ABSTRACT

NORMALCY OF SUMS AND PRODUCTS
OF NORMAL FUNCTIONS AND REAL AND
COMPLEX HARMONIC NORMAL FUNCTIONS

By

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Let f and g be normal functions from the unit disk into the Riemann sphere. A normal function is a uniformly continuous function when using the hyperbolic metric in the disk and the chordal metric in the sphere. Necessary and sufficient conditions are established for $f+g$ to be normal and for fg to be normal. From these conditions some simpler sufficient conditions for $f+g$ to be normal and fg to be normal are established. Also inequalities and quotients involving normal functions are investigated. Employing the product results several theorems concerning normal functions and Bers' pseudo-analytic functions of the first kind are presented.

Real valued normal functions are investigated and used to obtain information about normal functions in general. The logarithm of a normal function, if continuous and single valued, is normal and conditions guaranteeing that the exponential of a normal function is normal are developed. Using the exponential function a necessary and sufficient condition for a real harmonic function to be not normal is proved and applied to show that if u and v are real harmonic, normal and $u-K < w < v+K$, then w harmonic is normal. The exponential function is also used to connect sums of real normal functions and products of certain normal functions.

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A subclass of the class of normal functions is defined and characterized. The functions of this subclass, called very normal, are closed under additions and projections to the real and imaginary axes and they are compared with uniformly normal functions.

A type of sequence, called (A) sequence, is defined for an arbitrary function f and it is shown that the complex valued harmonic function f is not normal if and only if f has a special type of (A) sequence. Also (A) sequences are used to obtain several other properties of functions which are, in some sense, not normal.

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To Nancy

and

To Mom and Dad

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I. INTRODUCTION

Lehto and Virtanen [15, p. 47] defined the function $f(z)$ to be normal as follows: If $f(z)$ is meromorphic in a simply connected domain G, then $f(z)$ is normal if and only if the family $\{f(S(z))\}$ is normal in the sense of Montel where $z' = S(z)$ denotes an arbitrary one-one conformal mapping of G onto itself. Lappan in [13] essentially defined a complex valued function $f(z)$ to be normal in a simply connected hyperbolic region G if and only if $f(z)$ is uniformly continuous using the hyperbolic metric in G and taking as the range of $f(z)$ the Riemann sphere W with chordal metric. Let D be the unit disk and C the unit circle.

DEFINITION 1.A. The <u>hyperbolic distance</u>, $\rho(z_1,z_2)$, between the points z_1 and z_2 in D is given by $(1/2)\log((1+u)/(1-u)) = \tanh^{-1}u$, where $u = |z_1-z_2|/|1-\bar{z}_1z_2|$. (See [9, Chapter 15].)

Since the hyperbolic distance between points in a simply connected hyperbolic domain G is defined to be the hyperbolic distance between their images in D under any conformal map from G onto D, we shall usually have D as the domain of our functions. Now we state more precisely Lappan's definition of a normal function.

DEFINITION 1.B. [13, p. 156]. A complex valued function $f(z)$ in D is a <u>normal function</u> if and only if, for each pair of sequences $\{z_n\}$ and $\{z'_n\}$ of points in D such that $\rho(z_n,z'_n) \to 0$, the convergence of $\{f(z_n)\}$ to a value α in W implies the convergence of $\{f(z'_n)\}$ to α.

In the case of meromorphic functions the above definition is equivalent to Lehto and Virtanen's definition of normality (also see theorem 3.A). We now state a few definitions and theorems we will use frequently in this thesis.

DEFINITION 1.C. (Lappan [12, p. 43].) Two sequences $\{z_n\}$ and $\{z_n'\}$ of points in D such that $\rho(z_n, z_n') \to 0$ as $n \to \infty$ are called close sequences.

DEFINITION 1.D. (Lappan [12, p. 44].) The meromorphic function $f(z)$ in D is said to have property (D) on a sequence $\{z_n\}$ of points in D if $\{f(z_n)\}$ converges and, for each complex number δ, there exists a sequence $\{z_n'\}$ close to $\{z_n\}$ such that $f(z_n') \to \delta$.

THEOREM 1.E. (Lappan [12, p. 45].) The meromorphic function $f(z)$ in D is not normal if and only if there exists a sequence of points in D on which $f(z)$ has property (D).

DEFINITION 1.F. (Lappan [12, p. 46].) A holomorphic function $f(z)$ in D is said to be uniformly normal if, for each M, there exists a finite number K such that, for each z_o in D, $\rho(z, z_o) < M$ implies $|f(z) - f(z_o)| < K$.

THEOREM 1.G. (Lappan [12, p. 46].) A uniformly normal holomorphic function is a normal function.

DEFINITION 1.H. By the cluster values of a function $f(z)$ on a sequence $\{z_n\}$ of points in D we mean those complex numbers α for which there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $f(z_{n_k}) \to \alpha$ as $k \to \infty$.

A function in a Hardy p-class, $p > 0$, or a function of bounded characteristic can be written as a sum or product of two normal functions, but sums and products of normal functions need not be normal (see Lappan [11]). It would be desirable to know when the sum or product of two or more normal functions is normal. The basic results in this direction are: (1) if $f(z)$ is normal and meromorphic in D, $g(z)$ a bounded holomorphic function in D, then $f(z)+g(z)$ is normal in D (Lehto and Virtanen [15, p. 53]); and (2) if $f(z)$ is normal and meromorphic in D, $g(z)$ a holomorphic function in D such that $0 < M_1 < |g(z)| < M_2 < \infty$, then $f(z)g(z)$ is normal in D (Lappan [11, p. 188]). The sum of two uniformly normal functions is normal (Lappan [12, p. 46]) but the product of uniformly normal functions may not be normal. For example, let $f(z) = \mathrm{Log}(1-z)$ and let $B_f(z)$ be a Blaschke product such that $f(z)B_f(z)$ is not normal (see Lappan [12, p. 48] and [11, p. 190]). In chapter two some necessary and sufficient conditions for sums and products of normal functions to be normal and sufficient conditions for functions bounded by normal functions to be normal will be presented.

In an effort to obtain a better understanding of holomorphic normal functions, chapter three investigates normal real valued harmonic functions (we drop the property of being harmonic when convenient) and their relationships to normal complex valued functions and some simple expressions involving normal functions. Chapter four develops a subclass of the class of normal functions which are closed under addition, and chapter five investigates normalcy and non-normalcy of real and complex harmonic functions through behavior on close sequences.

II. SUMS AND PRODUCTS OF NORMAL FUNCTIONS

We start with some theorems on sums and products of normal functions whose proofs employ Lappan's definition of normalcy through convergence on close sequences.

THEOREM 2.1. *Let* $f(z)$ *and* $g(z)$ *be normal functions in* D. *Then* $f(z)+g(z)$ *is normal in* D *if and only if for each sequence* $\{z_n\}$ *in* D *such that* $f(z_n) \to \infty$, $g(z_n) \to \infty$, *and* $\{f(z_n)+g(z_n)\}$ *converges to a complex value* α *(possibly* ∞*), the sum* $\{f(z_n')+g(z_n')\}$ *converges to* α *for each sequence* $\{z_n'\}$ *close to* $\{z_n\}$.

Proof. The necessity is obvious from the definition of normal functions. For sufficiency, let $\{z_n\}$ be a sequence where $f(z_n)+g(z_n) \to \alpha$. By considering appropriate subsequences, if necessary, we may assume that $f(z_n) \to \beta$ and $g(z_n) \to \gamma$, β and γ complex numbers. Let $\{z_n'\}$ be close to $\{z_n\}$. If $\beta = \infty = \gamma$, then $f(z_n')+g(z_n') \to \alpha$ by the condition of the theorem. Otherwise $\beta+\gamma$ is a well-defined complex number (possibly ∞) and by the normalcy of $f(z)$ and $g(z)$ we get that $g(z_n')+g(z_n') \to \beta+\gamma$. Hence $f(z)+g(z)$ is normal.

Using theorem 2.1 we can prove several sufficient conditions for $f(z)+g(z)$ to be normal in D where $f(z)$ and $g(z)$ are normal in D.

COROLLARY 2.2. *If* $f(z)$ *and* $g(z)$ *are normal in* D *and there is a number* M, $0 < M < \infty$, *such that the sets* $\{z:|f(z)| > M\}$ *and* $\{z:|g(z)| > M\}$ *are disjoint; then* $f(z)+g(z)$ *is normal in* D.

Proof. The hypotheses of the condition of theorem 2.1 are satisfied vacuously.

Let $f(z)$ and $g(z)$ be holomorphic functions in D which omit the value zero, and let $Q_n(f) = \{z: n < |f(z)| < 2n\}$, $R_n(f,g) = Q_n(f) \cap \{z: 1-1/n^{\frac{1}{2}} < |f(z)/g(z)| < 1+1/n^{\frac{1}{2}}\}$ and $T_n(f,g,\mu) = \{z: |\arg f(z)-\arg g(z)| \le \pi-\mu(n)\}$ where $\mu(n)$ is a non-increasing function on the positive integers such that $0 < \mu(n) < \pi$. For each z the arguments of $f(z)$ and $g(z)$ are chosen so as to have the difference, in absolute values, less than or equal to π. Hence the "arguments" in this chapter are not continuous functions of z.

COROLLARY 2.3. *Let $f(z)$ and $g(z)$ be normal holomorphic functions in D, each of which omits the value zero, and $\mu(n)$ such that $(n \sin \mu(n))/(2 \cos(\mu(n)/2))$ increases to infinity monotonically as n increases to infinity. If there is a positive integer N such that $n \ge N$ implies $R_n(f,g)$ is contained in $T_n(f,g,\mu)$, then $f(z)+g(z)$ is a normal holomorphic function in D.*

Proof. It suffices to show that if $\{z_m\}$ is a sequence of points in D and $f(z_m) \to \infty$, then $f(z_m)+g(z_m) \to \infty$. Let $\epsilon > 0$ be given and let N satisfy the hypotheses, and let $m > P \ge N$ and both $(P \sin \mu(P))/(2 \cos(\mu(P)/2)) > 1/\epsilon$ and $P(1/(P^{\frac{1}{2}}-1)) > 1/\epsilon$.

We may assume without loss of generality that z_m is in $Q_m(f)$. Either z_m is in R_m, or $|f(z_m)/g(z_m)| \le 1-1/m^{\frac{1}{2}}$, or $|f(z_m)/g(z_m)| \ge 1+1/m^{\frac{1}{2}}$. If z_m is in R_m, then z_m is in $T_m(f,g,\mu)$. Since $|g(z_m)| > (1/2)m$ for $m > 4$, by computing the length of the diagonal of the parallelogram of sides m and $(1/2)m$, with interior angles $\mu(m)$ and $\pi-\mu(m)$, and noting that this diagonal is shorter than $|f(z_m)+g(z_m)|$, we see that

$|f(z_m)+g(z_m)| > (m \sin \mu(m))/(2 \cos(\mu(m)/2)) > (P \sin \mu(P))/(2 \cos(\mu(P))/2)$ $> 1/\varepsilon$.

If $|f(z_m)/g(z_m)| \le 1-1/m^{\frac{1}{2}} = \frac{m^{\frac{1}{2}}-1}{m^{\frac{1}{2}}}$, then $|f(z_m)| \le ((m^{\frac{1}{2}}-1)/m^{\frac{1}{2}})|g(z_m)| < |g(z_m)|$ and $|g(z_m)| > (m^{\frac{1}{2}}/(m^{\frac{1}{2}}-1))|f(z_m)|$. So $|f(z_m)+g(z_m)| \ge |g(z_m)| - |f(z_m)| \ge |f(z_m)|(m^{\frac{1}{2}}/(m^{\frac{1}{2}}-1)-1) >$ $m(1/(m^{\frac{1}{2}}-1)) > P(1/(P^{\frac{1}{2}}-1)) > 1/\varepsilon$. If $|f(z_m)/g(z_m)| \ge 1+1/m^{\frac{1}{2}}$, then $|f(z_m)+g(z_m)| > 1/\varepsilon$ similarly.

We note that the $\frac{1}{2}$ in the definition of $R_n(f,g,\mu)$ could be replaced by any number between zero and one to obtain a similar result. In the proof we show that if $f(z_m) \to \infty$, then $f(z_m)+g(z_m) \to \infty$ also. In doing this, we use the arguments of $f(z)$ and $g(z)$ only when $|f(z)|$ and $|g(z)|$ are large and hence we don't need to require that $f(z)$ and $g(z)$ be non-zero. If $f(z)$ and $g(z)$ are meromorphic, we exclude those $\{z_n\}$ in D which are poles of $f(z)$ or $g(z)$ and consider neighborhoods $\{U_n\}$ in D of $\{z_n\}$ such that $P < |f(z)| < \infty$, $g(z) \ne \infty$ for all z in $U_n - \{z_n\}$, where P is as in above proof. As before with given $\varepsilon > 0$, $|f(z)+g(z)| > 1/\varepsilon$ for every z in $U_n - \{z_n\}$. Since $f(z)$ and $g(z)$ are continuous at z_n, $|f(z_n)+g(z_n)| \ge 1/\varepsilon$.

There exist normal holomorphic functions $f(z)$ and $g(z)$ whose sum is normal but the functions fail to satisfy the hypotheses of corollary 2.3. Hence the hypotheses of corollary 2.3 are not necessary. For example, let $f(z) = (z+1)/(z-1)$, and $g(z) = (-1)(z+1)/(z-1) + z$ in D. Then $f(z)+g(z) = z$ which is normal; but, as evident, there is no positive integer N such that $n \ge N$ implies $R_n(f,g)$ is contained in $T_n(f,g,\mu)$ for any $\mu(n)$ satisfying the hypotheses.

COROLLARY 2.4. *Let* $f(z)$ *and* $g(z)$ *be normal holomorphic functions in* D. *If there exist* δ, M, t, R *where* $0 < \delta < \pi$, $1 < t < 2$, $0 < M < \infty$, $0 \le R < 1$ *such that the set*

$\{z\colon M < |f(z)| < \infty\} \cap \{z\colon 1/t < |f(z)/g(z)| < t\} \cap \{z\colon R < |z| < 1\}$ *is contained in* $\{z\colon |\arg f(z) - \arg g(z)| \le \pi - \delta\}$; *then* $f(z)+g(z)$ *is a normal holomorphic function in* D.

COROLLARY 2.5. *Let* $f(z)$ *and* $g(z)$ *be normal holomorphic functions in* D. *If there exist* δ, M, R *where* $0 < \delta < \pi$, $0 < M < \infty$, $0 \le R < 1$ *such that the set*

$\{z\colon M < |f(z)| < \infty\} \cap \{z\colon M < |g(z)| < \infty\} \cap \{z\colon R < |z| < 1\}$ *is contained in* $\{z\colon |\arg f(z) - \arg g(z)| \le \pi - \delta\}$; *then* $f(z)+g(z)$ *is a normal holomorphic function in* D.

As before, the arguments of $f(z)$ and $g(z)$ in corollaries 2.4 and 2.5 are chosen so as to have the difference, in absolute values, less than or equal to π. Then the corollaries 2.4 and 2.5 follow easily from corollary 2.3.

THEOREM 2.6. *Let* $f(z)$, $g(z)$, $a(z)$, *and* $b(z)$ *be normal holomorphic functions in* D *with* $a(z)f(z)$ *and* $b(z)g(z)$ *normal*, $a(z)$ *and* $b(z)$ *uniformly bounded in* D *by* A *and* B *respectively. Let* $D_n = \{z\colon n < |f(z)| < \infty,\ n < |g(z)| < \infty,\ |z| > 1-1/n\}$ *and* $E_n = \{z\colon |\arg a(z) - \arg b(z)| \le \beta(n)\}$, *where* $\beta(n)$ *is a real valued function on the positive integers such that* $\mu(n)-\beta(n)$ *is non-increasing*, $0 \le \mu(n)-\beta(n) \le \pi$, *with* $\mu(n)$ *and* $T_n(f,g,\mu)$ *as in corollary* 2.3. *If* $n \sin(\mu(n)-\beta(n))$ *montonically increases to infinity with* n *and there exists a positive integer* N *such that* $D_N \cap (D - (E_N \cup T_N(f,g,\mu)))$ *is empty, then* $a(z)f(z) + b(z)g(z)$ *is a normal holomorphic function in* D.

Proof. Let $N^* = \max(AN, BN, N)$ and let z be in $\{z\colon N^* < |a(z)f(z)| < \infty,\ N^* < |b(z)g(z)| < \infty,\ |z| > 1-1/N^*\}$. Then z is in D_N so z is not in $D - (E_N \cup T_N(f,g,\mu))$. Hence $|\arg a(z)f(z) - \arg b(z)g(z)| \le \pi - (\mu(N)-\beta(N))$ and we may complete the proof by applying corollary 2.5 to $a(z)f(z)$ and $b(z)g(z)$ with $M = N^*$, $R = 1-1/N^*$, and $\delta = \mu(N)-\beta(N)$.

We can also get a necessary and sufficient condition for a product of normal functions to be normal by considering again convergence on close sequences.

THEOREM 2.7. *Let $f(z)$ and $g(z)$ be normal functions in D. Then $f(z)g(z)$ is normal in D if and only if for each sequence $\{z_n\}$ in D such that $f(z_n) \to 0$, $g(z_n) \to \infty$ (or $f(z_n) \to \infty$, $g(z_n) \to 0$) and $\{f(z_n)g(z_n)\}$ converges to a complex value α (possibly ∞), the product $\{f(z_n')g(z_n')\}$ converges to α for each sequence $\{z_n'\}$ close to $\{z_n\}$.*

Proof. The necessity is immediate. For sufficiency, let $\{z_n\}$ be a sequence where $f(z_n)g(z_n) \to \alpha$. By considering appropriate subsequences, if necessary, we may assume that $f(z_n) \to \beta$ and $g(z_n) \to \gamma$, β and γ complex numbers. Let $\{z_n'\}$ be close to $\{z_n\}$. If $\beta = 0$, $\gamma = \infty$, or if $\beta = \infty$, $\gamma = 0$, we have $f(z_n')g(z_n') \to \alpha$ from the condition of the theorem. Otherwise $\beta\gamma$ is a well defined complex number (possibly ∞) and by the normalcy of $f(z)$ and $g(z)$, $f(z_n')g(z_n') \to \beta\gamma$ also. Hence $f(z)g(z)$ is normal in D.

The following corollary follows easily from theorem 2.7.

COROLLARY 2.8. *Let $f(z)$ and $g(z)$ be normal functions in D. If there exists finite positive constants K_f, K_g, M_f, M_g such that the sets $D_1 = \{z\colon |f(z)| > K_f;\ |g(z)| > K_g\}$ and*

$D_2 = \{z: |f(z)| < M_f; |g(z)| < M_g\}$ cover D (i.e. $D = D_1 \cup D_2$), then $f(z)g(z)$ is a normal function in D.

Also the hypotheses of the following theorem of Zinno [18, pp. 160-161] vacuously satisfy the hypotheses of theorem 2.7.

THEOREM 2.A. *Let $f(z)$ and $g(z)$ be two normal meromorphic functions in D. Let a_v and a_v' be zeros of $f(z)$ and $g(z)$ respectively and let b_v and b_v' be poles of $f(z)$ and $g(z)$ respectively. Suppose that*

(1) $$\inf_{\substack{v=1,2,.\\u=1,2,.}} \rho(a_v,b_u') > 0 \text{ and } \inf_{\substack{v=1,2,.\\u=1,2,.}} \rho(a_v',b_u) > 0$$

(inf over an empty set is infinite)

and

(2) *for any positive number ρ there exists a positive number m_ρ such that*

$$|f(z)| < m_\rho \text{ for } z \text{ in } D - \bigcup_{v=1}^{\infty} U(b_v,\rho)$$

$$|g(z)| < m_\rho \text{ for } z \text{ in } D - \bigcup_{v=1}^{\infty} U(b_v',\rho)$$

$$|f(z)| > 1/m_\rho \text{ for } z \text{ in } D - \bigcup_{v=1}^{\infty} U(a_v,\rho)$$

and

$$|g(z)| > 1/m_\rho \text{ for } z \text{ in } D - \bigcup_{v=1}^{\infty} U(a_v',\rho)$$

$$(U(z,\delta) = \{\zeta: \rho(z,\zeta) < \delta\})$$

Then the product $f(z)g(z)$ is a normal meromorphic function in D.

By excluding the possibility of sequences satisfying property (D), we can get another condition to guarantee the sum of two normal meromorphic functions to be normal and get some ideas on what is sufficient for functions bounded by normal functions to be normal. Again in the following theorem the arguments of the functions $f(z)$ and $g(z)$ are chosen so as to have the difference in absolute values less than or equal to π.

THEOREM 2.9. *Let* $f(z)$ *and* $g(z)$ *be two normal meromorphic functions in* D. *Let* $\delta > 0$ *be arbitrary and for each sequence* $\{z_n\}$ *for which* $f(z_n) \to \infty$, $f(z_n) \neq \infty$, $g(z_n) \neq \infty$, *let there exist* $\alpha(n)$, *a non-increasing function on the positive integers, such that* $0 \le \alpha \le \pi$ *and* $|f(z_n)|\sin\alpha(n) \ge \delta$ *for all* n *greater than some positive integer* N. *If*

$$1-1/|f(z_n)| < |f(z_n)/g(z_n)| < 1+1/|f(z_n)| \quad \textit{implies}$$

$$|\arg f(z_n)-\arg g(z_n)| \le \pi-\alpha(n) \quad \textit{for } n > N, \textit{ then}$$

$f(z)+g(z)$ *is a normal meromorphic function in* D.

Proof. It is clear that if $\{z_n\}$ is a sequence where $\{f(z_n)\}$ has no unbounded subsequences and $\{f(z_n)+g(z_n)\}$ converges, then $\{z_n\}$ is not a sequence for which $f(z)+g(z)$ has property (D). Hence let $\{z_n\}$ be a sequence for which $\{f(z_n)\}$ diverges to ∞ and $\{f(z_n)+g(z_n)\}$ converges to a finite complex value or diverges to ∞. Let $\{z_n'\}$ be any sequence close to $\{z_n\}$ so that $f(z_n') \to \infty$ also. Fix $n > N$ and consider the following cases.

Case I. $f(z_n') \neq \infty$.

(a.) If $g(z_n') = \infty$, then $|g(z_n')+f(z_n')| = \infty$.

(b.) If $g(z_n') \neq \infty$ and $1-1/|f(z_n')| < |f(z_n')/g(z_n')| < 1+1/|f(z_n')|$, then $|f(z_n')+g(z_n')| \ge |f(z_n')|\sin\alpha(n) \ge \delta > 0$ for $n > N$, the δ and N in the hypotheses of the theorem.

(c.) If $g(z_n') \neq \infty$ and $|f(z_n')/g(z_n')| \ge 1+1/|f(z_n')|$, then $|g(z_n')| \le (|f(z_n')|/(1+|f(z_n')|))|f(z_n')|$. Hence $|f(z_n')+g(z_n')| \ge |f(z_n')| - |g(z_n')| \ge$

$|f(z_n')|(1-|f(z_n')|/(1+|f(z_n')|)) = 1/(1+1/|f(z_n')|) \geq$ $1/2 > 0$ for n sufficiently large.

(d.) If $g(z_n') \neq \infty$ and $|f(z_n')/g(z_n')| \leq 1-1/|f(z_n')$, then $(|f(z_n')|/(|f(z_n')|-1))|f(z_n')| \leq |g(z_n')|$. Hence $|f(z_n')+g(z_n')| \geq |g(z_n')| - |f(z_n')| \geq$ $|f(z_n')|(|f(z_n')|/(|f(z_n')|-1)) = 1/(1-1/|f(z_n')|) \geq$ $1 > 0$ for n sufficiently large.

Case II. $f(z_n') = \infty$.

(a.) If $g(z_n') \neq \infty$, then $|f(z_n')+g(z_n')| = \infty$.

(b.) If $g(z_n') = \infty$, then there exists a neighborhood U containing z_n' such that $f(z) \neq \infty$ and $g(z) \neq \infty$ for all z in $U - \{z_n'\}$. Then there exists $\{z_{n,m}\}_{m=1}^{\infty}$ contained in U, $z_{n,m} \to z_n'$ as $m \to \infty$, where $|f(z_{n,m})| > m$ and we may proceed as in case I. Thus there is no sequence on which $f(z)+g(z)$ has property (D).

We know that a meromorphic function bounded above (or below) by a normal meromorphic function is the product and sum of normal meromorphic functions. We can say more than this if the function is appropriately bounded both above and below.

THEOREM 2.10. Let $f(z)$, $g(z)$, and $h(z)$ be meromorphic functions in D such that $g(z)$ and $h(z)$ are normal; and let K_1 and K_2 be constants such that $|g(z)|-K_1 \leq |f(z)| \leq |h(z)|+K_2$ for each z in D. If $\liminf_{n\to\infty} |g(z_n)| > K_1$ for each sequence $\{z_n\}$ in D for which $h(z_n) \to \infty$, then $f(z)$ is normal in D.

Proof. Assume $f(z)$ is not normal. Then there exists a sequence $\{z_n\}$ such that $f(z)$ has property (D) on $\{z_n\}$. Then $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ for which there exist sequences $\{z_{n_k}'\}$ and

$\{z''_{n_k}\}$, both close to $\{z_{n_k}\}$, satisfying $f(z'_{n_k}) \to 0$, $f(z''_{n_k}) \to \infty$. But then $\liminf_{n\to\infty} |g(z'_{n_k})| \le K_1$ and $h(z''_{n_k}) \to \infty$. Since $g(z)$ and $h(z)$ are normal, this means that $\liminf_{n\to\infty} |g(z_{n_k})| \le K_1$ and $h(z_{n_k}) \to \infty$, in violation of the hypotheses. Thus there is no sequence $\{z_n\}$ on which $f(z)$ has property (D), and the theorem is proved.

REMARK 2.11. Since there are non-normal functions of bounded characteristic the condition above on sequences is needed. For example, Bagemihl and Seidel [4, p. 7] construct a holomorphic function $f(z)$ of bounded characteristic such that $f(z_n) = 0$ for $z_n = 1-1/n^2$ and $f(z'_n) \to \infty$ where $z'_n = (1/2)(z_n+z_{n+1})$ and an elementary calculation shows that $\{z_n\}$ and $\{z'_n\}$ are close. Hence $f(z)$ is not normal. Their function is $B(z)/\exp((-1-z)/(1-z))$ where $B(z) = \prod_{n=1}^{\infty} (z_n-z)/(1-z_n z)$. But $|B(z)| \le |f(z)| \le |1/\exp((-1-z)/(1-z))|$, and $B(z)$ and $1/\exp((-1-z)/(1-z))$ are normal holomorphic functions. Since $B(z'_n) \to 0$ and $1/\exp((-1-z'_n)/(1-z'_n)) \to \infty$, there exists a sequence, namely $\{z'_n\}$, not fullfilling the hypotheses of theorem 2.10.

COROLLARY 2.12. Let $f(z)$ and $h(z)$ be meromorphic functions in D and $|h(z)|-K_1 \le |f(z)| \le |h(z)|+K_2$. Then $f(z)$ is normal if and only if $h(z)$ is normal.

THEOREM 2.13. Let $f(z) = h_1(z)/h_2(z)$ be the quotient of two bounded holomorphic functions in D with no common zeros. If $f(z)$ is not normal, then there is a sequence $\{z_n\}$ of points in D such that $h_1(z_n) \to 0$ and $h_2(z_n) \to 0$.

Proof. We prove the contrapositive. Let $M_1 \ge |h_1(z)|$ and $M_2 \ge |h_2(z)|$. Then $|h_1(z)/M_2| \le |f(z)| \le |M_1/h_2(z)|$ and $h_1(z)/M_2$ and $M_1/h_2(z)$ are normal. Since there is no sequence $\{z_n\}$ where $h_1(z_n)/M_2 \to 0$ and $M_1/h_2(z) \to \infty$, $f(z)$ is normal by theorem 2.10.

COROLLARY 2.14. *Let* $h_1(z) = B_1(z)\exp(g_1(z))$, $h_2(z) = B_2(z)\exp(g_2(z))$ *be holomorphic functions with* $B_1(z)$ *and* $B_2(z)$ *Blaschke products; and* M, $0 < M < \infty$, *such that* $-M \le \operatorname{Re}(g_1(z)-g_2(z)) \le M$. *Let* $L_i(\delta) = \{z: |B_i(z)| < \delta\}$, $i=1,2$. *If there exists* δ' *such that*

$$\rho(L_1(\delta'),L_2(\delta')) = \inf\{\rho(z_1,z_2): z_1 \text{ in } L_1(\delta'),\ z_2 \text{ in } L_2(\delta')\} \ge n > 0,$$

then $f(z) = h_1(z)/h_2(z)$ *is normal*.

The following corollary gives a restatement of a result of Cima [7, p. 769].

COROLLARY 2.15. *Let* $F(z) = B_1(z,a_n)/B_2(z,b_n)$, $B_1(z,a_n)$ *and* $B_2(z,b_n)$ *are Blaschke products with zeros* $\{a_n\}$ *and* $\{b_n\}$ *respectively*. *If the cluster points of the zeros of* $B_1(z,a_n)$ *and* $B_2(z,b_n)$ *are disjoint, then* $F(z)$ *is normal*.

Theorem 2.7 may be applied to obtain some properties of normal pseudoanalytic functions in D. The following material may be found in detail in [5] and [6].

DEFINITION 2.B. [5, p. 18]. Let w(z) be a function from D into the complex w-plane which possesses continuous partial derivatives. If there exists a constant K such that $|w_{\bar{z}}(z)| \le K|w(z)|$, we say w(z) is *approximately analytic*.

Let D_o be a domain containing D.

DEFINITION 2.C. [6, p. 215]. Two continuous functions F(z), G(z) defined in D_o are said to form a *generating pair* if $\operatorname{Im}\{\overline{F(z)}G(z)\} > 0$ for z in D_o.

Every function w(z) defined in D admits the unique representation $w(z) = \phi(z)F(z)+\Psi(z)G(z)$ with real functions $\phi(z)$, $\Psi(z)$, [6, p. 216].

DEFINITION 2.D. [6, p. 217]. The function $w(z) = \phi(z)F(z)+\Psi(z)G(z)$ is said to possess at the point z_o in D the (F,G)-derivative, denoted $\dot{w}(z_o)$ or $\left.\frac{d_{(F,G)}w(z)}{dz}\right|_{z=z_o}$ if the limit

$$\dot{w}(z_o) = \lim_{z\to z_o} \frac{w(z)-\phi(z_o)F(z)-\Psi(z_o)G(z)}{z-z_o}$$ exists and is finite.

DEFINITION 2.E. [6, p. 219]. A function w(z) possessing an (F,G)-derivative at all points of the domain D is called <u>regular (F,G)-pseudoanalytic of the first kind in</u> D or simply <u>pseudoanalytic</u> if there is no danger of confusion.

THEOREM 2.F. [5, p. 18]. <u>Every pseudoanalytic function is also approximately analytic</u>.

DEFINITION 2.G. [5, p. 24]. We call two functions w(z) and f(z) defined in D <u>similar</u> if there exists a function S(z) which is continuous and different from zero on the closure of D and such that f(z) = S(z)w(z) in D.

THEOREM 2.H. [5, p. 24-25]. SIMILARITY PRINCIPLE <u>Every approximately analytic function</u> w(z) <u>in</u> D <u>possesses a similar function</u> f(z) <u>which is analytic in</u> D.

THEOREM 2.16. <u>If</u> f(z) <u>and</u> w(z) <u>are similar functions in</u> D, <u>and if one of them is normal, then the other one is normal also</u>.

<u>Proof</u>. Let S(z) be such that f(z) = S(z)w(z) where S(z) is continuous non-zero on the closure of D. There exists an M such that $1/M < |S(z)| < M$ for all z in the closure of D. Hence theorem 2.7 implies f(z) is normal if and only if w(z) is normal since S(z) and 1/S(z) are normal in D.

COROLLARY 2.17. <u>If</u> w(z) <u>is approximately analytic and bounded (or bounded from zero) in</u> D, <u>then</u> w(z) <u>is normal in</u> D.

Proof. Let $f(z)$ be the analytic function in D similar to $w(z)$. Then $f(z) = S(z)w(z)$ is bounded (or bounded from zero) in D. Thus $f(z)$ is normal and so is $w(z)$ normal.

COROLLARY 2.18. *If* $w(z)$ *is a normal approximately analytic function in* D *and has an asymptotic value* α *at* ζ *on* C, *then* $w(z)$ *has a Fatou value* α *at* ζ *on* C.

Proof. Let Γ be the Jordan arc in $D \cup \{\zeta\}$ on which $w(z)$ tends to the limit α. Let $f(z)$ be the analytic function similar to $w(z)$. Since $S(z)$ in $f(z) = S(z)w(z)$ is uniformly continuous on $D \cup C$, $S(z)$ tends to a limit, say β, (finite and non-zero) on Γ also. Hence $f(z)$ tends to a limit, $\beta\alpha$, on Γ and so has a Fatou value $\beta\alpha$ at ζ on C. $S(z)$ also has β as a Fatou value at ζ. Therefore, since $w(z) = f(z)/S(z)$, $w(z)$ has a Fatou value α at ζ.

We could show, with proofs similar to those above, that normal approximately analytic functions have many of the properties related to Fatou values of normal analytic functions and that normal pseudoanalytic functions have some of the same identity and uniqueness properties as normal analytic functions.

III. REAL NORMAL FUNCTIONS

It is easy to see that some simple expressions of normal functions are still normal functions. For example:

LEMMA 3.1. *Let* $f(z)$ *be a normal function in* D *into the* ξ-*plane. If* $\phi(\xi)$ *is a continuous function on the closure of* $f(D)$ *into the* w-*plane, then the function* $w(z) = \phi(f(z))$ *mapping* D *into the* w-*plane is normal.*

Proof. Let $\{z_n\}$ and $\{z_n'\}$ be any two close sequences in D and $\phi(f(z_n))$ converge to some complex value α in the w-plane. Since $f(z)$ is normal, the cluster values of $f(z)$ on $\{z_n\}$ and $\{z_n'\}$ are identical and thence, since $\phi(\xi)$ is continuous, $\phi(f(z_n')) \to \alpha$.

LEMMA 3.2. *If* $u(z)$ *and* $v(z)$ *are real valued functions in* D, *then* $f(z) = \exp(u(z)+iv(z))$ *normal in* D *implies* $u(z)$ *is normal in* D.

REMARK 3.3. An equivalent formulation of lemma 3.2 is that $f(z)$ normal implies $\log|f(z)|$ is normal.

Proof of Lemma 3.2. The mappings ϕ_1 on the $\xi = f(z)$ plane to the non-negative real axis (including $+\infty$) by $\phi_1(\xi) = |\xi|$ and ϕ_2 on the non-negative real axis $x = |\xi|$ to the closed real numbers by $\phi_2(x) = \log x$ are continuous. Hence $u(z) = \phi_2(\phi_1(f(z)))$ is normal by lemma 3.1.

THEOREM 3.4. *If* $u(z)$ *and* $v(z)$ *are real valued normal functions in* D *such that* $v(z)$ *is bounded, then* $f(z) = \exp(u(z)+iv(z))$ *is normal in* D.

Proof. Since exp x is a continuous function whose domain is the real line, exp(u(z)) is normal. If exp(iv(z)) is normal, then f(z) = exp(u(z))exp(iv(z)) is normal by corollary 2.8. Let $\{z_n\}$ and $\{z'_n\}$ be two close sequences in D and $\exp(iv(z_n))$ converge to a complex value α. Let I denote the cluster values of $\{v(z_n)\}$ and I' the cluster values of $\{v(z'_n)\}$. If x is in I, then $\exp(ix) = \alpha$. Since v(z) is normal, I = I', and thus $\exp(iv(z'_n)) \to \alpha$ by x in I' implies $\exp(ix) = \alpha$. Therefore exp(iv(z)) is normal and the theorem is proved.

The following example shows that the condition v(z) be bounded is needed.

EXAMPLE 3.5. There exist u(z) and v(z) normal real valued such that f(z) = exp(u(z)+iv(z)) is not normal. Let $v(z) = \mathrm{Re}(1/(1-z))$ for z in D. Then v(z) is harmonic and normal since $v(z) > 1/2$ for all z in D. Let $z_n = 1-1/2n\pi$ and $z'_n = 1-1/(2n+\eta)\pi$, $0 < \eta < 2$. Then $v(z_n) = 2n\pi \to +\infty$, $v(z'_n) = (2n+\eta)\pi \to +\infty$, and by a simple computation $\rho(z_n, z'_n) = (1/2)\log(4n\pi+2\eta-1)/(4n\pi-1) \to 0$ as $n \to \infty$ independent of η. If $u(z) = 1$, then $f(z) = \exp(u(z)+iv(z))$ is such that $f(z_n) = e$ and, for $\eta = 1/2$, $f(z'_n) = ie$. Therefore f(z) is not normal even though u(z) and v(z) are normal and harmonic in D.

The condition on the boundedness may be dropped if we are given some other appropriate information. Note we don't even require v(z) to be normal.

THEOREM 3.6. Let u(z) be a real harmonic function in D and v(z) its harmonic conjugate. Then f(z) = exp(u(z)+iv(z)) is normal in D if and only if u(z) is normal in D.

Proof. The "if" part was done by Lappan in [14, p. 110] and the "only if" is lemma 3.2.

The following lemma is an obvious corollary to theorem 2.1 which we will refer to in the discussion on logarithms of normal functions.

LEMMA 3.7. _If_ $u(z)$ _and_ $v(z)$ _are real valued normal functions in_ D, _then_ $f(z) = u(z)+iv(z)$ _is also normal in_ D.

THEOREM 3.8. _If_ $f(z)$ _is a normal function in_ D _such that_ $\log f(z)$ _is a well defined, single valued, continuous function in_ D, _then_ $\log f(z)$ _is normal in_ D.

Proof. Let $\{z_n\}$ and $\{z_n'\}$ be close sequences in D and $\log f(z_n) \to \alpha$, α a complex number. Taking subsequences if necessary, we assume $f(z_n)$ converges to some complex value a. If $a = 0$ or ∞, then $\alpha = \pm\infty+ib = \infty$ and $\log f(z_n') \to \infty$ also since $f(z_n') \to a$ by $f(z)$ normal. Hence, without loss of generality, we assume neither $f(z_n)$ nor $f(z_n')$ tend to zero or infinity; so $\arg f(z_n)$ and $\arg f(z_n')$ are defined for every n sufficiently large as well as any cluster values of $\{\arg f(z_n)\}$ or $\{\arg f(z_n')\}$. Assume $\log f(z_n')$ tends to some complex value β. Then $\mathrm{Re}\alpha = \mathrm{Re}\beta$ since $\log|f(z)|$ is normal (lemma 3.2); and $\arg f(z_n) \to \mathrm{Im}\alpha$, $\arg f(z_n') \to \mathrm{Im}\beta$. If $|\mathrm{Im}\alpha| = |\mathrm{Im}\beta| = \infty$, then $\alpha = \beta = \infty$; so assume one is finite, say $|\mathrm{Im}\alpha| < \infty$. Connect z_n and z_n' by the hyperbolic straight line Γ_n; the hyperbolic length of Γ_n, $l(\Gamma_n)$, equals $\rho(z_n,z_n')$ which converges to zero as $n \to \infty$. Since $f(z_n) \to a \neq 0$ and $f(z)$ is normal, there exists an N such that $n \geq N$ implies: If z is on Γ_n, then $|f(z)-a| < |a|/2$ as well as $|f(z_n')-a| < |a|/2$. As z on Γ_n, $n \geq N$, travels from z_n to z_n', $\arg f(z)$ changes continuously from arg $f(z_n)$ to arg $f(z_n')$. Since $|f(z)-a| < |a|/2$, $|\arg f(z)-\arg f(z_n)| < \pi/2$ for any z on Γ_n. Since $f(z_n') \to a \neq 0$ also, $\arg f(z_n') \to \operatorname{Im}\alpha$. So $\alpha = \beta$ and log f(z) is normal.

LEMMA 3.9. _If_ f(z) _has a normal logarithm with_ arg f(z) _bounded in_ D, _then_ $\log|f(z)|$ _and_ $|f(z)|$ _are normal in_ D.

Proof. Let $\{z_n\}$ and $\{z_n'\}$ be close sequences and $\log|f(z_n)| \to \alpha$, α a real number.

Case 1. $|\alpha| = \infty$. If $\alpha = +\infty$, then Re log $f(z_n) > M$, any preassigned real number M, for all n greater than some positive integer N. Hence Re log $f(z_n') \to \infty$ as $n \to \infty$, i.e. $\log|f(z_n')| \to +\infty$. The proof is similar if $\alpha = -\infty$.

Case 2. $|\alpha| < \infty$. Then assume log $f(z_n) \to \alpha+i\beta \neq \infty$ since arg f(z) is bounded. By log f(z) being normal, log $f(z_n') \to \alpha+i\beta$ and so $\log|f(z_n')| \to \alpha$. Therefore $\log|f(z)|$ is normal. But $\log|f(z)|$ normal implies $|f(z)|$ normal by lemma 3.1.

THEOREM 3.10. _If_ f(z) _has a normal logarithm with normal_ arg f(z) _bounded in_ D, _then_ f(z) _is normal in_ D.

Proof. The theorem is just a corollary of lemma 3.9 and theorem 3.4.

The function $f(z) = |2+z|\exp(i\operatorname{Re}(1/(1-z)))$ is not normal in D but $\log f(z) = \log|2+z|+i\operatorname{Re}(1/(1-z))$ is normal in D, from example 3.5 and lemma 3.7. Hence theorem 3.10 needs the hypothesis that arg f(z) be bounded.

Let us now consider some applications of using exponentials and logarithms in the study of real and complex valued normal functions.

LEMMA 3.13. _Let_ $u(z)$ _be a harmonic real valued function in_ D _and let there exist two close sequences_, $\{z_n\}$ _and_ $\{z_n'\}$, _such that_ $u(z_n) \to \alpha$, $u(z_n') \to \beta$, α _and_ β _unequal real numbers, as_ $n \to \infty$. _Then for each real value (including_ $+\infty$ _and_ $-\infty$), _there exists a sequence_ $\{z_k^\delta\}$ _close to a subsequence_ $\{z_{n_k}\}$ _of_ $\{z_n\}$ _such that_ $u(z_k^\delta) \to \delta$ _as_ $k \to \infty$.

REMARK 3.14. As is evident from the reference to be cited in the proof of lemma 3.13, we can require $\{z_k^\delta\}$ to be such that $u(z_k^\delta) = \delta$ except for $\delta = +\infty$ or $-\infty$.

REMARK 3.15. Calling the above divergence property (Dh), we have the following characterization of not normal real harmonic functions: $u(z)$ _has property_ (Dh) _in_ D _if and only if_ $u(z)$ _is not normal in_ D. We will elaborate on a variant of this in chapter five.

Proof of Lemma 3.13. Let $v(z)$ be a harmonic conjugate to $u(z)$ and $F(z) = \exp(u(z)+iv(z))$. $F(z)$ is a holomorphic function in D such that $|F(z_n)| \to \exp \alpha$ and $|F(z_n')| \to \exp \beta$, $\exp \alpha \neq \exp \beta$. So there are close subsequences $\{z_{n_p}\}$ and $\{z_{n_p}'\}$ of $\{z_n\}$ and $\{z_n'\}$ respectively where $F(z_{n_p}) \to \alpha^*$, $F(z_{n_p}') \to \beta^*$, and $\alpha^* \neq \beta^*$. By Lappan [12, p. 44] there is a sequence $\{w_k^\delta\}$ such that $\rho(w_k^\delta, z_{n_{p_k}}) \to 0$ and $F(w_k^\delta) \to \exp \delta$ $(\exp(+\infty) = +\infty, \exp(-\infty) = 0)$. So $u(w_k^\delta) \to \delta$.

With lemma 3.13 we may prove a theorem with real harmonic functions similar to theorem 2.10.

THEOREM 3.16. _Let_ $u(z)$, $v(z)$, $w(z)$ _be real harmonic functions in_ D _such that_ $v(z)$ _and_ $w(z)$ _are normal, and let_ K_1 _and_ K_2 _be constants such that_ $v(z)+K_1 \le u(z) \le w(z)+K_2$ _for each_ z _in_

D. *If* $\liminf_{n\to\infty} v(z_n) > -\infty$ *for each sequence* $\{z_n\}$ *of points in* D *for which* $w(z_n) \to +\infty$, *then* u(z) *is normal in* D.

Proof. Assume u(z) is not normal in D. Then there exists a sequence $\{z_n\}$ such that u(z) has property (Dh) on $\{z_n\}$. Then $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ for which there exist sequences $\{z'_{n_k}\}$ and $\{z''_{n_k}\}$ both close to $\{z_{n_k}\}$, satisfying $u(z'_{n_k}) \to -\infty$, $u(z''_{n_k}) \to +\infty$. But then $\liminf_{k\to\infty} v(z'_{n_k}) = -\infty$ and $w(z''_{n_k}) \to +\infty$. Since v(z) and w(z) are normal, this means that $\liminf_{k\to\infty} v(z_{n_k}) = -\infty$ and $w(z_{n_k}) \to +\infty$ in violation of the hypotheses. Thus there is no sequence $\{z_n\}$ on which u(z) has property (Dh), and the theorem is proved.

Theorem 3.6 above shows the equivalence of the normality of the sum of normal harmonic functions with the normality of the product of non zero normal holomorphic functions. To this end we prove the following theorem on the sum of two normal harmonic functions.

THEOREM 3.17. *Let* $u_1(z)$ *and* $u_2(z)$ *be normal harmonic functions in* D. *If there are* K_1, K_2, $(K_i > 1)$ *such that* $|u_i(z)|^2 \le K_i|u_1(z)+u_2(z)|^2$, $(i=1, 2)$, *for every* z *in* D, *then* $u_1(z)+u_2(z)$ *is normal in* D.

LEMMA 3.18. *Let* v(z) *be a harmonic conjugate to* u(z) *and* $F(z) = \exp(u(z)+iv(z))$. *Then*

$$\rho(F(z)) = \frac{|F'(z)|}{1+|F(z)|^2} \le \frac{\sqrt{(u_x(z))^2+(u_y(z))^2}}{1+|u(z)|^2} .$$

Proof of lemma 3.18. This computational fact follows from the fact that $a^2 \leq 2(\cosh a)-1$.

We will also use the following theorems.

THEOREM 3.A. (Lehto and Virtanen [15, p. 56]). _If_ $f(z)$ _is a meromorphic function in_ D, _then_ $f(z)$ _is a normal function if and only if there exists a positive number_ G _such that_

$$\frac{|f'(z)|}{1+|f(z)|^2} \leq \frac{G}{1-|z|^2}$$

for each z _in_ D.

THEOREM 3.B. (Lappan [13, p. 157]). _If_ $u(z)$ _is a harmonic function in_ D, _then_ $u(z)$ _is a normal function if and only if there exists a positive number_ G _such that_

$$\frac{\sqrt{(u_x(z))^2+(u_y(z))^2}}{1+|u(z)|^2} \leq \frac{G}{1-|z|^2}$$

for each z _in_ D.

Proof of Theorem 3.17. With $F(z) = \exp(u_1(z)+u_2(z)+iv(z))$ where $v(z)$ is a harmonic conjugate to $u_1(z)+u_2(z)$, lemma 3.18 gives

$$\frac{|F'(z)|}{1+|F(z)|^2} \leq \frac{\sqrt{((u_1(z)+u_2(z))_x)^2+((u_1(z)+u_2(z))_y)^2}}{1+|u_1(z)+u_2(z)|^2} .$$

Since $\sqrt{(a+b)^2+(c+d)^2} \leq \sqrt{a^2+c^2} + \sqrt{b^2+d^2}$ for a, b, c, d real,

$$\frac{|F'(z)|}{1+|F(z)|^2} \leq \frac{\sqrt{(u_1(z)_x)^2+(u_1(z)_y)^2}+\sqrt{(u_2(z)_x)^2+(u_2(z)_y)^2}}{1+|u_1(z)+u_2(z)|^2} .$$

Then, it is easy to check that for $K_i > 1$,

$$\frac{1}{1+|u_1(z)+u_2(z)|^2} \leq \frac{K_i}{1+|u_i(z)|^2} \quad (i=1,\ 2).$$

Since $u_1(z)$ and $u_2(z)$ are normal, there exist G_1 and G_2 such that

$$\frac{|F'(z)|}{1+|F(z)|^2} \le (K_1G_1+K_2G_2)\frac{1}{1-|z|^2}$$

for all z in D. Hence F(z) is normal and theorem 3.6 implies $u_1(z)+u_2(z)$ is normal in D.

REMARK 3.19. By proper modification of the above proof we could show: If $u_i(z)$, i=1, 2,...,n, are normal harmonic functions and if there exist K_i, i=1, 2,...,n, $K_i > 1$, such that $|u_i(z)|^2 \le K_i|\sum_{j=1}^{n} u_j(z)|^2$, then $\sum_{j=1}^{n} u_j(z)$ is normal. Also, by using the Cauchy Riemann Differential Equations, with $v_i(z)$ a normal harmonic conjugate to $u_i(z)$ and $u_i(z)$ not necessarily normal for i in some subset J of $\{1, 2,...,n\}$, we can show: If $|v_i(z)|^2 \le K_i|\sum_{j=1}^{n} u_j(z)|^2$ for i in J and $|u_i(z)|^2 \le K_i|\sum_{j=1}^{n} u_j(z)|^2$ for i not in J, then $\sum_{j=1}^{n} u_j(z)$ is normal.

The above remark hints at a relationship between the normality of a harmonic function and the normality of its harmonic conjugate. The next few results make the relationship clearer.

THEOREM 3.20. <u>Let</u> u(z) <u>and</u> v(z) <u>be harmonic conjugates such that</u> $|v(z)| \ge k|u(z)|$, k <u>a fixed positive constant, for each</u> z <u>in</u> D. <u>If</u> u(z) <u>is normal in</u> D, <u>then</u> v(z) <u>is also normal in</u> D.

<u>Proof</u>. $\sqrt{(v_x(z))^2+(v_y(z))^2} = |f'(z)|$ and $\sqrt{(ku_x(z))^2+(ku_y(z))^2} = k|f'(z)|$ where f(z) = u(z)+iv(z). Since $|v(z)| \ge k|u(z)|$, we have $1/(1+|v(z)|^2) \le 1/(1+|ku(z)|^2)$. Therefore $\frac{\sqrt{(v_x(z))^2+(v_y(z))^2}}{1+v(z)^2} \le \frac{\sqrt{(ku_x(z))^2+(ku_y(z))^2}}{k(1+|ku(z)|^2} \le \frac{G}{k(1-|z|^2)}$

for each z in D, some constant G, since $ku(z)$ is normal and because of theorem 3.B. Hence $v(z)$ is normal.

THEOREM 3.21. *If $u(z)$ and $v(z)$ are harmonic conjugates in D, and if $u(z)$ is bounded in D, then $v(z)$ is normal in D.*

Proof. Since $u(z)$ is normal, $F(z) = \exp(u(z)+iv(z))$ is normal. If $v(z)$ is not normal, then there exist close sequences $\{z_n\}$ and $\{z_n'\}$ in D such that $v(z_n) \to 0$, $v(z_n') \to \pi$ as $n \to \infty$ from remark 3.15. Pick a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ so that $\{F(z_{n_k})\}$ converges, say to the real number α. If α is so that $0 < |\alpha| < \infty$, then $\alpha = \exp(\beta+i0)$ for some β $(\beta = \log|\alpha|)$. But $F(z_{n_k}') \to \exp(\beta+i\pi) \neq \alpha$, a contradiction of $F(z)$ normal. Hence $\alpha = 0$ or ∞, or $u(z_{n_k}) \to -\infty$ or $u(z_{n_k}) \to +\infty$. Since this contradicts $u(z)$ bounded, $v(z)$ must be normal.

COROLLARY 3.22. *If $f(z)$ has an analytic logarithm in D with $\arg f(z)$ bounded, then $f(z)$ is normal in D.*

Proof. This corollary follows directly from theorems 3.21 and 3.10.

EXAMPLE 3.23. *There exist normal holomorphic functions with normal real part but not normal imaginary part*. Let $f(z) = 1/(1-z) = u(z)+iv(z)$ for z in D. Since $f(D) = \{w\colon \operatorname{Re} w > 1/2\}$, $f(z)$ and $u(z)$ are normal in D. Consider the inverse images of $\operatorname{Im} w = 0$ and $\operatorname{Im} w = 1/2$ from the range of $f(z)$ for z in D. The first is the interval $(-1,1)$ while the second is the circle of radius one, center $1+i$, intersect D. Since the two inverse images are tangent, there exists a sequence $\{z_n\}$ contained in the first inverse image and a sequence $\{z_n'\}$ contained in the second inverse image which are close but

$v(z_n) = 0$, $v(z_n') = 1/2$ for all n. Hence $v(z)$ is not normal in D while its conjugate $u(z)$ is normal.

One could also obtain theorems concerning the boundary values of normal harmonic functions of somewhat similar nature as Lehto and Virtanen in [15, pp. 58-62] and Väisälä in [17, pp. 29-30] employing theorem 3.6, lemma 3.18, and the results of Lehto and Virtanen.

IV. VERY NORMAL FUNCTIONS

DEFINITION 4.1. A complex, finite valued function $f(z)$ in D is called <u>very normal</u> if there exists a positive number M such that, for each pair of points z_1 and z_2 in D, $|f(z_1)-f(z_2)| < M\,\rho(z_1,z_2)$.

THEOREM 4.2. *If $f(z)$ is very normal in D, then $f(z)$ is normal in D.*

Proof. The theorem is obvious from the fact that $\chi(a,b) \le |a-b|$ and the definition of a normal function.

THEOREM 4.3. *If $f(z)$ is a continuous, complex, finite valued function in D, then $f(z)$ is very normal if and only if there exists a positive number K such that*

$$M(f(z)) = \limsup_{z'\to z} \left|\frac{f(z')-f(z)}{z'-z}\right| \le \frac{K}{1-|z|}$$

for each z in D.

Proof. Necessity. Fix z in D and let P be such that $|f(z')-f(z)| < P\,\rho(z',z)$ for each z' in D. Since

$$\rho(z',z) = \frac{|z'-z|}{(1-|z|^2)(1+\varepsilon(z',z))} \tag{4.1}$$

where $\varepsilon(z',z) \to 0$ as $z' \to z$, we have

$$\frac{|f(z')-f(z)|}{|z'-z|} < \frac{P}{(1-|z|^2)(1+\varepsilon(z',z))} \qquad \text{for all}$$

z' sufficiently close to z so that $1+\varepsilon(z',z) > 0$. Hence, as $z' \to z$,

$$M(f(z)) \le \frac{P}{1-|z|^2} \le \frac{P}{1-|z|}$$

for each z in D.

Sufficiency. Let f(z) be continuous and $M(f(z)) \le \frac{K}{1-|z|}$ for each z in D. We have

$$(1-|z|^2)M(f(z)) = \limsup_{z'\to z} |f(z')-f(z)| \frac{1-|z|^2}{|z'-z|}$$

and, from (4.1),

$$(1-|z|^2)M(f(z)) = \limsup_{z'\to z} \frac{|f(z')-f(z)|}{\rho(z',z)}$$

From the hypothesis and $\frac{K}{1-|z|} \le \frac{2K}{1-|z|^2}$, we get

$$\limsup_{z'\to z} \frac{|f(z')-f(z)|}{\rho(z',z)} < 2K.$$

Let z_1 and z_2 be points in D and let L be the hyperbolic geodesic between z_1 and z_2. For each z on L, there is a $\delta = \delta(z) > 0$ such that $\rho(z',z) < \delta(z)$ implies $|f(z')-f(z)| < 4K\,\rho(z',z)$. Since L is compact, there exists a finite positive integer N and points $z_1' = z_1, z_2', \ldots, z_N' = z_2$, z_i' on L for i=1, 2,...,N, such that $|f(z_i')-f(z_{i-1}')| < 4K\,\rho(z_i',z_{i-1}')$, i=2,...,N. Since L is a geodesic, we find $|f(z_1)-f(z_2)| < 4K\,\rho(z_1,z_2)$. Hence f(z) is very normal in D.

COROLLARY 4.4. *If f(z) is holomorphic in D, then f(z) is very normal if and only if there exists a positive number A_f such that*

$$|f'(z)| < \frac{A_f}{1-|z|}$$

for each z in D.

Proof. Since $f(z)$ is holomorphic in D, $|f'(z)| = M(f(z))$. From Lappan [12, p. 47] we see in the case of holomorphic functions that the class of uniformly normal functions and the class of very normal functions are the same. We now restate Lappan's definition of a uniformly normal function, dropping the requirement that the function be holomorphic.

DEFINITION 4.5. A complex, finite valued function $f(z)$ in D is said to be uniformly normal if, for each $M > 0$, there exists a finite number $K > 0$ such that for each z_0 in D, $\rho(z,z_0) < M$ implies $|f(z)-f(z_0)| < K$.

THEOREM 4.6. If $f(z)$ is very normal in D, then $f(z)$ is uniformly normal in D.

Proof. Let $f(z)$ be very normal and M, a positive constant, given. If z_1 and z_2 are such that $\rho(z_1,z_2) < M$, then $|f(z_1)-f(z_2)| < PM$ where P is the constant of definition 4.1. Letting $K = PM$ for the K of definition 4.5, we see that $f(z)$ is uniformly normal.

Since any bounded discontinuous function in D is uniformly normal but neither normal nor very normal, the converse of Theorem 4.6 certainly in not valid. Moreover, the function $f(z) = \min(1,\sqrt{\rho(z,0)})$ defined in D is normal, uniformly normal, but not very normal since

$$\limsup_{z\to 0} \frac{|\sqrt{\rho(z,0)}-0|}{|z-0|} = \infty .$$

COROLLARY 4.7. If $u(z)$ is a real harmonic function in D, then $u(z)$ is very normal if and only if there exists a positive number K such that

$$\sqrt{(u_x(z))^2+(u_y(z))^2} \le \frac{K}{1-|z|}$$

for each z in D.

Proof. By an easy calculation, $\sqrt{(u_x(z))^2+(u_y(z))^2} = M(f(z))$.

THEOREM 4.8. The sum of two very normal functions is very normal. If $f(z) = u(z)+iv(z)$ where $u(z)$ and $v(z)$ are real functions in D, then $f(z)$ is very normal if and only if both $u(z)$ and $v(z)$ are very normal.

Proof. The theorem follows by standard triangle inequality arguments.

COROLLARY 4.9. If $f(z) = u(z)+iv(z)$ is harmonic in D, then $f(z)$ is very normal if and only if there exists a positive number K such that

$$\sqrt{(u_x(z))^2+(u_y(z))^2+(v_x(z))^2+(v_y(z))^2} \le \frac{K}{1-|z|}$$

for each z in D.

Proof. The sufficiency follows from $M(f(z)) \le \sqrt{(u_x(z))^2+(u_y(z))^2+(v_x(z))^2+(v_y(z))^2}$. For necessity, let $f(z) = u(z)+iv(z)$ be very normal in D. By theorem 4.8, both $u(z)$ and $v(z)$ are very normal in D and hence by corollary 4.7 there are constants K_u and K_v so that

$$\sqrt{(u_x(z))^2+(u_y(z))^2+(v_x(z))^2+(v_y(z))^2} \le$$
$$\sqrt{(u_x(z))^2+(u_y(z))^2} + \sqrt{(v_x(z))^2+(v_y(z))^2} \le \frac{K_u+K_v}{1-|z|}.$$

The proof is completed by setting $K = K_u+K_v$.

THEOREM 4.10. Let $f(z) = u(z)+iv(z)$ be holomorphic in D. The following are equivalent:

(1.) $f(z)$ is very normal

(2.) $u(z)$ is very normal

(3.) $v(z)$ is very normal.

Proof. Since $f(z)$ is holomorphic, we have that $|f'(z)| = \sqrt{(u_x(z))^2+(u_y(z))^2} = \sqrt{(v_x(z))^2+(v_y(z))^2}$, and the theorem follows directly from corollaries 4.7 and 4.4.

In theorem 4.10 it is clear that one can't replace the word "holomorphic" by "harmonic".

THEOREM 4.11. If $\log f(z)$ is very normal in D, then $f(z)$ is normal in D.

Proof. Let $\log f(z)$ be very normal in D and $\{z_n\}$ and $\{z_n'\}$ close sequences in D where $f(z_n)$ converges to some complex value α. Since $\log f(z) = \log |f(z)|+i \arg f(z)$, where $\arg f(0)$ is fixed and $\arg f(z)$ continuously defined, theorems 4.8 and 4.2 imply that $\log |f(z)|$ and $\arg f(z)$ are both very normal and normal. If α is zero or infinity then $\log |f(z_n)|$ tends to $-\infty$ or $+\infty$ respectively. Since $\log |f(z)|$ is normal, $\log |f(z_n')|$ tends to $-\infty$ or $+\infty$ respectively and hence $f(z_n')$ converges to α also. Therefore we may assume $0 < |\alpha| < \infty$. Since $\log |f(z)|$ is normal, $\log |f(z_n')|$ converges to $\log |\alpha|$ and hence $|f(z_n')| \to \alpha$. Also, from $\arg f(z)$ being very normal, for every ϵ positive there exists a positive integer $N(\epsilon)$ such that $n > N$ implies $|\arg f(z_n) - \arg f(z_n')| < \epsilon$. Then as n tends to infinity, $\{|f(z_n)|\}$ and $\{|f(z_n')|\}$ have the same cluster values and so do $\{\arg f(z_n)\}$ and $\{\arg f(z_n')\}$ have the same cluster values. Thus $f(z_n') \to \alpha$ also and $f(z)$ is normal.

V. (A) SEQUENCES FOR NOT NORMAL FUNCTIONS

In this chapter finite valued functions $f(z)$ with range in the plane are considered; $f(z)$ could be real valued. Unless otherwise specified, $f(z)$ is harmonic means $f(z) = u(z)+iv(z)$ and $u(z)$ and $v(z)$ are real valued harmonic functions. Most of the early material of this chapter can be generalized to functions with range on the Riemann sphere with chordal metric replacing absolute values.

DEFINITION 5.1. Let $\{z_n\}$ be a sequence in D. Then $\{z_n\}$ is an (A) *sequence for* $f(z)$ if there exists a sequence $\{z_n'\}$ close to $\{z_n\}$ such that $\limsup_{n\to\infty} |f(z_n)-f(z_n')| = \infty$

LEMMA 5.2. *Let* $w = f(z)$ *be harmonic in* D *and* $\{z_n\}$ *a sequence in* D *such that* $\lim_{n\to\infty} f(z_n) = \alpha$. *If there is a sequence* $\{z_n'\}$ *close to* $\{z_n\}$ *such that* $|f(z_n')-\alpha| > b$ (*or* $|f(z_n')| < 1/b$ *if* $\alpha = \infty$) *for some fixed* $b > 0$, *then* $\{z_n\}$ *is an* (A) *sequence for* $f(z)$.

Proof. If $\alpha \neq \infty$, then either $u(z_n') \not\to \mathrm{Re}\alpha$ or $v(z_n') \not\to \mathrm{Im}\alpha$. Without loss of generality we assume that $u(z_n') \not\to \mathrm{Re}\alpha$. Then lemma 3.13 implies that there exists a sequence $\{z''_{n_k}\}$ close to $\{z_{n_k}\}$, where $\{z_{n_k}\}$ is a subsequence of $\{z_n\}$, such that $u(z''_{n_k}) \to +\infty$. Hence $\{z_n\}$ is an (A) sequence for $f(z)$.

If $\alpha = \infty$, the result follows immediately from the definition.

THEOREM 5.3. *If* $f(z)$ *is harmonic and not normal in* D, *then there exists an* (A) *sequence for* $f(z)$.

Proof. Since $f(z)$ is harmonic and not normal, remark 3.15 implies the hypotheses of lemma 5.2 are fullfilled for some sequence $\{z_n\}$ in D.

THEOREM 5.4. _Let_ $f(z)$ _be an arbitrary function in_ D _and_ $\{z_n\}$ _a sequence in_ D _on which_ $f(z)$ _converges to_ $\alpha \neq \infty$. _If there exists a sequence_ $\{z_n'\}$ _close to_ $\{z_n\}$ _such that_ $\limsup_{n\to\infty} |f(z_n')-\alpha| > 0$, _then_ $f(z)$ _is not normal_.

Proof. The theorem follows immediately from the definition of a normal function.

COROLLARY 5.5. _Let_ $f(z)$ _be harmonic in_ D. _Then_ $f(z)$ _is not normal if and only if there exists an_ (A) _sequence for_ $f(z)$, _denoted by_ $\{z_n\}$, _such that_ $f(z)$ _is bounded on_ $\{z_n\}$.

Proof. The "if" part is a corollary of theorem 5.4 and the "only if" part is a corollary to theorem 5.3 and the fact that $f(z)$ is not normal.

EXAMPLE 5.6. _The boundedness of_ $f(z)$ _on_ $\{z_n\}$ _in theorem_ 5.4 _is needed, even if_ $f(z)$ _is real valued and harmonic in_ D. For example, let $f(z) = \mathrm{Re}(1/(1-z))$, $z_n = 1-\exp(-n)$, and $z_n' = 1-a_n\exp(-n)$ where $0 < a_n < a_{n+1} < 1$, $a_n \to 1$, and $(1-a_n)\exp(n) \to \infty$ as $n \to \infty$. Then

$\rho(z_n,z_n') = (1/2)\log((2-a_n\exp(-n))/(2a_n-a_n\exp(-n))) \to 0$ as $n \to \infty$

so that $\{z_n\}$ and $\{z_n'\}$ are close sequences. Then

$|f(z_n)-f(z_n')| = (1-a_n)\exp(n) \to \infty$ as $n \to \infty$. Hence $\{z_n\}$ is an (A) sequence for $f(z)$, but $f(z)$ is normal since it is real valued, harmonic, and bounded below.

THEOREM 5.7. _Let_ $f(z) = u(z)+iv(z)$ _be a harmonic function in_ D _and bounded on the sequence_ $\{z_n\}$. _Then_ $\{z_n\}$ _is an_ (A)

<u>sequence</u> <u>for</u> f(z) <u>if</u> <u>and</u> <u>only</u> <u>if</u>

$$\limsup_{n\to\infty}\frac{(1-|z_n|^2)\sqrt{(u_x(z_n))^2+(u_y(z_n))^2+(v_x(z_n))^2+(v_y(z_n))^2}}{1+|u(z_n)+iv(z_n)|^2}=\infty.$$

<u>Proof</u>. Let $\{z_n\}$ be a sequence on which, we assume without loss of generality, $f(z_n)\to 0$. Let $S_n(z) = (z+z_n)/(1+\bar{z}_n z)$. If $\{z_n\}$ is an (A) sequence for f(z), then $\{g_n(z)\}$, where $g_n(z) = f(S_n(z))$, is a family of complex harmonic functions in D with a subsequence $\{g_{n_k}(z)\}$ such that, in any neighborhood of $z = 0$, $\{g_{n_k}(z)\}$ assumes values arbitrarily large and arbitrarily small for an infinite number of functions. Hence $\{g_n(z)\}$ is not a normal family in any neighborhood of $z = 0$. So, with standard normal family arguments found in Ahlfors [1, p. 169] along with Lehto and Virtanen [15, pp. 54-55], we have

$$\infty = \limsup_{n\to\infty}\frac{\sqrt{((\mathrm{Re}g_n(0))_x)^2+((\mathrm{Re}g_n(0))_y)^2+((\mathrm{Im}g_n(0))_x)^2+((\mathrm{Im}g_n(0))_y)^2}}{1+|g_n(0)|^2}$$

$$=\limsup_{n\to\infty}\frac{(1-|z_n|^2)\sqrt{(u_x(z_n))^2+(u_y(z_n))^2+(v_x(z_n))^2+(v_y(z_n))^2}}{1+|u(z_n)+iv(z_n)|^2}.$$

Now assume $\{z_n\}$ is not an (A) sequence for f(z). With $\{g_n(z)\}$ as above, there exists an ε so that $\{g_n(z)\}$ is bounded on $D_o = \{z\colon |z| \le \varepsilon\}$. Hence $\{g_n(z)\}$ is a normal family in D_o. So, as before,

$$\infty > \limsup_{n\to\infty}\frac{\sqrt{((\mathrm{Re}g_n(0))_x)^2+((\mathrm{Re}g_n(0))_y)^2+((\mathrm{Im}g_n(0))_x)^2+((\mathrm{Im}g_n(0))_y)^2}}{1+|g_n(0)|^2}$$

$$=\limsup_{n\to\infty}\frac{(1-|z_n|^2)\sqrt{(u_x(z_n))^2+(u_y(z_n))^2+(v_x(z_n))^2+(v_y(z_n))^2}}{1+|u(z_n)+iv(z_n)|^2}.$$

COROLLARY 5.8. *Let* $f(z) = u(z)+iv(z)$ *be a harmonic function in* D. *Then* $f(z)$ *is normal if and only if there exists a constant* K *such that*

$$\frac{\sqrt{(u_x(z))^2+(u_y(z))^2+(v_x(z))^2+(v_y(z))^2}}{1+|u(z)+iv(z)|^2} \leq \frac{K}{1-|z|^2}$$

for each z *in* D.

DEFINITION 5.9. A function $f(z)$ defined in D is said to be a R-*normal function* if the family $F = \{f(S(z))\}$, where $z' = S(z)$ is an arbitrary one-one conformal map of D onto D, has the property that for every sequence $\{f_n\}$, f_n in F, there exists a subsequence which either converges uniformly on every compact subset of D, or else converges uniformly to infinity on every compact subset of D.

Every normal function that is harmonic, holomorphic, or bounded is R-normal and every R-normal function is normal. See Rung [16, p. 14] for an example of a normal function which is not R-normal.

DEFINITION 5.10. Two sequences $\{z_n\}$ and $\{z'_n\}$ of points in D such that there exists a positive constant K where $\limsup_{n\to\infty} \rho(z_n, z'_n) \leq K$ are called *essentially close sequences*.

DEFINITION 5.11. Let $\{z_n\}$ be a sequence in D. Then $\{z_n\}$ is an (AE) *sequence for* $f(z)$ if there exists a sequence $\{z'_n\}$ essentially close to $\{z_n\}$ such that $\limsup_{n\to\infty} |f(z_n)-f(z'_n)| = \infty$.

Naturally every (A) sequence for a function $f(z)$ is also an (AE) sequence for $f(z)$.

THEOREM 5.12. *Let* $f(z)$ *be an arbitrary function in* D. *If there exists an* (AE) *sequence for* $f(z)$, *denoted by* $\{z_n\}$, *with*

$f(z)$ *bounded on* $\{z_n\}$, *then* $f(z)$ *is not* R-*normal*.

Proof. Let $\{z_n\}$ be the (AE) sequence, with $\{z'_n\}$ the essentially close sequence such that $\limsup_{n\to\infty} |f(z_n)-f(z'_n)| = \infty$ and let K be the constant in the definition of essentially close sequence. Without loss of generality we assume $f(z_n) \to 0$ and $f(z'_n) \to \infty$. Consider the sequence $\{g_n(z')\}$, $g_n(z') = f(S_n(z'))$, where $S_n(z') = (z'+z_n)/(1+\bar{z}_n z')$. Let $D_o = \{z: |z'| \leq \tanh(K+1)\}$. Since $g_n(0) \to 0$ and $g(S_n^{-1}(z'_n)) \to \infty$ with $S_n^{-1}(z'_n)$ in D_o for all n, the sequence $\{g_n(z')\}$ contains no infinite subsequence which converges uniformly to a finite valued function or to infinity on D_o. Therefore $f(z)$ is not R-normal.

Let $\Gamma_i (i=1,2)$ be simple continuous curves $z_i = z_i(t)$ $(0 \leq t < 1)$ in D such that $|z_i(t)| \to 1$ as $t \to 1$. The *non-Euclidean Frechet distance* between Γ_1 and Γ_2 is

$$\mathcal{D}(\Gamma_1,\Gamma_2) = \max\Big(\lim_{\substack{z_1\in\Gamma_1 \\ |z_1|\to 1}} \sup \rho(z_1,\Gamma_2),\ \lim_{\substack{z_2\in\Gamma_2 \\ |z_2|\to 1}} \sup \rho(z_2,\Gamma_1)\Big).$$

COROLLARY 5.13. *Let* Γ_1 *and* Γ_2 *be simple continuous curves in* D *tending to the boundary such that* $\mathcal{D}(\Gamma_1,\Gamma_2)$ *is finite*. *If* $f(z)$ *is* R-*normal in* D *and* $f(z) \to \infty$ *on* Γ_1, *then* $f(z) \to \infty$ *on* Γ_2.

Proof. If $\{z_n^2\}$ is any sequence on Γ_2 tending to the boundary of D, there exists a sequence $\{z_n^1\}$ on Γ_1 and a finite constant K such that $\rho(z_n^1,z_n^2) < K$ for all n sufficiently large. Since $f(z_n^1) \to \infty$ and by theorem 5.12, we have $f(z_n^2) \to \infty$.

COROLLARY 5.14. *Let* $f(z)$ *be a* R-*normal function in* D. *If* Γ_1 *tends non-tangentially to* $\exp(i\theta)$ *on* C *with* $f(z) \to \infty$ *on* Γ_1, *then* $\exp(i\theta)$ *is a Fatou point of* $f(z)$ *with a Fatou value* ∞.

COROLLARY 5.15. *Let f(z) be a R-normal function in D. If Γ_1 tends non-tangentially to exp(iθ) on C with f(z) bounded on Γ_1, then f(z) is bounded in every Stolz angle at exp(iθ).*

Corollaries 5.13, 5.14, and 5.15 are analogues or generalizations to Theorem 3, Bagemihl and Seidel [5]; Theorem 1, Rung [16]; and Theorem 4, Bagemihl [3], respectively.

THEOREM 5.16. *If f(z) is defined in D and there exists an (AE) sequence for f(z), then f(z) is neither uniformly normal nor very normal.*

Proof. The theorem follows immediately from definitions 4.5, 4.1, 5.11, and theorem 4.6.

THEOREM 5.17. *If f(z) is not uniformly normal in D, then there exists an (AE) sequence for f(z).*

Proof. Let f(z) be not uniformly normal. Then there exists K, $0 < K < \infty$, such that for each positive integer n one can find z_n and z'_n in D where $\rho(z_n, z'_n) < K$ and $|f(z_n) - f(z'_n)| > n$. Hence $\{z_n\}$ is an (AE) sequence for f(z).

Naturally then theorems 5.16 and 5.17 yield the following:

COROLLARY 5.18. *Let f(z) be defined in D. Then f(z) is not uniformly normal if and only if there exists an (AE) sequence for f(z).*

Following the lead of Lange [10] and Gauthier [8], but with cluster sets instead of range sets, we easily obtain some results concerning the cluster sets along hyperbolic disks with centers on (A) sequences. For a sequence $\{z_n\}$, define the sequence of disks $\{\Delta_n^\epsilon\}$ by $\Delta_n^\epsilon = \Delta_n^\epsilon(z_n) = \{z: \rho(z, z_n) < \epsilon\}$ for each positive integer n.

THEOREM 5.19. *If* $\{z_n\}$ *is an* (A) *sequence for* $f(z) = u(z)+iv(z)$ *harmonic in* D *with* $f(z)$ *bounded on* $\{z_n\}$, *and for each* $\epsilon > 0$ *letting* $C(f, \bigcup_{i=1}^{\infty} \Delta_n^{\epsilon}(z_n)) = C(f,\Delta_n^{\epsilon}) = \bigcap_{j=1}^{\infty} (\overline{\bigcup_{n=j}^{\infty} f(\Delta_n^{\epsilon}(z_n))})$, *then* $C(f,\Delta_n^{\epsilon})$ *is either a line parallel to one of the axes for each* ϵ *sufficiently small or for every real number* x, *there are* w_1, w_2 *in* $C(f,\Delta_n^{\epsilon})$ *such that* $\text{Re } w_1 = x$ *and* $\text{Im } w_2 = x$.

Proof. Consider $C(u,\Delta_n^{\epsilon})$ and $C(v,\Delta_n^{\epsilon})$. If either one, say $C(u,\Delta_n^{\epsilon})$, is a singleton for some ϵ_o, then $C(v,\Delta_n^{\epsilon})$ will at least be a closed half line for every $\epsilon \le \epsilon_o$ since $f(z)$ is continuous and $\{z_n\}$ is an (A) sequence for $f(z)$. Lemma 3.13 and remark 3.15 however imply $v(z)$ is not normal and $C(v,\Delta_n^{\epsilon})$ is the whole (closed) real line. Therefore we have the first situation of the theorem's conclusion.

The only other situation is if for every $\epsilon > 0$, neither $C(u,\Delta_n^{\epsilon})$ nor $C(v,\Delta_n^{\epsilon})$ is a singleton. We know there exists a subsequence $\{z_{n_k}\}$ where $f(z_{n_k}) \to \beta = a+ib \neq \infty$ from hypotheses. From the "neither nor" statement there exist $\{z'_{n_{k_1}}\}$ and $\{z''_{n_{k_1}}\}$ close to $\{z_{n_{k_1}}\}$, a subsequence of $\{z_{n_k}\}$, where $u(z'_{n_{k_1}}) \to c \neq a$ and $v(z''_{n_{k_1}}) \to d \neq b$. Hence neither $u(z)$ nor $v(z)$ is normal and the conclusion follows.

EXAMPLE 5.20. *There exists a harmonic function* $f(z)$ *which is not normal and, for some* (A) *sequence* $\{z_n\}$ *for* $f(z)$ *and each* ϵ, $0 < \epsilon < \infty$, $C(f,\Delta_n^{\epsilon})$ *properly contains just one line and so is not the whole plane*. Let $f(z) = \text{Re}(z)+i\text{Re}(\exp(z))$ with the upper half plane $H = \{z: z = x+iy,\ x,y \text{ real},\ y > 0\}$ as its domain and

$\rho^*(z_1,z_2)$ as the hyperbolic distance between z_1 and z_2 in H. Let Γ_1 and Γ_2 be two distinct rays in the upper half plane perpendicular to the real axis. Then, as $\{z_n\}$ on Γ_1 and $\{z'_n\}$ on Γ_2 go to infinity such that $\text{Im} z_n = \text{Im} z'_n$, we have $\rho^*(z_n,z'_n) \to 0$ and $\{f(z_n)\}$, $\{f(z'_n)\}$ have bounded disjoint cluster sets. Hence f(z) is not normal. From the hyperbolic metric in H and $\text{Re}(f(x+iy)) = x$ and using the natural definition of an (A) sequence in H we see that $\{z_n = 2n\pi i\}$ is an (A) sequence for f(z); i.e. $C(x,\Delta_n^\epsilon)$ is the whole closed real line for any $\epsilon > 0$. Let w_o be in $C(f,\Delta_n^\epsilon)$. This implies there exists a sequence $\{z'_n\}$, $z'_n = z'_n+iy'_n$ in Δ_n^ϵ, such that $x'_n \to \text{Re } w_o$. Since $-1 \le \cos y'_n \le 1$, $\{\text{Re}(\exp(z'_n))\}$ has a cluster set contained in $\{w: w = b \exp(\text{Re } w_o), -1 \le b \le 1\}$. Hence Im w_o obeys the inequalities $-\exp(\text{Re } w_o) \le \text{Im } w \le \exp(\text{Re } w_o)$. One also could easily choose y'_n so that Im w_o is any number in the closed interval from $-\exp(\text{Re } w_o)$ to $\exp(\text{Re } w_o)$. Hence $\{w: w = a+i0, a \text{ real}\}$ is properly contained in $C(f,\Delta_n^\epsilon) = \{w: w = a+ib \exp(a), a \text{ real}, -1 \le b \le 1\}$ and this is properly contained in the closed plane. Clearly only one line is contained in $C(f,\Delta_n^\epsilon)$.

There are many such functions with the properties of example 5.20. For example, any function of the form

$$f(x+iy) = x + i \sum_{j=1}^{N} \exp(c_j x)(a_j \cos c_j y + b_j \sin c_j y),$$ where N is a positive integer, a_j and b_j real and not all zero, and c_j rational with at least one non-zero, also satisfies example 5.20.

THEOREM 5.21. <u>Let</u> f(z) <u>be harmonic and not normal in</u> D. <u>Then, for each</u> (A) <u>sequence</u> $\{z_n\}$ <u>for</u> f(z) <u>with</u> f(z) <u>bounded on</u> $\{z_n\}$, <u>either</u> $C(f,\Delta_n^\epsilon)$ <u>is a line for</u> ϵ <u>sufficiently small or</u>

$C(f, \Delta_n^{\epsilon})$ *intersects every half plane*.

Proof. The theorem is clear from the fact that $\exp(i\theta)f(z)$ is harmonic and not normal for all real θ.

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