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COHEN-MACAULAY BLOWING-UP ALGEBRAS AND CONSTRUCTIONS IN LINKAGE

By

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ABSTRACT

COHEN-MACAULAY BLOWING-UP ALGEBRAS AND CONSTRUCTIONS IN LINKAGE

By

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In this work, we study the Rees algebra and the associated graded ring of an ideal in a local Cohen-Macaulay ring. For ideals with small enough reduction number and sufficiently good residual intersection properties, we show that these blowing-up algebras are Cohen-Macaulay and we compute the number and degrees of their defining equations.

It is shown that a power of an ideal (locally a complete intersection in codimension one) coincides with its symbolic power whenever they coincide after deformation of the ideal, if the symbolic power defines a Cohen-Macaulay ring.

We give various constructions, including the tensor product of algebras of finite type over a field and the sum of two geometrically linked ideals, which produce Cohen-Macaulay ideals which are either strongly Cohen-Macaulay (or the entire linkage class has this property), strongly nonobstructed, or both, but are not in the linkage class of a complete intersection.

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INTRODUCTION

Let R be a noetherian local ring and let I be an R-ideal. The Rees algebra $\mathcal{R} = R[It]$ and the associated graded ring $G = gr_I(R)$ are two graded algebras that encode various algebraic and geometric properties of the ideal I. For example, Proj($\mathcal{R}$) is the blow-up of Spec(R) along V(I) and Proj(G) corresponds to the exceptional fiber of the blow-up. It is particularly interesting to know when these "blowing-up algebras" $\mathcal{R}$ and G are Cohen-Macaulay. Other than being important in its own right, this property often facilitates the study of various other properties of these algebras, such as their normality ([9]), the depth of their graded pieces ([17]), or the number and degrees of their defining equations ([46],[5], or Section 1.3).

The relationship between the Cohen-Macaulay property of $\mathcal{R}$ and G is fairly well-understood: Huneke observed some time ago that G is Cohen-Macaulay whenever $\mathcal{R}$ is (at least when R is Cohen-Macaulay and I is not nilpotent), but recently various criteria have been found for the converse to hold as well (e.g. [46], [5], [61], [49]). This shifts the focus of attention, at least in principle, to the study of the Cohen-Macaulayness of G.

Using their so-called approximation complexes ([27]), Herzog, Simis and Vasconcelos established the Cohen-Macaulayness of the associated graded ring of any ideal, in a local Cohen-Macaulay ring, satisfying sliding depth and the condition G_∞. (We refer to Chapter 0 for the definition of the terminology.) At around the same time, Huneke ([32]) proved a similar result, using his theory of d-sequences. These works demonstrated a principle that the Cohen-Macaulay property of blowing-up algebras is related to Cohen-Macaulay properties of the Koszul homology of the ideal. Other than just satisfying the condition G_∞, these ideals have an even more notable property: their Rees algebras coincides with their symmetric algebras; ideals with this property were then said to be of linear type. Until fairly recently, relatively little was known about the Cohen-Macaulayness of blowing-up algebras of ideals which are not of linear type (except for equimultiple ideals, which were studied by Sally (e.g. [57], [58]) sometime earlier; see [24] for a comprehensive treatment of this case).

One approach, which goes back to a classic paper of Northcott and Rees ([54]), is to pass to a minimal reduction J of I and study the finite extension of Rees algebras $R[Jt] \subset R[It]$. Philosophically, the reduction J is "simpler" in that its minimal number of generators is no worse than that of I, being at most the analytic spread ℓ of I (at least if the residue field of R is infinite). The reduction number r is then seen as an important way to measure how far I and J differ. Of course, this approach is fruitful only if $r > 0$.

The recent work [30] and [31] of Huckaba and Huneke made successful use of this approach in the first nontrivial cases: they proved the Cohen-Macaulayness of G (and $\mathcal{R}$) for ideals with $r = 1$ and having analytic deviation one and two. This work quickly inspired many others: under various additional assumptions (see Section 1.2 for precise statements) G was shown to be Cohen-Macaulay by Vasconcelos ([74]) and Ulrich ([68]) when $r \leq 1$, by Goto and Nakamura ([20],[21]), Aberbach and Huneke ([4]), and Aberbach, Huckaba and Huneke ([3]) when $r \leq 2$ and the analytic deviation is at most two, by Aberbach and Huckaba ([2]) when

$r \leq 3$ and the analytic deviation is at most two, by Simis, Ulrich and Vasconcelos ([61]) when $\nu(I) = \ell + 1$ and $r \leq \ell - g + 1$ (where $g =$ grade I), and by Tang ([63]) when $r \leq \ell - g + 1$ (and sufficiently many powers have high depth). One of our main results is the following theorem which includes essentially all of the above results as a special case (subsequently, futher generalizations were also obtained by Aberbach ([1]) and Goto, Nakamura, and Nishida ([22],[23])):

Theorem 1.2.8 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r, let $k \geq 1$ be an integer, and assume that $r \leq k$, that I satisfies sliding depth and G_∞ locally in codimension $\ell - 1$, that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$, and that I satisfies $AN^-_{\ell-\max\{2,k\}}$.

Then G is Cohen-Macaulay.

The condition "$AN^-_\bullet$" in the statement of the theorem denotes the requirement that certain residual intersections of I are Cohen-Macaulay. This property was first observed for ideals satisfying sliding depth ([28], [36]) and now better explains the philosophy involved here: "The Cohen-Macaulayness of a blowing-up algebra is related to the Cohen-Macaulayness of the residual intersections of the ideal." Indeed, our main technique is to exploit the residual intersection properties of the ideal, which facilitate the computation of various intersections and ideal quotients. Although most of the previous works made no explicit reference to residual intersections, they often made at least implicit use of these residual intersection techniques. We will study these so-called Artin-Nagata properties in Section 1.1. Section 1.2 is devoted to the proof of Theorem 1.2.8.

Once one knows that the blowing-up rings are Cohen-Macaulay, one can ask about the nature of their defining equations; we study this question in Section 1.3. Of course, if I is of linear type, then $\mathcal{R}$, being a symmetric algebra, has its equations completely determined by a presentation matrix of I. In general there will be higher order relations $\mathcal{A}$, namely elements in the kernel of the natural surjection from $\mathcal{R}$ to the symmetric algebra of I, which are usually difficult to determine explictly. However, one could at least ask about the number and degrees of these relations. The latter question was answered under the assumptions of the works [30] and [61]. Similarly, under somewhat stronger assumptions than in Theorem 1.2.8, we are able to compute the number and degrees of the defining equations:

Theorem 1.3.3 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r and assume that I satisfies G_∞ locally in codimension $\ell - 1$, that $S_j(I) \cong I^j$ and depth $R/I^j \geq d - \ell + r - j$ whenever $1 \leq j \leq r$, and that I satisfies $AN^-_{\ell-2}$.

Then $\mathcal{A}$ is minimally generated by $\binom{\nu(I)-\ell+r}{\nu(I)-\ell-1}$ forms of degree $r + 1$.

In particular, $\mathcal{A}$ is cyclic if in addition I has second analytic deviation one. In some special circumstances, one can find explicitly the generator of $\mathcal{A}$: Vasconcelos studied this problem for perfect ideals of grade 2 ([73]), in which case the required equation is essentially the determinant of a Jacobian dual of a presentation matrix. This cannot hold for ideals of larger grade simply by degree reasons, but we are able to show an analogous result for perfect Gorenstein ideals of grade 3 which satisfy the so-called row condition ([4],[61]) (we later learned that a similar result was shown by S. Morey ([52]) in case I has linear presentation):

Theorem 1.3.7 Let R be a local Gorenstein ring with infinite residue field, let I be a perfect Gorenstein ideal of grade 3, with analytic spread ℓ, minimal number of generators $n = \ell + 1$, assume that I satisfies G_∞ locally in codimension $\ell - 1$, let ϕ be an n by n alternating matrix presenting I with last row $(-x_1, ..., -x_\ell, 0)$ which generates the ideal of entries of ϕ, let ψ be the ℓ by ℓ alternating submaxtrix of ϕ obtained by deleting the last row and column, let ψ_j denote the jth column of ψ, for $1 \le j \le \ell$, write $\psi_j = A_j(\underline{x})^t$, where A_j is an ℓ by ℓ matrix whose jth row consists of zeros and whose ith row, for any $1 \le i \le \ell$, is the negative of the jth row of A_i, let $T_1, ..., T_\ell$ be variables over R, and let B be the matrix whose jth column is $A_j^t(\underline{T})^t$.

Then $\mathcal{A}$ is generated by $F = T_n^{-1}\chi_B(T_n)$, where χ denotes the characteristic polynomial of B in the variable T_n.

For ideals having second analytic deviation one, one can prove converses to the results of Section 1.2 asserting that G and $\mathcal{R}$ are Cohen-Macaulay. This was done earlier by Aberbach, Huckaba and Huneke ([3]), and by Aberbach and Huckaba ([2]) for ideals of small analytic deviation. The idea is that one can relate $(n-1)$-residual intersections of I to the ideal $I_1(\phi)$ of entries of a minimal presentation matrix of I. For example, one can show the following, which builds on one of the main results of [61]:

Theorem 1.4.10 Let R be a local Gorenstein ring with infinite residue field, let I be a strongly Cohen-Macaulay R-ideal of grade $g \ge 2$, analytic spread ℓ, and minimal number of generators $n = \ell + 1$ satisfying G_∞ locally in codimension $\ell - 1$, and assume that $I \subset I_1(\phi)^2$, where ϕ is a matrix with n rows presenting I. Then the following are equivalent.

(a) After elementary row operations, $I_1(\phi)$ is generated by the last row of ϕ;

(b) I has reduction number $\le \ell - g + 1$;

(c) $\mathcal{A}$ is generated by forms of degree $\leq \ell - g + 2$;

(d) $\mathcal{A}$ is generated by a single form of degree $\ell - g + 2$;

(e) $\mathcal{R}$ is Cohen-Macaulay;

(f) G is Cohen-Macaulay;

This result, which can be applied in particular to perfect ideals of grade 2 and to perfect Gorenstein ideals of grade 3, is proved in Section 1.4 as well as other weaker results requiring only sufficiently good Artin-Nagata properties rather than strong Cohen-Macaulayness.

In Section 1.5 we make some remarks about ideals generated by analytically independent elements, and take the opportunity to show that there exist Cohen-Macaulay homogeneous prime ideals of grade 3 and deviation 3 in $k[x_1, ..., x_6]$ which are locally generated by analytically independent elements but are not of linear type, answering a question of Ulrich ([66]) in the negative.

Let I be a prime ideal in a regular local ring R. An important problem is to determine when the power I^n of I coincides with its symbolic power $I^{(n)}$ (e.g. [29],[33]). One might expect the symbolic power, being at least unmixed, to have better depth properties than the ordinary power in general. For example, when R/I has dimension one, $R/I^{(n)}$ is Cohen-Macaulay, a trivial but often useful fact to know. One could ask in general: when is $R/I^{(n)}$ Cohen-Macaulay?

In Chapter 2 we study this question in case R/I has a deformation S/J for which $J^{(n)} = J^n$. (For example, any ideal in the linkage class of a complete intersection, licci for short, in particular any perfect ideal of grade 2, or any perfect Gorenstein ideal of grade 3, always admits such a deformation.) This problem is related to asking when the property that the power coincides with the symbolic power is preserved after specialization. In that sense, it is somewhat analogous to the study of the arithmetically Cohen-Macaulayness of an algebraic variety via its hyperplane sections. Indeed, our approach was motivated by the recent work of Huneke and Ulrich ([42]) on that topic. Our result shows that in many cases symbolic powers do not possess better depth properties (beyond being unmixed) than the ordinary powers:

Corollary 2.4 Let R be a local Cohen-Macaulay ring, let I be an R-ideal of height g, assume that I is a complete intersection locally in codimension $g+1$, that R/I has a deformation S/J which is equidimensional and satisfies $J^{(n)} = J^n$ for some n, and that $I^{(n)} \neq I^n$.

Then $R/I^{(n)}$ does not satisfy (S_2).

One application is the following analogue of an older result of Huneke and Ulrich: If I is a licci ideal satisfying (CI_1) but not a complete intersection, then for every $n \geq 3$, $R/I^{(n)}$ is not Cohen-Macaulay.

In Chapter 3 we study some constructions which produce examples of Cohen-Macaulay ideals having certain specified properties with respect to linkage. We are interested in the property of being strongly nonobstructed, which was proved to be a linkage invariant by Buchweitz in his Paris thesis ([12]), and the property of being strongly Cohen-Macaulay, which was shown to be an invariant of even linkage by Huneke ([34]). We have seen that the latter property is important in the study of blowing-up algebras. On the other hand, the property of being strongly nonobstructed plays a role in deformation theory: it implies that there are no obstructions to lifting infinitesimal deformations ([12],[25]).

The best known examples of strongly Cohen-Macaulay and strongly nonobstructed ideals are the licci ideals. Although a great deal is known about licci ideals, there are otherwise relatively few known classes of strongly Cohen-Macaulay and strongly nonobstructed ideals. This work arose in trying to better understand how these properties are related. It was known for some time that (perfect) strongly Cohen-Macaulay ideals are not necessarily licci, while Ulrich ([67]) constructed examples demonstrating that the properties of being strongly Cohen-Macaulay (or even that the entire linkage class enjoys this property) and strongly nonobstructed do not imply each other. There was, however, no known example of a strongly Cohen-Macaulay, strongly nonobstructed ideal that was not licci.

We will answer this question in a particularly natural way. After recalling in Section 3.1 some of the basic machinery of linkage, developed mainly by Huneke and Ulrich, in Section 3.2 we study the tensor product of two algebras over a field. It is well-known that the tensor product inherits many good homological properties from each of its factors. Despite this, however, we show the following:

Corollary 3.2.3 Let A and B be complete local licci algebras over a field k which are not complete intersections and let $C = A\hat{\otimes}_k B$.

Then C is strongly Cohen-Macaulay and strongly nonobstructed, but not licci.

This result demonstrates that homological conditions alone, like being strongly Cohen-Macaulay or strongly nonobstructed, will never be sufficient to guarantee that an ideal lies in the linkage class of a complete intersection. One obtains explicit counterexamples in any codimension at least 4. We leave it as an open question whether such an example can exist in codimension 3. The proof of 3.2.3, although using a standard reduction technique of [38], is fairly technical (partly because we prove the result in the complete case); a much simpler proof is available in case neither A nor B is Gorenstein: in that case, the tensor product is directly linked to an algebra which is not strongly Cohen-Macaulay (Theorem 3.2.2(a)).

In Section 3.2 we observe some new invariants of geometric linkage. It is known that the depth of the twisted conormal module $I \otimes \omega_{R/I}$ (the first Koszul homology module $H_1(I)$, respectively) is an invariant of the linkage class ([13]) (the even linkage class ([34]), respectively). It turns out that the depth of these modules modulo their torsion submodule is an invariant of their geometric linkage classes (of their even geometric linkage class, respectively).

In Section 3.4 we study the sum of two geometrically linked Cohen-Macaulay ideals. Peskine and Szpiro observed that such an ideal is Gorenstein, and it turns out to be an interesting way to construct ideals with the properties we are concerned with. Kustin and Miller ([47]) used this construction in their study of Gorenstein ideals of codimension 4 and Ulrich ([67]) showed that if $I \sim J$ are geometrically linked licci ideals then the sum $I + J$ is also licci, giving a nontrivial way to obtain new licci ideals from old ones. He also showed that if I is strongly nonobstructed and $H_1(I)$ and $H_1(J)$ are Cohen-Macaulay, then $K = I + J$ is strongly nonobstructed (equivalently, the conormal module K/K^2 is Cohen-Macaulay, as K is Gorenstein). Using this, and other results from [67], Ulrich constructed a codimension 5 Gorenstein ideal which is strongly nonobstructed but is not even syzygetic. He does not obtain such an example as a sum of links however, as his assumptions force K to be syzygetic.

We are able to construct many such examples directly as sums of links. To do this, we needed to generalize Ulrich's result to determine the precise conditions for a sum of links to be strongly nonobstructed and to be syzygetic:

Corollary 3.4.6 Let R be a local Gorenstein ring and let $K = I + J$ be a sum of two geometrically linked Cohen-Macaulay ideals.

(a) K is strongly nonobstructed if and only if $H_1(I)/\tau$, $H_1(J)/\tau$ and $(I \otimes \omega_{R/I})/\tau$ are Cohen-Macaulay.

(b) If I satisfies (CI_1), then K is syzygetic if and only if I and J are syzygetic, and $I \otimes \omega_{R/I}$ is torsion-free.

In fact we prove a more general result (Theorem 3.4.4) which gives the depth of K/K^2 and the depth of $H_1(K)$ directly in terms of invariants of I and J. Using the tensor product construction of Section 3.2 together with Corollary 3.4.6, one may construct fairly general strongly nonobstructed Gorenstein ideals which are not syzygetic.

We conclude in Section 3.5 with a naive construction, which is somewhat dual to linkage, which produces Cohen-Macaulay ideals of type 2: given any Cohen-Macaulay ideal I of codimension g whose deviation is at most g, intersect any two complete intersections inside I of codimension g which together generate I. Although this may not seem very promising, it actually turns out to be related to a construction of Section 3.2 describing a link of the tensor product. We make use of this construction to give a more "generic" example of a perfect homogeneous ideal of codimension 3 and type 2 having a pure resolution which is not strongly Cohen-Macaulay ([48]).

CHAPTER 0

PRELIMINARIES

In this short preliminary chapter, we establish the notation and terminology that we will use throughout this work. As general references for any undefined terminology, we refer to [10] and [75].

0.1 Basic Invariants

Let R be a noetherian local ring with maximal ideal m and residue field k and let I be an R-ideal, by which we always mean a proper ideal. The minimal number of generators of I is denoted by $\nu(I)$; equivalently, $\nu(I) = \dim_k I \otimes_R k$. We let $g = \text{grade } I$, and let $d(I) = \nu(I) - g$ be the *deviation* of I. If $d(I) = 0$ (respectively, $d(I) \leq 1$), I is called a *complete intersection* (respectively, an *almost complete intersection*). We will virtually always be working in a local Cohen-Macaulay ring. In this case the *codimension* and *height* coincide with the grade.

If R is Cohen-Macaulay, the canonical module of R is denoted by ω_R (if it exists); if R is Gorenstein, I has grade g, and $A = R/I$ is Cohen-Macaulay then one may take $\omega_A = \text{Ext}^g_R(A, R)$. The *type* of R is defined by (where $d = \dim R$)

$$r(R) = \nu(\omega_R) = \dim_k \text{Ext}^d_R(k, R).$$

0.2 Local Properties

We often say that I has a property $\mathcal{P}$ if R/I has property $\mathcal{P}$. For example, we say that I is a *Cohen-Macaulay R-ideal* if R/I is Cohen-Macaulay, and I is a *Gorenstein R-ideal* if R/I is Gorenstein. Similarly, we say I has type t if R/I does. An ideal is called *perfect* if it is Cohen-Macaulay and has finite projective dimension. Recall that an ideal is *unmixed* if every associated prime has the same height.

Now let s be an integer. We say that I has a property $\mathcal{P}$ *locally in codimension* s if I_p has property $\mathcal{P}$ for every $p \in V(I)$ with dim $R_p \leq s$. If I is a complete intersection locally in codimension s+ht I, we say that I satisfies (CI_s). An ideal is *generically a complete intersection* if it is a complete intersection locally at every associated prime; if it is unmixed, then this is equivalent to saying that I satisfies (CI_0). We say that I satisifes the condition G_s if its number of generators is at most the dimension locally in codimension $s-1$, i.e. if $\nu(I_p) \leq \dim R_p$ for all $p \in V(I)$ with dim $R_p \leq s-1$. We say that I satisfies G_∞ if I satisfies G_s for every s.

0.3 Blowing-up Algebras

The *Rees algebra* of I is

$$\mathcal{R} \;=\; R[It] \;\cong\; \bigoplus_{j\geq 0} I^j,$$

and the *associated graded ring* of I is

$$G \;=\; gr_I(R) \;=\; \mathcal{R} \otimes_R R/I \;\cong\; \bigoplus_{j\geq 0} I^j/I^{j+1}.$$

The *analytic spread* ℓ of I is defined by

$$\ell(I) = \dim \mathcal{R} \otimes_R k = \dim G \otimes_R k,$$

and satisfies the inequalities

$$\text{ht } I \le \ell(I) \le \min\{\dim R, \nu(I)\}.$$

One also defines the *analytic deviation* to be the difference $\ell(I) - \text{ht } I$, and the *second analytic deviation* to be $\nu(I) - \ell(I)$.

Burch's inequality ([15]) states that

$$\inf_j \text{ depth } R/I^j \le \dim R - \ell(I);$$

however we will also use this to refer to the improved version due to Brodmann (e.g. [24, 23.11]) which replaces the "inf" by "lim inf". It is also the case that equality holds in Burch's inequality whenever G is Cohen-Macaulay ([17]).

Let $S_j(I)$ denote the jth symmetric power of I and let

$$S(I) = \bigoplus_{j \ge 0} S_j(I)$$

be the symmetric algebra of I. There is a natural surjective homomorphism $\alpha : S(I) \to \mathcal{R}$; let $\mathcal{A}$ denote the kernel. If $\mathcal{A} = 0$, equivalently if α is an isomorphism, we say that I is of *linear type*. Otherwise, we define the *relation type*, $rt(I)$, to be the maximal degree occuring in a homogeneous minimal generating set of $\mathcal{A}$. If α is an isomorphism in degree 2, equivalently if $S_2(I) \cong I^2$, then I is said to be *syzygetic*.

0.4 Reductions

An R-ideal $J \subset I$ is called a *reduction* of I if $R[It]$ is a finite $R[Jt]$-module, or equivalently if $I^{r+1} = JI^r$ for some integer $r \ge 0$. Denote the smallest such r by $r_J(I)$; it is the *reduction number of I with respect to the reduction J*. A reduction $J \subset I$ is called a *minimal reduction* if it is minimal with respect to inclusion among all reductions of I. If the residue field k is infinite, every minimal reduction is minimally generated by $\ell(I)$ elements; in this case we define the *reduction number* r of I by

$$r(I) = \min\{r_J(I) \mid J \text{ is a minimal reduction of } I\}.$$

It holds that $r(I) = 0$ if and only if $\nu(I) = \ell$.

0.5 Ideals of Minors

If $\mathcal{M}$ is a matrix with entries in R, $I_t(\mathcal{M})$ denotes the R-ideal generated by all t by t minors of $\mathcal{M}$. If $\mathcal{M}$ is an alternating matrix, and t is even, $\mathrm{Pf}_t(\mathcal{M})$ denotes the R-ideal generated by all t-th order Pfaffians of $\mathcal{M}$ (obtained by deleting the same rows and columns of $\mathcal{M}$).

A matrix (respectively, an alternating matrix) $\mathcal{M}$ is said to be *generic* (over a ring A) if its entries (respectively, its upper triangular entries) are indeterminants over A. If $\mathcal{M}$ is a generic $n+1$ by n matrix, then $I_n(\mathcal{M})$ is a perfect ideal of grade 2, and if $\mathcal{M}$ is a generic alternating n by n matrix, and n is odd, then $\mathrm{Pf}_{n-1}(\mathcal{M})$ is a perfect Gorenstein ideal of grade 3.

0.6 Pairs

Let (R, I) and (S, J) be pairs, where R and S are noetherian local rings, and $I \subset R$ and $J \subset S$ are ideals (possibly $I = R$ and $J = S$). We say that the pairs (R, I) and (S, J) are *isomorphic*, and write $(R, I) \cong (S, J)$, if there is an isomorphism $\phi : R \to S$ with $\phi(I) = J$. The pairs are said to be *generically equivalent*, written $(R, I) \approx (S, J)$ if there are finite sets of variables X over R and Y over S such that $(R(X), IR(X)) \cong (S(Y), JS(Y))$, where $R(X) = R[X]_{mR[X]}$.

We say that (S, J) is a *deformation* of (R, I) if there is a sequence $\underline{a} = a_1, ..., a_s$ in S, which is regular on S and S/J, such that $(S/(\underline{a}), (J, \underline{a})/(\underline{a})) \cong (R, I)$. Then we also say that S/J is a deformation of R/I (or even that J is a deformation of I), while the canonical surjection $S/J \to R/I$ is called *specialization.*

We say that a pair (S, J) is *essentially a deformation* of (R, I) if there is a sequence of pairs (S_i, J_i), for $0 \leq i \leq n$, with $(S_0, J_0) = (R, I)$ and (S_n, J_n), such that for all $0 \leq i \leq n-1$, one of the following conditions holds:

(a) (S_{i+1}, J_{i+1}) is a deformation of (S_i, J_i);

(b) $(S_{i+1}, J_{i+1}) = ((S_i)_p, (J_i)_p)$ for some $p \in \text{Spec } S_i$;

(c) $(S_{i+1}, J_{i+1}) \approx (S_i, J_i)$.

0.7 Koszul Homology

If $a_1, ..., a_n \in R$, we let $H_i = H_i(a_1, ..., a_n)$ denote the ith Koszul homology of the Koszul complex built on $a_1, ..., a_n$. If $I = (a_1, ..., a_n)$, then we also write $H_i(I)$ for H_i, although this module depends on the generators $a_1, ..., a_n$. However, if R is Cohen-Macaulay, then the property that H_i is Cohen-Macaulay in a range $0 \leq i \leq k$, for some fixed k, is independent of the generating set of I. In addition, $H_i(I)$ is an R/I-module, either $H_i = 0$ or dim H_i = dim R/I, and $H_i = 0$ for every $i > n-\text{ht } I$.

We say that I satisfies *sliding depth* if depth $H_i(I) \geq \dim R - n + i$ for every i, and that I is *strongly Cohen-Macaulay* if $H_i(I)$ is Cohen-Macaulay for every i. The class of strong Cohen-Macaulay ideals contains the class of licci ideals ([34]). If I is strongly Cohen-Macaulay and satisfies G_s then by [27, proof of 4.6] one has that depth $R/I^j \geq \dim R/I - j + 1$ and that $S_j(I) \cong I^j$ whenever $1 \leq j \leq s - g + 1$.

There is a natural exact sequence

$$H_1(I) \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0$$

induced by the first syzygies of I. It turns out that I is syzygetic if and only if the first map in this sequence is injective. This implies that $H_1(I)$ is torsionfree as an R/I-module, and the converse holds if I is generically a complete intersection.

Elements $a_1, ..., a_n$ generating I form a *d-sequence* if

$$((a_1, ..., a_i) : (a_{i+1})) \cap I = (a_1, ..., a_i)$$

for every $0 \leq i \leq n-1$. Any ideal satisfying sliding depth and G_∞ is generated by a d-sequence, and any ideal generated by a d-sequence is of linear type ([27]).

CHAPTER 1

COHEN-MACAULAY BLOWING-UP ALGEBRAS

In this chapter we study the blowing-up algebras $\mathcal{R}$ and G of ideals having good residual intersection properties. In Section 1 we develop certain technical results about residual intersections that we will make use of throughout the chapter. In Section 2, for ideals having sufficiently many Cohen-Macaulay residual intersections, we prove the Cohen-Macaulayness of $\mathcal{R}$ and G when the the reduction number is sufficiently small. We study the defining equations of $\mathcal{R}$ is Section 3. In particular, under conditions similar to those of Section 2 that guarantee that $\mathcal{R}$ is Cohen-Macaulay, we compute the degrees and the number of defining equations of $\mathcal{R}$. For certain Gorenstein ideals of grade 3, we find the equation defining $\mathcal{R}$ explicitly in terms of the presentation matrix. In Section 4, for ideals having second analytic deviation one, we obtain partial converses to the results of Section 2 by showing that, under certain assumptions, the Cohen-Macaulayness of G forces the reduction number to be "small". We make some remarks about ideals generated by analytically independent elements in Section 5.

The results of the first two sections will appear in the joint paper [44], while the material of the later sections will appear in [45].

1.1 Artin-Nagata Properties

We begin by defining the notion of a residual intersection, the properties of which will play an important role in this chapter.

Definition 1.1.1 Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g, let K be a proper R-ideal, and let $s \geq g$ be an integer.

(a) K is an s-residual intersection of I if there exists an R-ideal $\mathfrak{a} \subset I$ with $K = \mathfrak{a} : I$ and ht $K \geq s \geq \nu(\mathfrak{a})$.

(b) K is a geometric s-residual intersection of I if K is an s-residual intersection of I and if in addition ht $I + K > s$.

If R is Gorenstein, I is unmixed and $s = g$, then the notion of residual intersection corresponds to "linkage", and geometric residual intersections correspond to "geometric linkage". Thus the study of residual intersections is a generalization of the study of linkage.

It is particularly interesting to know when residual intersections are Cohen-Macaulay. This problem was studied by Artin and Nagata ([7]), and more recently by Huneke and Ulrich ([36], [40], [68]). The term "Artin-Nagata", for this property, was coined by Ulrich in [68].

Definition 1.1.2 Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g and let s be an integer.

(a) I satisfies AN_s if for every $g \leq i \leq s$ and every i-residual intersection K of I, R/K is Cohen-Macaulay.

(b) I satisfies AN_s^- if for every $g \leq i \leq s$ and every geometric i-residual intersection K of I, R/K is Cohen-Macaulay.

Clearly the Artin-Nagata property AN_s^- is weaker than the property AN_s. It can be strictly weaker in that it can happen that an ideal admits s-residual intersections but no geometric s-residual intersections (for $s = g$ take any unmixed ideal which is not generically a complete intersection). Of course, if s is sufficiently large an ideal will not even admit any s-residual intersections. To avoid this type of trivial obstruction, one usually assumes the condition G_s of [7] which guarantees the existence of s-residual intersections (and geometric $(s-1)$-residual intersections). The following two theorems are the major results known to guarantee that an ideal has Artin-Nagata properties.

Theorem 1.1.3 (Herzog-Vasconcelos-Villarreal [28]) Let R be a local Cohen-Macaulay ring and let I be an R-ideal satisfying G_s and sliding depth.

Then I satisfies AN_s.

Theorem 1.1.4 (Ulrich [68]) Let R be a local Gorenstein ring of dimension d, let I be an R-ideal of grade g, and assume that I satisfies G_s and that depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq s - g + 1$. Then

(a) I satisfies AN_s;

(b) for every $g \leq i \leq s$ and every i-residual intersection $K = \mathfrak{a} : I$ of I, $\omega_{R/K} \cong I^{i-g+1}/\mathfrak{a}I^{i-g}$.

In particular the assumptions of Theorem 1.1.4 hold for a strongly Cohen-Macaulay satisfying G_s. In that case the result was first proved by Huneke [36] (for geometric residual intersections).

We will need the following basic, but important, lemma. It states roughly that if an ideal has a sufficiently good Artin-Nagata property, then one can build up any residual intersection as a sequence of "links."

Lemma 1.1.5 ([68, 1.7]) Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade g satisfying G_s, and let $K = \mathfrak{a} : I$ be an s-residual intersection or let $K = R$ and $\nu(I) \leq s$.

(a) There exists a generating set $a_1, ..., a_s$ of $\mathfrak{a}$ such that for every $g \leq i \leq s-1$ with $\mathfrak{a}_i = (a_1, ..., a_i)$, one has that $K_i = \mathfrak{a}_i : I$ is a geometric i-residual intersection or $K_i = R$. Moreover, any permutation of $a_1, ..., a_i$ enjoys the same property.

(b) If I satisfies AN_{s-2}^- then $a_1, ..., a_s$ forms a d-sequence.

(c) If I satisfies AN_t^- for some $t \leq s-1$ and $K \neq R$, then the following hold for $0 \leq i \leq t+1$:

(i) $K_i = \mathfrak{a}_i : (a_{i+1})$ and $\mathfrak{a}_i = K_i \cap I$ for $i \leq s-1$.

(ii) depth $R/\mathfrak{a}_i = d - i$.

(iii) K_i is unmixed of height i.

(iv) ht $\overline{I} = 1$ where " $-$ " denotes images in R/K_i, for $i \leq s-1$.

The next two lemmas show that the Artin-Nagata property is preserved after performing certain standard operations. These are analogous to results observed by Huneke [37] in the case of strongly Cohen-Macaulay ideals. We will state them for the case AN_s^-, but they also hold for AN_s.

Lemma 1.1.6 Let R be a local Cohen-Macaulay ring, let I be an R-ideal, let $x \in I$ be R-regular, let " $*$ " denote images in $R/(x)$, and assume that I satisfies AN_s^-.

Then I^* satisfies AN_{s-1}^-.

Proof. Let $(a_1^*, ..., a_i^*) : I^*$ be a geometric i-residual intersection of I^*, with ht $I^* \leq i \leq s-1$, and let $K = (a_1, ..., a_i, x) : I$. Since K^* coincides with the given residual intersection of I^*, it follows that K is a geometric i-residual intersection of I. Hence $R^*/(a_1^*, ..., a_i^*) : I^* \cong R/K$ is Cohen-Macaulay. □

Lemma 1.1.7 Let R be a local Cohen-Macaulay ring, let I be an R-ideal, let " $-$ " denote images in $R/0 : I$, assume that $I \cap (0 : I) = 0$, that I satisfies G_1 and that I satisfies AN_s^-.

Then $\overline{I}$ satisfies AN_s^-.

Proof. We may assume that $s \geq 0$. Since $I \cap (0 : I) = 0$ and I satisfies G_1, $0 : I$ is a geometric 0-residual intersection of I, and $\overline{R}$ is Cohen-Macaulay as I satisfies AN_0^-. Let $(\overline{a}_1, ..., \overline{a}_i) : \overline{I}$ be a geometric i-residual intersection of $\overline{I}$, with ht $\overline{I} \leq i \leq s$. We may assume that $a_i \in I$. Set $\mathfrak{a} = (a_1, ..., a_i)$ and $K = \mathfrak{a} : I$. Then

$$0 : I \subset K \subset (\mathfrak{a} + 0 : I) : I = (\mathfrak{a} + I \cap (0 : I)) : I = K,$$

hence $R/K \cong \overline{R}/\overline{\mathfrak{a}} : \overline{I}$ and $R/I + K \cong \overline{R}/\overline{I} + \overline{\mathfrak{a}} : \overline{I}$. It follows that K is a geometric i-residual intersection and that $\overline{R}/\overline{\mathfrak{a}} : \overline{I} \cong R/K$ is Cohen-Macaulay. □

The following lemma is a generalization of a similar result of [68]. It extends Lemma 1.1.5(c) to the case involving higher powers of I.

Lemma 1.1.8 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal, let k and t be integers, assume that I satisfies G_s and AN_{s-3}^- locally in codimension $s-1$, that I satisfies AN_t^-, and that depth $R/I^j \geq d - s + k - j$ for $1 \leq j \leq k$, and let $K = \mathfrak{a} : I$ be an s-residual intersection (or $K = R$ and $\nu(I) \leq s$), and let $\mathfrak{a}_i$ be the ideals as defined in Lemma 1.1.5. Then the following conditions hold:

(a) depth $R/\mathfrak{a}_i I^j \geq \min\{d-i, d-s+k-j\}$ for $0 \leq i \leq s$, $\max\{0, i-t-1\} \leq j \leq k$.

(b) $(\mathfrak{a}_i : (a_{i+1})) \cap I^j = \mathfrak{a}_i I^{j-1}$ for $0 \leq i \leq s-1$, $\max\{1, i-t\} \leq j \leq k$.

Proof. We first show that if (a) holds for i, then so does (b). However, to prove (b) one may check the equality locally at every $p \in \operatorname{Ass}(R/\mathfrak{a}_i I^{j-1})$. By (a), p has height at most $d-\min\{d-i, d-s+k-j+1\} = \max\{i, s-k+j-1\} \leq s-1$. By assumption I_p satisfies AN^-_{s-3}. But then by Lemma 1.1.5 (c.iii), any geometric $(s-2)$-residual intersection is unmixed of height $s-2$, hence is Cohen-Macaulay because it is at most one-dimensional. We conclude that I_p even satisfies AN^-_{s-2}. Now by Lemma 1.1.5(b), $a_1, ..., a_s$ form a d-sequence in R_p. Moreover, since $\operatorname{ht} K \geq s$, we have that $I_p = \mathfrak{a}_p \neq R_p$. Replacing R_p by R, we have that $(\mathfrak{a}_i : (a_{i+1})) \cap \mathfrak{a} = \mathfrak{a}_i$ for $0 \leq i \leq s-1$ since $a_1, ..., a_s$ form a d-sequence. Hence

$$(\mathfrak{a}_i : (a_{i+1})) \cap \mathfrak{a}^j = (\mathfrak{a}_i : (a_{i+1})) \cap \mathfrak{a} \cap \mathfrak{a}^j$$
$$= \mathfrak{a}_i \cap \mathfrak{a}^j.$$

Thus to prove (b) it is enough to show that

$$\mathfrak{a}_i \cap \mathfrak{a}^j = \mathfrak{a}_i \mathfrak{a}^{j-1}. \tag{1.1.9}$$

But this follows by standard properties of d-sequences. Indeed, if we let $\mathfrak{b} = (a_{i+1}, ..., a_s)$ then to show (1.1.9) it is enough to show that

$$\mathfrak{a}_i \cap \mathfrak{b}^j \subset \mathfrak{a}_i \mathfrak{b}^{j-1}.$$

Since $a_{i+1}, ..., a_s$ form a d-sequence modulo $\mathfrak{a}_i$, this follows from [35, 2.1].

Thus it is enough to prove (a), which we do by induction on i. Since the result is trivial for $i = 0$, we may assume that $0 \leq i \leq s-1$ and that (a) and (b) hold for i. We show that (a) holds for $i+1$. If $j = 0$, then $i \leq t$ and the result follows from

Lemma 1.1.5(c.ii). Thus we may assume that $j \geq 1$. Now using (b) for (i),

$$\begin{aligned} a_{i+1}\mathfrak{a}_i I^{j-1} &\subset \mathfrak{a}_i I^j \cap a_{i+1} I^j \\ &= a_{i+1}[(\mathfrak{a}_i I^j : (a_{i+1})) \cap I^j] \\ &\subset a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^j] \\ &= a_{i+1}\mathfrak{a}_i I^{j-1}, \end{aligned}$$

from which it is clear that all the containments are equalities.

But using part (b) with $i = 0$ shows that

$$(0 : (a_{i+1})) \cap \mathfrak{a}_i I^{j-1} \subset (0 : (a_{i+1})) \cap I^j = 0,$$

and hence that $a_{i+1}\mathfrak{a}_i I^{j-1} \cong \mathfrak{a}_i I^{j-1}$ and $a_{i+1} I^j \cong I^j$. Thus the required depth estimate for $R/\mathfrak{a}_{i+1} I^j$ follows if $i = 0$ from the latter isomorphism, while if $i > 0$ it follows by induction from the exact sequence

$$0 \longrightarrow \mathfrak{a}_i I^{j-1} \longrightarrow \mathfrak{a}_i I^j \oplus I^j \longrightarrow \mathfrak{a}_{i+1} I^j \longrightarrow 0. \quad \square$$

We want to apply these results to minimal reductions.

Remark 1.1.10 Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with analytic spread ℓ which satisfies G_ℓ, and let J be a minimal reduction of I such that ht $J : I \geq \ell$. Then there exists a generating set $a_1, ..., a_\ell$ of J with the following property: $\mathfrak{a}_i : I$ is a geometric i-residual intersection of I for ht $I \leq i \leq \ell - 1$, where $\mathfrak{a}_i = (a_1, ..., a_i)$. Moreover, any permutation of $a_1, ..., a_\ell$ enjoys the same property. If in addition, I satisifes $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, then ht $J : I \geq \ell$ holds for every minimal reduction J of I.

Proof. Using Lemma 1.1.5(a), we only have to prove the last statement, which follows from [68, 1.11] since I is of linear type locally in codimension $\ell - 1$. $\square$

Using Lemma 1.1.8, and applying the previous remark, we prove the following important technical result, which generalizes a result of [63].

Lemma 1.1.11 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, let k and t be integers with $r \leq k$ and $t \geq \ell - k - 1$, assume that I satisfies G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell - 1$, that I satisfies AN_t^-, and that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$, let J be a minimal reduction of I with $r_J(I) = r$, and let $\mathfrak{a}_i$ be the ideals in Remark 1.1.10.

Then $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$ for $0 \leq i \leq \ell - 1$ and $j \geq \max\{1, i - t\}$.

Proof. If $j \leq k$ the result follows from Lemma 1.1.8(b) with $s = \ell$. Hence we may assume that $j \geq k + 1$. We will prove the result by decreasing induction on i.

For $i = \ell$ note that the result holds automatically since $j \geq k + 1 \geq r + 1$ and hence $I^j = JI^{j-1}$. For $0 \leq i \leq \ell - 1$, note that $i - t \leq \ell - 1 - t \leq k$. Since the result clearly holds for $j = 1$, we may assume that $j \geq \max\{2, i - t + 1\}$, and that the equality holds for $i + 1$ by decreasing induction on i, and that $\mathfrak{a}_i \cap I^{j-1} = \mathfrak{a}_i I^{j-2}$ holds by increasing induction on j. Then

$$
\begin{aligned}
\mathfrak{a}_i \cap I^j &= \mathfrak{a}_i \cap \mathfrak{a}_{i+1} \cap I^j \\
&= \mathfrak{a}_i \cap (\mathfrak{a}_{i+1} I^{j-1}) \\
&= \mathfrak{a}_i \cap [\mathfrak{a}_i I^{j-1} + a_{i+1} I^{j-1}] \\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}[\mathfrak{a}_i : (a_{i+1}) \cap I^{j-1}] \\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^{\max\{i-t,1\}} \cap I^{j-1}] \\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}[\mathfrak{a}_i \cap I^{j-1}] \text{ by 1.1.8(b)} \\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}[\mathfrak{a}_i I^{j-2}] \\
&= \mathfrak{a}_i I^{j-1}. \quad \square
\end{aligned}
$$

We point out that one may compute the reduction number by checking it locally in codimension ℓ.

Remark 1.1.12 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ, let $k \geq 1$ be an integer, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-k-1}$, that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$, and that J is a minimal reduction of I with $r_{J_p}(I_p) \leq k$ for all $p \in V(I)$ with dim $R_p = \ell$.

Then $r(I) \leq k$.

Proof. It is enough to show that $I^{k+1} = JI^k$, which may be checked locally at every $p \in$ Ass (R/JI^k). But by Lemma 1.1.8(a) depth $R/JI^k \geq d - \ell$, and thus any such prime has height at most ℓ. Since I satisfies $AN^-_{\ell-3}$, I and J coincide locally in codimension $\ell - 1$ by Remark 1.1.10. Hence it is enough to check the equality at primes of height exactly ℓ, in which case the result holds by assumption. □

1.2 Cohen-Macaulayness of the associated graded ring

For the rest of this chapter we will fix the following notation: R will be a local Cohen-Macaulay ring of dimension d with infinite residue field, and I will be a proper R-ideal with grade g, minimal number of generators n, analytic spread ℓ, and reduction number r; G and $\mathcal{R}$ will denote the associated graded ring and the Rees algebra of I. Moreover, we will always assume that I satisfies the condition G_ℓ.

The passage of the Cohen-Macaulay property from G to $\mathcal{R}$ is well-understood by the following result:

Theorem 1.2.1 (Simis-Ulrich-Vasconcelos [61, 3.6]) Let R be a local Cohen-Macaulay ring with infinite residue field, and let I be an R-ideal of grade g and analytic spread ℓ, and assume that I satisfies G_ℓ and is not nilpotent. Then the following conditions are equivalent.

(a) $\mathcal{R}$ is Cohen-Macaulay;

(b) G is Cohen-Macaulay, $g > 0$ and $r(I) \leq \ell - 1$.

Slightly different results have also been obtained by Johnston and Katz ([46]) and by Aberbach, Huneke and Trung ([5]), without the assumption of G_ℓ.

These results allow us to focus our study on the Cohen-Macaulayness of the associated graded ring G; it has been shown to be Cohen-Macaulay under any of the following additional assumptions:

- $r \leq 1$, $\ell \leq g+1$ and depth $R/I \geq d - \ell$ ([30])
- $r \leq 1$, $\ell \leq g+2$, R/I is Cohen-Macaulay, I satisfies (CI_1) and R is Gorenstein ([31])
- $r \leq 1$ and I satisfies sliding depth ([74])
- $r \leq 1$, I satisfies $AN^-_{\ell-2}$ and depth $R/I \geq d - \ell$ ([68])
- $r \leq 2$, $\ell = g+1$, R/I is Cohen-Macaulay and depth $R/I^2 \geq d - \ell$ ([20])
- $r \leq 2$, $\ell = g+2$, R/I is Cohen-Macaulay, depth $R/I^2 \geq d - \ell$, I satisfies (CI_1) and R is Gorenstein ([21])
- $r \leq 2$, $\ell = 3$, $n \leq 4$ and I is perfect of grade 2 ([4])
- $r \leq 2$, $\ell = 4$, $n \leq 5$ and I is perfect Gorenstein of grade 3 ([3])
- $r \leq 3$, $\ell = g+2$ and depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq 3$ ([2])
- $r \leq \ell - g$, $\ell \geq g+3$, depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g - 1$, depth $R/I^{\ell-g} \geq d - \ell$, I satisfies $(CI_{\ell-g-1})$ and R is Gorenstein ([63])

- $r \leq \ell - g + 1$, $n \leq \ell + 1$, I is strongly Cohen-Macaulay and R is Gorenstein ([61])
- $r \leq \ell - g + 1$, $\ell \geq g + 2$, depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g + 1$, I satisfies $(CI_{\ell - g - 1})$ and R is Gorenstein ([63])

One of our main results will be a theorem (Theorem 1.2.8), which includes essentially all of the above results as special cases. The key is to systematically exploit the Artin-Nagata properties of the ideal. Indeed, although most of the above results make no specific mention of Artin-Nagata properties, they are implied by Theorems 1.1.3 and 1.1.4. (Subsequently, futher generalizations have recently been obtained by Aberbach ([1]) and by Goto, Nakamura and Nishida ([22]), ([23]).)

We begin with a general result about when a graded ring is Cohen-Macaulay. By $[M]_{\geq i}$ we denote the truncated submodule $\oplus_{j \geq i} M_j$ of a graded module $M = \oplus_j M_j$.

Proposition 1.2.2 Let S be a homogeneous noetherian ring of dimension d with S_0 local, let $I = S_+$, let $b_1, ..., b_\ell$ be linear forms in S, set $\mathfrak{b}_i = (b_1, ..., b_i)$ for $-1 \leq i \leq \ell$ (where $(\emptyset) = 0$), $J = \mathfrak{b}_\ell$, let g be an integer with $0 \leq g \leq \ell$, assume that $I^{k+1} \subset J$ (i.e. J is a reduction of I with $r_J(I) \leq k$) and that the following conditions are satisfied:

(a) $[\mathfrak{b}_i : (b_{i+1})]_{\geq i-g+1} = [\mathfrak{b}_i]_{\geq i-g+1}$ for $0 \leq i \leq \ell - 1$,

(b) depth $[S/\mathfrak{b}_i]_{i-g+1} \geq d - i - 1$ for $g - 1 \leq i \leq \ell - 1$,

(c) depth $[S/J]_j \geq d - \ell$ for $\ell - g + 1 \leq j \leq k$.

Then S is Cohen-Macaulay.

Proof. Note that by (a), $b_1, ..., b_g$ form an S-regular sequence and that one has $[S/\mathfrak{b}_{g-1}]_0 = [S/\mathfrak{b}_{-1}]_0 = S_0$. Hence we may factor out $\mathfrak{b}_g$ and assume that $g = 0$. Now conditions (a), (b), (c) are now

(a′) $[\mathfrak{b}_i : (b_{i+1})]_{\geq i+1} = [\mathfrak{b}_i]_{\geq i+1}$ for $0 \leq i \leq \ell - 1$, (b′) depth $[S/\mathfrak{b}_{i-1}]_i \geq d-i$ for $0 \leq i \leq \ell$,

(c′) depth $[S/J]_j \geq d-\ell$ for $\ell+1 \leq j \leq k$.

For $0 \leq i \leq \ell$ consider the graded S-modules $M_{(i)} = [S/\mathfrak{b}_i]_{\geq i+1} = I^{i+1}/\mathfrak{b}_i I^i$ and $N_{(i)} = I^i/\mathfrak{b}_{i-1}I^{i-1} + b_i I^i$ (where $I^{-1} = I^0 = S$). Observe that $M_{(i)}$ can be obtained as a trunction of $N_{(i)}$, namely that $M_{(i)} = [N_{(i)}]_{\geq i+1}$. In addition, $M_{(i-1)}$ coincides with $N_{(i)}$ in degree i, that is to say $[N_{(i)}]_i = [S/\mathfrak{b}_{i-1}]_i$. Thus we have a exact sequence

$$0 \longrightarrow M_{(i)} \longrightarrow N_{(i)} \longrightarrow [S/\mathfrak{b}_{i-1}]_i \longrightarrow 0. \tag{1.2.3}$$

Alternatively, we may view $N_{(i+1)}$ as a quotient of $M_{(i)}$, namely $N_{(i+1)} = M_{(i)}/b_{i+1}M_{(i)}$, for $0 \leq i \leq \ell-1$. Moreover, b_{i+1} is regular on $M_{(i)}$ by (a′). Hence for $0 \leq i \leq \ell-1$ we have an exact sequence

$$0 \longrightarrow M_{(i)}(-1) \xrightarrow{b_{i+1}} M_{(i)} \longrightarrow N_{(i+1)} \longrightarrow 0. \tag{1.2.4}$$

We will prove by decreasing induction on i, $0 \leq i \leq \ell$, that $\operatorname{depth}_S N_{(i)} \geq d-i$. Since $N_0 = S$, it will follows that S is Cohen-Macaulay.

First consider the case where $i = \ell$. Then $N_{(\ell)} = [S/\mathfrak{b}_{\ell-1}]_\ell \oplus \bigoplus_{j=\ell+1}^{k}[S/J]_j$. By (b′) and (c′), this has depth at least $d-\ell$ as an S_0-module and hence as an S-module.

Now let $0 \leq i \leq \ell-1$ and suppose that we know the induction assumption holds for $i+1$, i.e. that $\operatorname{depth}_S N_{(i+1)} \geq d-i-1$. Then $\operatorname{depth}_S M_{(i)} \geq d-i$ by (1.2.4). But by (b′) we also have $\operatorname{depth}_S [S/\mathfrak{b}_{i-1}]_i = \operatorname{depth}_{S_0} [S/\mathfrak{b}_{i-1}]_i \geq d-i$. Hence by the sequence (1.2.3) we conclude that $\operatorname{depth}_S N_{(i)} \geq d-i$. □

If we let $S = G$ be the associated graded ring, then this proposition gives a criterion for G to be Cohen-Macaulay. Moreover, in the previous section we have essentially shown that conditions like those of the proposition hold under suitable conditions. We obtain the following theorem:

Theorem 1.2.5 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, and assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell-1$, that depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell-g+1$, and that $r \leq \ell-g+1$.

Then G is Cohen-Macaulay.

Proof. Let J be a minimal reduction of I with $r_J(I) = r$, let $a_1, ..., a_\ell$ and $\mathfrak{a}_i$ be as in Lemma 1.1.10, and let a_i' be the image of a_i in $[G]_1$. We apply Proposition 1.2.2 to the ring $S = G$, linear forms $b_i = a_i'$ and with $k = \max\{r, \ell - g\}$.

We first check that $[\mathfrak{b}_i : (b_{i+1})]_{\geq i-g+1} = [\mathfrak{b}_i]_{\geq i-g+1}$ for $0 \leq i \leq \ell - 1$. Let $u \in [\mathfrak{b}_i : (b_{i+1})]_j$, with $j \geq i - g + 1$, and write $u = x + I^{j+1}$ for some element $x \in I^j$. Then $a_{i+1}x \in \mathfrak{a}_i + I^{j+2}$. If $i \leq \ell - 2$ then by Lemma 1.1.11 (with $t = g - 1$) we have

$$\begin{aligned} a_{i+1}x &\in \mathfrak{a}_{i+1} \cap (\mathfrak{a}_i + I^{j+2}) \\ &= \mathfrak{a}_i + \mathfrak{a}_{i+1} \cap I^{j+2} \\ &= \mathfrak{a}_i + \mathfrak{a}_{i+1} I^{j+1} \\ &= \mathfrak{a}_i + a_{i+1} I^{j+1}. \end{aligned}$$

On the other hand if $i = \ell - 1$ then this holds automatically since $I^{j+2} = JI^{j+1}$ as $j+2 \geq \ell-g+2 \geq r+1$. In any event, we conclude that $a_{i+1}(x-y) \in \mathfrak{a}_i$ for some element $y \in I^{j+1}$. But since $x - y$ coincides with x modulo I^{j+1} we may as well assume that $a_{i+1}x \in \mathfrak{a}_{i+1}$. Now we may use Lemma 1.1.8(b) and Lemma 1.1.11 to conclude that

$$\begin{aligned} x &\in (\mathfrak{a}_i : (a_{i+1})) \cap I^j \\ &= (\mathfrak{a}_i : (a_{i+1})) \cap I^{i-g+1} \cap I^j \\ &= \mathfrak{a}_i I^{i-g} \cap I^j \\ &\subset \mathfrak{a}_i \cap I^j \\ &= \mathfrak{a}_i I^{j-1}. \end{aligned}$$

It follows that $u \in [\mathfrak{b}_i]_j$.

Finally, we have to check that the conditions depth $[S/\mathfrak{b}_i]_{i-g+1} \geq d-i-1$ for $g-1 \leq i \leq \ell-1$ and depth $[S/J]_{\ell-g+1} \geq d-\ell$ hold. The assumption that depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq k$ and the exact sequences

$$0 \longrightarrow I^j/I^{j+1} \longrightarrow R/I^{j+1} \longrightarrow R/I^j \longrightarrow 0$$

give by induction the estimates depth I^j/I^{j+1} = depth $[S]_j \geq d-g-j$ for $1 \leq j \leq \ell-g$.

Since $[\mathfrak{b}_i : (b_{i+1})]_{\geq i-g+1} = [\mathfrak{b}_i]_{\geq i-g+1}$ for $0 \leq i \leq \ell-1$, there are exact sequences

$$0 \longrightarrow [S/\mathfrak{b}_i]_j \longrightarrow [S/\mathfrak{b}_i]_{j+1} \longrightarrow [S/\mathfrak{b}_{i+1}]_{j+1} \longrightarrow 0$$

for $0 \leq i \leq \ell-1$ and $j \geq i-g+1$. These sequences then give by induction that depth $[S/\mathfrak{b}_i]_j \geq d-g-j$ for $0 \leq i \leq \ell-1$ and $i-g+1 \leq j \leq \ell-g$. In particular we obtain that depth $[S/\mathfrak{b}_i]_{i-g+1} \geq d-i-1$ for $0 \leq i \leq \ell-1$ as required. Finally if $i = \ell$, then

$$\begin{aligned} [S/J]_{\ell-g+1} &= I^{\ell-g+1}/JI^{\ell-g} + I^{\ell-g+2} \\ &= I^{\ell-g+1}/JI^{\ell-g} + JI^{\ell-g+1} \\ &= I^{\ell-g+1}/JI^{\ell-g} \end{aligned}$$

since I has reduction number $r \leq \ell - g + 1$. Since by assumption depth $R/I^{\ell-g+1} \geq d - \ell$, the required depth estimate follows from Lemma 1.1.8(a). □

The above result was proved by Tang ([63, 5.5]) when R is Gorenstein (in which case one can delete the local Artin-Nagata condition by Theorem 1.1.4) and I satisfies $(CI_{\ell-g-1})$. This latter condition is much stronger than the condtion G_ℓ when ℓ is not very small. Theorem 1.2.5 would actually follow from Tang's proof, which was a major motivation for our work. But as his proof is rather computational, we have instead used Proposition 1.2.2.

We immediately obtain the following application.

Corollary 1.2.6 Let R be a local Cohen-Macaulay with infinite residue field and let I be a strongly Cohen-Macaulay R-ideal satisfying G_ℓ and $r(I) \leq \ell - g + 1$. Then G is Cohen-Macaulay.

Proof. As I is strongly Cohen-Macaulay and satisfies G_ℓ, one knows that depth $R/I^j \geq \dim R - g - j + 1$ for $1 \leq j \leq \ell - g + 1$ and by Theorem 1.1.3, I satisfies AN_ℓ. The result now follows from Theorem 1.2.5. □

When the ambient ring R is Gorenstein, it turns out that the assumptions in Theorem 1.2.5 force the equality $r = \ell - g + 1$ (or $r = 0$) (Corollary 1.2.12). But it is important to know that the Theorem holds in any *Cohen-Macaulay* ring, even if one is solely interested in a regular ambient ring R. This is because we want to use this result as a base case of an even more general result, in which we will reduce to Theorem 1.2.5 by factoring out certain residual intersections. (This method has been used by several authors, and goes back to Huneke ([32]).) Under suitable Artin-Nagata properties, we can maintain only the Cohen-Macaulayness of the ambient ring, and thus one needs to know results in that generality.

We will need to isolate a special case of [30, 2.9]. For convenience we include a proof, using an argument of [74].

Proposition 1.2.7 Let R be a local Cohen-Macaulay ring of dimension d, let I be an R-ideal with $I^2 = aI$ for some $a \in I$, assume that $I_p = 0$ for every $p \in \mathrm{Ass}(R) \cap V(I)$, and that depth $R/I \geq d-1$.

Then G is Cohen-Macaulay.

Proof. The R-homomorphism from the polynomial ring $R[T]$ to the Rees algebra $R[It]$, mapping T to at, induces an $R[T]$-homomorphism $IR[T] \to IR[It]$ which is surjective because $I^2 = aI$. To see that it is also injective it is enough to show that $(0 : (a^j)) \cap I = 0$ for all j. But this is clear, as one may check it locally at every $p \in \mathrm{Ass}(R) \cap V(I)$, and I is zero at any such prime. It follows that $IR[It] \cong IR[T]$ and hence has depth at least $d+1$. But then from the fundamental exact sequences

$$0 \longrightarrow I\mathcal{R}(-1) \longrightarrow \mathcal{R} \longrightarrow R \longrightarrow 0$$

$$0 \longrightarrow I\mathcal{R} \longrightarrow \mathcal{R} \longrightarrow G \longrightarrow 0$$

and a depth chase, it follows that G has depth at least d. □

We are now ready to prove one of our main results. As we mentioned, this result generalizes most of the previously known work. By performing certain standard operations, we will reduce this result to Theorem 1.2.5. The important fact is that the Artin-Nagata property is preserved in the process, which we previously verified in Lemmas 1.1.6 and 1.1.7.

Theorem 1.2.8 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r, let $k \geq 1$ be an integer, and assume that $r \leq k$, that I satisifes G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$, and that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$.

Then G is Cohen-Macaulay.

Proof. By setting $j = 1$ we see that depth $R/I \geq d - \ell + k - 1$. It follows that $k \leq \ell - g + 1$ since depth $R/I \leq d - g$. We set $\delta = \delta(I) = \ell - g + 1 - k \geq 0$. We will proceed by induction on δ. If $\delta = 0$ then $k = \ell - g + 1$ and the result is precisely Theorem 1.2.5. Hence we may assume that $\delta > 0$ and that the Theorem holds for all smaller values of δ. Note that we now have that $\ell \geq g + k \geq g + 1$.

Let J be a minimal reduction of I with $r_J(I) = r$, and let $\mathfrak{a}_i$ be the ideals as in Lemma 1.1.10. If we set $t = \ell - \max\{2, k\}$ then $g - t = \max\{g + 2 - \ell, 1 - \delta\} \leq 1$. Hence by Lemma 1.1.11, we have that $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$ for $1 \leq i \leq g$ and all $j \geq 1$. It follows that the images $a'_1, ..., a'_g$ of $a_1, ..., a_g$ in $[G]_1$ form a G-regular sequence ([71]). We first show that we may factor out $\mathfrak{a}_g$ and assume that $g = 0$. Let us denote by " * " images in $R/\mathfrak{a}_g$.

As $G(I^*) \cong G/(a'_1, ..., a'_g)G$, it follows that G is Cohen-Macaulay if and only if $G(I^*)$ is Cohen-Macaulay. Now dim $R^* = d - g$ and $\ell(I^*) = \ell(I) - g$, whereas k may be taken as unchanged. By Remark 1.1.10, $\mathfrak{a}_i : I$ is a geometric i-residual intersection for every $g \leq i \leq \ell - 1$, hence $I_p = (a_1, ..., a_i)_p$ for all $p \in V(I)$ with dim $R_p \leq i$. It follows that I^* satisfies $G_{\ell(I^*)}$. We next show that the condition on the depth of the powers is preserved. Since $R^*/I^{*j} \cong R/\mathfrak{a}_g + I^j$, by Lemma 1.1.11 we have an exact sequence

$$0 \longrightarrow R/\mathfrak{a}_g I^{j-1} \longrightarrow R/\mathfrak{a}_g \oplus R/I^j \longrightarrow R^*/I^{*j} \longrightarrow 0.$$

Now by Lemma 1.1.8(a) and since $\delta > 0$, we have that depth $R/\mathfrak{a}_g I^{j-1} \geq$

$d - \ell + k - j + 1$ for $1 \leq j \leq k$. Hence our assumption that depth $R/I^j \geq d - \ell + k - j$ together with this sequence implies that depth $R^*/I^{*j} \geq d - \ell + k - j = \dim R^* - \ell(I^*) + k - j$ for $1 \leq j \leq k$. Finally the Artin-Nagata conditions are preserved by Lemma 1.1.6. Hence we have reduced to the case where $g = 0$. Now $\delta(I) = \ell + 1 - k$.

Now we have that $\ell \geq k \geq 1$. Suppose that $\ell = 1$. Then $r \leq k = 1$ and hence $I^2 = aI$ for some $a \in I$. Further, since I satisfies G_1, it holds that I is locally zero at every associated prime of R containing I. In this case, we are done by Proposition 1.2.7.

Thus we may assume that $\ell \geq \max\{2, k\}$. In particular, we have that I satisfies AN_0^-. Now let $K = 0 : I$ and let " $^-$ " denote images in R/K. We will show that our assumptions are again preserved, and that δ decreases when passing from R to $\overline{R}$. First, note that by Lemma 1.1.5(c.i), we have that $I \cap K = 0$. Since I satisfies G_1, again it holds that $I_p = 0$ for every prime p containing I with dim $R_p = 0$. It follows that ht $I + K > 0$, and hence that K is a geometric 0-residual intersection of I. Since I satisfies AN_0^-, it holds that $\overline{R}$ is Cohen-Macaulay. We have $\dim \overline{R} = d$, $\ell(\overline{I}) = \ell$, and again k may be taken as unchanged. To see these last claims, it is enough to see that any minimal reduction of $\overline{I}$ lifts to a minimal reduction of I. But that is clear since given an equation $(\overline{I})^{s+1} = \overline{J_1}(\overline{I})^s$ in $\overline{R}$ for some integer $s \geq 0$ and where we may assume $J_1 \subset I$, then it follows that

$$I^{s+1} \subset (J_1 I^s + K) \cap I = J_1 I^s + I \cap K = J_1 I^s.$$

Now by Lemma 1.1.5(c.iv), $\overline{I}$ has grade 1, and clearly still satisfies G_ℓ since the dimension is unchanged. Again since $I \cap K = 0$ we have an exact sequence

$$0 \longrightarrow K \to gr_I(R) \longrightarrow gr_{\overline{I}}(\overline{R}) \longrightarrow 0. \qquad (1.2.9)$$

Since $\overline{R}$ is Cohen-Macaulay, it follows that depth $K = d$. Hence the degree 0 piece of this sequence gives us the estimate depth $\overline{R}/\overline{I} \geq \min\{\text{depth } K - 1, \text{ depth } R/I\}$ $\geq \min\{d-1, d-\ell+k-1\} = d-\ell+k-1$ since $k \leq \ell$. On the other hand, for all $j \geq 2$ we have the isomorphisms

$$\overline{I}^{j-1}/\overline{I}^{j} \cong I^{j-1}/I^{j} + I^{j-1} \cap K \cong I^{j-1}/I^{j-1}.$$

It follows that $\overline{I}$ satisfies the same depth estimate on its powers, namely that depth $\overline{R}/\overline{I}^{j} \geq d-\ell+k-j$ for $1 \leq j \leq k$. Finally, the Artin-Nagata properties are preserved by Lemma 1.1.7. Thus we have shown that $\overline{I}$ satisfies all of the assumptions of the theorem.

We have that $\delta(\overline{I}) = \ell(\overline{I})-$ grade $\overline{I} + 1 - k = \ell - k < \ell - k + 1 = \delta(I)$. Hence we may conclude by our induction on δ that $gr_{\overline{I}}(\overline{R})$ is Cohen-Macaulay. But as depth $K = d$, by the exact sequence (1.2.9) it then follows that $G = gr_I(R)$ is Cohen-Macaulay. □

We now give several corollaries.

Corollary 1.2.10 Under the assumptions of Theorem 1.2.8, $\mathcal{R}$ is Cohen-Macaulay if and only if $g \geq 2$, $g = 1$ and $r \neq \ell$, or I is nilpotent.

Proof. We know that G is Cohen-Macaulay and that $r \leq \ell - g + 1$. Hence by Theorem 1.2.1 the result if clear if I is not nilpotent. However, if I is nilpotent, one has $\ell = 0$, hence $r \leq 1$, and thus $I^2 = 0$. In this case $\mathcal{R} = R \oplus I$ is Cohen-Macaulay since dim $\mathcal{R} = d$ and depth $I \geq$ depth $R/I \geq d - \ell = d$. □

Corollary 1.2.11 Let R be a local Cohen-Macaulay ring with infinite residue field and let I be a perfect R-ideal of grade 2 satisfying G_ℓ.

Then $\mathcal{R}$ is Cohen-Macaulay if and only if $r(I) \leq \ell - 1$.

Proof. Since I is strongly Cohen-Macaulay, this follows immediately from Theorem 1.2.1 and Corollaries 1.2.6 and 1.2.10. □

Corollary 1.2.12 Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal of grade g and analytic spread ℓ and assume that I satisfies G_ℓ and that depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell - g$ and that $r(I) \leq \ell - g$.

Then I is strongly Cohen-Macaulay and satisfies G_∞ (and $r=0$).

Proof. It will be enough to show that $r=0$, for then I satisfies G_∞ and the result follows from [68, 2.13]. Thus we may assume that $\ell > g$.

By Theorem 1.1.4 we know that I satisfies $AN_{\ell-1}$, and hence G is Cohen-Macaulay by Theorem 1.2.8. Since then equality holds in Burch's inequality, one knows that in particular that depth $R/I^{\ell-g+1} \geq d-\ell$. Together with our assumption, we now have that depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell-g+1$.

Let J be a minimal reduction of I with $r_J(I) = r$. We must show that $J = I$. Suppose that this is not the case. Then by Remark 1.1.10, $K = J : I$ is an ℓ-residual intersection of I. We may now apply Theorem 1.1.4 to compute the canonical module of R/K; we find that $\omega_{R/K} \cong I^{\ell-g+1}/JI^{\ell-g}$. Since this module cannot vanish, it follows that $r \geq \ell - g + 1$, which is a contradiction. Hence $I = J$ holds and thus $r = 0$. □

Example 1.2.13 Let k be an infinite field, let X be a generic alternating 5 by 5 matrix, let Y be a generic 5 by 1 matrix, put $R = k[X, Y]$ (possibly localized at the irrelevant maximal ideal), and let $I = Pf_4(X) + I_1(XY)$ be the R-ideal generated by the 4 by 4 Pfaffians of X and the entries of the product matrix XY.

It is well-known that R/I is the associated graded ring of the ideal $Pf_4(X)$ in

$k[X]$ ([34, 2.2]) and it follows that I has grade 5, that I satisfies (CI_4) ([33, proof of 2.1]), and that R/I is Gorenstein ([34, 2.2]). Futhermore, $\ell(I) = 9$, and a computation using the computer algebra system MACAULAY shows that R/I^2 and R/I^3 are Cohen-Macaulay. By Theorem 1.1.4, it follows that I satisifes AN_7, and that $r(I) = 1$ by Remark 1.1.12. We may thus apply Theorem 1.2.8 to conclude that $\mathcal{R}$ is Cohen-Macaulay.

1.3 Number of Defining Equations

In this section we study the defining equations of the Rees algebra via the canonical homomorphism

$$\alpha : S(I) \longrightarrow \mathcal{R}(I)$$

from the symmetric algebra of I onto the Rees algebra. Recall that $\mathcal{A}$ denotes the kernel of this homomorphism.

Now let I be an ideal with analytic spread ℓ which satisfies G_ℓ and we fix a choice of a minimal reduction J (say with $r_J(I) = r$) satisfying ht $J : I \geq \ell$, and ideals $\mathfrak{a}_i$, as in Remark 1.1.10. We may extend $a_1, ..., a_\ell$ to a minimal generating set $a_1, ..., a_n$ of I. Let $S = R[T_1, ..., T_n]$ be a polynomial ring over R, and present the Rees algebra $\mathcal{R} = R[It] \cong S/Q$, by mapping T_i to $a_i t$.

Lemma 1.3.1 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I an R-ideal with analytic spread ℓ, let $k \geq 1$ be an integer and assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-k-1}$ and that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$.

Then $[(T_1, ..., T_\ell) \cap Q]_{k+1} \subset [Q]_k S$.

Proof. It will be enough to prove that $[(T_1, ..., T_i) \cap Q]_{k+1} \subset [Q]_k S$, for every $0 \leq i \leq \ell$, which we do by induction on i. Since the case $i = 0$ is trivial, we may assume that $i \geq 1$ and that the result holds for smaller i. Let $F \in (T_1, ..., T_i) \cap Q$ be a form of degree $k+1$ and write $F = \sum_{j=1}^{i} G_j T_j$ where $G_j \in [S]_k$. Evaluating F at $(a_1, ..., a_n)$ gives $0 = \sum_{j=1}^{i} G_j(a_1, ..., a_n) a_j$, hence

$$G_i(\underline{a}) \in (\mathfrak{a}_{i-1} : (a_i)) \cap I^k = \mathfrak{a}_{i-1} I^{k-1}$$

by Lemma 1.1.8(b). It follows that there are forms $H_1, ..., H_{i-1} \in [S]_{k-1}$ for which $P = G_i - \sum_{j=1}^{i-1} H_j T_j \in [Q]_k S$. But then

$$F - T_i P = \sum_{j=1}^{i-1} (G_j + T_i H_j) T_j \in [(T_1, ..., T_{i-1}) \cap Q]_{k+1} \subset [Q]_k S$$

by induction. It follows that $F \in [Q]_k S$. □

Proposition 1.3.2 In addition to the assuptions of Lemma 1.3.1, set $n = \nu(I)$ and assume that $S_j(I) \cong I^j$ for $1 \leq j \leq k$ and that $r(I) \leq k$.

Then $[\mathcal{A}]_{k+1}$ is minimally generated by $\binom{n-\ell+k}{n-\ell-1}$ forms.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow S(I) \longrightarrow \mathcal{R}(I) \longrightarrow 0,$$

which in degree $k+1$ is

$$0 \longrightarrow [\mathcal{A}]_{k+1} \longrightarrow S_{k+1}(I) \longrightarrow I^{k+1} \longrightarrow 0.$$

Since we may present $\mathcal{R} \cong S/Q$ and $S(I) \cong S/L$, where L is the ideal generated by the linear forms in Q, we have $\mathcal{A} \cong Q/L$. This will induce an exact sequence

$$0 \longrightarrow [\mathcal{A}]_{k+1} \longrightarrow S_{k+1}(I/J) \longrightarrow I^{k+1}/JI^k \longrightarrow 0$$

once we have shown that $[(L, T_1, ..., T_\ell) \cap Q]_{k+1} \subset L$ or equivalently that $[(T_1, ..., T_\ell) \cap Q]_{k+1} \subset L$. But this is clear from Lemma 1.3.1, since by the assumption on the symmetric powers we have $[Q]_k S \subset [Q]_1 S = L$. The result is now clear since if $r(I) \leq k$ then the module I^{k+1}/JI^k vanishes. □

This proposition allows us to compute the number of defining equations of the Rees algebras of ideals having the *minimal reduction number.* We take this to mean that $S_j(I) \cong I^j$ for $1 \leq j \leq r$, or equivalently that $\mathcal{A}_{r+1}$ is the first nonvanishing component of $\mathcal{A}$, where $r = r(I)$ is the reduction number.

Theorem 1.3.3 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ, minimal number of generators n, and reduction number r and assume that I satisfies G_ℓ and $AN^-_{\ell-2}$, and that $S_j(I) \cong I^j$ and depth $R/I^j \geq d - \ell + r - j$ whenever $1 \leq j \leq r$.

Then $\mathcal{A}$ is minimally generated by $\binom{n-\ell+r}{n-\ell-1}$ forms of degree $r+1$.

Proof. If $r = 0$ then $n = \ell$ and I is generated by a d-sequence by Proposition 1.1.5; since then $\mathcal{A} = 0$, the result holds. Hence we may assume $r \geq 1$. By Proposition 1.3.2, it is enough to show that the relation type $rt(I)$ is at most $r+1$ since then $\mathcal{A}$ will be generated by its degree $r+1$ component. By [56, 2.3] it is enough to show

$$((a_1, ..., a_i) : (a_{i+1})) \cap I^{r+1} = (a_1, ..., a_i)I^r$$

for all $0 \leq i \leq n-1$. Since I satisfies $AN^-_{\ell-2}$, this follows from Lemma 1.1.11 and Lemma 1.1.5(c.i) if $0 \leq i \leq \ell - 1$. However, if $\ell \leq i \leq n-1$ then

$$((a_1, ..., a_i) : (a_{i+1})) \cap I^{r+1} \subset I^{r+1} = JI^r \subset (a_1, ..., a_i)I^r. \quad \Box$$

Corollary 1.3.4 Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a strongly Cohen-Macaulay R-ideal of grade g, analytic spread ℓ and minimal number of generators n, and assume that I satisfies G_ℓ and has reduction number $r = \ell - g + 1$.

Then $\mathcal{A}$ is minimally generated by $\binom{n-g+1}{\ell-g+2}$ forms of degree $\ell - g + 2$.

Proof. Since I is strongly Cohen-Macaulay and satisfies G_ℓ, one has depth $R/I^j \geq d - g - j + 1$ and $S_j(I) \cong I^j$ whenever $1 \leq j \leq \ell - g + 1$. As I satisfies AN_ℓ by Theorem 1.1.3, the result follows from Theorem 1.3.3. □

Corollary 1.3.5 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ, minimal number of generators n, assume that I satisfies $G_{\ell+1}$ and $AN^-_{\ell-2}$ and that depth $R/I \geq d - \ell$.

Then $\mathcal{A}$ is minimally generated by $\binom{n-\ell+1}{2}$ quadrics.

Proof. By Theorem 1.3.3, it will be enough to show that $r \leq 1$. However, since I satisfies $AN^-_{\ell-2}$ and $G_{\ell+1}$, it follows that I is of linear type locally in codimension ℓ by [68, 1.11]. Since in particular I has reduction number at most one locally in codimension ℓ, the result follows from Remark 1.1.12. □

The previous corollary had been observed for the class of monomial curves in $\mathbf{P}^3$ lying on a quadric, i.e. the homogeneous codimension two prime ideals in $k[x, y, z, w]$ defining $x = t^{a+b}, y = s^a t^b, z = s^b t^a, w = t^{a+b}$, with $(a, b) = 1$. It is well-known that they are minimally generated by $b - a + 2$ equations. Huckaba and Huneke ([30]) showed that $\ell \leq 3$ and that the Rees algebra is defined by at most quadrics. Schenzel ([59]) and Morales and Simis ([51]) actually compute the defining equations explicitly. Since the curves are smooth we may immediately apply Corollary 1.3.5 (note that the condition $AN^-_{\ell-2}$ is vacuous since $\ell - 2 \leq 1 < 2 = g$) to conclude that $\mathcal{A}$ is minimally generated by $\binom{b-a}{2}$ quadrics.

More generally, now by Corollary 1.3.5 and [25, 4.1] (and its proof), for any saturated homogeneous ideal I defining a smooth curve in $\mathbf{P}^3$ lying on the surface $xy = zw$, generated by n equations, its Rees algebra is defined by linear and exactly $\binom{n-2}{2}$ quadratic equations.

Corollary 1.3.6 In addition to the assumptions of Theorem 1.3.3, assume that $n = \ell + 1$.

Then $\mathcal{A}$ is cyclic.

It is natural to ask, at least in the case $n = \ell + 1$, where does this single equation come from. In other words, can one compute the generator of $\mathcal{A}$ *explicitly*? This has been studied when I is a perfect ideal of grade 2 by Vasconcelos ([73]), and the answer is completely natural, just given by a determinant of a Jacobian dual of a presentation matrix of I. When the grade is larger than two, this will not work, as one can easily see by degree reasons. However, this still seems to be a good place to look for the equations, so we describe the general idea. We will then show how it leads to the explicit defining equation of the Rees algebra for Gorenstein ideals of grade 3.

Let R be a local Gorenstein ring with infinite residue field, and let I be a strongly Cohen-Macaulay R-ideal satisfying G_ℓ, having second analytic deviation one, i.e. $n = \ell + 1$, and having the expected reduction number $r = \ell - g + 1$ (we will show in the next section that often this is equivalent to assuming that $\mathcal{R}$ is Cohen-Macaulay). By Corollary 1.3.4 we know that $\mathcal{A}$ is cyclic, generated by a form of degree $\ell - g + 2$.

Let

$$R^m \xrightarrow{\phi} R^n \longrightarrow I \longrightarrow 0$$

be a presentation of I, where ϕ is an m by n matrix with entries in the maximal ideal of R. Recall that the symmetric algebra $S(I)$ admits a presentation

$$S(I) \cong R[T_1, ..., T_n]/(\ell_1, ..., \ell_m)$$

defined by the equations

$$(\ell_1, ..., \ell_m) = (T_1, ..., T_n)\phi.$$

Our assumptions imply that I satisfies the *row condition*: after elementary row operations, the entries of $I_1(\phi)$ can be generated by the last row of ϕ ([61], or Proposition 1.4.2). Moreover, it is known that $I_1(\phi)$ is Gorenstein ([61, 4.13]) and has height ℓ ([27, 9.1]), but let us assume further that it is a complete intersection and choose a generating set $I_1(\phi) = (x_1, ..., x_\ell)$. Now we may assume after row and column operations that $I_1(\phi)$ is generated by the entries of the last row of the submatrix consisting of the first ℓ columns. Let ψ be the ℓ by ℓ submatrix of ϕ obtained by deleting the last row and the last $m - \ell$ columns, and over the polynomial ring $R[T_1, ..., T_\ell]$ consider a Jacobian dual ([60]) of ψ:

$$(T_1, ..., T_\ell)\psi = (x_1, ..., x_\ell)B(\psi).$$

Here $B = B(\psi)$ is an ℓ by ℓ matrix whose entries are linear forms in the variables $T_1, ..., T_\ell$. The characteristic polynomial of B seems to be a good place to look for equations of the Rees algebra.

To give evidence to this claim, assume now that I is a perfect Gorenstein ideal of grade 3 satisfying G_ℓ and the row condition. After elementary row operations, $I_1(\phi)$ is generated by the last row of the presentation matrix ϕ, and by the structure theorem of Buchsbaum and Eisenbud ([11]), ϕ can be chosen as an alternating n by n matrix. Since ϕ is alternating, it follows that $I_1(\phi)$ is automatically a complete intersection. Hence we may apply the arguments above; we obtain the following:

Theorem 1.3.7 Let R be a local Gorenstein ring with infinite residue field, let I be a perfect Gorenstein ideal of grade 3, with analytic spread ℓ, minimal number of generators $n = \ell + 1$, and assume that I satisfies G_ℓ, let ϕ be an n by n alternating matrix presenting I with last row $(-x_1, ..., -x_\ell, 0)$ which generates the ideal of entries of ϕ and let ψ be the ℓ by ℓ alternating submaxtrix of ϕ obtained by deleting the last row and column. Then there exists a Jacobian dual $B = B(\psi)$:

$$(T_1, ..., T_\ell)\psi = (x_1, ..., x_\ell)B(\psi).$$

such that $\mathcal{A}$ is generated by $F = T_n^{-1}\chi_B(T_n)$, where χ denotes the characteristic polynomial of B in the variable T_n. Moreover, if we let ψ_j denote the jth column of ψ, for $1 \leq j \leq \ell$, and write $\psi_j = A_j(\underline{x})^t$, where A_j is an ℓ by ℓ matrix whose jth row consists of zeros and whose ith row, for any $1 \leq i \leq \ell$, is the negative of the jth row of A_i, then B may be taken to be the matrix whose jth column is $A_j^t(\underline{T})^t$.

Proof. It is enough to prove the second statement. Let a_{ij} and b_{ij} be the ijth entry of the matrices ψ and B respectively, and let a_{ijk} be the ikth entry of A_j. Then write $a_{ij} = \sum_{k=1}^{\ell} a_{ijk}x_k$ and $b_{ij} = \sum_{k=1}^{\ell} a_{kji}T_k$, where $a_{ijk} = -a_{jik}$ for $i \neq j$ and $a_{ijk} = 0$ for $i = j$. Let $\{z_{ijk}\}$ be a new set of variables with $z_{ijk} = -z_{jik}$ for $i \neq j$ and $z_{ijk} = 0$ for $i = j$, and define $\tilde{a}_{ij} = \sum_{k=1}^{\ell} z_{ijk}x_k$ and $\tilde{b}_{ij} = \sum_{k=1}^{\ell} z_{kji}T_k$. Denote by $\tilde{\psi}$ and $\tilde{B}$ the matrices whose ijth entry is $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$, respectively. It follows that $\tilde{B}$ is a Jacobian dual of $\tilde{\psi}$:

$$(T_1, ..., T_\ell)\tilde{\psi} = (x_1, ..., x_\ell)\tilde{B}. \tag{1.3.8}$$

Now it will be enough to show that $\det(\tilde{B}) = 0$. For then by specializing, it holds that $\det(B) = 0$, and hence that T_n divides the characteristic polynomial $\chi_B(T_n)$ of B. Since the characteristic polynomial may be obtained as a minor of a

Jacobian dual of ϕ, it is clear that $\chi_B(T_n)$ is a relation on the Rees algebra. But by Corollary 1.3.4, $\mathcal{A}$ is cyclic, hence since the quotient $F = T_n^{-1}\chi_B(T_n)$ is monic, it is the required form of degree $\ell - 1 = \ell - g + 2$.

Now to show the claim, multiply equation (1.3.8) on the right by the column $(T_1, ..., T_\ell)^t$. Since $\tilde{\psi}$ is an alternating matrix,

$$(\underline{x})\tilde{B}(T_1, ..., T_\ell)^t = 0.$$

Since the x's form a regular sequence, the entries of the matrix $\tilde{B}(T_1, ..., T_\ell)^t$ belong to the ideal generated by $(x_1, ..., x_\ell)$ and the subring over k generated by the T's and the z's. It follows that $\tilde{B}(T_1, ..., T_\ell)^t = 0$ and hence that $\det(\tilde{B}) = 0$. □

Example 1.3.9 Let $I \subset k[[x, y, z, w]]$ be the defining ideal of the Gorenstein monomial curve $k[[t^5, t^6, t^7, t^8]]$. Then I has a presentation matrix ϕ:

$$\begin{pmatrix} 0 & z & w & y & y \\ -z & 0 & x^2 - w & w - y & w \\ -w & w - x^2 & 0 & 0 & z \\ -y & y - w & 0 & 0 & x \\ -y & -w & -z & -x & 0 \end{pmatrix}$$

Since $I_1(\phi) = (x, y, z, w)$, ϕ satisfies the row condition. Deleting the last row and column, we obtain the Jacobian dual B :

$$\begin{pmatrix} -T_4 & T_4 & 0 & T_1 - T_2 \\ -T_3 & T_3 - T_4 & T_1 - T_2 & T_2 \\ -T_2 & T_1 & 0 & 0 \\ 0 & -xT_3 & xT_2 & 0 \end{pmatrix}$$

Dividing the characteristic polynomial $\chi_B(T_5)$ by T_5 gives us the nontrivial cubic relation on the Rees algebra:

$$T_5^3 + 2T_4T_5^2 + T_4^2T_5 + T_1T_2T_5 - T_1^2T_5 - T_2T_4^2 - T_2^2T_4 + 2T_1T_2T_4$$
$$-T_1^2T_4 + xT_2T_3T_5 + xT_2T_3T_4 + xT_2T_3^2 - xT_1T_3^2 - xT_2^3.$$

(This answers the query raised in [70, 2.11].)

1.4 Expected Reduction Number

In this section we give a converse of Theorem 1.2.5 for ideals having second analytic deviation one. It turns out that if the ideal in question is contained in a sufficiently high power of its content ideal $I_1(\phi)$, then one obtains the expected bound for the reduction number. We closely follow ideas from [3].

Lemma 1.4.1 With the assumptions as in Lemma 1.3.1, suppose in addition that $n = \ell + 1$ and let ϕ be a minimal presentation matrix of I.

Then $I_1(\phi)[Q]_{k+1} \subset [Q]_k S$.

Proof. Let $F \in [Q]_{k+1}$ and write $F = \alpha T_n^{k+1} + G$ where $G \in [(T_1, ..., T_\ell)]_{k+1}$ and $\alpha \in R$. Since Lemma 1.3.1 holds for any permutation of $T_1, ..., T_n$ (by Remark 1.1.10) and $x \in I_1(\phi)$, we may assume $x \in (a_1, ..., a_{n-1}) : (a_n)$. Hence there is a linear form $H = xT_n + \sum_{i=1}^{\ell} r_i T_i \in [Q]_1$, with $r_i \in R$. But then

$$\begin{aligned} xF &= \alpha x T_n^{k+1} + xG \\ &= \alpha T_n^k H - \alpha T_n^k (H - xT_n) + xG \\ &\in Q_1 + [(T_1, ..., T_\ell) \cap Q]_{k+1} \\ &\subset [Q]_k S \end{aligned}$$

by Lemma 1.3.1. □

The following shows that the row condition is satisfied, under mild conditions, when I has second analytic deviation one with minimal reduction number.

Proposition 1.4.2 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ, minimal number of generators $n = \ell + 1$, and reduction number r, assume that I satisfies G_ℓ and

$AN_{\ell-3}^-$ locally in codimension $\ell - 1$, that I satisfies $AN_{\ell-r-1}^-$ and that $S_j(I) \cong I^j$ and depth $R/I^j \geq d - \ell + r - j$ for $1 \leq j \leq r$ and let ϕ be a matrix presenting I with n rows.

Then, after elementary row operations, $I_1(\phi)$ is generated by the last row of ϕ.

Proof. Let J be a minimal reduction as in Lemma 1.1.10 and let $K = J : I = J : (a_n)$ which is, after elementary row operations, the ideal generated by the last row of ϕ. Hence it is enough to show that $K = I_1(\phi)$. By Lemma 1.4.1, $I_1(\phi)[Q]_{r+1} \subset [Q]_r S = [Q]_1 S$ by assumption on the symmetric powers (note that $r \geq 1$). Let $F \in [Q]_{r+1}$ with $F = T_n^{r+1} + G$ where $G \in [(T_1, ..., T_\ell)]_{r+1}$, and let $x \in I_1(\phi)$. Then $xF \in [Q]_1 S$ and hence there is an equation

$$xF = xT_n^{r+1} + xG = \sum_{i=1}^{m} L_i H_i$$

where $L_i \in [Q]_1$ and $H_i \in [S]_r$. But by comparing the coefficients of the term T_n^{r+1} it is clear that $x \in K$. □

We will need another version of Lemma 1.4.1 for large values of k. This holds whenever the associated graded ring G is Cohen-Macaulay.

Lemma 1.4.3 Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade g, and analytic spread ℓ, which satisfies G_ℓ, let J be any minimal reduction of I with ht $J : I \geq \ell$, let $\mathfrak{a}_i$ be the ideals defined in Remark 1.1.10 and assume that G is Cohen-Macaulay.

Then $(\mathfrak{a}_i : (a_{i+1})) \cap I^j = \mathfrak{a}_i I^{j-1}$ for $0 \leq i \leq \ell - 1$ and $j \geq i - g + 1$.

Proof. In the terminology of [4], J is a "special reduction" of I. Hence by [4, 5.10] and Remark 1.1.10 one has the equation

$$((\mathfrak{a}_i + I^{j+2}) : (a_{i+1})) \cap I^j = \mathfrak{a}_i I^{j-1} + I^{j+1}$$

whenever $0 \leq i \leq \ell - 1$ and $j \geq i - g + 1$. But then

$$(\mathfrak{a}_i : (a_{i+1})) \cap I^j \subset \mathfrak{a}_i I^{j-1} + I^{j+1}.$$

Since $\cap_{k\geq 1} I^{j+k} = 0$, the result follows. □

Lemma 1.4.4 Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade g, and analytic spread ℓ, which satisfies G_ℓ, let J be a minimal reduction of I with ht $J : I \geq \ell$ and assume that G is Cohen-Macaulay. Then

(a) $[(T_1, ..., T_\ell) \cap Q]_{k+1} \subset [Q]_k S$ for $k \geq \max\{\ell - g, 1\}$.

(b) If $\nu(I) = \ell + 1$ then $I_1(\phi)[Q]_{k+1} \subset [Q]_k S$ for $k \geq \max\{\ell - g, 1\}$.

Proof. This follows immediately from Lemma 1.4.3 as in the proof of Lemma 1.3.1 and Lemma 1.4.1. □

We are now ready to show one of the main results of this section.

Theorem 1.4.5 Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal, of grade g, analytic spread ℓ, and minimal number of generators $n = \ell + 1$, satisfying G_ℓ, let $k \geq 1$ be an integer such that $S_j(I) \cong I^j$ for $1 \leq j \leq k$, let ϕ be a minimal presentation matrix of I, assume that $I \subset I_1(\phi)^e$ for some $e \geq 2$, that G is Cohen-Macaulay, and that one of the following conditions holds:

(a) $k \geq \ell - g$;

(b) depth $R/I^j \geq d - g - j - 1$ for $1 \leq j \leq \ell - g - 2$, I satisfies $AN^-_{\ell-k-1}$, and I satisfies $AN^-_{\ell-3}$ locally in codimension $\ell - 1$.

Then $r(I) \leq \ell - g + 1 + \frac{\ell-g+1-k}{e-1}$.

Proof. If necessary, one may pass to the faithfully flat extension $R(X)$ of R to ensure that there is a minimal reduction J of I with ht $J : I \geq \ell$ ([4, 2.4]). Let $r = r(I) \geq k$ and choose a form $F \in [Q]_{r+1}$ with $F = T_n^{r+1} + G$ where $G \in [(T_1, ..., T_\ell)]_{r+1}$. Since $F \in [Q]_{r+1}$, repeated application of Lemma 1.4.4 (in case (a)) or Lemma 1.4.1 (in case (b)) gives $I_1(\phi)^{r-k+1}F \in [Q]_k S$. (Note that in case (b), we may assume that $k \leq \ell - g - 1$ and note the condition on the depth of $R/I^{\ell-g-1}$ is automatically satisfied since G is Cohen-Macaulay.) By the assumption on the symmetric powers we have $[Q]_k S \subset [Q]_1 S$, hence $I_1(\phi)^{r-k+1}F \in [Q]_1 S$. Now as in the proof of Proposition 1.4.2, it follows that $I_1(\phi)^{r-k+1} \subset J : I$. Let $\xi = \lceil \frac{r-k+1}{e} \rceil$. It will be enough to show that $r \leq \ell - g + \xi$. For then

$$\begin{aligned} r &\leq \ell - g + \xi \\ &= \ell - g + \lceil \frac{r-k+1}{e} \rceil \\ &\leq \ell - g + \frac{r-k+1}{e} + \frac{e-1}{e} \\ &= \ell - g + 1 + \frac{r-k}{e} \\ &= \ell - g + 1 + r/e - k/e \end{aligned}$$

from which it follows that

$$\begin{aligned} r &\leq \frac{\ell - g + 1 - k/e}{1 - 1/e} \\ &= \frac{(\ell - g + 1)e - k}{e - 1} \\ &= \frac{(\ell - g + 1)(e-1) + \ell - g + 1 - k}{e - 1} \\ &= \ell - g + 1 + \frac{\ell - g + 1 - k}{e - 1}. \end{aligned}$$

Now by the assumption that $I \subset I_1(\phi)^e$, we have

$$I^{\xi+1} = I^{\xi} I \subset (I_1(\phi)^e)^{\xi} I \subset I_1(\phi)^{r-e+1} I \subset (J : I) I \subset J.$$

In particular, $I^{\xi+\ell-g+1} \subset JI^{\ell-g}$. The result now follows from the next lemma. □

Lemma 1.4.6 Let R be a local Cohen-Macaulay ring with infinite residue field, let I an R-ideal with analytic spread ℓ which satisfies G_ℓ, let J be a minimal reduction of I with ht $J : I \geq \ell$, and assume that G is Cohen-Macaulay and that $I^{s+1} \subset JI^{\ell-g}$, for some integer $s \geq 1$.

Then $I^{s+1} = JI^s$.

Proof. Using Remark 1.1.10 this follows from [4, proof of 5.2]. □

Theorem 1.4.5 was inspired by the works of Aberbach, Huckaba and Huneke ([3]) and Aberbach and Huckaba ([2]), where it was shown when the analytic deviation is one or two, respectively.

Theorem 1.4.7 Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal of grade $g \geq 2$, analytic spread ℓ and minimal number of generators $n = \ell+1$ and assume that I satisfying G_ℓ, that depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell-g+1$ and that $I \subset I_1(\phi)^{\ell-g+2}$, where ϕ is a matrix with n rows presenting I. Then the following conditions are equivalent.

(a) After elementary row operations, $I_1(\phi)$ is generated by the last row of ϕ;

(b) $r(I) \leq \ell - g + 1$;

(c) $\mathcal{R}$ is Cohen-Macaulay.

Proof. Since I has second analytic deviation one, (a) and (b) are equivalent by [69, 5.1]. Now by Theorem 1.1.4, I satisfies AN_ℓ^- and thus (b) implies (c) by Corollary 1.2.10. Now assume (c) holds. Then in particular G is Cohen-Macaulay ([33, 1.1]) and hence (b) follows from Theorem 1.4.5. □

Naturally we can obtain stronger results by assuming the vanishing of the torsion of sufficiently many symmetric powers.

Corollary 1.4.8 Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, minimal number of generators $n = \ell + 1$, assume that I satisfies G_ℓ, that $S_j(I) \cong I^j$ for $1 \leq j \leq \ell - g + 1$, that G is Cohen-Macaulay and let ϕ be a minimal presentation matrix of I. Then

(a) $\mathcal{A}_{\ell-g+2} \neq 0$;

(b) if $I \subset I_1(\phi)^2$ then $r(I) = \ell - g + 1$.

Proof. To prove (a) put $k = \max\{j \mid S_j(I) \cong I^j\}$. Since $k \geq \ell - g + 1$, I satisfies the assumptions of Theorem 1.4.5(a). But by the proof, putting $e = 1$ shows that $k \leq \ell - g + 1$. As for (b), by the assumption on the symmetric powers, $r(I) = 0$ or $r(I) \geq \ell - g + 1$. Since $n > \ell$, the result follows from Theorem 1.4.5. □

Remark 1.4.9 Under the assumptions of Theorem 1.4.5 (a) or (b), assume that $e \geq [\frac{\ell-k}{g-1} + 1]$ and that $g \geq 2$.

Then $\mathcal{R}$ is Cohen-Macaulay.

Proof. Since I satisfies G_ℓ, it is enough to show that $r < \ell$ by Theorem 1.2.1. Since $\frac{\ell-k}{g-1} \geq 1$ by Corollary 1.4.8(a), this follows immediately from Theorem 1.4.5. □

We should point out that J. Lipman has shown that G Cohen-Macaulay implies that $\mathcal{R}$ is Cohen-Macaulay whenever R is a *regular* local ring (or more generally is pseudo-rational ([49])).

The following theorem complements one of the main results of [61]. It applies immediately to grade 2 perfect ideals and grade 3 Gorenstein ideals satifying G_ℓ and having second analytic deviation one.

Theorem 1.4.10 Let R be a local Gorenstein ring with infinite residue field, let I be a strongly Cohen-Macaulay R-ideal of grade $g \geq 2$, analytic spread ℓ, and minimal number of generators $n = \ell + 1$, and assume that I satisfies G_ℓ and that $I \subset I_1(\phi)^2$, where ϕ is a matrix with n rows presenting I. Then the following conditions are equivalent.

(a) After elementary row operations, $I_1(\phi)$ is generated by the last row of ϕ;

(b) $r(I) = \ell - g + 1$;

(c) $rt(I) = \ell - g + 2$;

(d) $\mathcal{A}$ is generated by a single form of degree $\ell - g + 2$;

(e) $\mathcal{R}$ is Cohen-Macaulay;

(f) G is Cohen-Macaulay.

Proof. Since I is strongly Cohen-Macaulay and satisfies G_ℓ, we have that $S_j(I) \cong I^j$ and depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq \ell-g+1$, and I satisfies AN_ℓ by Theorem 1.1.4. In particular, $r(I) \geq \ell-g+1$. Now (a) and (b) are equivalent as in Theorem 1.4.7, (b) implies (c) and (d) from Corollary 1.3.4, (c) implies (b) since I has second analytic deviation one, and trivially (d) implies (c). By Corollary 1.2.10, (b) implies (e), while (e) implies (f) by [33, 1.1]. Finally (f) implies (b) by Corollary 1.4.8(b). □

Remark 1.4.11 ([69, 2.11]) Under the assumptions of Theorem 1.4.10, $r(\mathcal{R}) = \nu(I_1(\phi)) + g - 2$ and $r(G) = \nu(I_1(\phi)) + 1$.

Remark 1.4.12 The condition $I \subset I_1(\phi)^2$ in Theorem 1.4.10 is really essential. A. Simis and B. Ulrich have discovered examples of strongly Cohen-Macaulay generically complete intersection prime ideals with second analytic deviation one in a Gorenstein ring (the diagonal ideal of a certain codimension 3 Gorenstein algebra) for which G is Cohen-Macaulay but $\mathcal{R}$ is not, and (after adjoining variables to the ideal) $\mathcal{R}$ is Cohen-Macaulay but the reduction number is not the expected value.

Example 1.4.13 Let I be the prime ideal defining the Gorenstein monomial curve $k[[t^{10}, t^{11}, t^{14}, t^{19}]]$. A computation on MACAULAY shows that $\mathcal{R}$ is normal. However, using Theorem 1.4.10, it is not hard to show that it is not Cohen-Macaulay.

1.5 Analytically Independent Elements

Suprisingly, one can use the results of the previous sections to say something about ideals having second analytic deviation zero, that is to say ideals generated by analytically independent elements. We also show that a question of Ulrich, prompted by a prior one of Valla about when such ideals have linear type, has a negative answer.

Proposition 1.5.1 Let R be a local Gorenstein ring of dimension d, let I be an R-ideal of grade g and assume that I satisfies G_∞, that I is generated by analytically independent elements and that G is Cohen-Macaulay. Then

(a) $rt(I) \leq d(I)$.

(b) Assume further that depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq d(I) - 2$.

Then I is of linear type if and only if I is syzygetic.

Proof. We may assume the residue field is infinite. Part (a) follows from Lemma 1.4.3 and [64, 3.3] (or Lemma 1.4.3 and the proof of Theorem 1.3.3) since $n = \nu(I) = \ell(I)$. Now for (b), since G is Cohen-Macaulay, depth $R/I^{n-g-1} \geq d - \ell$, and I satisfies $AN^-_{\ell-3}$ by Theorem 1.1.4. Since I is syzygetic, by part (a) it is enough to show $S_j(I) \cong I^j$ whenever $3 \leq j \leq d(I)$. But since $I = J$ for any reduction J of I, this follows from Lemma 1.3.1 by induction on j. □

Corollary 1.5.2 Let R be a local Gorenstein ring and let I be a Cohen-Macaulay ideal of deviation three satisfying G_∞ with G Cohen-Macaulay.

Then I is of linear type if and only if I is syzygetic.

Proof. It is enough to show by Proposition 1.5.1(b) that $\nu(I) = \ell(I)$. But if $\nu(I) = g + 3 > \ell$ then I would automatically satisfy $AN^-_{\ell-2}$ by Theorem 1.1.4. But since I satisfies G_∞, it would follow that I has reduction number at most one by Remark 1.1.12. Since I is syzygetic, this would contradict the fact that $r(I) \neq 0$. □

G. Valla asked if a prime ideal in a regular local ring which is generated by analytically independent elements is necessarily of linear type. A counterexample was produced in [60, 4.5]. It is a normal homogeneous Cohen-Macaulay prime ideal of codimension three and deviation three in a polynomial ring in nine variables which satisfies G_∞ and whose Rees algebra is Cohen-Macaulay. By Corollary 1.5.2 the ideal is not even syzygetic.

Ulrich asked in [66] whether such an ideal is of linear type if it is *locally* generated by analytically independent elements.

Proposition 1.5.3 There exist homogeneous perfect prime ideals in $k[x_1, ..., x_6]$ of codimension three and deviation three which are locally generated by analytically independent elements but are not of linear type.

Proof. Take the above counterexample of [60] and specialize it by three general linear forms. This produces by Bertini's theorem ([18]) the required prime ideal, in a six dimensional polynomial ring, which still satisfies G_∞, is generated by analytically independent elements and is not of linear type since the associated graded ring specializes ([17]). To check that the ideal is locally generated by analytically independent elements, note that the ideal has deviation at most two on the punctured spectrum, hence is strongly Cohen-Macaulay ([8]). It follows that the ideal is even of linear type on the punctured spectrum. □

We include one other criterion for linear type, for ideals having second analytic deviation at most one.

Proposition 1.5.4 Let R be a local Cohen-Macaulay ring and let I be an R-ideal of grade g and analytic spread ℓ with G Cohen-Macaulay. Then the following are equivalent:

(a) I is of linear type;

(b) $\nu(I) \leq \ell(I) + 1$, I satisfies G_ℓ and $S_j(I) \cong I^j$ whenever $1 \leq j \leq \ell - g + 2$.

Proof. We may assume the residue field is infinite. It is enough to show that (b) implies (a). By Corollary 1.4.8(a) it follows immediately that $n = \nu(I) \neq \ell + 1$. Hence $n = \ell$, and by Proposition 1.5.1(a) we conclude that I has relation type at most $\ell - g$. But by the assumption on the symmetric powers it holds that $rt(I) = 1$ or $rt(I) \geq \ell - g + 3$. Hence I is of linear type. □

CHAPTER 2

SYMBOLIC POWERS AND DEFORMATIONS

In this chapter we prove a criterion for the power of an ideal to coincide with its symbolic power: they should coincide locally in codimension one, they should coincide after deformation, and (the quotient ring by) the symbolic power should satisfy Serre's condition (S_2). Requiring the power to coincide with its symbolic power after deformation is in general much weaker than requiring them to coincide on the nose.

Definition 2.1 Let R be a noetherian ring, let I be an R-ideal, and let n be a positive integer. The nth symbolic power of I is

$$I^{(n)} = I^n R_W \cap R$$

where W is the complement in R of the union of the minimal primes of I.

The following result is well-known.

Remark 2.2 Let R be a local Cohen-Macaulay ring and let I be a complete intersection R-ideal.

Then $I^{(n)} = I^n$ for all n.

Proof. Since I is a complete intersection, the associated graded ring $gr_I(R)$ is a polynomial ring over R/I, and hence the modules I^n/I^{n+1} are R/I-free for all n. It follows that R/I^n is Cohen-Macaulay, hence unmixed, for all n. □

Proposition 2.3 Let R be a local Cohen-Macaulay ring, let I be an R-ideal and let (S, J) be a deformation of (R, I) with S/J equidimensional. Then

(a) R/I is equidimensional;

(b) if R/I satisfies (S_k) then so does S/J;

(c) if I satisfies (CI_k) then so does J.

Proof. Let $\underline{a} = a_1, ..., a_r \subset S$ be a sequence regular on S and on S/J with $(R, I) = (S/(\underline{a}), (J, \underline{a})/(\underline{a}))$. Since $R = S/(\underline{a})$ is Cohen-Macaulay, so is S, and in particular is equidimensional and catenary. Hence to show (a), it will be enough to show that every minimal prime p of $(J, \underline{a})$ has height at most ht $J + r$. But since both S and S/J are equidimensional and catenary, it holds that

$$\text{ht } p = \dim S - \dim S/p = \dim S - (\dim S/J - \text{ht } p/J) = \text{ht } J + \text{ht } p/J \leq \text{ht } J + r.$$

To prove (b), let $p \in \text{Spec } S/J$ with $\dim (S/J)_p = s$ and let q be a minimal prime of $(p, \underline{a})$. Since S/J is equidimensional, as above ht $q \leq s + r$.Then $\overline{q} = q/(\underline{a}) \in \text{Spec } R/I$ and ht $\overline{q} \leq s$. Now (S_q, J_q) is a deformation of $(R_{\overline{q}}, I_{\overline{q}})$, and $(S/J)_p$ is a localization of $(S/J)_q$. If $s \leq k$ then $(R/I)_{\overline{q}}$ is Cohen-Macaulay and hence so is $(S/J)_p$. Otherwise, depth $(R/I)_{\overline{q}} \geq k$ and hence depth $(S/J)_q \geq k + r$. The inequality $\dim (S/J)_p -$ depth $(S/J)_p \leq \dim (S/J)_q -$ depth $(S/J)_q$ now implies that depth $(S/J)_p \geq k$. This proves that S/J satisfies (S_k).

Since the property of being a complete intersection is preserved under deformation or localization, (c) follows as in the proof of (b). □

Our main result depends on an observation of Huneke and Ulrich; we include the short proof for convenience. Recall that H^i_m is the ith local cohomology functor.

Proposition 2.4 ([42, 2.1]) Let (B, m) be a local ring such that $\hat{B}$ is unmixed, let $x \in B$ be a regular element, set $A = B/xB$, and assume that depth $A/H^0_m(A) \geq 2$.

Then $H^0_m(A) = 0$.

Proof. We may assume that B is complete. Since $H^0_m(A)$ is supported at $\{m\}$, the exact sequence

$$0 \longrightarrow H^0_m(A) \longrightarrow A \longrightarrow A/H^0_m(A) \longrightarrow 0$$

shows that $H^i_m(A/H^0_m(A)) = H^i_m(A)$ for all $i \geq 1$. Since depth $A/H^0_m(A) \geq 2$, we have that $H^1_m(A) = 0$.

The exact sequence

$$0 \longrightarrow B \xrightarrow{x} B \longrightarrow A \longrightarrow 0$$

induces an exact sequence in local cohomology

$$0 \longrightarrow H^0_m(A) \longrightarrow H^1_m(B) \xrightarrow{x} H^1_m(B) \longrightarrow H^1_m(A).$$

Since $H^1_m(A) = 0$, it follows that $H^1_m(B) = xH^1_m(B)$. Thus to show $H^0_m(A) = 0$, it is enough by Nakayama's lemma to show that $H^1_m(B)$ is finitely generated. Since B is complete and unmixed, this follows from local duality (e.g. [10, 3.5.8]). $\square$

The following is the main result of this chapter.

Theorem 2.5 Let R be a local Cohen-Macaulay ring, let I be an R-ideal satisyfying (CI_0), let n and t be positive integers, assume that (R, I) has a deformation (S, J) with S/J equidimensional and $J^{(n)} = J^n$, that $I_p^{(n)} = I_p^n$ holds for all $p \in$ Spec R/I with dim $(R/I)_p \leq t$ and that depth $(R/I^{(n)})_p \geq 2$ for all $p \in$ Spec R/I with dim $(R/I)_p > t$.

Then $I^{(n)} = I^n$.

Proof. Note that the assumptions are preserved after localization at a prime ideal (the condition (CI_0) is preserved since by Proposition 2.3(a) R/I is equidimensional). We may proceed via induction on the dimension of R/I and assume that $I^{(n)} = I^n$ holds locally on the punctured spectrum. We may also assume that $\dim R/I > t$.

Let $\underline{a} = a_1, \ldots, a_r \subset S$ be a sequence regular on S and on S/J with $(R, I) = (S/(\underline{a}), (J, \underline{a})/(\underline{a}))$. For $0 \leq i \leq r$ consider the sequence of deformations $(S_i, J_i) = (S/(a_1, \ldots, a_i), (J, a_1, \ldots, a_i)/(a_1, \ldots, a_i))$. Then $(S_r, J_r) = (R, I)$ and $(S_0, J_0) = (S, J)$. We will show that the pair $(S_{i-1}, J_{i-1}^{(n)})$ is a deformation of $(S_i, J_i^{(n)})$ and that the assumptions are preserved from J_i to J_{i-1}. Each S_i is Cohen-Macaulay, and (S, J) is a deformation of every pair (S_i, J_i). By Proposition 2.3, S_i/J_i is equidimensional and J_i satisfies (CI_0) for all i.

We first consider the case say $i = r$, i.e. the pairs (R, I) and $(T, K) = (S_{r-1}, J_{r-1})$. We will denote by " $^-$ " reduction modulo $a = a_r$.

There are inclusions

$$I^n \subset \overline{K^{(n)}} \subset I^{(n)}.$$

To see the rightmost containment, it is enough to show it locally at every $p \in \mathrm{Ass}\, R/I^{(n)} = \mathrm{Min}\, R/I$. Let q be the preimage in T of p. Since I satisfies (CI_0), K_q is a complete intersection. It follows that, locally at p, both ideals in question coincide with the power I^n.

Moreover, it holds that

$$(*) \qquad \overline{K^{(n)}}^{(1)} = I^{(n)}.$$

The same argument above gives the containment "$\subset$", from which equality follows.

Now we are in the position to apply Proposition 2.4. Let $B = T/K^{(n)}$. Since $K^{(n)}$ is unmixed (as T/K is equidimensional) it follows that $\hat{B}$ is unmixed (e.g. [53, 34.10]), and a is regular on B since it is regular on T/K. Because $I^{(n)} = I^n$ holds on the punctured spectrum, the inclusions $I^n \subset \overline{K^{(n)}} \subset I^{(n)}$ imply that $\overline{K^{(n)}}$ is unmixed on the punctured spectrum. Hence if we set $A = B/xB \cong R/\overline{K^{(n)}}$ it holds that

$$A/H^0_m(A) = R/\overline{K^{(n)}}^{(1)}$$

which by (*) implies that

$$A/H^0_m(A) = R/I^{(n)}.$$

By hypothesis, depth $R/I^{(n)} \geq 2$. Thus we may apply Proposition 2.4 to conclude that $H^0_m(A) = 0$, or in other words that $\overline{K^{(n)}} = I^{(n)}$. It follows that $(T, K^{(n)})$ is a deformation of $(R, I^{(n)})$ via a.

We now observe that K satisfies the same hypothesis as I. Let $p \in \operatorname{Spec} T/K$ and let q be a minimal prime of (p, a). If dim $(T/K)_p > t$ then since $T/K^{(n)}$ is a deformation of $R/I^{(n)}$ it follows as in the proof of Proposition 2.3(b) that depth $(T/K^{(n)})_p \geq 2$. Now suppose that dim $(T/K)_p \leq t$. We must show that $K^{(n)}_p = K^n_p$. By localizing at q, we may assume that $q = m$. Thus we have a deformation $(T, K^{(n)})$ of $(R, I^{(n)})$. But then as dim $R/I \leq t$ we have by hypothesis that $I^{(n)} = I^n$. Since $K^{(n)}$ and K^n then have the same image in R, it follows that $K^{(n)} \subset (K^n, a)$, and as a is regular on $T/K^{(n)}$,

$$K^{(n)} = (K^n, a) \cap K^{(n)} = K^n + (a) \cap K^{(n)} = K^n + aK^{(n)}.$$

Hence by Nakayama's lemma, we conclude that $K^{(n)} = K^n$ and hence this equality holds locally at p.

By descending induction on i, we conclude that $(S, J^{(n)})$ is a deformation of $(R, I^{(n)})$ via the sequence $\underline{a}$. However, by assumption $J^{(n)} = J^n$, and hence by specializing we must have that $I^{(n)} = I^n$. □

We separate the two important extreme cases of the theorem.

Corollary 2.6 Let R be a local Cohen-Macaulay ring, let I be an R-ideal satisfying (CI_1), assume that (R, I) has a deformation (S, J) such that S/J is equidimensional and that $J^{(n)} = J^n$ for some n, and further assume one of the following conditions holds:

(a) $R/I^{(n)}$ satisfies (S_2) ;

(b) depth $R/I^{(n)} \geq 2$ and $I^{(n)} = I^n$ holds on the punctured spectrum.

Then $I^{(n)} = I^n$.

An unmixed ideal I is called *normally torsionfree* if $I^{(n)} = I^n$ for every n.

Corollary 2.7 Let R be a local Cohen-Macaulay ring, let I be an R-ideal satisfying (CI_1) and having a normally torsionfree deformation, and assume that $R/I^{(n)}$ satisfies (S_2) for infinitely many n.

Then $\ell(I_p) \leq \max\{\text{ht } I, \dim R_p - 2\}$ for all $p \in V(I)$.

Proof. By Corollary 2.6, we can conclude that R/I^n satisfies (S_2) for infinitely many n. The result now follows from Burch's inequality. □

For an ideal I, which satisfies (CI_0), in a local Cohen-Macaulay ring R, one can characterize the property that I is a complete intersection by the condition that R/I^n is Cohen-Macaulay for infinitely many n ([16]). Using Theorem 2.5, we obtain an analogous result for symbolic powers:

Corollary 2.8 Let R be a local Cohen-Macaulay ring and let I be an R-ideal satisfying (CI_1) and having a normally torsionfree deformation. Then the following are equivalent:

(a) I is a complete intersection ;

(b) $R/I^{(n)}$ is Cohen-Macaulay for infinitely many n.

Proof. By Remark 2.2 (and its proof), it is enough to show that (b) implies (a). But if (b) holds then by Corollary 2.6 it follows that R/I^n is Cohen-Macaulay for infinitely many n. By Burch's inequality one has that $\ell(I) = \operatorname{ht} I$. Since I satisfies (CI_0), and has a reduction which is a complete intersection, by localizing it follows that I is a complete intersection. □

Under stronger assumptions, one can prove sharper versions of Corollary 2.8. The following partly generalizes a result of Huneke and Ulrich on the powers of licci ideals ([41]).

Theorem 2.9 Let R be a regular local ring and let I be a licci R-ideal satisfying (CI_1). Then the following are equivalent:

(a) I is a complete intersection;

(b) $R/I^{(n)}$ is Cohen-Macaulay for some $n \geq 3$.

Proof. Since I is licci, (R, I) has a normally torsionfree deformation by [40, 2.6 and proof of 5.3]. The result now follows from [41, 2.8] using Corollary 2.6. □

We now illustrate how one may actually compute the depth of symbolic powers in certain cases.

Theorem 2.10 Let (R, m) be a local Gorenstein ring and let I be a licci ideal with dim $R/I > 0$ satisfying $\nu(I_p) \leq \max\{\text{ ht } I, \text{ dim } R_p - 1\}$ for all $p \in V(I) - \{m\}$.

Then depth $R/I^{(n)} = \max\{\dim R - \nu(I), 1\}$ for all sufficiently large n, and for $n > d(I)$ if $\nu(I) \leq \dim R$.

Proof. Since I is licci it is strongly Cohen-Macaulay and has a normally torsionfree deformation ([40, 2.6 and proof of 5.3]). By the numerical condition on the local number of generators, it follows that I is normally torsionfree, and satisfies G_∞, on the punctured spectrum ([70, 4.2]).

First, if $\nu(I) \leq \dim R - 1$, then again by [70, 4.2] I is normally torsionfree and I satisfies G_∞. Now the result follows from [41, 2.7]. Thus we may assume that $\nu(I) \geq \dim R$.

Now suppose to the contrary that depth $R/I^{(n)} \geq 2$ for infinitely many n. Then by Corollary 2.6, $I^{(n)} = I^n$ and Burch's inequality implies that $\ell(I) \leq \dim R - 2$. But then [70, 4.2] implies that $\nu(I) = \ell(I) \leq \dim R - 2$, which is a contradiction. Finally, if $\nu(I) \leq \dim R$, then again by [41, 2.7] the equality must hold for any $n > d(I)$. □

We now give an application to the module of differentials.

Let $R = k[[x_1, ..., x_n]]$ be a formal power series ring over a perfect field k, and let I be a reduced R-ideal. There is a natural exact sequence

$$0 \longrightarrow I/I^{(2)} \longrightarrow \Omega_k(R) \otimes_R R/I \longrightarrow \Omega_k(R/I) \longrightarrow 0$$

where $\Omega_k(-)$ denotes the universally finite module of differentials. From this sequence it is clear that the depth of $R/I^{(2)}$ is intimitely related to the depth of the module of differentials $\Omega(R/I)$.

Proposition 2.11 Let k be a perfect field, let $A = k[[x_1, ..., x_n]]/I$ be normal, and assume that A has a deformation $B = k[[y_1, ..., y_m]]/J$ such that the conormal module J/J^2 is a torsionfree B-module. Then the following hold:

(a) I/I^2 is a reflexive A-module if and only if $\Omega_k(A)$ is torsionfree.

(b) If A is Cohen-Macaulay, then I/I^2 is a Cohen-Macaulay A-module if and only if depth $\Omega_k(A) \geq \dim A - 1$.

Proof. Let $R = k[[x_1, ..., x_n]]$. Now if I/I^2 is reflexive or Cohen-Macaulay, it is torsionfree and hence $I^{(2)} = I^2$. It then follows from the fundamental exact sequence above and the fact that A is normal that $\Omega_k(A)$ is torsionfree (respectively satisfies depth $\Omega_k(A) \geq \dim A - 1$ if A is Cohen-Macaulay).

Conversely, assume that $\Omega_k(A)$ is torsionfree (respectively satisfies depth $\Omega_k(A) \geq \dim A - 1$ and that A is Cohen-Macaulay). Since A is normal, $\Omega_k(A)$ satisfies (S_1) and hence $I/I^{(2)}$ satisfies (S_2). It follows that $R/I^{(2)}$ satisfies (S_2), and hence by Corollary 2.6 that $I^{(2)} = I^2$. It then follows that I/I^2 is reflexive (respectively Cohen-Macaulay). □

We point out an example where the above equivalences fail.

Example 2.12 Let k be a perfect field, let X be a symmetric 3 by 3 matrix of variables, put $R = k[[X]]$, and let $I = I_2(X)$ be the R-ideal generated by the 2 by 2 minors of X. Then $A = R/I$ has an isolated singularity and $\Omega_k(A)$ is torsionfree (e.g. [68, 3.6]). However, the conormal module I/I^2 is not even torsionfree; in fact $\det(X) \in I^{(2)} - I^2$.

Finally, we give an application to the study of the Artin-Nagata properties of Section 1.1.

Proposition 2.13 Let R be a local Gorenstein ring of dimension d, let I be an unmixed R-ideal of grade $g \leq s$ satisfying (CI_0) and $\ell(I_p) \leq \max\{g, \dim R_p - 1\}$ for all $p \in V(I)$ with dim $R_p \leq s+1$, and assume that (R, I) has a deformation (S, J) with $J^{(j)} = J^j$ for $1 \leq j \leq s-g+1$. Then the following are equivalent:

(a) I satisfies AN_s ;

(b) depth $R/I^j \geq d-g-j+1$ for $1 \leq j \leq s-g+1$.

Proof. By [68, 2.9], it is enough to show that (a) implies (b). Now if I satisfies AN_s, the local assumption on the analytic spread implies by [68, 3.3] that depth $R/I^{(j)} \geq d-g-j+1$ for $1 \leq j \leq s-g+1$. Note that this already implies that R/I is Cohen-Macaulay since I is unmixed. Thus to prove (b), it will be enough to show that $I^{(j)} = I^j$ for $1 \leq j \leq s-g+1$. It follows that depth $(R/I^{(j)})_p \geq 2$ for $1 \leq j \leq s-g+1$ whenever dim $(R/I)_p > s-g+1$. But by [68, 1.11] and [70, 4.2], I is normally torsionfree locally in codimension $s+1$. Hence Theorem 2.5 with $t = s-g+1$ implies that $I^{(j)} = I^j$ for $1 \leq j \leq s-g+1$. □

CHAPTER 3

CONSTRUCTIONS IN LINKAGE

In this chapter we present ways to construct Cohen-Macaulay ideals having certain linkage properties. In Section 1 we review some of the basic facts about linkage, and in particular about the class of ideals which lie in the linkage class of a complete intersection. In Section 2, taking the tensor product of two algebras over a field which lie in the linkage class of a complete intersection, we obtain algebras which are strongly Cohen-Macaulay and strongly nonobstructed, but which do not lie in the linkage class of a complete intersection. The depth of the twisted conormal module and the first Koszul homology module are well-known to be invariants of the linkage class (respectively even linkage class), and we observe in Section 3 that moreover the depth of these modules modulo their torsion submodule is an invariant of the geometric linkage class. In Section 4, we study the sum of two geometrically linked ideals. We prove results which relate the depths of the first Koszul homology modules and the twisted conormal modules of the two linked ideals to that of their sum. In particular, we are able to obtain classes of Gorenstein ideals which are strongly nonobstructed, but are not syzygetic. In Section 5 we construct, by intersecting two complete intersections, some Cohen-Macaulay ideals of type 2.

3.1 Licci Ideals

In this section, we recall the most important results about linkage, mainly due to Huneke and Ulrich, that we will need in this chapter. We first recall the basic notion:

Definition 3.1.1 Let R be a local Gorenstein ring, and let I and J be two R-ideals.

(a) I and J are linked, denoted by $I \sim J$, if there is an R-regular sequence $\underline{\alpha} = \alpha_1, ..., \alpha_g$ in $I \cap J$ such that $J = (\underline{\alpha}) : I$ and $I = (\underline{\alpha}) : J$.

(b) I and J are geometrically linked if I and J are linked and have no common associated primes.

It follows that if I and J are linked, then I and J are unmixed ideals of grade g. In addition, if they are geometrically linked then $I \cap J = (\underline{\alpha})$.

Proposition 3.1.2 (Peskine-Szpiro [55]) Let R be a local Gorenstein ring, let I be an unmixed R-ideal, let $\underline{\alpha}$ be a maximal regular sequence properly contained in I and set $J = (\underline{\alpha}) : I$. Then

(a) I and J are linked (via the sequence $\underline{\alpha}$);

(b) R/I is Cohen-Macaulay if and only if R/J is Cohen-Macaulay;

(c) if R/I is Cohen-Macaulay then $\omega_{R/I} \cong J/(\underline{\alpha})$; in particular, $r(R/I) = \nu(J/(\underline{\alpha}))$.

By iterating this process, one is led to the following:

Definition 3.1.3 Let R be a local Gorenstein ring and let I and J be two R-ideals.

(a) I and J are in the same linkage class if there is a sequence of links

$I = I_0 \sim I_1 \sim \ldots \sim I_n = J$. If n is even, I and J are in the same even linkage class (or are evenly linked, for short). If all the links are geometric, I and J are in the same geometric linkage class.

(b) I is licci if I is in the linkage class of a complete intersection ideal.

Proposition 3.1.2(b) can now be rephrased as saying that "Cohen-Macaulayness is an invariant of the linkage class." We will study various other linkage invariants in the sequel. The class of licci ideals turns out to be a particularly nice one, inheriting many good properties trivially observed for complete intersections. We recall the classic examples:

Theorem 3.1.4 Let R be a local Gorenstein ring and let I be an R-ideal.

(a) (Apéry [6], Gaeta [9]) If I is perfect of grade 2, then I is licci.

(b) (J. Watanabe [77]) If I is a perfect Gorenstein ideal of grade 3, then I is licci.

Indeed, one also has structure theorems for these ideals. By the Hilbert-Burch theorem ([14]) any ideal in (a) is the ideal of n-sized minors of an $n+1$ by n matrix, while by Buchsbaum-Eisenbud ([11]) any ideal in (b) is the ideal of $2n$th order Pfaffians of an alternating $2n+1$ by $2n+1$ matrix.

The following two results give some of the most important properties which are invariant under linkage:

Theorem 3.1.5 (Huneke [34]) Let I and J be two ideals which are evenly linked. Then I is strongly Cohen-Macaulay if and only if J is strongly Cohen-Macaulay.

However, strongly Cohen-Macaulayness is usually not an invariant of the entire linkage class. For example, any non-strongly Cohen-Macaulay Gorenstein ideal is linked to an almost complete intersection (by 3.1.2(c)) which is of course strongly Cohen-Macaulay.

Theorem 3.1.6 (Buchweitz-Ulrich [13]) Let I and J be two perfect R-ideals in the same linkage class.

Then depth $I \otimes \omega_{R/I}$ = depth $J \otimes \omega_{R/J}$.

In particular, the property that the so-called twisted conormal module $I \otimes \omega_{R/I}$ is Cohen-Macaulay is a linkage invariant. This was first shown (for generic complete intersections in a regular k-algebra) by Buchweitz ([12]). This property has been shown to be important in deformation theory (e.g. [12], [25]) and that motivated the following definition:

Definition 3.1.7 Let R be a local Gorenstein ring. A Cohen-Macaulay R-ideal I is strongly nonobstructed if $I \otimes \omega_{R/I}$ is Cohen-Macaulay.

If I is Gorenstein, then $I \otimes \omega_{R/I} = I/I^2$; in this case strongly nonobstructedness is simply the Cohen-Macaulayness of the conormal module.

Since any complete intersection is trivially both strongly Cohen-Macaulay and strongly nonobstructed, we obtain the following (using the fact that any complete intersection is linked to itself):

Corollary 3.1.8 Every licci ideal is strongly nonobstructed and its entire linkage class is strongly Cohen-Macaulay.

Part of our motivation was to understand whether the converse of this corollary holds. Before our investigation began, there was apparantly no known examples of nonlicci ideals with these properties. We will construct such examples in the next section.

The following main result of [38] allows one to reduce many questions about licci ideals to the simplest examples described in Theorem 3.1.4.

Theorem 3.1.9 (Huneke-Ulrich [38, 4.3]) Let R be a regular local ring and let I be a licci R-ideal which is not a complete intersection. Then (R, I) has a essentially a deformation (S, J) with $S/J \cong (P[X]/K)_{(m_P, X)}$, where P is a regular local ring, K is a P-ideal and either

(a) X is a generic 2 by 3 matrix and $K = I_2(X)$, or

(b) X is a generic alternating 5 by 5 matrix and $K = Pf_4(X)$.

We will need the following results as well.

Proposition 3.1.10 ([39]) Let R be a local Gorenstein ring with infinite residue field, let S be a faithfully flat local Gorenstein extension, and let I be an R-ideal. Then I is licci if and only if IS is licci.

Proposition 3.1.11 ([38, 2.16]) Let R be a local Gorenstein ring, let I be an R-ideal, let (S, J) be a deformation of (R, I) and let $I = I_0 \sim I_1 \sim \cdots \sim I_n$ be a sequence of links in R. Then there exists a sequence $J = J_0 \sim J_1 \sim \cdots \sim J_n$ in S such that (S, J_i) is a deformation of (R, I_i) for all $0 \leq i \leq n$.

It follows from the above result that if I is licci and (S, J) is a deformation of (R, I), then J is also licci. Moreover, one has the following:

Theorem 3.1.12 ([40, 5.1]) Let k be a field, $R = k[[x_1, .., x_n]]$, and let I be a licci R-ideal. Then (R, I) has a deformation (S, J), where S is a power series ring over k, such that J satisfies G_∞.

Finally we recall the "shift condition" satisfied by any homogeneous licci ideal.

Theorem 3.1.13 ([38, 5.13]) Let $S = k[x_1, ..., x_n]$ be a positively graded polynomial ring over a field k, let I be a homogeneous S-ideal, such that $I_{(x_1, ..., x_n)}$ is licci, having homogeneous resolution

$$0 \longrightarrow \oplus_{i=1}^{b_g} S(-n_{gi}) \longrightarrow \cdots \longrightarrow \oplus_{i=1}^{b_1} S(-n_{1i}) \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Then max $\{n_{gi}\} > (g-1)$ min $\{n_{1i}\}$.

3.2 Tensor Products

In this section we work in the category $\mathcal{CM}(k)$ of complete local Cohen-Macaulay k-algebras with residue field k. Given any $A \in \mathcal{CM}(k)$, we say that A is licci, strongly Cohen-Macaulay or strongly nonobstructed if there exists a presentation $A \cong k[[x_1, \ldots, x_n]]/I$ where I has the corresponding property. For the property of being strongly Cohen-Macaulay and strongly nonobstructed, this definition is independent of the given presentation. The property of being licci is independent of the presentation if k is infinite, as can be seen from [39, 4.7].

We say that two algebras A and B in $\mathcal{CM}(k)$ are linked if there exists a regular $R \in \mathcal{CM}(k)$ and presentations $A \cong R/I$, $B \cong R/J$, where I and J are linked R-ideals.

Given $A, B \in \mathcal{CM}(k)$, one may form the (complete) tensor product $A \hat{\otimes}_k B$. The following statements are by and large well-known.

Proposition 3.2.1 Consider the following properties of an algebra $A \in \mathcal{CM}(k)$:

(a) strong Cohen-Macaulayness;

(b) strong nonobstructedness;

(c) Cohen-Macaulayness of $\mathrm{Tor}_i^R(A, \omega_A)$ for every i.

If two algebras in $\mathcal{CM}(k)$ enjoy any of the above properties, then so does their complete tensor product.

Proof. Let $A, B \in \mathcal{CM}(k)$ and let $R, S \in \mathcal{CM}(k)$ be regular local rings mapping onto A and B respectively. Now $A\hat{\otimes}_k B$ has a resolution over $R\hat{\otimes}_k S$ given by the complete tensor product of the resolutions of A and B. In particular it follows that $A\hat{\otimes}_k B$ is Cohen-Macaulay so $A\hat{\otimes}_k B \in \mathcal{CM}(k)$.

Now (a) is proved in [36, 1.9] and (c) follows from the Kunneth formula ([50])

$$\mathrm{Tor}_i^{R\hat{\otimes}_k S}(A\hat{\otimes}_k B, \omega_{A\hat{\otimes}_k B}) \cong \bigoplus_{j=0}^{i} \mathrm{Tor}_j^R(A, \omega_A)\hat{\otimes}_k \mathrm{Tor}_{i-j}^S(B, \omega_B).$$

Since $I \otimes \omega_A \cong \mathrm{Tor}_1^R(A, \omega_A)$, (b) follows from (c) with $i = 1$. □

One might expect that virtually any reasonable homological property would be preserved from A and B to the tensor product. However one has the following:

Theorem 3.2.2 Let A and B be complete local licci algebras over the residue field k.

(a) If A and B are generically complete intersections, then the entire linkage class of $A\hat{\otimes}_k B$ is strongly Cohen-Macaulay if and only if A or B is Gorenstein.

(b) $A\hat{\otimes}_k B$ is licci if and only if A or B is a complete intersection.

Corollary 3.2.3 Let A and B be complete local licci algebras over the residue field k which are not complete intersections and let $C = A\hat{\otimes}_k B$. Then

(a) C is strongly Cohen-Macaulay and strongly nonobstructed, but not licci;

(b) $\mathrm{Tor}_i^{R\hat{\otimes}_k S}(C, \omega_C)$ is Cohen-Macaulay for every i.

Proof. This follows from Proposition 3.2.1 using Theorems 3.1.5, 3.1.6 (and [13] for part (b)), and Theorem 3.2.2(b). □

This gives a natural construction of algebras which are not licci but enjoy many of the properties usually only observed for that class. This lends support to the belief that it is very difficult to find necessary and sufficient conditions for an ideal to lie in the linkage class of a complete intersection. Before giving the proof of Theorem 3.2.2, which is somewhat technical, we give two explicit examples.

Example 3.2.4 Let $A = k[[X]]/I_2(X)$ where X is a generic 2 by 3 matrix, and let $B = k[[Y]]/Pf_4(Y)$ where Y is a generic 5 by 5 alternating matrix. Then $A\hat{\otimes}_k B$ is a Cohen-Macaulay k-algebra of embedding codimension 5, deviation 3 and type 2 which is strongly nonobstructed and whose entire linkage class is strongly Cohen-Macaulay, but which is not licci.

Example 3.2.5 Let $A = k[[X]]/Pf_4(X)$ and $R = k[[X]]$, where X is a generic 5 by 5 alternating matrix. Then $C = A\hat{\otimes}_k A$ is a Gorenstein k-algebra of embedding codimension 6 and deviation 4 which is strongly nonobstructed, strongly Cohen-Macaulay, satisfies $\text{Ext}^i_{R\hat{\otimes}_k R}(C, C)$ is Cohen-Macaulay for every i, but is not licci.

We introduce an explicit construction, which we call the *transversal link*, which we will use in the proof of Theorem 3.2.2(a).

Let $I_1 \subset R_1 = k[[X]]$ and $I_2 \subset R_2 = k[[Y]]$ be unmixed ideals which are generically complete intersections and let $R = k[[X,Y]]$. Let $I_1 \sim J_1$ and $I_2 \sim J_2$ be any two geometric links, linked by the regular sequences $\underline{\alpha}_1$ and $\underline{\alpha}_2$ respectively. Set $L = ((\underline{\alpha}_1)R, (\underline{\alpha}_2)R) : (I_1R + I_2R)$. By Proposition 3.1.2(a), L is linked to $I_1R + I_2R$. Moreover, this link is again geometric (by transversality). We call L a transversal link. We summarize its properties in the following proposition.

Proposition 3.2.6 Let $R_1 = k[[X]], R_2 = k[[Y]], R = k[[X,Y]]$, let $I_1 \subset R_1$ and $I_2 \subset R_2$ be Cohen-Macaulay ideals, let $J_i = (\underline{\alpha}_i) : I_i$ be geometric links and set $A_i = R_i/I_i$ and $B_i = R_i/J_i$ for $i = 1,2$. Let $I = I_1R + I_2R$ and let $L = ((\underline{\alpha}_1)R, (\underline{\alpha}_2))R : I$ be its transversal link. Then

(a) $d(L) = r(A_1)r(A_2)$.

(b) $r(R/L) = r(B_1) + r(B_2)$.

(c) If depth $R_1/J_1^2 \geq \dim B_1 - 1$ and depth $R_2/J_2^2 \geq \dim B_2 - 1$ then depth $R/L^2 \geq \dim R/L - 1$.

(d) If A_1 and A_2 are not Gorenstein, then L is not syzygetic.

(e) If A_1 is Gorenstein, $r((A_2)_p) \leq \dim(A_2)_p$ for every $p \in V(J_2)$, and J_2 is strongly Cohen-Macaulay, then L is strongly Cohen-Macaulay.

Proof. If " $'$ " denotes reduction modulo $\underline{\alpha}_1, \underline{\alpha}_2$ then we have $L' = J_1' J_2'$ by transversality. Since $\underline{\alpha}_1, \underline{\alpha}_2$ are part of a minimal generating set of L (as can be seen by factoring out the variables X or Y), one has

$$d(L) = \nu(L') = \nu(J_1')\nu(J_2') = r(A_1)r(A_2)$$

by Proposition 3.1.2(c). This proves (a). Similarly, for (b) we have that

$$r(R/L) = \nu(I/(\underline{\alpha}_1, \underline{\alpha}_2)) = \nu(I_1/(\underline{\alpha}_1)) + \nu(I_2/(\underline{\alpha}_2)) = r(B_1) + r(B_2).$$

Now we show (c). Let $g_i = \text{grade } I_i$, $g = g_1 + g_2$ and $\underline{\alpha} = \underline{\alpha}_1, \underline{\alpha}_2$. Since

$$(\underline{\alpha}) + L^2/L^2 \cong (\underline{\alpha})/(\underline{\alpha}) \cap L^2 = (\underline{\alpha})/(\underline{\alpha})L \cong (R/L)^g,$$

as $\underline{\alpha}$ generically generates L, it follows that there is an exact sequence

$$0 \longrightarrow (R/L)^g \longrightarrow L/L^2 \longrightarrow L'/(L')^2 \longrightarrow 0.$$

If we let " $*$ " denote reduction modulo $\underline{\alpha}_i$, then similarly there is an exact sequence for $i = 1, 2$

$$0 \longrightarrow (R_i/J_i)^{g_i} \longrightarrow J_i/J_i^2 \longrightarrow J_i^*/(J_i^*)^2 \longrightarrow 0.$$

But since $(L')^2 = (J_1')^2(J_2')^2$, there is also an exact sequence

$$0 \longrightarrow R'/(L')^2 \longrightarrow R'/(J_1')^2 \oplus R'/(J_2')^2 \longrightarrow R'/(J_1')^2 + (J_2')^2 \longrightarrow 0.$$

By the transversality of J_1 and J_2, the result follows by chasing depths in these sequences.

To show (d), it will be enough to show that L' is not syzygetic ([34, 1.4]). Changing notation, we write R for R'. Then $L = J_1J_2$ is a product of two transversal ideals. Moreover, since neither I_1 nor I_2 is Gorenstein and $\nu(J_i) = r(R_i/I_i)$ by Proposition 3.1.2(c), it follows that neither J_i is principal. Suppose that a_1, b_1 and a_2, b_2 are part of a minimal generating set of J_1 and J_2 respectively. Then $a_1a_2, a_1b_2, a_2b_1, b_1b_2$ is part of a minimal generating set of L. We conclude that L is not syzygetic since it admits the quadratic relation $T_1T_4 = T_2T_3$ on these minimal generators.

It remains to show (e). It will be enough to show that L' is strongly Cohen-Macaulay ([34, 1.6]). Changing notation again, we write R for R'. It follows from Proposition 3.1.2(c) that J_2 satisfies G_∞. Now since I_1 is Gorenstein, by Proposition 3.1.2(c) we have that J_1 is cyclic. Write $J_1 = (b)$ for some element $b \in R_1$. Then $L = bJ_2$ also satisfies G_∞. By [68, 2.13] it will be enough to show that depth $R/L^j \geq \dim R - g - j + 1$ for $1 \leq j \leq d(L) - 1$. But since J_1 is an almost complete intersection, it holds that depth $R_1/J_1^j \geq \dim B_1 - 1$ for all $j \geq 1$, and as J_2 is strongly Cohen-Macaulay and satisfies G_∞ it follows that depth $R_2/J_2^j \geq \dim B_2 - j + 1$ for all $j \geq 1$. Hence from the exact sequence

$$0 \longrightarrow R/L^j \longrightarrow R/J_1^jR \oplus R/J_2^jR \longrightarrow R/J_1^jR + J_2^jR \longrightarrow 0,$$

we conclude that depth $R/L^j \geq \dim R/L - j + 1$ for all $j \geq 1$. This completes the proof. □

Corollary 3.2.7 With the notation of Proposition 3.2.6, assume that I_1 and I_2 are licci ideals which are not Gorenstein. Then L is strongly nonobstructed, depth $R/L^2 \geq \dim R/L - 1$, but L is not syzygetic.

Proof. Since I_1 and I_2 are licci, they are strongly nonobstructed (Corollary 3.1.8). By Proposition 3.2.1 it follows that $I = (I_1 + I_2)R$ is strongly nonobstructed since it defines the complete tensor product $A_1 \hat{\otimes}_k A_2$. Now Theorem 3.1.6 implies that its transversal link L is also strongly nonobstructed, and Proposition 3.2.6(d) implies that L is not syzygetic.

By Theorem 3.1.5, it holds that J_i are strongly Cohen-Macaulay and since they are generically complete intersections, we have the exact sequence

$$0 \longrightarrow H_1(J) \longrightarrow B^n \longrightarrow J/J^2 \longrightarrow 0$$

with $J = J_i$, $B = B_i$ and $n = \nu(J_i)$ for $i = 1, 2$. Since $H_1(J_i)$ is Cohen-Macaulay we have that depth $R_i/J_i^2 \geq \dim B_i - 1$. Hence by Proposition 3.2.6(c), depth $R/L^2 \geq \dim R/L - 1$. □

Example 3.2.8 Let $R = k[[x_1, ..., x_4, y_1, ..., y_4]]$ and let $I_1 = (x_1x_2, x_1x_3, x_3x_4)$ and $I_2 = (y_1y_2, y_1y_3, y_3y_4)$, which are grade 2 perfect ideals, and in particular are licci. Then $I = I_1 + I_2$ is a perfect R-ideal of grade 4 and deviation 2, which is strongly Cohen-Macaulay, strongly nonobstructed, but not licci by Corollary 3.2.3.

Linking via the regular sequences x_1x_2, x_3x_4 and y_1y_2, y_3y_4 produces a transversal link $L = (x_1x_2, x_2y_2, x_2y_4, x_3x_4, x_4y_2, x_4y_4, y_3y_4, y_1y_2)$. By Proposition 3.2.6 and Corollary 3.2.7, L is a perfect R-ideal of grade 4, deviation 4 and type 2 which is strongly nonobstructed, depth $R/L^2 \geq \dim R/L - 1$, but L is not syzygetic.

Proof of Theorem 3.2.2. We first prove part (a). Since A and B are licci, by Proposition 3.1.12, they have deformations $\tilde{A}$ and $\tilde{B}$ in $\mathcal{CM}(k)$ which satisfy G_∞.

Then $\tilde{A}\hat{\otimes}_k\tilde{B}$ is a deformation of $A\hat{\otimes}_k B$. Now A or B is Gorenstein if and only if $\tilde{A}$ or $\tilde{B}$ is Gorenstein. Since the property of being strongly Cohen-Macaulay is preserved under deformation (by [41, proof of 2.1], since A and B are generically complete intersections), by Proposition 3.1.11 it follows that the entire linkage class of $A\hat{\otimes}_k B$ is strongly Cohen-Macaulay if and only if the entire linkage class of $\tilde{A}\hat{\otimes}_k\tilde{B}$ is strongly Cohen-Macaulay. Hence we may assume that $A \cong R_1/I_1$ and $B \cong R_2/I_2$ where $R_1 = k[[X]]$, $R_2 = k[[Y]]$ and I_1 and I_2 satisfy G_∞. Let $R = k[[X,Y]]$ and let L be a transversal link of $I_1R + I_2R$.

Now assume that A or B is Gorenstein. By symmetry, let us assume that A is Gorenstein. To show that the entire linkage class of $A\hat{\otimes}_k B$ is strongly Cohen-Macaulay, it is enough to show by Theorem 3.1.5 that a single link is strongly Cohen-Macaulay. In particular, it is enough to show that L is strongly Cohen-Macaulay. Since I_1 is Gorenstein and J_2 is strongly Cohen-Macaulay, this will follow from Proposition 3.2.6(e) if I_2 satisfies the local condition on the type. However, as I_2 is licci, by [40, 2.5] and [38, 2.17] it always has a deformation satisfying this condition. By the previous deformation argument, it follows that L is strongly Cohen-Macaulay.

For the converse, if the entire linkage class of $A\hat{\otimes}_k B$ is strongly Cohen-Macaulay then L is strongly Cohen-Macaulay, and in particular L is syzygetic. But then by Proposition 3.2.6(d), we immediately obtain that either A or B is Gorenstein.

It remains to prove part (b). First observe that by transversality that if A or B is a complete intersection, then $A\hat{\otimes}_k B$ is licci. Indeed, if say B is a complete intersection and $A \sim A_1 \cdots \sim A_n$ where A_n is a complete intersection, then $A\hat{\otimes}_k B \sim A_1\hat{\otimes}_k B \cdots \sim A_n\hat{\otimes}_k B$ and of course $A_n\hat{\otimes}_k B$ is a complete intersection. Conversely, we must show that if A and B are not complete intersections then

$A\hat{\otimes}_k B$ is not licci.

If k' denotes the algebraic closure of k, then let $A' = A\hat{\otimes}_k k'$ and $B' = B\hat{\otimes}_k k'$. If $A\hat{\otimes}_k B$ is licci then so is $A'\hat{\otimes}_{k'} B'$. If one of A' or B' is a complete intersection then the same holds for A and B. Hence we may assume that k is algebraically closed. In particular, since k is infinite, the property of being licci is independent of the presentation of the algebra.

Choose presentations $A \cong R/I$ and $B \cong S/J$, where $R = k[[x_1, ..., x_d]]$ and $S = k[[y_1, ..., y_e]]$. Since I is licci, there is a sequence of links

$$(x_1, ..., x_g) \sim I_1 \sim \cdots \sim I_{n-1} \sim I_n = I.$$

By [38, 2.17], there is a sequence of links in a polynomial extension $T = R[Z]$

$$(x_1, ..., x_g)T \sim L_1 \sim \cdots \sim L_{n-1} \sim L_n = L$$

with the property that (T_q, L_q) is a deformation of (R, I) for some prime $q \in \text{Spec}(T)$. Similarly, there is an ideal $L' \subset U = S[Z']$ and a prime $q' \in \text{Spec}(U)$ such that $(U_{q'}, L'_{q'})$ is a deformation of (S, J). It follows that there is a deformation of tensor products

$$\widehat{(T/L)_q}\hat{\otimes}_k\widehat{(U/L')_{q'}} \longrightarrow A\hat{\otimes}_k B.$$

Since the property of being licci is preserved under deformation (3.1.11), it will be enough to show that the former ring is not licci.

Now by construction of the T-ideal L and U-ideal L' (cf. [38]), we may view them as an extended ideals from the polynomial subrings $k[X, Z]$ and $k[Y, Z']$ respectively. If we let $p = q \cap k[X, Z]$, $p' = q' \cap k[Y, Z']$, $A_0 = (k[X, Z]/L)_p$ and $B_0 = (k[Y, Z']/L')_{p'}$ then there is a faithfully flat morphism of local rings

$$(A_0 \otimes_k B_0)_M \longrightarrow \widehat{(T/L)_q}\hat{\otimes}_k\widehat{(U/L')_{q'}},$$

where M is the maximal ideal $m_{A_0} \otimes B + A_0 \otimes m_{B_0}$. Since the property of being licci descends from faithfully flat extensions by Proposition 3.1.11, it is then enough to show that (after localizing at M) that $A_0 \otimes_k B_0$ is not licci, i.e. that there is some defining ideal which is not licci. Hence we have reduced the problem to the case where A and B are essentially of finite type over k and have replaced the complete tensor product by the ordinary tensor product.

Now we would like to apply Theorem 3.1.9. Since $A \cong R/I$ and $B \cong S/J$ are licci and not complete intersections, they have essentially a deformation to a pair $(P[W]_{(m,W)}, K)$ where P is a regular local ring and K is either the ideal of 2 by 2 minors of a generic 2 by 3 matrix or the ideal of 4 by 4 Pfaffians of a generic alternating 5 by 5 matrix. Further, if A' is essentially a deformation of A and B' is essentially a deformation of B, where all the rings are essentially of finite type over k, then $(A' \otimes_k B')_{M'}$ is essentially a deformation of $(A \otimes_k B)_M$, where $M' = m_{A'} \otimes B' + A' \otimes m_{B'}$; here we use the fact that k is algebraically closed to ensure that if $p \in \mathrm{Spec}(A)$ and $q \in \mathrm{Spec}(B)$ then $p \otimes B + A \otimes q$ is a prime ideal in $A \otimes_k B$ (e.g. [78, p.198]). Since by Proposition 3.1.11 the property of being licci is preserved under essentially a deformation, we have reduced to the case where A and B are either of the algebras described in Theorem 3.1.9. However, if $A \cong (P[W]/K)_{(m,W)}$ and $B \cong (P'[W']/K')_{(m',W')}$ then there is a faithfully flat morphism

$$(k[W]/K \otimes_k k[W']/K')_{M'} \longrightarrow (A \otimes_k B)_M.$$

Now by faithfully flat descent, we are reduced to showing that $A \otimes_k B$ is not licci (after localizing at the maximal ideal generated by the variables) where A and B are one of the algebras $C = k[X]/I_2(X)$, where X is a generic 2 by 3 matrix, or $D = k[Y]/Pf_4(Y)$, where Y is a generic 5 by 5 alternating matrix. By part (a) of the theorem we know that one of A or B is Gorenstein. Thus it only remains to show that both $(C \otimes_k D)_M$ and $(D \otimes_k D)_M$ are not licci. We show this by checking that they do not satisfy the "shift condition."

Now C and D have homogeneous resolutions over $R = k[X]$ and $S = k[Y]$ of the form

$$0 \longrightarrow R^2(-3) \longrightarrow R^3(-2) \longrightarrow R \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow S(-5) \longrightarrow S^5(-3) \longrightarrow S^5(-2) \longrightarrow S \to D \longrightarrow 0.$$

Thus $C \otimes_k D$ and $D \otimes_k D$ have respectively graded resolutions over $T = R \otimes_k S$ and $U = S \otimes_k S$ of the form

$$0 \longrightarrow T^2(-8) \cdots \longrightarrow T^8(-2) \longrightarrow T \longrightarrow C \otimes_k D \longrightarrow 0$$

$$0 \longrightarrow U(-10) \longrightarrow \cdots \longrightarrow U^{10}(-2) \longrightarrow U \longrightarrow D \otimes_k D \longrightarrow 0.$$

Now $C \otimes_k D$ has projective dimension 5 and $\max\{n_{gi}\} = 8 = (g-1)\min\{n_{1i}\}$. Similarly $D \otimes_k D$ has projective dimension 6 and $\max\{n_{gi}\} = 10 = (g-1)\min\{n_{1i}\}$. It now follows from Theorem 3.1.13 that $(C \otimes_k D)_M$ and $(D \otimes_k D)_M$ are not licci. This completes the proof of Theorem 3.2.2. $\square$

One could ask if the tensor product is essentially the only way to construct the examples of this section. More specifically, for low codimensional ideals one could ask the following:

Question 3.2.9 Let R be a regular local ring and let I be a perfect R-ideal of grade g. Then is I licci if I satisfies any of the following conditions?

(a) $g = 3$, I is strongly nonobstructed and strongly Cohen-Macaulay.

(b) $g = 4$, I is strongly nonobstructed and the entire linkage class is strongly Cohen-Macaulay.

(c) $g = 4$, I is Gorenstein and strongly Cohen-Macaulay.

(d) $g = 5$, I is Gorenstein, strongly nonobstructed and strongly Cohen-Macaulay.

3.3 Invariants of Geometric Linkage

In this section we point out that the depth of certain modules modulo their torsion is an invariant of the geometric linkage class. We will often make use of the following well-known fact, which can be shown by tensoring with the total ring of quotients. For an R/I-module built from an ideal I, τ always denotes the R/I-torsion.

Lemma 3.3.1 Let A be a noetherian local ring, let

$$M_1 \xrightarrow{\phi} M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence of finitely generated A-modules having a rank, assume that M_2 is torsionfree and that rank M_2 = rank M_1 + rank M_3.

Then $\tau(M_1) = \ker \phi$.

The following is an analog of Theorem 3.1.6.

Proposition 3.3.2 Let R be a local Gorenstein ring and let I and J be two Cohen-Macaulay R-ideals which are geometrically linked.

Then depth $(I \otimes \omega_{R/I})/\tau$ = depth $(J \otimes \omega_{R/J})/\tau$.

Proof. If the ideals have grade 0 then $\omega_{R/I} \cong J$ by Proposition 3.1.2(c) and the result is trivial. Also, if I is a complete intersection then the result would just follow from Theorem 3.1.6. Hence we may assume that I and J have positive grade g and are both not complete intersections. Under these assumptions we will show that depth $(I \otimes \omega_{R/I})/\tau$ = depth IJ. By symmetry, this will prove the proposition.

Consider the natural exact sequence

$$0 \longrightarrow \omega_{R/I} \longrightarrow R/(\underline{\alpha}) \longrightarrow R/J \longrightarrow 0.$$

Tensoring this sequence with R/I gives an exact sequence

$$\mathrm{Tor}_1^R(\omega_{R/I}, R/I) \longrightarrow \mathrm{Tor}_1^R(R/(\underline{\alpha}), R/I) \longrightarrow \mathrm{Tor}_1^R(R/J, R/I)$$

which after natural identifications corresponds to the sequence

$$I \otimes \omega_{R/I} \longrightarrow (\underline{\alpha})/(\underline{\alpha})I \longrightarrow (\underline{\alpha})/IJ \longrightarrow 0.$$

Since it holds that

$$(\underline{\alpha})/(\underline{\alpha})I \cong (\underline{\alpha}) \otimes_R R/I \cong (\underline{\alpha})/(\underline{\alpha})^2 \otimes_{R/(\underline{\alpha})} R/I \cong (R/(\underline{\alpha}))^g \otimes_{R/(\underline{\alpha})} R/I \cong (R/I)^g$$

is free over R/I and I is generically generated by $\underline{\alpha}$ (since the given link is geometric), by Lemma 3.3.1 there is a short exact sequence

$$0 \longrightarrow (I \otimes \omega_{R/I})/\tau \longrightarrow (R/I)^g \longrightarrow (\underline{\alpha})/IJ \longrightarrow 0.$$

Now from this sequence it will follow that

$$\text{depth } (I \otimes \omega_{R/I})/\tau = \text{depth } (\underline{\alpha})/IJ + 1 = \text{depth } R/IJ + 1 = \text{depth } IJ,$$

as long as R/IJ is not Cohen-Macaulay. However, if R/IJ would be Cohen-Macaulay then it would be unmixed and it would follow that $IJ = I \cap J = (\underline{\alpha})$; but cutting down to the grade one case, we obtain that IJ is principal, which implies I or J is principal since R is local. Since I and J are not complete intersections, this gives our contradiction and completes the proof. □

Corollary 3.3.3 Let R be a local Gorenstein ring and let I and J be two Gorenstein R-ideals in the same geometric linkage class.

Then depth $R/I^{(2)}$ = depth $R/J^{(2)}$.

Proof. This follows immediately from Proposition 3.3.2 since if I is Gorenstein then $I \otimes \omega_{R/I} = I/I^2$ and $(I \otimes \omega_{R/I})/\tau \cong I/I^{(2)}$. □

The following application shows that the depth of $\Omega(A) \otimes \omega_A$ is invariant for two geometrically linked *rigid* algebras. Recall that a noetherian local k-algebra is rigid if every infinitesimal deformation is trivial.

Corollary 3.3.4 Let A and B be complete reduced local Cohen-Macaulay algebras over a perfect residue field which are geometrically linked and assume that A and B are both either rigid or not rigid.

Then depth $\Omega(A) \otimes \omega(A)$ = depth $\Omega(B) \otimes \omega(B)$.

Proof. Choose presentations $A \cong S/I$, $B \cong S/J$ where $S = k[[x_1, ..., x_n]]$ and I and J are linked S-ideals. The natural exact sequence of A-modules

$$I/I^2 \longrightarrow A^n \longrightarrow \Omega(A) \longrightarrow 0$$

induces by Lemma 3.3.1 the short exact sequence

$$0 \longrightarrow (I/I^2 \otimes \omega_A)/\tau \longrightarrow \omega_A^n \longrightarrow \Omega(A) \otimes \omega_A \longrightarrow 0,$$

and analogously there is an exact sequence of B-modules

$$0 \longrightarrow (J/J^2 \otimes \omega_B)/\tau \longrightarrow \omega_B^n \longrightarrow \Omega(B) \otimes \omega_B \longrightarrow 0.$$

The result now follows from Proposition 3.3.2 by chasing depths in these sequences, using the fact ([25, 1.3]) that A is rigid if and only if $\mathrm{Ext}^1_A(\Omega(A) \otimes \omega_A, \omega_A) = 0$. □

Now we give the analog of Theorem 3.1.5.

Proposition 3.3.5 Let R be a local Gorenstein ring and let I and J be two Cohen-Macaulay R-ideals which are geometrically evenly linked.

Then depth $H_1(I)/\tau =$ depth $H_1(J)/\tau$.

Proof. We follow the proof [34, 1.11] of the corresponding result for $H_1(I)$. We may assume that I and J are doubly linked, say $I = (\underline{\alpha}) : K$, $K = (\underline{\beta}) : J$, where both links are geometric. Since I is generically a complete intersection, Lemma 3.3.1 induces the exact sequence

$$0 \longrightarrow H_1(I)/\tau \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0.$$

Put $t =$ depth $H_1(I)/\tau$. Then $t = \min\{\text{depth } R/I^2 + 1, \dim R/I\}$. We induct on the grade g. If $g = 0$ then $I = J$ so the result is trivial. If $g = 1$ then I and J are isomorphic as ideals and the result follows as well. So we may assume that $g \geq 2$ and that the result is known for smaller g. By prime avoidance, one may choose an element γ_g, which is a minimal generator of $(\underline{\beta})$, such that $\alpha_1, ..., \alpha_{g-1}, \gamma_g$ is an R-regular sequence which generically generates K. Extend γ_g to $\gamma_1, ..., \gamma_g$, a minimal generating set of $(\underline{\beta})$ and let $L = (\alpha_1, ..., \alpha_{g-1}, \gamma_g) : K$. Note that this link is geometric, and I and J are both doubly linked to L. Hence we may now assume that I and J are doubly linked and that both linking sequences contain a common element, say x. Since the links are geometric we have that $(x) \cap I^2 = xI$, which can be checked by localizing at the associated primes of I.

Now let " $*$ " denote reduction modulo x. Then there is an exact sequence

$$0 \longrightarrow R/I \longrightarrow R/I^2 \longrightarrow R^*/(I^*)^2 \longrightarrow 0.$$

It follows from this sequence that $t(I^*) = t$ and by induction we know that $t(J^*) = t(I^*)$. Hence $t(J) = t(J^*) = t$. $\square$

One may restate Theorem 3.1.6 superficially as follows:

Remark 3.3.6 Let R be a local Gorenstein ring and let I and J be two perfect R-ideals in the same linkage class which are both generically complete intersections. Then depth $(H_1(I) \otimes \omega_{R/I})/\tau =$ depth $(H_1(J) \otimes \omega_{R/J})/\tau$.

Proof. The exact sequence

$$H_1(I) \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0$$

induces by Lemma 3.3.1 an exact sequence

$$0 \longrightarrow (H_1(I) \otimes \omega_{R/I})/\tau \longrightarrow \omega_{R/I}^n \longrightarrow I/I^2 \otimes \omega_{R/I} \longrightarrow 0.$$

But since depth $I/I^2 \otimes \omega_{R/I} =$ depth $J/J^2 \otimes \omega_{R/J}$ by Theorem 3.1.6, the result follows. □

We point out that, despite the previous results, depth $H_1(I) \otimes \omega_{R/I}$ is *not* a linkage invariant. By [65, 2.19] if R is a regular local ring containing the rational numbers and if I is a licci R-ideal which is generically a complete intersection, then $H_1(I) \otimes \omega_{R/I}$ is torsionfree if and only if I is Gorenstein. However, by Remark 3.3.6 it holds that any for any licci ideal I which is generically a complete intersection, $(H_1(I) \otimes \omega_{R/I})/\tau$ is Cohen-Macaulay.

3.4 Sums of Links

In this section we use sums of links to construct more nonlicci ideals with certain specified properties. We begin with the following observation from [55], whose proof is short enough to be repeated.

Proposition 3.4.1 (Peskine-Szpiro) Let R be a local Gorenstein ring and let I and J be two Cohen-Macaulay R-ideals of grade g which are geometrically linked. Then $I + J$ is a Gorenstein ideal of grade $g + 1$.

Proof. By factoring out the regular sequence defining the link, we may assume that $g = 0$. Since then $I \cap J = 0$ there is the exact sequence

$$0 \longrightarrow R \longrightarrow R/I \oplus R/J \longrightarrow R/I + J \longrightarrow 0.$$

It follows that dim $R/I + J = d - 1$ where $d =$ dim R. Applying the functor Hom$(k, -)$, where k is the residue field, we obtain an exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^{d-1}(k, R/I + J) \longrightarrow \mathrm{Ext}_R^d(k, R) \longrightarrow \mathrm{Ext}_R^d(k, R/I) \oplus \mathrm{Ext}_R^d(k, R/J).$$

But then $r(R/I + J) = \dim_k \mathrm{Ext}^{d-1}(k, R/I + J) \leq \dim_k \mathrm{Ext}^d(k, R) = r(R) = 1$, hence $R/I + J$ is Gorenstein. $\square$

This turns out to be an interesting way to construct new examples of Gorenstein ideals. For example, Ulrich has shown the following:

Theorem 3.4.2 ([67, 2.1]) Let R be a local Gorenstein ring and let I and J be two licci R-ideals which are geometrically linked.

Then $I + J$ is licci.

In [67], Ulrich also gives conditions for $I + J$ to be strongly nonobstructed, but his assumptions force $I + J$ to be syzygetic as well. We generalize his result, making use of the following result whose proof may be found in [67, proof of 3.1].

Lemma 3.4.3 ([67]) Let R be a local Gorenstein ring and let I be a Cohen-Macaulay R-ideal. Then exactly one of the following conditions holds:

(a) depth $H_1(I) =$ depth $S_2(I)$;

(b) $H_1(I)$ is Cohen-Macaulay and depth $S_2(I) \geq \dim R/I + 1$.

We are now ready to prove the main result of this section.

Theorem 3.4.4 Let R be a local Gorenstein ring, let I and J be two Cohen-Macaulay ideals which are geometrically linked and let $K = I + J$. Then

(a) depth $K/K^2 = \min\{\text{depth } I/I^2, \text{depth } J/J^2, \text{depth } (I \otimes \omega_{R/I})/\tau - 1\}$;

(b) depth $H_1(K) = \min\{\text{depth } H_1(I), \text{depth } H_1(J), \text{depth } I \otimes \omega_{R/I}, \dim R/I - 1\}$.

Proof. We first prove (a). If I has grade 0 then $K = I \oplus J$ and hence $K^2 = I^2 \oplus J^2$, since $I \cap J = 0$. As $(I \otimes \omega_{R/I})/\tau = 0$, the claim follows. Hence we may assume that I has positive grade g, and we may also assume that I and J are both not complete intersections (for else K is a hypersurface section of either I or J by Proposition 3.1.2(c)). Let I and J be geometrically linked by $\underline{\alpha}$. Since $\underline{\alpha}$ form a regular sequence, $(\underline{\alpha})/(\underline{\alpha})I \cong (R/I)^g$ and thus depth $R/(\underline{\alpha})I = \dim R/I$ (and similarly for J, depth $R/(\underline{\alpha})J = \dim R/I$). It also follows that $I^2 \cap IJ = (\underline{\alpha})I$ and $IK \cap J^2 = (\underline{\alpha})J$, since these equalities are easily seen to hold locally at every associated prime of $(\underline{\alpha})$. Hence there are exact sequences

$$0 \longrightarrow (\underline{\alpha})I \longrightarrow I^2 \oplus IJ \longrightarrow IK \longrightarrow 0$$

$$0 \longrightarrow (\underline{\alpha})J \longrightarrow IK \oplus J^2 \longrightarrow K^2 \longrightarrow 0.$$

But since depth IJ = depth $(I \otimes \omega_{R/I})/\tau <$ depth $(\underline{\alpha})I$ by the proof of Proposition 3.3.2, (a) now follows by chasing depths in these sequences.

To prove (b), let " $^-$ " denote reduction modulo the linking sequence $\underline{\alpha}$ and note that depth $H_1(I)$ = depth $H_1(\overline{I})$, depth $H_1(J)$ = depth $H_1(\overline{J})$, and depth $H_1(K)$ = depth $H_1(\overline{K})$. Since I is generically a complete intersection, there is an exact sequence

$$0 \longrightarrow \omega^g_{R/I} \longrightarrow I \otimes \omega_{R/I} \longrightarrow \overline{I} \otimes \omega_{\overline{R}/\overline{I}} \longrightarrow 0.$$

It follows that (b) remains unchanged by factoring out $(\underline{\alpha})$, and thus we may assume that $g = 0$.

Now since $K = I \oplus J$ and $\omega_{R/I} \cong J$ by Proposition 3.1.2(c), we have that

$$S_2(K) \cong S_2(I) \oplus S_2(J) \oplus (I \otimes \omega_{R/I}). \tag{3.4.5}$$

We now use Lemma 3.4.3 to replace the depth of the symmetric square by the depth of H_1. Indeed, depth $H_1(I) =$ depth $S_2(I)$ (and similarly for J) since the condition of Lemma 3.4.3(b) would imply depth $S_2(I) \geq \dim R/I + 1 > \dim R$. Similarly we claim that depth $H_1(K) =$ depth $S_2(K)$. For otherwise, Lemma 3.4.3 implies that depth $S_2(K) \geq \dim R/K + 1 = \dim R$, and thus $S_2(K)$ would be torsionfree. But then (3.4.5) implies that $I \otimes \omega_{R/I} = 0$, hence that $I = 0$ or $J = 0$, which is impossible. Now (3.4.5) implies that

$$\text{depth } H_1(K) = \min\{\text{depth } H_1(I), \text{ depth } H_1(J), \text{ depth } I \otimes \omega_{R/I}\}. \quad \square$$

Corollary 3.4.6 Let R be a local Gorenstein ring of dimension d and let $K = I + J$ be a sum of two geometrically linked Cohen-Macaulay ideals of grade g.

(a) K is strongly nonobstructed if and only if depth $R/I^2 \geq d - g - 1$, depth $R/J^2 \geq d - g - 1$ and $(I \otimes \omega_{R/I})/\tau$ is Cohen-Macaulay.

(b) If I satisfies (CI_1), then K is syzygetic if and only if I and J are syzygetic, and $I \otimes \omega_{R/I}$ is torsion-free.

Proof. Part (a) follows immediately from Theorem 3.4.4 since by Proposition 3.4.1 K is Gorenstein, hence is strongly nonobstructed if and only if K/K^2 is Cohen-Macaulay.

For (b), note that K is generically a complete intersection as I satisfies (CI_1). Now if I and J are syzygetic, then $H_1(I)$ and $H_1(J)$ are torsion-free, and if $I \otimes \omega_{R/I}$ is torsionfree, so is $H_1(K)$ by Theorem 3.4.4(b). It follows that K is syzygetic. Conversely, assume that K is syzygetic, and hence that $H_1(K)$ is torsionfree. Since I satisfies (CI_1), it follows from Theorem 3.4.4(b) that $H_1(I)$ and $I \otimes \omega_{R/I}$ are torsionfree. To check that $H_1(J)$ is torsionfree, it remains to show that J is syzygetic locally in codimension one. But in codimension one J is linked to I, which is a complete intersection, hence J is an almost complete intersection (Proposition 3.1.2). Since J is also generically a complete intersection it follows that J is syzygetic. □

Theorem 3.4.4 gives a concrete method to construct Gorenstein ideals whose conormal module and first Koszul homology module can have prescribed depths. We point out that Corollary 3.4.6(b) was obtained in [67, 3.10, 3.11], where the converse was shown under the stronger assumptions that $H_1(I)$ and $I \otimes \omega_{R/I}$ satisfy (S_2).

In [67] Ulrich constructed an example of a grade 5 Gorenstein ideal which is strongly nonobstructed but is not syzygetic. We can now give a general method to construct such examples:

Construction 3.4.7 (Sums of transversal links) Let $I_1 \subset k[[X]]$ and $I_2 \subset k[[Y]]$ be two licci ideals that satisfy (CI_1) but are not Gorenstein and let L be any transversal link of $I = (I_1, I_2)$. Then $K = I + L$ is a strongly nonobstructed Gorenstein ideal which is not syzygetic. Equivalently, the conormal module K/K^2 is Cohen-Macaulay, but the first Koszul homology module $H_1(K)$ has torsion.

Proof. Since I is licci, by Corollary 3.1.8, it is strongly Cohen-Macaulay and strongly nonobstructed, and by Corollary 3.2.7 it follows that L is not syzygetic and depth $R/L^2 \geq \dim R/L - 1$. Hence the result follows by applying Corollary 3.4.6 to the geometric link $I \sim L$. □

Example 3.4.8 Applying Construction 3.4.7 to Example 3.2.8 produces a sum of links $K = (x_1x_2, x_2y_2, x_2y_4, x_3y_4, x_4y_2, x_4y_4, y_3y_4, y_1y_2, x_1x_3, y_1y_3)$, which is a grade 5 Gorenstein ideal generated by 10 quadrics, which is strongly nonobstructed but is not syzygetic. This is quite similar to the example obtained in [67] by a somewhat different method.

Examples 3.4.9 Let k be a field, let X be a generic alternating 5 by 5 matrix, let Y be a generic 5 by 1 matrix, put $R = k[[X, Y]]$ and consider the R-ideal $I = Pf_4(X) + I_1(XY)$. We have stated in Example 1.2.13 that R/I^2 is Cohen-Macaulay (i.e. that I is strongly nonobstructed). We will use Example 3.4.8 to give a proof of this fact.

Write $I = (f_1, .., f_5, \ell_1, ..., \ell_5)$, where f_i denotes the Pfaffian obatined by deleting the ith row and column, and ℓ_i is the product of the ith row of X with Y. The elements $f_5, \ell_2, \ell_3, \ell_4, \ell_5$ form a regular sequence and linking via this sequence gives a link $J = (f_5, \ell_2, \ell_3, \ell_4, \ell_5, y_1y_5)$. Now $f_5, \ell_3, \ell_4, \ell_5, y_1y_5$ is a regular sequence and linking via this sequence produces the ideal $\tilde{K}$ generated by the polynomials

$$f_5, \ell_5, y_1y_5, y_1y_4, x_{34}y_4, x_{34}y_5, x_{23}y_4 - x_{13}y_5, x_{24}y_4 - x_{14}y_5, x_{45}y_1 - x_{34}y_3, x_{35}y_1 - x_{34}y_2.$$

It is clear that $(x_{11}, x_{12}, x_{15}, x_{24}, x_{35}, y_3) + \tilde{K} = (x_{11}, x_{12}, x_{15}, x_{24}, x_{35}, y_3) + K$, where K is the ideal of Example 3.4.6 (properly relabed). It follows that $\tilde{K}$ specializes to K. Since K/K^2 is Cohen-Macaulay, it follows that $\tilde{K}/\tilde{K}^2$ is Cohen-Macaulay ([41, 2.2]), and hence $\tilde{K}$ is strongly nonobstructed since it is Gorenstein. But as I is doubly linked to $\tilde{K}$, I is also strongly nonobstructed by Theorem 3.1.6. As I is Gorenstein, this shows that R/I^2 is Cohen-Macaulay.

Contrary to the examples above, Gorenstein ideals of grade 4 are much more well-behaved:

Theorem 3.4.10 (Vasconcelos-Villarreal [76]) Let R be a local Gorenstein ring and let I be a perfect Gorenstein R-ideal of grade 4.

Then $H_1(I)$ is Cohen-Macaulay if and only if I is strongly nonobstructed.

By considering a sum of links, this result imposes some genuine restrictions on two linked perfect ideals of grade 3. One example is as follows.

Corollary 3.4.11 Let R be a local Gorenstein ring, let I and J be two geometrically linked perfect R-ideals of grade 3, and assume that I is strongly nonobstructed and satisfies (CI_1). Then the following conditions are equivalent.

(a) $H_1(I)$ and $H_1(J)$ are Cohen-Macaulay;

(b) depth $H_1(I) \geq \dim R/I - 1$ and depth $H_1(J) \geq \dim R/J - 1$;

(c) $H_1(I)/\tau$ and $H_1(J)/\tau$ are Cohen-Macaulay.

Proof. Consider $K = I + J$, which is a perfect Gorenstein ideal of grade 4. Clearly (a) implies (b). Now by Theorem 3.4.4(b), (b) holds if and only if $H_1(K)$ is Cohen-Macaulay which holds if and only if K is strongly nonobstructed by Theorem 3.4.10, which holds if and only if (c) holds by Corollary 3.4.6(a). Hence (b) and (c) are equivalent. Now assume that (b) and (c) hold. By (b) it follows that $H_1(I)$ and $H_1(J)$ are torsionfree, hence are Cohen-Macaulay by (c), and this proves (a). □

Using different methods, the direction (b) ⇒ (a) in the above result can be shown to hold in greater generality: if I is a syzygetic grade 3 perfect ideal then one has depth $H_1(J) \neq \dim R/J - 1$ for any geometric link J of I ([72, 2.1(c)]). This result somewhat suggests that, for perfect ideals of grade 3, some properties of the Koszul homology H_1 may be preserved throughout the *entire* linkage class. This is in fact the case: Vasconcelos has shown the property that H_1 is Cohen-Macaulay in an invariant of the entire linkage class:

Theorem 3.4.12 (Vasconcelos [72, 2.4]) Let R be a local Gorenstein ring and let I and J be two perfect R-ideals of grade 3 which are linked.

Then $H_1(I)$ is Cohen-Macaulay if and only if $H_1(J)$ is Cohen-Macaulay.

Using this, one can show a certain rigidity of the twisted conormal module.

Corollary 3.4.13 Let R be a local Gorenstein ring and let I be a perfect R-ideal of grade 3 satisfying (CI_1) and with $H_1(I)$ Cohen-Macaulay.

Then depth $I \otimes \omega_{R/I} \neq \dim R/I - 1$.

Proof. Let J be any geometric link of I. Then by Theorem 3.4.12, $H_1(J)$ is Cohen-Macaulay. Now if depth $I \otimes \omega_{R/I} = \dim R/I - 1$ then by Theorem 3.4.4, letting $K = I + J$, one has $H_1(K)$ Cohen-Macaulay and that K is not strongly nonobstructed. This contradicts Theorem 3.4.10. □

In his recent book [75, p.68], Vasconcelos asked the following question.

Question 3.4.14 (Vasconcelos) Let R be a regular local ring and let I be a Gorenstein ideal of deviation 3. Is I strongly Cohen-Macaulay?

This is equivalent to just asking whether $H_1(I)$ is Cohen-Macaulay. Let us analyze what this would mean for a sum of links $I + J$. However, we will replace this sum of links by the ideal $K = I + xJ$, where x is an element of R that is regular on R and on R/I ([47]). Virtually everything we have said for sums of links holds for K and one has the equation $d(K) = d(I) + r(R/I) - 1$. Thus if K has deviation 3, then either I is Gorenstein of deviation 3, or is linked to such an ideal, or else I has type 2 and deviation 2. Of course the latter ideal is automatically strongly Cohen-Macaulay ([8]). An affirmative answer to Vasconcelos' question, by Theorem 3.4.4, would thus imply that any perfect R-ideal I of type 2 and deviation 2 satisfies depth $I \otimes \omega_{R/I} \geq \dim R/I - 1$. We have seen in Corollary 3.4.12 that in grade 3 this would already imply that I is strongly nonobstructed. This suggests the following question:

Question 3.4.15 Let R be a regular local ring and let I be a perfect ideal of type 2 and deviation 2. Is I is licci?

This actually includes an older question which is still open in general: Is every Gorenstein ideal of deviation 2 licci? (For some information about this problem, see [26] and more recently [43]).

We conclude with an example of a grade 4 perfect ideal with deviation 3 and type 3, whose entire linkage class is strongly Cohen-Macaulay but whose twisted conormal module has torsion.

Example 3.4.16 Let $R = k[[x_1, ..., x_8]]$ and let I be the Cohen-Macaulay R-ideal $I = (x_1x_2, x_2x_4, x_3x_4, x_4x_6, x_5x_6, x_6x_8, x_7x_8)$ which may be viewed as the edge-ideal associated to a tree. By [62, 3.11], I is strongly Cohen-Macaulay and of linear type. Linking via the regular sequence $x_1x_2, x_3x_4, x_5x_6, x_7x_8$ gives an ideal J with the same graph, hence also strongly Cohen-Macaulay. It follows that the entire linkage class is strongly Cohen-Macaulay. However, $K = I + J$ is a Gorenstein ideal of grade 5 generated by 10 quadrics. In fact, it is the ideal of Example 3.4.8. Since K is not syzygetic, and $H_1(I)$ and $H_1(J)$ are Cohen-Macaulay, it follows that $I \otimes \omega_{R/I}$ has torsion by Corollary 3.4.6(b). Since K is strongly nonobstructed, it follows however that $(I \otimes \omega_{R/I})/\tau$ is Cohen-Macaulay by Corollary 3.4.6(a).

3.5 Intersections of Complete Intersections

We give a naive method, which is somewhat dual to linkage, to produce Cohen-Macaulay ideals of type 2.

Construction 3.5.1 (Intersections of complete intersections) Let R be a local Gorenstein ring, let I be a Cohen-Macaulay R-ideal of grade g satisfying $\nu(I) \leq 2g$ (which always holds after adjoining variables to I), let $\underline{\alpha}$ and $\underline{\beta}$ be regular sequences of length g properly contained in I such that $I = (\underline{\alpha}) + (\underline{\beta})$, and set $\mathcal{I} = (\underline{\alpha}) \cap (\underline{\beta})$.

Then $\mathcal{I}$ is a Cohen-Macaulay ideal of type 2.

Proof. The exact sequence

$$0 \longrightarrow R/\mathcal{I} \longrightarrow R/(\underline{\alpha}) \oplus R/(\underline{\beta}) \longrightarrow R/I \longrightarrow 0$$

shows that $\mathcal{I}$ is Cohen-Macaulay and induces an exact sequence

$$0 \longrightarrow \omega_{R/I} \longrightarrow R/(\underline{\alpha}) \oplus R/(\underline{\beta}) \longrightarrow \omega_{R/\mathcal{I}} \longrightarrow 0$$

from which the result is immediate. □

Another construction, which can be shown similarly, produces ideals of arbitrary type, but of smaller grade.

Construction 3.5.2 Let R be a local Gorenstein ring, let I be a Cohen-Macaulay R-ideal of grade g satisfying $\nu(I) \leq 2g - 2$ (which always holds after adjoining variables to I), let $\underline{\alpha}$ and $\underline{\beta}$ be regular sequences of length $g - 1$ in I such that $I = (\underline{\alpha}) + (\underline{\beta})$, and set $\mathcal{J} = (\underline{\alpha}) \cap (\underline{\beta})$.

Then $\mathcal{J}$ is a Cohen-Macaulay ideal of grade $g - 1$ with $r(R/\mathcal{J}) \leq r(R/I) + 2$.

The proof of Construction 3.5.1 shows moreover that one can construct a resolution of $R/\mathcal{I}$ in terms of a resolution of R/I. This is especially interesting in the graded case since, as the resolution is essentially built from the Koszul complexes on $\underline{\alpha}$ and $\underline{\beta}$, the shifts at the tail end will tend to be fairly large.

Example 3.5.3 Let $R = k[X]$, where X is a generic 2 by 4 matrix, and let $I = I_2(X)$. Write $I = (f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34})$, where f_{ij} denotes the minor involving columns i and j. Then f_{12}, f_{13}, f_{24} and f_{14}, f_{23}, f_{34} are regular sequences and let $\mathcal{I}$ be the intersection of these two complete intersections. Since I has a linear resolution

$$0 \longrightarrow R^3(-4) \longrightarrow R^8(-3) \longrightarrow R^6(-2) \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

$\mathcal{I}$ has a resolution of the form

$$0 \longrightarrow R^2(-6) \longrightarrow R^9(-4) \longrightarrow R^8(-3) \longrightarrow R \longrightarrow R/\mathcal{I} \longrightarrow 0.$$

In particular, $\mathcal{I}$ is a perfect ideal of grade 3 and type 2 with a *pure* resolution. It follows that $H_1(I)$ is Cohen-Macaulay ([72, 2.9]), but $H_2(I)$ is not Cohen-Macaulay ([72, 4.2.4]). This ideal has essentially the same properties as the example constructed in [48, 2.6].

It turns out, however, that Construction 3.5.1 is somewhat related to the transversal link of Section 3.1.

Example 3.5.4 Let $R = k[[x_1, ..., x_n]]$ be a power series ring over a field k, let $I = (x_1, ..., x_g)$, consider regular sequences $\underline{\alpha} = x_1, ..., x_k, q_1, ..., q_{g-k}$ and $\underline{\beta} = q'_1, ..., q'_k, x_{k+1}, ..., x_g$ where $1 \leq k \leq g-1$ and q_i (respectively q'_i) are general quadrics in $x_{k+1}, ..., x_g$ (respectively $x_1, ..., x_k$) and let $\mathcal{I} = (\underline{\alpha}) \cap (\underline{\beta})$.

Then $\mathcal{I}$ is licci if and only if $k = 1$ or $k = g-1$.

Proof. Let $q = (q_1, ..., q_{g-k})$ and $q' = (q'_1, ..., q'_k)$. Then

$$\mathcal{I} = (x_1, ..., x_k, q) \cap (q', x_{k+1}, ..., x_g)$$
$$= q + q' + (x_1, ..., x_k)(x_{k+1}, ..., x_g).$$

This may be viewed as a transversal link of the ideals $I_1 = q : (x_{k+1}, ..., x_g)$ and $I_2 = q' : (x_1, ..., x_g)$. It follows from Theorem 3.2.2 that $\mathcal{I}$ is licci if and only if I_1 or I_2 is a complete intersection, which holds if and only if $k = 1$ or $k = g - 1$ (as q and q' are general). $\square$

For example, if we consider the regular sequences $\underline{\alpha} = x_1, x_2y_2, x_3y_3, ..., x_gy_g$ and $\underline{\beta} = x_1y_1, x_2, ..., x_g$ in $k[[x_1, ..., x_g, y_1, ..., y_g]]$ (corresponding to $k = 1$) then we obtain the ideals

$$\mathcal{I} = (x_1y_1, ..., x_gy_g, x_1x_2, ..., x_1x_g).$$

The fact that these ideals are licci would also follow from [62, 2.3].

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