### ANALYSIS OF THE USE OF LOW-PASS FILTERS WITH HIGH-GAIN OBSERVERS

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High-gain observers are an important technique for state estimation in nonlinear systems. This technique utilizes a high gain in order to achieve fast reconstruction of the system states. While the high gain enables fast recovery of the states and good disturbance rejection properties, it can also act to amplify measurement noise. The trade-off between robustness to measurement noise and fast recovery of states is an important and well known problem within the study of high-gain observers.

One method to attenuate the effects of measurement noise is to use a low-pass filter in conjunction with the high-gain observer. The focus of this work will be an analysis of this technique. Four different configurations will be considered. The first three configurations place the filter before the observer and study the outcome of making the filter order lower than that of the plant, the same as that of the plant, or higher than that of the plant. The fourth configuration places the filter after the observer. Simulations of all four configurations and comparisons between them will be presented. The low-pass filter is shown to be a simple and effective means of reducing the effects of measurement noise on systems that utilize highgain observers.

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# Chapter 1

# Introduction

Derivative estimation is an important part of the control of physical systems. This is because, due to various constraints such as convenience and cost, it is often best not to measure all states. Instead, some states are measured and then used to estimate the remaining states. This is typical in many electromechanical systems where position is measured, and velocity is estimated. One such system is control of an inverted pendulum, where the angle of the pendulum is measured but its velocity is not. Another such system is a magnetic suspension system where the control task is to levitate a magnetic ball using an electromagnet. In this case, ball position may be measured and the ball's velocity and the electromagnet's current can be estimated.

Due to the strong need for derivative estimation, many techniques have been developed for this purpose. Among these methods are differentiation via a sliding mode technique [11], algebraic derivative estimation techniques which rely on a Taylor series expansion [6] [14] [17], derivative estimation based on an inverse Taylor series method [12], and high-gain observers [8]. In particular, the algebraic derivative estimation technique has gained attention in recent years including applications to DC motor control [15] and a MATLAB derivative estimation toolbox [20].

Due to the fact that measurements are often noisy, any derivative estimation technique must take the effects of measurement noise into account. The effects of measurement noise on the sliding mode technique, the algebraic derivative estimation technique, the inverse Taylor series technique, and high-gain observers are considered in [11], [17], [12], and [2] respectively. As is shown in [2], high-gain observers are not robust to measurement noise. To understand why this is the case, consider the nonlinear system

$$\dot{z} = \psi(z, x, d, u) \tag{1.1}$$

$$\dot{x} = Ax + B\phi(z, x, d, u) \tag{1.2}$$

$$y = Cx + v \tag{1.3}$$

where  $u \in \mathbb{R}$  is the control input,  $z \in \mathbb{R}^l$  and  $x \in \mathbb{R}^r$  are states,  $y \in \mathbb{R}$  is the measured output, and  $v(t) \in \mathbb{R}$  represents measurement noise. The vector  $d(t) \in \mathbb{R}^p$  is used to represent exogenous signals which may contain both disturbance and reference signals, however the control relies only upon the reference signals. A, B, and C, are given by

$$A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$
(1.4)  
$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \qquad (1.5)$$

Systems that fit this form include nonlinear systems in the normal form and any system that can be described by a chain of integrators, such as the aforementioned inverted pendulum and magnetic suspension systems. The state feedback for this system is designed as

$$\dot{\vartheta} = \Gamma(\vartheta, x, d)$$
 (1.6)

$$u = \gamma(\vartheta, x, d) \tag{1.7}$$

The high-gain observer is given by

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, d, u) + H(y - \hat{x}_1)$$
(1.8)

where the observer gain H is given by

$$H = \left[\frac{\alpha_1}{\varepsilon}, \frac{\alpha_2}{\varepsilon^2}, \dots, \frac{\alpha_r}{\varepsilon^r}\right]^T$$
(1.9)

Here,  $\varepsilon$  is a small positive parameter, and the roots of

$$s^{r} + \alpha_{1}s^{r-1} + \dots + \alpha_{r-1}s + \alpha_{r} = 0$$
(1.10)

have negative real parts. The function  $\phi_0$  is a nominal model of  $\phi$ . At this point it is clear why measurement noise is a problem when using a high-gain observer, namely that the measurement noise in (1.8) is multiplied by the high gain of the observer. Multiple techniques have been developed to reduce this noise amplification problem. In the next section we will discuss some of them, however the focus of this work is the low-pass filter method. In this method we use a low-pass filter to reduce the effects of measurement noise amplification by the high-gain observer. While this method has been used frequently as a source of comparison for other methods as in [4], a deeper analysis of the stability properties of the use of a low-pass filter in conjunction with a high-gain observer has not been considered.

There have been similar analyses in the past, but each is slightly different than what is presented here. In [1], the measurement noise is assumed to be the output of a known transfer function, and the analysis of stability is carried out from there. Similarly, in this work, we will consider measurement noise that is passed through a low-pass filter. In both cases, the resulting input to the observer is some sort of filtered measurement noise. The difference is that in this work, a low-pass filter is applied, while in [1], a band-pass filter was applied.

Another similar analysis is presented in [3]. In [3], a multirate discrete time high-gain observer system is presented. The idea is to sample the sensor to obtain measured data at one rate, and then to pass the measurement and corresponding estimates to the control at another, slower rate. Although this system is clearly different in that it is a discrete time system, and we only consider continuous time systems in this work, there are still some similarities. In the case of [3], the slower sample rate used for the control signal operates in a manner similar to that of a low-pass filter. In this sense, the analysis presented in [3] would be similar to a discrete time analysis of what is presented here.

We will begin our analysis in Chapter 2 by considering systems where the low-pass filter is placed before the observer. We will start with the special case when the filter is of the same order as the observer. The results of this analysis will be demonstrated via simulation. Additionally, simulation will be used to compare the efficacy of using a low-pass filter to another technique for reducing the effects of measurement noise on high-gain observers, a nonlinear-gain observer.

In Chapter 3 we will extend the analysis of Chapter 2 to filters of higher and lower orders. Simulation will again be used to illustrate the analytic results. Additionally, a comparison of different filter orders will be presented. We will conclude our analysis in Chapter 4 by examining the case where the low-pass filter is placed after the observer. Analytic and simulation results will be presented, and we will finish the chapter with a comparison between placing the filter before the observer versus placing the filter after the observer.

# Chapter 2

# Analysis of Low-Pass Filters of the Same Order as the Nonlinear System

## 2.1 Introduction

The use of high-gain observers is a well-known and effective method for state estimation and output feedback control of nonlinear systems. One problem associated with high-gain observers is the trade-off that exists between the fast convergence of state estimates and robustness to measurement noise. A number of techniques have been proposed to attenuate the effects of measurement noise on systems utilizing a high-gain observer.

One such approach is the switched-gain technique proposed in [2]. In this approach, a highgain is used during the transient period to ensure quick convergence of state estimates. Once the system reaches steady-state, the observer switches to a low-gain, so as not to amplify measurement noise. While this approach is effective, there are design issues associated with choosing a switching time and a delay time to ensure that switching happens only once. In [5] and [18], an adaptive gain is used to solve the problem of measurement noise. In [5], the adaptive gain parameter is applied to an extended Kalman filter, allowing the extended Kalman filter to vary between acting as a high-gain observer when the estimation error is large, and a standard Kalman filter when the estimation error is small. In [18], the adaptive high-gain observer is extended to systems with poorly known dynamics which may not have locally Lipschitz functions. The authors of [16] consider a nonlinear-gain observer. The nonlinear-gain is a piece-wise linear function, that varies between applying a high-gain when the estimation error is large and a low-gain when the estimation error is small. Similar to the switching method of [2], an additional design parameter is required to determine when the high-gain is applied and when the low-gain is applied.

Another method of reducing the effects of measurement noise, and the method that will be discussed in detail in this thesis, is to use a low-pass filter in conjunction with the high-gain observer. While this technique has been used in the past, in papers such as [19], and it is common in experimental applications of high-gain observers, there is no previous analysis of this method. This method is especially effective for systems that may be corrupted by high-frequency, high-amplitude noise, as will be demonstrated in the simulation results.

## 2.2 System Description

We consider the nonlinear system

$$\dot{z} = \psi(z, x, d, u) \tag{2.1}$$

$$\dot{x} = Ax + B\phi(z, x, d, u) \tag{2.2}$$

$$y = Cx + v \tag{2.3}$$

where  $u \in \mathbb{R}$  is the control input,  $z \in \mathbb{R}^l$  and  $x \in \mathbb{R}^r$  are states,  $y \in \mathbb{R}$  is the measured output, and  $v(t) \in \mathbb{R}$  represents measurement noise. The vector  $d(t) \in \mathbb{R}^p$  is used to represent exogenous signals which may contain both disturbance and reference signals, however the control relies only upon the reference signals. A, B, and C, are given by

$$A = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$
(2.4)  
$$C = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \qquad (2.5)$$

Possible sources for this model include the normal form of input-output linearizable systems as shown in [7]. Additional sources include electro-mechanical systems where position is measured but velocity and acceleration are not. While this model does not include additional measured outputs that are not included in the nonlinear system output y, this analysis can be extended to include those measured outputs as in [2] and [16].

- Assumption 1. 1. d(t) is continuously differentiable and bounded.  $d(t) \in \mathcal{D}$ , which is a compact subset of  $\mathbb{R}^p$ .
  - ψ and φ are locally Lipschitz in their arguments, uniformly in d, over the domain of interest; that is, for each compact subset of (z, x, u) in the domain of interest, ψ and φ satisfy the Lipschitz inequality with a Lipschitz constant independent of d for all d ∈ D.
  - 3. Measurement noise v(t) is a measureable function of t and is bounded, i.e.  $|v(t)| \leq \mu$ .

The state feedback control is designed as

$$\dot{\vartheta} = \Gamma(\vartheta, x, d) \tag{2.6}$$

$$u = \gamma(\vartheta, x, d) \tag{2.7}$$

The closed loop system under (2.6)-(2.7) meets the following assumption.

- **Assumption 2.** 1.  $\Gamma$  and  $\gamma$  are locally Lipschitz in their arguments, uniformly in d, over the domain of interest.
  - 2.  $\Gamma$  and  $\gamma$  are globally bounded in x.
  - 3.  $\phi$  is zero in  $\mathcal{A}$ , uniformly in d.
  - 4. The closed loop system under state feedback is globally uniformly asymptotically stable with respect to a compact positively invariant set  $\mathcal{A}$ , uniformly in d.

For the purposes of generality the problem is framed around a positively invariant and asymptotically attractive set as opposed to an equilibrium point. This allows us to address problems where we are attempting to stabilize to an equilibrium point, in which case the set  $\mathcal{A}$  would simply be  $\mathcal{A} = 0$ , or problems where we are attempting to regulate the output of a system to zero or to track a reference trajectory. In these cases the set  $\mathcal{A}$  would be the zero-order manifold. For further generality, the analysis of this chapter can be extended to systems that have a finite region of attraction, similar to the analysis shown in [2].

## 2.3 Filter and Observer Design

The nonlinear system output y is passed through a low-pass filter of the form

$$\tau \dot{w} = A_f w + B y \tag{2.8}$$

$$w_1 = Cw \tag{2.9}$$

where  $\tau$  is the filter time constant,  $w \in \mathbb{R}^r$  is the filter state, and the Hurwitz matrix  $A_f$  is given by

$$A_{f} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ -\beta_{1} & -\beta_{2} & \dots & \dots & -\beta_{r} \end{bmatrix}$$
(2.10)

where the  $\beta_i{'}\!\!\!\mathrm{s}$  are chosen such that the roots of

$$s^{r} + \beta_{r}s^{r-1} + \dots + \beta_{2}s + \beta_{1} = 0$$
(2.11)

have negative real parts, with  $\beta_1 = 1$ . We require  $\beta_1 = 1$  to ensure that the filter has a unity DC gain, a need that will be shown later. The high-gain observer is given by

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, d, u) + H(w_1 - \hat{x}_1)$$
(2.12)

where the observer gain H is given by

$$H = \left[\frac{\alpha_1}{\varepsilon}, \frac{\alpha_2}{\varepsilon^2}, \dots, \frac{\alpha_r}{\varepsilon^r}\right]^T$$
(2.13)

Here,  $\varepsilon$  is a small positive parameter, and the roots of

$$s^{r} + \alpha_{1}s^{r-1} + \dots + \alpha_{r-1}s + \alpha_{r} = 0$$
(2.14)

have negative real parts. The function  $\phi_0$  is a nominal model of  $\phi$ .

**Assumption 3.** 1.  $\phi_0$  is locally Lipschitz in its arguments, uniformly in d, over the domain of interest.

2.  $\phi_0$  is globally bounded in  $\hat{x}$  and zero in  $\mathcal{A}$ .

The output feedback controller is obtained by replacing x with  $\hat{x}$  in the state feedback controller (2.6)-(2.7).

## 2.4 Closed Loop System Analysis

For the analysis of the closed loop system, we apply the change of variables

$$\eta_i = \frac{\frac{1}{\tau^{i-1}} w_i - \hat{x}_i}{\varepsilon^{r-i}}$$
(2.15)

for  $i = 1, \ldots, r$ . The closed loop system under output feedback control can be written as

 $\dot{\chi}=f(\chi,d,D(\varepsilon)\eta)$  where  $\chi=[z,x,w,\vartheta]^T\in\mathbb{R}^M$  and

$$\dot{\chi} = \begin{bmatrix} \psi \left( z, x, d, \gamma \left( \vartheta, E^{-1}(\tau) w - D(\varepsilon) \eta, d \right) \right) \\ Ax + B\phi \left( z, x, d, \gamma \left( \vartheta, E^{-1}(\tau) w - D(\varepsilon) \eta, d \right) \right) \\ \frac{1}{\tau} \left( A_f w + B(x_1 + v) \right) \\ \Gamma \left( \vartheta, E^{-1}(\tau) w - D(\varepsilon) \eta, d \right) \end{bmatrix}$$
(2.16)  
$$\varepsilon \dot{\eta} = A_0 \eta + \varepsilon Bg \left( \chi, d, D(\varepsilon) \eta \right) + \frac{\varepsilon}{\tau^r} Bv$$
(2.17)

where

$$A_{0} = \begin{bmatrix} -\alpha_{1} & 1 & \dots & \dots & 0 \\ -\alpha_{2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ -\alpha_{r-1} & 0 & \dots & \dots & 1 \\ -\alpha_{r} & 0 & \dots & \dots & 0 \end{bmatrix}$$
(2.18)  
$$g\left(\chi, d, D(\varepsilon)\eta\right) = \frac{1}{\tau^{r}} \left(-\beta_{1}w_{1} - \dots - \beta_{r}w_{r} + x_{1}\right)$$
$$-\phi_{0}\left(D(\varepsilon)\eta, d, \gamma\left(\vartheta, E^{-1}(\tau)w - D(\varepsilon)\eta, d\right)\right)$$
$$D(\varepsilon) = diag\left[\varepsilon^{r-1}, \dots, \varepsilon, 1\right]$$
(2.20)

$$E(\tau) = diag\left[1, \tau, \dots, \tau^{r-1}\right]$$
(2.21)

The matrix  $A_0$  is Hurwitz by design, and (2.16)-(2.17) is a standard singularly perturbed system. When  $\varepsilon = 0$ , the unique solution of (2.17) is  $\eta = 0$ . With  $\eta = 0$ , (2.16)-(2.17) reduce to the closed loop system under (2.6)-(2.7) with x in (2.6)-(2.7) replaced by  $E^{-1}(\tau)w$ , written as  $\dot{\chi} = f(\chi, d, 0) = f_r(\chi, d)$ ,

$$f_r(\chi, d) = \begin{bmatrix} \psi(z, x, d, \gamma(\vartheta, E^{-1}(\tau)w, d)) \\ Ax + B\phi(z, x, d, \gamma(\vartheta, E^{-1}(\tau)w, d, )) \\ \frac{1}{\tau}(A_f w + B(x_1 + v)) \\ \Gamma(\vartheta, E^{-1}(\tau)w, d) \end{bmatrix}$$
(2.22)

**Assumption 4.** 1. The reduced order system (2.22) is globally uniformly asymptotically stable with respect to  $\mathcal{A}$ , uniformly in d.

With this assumption, by a converse Lyapunov theorem in [13], there exists a smooth Lyapunov function  $V(\chi)$  and three positive definite functions  $U_1(\chi)$ ,  $U_2(\chi)$ , and  $U_3(\chi)$ , such that

$$V(\chi) = 0 \Leftrightarrow \chi \in \mathcal{A} \tag{2.23}$$

$$U_1(\chi) \le V(\chi) \le U_2(\chi) \tag{2.24}$$

$$\lim_{|\chi| \to \infty} U_1(\chi) = \infty \tag{2.25}$$

$$\frac{\partial V}{\partial \chi} f(\chi, d, 0) \le -U_3(\chi) \qquad \forall \ d \in \mathcal{D}$$
(2.26)

In assumption 4, we assume that the filter time constant  $\tau$  is chosen small enough that the closed loop system where x in (2.6)-(2.7) has been replaced by  $E^{-1}(\tau)w$  maintains stability. To see that it is possible to choose  $\tau$  small enough to maintain stability in the reduced system (2.22) we place the system into standard singularly perturbed form. In order to do this, we begin by applying the change of variables

$$w^{(i)} = w^{(i-1)} + \tau^{i-1} (A_f^{-1})^i B x_i$$
(2.27)

for i = 1, ..., r - 1, where  $w^{(0)} = w$ . Then,

$$\tau \dot{w}^{(i)} = A_f w^{(i)} + Bv + \tau^i (A_f^{-1})^i Bx_{i+1}$$
(2.28)

for  $i = 1, \ldots, r - 1$ . Next, we apply the scaling

$$\zeta = \frac{1}{\tau^{r-1}} w^{(r-1)} \tag{2.29}$$

substituting into  $E^{-1}(\tau)w$  gives

$$E^{-1}(\tau)w = E(\tau)\zeta + \frac{1}{\beta_1}x + \tau G(\tau)x$$
(2.30)

where  $G(\tau)$  is a polynomial in  $\tau$ . Then, the reduced system is transformed into

$$\dot{z} = \psi(z, x, d, \gamma(\vartheta, E(\tau)\zeta + \frac{1}{\beta_1}x + \tau G(\tau)x, d))$$
(2.31)

$$\dot{x} = Ax + B\phi(z, x, d, \gamma(\vartheta, E(\tau)\zeta + \frac{1}{\beta_1}x + \tau G(\tau)x, d))$$
(2.32)

$$\dot{\vartheta} = \Gamma(\vartheta, E(\tau)\zeta + \frac{1}{\beta_1}x + \tau G(\tau)x, d)$$
(2.33)

$$\tau \dot{\zeta} = A_f \zeta + \frac{1}{\tau^{r-1}} Bv + (A_f^{-1})^{r-1} Bx_r$$
(2.34)

which is in the standard singularly perturbed form. Consider (2.30) with measurement noise

excluded (v = 0) and let  $\tau = 0$ . Then,  $E^{-1}(\tau)w$  becomes

$$E^{-1}(\tau)w = \frac{1}{\beta_1}x$$
 (2.35)

In order for the singularly perturbed system (2.31)-(2.34) to reduce to the closed loop system under (2.6)-(2.7), we must have  $\beta_1 = 1$ , which corresponds to a unity DC gain for the low-pass filter.

Proving that the system (2.31)-(2.34) is globally uniformly asymptotically stable for sufficiently small  $\tau$  is similar to [9]. Although  $\tau$  is chosen small enough to maintain stability, it should not be chosen arbitrarily small, because smaller values of  $\tau$  will not be as effective at filtering measurement noise. This trade-off is similar to what is discussed in [2] and [16].

**Theorem 1.** Let Assumptions 1-4 hold and consider the closed-loop system (2.16)-(2.17). Let S be any compact subset of  $\mathbb{R}^M$  such that  $\chi(t_0) \in S$  and let Q be any compact subset of  $\mathbb{R}^r$  such that  $\hat{x}(t_0) \in Q$ . Then, given any  $\xi > 0$ , there exists  $\varepsilon^* > 0$ , and  $T^* > 0$ , dependent on  $\xi$ , such that for every  $0 < \varepsilon \leq \varepsilon^*$ , the trajectories  $(\chi, \hat{x})$  of the closed-loop system starting in  $S \times Q$  are bounded for all  $t \geq t_0$  and satisfy

 $\|\chi(t,\varepsilon)\| \le \xi \qquad and \qquad \|\hat{x}(t,\varepsilon)\| \le \xi \qquad \forall \ t \ge T^*$ (2.36)

$$\|\chi(t) - \chi_r(t)\| \le \xi \qquad \forall \ t \ge t_0 \tag{2.37}$$

where  $\chi_r$  is the solution of (2.22) with  $\chi_r(t_0) = \chi(t_0)$ .

Remark 1. Theorem 1 shows boundedness of trajectories, ultimate boundedness of trajectories as they approach the set  $\mathcal{A}$ , and closeness of the trajectories under output feedback to those of the reduced system.

Proof. Let  $c > \max_{\chi \in \mathcal{S}} V(\chi)$ , then S is in the interior of  $\Omega = \{\chi : V(\chi) \leq c\} \subset \mathbb{R}^M$ . The set  $\Omega$  is compact for any choice of c. Due to the global boundedness of g in  $D(\varepsilon)\eta$ , there exists a constant  $k_g$ , independent of  $\varepsilon$ , such that  $||g(\chi, d, D(\varepsilon)\eta)|| \leq k_g$  for all  $\chi \in \Omega$  and  $\eta \in \mathbb{R}^r$ . Let  $W(\eta) = \eta^T P_0 \eta$ , where  $P_0$  is the positive definite symmetric solution of  $P_0 A_0 + A_0^T P_0 = -I$ .  $P_0$  is guaranteed to exist because  $A_0$  is Hurwitz by design. It can be shown that

$$\dot{W}(\eta) \le \frac{-1}{\varepsilon} \|\eta\|^2 + 2k_g \|BP_0\| \|\eta\| + \frac{2\mu}{\tau^r} \|BP_0\| \|\eta\|$$
(2.38)

and for  $W \ge \sigma_1^2 \left(k_g + \frac{\mu}{\tau^T}\right)^2 \varepsilon^2$  where  $\sigma_1 = 4 \|BP_0\| \sqrt{\|P_0\|}$ , we have

$$\dot{W}(\eta) \le \frac{-1}{2\varepsilon \|P_0\|} W(\eta) \tag{2.39}$$

So, the set  $\Sigma = \{W \le \sigma_1^2 \left(k_g + \frac{\mu}{\tau^r}\right)^2 \varepsilon^2\}$  is positively invariant. If  $\eta(t_0)$  is not in  $\Sigma$ , we have

$$W(\eta(t)) \le W(\eta(t_0)) \exp\left(\frac{-\sigma_2(t-t_0)}{\varepsilon}\right)$$
(2.40)

where  $\sigma_2 = \frac{1}{2\|P_0\|}$ . For bounded w(0) and  $\hat{x}(0)$ , we have  $\|\eta(t_0)\| \leq \frac{k_0}{\varepsilon^{r-1}}$  for some constant  $k_0$ , which gives

$$W(\eta(t_0)) \le \frac{\sigma_3^2}{\varepsilon^{2(r-1)}} \tag{2.41}$$

where  $\sigma_3 = k_0 \sqrt{\|P_0\|}$ . So,  $\eta$  will reach  $\Sigma$  in the interval  $[t_0, t_0 + T(\varepsilon)]$ , where

$$T(\varepsilon) \le \frac{2\varepsilon}{\sigma_2} \ln\left(\frac{\sigma_3}{\sigma_1(k_g + \frac{\mu}{\tau^r})\varepsilon^r}\right)$$
(2.42)

and, inside  $\Sigma$ ,  $\|\eta\|$  is bounded by  $\|\eta\| \leq \sigma_1 \left(k_g + \frac{\mu}{\tau^r}\right) \varepsilon \sqrt{\frac{1}{\lambda_{min}(P_0)}}$ . Since f is globally bounded in  $D(\varepsilon)\eta$ , there exists  $k_f > 0$ , independent of  $\varepsilon$ , such that  $\|f(\chi, d, D(\varepsilon)\eta)\| \leq k_f$ .

Moreover, for any  $0 < \tilde{\varepsilon} < 1$ , there is  $L_1$ , independent of  $\varepsilon$ , such that

$$||f(\chi, d, D(\varepsilon)\eta) - f(\chi, d, 0)|| \le L_1 ||\eta||$$
 (2.43)

for all  $(\chi, \eta) \in \Omega \times \Sigma$ . Let  $L_2$  be an upper bound on  $\|\frac{\partial V}{\partial \chi}\|$ , and set  $L = L_1 L_2$ , then

$$\dot{V} \le -U_3(\chi) + k_1 \left(k_g + \frac{\mu}{\tau^r}\right)\varepsilon$$
(2.44)

where  $k_1 = L\sigma_1 \sqrt{\frac{1}{\lambda_{\min}(P_0)}}$ . Now, let  $\beta = \min_{\chi \in \partial \Omega} U_3(\chi)$  and  $\varepsilon_1^* = \beta / \left(k_1(k_g + \frac{\mu}{\tau^r})\right)$ , then for  $0 < \varepsilon \leq \varepsilon_1^*$  we have  $\dot{V} \leq 0$  for all  $(\chi, \eta) \in \partial \Omega \times \Sigma$ , and from the forgoing analysis, we have  $\dot{W} \leq 0$  for all  $(\chi, \eta) \in \Omega \times \partial \Sigma$ , so the set  $\Omega \times \Sigma$  is positively invariant. Furthermore, because  $\chi(t_0)$  is in the interior of  $\Omega$ , we have

$$\|\chi(t) - \chi(t_0)\| \le k_2(t - t_0) \tag{2.45}$$

for some constant  $k_2 > 0$ , independent of  $\varepsilon$ , as long as  $\chi(t) \in \Omega$ . So, there exists finite  $T_1$  independent of  $\varepsilon$ , such that  $\chi(t) \in \Omega$  for all  $t \in [t_0, t_0 + T_1]$ . Since  $\eta$  enters  $\Sigma$  in the interval  $[t_0, t_0 + T(\varepsilon)]$ , where  $T(\varepsilon) \to 0$  as  $\varepsilon \to 0$ , there exists  $\varepsilon_2^*$  such that  $0 < \varepsilon \leq \varepsilon_2^*$  gives  $T(\varepsilon) \leq T_1$ . Taking  $\varepsilon_1 = \min(\tilde{\varepsilon}, \varepsilon_1^*, \varepsilon_2^*)$  guarantees that the trajectory  $(\chi, \eta)$  enters  $\Omega \times \Sigma$  in the interval  $[t_0, t_0 + T(\varepsilon)]$  and stays there for all  $t \geq T(\varepsilon)$ . Inside the interval  $[t_0, t_0 + T(\varepsilon)]$ ,  $\eta$  and  $\chi$  are bounded by (2.41) and (2.45) respectively, so the closed loop trajectories are bounded.

From the forgoing analysis, for every  $0 < \varepsilon \leq \varepsilon_1$ , the trajectory of the closed loop system is inside the set  $\Omega \times \Sigma$  for  $t \geq t_0 + T(\varepsilon)$ , where  $\Omega \times \Sigma$  is  $\mathcal{O}\left(\frac{\mu\varepsilon}{\tau}\right)$  in the direction of  $\eta$ . Since  $\|\hat{x}\| \leq \|D(\varepsilon)\eta\| \leq \|\eta\|$ , there exists  $\varepsilon_3^* = \varepsilon_3^*(\xi) \leq \varepsilon_1$  such that for every  $0 < \varepsilon \leq \varepsilon_3^*$  we have

$$\|\hat{x}\| \le \xi \qquad \forall \ t \ge T(\varepsilon_3^*) := T_a(\xi) \tag{2.46}$$

and from the previous analysis we have  $\dot{V} \leq -U_3(\chi) + k_1 \left(k_g + \frac{\mu}{\tau^T}\right)\varepsilon$ , so

$$\dot{V} \le -\frac{1}{2}U_3(\chi) \text{ for } \chi \notin \{U_3(\chi) \le 2k_1 \left(k_g + \frac{\mu}{\tau^r}\right)\varepsilon := \theta(\varepsilon)\}$$

$$(2.47)$$

Since  $U_3(\chi)$  is positive definite and continuous, the set  $\{U_3(\chi) \leq \theta(\varepsilon)\}$  is compact for small enough  $\varepsilon$ . Let  $c_0(\varepsilon) = \max_{U_3(\chi) \leq \theta(\varepsilon)} \{V(\chi)\}, c_0(\varepsilon)$  is nondecreasing and  $\lim_{\varepsilon \to 0} c_0(\varepsilon) = 0$ . The set  $\{V(\chi) \leq c_0(\varepsilon)\}$  is compact and  $\{U_3(\chi) \leq \theta(\varepsilon)\} \subset \{V(\chi) \leq c_0(\varepsilon)\}$ . Now, choose  $\varepsilon_4^* = \varepsilon_4^*(\xi) \leq \varepsilon_1$  such that  $\{U_3(\chi) \leq \theta(\varepsilon)\}$  is compact,  $\{V(\chi) \leq c_0(\varepsilon)\}$  is in the interior of  $\Omega$ , and

$$\{V(\chi) \le c_0(\varepsilon)\} \subset \{\|\chi\| \le \xi\}$$
(2.48)

then,  $\{V(\chi) \leq c_0(\varepsilon)\} \times \Sigma$  is positively invariant and every trajectory starting in  $\Omega \times \Sigma$ reaches  $\{V(\chi) \leq c_0(\varepsilon)\} \times \Sigma$  in finite time. So, there exists a time  $T_b = T_b(\xi)$  such that for every  $0 < \varepsilon \leq \varepsilon_4^*$ 

$$\|\chi\| \le \xi \qquad \forall \ t \ge T_b \tag{2.49}$$

take  $\varepsilon_2 = \min(\varepsilon_3^*, \varepsilon_4^*)$  and  $T^*(\xi) = \max(T_a, T_b)$  to obtain (2.36).

From earlier analysis we have  $\|\chi(t) - \chi(t_0)\| \le k_2(t - t_0) \ \forall t \in [t_0, t_0 + T(\varepsilon)]$ . Similarly, it can be shown that  $\|\chi_r(t) - \chi(t_0)\| \le k_2(t - t_0) \ \forall t \in [t_0, t_0 + T(\varepsilon)]$ . Hence,

$$\|\chi(t) - \chi_r(t)\| \le 2k_2 T(\varepsilon) \qquad \forall \ t \in [t_0, t_0 + T(\varepsilon)]$$

$$(2.50)$$

and, since  $T(\varepsilon) \to 0$  as  $\varepsilon \to 0$ , there exists  $0 < \varepsilon_5^* \le \varepsilon_2$  such that for every  $0 < \varepsilon \le \varepsilon_5^*$ , we have

$$\|\chi(t) - \chi_r(t)\| \le \xi \qquad \forall \ t \in [t_0, t_0 + T(\varepsilon)]$$

$$(2.51)$$

Next, we consider the interval  $[t_0+T(\varepsilon), T_2]$ , for  $T_2$  to be defined later. Over this interval,  $\chi(t)$  satisfies (2.16) which can be viewed as a perturbation of (2.22) in both the initial condition and the parameters. Then, by the continuous dependence of solutions of differential equations on their parameters and initial conditions over compact time intervals [7], we have

$$\|\chi(t) - \chi_r(t)\| \le 2c_1k_2T(\varepsilon) + c_2k_1\left(k_g + \frac{\mu}{\tau^r}\right)\varepsilon$$
(2.52)

for some constants  $c_1 \ge 1$  and  $c_2 > 0$ . Since  $T(\varepsilon) \to 0$  as  $\varepsilon \to 0$ , there exists  $0 < \varepsilon_6^* \le \varepsilon_2$ such that for every  $0 < \varepsilon \le \varepsilon_6^*$ , we have

$$\|\chi(t) - \chi_r(t)\| \le \xi \qquad \forall \ t \in [t_0 + T(\varepsilon), T_2]$$

$$(2.53)$$

Finally, from previous analysis and the uniform asymptotic stability of (2.22), we know that for some  $\xi > 0$  there exists  $T_2 = T_2(\xi) \ge T(\varepsilon)$  and  $0 < \varepsilon_7^* \le \varepsilon_2$  such that for  $0 < \varepsilon \le \varepsilon_7^*$ 

$$\|\chi(t)\| \le \xi/2, \qquad \|\chi_r(t)\| \le \xi/2$$
 (2.54)

so,  $\|\chi(t) - \chi_r(t)\| \leq \xi \ \forall \ t \geq T_2$ . Choose  $\varepsilon_3 = \min(\varepsilon_5^*, \varepsilon_6^*, \varepsilon_7^*)$ , then, for each  $0 < \varepsilon \leq \varepsilon_3$  we have  $\|\chi(t) - \chi_r(t)\| \leq \xi, \ \forall \ t \geq t_0$ .

To conclude the proof take  $\varepsilon^* = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ .

## 2.5 Simulation Results

To illustrate the efficacy of the low-pass filter in conjunction with the high-gain observer, we simulate the nonlinear system shown in [16] so as to compare to the nonlinear-gain method presented there. Consider the nonlinear system

$$\dot{x}_1 = x_2 \tag{2.55}$$

$$\dot{x}_2 = -x_1^3 + u \tag{2.56}$$

$$y = x_1 + v \tag{2.57}$$

where the  $x_i$ 's are the states, y is the output, and v is the bounded measurement noise. The system (2.55)-(2.57) is passed through the low-pass filter

$$\tau \dot{w_1} = w_2 \tag{2.58}$$

$$\tau \dot{w_2} = -w_1 - 2w_2 + y \tag{2.59}$$

where  $\tau$  is the filter time constant and  $w_i$ 's are the filter states. The filter time constant is set at  $\tau = 0.03$ , and the output from the filter is  $w_1$ . The system is stabilized with the controller  $u = -x_2$ . As in [16] the control is saturated outside of [-10, 10]. The observer parameters are chosen to be  $\alpha_1 = 2$ ,  $\alpha_2 = 1$ , and the initial conditions are  $x_1(0) = 2$ and  $x_2(0) = w_1(0) = w_2(0) = \hat{x}_1(0) = \hat{x}_2(0) = 0$ . The output is corrupted with lowamplitude additive white noise generated using the Simulink "Uniform Random Number" block with limits of  $\pm 0.001$  and a sampling time of 0.0008 seconds. Figure 2.1 shows that the steady-state trajectories of the filtered output feedback system become closer to those of the reduced system as epsilon decreases. The transient trajectories are not shown here as all three trajectories are too close to distinguish.



Figure 2.1: Steady-state trajectories for the reduced system (solid blue), filtered output feedback with  $\varepsilon = 0.004$  (dotted green), and filtered output feedback with  $\varepsilon = 0.001$  (dashed red).

To compare between the filter and the nonlinear-gain system, we choose the observer parameters for the nonlinear-gain system as  $\alpha_1 = 11$ ,  $\alpha_2 = 10$ ,  $\varepsilon_1 = 0.0005$ ,  $\varepsilon_2 = 0.9$ , and d = 0.15. The values for  $\alpha_1$  and  $\alpha_2$  are chosen such that one observer pole is significantly faster than the other, which is a requirement of [16]. The remaining values,  $\varepsilon_1$ ,  $\varepsilon_2$ , and d are chosen following the procedure outlined in [16]. In addition to the filter and the nonlineargain systems, a combination of the two was also simulated. For this simulation the output of the nonlinear system was passed through the previously described low-pass filter, and then the output of the low-pass filter was input into the nonlinear high-gain observer. The initial conditions for all systems are the same as those listed previously for the filter system.

In addition to the case when the output is corrupted by low-amplitude additive white noise we also consider the case when the output is corrupted by high-frequency, highamplitude noise. This noise is generated by passing white noise from the Simulink "Uniform Random Number" block through a high-pass filter. The bounds on the high-frequency, highamplitude noise are  $\pm 0.1$ .

Figure 2.2 shows the trajectory  $x_2$  for each system when the output is corrupted by the low-amplitude white noise. Although there is very little difference between the three systems, the filter and combination systems have a slightly larger overshoot. A larger difference between the systems can be seen in the control signal, u, shown in Figures 2.3 and 2.4. These figures show that both the filter and combination systems have a slower control response, and the filter system control signal is noisier. So, while the performance of the three systems is very similar when their output is corrupted by low-amplitude white noise, the nonlinear-gain observer does perform slightly better.



Figure 2.2: Trajectory  $x_2$  for all systems when the output is corrupted by low-amplitude additive white noise



Figure 2.3: Control signal u for all systems during the transient period when the output is corrupted by low-amplitude additive white noise



Figure 2.4: Close up of control signal u for all systems during the steady-state period when the output is corrupted by low-amplitude additive white noise

Figures 2.5 and 2.6 respectively show the transient and steady-state response of the  $x_2$  trajectory of all three systems when the output is corrupted by high-frequency, high-amplitude noise. The nonlinear-gain system has a noisy and undesirable transient, but provides good noise reduction at steady-state. In contrast, the filter and combination systems have nice transient behavior, but the filter steady-state response is delayed and somewhat noisy. As with the low-amplitude white noise, a larger difference between the systems can be seen when comparing their control signals, u, as shown in Figure 2.7. In this case, the nonlineargain observer transient control signal demonstrates extremely undesirable behavior. Both the filter system and combined system have a good response time, however the filter system control signal is fairly noisy. Figure 2.8 shows a close-up of the steady-state response of the nonlinear-gain and combined systems. The filter system is not included in this comparison because it is too noisy. Although the combined system is slightly slower it has far less measurement noise in its steady-state response than the nonlinear-gain system.



Figure 2.5: Trajectory  $x_2$  for all systems during the transient period when the output is corrupted by high-frequency, high-amplitude noise



Figure 2.6: Close up of the trajectory  $x_2$  for all systems during the steady-state period when the output is corrupted by high-frequency, high-amplitude noise



Figure 2.7: Control signal u for all systems during the transient period when the output is corrupted by high-frequency, high-amplitude noise



Figure 2.8: Close up of control signal u for the nonlinear-gain and combination systems during the steady-state period when the output is corrupted by high-frequency, high-amplitude noise

These simulations show that the filter system gives a good transient response when the output is corrupted by either low-amplitude white noise or high-frequency, high-amplitude noise, however the steady-state performance of the nonlinear-gain system is generally better. Overall, the best performance is provided by the combination nonlinear-gain and filter system, however this system increases the design complexity, a factor which should be considered when choosing a method.

## 2.6 Conclusions

We have presented analysis of the use of a low-pass filter in conjunction with a high-gain observer as a means to attenuate the effects of measurement noise on high-gain observer systems. It was shown that the filtered output feedback system recovers the performance of the reduced order system. The low-pass filter was effective at minimizing the effects of
both additive white noise and high-frequency, high-amplitude noise. In addition, it was compared with the performance of a nonlinear-gain observer, and a nonlinear-gain observer combined with a low-pass filter. The low-pass filter and the combination system performed better than the nonlinear-gain system when exposed to high-frequency, high-amplitude noise. When exposed to additive white noise, the three systems all performed similarly.

## Chapter 3

# Filters of Higher or Lower Order

## 3.1 Introduction

In Chapter 2 the analysis was limited to filters of the same order as the system. While this was convenient for the purposes of analysis, filters are not chosen based on the order of the system they are filtering. For this reason, it is natural to extend the analysis of the previous chapter to filters that have either a higher or lower order than the nonlinear system. We will continue to consider low-pass filters whose transfer function is that of an all pole plant. This form includes many common low-pass filters such as Butterworth, Chebyshev Type I, and Elliptic filters [10].

## 3.2 Higher Order Filters

### 3.2.1 Formulation

In this section we will consider filters whose order is higher than that of the system. Returning to the nonlinear system of Chapter 2

$$\dot{z} = \psi(z, x, d, u) \tag{3.1}$$

$$\dot{x} = Ax + B\phi(z, x, d, u) \tag{3.2}$$

$$y = Cx + v \tag{3.3}$$

where  $u \in \mathbb{R}$  is the control input,  $z \in \mathbb{R}^l$  and  $x \in \mathbb{R}^r$  are states,  $y \in \mathbb{R}$  is the measured output,  $d(t) \in \mathbb{R}^p$  is a vector of exogenous signals, and  $v(t) \in \mathbb{R}$  represents measurement noise. The matrices A, B, and C are the same as (2.4)-(2.5). The state feedback control for this system is given by

$$\dot{\vartheta} = \Gamma(\vartheta, x, d) \tag{3.4}$$

$$u = \gamma(\vartheta, x, d) \tag{3.5}$$

The nonlinear system and the closed loop system under state feedback meet Assumptions 1 and 2. We then design the low-pass filter as

$$\tau \dot{w} = A_f w + B_f y \tag{3.6}$$

$$w_1 = C_f w \tag{3.7}$$

where  $\tau$  is the filter time constant,  $w \in \mathbb{R}^m$  is the filter state, and m > r since we are considering higher order filters. The matrix  $A_f$  is the same as (2.10). The  $m \times 1$  matrix  $B_f$ and the  $1 \times m$  matrix  $C_f$  have the same form as (2.4) and (2.5), but have larger dimensions due to the extra filter states. The high-gain observer is

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, d, u) + H(w_1 - \hat{x}_1)$$
(3.8)

where  $\hat{x} \in \mathbb{R}^r$  are state estimates, and H is the same as (2.13). The function  $\phi_0$  meets Assumption 3.

### 3.2.2 Closed Loop System Analysis

We begin the analysis of the closed loop system by applying the change of variables from Chapter 2

$$\eta_i = \frac{\frac{1}{\tau^{i-1}} w_i - \hat{x}_i}{\varepsilon^{r-i}} \tag{3.9}$$

for i = 1, ..., r. With this change of variables, and defining  $\chi = [z, x, w, \vartheta]^T \in \mathbb{R}^M$ , the closed loop system under output feedback control becomes  $\dot{\chi} = f(\chi, d, D(\varepsilon)\eta)$ .

$$\dot{\chi} = \begin{bmatrix} \psi \left( z, x, d, \gamma \left( \vartheta, E^{-1}(\tau) \bar{w} - D(\varepsilon) \eta, d \right) \right) \\ Ax + B\phi \left( z, x, d, \gamma \left( \vartheta, E^{-1}(\tau) \bar{w} - D(\varepsilon) \eta, d \right) \right) \\ \frac{1}{\tau} \left( A_f w + B(x_1 + v) \right) \\ \Gamma \left( \vartheta, E^{-1}(\tau) \bar{w} - D(\varepsilon) \eta, d \right) \end{bmatrix}$$
(3.10)  
$$\varepsilon \dot{\eta} = A_0 \eta + \varepsilon Bg \left( \chi, d, D(\varepsilon) \eta \right)$$
(3.11)

where  $A_0$  and  $D(\varepsilon)$  are the same as (2.18) and (2.20) respectively.  $E(\tau)$  in this case remains

the same as (2.21) despite the fact that the filter state vector w is larger than the system state vector x. This is due to the fact that  $E(\tau)$  comes from the scaled estimation error

$$\eta = D^{-1}(\varepsilon)(E(\tau)\bar{w} - \hat{x})$$
(3.12)

Since  $\eta \in \mathbb{R}^r$ ,  $E(\tau)$  is an  $r \times r$  matrix. For clarity as to the dimension of  $E(\tau)$ , we have defined  $\bar{w} = [w_1, \ldots, w_r]$ .

Due to the higher order of the filter, the function  $g(\chi, d, D(\varepsilon)\eta)$  is given by

$$g(\chi, d, D(\varepsilon)\eta) = \frac{1}{\tau^r} w_{r+1} - \phi_0(E^{-1}(\tau)\bar{w} - D(\varepsilon)\eta, d, \gamma(\vartheta, E^{-1}(\tau)\bar{w} - D(\varepsilon)\eta, d))$$
(3.13)

which is different from (2.19). A further difference between this closed loop system and the closed loop system from Chapter 2, are the fast equations (2.17) and (3.11). For the closed loop system where the filter is the same order as the system, the noise appeared in (2.17) with a coefficient of  $\frac{\varepsilon}{\tau T}$ . In (3.11), the noise does not appear at all due to the higher order of the filter. The reason for this can be seen by briefly considering the  $r^{th}$  derivative of the filter output  $w_1$  for both cases. In the case where the system and filter are of the same order we have

$$\dot{w_1}^{(r)} = \frac{1}{\tau^r} \left( -\beta_1 w_1 - \dots - \beta_r w_r + x_1 + v \right)$$
(3.14)

while in the case where the order of the filter is higher than that of the system, we have

$$\dot{w_1}^{(r)} = \frac{1}{\tau^r} w_{r+1} \tag{3.15}$$

When the change of variables (3.9) is applied, the  $r^{th}$  derivative of the filter output appears

in the fast equation (2.17), where it is multiplied by  $\varepsilon$ . This means the noise term appears in the fast equation only when the order of the filter is the same as the order of the system. As in the analysis in Chapter 2, the closed loop system (3.10)-(3.11) is a standard singularly perturbed system. Setting  $\varepsilon = 0$  in (3.11) results in the reduced system  $f(\chi, d, 0) = f_r(\chi, d)$ , where

$$f_r(\chi, d) = \begin{bmatrix} \psi(z, x, d, \gamma(\vartheta, E^{-1}(\tau)\bar{w}, d)) \\ Ax + B\phi(z, x, d, \gamma(\vartheta, E^{-1}(\tau)\bar{w}, d,)) \\ \frac{1}{\tau}(A_f w + B_f(x_1 + v)) \\ \Gamma(\vartheta, E^{-1}(\tau)\bar{w}, d) \end{bmatrix}$$
(3.16)

This system is the same as the closed loop system under (3.4)-(3.5) with x in (3.4)-(3.5)replaced with  $E^{-1}(\tau)\bar{w}$ . The reduced system (3.16) meets Assumption 4. As stated in Chapter 2, Assumption 4 assumes that  $\tau$  is chosen small enough that the reduced system (3.16) maintains stability. In Chapter 2, in order to see that it is possible to choose such a  $\tau$ , the system was placed in standard singularly perturbed form. We will repeat the same process of transforming the reduced system (3.16) into standard singularly perturbed form in order to illustrate where the reduced system for the case of a higher order filter differs from the reduced system when the filter order is the same as the system order. To begin we apply the change of variables

$$w^{(i)} = w^{(i-1)} + \tau^{i-1} (A_f^{-1})^i B_f x_i$$
(3.17)

for  $i = 1, \dots, r$ , where  $w^{(0)} = w$ . Then we apply the scaling

$$\zeta = \frac{1}{\tau^{r-1}} w^{(r-1)} \tag{3.18}$$

to get

$$\tau \dot{\zeta} = A_f \zeta + \frac{1}{\tau^{r-1}} B_f v + (A_f^{-1})^{r-1} B_f x_r \tag{3.19}$$

As when the filter and system have the same order, we apply the change of variables for  $i = 1, \dots, r$ , however at each step  $w^{(i)} \in \mathbb{R}^m$ . Thus, the steps in both cases are the same, but the dimensions are different. For the sake of clarity in dimensions, we define  $\bar{\zeta} = [\zeta_1, \dots, \zeta_r]^T$ . We then substitute  $\bar{\zeta} = \frac{1}{\tau^{r-1}} \bar{w}^{(r-1)}$  into  $E^{-1}(\tau) \bar{w}$  to get

$$E^{-1}(\tau)\bar{w} = E(\tau)\bar{\zeta} + \frac{1}{\beta_1}x + \tau G(\tau)x$$
(3.20)

where  $G(\tau)$  is a polynomial in  $\tau$ . Now, the reduced system in standard singularly perturbed form is

$$\dot{z} = \psi(z, x, d, \gamma(\vartheta, E(\tau)\bar{\zeta} + \frac{1}{\beta_1}x + \tau G(\tau)x, d))$$
(3.21)

$$\dot{x} = Ax + B\phi(z, x, d, \gamma(\vartheta, E(\tau)\bar{\zeta} + \frac{1}{\beta_1}x + \tau G(\tau)x, d))$$
(3.22)

$$\dot{\vartheta} = \Gamma(\vartheta, E(\tau)\bar{\zeta} + \frac{1}{\beta_1}x + \tau G(\tau)x, d)$$
(3.23)

$$\tau \dot{\zeta} = A_f \zeta + \frac{1}{\tau^{r-1}} B_f v + (A_f^{-1})^{r-1} B_f x_r$$
(3.24)

In summary the only differences between the closed loop systems (2.16)-(2.17) and (3.10)-(3.11) are the absence of the noise term in (3.11), the slight modification to  $g(\chi, d, D(\varepsilon)\eta)$ , and the dimensions of the filter. Additionally, the only difference between the reduced systems (2.31)-(2.34) and (3.21)-(3.24) is the dimension of (3.24), which corresponds to the dimension of the filter. Given these similarities and the fact that Assumptions 1-4 are met for the case where the order of the filter is higher than the order of the system, the results from Chapter 2 hold for this case. The proof of this would be identical to the proof of Theorem 1 with the noise term dropped from the fast equation (2.17).

### 3.2.3 Simulation

To verify the results of this chapter, we present a simulation of the nonlinear magnetic suspension system shown in [7]. The system is comprised of a steel ball that is levitated via an electromagnet. The position of the ball is measured using an optical sensor, while the velocity and current are estimated by a high-gain observer. The state equations for this system are

$$\dot{x}_1 = x_2 \tag{3.25}$$

$$\dot{x}_2 = g - \frac{k}{m} x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2}$$
(3.26)

$$\dot{x}_3 = \frac{a+x_1}{L_0a+L_1a+L_1x_1} \left( -Rx_3 + \frac{L_0ax_2x_3}{(a+x_1)^2} + u(x) \right)$$
(3.27)

$$y = x_1 + v \tag{3.28}$$

where  $x_1$  is the ball position,  $x_2$  is the ball velocity,  $x_3$  is the current applied to the electromagnetic, and u is the control voltage. The constant parameters are: R, the circuit series resistance, k, a viscous friction coefficient, and m, the mass of the ball. Additionally,  $L_0$ ,  $L_1$ , and a are positive parameters used in calculating the inductance of the electromagnet. In order to design the control, we apply the following change of variables

$$z_1 = x_1 \tag{3.29}$$

$$z_2 = x_2 \tag{3.30}$$

$$z_3 = g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2}$$
(3.31)

which transforms the system into

$$\dot{z}_1 = z_2 \tag{3.32}$$

$$\dot{z}_2 = z_3 \tag{3.33}$$

$$\dot{z}_3 = f(x) + g(x)u \tag{3.34}$$

where

$$f(x) = \frac{L_0 a x_3 (L_1 x_2 + Ra + Rx_1)}{m(a+x_1)^2 (L_0 a + L_1 a + L_1 x_1)} - \frac{k}{m} \left( g - \frac{k}{m} x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2} \right)$$
(3.35)

$$g(x) = \frac{L_0 a x_3}{-m(a+x_1)(L_0 a + L_1 a + L_1 x_1)}$$
(3.36)

Here, f(x) and g(x) have been left in terms of x for the sake of convenience in designing a state feedback control. Using feedback linearization, the control is designed as

$$u(x) = \frac{1}{g(x)}(v - f(x))$$
(3.37)

where

$$v = -k_1 z_1 - k_2 z_2 - k_3 z_3 \tag{3.38}$$

and, in terms of x

$$v = -k_1 x_1 - k_2 x_2 - k_3 \left( g - \frac{k}{m} x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2} \right)$$
(3.39)

The fifth order low-pass filter is designed as

$$\tau \dot{w_1} = w_2 \tag{3.40}$$

$$\tau \dot{w_2} = w_3 \tag{3.41}$$

$$\tau \dot{w_3} = w_4 \tag{3.42}$$

$$\tau \dot{w_4} = w_5 \tag{3.43}$$

$$\tau \dot{w_5} = -w_1 - 5w_2 - 10w_3 - 10w_4 - 5w_5 + y \tag{3.44}$$

The linear high-gain observer is given by

$$\dot{\hat{z}}_1 = \hat{z}_2 + \frac{\alpha_1}{\varepsilon} (w_1 - \hat{z}_1)$$
 (3.45)

$$\dot{\hat{z}}_1 = \hat{z}_3 + \frac{\alpha_2}{\varepsilon^2} (w_1 - \hat{z}_1)$$
 (3.46)

$$\dot{\hat{z}}_1 = \frac{\alpha_3}{\varepsilon^3} (w_1 - \hat{z}_1)$$
 (3.47)

The observer is designed and implemented in the z coordinates because Theorem 1 applies only to systems in a chain of integrators form. Since the high-gain observer is robust to uncertainties in  $\phi_0$ , this term can be neglected for convenience, giving the linear high-gain observer in (3.45)-(3.47). The parameter values used are given in Table 3.1. The measurement noise is white noise given by  $v \sim \mathcal{U}(-1 \times 10^{-6}, 1 \times 10^{-6})$ . The initial conditions are  $x_1(0) = 0.1, \hat{x}_1(0) = 0.05$ , and all other states start at 0. The control is saturated outside

Parameter	Value	Parameter	Value
R	$1\Omega$	$k_1$	82
$L_0$	$0.01~\mathrm{H}$	$k_2$	161
$L_1$	$0.02~\mathrm{H}$	$k_3$	80
k	0.001	$\alpha_1$	3
m	$0.1 \ \mathrm{kg}$	$\alpha_2$	3
g	$9.81 \text{ m/s}^2$	$\alpha_3$	1
a	$0.05 \mathrm{m}$	au	0.001

Table 3.1: Parameter Values Used for Simulation of Magnetic Suspension System

the set [-20, 20]. The control task is to keep the ball suspended at y = 0.05m. As shown in Figure 3.1, decreasing  $\varepsilon$  causes the trajectories of the closed loop system under output feedback to approach those of the reduced system.



Figure 3.1: Trajectories for the reduced system (solid blue), filtered output feedback with  $\varepsilon = 8 \times 10^{-4}$  (dotted red) and  $\varepsilon = 1 \times 10^{-4}$  (dashed green) when using a higher order filter.

## 3.3 Lower Order Filters

### 3.3.1 Formulation

We now move on to consideration of the case where the order of the filter is lower than the order of the plant. We continue to consider the nonlinear system (2.1)-(2.3), and the state feedback control (3.4)-(3.5). The lower order filter is given by

$$\tau \dot{w} = A_f w + B_f y \tag{3.48}$$

$$w_1 = C_f w \tag{3.49}$$

where  $w \in \mathbb{R}^m$  and m < r. The matrix  $A_f$  is the same as (2.10). The  $m \times 1$  matrix  $B_f$ and the  $1 \times m$  matrix  $C_f$  are of the same form as (2.4) and (2.5), however with a smaller dimension due to the lower number of filter states. The observer is the same as (3.8).

### 3.3.2 Closed Loop System Analysis

If we attempt to apply the same change of variables we have used previously, namely

$$\eta_i = \frac{\frac{1}{\tau^{i-1}} w_i - \hat{x}_i}{\varepsilon^{r-i}} \tag{3.50}$$

we quickly come across a problem. When we attempt to find  $\eta_{m+1}$ , we have

$$\eta_{m+1} = \frac{\frac{1}{\tau^m} w_{m+1} - \hat{x}_m}{\varepsilon^{r - (m+1)}} \tag{3.51}$$

However,  $w_{m+1}$  does not exist. If we try to circumvent this problem by instead applying the scaling

$$\eta_i = \frac{w_1^{(i)} - \hat{x}_i}{\varepsilon^{r-i}}$$
(3.52)

where  $w_1^{(i)}$  is the *i* derivative of the filter output  $w_1$ , we again find a problem. At i = m + 1 we have

$$\eta_{m+1} = \frac{\frac{1}{\tau^m} (-\beta_1 w_1 - \dots - \beta_m w_m + x_1 + v - \hat{x}_1)}{\varepsilon^{r - (m+1)}}$$
(3.53)

and

$$\varepsilon \dot{\eta}_{m+1} = \frac{\frac{1}{\tau^m} (-\beta_1 \dot{w}_1 - \dots - \beta_m \dot{w}_m + \dot{x}_1 + \dot{v} - \dot{\hat{x}}_{m+1})}{\varepsilon^{r-m}}$$
(3.54)

Hence, in order to put the closed loop system under output feedback in standard singularly perturbed form, we would have to be able to differentiate the measurement noise, which is an unreasonable assumption to make for a general system.

If we consider the special case where measurement noise is differentiable, we can complete the analysis. Using the change of variables given in (3.52), the closed loop system under output feedback is given by

$$\dot{\chi} = \begin{bmatrix} \psi \left( z, x, d, \gamma \left( \vartheta, E^{-1}(\tau) \bar{w} - D(\varepsilon) \eta, d \right) \right) \\ Ax + B\phi \left( z, x, d, \gamma \left( \vartheta, E^{-1}(\tau) \bar{w} - D(\varepsilon) \eta, d \right) \right) \\ \frac{1}{\tau} \left( A_f w + B_f(x_1 + v) \right) \\ \Gamma \left( \vartheta, E^{-1}(\tau) \bar{w} - D(\varepsilon) \eta, d \right) \end{bmatrix}$$
(3.55)  
$$\varepsilon \dot{\eta} = A_0 \eta + \varepsilon Bg \left( \chi, d, D(\varepsilon) \eta \right)$$
(3.56)

where  $\chi \in \mathbb{R}^M$ , the  $r \times 1$  vector  $\bar{w}$ , and the  $r \times r$  block diagonal matrix  $E(\tau)$  are given by

$$\chi = [z, x, w, \vartheta]^T \tag{3.57}$$

$$\bar{w} = [w_1, \dots, w_m, w_1^{(m)}, \dots, w_1^{(r)}]^T$$
(3.58)

$$E(\tau) = \begin{bmatrix} H(\tau) & 0\\ 0 & I \end{bmatrix}$$
(3.59)

where  $H(\tau) = diag(1, \tau, \dots, \tau^{r-m})$ . We define the right hand side of (3.55) as  $f(\chi, d, D(\varepsilon)\eta)$ . For this case,

$$g(\chi, d, D(\varepsilon)\eta) = w_1^{(r+1)} - \phi(E^{-1}(\tau)\bar{w} - D(\varepsilon)\eta, d, \gamma(\vartheta, E^{-1}(\tau)\bar{w} - D(\varepsilon)\eta, d))$$
(3.60)

where

$$w_1^{(r+1)} = \frac{1}{\tau^{r-m}} \left( -\beta_1 w_1^{(r-m)} - \dots - \beta_m w_m^{(r-m)} + \phi(z, x, d, \gamma(\vartheta, \bar{w} - D(\varepsilon)\eta, d)) + v^{(r-m)} \right)$$
(3.61)

Setting  $\varepsilon = 0$  again results in  $\eta = 0$ , giving the following reduced system.

$$f_r(\chi, d) = \begin{bmatrix} \psi(z, x, d, \gamma(\vartheta, E^{-1}(\tau)\bar{w}, d)) \\ Ax + B\phi(z, x, d, \gamma(\vartheta, E^{-1}(\tau)\bar{w}, d,)) \\ \frac{1}{\tau}(A_f w + B_f(x_1 + v)) \\ \Gamma(\vartheta, E^{-1}(\tau)\bar{w}, d) \end{bmatrix}$$
(3.62)

The reduced system (3.62) is not in the standard singularly perturbed form, due to the negative powers of  $\tau$ . In order to transform (3.62) into standard singularly perturbed form, we begin with the same change of variables we have used previously

$$w^{(i)} = w^{(i-1)} + \tau^{i-1} (A_f^{-1})^i B_f x_i$$
(3.63)

for  $i = 1, \dots, r$ , where  $w^{(0)} = w$ . Then we apply the scaling

$$\zeta = \frac{1}{\tau^{r-1}} w^{(r-1)} \tag{3.64}$$

to get

$$\tau \dot{\zeta} = A_f \zeta + \frac{1}{\tau^{r-1}} B_f v + (A_f^{-1})^{r-1} B_f x_r \tag{3.65}$$

As in the case of a higher order filter, it is important to highlight the dimensions of the quantities above. In this case, each  $w^{(i)}$  equation will have the same order as the filter,

which will be less than the nonlinear system. Similarly,  $\zeta$  will have the same order as the filter. We will, however, have r - 1 different  $w^{(i)}$  equations. In standard form, the reduced system is given by

$$f_r(\chi, d) = \begin{bmatrix} \psi(z, x, d, \gamma(\vartheta, P(\zeta, x), d)) \\ Ax + B\phi(z, x, d, \gamma(\vartheta, P(\zeta, x), d, )) \\ \Gamma(\vartheta, P(\zeta, x), d) \\ \tau\dot{\zeta} = A_f\zeta + \frac{1}{\tau^T - 1}B_fv + G(\tau) \end{bmatrix}$$
(3.66)

where  $G(\tau)$  is a polynomial in  $\tau$  and the  $r \times 1$  vector  $P(\zeta, x)$  is given by

$$P(\zeta, x) = \begin{bmatrix} H(\tau)\zeta + \frac{1}{\beta_1}\bar{x} + \tau G(\tau)\bar{x} \\ \frac{1}{\beta_1}\tilde{x} + M^{-1}(\tau)\bar{v} \end{bmatrix}$$
(3.67)

Here,  $\bar{x}$ ,  $\tilde{x}$ ,  $\bar{v}$ , and  $M(\tau)$  are given by

$$\bar{x} = [x_1, \dots, x_m]^T \tag{3.68}$$

$$\tilde{x} = [x_{m+1}, \dots, x_r]^T$$
 (3.69)

$$\bar{v} = [v, \dot{v}, \dots, v^{(m-r)}]^T$$
(3.70)

$$M(\tau) = \begin{bmatrix} \tau & 0 & \cdots & 0 & 0 \\ -1 & \tau & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & \tau & 0 \\ 0 & 0 & \cdots & -1 & \tau \end{bmatrix}$$
(3.71)

From this point forward, as long as the additional assumption that measurement noise is

differentiable is met, the analysis is the same as Chapter 2.

### 3.3.3 Simulation

We return to the magnetic suspension example, however we now use a first order low-pass filter given by

$$\tau \dot{w}_1 = -w_1 + y \tag{3.72}$$

The measurement noise used for this simulation is not uniform white noise, because we assumed differentiable measurement noise. Instead, following the procedure outlined in [1], we use wide-band noise that is generated by passing the output of the Simulink "Uniform Random Number" block through the transfer function

$$H(s) = \frac{.005^2}{.005^2 s^2 + .01s + 3} \tag{3.73}$$

The output signal from this transfer function is differentiable noise that can be used to simulate the reduced system (3.62). The limits on the the Simulink "Uniform Random Number" block are  $\pm 0.3$ , which, after being passed through the transfer function (3.73), corresponds to differentiable wide-band noise that is  $\mathcal{O}(10^{-6})$ . This differentiable wideband noise allows us to compare the trajectories under output feedback with those of the reduced system. The differentiable measurement noise was used in simulating both the reduced system and the system under output feedback so that the comparison is under the same conditions. Figure 3.2 shows that as  $\varepsilon$  decreases, the trajectories of the closed loop system under output feedback approach the trajectories of the reduced system. This serves to confirm the extension of the analytic result from Chapter 2 to the case of a lower order filter with differentiable measurement noise.



Figure 3.2: Trajectories for the reduced system (solid blue), filtered output feedback with  $\varepsilon = 8 \times 10^{-4}$  (dotted red) and  $\varepsilon = 5 \times 10^{-4}$  (dashed green) with a lower order filter.

Finally, it is useful to do a comparison between the three different filter possibilities. For this comparison, the lower order filter was simulated with the same uniform white noise as the other two cases. Although the analytic result requires differentiable noise, it is still possible for the system to be simulated with nondifferentiable noise. There is no guarantee in this case that the system would be stable, however we can see from the simulation that we still have stability.

Figure 3.3 shows the trajectories for the magnetic suspension system with a filter that is the same order as the plant, one that is a higher order, and one that is a lower order. Although there is little difference between the three filters when we compare the full trajectories, looking at the steady-state response shown in Figure 3.4 shows a slight advantage to using a filter that is the same order or higher than that of the plant. The trajectories when using a filter that is the same order or higher than the nonlinear system have a less oscillatory response than when the lower order filter is used.



Figure 3.3: Trajectories for the system under state feedback (blue "x" marks), and output feedback with a lower order filter (solid green), a filter the same order as the plant (dashed red), and a higher order filter (dotted turquoise).



Figure 3.4: Steady-state  $x_2$  trajectories for the system under output feedback with a lower order filter (solid blue), a filter the same order as the plant (dashed green), and a higher order filter (dotted red).

## Chapter 4

# Filters After the Observer

## 4.1 Introduction

Until now, we have been considering systems where filter is placed before the observer, as shown in Figure 4.1. It is also possible for the filter to be placed after the observer, as shown in Figure 4.2.

When the filter is placed after the observer, it filters the control signal rather than the system output with noise. This configuration makes a good deal of sense if one thinks of replacing the observer with a sensor that measures states instead of estimating them. To filter a measured state, the filter is placed after the measurement is taken. It would be nonsensical to place the filter before the measurement it was filtering. In the same way, if the observer is thought of as a soft sensor, it makes sense to place the filter after the observer.



Figure 4.1: Block Diagram of System with Filter Before Observer



Figure 4.2: Block Diagram of System with Filter After Observer

## 4.2 Formulation

In this section and the next, we will show that we can extend the results of [2] to this case. We will again consider the nonlinear system of Chapter 2

$$\dot{z} = \psi(z, x, d, u) \tag{4.1}$$

$$\dot{x} = Ax + B\phi(z, x, d, u) \tag{4.2}$$

$$y = Cx + v \tag{4.3}$$

where  $u \in \mathbb{R}$  is the control input,  $z \in \mathbb{R}^l$  and  $x \in \mathbb{R}^r$  are states,  $y \in \mathbb{R}$  is the measured output,  $d(t) \in \mathbb{R}^p$  is a vector of exogenous signals, and  $v(t) \in \mathbb{R}$  represents measurement noise. The nonlinear system meets Assumption 1 of Chapter 2. The state feedback control is once again designed as

$$\dot{\vartheta} = \Gamma(\vartheta, x, d)$$
 (4.4)

$$u = \gamma(\vartheta, x, d) \tag{4.5}$$

We will modify our assumptions of the previous chapters in order to apply the results of [2], which does not require global uniform asymptotic stability. Although we could continue to require global uniform asymptotic stability and still apply the results of [2], there is no good reason to do so, as golabl uniform asymptotic stability is a more strict assumption.

#### Assumption 5.

The closed loop system under (4.4)-(4.5) is uniformly asymptotically stable with respect to a compact positively invariant set  $\mathcal{A}$ , uniformly in d.

The filter takes a similar form to the filter used in Chapters 2 and 3

$$\tau \dot{w} = A_f w + B_f u \tag{4.6}$$

$$w_1 = C_f w \tag{4.7}$$

where  $\tau$  is the filter time constant,  $w \in \mathbb{R}^m$  are filter states, and  $w_1$  is the filter output. In this case, the dimension of the filter m, is not constrained in any way, as it won't effect the analysis. The  $m \times m$  matrix  $A_f$  has the same structure as (2.10), the  $m \times 1$  matrix  $B_f$ , and the  $1 \times m$  matrix  $C_f$  have the same structure as (2.4)-(2.5). The other difference between this filter and the previously examined filter is that the input to this filter is u, while the input to the previous filter was y. The output feedback takes the form

$$\dot{\hat{x}} = A\hat{x} + B\phi_0(\hat{x}, d, u) + H(y - \hat{x}_1)$$
(4.8)

where  $\phi_0$  is a nominal model of  $\phi$  and meets Assumption 3 of Chapter 2. This observer is slightly different from the observer of the previous chapters in that the high gain is multiplied by the error between the nonlinear system output y and the estimate  $\hat{x}_1$ . In previous chapters, the high gain was multiplied by the error between the filter output  $w_1$  and the estimate  $\hat{x}_1$ . The difference here is due to the fact that the signal coming into the observer is now the nonlinear system output instead of the filter output.

## 4.3 Closed Loop System Analysis

In this section, we begin by applying the same change of variables as was used in [2]

$$\eta_i = \varepsilon^{i-1} \left( x_i - \hat{x}_i \right) \tag{4.9}$$

This change of variables is clearly different from the change of variables used to transform the system into standard singularly perturbed form in previous chapters. This difference is due to the fact that the input to the observer is now the nonlinear system output. We define  $\chi = [z, x, w, \vartheta]^T \in \mathbb{R}^M$ , and then the closed loop system under output feedback becomes

$$\dot{\chi} = f(\chi, d, x - D^{-1}(\varepsilon)\eta) = \begin{bmatrix} \psi\left(z, x, d, \gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right)\right) \\ Ax + B\phi\left(z, x, d, \gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right)\right) \\ \frac{1}{\tau}\left(A_{f}w + B_{f}\gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right)\right) \\ \Gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right) \end{bmatrix}$$
(4.10)  
$$\varepsilon\dot{\eta} = A_{0}\eta + \varepsilon^{r}Bg\left(\chi, d, x - D^{-1}(\varepsilon)\eta\right) + B_{0}v$$
(4.11)

where  $A_0$  has the same form as (2.18),  $B_0 = [-\alpha_1, \cdots, -\alpha_r]^T$ , and

$$g\left(\chi, d, x - D^{-1}(\varepsilon)\eta\right) = \phi\left(z, x, d, \gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right)\right) - \phi_0\left(x - D^{-1}(\varepsilon)\eta, d, \gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right)\right)$$
(4.12)

In order to make the analysis easier, we recognize that the filter can be considered part of the dynamic control  $\Gamma(\vartheta, x, d)$ . The new combined control can be written as

$$\dot{\vartheta} = \Gamma\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right) \tag{4.13}$$

$$\tau \dot{w} = A_f w + B_f \gamma \left(\vartheta, x - D^{-1}(\varepsilon)\eta, d\right)$$
(4.14)

$$u_0 = C_f w \tag{4.15}$$

or in a more compact form

$$\dot{\vartheta}_0 = \Gamma_0\left(\vartheta, x - D^{-1}(\varepsilon)\eta, d, w\right) \tag{4.16}$$

$$u_0 = C_f w \tag{4.17}$$

Items 1 and 2 of Assumption 2, which apply to the original state feedback control (4.4)-(4.5), are assumed to apply to the new combined control as well. The closed loop system under output feedback can now be written as

$$\dot{\chi} = f(\chi, d, D^{-1}(\varepsilon)\eta) = \begin{bmatrix} \psi(z, x, d, C_f w) \\ Ax + B\phi(z, x, d, C_f w) \\ \Gamma_0(\vartheta, x - D^{-1}(\varepsilon)\eta, d, w) \end{bmatrix}$$
(4.18)  
$$\varepsilon \dot{\eta} = A_0 \eta + \varepsilon^r Bg_0\left(\chi, d, x - D^{-1}\eta, C_f w\right) + B_0 v$$
(4.19)

where

$$g_0\left(\chi, d, x - D^{-1}(\varepsilon)\eta, C_f w\right) = \phi\left(z, x, d, C_f w\right) - \phi_0\left(x - D^{-1}(\varepsilon)\eta, d, C_f w\right)$$
(4.20)

With the exception of the negative powers of  $\varepsilon$ , (4.18)-(4.19) are in the standard singularly perturbed form. As observed in [2], due to the global boundedness of the functions f and gin  $D^{-1}(\varepsilon)\eta$ , we can extend the behavior of standard singularly perturbed systems to (4.18)-(4.19). The reduced system can be found by setting  $\eta = 0$  to get

$$f(\chi, d, 0) = f_r(\chi, d) = \begin{bmatrix} \psi(z, x, d, C_f w) \\ Ax + B\phi(z, x, d, C_f w) \\ \Gamma_0(\vartheta, x, d, w) \end{bmatrix}$$
(4.21)

The reduced system is the same as the closed loop system under (4.4)-(4.5) with u in (4.1)-(4.2) replaced by  $u_0$ . As with our state feedback control, rather than require global uniform asymptotic stability, we will instead require the following assumption to be met.

#### Assumption 6.

The reduced order system (4.21) is uniformly asymptotically stable with respect to  $\mathcal{A}$ , uniformly in d.

As before, this assumes that  $\tau$  is chosen small enough that the reduced system maintains the stability of the closed loop system under state feedback. We can see that this is possible simply by recognizing that, in this case, the reduced system is already in standard singularly perturbed form. This is clear if we consider an expanded from of (4.21).

$$\dot{z} = \psi \left( z, x, d, C_f w \right) \tag{4.22}$$

$$\dot{x} = Ax + B\phi\left(z, x, d, C_f w\right) \tag{4.23}$$

$$\dot{\vartheta} = \Gamma\left(\vartheta, x, d\right) \tag{4.24}$$

$$\tau \dot{w} = A_f w + B\gamma \left(\vartheta, x, d\right) \tag{4.25}$$

In this system, if  $\tau = 0$ , we will have  $C_f w = \gamma(\vartheta, x, d)$ , which will give the closed loop system under state feedback.

From this point forward, the analysis will be identical to the analysis of [2], with the additional assumption that  $\tau$  is made small enough that the reduced system maintains stability. The other difference between this system and that of [2] is that the reduced system here will contain the filter with the dynamic control. For convenience, we will repeat here the result of [2].

**Theorem 2.** Let Assumptions 1, 2.1 - 2.3, 3, and 5-6 hold and consider the closed-loop system (4.18)-(4.19). Let  $\mathcal{M}$  be any compact set in the interior of  $\mathcal{R}$  and  $\mathcal{N}$  be any compact subset of  $\mathbb{R}^r$ , and suppose that  $\chi(t_0) \in \mathcal{M}$  and  $\hat{x}(t_0) \in \mathcal{N}$ . Then:

- There exist positive constants  $c_a$  and  $\mu^*$  such that for each  $\mu < \mu^*$  there is a constant  $\varepsilon_a = \varepsilon_a(\mu) > c_a \mu^{1/r}$ , with  $\lim_{\mu \to 0} \varepsilon_a(\mu) = \varepsilon_a^* > 0$ , such that for each  $\varepsilon \in (c_a \mu^{1/r}, \varepsilon_a]$  the trajectories of the closed-loop system are bounded for all  $t \ge t_0$ .
- There exist  $\mu_1^* > 0$  and a class  $\mathcal{K}$  function  $\rho_1$  such that for every  $\mu < \mu_1^*$  and every  $\xi_1 > \rho_1(\mu)$  there are constants  $T_1 = T_1(\xi_1) \ge t_0$  and  $\varepsilon_b = \varepsilon_b(\mu, \xi_1) > c_a \mu^{1/r}$ , with  $\lim_{\mu \to 0} \varepsilon_b(\mu, \xi_1) = \varepsilon_b^*(\xi_1) > 0$ , such that for each  $\varepsilon \in (c_a \mu^{1/r}, \varepsilon_b]$ , we have

$$\max\{|\chi(t)|_{\mathcal{A}}, \|x(t) - \hat{x}(t)\|\} \le \xi_1, \qquad \forall t \ge T_1.$$
(4.26)

• There exist  $\mu_2^* > 0$  and a class  $\mathcal{K}$  function  $\rho_2$  such that for every  $\mu < \mu_2^*$  and every  $\xi_2 > \rho_2(\mu)$  there is a constant  $\varepsilon_c = \varepsilon_c(\mu, \xi_2) > c_a \mu^{1/r}$ , with  $\lim_{\mu \to 0} \varepsilon_c(\mu, \xi_2) = \varepsilon_c^*(\xi_2) > 0$ ,

such that for each  $\varepsilon \in (c_a \mu^{1/r}, \varepsilon_c]$  we have

$$\|\chi(t) - \chi_r(t)\| \le \xi_2, \quad \forall t \ge t_0$$
(4.27)

where  $\chi_r(t)$  is the solution of (4.21) with  $\chi_r(t_0) = \chi(t_0)$ .

The three bullet points of Theorem 2 respectively state boundedness of the trajectories under output feedback, ultimate boundedness of the trajectories under output feedback where they come close to  $\mathcal{A}$  as time progresses, and closeness of the trajectories under output feedback to those of the reduced system. This result is somewhat different than the result of Theorem 1. While Theorem 1 puts no restrictions on the relationship between the bound on measurement noise  $\mu$  and the small parameter  $\varepsilon$ , here, we are requiring  $\frac{\mu}{\varepsilon}$  to be small. Additionally, in Theorem 1, there was no lower bound placed on the value of  $\varepsilon$ , while here  $\varepsilon$  is restricted to being greater than some constant that is a function of  $\mu$ . Finally, in this case, the trajectories under output feedback cannot be made arbitrarily close to the trajectories of the reduced system and the ultimate bound cannot be made arbitrarily small. Instead, in both cases, there is a lower bound that is dependent on  $\mu$  that prevents arbitrary closeness or arbitrarily small ultimate bound.

## 4.4 Simulation

In this section we continue to use the magnetic suspension system of the previous chapter. All parameters are the same as the previous chapter, however, in this case we have used a third order filter of the form

$$\tau \dot{w}_1 = w_2 \tag{4.28}$$

$$\tau \dot{w}_2 = w_3 \tag{4.29}$$

$$\tau \dot{w}_2 = -w_1 - 3w_2 - 3w_3 + u \tag{4.30}$$

To compare simulation results, we will also consider the configuration where the filter is before the observer. For this case, the filter is given by

$$\tau \dot{w}_1 = w_2 \tag{4.31}$$

$$\tau \dot{w}_2 = w_3 \tag{4.32}$$

$$\tau \dot{w}_2 = -w_1 - 3w_2 - 3w_3 + y \tag{4.33}$$

In both cases we have used the same parameters as the previous chapter and chosen  $\varepsilon = 8 \times 10^{-4}$ . The noise is given by  $v \sim \mathcal{U}(-1 \times 10^{-6}, 1 \times 10^{-6})$ . Figure 4.3 shows a comparison of the trajectories for both systems. Clearly, there is little difference as far as the transient response is concerned. Considering the steady-state response of the  $x_2$  trajectory in Figure 4.4 shows a slight advantage to placing the filter before the observer. In this case, the  $x_2$  trajectory when placing the filter before the observer has a less oscillatory steady-state response than when the filter is placed after the observer.



Figure 4.3: Trajectories for the configuration where the filter is placed before the observer (solid blue) and after the observer (dashed green)



Figure 4.4: Steady-state  $x_2$  trajectories for the configuration where the filter is placed before the observer (solid blue) and after the observer (dashed green)

For another comparison, consider the example system of Chapter 2. For convenience, it is repeated here

$$\dot{x}_1 = x_2 \tag{4.34}$$

$$\dot{x}_2 = -x_1^3 + u \tag{4.35}$$

$$y = x_1 + v \tag{4.36}$$

where the  $x_i$ 's are the states, y is the output, and v is the bounded measurement noise. The parameters are again kept the same as Chapter 2 with  $\varepsilon = 5 \times 10^{-4}$ . For this simulation, the amplitude of the measurement noise was increased, so that  $v \sim \mathcal{U}(-0.01, 0.01)$ . The filter after the observer is

$$\tau \dot{w_1} = w_2 \tag{4.37}$$

$$\tau \dot{w_2} = -w_1 - 2w_2 + u \tag{4.38}$$

and the filter before the observer is given by

$$\tau \dot{w_1} = w_2 \tag{4.39}$$

$$\tau \dot{w_2} = -w_1 - 2w_2 + y \tag{4.40}$$

Figure 4.5 shows the trajectories of both configurations as well as the trajectories of the system under state feedback. In this case, it is clear that placing the filter after the observer results in a response that is closer to the system under state feedback. However, Figure 4.6 clearly shows that the steady-state  $x_2$  trajectory of the configuration where the filter is

placed after the observer is noisier than when the filter is placed before the observer. For this reason, the application should always be considered when choosing which configuration to use. In an application where a closer response is more important, or there is less measurement noise within the system, the filter should be placed after the observer. When sensitivity to measurement noise is a larger factor, the filter should be placed before the observer.



Figure 4.5: Trajectories for the system under state feedback (solid blue) and the system under output feedback with the configuration where the filter is placed before the observer (dotted red) and after the observer (dashed green)



Figure 4.6: Steady-state  $x_2$  trajectories for the system under state feedback (solid blue) and the system under output feedback with the configuration where the filter is placed before the observer (dotted red) and after the observer (dashed green)

# Chapter 5

# Conclusion

In this work we have presented an analysis of the use of low-pass filters in conjunction with high-gain observers as a method to attenuate the effects of measurement noise. Specifically, we have looked at four different combinations of low-pass filter and high-gain observer. The first case considered was the case where the low-pass filter is placed before the observer and is of the same order as the nonlinear system. For this case we were able to analytically prove that the trajectories of the closed loop system under output feedback were bounded, ultimately bounded, and close to the trajectories of the reduced system. Through simulation, we compared this method with a nonlinear-gain observer. It was shown that for low-amplitude white noise, the nonlinear-gain observer performed slightly better than the high-gain observer with low-pass filter system. For a high-amplitude, high-frequency noise condition, it was found that the nonlinear-gain observer yielded an unacceptable response, while the low-pass filter with high-gain observer performed adequately, and the low-pass filter with nonlinear-gain observer performed adequately, and the low-pass filter with nonlinear-gain observer performed best.

Continuing the analysis of filters placed before the observer, we extended the results of the case where the filter is the same order as the nonlinear system to a higher order filter. We also considered the case of a lower order filter. We were able to extend the analytic results only for the special case where measurement noise was differentiable. A magnetic suspension system was simulated in order to confirm the analytic results in each case and also to draw a comparison between the different filter orders. It was shown that a filter that
was of the same order or a higher order than the nonlinear system performed better than the case where the filter was of a lower order than the nonlinear system.

To complete our study, we considered the case where the filter is placed after the observer. This placement was justified by considering the high-gain observer as a soft sensor. We showed that by considering the filter to be part of the dynamic control, the results of [2] could be extended to this system. Through simulation, it was shown that placing the filter after the observer resulted in trajectories under output feedback that were closer to the trajectories under state feedback than when the filter was placed before the observer. Additionally, the simulations demonstrated that the filter before the observer configuration was more adept than the filter after the observer configuration at reducing the effects of measurement noise.

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