ABSTRACT

FREE PERIODIC VIBRATIONS OF CONTINUOUS SYSTEMS GOVERNED BY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by Paul Thomas Blotter

Approximate expressions are obtained for the frequency-amplitude relations and for the nonlinear mode shapes for a general class of continuous systems governed by nonlinear partial differential equations. The formulation applies to problems in one space variable and one time variable, in which nonlinearities in the displacement and its spatial derivatives are involved. Some typical systems in this general class include strings, circular membranes, beams and circular plates on nonlinear elastic foundations or with immovable boundary supports vibrating at large amplitudes, as well as elastic media with nonlinear constitutive equations.

Two different techniques are developed and used. The first involves a modified perturbation approach. The second approach involves a linearization using ultraspherical polynomials.

The general expressions obtained are applied to several dynamic systems. Numerical results are cataloged in the form of graphs and tables and compared with those obtained by other authors using different methods. The results include those for several nonlinear continuous systems whose solutions are not available in the literature.

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements

DOCTOR OF PHILOSOPHY

Department of Metallurgy, Mechanics and
Materials Science

1968

ACKNOWLEDGMENTS

To all the individuals who have helped and encouraged me during the research and writing of this dissertation, I offer my sincere thanks. Foremost appreciation is extended to my committee chairman, Dr. David H. Y. Yen, who supplied the original motivation and continued to give generously of his time and valuable counsel throughout the study. The guidance of other committee members, Dr. R.W. Little, Dr. W.A. Bradley, Dr. G.E. Mase and Dr. N.J. Hills is also greatly appreciated.

Grateful thanks are extended to the National Science Foundation and the Department of Metallurgy, Mechanics and Materials Science of Michigan State University for providing funds to finance my graduate program. Also, the sacrifice by my wife and three boys is acknowledged.

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I. INTRODUCTION

The primary objective of this research is to develop approximate expressions for the frequency-amplitude relations and for the nonlinear mode shapes for a general class of nonlinear continuous systems, the free periodic motions of which are governed by nonlinear partial differential equations. The formulation is sufficiently general and applies to a wide class of nonlinear elastic continua problems in one space variable and one time variable. The governing equation of motion is assumed to contain a restoring function that is nonlinear in the transverse displacement and its spatial derivatives. Some typical systems in this general class, to which the analytical results are eventually applied, include continuous structures such as strings, membranes, beams, and plates on nonlinear elastic foundations, or with immovable boundary supports vibrating at large amplitudes, as well as elastic media with nonlinear constitutive relations.

Essentially two different techniques are developed. The first is a modified perturbation approach and the second involves a linearization using ultraspherical polynomials.

In Chapter II the modified perturbation method is presented. The historical development of the perturbation technique as applied to nonlinear dynamic systems and other more recent contributions are briefly discussed. The fundamental differences of the method developed in this paper are compared with those approaches used by

other authors. Since a more detailed introduction to the analytical procedure forms part of Chapter II, it is only emphasized at this point that the final integral expressions for frequency and displacement are of a general nature and require only that the type nonlinearity be specified and that the eigenfunctions and eigenvalues for the associated linear problem be known. Both first and second order approximations to frequency-amplitude response and transverse displacement are found. The linear case is taken as the zeroth order.

In Chapter III these general expressions are applied to particular continuous systems. Solutions are found for uniform strings, prismatic and variable cross-section beams, and circular plates resting on nonlinear elastic foundations. As examples of dynamic systems having a restoring function that is nonlinear in the derivatives of the transverse displacement, solutions for vibrating beams with immovable or springed end supports, circular plates and membranes vibrating at large amplitudes, and beams made of materials with nonlinear constitutive equations are also considered. The applications demonstrate the relative simplicity of the perturbation method developed here as compared with other existing perturbation approaches.

The particular results obtained in Chapter III are further specialized in Chapter IV by specifying various boundary conditions for the above mentioned systems, which then lead to the linear eigenfunctions and eigenvalues. Numerical results are then presented. These numerical approximations are compared with those obtained by other authors using different methods. The results also include those for several nonlinear systems whose solutions are not available in the literature.

In Chapter V a method independent of the perturbation theory is presented whereby a class of nonlinear partial differential equations are approximated by equivalent linear partial differential equations with variable coefficients. It is then necessary to solve the linear equations. The linearization is achieved by approximating the nonlinear restoring force over the amplitude span by the linear term of a set of ultraspherical polynomials. This method is an extension of a similar one previously used for problems of a single degree of freedom. General expressions obtained are again applied to the dynamic systems considered in Chapter III and the results are found to be in good agreement with those obtained by the perturbation methods.

A brief summary of results as well as conclusions are contained in Chapter VI.

II. PERTURBATION METHOD

2.1. Introduction and Historical Background

A classical approach for treating nonlinear partial differential equations is the method of perturbation. The method is reliable in finding an approximation to some unknown solution in the neighborhood of another solution which either is known or can be found with relative ease. The method is generally very cumbersome when applied to nonlinear partial differential equations.

Several variants of the perturbation method have recently appeared in the literature dealing with the nonlinear vibrations of continuous systems. Stoker [1] studied the problem of a tightly stretched elastic vibrating string with fixed end points, embedded in nonlinear restoring springs distributed continuously along its length. He developed the nonlinear infinite degree of freedom problem as an analogue of the one treated in Poincare's theory [41] with finite degree of freedom and governed by nonlinear ordinary differential equations. As a specific application, a procedure customarily followed for treating Duffing's equation was used by Stoker to establish the first order frequency-amplitude relationship in the case of a continuous string on a Duffing-type nonlinear elastic foundation. The partial differential equation of motion was modified slightly by adding frequency terms to both sides of the equation, whereby in effect the homogeneous part was in resonance with the external periodic forcing function. This particular maneuver, however, was avoided in his book [2] published sometime later, where this so-called degenerate case was solved by assuming a linear mode shape as a first approximation and following the Lindstedt [2] perturbation scheme, terms contributing to aperiodic motion were made to disappear. (It is interesting to note that higher order free periodic vibrations described by Stoker's equation of motion with the forcing function evanescent exist only if the coefficient on the cubic nonlinearity is identically zero.)

Han [3] made a somewhat different modification of the classical Lindstedt method to study a simply supported beam on a nonlinear elastic foundation vibrating at amplitudes compatible with the assumptions of the small displacement theory. First order terms were obtained through the usual concept of balancing coefficients of equal powers of a perturbation parameter. To obtain second and higher order terms would require additional series expansions involving the perturbation parameter. The approach would become unwieldy when applied to more complicated systems.

Carrier [5] applied the perturbation method to study a vibrating string having fixed ends and at displacements sufficiently large to induce a variable tension. Chu and Herrmann [6] solved the nonlinear coupled equations of motion of a vibrating rectangular elastic plate with hinged immovable edges. Eringen [7], along with Chobotov and Binder [8], developed an analogous procedure in order to solve the coupled equations describing the vibration of membranes at large amplitudes.

Keller and Ting [9] presented another perturbation approach to nonlinear problems. Rather than substituting power expansions of some small parameter directly and equating coefficients of like powers, they used Taylor's series and repeated differentiations to establish a system of inhomogeneous linear equations. Orthogonality conditions of the solution of the homogeneous equation and the inhomogeneous parts of equations that are necessary for the existence of solutions of the system of equations then yield higher order approximations for frequency-amplitude relations and mode configurations. Evensen [10] has recently applied the approach developed by Keller and Ting to obtain first order frequency-amplitude relations for uniform beams with clamped-clamped and clamped-supported immovable boundary conditions.

The perturbation scheme considered in this section is a generalization of an approach initially developed by McQueary and Clark [4], who determined first and second order approximations to the nonlinear frequency and mode shape of a continuous string and first order approximations to a membrane, both supported by nonlinear elastic foundations. The approach was later followed by Mack and McQueary [11] to obtain second order results for a membrane on a Duffing-type nonlinear elastic foundation. The set of recursive inhomogenous linear equations, found through the usual power expansion technique or by the method of Keller and Ting, is solved by series expansions in the product space of linear spatial eigenfunctions and trigonometric time functions. First and second order frequency-amplitude relations and mode shapes for nonlinear dynamic systems are found by substituting this general series type solution into the linear recursion formulae and using the orthogonality properties of both the spatial and time functions. No restrictions are made a priori to limit th

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to limit the procedure to a particular type of vibrating elastic continua, such as a beam, plate, membrane or string. Furthermore, the expressions developed allow a general nonlinear function of the displacement and its derivatives and are applicable to a broad class of dynamic systems to be solved.

The final results are explicit once a knowledge of the spatial eigenfunctions of the associated linear problem, along with the linear frequencies is available. Particular applications of the general solutions are made to vibrating elastic continua on nonlinear foundations, continua experiencing large deflections, systems with immovable supports and materials having nonlinear constitutive equations.

2.2 General Perturbation Method for a Class of Nonlinear Partial Differential Equations.

A method is developed to determine periodic solutions and frequency-amplitude relations for equations of motion governing free vibrations of nonlinear continuous systems. The dynamic system includes a general type restoring force nonlinear in the displacement function and its spatial derivatives.

Consider the periodic motion governed by the following nondimensional equation

$$L_x u + \omega^2 u_{tt} + \epsilon \sum_{j=1}^{M} \alpha_j N_j u = 0 \tag{2.2.1}$$

where L_x is an autonomous* linear differential operator of order 2n, u is a dependent function of the spatial variable x and time t, ω^2 is a frequency parameter, ϵ is a small parameter which depends upon the physical constants of the system and either occurs naturally or is artificially introduced. The α_j are coefficients dependent upon x and the N_j are nonlinear autonomous differential operators given by

$$N_j u = f_j \; (u, u_x, u_{xx} \text{--------} u_\xi) \tag{2.2.2}$$

where f_j is a polynomial of finite degree in u, u_x, u_{xx} -----u_ξ with the notation that $u_\xi \equiv u_{\underbrace{xxx\ldots}_{2n}}$.

It will be assumed that the operator L_x is self adjoint for every t in the space of functions defined by the homogeneous boundary conditions

$$\begin{aligned} D_i u(0, t) &= 0 \qquad & i &= 1, 2 \text{-------} p \\ D_j u(L, t) &= 0 \qquad & j &= p+1, \text{-----} 2n \end{aligned} \tag{2.2.3}$$

---

* An operator in which the time t does not appear explicitly, but only as a differential dt, is called autonomous.

where the D's are also autonomous linear differential operators of degree $< 2n$. For periodicity it is assumed that

$$u(x, t) = u(x, t + 2\pi) \tag{2.2.4}$$

Furthermore, without loss of generality, the origin of the time scale will be selected so that

$$u_t(x, 0) = 0 \tag{2.2.5}$$

i.e. the system has zero initial velocity. The initial configuration will not be specified.

Now to apply the perturbation method the dependent variable and the square of the nonlinear frequency parameter are expanded in infinite series in ϵ as follows

$$u = \sum_{i=0}^{\infty} \epsilon^i u_i = u_0 + \epsilon u_1 + \epsilon^2 u_2 + -- \tag{2.2.6}$$

$$\omega^2 = \sum_{i=0}^{\infty} \epsilon^i \omega_i^2 = \omega_0^2 + \epsilon\omega_1^2 + \epsilon^2\omega_2^2 \tag{2.2.7}$$

where ϵ is the so called perturbation parameter.

Before the above expansions are substituted into the equation (2.2.1), special attention must be given to the nonlinear function f_j. Each f_j may be expressed as a Taylor's series for a function of several variables about a given geometric configuration, which is taken to be the solution or mode shape of the associated linear problem defined by setting ϵ equal to zero in equation (2.2.1). The function f_j is then written as

$$f_j(u, u_x --- u_\xi) = \sum_{k=0}^{N-1} \frac{1}{k!} \Big[(u - u_0)\frac{\partial}{\partial u_0} + (u_x - u_{ox})\frac{\partial}{\partial u_{ox}}$$

$$+ --------- (u_\xi - u_{o\xi}) \Big]^k f_j(u_o, u_{ox} ----- u_{o\xi}) + R_n \tag{2.2.8}$$

where R_n is some remainder defined in the usual sense of Taylor's theorem. The following notation has been implied in equation (2.1.8), namely

$$\frac{\partial f_j\,(u_o, u_{ox}, ---u_{o\xi})}{\partial u_{ox}} = \frac{\partial f_j\,(u, u_x, ----u_\xi)}{\partial u_x}\Bigg|_{\substack{u = u_o \\ u_x = u_{ox} \\ u_\xi = u_{o\xi}}} \qquad (2.2.9)$$

where u_o, u_{ox}, --- $u_{o\xi}$ refer to the linear mode and its derivatives with respect to x.

Replace the displacement function u by the perturbation expansion (2.2.6) and f_j is written

$$f_j(u,\ u_x, ---u_\xi) = f_j\,(u_o + \epsilon u_1 + \epsilon^2 u_2 ---,\ u_{ox} + \epsilon u_{1x} + \epsilon^2 u_{2x} + ---,$$

$$----,\ u_{o\xi} + \epsilon u_{1\xi} + \epsilon^2 u_{2\xi} + ----) \qquad (2.2.10)$$

Upon substituting the function (2..2.10) into (2.2.8) Taylor's expansion about the linear mode u_o follows as

$$f_j = \bar{f}^j + \bar{f}^j_u \left[\epsilon u_1 + \epsilon^2 u_2 + ----\right] + \bar{f}^j_{u_x}\left[\epsilon u_{1x} + \epsilon^2 u_{2x} +---\right)$$

$$+ \bar{f}^j_{u_{xx}} \left[\epsilon u_{1xx} + \epsilon^2 u_{2xx} + ---\right] + -----------$$

$$+ \tfrac{1}{2} \bar{f}^j_{uu}\left[\epsilon u_1 + \epsilon^2 u_2 + ----\right]^2 + \tfrac{1}{2}\bar{f}^j_{u_x u_x}\left[\epsilon u_{1x} + \epsilon^2 u_{2x} + ----\right]^2$$

$$+ \tfrac{1}{2} \bar{f}^j_{u_{xx} u_{xx}} \left[\epsilon u_{1xx} + \epsilon^2 u_{2xx} + ---\right]^2 + --------------$$

$$+ \bar{f}^j_{uu_x} \left[\epsilon u_1 + \epsilon^2 u_2 + ---\right]\left[\epsilon u_{1x} + \epsilon^2 u_{2x} + ---\right] + --------$$

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$$+ \bar{f}^{j}_{uu_{xx}} \left[\epsilon u_1 + \epsilon^2 u_2 + ---\right]\left[\epsilon u_{1xx} + \epsilon^2 u_{2xx} + ---\right] + ---- \qquad (2.2.11)$$

with the notation

$$\bar{f}^{j}_{uu_x} = \left. \frac{\partial^2 f_j (u, u_x, ---u_\xi)}{\partial u \, \partial u_x} \right|_{\begin{array}{l} u = u_o \\ u_x = u_{ox} \\ u_\xi = u_{o\xi} \end{array}}$$

being understood.

Return now and consider the equation of motion in its entirety. Substitute the expansions (2. 2. 6) and (2. 2. 7) into equation (2. 2. 1), remembering that f_j have already been expanded as given in equation (2. 2. 11). Collecting coefficients of like powers of ϵ and equating these to zero lead to a system of equations as follows

$$\epsilon^0 : \; L_x u_o + \omega_o^2 u_{ott} = 0 \qquad (2.2.12a)$$

$$\epsilon^1 : \; L_x u_1 + \omega_o^2 u_{1tt} = -\omega_1^2 u_{ott} - \sum_{j=1}^{M} \alpha_j \bar{f}^j \qquad (2.2.12b)$$

$$\epsilon^2 : \; L_x u_2 + \omega_o^2 u_{2tt} = -\omega_2^2 u_{ott} - \omega_1^2 u_{1tt}$$

$$- \sum_{j=1}^{M} \alpha_j \left[\bar{f}^j_u u_1 + \bar{f}^j_{u_x} u_{1x} + ---- \bar{f}^j_{u_\xi} u_{1\xi} \right] \qquad (2.2.12c)$$

$$\epsilon^3 : \; L_x u_3 + \omega_o^2 u_{3tt} = -\omega_3^2 u_{ott} - \omega_2^2 u_{1tt} - \omega_1^2 u_{2tt}$$

$$- \sum_{j=1}^{M} \alpha_j \left[\bar{f}^j_u u_2 + \bar{f}^j_{u_x} u_{2x} + ----- + \tfrac{1}{2} \bar{f}^j_{uu} u_1^2 + \tfrac{1}{2} \bar{f}^j_{u_x u_x} u_{1x}^2 + ---- \right.$$

$$\left. + \bar{f}^j_{uu_x} u_1 u_{1x} + \bar{f}^j_{uu_{xx}} u_1 u_{xx} + ---- \right]$$

$$\epsilon^i = L_x u_i + \omega_o^2 u_{itt} = N_i \qquad (2.2.12d)$$

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In the recursion formulae (2.2.12), all the displacement functions u_i will be assumed to satisfy the same boundary conditions as given in equations (2.2.3), the same initial condition (2.2.5), and (2.2.4) for periodicity. The initial configurations $u_i(x, 0)$ are assumed unspecified. It is noted that the inhomogeneous part N_i appearing in the $(i + 1)$ th equation only depends upon the solutions u_j; $j<i$ and that $N_o = 0$. It is therefore possible to solve these equations in a sequential manner for u_i and ω_i^2; $i = 0, 1, 2, ---$, beginning with the linear equation (2.2.12a).

The linear problem as governed by equation (2.2.12a) can be solved by the method of separation of variables. It has periodic solutions of the form

$$u_o(x, t) = V_k(x) \cos t \qquad k = 1, 2---- \qquad (2.2.13)$$

where V_k is the k^{th} eigenfunction satisfying the equation

$$L_x V_k - \Omega_k^2 V_k = 0 \qquad (2.2.14)$$

and the boundary conditions (2.2.3). In equation (2.2.14), Ω_k^2 is the corresponding eigenvalue. Upon substitution of (2.2.13) into (2.2.12a) and then comparing it with (2.2.14), it follows that

$$\omega_o^2 = \Omega_k^2 \qquad (2.2.15)$$

It is to be noted that in expressing $u_o(x, t)$ in equation (2.2.12) it has been assumed that the dimensionless t has been scaled so that the period of vibration in the k^{th} mode is 2π. The corresponding frequency is then given through Ω_k^2.

By the assumption of the self-adjointness of the operator L_x on the space of functions defined by (2.2.14), the set of eigenfunctions V_k

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are orthogonal in the sense

$$\int_0^L r(x) V_k(x) V_q(x) dx = 0 \qquad k \neq q \tag{2.2.16}$$

where r (x) is some weighting function. Without loss of generality it will also be assumed that the V_k are normalized such that

$$\int_0^L r(x) V_k(x) V_k(x) dx = 1 \tag{2.2.17}$$

In the above it has been tacitly assumed that the eigenvalues Ω_k^2 are simple. In what follows it will also be assumed that the V_k are complete in the usual sense of eigenfunction expansions [40]. Finally, it will be assumed that all the Ω_k^2 are positive, which is equivalent to the assumption that the operator L_x be positive definite.

In order to solve the nonlinear problem in the neighborhood of the k^{th} linear mode, the linear solution $u_o(x, t)$ is taken in the form

$$u_o(x, t) = A_{1k} V_k(x) \cos t \tag{2.2.18}$$

where A_{1k} is a constant and $\omega_o^2 = \Omega_k^2$. To solve the equations (2.2.12a), (2.2.12b) ---- and in view of the boundary conditions, initial conditions and periodicity, solutions are expressed in the form

$$u_i(x, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{(i)} V_n(x) \cos mt \qquad i = 1, 2 ---- \tag{2.2.19}$$

where $A_{1k}^{(i)} = 0$ for k = 1, 2, ------------, and i = 1, 2, 3, ---------.

These series given in (2.2.19) are substituted into the differential equations (2.2.12) and solved recursively. The only unknowns in the (i + 1) th eq

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th equation are u_i and ω_i^2. All u_j and ω_j^2 $j<i$ are known. As will be illustrated below, in order to solve this one equation with two unknowns, the orthogonality properties of both the spatial eigenfunctions and trigonometric time functions will be utilized.

Substituting $u_1(x,t)$ as given by (2.2.19) into (2.2.12b) yields

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\Omega_n^2 - m^2\omega_o^2)\,A_{mn}^{(1)}V_n(x)\cos mt - \omega_1^2 A_{1k}V_k(x)\cos t$$

$$+\sum_{j=1}^{M}\alpha_j f^{-j} = 0 \qquad (2.2.20)$$

Multiply both sides of equation (2.2.20) by $r(x)\,V_q(x)$ and $\cos pt$ and integrate with respect to x and t, from x to L and from 0 to 2π respectively. By virtue of the orthogonality and normality conditions (2.2.17) of the spatial functions and the orthogonality property

$$\int_0^{2\pi} \cos pt \cos mt = \pi\,\delta_{mp} \qquad (2.2.21)$$

the equation follows as

$$(\Omega_q^2 - p^2\Omega_k^2)A_{pq}^{(1)} - \omega_1^2 A_{1k}\delta_{kq}\delta_{1p} + \frac{1}{\pi}\int_0^L\int_0^{2\pi}\sum_{j=1}^{M} f^{-j}\, r(x)V_q(x)\cos pt\,dxdt = 0$$

$$(2.2.22)$$

where δ_{kp} is the Kronecker delta.

For $p = 1$, $q = k$ and recalling the conditions upon the amplitude parameter (2.2.19), a closed form expression for the first order frequency-amplitude relation is obtained as follows

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$$\omega_1^2 = \frac{1}{\pi A_{1k}} \int_0^L \int_0^{2\pi} \sum_{j=1}^{M} \alpha_j f^{-j} r(x) V_k(x) \cos t \, dx\, dt \qquad (2.2.23)$$

The amplitude parameters $A_{mn}^{(1)}$ other than $A_{1k}^{(1)}$, which is zero by (2.2.15) are also given by equation (2.2.23) as

$$A_{mn}^{(1)} = \frac{1}{\pi(m^2\Omega_k^2 - \Omega_n^2)} \int_0^L \int_0^{2\pi} \sum_{j=1}^{M} \alpha_j f^{-j} r(x) V_n(x) \cos mt \, dx\, dt \quad (2.2.24)$$

Thus the first order nonlinear correction ω_1^2 and u_1 (x, t) are completely determined as

$$u_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{mn}^{(1)} V_n(x) \cos mt \qquad m \text{ or } n \neq 1 \qquad (2.2.25)$$

provided that $m^2\Omega_k^2 - \Omega_n^2 \neq 0$. When the provision fails, the nonlinear problem has no solution unless the integral on the right hand side of equation (2.2.24) also vanishes. This degenerate case may be treated by the method of Keller and Ting [9] and will not be considered here.

The second order corrections for the nonlinear mode shape and frequency may be found in exactly the same manner by solving equation (2.2.12c). Substitution of equations (2.2.18), (2.2.19) for i = 2 and (2.2.25) into equation (2.2.12c), along with equation (2.2.14) yields the following equation for u_2 and ω_2^2.

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (\Omega_n^2 - m^2\omega_0^2) A_{mn}^{(2)} V_n(x) \cos mt - \omega_2^2 A_{1k} V_k(x) \cos t$$

$$-\omega_1^2 \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} p^2 A_{pq}^{(1)} V_q(x) \cos pt + \sum_{j=1}^{M} \alpha_j \Big[f_u^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(x) \cos pt$$

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$$+ f_{u_x}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{dV_q(x)}{dx} \cos pt + \text{------} f_{u_\xi}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)}$$

$$\left. \frac{d^{2n} V_q(x)}{dx^{2n}} \cos pt \right] = 0 \qquad (2.2.26)$$

Now multiply both sides above by $r(x)\, V_i(x)$ and $\cos kt$, and integrate with respect to x and t as before. Owing to the orthogonality conditions (2.2.16) and (2.2.21) the result follows as

$$(\Omega_i^2 - k^2 \Omega_k^2)\, A_{ki}^{(2)} - \omega_2^2 A_{1k} \delta_{ki} \delta_{1k} - \omega_1^2 k^2 A_{ki}^{(1)}$$

$$+ \frac{1}{\pi} \int_0^L \int_0^{2\pi} \sum_{j=1}^{M} \alpha_j \left[f_u^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q(x) \cos pt \right.$$

$$+ f_{u_x}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{dV_q(x)}{dx} \cos pt + \text{------}$$

$$\text{---}\ f_u^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{d^{2n} V_q(x)}{dx^{2n}} \left. \cos pt \right] r(x) V_i(x) \cos kt \; dx\, dt = 0$$

$$(2.2.27)$$

Setting i = k and k = 1 and by the conditions of $A_{1k}^{(i)} = 0$ by (2.2.19) the second order frequency-amplitude relation is determined as

$$\omega_2^2 = \frac{1}{\pi A_{1k}} \int_0^L \int_0^{2\pi} \sum_{j=1}^{M} \alpha_j r(x) \left[f_u^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q V_k \cos pt \cos t \right.$$

$$+ f_{u_x}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q' V_k \cos pt \cos t$$

$$+ f_{u_{xx}}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q'' V_k \cos pt \cos t$$

$$+ -$$

$$+ f_{u_{\xi}}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{d^{2n}V_q}{dx^{2n}} V_k \cos pt \cos t \Big] dx\, dt$$

(2.2.28)

The corresponding amplitude parameters necessary for determining the second order terms are likewise found to be

$$A_{mn}^{(2)} = \frac{1}{(\Omega_n^2 - m^2 \Omega_k^2)} \left[\omega_1^2 m^2 A_{mn}^{(1)} \right]$$

$$\frac{-1}{(\Omega_n^2 - m^2 \Omega_k^2)\,\pi} \int_0^L \int_0^{2\pi} \sum_{j=1}^{M} \alpha_j r(x) \Big[f_u^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q V_n \cos pt \cos mt$$

$$+ f_{u_x}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} V_q' V_n \cos pt \cos mt$$

$$+ -$$

$$+ f_{u_{\xi}}^{-j} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} A_{pq}^{(1)} \frac{d^{2n}V_q}{dx^{2n}} V_n \cos pt \cos mt \Big] dx\, dt = 0$$

(2.2.29)

Third, fourth and higher order approximations can also be found by continuing this procedure, but the results will not be presented.

It is seen that the general expressions developed for frequency-amplitude relations and nonlinear mode shape approximations involve only the linear eigenfunctions and eigenvalues of the associated linear problem. Consequently, the nonlinear results for vibrating elastic media such as beams, strings, membranes and plates are readily obtainable by direct substitution once the linear eigenfunctions and eigenvalues are available. Several applications will be made in the next chapter, following which numerical results will be presented.

2. 3. Mode Configuration and Amplitude for Nonlinear Dynamic Systems.

The definition of normal modes in nonlinear vibrations must be clearly distinguished from that in the linear theory. In the linear theory the terminology "normal solutions" refers to a fundamental set of mathematical solutions, which are orthogonal and span the solution space of the system. The normal mode depicts a geometrical configuration that is maintained throughout one period of oscillation. Nonlinear systems, however, do not have such superposition properties and the geometrical configuration changes with respect to time. For example, it will be shown in a later chapter that the location of the maximum amplitude of vibration of a clamped-supported beam does not even remain fixed in the nonlinear theory.

Normal modes for nonlinear vibrating systems with a finite number of degrees of freedom have been verbally defined by Rosenberg [12] as occuring when (a) all masses execute periodic motion of the same period, (b) all the masses pass through the equilibrium position at the same instant, and (c) at any time t, the position of all masses is uniquely defined by the position of any one of them.

Thein Wah [13] concluded that in a nonlinear continuous system the separation of space and time variables is a sufficient condition for satisfying Rosenberg's criteria. Furthermore, for separable equations of motion it was noted that (a) the commonly called normal modes are mathematically orthogonal, (b)the principle of superposition is not valid, (c) the nonlinear frequencies are functions of the amplitude, and (d) the linear frequency is approached in the limit as the amplitude parameter tends to zero. Although Wah has defined the normal mode for a class of nonlinear systems, his criteria are not applicable for systems considered in this paper since the governing nonlinear equations treated here are not separable.

A less restrictive definition of normal modes for nonlinear dynamic systems is that given by McQueary and Clark [4] as follows: "A nonlinear periodic mode is any state configuration, of a nonlinear system, that is periodically repeated in finite time."

The amplitude parameters introduced in the previous sections are for dimensional lateral displacement. From equations(2.2.6) and (2.2.19) it follows that

$$u(x,t) = \sum_{i=o}^{\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \epsilon^i A_{mn}^{(i)} V_n(x) \cos mt \qquad (2.3.1)$$

Let $\overline{A}$ represent the maximum displacement of the nonlinear mode and denote the spatial point at which this maximum occurs by $x = x_o$ and at time $t = 0$. As will become more evident in later applications, $A_{mn}^{(i)} = C_{mn}^{(i)} A_{1k}^{2i+1}$, where $C_{mn}^{(i)}$ is some constant and A_{1k} is associated with the linear mode such that

$$u_o = A_{1k} V_k(x) \cos t \qquad (2.3.2)$$

Equation (2.3.1) can now be rewritten as

$$\overline{A} = A_{1k} + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \epsilon^i C_{mn}^{(i)} A_{1k}^{2i+1} V_n(x_o) \qquad (2.3.3)$$

For small ϵ the maximum lateral displacement of the linear mode is therefore

$$A = A_{1k} \left| V_k \right|_{max} \qquad (2.3.4)$$

Since the linear eigenfunctions $V_k^{(x)}$ are normalized according to equation (2.2.17),

$$\left| V_1 \right|_{max} \neq 1 \qquad (2.3.5)$$

in general, but (2.3.4) clearly indicates that the amplitude parameter A_{1k} is related to the maximum lateral displacement $\overline{A}$. A_{1k} will be regarded as a normalized amplitude.

It should be pointed out that there is no loss of generality in omitting the term linear in u in this series (3.1.1), for it may be absorbed into the operator L_x. The problem is equivalent to continuous media resting on Duffing-type nonlinear elastic foundations.

The first order frequency amplitude relation follows from equation (2.2.23) established in Chapter II. Upon setting

$$\begin{aligned} M &= 1 \\ f_j &= u^3 && j=1 \\ \bar{f}^j &= u_o^3 = A_{1k}^3 V_k^3 \cos^3 t && j=1 \\ \omega_o^2 &= \Omega_k^2 \end{aligned} \tag{3.1.4}$$

and integrating with respect to time, the result follows as

$$\omega_1^2 = \frac{3}{4} A_{1k}^3 \int_0^L \alpha_1 r(x) V_k^4(x)\,dx \tag{3.1.5}$$

First order amplitude parameters $A_{mn}^{(1)}$ are determined from equation (2.2.24) to be

$$\begin{aligned} A_{1n}^{(1)} &= \frac{3 A_{11}^3 \int_0^L \alpha_1 r(x) V_k^3 V_n(x)\,dx}{-4(\Omega_n^2 - \Omega_k^2)} \\ A_{3n}^{(1)} &= \frac{A_{11}^3 \int_0^L \alpha_1 r(x) V_k^3 V_n(x)\,dx}{-4(\Omega_n^2 - 9\Omega_k^2)} \end{aligned} \tag{3.1.6}$$

with all other $A_{mn}^{(1)} = 0$. The first order approximation to the non-linear mode shape follows as

III. APPLICATION OF PERTURBATION RESULTS

3.1. Restoring Force Nonlinear in Displacement

Several continuous systems governed by partial differential equations containing nonlinear forcing functions in the displacement will be considered in this section. The systems include strings, beams, plates, etc., which are attached to nonlinear elastic foundations of nonlinear springs. Now in the general equation of motion (2.2.1), the nonlinear restoring function is a function of u which may be expressed as

$$\epsilon \sum_{j=1}^{M} \alpha_j N_j u = \epsilon \sum_{j=1}^{M} \alpha_j u^{2j+1} \tag{3.1.1}$$

Observe that in arriving at the series (3.1.1) the nonlinear function has been assumed to be odd in u so that

$$u \, \epsilon \sum_{j=1}^{M} \alpha_j u^{2j+1} > 0 \tag{3.1.2}$$

For practical purposes, the series (3.1.1) is now truncated after the first term. Thus the equation (2.2.1) becomes

$$L_x u + \omega^2 u_{tt} + \epsilon \, \alpha_1 u^3 = 0 \tag{3.1.3}$$

$$u = A_{1k}V_k(x)\cos t + \epsilon\left[\sum_{n=2}^{\infty} A_{in}^{(1)}V_n\cos t + \sum_{n=1}^{\infty} A_{3n}^{(1)}V_n\cos 3t\right]$$

(3.1.7)

In addition to the above, the following functions as defined in Chapter II will be needed to determine the second order frequency-amplitude relations for continua on Duffing-type foundations;

$$\bar{f}^j_{u_x} = \bar{f}^j_{u_{xx}} = - - - - - - - - - = \bar{f}^j_{u_\xi} = 0$$

(3.1.8)

Upon substitution of equations (3.1.4) and (3.1.8) into equation (2.2.28), the second order frequency-amplitude relation follows as

$$\omega_2^2 = 3A_{1k}\left[\sum_{q=2}^{\infty} A_{1q}^{(1)}\frac{3}{4}\int_0^L \alpha_1 rV_k^3 V_q dx + \sum_{q=1}^{\infty} A_{3q}^{(1)}\frac{1}{4}\int_0^L \alpha_1 rV_k^3 V_q dx\right]$$

(3.1.9)

If the first order expressions for amplitude (3.1.6) are substituted into equation (3.1.9), the second order frequency-amplitude relation becomes

$$\omega_2^2 = -\frac{3}{16}A_{1k}^4\left[\sum_{q=2}^{\infty} 9\,\frac{\left(\int_0^L \alpha_1 rV_k^3 V_q dx\right)^2}{\left(\Omega_q^2 - \Omega_k^2\right)} + \sum_{q=1}^{\infty}\frac{\left(\int_0^L \alpha_1 rV_k^3 V_q dx\right)^2}{\left(\Omega_k^2 - 9\Omega_k^2\right)}\right]$$

(3.1.10)

These above expressions for continuous media vibrating on non-linear elastic foundations will be further applied to more specific systems, such as strings, beams, plates, and membranes, in the remaining parts of this section.

3.1.1. Uniform String

Let us consider a taut uniform string, fixed at both ends and attached to an elastic foundation which has both linear and a cubic spring response. It is assumed that the initial tension and subsequent displacements are of such magnitude that the tension is considered constant throughout the motion. The displacements, however, may be sufficiently large relative to the supporting foundation to warrant the inclusion of a small nonlinear term in the restoring function.

The equation of motion is written as

$$-T \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \rho \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + K\bar{u} + K\eta \bar{u}^3 = 0 \qquad (3.1.1.1)$$

where T is the constant tension, ρ mass per unit length, K the linear spring parameter, $K\eta$ the nonlinear spring parameter and u represents the transverse displacement that depends on the spatial variable x and time t.

The equation (3.1.1.1) may be nondimensionalized so that the string is of length π and the period of vibration fixed at 2π by introducing

$$\bar{u} = Lu \qquad t = \bar{\omega}\bar{t} \qquad \alpha_1 = \frac{KL^2}{T\pi^2}$$

$$x = \frac{\pi}{L}\bar{x} \qquad \omega = \left(\frac{\rho L^2}{T\pi^2}\right)^{\frac{1}{2}} \bar{\omega} \qquad \epsilon = \eta L^2$$

$$(3.1.1.2)$$

The nondimensional equation of motion takes the form

$$-u_{xx} + \omega^2 u_{tt} + \alpha_1 u + \alpha_1 \epsilon u^3 = 0 \qquad (3.1.1.3)$$

which is similar to equation (3.1.3) if the operator L_x is defined as

$$L_x = -\frac{\partial^2}{\partial x^2} + \alpha_1 \qquad (3.1.1.4)$$

To determine the first order frequency-amplitude relation one simply employs equation (3.1.5). For the boundary conditions of the problem, the normalized linear eigenfunctions are

$$V_k(x) = \sqrt{2/\pi}\,\sin kx \qquad (3.1.1.5)$$

Let us consider perturbation near the first linear mode. By taking $u_o(x,t) = A_{11}V_1(x)\cos t$ so that $\omega_o^2 = \Omega_1^2 = 1$, the first order is given by

$$\omega_1^2 = \frac{3}{4} A_{11}^2 \int_0^L \alpha_1 \left(\sqrt{2/\pi}\,\sin x\right)^4 dx \qquad (3.1.1.6)$$

For constant α_1 the equation (3.1.1.6) is easily integrated and the frequency-amplitude response is given by

$$\omega^2 \simeq \omega_o^2 + \epsilon\,\omega_1^2 = 1 + \alpha_1 + \epsilon\,\alpha_1 A_{11}^2 \frac{9}{8\pi} \qquad (3.1.1.7)$$

which agrees with those of Stoker [1], Keller and Ting [9] and McQueary and Clark [4] obtained previously upon replacing A_{11} by $\sqrt{\pi/2}\,A$, where A is the maximum amplitude.

The amplitude parameters for the first order nonlinear mode are obtained through a straight-forward substitution of equation (3.1.1.5) into equation (3.1.6). The only nonzero terms remaining are

$$A_{13}^{(1)} = -\ 3/16\ A^3/(-9 + \omega_o^2 - \alpha_1)$$
$$A_{31}^{(1)} = 3/16\ A^3/(-1 + 9\omega_o^2 - \alpha_1) \qquad (3.1.1.8)$$
$$A_{33}^{(1)} = -1/16\ A^3/(-9 + 9\omega_o^2 - \alpha_1)$$

and the mode shape is written as

$$u = u_o + \epsilon u_1 = A \sin x \cos t + \epsilon A_{13}^{(1)} \sin 3x \cos t$$
$$+ \epsilon A_{31}^{(1)} \sin x \cos 3t + \epsilon A_{33}^{(1)} \sin 3x \cos 3t \qquad (3.1.1.9)$$

The second order frequency-amplitude relation may be determined in a similar manner. Recall that

$$V_k = \sqrt{2/\pi} \sin kx \qquad (3.1.1.10)$$

and

$$\Omega_k^2 = k^2 + \alpha_1 \qquad (3.1.1.11)$$

Equation (3.1.10) then yields

$$\omega_2^2 = 3/256\ (\sqrt{2/\pi}\ A_{11})^4 \left[\frac{9}{(-9 + \omega_o^2 - \alpha_1)} \right. \qquad (3.1.1.12)$$
$$\left. + \frac{9}{(1 - 9\omega_o^2 - \alpha_1)} + \frac{1}{(-9 + 9\omega_o^2 - \alpha_1)} \right]$$

In order to reduce (3.1.1.12) to its previously published form

$$\omega^2 = \omega_o^2 + \frac{9}{16} \epsilon A - \frac{21}{4096} \epsilon^2 A^4 \qquad (3.1.1.13)$$

one simply makes the substitution $A_{11} = \sqrt{\pi/2\,A}$ and sets $\alpha_1 = 1$.

3.1.2. Prismatic Beams

The dimensional nonlinear differential equation of motion of a vibrating prismatic beam resting upon a nonlinear elastic foundation and restricted to small displacement theory may be written as

$$E\,I\,\frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + \frac{\gamma A}{g}\,\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + K\bar{u} + K\eta\,\bar{u}^3 = 0 \tag{3.1.2.1}$$

where E is the elastic modulus of the material, I the second moment of area, γ the weight per unit volume, A the cross sectional area, $\bar{u}$ the transverse displacement as a function of $\bar{x}$ measured along the beam and time $\bar{t}$, K and $K\eta$ the linear and nonlinear foundation parameters respectively and g the acceleration due to gravity.

Performing the variable changes

$$\bar{u} = L\,u \qquad t = \bar{\omega}\,\bar{t} \qquad \alpha_1 = \frac{K L^4}{E I \pi^4}$$

$$x = \frac{\pi}{L}\,x \qquad \bar{\omega} = \left(\frac{\gamma E I \pi^4}{\gamma A L^4}\right)^{\frac{1}{2}} \omega \qquad \epsilon = \eta\,L^2 \tag{3.1.2.2}$$

one obtains from (3.1.2.1) the following nondimensional equation of motion for a beam of length π and fixed period of vibration of 2π

$$u_{xxxx} + \omega^2 u_{tt} + \alpha_1 u + \alpha_1\,\epsilon\,u^3 = 0 \tag{3.1.2.3}$$

which is of the general form of equation (3.1.3) if the operator L_x is defined as

$$L_x = \frac{\partial^4}{\partial x^4} + \alpha_1 \tag{3.1.2.4}$$

The corresponding linear equation is found by setting ϵ equal to zero in (3.1.2.3).

$$u_{xxxx} + \omega^2 u_{tt} + \alpha_1 u = 0 \tag{3.1.2.5}$$

Assuming that harmonic motions of the beam exist, one finds that by separation of variables the mode shapes of the motions must satisfy the following ordinary differential equation

$$V^{IV} - (\Omega^2 - \alpha_1)\, V = 0 \tag{3.1.2.6}$$

where Ω is the vibration frequency.

It is easily shown that for nontrivial solutions V(x) of (3.1.2.6) to exist, Ω^2 must exceed α_1. To see this let us multiply both sides of (3.1.2.6) by V and integrate over the span of the beam

$$\int_0^L V\, V^{IV}\, dx - (\Omega^2 - \alpha_1) \int_0^L V^2 dx = 0 \tag{3.1.2.7}$$

which may be rewritten as

$$(\Omega^2 - \alpha_1) = \frac{\int_0^L V\, V^{IV}\, dx}{\int_0^L V^2 dx} \tag{3.1.2.8}$$

Upon integrating the numerator by parts, it follows that

$$\int_0^L V\, V^{IV}\, dx = V\, V^{III} \Big|_0^L - V^{I} V^{II} \Big|_0^L + \int_0^L (V^{II})^2 dx \tag{3.1.2.9}$$

Now the first two terms on the right vanish upon applying the boundary conditions, a fact that is implied by the assumption of self-adjointness of the operator L_x. Hence

$$\Omega^2 - \alpha_1 = \frac{\int_0^L (V^{II})^2 \, dx}{\int_0^L V^2 \, dx} \geq 0 \qquad (3.1.2.10)$$

Thus the numerical value of α_1, if it is positive, determines a lower bound of the vibration frequencies of the system.

Results on beams available in the literature obtained by perturbation techniques have been limited to first order only when boundary conditions are other than simply-supported. However, beams with other common boundary conditions such as fixed-fixed, fixed-hinged and cantilevers, even though they involve complicated eigenfunctions, can be easily programmed to the computer and higher order nonlinear approximations are readily obtained by using the integral form of Chapter II. The numerical results of several beam problems will be presented in the following sections.

3.1.3. Beams with Variable Geometry

Beginning with Kirchhoff's work in 1879 the literature contains several investigations of linear transverse vibrations of beams with variable cross sections. A synopsis of the historical development with references may be found in the work of Wang [14], who applied hypergeometric series to such problems. Relatively little, however, is available concerning nonlinear oscillations of such beams, Variations in cross section offer no restriction to the general expressions of the previous chapter, and section 3.1, provided that knowledge of the linear problem is available.

The dimensional nonlinear partial differential equation of motion for a beam with a variable cross section with linear and cubic restoring forces is given by

$$\frac{\partial^2}{\partial \bar{x}^2}\left(E I \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right) + \frac{\gamma A}{g}\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + k\bar{u} + k\eta\bar{u}^3 = 0 \qquad (3.1.3.1)$$

Let us consider the case in which cross section varies in the following manner

$$\frac{\gamma A}{g} = \frac{\gamma_o A_o}{g}\left(\frac{x}{L}\right)^n \qquad (3.1.3.2)$$

$$E I = E_o I_o\left(\frac{x}{L}\right)^{n+2} \qquad (3.1.3.3)$$

$$k = k_o\left(\frac{x}{L}\right)^n \qquad (3.1.3.4)$$

where the constants are previously defined and n, which specifies the particular geometry, may be either an integer or noninteger.

Upon making the changes of variables

$$\bar{u} = L u \qquad t = \bar{\omega}\,\bar{t} \qquad \alpha_1 = \frac{k_o L^4}{E_o I_1 \pi^2}$$

$$x = \frac{\pi}{L}\bar{x} \qquad \omega^2 = \frac{\gamma_o A_o L^4 \bar{\omega}^2}{E_o I_o g \pi^2} \qquad \epsilon = \eta L^2 \qquad (3.1.3.5)$$

the nondimensional equation becomes

$$\frac{\partial^2}{\partial x^2}\, x^{n+2}\frac{\partial^2 u}{\partial x^2} + \omega^2 x^n \frac{\partial^2 u}{\partial t^2} + \alpha_1 x^n u + \alpha_1 \epsilon\, x^n u^3 = 0 \qquad (3.1.3.6)$$

The equation is again similar to equation (3.1.3), after being divided through by x^n, if L_x is taken to be

$$L_x = \frac{1}{x^n}\frac{\partial^2}{\partial x^2} x^{n+2} \frac{\partial^2}{\partial x^2} + \alpha_1 \qquad (3.1.3.7)$$

The linear equation corresponding to (3.1.36) merits some attention at this point, which is found by setting ϵ equal to zero. After separation of variables the spatial part of the linear equation is

$$x^2 \frac{d^4V}{dx^4} + 2(n+2)\ x \frac{d^3V}{dx^3} + (n+1)(n+2)\frac{d^2V}{dx^2} - \tilde{K}^4 V = 0 \qquad (3.1.3.8)$$

where $\tilde{K}^4 = (\Omega^2 - \alpha_1)$, and Ω the frequency, and V describes the mode shape of vibration. Upon introducing the operator notation $D \equiv \frac{d}{dx}$ the above equation is factorable to the form

$$(xD^2 + PD + \tilde{K}^2)(xD^2 + QD - \tilde{K}^2)V = 0 \qquad (3.1.3.9)$$

where P and Q are constants to be determined. When the factorizations are expanded, one can show that

$$P = Q = n + 1 \qquad (3.1.3.10)$$

Substitution of P, Q, and D into the factored equation (3.1.3.9) yields the equivalent pair of equations to be solved as

$$x\frac{d^2V}{dx^2} + (n+1)\frac{dV}{dx} + \tilde{K}^2\ V = 0 \qquad (3.1.3.11)$$

$$x\frac{d^2V}{dx^2} + (n+1)\frac{dV}{dx} - \tilde{K}^2\ V = 0 \qquad (3.1.3.12)$$

These expressions are forms of Bessel's equation and the solutions follow as

$$V(x) = x^{-n/2}\left[C_1 J_n(2kx^{\frac{1}{2}}) + C_2 Y_n(2kx^{\frac{1}{2}}) + C_3 I_n(2kx^{\frac{1}{2}}) + C_4 K_n(2kx^{\frac{1}{2}})\right] \tag{3.1.3.13}$$

if n is zero or a positive integer and

$$V(x) = x^{-n/2}\left[C_1 J_n(2kx^{\frac{1}{2}}) + C_2 J_{-n}(2kx^{\frac{1}{2}}) + C_3 I_n(2kx^{\frac{1}{2}}) + C_4 I_{-n}(2kx^{\frac{1}{2}})\right] \tag{3.1.3.14}$$

if n is neither zero nor a positive integer.

Further development of the linear problem to obtain eigenfunctions and eigenvalues requires a knowledge of the boundary conditions. Several particular examples will be considered later. However, one important point to be emphasized here is that well known solutions of variable section beams without a foundation term can be readily adapted to the similar problem with a foundation only if the foundation parameter varies in the same manner as the mass or cross sectional area, the density of the material being assumed constant. In particular, a Winkler type foundation can be easily coped with only when the beam height is constant with respect to length. With $n = 1$ the cross section is a wedge of constant depth and it represents a Winkler type problem, in the case $n = 2$ the cross section is a double wedge or pyramid and a Winkler foundation is not implied. For $n - 3/2$ a parabolic-wedge type section again excludes the Winkler definition. The nonlinear problem is by no means restricted by the preceding remarks, and several examples are worked out in the sequel, including other cross sections, such as those with an exponential varying geometry.

3.1.4. Thin Circular Plates

The partial differential equation for vibrating plates on nonlinear foundations under the classical small displacement assumptions is given by

$$D \nabla^4 \bar{u} + \rho \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + k \bar{u} + k \eta \bar{u}^3 = 0 \qquad (3.1.4.1)$$

where, in addition to the constants defined earlier, D is the conventional plate stiffness $\frac{E h^3}{12(1-\nu^2)}$, ρ is the mass per unit area and ∇^4 stands for the biharmonic operator. To nondimensionalize the equation the following variable changes are made

$$\bar{u} = a u \qquad t = \bar{\omega} t \qquad \alpha_1 = \frac{k a^4}{D}$$

$$\bar{r} = a r \qquad \bar{\omega}^2 = \frac{D}{\rho a^4} \omega^2 \qquad \epsilon = \eta a^2 \qquad (3.1.4.2)$$

where a is the radius of the plate. The nondimensional equation of motion of a circular plate of unit radius becomes

$$\nabla^4 u + \omega^2 u_{tt} + \alpha_1 u + \alpha_1 \epsilon u^3 = 0 \qquad (3.1.4.3)$$

which again is similar to equation (3.1.3) with the operator

$$L_x = \nabla^4 + \alpha_1 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) + \alpha_1$$

(3.1.4.4)

The linear mode shape and calculated frequencies for plates without a foundation for various boundary conditions are well known in the literature [21, 22, 26, 31]. Direct substitution of these well known results on the corresponding linear problem into equations (3.1.5), (3.1.6), and (3.1.10) yields first order frequency-amplitude relations, mode shapes and second order frequency - amplitude relations for the nonlinear problem.

3.2. Continua Having Immovable Supports and Large Deflections

3.2.1. Elastic Bars

In the classical theory for the transverse vibration of elastic bars, axial extensions of the bars are not considered. One end of the bar is usually considered free to move such that the effect of the changes in axial tension during motions is negligible. Woinowsky-Krieger [15] studied the transverse vibrations of hinged bars using elliptic functions and showed that the axial tension increased the frequency of vibration. Burgreen [16], Eringen [17] and McDonald [18] studied similar problems, for simply supported beams. Recently, Evensen [10] obtained first order perturbation approximations for the frequency-amplitude relations for clamped-clamped and clamped-hinged beams of uniform cross sections, as well as for hinged-hinged beams.

Periodic vibrations studied in this section include systems for which the above-mentioned classical theory is not applicable. Both the initial tension and that induced by deflections are considered. The present approach leads to results that agree with those in the existing literature. In addition, second order frequency terms and nonlinear mode configurations are also presented.

When longitudinal inertia is neglected the free vibration of a uniform beam with end conditions ranging from spring-supported to immovable is [15].

$$E\,I\,\frac{\partial^4 \bar{u}}{\partial \bar{x}^4} - (T_o + T)\,\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \rho\,\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = 0 \qquad (3.2.1.1)$$

where T_o is the initial axial tension and T the induced tension, which is approximated

$$T = \frac{E\,\tilde{A}}{2\,L}\int_0^L \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 d\bar{x} \qquad (3.2.1.2)$$

with

$$\tilde{A} = A\left(1 - \frac{E\,A}{L\,K}\right) \qquad (3.2.1.3)$$

where K is the spring constant of the supports relative to the axial displacement. Other symbols are defined a priori. Note that for immovable supports K is infinite and $\tilde{A} = A$.

Upon introducing the following variable changes and definitions

$$\bar{u} = L\,u \qquad t = \bar{\omega}\,\bar{t} \qquad \beta = \frac{T_o L^2}{E\,I\,\pi^2}$$

$$x = \frac{\pi}{L}\,\bar{x} \qquad \omega^2 = \left(\frac{\rho\,L^4}{E\,I\,\pi^4}\right)\bar{\omega}^{-2} \qquad \epsilon = \frac{A\,L^2}{2\,\pi\,I} \qquad (3.2.1.4)$$

and the dimensionless equation of motion becomes

$$u_{xxxx} - \beta\,u_{xx} - \epsilon\int_0^\pi (u_x)^2 dx\; u_{xx} + \omega^2 u_{tt} = 0 \quad (3.2.1.5)$$

which is in the form of equation (2.2.1)

$$L_x u + \omega^2 u_{tt} + \epsilon\,\alpha_1 N_1 u = 0$$

where

$$L_x = \left(\frac{\partial^4}{\partial x^4} - \beta\,\frac{\partial^2}{\partial x^2}\right) \qquad (3.2.1.6)$$

$$N_1 u = f_1^j\,(u_x, u_{xx}) = \int_0^\pi (u_x)^2 dx\; u_{xx} \qquad (3.2.1.7)$$

and

$$\alpha_1 = -1 \qquad (3.2.1.8)$$

To determine first order frequency-amplitude results use is now made of the general form given by equation (2.2.23). With the weighting function taken as unity one obtains

$$\omega_1^2 = \frac{1}{\pi A_{11}} \int_0^{2\pi} \int_0^{\pi} \alpha_1 \bar{f}^j (u_x, u_{xx}) V_1(x) \cos t \ dxdt \quad j=1$$

$$(3.2.1.9)$$

Note that the linear mode vibration is written as

$$u_o = A_{11} V_1(x) \cos t \qquad (3.2.1.10)$$

i.e. perturbations in the neighborhood of the first linear mode are considered. The nonlinear function $\bar{f}^j$ becomes

$$\bar{f}^j (u_x, u_{xx}) = A_{11}^3 V_{1,xx} \cos^3 t \int_0^{\pi} (V_{1,x})^2 \ dx \qquad (3.2.1.11)$$

$$j=1$$

After direct substitution and integration with respect to time, (3.2.1.9) reduces to

$$\omega_1^2 = -\frac{3}{4} A_{11}^2 \int_0^{\pi} V_1 V_{1,xx} dx \int_0^{\pi} (V_{1,x})^2 dx \qquad (3.2.1.12)$$

which can be shown to agree with those of Evensen.

The complete set of nonzero amplitude parameters is obtained from equations (2.2.24) as

$$q > 1 \qquad A_{q1}^{(1)} = \frac{\frac{3}{4} A_{11}^3}{(\omega_q^2 - \omega_o^2)} \int_0^{\pi} V_q V_{1,xx} dx \int_0^{\pi} (V_{1,x})^2 dx \tag{3.2.1.13}$$

$$q > 0 \qquad A_{q3}^{(1)} = \frac{\frac{1}{4} A_{11}^3}{(\omega_q^2 - 9\omega_o^2)} \int_0^{\pi} V_q V_{1,xx} dx \int_0^{\pi} (V_{1,x})^2 dx \tag{3.2.1.14}$$

Thus, the first nonlinear mode of vibration is

$$u = A_{11} V_1 \cos t + \sum_{q=2}^{\infty} A_{q1}^{(1)} V_q \cos t + \sum_{q=1}^{\infty} A_{q3}^{(1)} V_q \cos 3t \tag{3.2.1.15}$$

To compute second order terms derivatives of the nonlinear restoring function f_j are needed. These are

$$\bar{f}^j = f_1(u_x, u_{xx}) \Big|_{u_{ox}, u_{oxx}} = u_{oxx} \int_0^{\pi} u_{ox}^2 \, dx \qquad j=1 \tag{3.2.1.16}$$

$$\bar{f}^j_{u_x} = \frac{\partial f_1}{\partial u_x} \Big| u_{ox}, u_{oxx} = u_{oxx} \int_0^{\pi} 2 u_{ox} \, dx \qquad j=1 \tag{3.2.1.17}$$

$$\bar{f}^j_{u_{xx}} = \frac{\partial f_1}{\partial u_{xx}} \Big| u_{ox}, u_{oxx} = \int_0^{\pi} (u_{ox})^2 \, dx \qquad j=1 \tag{3.2.1.18}$$

with all other derivatives of f_j being zero. The second order frequency-amplitude relations now follows from equation (2. 2. 28) with the weighting function again being equal to unity as

$$\omega_2^2 = \alpha_1 A_{11} \Big[\frac{3}{2} \int_0^{\pi} V_{1,x} dx \int_0^{\pi} V_{1,xx} \sum_{q=2}^{\infty} A_{q1}^{(1)} V_{q,x} V_1 \, dx$$

$$+ \frac{1}{2}\int_0^{\pi} V_{1,x}dx \int_0^{\pi} V_{1,xx} \sum_{q=1}^{\infty} A_{q3}^{(1)} \; V_{q,x} \; V_1 \; dx$$

$$+ \frac{3}{4}\int_0^{\pi} (V_{1,x})^2 dx \int_0^{\pi} \sum_{q=2}^{\infty} A_{q1}^{(1)} \; V_{q,xx} \; V_1 \; dx$$

$$+ \frac{1}{4}\int_0^{\pi} (V_{1,x})^2 \; dx \int_0^{\pi} \; \sum_{q=1}^{\infty} A_{q3}^{(1)} \, V_{q,xx} \; V_1 \; dx \;] \qquad (3.2.1.19)$$

Numerical results corresponding to several particular boundary conditions will be discussed later.

3.2.2. Elastic Plates

The dynamic analogue of the von Karman large deflection plate theory of equilibrium was proposed by Herrmann [19]. In a later paper [6] Herrmann applied these earlier results to study free vibrations of rectangular plates with hinged immovable supports.. The coupled nonlinear equations of motion were solved by a perturbation method and frequency-amplitude relations for moderately large deflections were obtained. However, the recursion formulae involved in the perturbation scheme remained coupled and the iterative process became somewhat involved.

Berger [20] decoupled the static nonlinear deflection equations such that one of them assumes a quasilinear form and is integrable by assuming that the strain energy due to the second invariant of the middle surface strains may be neglected. Wah [21] extended the

Berger formulation to large amplitude vibrations of rectangular and circular plates. A modified Galerkin approach was proposed whereby the first of a system of nonlinear equations was solved in terms of elliptic functions to approximate some salient parameters of the nonlinear system, such as the frequency of vibration. Gajendar [22] followed the same method of Wah for large amplitude vibrations of plates on elastic foundations.

In the following, the Wah decoupled nonlinear equations of motion describing axisymmetric plate vibrations are taken to be

$$\nabla^4\bar{u} - \frac{N}{D}\nabla^2\bar{u} + \frac{\rho}{D}\frac{\partial^2\bar{u}}{\partial\bar{t}^2} = 0 \qquad (3.2.2.1)$$

where

$$\frac{N}{D} = \frac{12}{a^2h^2}\int_o^a {\frac{d\bar{u}}{d\bar{r}}}^{2}\,\bar{r}d\bar{r} \qquad (3.2.2.2)$$

which further reduces to

$$\frac{N}{D} = \frac{-12}{a^2h^2}\int_o^a \bar{u}(\nabla^2\bar{u})\,\bar{r}d\bar{r} \qquad (3.2.2.3)$$

for simply supported or clamped edge conditions. The independent spatial variable is now r instead of x while other symbols remain as previously defined.

The following variable changes and dimensionless constants

$$\bar{u} = a\,u \qquad t = \bar{\omega}\bar{t} \qquad \epsilon = \frac{12a^2}{h^2}$$
$$\bar{r} = a\,r \qquad \bar{\omega}^2 = \frac{D}{\rho a^4}\,\omega^2 \qquad (3.2.2.4)$$

are introduced so that the nondimensional equation of motion is

$$\nabla^4 u + \omega^2 u_{tt} - \epsilon\, f_1 = 0 \tag{3.2.2.5}$$

where

$$f_1 = \int_0^1 (u_r)^2 \, r dr \, \nabla^2 u \tag{3.2.2.6}$$

The above is of the form of equation (2. 2. 1) with

$$\alpha_1 = -1 \qquad\qquad M = 1$$

$$L_x = \nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) \tag{3.2.2.7}$$

With the nonlinear function defined as

$$f_1(u_r, u_{rr}) = (u_{rr} + \frac{1}{r} u_r) \int_0^1 (u_r)^2 \, r dr \tag{3.2.2.8}$$

the first order frequency-amplitude relations follows immediately from equation (2. 2. 23) with the weighting function now equal to r. After perfoming integration with respect to time the result is

$$\omega_1^2 = -\frac{3}{4} A_{11}^2 \int_0^1 r(V_{1,r})^2 dr \int_0^1 (r\, V_1 V_{1,rr} + V_1 V_{1,r}) \, dr \tag{3.2.2.9}$$

The amplitude parameters follow from the expression (2. 2. 24)

$$A_{qp}^{(1)} = \frac{\alpha_1}{\pi(q^2\omega_o^2 - \omega_q^2)} \int_0^1 (V_{1,r})^2 \, r dr \int_o^{2\pi} \int_o^1 (r\, V_q V_{1,rr} + V_q V_{1,r}) dr \cos pt \cos^3 t \, dt \tag{3.2.2.10}$$

and after integration with respect to time one has

$$q > 1 \quad A^{(1)}_{q1} = \frac{\frac{3}{4}\alpha_1}{(\omega_o^2 - \omega_q^2)} \int_0^1 (V_{1,r})^2 \, rdr \int_0^1 (r \, V_q V_{1,rr} + V_q V_{1,r}) dr \tag{3.2.2.11}$$

$$q > 0 \quad A^{(1)}_{q3} = \frac{\frac{1}{4}\alpha_1}{(9\omega_o^2 - \omega_q^2)} \int_0^1 (V_{1,r})^2 \, rdr \int_0^1 (r \, V_q V_{1,rr} + V_q V_{1,r}) \, dr \tag{3.2.2.12}$$

The first order mode shape is of the form of equation (2.2.25) with the constants defined as above.

For the second order approximation, the following nonzero derivatives of the function f_j are needed

$$\frac{\partial f}{\partial u_r} = u_{rr} \int_0^1 2 \, u_r \, rdr + \frac{1}{r} u_r \int_0^1 2 \, u_r \, rdr + \frac{1}{r} \int_0^1 u_r^2 \, rdr \tag{3.2.2.13}$$

$$\frac{\partial f}{\partial u_{rr}} = \int_0^1 u_r^2 \, rdr \tag{3.2.2.14}$$

Upon substitution into equation (2.2.28) and factoring out A^3_{11} one finds

$$\omega_2^2 = \alpha_1 A^4_{11} \left[\frac{3}{4} \delta_{1p} + \frac{1}{4} \delta_{3p} \right] \Big[\int_0^1 2 \, V_{1,r} \, rdr \int_0^1 (r \, V_{1,rr} + V_{1,r}) \sum_q^\infty \sum_p^\infty A^{(1)}_{pq} \, V_{q,r} \, V_1 \, dr + \int_0^1 (V_{1,r})^2 \, rdr \int_0^1 \sum_q^\infty \sum_p^\infty A^{(1)}_{pq} \, V_{q,r} V_1 \, dr$$

$$+ \int_0^1 V_1^2 \, rdr \int_0^1 \sum_q^\infty \sum_p^\infty A^{(1)}_{pq} \, V_{q,rr} \, V_1 \, rdr \Big] \tag{3.2.2.15}$$

Obviously, the normalized eigenfunctions V_n are the familar Bessel functions for circular plates. Detailed computations are programmed on a digital computer and the results will be presented in the next chapter.

3.2.3. Membranes

The question naturally arises as to whether the relative magnitude of the strain energy introduced by the second strain invariant is negligible in large amplitude vibrations of membranes as was found to be the case for vibrating plates. Since the formulation of the membrane problem excludes any contribution due to bending, the reliability of results obtained in such a manner is speculative from a theoretical point of view. The fact that the Berger - Wah development of the plate problem found justification upon comparison with known results suggests that the validity of the membrane analogy be studied in a similar manner. So Eringen [7] and Chobotov [8] have studied the membrane problems using the coupled nonlinear equations. Their results can be used for comparison purposes.

Timoshenko [23] gives the strain energy due to stretching of a membrane as

$$\phi = \frac{Eh}{2(1-\upsilon^2)} \int\int \left[e^2 - 2(1-\upsilon)e_2\right] dxdy \qquad (3.2.3.1)$$

where e = the first invariant of strain

$= \epsilon_x + \epsilon_y$ in rectangular coordinates

$= \epsilon_r + \epsilon_\theta$ in cylindrical coordinates

and e_2 = the second invariant of strain

$e_2 = \epsilon_x \epsilon_y - \frac{1}{4}\gamma_{xy}$ in rectangular coordinates

$= \epsilon_r \epsilon_\theta$ in cylindrical coordinates with circular symmetry.

Upon neglecting the second strain invariant and including the work of some external load q (3.2.3.1) reduces to

$$\phi = \frac{Eh}{2(1-\upsilon^2)} \int\int e^2 \, dxdy = \int\int q\, w\, dxdy \qquad (3.2.3.2)$$

The first strain invariant expressed in rectangular coordinates is

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (3.2.3.3)$$

where u and v are displacements in the plane of the plate and w the transverse deflection. By principles of the calculus of variations and integration by parts

$$\delta\phi = \frac{Eh}{1-\upsilon^2} \int\int \left[-\delta u \frac{\partial e}{\partial x} - \delta v \frac{\partial e}{\partial y} - \delta w \frac{\partial}{\partial x}\left(e \frac{\partial w}{\partial x}\right) - \delta w \frac{\partial}{\partial y}\left(e \frac{\partial w}{\partial y}\right) \right] dxdy - \int\int q\, \delta\, w\, dxdy = 0$$

$$(3.2.3.4)$$

Since δu, δv and δw are arbitrary, the coefficients must vanish independently. Therefore

$$\frac{\partial e}{\partial x} = 0 \qquad \frac{\partial e}{\partial y} = 0 \qquad (3.2.3.5)$$

$$\frac{\partial}{\partial x}\left(e \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(e \frac{\partial w}{\partial y}\right) + \frac{q(1-\upsilon^2)}{Eh} = 0 \qquad (3.2.3.6)$$

From the first set of equations one deduces that e is constant with respect to x and y . Defining q as the inertia term one then has the following equation of motion

$$- e \nabla^2 w + \frac{\rho(1-\upsilon^2)}{Eh} \frac{\partial^2 w}{\partial t^2} = 0 \qquad (3.2.3.7)$$

It is now convenient to consider cylindrical coordinates and write Hooke's law as

$$\begin{aligned} \epsilon_r &= \frac{1}{hE} (N_r - \upsilon N_\theta) \\ \epsilon_\theta &= \frac{1}{hE} (N_\theta - \upsilon N_r) \end{aligned} \qquad (3.2.3.8)$$

where N designates the stress and ϵ the strain. The total strain e can be written as

$$e = \epsilon_r + \epsilon_\theta = \epsilon_r{}^1 + \epsilon_\theta{}^1 + \epsilon_o = \text{constant} \qquad (3.2.3.9)$$

where ϵ_o is initial strain and the primed quantities refer to the induced stress and strain. Since the sum is constant it follows that

$$\epsilon_r{}^1 + \epsilon_\theta{}^1 = \frac{(1-\upsilon)}{hE} (N_r{}^1 + N_\theta{}^1) \qquad (3.2.3.10)$$

which may again be written as

$$\frac{(1-\upsilon)}{hE} (N_r{}^1 + N_\theta{}^1) = \frac{\partial u}{\partial r} + \frac{1}{2} \left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^{-2}} \left(\frac{\partial w}{\partial \theta}\right)^2$$

$$(3.2.3.11)$$

Equation (3. 2. 3. 11) is now multiplied by rdrdθ and integrated over the area. Assuming circular symmetry and that on the boundary v is continuous and u evanescent, one obtains

$$\epsilon_r^{\,1} + \epsilon_\theta^{\,1} = \frac{1}{2\pi a^2} \int_A \left(\frac{\partial w}{\partial r}\right)^2 \bar{r} d\bar{r} \qquad (3.2.3.12)$$

By equation (3.2.3.9) and (3.2.3.12) the equation of motion including the initial stress is written as

$$-N_o \nabla^2 w - \frac{Eh}{(1-\upsilon)\, 2\pi a^2} \int_A \left(\frac{\partial w}{\partial r}\right)^2 \bar{r} d\bar{r}\, \nabla^2 w + \rho \frac{\partial^2 w}{\partial t^2} = 0$$

(3.2.3.13)

where N_o is the initial tension in the membrane.

After the following variable changes and definitions are introduced

$$w = a u \qquad t = \bar{\omega} \bar{t}$$

$$\bar{r} = a r \qquad \epsilon = \frac{Eh}{(1-\upsilon) N_o} \qquad (3.2.3.14)$$

the nondimensional equation of motion takes the form

$$-\nabla^2 u + \omega^2 u_{tt} + \alpha_1 \epsilon f_1 = 0 \qquad (3.2.3.15)$$

where

$$f_1 = \int_0^1 \left(\frac{\partial u}{\partial r}\right)^2 r dr \,\nabla^2 u \qquad (3.2.3.16)$$

$$\alpha_1 = 1$$

and u is now the transverse displacement. The remaining manipulations to obtain the nonlinear dynamic results are identical with those in the plate problem.

The results will involve the expressions V_r and ω_n, the linear membrane eigenfunctions and eigenvalues respectively.

3.3. Nonlinear Material Properties

Free vibrations of continuous structures with nonlinear material properties represent another class of problems that can be treated by the general theory presented in Chapter II . In this section the motion of a homogeneous slender beam of uniform cross section and having a longitudinal plane of symmetry passing through the centroid will be considered as an example. The usual small deflection theory is also assumed. Obviously, the method could be extended to other elastic structures, and continua with combinations of large deflections, elastic foundations etc.

The equation of motion for the free vibrations of a beam with nonlinear viscoelastic material properties as presented by Sethna [24] is modified such that material time dependency is excluded. For a material described by

$$N = E(\epsilon + h\epsilon^3) \tag{3.3.1}$$

where N is stress, ϵ strain in the x direction, E and h constants, the equation of motion is written as

$$EI_1 \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + \frac{\rho A}{g} \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + EI_2 h \left[3 \left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \right)^2 \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} + 6 \frac{\partial^2 \bar{u}}{\partial \bar{x}^3} \left(\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} \right)^2 \right]$$
$$= 0 \tag{3.3.2}$$

The following variable changes and definitions are introduced

$$\bar{u} = L u \qquad t = \bar{\omega}\bar{t} \qquad \epsilon = \frac{I_2 h 3\pi^4}{I_1 L^2} \tag{3.3.3}$$

$$x = \frac{\pi}{L}\bar{x} \qquad \bar{\omega} = \left(\frac{g E I_1 \pi^4}{\rho A L^4} \right)^{1/2} \omega$$

where I_1 and I_2 are the second and so called fourth moments of inertia and other notation has already been defined. The nondimensional equation of motion now becomes

$$u_{xxxx} + \omega^2 u_{tt} + \epsilon\left[(u_{xx})^2 u_{xxxx} + 2 u_{xx}(u_{xxx})^2\right] = 0 \tag{3.3.4}$$

which is again similar to the general equation (2. 2. 1) with

$$L_x = \frac{\partial^4}{\partial x^4} \qquad \alpha_1 = 1 \qquad M = 1 \tag{3.3.5}$$

and the nonlinear function f, is defined as

$$f_1 = (u_{xx})^2 u_{xxxx} + 2 u_{xx}(u_{xxx})^2 \tag{3.3.6}$$

With a knowledge of the nonlinear restoring function and the linear mode, the first order frequency-amplitude relations follows directly from equation (2. 2. 23) as

$$\omega_1^2 = \frac{3}{4} A_{11}^2 \int_0^\pi \left[V_1 (V_{1,xx})^2 V_{1,xxxx} + 2 V_1 V_{1,xx}(V_{1,xxx})^2\right] dx \tag{3.3.7}$$

The nonzero amplitude parameters for the nonlinear mode shape are easily determined to be

$$q>1 \quad A_{q1}^{(1)} = \frac{\frac{3}{4} A_{11}^3}{(\omega_o^2 - \omega_q^2)} \int_0^\pi \left[V_q (V_{1,xx})^2 V_{1,xxxx} + 2 V_q V_{1,xx} \right.$$

$$\left. (V_{1,xxx})^2\right] dx$$

$$q>0 \quad A_{q3}^{(1)} = \frac{\frac{1}{4} A_{11}^{3}}{(9\omega_o^2 - \omega_q^2)} \int_o^{\pi} \Big[V_q (V_{1,xx})^2 \, V_{1,xxxx} + 2 \, V_q \, V_{1,xx} (V_{1,xxx})^2 \Big] dx \tag{3.3.8}$$

After differentiating the function f with respect to u, u_x, u_{xx} --- and evaluating the results at points along the linear mode shape as before, the second order frequency-amplitude relations follow from equation (2.2.28) directly as

$$\begin{aligned} \omega_2^2 = A_{11} \int_o^{\pi} \Bigg\{ & \Big[2 \, V_{1,xx} V_{1,xxxx} + 2 (V_{1,xxx})^2 \Big] \sum_{q=1}^{\infty} A_{qp}^{(1)} \, V_{q,xx} V_1 \\ & + \Big[\, 4 V_{1,xx} V_{1,xxx} \Big] \sum_{q=1}^{\infty} A_{qp}^{(1)} \, V_{q,xxx} V_1 \\ & + \Big[(V_{1,xx})^2 \Big] \sum_{q=1}^{\infty} A_{qp}^{(1)} \, V_{q,xxxx} V_1 \Bigg\} dx \left(\frac{3}{4} \delta_{1p} + \frac{1}{4} \delta_{3p} \right) \end{aligned} \tag{3.3.9}$$

It is important to note that a third power of the amplitude is implicitly contained in the amplitude parameters, consequently the first order frequency depends on the square of the amplitude, the second order on the quadruple and so forth.

IV NUMERICAL RESULTS

Numerical results obtained through using the expressions derived in the previous chapters are presented in this chapter. Whenever possible, the results are compared with those existing in the literature. Higher order approximations to the nonlinear frequency-amplitude relations and to the nonlinear mode shapes are given which complement the existing solutions. In addition, numerical results are also presented for a number of nonlinear continuous systems for which nothing has been published in the literature. A CDC 3600 digital computer and a Newton-Cotes numerical integration technique were used in obtaining the results.

4.1 Solutions for Equations of Motion with Restoring Forces Nonlinear in Displacement.

4.1.1. String

Perhaps the simplest example to which the methods developed in the previous chapter may be applied is that of a vibrating string. In section 3.1.1 the first order frequency-amplitude relation is found from equation (3.1.1.7) to be

$$\omega^2 = \omega_0^2 + \alpha_1 \frac{9}{16} A^2 \qquad (4.1.1.1)$$

where α_1 is a dimensionless quantity and A is the amplitude of vibration defined such that

$$A = A_{11} \, | V_1 |_{max} \qquad (2.3.4)$$

It is to be noted that the amplitude A represents the maximum displacement of the linear mode and since ϵ and A_{11}^2 occur only through the product ϵA_{11}^2, we may for convenience set $\epsilon = 1$ and regard A_{11}^2 as being small.

The first order nonlinear mode shape as written in equation (3.7) with $\alpha_1 \equiv 1$ has nonzero amplitude parameters as contained in Table 4.1-1. The normalized linear and nonlinear mode shapes are plotted in figure 4.1-1, where the amplitude is not taken as A but defined such that

$$\epsilon A_{11} = \sqrt{\pi/2}\, \epsilon A \qquad (4.1.1.2)$$

The determination of the second order correction term to the nonlinear frequency requires additional eigenfunctions and eigenvalues from the linear problem. However, in the case of a string, spatial eigenfunctions beyond n = 3 contribute nothing as a consequence of the special nonlinearity assumed. With the information from table 4.1-2, the second order frequency-amplitude response is

$$\omega^2 = \omega_0^2 + \frac{9}{16}\alpha_1 A^2 - \frac{21}{4096}\alpha_1^2 A^4 \qquad (4.1.1.3)$$

Figure 4.1-1 compares the first and second order approximations for small amplitudes. As mentioned earlier, the first order result given as (4.1.1.1) agrees with those obtained by other authors.

Table 4.1.1-1. Frequency numerical values for a uniform string resting on a cubic nonlinear elastic foundation.

LINEAR MODE n	LINEAR FREQ $\omega_0^2 = n^2 + \alpha$	FIRST ORDER A^2	FIRST ORDER A_{11}^2	SECOND ORDER $A^4\,10^{-3}$	SECOND ORDER $A_{11}^4\,10^{-3}$
1	2	0.5625	0.3581		
2	5			6.59179	2.67155
3	10			-5.12695	-2.07787
4	17			-5.12695	-2.07787
5	26			-5.12695	-2.07787
6					

Table 4.1.1-2. Nonlinear amplitude parameters for a uniform string resting on a cubic nonlinear elastic foundation.

n = spatial function
m = time function

$A_{mn}^{(1)} = a_{mn} A_{11}^3$

$A_{mn}^{(1)} = d_{mn} A^3$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		0.0	1.4920 10^{-2}	0.0	0.0	0.0	0.0
3	7.4603 10^{-3}	0.0	-4.9735 10^{-3}	0.0	0.0	0.0	0.0

d_{mn}

m \ n	1	2	3	4	5	6	7
1		0.0	2.3437 10^{-2}	0.0	0.0	0.0	0.0
3	1.1718 10^{-2}	0.0	-7.8125	0.0	0.0	0.0	0.0

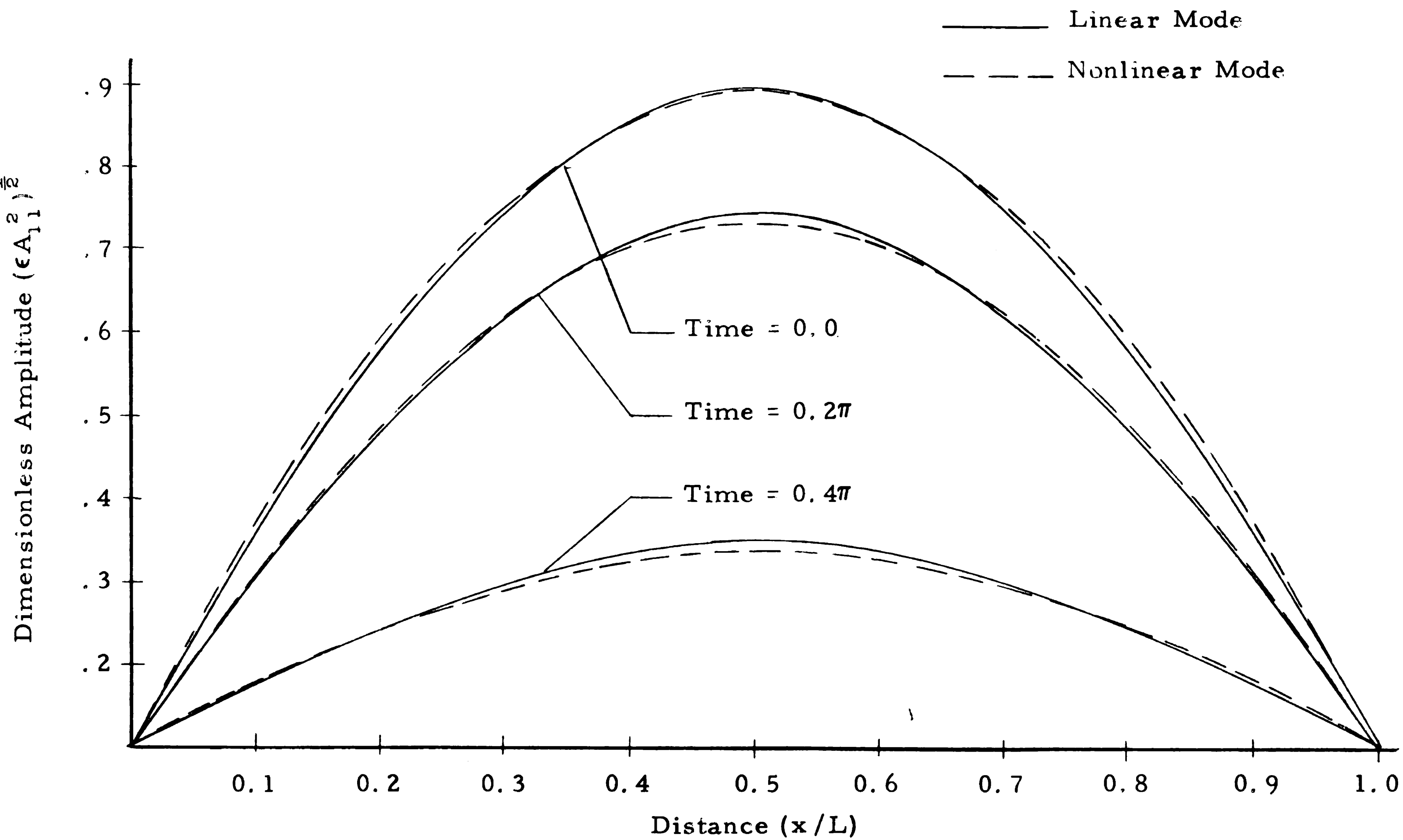

Figure 4.1.1-1. Normalized mode shape of a uniform string resting on a Duffing type nonlinear foundation.

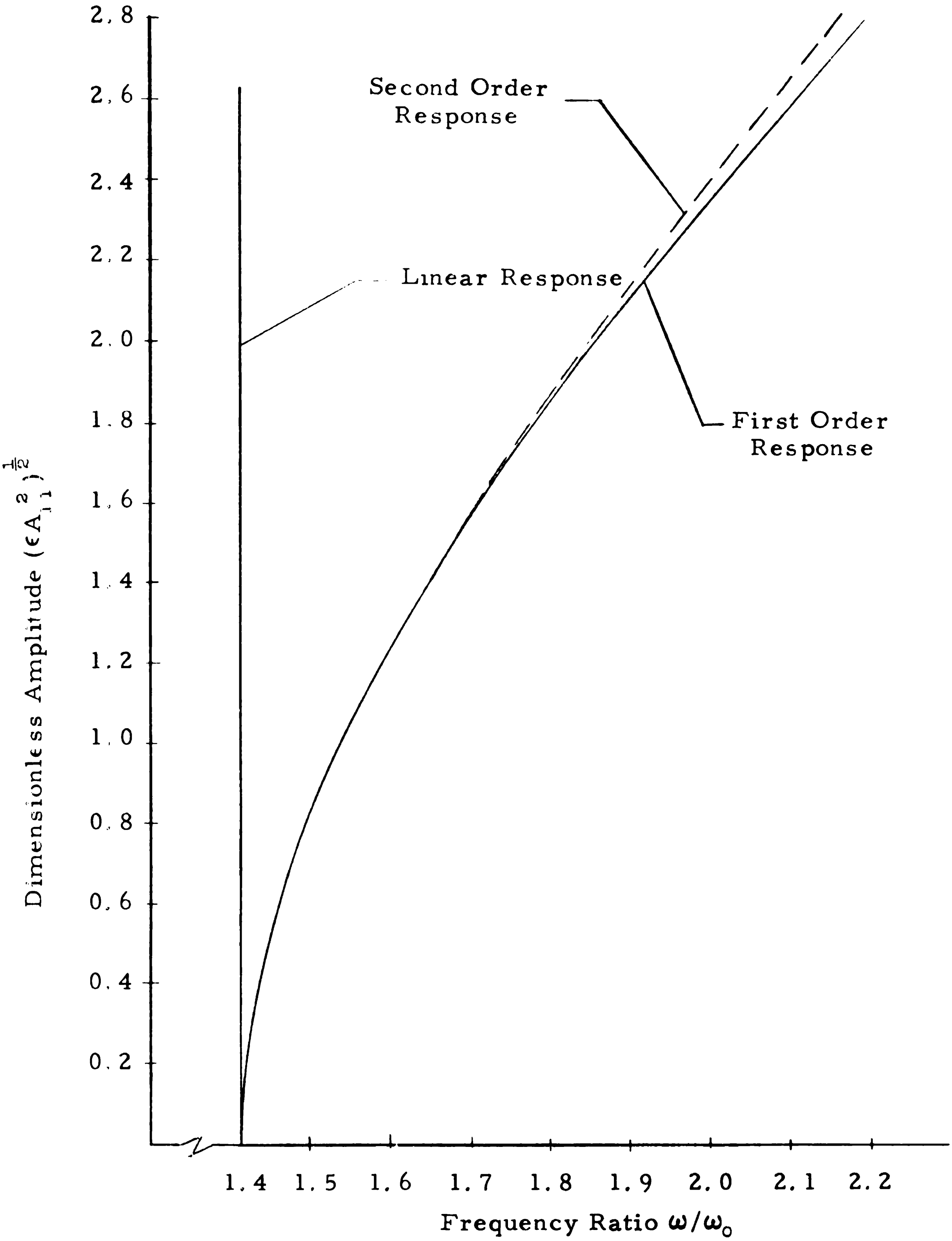

Figure 4.1.1-2. Frequency-amplitude response for a uniform string resting on a Duffing type nonlinear foundation.

4.1.2. Prismatic Beams

Uniform beams on nonlinear Winkler type foundations as described in section 3.1.2 are considered here for simply supported, clamped-clamped, clamped-hinged and cantilevered boundary conditions. The first order nonlinear frequency-amplitude relation with $r(x) = 1$ was found to be

$$\omega_1^2 = \frac{3}{4} A_{11}^2 \int_0^{\pi} \alpha_1 V_1 x^4 dx \tag{3.1.5}$$

The dimensionless parameter α_1 may be taken as unity and the maximum amplitude of vibration A is related to A_{11} by

$$A = A_{11} \left| V_1 \right|_{max} \tag{2.3.4}$$

Wylie [25], along with several other authors gives the exact linear eigenfunctions for the above-mentioned boundary conditions. Substitution of these eigenfunctions into equation (3.1.5) gives the correction term ω_1^2 to be added to the linear frequency in order to obtain the nonlinear response. After substituting these linear eigenfunctions into (3.1.5) and performing the integrations the first order frequency-amplitude relations for beams with the various boundary conditions are, simply supported

$$\omega^2 = \omega_0^2 + .35812 \, \epsilon \, A_{11}^2 \tag{4.1.2.1}$$

clamped-supported

$$\omega^2 = \omega_0^2 + .40324\, \epsilon A_{11}^2 \tag{4.1.2.2}$$

clamped-clamped

$$\omega^2 = \omega_0^2 + .44210\, \epsilon A_{11}^2 \tag{4.1.2.3}$$

and for a cantilever beam

$$\omega^2 = \omega_0^2 + .56070\, \epsilon A_{11}^2 \tag{4.1.2.4}$$

Linear eigenfunctions and frequencies beyond the fundamental mode necessary for nonlinear mode shapes and second order nonlinear frequency terms are taken from reference [26]. To determine the amplitude parameters for the nonlinear configuration it is recalled that

$$A_{1n}^{(1)} = \frac{-3}{4} \frac{A_{11}^3 \int_0^{\pi} \alpha_1 V_1^3 V_n dx}{(\omega_n^2 - \omega_0^2)} \qquad n > 1 \tag{3.1.6}$$

$$A_{3n}^{(1)} = -\frac{1}{4} A_{11}^3 \int_0^{\pi} \frac{\alpha_1 V_1^3 V_n dx}{(\omega_n^2 - 9\omega_0^2)} \qquad n > 0$$

with α_1, ω_n^2, ω_0^2 and V_n known, the constants are determined as shown in tables 4.1.2-1, 4.1.2-2, 4.1.2-3, and 4.1.2-4. A continuous graph of these nonlinear mode shapes appears in figure 4.1.2-1.

The second order frequency-amplitude results are obtained from equation (3.1.10) by direct substitution of the linear information and then followed by integration. Since the integration involves a series, a sufficient number of linear eigenfunctions must be taken to insure convergence is relatively good using only a limited number of linear functions. Nonlinear frequency-amplitude relations that include the second order approximation are taken to be

$$\omega^2 = \omega_0^2 + 0.35812\epsilon A_{11}^2 + .002063\epsilon^2 A_{11}^4 \qquad (4.1.2.5)$$

for the simply-supported,

$$\omega^2 = \omega_0^2 + 0.40324\epsilon A_{11}^2 + .001376\epsilon^2 A_{11}^4 \qquad (4.1.2.6)$$

for clamped-supported,

$$\omega^2 = \omega_0^2 + 0.44210\epsilon A_{11}^2 + .000743\epsilon^2 A_{11}^4 \qquad (4.1.2.7)$$

for clamped-clamped, and

$$\omega^2 = \omega_0^2 + 0.56070\epsilon A_{11}^2 - .01995\epsilon^2 A_{11}^4 \qquad (4.1.2.8)$$

for a cantilever beam. These results are plotted as continuous curves for small amplitudes in figure 4.1.2-2.

Den Hartog [27], by applying Rayleigh's energy method, has approximated the linear frequencies of vibration by assuming some spatial function which does not satisfy the differential equation of motion, but satisfies the boundary conditions. For the case of a cantilever beam without a foundation term, a quarter cosine wave approximation yields an approximate linear frequency only 4% above the exact value. A full

cosine wave approximates the linear frequency of a clamped-clamped beam within 1.3%. These results can be extended to linear beams on linear elastic foundations by including an additive constant, i.e. the linear frequency p^2 now becomes $p^2 - \alpha_1$, and close agreement between the approximate and the exact is again obtained.

For the case of beams on nonlinear elastic foundations, approximate spatial functions also yield results similar to those obtained from exact eigenfunctions. The nonlinear analogues of the above-mentioned linear systems result in the frequency-amplitude relations as

$$\omega^2 = \omega_0^2 + 0.6075\,\epsilon A_{11}^2 \qquad (4.1.2.9)$$

for the cantilever with a quarter cosine wave approximation and

$$\omega^2 = \omega_0^2 + 0.4642\,\epsilon A_{11}^2 \qquad (4.1.2.10)$$

for the clamped-clamped beam with a full cosine approximation. These results are plotted in figure 4.2-3.

Table 4.1.2-1. Nonlinear amplitude parameters for a uniform beam simply supported and resting on a cubic nonlinear elastic foundation.

n = spatial function m = time function							$A_{mn}^{(1)} = a_{mn}A_{11}^{3}$ $A_{mn}^{(1)} = d_{mn}A^{3}$
			a_{mn}				
m \ n	1	2	3	4	5	6	7
1		0.0	1.4920 10^{-3}	0.0	0.0	0.0	0.0
3	7.4603 10^{-3}	0.0	6.2169	0.0	0.0	0.0	0.0
			d_{mn}				
1		0.0	2.3437	0.0	0.0	0.0	0.0
3	1.1718 10^{-2}	0.0	9.7656 10^{-4}	0.0	0.0	0.0	0.0

Table 4.1.2-2. Nonlinear amplitude parameters for a uniform beam clamped-supported and resting on a cubic nonlinear elastic foundation.

n = spatial function m = time function							$A_{mn}^{(1)} = a_{mn}A_{11}^{3}$ $A_{mn}^{(1)} = d_{mn}A^{3}$
			a_{mn}				
m \ n	1	2	3	4	5	6	7
1		1.1718 10^{-3}	1.1952 10^{-3}	-1.2506 10^{-4}	-5.1047 10^{-6}		
3	4.8836 10^{-3}	-2.0894 10^{-3}	5.3277 10^{-4}	-4.5560 10^{-5}	-1.7657 10^{-6}		
			d_{mn}				
1		-1.8260 10^{-6}	-8.0384 10^{-8}	3.6351 10^{-10}	6.4116 10^{-13}		
3	-1.7125 10^{-4}	3.2558 10^{-6}	-3.5832 10^{-8}	1.3242 10^{-10}	2.2178 10^{-13}		

Table 4.1.2-3. Nonlinear amplitude parameters for a uniform beam clamped-clamped and resting on a cubic nonlinear elastic foundation.

n = spatial function
m = time function

$A_{mn}^{(1)} = a_{mn} A_{11}{}^{3}$

$A_{mn}^{(1)} = d_{mn} A^{3}$

m \ n	1	2	3	4	5	6	7
			a_{mn}				
1		0.0	1.0695 10^{-3}	0.0	-2.9363 10^{-5}	0.0	-3.9313 10^{-7}
3	3.0007 10^{-3}	0.0	5.3925 10^{-4}	0.0	-1.0346 10^{-5}	0.0	-1.3311 10^{-7}
			d_{mn}				
1		0.0	2.8144 10^{-8}	0.0	-1.4429 10^{-12}	0.0	-3.6174 10^{-17}
3	4.0827 10^{-5}	0.0	1.4189 10^{-8}	0.0	-5.0842 10^{-13}	0.0	-1.2248 10^{-17}

Table 4.1.2-4. Nonlinear amplitude parameters for a uniform cantilever beam resting on a cubic nonlinear elastic foundation.

n = spatial function
m = time function

$A_{mn}^{(1)} = a_{mn} A_{11}{}^{3}$

$A_{mn}^{(1)} = d_{mn} A^{3}$

m \ n	1	2	3	4	5	6	7
			a_{mn}				
1		4.9370 10^{-2}	-2.1920 10^{-3}	1.9292 10^{-4}	-3.4120 10^{-5}		
3	2.0731 10^{-2}	-1.9225 10^{-2}	-9.5072 10^{-4}	6.8424 10^{-5}	-1.1629 10^{-5}		
			d_{mn}				
1		-3.6140 10^{-4}	6.6730 10^{-7}	-2.5422 10^{-9}	1.9427 10^{-11}		
3	-1.9673 10^{-3}	1.4073 10^{-4}	2.8942 10^{-7}	-9.0163 10^{-10}	6.6214 10^{-12}		

Table 4.1.2-5. Frequency numerical values for a uniform beam simply supported and resting on a cubic nonlinear elastic foundation.

LINEAR MODE n	LINEAR FREQ. $\omega_o^2 = n^4 + \alpha$	FIRST ORDER ω_1^2		SECOND ORDER ω_2^2	
		A^2	A_{11}^2	$A^4(10)^{-3}$	$A_{11}^4(10)^{-3}$
1	2	0.56250	0.35812		
2	17			6.59179	2.67155
3	82			5.09033	2.06303
4	257			5.09033	2.06303
5	626			5.09033	2.06303

Table 4.1.2-6. Frequency numerical values for a uniform beam clamped - supported and resting on a cubic nonlinear elastic foundation.

LINEAR MODE n	LINEAR FREQ. $\omega_o^2 = n^4 + \alpha$	FIRST ORDER ω_1^2		SECOND ORDER ω_2^2	
		A^2	A_{11}^2	$A^4(10)^{-3}$	$A_{11}^4(10)^{-3}$
1	2	0.55635	0.40324		
2	17			3.67310	1.93057
3	82			2.65111	1.39341
4	257			2.61869	1.37637
5	626			2.61856	1.37630

Table 4.1.2-7. Frequency numerical values for a uniform beam clamped-clamped and resting on a cubic nonlinear elastic foundation.

LINEAR MODE n	LINEAR FREQ.	FIRST ORDER ω_1^2 A^2	FIRST ORDER ω_1^2 A_{11}^2	SECOND ORDER ω_2^2 $A^4(10)^{-3}$	SECOND ORDER ω_2^2 $A_{11}^4(10)^{-3}$
1	2	0.55067	0.44210		
2	17			2.05826	1.32667
3	82			1.15694	.745720
4	257			1.15694	.745720
5	626			1.15286	.743089
9	6562			1.15286	.743088

Table 4.1.2-8. Frequency numerical values for a uniform beam cantilevered and resting on a cubic nonlinear elastic foundation.

LINEAR MODE n	LINEAR FREQ.	FIRST ORDER ω_1^2 A^2	FIRST ORDER ω_1^2 A^2	SECOND ORDER ω_2^2 $A^4(10)^{-3}$	SECOND ORDER ω_2^2 $A_{11}^4(10)^{-3}$
1	2	0.44037	0.56070		
2	17			-1.18956	-1.92844
3	82			-1.229[illegible]0	-1.99271
4	257			-1.23035	-1.99458
5	626			-1.23045	-1.99474

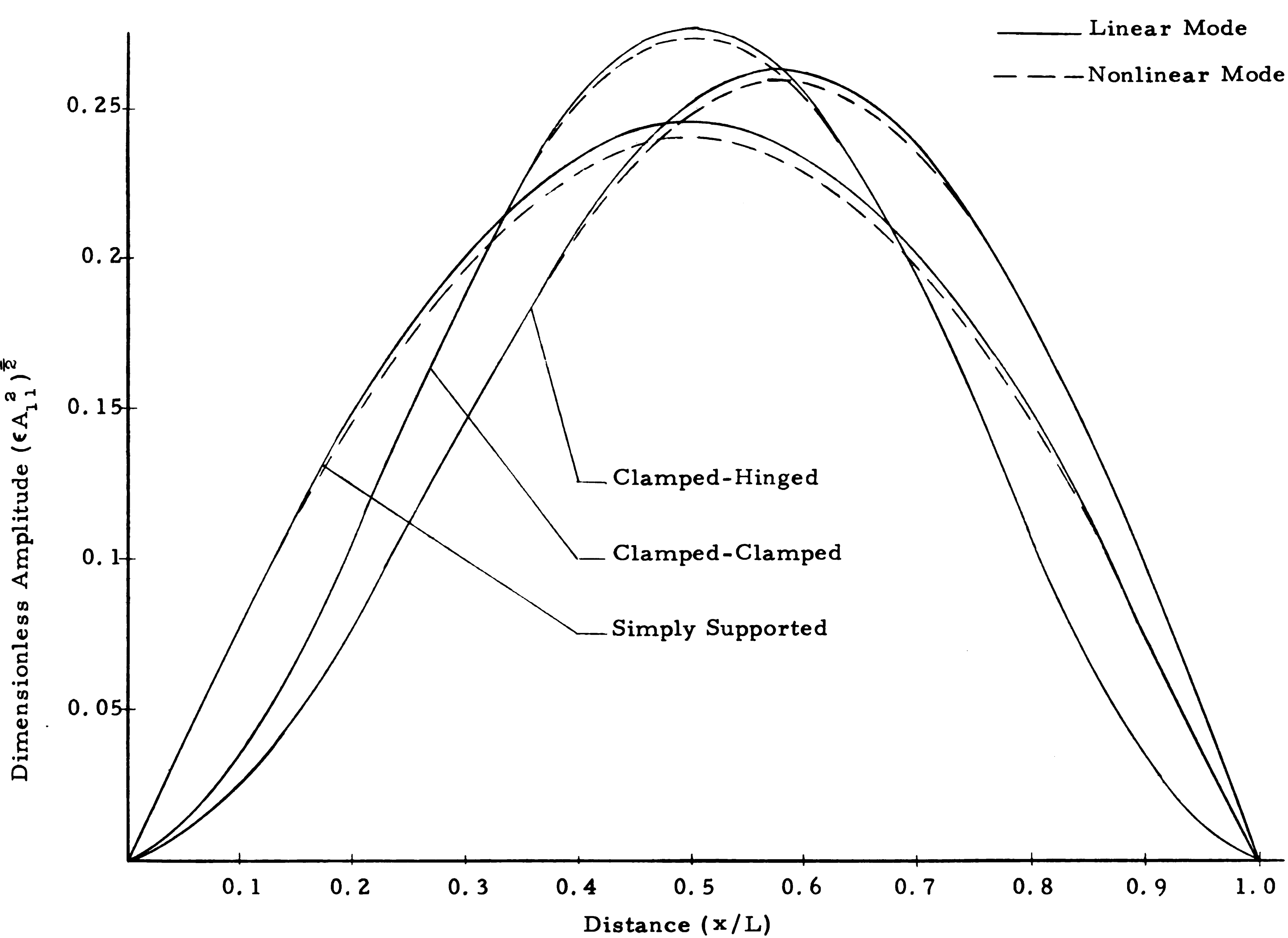

Figure 4.1.2-1. Normalized mode configurations for beams on nonlinear foundations.

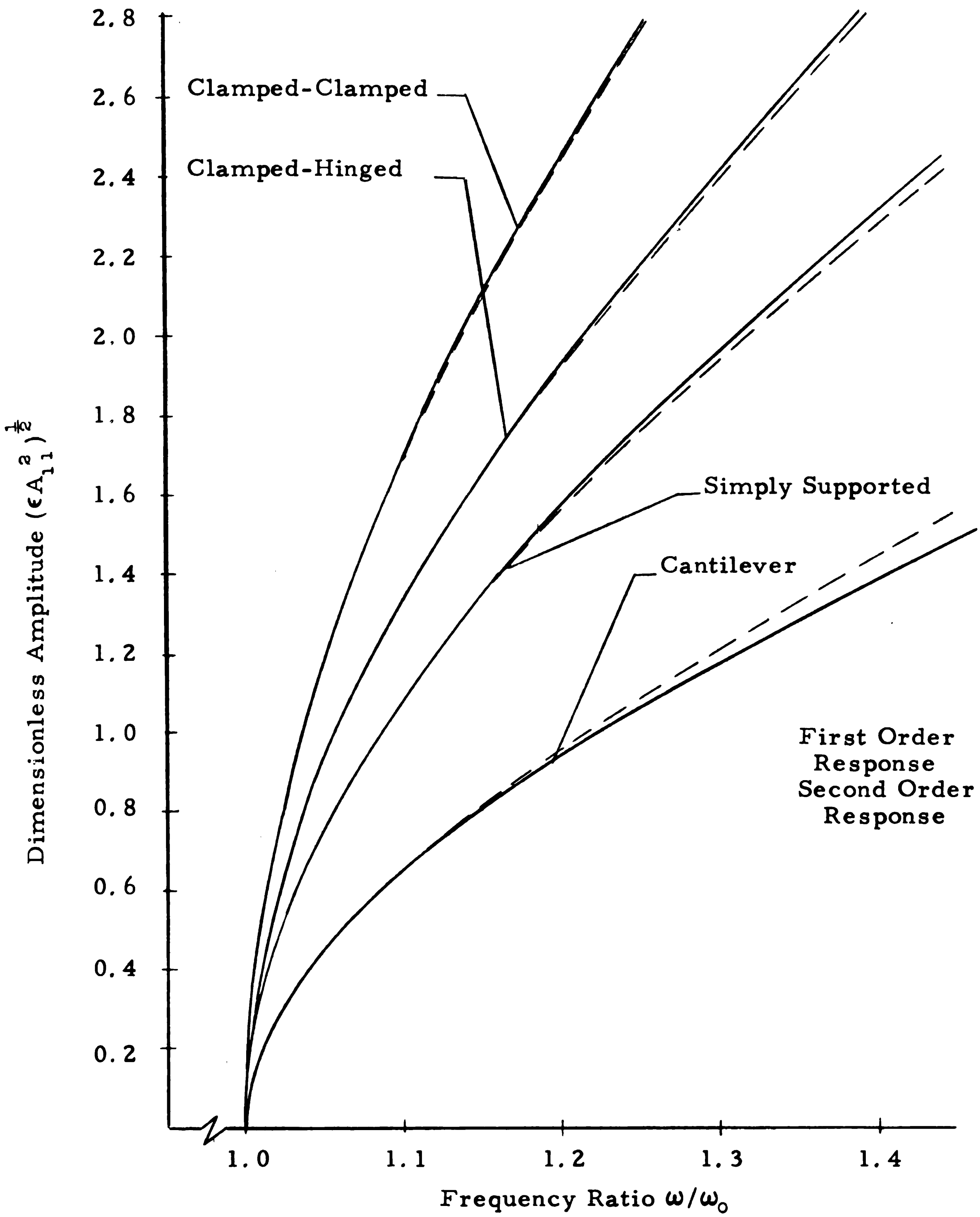

Figure 4.1.2-2. Frequency-amplitude curves for various beams resting on Duffing type nonlinear elastic foundations.

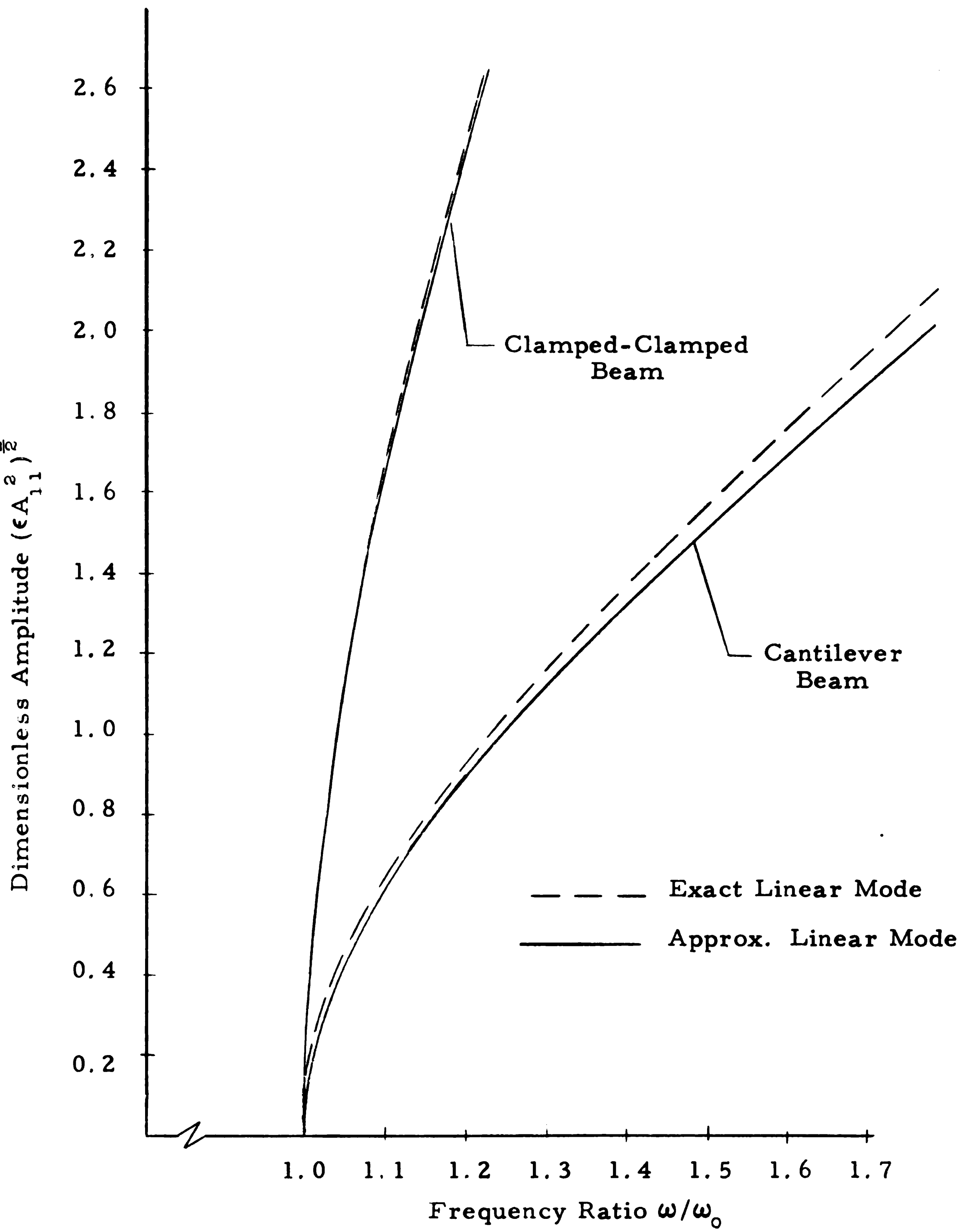

Figure 4.1.2-3. Nonlinear frequency-amplitude responses for various beams resting on Duffing type nonlinear elastic foundations. Results from using approximate linear mode shapes are compared with results from linear eigenfunctions.

4.1.3. Beams with Variable Cross Section

The following nonlinear results are examples of the theory and application of section (3.1.3). Beams and restoring functions as described by taking particular values of n in equations (3.1.3.2), (3.1.3.3), and (3.1.3.4) are considered. Linear transverse vibrations of beams of variable section without any restoring force are treated by Cranch and Adler [28] using simple beam thory and the solutions involve Bessel functions. These linear results are extended to include a linear restoring function and thereby supply the necessary linear eigenfunctions and eigenvalues for computing the results.

Let us consider a cantilever beam with a coordinate system so chosen that the distance x measured along the beam originates at the free end. The boundary conditions at the clamped end give the frequency equation

$$J_n\,(2kL^{\frac{1}{2}})\, I_{n+1}(2kL^{\frac{1}{2}}) + J_{n+1}\,(2kL^{\frac{1}{2}}) I_n(2kL^{\frac{1}{2}}) = 0 \qquad (4.1.3.1)$$

and the linear mode configuration follows as

$$V_n(x) = C_x^{-\frac{n}{2}} \left[I_n(2kL^{\frac{1}{2}})\, J_n\,(2kx^{\frac{1}{2}}) - J_n\,(2kL^{\frac{1}{2}}) I_n(2kx^{\frac{1}{2}}) \right] \qquad (4.1.3.2)$$

where $k^4 = \omega^2 - \alpha_1$, $L = \pi$, x is the dimensionless length variable, J_n and I_n are the Bessel and Modified Bessel functions of the first kind.

Example 1. We now consider a cantilever beam with rectangular cross section and set n = 1. If α_1 and E in equations (3.1.3.2), (3.1.3.3) and (3.1.3.4) are both constant, then the cross section varies in such a manner that the beam depth is constant, the height varies linearly and the foundation parameter varies as the area. Upon substitution of the linear eigenfunction into equation (3.1.5) with n = 1 and arbitrarily setting $\alpha_1 = 1$, the first order nonlinear frequency approximation is found as

$$\omega^2 = \omega_0^2 + 0.7665\,\epsilon A_{11}^2 \qquad (4.1.3.3)$$

To compute the nonlinear mode shapes, the first five linear eigenfunctions are taken. The amplitude parameters for the nonlinear mode shape are given in table 4.1.3-1 . Furthermore, the normalized nonlinear mode shape is graphed in figure 4.1.3-1.

Inspection of the results given in table 4.1.3-3 indicates good convergence of the second order frequency (3.1.10) including second order corrections is given as

$$\omega^2 = \omega_0^2 + 0.7665\,\epsilon A_{11}^2 - .0861\,\epsilon^2 A_{11}^4 \qquad (4.1.3.4)$$

The results are plotted in figure 4.1.3-3.

<u>Example 2</u>. By setting n = 2 both the width and depth are a linearly tapered if a rectangular cross section is assumed and the beam represents a pyramid. Also, by the change of variables

$$r = r_0(x/L) \qquad A_0 = \pi r_0^2 \qquad I_0 = \frac{\pi}{4} r_0^4 \qquad (4.1.3.5)$$

the results with n = 2 represent a circular cross section with a uniform taper. Both α_1 and E are again assumed constant.

The first order nonlinear frequency-amplitude response is

$$\omega^2 = \omega_0^2 + 0.8257\,\epsilon\,A_{11}^2 \tag{4.1.3.6}$$

The nonlinear mode shape is described by equation (3.1.7) with the constants defined in table 4.1.3-2. These mode shapes are plotted in figure 4.1.3-1.

The second order frequency-amplitude relation, again computed by truncating after the fifth linear eigenfunction, is found from table 4.1.3-4 as

$$\omega^2 = \omega_0^2 + 0.8257\,\epsilon A_{11}^2 - .08257\,\epsilon^2 A_{11}^4 \tag{4.1.3.7}$$

A frequency graph appears in figure 4.1.3-4.

<u>Example 3.</u> If one sets n = 3/2, the beam has a parabolic width and linearly tapered height with γ_1 and E constant. Again the restoring parameter varies as the area. Upon substitution into equation (3.1.5) with $\alpha \equiv 1$, the first order approximation of frequency is

$$\omega^2 = \omega_0^2 + .7961\,\epsilon\,A_{11}^2 \tag{4.1.3.8}$$

The nonlinear mode shape has constants according to table 4.1.3-6 and the nonlinear frequency-amplitude relation including the second order perturbation from table 4.1.3-5 is

$$\omega^2 = \omega_0^2 + .7961\,\epsilon A_{11}^2 - .0876\,\epsilon^2 A_{11}^4 \qquad (4.1.3.9)$$

These results are graphed in figure 4.1.3-5.

Table 4.1.3-1. Nonlinear amplitude parameters for a variable section cantilever beam with a nonlinear displacement dependent forcing function as described in Example 1.

n = spatial function $\qquad A^{(1)}_{mn} = a_{mn} A^3_{11}$

m = time function $\qquad A^{(1)}_{mn} = d_{mn} A^3$

m \ n	1	2	3	4	5	6	7
				a_{mn}			
1		-3.8703 10^{-2}	-7.1743 10^{-3}	-2.0219 10^{-3}	-6.9666 10^{-4}		
3	8.2756 10^{-3}	2.5774 10^{-2}	-3.6437 10^{-3}	-7.7010 10^{-4}	-2.4566 10^{-4}		
				d_{mn}			
1		-4.6487 10^{-5}	-4.0444 10^{-7}	-5.6202 10^{-9}	-8.1320 10^{-11}		
3	1.9388 10^{-4}	3.0958 10^{-5}	-2.0540 10^{-7}	-2.1405 10^{-9}	-2.8676 10^{-11}		

Table 4.1.3-2. Nonlinear amplitude parameters for a variable section cantilever beam with a nonlinear displacement dependent forcing function as described in Example 2.

n = spatial function $\qquad A^{(1)}_{mn} = a_{mn} A^3_{11}$

m = time function $\qquad A^{(1)}_{mn} = d_{mn} A^3$

m \ n	1	2	3	4	5	6	7
				a_{mn}			
1		-2.4515 10^{-2}	-6.0515 10^{-3}	-2.0445 10^{-3}	-8.2047 10^{-4}		
3	3.9439 10^{-3}	9.5634 10^{-3}	-3.9678 10^{-3}	-8.4143 10^{-4}	-3.0024 10^{-4}		
				d_{mn}			
1		-5.2391 10^{-6}	-4.2723 10^{-8}	-5.0183 10^{-10}	-7.0747 10^{-12}		
3	3.1824 10^{-5}	2.0437 10^{-6}	-2.8012 10^{-8}	-2.0652 10^{-10}	-2.5889 10^{-12}		

Table 4.1.3-3. Frequency numerical values for a variable section cantilever beam with a nonlinear displacement dependent forcing function as described in Example 1.

LINEAR MODE n	LINEAR FREQ. $\omega_0^2 = (\frac{\gamma_n}{2\pi^{\frac{1}{2}}})^4 + \alpha$	FIRST ORDER ω_1^2 A^2	A_{11}^2	SECOND ORDER ω_2^2 $A^4 10^{-2}$	$A_{11}^4 10^{-2}$
1	γ_1 = 4.61	0.4035	0.7665		
2	γ_2 = 7.80			-1.81787	-6.56106
3	γ_3 = 11.0			-2.26738	-8.18345
4	γ_4 = 14.1			-2.36215	-8.52547
5	γ_5 = 17.3			-2.38759	-8.61730

Table 4.1.3-4. Frequency numerical values for a variable section cantilever beam with a nonlinear displacement dependent forcing function as described in Example 2.

LINEAR MODE n	LINEAR FREQ. $\omega_0^2 = (\frac{\gamma_n}{2\pi^{\frac{1}{2}}})^4 + \alpha$	FIRST ORDER ω_1^2 A^2	A_{11}^2	SECOND ORDER ω_2^2 $A^4 10^{-2}$	$A_{11}^4 10^{-2}$
1	γ_1 = 5.91	0.3988	0.8257		
2	γ_2 = 9.20			-1.30134	-5.57869
3	γ_3 = 12.4			-1.74476	-7.47958
4	γ_4 = 15.6			-1.86696	-8.00342
5	γ_5 = 18.8			-1.90836	-8.18092

Table 4.1.3-5. Frequency numerical values for a variable section cantilever beam with a nonlinear displacement dependent forcing function as described in Example 3.

LINEAR MODE n	LINEAR FREQ $\omega_0^2=\left(\frac{\gamma_n}{2\pi^{\frac{1}{2}}}\right)^4+\alpha$	FIRST ORDER ω_1^2 A^2	FIRST ORDER ω_1^2 A^2	SECOND ORDER ω_2^2 $A^4 10^{-2}$	SECOND ORDER ω_2^2 $A_{11}^4 10^{-2}$
1	γ_1 = 5.27	0.4010	0.7961		
2	γ_2 = 8.51			-1.58874	-6.25986
3	γ_3 = 11.7			-2.06616	-8.14098
4	γ_4 = 14.84			-2.18589	-8.61271
5	γ_5 = 18.01			-2.22290	-8.75856

Table 4.1.3-6. Nonlinear amplitude parameters for a variable section cantilever beam with a nonlinear displacement dependent forcing function as described in Example 3.

n = spatial function
m = time function

$$A_{mn}^{(1)} = a_{mn} A_{11}^3$$
$$A_{mn}^{(1)} = d_{mn} A^3$$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		-3.0800×10^{-2}	-6.8065×10^{-3}	-2.1441×10^{-3}	-8.1026×10^{-4}		
3	5.6373×10^{-3}	1.5512×10^{-2}	-3.8700×10^{-3}	-8.4659×10^{-4}	-2.9078×10^{-4}		

d_{mn}

m \ n	1	2	3	4	5	6	7
1		-1.4963×10^{-5}	-1.3062×10^{-7}	-1.7203×10^{-9}	-2.5772×10^{-11}		
3	7.5420×10^{-5}	7.5360×10^{-6}	-7.4272×10^{-8}	-6.7924×10^{-10}	-9.2493×10^{-12}		

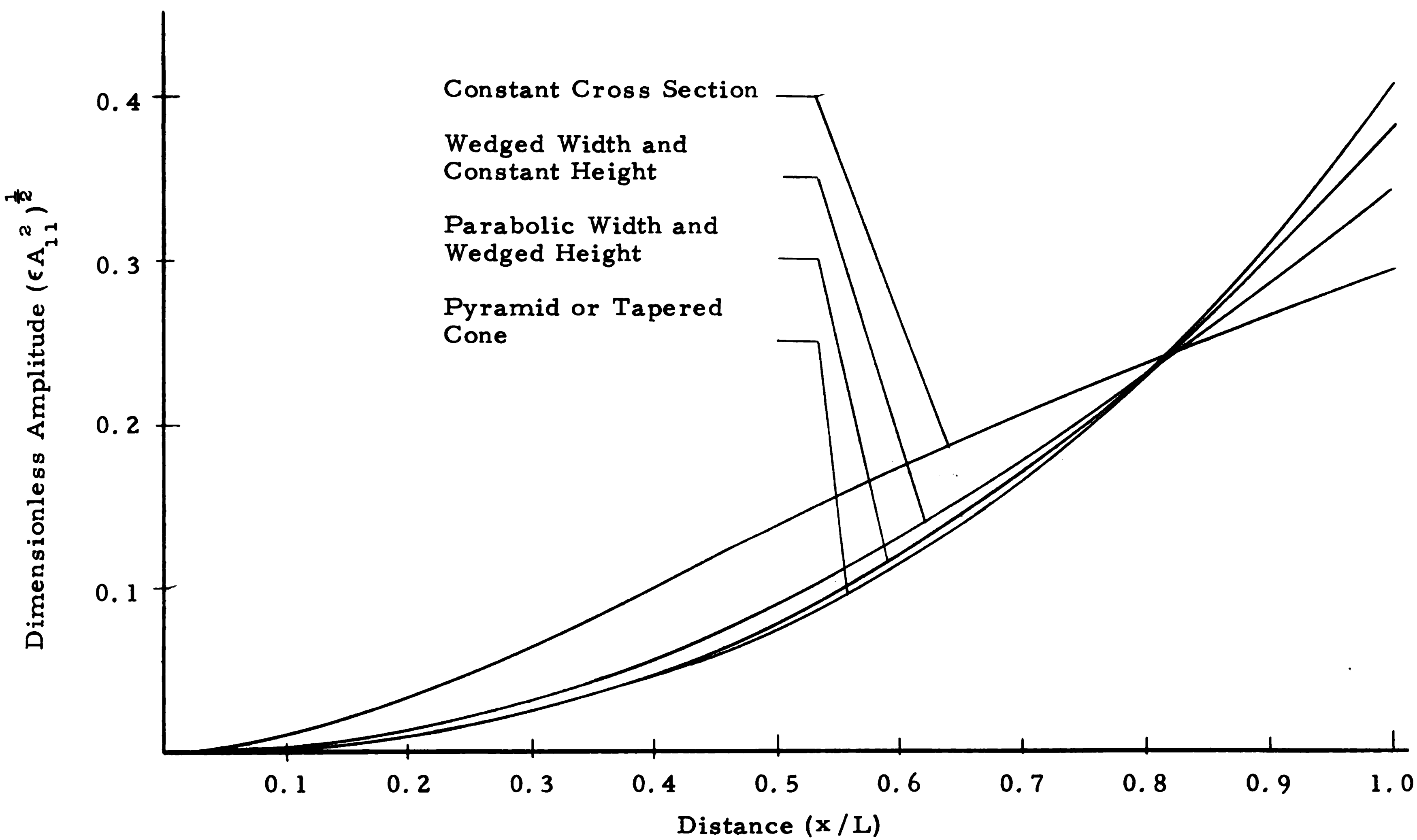

Figure 4.1.3-1. Normalized nonlinear modes for cantilever beams with variable cross-sections resting on nonlinear foundations. Time = 1.6π.

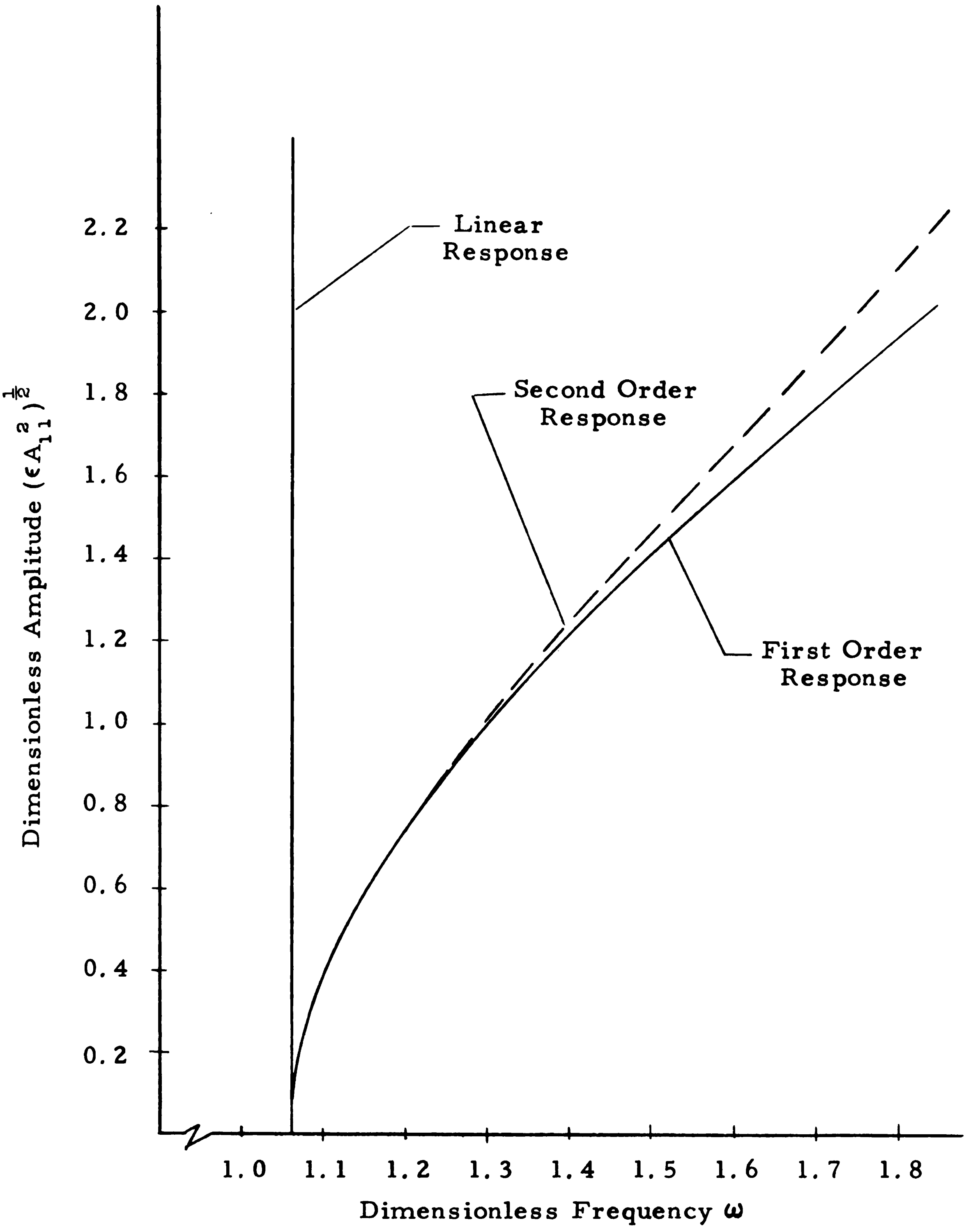

Figure 4.13-2. Frequency-amplitude response curves for cantilever beam with constant cross section.

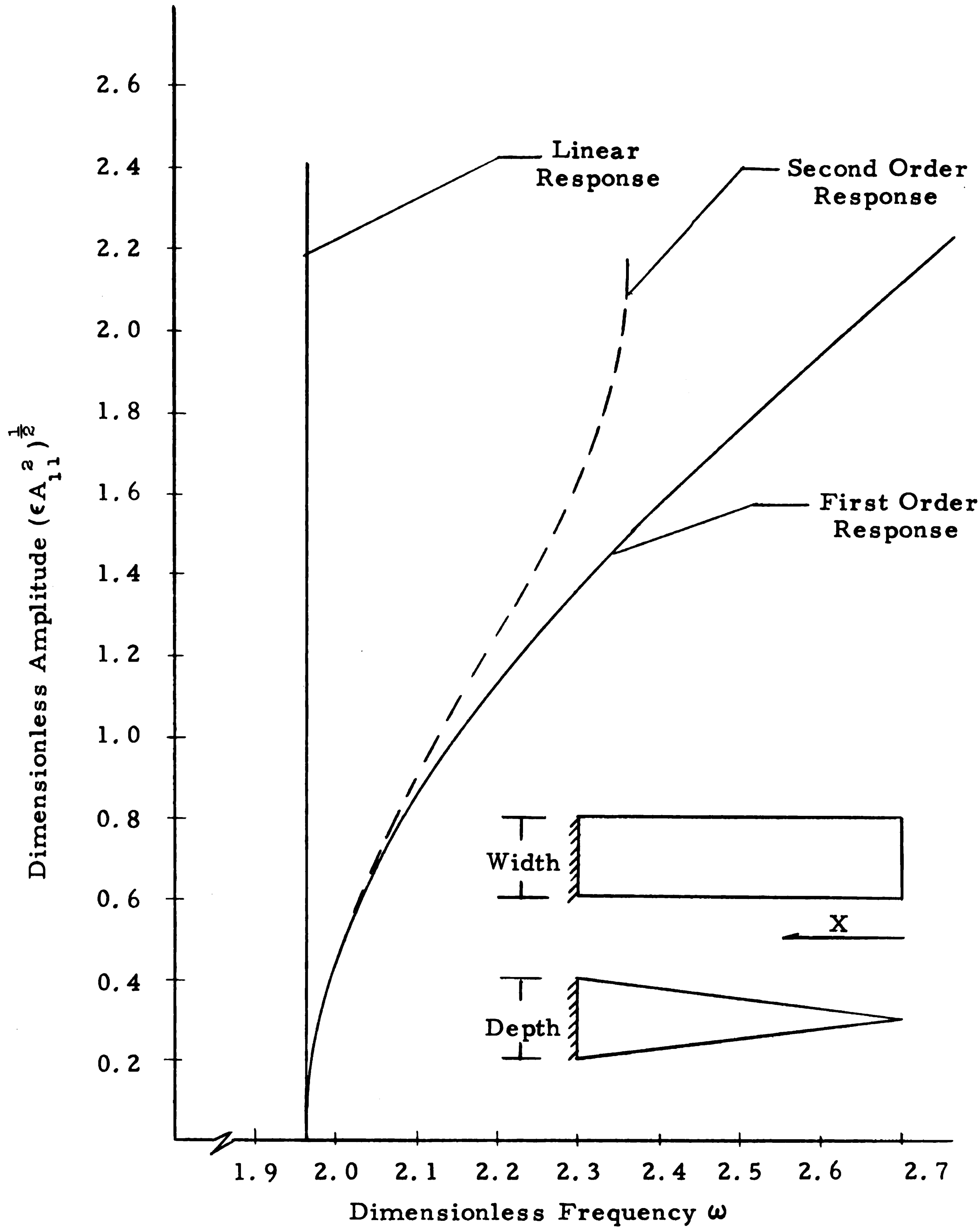

Figure 4.1.3-3. Frequency-amplitude response curves for cantilever beam with variable cross section as described in Example 1.

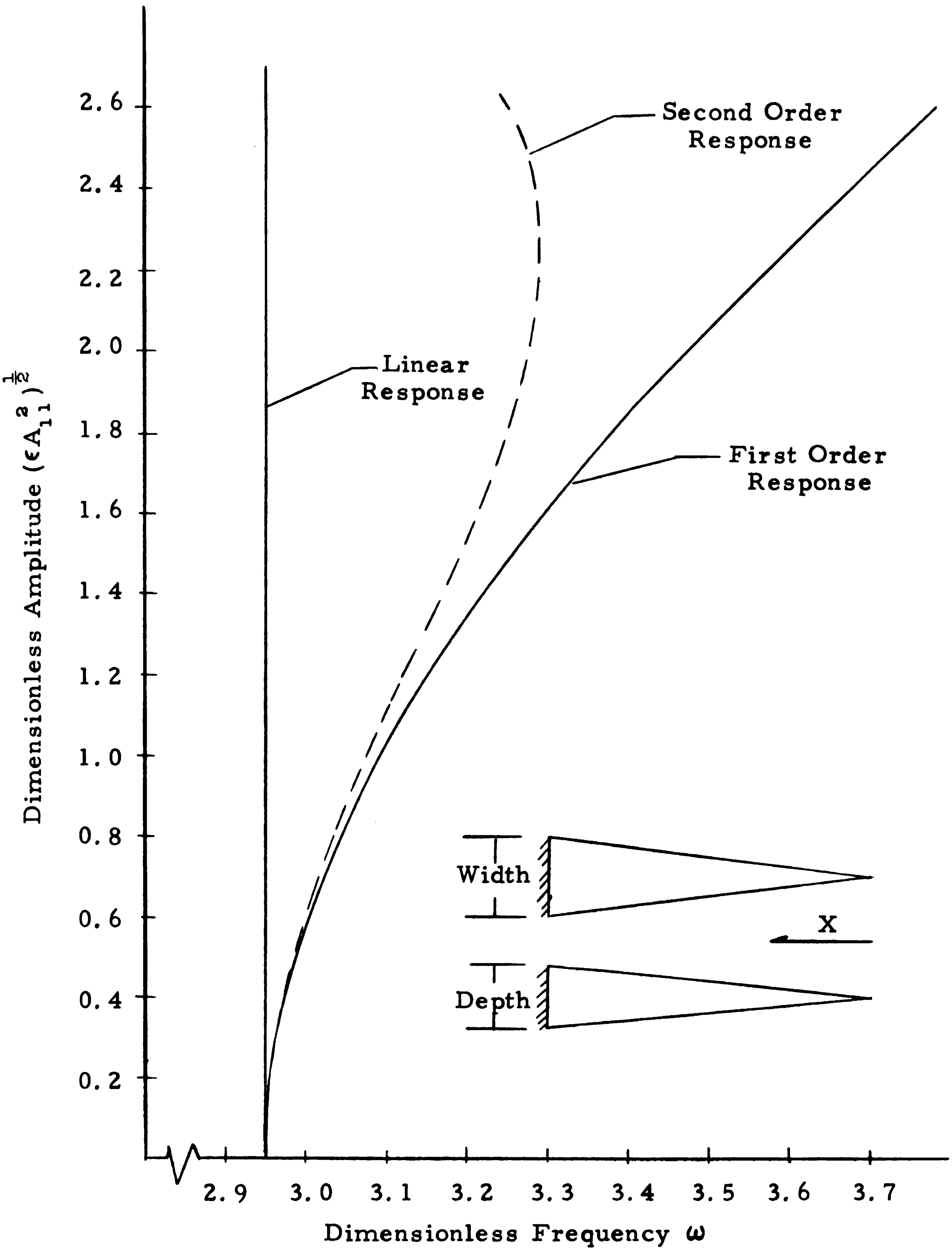

Figure 4.1.3-4. Frequency-amplitude response curves for cantilever beam with variable cross section as described in Example 2.

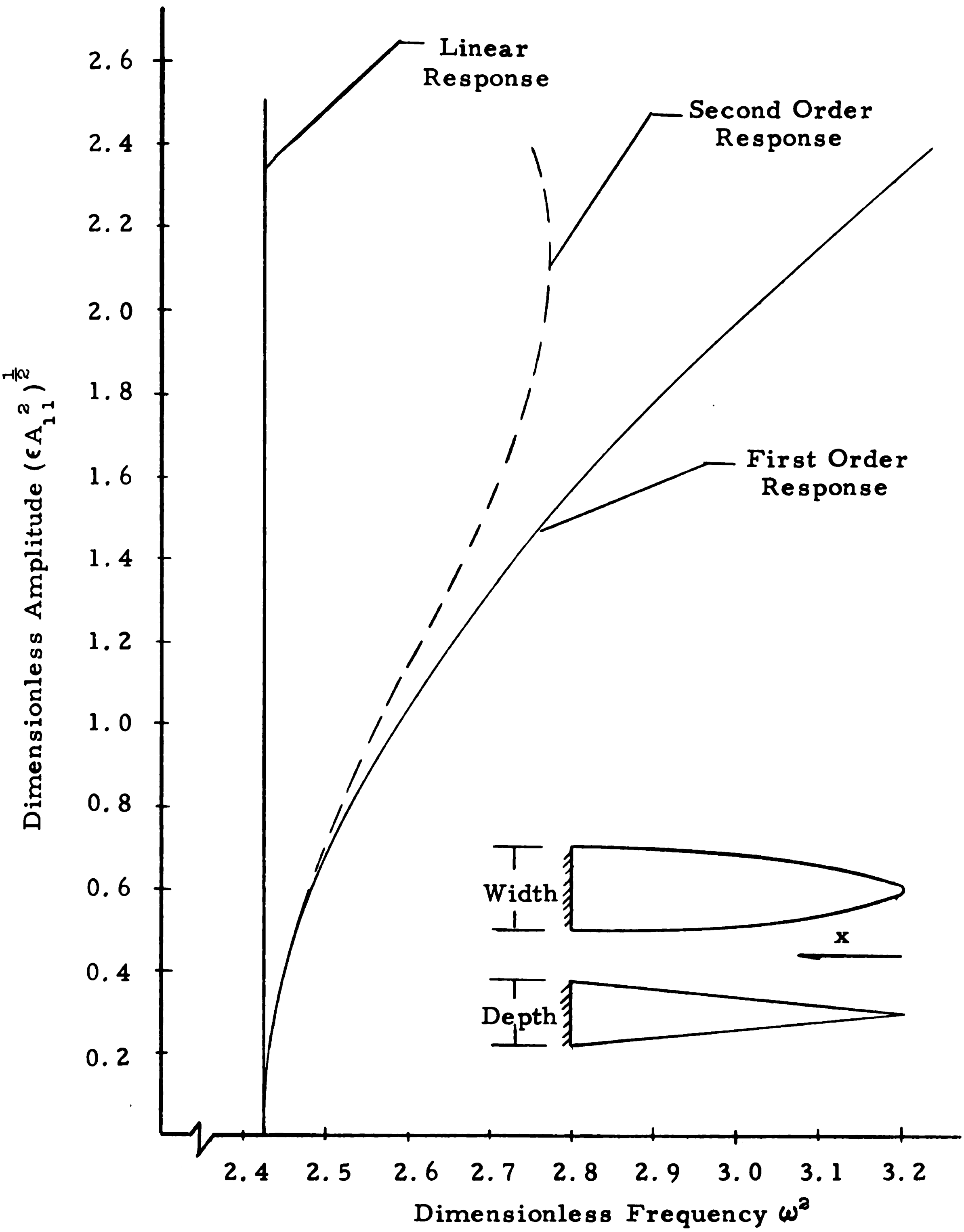

Figure 4.1.3-5. Frequency-amplitude response curves for a cantilever beam with variable cross section as described in Example 3.

In order to include examples of variable section beams involving trigonometric solutions rather than the Bessel functions, let us consider the nonlinear vibration of beams with cross sectional areas, moments of inertia and displacement dependent restoring functions that vary in an exponential manner. The nonlinear partial differential equation of motion is taken as equation (3.1.3.1). By making the variable changes

$$u = L\bar{u} \qquad t = \bar{\omega}\,\bar{t} \qquad \alpha = \frac{K_0 L^4}{EI_0 \pi^4} \qquad A = A_0 e^{x/L}$$

$$I = I_0 e^{x/L}$$

$$x = \frac{\pi}{L}\bar{x} \qquad \bar{\omega}^2 = \frac{gEI_0 \pi^4 \omega^2}{\gamma A_0 L^4} \qquad \epsilon = \eta L^2 \qquad K = K_0 e^{x/L}$$

The dimensionless equation of motion follows as

$$\frac{\partial^2}{\partial x^2}\left(e^{x/L}\frac{\partial^2 u}{\partial x^2}\right) + \omega^2 e^{x/L}\frac{\partial^2 u}{\partial t^2} + \alpha_1 e^{x/L} u + \alpha_1 \epsilon e^{x/L} u^3 = 0$$

The linear frequencies and mode shapes as given by Suppiger [29] are extended to include foundation terms. They are

$$\omega_0^2 = \alpha + \left(\frac{3.12}{\pi}\right)^4$$

$$V_1(x) = Ce^{-\frac{x}{2\pi}}\left[\cos h\frac{3.17}{\pi}x - 1.115\sin h\frac{3.17}{\pi}x - \cos\frac{3.09}{\pi}x + 7.43\sin\frac{3.09}{\pi}x\right]$$

for a simply supported beam, and

$$\omega_0^2 = \alpha + \left(\frac{4.73}{\pi}\right)^4$$

$$V_1(x) = Ce^{-\frac{x}{2\pi}}\left[\cos h\frac{4.75}{\pi}x - 0.98 \sin h\frac{4.75}{\pi}x + 0.99 \sin\frac{4.71}{\pi}x - \cos\frac{4.71}{\pi}x\right]$$

for a clamped-clamped beam.

The first order nonlinear frequency for the simply supported system follows from equation (3.1.5) upon integration as

$$\omega^2 = \omega_0^2 + 0.3578\,\epsilon A_{11}^2$$

and for the clamped-clamped beam

$$\omega^2 = \omega_0^2 + 0.4325\,\epsilon A_{11}^2$$

where the amplitude A_{11} is again defined in the normalized sense. These results are plotted with $\alpha \equiv 1$ in figure 4.1.3-6.

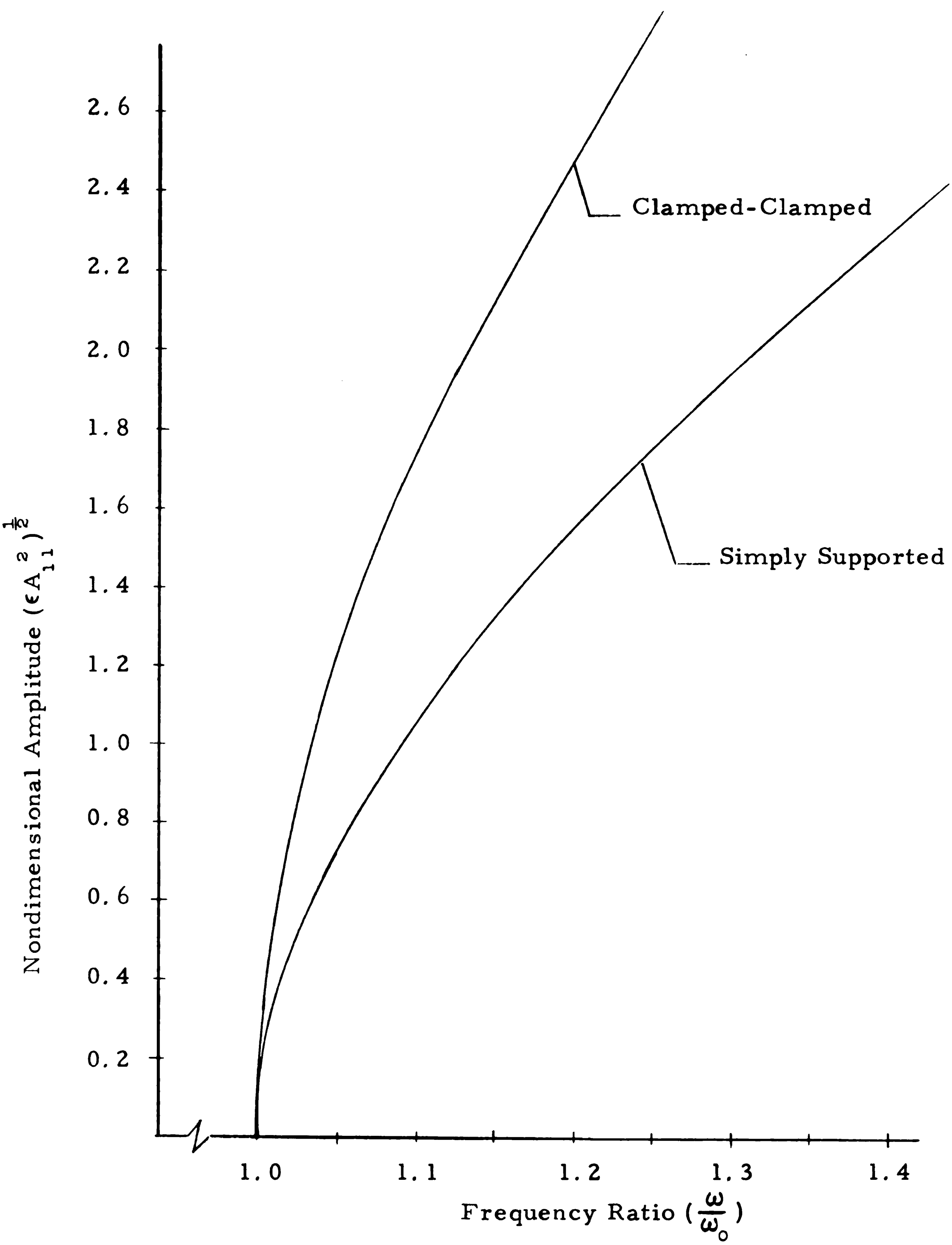

Figure 4.1.3-6. First order amplitude-frequency curves for beams with exponential varying cross sections resting on non-linear elastic foundations.

4.2. Numerical Solutions for Continua Having Immovable Supports and Large Amplitudes of Vibration.

4.2.1. Elastic Beams with Immovable Supports

Numerical results for the nonlinear problem of a uniform beam with immovable end supports are given in this section. The system is described in section 3.2.1 and the dimensionless equation of motion is taken as equation (3.2.1.5). Linear frequencies and eigenfunctions are again taken from Wylie [25] for simply supported, clamped-supported and clamped-clamped boundary conditions.

The first order nonlinear frequency-amplitude relation is given by equation (3.2.1.12) as

$$\omega_1^2 = -\frac{3}{4} A_{11}^2 \int_0^{\pi} V_1 V_{1,xx} \, dx \int_0^{\pi} (V_{1,x})^2 \, dx \tag{3.2.1.1.12}$$

Upon substituting the linear eigenfunctions and performing the integrations on the computer one obtains

$$\omega^2 = \omega_0^2 + 1.17809 \epsilon A^2 \tag{4.2.1.1}$$

for the simply supported,

$$\omega^2 = \omega_0^2 + 1.40748 \epsilon A^2 \tag{4.2.1.2}$$

for the clamped-supported, and

$$\omega^2 = \omega_0^2 + 1.45168\ \epsilon A^2 \qquad (4.2.1.3)$$

for the clamped-clamped end conditions. The amplitudes are defined as before, where

$$A = A_{11} \mid V_1 \mid_{max} \qquad (2.3.4)$$

i.e. A is the maximum dimensionless deflection and A_{11} is associated with the normalized eigenfunction. It is to be noted that the product ϵA^2 is equivalent to $\frac{1}{2}\pi$ multiplying the dimensionless ratio of the deflection to the radius of gyration of the beam cross section. These first order results are identical with those of Evensen [10].

The nonlinear mode shapes are determined upon substituting higher modes and corresponding frequencies of the linear problem into equation (3.2.1.15). The amplitude parameters are given in tables 4.2.1-1, and 4.2.1-2. The mode shapes are graphed in figures 4.2.1-2 and 4.2.1-2.

Second order frequency-amplitude results follow directly from equation (3.2.1.19) and tables 4.2.1-3, 4.2.1-4 and 4.2.1-5 as

$$\omega^2 = \omega_0^2 + 1.17809\ \epsilon A^2 - .0192765\ \epsilon^2 A^4 \qquad (4.2.1.4)$$

for simply supported,

$$\omega^2 = \omega_0^2 + 1.40748\ \epsilon A^2 - .0119564\ \epsilon^2 A^4 \qquad (4.2.1.5)$$

for clamped-supported, and

$$\omega^2 = \omega_0{}^2 + 1.45168\,\epsilon\, A^2 - .00602962\,\epsilon^2 A^4 \qquad (4.2.1.6)$$

for clamped-clamped end conditions. These results are plotted in figure 4.2.1-3. There are no second order approximations available for comparison.

Table 4.2.1-1. Nonlinear amplitude parameters for a uniform beam clamped-hinged with immovable supports.

n = spatial function
m = time function

$A_{mn}^{(1)} = a_{mn}A_{11}{}^3$
$A_{mn}^{(1)} = d_{mn}A^3$

m \ n	1	2	3	4	5	6	7
				a_{mn}			
1		1.6373×10^{-2}	3.0874×10^{-3}	9.0027×10^{-4}	3.3543×10^{-4}	1.4718×10^{-4}	7.2679×10^{-5}
3	1.7423×10^{-2}	3.4531×10^{-2}	1.2534×10^{-3}	3.1934×10^{-4}	1.1477×10^{-4}	4.9697×10^{-4}	2.4399×10^{-5}
				d_{mn}			
1		-2.5514×10^{-5}	-2.0765×10^{-7}	-2.6168×10^{-9}	-4.2139×10^{-11}	-7.9930×10^{-13}	-1.7074×10^{-14}
3	-6.1097×10^{-4}	-5.3810×10^{-5}	-8.4298×10^{-8}	-9.2823×10^{-10}	-1.4418×10^{-11}	-2.6989×10^{-13}	-5.77319×10^{-15}

Table 4.2.1-2. Frequency numerical values for a uniform beam with simply supported immovable supports.

LINEAR MODE n	LINEAR FREQ $\omega_o^2=n^4$	FIRST ORDER ω_1^2 A^2	SECOND ORDER ω_2^2 $A^4 10^{-2}$
1	2	1.17809	
2	16		1.92765
3	81		1.92765
4	256		1.92765
5	625		1.92765
6	1296		1.92765
7	2508		1.92765

Table 4.2.1-3. Frequency numerical values for a uniform beam with clamped-supported immovable supports.

LINEAR MODE n	LINEAR FREQ. $\omega_o^2=(\frac{\gamma_n}{\pi})^4$	FIRST ORDER ω_1^2 A^2	SECOND ORDER ω_2^2 $A^4 10^{-2}$
1	$\gamma_1=$ 3.9266023	1.40748	
2	$\gamma_2=$ 7.0685827		- .886755
3	$\gamma_3=$10.2101761		-1.111530
4	$\gamma_4=$13.3517688		-1.167424
5	$\gamma_5=$16.4933614		-1.185525
6	$\gamma_6=$19.6349541		-1.192560
7	$\gamma_7=$22.7765468		-1.195649

Table 4.2.1-4. Frequency numerical values for a uniform beam with clamped-clamped immovable supports.

LINEAR MODE n	LINEAR FREQ $\omega_0^2 = (\frac{\gamma_n}{\pi})^4$	FIRST ORDER ω_1^2 A^2	SECOND ORDER ω_2^2 $A^4\, 10^{-3}$
1	γ_1 = 4.7300408	1.45168	
2	γ_2 = 7.8532046		5.695165
3	γ_3 = 10.9956078		-4.837814
4	γ_4 =14.1371655		-4.837814
5	γ_5 =17.2787596		-5.839504
6	γ_6 =20 4203525		-5.839504
7	γ_7 =23.561945		-6.029620

Table 4.2.1-5. Nonlinear amplitude parameters for a uniform beam clamped-clamped with immovable supports.

n = spatial function
m = time function

$A_{mn}^{(1)} = a_{mn} A_{11}^3$

$A_{mn}^{(1)} = d_{mn} A^3$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		-3.6299 10^{-7}	6.3668 10^{-3}	-2.3333 10^{-8}	7.9605 10^{-4}	1.1222 10^{-8}	1.8453 10^{-4}
3	9.4495 10^{-3}	5.6967 10^{-7}	2.9627 10^{-3}	-8.6565 10^{-9}	2.7790 10^{-4}	3.8291 10^{-9}	6.2323 10^{-5}

d_{mn}

m \ n	1	2	3	4	5	6	7
1		-2.2138 10^{-10}	1.6753 10^{-7}	-2.6538 10^{-14}	3.9139 10^{-11}	2.3862 10^{-17}	1.7004 10^{-14}
3	1.2856 10^{-4}	3.4743 10^{-10}	7.7960 10^{-8}	-9.8456 10^{-15}	1.3663 10^{-11}	8.1422 10^{-18}	5.7428 10^{-15}

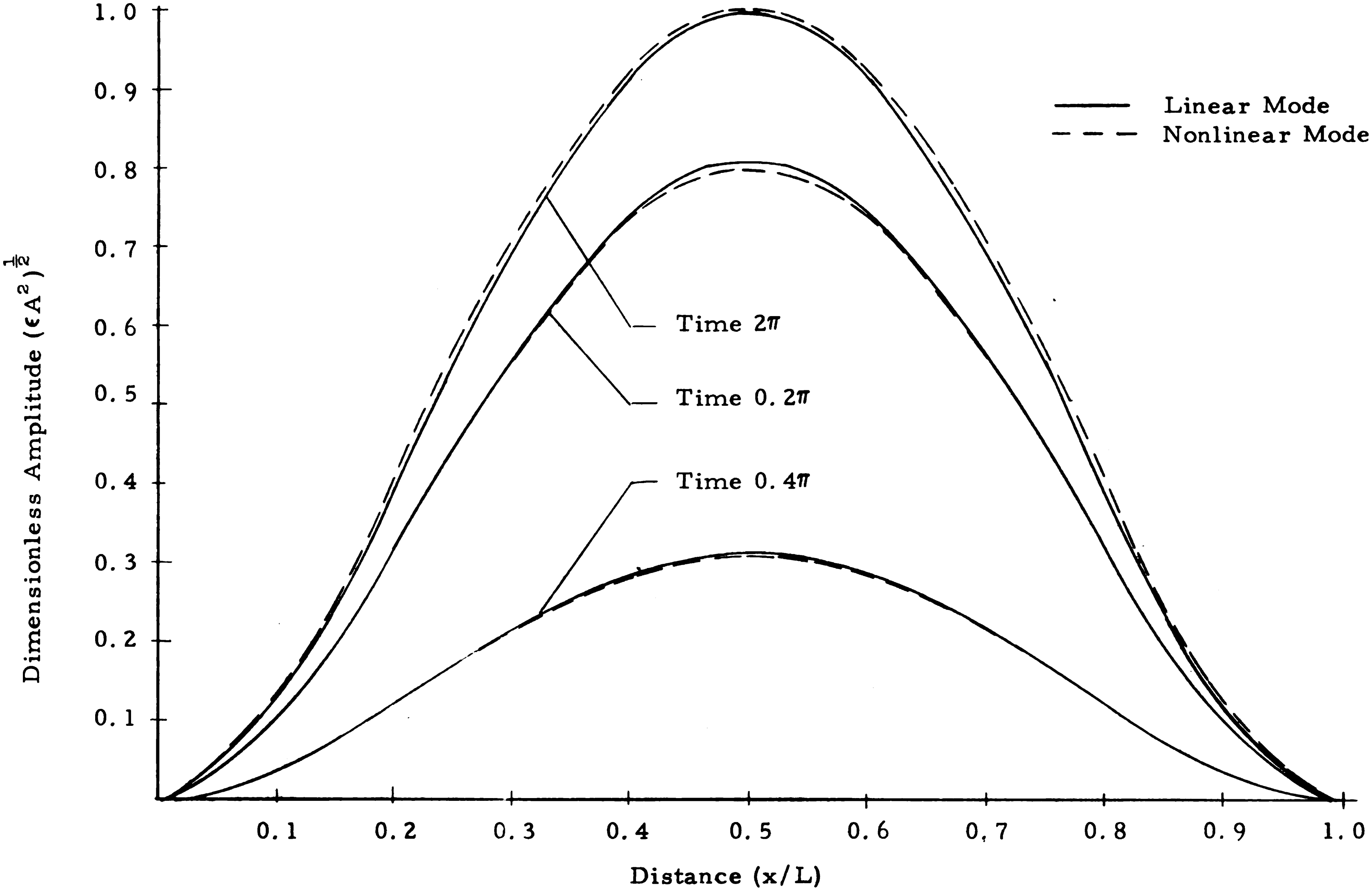

Figure 4.2.1-1. Linear and nonlinear mode shapes for a uniform beam with clamped-clamped immovable supports. Amplitude corresponds to displacement.

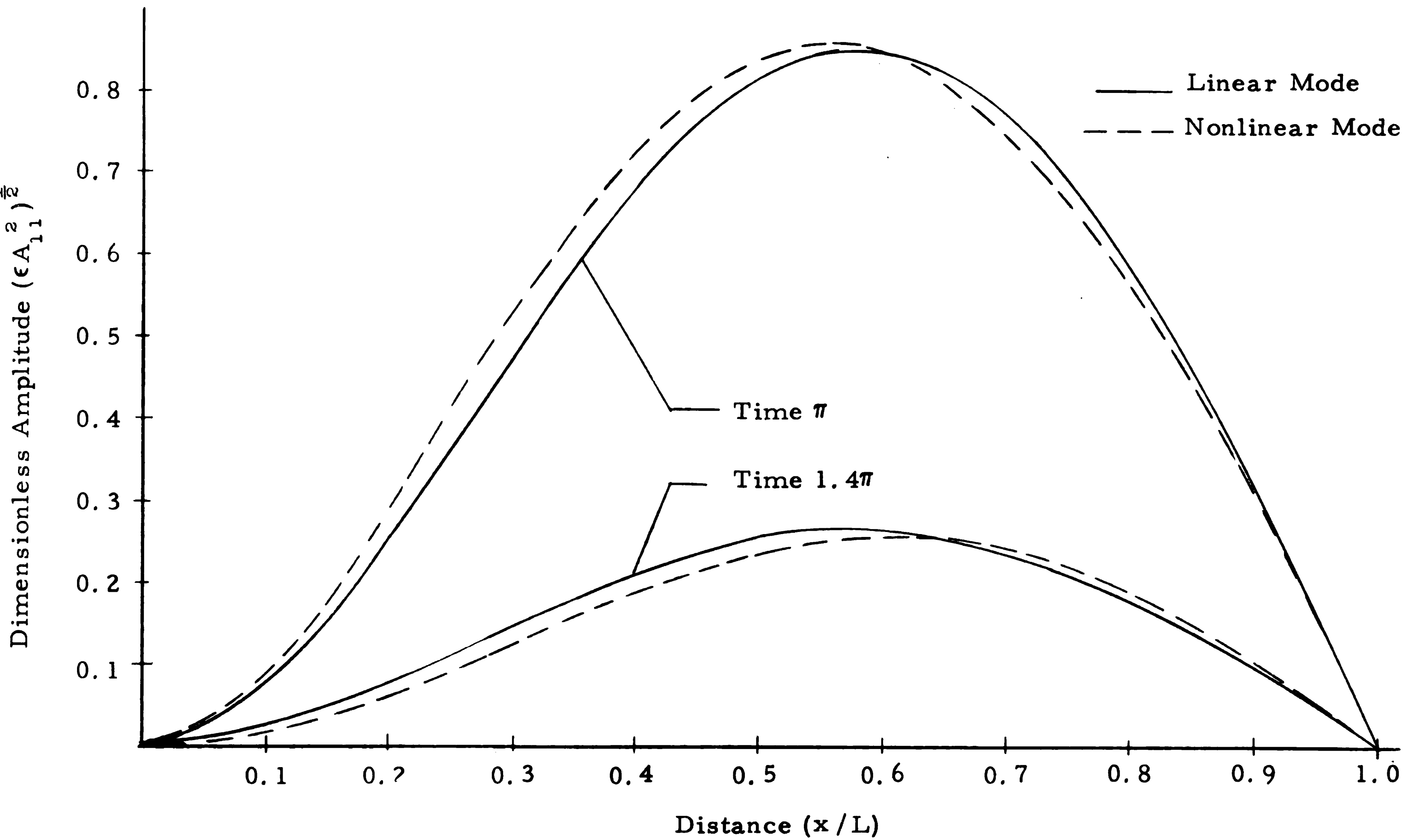

Figure 4.2.1-2. Normalized mode shape for a uniform beam clamped-supported and vibrating at large amplitudes.

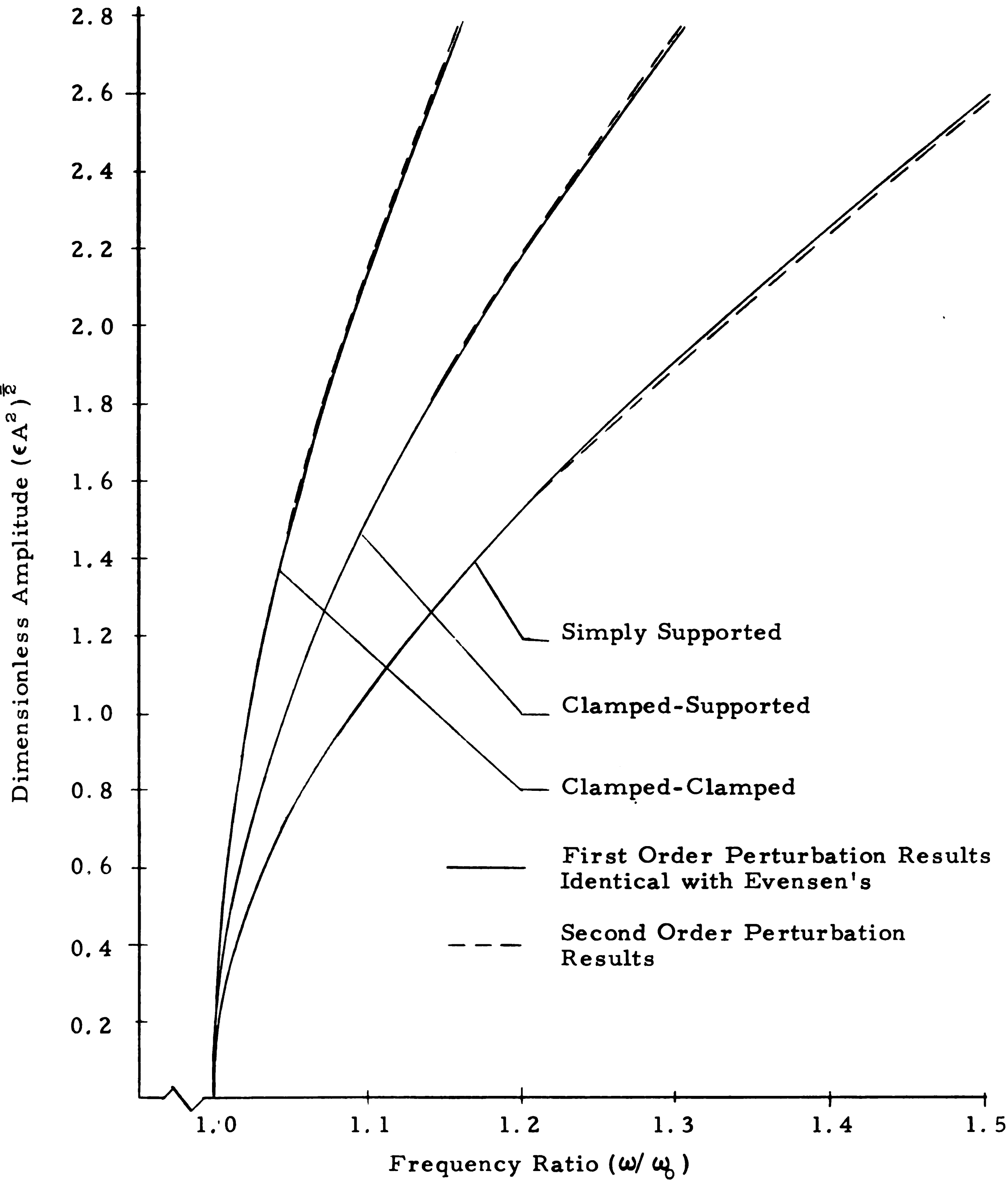

Figure 4.2.1-3. Frequency-amplitude curves for beams vibrating with large amplitudes and having various boundary conditions.

4.2.2 Vibration of Circular Plates at Large Amplitudes

The numerical solutions contained in this section pertain to axisymmetric clamped and simply supported plates vibrating at large amplitudes as described in section 3.2.2, where the dimensionless equation of motion is given as equation (3.2.2.5).

Information on the corresponding linear problem is taken from reference [26]. For a clamped axisymmetric plate with zero initial radial tension the frequency equation is

$$\frac{J_{n+1}(\gamma)}{J_n(\gamma)} + \frac{I_{n+1}(\gamma)}{I_n(\gamma)} = 0 \qquad (4.2.2.1)$$

and for the simply supported plate the frequency equation is taken as

$$\frac{J_{n+1}(\gamma)}{J_n(\gamma)} + \frac{I_{n+1}(\gamma)}{I_n(\gamma)} = \frac{2}{1-\upsilon} \qquad (4.2.2.2)$$

The corresponding linear eigenfunction for both boundary conditions follows in the form

$$V_n(r) = C\left[I_0(\gamma_n a) J_0(\gamma_n r) - J_0(\gamma_n a) I_0(\gamma_n r)\right] \qquad (4.2.2.3)$$

Notice that Poisson's ratio appears explicitly in the frequency equation for a simply supported plate, but not in the linear frequency equation for a clamped plate. Consequently, clamped plate vibrational response is independent of υ, but this is not true for the simply supported plate. As pointed out by Berger [20] in the case of the static analogue, neglecting the strain energy due to the second strain invariant can be interpreted as neglecting part of the variation of the deflection caused by a change in ν.

Cognizant of the linear results for the problem, the first order approximation to the nonlinear frequency follows from equation (3.2.2.9) as

$$\omega^2 = \omega_0^2 + 3.28138\,\epsilon A^2 \qquad (4.2.2.4)$$

for the clamped plate, and

$$\omega^2 = \omega_0^2 + 3.59788\,\epsilon A^2 \qquad (4.2.2.5)$$

for the simply supported plate.

The amplitude parameters are computed according to equations (3.2.2.11) and (2.2.2.12) and appear in table 4.2.2-1 and 4.2.2-2. The mode shapes for both boundary conditions are plotted in figure 4.2.2-1.

Substitutions of the linear frequencies and eigenfunctions into equation (3.2.2.15), along with the already computed amplitude parameters yields the second order approximation to the nonlinear frequency as

$$\omega^2 = \omega_0^2 + 3.28138\,\epsilon A^2 - .03258\,\epsilon^2 A^4 \qquad (4.2.2.6)$$

in the case of a clamped plate, and

$$\omega^2 = \omega_0^2 + 3.59788\,\epsilon A^2 + .02572\,\epsilon^2 A^4 \qquad (4.2.2.7)$$

for a simply-supported plate. In order to compare these results with those obtained by Wah using a modified Galerkin approach, the dimensionless amplitude is plotted against the ratio of the nonlinear period to the linear period in figure 4.2.2-2. Tables 4.2.2-3 and 4.2.2-4 contain the nonlinear results corresponding to the order of the linear eigenfunction used in the series of equation (3.2.2.15).

Table 4.2.2-1. Nonlinear amplitude parameters for a clamped circular plate vibrating at large amplitudes.

n = spatial function
m = time function

$A_{mn}^{(1)} = a_{mn} A_{11}^{3}$
$A_{mn}^{(1)} = d_{mn} A^{3}$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		-1.2213 10^{-2}	1.9440 10^{-3}	-5.2062 10^{-4}			
3	1.4371 10^{-2}	-9.3599 10^{-3}	7.2529 10^{-4}	-1.7955 10^{-4}			

Table 4.2.2-2. Nonlinear amplitude parameters of a simply supported circular plate vibrating at large amplitudes.

n = spatial function m = time function							$A_{mn}^{(1)} = a_{mn}A_{11}^{3}$ $A_{mn}^{(1)} = d_{mn}A^{3}$
a_{mn}							
m \ n	1	2	3	4	5	6	7
1		2.9392 10^{-3}	-3.2279 10^{-4}				
3	4.3281 10^{-2}	1.2666 10^{-3}	-1.1155 10^{-4}				

Table 4.2.2-3. Frequency numerical values for a clamped circular plate vibrating at large amplitudes.

LINEAR MODE n	LINEAR FREQ $\omega_0^2 = \gamma_n^4$	FIRST ORDER ω_1^2		SECOND ORDER ω_2^2	
		A^2	A_{11}^2	A^4	A_{11}^4
1	γ_1 = 3.1961	3.281.38	35.7989		
2	γ_2 = 6.3064			-.036912	-4.35053
3	γ_3 = 9.4395			-.031483	-3.70157
4	γ_4 = 12.577			-.032580	-3.84009

Table 4.2.2-4. Frequency numerical values for a simply supported plate vibrating at large amplitudes.

LINEAR MODE n	LINEAR FREQ $\omega_0^2 = \gamma_n^4$	FIRST ORDER ω_1^2		SECOND ORDER ω_2^2	
		A^2	A_{11}^2	A^4	A_{11}^4
1	γ_1 = 2.22	3.59788	25.2322		
2	γ_2 = 5.45			.026509	1.30368
3	γ_3 = 8.61			.025724	1.26519

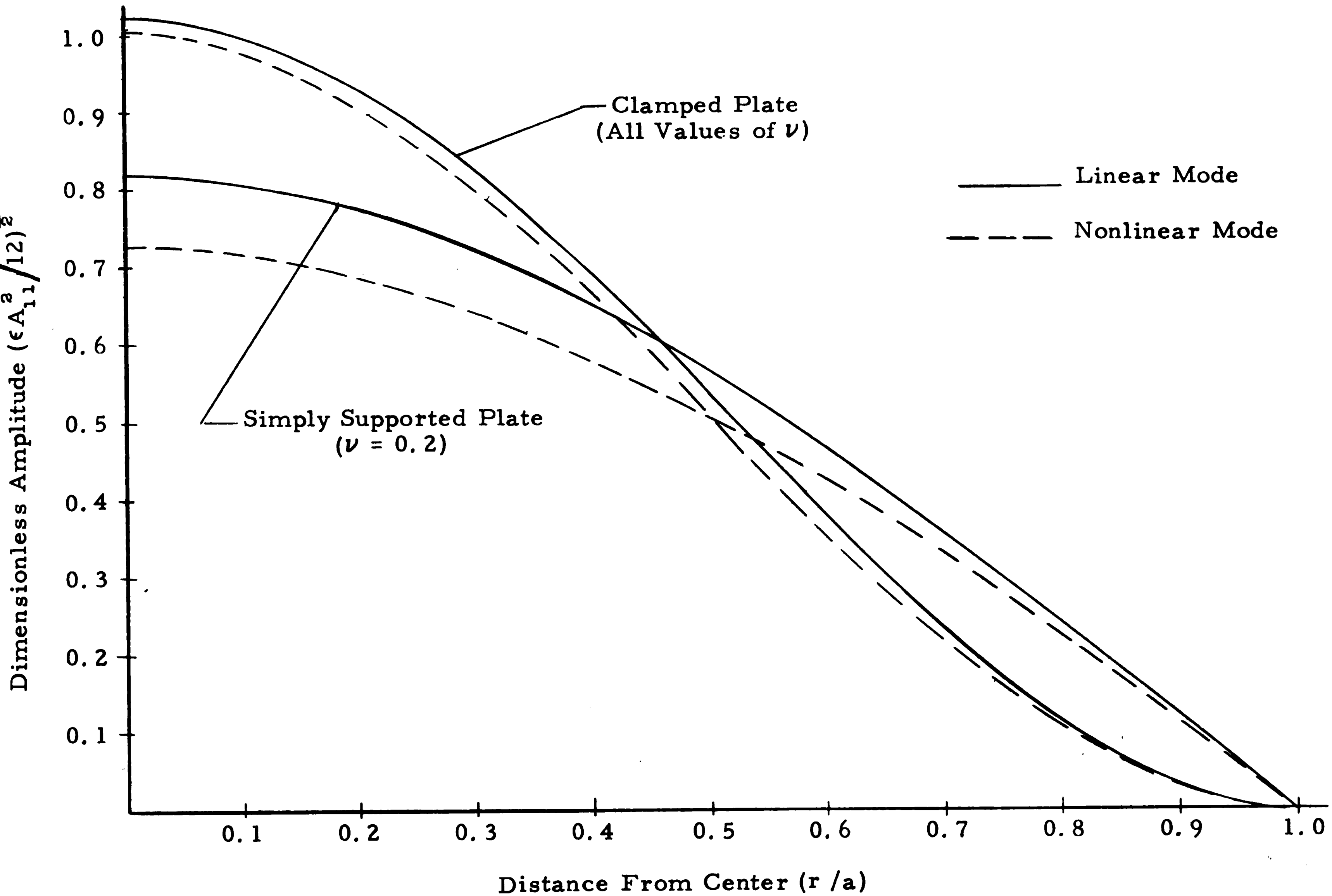

Figure 4.2.2-1. Normalized mode configurations for various plates with large amplitudes of vibration time = 0.4π

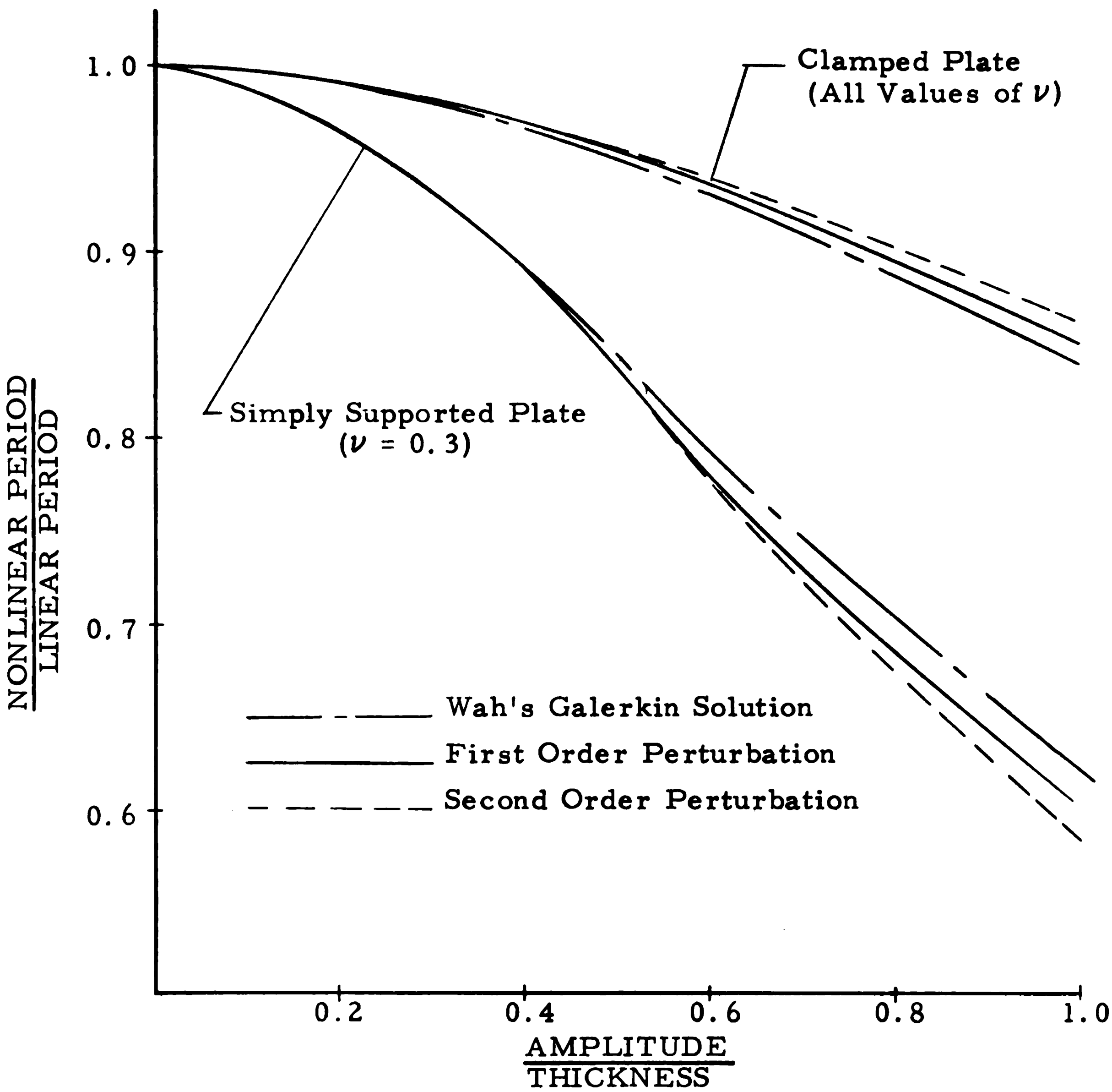

Figure 4.2.2-2. Ratio nonlinear-linear period vs ratio amplitude-thickness for circular plates with various boundary conditions.

4.2.3. Vibration of Membranes at Large Amplitudes

The following numerical results are obtained through the expressions developed in section 3.2.3 describing large amplitude vibrations of circular membranes. Using the linear theory from reference [31] and substituting into equation (3.2.2.9), one obtains the first order frequency-amplitude relation

$$\omega^2 = \omega_0^2 + 3.375\,\epsilon A^2 \tag{4.2.3.1}$$

The nonlinear modal constants are found from equations (3.2.2.11) and (3.2.2.12). Numerical values are given in table 4.2.3-1 and the mode configuration at different times are plotted in figure 4.2.3-1.

By equation (3.2.2.15) and after some manipulation the second order frequency-amplitude relation is

$$\omega^2 = \omega_0^2 + 3.375\,\epsilon A^2 + .0609\,\epsilon^2 A^4 \tag{4.2.3.2}$$

The relationship between the amplitudes is again

$$A = A_{11} \, |V_1|_{max} \tag{2.3.4}$$

where A is the maximum deflection of the linear mode, $|V_1|_{max}$ is the maximum value of the normalized eigenfunction, ϵ the perturbation parameter and A_{11} the amplitude associated with the normalized linear mode. In order to compare the results with those of Chobotov and Binder [8] the ratio of the nonlinear to linear period is plotted against a dimensionless amplitude parameter defined as

$$\lambda^2 = \frac{6 W_0^2 \, Eh}{a^4 \bar{\omega}_0^2 \, \rho (1-\upsilon)} \tag{4.2.3.3}$$

where in the notation of Chobotov, W_0 is the maximum central displacement of the nonlinear mode. Figure 4.2.3-2 indicates exceptionally good agreement even at large amplitudes.

Table 4.2.3-1. Frequency numerical values for a circular membrane vibrating at large amplitudes.

LINEAR MODE n	LINEAR FREQ $\omega_0^2 = \gamma_n^2$	FIRST ORDER ω_1^2		SECOND ORDER ω_2^2	
		A^2	A_{11}^2	A^4	A_{11}^4
1	$\gamma_1 = 2.404$	3.3756	25.0323		
2	$\gamma_2 = 5.520$			.06219	3.42018
3	$\gamma_3 = 8.654$			.06142	3.37775
4	$\gamma_4 = 11.792$			.06097	3.35286

Table 4.2.3-2. Nonlinear amplitude parameters for a circular membrane vibrating at large amplitudes.

n = spatial function
m = time function

$$A_{mn}^{(1)} = a_{mn} A_{11}^3$$
$$A_{mn}^{(1)} = d_{mn} A^3$$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		1.9470 10^{-4}	-2.5605 10^{-4}	-1.7831 10^{-4}			
3	1.8046 10^{-1}	-7.4388 10^{-5}	-2.5783 10^{-4}	-9.1010 10^{-5}			

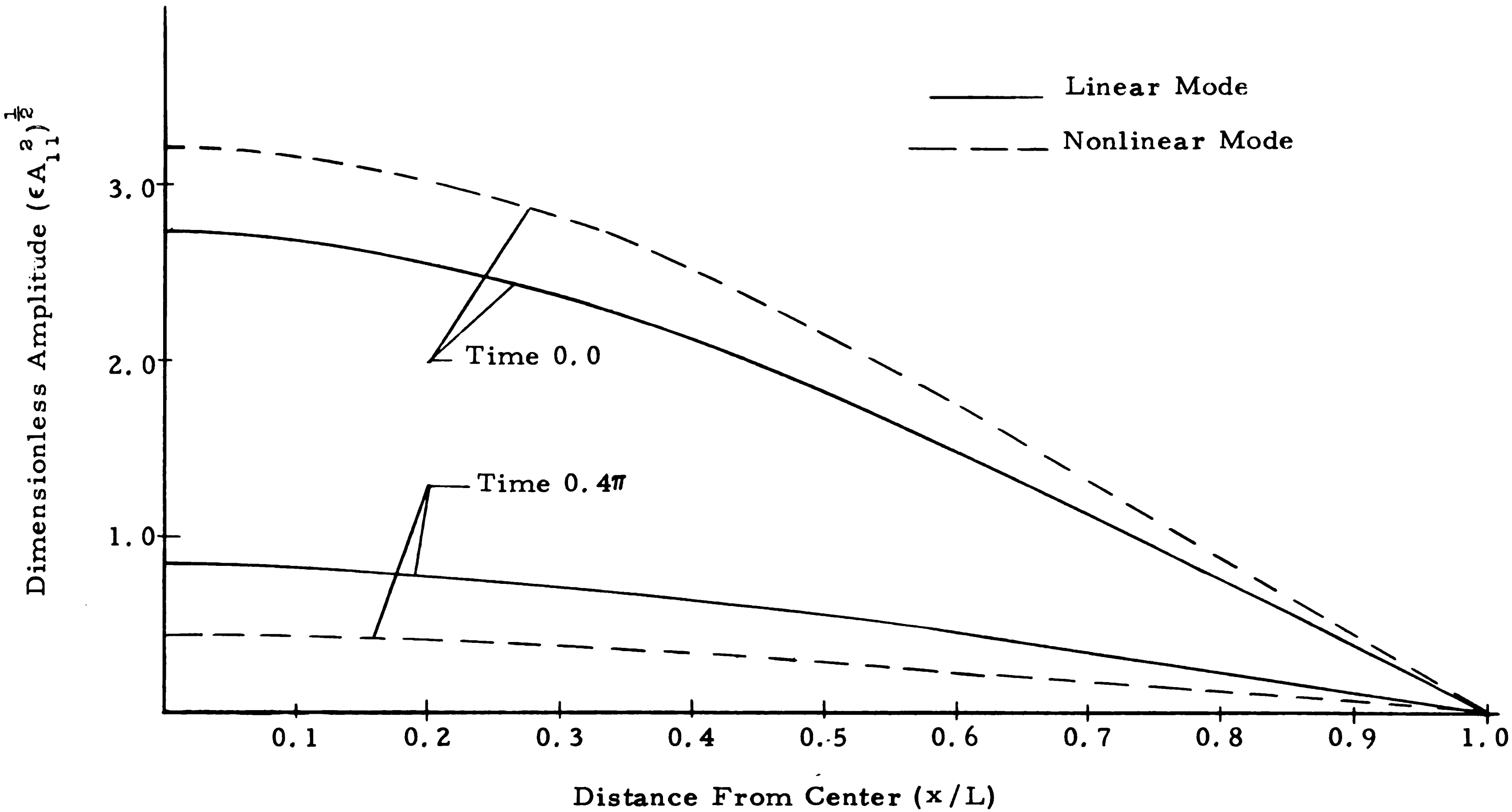

Figure 4.2.3-1. Normalized mode configurations for a membrane vibrating with large amplitudes.

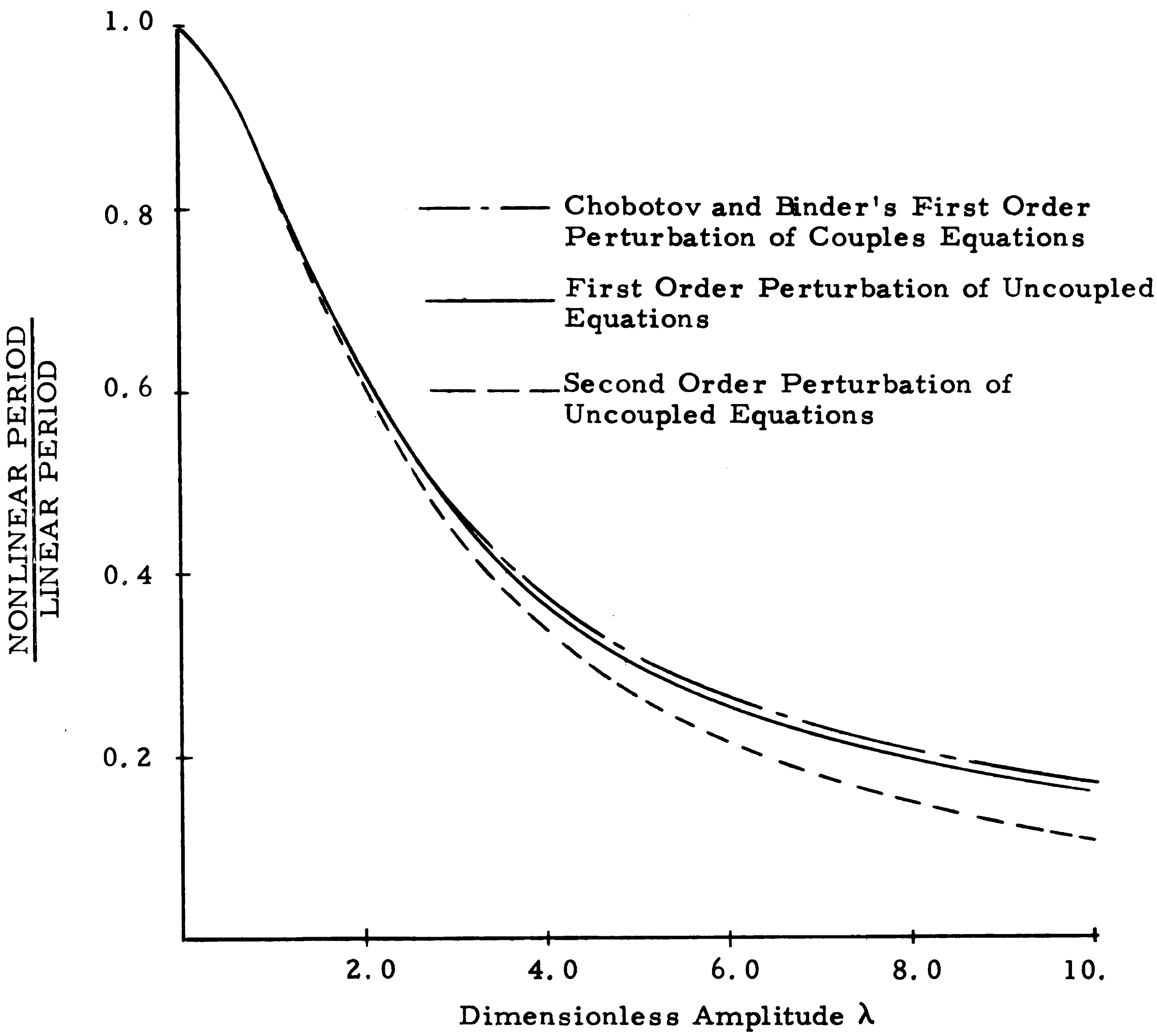

Figure 4.2.3-2. Ratio of the nonlinear to linear period vs nondimensional displacement for vibration of a circular membrane with large amplitudes.

4.3 Numerical Solutions for Continuous Media Having Nonlinear Constitutive Equations.

Several different types of dynamic systems including plates, membranes and beams with nonlinear elastic materials are solvable by means of the general theory presented in Chapter II. However, in order to illustrate the numerical procedure, only application to beams as developed in section 3.3 is considered. The motion is assumed to be defined by the dimensionless equation of motion (3.3.4).

The necessary results on the linear problem for beams with various boundary conditions are contained in Wylie [25]. Substituting these results into equation (3.3.7) and performing the integrations on the computer, one obtains the first order frequency-amplitude relations as

$$\omega^2 = \omega_0^2 + .18750\,\epsilon A^2 \tag{4.3.1}$$

for the simply supported beam,

$$\omega^2 = \omega_0^2 + 1.1042\,\epsilon A^2 \tag{4.3.2}$$

for the clamped-supported, and

$$\omega^2 = \omega_0^2 + 4.8474\,\epsilon A^2 \tag{4.3.3}$$

for the clamped-clamped beam. The amplitude A has been previously defined as the maximum deflection of the linear mode.

After truncating the series in equation (3.3.8) to include the sixth linear eigenfunction, the amplitude parameters follow as in tables 4.3-1, 4.3-2, and 4.3-6, where A_{11} is associated with the normalized eigenfunction as before.

Second order nonlinear frequency-amplitude relations follow from equation (3.3.9) as

$$\omega^2 = \omega_0^2 + .18750\,\epsilon A^2 - .011865\epsilon^2 A^4 \qquad (4.3.4)$$

for the simply supported case,

$$\omega^2 = \omega_0^2 + 1.1042\,\epsilon A^2 - .38317\epsilon^2 A^4 \qquad (4.3.5)$$

for the clamped-supported, and

$$\omega^2 = \omega_0^2 + 4.8474\,\epsilon A^2 - 4.12345\epsilon^2 A^4 \qquad (4.3.6)$$

for the clamped-clamped beam.

Kauderer [32] has approximated the case of the simply supported beam by assuming a particular solution of higher harmonics and equating coefficients. The first order frequency relation obtained here is identical with his approximation. Figure 4.3-4 shows the frequency response for all three boundary conditions. Figures 4.3-1, 4.3-2, and 4.3-3 are graphs of the nonlinear mode shapes compared with the corresponding linear configuration at different times. It is interesting to note how the nonlinear contribution changes the algebraic sign during a period of vibration and the maximum deflection of the clamped-supported beam changes position along the span at different times.

Table 4.3-1. Nonlinear amplitude parameters for a uniform beam simply supported and having the nonlinear constitutive equation as $N = E(\epsilon + h\epsilon^3)$

n = spatial function $A^{(1)}_{mn} = a_{mn} A^3_{11}$

m = time function $A^{(1)}_{mn} = d_{mn} A^3$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		0.0	4.4762 10^{-3}	0.0	0.0	0.0	
3	4.9735 10^{-3}	0.0	1.6578 10^{-3}	0.0	0.0	0.0	

d_{mn}

m \ n	1	2	3	4	5	6	7
1		0.0	7.0312 10^{-3}	0.0	0.0	0.0	
3	7.8125 10^{-3}	0.0	2.6041 10^{-3}	0.0	0.0	0.0	

Table 4.3-2. Nonlinear amplitude parameters for a uniform beam clamped-supported and having the nonlinear constitutive equation as $N = E(\epsilon + h\epsilon^3)$

n = spatial function $A^{(1)}_{mn} = a_{mn} A^3_{11}$

m = time function $A^{(1)}_{mn} = d_{mn} A^3$

a_{mn}

m \ n	1	2	3	4	5	6	7
1		-2.1208 10^{-2}	-5.1031 10^{-3}	-1.0420 10^{-2}	-3.0343 10^{-3}	-1.2659 10^{-3}	
3	1.3670 10^{-2}	-4.4729 10^{-2}	-2.0716 10^{-3}	-3.6963 10^{3}	-1.0382 10^{-3}	-4.2744 10^{-4}	

d_{mn}

m \ n	1	2	3	4	5	6	7
1		3.3048 10^{-5}	3.4321 10^{-7}	3.0288 10^{-8}	3.8111 10^{-10}	6.8748 10^{-12}	
3	-4.7938 10^{-4}	6.9701 10^{-5}	1.3933 10^{-7}	1.0744 10^{-8}	1.3039 10^{-10}	2.3213 10^{-12}	

Table 4.3-3. Frequency numerical values for a uniform beam simply supported and having the nonlinear constitutive equation as $N = E(\epsilon + h\epsilon^3)$

LINEAR MODE n	LINEAR FREQ. $\omega_0^2 = n^4$	FIRST ORDER ω_1^2 A^2	A_{11}^2	SECOND ORDER ω_2^2 $A^4 10^{-2}$	$A_{11}^4 10^{-3}$
1	1	0.18750	0.1194		
2	16			-1.46484	.593678
3	81			-1.18652	-4.80879
4	256			-1.18652	-4.80879
5	625			-1.18652	-4.80879
6	1296			-1.18652	-4.80879

Table 4.3-4. Frequency numerical values for a uniform beam clamped-supported and having the nonlinear constitutive equation as $N = E(\epsilon + h\epsilon^3)$

LINEAR MODE n	LINEAR FREQ. $\omega_0^2 = (\frac{\gamma_n}{\pi})^4$	FIRST ORDER ω_1^2 A^2	A_{11}^2	SECOND ORDER ω_2^2 $A^4 10^{-1}$	$A_{11}^4 10^{-1}$
1	γ_1 = 3.9266023	1.1042	0.8006		
2	γ_2 = 7.0685827			-.805985	-.423622
3	γ_3 =10.2101761			-.990191	-.520440
4	γ_4 =13.3517688			-3.23279	-1.69914
5	γ_5 =16.4933614			-3.67639	-1.93229
6	γ_6 =19.6349541			-3.83178	-2.01396

Table 4.3-5. Frequency numerical values for a uniform beam clamped-clamped and having the nonlinear constitutive equation as $N = E(\epsilon + h\epsilon^3)$

LINEAR MODE n	LINEAR FREQ. $\omega_0^2 = (\frac{\gamma_n}{\pi})^4$	FIRST ORDER ω_1^2		SECOND ORDER ω_2^2	
		A^2	A_{11}^2	A^4	A_{11}^4
1	γ_1 = 4.7300408	4.8474	3.8953		
2	γ_2 = 7.8532046			.19074	.12294
3	γ_3 = 10.9956078			-1.26516	-.81547
4	γ_4 = 14.1371655			-1.26516	-.81547
5	γ_5 = 17.2787596			-3.54851	-2.28723
6	γ_6 20.4203525			-4.12345	-2.65782

Table 4.3-6. Nonlinear amplitude parameters for a uniform beam clamped-clamped and having the nonlinear constitutive equation as $N = E(\epsilon + h\epsilon^3)$

n = spatial function; m = time function

$A_{mn}^{(1)} = a_{mn} A_{11}^3$

$A_{mn}^{(1)} = d_{mn} A^3$

m \ n	1	2	3	4	5	6	7
a_{mn}							
1		2.7386 10^{-6}	-4.3225 10^{-2}	8.5521 10^{-7}	-2.1976 10^{-2}	-7.8798 10^{-3}	
3	3.1573 10^{-2}	-4.2980 10^{-6}	-2.0114 10^{-2}	3.1728 10^{-7}	-7.6719 10^{-3}	-2.6887 10^{-3}	
d_{mn}							
1		1.6702 10^{-9}	-1.1374 10^{-6}	9.7250 10^{-13}	-1.0799 10^{-9}	-7.2610 10^{-13}	
3	4.2956 10^{-4}	-2.6212 10^{-9}	-5.2927 10^{-7}	3.6079 10^{-13}	-3.7700 10^{-10}	-2.4775 10^{-13}	

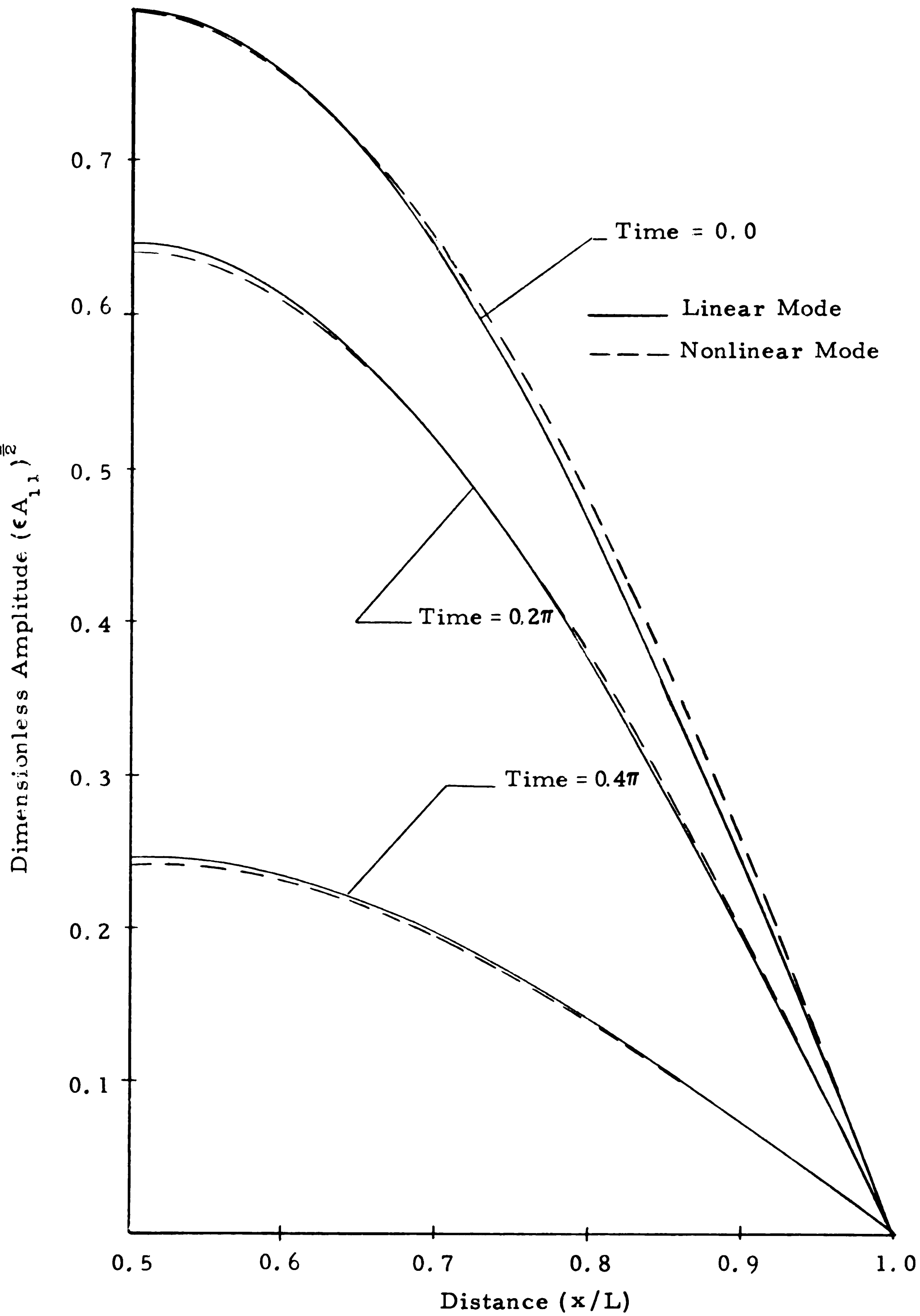

Figure 4.3-1. Normalized mode shape for uniform beam simply supported and having a nonlinear constitutive equation of $N = E(\epsilon + h\epsilon^2)$.

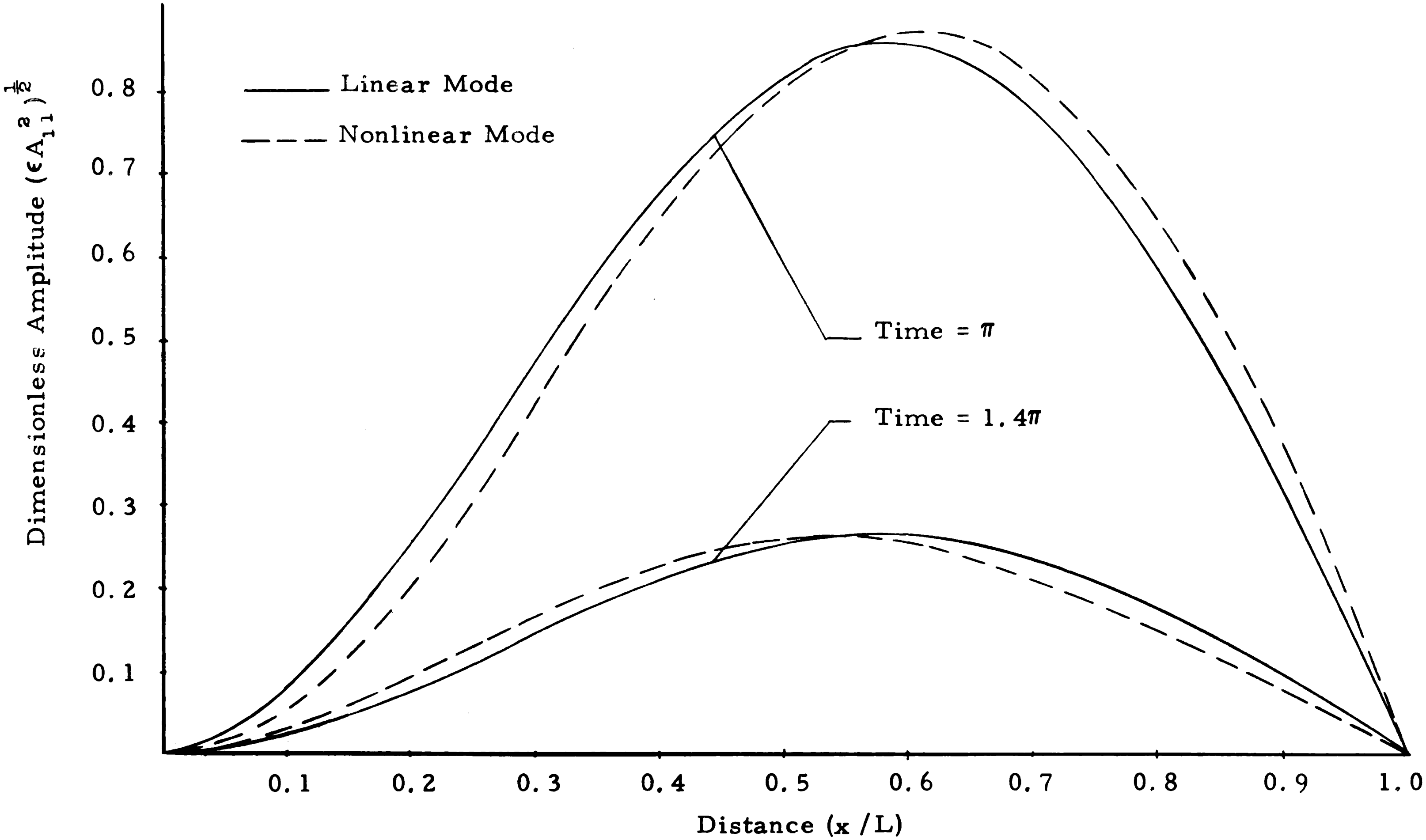

Figure 4.3-2. Normalized mode shape for a uniform beam clamped-supported and having a nonlinear constitutive equation $N = E(\epsilon + h\epsilon^3)$.

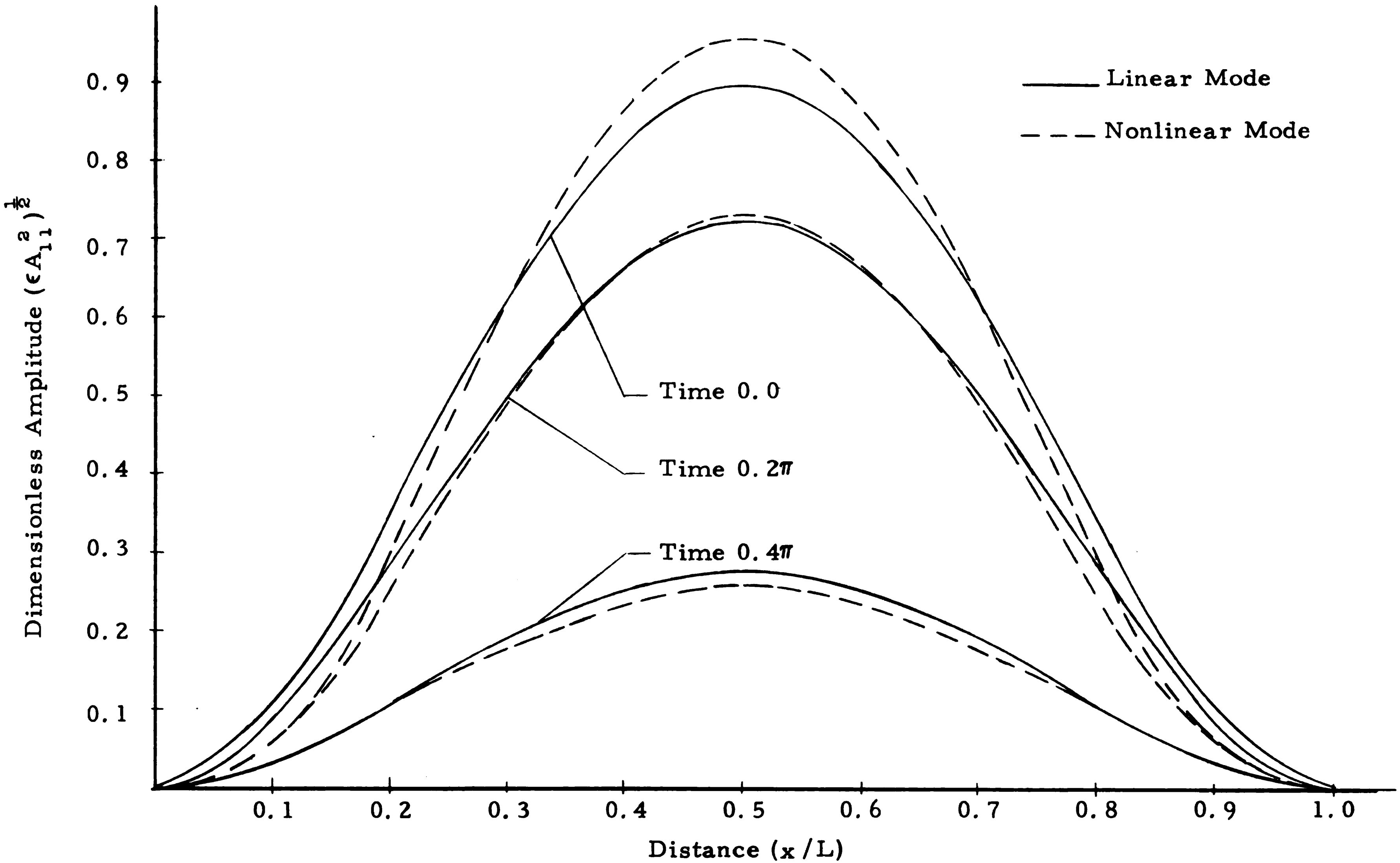

Figure 4.3-3. Normalized mode shape for a uniform beam clamped-clamped and having a nonlinear constitutive equation $N = E(\epsilon + h\,\epsilon^3)$.

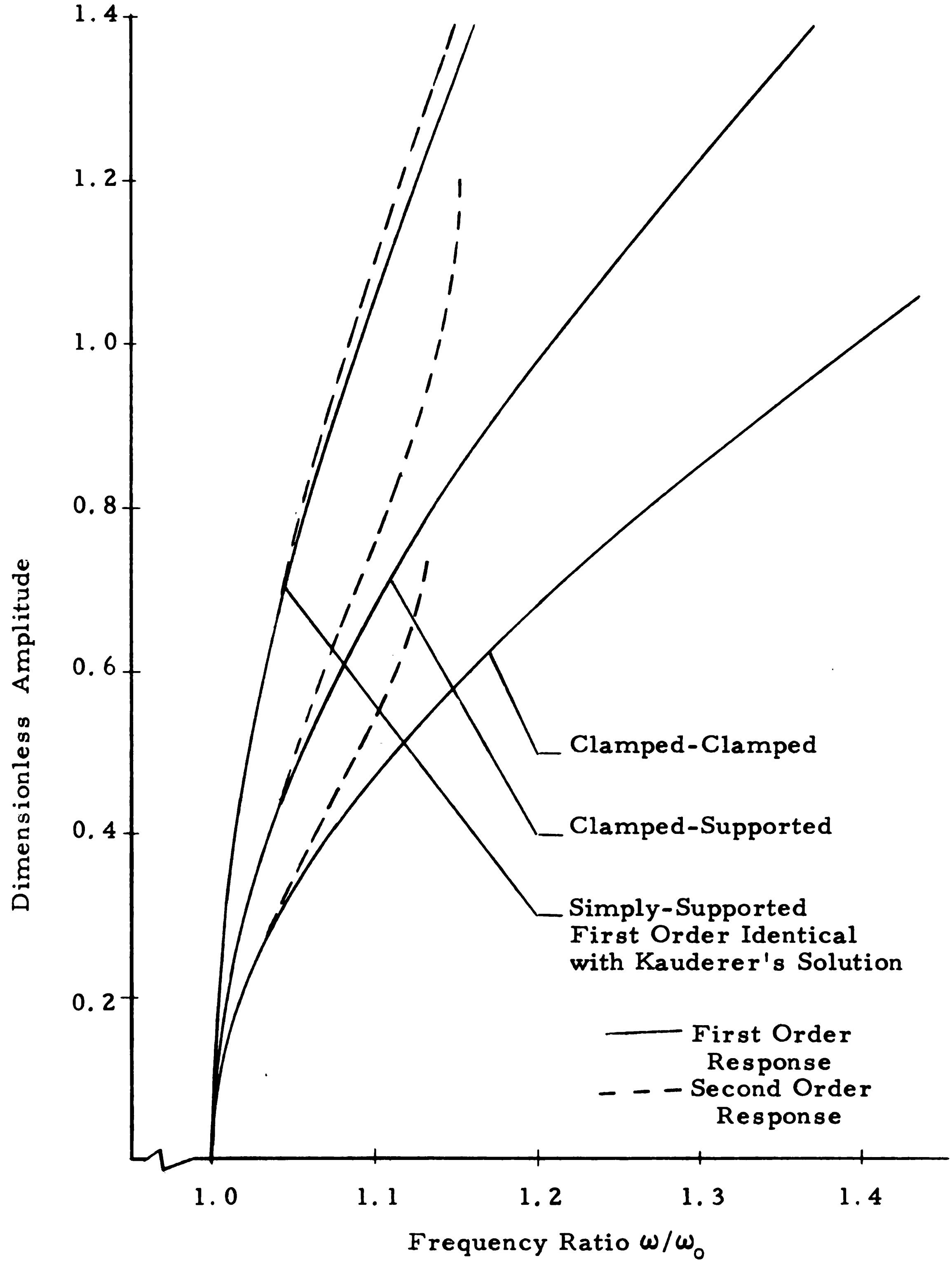

Figure 4.3-4. Frequency-amplitude curves for uniform beams having a nonlinear constitutive equation of the form $N = E(\epsilon + h\epsilon^3)$.

V. A GENERAL SOLUTION USING ULTRASPHERICAL POLYNOMIALS

5.1. Introduction

In this chapter a method is presented for determining the amplitude-frequency relations for a class of nonlinear continuous systems undergoing periodic motions. The method applies to systems governed by nonlinear partial differential equations in one space variable and one time variable, in which the nonlinear terms are assumed due to nonlinear forcing functions which depend on the displacement and its spatial derivatives, but do not depend on time explicitly.

Recently Denman et al [33, 34] developed a method for treating nonlinear vibration problems with one degree of freedom. By linearizing the nonlinear spring forces using a set of ultraspherical polynomials over the interval over which the motion takes place, they were able to obtain approximate amplitude-frequency relations with fair accuracy. Some attempt has also been made to extend the method to systems with two degrees of freedom [36].

In this present chapter, the method of ultraspherical polynomials is extended to nonlinear continuous systems of the type described above. An obvious difficulty immediately arises because the maximum displacement varies from point to point and these maximum displacements are not known in advance. To overcome this difficulty and to achieve the linearization of the nonlinear forces, one must initially assume some appropriate "mode of deflection." In cases where the linear mode of small vibration is known, this linear mode is taken to be the mode of deflection. Otherwise some suitable approximation to the linear mode has to be used. Next, an amplitude parameter is introduced so that the maximum displacements are determined by the product of the amplitude parameter and the normalized linear mode. Using ultraspherical polynomials, the nonlinear force at each point is replaced by a force that is linear in the displacement. This results in a linear partial differential equation. Together with the initial and boundary conditions, one is thus led to solve a linear eigenvalue problem and it is the determination of the eigenvalue that leads to the desired amplitude-frequency relation.

In order to illustrate the procedure, the general form of the nonlinear equation of motion is linearized and a frequency-amplitude relationship is established in section 5.2. The developed expressions are then applied in section 5.3 to typical systems of strings, bars, circular membranes and plates on nonlinear foundations, vibrating with large amplitudes, vibrating with immovable end supports, or consisting of nonlinear elastic materials as described in the previous sections. It is found that for these cases, if the Tchebycheff polynomials of the first kind (a special case of the ultraspherical polynomials found by setting $\lambda = 0$) are used, the frequency results agree exactly with those of the first order perturbation solutions.

In addition to the general solution of the linearized equation of motion by eigenfunction expansions, a uniform string on a Duffing type foundation is solved by reducing the linearized equation to the form

of the well known Mathieu equation, and a circular membrane supported by the same type foundation is solved by the application of Hankel transforms. The latter two procedures simply suggest that once the nonlinear equation of motion has been linearized, several approaches could be used to obtain the solution to the linearized problem, depending on the type of physical dynamic system.

5.2. Linearization of a Class of Nonlinear Equations of Motion and a Corresponding Frequency-Amplitude Relationship

Consider the following dimensionless form of a nonlinear partial differential equation of motion

$$L_x u + \omega^2 u_{tt} + N_x u = 0 \qquad (5.2.1)$$

where L_x denotes some linear differential operator of degree 2n in the spatial variable x. N_x is an operator such that $N_x u$ denotes a nonlinear restoring function with the property that

$$u N_x u \geq 0 \qquad (5.2.2)$$

for all u, and is assumed representable in the form

$$N_x u = \sum_{n=1}^{\infty} \bar{\alpha}_{2n+1} u^{2n+1} \qquad (5.2.3)$$

where $\bar{\alpha}_{2n+1}$ is either constant or at most a functional of the associated linear spatial eigenfunctions or derivatives thereof. $N_x u$ must be mathematically well defined at all points along the continua.

Eventually periodic solutions $u(x,t) = u(x, t+2\pi)$ are sought that satisfy 2n time-independent homogeneous boundary conditions of the form

$$\begin{aligned} D_i u(0,t) &= 0 \qquad i = 1,2,\ldots p \\ D_i u(L,t) &= 0 \qquad i = p+1, \ldots . 2n \end{aligned} \tag{5.2.4}$$

where D_i are linear differential operators of order $<2n$ in x, so that the associated linear time-reduced equation

$$L_x V - \Omega^2 V = 0 \tag{5.2.5}$$

where $V \equiv V(x)$, together with the boundary conditions

$$\begin{aligned} D_i V(0) &= 0 \qquad i = 1,2,\ldots p \\ D_i V(L) &= 0 \qquad i = p+1, \ldots . 2n \end{aligned} \tag{5.2.6}$$

form a properly posed self adjoint boundary value problem.

Henceforth it is assumed also that the boundary value problem posed in (5. 2. 5) and (5. 2. 6) admits nontrivial solutions when Ω^2 is equal to the eigenvalues

$$\Omega^2 = \Omega_i^2 \qquad i = 1,2,\ldots \tag{5.2.7}$$

which are all positive and form an infinite discrete set increasing monotonically to infinity, and that the corresponding orthonormal eigenfunctions $V_i(x)$ satisfying

$$\int_0^L r(x)V_i(x)V_j(x)\,dx = \delta_{ij} \tag{5.2.8}$$

are complete in the usual sense of eigenfunction expansions.

The first step in solving the system as previously described is to linearize the nonlinear equation of motion (5.2.1) by approximating $N_x u$ by a function that is linear in u. This is done by first assuming that the geometrical configuration of the continuous system is given by a linear mode $AV_i(x)$ and then treating $N_x u$ at each spatial point x along the continua, as an analogue to Denmans's single degree of freedom problem. Thus $N_x u$ is approximated by some linear ultraspherical polynomial in the following manner

$$N_x u = g(x,\lambda,A)\ P_1^{\lambda}\left(\frac{u}{AV_i(x)}\right) \tag{5.2.9}$$

where P_1^{λ} is the linear ultrashperical polynomial of degree λ, and $g(x,\lambda,A)$ is some function yet to be determined.

In order to determine $g(x,\lambda,A)$, equation (5.2.9) is multiplied by

$$\left[1 - \left(\frac{u}{AV_i(x)}\right)^2\right]^{\lambda-\frac{1}{2}}\ P_1^{\lambda}\left(\frac{u}{AV_i(x)}\right) \tag{5.2.10}$$

and integrated with respect to u from $-AV_i(x)$ to $AV_i(x)$, x being treated as a parameter. It then follows that

$$g(x,\lambda,A) = \frac{\int_{-AV_i(x)}^{+AV_i(x)} N_x u \left[1 - \left(\frac{u}{AV_i(x)}\right)^2\right]^{\lambda-\frac{1}{2}} P_1^{\lambda}\left(\frac{u}{AV_i(x)}\right) du}{\int_{-AV_i(x)}^{+AV_i(x)} \left[1 - \left(\frac{u}{AV_i(x)}\right)^2\right]^{\lambda-\frac{1}{2}} \left[P_1^{\lambda}\left(\frac{u}{AV_i(x)}\right)\right]^2 du} \tag{5.2.11}$$

which can be reduced to

$$g(x,\lambda,A) = \frac{\int_0^1 N_x sAV_i \left[1 - s^2\right]^{\lambda-\frac{1}{2}} P_1^{\lambda}(s)\, ds}{\int_0^1 \left[1 - s^2\right]^{\lambda-\frac{1}{2}} \left[P_1^{\lambda}(s)\right]^2 ds} \tag{5.2.12}$$

upon setting $s = u/AV_i(x)$ and using the fact that both $N_x u$ and $P_1(u/AV_i(x))$ are odd functions of u.

When the terms in the series expansion for $N_x u$ given in equation (5. 2. 3) along with the normalized $P_1^{\lambda}(s)$ * written as

$$P_1^{\lambda}(s) = 2\lambda s \tag{5.2.13}$$

are substituted into equation (5. 2. 12) one obtains

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* Appendix A contains a more complete discussion of ultraspherical polynomials and normalization.

$$g_{2n+1}(x,\lambda,A) = \frac{\bar{\alpha}_{2n+1}(AV_i)^{2n+1} \int_0^1 s^{2n+2}\left[1 - s^2\right]^{\lambda-\frac{1}{2}} ds}{2\lambda \int_0^1 \left[1 - s^2\right]^{\lambda-\frac{1}{2}} s^2\, ds} \tag{5.2.14}$$

for nonnegative integer n. After making the variable change $s = y^{1/2}$ and some manipulation, the integrated result follows as

$$g_{2n+1}(x,\lambda,A) = \frac{\bar{\alpha}_{2n+1} H_{2n+1}\left[AV_i(x)\right]^{2n+1}}{2\lambda} \tag{5.2.15}$$

$$\lambda > -1/2$$

where the constant H_{2n+1} are given in terms of Gamma functions as

$$H_{2n+1} = \frac{\Gamma(n+3/2)\ \Gamma(\lambda+2)}{\Gamma(3/2)\ \Gamma(\lambda+n+2)} \tag{5.2.16}$$

Thus, with

$$g(x,\lambda,A) = \sum_{n=o}^{\infty} g_{2n+1}(x,\lambda,A) \tag{5.2.17}$$

and from equations (5.2.9) and (5.2.13) it follows that

$$N_x u \simeq \sum_{n=o}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1}\left[AV_i(x)\right]^{2n} u \tag{5.2.18}$$

Substitution of (5.2.18) into equation (5.2.1) reduces the latter to the following linear form

$$L_x u + \omega^2 u_{tt} + \sum_{n=o}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1} \left[AV_i(x) \right]^{2n} u = 0 \qquad (5.2.19)$$

The equation (5.2.19) is separable. To solve the self adjoint boundary value problem now described by the linear equation (5.2.19) along with the boundary conditions (5.2.4), initial conditions (2.2.5) and periodicity requirement (2.2.4) as discussed before it is sufficient to represent the solution in the form

$$u(x,t) = \left(\sum_{p=1}^{\infty} A_p V_p(x) \right) \cos t \qquad (5.2.20)$$

where the series in the parentheses is the expansion of the spatial part of $u(x,t)$ with respect to the linear eigenfunctions $V_p(x)$. The coefficients A_p are yet to be determined. Upon substitution of (5.2.20) into (5.2.19) one obtains

$$\sum_{p=1}^{\infty} L_x A_p V_p(x) \cos t - \omega^2 \sum_{p=1}^{\infty} A_p V_p(x) \cos t$$
$$+ \sum_{n=o}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1} \left[AV_i(x) \right]^{2n} \sum_{p=1}^{\infty} A_p V_p(x) \cos t = 0 \qquad (5.2.21)$$

It is to be noted that ω^2 appearing above yields the approximate nonlinear vibration frequency.

Dividing through (5.2.21) by cos t and using equation (5.2.5) one obtains

$$\sum_{p=1}^{\infty} (\Omega_p^2 - \omega^2) A_p V_p(x) + \sum_{n=o}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1} \left[A V_i(x) \right]^{2n}$$

$$\sum_{p=1}^{\infty} A_p V_p(x) = 0 \qquad (5.2.22)$$

Upon multiplying by $r(x)V_k(x)$, and integrating with respect to x from 0 to L equation (5.2.22) reduces, in view of the orthogonality condition (5.2.8) to

$$\sum_{p=1}^{\infty} (\Omega_p^2 - \omega^2) A_p \delta_{pk} + \int_0^L \sum_{n=o}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1} A^{2n} V_i^{2n}(x) \sum_{p=1}^{\infty} A_p V_p(x) r(x)$$

$$V_k(x)\,dx = 0 \qquad (5.2.23)$$

Thus

$$\omega^2 = \Omega_k^2 + \frac{1}{A_k} \int_0^L \sum_{n=o}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1} A^{2n} V_i^{2n}(x) \sum_{p=1}^{\infty}$$

$$(5.2.24)$$

$$A_p V_p(x) r(x) V_k(x)\,dx \qquad k=1,2,3,\ldots$$

The ω^2 above can be interpreted as the square of the nonlinear frequency found by assuming the ith linear mode in reducing the nonlinear $N_x u$. It is expected that as the nonlinearity tends to zero, u(x,t) tends to $AV_i(x)$ cos t and ω^2 tends to Ω_i^2. Consequently, only when k = i in (5.2.24) does equation (5.2.24) yield a meaningful solution. Upon setting $k = i$ and $A = A_i$, the square of the ith nonlinear frequency ω_i is given by

$$\omega_i^2 = \Omega_i^2 + \int_0^L \sum_{n=0}^{\infty} \bar{\alpha}_{2n+1} H_{2n+1} A_i^{2n} V_i^{2n+2}(x)\, r(x)\, dx \qquad (5.2.25)$$

$$i=1,2\ldots.$$

The expressions given in (5.2.25) for the nonlinear frequencies will be applied to a number of particular continuous systems in the subsequent sections and the results will be compared with those obtained by the perturbation method in the previous chapters.

5.3. The Application of Ultraspherical Polynomials to Approximate Frequency-Amplitude Relationships

5.3.1. The Restoring Force Nonlinear in Lateral Displacement

In this section the frequency-amplitude relationships for continuous systems supported by nonlinear foundations whose resistances depend upon the lateral displacements, will be determined by using the ultraspherical polynomial linearization method as developed in section 5.2. The motions of the vibrating systems are governed by equation (5.2.1), with the boundary conditions in the form (5.2.4), and initial conditions (2.2.5), with the initial configuration unspecified. The linear eigenvalues and eigenfunctions follow from the solution of the associated linear problem as posed by the time reduced equation (5.2.5) and the boundary conditions (5.2.6).

As mentioned above the nondimensional equation of motion is

$$L_x u + \omega^2 u_{tt} + N_x u = 0 \qquad (5.2.1)$$

where, for example, $L_x = \frac{-\partial^2}{\partial x^4} + \alpha_1$ for a string, $L_x = \nabla^4 + \alpha_1$ in the case of a circular plate, $L_x = -\nabla^2 + \alpha_1$ denotes a membrane and $L_x = \frac{\partial^4}{\partial x^4} + \alpha_1$ for a uniform beam. The constant α_1 is the linear foundation constant and ω^2 is a dimensionless frequency parameter as defined in section 3.1.2 for a beam of length π.

In order to compare results with those found using the perturbation approach, a Duffing-type foundation is considered by truncating equation (5.2.3) at $n = 1$. The nonlinear restoring function is written as

$$N_x u = \bar{\alpha}_3 u^3 \qquad (5.3.1.1)$$

where $\bar{\alpha}_3$ is a dimensionless constant similar to α_1 as defined in section 3.1.

The lateral displacement at any point is approximated by assuming the deflected configuration to be one of the linear mode shapes, say, for example, the first linear mode $V_1(x)$

$$u_o = A_{11} V_1(x) \cos t \qquad (5.3.1.2)$$

where A_{11} is the so-called normalized amplitude, $V_1(x)$ is the spatial eigenfunction that corresponds to the square of the linear frequency Ω_1^2 satisfying (5.2.5). From equation (5.2.25) with $n = 1$ the frequency-amplitude relationship follows as

$$\omega^2 = \Omega^2 + H_3\bar{\alpha}_3 A_{11}^2 \int_0^L r(x) V_1^4(x)\, dx \qquad (5.3.1.3)$$

where H_3 is found from equation (5.2.16) with n = 1 as

$$H_3 = \frac{3}{2(\lambda+2)} \qquad (5.3.1.4)$$

If $\lambda = 0$ (Tchebycheff polynomials of the first kind), the frequency-amplitude response as determined by the method of ultra-shperical polynomials is identical to the first order frequency-amplitude relationship as found by the perturbation approach, which was given as equation (2.2.23) in section 2.2.

5.3.2 Elastic Beams with Immovable Supports

A prismatic vibrating beam having immovable or axially springed end supports represents a dynamic system whose governing equation of motion contains a restoring function nonlinear in the derivatives of the lateral displacement. The system was considered previously in section (3.2.1), and the nondimensional equation of motion given as

$$u_{xxxx} - \beta u_{xx} + \omega^2 u_{tt} - \epsilon \int_0^{\pi} u_x^2\, dx\; u_{xx} = 0 \qquad (3.2.1.5)$$

Equation (3.2.1.5) may be brought to the general equation (5.2.1) if one sets

$$L_x = \frac{\partial^4}{\partial x^4} - \beta \frac{\partial^2}{\partial x^2} \tag{5.3.2.1}$$

and

$$N_x u = - \epsilon \int_o^{\pi} u_x^2 \, dx \; u_{xx} \tag{5.3.2.2}$$

In order to represent $N_x u$ in the desired form of equation (5. 2. 3) the linear mode shape for u_o is again assumed as

$$u_o = A_{11} V_1(x) \cos t \tag{3.2.1.10}$$

where $V_1(x)$ together with its corresponding Ω_1^2 satisfy (5. 2. 5).

If one takes the first and second spatial derivatives of equation (3. 2. 1. 10) and then substitutes the results into equation (5. 3. 2. 2), it follows that

$$N_x u = \epsilon A_{11}^3 \int_o^{\pi} V_{1x}^2 \, dx \; V_{1xx} \cos^3 t = \frac{- \epsilon \int_o^{\pi} V_{1x}^2 \, dx \; V_{1xx} \; u^3}{V_1^3} \tag{5.3.2.3}$$

which can also be written as

$$N_x u = \bar{\alpha}_3 u^3 \tag{5.3.2.4}$$

where

$$\bar{\alpha}_3 = \frac{- \epsilon \int_o^{\pi} V_{1x}^2 \, dx \; V_{1xx}}{V_1^3} \tag{5.3.2.5}$$

When one substitutes equation (5. 3. 2. 5) into the expression (5. 3. 1. 3) and sets both n = 1 and r(x) = 1, the frequency-amplitude relationship follows as

$$\omega^2 = \Omega^2 - \epsilon A_{11}^2 H_3 \int_0^{\pi} V_1 V_{1x}^2 \, dx \int_0^{\pi} r(x) V_{1xx} \, dx \qquad (5.3.2.6)$$

where

$$H_3 = \frac{3}{2(\lambda+2)} \qquad (5.3.2.6)$$

If $\lambda = 0$, the ultraspherical polynomial results agree with the perturbation first order approximation to frequency-amplitude response as given in equation (3.2.1.12) of section 3.2.1 and equation (2.2.7).

5.3.3. Circular Plates Vibrating with Large Amplitudes

The vibrating plate systems considered in this section are those described previously in section 3.2.2. However, frequency-amplitude relationships are now found by the ultraspherical polynomial method. The nondimensional equation of motion is

$$\nabla^4 u + \omega^2 u_{tt} - \epsilon \int_0^1 u_r r dr \, \nabla^2 u = 0 \qquad (3.2.2.5)$$

as given in section 3.2.2. Equation (3.2.2.5) takes the form of equation (5.2.1) if r is now considered as the spatial variable instead of x and

$$L_x \equiv L_r = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) = \nabla^4 \qquad (5.3.3.1)$$

$$N_x u \equiv N_r u = -\epsilon \int_0^1 u_r^2 \, r dr \, \nabla^2 u \qquad (5.3.3.2)$$

The nonlinear restoring function $N_r u$ is represented in the form of equation (5.2.3) by first assuming the mode configuration to be

$$u_o = A_{11} V_1(r) \cos t \qquad (2.3.2)$$

After the first and second spatial derivatives of the lateral displacement as approximated by the linear mode are substituted into equation (5. 3. 3. 2), the restoring function reduces to

$$N_r u = \bar{\alpha}_3 u^3 \tag{5.3.3.3}$$

where

$$\bar{\alpha}_3 = -\epsilon \int_0^1 \frac{V_{1r}^2 r dr}{V_1^3} \left(V_{1rr} + \frac{1}{r} V_{1r}\right) \tag{5.3.3.4}$$

If equation (5. 3. 3. 4) is substituted into equations (5. 2. 25), with n = 1 and the weighting function r(r) = r, the frequency amplitude relationship follows as

$$\omega^2 = \Omega_1^2 - \epsilon A_{11}^2 H_3 \int_0^1 r V_{1r}^2 \, dr \int_0^1 (r V_1 V_{1rr} + V_1 V_{1r}) dr \tag{5.3.3.5}$$

where

$$H_3 = \frac{3}{2(\lambda+2)} \tag{5.3.3.6}$$

and Ω_1^2 is the eigenvalue corresponding to $V_1(x)$.

Again if the Tchebycheff polynomials are used, i. e. $\lambda=0$ the approximation to frequency-amplitude is identical with the results obtained from the perturbation method given in section 3. 2. 2.

5. 3. 4. Beams Having Nonlinear Constitutive Equations

Another class of dynamic systems having nonlinear restoring forces depending on the displacement and its derivatives is that of vibrating beams with nonlinear material properties. The particular type of material stress-strain relationship to be considered in this section is taken to be that already discussed in section 3.3. The nondimensional equation of motion is given as

$$u_{xxxx} + \omega^2 u_{tt} + \epsilon\left[u_{xx}^2 u_{xxxx} + 2\, u_{xx} u_{xxx}^2\right] = 0 \qquad (3.3.4)$$

which is equivalent to equation (5.2.1) with

$$L_x = \frac{\partial^4}{\partial x^4} \qquad (5.3.4.1)$$

and

$$N_x u = \epsilon\left[u_{xx}^2 u_{xxxx} + 2\, u_{xx} u_{xxx}^2\right] \qquad (5.3.4.2)$$

The configuration of the nonlinear system is again approximated by the linear mode and after taking spatial derivatives up to the fourth order, the restoring function of equation (5.2.3) is written as

$$N_x u = \bar{\alpha}_3 u^3 \qquad (5.3.4.3)$$

where

$$\bar{\alpha}_3 = \frac{\epsilon}{V_1^3}\left[V_{1xx}^2 V_{1xxxx} + 2\, V_{1xx} V_{1xxx}^2\right] \qquad (5.3.4.4)$$

Upon substitution of equation (5.3.4.4) into equation (5.2.2.5) and with n = 1, the frequency-amplitude relationship follows as

$$\omega^2 = \Omega_1^2 + \frac{3\epsilon}{2(\lambda+2)} A_{11}^2 \int_0^{\pi} r(x) \left[V_{1xx}^2 V_{1xxxx} + 2\, V_{1xx} V_{1xxx}^2\right] \qquad (5.3.4.5)$$

If $\lambda = 0$, the Tchebycheff approximation is identical with first order frequency-amplitude responses given by equation (3.3.7), which follow from the perturbation theory.

5.4. Alternate Methods to Approximate Frequency-Amplitude Relations by using Ultraspherical Polynomials

After the nonlinear equation of motion (5.2.1) has been reduced to an approximate linear form by using the ultraspherical polynomial expansion and retaining only the linear term, the so called "linearized" equation (5.2.19) may be solved by a variety of methods. In the previous sections of this chapter a general type solution for frequency-amplitude relations was developed in terms of the eigenfunctions and eigenvalues of an associated linear problem. There are, however, occasions in which the particular linear systems may be solved more directly by applying other methods to the linearized equation of motion. As illustrations, two examples are given below.

5.4.1. Nonlinear Vibrating Spring Reduced to the Mathieu Equation

The nondimensional "linearized" equation of motion governing a stretched string of length π undergoing periodic vibrations in the presence of a nonlinear Duffing-type foundation force follows from equation (5.2.19) as

$$\frac{\partial^2 u}{\partial x^2} - \omega^2 \frac{\partial^2 u}{\partial t^2} - \left[\alpha_1 + \frac{3\alpha_1}{2(\lambda+2)} A^2 \sin^2 x \right] u = 0 \qquad (5.4.1.1)$$

where $L_x = \frac{-\partial^2}{\partial x} + \alpha_1$, the linear mode $V_1(x) = \sqrt{2/\pi}\ \sin x$, n is truncated after unity, $H_1 = 1$, $H_3 = \frac{3}{2(\lambda+2)}$, $\bar{\alpha}_3$ is taken to be equal to α_1, along with $A = \sqrt{2/\pi}\ A_{11}$, and ω^2 as defined previously. The equation (5. 4. 1. 1) is justified on the physical ground if the original length of the string is sufficiently long so that even though the slope $\frac{\partial u}{\partial x}$ is small compared with unity, the deflection u may become moderately large.

Since equation (5. 4. 1. 1) is linear, it follows that the motions are harmonic in time. Writing

$$u(x,t) = X(x) \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix} \tag{5.4.1.2}$$

for motions that have a fundamental period of 2π, one finds that $X(x)$ satisfies

$$\frac{d^2X}{dx^2} + \left[\omega^2 - \alpha_1 - \frac{3\alpha_1}{2(\lambda+2)} A^2 \sin^2 x\right] X = 0 \tag{5.4.1.3}$$

with the boundary conditions $X(0) = X(\pi) = 0$. Now since $\sin^2 x = (1 - \cos 2x)/2$, equation (5. 4. 1. 3) may be written as

$$\frac{d^2X}{dx^2} + (P - 2q \cos 2x)\ X = 0 \tag{5.4.1.4}$$

with the constant being defined by

$$P = \omega^2 - \alpha_1 - \frac{3\alpha_1 A^2}{4(\lambda+2)} \qquad q = -\frac{3\alpha_1 A^2}{8(\lambda+2)} \tag{5.4.1.5}$$

Equation (5.4.1.4) will now be recognized as the Mathieu equation, and the solution which vanishes at $x = 0$ and $x = \pi$ and reduces to $\sin x$ for small q (or A) occurs when

$$P = b_1(q) = 1 - q - \frac{1}{8}q^2 + \frac{1}{64}q^3 \ldots \tag{5.4.1.6}$$

The corresponding solution of $X(x)$ is given by

$$X(x) = se_1(x,q) = \sin x - \frac{q}{8}\sin 3x + q^2\left[\frac{\sin 5x}{192} + \frac{\sin 3x}{64} - \frac{\sin x}{128}\right] + \ldots \tag{5.4.1.7}$$

Both functions $b_1(q)$ and $se_1(x, q)$ are plotted in [37].

Combining (5.4.1.6) and (5.4.1.5) one obtains the following amplitude-frequency relation

$$\omega^2 = \Omega_1^2 + \frac{9}{8}\frac{\alpha_1 A^2}{(\lambda+2)} - \frac{9}{512}\frac{\left(\alpha_1 A^2\right)^2}{(\lambda+2)^2} + \ldots \tag{5.4.1.8}$$

where Ω_1 is the fundamental frequency for the linear problem

$$\Omega_1^2 = 1 + \alpha_1 \tag{5.4.1.9}$$

Stoker [1]has given the first order perturbation solution for this same problem. Except for some differences in the notations he obtained

$$\omega^2 = \Omega_1^2 + \frac{9}{16}\alpha_1 A^2 \tag{4.1.1.3}$$

which coincides with the first two terms of the results given in (5.4.1.8) if one sets $\lambda = 0$, i.e., if the Tchebycheff polynomials of the first kind are used.

5.4.2. Integral Transform Methods

The frequency-amplitude relations for nonlinear systems can also be obtained by applying integral transform techniques to the linearized equation of motion. For example, the nonlinear vibrating string of the previous section, or likewise a simply supported uniform beam, could be investigated by using the sine integral transform. In order to illustrate the general procedure, the Hankel transform is used in this section to predict frequency-amplitude relations for a nonlinear vibrating membrane.

The dimensionless dynamic equation governing the axisymmetric motion of a vibrating circular membrane attached to a cubic foundation is taken to be

$$-\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \omega^2 \frac{\partial^2 u}{\partial t^2} + \alpha_o u + \alpha_3 u^3 = 0 \tag{5.4.2.1}$$

Equation (5.4.2.1) is of the form (5.2.1) if

$$L_x = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \alpha_o \tag{54.2.2}$$

and

$$N_x u = \alpha_3 u^3 \qquad (5.4.2.3)$$

It is observed that for $\alpha_3 = 0$, the first linear mode of vibration is $J_o(k_1 r)$, where J_o is the Bessel function of the first kind of order zero and $k_1 = 2.40483\ldots$ is the first zero of J_o. Introducing the amplitude parameter A one may represent the maximum displacement at r by $AJ_o(k_1 r)$. The nonlinear force given in (5.4.2.3) is then approximated by the linear ultraspherical polynomial $P_1^{\lambda}\left(\frac{u}{AJ_o(k_1 r)}\right)$ in the same manner as was done in Section 5.1. Thus by equation (5.2.19) with $V_1(x) = J_o(k_1 r)$ one has

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \omega^2 \frac{\partial^2 u}{\partial t^2} - \left[\alpha_o + \frac{3\alpha_3}{2(\lambda+2)} A^2 J_o^2 (k_1 r)\right] u = 0 \qquad (5.4.2.4)$$

Upon separating variables by writing

$$u(r,t) = R(r) \begin{Bmatrix} \sin t \\ \cos t \end{Bmatrix} \qquad (5.4.2.5)$$

one finds that R(r) satifies

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[P + q\, J_o^2 (k_1 r)\right] R = 0 \qquad (5.4.2.6)$$

where $p = \omega^2 - \alpha_o$ and $q = -\frac{3\alpha_3 A^2}{2(\lambda+2)}$. In addition R(r) satisfies the boundary conditions

$$R(0) = \text{Finite} \qquad R(1) = 0 \qquad (5.4.2.7)$$

It is noted that (5.4.2.6) plays the same role as the Mathieu equation (5.4.1.4) does in the string problem.

To solve the eigenvalue problem posed by (5.4.2.6) and (5.4.2.7) the method of Hankel transforms is employed. Let

$$\bar{R}_n = \int_0^1 rR(r)J_o(k_n r)\,dr \tag{5.4.2.8}$$

where k_n denotes the nth zero of J_o. Equation (5.4.2.6) is now multiplied by $rJ_o(k_n r)$ and integrated with respect to r from 0 to 1. After integrating the first two terms by parts one obtains

$$(p - k_n^2)\bar{R}_n + q\int_0^1 r\,R(r)J_o^2(k_1 r)J_o(k_n r)\,dr = 0 \tag{5.4.2.9}$$

To simplify the integral on the left it is noted that R(r) has the following eigenfunction expansion in terms of $J_o(k_n r)$

$$R(r) = 2\sum_{m=1}^{\infty} \frac{\bar{R}_m J_o(k_m r)}{J_1^2(k_m)} \tag{5.4.2.10}$$

Substituting (5.4.2.10) into (5.4.2.9) and interchanging the order of summation and integration, one obtains

$$(p - k_n^2)\,\bar{R}_n + 2q\sum_{m=1}^{\infty} a_{mn}\,\bar{R}_m = 0 \tag{5.4.2.11}$$

where

$$a_{mn} = \frac{1}{J_1^2(k_m)}\int_0^1 rJ_o(k_m r)J_o^2(k_1 r)J_o(k_n r)\,dr \tag{5.4.2.12}$$

The integrals that appear above have been computed and tabulated by McQueary and Mack in [38].

In order to obtain the desired amplitude-frequency relation it is necessary to determine the relations between p and q in (5.4.2.11) under which nontrivial solutions for $\bar{R}_n$ occur. To achieve this the series in (5.4.2.11) is now truncated after m = N. Equation (5.4.2.11) then becomes a set of N homogeneous linear algebraic euqations in $\overline{R}_n$, n = 1, 2, ..., N. By setting the determinant of the coefficients equal to zero N general relations between p and q result. The desired relation is the one for which all the nontrivial $\bar{R}_n$ except $\bar{R}_1$ should tend to zero as q(proportional to A^2) tends to zero. The relations may be improved by taking a larger N.

Let us instead describe here an alternative iterative procedure by which the desired relation between p and q may be developed as a power series in q for p. It is observed that by setting N = 1 one obtains from (5.4.2.11) and (5.4.2.12) the following solution.

$$\bar{R}_n = 0 \qquad n > 1 \;, \quad \bar{R}_1 \neq 0 \tag{5.4.2.13}$$

and

$$p = k_1^2 \tag{5.4.2.14}$$

Equation (5.4.2.14) simply states that for linear vibrations in the first mode the frequency

$$\omega^2 = \omega_o^2 = k_1^2 + \alpha_o \tag{5.4.2.14}$$

is independent of the amplitude A. Now taking N = 2 one obtains from (5.4.2.11) the following second order determinantal equation

$$\left[(p - k_1^2) + 2q\,a_{11}\right]\left[(p - k_2^2) + 2q\,a_{22}\right]$$

$$- 4q^2 a_{12} a_{21} = 0 \qquad (5.4.2.15)$$

which may be rewritten as

$$p - k_1^2 = -2q\,a_{11} - \frac{2q\,a_{22}\,(p - k_1^2)}{(p - k_2^2)} + O(q^2)$$

$$(5.4.2.16)$$

where $0(q^2)$ stands for terms which are at least quadratic in q. By (5.4.2.14) as a first approximation the second term on the right may be dropped. Equation (5.4.2.16) then states that $p - k_1^2 = 0(q)$, which in turn shows that the second term on the right is of the order $0(q^2)$. Thus

$$p = k_1^2 - 2q\,a_{11} + O(q^2) \qquad (5.4.2.17)$$

Or, in terms of ω^2 and A^2.

$$\omega^2 = \omega_o^2 + \frac{3\alpha_3 a_{11} A^2}{(\lambda+2)} + O(A^4) \qquad (5.4.2.18)$$

By increasing N, higher order terms on the right hand side of (5.4.2.16) can likewise be determined. However, it can be shown that terms involving lower powers of A^2 which have already been determined will not be affected.

Mack and McQueary have obtained the first order perturbation solution for the same membrance problem [11]. Again, except for some differences in notations, their result is

$$\omega^2 = \omega_o^2 + \frac{3\alpha_3 \int_0^1 r J_o^4 (k_1 r)\, dr\, A^2}{2\, J_1^2 (k_1)} \qquad (5.4.2.19)$$

Recalling the definition of a_{11}, one sees that (5.4.2.20) agrees with (5.4.2.19) up to terms in A^2, provided that λ is taken to be zero, i.e., provided that we use the Tchebycheff polynomial of the first kind.

VI. SUMMARY AND CONCLUSIONS

Oscillations of both discrete and continuous systems outside the classical linear domain are no longer independent of amplitude. Two approximate formulations were developed in this research to determine the frequency-amplitude relations and solutions for a general class of nonlinear continuous vibrating systems whose motions are governed by nonlinear partial differential euqations in one spatial variable and one time variable. The nonlinearity was assumed in the form of a restoring function of the dependent variable and its spatial derivatives.

The first approach involved a modified version of the usual perturbation theory. The nonlinear equation was first reduced to a system of linear equations which were then solved in a recursive manner by expanding the solutions as series of products of some set of spatial eigenfunctions and time harmonic functions. Both first and second order approximations to the frequency-amplitude relations and to the vibration configurations were obtained for a number of nonlinear continuous systems.

A second approximate formulation leading to the solution of the above-mentioned systems was also presented. The nonlinear functional in the equation of motion was reduced to an approximately equivalent linear form by using the linear term of a set of ultraspherical polynomials over the interval described by the amplitude of the motion. The linearized equation was then solved for the frequency-amplitude relations. The results were found to be identical with the first order results obtained by perturbation theory if the Tchebycheff polynomials, a subset of the ultraspherical, were used. Although the numerical results were given only for the first nonlinear mode, the analytical expressions are sufficiently general to be extended to study higher modes of vibration.

The general expressions developed through both the perturbation and ultraspherical polynomial approaches for the frequency-amplitude relations and the mode shapes of vibrations were applied to a number of dynamic systems. One need specify the differential operator, nonlinear restoring function, boundary conditions, and linear spatial eigenfunction. A total of twenty-eight solutions to different nonlinear systems were programmed on the computer and the results were catalogued in the form of graphs and tables. As the literature contains some solutions obtained by other authors using different methods, comparisons and justifications of the results presented herein are thus possible.

For the particular case of continuous systems on nonlinear elastic foundations, our first order frequency-amplitude relations for the uniform string and simply supported beam agree with those of Stoker and of Han respectively. Second order string and membrane results agree with those of McQueary, Mack and Clark. In addition, first and second order approximations to the frequency-amplitude relations and first order nonlinear mode shapes were graphed for prismatic beams with fixed-fixed, fixed-hinged and simply-supported boundary conditions as well as for uniform cantilever beams. Another contribution includes beams with variable cross sections. Cantilever beams with single and double wedge, circular and parabolic type cross sections were solved. First order results for fixed-fixed and simply-supported beams with exponentially varying cross sections were also presented. For the class of problems just mentioned, our results indicated that the general trend is for the first order nonlinear frequency to increase greatly as compared with the linear problem for a specified amplitude and the second order response to decrease only slightly from the first order frequency, while still remaining greater than the linear response. The increase in frequency for a specified amplitude due to the nonlinear foundation was inversely proportional to the stiffening imposed by the boundary conditions, i.e. the simply supported beam showed the greatest increase. The mode shape was generally flattened as compared to the linear problem. The location of the maximum displacement was found to shift in the case of fixed-hinged beams, supporting the time dependency concept of nonlinear modes. It was also noted, that by approximating the linear mode shape for fixed-fixed and cantilever beams with a cosine wave type solution instead of the exact eigenfunction, the approximation to the geometrical configuration is not extremely critical insofar as obtaining the frequency-amplitude results is concerned.

In the special case of continuous systems vibrating with immovable end supports, the first order frequency-amplitude results were identical with those of Evensen for simply supported, fixed-fixed and fixed-hinged beams. The second order frequency approximations presented are new but were found almost negligible. The nonlinear mode shapes were also presented and found to flatten at small amplitudes and more bell shaped at larger amplitudes. Again the maximum amplitude of the fixed-hinged beam shifted along the beam as the motion went through a complete period.

The numerical results obtained for both simply supported and clamped circular plates vibrating at large amplitudes were in good agreement with results obtained by Wah who used a modified Galerkin approach. Our first order results predict a period slightly less for a specified amplitude as compared with those results of Wah. Wah did not determine the second order results. Our second order results showed an even greater trend to a lower period for a specified amplitude. The nonlinear mode shape again was flattened, with the simply supported plate being more so than the clamped plate.

In the case of a membrane vibrating at large amplitudes, our results were compared with those obtained by Chobotov and Binder. The latter authors applied a perturbation approach and reduced the nonlinear coupled equations to a system of rather complicated linear equations. They then solved the first few linear equations by a Galerkin technique. The dynamic analogue of Berger's assumption for static plates with large deflections as applied to vibrating plates by Wah was used here to decouple the governing equations of motion. The results obtained in a relatively simple manner agree exceptionally with those of Chobotov.

Another interesting case investigated was that of vibrating systems with nonlinear material equations. The complexity of governing equations of motion has in the past greatly limited research in this area. However, in the simple case of a simply supported beam with a constitutive equation of the form $N = E(\epsilon + h\epsilon^3)$, Kauderer contributed a lower order approximation to the frequency and to the nonlinear mode shape. The general equations developed in this study produced the same results when applied to Kauderer's problem. Furthermore, both the first and the second order frequency-amplitude relations as well as the first order nonlinear mode shapes were determined for beams of a similar constitutive equation with fixed-fixed, fixed-clamped and simply supported boundary conditions. Our results are qualitatively comparable to those for beams with immovable supports, but the differences from the linear solution are more pronounced.

It is to be noted that the ultraspherical approach was used only for first order results. An attempt was made to reduce the ultraspherical polynomial approximations so that they may agree with higher order results obtained by the perturbation theory. The attempt was not successful.

Several avenues of research suggested by this study are as follows. Systems involving more than one variable, such as rectangular plates vibrating at large amplitudes, could be investigated. Also, with suitable adaptations, systems with time dependent nonlinearities, such as those involving small damping and external forcing fuctions might also be studied. Finally, systems with complicated material equations could perhaps be studied by considering their vibratory behavior. Comparison of the analytical findings with experiments , might yield fruitful results on the material properties.

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APPENDIX A

THE ULTRASPHERICAL POLYNOMIALS

The ultraspherical polynomials denoted by $P_n^{\lambda}(x)$ represent a subset of the more general Jacobi polynomials $P_n^{\alpha,\beta}(x)$ with $\alpha=\beta$ Another more specialized case of the Jakobi set is the Legendre polynomials where $\alpha=\beta=0$. The later may be generalized to the so-called Gegenbauer polynomials $C_n^{\upsilon}(x)$. It can be shown that the ultraspherical and Gegenbauer polynomials are essentially equivalent [39] and consequently, both names are commonly used interchangeably in the literature.

The ultraspherical polynomials are orthogonal on the interval $[-1, 1]$ with respect ot the weighting function $(1-x^2)^{\lambda-\frac{1}{2}}$ and may be obtained from Rodrigue's formula [40] as

$$P_n^{\lambda}(x) = A_n^{\lambda}(1-x^2)^{-\lambda+\frac{1}{2}}\left(\frac{d}{dx}\right)^n(1-x^2)^{n+\lambda-\frac{1}{2}} \tag{A.1}$$

where A_n^{λ} is some normalization constant, n is any nonnegative integer commonly referred to as the order of the polynomial and the index λ takes on the values $-\frac{1}{2} < \lambda < \infty$. Other sets of polynomials may be determined as a subset of the ultraspherical polynomial by assigning particular values of λ . For example, if $\lambda=0$ the Tchebycheff polynomials of the first kind are defined, if $\lambda=1$ the Tchebycheff polynomials of the second kind are determine, for $\lambda=\frac{1}{2}$ the Legendre polynomials and if $\lambda\to\infty$ the expansion corresponds to a Taylor's series of an analytic function about the origin.

A function f(x) expandable over the interval [-A, A] in these polynomials may be written (33).

$$f(x) = \sum_{n=0}^{\infty} A_n^{\lambda} P_n^{\lambda} (x/A) \qquad (A.2)$$

where the coefficients are obtained in the usual way by multiplication of the weighting function and integration over the span as

$$A_n^{\lambda} = \frac{\int_{-1}^{+1} f(Ax) P_n^{\lambda}(x/A)(1 - x^2)^{\lambda - \frac{1}{2}}\, dx}{\int_{-1}^{+1} \left[P_n^{\lambda}(x/A)\right]^2 (1 - x)^{\lambda - \frac{1}{2}}\, dx} \qquad (A.3)$$

It is important to observe that $a_n^{\lambda} P_n^{\lambda}(x/A)$ is unchanged if $P_n^{\lambda}(x/A)$ is multiplied by some constant. Consequently, the normalization factor A_n^{λ} from equation (A. 2) is not unique and any convenient normalization constant is permissible. In this treatise A_n^{λ} is taken as

$$A_n^{\lambda} = \frac{(-1)^n\, \Gamma(\lambda + \frac{1}{2})\, \Gamma(n + 2\lambda)}{2^n n!\, \Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} \qquad (A.4)$$

for all subsets except $\lambda=0$. Since $\Gamma(0)$ is undefined, for the case of the Tchebycheff polynomials for the first kind the constant is taken to be

$$A_n^{\circ} = \frac{(-1)^n \, 2^n \, n!}{(2n)!} \tag{A.5}$$

As already mentioned, the approximating function is not altered by redefining A_n^{λ}.