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ABSTRACT

RIEMANNIAN GEOMETRY OF
VECTOR BUNDLES

By

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A natural metric structure on the tangent bundle of a manifold, considered as a manifold, was introduced by S. Sasaki. The curvature of this metric was studied by Kowalski and he answered the question of locally symmetric tangent bundles. Naturally, similar questions were raised and D. Blair and others provided answers concerning locally symmetric tangent sphere bundles and conformally flat tangent sphere bundles. In this line of study, the Sasaki metric on the normal bundle of a submanifold was studied by Borisenko and Yampol'skii and they showed that the Sasaki metric on the normal bundle is flat if and only if the submanifold is flat with flat normal connection.

In this thesis, we attempt to extend this to general vector bundles over a manifold and define a metric via a similar method. We compute the curvature of this metric on general vector bundles and obtain some differential geometric results. We prove that the Sasaki metric on a general vector bundle is locally symmetric if and only if the base manifold is locally symmetric and the connection ∇ of this metric is flat. It is also proved that a vector bundle is conformally flat if and only if either the base manifold is flat with flat connection, or it has constant curvature with flat connection and rank 1. The unit vector bundle of a vector bundle of rank 2 is also studied.

Then, the normal bundle of an integral submanifold M in a Sasakian manifold is studied and we show that the normal bundle has a contact metric structure satisfying $R._\xi\xi = 0$, where ξ is the characteristic vector field and R denotes the Riemannian curvature tensor. Moreover, $R..\xi$ depends only on the induced metric of the submanifold M.

Motivated by this, we consider the contact metric manifolds with $R._\xi\xi = 0$ and prove that a locally symmetric contact metric manifold with $R._\xi\xi = 0$ is locally the product of a flat $(n+1)$-dimensional manifold and a manifold of constant curvature 4. It is also shown that a contact metric manifold of dimension ≥ 5 with $R._\xi\xi = 0$ cannot be conformally flat.

Finally, we investigate the normal bundle NL of a Lagrangian submanifold L in a Kähler manifold and show that NL has a natural symplectic structure and provide equivalent conditions for NL to be Kähler.

DEDICATION

To my parents and my wife.

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Introduction

Let M^n be an n-dimensional differentiable manifold. The set of all tangent vectors of M^n form, with a natural topology, the ***tangent bundle*** of M^n, denoted by TM^n. The set of all unit vectors of M^n constitutes a hypersurface of TM^n, called the ***tangent sphere bundle*** of M^n, denoted T_1M^n. The tangent bundle of a given manifold M^n and more generally a vector bundle over a given manifold are among fundamental objects in modern differential geometry.

H. Poincarè first introduced a notion of Riemannian metrics on the tangent sphere bundles when regarded as manifolds. (See e.g. [Sa58].)

In 1958, S. Sasaki [Sa58] studied the differential geometry of tangent bundles of Riemannian manifolds by introducing a natural Riemannian metric structure on the tangent bundle of a manifold.

Let (M^n, G) be a Riemannian manifold. Given the line element $ds^2 = G_{ij}dx^i dx^j$ of the manifold M^n, the line element of the tangent bundle TM^n is defined by

$$d\sigma^2 = G_{ij}dx^i dx^j + G_{ij}Dv^i Dv^j \tag{0.1}$$

where Dv^i is the covariant differential of v^i, i.e.,

$$Dv^i = dv^i + \Gamma^i_{jk}v^j dx^k,$$

Γ^i_{jk} being the Christoffel symbols of G and v^i the fiber coordinates. This metric g, called the *Sasaki metric*, is canonically defined on naturally lifted vectors on M.

In 1961, the Sasaki metric on tangent bundles was determined in an invariant manner by Dombrowski [Do]. He studied the Sasaki metric on tangent bundles in terms of the *connection map* $K : TTM \to TM$. Due to his work, the classical Sasaki metric g on the tangent bundle is expressed in vector form by

$$g(X,Y) = G(\pi_* X, \pi_* Y) + G(KX, KY) \tag{0.2}$$

where $\pi : TM \to M$ is the projection map.

Then, Kowalski [Ko] began studying the curvature of the Sasaki metric on the tangent bundle of a Riemannian manifold and answered some geometric questions. In particular, he proved the following theorem.

Theorem 0.1 *Let M^n be a Riemannian manifold with Riemannian metric G. The classical Sasaki metric on the tangent bundle is locally symmetric if and only if the metric G of the base manifold M is flat.*

The tangent sphere bundle T_1M has an induced metric considered as a hypersurface of TM. It is of interest as a contact manifold and the induced metric here is homothetic to an associated metric of the contact structure. The question of locally symmetric tangent sphere bundles was studied by D. Blair [Bl89] and he obtained the following result.

Theorem 0.2 *The tangent sphere bundle T_1M^n with the Sasaki metric g is locally symmetric if and only if either (M,G) is flat, or M is 2-dimensional and of constant curvature 1.*

D. Blair and T. Koufogiorgos also studied conformally flat tangent sphere bundle and proved the following theorem [BlK].

Theorem 0.3 *Let M be an $(n+1)$-dimensional Riemannian manifold and T_1M its tangent sphere bundle with the standard contact metric structure. Then, T_1M is conformally flat if and only if M is a surface of constant Gaussian curvature* 0 *or* $+1$.

We now turn to normal bundles. Let M^n be a submanifold of $(\tilde{M}^{n+k}, \tilde{g})$. Then, the Sasaki metric g of the normal bundle NM^n is similarly defined as follows: The line element du^2 of the Sasaki metric in naturally induced local coordinates (x^i, ξ^α) are defined by

$$du^2 = G_{ij}dx^i dx^j + \tilde{g}^{\perp}_{\alpha\beta} D^{\perp}\xi^\alpha D^{\perp}\xi^\beta$$

where G is the induced metric on M^n, $\tilde{g}^{\perp}$ is the fiber metric induced from $\tilde{g}$ and $D^{\perp}\xi^\alpha = d\xi^\alpha + \mu^\alpha_{\beta i}\xi^\beta dx^i$ the covariant differential of the normal ξ in the normal connection. The Sasaki metric g on the normal bundle was determined in an invariant manner by H. Reckziegel [Re] again using the concept of the connection map $K : TNM^n \longrightarrow NM^n$ and can be expressed in the form

$$g(\tilde{X}, \tilde{Y}) = G(\pi_*\tilde{X}, \pi_*\tilde{Y}) + \tilde{g}^{\perp}(K\tilde{X}, K\tilde{Y})$$

Borisenko and Yampol'skii [BoY] studied this metric structure and as an analogue of a result of Kowalski, they showed the following theorem.

Theorem 0.4 *The Sasaki metric of NM^n is flat if and only if M^n is flat with a flat normal connection.*

In the first chapter of this thesis, we define the Sasaki metric on a vector bundle over a manifold equipped with fiber metric and a metric connection on it. Then, we compute the covariant derivatives with respect to the Riemannian connection of the Sasaki metric on the vector bundle and calculate the curvature on various lifted vector fields. Using this, we study locally symmetric and conformally flat vector bundles and prove the following theorems.

Theorem 1.6 *Let $\pi : E^{n+k} \longrightarrow M^n$ be a vector bundle over a manifold M with fiber metric $g^\perp$ and a metric connection ∇. Then, the Sasaki metric on E is locally symmetric if and only if the connection ∇ is flat and M is locally symmetric.*

Theorem 1.7 *Let $\pi : E^{n+k} \longrightarrow M^n$, $n \geq 3$, be a vector bundle over a manifold M^n with fiber metric $g^\perp$ and a metric connection ∇. Then, E^{n+k} is conformally flat if and only if either, M^n is flat with flat connection ∇, or M^n has (nonzero) constant curvature with flat connection ∇ and $k = 1$.*

Theorem 1.8 *Let $\pi : E^{2+k} \longrightarrow M^2$ be a vector bundle over a manifold M^2 with fiber metric $g^\perp$ and a metric connection ∇. Then, E^{2+k} is conformally flat if and only if either, M^2 is flat with flat connection ∇ and $k \geq 2$, or M^2 has constant curvature with flat connection ∇ and $k = 1$.*

We also study the unit vector bundle of a general vector bundle of rank 2 and prove the following theorem.

Theorem 1.10 *Let $\pi : E^{n+2} \longrightarrow M^n$, $n \geq 3$, be a vector bundle over an Einstein manifold M with fiber metric $g^\perp$ and a metric connection ∇. Suppose the unit vector bundle E_1 is conformally flat and is of constant scalar curvature. Then, either the connection ∇ is flat, or (M, G) admits an almost Hermitian structure.*

Chapter 2 is a preliminary to the remainder of the thesis. We review definitions and some well known results on contact manifolds. The Sasakian structures and some formulas related to them will also be discussed.

In Chapter 3, we study the normal bundle of an integral submanifold in a contact manifold and curvature properties of it associated with the Sasaki metric of the normal bundle. We define a linear operator l by $lX = R_{X\xi}\xi$ and obtain the following results.

Theorem 3.1 *Let M^n be an integral submanifold of a Sasakian manifold $\tilde{M}^{2n+1}$ with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$. Then, NM has the contact metric structure (ϕ, ξ, η, g) with $l = 0$.*

Theorem 3.2 *Let M^n be an integral submanifold of a Sasakian manifold $\tilde{M}^{2n+1}$. Then, for the contact metric structure (ϕ, ξ, η, g) on NM, $R_{..}\xi$ is intrinsic, i.e., it depends only on the induced metric on M.*

Motivated by Theorem 3.1, we also study contact metric manifolds satisfying $l = 0$ and answer a question raised by Perrone [Pe].

We prove the following theorems.

Theorem 3.6 *Let M^{2n+1} be a locally symmetric contact metric manifold with $l = 0$. Then, M is locally isometric to $\mathbf{E}^{n+1} \times S^n(4)$.*

Theorem 3.7 *Let M^{2n+1}, $n \geq 2$, be a contact metric manifold satisfying $l = 0$. Then, M^{2n+1} can not be conformally flat.*

In the last chapter of this thesis, we will study Lagrangian submanifolds in a Kähler manifold and the normal bundle of the submanifolds using the Sasaki metric of the normal bundle. We obtain the following results.

Theorem 4.1 *Let L be a Lagrangian submanifold of a Kähler manifold (M^{2n}, J, g). Then, $(NL, \tilde{J}, \tilde{g})$ is a symplectic manifold.*

Theorem 4.2 *Let L be a Lagrangian submanifold of a Kähler manifold (M^{2n}, J, g). Then, the following are equivalent:*

(1) *NL is Kähler.*
(2) *L has flat normal connection.*
(3) *L is flat.*

Chapter 1

Geometry of Vector Bundles

We define the Sasaki metric on general vector bundles and compute its Riemannian curvature in Section 1. In Section 2 and Section 3, we study vector bundles over a manifold and provide the necessary and sufficient conditions for the bundle to be locally symmetric and conformally flat, respectively. In the final section, we study conformally flat unit vector bundles of rank 2.

1.1 Vector bundles and their Sasaki metrics

We consider the vector bundle $\pi : E^{n+k} \longrightarrow M^n$ of rank k equipped with fiber metric $g^{\perp}$ and a metric connection ∇ where (M^n, G) is a Riemannian manifold. Let D be the Riemannian connection and $\underline{R}$ the curvature tensor of M. Elements of E can be identified as (x, U) where x is a point in M and U is a vector in its fiber $\pi^{-1}(x)$. Let $\{e_\alpha\}$ be a local orthonormal basis of the sections of E. Then, $(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_k)$ form local coordinates for E where $q_i = x_i \circ \pi$ and u_α are coordinates of U with respect to $\{e_\alpha\}$. For a section $U = U^\alpha e_\alpha$ of the bundle E,

$$\nabla_X U = X^i\left(\frac{\partial U^\alpha}{\partial x_i} + \mu^\alpha_{\beta i} U^\beta\right) e_\alpha$$

where $\nabla_{\frac{\partial}{\partial x_i}} e_\beta = \mu^\alpha_{\beta i} e_\alpha$.

We say that the connection ∇ is *flat* if the curvature tensor

$$R_{XY}U = \nabla_X\nabla_Y U - \nabla_Y\nabla_X U - \nabla_{[X,Y]}U$$

vanishes for any X, Y, and U.

$\pi_* : TE \to TM$ is a fiber-preserving linear transformation and is onto. Let $(\tilde{X}^i, \tilde{X}^{n+\alpha})$ be the local components of the tangent vector $\tilde{X}$ to E at (x, U) with respect to the basis $(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial u_\alpha})$. Then, $\pi_*\tilde{X} = \tilde{X}^i\frac{\partial}{\partial x_i}$. We define a linear map $K : TE \to E$ by

$$K\tilde{X} = (\tilde{X}^{n+\alpha} + \mu^{\alpha}_{\beta\,i}u^\beta\tilde{X}^i)e_\alpha. \tag{1.1}$$

Clearly, K is fiber-preserving and is also onto.

We define an inner product g of the vectors $\tilde{X}$ and $\tilde{Y}$ tangent to E at (x, V) by

$$g(\tilde{X}, \tilde{Y}) = G(\pi_*\tilde{X}, \pi_*\tilde{Y}) + g^\perp(K\tilde{X}, K\tilde{Y}). \tag{1.2}$$

This metric is called the *Sasaki metric* of the bundle E.

We call the kernels of the mappings π_* and K the vertical subspace VE and the horizontal space HE, respectively. Then, there is a splitting

$$TE = HE \oplus VE$$

and HE and VE are orthogonal.

For a vector field $X = X^i\frac{\partial}{\partial x^i}$ on M, we define

$$X^H = X^i\frac{\partial}{\partial q_i} - \mu^{\alpha}_{\beta\,i}u^\beta X^i\frac{\partial}{\partial u_\alpha}. \tag{1.3}$$

For a section $U = U^\alpha e_\alpha$ of E, we define

$$U^V = U^\alpha\frac{\partial}{\partial u_\alpha}. \tag{1.4}$$

Then,

$$
\begin{aligned}
\pi_* X^H &= X^i \frac{\partial}{\partial x^i} = X \\
\pi_* V^V &= 0 \\
K X^H &= (-\mu^{\alpha}_{\beta\, i} u^\beta X^i + \mu^{\alpha}_{\beta\, i} u^\beta X^i) e_\alpha = 0 \\
K V^V &= V^\alpha e_\alpha = V
\end{aligned}
$$

i.e., $X^H \in HE$ and $V^V \in VE$.

Thus, we note that at the point (x, W)

$$
\begin{aligned}
g(X^H, Y^H)_W &= G(\pi_* X^H, \pi_* Y^H)_x = G(X, Y)_x \\
g(X^H, U^V)_W &= G(\pi_* X^H, \pi_* U^V)_x + g^\perp(K X^H, K U^V)_W = 0 \\
g(U^V, V^V)_W &= g^\perp(K U^V, K V^V)_W = g^\perp(U, V)_W
\end{aligned}
$$

Now, we let $\tilde{X} = (\tilde{X}^i, \tilde{X}^{n+\alpha})$ be a tangent vector to E. Then,

$$
\begin{aligned}
(\pi_* \tilde{X})^H &= (\tilde{X}^i \frac{\partial}{\partial x_i})^H = \tilde{X}^i \frac{\partial}{\partial q_i} - \mu^{\alpha}_{\beta\, i} v^\beta \tilde{X}^i \frac{\partial}{\partial v_\alpha} \\
(K \tilde{X})^V &= [(\tilde{X}^{n+\alpha} + \mu^{\alpha}_{\beta\, i} v^\beta \tilde{X}^i) e_\alpha]^V = (\tilde{X}^{n+\alpha} + \mu^{\alpha}_{\beta\, i} v^\beta \tilde{X}^i) \frac{\partial}{\partial v_\alpha}
\end{aligned}
$$

and, hence, we can write

$$
\tilde{X} = (\pi_* \tilde{X})^H + (K \tilde{X})^V.
$$

Thus, it is enough to consider various combinations of horizontally and vertically lifted vector fields. We now prove in general three lemmas that were stated in the normal bundle case by Borisenko and Yampol'skii [BoY].

Lemma 1.1 *Let X and Y be vector fields on M, and U and V sections of the bundle E. Then, the Lie brackets at the point (x, W) are as follows:*

$$
[U^V, V^V] = 0, \qquad [X^H, U^V] = (\nabla_X U)^V,
$$

$$
\pi_*[X^X, Y^H] = [X, Y], \quad K[X^H, Y^H] = -R_{XY} W.
$$

Outline of Proof: The proof can be done by direct calculations using definitions of horizontal and vertical lifts. For example, using (1.3), we have at $W = u^\alpha e_\alpha$

$$
\begin{aligned}
[X^H, Y^H] &= [X^i \frac{\partial}{\partial q_i} - \mu^\alpha_{\beta i} u^\beta X^i \frac{\partial}{\partial u_\alpha}, Y^j \frac{\partial}{\partial q_j} - \mu^\gamma_{\delta j} u^\delta Y^j \frac{\partial}{\partial u_\gamma}] \\
&= (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial q^j} + \mu^\alpha_{\beta i} u^\beta Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial u^\alpha} - \mu^\gamma_{\delta j} u^\delta X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial u^\gamma} \\
&\quad + Y^j \frac{\partial \mu^\alpha_{\beta i}}{\partial x^j} u^\beta X^i \frac{\partial}{\partial u^\alpha} - X^i \frac{\partial \mu^\gamma_{\delta j}}{\partial x^i} u^\delta Y^j \frac{\partial}{\partial u^\gamma} \\
&\quad + \mu^\alpha_{\beta i} \mu^\gamma_{\alpha j} X^i Y^j u^\beta \frac{\partial}{\partial u^\gamma} - \mu^\gamma_{\delta j} \mu^\alpha_{\gamma i} X^i Y^j u^\delta \frac{\partial}{\partial u^\alpha} \\
&= [X, Y]^H_W - (R_{XY} W)^V.
\end{aligned}
$$

The other two cases are easy. $\square$

By definition, $R_{XY}U$ is a section of the total space E^{n+k} such that at any point $x \in M$, $R_{XY}U$ is tangent to the fiber $\pi^{-1}(x)$. If V is another section of E, it is possible to compute the inner product $g^\perp(R_{XY}U, V)$. We define the adjoint $\hat{R}_{UV}X$ by the equality

$$G(\hat{R}_{UV}X, Y) = g^\perp(R_{XY}U, V). \tag{1.5}$$

We now compute the covariant derivatives with respect to the Riemannian connection $\tilde{\nabla}$ of the Sasaki metric g on E.

Lemma 1.2 *Let X and Y be tangent vector fields on M, and U and V sections of the bundle E. Then, at each point (x, W)*

$$
\begin{aligned}
&\tilde{\nabla}_{U^V} V^V = 0, && \tilde{\nabla}_{X^H} V^V = (\nabla_X V)^V + \tfrac{1}{2}(\hat{R}_{WV}X)^H, \\
&\tilde{\nabla}_{U^V} Y^H = \tfrac{1}{2}(\hat{R}_{WU}Y)^H, && \tilde{\nabla}_{X^H} Y^H = (D_X Y)^H - \tfrac{1}{2}(R_{XY}W)^V.
\end{aligned}
$$

Outline of Proof: We use Lemma 1.1 and a well known formula for the Riemannian connection

$$2g(\tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) = \tilde{X}g(\tilde{Y}, \tilde{Z}) + \tilde{Y}g(\tilde{Z}, \tilde{X}) - \tilde{Z}g(\tilde{X}, \tilde{Y})$$

$$+ g([\tilde{X},\tilde{Y}],\tilde{Z}) - g([\tilde{Y},\tilde{Z}],\tilde{X}) + g([\tilde{Z},\tilde{X}],\tilde{Y})$$

to compute various combinations of covariant derivatives. For example, we can compute as follows,

$$\begin{aligned} 2g(\tilde{\nabla}_{U^V}Y^H, X^H) &= U^V g(Y^H, X^H) + g([U^V, Y^H], X^H) \\ &\quad -g([Y^H, X^H], U^V) + g([X^H, U^V], Y^H) \\ &= U^V g(X^H, Y^H) + g((R_{YX}W)^V, U^V) \\ &= U^V g(X^H, Y^H) + g^\perp(R_{YX}W, U) \\ &= U^V g(X^H, Y^H) + G(\hat{R}_{WU}Y, X). \end{aligned}$$

But, since $g(X^H, Y^H)$ is a constant along each fiber, $U^V g(X^H, Y^H)$ vanishes. We also have

$$\begin{aligned} 2g(\tilde{\nabla}_{U^V}Y^H, V^V) &= Y^H g(U^V, V^V) + g([U^V, Y^H], V^V) - g([Y^H, V^V], U^V) \\ &= Y g^\perp(U, V) - g((\nabla_Y U)^V, V^V) - g((\nabla_Y V)^V, U^V) \\ &= Y g^\perp(U, V) - g^\perp(\nabla_Y U, V) - g^\perp(\nabla_Y V, U) \\ &= 0. \end{aligned}$$

Therefore, we get $\tilde{\nabla}_{U^V}Y^H = \frac{1}{2}(\hat{R}_{WU}Y)^H$.

Other cases follow by similar calculations. $\square$

We define the covariant derivatives of the tensors R and $\hat{R}$ as usual:

$$\begin{aligned} (\nabla_Z R)_{XY}U &= \nabla_Z R_{XY}U - R_{D_Z X\, Y}U - R_{X\, D_Z Y}U - R_{XY}\nabla_Z U \\ (D_X \hat{R})_{UV}Z &= D_X \hat{R}_{UV}Z - \hat{R}_{\nabla_X U\, V}Z - \hat{R}_{U\, \nabla_X V}Z - \hat{R}_{UV}D_X Z. \end{aligned}$$

Lemma 1.3 *The curvature tensor of the Sasaki metric of the bundle E at the point (x, W) is given by*

$$\tilde{R}_{X^H Y^H}Z^H = [\underline{R}_{XY}Z + \frac{1}{4}\hat{R}_{W\, R_{ZY}W}X + \frac{1}{4}\hat{R}_{W\, R_{XZ}W}Y + \frac{1}{2}\hat{R}_{W\, R_{XY}W}Z]^H$$

$$
\begin{aligned}
& + \frac{1}{2}[(\nabla_Z R)_{XY} W]^V, \\
\tilde{R}_{X^H Y^H} U^V &= \frac{1}{2}[(D_X \hat{R})_{WU} Y - (D_Y \hat{R})_{WU} X]^H \\
& + [R_{XY} U + \frac{1}{4} R_{\hat{R}_{WU} Y\, X} W - \frac{1}{4} R_{\hat{R}_{WU} X\, Y} W]^V, \\
\tilde{R}_{X^H U^V} Z^H &= \frac{1}{2}[(D_X \hat{R})_{WU} Z]^H + [\frac{1}{2} R_{XZ} U + \frac{1}{4} R_{\hat{R}_{WU} Z\, X} W]^V, \\
\tilde{R}_{X^H U^V} V^V &= -[\frac{1}{2} \hat{R}_{UV} X + \frac{1}{4} \hat{R}_{WU} \hat{R}_{WV} X]^H, \\
\tilde{R}_{U^V V^V} Z^H &= [\hat{R}_{UV} Z + \frac{1}{4} \hat{R}_{WU} \hat{R}_{WV} Z - \frac{1}{4} \hat{R}_{WV} \hat{R}_{WU} Z]^H, \\
\tilde{R}_{U^V V^V} S^V &= 0.
\end{aligned}
$$

Outline of Proof: We will outline the proof of the first three identities. The above two lemmas will be used freely. At the point $W = u^\alpha e_\alpha$, we have

$$
\begin{aligned}
\tilde{R}_{X^H U^V} Z^H &= \tilde{\nabla}_{X^H} \tilde{\nabla}_{U^V} Z^H - \tilde{\nabla}_{U^V} \tilde{\nabla}_{X^H} Z^H - \tilde{\nabla}_{[X^H, U^V]} Z^H \\
&= \frac{1}{2} \tilde{\nabla}_{X^H} (\hat{R}_{WU} Z)^H - \tilde{\nabla}_{U^V} ((D_X Z)^H - \frac{1}{2} (R_{XZ} W)^V) \\
& \quad - \tilde{\nabla}_{(\nabla_X U)^V} Z^H.
\end{aligned} \tag{1.6}
$$

But,

$$
\begin{aligned}
\tilde{\nabla}_{X^H} (\hat{R}_{WU} Z)^H &= \tilde{\nabla}_{X^H} u^\alpha (\hat{R}_{e_\alpha U} Z)^H \\
&= (X^H u^\alpha)(\hat{R}_{e_\alpha U} Z)^H + u^\alpha \tilde{\nabla}_{X^H} (\hat{R}_{e_\alpha U} Z)^H \\
&= -\mu^\alpha_{\beta i} u^\beta X^i (\hat{R}_{e_\alpha U} Z)^H + u^\alpha \{(D_X \hat{R}_{e_\alpha U} Z)^H - \frac{1}{2} (R_{X\, \hat{R}_{e_\alpha U} Z} W)^V\} \\
&= -(\hat{R}_{\nabla_X W\, U} Z)^H + u^\alpha (D_X \hat{R}_{e_\alpha U} Z)^H - \frac{1}{2} (R_{X\, \hat{R}_{WU} Z} W)^V \\
&= -(\hat{R}_{\nabla_X W\, U} Z)^H + (D_X \hat{R}_{WU} Z)^H - \frac{1}{2} (R_{X\, \hat{R}_{WU} Z} W)^V
\end{aligned} \tag{1.7}
$$

where we have the last equality since

$$
D_X \hat{R}_{WU} Z = D_X u^\alpha \hat{R}_{e_\alpha U} Z = (X u^\alpha) \hat{R}_{e_\alpha U} Z + u^\alpha D_X \hat{R}_{e_\alpha U} Z,
$$

and since the u^α are the fiber coordinates.

Similarly, we can compute

$$\tilde{\nabla}_{U^V}(R_{XZ}W)^V = (R_{XZ}U)^V. \tag{1.8}$$

Continuing our computation of (1.6) with (1.7) and (1.8), we have

$$\begin{aligned}
\tilde{R}_{X^H U^V} Z^H &= -\frac{1}{2}(\hat{R}_{\nabla_X W U}Z)^H + \frac{1}{2}(D_X \hat{R}_{WU}Z)^H - \frac{1}{4}(R_{X\,\hat{R}_{WU}Z}W)^V \\
&\quad - \frac{1}{2}(\hat{R}_{WU}D_X Z)^H + \frac{1}{2}(R_{XZ}U)^V - \frac{1}{2}(\hat{R}_{W\,\nabla_X U}Z)^H \\
&= \frac{1}{2}[(D_X\hat{R})_{WU}Z]^H + [\frac{1}{2}R_{XZ}U + \frac{1}{4}R_{\hat{R}_{WU}Z\,X}W]^V
\end{aligned} \tag{1.9}$$

as desired.

The second identity follows easily from the Bianchi identity and (1.9).

To show the first identity, we compute

$$\begin{aligned}
\tilde{R}_{X^H Y^H} Z^H &= \tilde{\nabla}_{X^H}\tilde{\nabla}_{Y^H}Z^H - \tilde{\nabla}_{Y^H}\tilde{\nabla}_{X^H}Z^H - \tilde{\nabla}_{[X^H,Y^H]}Z^H \\
&= \tilde{\nabla}_{X^H}((D_Y Z)^H - \frac{1}{2}(R_{YZ}W)^V) - \tilde{\nabla}_{Y^H}((D_X Z)^H - \frac{1}{2}(R_{XZ}W)^V) \\
&\quad - \tilde{\nabla}_{[X,Y]^H}Z^H + \tilde{\nabla}_{(R_{XY}W)^V}Z^H
\end{aligned} \tag{1.10}$$

Here, we do a similar calculation to the one in (1.7) and obtain the following:

$$\tilde{\nabla}_{X^H}(R_{YZ}W)^V = (\nabla_X R_{YZ}W)^V + \frac{1}{2}(\hat{R}_{W\,R_{YZ}W}X)^H - (R_{YZ}\nabla_X W)^V \tag{1.11}$$

and

$$\tilde{\nabla}_{Y^H}(R_{XZ}W)^V = (\nabla_Y R_{XZ}W)^V + \frac{1}{2}(\hat{R}_{W\,R_{XZ}W}Y)^H - (R_{XZ}\nabla_Y W)^V. \tag{1.12}$$

Thus, using (1.11) and (1.12), the equation (1.10) can be written as follows:

$$\begin{aligned}
\tilde{R}_{X^H Y^H} Z^H &= (D_X D_Y Z)^H - \frac{1}{2}(R_{X\,D_Y Z}W)^V - \frac{1}{2}(\nabla_X R_{YZ}W)^V - \frac{1}{4}(\hat{R}_{W\,R_{YZ}W}X)^H \\
&\quad + \frac{1}{2}(R_{YZ}\nabla_X W)^V - (D_Y D_X Z)^H + \frac{1}{2}(R_{Y\,D_X Z}W)^V \\
&\quad + \frac{1}{2}(\nabla_Y R_{XZ}W)^V + \frac{1}{4}(\hat{R}_{W\,R_{XZ}W}Y)^H - \frac{1}{2}(R_{XZ}\nabla_Y W)^V
\end{aligned}$$

$$
\begin{aligned}
&- (D_{[X,Y]}Z)^H + \frac{1}{2}(R_{[X,Y]Z}W)^V + \frac{1}{2}(\hat{R}_{W\,R_{XY}W}Z)^H \\
= \;& [\underline{R}_{XY}Z + \frac{1}{4}\hat{R}_{W\,R_{ZY}W}X + \frac{1}{4}\hat{R}_{W\,R_{XZ}W}Y + \frac{1}{2}\hat{R}_{W\,R_{XY}W}Z]^H \\
&+ \frac{1}{2}[-R_{X\,D_YZ}W - \nabla_X R_{YZ}W + R_{YZ}\nabla_X W + R_{Y\,D_XZ}W \\
&+ \nabla_Y R_{XZ}W - R_{XZ}\nabla_Y W + R_{[X,Y]Z}W]^V \qquad (1.13)
\end{aligned}
$$

Now, using the Jacobi identity, it is straightforward to see that the vertical part of (1.13) is equal to $\frac{1}{2}[(\nabla_Z R)_{XY}W]^V$. Thus, we obtain the first identity.

The remaining identities can be proved by the similar arguments and simple computations. $\square$

1.2 Locally symmetric vector bundles

The locally symmetric tangent bundle was first studied by Kowalski, who showed [Ko] that the classical Sasaki metric on the tangent bundle is locally symmetric if and only if the metric of the base space G is flat.

We now study the local symmetry of a general vector bundle.

Proposition 1.4 *Let $\pi : E^{n+k} \longrightarrow M^n$ be a vector bundle over a manifold M with fiber metric $g^\perp$ and a metric connection ∇. Suppose the connection ∇ is flat. Then, for the Sasaki metric g on E, we have*

$$(\tilde{\nabla}_{A^H}\tilde{R})(X^H, Y^H, Z^H) = [(D_A\underline{R})(X, Y, Z)]^H$$

for any vectors $A, X, Y,$ and Z tangent to M^n.

Proof: Since the connection ∇ is flat, we note, from Lemma 1.3, that $\tilde{R}_{X^HY^H}U^V$, $\tilde{R}_{X^HU^V}Z^H$, $\tilde{R}_{X^HU^V}V^V$, and $\tilde{R}_{U^VV^V}Z^H$ all vanish. Using this together with Lemma 1.2

and Lemma 1.3, we have

$$\begin{aligned}
(\tilde{\nabla}_{A^H}\tilde{R})(X^H,Y^H,Z^H) &= \tilde{\nabla}_{A^H}\tilde{R}_{X^H Y^H}Z^H - \tilde{R}_{\tilde{\nabla}_{A^H}X^H Y^H}Z^H \\
&\quad - \tilde{R}_{X^H\,\tilde{\nabla}_{A^H}Y^H}Z^H - \tilde{R}_{X^H Y^H}\tilde{\nabla}_{A^H}Z^H \\
&= \tilde{\nabla}_{A^H}[\underline{R}_{XY}Z]^H - \tilde{R}_{(D_AX)^H Y^H}Z^H + \frac{1}{2}\tilde{R}_{(R_{AX}W)^V Y^H}Z^H \\
&\quad - \tilde{R}_{X^H\,(D_AY)^H}Z^H + \frac{1}{2}\tilde{R}_{X^H\,(R_{AY}W)^V}Z^H \\
&\quad - \tilde{R}_{X^H Y^H}(D_AZ)^H + \frac{1}{2}\tilde{R}_{X^H Y^H}(R_{AZ}W)^V \\
&= [D_A\underline{R}_{XY}Z - \underline{R}_{D_AX\,Y}Z - \underline{R}_{X\,D_AY}Z - \underline{R}_{XY}D_AZ]^H \\
&= [(D_A\underline{R})(X,Y,Z)]^H
\end{aligned}$$

as desired. □

We will use the following lemma of Cartan [Ca] pp.257-258.

Lemma 1.5 *Let (M,g) be a Riemannian manifold, ∇ the Riemannian connection of g and R its curvature tensor. Then, (M,g) is locally symmetric if and only if*

$$(\nabla_X R)(Y,X,Y,X) = 0 \tag{1.14}$$

for any orthonormal pairs $\{X,Y\}$.

We now prove the following theorem.

Theorem 1.6 *Let $\pi : E^{n+k} \to M^n$ be a vector bundle over a manifold M with fiber metric $g^\perp$ and a metric connection ∇. Then, the Sasaki metric on E is locally symmetric if and only if the connection ∇ is flat and M is locally symmetric.*

Proof: Suppose that E is locally symmetric, i.e., $\tilde{\nabla}\tilde{R} = 0$. First, we show that the connection ∇ is flat. Using Lemma 1.2 and Lemma 1.3, we have at (x,W) on E

$$(\tilde{\nabla}_{Y^H}\tilde{R})(X^H,U^V,V^V) = \tilde{\nabla}_{Y^H}\tilde{R}_{X^H U^V}V^V - \tilde{R}_{\tilde{\nabla}_{Y^H}X^H\,U^V}V^V$$

$$
\begin{aligned}
& - \tilde{R}_{X^H \tilde{\nabla}_{Y^H} U^V} V^V - \tilde{R}_{X^H U^V} \tilde{\nabla}_{Y^H} V^V \\
= & -\frac{1}{2}\tilde{\nabla}_{Y^H}(\hat{R}_{UV}X)^H - \frac{1}{4}\tilde{\nabla}_{Y^H}(\hat{R}_{WU}\hat{R}_{WV}X)^H \\
& - \tilde{R}_{(D_Y X)^H U^V} V^V + \frac{1}{2}\tilde{R}_{(R_{YX}W)^V U^V} V^V \\
& - \tilde{R}_{X^H (\nabla_Y U)^V} V^V - \frac{1}{2}\tilde{R}_{X^H (R_{WU}Y)^H} V^V \\
& - \tilde{R}_{X^H U^V} (\nabla_Y V)^V - \frac{1}{2}\tilde{R}_{X^H U^V} (R_{WV}Y)^H \\
= & -\frac{1}{2}[D_Y \hat{R}_{UW} X]^H + \frac{1}{4}[R_{Y\,\hat{R}_{UV}X} W]^V \\
& - \frac{1}{4}[D_Y \hat{R}_{WU}\hat{R}_{WV}X]^H + \frac{1}{8}[R_{Y\,\hat{R}_{WU}\hat{R}_{WV}X} W]^V \\
& + \frac{1}{2}[\hat{R}_{UV} D_Y X]^H + \frac{1}{4}[\hat{R}_{WU}\hat{R}_{WV} D_Y X]^H \\
& + \frac{1}{2}[\hat{R}_{\nabla_Y U\, V} X]^H + \frac{1}{4}[\hat{R}_{W\,\nabla_Y U}\hat{R}_{WV} X]^H \\
& - \frac{1}{4}[(D_X \hat{R})_{WV}\hat{R}_{WU}Y - (D_{\hat{R}_{WU}Y}\hat{R})_{WV}X]^H \\
& - \frac{1}{2}[R_{X\,\hat{R}_{WU}Y} V + \frac{1}{4}R_{\hat{R}_{WV}\hat{R}_{WU}Y\,X} W - \frac{1}{4}R_{\hat{R}_{WV}X\,\hat{R}_{WU}Y} W]^V \\
& + [\frac{1}{2}\hat{R}_{U\nabla_Y V} X + \frac{1}{4}\hat{R}_{WU}\hat{R}_{W\,\nabla_Y V} X]^H \\
& - \frac{1}{4}[(D_X \hat{R})_{WU} Y]^H \\
& - \frac{1}{2}[\frac{1}{2}R_{X\,\hat{R}_{WV}Y} U + \frac{1}{4}R_{\hat{R}_{WU}\hat{R}_{WV}Y\,X} W]^V \\
= & \ \frac{1}{4}[R_{Y\,\hat{R}_{UV}X} W]^V + \frac{1}{8}[R_{Y\,\hat{R}_{WU}\hat{R}_{WV}X} W]^V \\
& - \frac{1}{2}[R_{X\,\hat{R}_{WU}Y} V]^V - \frac{1}{8}[R_{\hat{R}_{WV}\hat{R}_{WU}Y\,X} W]^V \\
& + \frac{1}{8}[R_{\hat{R}_{WV}X\,\hat{R}_{WU}Y} W]^V - \frac{1}{4}[R_{X\,\hat{R}_{WV}Y} U]^V \\
& - \frac{1}{8}[R_{\hat{R}_{WU}\hat{R}_{WV}Y\,X} W]^V + [\ldots\ldots\ldots]^H \qquad (1.15)
\end{aligned}
$$

Applying K to this, we get

$$
\begin{aligned}
8K[(\tilde{\nabla}_{Y^H}\tilde{R})(X^H, U^V, V^V)] = & \ 2R_{Y\,\hat{R}_{UV}X} W + R_{Y\,\hat{R}_{WU}\hat{R}_{WV}X} W \\
& - 4R_{X\,\hat{R}_{WU}Y} V - R_{\hat{R}_{WV}\hat{R}_{WU}Y\,X} W \\
& + R_{\hat{R}_{WV}X\,\hat{R}_{WU}Y} W - 2R_{X\,\hat{R}_{WV}Y} U \\
& - R_{\hat{R}_{WU}\hat{R}_{WV}Y\,X} W \qquad (1.16)
\end{aligned}
$$

Since ∇ is compatible to $g^{\perp}$, we have that

$$g^{\perp}(R_{XY}U,V) = -g^{\perp}(R_{XY}V,U) \tag{1.17}$$

for any X, Y, U, and V.

Since $\tilde{\nabla}\tilde{R} = 0$, taking the inner product of (1.16) with W, we have, using (1.17) and the definition (1.5) of $\hat{R}$, that

$$\begin{aligned} 0 &= 4g^{\perp}(R_{X\,\hat{R}_{WU}Y}V, W) + 2g^{\perp}(R_{X\,\hat{R}_{WV}Y}U, W) \\ &= 4G(\hat{R}_{VW}X, \hat{R}_{WU}Y) + 2G(\hat{R}_{UW}X, \hat{R}_{WV}Y) \end{aligned} \tag{1.18}$$

We choose $X = Y$ and $U = V$ in (1.18) and then, we have

$$\begin{aligned} 0 &= 4G(\hat{R}_{UW}X, \hat{R}_{WU}X) + 2G(\hat{R}_{UW}X, \hat{R}_{WU}X) \\ &= -6G(\hat{R}_{WU}X, \hat{R}_{WU}X) \end{aligned}$$

that is, $6|\hat{R}_{WU}X|^2 = 0$ for any X, W, and U. Then, by the definition (1.5) of $\hat{R}$, we have

$$R_{XY}W = 0$$

for any X, Y, and W, i.e., the connection ∇ is flat.

Finally, by Proposition 1.4, $D\underline{R} = 0$, i.e., M is locally symmetric.

For the converse, in view of the lemma of Cartan, it is enough to check if the equation (1.14) holds for an orthonormal pairs which we may decompose into horizontal and vertical parts.

Suppose the connection ∇ is flat. Then, from Lemma 1.2, we see that

$$\tilde{\nabla}_{X^H}U^V = (\nabla_X U)^V \text{ and } \tilde{\nabla}_{U^V}X^H = 0.$$

So, using Lemma 1.3, we see that all of the types $\tilde{R}_{X^HY^H}U^V$, $\tilde{R}_{X^HU^V}Z^H$, $\tilde{R}_{X^HU^V}V^V$, $\tilde{R}_{U^VV^V}Z^H$, and $\tilde{R}_{U^VV^V}S^V$ vanish.

Therefore, using Lemma 1.3 and Proposition 1.4 again, it is straightforward to see the equation (1.14). $\square$

1.3 Conformally flat vector bundles

We now study conformally flat vector bundles. It is well known that a Riemannian manifold (M^n, g) is conformally flat if and only if

$$\begin{aligned} R_{XY}Z &= \frac{1}{n-2}(g(Y,Z)QX - g(Z,X)QY + g(QY,Z)X - g(Z,QX)Y) \\ &\quad - \frac{R}{(n-1)(n-2)}(g(Y,Z)X - g(Z,X)Y) \qquad \text{for } n \geq 4 \end{aligned} \tag{1.19}$$

and

$$(\nabla_X P)Y = (\nabla_Y P)X \qquad \text{for } n = 3 \tag{1.20}$$

where Q is the Ricci operator, $R = TrQ$ is the scalar curvature of M^n and P is the tensor field defined by

$$P = -Q + \frac{R}{4}Id.$$

We note that the equation (1.19) (with $n = 3$) is valid on any 3-dimensional Riemannian manifold.

We now present the following theorems.

Theorem 1.7 *Let $\pi : E^{n+k} \to M^n$, $n \geq 3$, be a vector bundle over a manifold M^n with fiber metric $g^\perp$ and a metric connection ∇. Then, E^{n+k} with the Sasaki metric g is conformally flat if and only if either, M^n is flat with flat connection ∇, or M^n has (nonzero) constant curvature with flat connection ∇ and $k = 1$.*

Proof: Suppose that E^{n+k} is conformally flat. From Lemma 1.3, we have at (x, W)

$$\tilde{R}_{X^H Y^H} W^V = \frac{1}{2}[(D_X \hat{R})_{WW} Y - (D_Y \hat{R})_{WW} X]^H$$

$$
\begin{aligned}
& + [R_{XY}W + \frac{1}{4}R_{\hat{R}_{WWY}X}W - \frac{1}{4}R_{\hat{R}_{WWX}Y}W]^V \\
= \; & \frac{1}{2}[(D_X\hat{R})_{WW}Y - (D_Y\hat{R})_{WW}X]^H + [R_{XY}W]^V \qquad (1.21)
\end{aligned}
$$

On the other hand, since E^{n+k} is conformally flat, we also have

$$\tilde{R}_{X^HY^H}W^V = \frac{1}{n+k-2}[g(\tilde{Q}Y^H, W^V)X^H - g(\tilde{Q}X^H, W^V)Y^H] \qquad (1.22)$$

where $\tilde{Q}$ is the Ricci operator of the total space E^{n+k}.

Comparing vertical components of (1.21) and (1.22), we conclude that $R_{XY}W = 0$, i.e., the connection ∇ is flat. We now take an orthonormal basis $\{X_i^H, V_\alpha^V\}, i = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, k$, so that $\{X_i\}$ form an orthonormal basis of M. Then, since the connection ∇ is flat, we compute, using Lemma 1.3,

$$
\begin{aligned}
\tilde{Q}X^H &= \sum_i \tilde{R}_{X^HX_i^H}X_i^H + \sum_\alpha \tilde{R}_{X^HV_\alpha^V}V_\alpha^V \\
&= [\sum_i \underline{R}_{XX_i}X_i]^H \\
&= (\underline{Q}X)^H, \qquad (1.23)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{Q}W^V &= \sum_i \tilde{R}_{W^VX_i^H}X_i^H + \sum_\alpha \tilde{R}_{W^VV_\alpha^V}V_\alpha^V \\
&= 0, \qquad (1.24)
\end{aligned}
$$

and

$$\tilde{R} = \sum_{i,j} g(\tilde{R}_{X_i^HX_j^H}X_j^H, X_i^H) = \underline{R} \qquad (1.25)$$

where $\tilde{R}$ and $\underline{R}$ are the scalar curvatures of E and M, respectively.

We write $a = \frac{1}{n+k-2}$ and $b = \frac{1}{(n+k-1)(n+k-2)}$. Then, since E is conformally flat, we have, taking the horizontal projection in view of (1.19),

$$
\begin{aligned}
\underline{R}_{XY}Z &= a\{G(Y,Z)\underline{Q}X - G(Z,X)\underline{Q}Y + G(\underline{Q}Y,Z)X - G(Z,\underline{Q}X)Y\} \\
&\quad - b\underline{R}\{G(Y,Z)X - G(Z,X)Y\} \qquad (1.26)
\end{aligned}
$$

where G is a Riemannian metric of the base manifold M. So, taking trace of this, we have

$$\begin{aligned}
\underline{Q}X &= a\{n\underline{Q}X - \sum_i G(X, X_i)\underline{Q}X_i + \sum_i G(\underline{Q}X_i, X_i)X - \sum_i G(\underline{Q}X, X_i)X_i\} \\
&\quad - b\underline{R}\{nX - \sum_i G(X, X_i)X_i\} \\
&= a\{(n-2)\underline{Q}X + \underline{R}X\} - b(n-1)\underline{R}X \qquad (1.27)
\end{aligned}$$

or,

$$\underline{Q}X = c\underline{R}X \qquad (1.28)$$

where $c = \frac{b(n-1)-a}{a(n-2)-1}$. Hence, M is an Einstein manifold. Moreover, $\underline{R}$ is a constant since $n \geq 3$.

If $\underline{R} = 0$, then $\underline{Q} = 0$. Thus, from (1.26), we conclude that M is flat.

If $\underline{R} \neq 0$, then, using (1.26) and (1.28), we have

$$\underline{R}_{XY}Z = A\{G(Y, Z)X - G(Z, X)Y\}$$

where $A = 2(ac - b)\underline{R}$ is a nonzero constant. Hence, M has a constant curvature A.

Now, we take trace in (1.27) and get

$$\begin{aligned}
\underline{R} &= a\{(n-2)\underline{R} + n\underline{R}\} - bn(n-1)\underline{R} \\
&= (n-1)(2a - bn)\underline{R}
\end{aligned}$$

or, equivalently,

$$1 = (n-1)(2a - bn) = (n-1)(2 - \frac{n}{n+k-1})\frac{1}{n+k-2}. \qquad (1.29)$$

By a simple computation, we see, from this, that $k(k-1) = 0$. Thus, $k = 1$.

We now prove the converse. In either case, since the connection is flat, we note that the equations (1.23) - (1.25) remain. We also recall that on a manifold (M, G) of constant curvature C, we have

$$\underline{R}_{XY}Z = C(G(Y, Z)X - G(Z, X)Y) \qquad (1.30)$$

Case 1: M^n is flat with flat connection ∇.

Since $C = 0$, from (1.30), $\tilde{R}_{\tilde{X}\tilde{Y}}\tilde{Z} = 0$ for any vectors $\tilde{X}, \tilde{Y}$, and $\tilde{Z}$ tangent to E, E^{n+k} is flat and hence, conformally flat.

Case 2: M^n has (nonzero) constant curvature, say C, with flat connection ∇ and $k = 1$.

We shall see that the equation (1.19) holds for various combinations of horizontally and vertically lifted vector fields. This will be mainly simple computations using Lemma 1.3, (1.19) and (1.30). First of all, we look at the case $\{X^H, Y^H, Z^H\}$. Since the manifold M^n has constant curvature, we have, from (1.30),

$$\underline{Q}X = C(nX - \sum_{i=1}^{n} G(X_i, X)X_i) = C(n-1)X \tag{1.31}$$

where $\{X_i\}$ is an orthonormal basis of M^n. Thus,

$$\underline{R} = Cn(n-1). \tag{1.32}$$

Hence, the RHS of (1.19) is, using (1.23), (1.25), (1.31), and (1.32), equal to

$$\begin{aligned} & \frac{1}{n-1} 2C(n-1)(G(Y,Z)X^H - G(Z,X)Y^H) \\ & - \frac{\tilde{R}}{n(n-1)}(G(Y,Z)X^H - G(Z,X)Y^H) \\ = \; & \{2C - \frac{Cn(n-1)}{n(n-1)}\}(G(Y,Z)X^H - G(Z,X)Y^H) \\ = \; & C(G(Y,Z)X^H - G(Z,X)Y^H), \end{aligned}$$

which is equal to $[\underline{R}_{XY}Z]^H$ by (1.30). On the other hand, since the connection ∇ is flat, we have, from Lemma 1.3, that $\tilde{R}_{X^HY^H}Z^H = [\underline{R}_{XY}Z]^H$. Hence, the equation (1.19) holds for this case. For the remaining cases, since the connection ∇ is flat, we see, from Lemma 1.3, that the LHS of (1.19) is zero. Moreover, using (1.24), (1.31), and (1.32), it is easy to see that the RHS of (1.19) vanishes. For example, we can see that the RHS of (1.19) vanishes for the cases with $\{X^H, U^V, Z^H\}$ and $\{X^H, U^V, V^V\}$ as follows: Using (1.23) and (1.24), for the case $\{X^H, U^V, Z^H\}$, we have

$$\begin{aligned}
\text{RHS of (1.19)} &= \frac{1}{n-1}\{-g(Z^H, \tilde{Q}X^H)U^V\} - \frac{\tilde{R}}{n(n-1)}\{-g(Z^H, X^H)U^V\} \\
&= \{-\frac{C(n-1)}{n-1} + \frac{Cn(n-1)}{n(n-1)}\}g(Z^H, X^H)U^V = 0
\end{aligned}$$

and, for $\{X^H, U^V, V^V\}$, we have

$$\begin{aligned}
\text{RHS of (1.19)} &= \frac{1}{n-1}g(U^V, V^V)\tilde{Q}X^H - \frac{\tilde{R}}{n(n-1)}g(U^V, V^V)X^H \\
&= \{\frac{C(n-1)}{n-1} - \frac{Cn(n-1)}{n(n-1)}\}g(U^V, V^V)X^H
\end{aligned}$$

This completes the proof. $\square$

Theorem 1.8 *Let $\pi : E^{2+k} \to M^2$ be a vector bundle over a manifold M^2 with fiber metric $g^\perp$ and a metric connection ∇. Then, E^{2+k} with the Sasaki metric g is conformally flat if and only if either, M^2 is flat with flat connection ∇ and $k \geq 2$, or M^2 has constant curvature with flat connection ∇ and $k = 1$.*

Proof: We suppose that E^{2+k} is conformally flat. Then, we observe that the equations (1.21) and (1.22) are still valid and so, the connection is also flat.

Case 1: $k \geq 2$.

Notice also that the equation (1.27) holds. However, $\underline{R}$ may not be a constant but we shall show that $\underline{R}$ is, in fact, identically zero on M^2. So assume that $\underline{R} \neq 0$ at some point $x \in M^2$. Then, we can choose a neighborhood U of x such that $\underline{R} \neq 0$ on U. Since the equation (1.27) holds, we infer, by the same computation as above, that $k = 1$. This is a contradiction.

Hence, $\underline{R} \equiv 0$, i.e., the Gaussian curvature $K \equiv 0$ on M^2 when $k \geq 2$.

Case 2: $k = 1$.

Recall that the connection is still flat in this case. Then, we compute, using (1.24),

$$\begin{aligned}
&(\tilde{\nabla}_{X^H}\tilde{P})W^V - (\tilde{\nabla}_{W^V}\tilde{P})X^H \\
=\;& \tilde{\nabla}_{X^H}\tilde{P}W^V - \tilde{P}\tilde{\nabla}_{X^H}W^V - \tilde{\nabla}_{W^V}\tilde{P}X^H + \tilde{P}\tilde{\nabla}_{W^V}X^H \\
=\;& -\tilde{\nabla}_{X^H}\tilde{Q}W^V + \frac{1}{4}\tilde{\nabla}_{X^H}\tilde{R}W^V - \tilde{P}(\nabla_X W)^V + \tilde{\nabla}_{W^V}\tilde{Q}X^H - \frac{1}{4}\tilde{\nabla}_{W^V}\tilde{R}X^H \\
=\;& \frac{1}{4}X^H(\tilde{R})W^V + \frac{1}{4}\tilde{R}(\nabla_X W)^V - \frac{1}{4}\tilde{R}(\nabla_X W)^V - \frac{1}{4}W^V(\tilde{R})X^H - \frac{1}{4}\tilde{R}\tilde{\nabla}_{W^V}X^H \\
=\;& \frac{1}{4}X^H(\tilde{R})W^V - \frac{1}{4}W^V(\tilde{R})X^H \qquad (1.33)
\end{aligned}$$

Thus, if E^{2+k} is conformally flat, we have, using (1.20),

$$0 = \frac{1}{4}X^H(\tilde{R})W^V - \frac{1}{4}W^V(\tilde{R})X^H.$$

But, since X^H and W^V are linearly independent, we have $X^H(\tilde{R}) = X(\underline{R}) = 0$, that is, $\underline{R}$ is a constant. Hence, the Gaussian curvature K is a constant.

To prove the converse, we first observe that the equations (1.19), (1.23), and (1.25) are still valid since the connection ∇ is flat. Thus, the equations (1.26) - (1.28) remain valid.

Case 1: M^2 is flat with flat connection ∇ and $k \geq 2$.

From our observation above, the same argument as in the Case 1 of the converse of Theorem 1.7 proves this case.

Case 2: M^2 has a constant curvature and $k = 1$.

From (1.30), we see that $\underline{R}$ is a constant and so, from (1.25), $\tilde{R}$ is a constant. Hence, this shows, in view of (1.33), that

$$(\tilde{\nabla}_{X^H}\tilde{P})W^V - (\tilde{\nabla}_{W^V}\tilde{P})X^H = 0.$$

Moreover, since M has a constant curvature, $QX = cX$ for a constant c. Using this and (1.23), it is a routine computation to see

$$(\tilde{\nabla}_{X^H}\tilde{P})Y^H - (\tilde{\nabla}_{Y^H}\tilde{P})X^H = 0.$$

Therefore, E^{2+k} is conformally flat. $\square$

Corollary 1.9 *The classical Sasaki metric g on the tangent bundle of a Riemannian manifold (M^n, G), $n \geq 2$, is conformally flat if and only if (M^n, G) is flat in which case (TM^n, g) is flat.*

1.4 Conformally flat unit vector bundles

Let $\pi : E^{n+k} \longrightarrow M^n$ be a vector bundle equipped with fiber metric $g^{\perp}$ and a metric connection ∇ where (M^n, G) is a Riemannian manifold. Let D be the Riemannian connection and $\underline{R}$ the curvature tensor of M. We consider a hypersurface E_1 of E defined by

$$E_1 = \{U \in E : |U| = 1\},$$

called the unit vector bundle. The metric on E_1 induced from the Sasaki metric on E is denoted by g', the Riemannian connection of g' by ∇', and it's Riemannian curvature tensor by $R'_{\tilde{X}\tilde{Y}}\tilde{Z}$.

Notice that the vector field $W = u^{\alpha}(e_{\alpha})^V$ is a unit normal and the position vector of a point W in E_1. Then, we consider the Weingarten map A, defined by $A\tilde{X} = -\tilde{\nabla}_{\tilde{X}}W$, of the immersion $\iota : E_1 \rightarrow E$.

For any vertical vector field V tangent to E_1, we have using Lemma 1.2

$$\tilde{\nabla}_{\iota_* V}W = (\iota_* V u^{\alpha})(e_{\alpha})^V + u^{\alpha}\tilde{\nabla}_{\iota_* V}(e_{\alpha})^V = \iota_* V, \tag{1.34}$$

and for $X^H = (X^i, X^{n+\alpha})$ tangent to E_1,

$$\begin{aligned}
\tilde{\nabla}_{X^H} W &= \tilde{\nabla}_{X^H} u^\alpha (e_\alpha)^V \\
&= (X^H u^\alpha)\frac{\partial}{\partial u^\alpha} + u^\alpha \tilde{\nabla}_{X^H}(e_\alpha)^V \\
&= -\mu^\alpha_{\beta\,i} u^\beta X^i \frac{\partial}{\partial u^\alpha} + u^\alpha((\nabla_X e_\alpha)^V + \frac{1}{2}(R_{W\,e_\alpha} X)^H) \\
&= -\mu^\alpha_{\beta\,i} u^\beta X^i \frac{\partial}{\partial u^\alpha} + u^\alpha X^i (0 + \mu^\beta_{\alpha\,i})\frac{\partial}{\partial u^\beta} + \frac{1}{2}(R_{WW} X)^H \\
&= 0
\end{aligned} \tag{1.35}$$

Hence, $A = -Id$ on vertical vectors and $A = 0$ on horizontal vectors. From this and the well-known identity for the second fundamental form σ

$$g(\sigma(\tilde{X}, \tilde{Y}), \tilde{V}) = g(A_{\tilde{V}} \tilde{X}, \tilde{Y}),$$

we have that

$$\sigma(\tilde{X}, \tilde{Y}) = 0 \tag{1.36}$$

if at least one of $\tilde{X}$ and $\tilde{Y}$ is horizontal.

In this section, we consider the vector bundles $\pi : E^{n+k} \to M^n$ with $k = 2$. Since each fiber has dimension 2, we can choose orthonormal sections $\{U, V\}$. Then, we can write

$$\nabla_X U = k(X)V \text{ and } \nabla_X V = -k(X)U,$$

where k is a 1-form. Thus,

$$\begin{aligned}
R_{XY} U &= \nabla_X k(Y)V - \nabla_Y k(X)V - k([X,Y])V \\
&= 2dk(X,Y)V.
\end{aligned}$$

We define a linear operator L by

$$G(LX, Y) = 2dk(X, Y).$$

Then, we have

$$G(LX,Y) = 2dk(X,Y) = g^{\perp}(R_{XY}U,V) = G(\hat{R}_{UV}X,Y) \tag{1.37}$$

and

$$G(L^2X,Y) = G(\hat{R}_{UV}\hat{R}_{UV}X,Y) = -G(\hat{R}_{UV}X,\hat{R}_{UV}Y) = -G(LX,LY). \tag{1.38}$$

Thus, from (1.37), we can write $L = \hat{R}_{UV}$.

We prove the following theorem.

Theorem 1.10 *Let $\pi : E^{n+2} \to M^n$, $n \geq 3$, be a vector bundle over an Einstein manifold M with fiber metric $g^{\perp}$ and a metric connection ∇. Suppose the unit vector bundle E_1 is conformally flat and is of constant scalar curvature. Then, either the connection ∇ is flat, or (M,G) admits an almost Hermitian structure.*

Proof: We take an orthonormal basis $\{X_i^H, V\}$, $i = 1,\ldots,n$, tangent to E_1 so that $\{X_i\}$ form an orthonormal basis of M. Then, using the Gauss equation for E_1 in E and (1.36), we have

$$\begin{aligned}
g'(Q'X^H,Y^H) &= \sum_{i=1}^{n} g'(R'_{X^H X_i^H}X_i^H,Y^H) + g'(R'_{X^H V}V,Y^H) \\
&= \sum_{i=1}^{n}(g(\tilde{R}_{X^H X_i^H}X_i^H,Y^H) + g(\sigma(X^H,Y^H),\sigma(X_i^H,X_i^H)) \\
&\quad - g(\sigma(X_i^H,Y^H),\sigma(X_i^H,X^H)) + g(\tilde{R}_{X^H V}V,Y^H) \\
&\quad + g(\sigma(X^H,Y^H),\sigma(V,V)) - g(\sigma(V,Y^H),\sigma(V,X^H)) \\
&= \sum_{i=1}^{n} g(\tilde{R}_{X^H X_i^H}X_i^H,Y^H) + g(\tilde{R}_{X^H V}V,Y^H)
\end{aligned}$$

Continuing the computation using Lemma 1.3, we have at U

$$g'(Q'X^H,Y^H) = \sum_{i=1}^{n} g(\tilde{R}_{X^H X_i^H}X_i^H,Y^H) + g(\tilde{R}_{X^H V}V,Y^H)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} g([R_{X\,X_i}X_i + \frac{3}{4}\hat{R}_{U\,R_{X\,X_i}U}X_i]^H, Y^H) \\
&\quad + g(-\frac{1}{4}[\hat{R}_{UV}\hat{R}_{UV}X]^H, Y^H) \\
&= G(\underline{Q}X, Y) \\
&\quad - \frac{3}{4}\sum_{i=1}^{n} g^{\perp}(R_{X\,X_i}U, R_{Y\,X_i}U) + \frac{1}{4}G(\hat{R}_{UV}X, \hat{R}_{UV}Y) \quad (1.39)
\end{aligned}
$$

and

$$
\begin{aligned}
g'(Q'V, V) &= \sum_{i=1}^{n} g'(R'_{V\,X_i^H}X_i^H, V) \\
&= \sum_{i=1}^{n} g(\tilde{R}_{X_i^H V}V, X_i^H) \\
&= \sum_{i=1}^{n} g(-\frac{1}{4}[\hat{R}_{UV}\hat{R}_{UV}X_i]^H, X_i^H) \\
&= \frac{1}{4}\sum_{i=1}^{n} G(\hat{R}_{UV}X_i, \hat{R}_{UV}X_i). \quad (1.40)
\end{aligned}
$$

Using (1.39) and (1.40), we also have

$$
\begin{aligned}
R' &= \sum_{i=1}^{n} g'(Q'X_i^H, X_i^H) + g'(Q'V, V) \\
&= \sum_{i=1}^{n} G(\underline{Q}X_i, X_i) - \frac{3}{4}\sum_{i,j=1}^{n} g^{\perp}(R_{X_i\,X_j}U, R_{X_i\,X_j}U) + \frac{1}{2}\sum_{i=1}^{n} G(\hat{R}_{UV}X_i, \hat{R}_{UV}X_i) \\
&= \underline{R} + \frac{1}{2}\sum_{i=1}^{n} |\hat{R}_{UV}X_i|^2 - \frac{3}{4}\sum_{i,j=1}^{n} |R_{X_iX_j}U|^2 \quad (1.41)
\end{aligned}
$$

But, due to (1.38), we have

$$
\begin{aligned}
trL^2 &= \sum_{i=1}^{n} G(L^2X_i, X_i) \\
&= -\sum_{i=1}^{n} G(LX_i, LX_i) \\
&= -\sum_{i=1}^{n} G(\hat{R}_{UV}X_i, \hat{R}_{UV}X_i). \quad (1.42)
\end{aligned}
$$

Thus, from (1.41) and (1.42), we get

$$R' = \underline{R} - \frac{1}{2}trL^2 - \frac{3}{4}\sum_{i,j=1}^{n} |R_{X_iX_j}U|^2 \tag{1.43}$$

Now, since E_1 is conformally flat and since $\dim E_1 = n+1$ is at least 3, we have, in view of (1.19),

$$\begin{aligned} g'(R'_{\tilde{X}\tilde{Y}}\tilde{Z},\tilde{W}) &= \frac{1}{n-1}(g'(\tilde{Y},\tilde{Z})g'(Q'\tilde{X},\tilde{W}) - g'(\tilde{Z},\tilde{X})g'(Q'\tilde{Y},\tilde{W}) \\ &\quad + g'(Q'\tilde{Y},\tilde{Z})g'(\tilde{X},\tilde{W}) - g'(Q'\tilde{X},\tilde{W})g'(\tilde{Y},\tilde{W})) \\ &\quad - \frac{R'}{n(n-1)}(g'(\tilde{Y},\tilde{Z})g'(\tilde{X},\tilde{W}) - g'(\tilde{X},\tilde{Z})g'(\tilde{Y},\tilde{W})) \end{aligned} \tag{1.44}$$

From this together with (1.39), we have at U, for X and Y orthogonal,

$$\begin{aligned} g'(R'_{X^HV}V,Y^H) &= \frac{1}{n-1}g'(Q'X^H,Y^H) \\ &= \frac{1}{n-1}(G(\underline{Q}X,Y) \\ &\quad - \frac{3}{4}\sum_{i=1}^{n} g^{\perp}(R_{XX_i}U,R_{YX_i}U) + \frac{1}{4}G(LX,LY)) \end{aligned} \tag{1.45}$$

On the other hand, again using the Gauss equation, (1.36) and Lemma 1.3 successively, we have

$$\begin{aligned} g'(R'_{X^HV}V,Y^H) &= g(\tilde{R}_{X^HV}V,Y^H) \\ &= -\frac{1}{4}G(\hat{R}_{UV}\hat{R}_{UV}X,Y) \\ &= \frac{1}{4}G(\hat{R}_{UV}X,\hat{R}_{UV}Y) \\ &= \frac{1}{4}G(LX,LY) \end{aligned} \tag{1.46}$$

So, comparing (1.45) and (1.46), we get

$$\frac{1}{4}(1-\frac{1}{n-1})G(LX,LY) = \frac{1}{n-1}(G(\underline{Q}X,Y) - \frac{3}{4}\sum_{i=1}^{n} g^{\perp}(R_{XX_i}U,R_{YX_i}U)) \tag{1.47}$$

But, from (1.37), we see that

$$\begin{aligned}\sum_{i=1}^{n} g^{\perp}(R_{X\,X_i}U, R_{Y\,X_i}U) &= 4\sum_{i=1}^{n} g^{\perp}(dk(X,X_i)V, dk(X,X_i)V) \\ &= \sum_{i=1}^{n} G(LX,X_i)G(LY,X_i) \\ &= G(LX,LY) \qquad (1.48)\end{aligned}$$

where we have the last equality since $\{X_i\}$ is an orthonormal basis. Therefore, we have, using (1.47) and (1.38)

$$G(L^2X,Y) = -\frac{4}{n+1}G(\underline{Q}X,Y) \qquad (1.49)$$

and hence,

$$L^2 + \frac{4}{n+1}\underline{Q} = \alpha I \qquad (1.50)$$

where α is a function and I is the identity transformation.

Now, from (1.38) and (1.48), we have

$$\sum_{i,j=1}^{n} |R_{X_iX_j}U|^2 = -trL^2.$$

Therefore, (1.43) gives

$$R' = \underline{R} + \frac{1}{4}trL^2.$$

Now, the trace of (1.50) yields

$$n\alpha = trL^2 + \frac{4}{n+1}\underline{R} = 4R' - \frac{4n}{n+1}\underline{R}.$$

Thus, since R' is a constant and M is Einstein with $\dim M \geq 3$, α is a constant.

Again, since M is Einstein, i.e., $\underline{Q} = \frac{\underline{R}}{n}I$, we have, from (1.50),

$$L^2 = -\beta I \qquad (1.51)$$

where $\beta = \frac{4\underline{R}}{n(n+1)} - \alpha$ is a constant.

Now, taking $X = Y$ in (1.38), we easily see that $\beta \geq 0$.

Case 1: $\beta = 0$.

In this case, $L^2 = 0$. Taking $X = Y$ in (1.38) again, we have that $|LX|^2 = 0$ for any X, that is, $L = 0$. Hence, by the definition (1.5) of $\hat{R}$, we have

$$R_{XY}W = 0$$

for any X, Y, and W, i.e., the connection ∇ is flat.

Case 2: $\beta > 0$.

We define a tensor field J by $J = \frac{1}{\sqrt{\beta}}L$. From the definition of J, it is clear that J is an almost complex structure on M. Moreover, we have, using (1.38) and (1.51),

$$\begin{aligned} G(JX, JY) &= \frac{1}{\beta}G(LX, LY) \\ &= -\frac{1}{\beta}G(L^2X, Y) \\ &= G(X, Y) \end{aligned}$$

This completes the proof. $\square$

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Chapter 2

Review of Contact Manifolds

In this chapter, we review definitions and some well known results on contact manifolds which will be used later in this thesis. Section 1 is an introduction to contact manifolds and their integral submanifolds. Section 2 will discuss mainly Sasakian structures and some formulas related to them. As for the notations, we basically follow those of [Bl76].

2.1 Contact manifolds and integral submanifolds of the contact distribution

An odd dimensional differentiable manifold M^{2n+1} is said to have an *almost contact structure* (ϕ, ξ, η) if it admits a (1,1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 \text{ and } \phi^2 = -I + \eta \otimes \xi \tag{2.1}$$

where I denotes the identity transformation, or equivalently, if the structural group of its tangent bundle is reducible to $U(n) \times 1$. A manifold M with an almost contact structure (ϕ, ξ, η) is called an *almost contact manifold* and is denoted by (M, ϕ, ξ, η). On an almost contact manifold, we have $\phi \circ \xi = 0, \eta \circ \phi = 0$, and $rank\phi = 2n$. If g is a Riemannian metric on an almost contact manifold M^{2n+1} with the structure

(ϕ, ξ, η) such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X and Y, then M^{2n+1} is said to have an *almost contact metric structure* (ϕ, ξ, η, g), and g is called a *compatible metric.*

Proposition 2.1 *An almost contact manifold M admits a compatible metric g such that $\eta(X) = g(X, \xi)$ for any vector field X.*

Proof: Let h be a Riemannian metric on M and define h' by

$$h'(X, Y) = h(\phi^2 X, \phi^2 Y) + \eta(X)\eta(Y).$$

Now, we define g by

$$g(X, Y) = \frac{1}{2}(h'(X, Y) + h(\phi X, \phi Y) + \eta(X)\eta(Y)) \tag{2.3}$$

It is easy to check that g is a compatible Riemannian metric. Now, setting $Y = \xi$ in (2.2), we have that $\eta(X) = g(X, \xi)$. $\square$

Let M^{2n+1} be an almost contact manifold with an almost contact metric structure (ϕ, ξ, η, g). Let U be a coordinate neighborhood and choose a unit vector field X_1 on U orthogonal to ξ. Then, by (2.1) and (2.2), ϕX_1 is also a unit vector field on U, orthogonal to ξ and X_1. Next, we chose a unit vector field X_2 orthogonal to ξ, X_1 and ϕX_1, then ϕX_2 is a unit vector field orthogonal to $\xi, X_1, \phi X_1$ and X_2. Proceeding in this way, we obtain an orthonormal basis $\{\xi, X_1, \phi X_1, X_2, \phi X_2, \cdots, X_n, \phi X_n\}$, called *$\phi$-basis.*

If (M, ϕ, ξ, η, g) is an almost contact metric manifold, we can define a 2-form Φ on M by $\Phi(X, Y) = g(X, \phi Y)$. This 2-form is called the *fundamental 2-form* of the almost contact metric structure.

A manifold M^{2n+1} is said to be a *contact manifold* if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. η is called the *contact form.* $\eta = 0$ defines a $2n$-dimensional distribution or subbundle $\mathcal{D}$ of the tangent bundle with the fibers $\mathcal{D}_p = \{X \in T_pM \mid \eta(X) = 0\}$. $\mathcal{D}$ is sometimes called the *contact distribution.* Since $\eta \wedge (d\eta)^n \neq 0$, $\mathcal{D}$ is not integrable and $d\eta$ has rank $2n$. The subspace $V_p = \{X \in T_pM \mid d\eta(X, T_pM) = 0\}$ of T_pM is of dimension 1. Let ξ_p be the element of V_p on which η has the value 1. Then, ξ is a vector field, which we call the *characteristic vector field,* defined on M^{2n+1} such that

$$d\eta(\xi, X) = 0 \text{ and } \eta(\xi) = 1 \tag{2.4}$$

for any tangent vector X to M.

Theorem 2.2 *Let M^{2n+1} be a contact manifold with the contact form η. Then, there exists an almost contact metric structure (ϕ, ξ, η, g) such that $\Phi = d\eta$.*

Proof: We choose the characteristic vector field ξ so that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any tangent vector X to M. Thus, if h' is a Riemannian metric on M^{2n+1}, h defined by

$$h(X, Y) = h'(-X + \eta(X)\xi, -Y + \eta(Y)\xi) + \eta(X)\eta(Y)$$

is a Riemannian metric such that $\eta(X) = h(X, \xi)$. Setting $\Phi = d\eta$, Φ is a symplectic form on $\mathcal{D}$ and hence, by polarization, there exists a metric g' and an endomorphism ϕ on $\mathcal{D}$ such that

$$g'(X, \phi Y) = d\eta(X, Y) \text{ and } \phi^2 = -I.$$

Extending g' to a metric g agreeing with h in the direction ξ and extending ϕ so that $\phi \circ \xi = 0$, we obtain an almost contact metric structure (ϕ, ξ, η, g) with $\Phi = d\eta$. $\square$

An almost contact metric structure with $\Phi = d\eta$ is called an *associated almost contact metric structure* for η, or simply, a *contact metric structure* (ϕ, ξ, η, g).

Let M^{2n+1} be a contact metric manifold with contact form η. Roughly speaking, the condition $\eta \wedge (d\eta)^n \neq 0$ means that $\mathcal{D}$ is as far from being integrable as possible. In particular, we have the following theorem.

Theorem 2.3 (Sasaki [Sa64]) *Let M^{2n+1} be a contact manifold with contact form η. Then, there exist integral submanifolds of the contact distribution $\mathcal{D}$ of dimension n but of no higher dimension.*

Proof: Since $\eta \wedge (d\eta)^n \neq 0$, we can choose, by the classical theorem of Darboux (see for example, [St] pp.137), local coordinates $(x^i, y^i, z), i = 1, 2, \ldots, n$, such that $\eta = dz - \sum_{i=1}^n y^i dx^i$ on the coordinate neighborhood. Then, for a point p with coordinates (x_0^i, y_0^i, z_0) in the coordinate neighborhood, $x^i = x_0^i, y^i = y_0^i, z_0$ defines an n-dimensional integral submanifold of $\mathcal{D}$ in the neighborhood and a maximal integral submanifold containing this coordinate slice is an integral submanifold of $\mathcal{D}$ in M^{2n+1}.

Now, we let M^r an r-dimensional integral submanifold of $\mathcal{D}$ and we suppose that $r > n$. We denote by $X_1, X_2, \ldots, X_r$ r linearly independent local vector fields tangents to M^r and extend these to a basis by $X_{r+1}, X_{r+2}, \ldots, X_{2n}, X_{2n+1} = \xi$. Then, for $i, j = 1, 2, \ldots, n$ we have

$$\eta(X_i) = 0 \text{ and } d\eta(X_i, X_j) = \frac{1}{2}(X_i\eta(X_j) - X_j\eta(X_i) - \eta([X_i, X_j])) = 0.$$

Thus, since $r > n$, we see that

$$(\eta \wedge (d\eta)^n)(X_1, X_2, \ldots, X_{2n+1}) = 0,$$

which is a contradiction. $\square$

We have just seen that if X and Y are vector fields tangent to an integral submanifold of $\mathcal{D}$, then

$$\eta(X) = \eta(Y) = 0 \text{ and } d\eta(X, Y) = 0.$$

Thus, for a submanifold M^r immersed in a contact metric manifold M^{2n+1}, we see that if M^r is an integral submanifold of $\mathcal{D}$, ϕX is normal to M^r in M^{2n+1} for any tangent vector X to M^r.

We now state a theorem which shows the abundance of integral submanifolds of $\mathcal{D}$. As we saw above, ϕX is normal to an integral submanifold for X tangent to it, so loosely speaking the geometry is normal to the submanifolds. We shall study integral submanifolds in contact manifolds later in this thesis.

Theorem 2.4 (Sasaki [Sa64]) *Let X be a vector at $p \in M^{2n+1}$ belonging to $\mathcal{D}$. Then, there exists an r-dimensional integral submanifold $M^r (1 \leq r \leq n)$ of $\mathcal{D}$ through p such that X is tangent to M^r.*

We remark that on a contact metric manifold M with structure (ϕ, ξ, η, g), the integral curves of the characteristic vector field ξ are geodesics. Indeed, as $|\xi| = 1$, $g(\nabla_X \xi, \xi) = 0$. And, $\frac{1}{2} g(\nabla_\xi \xi, X) = d\eta(\xi, X) = 0$ for all vector fields X orthogonal to ξ and hence,

$$\nabla_\xi \xi = 0. \tag{2.5}$$

2.2 K-contact and Sasakian structures

We introduce the concept of a normal almost contact manifold. Consider a product manifold $M^{2n+1} \times \mathbf{R}$ of an almost contact manifold (M, ϕ, ξ, η) with the real line $\mathbf{R}$. A vector field on $M^{2n+1} \times \mathbf{R}$ looks like $(X, f\frac{d}{dt})$ where $X \in TM^{2n+1}$, f is a function on $M \times \mathbf{R}$, and t is a coordinate of $\mathbf{R}$. We define a linear map J on the tangent spaces of $M^{2n+1} \times \mathbf{R}$ by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}).$$

Then, $J^2 = -I$, i.e., J is an almost complex structure on $M^{2n+1} \times \mathbf{R}$. Let $[J, J]$ be the Nijenhuis torsion of J and similarly $[\phi, \phi]$ the torsion of ϕ. We say that the almost contact structure is *normal* if this almost complex structure J is integrable i.e., $[J, J] = 0$.

On a contact metric manifold, from the following two well known formulas

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} 3d\Phi(X,Y,Z) &= X\Phi(Y,Z) + Y\Phi(Z,X) + Z\Phi(X,Y) \\ &\quad - \Phi([X,Y],Z) - \Phi([Y,Z],X) - \Phi([Z,X],Y), \end{aligned} \tag{2.7}$$

we have (e.g. see [Bl76])

$$2g((\nabla_X \phi)Y, Z) = g(N^{(1)}(Y,Z), \phi X) + 2d\eta(\phi Y, X)\eta(Z) - 2d\eta(\phi Z, X)\eta(Y) \tag{2.8}$$

where $N^{(1)}(X,Y) = [\phi,\phi](X,Y) + 2d\eta(X,Y)\xi$.

On a contact metric manifold, we define a tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$. It is shown in [Bl76] that h is a symmetric operator. We now have the following proposition.

Proposition 2.5 *On a contact metric manifold with structure (ϕ, ξ, η, g), we have*

(1) $\nabla_\xi \phi = 0$

(2) $\nabla_X \xi = -\phi X - \phi h X$

(3) $h = 0$ *if and only if* ξ *is a Killing vector field*

(4) $h\phi + \phi h = 0$

Outline of Proof: The proofs of (1) and (2) are straight forward using (2.8). For example, using (2.8), we have

$$\begin{aligned} 2g((\nabla_X\phi)\xi, Z) &= g(\phi^2[\xi, Z] - \phi[\xi, \phi Z], X) - 2d\eta(\phi Z, X) \\ &= -2g(\phi h Z, \phi X) - 2g(\phi Z, \phi X) \\ &= -2g(hZ, X) - 2g(X, Z) + 2g(\eta(X)\xi, Z) \end{aligned}$$

that is,

$$-\phi\nabla_X\xi = -X - hX + \eta(X)\xi.$$

Applying ϕ to both sides of this, we get (2).

Now, we have

$$0 = d\eta(X,\xi) = \frac{1}{2}(X\eta(\xi) - \xi\eta(X) - \eta([X,\xi]))$$

from which we obtain

$$(\mathcal{L}_\xi\eta)(X) = \xi\eta(X) - \eta([\xi, X]) = 0.$$

Therefore, we see that $(\mathcal{L}_\xi g)(X,\xi) = \xi\eta(X) - \eta([\xi, X]) = 0$, and, in turn, using $\mathcal{L}_\xi = \iota(\xi) \circ d + d \circ \iota(\xi)$, we have $\mathcal{L}_\xi d\eta = 0$. But, since $\Phi = d\eta$, we get

$$\begin{aligned} 0 &= (\mathcal{L}_\xi\Phi)(X, Y) = \xi g(X, \phi Y) - g([\xi, X], \phi Y) - g(X, \phi[\xi, Y]) \\ &= (\mathcal{L}_\xi g)(X, \phi Y) + g(X, (\mathcal{L}_\xi\phi)Y) \end{aligned}$$

Thus, $h = \frac{1}{2}\mathcal{L}_\xi\phi = 0$ if and only if ξ is Killing.

Finally, we see, using (1), that

$$\begin{aligned} 2hX &= \mathcal{L}_\xi(\phi X) - \phi(\mathcal{L}_\xi X) \\ &= \nabla_\xi\phi X - \nabla_{\phi X}\xi - \phi\nabla_\xi X + \phi\nabla_X\xi \\ &= \phi\nabla_X\xi - \nabla_{\phi X}\xi. \end{aligned}$$

Applying ϕ on both sides, we have, using (2)

$$\begin{aligned} 2\phi hX &= -\nabla_X\xi - \phi\nabla_{\phi X}\xi \\ &= \phi X + \phi hX - \phi X - h\phi X \\ &= \phi hX - h\phi X, \end{aligned}$$

which yields (4). $\square$

In Theorem 2.2, we showed that a contact manifold with contact form η inherits an almost contact metric structure (ϕ, ξ, η, g) with $\Phi = d\eta$. This structure was referred as an associated structure or simply as a contact metric structure. A contact metric manifold M is called a *Sasakian manifold* if the associated almost contact metric structure is normal. The associated structure (ϕ, ξ, η, g) is called a *Sasakian structure.* It should be noted that Sasakian structure is not to be confused with the Sasaki metric defined earlier in this thesis. It is shown [Bl76] that the integrability of J is equivalent to the vanishing of the tensor field $N^{(1)} = [\phi, \phi] + 2d\eta \otimes \xi$. So, we have the following proposition.

Proposition 2.6 *A contact metric manifold M is Sasakian if and only if*

$$[\phi, \phi] + 2d\eta \otimes \xi = 0.$$

A Sasakian structure is an odd dimensional analogue of a Kähler structure on an almost Hermitian structure. This point of view is suggested in the following theorem.

Theorem 2.7 *An almost contact metric structure (ϕ, ξ, η, g) is Sasakian if and only if*

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.9}$$

where ∇ denotes the Riemannian connection of g.

Outline of Proof: The necessity is immediate from (2.8). For the converse, we set $Y = \xi$ in (2.9) to get

$$-\phi\nabla_X\xi = \eta(X)\xi - X$$

and, applying ϕ subsequently, so

$$\nabla_X\xi = -\phi X.$$

By the skew-symmetry of ϕ, we see that ξ is Killing. From this, we can easily see that $\Phi = d\eta$. Thus, (ϕ, ξ, η, g) is a contact metric structure and now using the formula

$$[\phi, \phi](X, Y) = (\phi\nabla_Y\phi - \nabla_{\phi Y}\phi)X - (\phi\nabla_X\phi - \nabla_{\phi X}\phi)Y,$$

we can directly compute that $[\phi, \phi] + 2d\eta \otimes \xi = 0$. □

A contact metric manifold M^{2n+1} with structure (ϕ, ξ, η, g) is called a *K-contact manifold* if the characteristic vector field ξ is a Killing vector field with respect to g, i.e., if $\mathcal{L}_\xi g = 0$ or equivalently, if $g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 0$ for all vector fields X and Y. It is immediate from Proposition 2.5 that

$$\nabla_X\xi = -\phi X \tag{2.10}$$

if and only if the manifold is K-contact. In particular, a Sasakian manifold is K-contact. It is noted, however, that Sasakian and K-contact structures are equivalent on 3-dimensional manifolds. (For more details on this, refer to [Bl76].)

Here, we give a curvature property of K-contact manifolds.

Proposition 2.8 *Let M^{2n+1} be a K-contact manifold with structure tensors (ϕ, ξ, η, g). Then, the sectional curvature of any plane section containing ξ is equal to* 1.

In particular, Sasakian manifolds have this property. We introduce the notion of ϕ-sectional curvature, a notion similar to that of holomorphic sectional curvature on

Kähler manifolds. A plane section in T_pM^{2n+1} is called a *ϕ-section* if there exists a vector $X \in T_pM^{2n+1}$ orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. The sectional curvature $K(X, \phi X)$ is called a *ϕ-sectional curvature*, and is denoted by $H(X)$. It is known that the ϕ-sectional curvature determines the curvature completely on Sasakian manifolds just as the holomorphic sectional curvature of a Kähler manifold determines the curvature completely.

We now give some curvature properties of Sasakian manifolds.

Proposition 2.9 *On a Sasakian manifold, we have* $R_{XY}\xi = \eta(Y)X - \eta(X)Y$.

Proof: The proof is immediate due to Theorem 2.7. □

Proposition 2.10 *On a Sasakian manifold, we have* $R_{X\xi}\xi = X$ *for any vector field* X *orthogonal to* ξ.

Proof: We choose $Y = \xi$ in Proposition 2.9 □

Later in this thesis, we will study contact metric manifolds with $R_{X\xi}\xi = 0$. This suggests that we define an operator l by $lX = R_{X\xi}\xi$.

Proposition 2.11 *On a Sasakian manifold, we have*

$$R_{XY}\phi Z = \phi R_{XY}Z + g(\phi X, Z)Y - g(Y, Z)\phi X + g(X, Z)\phi Y - g(\phi Y, Z)X.$$

Proof: It is proved again using Theorem 2.7 and (2.10).

$$\begin{aligned}
R_{XY}\phi Z &= \nabla_X\nabla_Y\phi Z - \nabla_Y\nabla_X\phi Z - \nabla_{[X,Y]}\phi Z \\
&= \nabla_X(\nabla_Y\phi)Z + \nabla_X(\phi\nabla_Y Z) - \nabla_Y(\nabla_X\phi)Z - \nabla_Y(\phi\nabla_X Z) \\
&\quad - (\nabla_{[X,Y]}\phi)Z - \phi\nabla_{[X,Y]}Z \\
&= \nabla_X(g(Y, Z)\xi - \eta(Z)Y) + g(X, \nabla_Y Z)\xi - \eta(\nabla_Y Z)X
\end{aligned}$$

$$
\begin{aligned}
& - \nabla_Y(g(X,Z)\xi - \eta(Z)X) - g(Y,\nabla_X Z)\xi + \eta(\nabla_X Z)\xi \\
& - g([X,Y],Z)\xi + \eta(Z)[X,Y] + \phi R_{XY} Z \\
= \quad & \phi R_{XY} Z - g(Y,Z)\phi X + g(\nabla_X Y, Z)\xi + g(Y,\nabla_X Z)\xi \\
& - \eta(Z)\nabla_X Y + g(\phi X, Z)Y - g(\xi, \nabla_X Z)Y + g(X,\nabla_Y Z)\xi - \eta(\nabla_Y Z)X \\
& + g(X,Z)\phi Y - g(\nabla_Y X, Z)\xi - g(X,\nabla_Y Z)\xi \\
& + \eta(Z)\nabla_Y X - g(\phi Y, Z)X + g(\xi,\nabla_Y Z)X - g(Y,\nabla_X Z)\xi + \eta(\nabla_X Z)Y \\
& - g([X,Y],Z)\xi + \eta(Z)[X,Y] \\
= \quad & \phi R_{XY} Z - g(Y,Z)\phi X + g(\phi X, Z)Y + g(X,Z)\phi Y - g(\phi Y, Z)X,
\end{aligned}
$$

as desired. □

From this Proposition, we can easily derive

$$R_{XY} Z = -\phi R_{XY} \phi Z + g(Y,Z)X - g(X,Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y.$$

Finally, we include the following theorems, which will be used later in this thesis.

Theorem 2.12 (Blair [Bl76]) *Let M^{2n+1} be a contact metric manifold. Suppose that $R_{XY}\xi = 0$ for all vector fields X and Y. Then, M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature equal to 4.*

Theorem 2.13 (Blair [Bl76]) *Let M^{2n+1}, $n \geq 2$, be a contact manifold. Then, M^{2n+1} does not admit a contact metric structure of vanishing curvature.*

As an extension of Theorem 2.13, Olszak [Ol] proved the following.

Theorem 2.14 *Let M^{2n+1}, $n \geq 2$, be a contact metric manifold of constant curvature. Then, the sectional curvature of M^{2n+1} is equal to 1 and M^{2n+1} is Sasakian.*

Chapter 3

The Normal Bundle of a Submanifold in a Contact Manifold

In this chapter, the normal bundle NM of an integral submanifold M of a contact metric manifold $\tilde{M}$ is investigated. We show that when $\tilde{M}$ is Sasakian, NM has a contact metric structure satisfying $l = 0$, and that for the contact metric structure on NM, $R_{\cdot\cdot}\xi$ depends only on the induced metric on M. Thus, we have a large class of examples of contact metric manifolds with $l = 0$. Motivated by this, we also study the contact metric manifolds with $l = 0$ and show that such a manifold cannot be locally symmetric unless it is locally isometric to $\mathbf{E}^{n+1} \times S^n(4)$. We also prove that a contact metric manifold with $l = 0$ can never be conformally flat.

3.1 The normal bundle of a submanifold in a Sasakian manifold

Let M^n be an integral submanifold of a contact metric manifold $\tilde{M}^{2n+1}$ with structure tensors $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$. We consider the normal bundle NM of the submanifold M equipped with the Sasaki metric $\overline{g}$. This metric is not to be confused with a

Sasakian structure. On the normal bundle NM, we define

$$\begin{aligned}\overline{\phi}X^H &= (\tilde{\phi}X)^V \\ \overline{\phi}\tilde{\xi}^V &= 0 \\ \overline{\phi}\zeta^V &= (\tilde{\phi}\zeta)^H\end{aligned}$$

for all tangent vectors X and normal vectors ζ orthogonal to $\tilde{\xi}$.

Let $\overline{\xi} = \tilde{\xi}^V$ and $\overline{\eta}(X) = \overline{g}(X, \overline{\xi})$ for any vector X. Then,

$$\begin{aligned}\overline{\phi}^2 X^H &= (\tilde{\phi}^2 X)^H = -X^H \\ &= -X^H + \overline{\eta}(X^H)\overline{\xi} \\ \overline{\phi}^2\overline{\xi} &= 0 = -\overline{\xi} + \overline{\eta}(\overline{\xi})\overline{\xi} \\ \overline{\phi}^2\zeta^V &= (\tilde{\phi}^2\zeta)^V = -\zeta^V \\ &= -\zeta^V + \overline{\eta}(\zeta^V)\overline{\xi}\end{aligned}$$

So, NM has an almost contact structure $(\overline{\phi}, \overline{\xi}, \overline{\eta})$.

We consider the Weingarten map of the immersion $\iota : M^n \longrightarrow \tilde{M}^{2n+1}$. The induced metric G on M^n is given by $G(X,Y) \circ \iota = \tilde{g}(\iota_* X, \iota_* Y)$ for any tangent vectors X and Y. For brevity, however, we shall not distinguish notationally between X and $\iota_* X$. Recall the Weingarten equation

$$\tilde{\nabla}_X \tilde{\xi} = -A_{\tilde{\xi}} X + D^{\perp}_X \tilde{\xi} \tag{3.1}$$

and the Gauss formula

$$\tilde{\nabla}_X Y = D_X Y + \sigma(X,Y) \tag{3.2}$$

where $A_{\tilde{\xi}}$ is the Weingarten map and D is the Riemannian connection of the induced metric G.

Using these together with Lemma 1.2 and Lemma 1.3, we have at the point $\nu \in NM$

$$\begin{aligned}
2d\overline{\eta}(X^H, Y^H) &= X^H\overline{\eta}(Y^H) - Y^H\overline{\eta}(X^H) - \overline{\eta}([X^H, Y^H]) \\
&= -\overline{g}([X^H, Y^H], \overline{\xi}) \\
&= -\overline{g}(K[X^H, Y^H], \overline{\xi}) \\
&= \tilde{g}(R^{\perp}_{XY}\nu, \tilde{\xi}) \\
&= -\tilde{g}(R^{\perp}_{XY}\tilde{\xi}, \nu) \\
&= -\tilde{g}(\tilde{\nabla}_X(\tilde{\nabla}_Y\tilde{\xi} + A_{\tilde{\xi}}Y) - \tilde{\nabla}_Y(\tilde{\nabla}_X\tilde{\xi} + A_{\tilde{\xi}}X) - \tilde{\nabla}_{[X,Y]}\tilde{\xi}, \nu) \\
&= -\tilde{g}(\tilde{R}_{XY}\tilde{\xi}, \nu) - \tilde{g}(\sigma(X, A_{\tilde{\xi}}Y) - \sigma(Y, A_{\tilde{\xi}}X), \nu) \\
&= -\tilde{g}(\tilde{R}_{XY}\tilde{\xi}, \nu) - \tilde{g}(A_{\tilde{\xi}}Y, A_\nu X) + \tilde{g}(A_{\tilde{\xi}}X, A_\nu Y) \\
&= -\tilde{g}(\tilde{R}_{XY}\tilde{\xi}, \nu) + \tilde{g}([A_{\tilde{\xi}}, A_\nu]X, Y) \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
2d\overline{\eta}(X^H, \zeta^V) &= X^H\overline{\eta}(\zeta^V) - \zeta^V\overline{\eta}(X^H) - \overline{\eta}([X^H, \zeta^V]) \\
&= -\overline{g}([X^H, \zeta^V], \overline{\xi}) \\
&= -\tilde{g}(\nabla^{\perp}_X\zeta, \tilde{\xi}) \\
&= -\tilde{g}(\tilde{\nabla}_X\zeta, \tilde{\xi}) \\
&= \tilde{g}(\zeta, \tilde{\nabla}_X\tilde{\xi}) \\
&= -\tilde{g}(\zeta, \tilde{\phi}X + \tilde{\phi}\tilde{h}X) \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
2d\overline{\eta}(X^H, \overline{\xi}^V) &= X^H\overline{\eta}(\overline{\xi}^V) - \overline{\xi}^V\overline{\eta}(X^H) - \overline{\eta}([X^H, \overline{\xi}^V]) \\
&= -\overline{g}([X^H, \overline{\xi}], \overline{\xi}) \\
&= -\overline{g}((\nabla^{\perp}_X\tilde{\xi})^V, \overline{\xi}) \\
&= -\overline{g}(\tilde{\nabla}_X\tilde{\xi}, \tilde{\xi}) \\
&= 0
\end{aligned}$$

$$2d\overline{\eta}(\overline{\xi},\zeta^V) = \overline{\xi}\overline{\eta}(\zeta^V) - \zeta^V\overline{\eta}(\overline{\xi}) - \overline{\eta}([\overline{\xi},\zeta^V])$$
$$= 0$$

for any normal vector ζ orthogonal to ξ

and, finally

$$2d\overline{\eta}(\zeta^V,\nu^V) = -\overline{\eta}([\zeta^V,\nu^V]) = 0.$$

Suppose now that $\tilde{M}^{2n+1}$ is Sasakian. Then, using (3.1), (3.2) and the fact that $\tilde{\phi}X$ is normal to M^n for any tangent vector X to M^n, we see that

$$\begin{aligned} G(A_{\tilde{\xi}}X,Y) &= \tilde{g}(\sigma(X,Y),\tilde{\xi}) = \tilde{g}(\tilde{\nabla}_X Y,\tilde{\xi}) \\ &= -\tilde{g}(Y,\tilde{\nabla}_X\tilde{\xi}) = \tilde{g}(Y,\tilde{\phi}X) \\ &= 0. \end{aligned}$$

for any vectors X and Y tangent to M, that is,

$$A_{\tilde{\xi}} = 0 \tag{3.5}$$

Moreover, since $\tilde{M}^{2n+1}$ is Sasakian, using Theorem 2.7 and (3.1), we have

$$\begin{aligned} \tilde{g}(R^{\perp}_{XY}\tilde{\xi},\zeta) &= \tilde{g}(\nabla^{\perp}_X(-\tilde{\phi}Y) + \nabla^{\perp}_Y\tilde{\phi}X + \tilde{\phi}[X,Y],\zeta) \\ &= \tilde{g}(-\tilde{\nabla}_X\tilde{\phi}Y - A_{\tilde{\phi}Y}X + \tilde{\nabla}_Y\tilde{\phi}X + A_{\tilde{\phi}X}Y + \tilde{\phi}[X,Y],\zeta) \\ &= \tilde{g}(-(\tilde{\nabla}_X\tilde{\phi})Y - \tilde{\phi}\tilde{\nabla}_XY + (\tilde{\nabla}_Y\tilde{\phi})X + \tilde{\phi}\tilde{\nabla}_YX + \tilde{\phi}[X,Y],\zeta) \\ &= \tilde{g}(-(\tilde{\nabla}_X\tilde{\phi})Y + (\tilde{\nabla}_Y\tilde{\phi})X,\zeta) \\ &= -\tilde{g}(X,Y)\tilde{g}(\tilde{\xi},\zeta) + \tilde{\eta}(Y)\tilde{g}(X,\zeta) + \tilde{g}(X,Y)\tilde{g}(\tilde{\xi},\zeta) - \tilde{\eta}(X)\tilde{g}(Y,\zeta) \\ &= 0 \end{aligned}$$

for any ζ normal to M^n. Thus,

$$R^{\perp}_{XY}\tilde{\xi} = 0 \tag{3.6}$$

for any tangent vectors X and Y. So, by the definition of $\hat{R}$,

$$\hat{R}_{\bar{\xi}\psi}X = 0 \tag{3.7}$$

for all $\psi \in NM$.

Now, we let $\phi = \overline{\phi}$, $\xi = 2\overline{\xi}$, $\eta = \frac{1}{2}\overline{\eta}$ and $g = \frac{1}{4}\overline{g}$. If $\tilde{M}$ is Sasakian, we have, from (3.3), (3.4), and (3.5),

$$\begin{aligned} d\eta(X^H, Y^H) &= 0 = g(X^H, \phi Y^H) \\ d\eta(X^H, \zeta^V) &= -\frac{1}{4}\tilde{g}(\zeta, \tilde{\phi}X) = \frac{1}{4}\overline{g}(X^H, \phi\zeta^V) = g(X^H, \phi\zeta^V) \\ d\eta(X^H, \xi) &= 0 = g(X^H, \phi\xi) \end{aligned}$$

and other cases follow similarly. We now give the following theorems.

Theorem 3.1 *Let M^n be an integral submanifold of a Sasakian manifold $\tilde{M}^{2n+1}$ with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$. Then, NM has the contact metric structure (ϕ, ξ, η, g) with $l = 0$.*

Proof: We have just seen that when M^n is an integral submanifold of a Sasakian manifold $\tilde{M}^{2n+1}$, NM has the contact metric structure (ϕ, ξ, η, g). Now, by (3.7) and Lemma 1.3, we have

$$\begin{aligned} R_{X^H\xi}\xi &= R_{X^H\tilde{\xi}^V}\tilde{\xi}^V \\ &= -[\frac{1}{2}\hat{R}_{\bar{\xi}\bar{\xi}}X + \frac{1}{4}\hat{R}_{\psi\bar{\xi}}\hat{R}_{\psi\bar{\xi}}X]^H \\ &= 0 \end{aligned}$$

and

$$R_{\zeta^V\xi}\xi = 0.$$

This completes the proof. $\square$

Theorem 3.2 *Let M^n be an integral submanifold of a Sasakian manifold $\tilde{M}^{2n+1}$. Then, for the contact metric structure (ϕ, ξ, η, g) on NM, $R..\xi$ is intrinsic, i.e., it depends only on the induced metric G on M. In particular, at $\nu = \tilde{\phi}W \in NM$,*

$$g(R_{X^HY^H}\xi, Z^H) = \frac{1}{4}\{G(\underline{R}_{XY}Z, W) - G(Y,Z)G(W,X) + G(X,Z)G(Y,W)\} \quad (3.8)$$

where $\underline{R}$ denotes the curvature of D. The other cases of $R..\xi$ vanish.

Proof: Using Lemma 1.3 and (3.7), we have, at $\nu \in MN$,

$$\begin{aligned} g(R_{X^HY^H}\xi, Z^H) &= \frac{1}{2}\overline{g}(\overline{R}_{X^HY^H}\tilde{\xi}^V, Z^H) \\ &= \frac{1}{4}\tilde{g}((D_X\hat{R})(\nu,\tilde{\xi})Y - (D_Y\hat{R})(\nu,\tilde{\xi})X, Z) \\ &= \frac{1}{4}\tilde{g}(D_X\hat{R}_{\nu\tilde{\xi}}Y - \hat{R}_{D^\perp_X\nu\tilde{\xi}}Y - \hat{R}_{\nu D^\perp_X\tilde{\xi}}Y - \hat{R}_{\nu\tilde{\xi}}D_XY, Z) \\ &\quad -\frac{1}{4}\tilde{g}(D_Y\hat{R}_{\nu\tilde{\xi}}X - \hat{R}_{D^\perp_Y\nu\tilde{\xi}}X - \hat{R}_{\nu D^\perp_Y\tilde{\xi}}X - \hat{R}_{\nu\tilde{\xi}}D_YX, Z) \\ &= \frac{1}{4}\tilde{g}(\hat{R}_{\nu\, D^\perp_Y\tilde{\xi}}X - \hat{R}_{\nu\, D^\perp_X\tilde{\xi}}Y, Z) \end{aligned} \quad (3.9)$$

Continuing this computation at $\nu = \tilde{\phi}W \in MN$, using (3.1), the Ricci equation $\tilde{g}(\tilde{R}_{XY}U, V) = \tilde{g}(R^\perp_{XY}U, V) - \tilde{g}([A_U, A_V]X, Y)$, and Proposition 2.11,

$$\begin{aligned} g(R_{X^HY^H}\xi, Z^H) &= -\frac{1}{4}\tilde{g}(R^\perp_{YZ}\tilde{\phi}X, \nu) + \frac{1}{4}\tilde{g}(R^\perp_{XZ}\tilde{\phi}Y, \nu) \\ &= \frac{1}{4}\{\tilde{g}(\tilde{R}_{XZ}\tilde{\phi}Y, \tilde{\phi}W) - \tilde{g}(\tilde{R}_{YZ}\tilde{\phi}X, \tilde{\phi}W) \\ &\quad - \tilde{g}([A_{\tilde{\phi}W}, A_{\tilde{\phi}Y}]X, Z) + \tilde{g}([A_{\tilde{\phi}W}, A_{\tilde{\phi}X}]Y, Z) \\ &= \frac{1}{4}\{\tilde{g}(\tilde{R}_{XZ}Y, W) - \tilde{g}(Z,Y)\tilde{g}(\tilde{\phi}X, \tilde{\phi}W) + \tilde{g}(X,Y)\tilde{g}(\tilde{\phi}Z, \tilde{\phi}W) \\ &\quad - \tilde{g}(\tilde{R}_{YZ}X, W) + \tilde{g}(Z,X)\tilde{g}(\tilde{\phi}Y, \tilde{\phi}W) - \tilde{g}(Y,X)\tilde{g}(\tilde{\phi}Z, \tilde{\phi}W) \\ &\quad - \tilde{g}(A_{\tilde{\phi}Y}X, A_{\tilde{\phi}W}Z) + \tilde{g}(A_{\tilde{\phi}W}X, A_{\tilde{\phi}Y}Z) \\ &\quad + \tilde{g}(A_{\tilde{\phi}X}Y, A_{\tilde{\phi}W}Z) - \tilde{g}(A_{\tilde{\phi}W}Y, A_{\tilde{\phi}X}Z)\}. \end{aligned} \quad (3.10)$$

Comparing the tangential part of

$$g(X,Y)\tilde{\xi} = (\tilde{\nabla}_X\tilde{\phi})Y = \tilde{\nabla}_X\tilde{\phi}Y - \tilde{\phi}\tilde{\nabla}_XY,$$

we get

$$A_{\tilde{\phi}Y}X = -\tilde{\phi}\sigma(X,Y)$$

and therefore, by the symmetry of the second fundamental form,

$$A_{\tilde{\phi}Y}X = A_{\tilde{\phi}X}Y. \tag{3.11}$$

Also, since $\tilde{\eta}(\sigma(W,X)) = \tilde{g}(\sigma(W,X),\tilde{\xi}) = \tilde{g}(A_{\tilde{\xi}}W,X) = 0$,

$$\tilde{g}(A_{\tilde{\phi}W}X, A_{\tilde{\phi}Y}Z) = \tilde{g}(\tilde{\phi}\sigma(W,X),\tilde{\phi}\sigma(Y,Z)) = \tilde{g}(\sigma(W,X),\sigma(Y,Z)). \tag{3.12}$$

Similarly,

$$\tilde{g}(A_{\tilde{\phi}W}Y, A_{\tilde{\phi}X}Z) = \tilde{g}(\sigma(W,Y),\sigma(X,Z)). \tag{3.13}$$

Thus, by (3.11)-(3.13), (3.10) becomes

$$\begin{aligned} g(R_{X^HY^H}\xi, Z^H) &= \frac{1}{4}\{\tilde{g}(\tilde{R}_{XZ}Y,W) - \tilde{g}(\tilde{R}_{YZ}X,W) - \tilde{g}(Z,Y)\tilde{g}(X,W) \\ &\quad + \tilde{g}(X,Y)\tilde{g}(Z,W) + \tilde{g}(Z,X)\tilde{g}(Y,W) - \tilde{g}(Y,X)\tilde{g}(Z,W) \\ &\quad + \tilde{g}(\sigma(W,X),\sigma(Y,Z)) - \tilde{g}(\sigma(W,Y),\sigma(X,Z))\} \end{aligned}$$

But then, applying Bianchi identity and Gauss equation successively, we get

$$\begin{aligned} g(R_{X^HY^H}\xi, Z^H) &= \frac{1}{4}\{\tilde{g}(\tilde{R}_{XY}Z,W) - \tilde{g}(Z,Y)\tilde{g}(X,W) + \tilde{g}(Z,X)\tilde{g}(Y,W) \\ &\quad + \tilde{g}(\sigma(W,X),\sigma(Y,Z)) - \tilde{g}(\sigma(W,Y),\sigma(X,Z))\} \\ &= \frac{1}{4}\{G(\underline{R}_{XY}Z,W) - G(Y,Z)G(W,X) + G(X,Z)G(Y,W)\} \end{aligned}$$

It is immediate to see, from Lemma 1.3 and (3.7), that the other cases of $R_{..}\xi$ vanish. This completes the proof. □

Corollary 3.3 (Yano and Kon [YaK]) *Let M^n be an integral submanifold of a Sasakian manifold $\tilde{M}^{2n+1}$. Then, M^n has flat normal connection if and only if it has the constant curvature* 1.

Proof: If M^n has flat connection, then, by (3.9), (3.8) gives

$$G(\underline{R}_{XY}Z, W) = G(Y, Z)G(W, X) - G(X, Z)G(Y, W)$$

which shows that M has the constant curvature 1.

For the converse, we suppose that M^n has the constant curvature 1. Then, from (3.8), we see that

$$g(R_{X^H Y^H}\xi, Z^H) = 0.$$

Hence, from the first equality of (3.10), we have

$$-\tilde{g}(R^{\perp}_{YZ}\tilde{\phi}X, \nu) + \tilde{g}(R^{\perp}_{XZ}\tilde{\phi}Y, \nu) = 0. \tag{3.14}$$

So, choosing $Y = Z$ in (3.14), we see that

$$\tilde{g}(R^{\perp}_{XY}\tilde{\phi}Y, \nu) = 0,$$

from which we have, by linearization,

$$\tilde{g}(R^{\perp}_{XY}\tilde{\phi}Z, \nu) + \tilde{g}(R^{\perp}_{XZ}\tilde{\phi}Y, \nu) = 0. \tag{3.15}$$

Replacing $X, Y,$ and Z by $Z, X,$ and Y, respectively, in (3.15), we obtain

$$\tilde{g}(R^{\perp}_{XZ}\tilde{\phi}Y, \nu) + \tilde{g}(R^{\perp}_{YZ}\tilde{\phi}X, \nu) = 0. \tag{3.16}$$

Combining (3.14) and (3.16), we get

$$\tilde{g}(R^{\perp}_{XZ}\tilde{\phi}Y, \nu) = 0$$

for any X, Y, Z and ν. Therefore, we conclude that

$$R^{\perp}_{XZ}\tilde{\phi}Y = 0$$

for any $X, Y,$ and Z.

Finally, it was shown by (3.6) that

$$R^{\perp}_{XZ}\tilde{\xi} = 0.$$

Hence, the normal connection is flat. $\square$

Example 3.4 $S^n \subset S^{2n+1}$.

We consider the usual contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on S^{2n+1} induced from the usual almost complex structure on $\mathbf{C}^{n+1}$ by $\tilde{\xi} = -JZ$, $\tilde{\eta}(X) = \tilde{g}(X, \tilde{\xi})$, $\tilde{g}$ the standard metric of S^{2n+1} as a unit sphere and $\tilde{\phi}(X) =$ the tangential part of JX. Let L be an $(n+1)$-dimensional linear subspace of $\mathbf{C}^{n+1}$ passing through the origin and such that JL is orthogonal to L. Then, $S^n = S^{2n+1} \cap L$ is an integral submanifold of S^{2n+1} since $\tilde{\phi}X$ is normal, and the normal connection is flat. In this case, NS^n becomes $\mathbf{E}^{n+1} \times S^n(4)$.

Example 3.5 $T^2 \subset S^5$.

We write $S^5 = \{Z \in \mathbf{C}^3 : |Z| = 1\}$ and consider the embedding $X : T^2 \to S^5$ given by

$$X = \frac{1}{3}(\cos u_1, \sin u_1, \cos u_2, \sin u_2, \cos(u_1+u_2), -\sin(u_1+u_2)),$$

where $\{u_1, u_2\}$ are local coordinates on T^2 such that $\frac{\partial}{\partial u_i}$ are orthonormal. Let $\{v_1, v_2, v_3\}$ denote the fiber coordinates on NT^2. We may regard X as a position vector of T^2 in S^5. Putting $X_i = \frac{\partial_i X}{\partial U_i}$ for $i = 1, 2$, we have

$$\begin{aligned} X_1 &= \frac{1}{3}(-\sin u_1, \cos u_1, 0, 0, -\sin(u_1+u_2), -\cos(u_1+u_2)) \\ X_2 &= \frac{1}{3}(0, 0, -\sin u_2, \cos u_2, -\sin(u_1+u_2), \cos(u_1+u_2)). \end{aligned}$$

Moreover, the characteristic vector field $\tilde{\xi}$ given by

$$\tilde{\xi} = -JX = \frac{1}{3}(\sin u_1, -\cos u_1, \sin u_2, -\cos u_2, -\sin(u_1+u_2), -\cos(u_1+u_2))$$

is orthogonal to X_1 and X_2. Thus, $\tilde{\phi}X_i = JX_i$, from which we see that $\tilde{\phi}X_i$ is normal to T^2. Therefore, T^2 is a flat integral submanifold of S^5.

Now on NT^2, we have $\eta = \frac{1}{2}(dv_3 + v_1du_1 + v_2du_2)$ and

$$g = \frac{1}{4}\begin{pmatrix} 1+v_1^2+v_3^2 & v_1v_2 & -v_3 & 0 & v_1 \\ v_1v_2 & 1+v_2^2+v_3^2 & 0 & -v_3 & v_2 \\ -v_3 & 0 & 1 & 0 & 0 \\ 0 & -v_3 & 0 & 1 & 0 \\ v_1 & v_2 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that this metric has the similar form of the associated metric to the standard contact metric structure on $\mathbf{E}^{2n+1}$

$$g = \frac{1}{4}\begin{pmatrix} \delta_{ij}+y^iy^j+\delta_{ij}z^2 & \delta_{ij}z & -y^i \\ \delta_{ij}z & \delta_{ij} & 0 \\ -y^i & 0 & 1 \end{pmatrix},$$

(x^i, y^i, z) the coordinates of $\mathbf{E}^{2n+1}$, which was constructed as a formal generalization of the flat associated metric of the Darboux form on $\mathbf{E}^3$ (See [Bl76]).

Remark. In NM, for ζ orthogonal to $\tilde{\xi}$

$$\nabla_{\zeta^V}\xi = 2\nabla_{\zeta^V}\tilde{\xi}^V = 0$$

so from Proposition 2.5, $h\zeta^V = -\zeta^V$. Thus, -1 is an eigenvalue with multiplicity n. Since $h\phi + \phi h = 0$, $+1$ is an eigenvalue with multiplicity n and hence, $hX^H = X^H$. Now, from Lemma 1.1, we see that the distribution $[+1]$ in NM is integrable if and only if NM is locally the product $\mathbf{E}^{n+1} \times S^n(4)$, i.e., $M^n \subset \tilde{M}^{2n+1}$ has constant curvature 1.

3.2 Contact manifolds with $l = 0$

From Theorem 3.1, we see that there is a large class of examples of contact metric manifolds with $l = 0$. We will study properties of those manifolds with $l = 0$.

In [Pe], Perrone raised the question whether a locally symmetric contact manifold with $l = 0$ satisfies $R_{XY}\xi = 0$ for any vectors X and Y, i.e., if, in view of Theorem 2.12, M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature equal to 4. We answer this question in the following theorem.

Theorem 3.6 *Let M^{2n+1} be a locally symmetric contact metric manifold with $l = 0$. Then, M is locally isometric to $\mathbf{E}^{n+1} \times S^n(4)$.*

Proof: We first consider case when the manifold M^{2n+1} is irreducible. In this case, M^{2n+1} is Einstein. Since $l = 0$, $Q\xi = 0$ and therefore $R = 0$. Moreover, all sectional curvatures have the same sign. Hence, M^{2n+1} is flat, which is impossible for $n \geq 2$ in view of Theorem 2.13. In dimension 3, a locally symmetric contact metric manifold is of constant curvature 0 or $+1$, l being 0 in the flat case (See [BlS]).

So, M^{2n+1} is reducible, i.e., $M^{2n+1} = M_1 \times M_2 \times \cdots \times M_k \times \mathbf{E}^r$ where each M_i is not flat. Then, the characteristic vector field ξ must be tangent to the flat factor $\mathbf{E}^r$, for otherwise, ξ has a non-vanishing projecton tangent to some M_i and so, the sectional curvature containing ξ is nonzero. This is impossible since $l = 0$. Therefore, $R_{XY}\xi = 0$ for any X and Y tangent to M^{2n+1}. Hence, by Theorem 2.12, we get the conclusion. $\square$

Theorem 3.7 *Let M^{2n+1}, $n \geq 2$, be a contact metric manifold satisfying $l = 0$. Then, M^{2n+1} can not be conformally flat.*

Proof: The proof will be by contradiction. Suppose (ϕ, ξ, η, g) is a conformally flat contact metric structure. Then, in view of (1.19), we have

$$R_{X\xi}\xi = \frac{1}{2n-1}(QX - \eta(X)Q\xi + g(Q\xi,\xi)X - g(QX,\xi)\xi)$$

$$-\frac{R}{2n(2n-1)}(X-\eta(X)\xi) \tag{3.17}$$

Since $l=0$, the Ricci curvature in the direction of ξ vanishes and so, from (3.17), we get

$$QX=\eta(X)Q\xi+\eta(QX)\xi+\frac{R}{2n}(X-\eta(X)\xi) \tag{3.18}$$

We consider the tensor L defined by $L=\frac{1}{2n-1}(-Q+\frac{R}{4n}I)$ where I is the identity transformation.

Then, by (3.18),

$$\begin{aligned} LX &= \frac{1}{2n-1}(-\eta(X)Q\xi-\eta(QX)\xi-\frac{R}{2n}X+\frac{R}{2n}\eta(X)\xi+\frac{R}{4n}X) \\ &= \frac{1}{2n-1}(-\eta(X)Q\xi-\eta(QX)\xi+\frac{R}{2n}\eta(X)\xi-\frac{R}{4n}X) \end{aligned}$$

for any vector X.

In particular, for any X orthogonal to ξ,

$$LX=-\frac{1}{2n-1}(\eta(QX)\xi+\frac{R}{4n}X)$$

From this and that $(\nabla_X L)Y=(\nabla_Y L)X$ on a conformally flat manifold, we have, using (3.18), for X and Y orthogonal to ξ

$$\begin{aligned} 0 &= -\nabla_X(\eta(QY)\xi+\frac{R}{4n}Y)+\eta(\nabla_X Y)Q\xi+\eta(Q\nabla_X Y)\xi \\ &\quad -\frac{R}{2n}\eta(\nabla_X Y)\xi+\frac{R}{4n}\nabla_X Y \\ &\quad +\nabla_Y(\eta(QX)\xi+\frac{R}{4n}X)-\eta(\nabla_Y X)Q\xi-\eta(Q\nabla_Y X)\xi \\ &\quad +\frac{R}{2n}\eta(\nabla_Y X)\xi-\frac{R}{4n}\nabla_Y X \\ &= -\eta(QY)\nabla_X\xi-(X\eta(QY))\xi-\frac{R}{4n}\nabla_X Y-\frac{1}{4n}X(R)Y \\ &\quad +\eta(\nabla_X Y)Q\xi+\eta(Q\nabla_X Y)\xi-\frac{R}{2n}\eta(\nabla_X Y)\xi+\frac{R}{4n}\nabla_X Y \\ &\quad +\eta(QX)\nabla_Y\xi+(Y\eta(QX))\xi+\frac{R}{4n}\nabla_Y X+\frac{1}{4n}Y(R)X \end{aligned}$$

$$
\begin{aligned}
&-\eta(\nabla_Y X)Q\xi - \eta(Q\nabla_Y X)\xi + \frac{R}{2n}\eta(\nabla_Y X)\xi - \frac{R}{4n}\nabla_Y X \\
= \;& \eta(QY)(\phi X + \phi hX) - (X\eta(QY))\xi - \frac{1}{4n}X(R)Y \\
&+ \eta(\nabla_X Y)Q\xi + \eta(Q\nabla_X Y)\xi - \frac{R}{2n}\eta(\nabla_X Y)\xi \\
&- \eta(QX)(\phi Y + \phi hY) + (Y\eta(QX))\xi + \frac{1}{4n}Y(R)X \\
&- \eta(\nabla_Y X)Q\xi - \eta(Q\nabla_Y X)\xi + \frac{R}{2n}\eta(\nabla_Y X)\xi \qquad (3.19)
\end{aligned}
$$

Taking the component orthogonal to ξ from (3.19), we have

$$
\begin{aligned}
0 \;=\; & \eta(QY)(\phi X + \phi hX) + \eta([X,Y])Q\xi \\
& - \frac{1}{4n}X(R)Y + \frac{1}{4n}Y(R)X - \eta(QX)(\phi Y + \phi hY) \\
=\; & -2g(X,\phi Y)Q\xi - \frac{1}{4n}(X(R)Y - Y(R)X) \\
& + \eta(QY)(\phi X + \phi hX) - \eta(QX)(\phi Y + \phi hY) \qquad (3.20)
\end{aligned}
$$

If $Q\xi \neq 0$, then, since the dimension $2n+1 \geq 5$, we can choose $X = \phi Y$ and X and Y orthogonal to $Q\xi$ in (3.20), and so,

$$-2g(\phi Y, \phi Y)Q\xi = 0,$$

which gives

$$Q\xi = 0, \qquad (3.21)$$

a contradiction. Therefore, $Q\xi = 0$.

Now, using (1.19) again, we see, for X and Y orthogonal to ξ

$$R_{XY}\xi = 0.$$

But, $R_{X\xi}\xi = 0$ by hypothesis.

Hence, we have $R_{XY}\xi = 0$ for any vectors X and Y. Then, by Theorem 2.12, M is locally the product $\mathbf{E}^{n+1} \times S^n(4)$, which is, as is well known, not conformally flat, giving a contradiction. $\Box$

Chapter 4

The Normal Bundle of a submanifold in a Kähler manifold

In this chapter, the normal bundle NL of a Lagrangian submanifold L of a Kähler manifold is studied. We show that the normal bundle NL of such has a natural symplectic structure and provide the equivalent conditions for NL to be Kähler.

4.1 Lagrangian submanifolds in a Kähler manifold

An exterior 2-form τ on a manifold M is called a *symplectic structure* if τ is non-degenerate at each point of M and is closed, i.e., $d\tau = 0$. We say that (M, τ) is a symplectic manifold. It is well known that a symplectic manifold is even dimensional. Let W be a subspace of an even dimensional vector space V^{2n}. For a non-degenerate bilinear form τ on V^{2n}, we define $W_\tau^\perp$ by

$$W_\tau^\perp = \{v \in V \mid \tau(v, w) = 0 \text{ for all } w \in W\}.$$

The subspace W is said to be *Lagrangian* if $W = W_\tau^\perp$.

We consider an immersed submanifold L of M^{2n} with the immersion $\iota : L \to M$ where (M, τ) is a symplectic manifold. We say L is a *Lagrangian submanifold* of M^{2n} if T_pL is a Lagrangian subspace of (T_pM, τ) for each p in L.

We now consider a Lagrangian submanifold L of a Kähler manifold (M^{2n}, J, g). On the normal bundle NL of L, we define $\tilde{J}$ by

$$\begin{aligned} \tilde{J}X^H &= (JX)^V \\ \tilde{J}\xi^V &= (J\xi)^H \end{aligned}$$

for any tangent vectors X and normal vectors ξ to L. Then, $\tilde{J}$ is an almost complex structure on NL since

$$\tilde{J}^2X^H = \tilde{J}(JX)^V = (J^2X)^H = -X^H,$$

and

$$\tilde{J}^2\xi^V = \tilde{J}(J\xi)^H = (J^2\xi)^V = -\xi^V.$$

Moreover, we see that $\tilde{J}$ is compatible with the Sasaki metric $\tilde{g}$ on the normal bundle as follows:

$$\begin{aligned}
\tilde{g}(\tilde{J}X^H, \tilde{J}Y^H) &= g(\pi_*\tilde{J}X^H, \pi_*\tilde{J}Y^H) + g(K\tilde{J}X^H, K\tilde{J}Y^H) \\
&= g(JX, JY), \\
\tilde{g}(X^H, Y^H) &= g(\pi_*X^H, \pi_*Y^H) + g(KX^H, KY^H) \\
&= g(X, Y), \\
\tilde{g}(\tilde{J}X^H, \tilde{J}\xi^V) &= g(\pi_*\tilde{J}X^H, \pi_*\tilde{J}\xi^V) + g(K\tilde{J}X^H, K\tilde{J}\xi^V) = 0, \\
g(X^H, \xi^V) &= 0, \\
\tilde{g}(\tilde{J}\xi^V, \tilde{J}\eta^V) &= g(\pi_*\tilde{J}\xi, \pi_*\tilde{J}\eta^V) + g(K\tilde{J}\xi^V, K\tilde{J}\eta^V) \\
&= g(J\xi, J\eta), \\
\tilde{g}(\xi, \eta) &= g(\pi_*\xi^V, \pi_*\eta^V) + g(K\xi^V, K\eta^V) \\
&= g(\xi, \eta)
\end{aligned}$$

By the compatibility of J with g, $\tilde{J}$ is compatible with the Sasaki metric $\tilde{g}$.

Therefore, $(NL, \tilde{J}, \tilde{g})$ is an almost Hermitian manifold.

We let $\tilde{\nabla}$ and ∇ denote the Riemannian connections of $\tilde{g}$ and g, respectively. Let G be the induced metric on L and D its Riemannian connection. Then , we have at ξ, by Lemma 1.1,

$$[X^H, Y^H] = [X, Y]^H - (R^{\perp}_{XY}\xi)^V \tag{4.1}$$

We now prove the following theorems.

Theorem 4.1 *Let L be a Lagrangian submanifold of a Kähler manifold (M^{2n}, J, g). Then, $(NL, \tilde{J}, \tilde{g})$ is a symplectic manifold.*

Proof: We consider the fundamental 2-form Ω defined by $\Omega(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{J}\tilde{Y})$. Since $\tilde{g}$ is positive definite and $\tilde{J}$ is non-singular at each point, it follows that $\Omega^n \neq 0$, i.e., Ω is non-degenerate.

We will show that the fundamental 2-form Ω is closed.

We have

$$\begin{aligned} \Omega(X^H, Y^H) &= \tilde{g}(X^H, \tilde{J}Y^H) \\ &= g(\pi_* X^H, \pi_* \tilde{J}Y^H) + g(KX^H, K\tilde{J}Y^H) \\ &= 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \Omega(X^H, \eta^V) &= \tilde{g}(X^H, \tilde{J}\eta^V) \\ &= g(\pi_* X^H, \pi_* \tilde{J}\eta^V) + g(KX^H, K\tilde{J}\eta^V) \\ &= g(X, J\eta), \end{aligned} \tag{4.3}$$

and similarly,

$$\Omega(\eta^V, \zeta^V) = 0. \tag{4.4}$$

Recall the coboundary formula

$$\begin{aligned} 3d\Phi(X,Y,Z) &= X\Phi(Y,Z) + Y\Phi(Z,X) + Z\Phi(X,Y) \\ &\quad - \Phi([X,Y],Z) - \Phi([Y,Z],X) - \Phi([Z,X],Y). \end{aligned}$$

Using this and (4.1) - (4.4), we compute,

$$\begin{aligned} 3d\Omega(X^H,Y^H,Z^H) &= -\Omega([X^H,Y^H],Z^H) - \Omega([Y^H,Z^H],X^H) - \Omega([Z^H,X^H],Y^H) \\ &= \Omega((R^{\perp}_{XY}\xi)^V,Z^H) + \Omega((R^{\perp}_{YZ}\xi)^V,X^H) + \Omega((R^{\perp}_{ZX}\xi)^V,Y^H) \end{aligned}$$

Now, using the Gauss-Weingarten equations, we get

$$\nabla_X J\xi = D_X J\xi + \sigma(X, J\xi)$$

and

$$J\nabla_X \xi = -JA_\xi X + JD^{\perp}_X \xi.$$

Comparing the tangential parts of these, using the Kähler condition, we have that

$$JD^{\perp}_X\xi = D_X J\xi. \tag{4.5}$$

Continuing our computation, using this and the Bianchi identity, we get

$$\begin{aligned} 3d\Omega(X^H,Y^H,Z^H) &= -g(\underline{R}_{XY}J\xi,Z) - g(\underline{R}_{YZ}J\xi,X) - g(\underline{R}_{ZX}J\xi,Y) \\ &= g(\underline{R}_{XY}Z,J\xi) + g(\underline{R}_{YZ}X,J\xi) + g(\underline{R}_{ZX}Y,J\xi)) \\ &= 0 \end{aligned} \tag{4.6}$$

$$\begin{aligned} 3d\Omega(X^H,Y^H,\eta^V) &= X^H\Omega(Y^H,\eta^V) + Y^H\Omega(\eta^V,X^H) \\ &\quad - \Omega([X^H,Y^H],\eta^V) - \Omega([Y^H,\eta^V],X^H) - \Omega([\eta^V,X^H],,Y^H) \\ &= \tilde{g}(\tilde{\nabla}_{X^H}Y^H,(J\eta)^H) + \tilde{g}(Y^H,\tilde{\nabla}_{X^H}(J\eta)^H) + \tilde{g}(\tilde{\nabla}_{Y^H}\eta^V,(JX)^V) \end{aligned}$$

$$
\begin{aligned}
&+\tilde{g}(\eta^V, \tilde{\nabla}_{Y^H}(JX)^V) - \Omega([X,Y]^H - (R^\perp_{XY}\xi)^V, \eta^V) \\
&- \Omega((D^\perp_Y\eta)^V, X^H) + \Omega((D^\perp_X\eta)^V, Y^H) \\
= \;& \tilde{g}((D_XY)^H, (J\eta)^H) + \tilde{g}(Y^H, (D_XJ\eta)^H) + \tilde{g}((D^\perp_Y\eta)^V, (JX)^V) \\
&+ \tilde{g}(\eta^V, (D^\perp_Y JX)^V) - \Omega([X,Y]^H, \eta^V) \\
&- \Omega((D^\perp_Y\eta)^V, X^H) + \Omega((D^\perp_X\eta)^V, Y^H) \\
= \;& \tilde{g}((D_XY)^H, (J\eta)^H) + \tilde{g}(\eta^V, (D^\perp_Y JX)^V) - \tilde{g}([X,Y]^H, (J\eta)^H) \\
= \;& \tilde{g}((D_XY)^H, \tilde{J}\eta^V) + \tilde{g}(\tilde{J}\eta^V, -(D_XY)^H) - \tilde{g}([X,Y]^H, \tilde{J}\eta^V) \\
= \;& 0
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
3d\Omega(X^H, \eta^V, \zeta^V) = \;& \eta^V\Omega(\zeta^V, X^H) + \zeta^V\Omega(X^H, \eta^V) \\
&- \Omega([X^H, \eta^V], \zeta^V) - \Omega([\eta^V, \zeta^V], X^H) - \Omega([\zeta^V, X^H], \eta^V) \\
= \;& \tilde{g}(\tilde{\nabla}_{\eta^V}\zeta^V, \tilde{J}X^H) + \tilde{g}(\zeta^V, \tilde{\nabla}_{\eta^V}(JX)^V) + \tilde{g}(\tilde{\nabla}_{\zeta^V}X^H, (J\eta)^H) \\
&+ \tilde{g}(X^H, \tilde{\nabla}_{\zeta^V}(J\eta)^H) - \Omega((D^\perp_X\eta)^V, \zeta^V) + \Omega((D^\perp_X\zeta)^V, \eta^V) \\
= \;& \frac{1}{2}\tilde{g}((\hat{R}_{\xi\zeta}X)^H, (J\eta)^H) + \frac{1}{2}\tilde{g}((\hat{R}_{\xi\zeta}J\eta)^H, X^H) \\
= \;& 0
\end{aligned} \tag{4.8}
$$

Finally,

$$
3d\Omega(\eta^V, \zeta^V, \delta^V) = 0 \tag{4.9}
$$

is immediate. This completes the proof. $\square$

Theorem 4.2 *Let L be a Lagrangian submanifold of a Kähler manifold (M^{2n}, J, g). Then, the following are equivalent:*

(1) *NL is Kähler.*
(2) *L has flat normal connection.*
(3) *L is flat.*

Proof: We compute the Nijenhuis torsion.

$$\begin{aligned}
[\tilde{J},\tilde{J}](X^H,Y^H) &= -[X^H,Y^H]+[\tilde{J}X^H,\tilde{J}Y^H]-\tilde{J}[(JX)^V,Y^H]-\tilde{J}[X^H,(JY)^V] \\
&= -[X^H,Y^H]+(JD^{\perp}_{Y}JX)^V-(JD^{\perp}_{X}JY)^V \\
&= -[X,Y]^H+(R^{\perp}_{XY}\xi)^V+(JD^{\perp}_{Y}JX-JD^{\perp}_{X}JY)^V \qquad (4.10)
\end{aligned}$$

Using (4.5), we continue this computation and get

$$\begin{aligned}
[\tilde{J},\tilde{J}](X^H,Y^H) &= -[X,Y]^H+(R^{\perp}_{XY}\xi)^V-(D_YX-D_XY)^H \\
&= -[X,Y]^H+(R^{\perp}_{XY}\xi)^V-[Y,X]^H \\
&= (R^{\perp}_{XY}\xi)^V \qquad (4.11)
\end{aligned}$$

$$\begin{aligned}
[\tilde{J},\tilde{J}](X^H,\eta^V) &= -[X^H,\eta^V]+[(JX)^V,(J\eta)^H]-\tilde{J}[(JX)^V,\eta^V]-\tilde{J}[X^H,(J\eta)^H] \\
&= -(D^{\perp}_{X}\eta)^V-(D^{\perp}_{J\eta}JX)^V-\tilde{J}([X,J\eta]^H-R^{\perp}_{X\,J\eta}\xi)^V) \\
&= -(D^{\perp}_{X}\eta)^V-(D^{\perp}_{J\eta}JX)^V-(J[X,J\eta])^V+(JR^{\perp}_{X\,J\eta}\xi)^H \\
&= -(\nabla_X\eta)^V-(A_\eta X)^V-(\nabla_{J\eta}JX)^V-(A_{JX}J\eta)^V \\
&\quad -(J\nabla_XJ\eta)^V+(J\nabla_{J\eta}X)^V+(JR^{\perp}_{X\,J\eta}\xi)^H \\
&= -(\nabla_X\eta)^V-(A_\eta X)^V-(\nabla_{J\eta}JX)^V-(A_{JX}J\eta)^V \\
&\quad +(\nabla_X\eta)^V+(\nabla_{J\eta}JX)^V+(JR^{\perp}_{X\,J\eta}\xi)^H \\
&= -(A_\eta X)^V-(A_{JX}J\eta)^V+(JR^{\perp}_{X\,J\eta}\xi)^H \qquad (4.12)
\end{aligned}$$

Again, using the Kähler condition and (4.5), we see that

$$\begin{aligned}
J[X,J\eta] &= -\nabla_X\eta-\nabla_{J\eta}JX \\
&= -D^{\perp}_{X}\eta-D^{\perp}_{J\eta}JX+A_\eta X+A_{JX}J\eta.
\end{aligned}$$

But, since $J[X,J\eta]$ is normal to L, we have

$$A_\eta X+A_{JX}J\eta=0 \qquad (4.13)$$

Thus, we have, from (4.12), that

$$\begin{aligned}[\tilde{J},\tilde{J}](X^H,\eta^V) &= (JR^{\perp}_{X\,J\eta}\xi)^H \\ &= (\underline{R}_{X\,J\eta}J\xi)^H \qquad (4.14)\end{aligned}$$

$$\begin{aligned}[\tilde{J},\tilde{J}](\eta^V,\zeta^V) &= -[\eta^V,\zeta^V]+[(J\eta)^H,(J\zeta)^H]-\tilde{J}[(J\eta)^H,\zeta^V]-\tilde{J}[\eta^V,(J\zeta)^H] \\ &= [(J\eta)^H,(J\zeta)^H]-\tilde{J}(D^{\perp}_{J\eta}\zeta)^V+\tilde{J}(D^{\perp}_{J\zeta}\eta)^V \\ &= [J\eta,J\zeta]^H-(R^{\perp}_{J\eta\,J\zeta}\xi)^V-(JD^{\perp}_{J\eta}\zeta)^H+(JD^{\perp}_{J\zeta}\eta)^H \\ &= [J\eta,J\zeta]^H-(R^{\perp}_{J\eta\,J\zeta}\xi)^V-(D_{J\eta}J\zeta)^H+(D_{J\zeta}J\eta)^V \\ &= -(R^{\perp}_{J\eta\,J\zeta}\xi)^V \qquad (4.15)\end{aligned}$$

In view of (4.11), (4.14), and (4.15), we conclude that $[\tilde{J},\tilde{J}]$ vanishes if and only if L has flat normal connection.

This together with the equations (4.6) - (4.9) shows that NL is Kähler if and only if L has flat normal connection.

Moreover, using (4.5), we have

$$R_{XY}J\xi = JR^{\perp}_{XY}\xi,$$

from which we easily see that L is flat if and only if NL has flat normal connection.
□

Remark. We have also shown, in the proof of Theorem 4.2, that $\tilde{J}$ on NL is integrable if and only if L has flat connection.

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