary symmetry but contains a very detailed discussion of isospin, strangeness, and the discrete symmetries of elementary particles. The book by Gell-Mann and Ne'eman is a collection of basic papers in unitary symmetry published in or before 1964. The book by Sakurai, although written about the same time as [6] does not discuss unitary symmetry but covers much the same groups as Roman's book. The book by Lichtenberg is a recent, and readable, textbook on the applications of unitary symmetry to particle physics.

Some of the basic works dealing with special relativity and quantum mechanics can be found translated and reprinted in the following books.

[8] Einstein, A., et al., "Theory of Relativity," Dover reprint of 1923 edition published by Methuen and company, London.

[9] Van der Waerden, B.L., "Sources of Quantum Mechanics," Dover 1968 reprint of a book published by North Holland Publishing Company, Amstersdam in 1967.

[10] Schroedinger, E., "Collected Papers in Wave Mechanics," translated from the second German edition, Blackie and Son limited, London (1928).

Dirac's article on his relativistic wave equation appears in

[11] Dirac, P.A.M., "The Quantum Theory of the Electron," Proceedings of the Royal Society of London, Series A, volume 118, p. 351 (1928).

and is reprinted as paper 1 in

[12] Hofstadter, R., editor, "Electron Scattering and Nuclear and Nucleon Structure," W.A. Benjamin, New York (1963).

Two recent discussions on the history of the Dirac equation are

[13] Kragh, Helge, "The Genesis of Dirac's Relativistic Theory of Electrons," Arkive for the History of Exact Sciences, volume 24, p. 31 (1981).

[14] Moyer, Donald Franklin

(i) "Origins of Dirac's Electron 1925-1928," American Journal of Physics, volume 49, p. 944 (1981).

(ii) "Evolution of Dirac's Electron 1928-1932" ibid, volume 49, p. 1055 (1981).

(iii) "Vindication of Dirac's Electron 1932-1934" ibid, volume 49, p. 1120 (1981).

The following article

[15] Nakaniski, N., "The Bethe-Salpeter Equation," supplements to Progress in Theoretical Physics #43 (1969).

is a review of work dealing with the Bethe-Salpeter equation up to 1969. Reference [15] contains an extensive bibliography of papers dealing with the Bethe-Salpeter equation, and one of the original purposes of this review article was to compile and publish this bibliography in one place. The first attempts to construct a relativistic two body wave equation go back to G. Breit's work in 1929 on the hyperfine structure of helium.

[16] Breit, G., "The Effect of Retardation on the Interaction of Two Electrons," Physical Review, volume 34, p. 553 (1929).

Discussions of this early work of Breit and other authors can be found in the Encyclopedia of Physics article by H.A. Bethe and E.E. Salpeter, which was published separately in book form.

[17] Bethe, H.A. and E.E. Salpeter, "Quantum Mechanics of One and Two Electron Systems," Academic Press, New York (1957).

Also discussed in [17] is the more recent work of these two authors and others. The more recent work begins with Bethe and Salpeter's basic work on the relativistic two body wave equation.

[18] Salpeter, E.E. and H.A. Bethe, "A Relativistic Equation for Bound State Problems," Physical Review, volume 84, p. 1232 (1951).

A great deal of the literature on the Bethe-Salpeter equation deals with radiative corrections to positronium. Some of the first articles on this system dealing with radiative corrections are

[19] Karplus, Robert and Abraham Klein, "Electrodynamic Displacement of Atomic Energy Levels III. The Hyperfine Structure of Positronium," Physical Review, volume 87, p. 848 (1952).

[20] Fulton, Thomas and Robert Karplus, "Bound State Corrections in Two Body Systems," Physical Review, volume 93, p. 1109 (1954).

[21] Fulton, Thomas and Paul C. Martin, "Two Body Systems in Quantum Electrodynamics, Energy Levels of Positronium," Physical Review, Volume 95, p. 811 (1954).

A survey of work on the positronium system can be found in

[22] Stroscio, Michael A., "Positronium: A Review of the Theory," Physics Reports, volume 22C, p. 215 (1975).

This review also contains a good introduction to the Bethe-Salpeter equation.

The major advance in the study of relativistic multi-particle systems began with [17] and was based upon development of a covariant renormalization scheme by F.J. Dyson, R.P. Feynman, J. Schwinger and others in the late 1940's. Many of the basic articles detailing this development can be found in the collection

[23] Schwinger, J., "Quantum Electrodynamics," Dover, New York (1958)

Also reprinted in [23] is one of the early articles dealing with radiative corrections to positronium referred to previously, namely [18].

In determining higher order radiative corrections to the positronium system, a more systematic development of the Bethe-Salpeter equation and related perturbation expansions was needed. Work along this line can be found in

[24] Feldman, Gordon, and Thomas Fulton and John Townsend, "Simplified Bethe-Salpeter Equation for Positronium," Annals of Physics [N.Y.], volume 82, p. 501 (1974).

[25] Cung, V.K., Thomas Fulton, Wayne W. Repko, and Donald Schnitzler, "Complete Reduction of the Fermion - Antifermion Bethe-Salpeter Equation with Static Kernel I," Annals of Physics (N.Y.), volume 96, p. 264 (1976).

[26] Cung, V.K., Thomas Fulton, Wayne W. Repko, Alan Schaum, and Alberto Devoto, "Complete Reduction of the Fermion - Antifermion Bethe-Salpeter Equation with Static Kernel II," Annals of Physics (N.Y.), Volume 98, p. 516 (1976).

[27] Repko, Wayne W., "Exact Relativistic Hamiltonian Arising from the two Fermion Bethe-Salpeter Equation with a Static Kernel," Unpublished report prepared for Aspen Workshop on Quantum Electrodynamics (1976).

This thesis makes use of results in [26] and [27] as the starting point for this numerical investigation of the Bethe-Salpeter equation with a static kernel and harmonic oscillator potential. The review article of H. Grotch and D.R. Yennie, referred to in the text, is

[28] Grotch, H. and D.R. Yennie, "Effective Potential model for Calculating Nuclear Corrections to the Energy Levels of Hydrogen," Reviews of Modern Physics, volume 41, p. 350 (1969).

The above citations mainly deal with background material and serve to place this work in the context of current elementary particle physics development. Most of the theoretical calculations dealing with the charmonium system make use of the nonrelativistic Schroedinger wave equation with a potential of the form $V(r) = ar + b/r$. The most extensive treatment of this standard model is

[29] Eichten, E. and K. Gottfried, T. Kinoshita, K.D. Lane, and T.M. Yan, "Charmonium: The Model," Physical Review, third series, volume D17, p. 3090 (1978).

[30] Eichten, E. and K. Gottfried, T. Kinoshita, K.D. Lane, and T.M. Yan, "Charmonium: Comparison with Experiment," third series, volume D21, p. 203 (1980).

In these articles the authors make a great many predictions for the charmonium system involving mass spectra and various types of decay

rates for all of the allowed states. One item that hampered this earlier work was the incorrect identification of the ^1S_o states. The radiative decay rates predicted by these authors are also incorrect, being off by a factor of two or three from the experimentally determined decay rates.

In the following work, Hoestler and Repko tried to see how a proper treatment of relativistic effects would affect a charmonium-like system.

[31] Hoestler, Levere C. and Wayne W. Repko, "Bethe-Salpeter Equation with Instantaneous Harmonic Oscillator Exchange," Annals of Physics (N.Y.), volume 130, p. 329 (1980).

In [31] they assumed a harmonic potential and calculated the spectra and $\Gamma((3095) \rightarrow e^+ + e^-)$. Using these two quantities they determined, in exactly the same way as that described in chapter 2, the quark mass and the harmonic coupling constant. The convergence for an initially purely linear potential (which occurs when n = o) to a purely harmonic potential level structure (limiting case for $m \rightarrow \infty$) was shown to be very rapid. They made no attempt to determine wavefunctions or other quantities that could, in principle, be calculated for this system.

In our method of solution we choose to make use of the fact that a limiting form of the equations possessed Airy functions as their solutions. To make use of this fact we needed accurate values of these functions. To accomplish this we made use of the work of Yudell L. Luke [32].

[32] Luke, Yudell L., "Mathematical Functions and their approximations," Academic Press, New York (1975).

The method used in this book to determine the Airy function is to expand Bessel functions of orders $\pm$ 1/3 and $\pm$ 2/3 in a series of Chebyshev polynomials. The appropriate coefficients are found in [32] on pages 351 and 352 (for the $\pm$ 1/3 cases) and on pages 354 and 355 (for the $\pm$ 2/3

cases). The definitions and properties of the Chebyshev polynomials used here are given on pages 453 to 464 of [32]. The recurrence relations used in the Airy function computations are given on pages 453 and 459 of [32].

As a check on the wavefunctions at m = o and L $\neq$ 0 one can compare our results in figures 7 and 8 with those shown in

[33] Maurone, Philip A. and Alain J. Phares, "The Linear Potential Wavefunctions," Journal of Mathematical Physics, Volume 21, p. 830 (1980).

The method of solutions employed in [33] was developed only recently and can be used to solve recurrence relations that result from series solutions of differential equations involving three or more terms. This method of solution is described in the following articles,

[34] Antippa, Adel F. and Alain J. Phares, "General Formalism Solving Linear Recursion Relations," Journal of Mathematical Physics, Volume 18, p. 173 (1977).

[35] Antippa, Adel F. and Alain J. Phares, "The Linear Potential: A Solution in Terms of Combinatorics Functions," Journal of Mathematical Physics, volume 19, p. 308 (1978).

[36] Phares, Alain J., "The Energy Eigenvalue Equation for the Linear Potential," Journal of Mathematical Physics, volume 19, p. 2239 (1978).

[37] Antippa, Adel F. and Toan Nguyen Ky, "The Linear Potential Eigenenergy Equation. I: the coefficients K_n (3ℓ')," Canadian Journal of Physics, volume 57, p. 417 (1979).

and some unpublished Villanova University reports referred to in these articles.

The other limiting case of our equations is the three-dimensional Schroedinger equation with a harmonic oscillator potential. This equation is discussed in several places. Two convenient sources for this equation are

[38] Fong, Peter P., "Elementary Quantum Mechanics," Addison Wesley, Reading, Mass. (1962).

[39] Morse, P.M. and H. Feshbach, "Methods of Theoretical Physics," volume 2, p. 1662, McGraw-Hill, New York (1953).

Some of the numerical data used, e.g. masses, spins, etc. of the various charmonium states can be found in the periodically issued Particle Data tables compiled by the Particle Data Group at Berkeley and Cern. The last published compilation is

[40] Particle Data Group, "Review of Particle Properties," Reviews of Modern Physics, volume 52 (April 1980).

The more recent work dealing with the $^1S_0$ states as well as the radiative decays of various charmonium states can be found in

[41] Gaiser, John E., "Charmonium Spectroscopy from Inclusive Photons in J/ψ and ψ' Decays," preprint SLAC-PUB. 2887, invited talk presented at the XVIIth Rencontre de Moriond: Workshop on New Flavours, Les Arcs, France, January 24-30, 1982.

[42] Konigsmann, Kay C., "J/ψ Radiative Transitions to Pseudoscalars," preprint SLAC-PUB. 2910, invited talk presented at the XVIIth Rencontre de Moriond: Workshop on New Spectroscopy, Les Arcs, France, March 20-26, 1982.

Reference [41] includes a compilation of theoretical models and their predictions of the charmonium system properties along with the more recent Crystal Ball experimental results. [41] is the only source that I know that contains information on the radiative decay rates of the various triplet P states.

This work explicitly assumes the existence of quarks as constituents of elementary particles. Even though the quark model is discussed in many books and articles, e.g. [1], [2], [3], [7], [29], and [30] above, the only experiment giving any positive indication for the existence of free quarks is that by LaRue, et. al.,

[43] La Rue, G.S., and W.M. Fairbank and A.F. Hebard, "Evidence for the Existence of Fractional Charge on Matter," Physical Review Letters, volume 38, p. 1011 (1977).

Additional information on the quark model can be found in R.P. Feynman's book

[44] Feynman, Richard P., "Photon-Hadron Interactions," W.A. Benjamin, Reading, Mass. (1972).

In particular the ratio $(e^+ + e^- \rightarrow \text{hadrons})/(e^+ + e^- \rightarrow \mu^+ + \mu^-)$ is calculated in lecture 35 of this book.

A derivation of the radiative decay width given in appendix II.4 can be found in the book by Sakurai.

[45] Sakurai, J.J., "Advanced Quantum Mechanics," Addison-Wesley, Reading, Mass. (1967).

The two-gluon and two-gamma widths for the 1^1S_0, 2^1S_0, 1^3P_0, and 1^3P_2 states were computed from equations given in

[46] Bakbieri, R., R. Gatto, and R. Kogerler, "Calculation of the Annihilation Rate of P-wave quark anti-quark Bound States," Physics Letters, Volume 60B, p. 183 (1976).