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THE DONALDSON INVARIANT AND EMBEDDED 2-SPHERES

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ABSTRACT

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By

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In this thesis we study the Donaldson invariants for smooth 4–manifolds containing an embedded 2–sphere with an arbitrary negative self–intersection. In [FS3] Fintushel and Stern have described the relation between the Donaldson invariant of the manifold X and its blowup $X\#\overline{\mathbf{C}P^2}$. We prove the existence of a similar relation for $X = Y \cup_L N$, where N is a neighborhood of the sphere with self–intersection $-p$, and L is a lens space $L(p,1)$. We describe the most efficient way of covering by Taubes neighborhoods the points that contribute to Donaldson invariant. The technique we develop allows to compute explicitly the coefficients in this formula in some particular cases, which we ilustrate at the end of the thesis.

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1 Introduction.

For a long time Donaldson invariants were the most powerful tools for studying smooth 4–dimensional manifolds. For a simply connected 4–manifold with b_+ odd and bigger or equal to 3, these invariants are defined as linear maps $D_d : \mathbf{A}^d(X) \to \mathbf{R}$ where $\mathbf{A}^d(X)$ is the set of elements of $\mathbf{A}(X) = \mathrm{Sym}_*(\mathrm{H}_0(X) \oplus \mathrm{H}_2(X))$ having degree d. We provide the elements of $\mathrm{H}_i(X)$ with the degree $\frac{1}{2}(4-i)$. Donaldson's original definition of these invariants is subject to the restriction

$$d \equiv \frac{3(b_2^+(X)+1)}{2} \pmod 4 \tag{1.1}$$

and $d > \frac{3}{2}(b_2^+(X)+1)$. The first version of the blowup formula in [FM 2] allowed the removal of the second restriction. For a special class of manifolds (called *simple type*) one can define Donaldson's invariants for $d \equiv \frac{3(b_2^+(X)+1)}{2} \pmod 2$, but in our paper we shall stick to the original restriction. Since the discovery of Donaldson invariants there arose a question about computing them. There are many elliptic surfaces for which these invariants have been completely computed. Except for computing specific examples one would like to develop general techniques, like the Mayer-Vietoris type of argument: having the decomposition of the manifold Xonto $X_1 \cup_Y X_2$ and knowing the invariant for the components X_1, X_2 and some properties of Y (which in fact are related to the invariant of the 4–manifold $\mathbf{R} \times Y$), determine the invariant for the manifold X. When X_2 is a neighborhood of an embedded sphere with negative self–intersection, then R. Fintushel and R. Stern provided the following answer in [FS2]:

Theorem 1.2 *Let X be an oriented simply connected 4-manifold, which contains an embedded 2-sphere S representing a homology class σ with self-intersection $\sigma^2 \leq -2$. Then there are polynomials $B_{j,k} = B_{j,k}(x)$ and $A_{j,k}$ depending only on σ^2, such that for $z \in \mathbf{A}(\sigma^\perp)$*

$$D(\sigma^{2k-1}z) = B_{0,k}D_\sigma(z) + \sum_{j=1}^{k-1} D(\sigma^{2j-1}B_{j,k}(x)z) \qquad \text{if} \qquad \sigma^2 = -(2k+1),$$

and

$$D(\sigma^{2k}z) = A_{0,k}D_\sigma(z) + \sum_{j=1}^{k} D(\sigma^{2j-2}A_{j,k}(x)z) \qquad \text{if} \qquad \sigma^2 = -2k.$$

Here the degrees of the polynomials $B_{j,k}$ are equal to $k-j$, whereas the degrees of $A_{j,k}$ are $k-j+1$. The $A_{0,k}$ and $B_{0,k}$ are constants.

The proof of [FS2] uses the compactifications of certain open sets in the moduli spaces provided by C. Taubes [T1] and does not show how to compute $B_{j,k}(x)$ for specific examples. In this paper we would like to provide an alternative proof of a theorem that generalizes the above result. Namely we prove that for an arbitrary number of classes σ representing an embedded sphere with negative self–intersection we have the following result:

Theorem 1.3 *Let X be as above and suppose that X contains an embedded S^2 with self-intersection $\sigma^2 = -p \leq -1$. Then there is a formal power series $B_{(p);n}(x,t) =$*

$\sum_{j=0}^{\infty} \frac{1}{j!} B_{(p);n,j}(x)t^j$, *such that:*

$$D(\exp(\sigma \cdot t) \cdot z) = D(\sum_{n=1}^{p} B_{(p);n}(x,t)\sigma^n \cdot z)$$

There are few comments necessary about this formula:

1. Each $B_{(p);n,j}$ is a polynomial in x of a degree d satisfying the inequality $2d+n \leq j$.

2. For $j \leq p$ this formula provides nothing but the trivial statement $D(\sigma^j) = D(\sigma^j)$. Later though we shall define the relative Donaldson invariants, for which a similar formula will become nontrivial.

3. For given $z \in \mathrm{H}_2(X)$ only half of the terms in the above formula are nontrivial. Due to the restriction (1.1), either $D(\sigma^{even} \cdot z)$ or $D(\sigma^{odd} \cdot z)$ is always zero.

4. When $p = 1$ or 2, X admits an orientation preserving diffeomorphism that induces a map on $\mathrm{H}_2(X)$ sending σ onto $-\sigma$ and which is identity on $\sigma^\perp$. Thus in this case, $D(\sigma^{odd} z) = 0$ for all z.

5. In the case $p = 1$, Fintushel and Stern [FS3] have actually computed the function $B(x,t) = B_{(1);0}$. Using their result and Theorem 1.3 one can compute the functions $B_{(p);j}$ for higher p.

In the proof of the above theorem we split the manifold X into N, the neighborhood of the embedded sphere with negative self–intersection, and the remaining part Y. These spaces are connected along the lens space $L = L(p,1)$. Section 2 provides a review of the results on gluing. Roughly speaking by stretching the cylinder $L \times (-\epsilon, \epsilon)$ we obtain open sets U_i in the moduli space for the manifold X that can be described in terms of moduli spaces of X_1 and X_2. In order to combine these open sets back into a global one, we need to understand how these sets overlap. Section 3 describes the moduli spaces of connections on the cylinders $\mathbf{R} \times L(p,1)$, which are essential component of the sets $U_i \cap U_j$. In section 4 we describe the most efficient way of chosing the sets U_i, which cover the points of the moduli space over X which contribute nontrivially to the Donaldson invariant. We introduce a partial order in the set $\{U_i\}$ that will enable us to make inductive argument in the proof of the main theorem. The next section is devoted to computations inside the sets U_i. In the same section, by using relative Mayer–Vietoris sequence, we also show that the computations in U_i's done separately suffice for proving a global result. Finally, in section 6 we prove the main result. The techniques developed in this paper have wider application than just proving the existence of a general formula. In the final section we compute three examples of particular formulas. One of such formulas has been used by [FS3] as a necessary initial condition for their differential equation defining the function $B_{j,k}(X)$ in the blowup formula.

The author would like to thank Ronald Fintushel for introducing him to gauge theory, as well as for guidance and help while working on this problem.

2 Construction of the Donaldson Invariants.

In this chapter we will recall some main points of the theory of Donaldson invariants. The details can be found in [D1] and in [DK]. Let X be a simply connected four dimensional manifold and $P \to X$ be an $SU(2)$ bundle over X. These bundles are classified by their second Chern class $c_2(P) = k$. For a generic Riemannian metric on X, the space of gauge equivalence classes of anti-self-dual (ASD) connections on P is a manifold of dimension $8k - 3(1 + b_+)$. We will denote this manifold by $\mathcal{M}_k(X)$. Over the product $\mathcal{M}_k(X) \times X$ there is a universal $SO(3)$ bundle $\mathbf{P}$ which gives rise to a homomorphism $\mu : \mathrm{H}_i(X) \to \mathrm{H}^{4-i}(\mathcal{M}_k(X))$ obtained by assigning to $\sigma \in \mathrm{H}_i(X)$ the class $-\frac{1}{4}p_1(\mathbf{P})/\sigma$. To each cohomology class $\mu(\sigma)$ corresponds a codimension $4-i$ variety V_σ contained in $\mathcal{M}_k(X)$. Let $[x]$ denote the generator of $\mathrm{H}_0(X)$, and $\nu = \mu(x)$ with V_ν its dual divisor. The class ν can be also realized as $-\frac{1}{4}p_1(\mathcal{M}_k^o(X))$, where $\mathcal{M}_k^o(X)$ is the moduli space of based anti–self–dual connections. The Donaldson invariant is a linear function

$$D : \mathbf{A}(X) = \mathrm{Sym}_*(\mathrm{H}_0(X) \oplus \mathrm{H}_2(X)\ ) \to \mathbf{R}$$

which assigns to the classes $\sigma_1, \ldots, \sigma_r \in \mathrm{H}_2(X)$ and s copies of the generator $[1] \in \mathrm{H}_0(X)$ the intersection number

$$\#\,(\mathcal{M}_k(X) \cap V_{\sigma_1} \cap \cdots V_{\sigma_r} \cap_{i=1}^{s} V_\nu) \tag{2.1}$$

where the numbers k, r and s are related through $2r + 4s = 8k - 3(1 + b_+)$. In order to make sense of this intersection number, one has to compactify the manifold $\mathcal{M}_k(X)$. For that purpose we need to define *ideal connections* on X as a pair $([\omega], (x_1, x_2, \ldots, x_s))$, where $[\omega]$ is a point in $\mathcal{M}_{k-s}(X)$ and $(x_1, x_2, \ldots, x_s)$ is an unordered s–tuple of points in X. To an ideal connection $A = ([\omega], (x_1, x_2, \ldots, x_s))$ we associate its measure (or curvature density) $\mu_A = \|F_\omega\|^2 + 8\pi^2 \sum_{r=1}^{s} \delta_{x_r}$.

Definition 2.2 (Compare [DK]) *We say that a sequence of connections $\{\omega_n\}$ in $\mathcal{M}_k(X)$ converges to an ideal connection $A = ([\omega], (x_1, x_2, \ldots, x_s))$ if the following conditions are satisfied:*

1. *The measures μ_{ω_n} converge to μ_A.*
2. *There are bundle maps $\rho_n : P'|_{X \setminus \{x_1, \ldots, x_s\}} \to P|_{X \setminus \{x_1, \ldots, x_s\}}$ such that $\rho_n^*(\omega_n)$ converges on compact subsets of punctured manifold to ω.*

Let us denote by $s^n(X)$ the n^{th} symmetric product of X. Then the above definition of convergence provides the set

$$I\mathcal{M}_k(X) = \mathcal{M}_k(X) \cup \mathcal{M}_{k-1}(X) \times X \cup \mathcal{M}_{k-2}(X) \times s^2(X) \cup \ldots$$

with a topology. The Uhlenbeck compactification $\bar{\mathcal{M}}_k(X)$ of $\mathcal{M}_k(X)$ is the closure of $\mathcal{M}_k(X)$ in $I\mathcal{M}_k(X)$. This is a stratified space with singularities. We can define a fundamental class of a singular space (and thus introduce an intersection theory) if the singular set is of codimension at least 2. This is the case when $d > \frac{3}{4}(1 + b_+)$. In [FM 2] it is shown that the classes $\mu(\sigma)$ extend over the compactified moduli space, whereas the class ν extends only away from the lowest stratum, corresponding to a completely concentrated connection. Under the above restrictions on the dimension of the moduli space, the intersection of divisors V_σ takes place away from completely concentrated connections, thus the intersection (2.1) is well defined. One can extend the definition of the Donaldson invariants beyond the dimension restrictions by considering connections on $SU(2)$ bundles over $X \# \overline{\mathbf{C}P^2}$ and using the first version of the blowup formula from [FM 2].

2.1 Donaldson polynomials on manifolds with cylindrical ends

In order to prove Theorem 1.3 we need to separate the sphere S with its tubular neighborhood N from the rest of manifold X. As $\partial N = L(p, -1)$ (which we abbreviate as L), we can represent X as $Y \underset{L}{\cup} N$. From this point on, one could try to study spaces of connections on the manifolds Y and N with boundary L. However for boundaries as simple as lens spaces, it is more convenient to stretch the neck $L \times (-\epsilon, \epsilon)$ to an infinite length and study spaces of connections on manifolds with cylindrical ends (see the definition below). The advantage of this approach is that the restrictions over L of connections with finite energy on bundles over X may be almost arbitrary, whereas the connections with finite energy on bundles over a manifold with a cylindrical end limit to a flat connection. The whole set of flat connections is a finite dimensional variety, thus we have very limited set of possible "boundary values".

In our formal description of this procedure we shall follow the notation of [MMR]. We are going to apply this theory to both N and Y; thus for the presentation of general theory let us fix a closed, oriented, Riemannian 4-manifold (Z, g_Z) whose boundary is a 3-manifold (L, g_L).

Definition 2.3 *A Riemannian 4-manifold* (Z, g_Z) *is called* a manifold with cylindrical end *if* Z *has a subset isometric to* $[-1, \infty) \times L$. *This subset we will call* the cylindrical end.

We do not assume that the above is the only end of Z. We shall refer to manifolds with two cylindrical ends as *tubes*.

Definition 2.4 *On the manifold* Z *with cylindrical end let* $\tau_Z : Z \to \mathbf{R}$ *be a function which maps a cylindrical end onto* $[-1, \infty)$ *and is less than* -1 *on the complement of the cylindrical end.*
Define Z_a *to be* $\tau_Z^{-1}((-\infty, a])$ *and* $Z_{[a,b]} = Z_b \setminus Z_a = \tau^{-1}([a, b])$.

There is a corresponding theory of ASD connections on bundles over Z. Let $E \to Z$ be a C^∞ principal $SU(2)$ bundle. For any $\mathrm{L}^2_{k,\mathrm{loc}}$ connection ω and any real number

$\delta \geq 0$ define the $L^2_{k,\delta}$ norm on $C_0^\infty(Z, \Lambda^p(\text{ad E}))$ by

$$\|s\|_{k,\omega,\delta} = (\sum_{i=0}^{k} \int_Z |\nabla_\omega^i s|^2 e^{\delta \cdot \tau})^{1/2} \tag{2.5}$$

Set $L^2_{k,\omega,\delta}$ to be the completion of $C_0^\infty(Z, \Lambda^p(\text{ad E}))$ with respect to this norm. By $\mathcal{A}_{k,\delta}$ we denote the set of all connections with the norm defined above. When $\delta = 0$ we abbreviate $L^2_{k,\omega,0}$ by $L^2_{k,\omega}$. Let $\mathcal{G}_E$ be the group of L^2_3 gauge transformations of E . Then $\mathcal{B}_E$ denotes the quotient $\mathcal{A}_E/\mathcal{G}_E$. Define also $\mathcal{G}_E^o$ to be a group of those gauge transformations which are the identity on a fixed fiber E_x of E. If we divide $\mathcal{A}_E$ by $\mathcal{G}_E^o$, then we get the space of based connections denoted by $\mathcal{B}_E^o$, or $\mathcal{B}_E^x$ if we want to indicate the point over which we fix the fiber. The same space can be also obtained as

$$\mathcal{B}_E^x = \mathcal{A}_E \underset{\mathcal{G}_E}{\times} E_x \tag{2.6}$$

Every element of the space $\mathcal{B}_E^x$ can be represented as a pair $[\omega, \rho_x]$, where ω is a gauge equivalence of connection in $\mathcal{B}_E$, and ρ_x is a framing of E at x. For any complete Riemannian metric on Z, the energy k of an ASD connection ω is:

$$k = \frac{-1}{8\pi^2} \int_Z \text{tr } F_\omega \wedge *F_\omega = \frac{-1}{8\pi^2} \|F_\omega\|^2_{L^2}$$

(the last equality holds only when ω is ASD). Note that, unlike in the case of closed manifolds – the number k need not be integer. In fact we shall show in the next section that k is of the form $\frac{n}{p}$, where n is an integer and p is the self-intersection of S. With this understood, denote by $\mathcal{M}_k(Z)$ the space of gauge equivalences classes of $g-$ASD connections on E with the energy k. Again, if we divide the space of ASD-connections by $\mathcal{G}^o$ then we obtain the space of based connections denoted $\mathcal{M}_k^o(Z)$, or $\mathcal{M}_k^x(Z)$ when we will need to indicate the point at which the connections are based.

Let $\chi(L)$ denote the space of gauge equivalences of flat connections in $\mathcal{B}_E(L)$. This space is called *the character variety of* L and is identified with

$$\text{Hom}(\pi_1(L), SU(2))/\text{Ad}SU(2)$$

Similarly, the flat connections modulo based gauge transformations are identified with $\mathcal{R}(L) = \text{Hom}(\pi_1(L), SU(2))$, the *representation variety.* According to Theorem 4.6.1 of [MMR] there is a well-defined map $\partial^o : \mathcal{M}_k^x(Z) \to \mathcal{R}(L)$. This map descends to $\partial : \mathcal{M}_k(Z) \to \chi(L)$. These maps associate to every ASD connection ω (or gauge equivalence class $[\omega]$) the limit $\lim_{t\to\infty}[\omega|_{t\times L}]$. This limit is (the gauge equivalence class of) a flat connection on L. In the particular situation when $L = L(p,1)$ the character variety is just $\left[\frac{p}{2}\right] + 1$ isolated points. Let us denote the elements of the character variety $\chi(L)$ by numbers $\{0, 1, 2, \ldots \left[\frac{p}{2}\right]\}$. Notice that $\chi(L)$ is a quotient of $\mathbf{Z}_p$ by the $\mathbf{Z}_2$ action given by multiplying by -1. Thus for given p there is a map $\mathbf{Z} \to \chi(L)$ that is a composition of the modulo p quotient of integers with the above $\mathbf{Z}_2$ quotient.

Definition 2.7 *We will refer to the elements* 0 *and* $p/2 \in \chi(L)$ *as* trivial *elements of the character variety. Thus for p odd there is only one trivial element. The remaining elements of* $\chi(L)$ *we will call* nontrivial.

The representation variety consists of an isolated point or points corresponding to trivial elements in $\chi(L)$, and a copy of S^2 corresponding to every nontrivial element of $\chi(L)$. For given $m \in \chi(L)$ define $\mathcal{M}_k(X,m)$ as $\partial^{-1}(m)$. As we have seen before, the number k is of the form $\frac{n}{p}$. If the moduli space $\mathcal{M}_k(X,m)$ is nonempty, then the numbers m and k are not arbitrary. In order to describe their relation we shall define the Chern–Simons function. Every connection ω on the trivial bundle $\theta = SU(2) \times Y$ we can write as $\omega = \Theta + \alpha$, where Θ is the trivial connection and $\alpha \in \Omega^1(Y, ad\theta)$. Then define:

$$CS(\omega) = \int_Y \operatorname{tr}(\alpha \wedge d\alpha + \tfrac{2}{3}\alpha \wedge \alpha \wedge \alpha)$$

The gauge transformation may change the value of the function CS by an integer, thus Chern–Simons function descends to a function $CS : \mathcal{B}(Y) \to \mathbf{R}/\mathbf{Z}$. According to the Chern–Simons theorem we have

$$k \equiv CS(m) \bmod \mathbf{Z}$$

The formal dimension of $\mathcal{M}_k(X,m)$ is given by the formula:

$$8k \; - \; \frac{3}{2} \cdot (\sigma(Z) + \chi(Z)) \; - \; \frac{h^1_m + h^0_m}{2} \; + \; \frac{\rho(m)}{2} \tag{2.8}$$

Here $\sigma(Z)$ is the signature and $\chi(Z)$ is the Euler number of Z. The h^i_m are dimensions of cohomology groups $\mathrm{H}^i(L, \mathrm{ad}\, m)$ and $\rho(m)$ is the Atiyah–Patodi–Singer ρ invariant for the signature complex twisted by $\mathrm{ad}(m) \otimes \mathbf{C}$. The invariant $h_m = h^1_m + h^0_m$ is equal to:

$$h_m = \begin{cases} 1 & \text{when } m \text{ is nontrivial} \\ 3 & \text{when } m \text{ is trivial.} \end{cases} \tag{2.9}$$

It is important to notice that the invariant h_m does not depend on an orientation of L, whereas $\rho(L) = -\rho(\bar{L})$. We will give more explicit formula for $\rho(m)$ in Lemma 3.8. For the sake of later gluing theorems, we need some facts about ASD connections on Z in local coordinates.

Definition 2.10 *Let* Γ *be a flat connection on a* $SU(2)$ *bundle* η *over* L, *and* U_Γ *be an open neighborhood of* Γ *in the space of gauge equivalence classes of connections on* η. *Let* I *be a subinterval of* $[0,\infty)$. *Following [MMR] we say that an* L^2_2 *connection* $\omega = \Gamma + A(t) + \alpha(t)\, dt$ *on* $I \times \eta$ *is* in standard form with respect to U_Γ *if for all* $t \in I$

- $A(t) \in U_\Gamma$
- $\alpha(t) \in \operatorname{Ker}(\Delta_\Gamma)^\perp$

where Δ_Γ *is the Laplacian on* $\Omega^0(L, ad\,(\Gamma))$.

Theorem 2.11 (Corollary 4.3.3 in [MMR]) *There is a constant ε_0, depending only on L, such that for a generic metric g on Z and for all $1 \leq a \leq b \leq \infty$ and any $L^2_{2,loc}$ $g-$ASD connection ω of finite energy satisfying*

$$\int_{a-1}^{b+1} \|F_\omega\|_g^2 < \varepsilon_0,$$

there is a neighborhood U_Γ and an $L^2_{3,loc}$ bundle isomorphism $\phi : [a,b) \times \eta \to E|_{[a,b)\times L}$ such that $\phi^*\omega = \Gamma + A(t) + \alpha(t)\, dt$ is in standard form with respect to U_Γ.

An important consequence of the previous theorem is:

Lemma 2.12 *Let Z be a Riemannian manifold containing a submanifold isometric to $C_l = (-l,l) \times L$ for some 3-manifold L whose character variety $\chi(L)$ consists of finitely many points. Then there is an $\epsilon > 0$ such that for every ASD-connection ω on an $SU(2)$ bundle E over Z satisfying*

$$\int_{C_l} \|F_\omega\|_g^2 \leq \epsilon \tag{2.13}$$

there is a unique gauge equivalence class of flat connections $[\Gamma] \in \chi(L)$ which is closest to the restriction $\omega|_{\{0\}\times L}$ in the $L^2_{3/2}$ norm.

Proof: Let δ be the minimum distance between any two points in the character variety $\chi(L)$ in the $L^2_{3/2}$ norm. Consider the set U_Z of all ASD connections on Z satisfying (2.13), and the set U_{C_l} of corresponding restrictions to C_l. Because of the energy bound, any sequence of connections on $E|_{C_l}$ cannot bubble on C_l. Thus the set U_{C_l} is compact as is the corresponding set U_Z in the Uhlenbeck compactification. Hence by Theorem 2.11 we can cover U_Z with finite number of the sets $U_\Gamma^I = \{\omega | \omega|_{\{t\}\times L} \in U_\Gamma \text{ for all } t \in (-l,l)\}$. Now we want to estimate the distance $\|\omega|_{\{t\}\times L} - \Gamma\|_{L^2_{3/2}}$ by the curvature of ω on $E|_{C_l}$.

According to Lemma 4.1.1 of [MMR] there is a δ_1 such that for every connection A on the $SU(2)$ bundle over L and satisfying $\|F_A\|_{L^2_{1/2}} < \delta_1$, there is some flat connection Γ_A such that

$$\|A - \Gamma_A\|_{L^2_{3/2}} < \delta/2$$

which, because of our definition of δ, must be unique. Thus to conclude the lemma we need to bound the energy of the curvature of $\omega|_{\{t\}\times L}$ by the curvature of the connection ω on the cylinder C_l. This is done in Lemma 3.5.1 [MMR] which provides a constant $C_1 > 0$ such that the following estimate holds:

$$\left(\|F_{\omega|_{\{t\}\times L}}\|_{L^2_1}\right)^2 \leq C_1 \int_{[t-1,t+1]\times L} |F_\omega|^2$$

which combined with Sobolev inequality $\|F\|_{L^2_{1/2}} \leq C_2 \cdot \|F\|_{L^2_1}$ gives that

$$\left(\|F_{\omega|_{\{t\}\times L}}\|_{L^2_{1/2}}\right)^2 \leq C \int_{C_l} |F_\omega|^2$$

Thus when $\epsilon < \frac{\delta_1}{2C}$ every connection ω satisfying (2.13) has a unique flat connection Γ_ω. $\square$

Once we consider the based connections $[\omega, \rho_n] \in \mathcal{M}^n_{k_-,k_+}(X_l, C_{d_0})$, then we can "recover" the flat connection in representation variety.

2.2 The Gluing Theorem

A goal of this subsection is to set up tools for covering the moduli space $\mathcal{M}_k(X)$ by well–understood open sets. We next describe the gluing procedure which joins connections on two manifolds with cylindrical ends. We follow [MMR] and [MM]. For any $m \in \chi(L)$ and real positive numbers $T^\pm$ let us define open sets $\mathcal{M}_{k_\pm}(X^\pm, [T^\pm, \epsilon], m) \subset \mathcal{M}_{k_\pm}(X^\pm, m)$ as containing those ASD connections $\omega^\pm$ on the bundles $E^\pm$, for which

$$\int_{[T^\pm,\infty)\times L} \|F_{\omega^\pm}\|_g^2 < \epsilon$$

Whenever possible we are going to skip some of the indices in this notation. Thus the lack of the ϵ parameter means that $\epsilon$is equal to $\varepsilon_0/2$, where ε_0 is taken from Theorem 2.11. Analogously we can define a based version $\mathcal{M}^o_k(X, T, m)$ of the above sets. It follows from [MMR] that for generic metrics on $X^\pm$ and any $k_\pm > 0$ the spaces $\mathcal{M}_{k_\pm}(X^\pm, T^\pm, m)$ are smooth manifolds. Consider the fibered product

$$\tilde{U}^m_{k_-,k_+} = \mathcal{M}^p_{k_-}(X_-, T^-, m) \underset{\partial o}{\times} \mathcal{M}^p_{k_+}(X_+, T^+, m) \tag{2.14}$$

We now describe how to glue together two connections $[\omega_+, \omega_-] \in \tilde{U}^m_{k_-,k_+}$. Let $E_0 = SU(2) \times L$ be the restriction of $E^\pm$ to L. For a constant d_0 chose the numbers $l^\pm \geq T^\pm + d_0$ and construct a manifold X_l by identifying the cylinders $X^+_{[l^+ - d_0, l^+ + d_0]} \subset X_{(l^+ + d_0)}$ with $X^-_{[l^- - d_0, l^- + d_0]} \subset X_{(l^- + d_0)}$. Let us denote by C_{d_0} the cylinder on which the identification takes place. Note that X_l contains a subset isometric to a larger cylinder C_l of the length l. This construction provides gluing of manifolds $X^+_{l^+}$ and $X^-_{l^-}$. When it is clear from the context, we are going to skip subindexes $l_\pm$ indicating finiteness of the cylindrical end, thus using the symbol $X^\pm$ for the spaces with finite or infinite cylindrical ends. Theorem (2.11) gives bundle isomorphisms $\eta^\pm : [T^\pm, \infty) \times E_0 \to E^\pm|_{[T^\pm,\infty)\times L}$ such that $(\eta^\pm)^*(\omega^\pm|_{[l^\pm - d_0, \infty)}) = \Gamma + A^\pm(t) + \alpha^\pm(t)\, dt$, where Γ is a flat connection on E_0. These bundle isomorphisms provide a clutching of the bundles $E^\pm$ through an isomorphism $\eta^-(\eta^+)^{-1} : E^+|_{\{l^+\}\times L} \to E^-|_{\{l^-\}\times L}$. Also if both connections $\omega^\pm$ are based at a point $(l^\pm, n) \in \{l^\pm\} \times L$, then the above isomorphism provides a basing at a corresponding point in C_{d_0}. Using the partition of unity $\{\phi^+, \phi^-\}$ on C_{d_0} we can define a glued connection by:

$$\tilde{\gamma}_l(\omega^+, \omega^-) = \omega_l = \begin{cases} \omega^+ & \text{on } Z^+_{l^+ - d_0}; \\ \Gamma + \phi^+(A^+(t) + \alpha^+(t)\, dt) + \phi^-(A^-(t) + \alpha^-(t)\, dt) & \text{on } C_{d_0}; \\ \omega^- & \text{on } Z^-_{l^- - d_0}; \end{cases}$$

When the connections ω^+, ω^- vary in some open sets U^+, U^-, then $\tilde{\gamma}_l(U^+, U^-)$ describes an open set in the space $\mathcal{B}_E$. The goal of the gluing theorems is to project this set onto the space of ASD connections. The main result of [MM] answers when can we do it:

Theorem 2.15 (Propositions 4.1.1 and 4.2.1 in [MM]) *There is a constant l_0 depending only on the sets $\tilde{U}^m_{k_-,k_+}$ such that for every $([\omega^+,\omega^-]) \in \tilde{U}^m_{k_-,k_+}$ and $l \geq 4d_0$ and $d_0 \geq l_0$ there is a unique solution $u = u(\omega^+,\omega^-)$ to the equation*

$$F^+(\tilde{\gamma}_l(\omega^+,\omega^-) + u) = 0$$

This solution u satisfies the estimate

$$\|u\|_{L^2_2} \leq C_0 e^{-\frac{5}{2}\delta l}$$

where δ is the minimum of the lowest eigenvalues of Laplacians Δ_m, $m \in \chi(L)$. Moreover the assignment $([\omega^+,\omega^-]) \to \tilde{\gamma}_l(\omega^+,\omega^-) + u$ defines a unique smooth map γ^o_l : $\tilde{U}^m_{k_+,k_-} \to \mathcal{M}^n_{k_++k_-}(X_l)$ that is a diffeomorphism onto an open subset of $\mathcal{M}^n_{k_++k_-}(X_l)$. This diffeomorphism factors through to $\gamma_l : U^m_{k_+,k_-} = \tilde{U}^m_{k_+,k_-}/SO(3) \to \mathcal{M}_{k_++k_-}(X_l)$.

We can use the above theorem also for the purpose of reversing gluing. Let X_l contain a fixed cylinder $C_d = (-d_0, d_0) \times L$ for some $d \geq d_0$. Define $\mathcal{M}_{k_-,k_+}(X_l, C_d)$ to be the set of those $\omega \in \mathcal{M}_{k_-+k_+}(X_l)$, for which the following holds:

$$\int_{C_d} \|F_{\omega^\pm}\|^2_g < \varepsilon_0/2$$

$$\left| \int_{X^-_{l-}} \|F_{\omega^\pm}\|^2_g - 8\pi^2 k_- \right| < \varepsilon_0/2$$

$$\left| \int_{X^+_{l+}} \|F_{\omega^\pm}\|^2_g - 8\pi^2 k_+ \right| < \varepsilon_0/2$$

Let $\mathcal{M}^n_{k_-,k_+}(X_l, C_{d_0})$ denote the corresponding set of connections based at some point $n \in C_{d_0}$. The following claim describes the domain of the "ungluing":

Lemma 2.16 *Assume that l and d_0 satisfy the assumptions of the gluing theorem. Then there is a smooth map*

$$\mathcal{M}^n_{k_-,k_+}(X_l, C_{d_0}) \xrightarrow{\rho} \coprod_m \tilde{U}^m_{k_-,k_+}$$

the union taken over all $m \in \chi(L)$ such that $CS(m) \equiv k_+ \equiv -k_- \mod \mathbf{Z}$.

Proof: According to Lemma 2.12 for every connection ω from $\mathcal{M}_{k_-,k_+}(X_l, C_{d_0})$ we can find a unique flat connection Γ_s, $s \in \{0, 1, \ldots, [\frac{p}{2}] + 1\}$ such that Γ_s is the closest to $\omega|_{C_{d_0}}$. Define two approximately ASD connections $\tilde{\omega}^\pm = \tilde{\gamma}_l(\omega|_{X^\pm}, \xi^s)$ on the bundle $E_\pm$ over the spaces with infinite cylindrical ends $X^\pm$. The assumptions on l and d_0 guarantee that we can use the gluing theorem (2.15) for perturbing these approximate connections to a pair of ASD connections $[\omega^-,\omega^+] \in \tilde{U}^{\xi^s}_{k_-,k_+}$. □

Now we are interested in describing the image of the above "ungluing" map. Our main statement is the following:

Lemma 2.17 *For every ϵ satisfying $0 < \epsilon < \varepsilon_0$ there is a length l_0 such that for every $l \geq l_0$ the set*

$$\mathcal{M}^c_{k_+}(X^+, [l, \varepsilon_0 - \epsilon]) \underset{\partial o}{\times} \mathcal{M}^c_{k_-}(X^-, [l, \varepsilon_0 - \epsilon])$$

is in the image of ungluing map ρ defined on $\mathcal{M}^n_{k_-,k_+}(X_l, C_{d_0})$.

Proof: Let us begin with a simple topological lemma:

Lemma 2.18 *Let B_ε denote a ball in $\mathbf{R}^n$ centered at the origin and with a radius ε. Let $f : B_\varepsilon \to \mathbf{R}^n$ be a continuous map such that for every $x \in B_\varepsilon$, $\mathrm{dist}(f(x), x) < \varepsilon$. Then the center of B_ε is in the image of f.*

Proof: Let $S_\varepsilon = \partial B_\varepsilon$. Suppose that f maps B_ε to $\mathbf{R}^n \setminus \{0\}$. Then the restriction $f : S_\varepsilon \to \mathbf{R}^n \setminus \{0\}$ is homotopically trivial. On the other hand the map

$$F(x, t) = tx + (1 - t)f(x)$$

provides a homotopy between f and identity map (the interval $tx + (1 - t)f(x)$ never passes through 0 as $\mathrm{dist}(f(x), x) < \varepsilon$). This contradicts the triviality of f. □

Definition 2.19 *Let U be an open set in the Riemannian manifold M. For a positive number ϵ define the sets*

$$U_{+\epsilon} = \{m \in M | \mathrm{dist}(m, U) < \epsilon\}$$

$$U_{-\epsilon} = \{m \in M | B(m, \epsilon) \subset U\}$$

Corollary 2.20 *Let M be a manifold with a radius of injectivity bounded from below (for example if M is precompact) and let U be an open subset in M. Then there exists an ϵ_1 such that for every $\epsilon < \epsilon_1$ and every map $f : U \to M$ such that $\mathrm{dist}(f(x), x) < \epsilon$ the set $U_{-\epsilon}$ is in the image of f.*

Let us come back to the proof of Lemma 2.17. Consider the map

$$\rho \circ \gamma : \mathcal{M}^c_{k_+}(X^+, [l_+, \varepsilon_0]) \underset{\partial o}{\times} \mathcal{M}^c_{k_-}(X^-, [l_-, \varepsilon_0]) \to \mathcal{M}^c_{k_+}(X^+) \underset{\partial o}{\times} \mathcal{M}^c_{k_-}(X^-)$$

Both components γ and ρ are defined by the gluing theorem. If we replaced these maps by corresponding approximate gluing, then after restricting the connections over $X^+_{l_+}$ and $X^-_{l_-}$ we would get an identity. The theorem (2.15) tells us that while stretching the neck we can make the actual gluing as close to the approximate one as we want. Thus it remains to show that the L^2 norm of the curvature can be estimated by an L^2_2 norm of the connection, which is due to the Sobolev embedings theorems (see [P]):

$$\|F_A - F_{A+a}\|_{L^2} = \|d_A a + a \wedge a\| \leq \|a\|_{L^2_1} + C_1\|a\|_{L^4} \leq C_2\|a\|_{L^2_1}$$

3 The description of instantons on the cylindrical end.

Certain $SU(2)$ bundles over $\mathbf{R} \times L$ can be obtained from $SU(2)$ bundles E over a sphere S^4 by dividing by a $\mathbf{Z}_p$ action. It turns out that the condition for bundles to arise in this way is closely related to the existence of ASD connections on these bundles. Following [FL], we can describe the $\mathbf{Z}_p$-invariant bundles over S^4 as

$$(D^4_- \times SU(2), T_2) \bigcup_F (D^4_+ \times SU(2), T_1) \tag{3.1}$$

where $\xi = e^{\frac{2\pi i}{p}}$ acts on $D^4_\pm \subset \mathbf{C}^2$ via $\xi(z_1, z_2) = (z_1\xi, z_2\xi)$, and for $g \in SU(2)$,

$$T_1(g) = g \cdot \begin{pmatrix} \exp \xi m & 0 \\ 0 & \exp \bar{\xi} m \end{pmatrix}, \text{ and } T_2(g) = g \cdot \begin{pmatrix} \exp \xi m' & 0 \\ 0 & \exp \bar{\xi} m' \end{pmatrix}$$

We will refer to the numbers m and m' as the *weights* of the $\mathbf{Z}_p$ action. The transition function $F : S^3 \times SU(2) \to S^3 \times SU(2)$ of (3.1) can be written as $F(x, g) = (x, f(x)g)$ for $f : S^3 \to SU(2)$. The $\mathbf{Z}_p$ equivariance of F means that $f(x\xi) = T_2^{-1} f(x) T_1$. Notice that the degree k of the map f is equal to the second Chern class of the bundle E. We denote the bundle obtained from the above construction divided by a Z_p action by $E(k, m, m')$. [FL] adapted to the case of $SU(2)$ bundles over $\mathbf{R} \times L(p, 1)$, gives a construction of $\mathbf{Z}_p$ equivariant bundles with weights (m, m') and degree $k = m'^2 - m^2$. Let us now denote by $\mathcal{M}_{k/p}(\mathbf{R} \times L, [m, m'])$ the moduli space of ASD connections on a bundle $E(k, m, m')$. Before quoting the results of [Au] describing moduli spaces of ASD connections on $E(k, m, m')$, we need to comment on orientations. The construction of ASD connections is based on the identification of S^4 with quaternions $\mathbf{H}$; thus when $\mathbf{R} \times L$ is the boundary of a complex manifold M, the orientations in [Au] agree with the convention that the weight m is from "the manifold side" and m' is from the other side. In our particular application we are studying the spaces $\mathbf{R} \times L$ as the ends of the space $N = \overline{\mathbf{C}P^2}$, thus we have to reverse orientations coming from the complex structures. We can do it in two ways: either by changing the convention about the order of weights or by changing $L(p, q)$ to $L(p, -q)$. In this paper we have chosen the second way, thus even though the boundary of N is $L(p, -1)$, while quoting Austin's paper we fix $q = +1$. The main construction in [Au] (Lemma 5.1) gives a description of ASD connections on the bundles $E(k, m, m')$ for which there exists a solution (a, b) to a system:

$$\begin{aligned} a &\equiv m + m' \pmod p \\ b &\equiv m' - m \pmod p \\ a \cdot b &= k \end{aligned} \tag{3.2}$$

These bundles are called S^1 *equivariant*. (The reason for this name is that the $\mathbf{Z}_p$ action on these bundles extends to an S^1 action). To state the general result we need one more definition:

Definition 3.3 *A $\mathbf{Z}_p$ equivariant bundle $E(k,m,m')$ is a composite of $\mathbf{Z}_p$ equivariant bundles $\{E(k_i,m_i,m_i')\}_{i=1}^n$ if and only if for all $i=1,2,\ldots,n$ we have:*
$k_i > 0$; $k=\sum_i k_i$; $m \equiv m_1 \pmod p$; $m_i' \equiv m_{i+1} \pmod p$; $m_n' \equiv m' \pmod p$

Lemma 3.4 (Compare Lemma 5.2 in [Au].) *A $\mathbf{Z}_p$ equivariant bundle supports an invariant ASD connection if and only if it is a composite of S^1 equivariant bundles.*

The bundles over $\mathbf{R}\times L$ may have noninteger second Chern numbers (which we will often refer to as *energies*). In fact the minimal amount of energy on a nontrivial moduli space on $\mathbf{R}\times L$ is $\frac{1}{p}$. This ostensibly leads to large number of possible distributions of energy on the cylinder. We would like to use (3.4) to limit the number of possibilities.

Lemma 3.5 *For fixed m and m' let k be the minimal amount of energy for which $E(k,m,m')$ supports an ASD instanton. Then*

$$k = \begin{cases} m'^2 - m^2 & \text{if } m' > m \\ m'^2 - m^2 + p(m-m') & \text{if } m' < m \end{cases}$$

Proof: As $m, m' \le p/2$, then $(a,b) = (m+m', m'-m)$ is the smallest nonegative solution to (3.2) in the case $m' > m$, and $(a,b) = (m-m', p-m-m')$ is the smallest solution in the case $m' < m$. The product of these numbers gives the required result for S^1 equivariant bundles. Now assume that $E(k,m,m')$ is a composite of $\{E(k_i,m_i,m_i')\}_{i=1}^n$. The in the case of $m' > m$ we have:

$$k = \sum k_i \ge m_n'^2 - m_1^2 + \sum_{i\in I} p(m_i - m_i') \ge m_n'^2 - m_1^2$$

Here I is the set of those indexes i for which $m_i > m_i'$. In the case $m' < m$ we have: $k \ge m_n'^2 - m_1^2 + \sum_{i\in I} p(m_i - m_i') \ge m_n'^2 - m_1^2 + \sum_{i=1}^n p(m_i - m_i') = m_n'^2 - m_1^2 + p(m_1 - m_n')$
□

As an example we can make the following list representing the minimal energies required for "tunnelling" between flat connections m and m' on $\mathbf{R}\times L(4,1)$.

$m\backslash m'$	0	1	2
0	0	1/4	1
1	3/4	0	3/4
2	1	1/4	0

(3.6)

The dimension of the moduli space $\mathcal{M} = \mathcal{M}_{k/p}(\mathbf{R}\times L(p,1),[m,m'])$ is

$$\dim \mathcal{M} = \frac{8k}{p} - 3 + n + \frac{2}{p}\sum_{j=1}^{p-1} \cot^2(\frac{\pi j}{p}) \cdot (\sin^2(\frac{\pi j m'}{p}) - \sin^2(\frac{\pi j m}{p}))$$

where $n \in \{0,1,2\}$ is the number of m, m' different from $0, p$. By using Fourier expansions, this formula can be simplified to:

$$\dim \mathcal{M}_{m^2/p}(\mathbf{R}\times L(p,1),[0,m]) = 4m - 3 \quad \text{for } m \le p/2 \tag{3.7}$$

Using this result and the gluing theorem we can complete our calculations of the dimensions of moduli spaces $\mathcal{M}_\kappa(Y,m)$, where κ is in $\frac{1}{p}\mathbf{Z}$. A similar result has been observed in [MMR] in Lemma 13.4.1:

Lemma 3.8 *The dimension of $\mathcal{M}_\kappa(Y,m)$ is equal to*

$$8\kappa \;-\; \frac{3}{2}\cdot(\sigma(Y)+\chi(Y)) \;-\; \frac{1}{2} \;+\; \frac{8m^2}{p} + 1 - 4m$$

when m is nontrivial, and

$$8\kappa \;-\; \frac{3}{2}\cdot(\sigma(Y)+\chi(Y)) \;-\; \frac{3}{2}$$

if m is trivial.

Proof: By Lemma 3.5, there is a moduli space of ASD connections on N with boundary value m and with energy $\frac{m^2}{p}$. What is more, this is the smallest energy giving a nontrivial moduli space on N with this boundary value. Gluing together the bundles over Y and N we obtain a bundle over X with energy $\kappa + \frac{m^2}{p}$. From the gluing theorem (2.15) we have that dim $\mathcal{M}_\kappa(X,m) =$ dim $\mathcal{M}_\kappa(Y,m) + h_m +$ dim $\mathcal{M}_{m^2/p}(N,m)$, which for m nontrivial, by using (2.8), gives the following equation:

$$8\kappa - \frac{3}{2}\cdot(\sigma(X)+1+\chi(X)-2) - \frac{1}{2} + \frac{\rho_m}{2} + 1 + 4m - 3 = 8(\kappa + \frac{m^2}{p}) - \frac{3}{2}\cdot(\sigma(X)+\chi(X))$$

here we used $\sigma(X) = \sigma(Y) - 1$ and $\chi(X) = \chi(Y) + 2$. Solving this equation for ρ_m and plugging the result into (2.8) gives the first formula. The second formula can be obtained similarly after gluing in the trivial connection on N. □

Similar calculations give:

$$\dim \mathcal{M}_\kappa(N,m) = \begin{cases} 8\kappa - 3 + 4m - \frac{8m^2}{p} & \text{if } m \text{ is nontrivial} \\ 8\kappa - 3 & \text{if } m \text{ is trivial.} \end{cases} \tag{3.9}$$

One can notice that the first formula gives $8\kappa - 3$ for $m = 0$ and $\frac{p}{2}$, so it is valid in both cases.

In order to formulate the result which allows us to quantize moduli spaces on the cylinders, we need to notice a simple fact connected with gluing. This fact has also been mentioned in [FS1]:

Lemma 3.10 *Let $\mathcal{M}^o = \mathcal{M}^o_\kappa(X,m)$ for a nontrivial element m of the character variety. Then the base point fibration $\mathcal{M}^o \to \mathcal{M}$ reduces, i.e. there exists an S^1-bundle Q such that $\mathcal{M}^o = Q \times_{S^1} SO(3)$.*

Proof: Consider the map $\partial^o : \mathcal{M}^o \to S^2 \subset \mathcal{R}(L)$. This map restricts over each fiber as $\partial^o|_{SO(3)} : SO(3) \to S^2$ that sends $g \to g\xi g^{-1}$. Thus for fixed point $pt \in S^2$ $(\partial^o)^{-1}(pt)$ is an S^1 bundle Q over S^2 for which $\mathcal{M}^o = Q \times_{S^1} SO(3)$. □

The moduli spaces on the cylinders $\mathbf{R} \times L$ have two boundary value maps

$$\partial^o_\pm : \mathcal{M}^o_\kappa(\mathbf{R} \times L(p,1), [m,m']) \to \mathcal{R}(L)$$

with $\kappa \in \mathbf{Z}_p$. Let us recall that according to our convention on the cylinders we consider as positive the direction from N to Y. Thus ∂^o_- points toward N, and ∂^o_+ points toward Y. With this understood, define $\mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R} \times L(p,1),[m,m'])$ to be $(\partial^o_-)^{-1}(pt)$ for any $pt \in \mathrm{Im}\,(\partial^0_-)$. Thus when m is a trivial element of $\chi(L)$, then $\mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R}\times L(p,1),[m,m'])$ is the based moduli space $\mathcal{M}^o_\kappa(\mathbf{R}\times L(p,1),[m,m'])$. When m is nontrivial, then according to Lemma 3.10 the space $\mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R} \times L(p,1),[m,m'])$ is an S^1 reduction of the base–point fibration. This space should be thought of as the gluing parameter bundle over an unbased moduli space. This gluing parameter is attached at the end of the cylinder which is closer to N. The next lemma shows that these moduli spaces provide good quantization of the whole moduli space:

Lemma 3.11 $\dim \mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R} \times L(p,1),[m,m']) \geq 4\cdot|m-m'|$.

Proof: This is a direct consequence of Lemma 3.5 describing the lowest energy on the moduli space $\mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R} \times L(p,1),[m,m'])$ and the dimension formula (3.7).

Case 1: $m' > m$. The moduli space $\mathcal{M}_{m'^2/p}(\mathbf{R} \times L(p,1),[0,m'])$ has a boundary corresponding to a splitting

$$\mathcal{M}^o_{m^2/p}(\mathbf{R} \times L(p,1),[0,m]) \underset{\partial^o}{\times} \mathcal{M}^o_{(m'^2-m^2)/p}(\mathbf{R} \times L(p,1),[m,m'])$$

where all of the moduli spaces involved have the minimal energy required for each tunneling. From (3.7) we have: $4\cdot m'-3 = 4\cdot m - 3 + \dim \mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R}\times L(p,1),[m,m'])$, from which the lemma follows.

Case 2: $m' < m$. From the index formula we get that $\dim \mathcal{M}_m(\mathbf{R}\times L(p,1),[0,0]) = 8m-3$. This space allows a splitting $\mathcal{M}^o_{m^2/p}(\mathbf{R} \times L(p,1),[0,m]) \underset{\partial^o}{\times} \mathcal{M}^o_{m(p-m)/p}(\mathbf{R} \times L(p,1),[m,0])$. Analyzing the dimensions of both spaces we get:

$$8m-3 = 4m-3+\dim \mathcal{M}^{\mathbf{Z}_p}_{m(p-m)/p}(\mathbf{R} \times L(p,1),[m,0])$$

Thus $\dim \mathcal{M}^{\mathbf{Z}_p}_{m(p-m)/p}(\mathbf{R} \times L(p,1),[m,0]) = 4m$.

Now consider the space $\mathcal{M}_{(pm-m^2)/p}(\mathbf{R} \times L(p,1),[m,0])$ and its splitting

$$\mathcal{M}^o_{\frac{m'^2-m^2}{p}+m-m'}(\mathbf{R} \times L(p,1),[m,m']) \underset{\partial^o}{\times} \mathcal{M}^o_{\frac{pm'-m'^2}{p}}(\mathbf{R} \times L(p,1),[m',0])$$

Again the energies have been chosen according to Lemma 3.5 so as to be the smallest ones. Analyzing the dimensions of the involved spaces we get the equation:

$$4m = \dim \mathcal{M}^{\mathbf{Z}_p}_\kappa(\mathbf{R} \times L(p,1),[m,m'])+4m',$$

which concludes the proof of the lemma. $\square$

4 Dividing the moduli space into open sets.

Now we can come back to the computation of the Donaldson invariant $D(z\sigma^n)$. We assume here that the numbers p and n are fixed. The goal of this section is to list all open sets defined in (2.14) that cover the intersection

$$\mathcal{M}_{k_0}(X) \cap V_z \cap \bigcap_{i=1}^{n} V_{\sigma_i} \tag{4.1}$$

To fix notation let us assume that the class $z \in \mathbf{A}(Y)$ has the degree d, and

$$\dim \mathcal{M}_{k_0}(X) = 2\,(d+n)$$

Since we are using a generic metric on X, the intersection of divisors on Y–side is transverse, so the intersection $U^m_{k_Y,k_N} \cap V_z \cap \bigcap_{i=1}^{n} V_{\sigma_i}$ (see (2.14) for the definition of $U^m_{k_Y,k_N}$) is empty if dim $\mathcal{M}_{k_Y}(Y,m) \leq 2d$. This implies that dim $\mathcal{M}_{k_N}(N,m)$ must not exceed $2n-1$ if m is trivial nor $2n-3$ if m is trivial. In order to choose the sets $U^m_{k_Y,k_N}$ in the most efficient way, let us define the set

$$\begin{aligned}\mathcal{J} \;=\; &\{(m,k) \mid m \in \chi(L),\ k \in \mathbf{Z}[\tfrac{1}{p}], \mathcal{M}_k(N,m) \neq \emptyset, \\ &\text{and } 0 < \dim \mathcal{M}_k(N,m) + h_m \leq 2n\}\end{aligned}$$

(h_m is defined in (2.9)). We set also $\bar{\mathcal{J}} = \mathcal{J} \cup \{(0,0)\}$. Note that the bound on the dimension of the spaces $\mathcal{M}_k(N,m)$ implies that the set $\mathcal{J}$ is finite. In $\bar{\mathcal{J}}$ we define a partial order by saying that $(m_1,k_1) \leq (m_2,k_2)$ if there is a nonempty moduli space $\mathcal{M}_{k_2-k_1}(\mathbf{R} \times L, [m_1,m_2])$.

Definition 4.2 *To every element $(m,k) \in \bar{\mathcal{J}}$ assign its* degree deg (m,k) *to be the length of the longest chain in $\bar{\mathcal{J}}$ from $(0,0)$ to (m,k) linearly ordered by $\leq$. Define* $\deg(\mathcal{J}) = \max_{x \in \mathcal{J}} \deg(x)$.

The next lemma shows how to compute the above degrees.

Lemma 4.3 *For every $(m,k) \in \mathcal{J}$ we have the following relation:*

$$4 \cdot (\deg\,(m,k) - 1) + 1 = \dim \mathcal{M}_k(N,m)$$

Proof: Set $\alpha = \deg\,(m,k)$. The definition of α implies that the moduli space $\mathcal{M}_k(N,m)$ has a boundary component corresponding to a fibered product of $(\alpha-1)$ moduli spaces $\mathcal{M}_{k_j}(\mathbf{R} \times L, [m_j, m_j'])$ and the last α^{th} space $\mathcal{M}_{k_N}(N, m_N)$. Thus from (3.11) we have: dim $\mathcal{M}_k(N,m) \geq 4(\alpha-1)+1$.

We prove the inequality in the opposite direction by induction on the degree of elements of $\mathcal{J}$. Lemma 3.5 gives us the following inequalities in $\mathcal{J}$:

$$(m,k) \leq (m+1, k + \frac{2m+1}{p}) \leq (m, k+1)$$

Thus deg $(m,k+1) \geq$ deg $(m,k) + 2$. Using the dimension formula (3.9) we get:

$$\begin{aligned}\dim \mathcal{M}_{k+1}(N,k+1) &= \dim \mathcal{M}_k(N,k) + 8 = 4(\deg\,(m,k) - 1) + 1 + 8 \\ &\leq 4(\deg\,(m,k+1) - 3) + 1 + 8 = 4(\deg\,(m,k+1) - 1) + 1\end{aligned}$$

□

Corollary 4.4 *Let Γ be a graph representing the set $\mathcal{J}$ with the relation $\leq$. Then $(m,k) \leq (m',k')$ are immediate neighbors in the graph Γ if and only if $m' = m \pm 1$ and*

$$k' - k = \begin{cases} \frac{1}{p} \cdot [2m+1] & \text{if } m' = m+1 \\ \frac{1}{p} \cdot [-2m+1] + 1 & \text{if } m' = m-1 \end{cases}$$

Proof: The fact that $(m,k) \leq (m',k')$ for m' and k' defined above follows from Lemma 3.5. The Lemma 4.3 gives us:

$$\begin{aligned} \deg(m',k') - \deg(m,k) &= \tfrac{1}{4}\left[\dim \mathcal{M}_{k'}(N,m') - \dim \mathcal{M}_k(N,m)\right] \\ &= \tfrac{1}{4} \dim \mathcal{M}^{\mathbf{Z}_p}_{k'-k}(\mathbf{R} \times L, [m,m']) = 1 \end{aligned} \tag{4.5}$$

thus there are no other points of $\mathcal{J}$ between (m,k) and (m',k'). □

For each element of $\mathcal{J}$ we wish to assign an open set in the moduli space $\mathcal{M}_k(X)$. Let us fix $s = \deg(\mathcal{J})$. Notice that by changing the metric on X we can identify in X a subset isometric to a cylinder $[0,l] \times L$ for an arbitrary large l. We can adjust its length l so as to have a collection of disjoint subcylinders $C_1^0, \ldots, C_t^0, R_1, C_1^1, \ldots, C_t^1$, $R_2, C_1^2, \ldots C_t^2, \ldots R_s, C_1^s, \ldots C_t^s$ satisfying the following:

1. For each C_i and R_i there is some $r \in [0,l]$ such that these cylinders are equal to $[r, r+1] \times L$;

2. For every connection ω with energy $k+n$ on X there are $s+1$ numbers $i_0, i_1, i_2, \ldots, i_s$ $(1 \leq i_r \leq t)$ such that

$$\int_{C^r_{i_r}} |F_\omega|^2 < \varepsilon_0$$

where ε_0 is the number defined in Theorem (2.11).
Each cylinder R_i splits the manifold X into two manifolds with boundary, which we denote as Y_i and N_i. We also denote by $T_{i,j}$ the cylinder that is cut from $[0,l] \times L$ by the cylinders R_i and R_j.

Definition 4.6 *For every element $(m,k) \in \mathcal{J}$ we define an open set $U_k^m \subset \mathcal{M}_{k_0}(X)$ as the set of all ω that satisfy:*

1. $\int\limits_{R_i} |F_\omega|^2 < \varepsilon_0$, *where $i = deg(m,k)$.*

2. $\int\limits_{N_i} |F_\omega|^2 = k + \varepsilon$ *for some $|\varepsilon| < \varepsilon_0$*

3. *If (r,n) is a point in R_i then the character m is closest to $\omega|_{\{r\} \times L}$ in the sense of Lemma 2.12.*

Having this assignment in mind we shall understand that the collection of open sets $\{U_k^m\}$ has also defined a partial order $\leq$ and $\deg(U_k^m)$ makes sense as well. Each set U_k^m has uniquely assigned region R_i with $i = \deg(m,k)$ and let us choose one point $x_i \in R_i$. According to Lemma 2.17 each U_k^m can be described as the fibered product:

$$\mathcal{M}^{x_i}_{k_X-k}(Y,m) \times_{\partial} \mathcal{M}^{x_i}_k(N,m)/SO(3) \tag{4.7}$$

Thus every connection in U_k^m can be written as $\omega = [\omega_Y, \omega_N]$. In general, if a given connection ω allows the representation (4.7) for some i, then we say that ω allows splitting at the region R_i. In the next lemma we shall prove that the sets U_k^m suffice for covering the intersection of divisors $V_z \cap \bigcap_{i=1}^{n} V_\sigma$. As the proof of that lemma requires splittings of moduli space at multiple regions $R_1, R_2, \ldots, R_r$ on the cylinder $[-n,n]\times L$, we need to be more precise about the notation of cylindrical–end manifolds Y and N. Thus by Y and N we shall mean finite cylinder manifolds ending at the first splitting regions (including these regions). For a given region R we shall mean by the terms Y–side or N–side the corresponding connected components of the manifold X cut by R. (See the picture below):

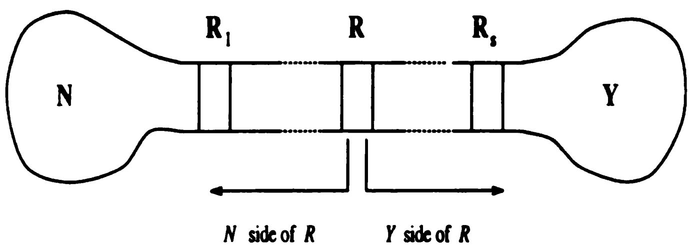


Figure 1. The splitting of the manifold X.

Lemma 4.8 *Every connection that belongs to the intersection $\mathcal{M}_{k_0}(X) \cap V_z \cap \bigcap_{i=1}^{n} V_\sigma$ is contained in one of the sets U_i^m, $(m,i) \in \mathcal{J}$. Moreover deg $(\mathcal{J}) \le \frac{n+1}{2}$.*

Proof: First we shall prove that $s = \deg \mathcal{J}$ is a sufficient number of the splitting regions. Assume that ω is a connection that does not allow splitting at any of the regions $R_1, \ldots, R_s$. According to the definition of the length l, there are cylinders $C^T_{i_r}$, $r = 0, 1, \ldots, s$ such that ω has energy less than ε_0 in these regions, thus it allows the splitting at $C^T_{i_r}$. Let m_r denote the gauge equivalence class of the limiting flat connection on $C^T_{i_r}$, and k_r the energy on the N–side of $C^T_{i_r}$. As $\omega \in \mathcal{M}_{k_0}(X) \cap V_z \bigcap_{i=1}^{n} V_\sigma$, then for every r the dimension bound $\dim \mathcal{M}_k(N, m_r) \le 2n$, defining the set $\mathcal{J}$, is satisfied, thus the pair $(m_r, k_r) \in \mathcal{J}$ for every r. By Lemma 2.17, ω can be represented as a fibered product of connections $[\omega_N, \omega_1, \ldots, \omega_s, \omega_Y]$. The first connection ω_Y lies in a moduli space of nonnegative dimension. If one of connections ω_i defined on $[l_i, l_{i+1}] \times L \supset R_i$ was flat, then it would allow the obvious splitting along R_i as $\omega_i = [\omega_{i,-}, \omega_{i,+}]$. In here $\omega_{i,\pm}$ are the restrictions of ω_i onto different connected components of $[l_i, l_{i+1}] \times L \setminus R_i$. As each ω_j is nonflat, then we have a sequence of strict inequalities in $\mathcal{J}$

$$(m_N, k_N) < (m_1, k_1) < \cdots < (m_r, k_r)$$

from which we get that the last pair (m_r, k_r) has the degree greater than or equal to $s+1$, thus contradicting the definition of s.

Next assume that ω allows splitting at R_i with k_i being its energy on the $N-$ side and m_i being the nearest flat connection on R_i in the sense of Definition 4.6. We still need to prove that for some i there is the equality $\deg(m_i, k_i) = i$. So assume there is an ω which satisfies: for every $i = 1, 2, \dots, s$ either ω does not split at R_i or $\deg(m_i, k_i) \neq i$ for (m_i, k_i) defined by a splitting at R_i. Consider first the case when ω splits at R_i with $\deg(m_i, k_i) < i$. We claim that there exists another region R_j for which the equality $\deg(m_j, k_j) = j$ holds. This is proved by induction with respect to i. For $i = 1$ the statement is obviously true, since each element of $\mathcal{J}$ has degree at least one. Thus we may assume that:

For every $r < i$ and every connection ω that allows a splitting at the region R_r with $\deg(m_r, k_r) < r$ *there exist $j < r$ such that ω splits at R_j and* $\deg(m_j, k_j) = j$.

Consider now a connection ω that splits at R_i with $\deg(m_i, k_i) < i$. Then we can write ω as $[\omega_Y, \omega_N]$. The connection ω_N must allow splitting at R_{j_1} for some $i > j_1 \geq 1$, as otherwise ω_N could be written as $\omega_N = [\omega^i_{N,1}, \omega^i_{T_1}, \dots, \omega^i_{T_i}]$, showing that $\deg(m_i, k_i) \geq i$ in contradiction to our assumption. Splitting ω_N at R_{j_1} we get: $\omega_N = [\omega_{T_1}, \omega_{N_1}]$ and the numbers(m_{j_1}, k_{j_1}) corresponding to that splitting. If $\deg(m_{j_1}, k_{j_1}) < j_1$, then we are done by an inductive assumption. If $\deg(m_{j_1}, k_{j_1}) = j_1$, then ω splits at R_{j_1} with the right equality $\deg(m_{j_1}, k_{j_1}) = j_1$. We claim that the last possibility: $\deg(m_{j_1}, k_{j_1}) > j_1$, implies that the connection ω_{T_1} splits at some other region R_{j_2} for $j_1 < j_2 < i$. If that claim is not true, then ω_{T_1} can be written as a composition of $i - j_1$ nontrivial connections on the cylinder between the regions R_{j_1} and R_i. Thus we have:

$$\deg(m_i, k_i) \geq \deg(m_{j_1}, k_{j_1})) + (i - j_1) > i$$

again contradicting our assumption. Thus we can write $\omega_{T_1} = [\omega_{T_2^+}, \omega_{T_2^-}]$. Define $\omega_{N_2} = [\omega_{N_1}, \omega_{T_2^-}]$. Same argument as above aplied for ω_{N_2} instead of ω_{N_1} shows that we either find the required splitting for ω, or we will find another splitting of $\omega_{T_2^+}$ at the region R_{j_3} for $j_2 < j_3 < i$. As there can be only i such splittings, this process must terminate by finding the R_j with the required properties.

The proof in the case when $deg(m_i, k_i) > i$ is the same as in the the previous case, exept that one has to reverse the "orientation" of the whole proof, i.e. start the induction from the point that is closest to the Y–side and replace all N's with Y's, and "+" with "−".

Let us now consider an element $(m, k) \in \mathcal{J}$ with maximal degree. There is a corresponding connection in $\mathcal{M}_k(N, m)$ with the splitting $[\omega_N, \omega_1, \dots, \omega_{s-1}]$ that defines its degree. According to Corollary 3.11 the smallest dimension of a moduli space $\mathcal{M}_k(\mathbf{R} \times L, [i, j])$ including the gluing parameter from the N-side is no less than $4 \cdot (i - j)$. Thus

$$\begin{aligned} 2(d+n) &= \dim \mathcal{M}_{k_Y}(Y) + h_{m_Y} + \sum_{j=1}^{s-1} \dim \mathcal{M}^{\mathbf{Z}_p}_{k_j}(\mathbf{R} \times L, [m_j, m'_j]) + \\ &+ \dim \mathcal{M}_{k_N}(N, m_N) \geq 2d + h_X + 4(s-1) + 1 \geq 2d + 4s - 2 \end{aligned}$$

from which the last statement of the lemma follows. $\square$

For example, when evaluating $D(z\sigma^4)$ for $\sigma \cdot \sigma = -4$ we get the following set $\bar{\mathcal{J}}$ encoding the covering of the moduli space:

$$\begin{tikzcd} & (0,0) \arrow[d] & \\ & (1,\frac{1}{4}) \arrow[dl] \arrow[dr] & \\ (2,1) & & (0,1) \end{tikzcd}$$

In general on the graph representing $\mathcal{J}$ we can denote only the values of the equivalence classes of the flat connections $m \in \chi(L)$. To each edge of the graph $\mathcal{J}$ we can assign the minimal energy for tunneling between the boundary values denoted at the ends of the edge that has been established in Lemma 3.5. Then the energy of an arbitrary vertex can be obtained by summing the energies of the edges while going from the top of $\mathcal{J}$ to the given vertex.

Example 4.9 When computing $D(z\sigma^n)$ for $\sigma \cdot \sigma = -6$ we obtain the following graph, that stabilizes:

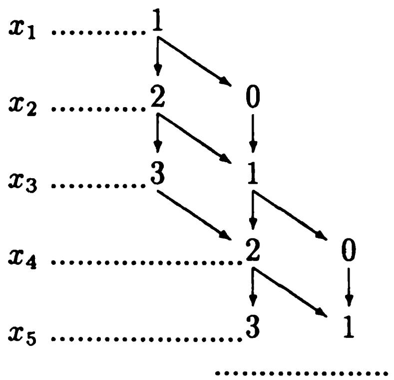


Figure 2. The graph $\mathcal{J}$ for $\sigma \cdot \sigma = -6$.

On this graph the points $x_1, x_2, \ldots$ denote the chosen points from the regions $R_1, R_2, ..$ All points on the graph lying on the level indicated by x_i correspond to open sets U_k^m that split at the region R_i. The above graph, according to Lemma 4.8, terminates at the level $[\frac{n+1}{2}]$. The meaning of the columns of this graph will be explained later.

Lemma 4.10 *The intersection of the divisors $V_z \bigcap_{i=1}^{s} V_{\sigma_i}$ is empty in $U_{k'}^m$ unless the moduli space $\mathcal{M}_{k'}(N, m)$ contains a reducible connection.*

Before proceeding with the proof we want to comment on the terminology. The set U_k^m for a generic metric does not contain reducible connections, as $b_+(X) \geq 3$.

Nonetheless *we shall informally say that A is a reducible connection in U_k^m* if the restriction of A to N is a reducible connection in $\mathcal{M}_k(N,m)$.
Proof: The open set $U_{k'}^m$ has dimension $2(d+n)$. On the other hand, this set viewed as fibered product has the dimension

$$\dim\mathcal{M}_{k_0-k'}(Y,m) + h_m + \dim\mathcal{M}_{k'}(N,m)$$

As $b_+(X) = b_+(Y) \geq 3$, then the intersection $\mathcal{M}_{k_0-k'}(Y,m) \cap V_z$ is transverse since it contains only irreducible connection. Thus $\dim\mathcal{M}_{k_1}(Y,m) \geq 2d$. If the divisors V_σ intersected along irreducible connection as well, then we would have: $2(d+n) \geq 2d + h_m + 2n$, in contradiction to the fact that $h_m = 1$ or 3. □

5 Computing intersection numbers in open sets U_k^m

5.1 Description of the neighborhoods of the reducible connections

Let us recall that our main purpose is to compute the intersection number

$$\mathcal{M}_k(X) \cap V_z \cap \bigcap_{i=1}^{n} V_{\sigma_i} \tag{5.1}$$

or dually, to evaluate the cohomology class $\mu(z)\mu^n(\sigma)$ on the moduli space $\mathcal{M}_k(X)$. As we saw in the previous paragraph, this intersection is included in the set covered by the sum of open sets U_k^m and takes place near connections whose restriction to N is reducible. Thus our first task is to describe the neighborhoods of reducible connections in the moduli spaces $\mathcal{M}_{k_N}(N,m_i)$, and then we will describe a procedure for keeping the intersection (5.1) inside these neighborhoods. Before proceeding, we want to notice that there is one kind of noncompactness of the sets U_k^m that can be taken care of immediately. This noncompactness comes from the completely concentrated connections (so–called *bubbles*). Recall that the sets U_k^m are obtained via fibered products $\mathcal{M}_{k_Y}^x(Y,m) \underset{\partial}{\times} \mathcal{M}_k^x(N,m)$, where the point x belongs to the region R, on which the energy estimate excludes bubbling. Thus both basepoint fibrations extend to the $SO(3)$ bundles over the Uhlenbeck compactifications of these moduli spaces, and from this point on we will assume that the sets U_k^m contain completely concentrated connections.

When a reducible connection occurs in the top stratum, its neighborhood is modelled on $\mathbf{C}^n/S^1$. For the sake of later generalizations we set the following notation:

Definition 5.2 • *Every neighborhood of a reducible connection is modeled on some stratified space divided by as S^1 action. Let K denote a compact neighborhood of the total space of this S^1 action. The space K/S^1 describes a compact neighborhood of the reducible connection. In this case $K = \{z \in \mathbf{C}^n \mid \|z\| \leq 1\}$; the general description of K is given below.*

- *Let M denote the link of reducible connection. Thus $M = \partial K$.*
- *Let $r(K/S^1)$ be the image of K/S^1 under the deformation retraction shrinking the cone parameter to zero. (Thus in the case of top stratum connection $r(K/S^1)$ is a point.) There is a corresponding deformation retraction of the set K, which we will denote by the same symbol. As S^1 acts trivially on $r(K)$, then we have $r(K/S^1) \cong r(K)$.*

The set M/S^1 is often referred to as the *link* of the reducible connection. In this case $M/S^1 = \mathbf{C}P^{n-1}$. In the proof of Proposition (5.1.21) in [DK] it is shown that the Pontrjagin class of the restriction of the adjoint bundle to the link of the reducible connection on the bundle $L \oplus \bar{L}$ is equal to

$$p_1(\mathbf{P}^{\mathrm{ad}}) = h^2 \times 1 + 4 \cdot h \times c_1(L) + 4 \cdot 1 \times c_1^2(L)$$

where h is the positive generator of $\mathrm{H}^2(\mathbf{C}P^{n-1})$. Thus we have the following:

Lemma 5.3 *On the link of a reducible connection on the top stratum in $\mathcal{M}_k(N,m)$ we have the following identities in $\mathrm{H}^2(M/S^1)$:*

$$\begin{aligned} \mu(\sigma) &= -\langle c_1(L), \sigma\rangle \cdot h \\ \mu(x) &= -\tfrac{1}{4} \cdot h^2 \end{aligned}$$

The second equation holds for points $x \in N$.

The description of reducible connections in the lower strata of $\mathcal{M}_k(N,m)$ has been worked out in [KoM]. For every sequence of integer numbers $n_1 \geq n_2 \geq \cdots \geq n_s$ there is a stratum St corresponding to s bubbles with multiplicities described by the numbers n_j. Let us denote by $k = k_i - \sum_{j=1}^{s} n_j$ and by $P = L \oplus \overline{L}$ the $SU(2)$ bundle with $c_2(P) = k$. Let FN be a principal $SO(4)$ bundle associated to a tangent bundle TN, and let us define $Fr = P \underset{N}{\times} FN$ to be a fibered product of the bundles P and FN over N. Then Fr is a principal $SO(3) \times SO(4)$ bundle over N. Following the notation from [KoM] and [FM 2], for each $n > 0$ define $\bar{Z}_n$ to be the space of gauge equivalence classes of ideal ASD connections on S^4 with standard metric that satisfy:

1. they are based at the south pole
2. they are centered at the north pole
3. they are concentrated in the ϵ ball around the north pole

The notions of "concentration of energy" and "ϵ ball" in the last statement should be understood in the following way: for a fixed annulus A contained in the complement of the ϵ ball, every ASD connection ω with the "energy concentrated in the ϵ ball" can be glued along A to another ASD connection on the manifold N. The spaces $\bar{Z}_n$ have a natural $SO(3)$ action that changes the framing at the south pole, and the $SO(4)$ action that rotates the sphere. In addition to that there is an $\mathbf{R}_+$ action on each $\bar{Z}_n$

corresponding to conformal contraction toward north pole. Thus there exists a space Z_n such that $\bar{Z}_n = cZ_n$. For every $j = 1, 2, \ldots, s$ define $Gl_j = Fr \underset{SO(3)\times SO(4)}{\times} cZ_{n_j}$. As is proven in [FM 2] the neighborhood K of the reducible connection in the strata St is

$$\left(\mathbf{C}^n \times \prod_{j=1}^{s} Gl_j \right) / S^1 \tag{5.4}$$

divided by the symmetry group permuting the Gl_j factors. The top stratum described above fits into the above description for $s = 0$. In Theorem 4.4.2 of [KoM] it has been shown that given a reducible connection ω, for $r = n_1 + n_2 + \cdots + n_s$, different strata of the form (5.4) can be glued together to form a smoothly stratified space $GP(\omega, r)$. This space allows the projection $\pi : GP(\omega, r) \to \Sigma^r(N)$ induced by an S^1 equivariant deformation retraction onto the r–fold symmetric product $\Sigma^r(N)$. The notation from Definition 5.2 extends to the case of a reducible connection in the lower strata. Lemma 4.7.4 in [KoM] gives a description of the class $\mu(\sigma)$ on the link of given reducible connection:

Theorem 5.5 *Let M be a link of reducible connection $\omega \in GP(\omega, r)$. Then the class $\mu(\sigma) \in \mathrm{H}^2(M/S^1)$ is equal to*

$$\mu(\sigma) = \pi^*\Sigma^r(\sigma) - \langle c_1(L), \sigma\rangle \cdot c_M$$

where

1. *c_M is the first Chern class of the S^1 bundle $M \to M/S^1$.*
2. *$\Sigma^r(\sigma) \in \mathrm{H}^2(\Sigma^r(N))$ is the class induced from the Poincare dual to σ in $\mathrm{H}^2(N)$ by symmetrization.*

Similarly we conclude that

$$-4\mu(x)|_{M/S^1} = -4\pi^*\Sigma^s(x) + c_M^2$$

Next we want to apply the gluing theorem to "attach" connections on the $Y-$ side to the neighborhoods of reducible connections on $N-$ side. Recall that the $SO(3)$ fibration $\mathcal{M}^x_{k_0-k}(Y, m)$ reduces to an S^1 fibration when m is nontrivial. In that case let Q denote this S^1–fibration. For the sake of uniform notation let Q be the whole total space of $\mathcal{M}^x_{k_0-k}(Y, m)$ when m is trivial. Notice that even in the later case we can still think of Q as of the total space of an S^1 fibration with the base Q/S^1.

When the boundary value m is nontrivial, the basepoint fibration on the $Y-$side is $Q \underset{S^1}{\times} SO(3)$ and the basepoint fibration $U_k^{m,x} \to U_k^m$ near a "reducible" connection can be described through the fibered product of:

$$\begin{array}{ccc} Q \underset{S^1}{\times} SO(3) & & SO(3) \underset{S^1}{\times} K \\ & \searrow \quad \swarrow & \\ & S^2 & \end{array}$$

which gives the basepoint fibration $Q \times_{S^1} (SO(3) \times K) \to Q \times_{S^1} K$. This $SO(3)$ fibration reduces to an S^1 bundle with a total space $Q \times_{S^1} (S^1 \times K) = Q \times K$.

The case when m is trivial is even easier to describe. The $SO(3)$ fibration over U_k^m is:

$$Q \times \left(SO(3) \times_{S^1} K\right) \to Q \times_{S^1} K$$

which again admits a reduction whose total space is $Q \times \left(S^1 \times_{S^1} K\right) = Q \times K$.

5.2 Desigularization of $Q \times_{S^1} K$

To make the future computations easier we want to "enlarge" the set $Q \times_{S^1} r(K) = Q/S^1 \times r(K)$, onto which $Q \times_{S^1} K$ deformation retracts. For any S^1 bundle Q let us denote by $Q^C = Q \times_{S^1} D^2$, the disc bundle associated to a circle bundle Q. Consider a projection $\pi : Q \times_{S^1} M^C \to Q \times_{S^1} K$ that is identity for every nonzero radius parameter of the disc D^2, and for the radius zero $\pi([q],[m]) = ([q], r(m))$. Here [.] denotes the equivalence class with the respect to the S^1 action. This map is well defined because of the S^1 equivariance of the retraction r. If α is any top dimensional cohomology class in $\mathrm{H}^{top}(Q \times_{S^1} K, Q \times_{S^1} M)$ then

$$\langle \alpha, [Q \times_{S^1} K, \partial] \rangle = \langle \alpha, \pi_*([Q \times_{S^1} M^C, \partial]) \rangle = \langle \pi^* \alpha, [Q \times_{S^1} M^C, \partial] \rangle \tag{5.6}$$

Thus we can evaluate the pullback $\pi^*\alpha$ in the enlarged space $Q \times_{S^1} M^C$, getting the same answer.

5.3 Closing neighborhoods of reducible connections

Next we wish to describe a procedure that will keep the intersection of divisors in (5.1) inside neighborhoods of reducible connections. Below, we describe compact spaces C containing neighborhoods $Q \times_{S^1} M^C$ together with a technique for translating the intersection problem from U_k^m into the sum of these compact sets.

In case m is trivial, let $\mathcal{N}$ denote the based moduli space $\mathcal{M}^{x}_{k_N}(N, m)$ and $G = SO(3)$, and in the case when m is nontrivial, let $\mathcal{N}$ be the S^1 reduction of the above basepoint fibration and $G = S^1$. Then the set $U_k^m = Q \times_G \mathcal{N}$.

Let $\mathcal{M}_k(N, m) = \mathcal{N}/G$ contain S^1 reducible connections $A_1, A_2, \ldots, A_s$, each contained in the compact nieghborhood K_i/S^1. In the set U_k^m each A_i is contained in a neighborhood of the form $Q \times_{S^1} K_i$. Let $q_i : Q \times_{S^1} M_i \to Q/S^1 \times M_i/S^1$ denote projections of the boundary of the above neighborhoods. The cohomology classes $\mu(\sigma)$ and $\mu(x)$ can be obtained from pullback via q_i of the first Chern class of the bundle $M_i \to M_i/S^1$. Since the dimension of $Q/S^1 \times M_i/S^1$ is 2 less than the dimension of U_k^m, the top dimensional cohomology class will evaluate trivially on it. Thus it is reasonable to compactify $Q \underset{S^1}{\times} M_i^C$ as:

$$C_i = Q \underset{S^1}{\times} M_i^C \underset{q_i|Q \underset{S^1}{\times} M_i}{\cup} Q/S^1 \times M_i/S^1 \tag{5.7}$$

The space that we added to $Q \underset{S^1}{\times} M_i^C$ is homotopy equivalent to either of the two spaces: $Q^C \underset{S^1}{\times} M_i$ or $Q \underset{S^1}{\times} M_i^C$. Thus the projection may be replaced by an inclusion giving:

$$\begin{aligned} C_i &\simeq Q \underset{S^1}{\times} M_i^C \underset{Q \underset{S^1}{\times} M_i}{\cup} Q^C \underset{S^1}{\times} M_i && (5.8)\\ &\simeq Q \underset{S^1}{\times} M_i^C \underset{Q \underset{S^1}{\times} M_i}{\cup} Q \underset{S^1}{\times} M_i^C && (5.9)\end{aligned}$$

To define the class $\mu(\sigma)$ we have to pull back the bundle $M_i \to M_i/S^1$ over the space $Q \underset{S^1}{\times} M_i$, thus it is more convenient to use the construction (5.8). Then $\mu(\sigma)$ is a multiple of the first Chern class of the S^1 bundle:

$$Q \times M_i^C \underset{Q \times M_i}{\cup} Q^C \times M_i \to Q \underset{S^1}{\times} M_i^C \underset{Q \underset{S^1}{\times} M_i}{\cup} Q^C \underset{S^1}{\times} M_i$$

The total space of this bundle is the boundary of $Q^C \times M_i^C$, which can be considered as the S^3 bundle $S\left(Q^C \times M_i^C\right)$ over $Q/S^1 \times M_i/S^1$. Thus C_i is $P\left(Q^C \times M^C\right)$, the projectivization of the bundle $Q^C \times M_i^C$. The first Chern class of the bundle $S\left(Q^C \times M_i^C\right) \to P\left(Q^C \times M_i^C\right)$ is the class h that restricts as a generator of each fiber $\mathbf{C}P^1$.

The next two lemmas justify our choice of the above compactification:

Lemma 5.10 *Let w be a fixed divisor in $\mathcal{M}_k(N,m)$. Let $\mathcal{M}^*$ be a complement of the sum of the neighborhoods $\coprod_i K_i/S^1$. Then there is a one-to-one correspondence between divisors V on U_k^m that pull back from the divisor w, and collections of divisors $\hat{V}_i$ on C_i such that $\hat{V}_i \cap (Q/S^1 \times M_i/S^1) = [Q/S^1] \times [\hat{v}_i]$, and $(\hat{v}_1, \ldots, \hat{v}_s) = \partial[w]$. Here ∂ is the boundary map ∂ from the Mayer–Vietoris sequence:*

$$\cdots \to \mathrm{H}_*(\mathcal{M}^*) \oplus \bigoplus_i \mathrm{H}_*(K_i) \to \mathrm{H}_*(\mathcal{M}_k(N,m)) \xrightarrow{\partial} \bigoplus_i \mathrm{H}_{*-1}(M_i/S^1) \to \cdots$$

Proof: Given the divisor $V \subset U_k^m$ that pulls back from $w \subset \mathcal{M}_k(N,m)$ define $v_i = w|_{M_i/S^1}$. Using the deformation retractions $r_i : C_i \setminus Q \underset{S^1}{\times} M_i^C \to Q/S^1 \times M_i/S^1$, we can construct the divisors $\hat{V}_i = V|_{Q \underset{S^1}{\times} M_i^C} \underset{Q \underset{S^1}{\times} M_i}{\cup} r^{-1}([Q/S^1] \times v_i)$.

Going in the other direction: given a collection of divisors $(\hat{V}_i, \hat{v}_i)$ define

$$V = \oplus_i \hat{V}|_{Q \underset{S^1}{\times} M_i^C} \bigcup_{Q \underset{S^1}{\times} M_i} p_N^{-1}(w|)$$

where $p_N : Q \underset{G}{\times} \mathcal{N} \to \mathcal{N}/G$ is the projection, and $w|$ is a restriction of the class w onto the set $\mathcal{N}/G \setminus \left(\coprod_i p_N(Q \underset{S^1}{\times} M_i^C)\right)$. It can be directly seen that the composition of these two assignments gives the identity. □

Lemma 5.11 *Let $w_1, \ldots, w_r$ be fixed divisors on $\mathcal{M}_k(N,m)$ and $V_1, \ldots, V_r$ denote pull-backs of w_i's to U_k^m. Assume also that the sum of codimensions of V_i's is equal to the dimension of U_k^m. Then the following signed intersections are equal:*

$$\bigcap_{i=1}^{r} V_i = \sum_{j=1}^{s} \bigcap_{i=1}^{r} \hat{V}_{i,j}$$

where $\hat{V}_{i,j}$ is the divisor on C_j corresponding to V_i as in Lemma 5.10.

Proof: Let $\hat{V}_{i,j}^t$ be a perturbation of $\hat{V}_{i,j}$ into transversal position. Since $Q/S^1 \times M_j/S^1$ has codimension 2 in C_j, the one dimensional set $\{\bigcap_{i=1}^{r} \hat{V}_{i,j}^t \mid t \in [0,1]\}$ misses $Q/S^1 \times M_j/S^1$, the set added in the compactification. As $\partial[w_i] = (\hat{v}_{i,1}, \ldots, \hat{v}_{i,s})$, and this is a homological condition, we can use Lemma 5.10 for all $\hat{V}_{i,j}^t$, to get a corresponding perturbation of the divisors $V_{i,j}^t$. Because the dimension of $\mathcal{M}_k(N,m)$ is lower than the dimension of U_k^m the perturbed divisors cannot intersect outside the set $Q \underset{S^1}{\times} (\mathcal{N} \setminus (\coprod_i K_i))$. Thus the intersection number of V_i is equal to the intersection number of $\hat{V}_{i,j}$. □

The top stratum of the moduli space $\mathcal{M}_{k_N}(N,m)$ has a reducible connection on the bundle $\mathbf{L}$ if and only if $c_1^2(\mathbf{L}) = k_N$. Thus if the reducible connection exists, it is unique. Similarly on the stratum in which the background connections have energy $k_N - s$ there exists a unique reducible connection if and only if $c_1^2(\mathbf{L}') = k_N - s$. Thus every reducible connection $A_j \in \mathcal{M}_{k_N}(N,m)$ has uniquely assigned l_j, the first Chern class of the reduction of the bundle $\mathbf{L}_j$ and $s_j = k_N - c_1^2(\mathbf{L}_j)$. We can perform the closing–up construction for every link M_j of A_j separately getting the following equivalent of Theorem 5.5 for the spaces C_j:

Corollary 5.12 *Under the above correspondence of cohomology classes we have the following equalities in $\mathrm{H}^*(C_j)$:*

$$\mu(\sigma) = \pi^* \Sigma^{s_j}(\sigma) - l_j \cdot h_j$$

where the class h_j restricts as the generator of each fiber $\mathbf{C}P^2 \hookrightarrow P(Q^C \times M_j^C)$. Similarly

$$-4\mu(x_r) = \begin{cases} -4\pi^* \Sigma^{s_j}(x_r) + h_j^2 & \text{if } r \leq i \\ c_Q^2 & \text{if } r > i \end{cases}$$

We will need some properties of the characteristic classes of the bundle $P(\mathcal{Q}^C \times M^C)$ which we now evaluate. The Künneth formula provides an embedding

$$\mathrm{H}^*(M/S^1) \to \mathrm{H}^*(\mathcal{Q}/S^1 \times M/S^1)$$

that sends $x \in \mathrm{H}^*(M/S^1)$ to $1 \otimes x$. In the description that follows by cohomology classes $x \in \mathrm{H}^*(M/S^1)$ or $y \in \mathrm{H}^*(\mathcal{Q}/S^1)$ we shall mean their images $1 \otimes x$ or $y \otimes 1$ in $\mathrm{H}^*(\mathcal{Q}/S^1 \times M/S^1)$. From the definition of characteristic classes we get the relation $h^2 = c_1(\mathcal{Q}^C \times M^C)h - c_2(\mathcal{Q}^C \times M^C)$. Notice that $c_1(\mathcal{Q}^C \times M^C) = c_M + c_Q$ and $c_2 = c_Q c_M$, which in vector notation $(a,b) = a \cdot h + b$ can be written as $h^2 = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $A = \begin{pmatrix} c_M + c_Q & 1 \\ -c_Q c_M & 0 \end{pmatrix}$. By elementary computations we obtain the following lemma:

Lemma 5.13 *We have*

$$h^{n+2} = \frac{c_M^{n+2} - c_Q^{n+2}}{c_M - c_Q} \cdot h - \frac{c_M^{n+1} - c_Q^{n+1}}{c_M - c_Q} \cdot c_M c_Q$$

Proof: Using the expression for h^2 we get:

$$h \cdot (ah + b) = ah^2 + bh = [a(c_M + c_Q) + b] \cdot h \; - \; a \cdot c_M c_Q = A \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus we have

$$h^{n+1} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Set

$$S = \begin{pmatrix} 1 & 1 \\ -c_M & -c_Q \end{pmatrix}$$

A direct check verifies the relation:

$$A = S \begin{pmatrix} c_Q & 0 \\ 0 & c_M \end{pmatrix} S^{-1}$$

thus

$$A^n = S \begin{pmatrix} c_Q^n & 0 \\ 0 & c_M^n \end{pmatrix} S^{-1}$$

from which the result follows. □

The situation is different when the reducible background connection is trivial. In that case there is no direct relation between $\mu(\sigma)^2$ and $\mu(x)$. We need to supply a further discussion with some more notation. Notice first that whenever the moduli space $\mathcal{M}_k(N,m)$ admits a trivial background connection, then k must be an integer and $m = 0$. Thus on the Y side we always have the $SO(3)$ fibraton $\mathcal{Q} = \mathcal{M}_{k_Y}^{x}(Y,0)$. Let M be a restriction of the basepoint fibration over the link of the trivial connection $M/SO(3) \subset \mathcal{M}_k(N,m)$ with the projection $\pi : M/SO(3) \to \Sigma^k(N)$.

We have the following analog of Theorem 5.5:

Lemma 5.14 *We have the following equalities of classes in* $\mathrm{H}^*(M/SO(3))$*:*

$$\mu(\sigma) = \pi^*\Sigma^s(\sigma)$$

$$\mu(x_j) = \pi^*\Sigma^s(x_j) - \tfrac{1}{4}P_M$$

where P_M *is the first Pontrjagin class of the fibration* $M \to M/SO(3)$.

For any $SO(3)$ fibration $\mathcal{Q}$ define $\mathcal{Q}^G = \mathcal{Q} \underset{SO(3)}{\times} cSO(3)$ and let p_Q denote its first Pontrjagin class. A similar construction to (5.8) produces an $SO(3)$ fibration:

$$\partial(\mathcal{Q}^G \times \mathcal{P}^G) \to \partial(\mathcal{Q}^G \underset{SO(3)}{\times} \mathcal{P}^G) \tag{5.15}$$

which we shall denote by $S_G(\mathcal{Q}^G \times \mathcal{P}^G) \to P_G(\mathcal{Q}^G \times \mathcal{P}^G)$, or even $S_G \to P_G$ when the two other $SO(3)$ bundles will be obvious from the context. Let H denote the Pontrjagin class of this fibration. As in the case of S^1 fibrations, there is a further projection $P_G(\mathcal{Q}^G \times M^G) \to \mathcal{Q}/G \times \mathcal{P}/G$, whose fiber is $\Sigma SO(3)$, the suspension of $SO(3)$. The fibration $S_G \to P_G$ restricts over each fiber of P_G as the fibration:

$$SO(3) * SO(3) \to \Sigma SO(3)$$

and the first Pontrjagin class is 2. As before, the computations of the intersections of the dual classes in the neighborhood of the trivial connection can be translated into the intersection of corresponding divisors in $P_G(\mathcal{Q}^G \times M^G)$. As in the case of S^1 reducible connections, there is a correspondence between the classes that pullback from the moduli space containing $M/SO(3)$ and the classes in the compactification (5.15). Using the above correspondence and Theorem 5.5 we get the following corollary:

Corollary 5.16 *Under the above correspondence of cohomology classes we have the following equalities:*

$$\mu(\sigma) = \pi^*\Sigma^s(\sigma) \tag{5.17}$$

where the class H *restricts as twice generator of each fiber* $P_G(\mathcal{Q}^G \times M^G)$. *Similarly*

$$-4\mu(x_j) = \begin{cases} -4\pi^*\Sigma^s(x_j) + H & \text{if } j \le i \\ p_Q & \text{if } j > i \end{cases}$$

It follows from the corollary that in this neighborhood, the top power of the classes $\mu(\sigma)$ evaluates trivially, so it seems that these neighborhoods are redundant. But our method of computing $\mu(\sigma)^{top}$ on the whole moduli space $\mathcal{M}_{k_X}(X)$ requires evaluating in $P_G(\mathcal{Q}^G \times M^G)$ terms of the form $\mu(\sigma)^\alpha \cdot \mu(x_1)^{\alpha_1} \cdots \mu(x_s)^{\alpha_s}$ for various $\alpha, \alpha_1, \ldots, \alpha_s$, which may give a nontrivial contribution in P_G. So let us state the following analog of Lemma 5.13:

Lemma 5.18 *With the notation above we have:*

$$H^{n+2} = \frac{p_P^{n+2} - p_Q^{n+2}}{p_M - p_Q} \cdot H - \frac{p_P^{n+1} - p_Q^{n+1}}{p_P - p_Q} \cdot p_P p_Q$$

Proof: As $S^1 \subset SO(3)$, then there is an S^1 fibration $\mathcal{Q} \to \mathcal{Q}/S^1$ with the first Chern class $c_Q \in \mathrm{H}^2(\mathcal{Q}/S^1)$. Let $p : \mathcal{Q}/S^1 \to \mathcal{Q}/SO(3)$ denote the projection of the fibration with the fiber S^2. Then since $p^*\left(\mathcal{Q} \underset{SO(3)}{\times} \mathbf{C}^3\right) = \mathcal{Q} \underset{S^1}{\times} \mathbf{C}^3$ we have:

$$p^*(-p_Q) = p^*\left(c_2(\mathcal{Q} \underset{SO(3)}{\times} \mathbf{C}^3)\right) = c_2\left(p^*(\mathcal{Q} \underset{SO(3)}{\times} \mathbf{C}^3)\right) = c_2(\mathcal{Q} \underset{S^1}{\times} \mathbf{C}^3) = -c_Q^2 \quad (5.19)$$

Observe also that for every fiber $S^2 \subset \mathcal{Q}/S^1$, $c_2[S^2] = 2$. Thus from Leray–Hirsh theorem we get that the map $\langle \mathrm{H}^0(S^2) \oplus \mathrm{H}^2(S^2) \rangle \otimes \mathrm{H}^*(\mathcal{Q}/SO(3)) \to \mathrm{H}^*(\mathcal{Q}/S^1)$ sending $i^*(c_Q) \otimes \alpha \to p^*(\alpha) \cup c_Q$ is monomorphism, and in particular also p^* is monomorphism.

After these general remarks about $SO(3)$ fibrations let us come back to the construction described in the Lemma. Let h denote the first Chern class of the bundle $\partial(\mathcal{Q}^G \times \mathcal{P}^G) \to \partial(\mathcal{Q}^G \underset{S^1}{\times} \mathcal{P}^G)$. Then using Lemma 5.13 and (5.19) we have:

$$p^*(H) = h^2 = (h(c_Q + c_P) - c_Q c_P) \quad (5.20)$$

Thus:

$$p^*(H^2) = h^4 = \frac{c_P^4 - c_Q^4}{c_P - c_Q} \cdot h \; - \; \frac{c_P^3 - c_Q^3}{c_P - c_Q} c_P c_Q =$$

$$\frac{c_P^4 - c_Q^4}{c_P^2 - c_Q^2} \cdot h(c_P + c_Q) \; - \; \frac{c_P^3 - c_Q^3}{c_P - c_Q} \cdot c_P c_Q$$

which by (5.20) can be written as:

$$\frac{c_P^4 - c_Q^4}{c_P^2 - c_Q^2} \cdot [p^* H + c_P c_Q] \; - \; \frac{c_P^3 - c_Q^3}{c_P - c_Q} \cdot c_P c_Q =$$

$$= p^*(H) \cdot (c_P^2 + c_Q^2) \; + \; (c_P c_Q) \cdot [-c_P c_Q] = p^*\left(H[p_P + p_Q] - p_P p_Q\right)$$

As p^* is a monomorphism, the above proves the same relation that we had in Lemma 5.13 with Chern classes replaced by Pontrjagin classes. Thus the result follows. □

Thus in this paragraph we have established the relations between the class $\mu(\sigma)$ and Pontrjagin classes of basepoint fibration in each of the sets U_k^m separately. The next chapter deals with the problem of matching these descriptions together.

5.4 Compactifications of the U_k^m

The results of the previous paragraph are based on the assumption that the spaces M/S^1 and $\mathcal{Q}/S^1$ are compact. However this assumption is not valid in general. Even after taking the Uhlenbeck compactification, the sets

$$U_k^m = \mathcal{M}_{k_Y}^x(Y) \underset{\partial}{\times} \mathcal{M}_k^x(N)$$

still remain open. This is because there are connections whose energy may escape through the cylindrical end. We want to describe the boundary of the sets U_k^m containing such connections. As we shall see in the next section, we need to understand the characteristic classes of the bundles over U_k^m that over the boundary ∂U_k^m pull back from certain set $r(U)$ that has the dimension at least 2 less than $\dim U_k^m$. Let us denote by $\bar{U}_k^m$ the compact space $U_k^m \cup_p r(U)$, where $p : \partial U_k^m \to r(U)$. Since there is an isomorphism of cohomology groups $\mathrm{H}^{top}(\bar{U}_k^m) \simeq \mathrm{H}^{top}(U_k^m, \partial)$, we shall refer to our construction as either compactifying the spaces U_k^m or as to splitting the global characteristic classes into the sum of relative ones.

We shall begin the description of the connections in the boundary of U_k^m with the generalization of the definition of based connections. Let us recall that if $\tilde{\mathcal{M}}_k(X)$ denotes the space of ASD connections in $\mathcal{A}$, then $\mathcal{M}_k^x(X) = \tilde{\mathcal{M}}_k(X) \times_{\mathcal{G}_E} E_x$. Thus we can similarly define a moduli space of connections based at two points as

$$\mathcal{M}_k^{x,y}(X) = \tilde{\mathcal{M}}_k(X) \times_{\mathcal{G}_E} (E_x \times E_y) \tag{5.21}$$

Note that $\mathcal{M}_k^{x,y}(X)$ can be also described as a fibered product $\mathcal{M}_k^x(X) \times_{\mathcal{M}_k(X)} \mathcal{M}_k^y(X)$. The projection $\mathcal{M}_k^{x,y}(X) \to \mathcal{M}_k^x(X)$ is in fact the quotiening $\mathcal{M}_k^{x,y}(X)$ by $SO(3)_y$ acting on E_y. We shall use this definition for the moduli spaces of connections on the cylinders $\mathbf{R} \times L$ based at the different ends of $\mathbf{R}$. We indicate this by ${}^x\mathcal{M}_k^y(\mathbf{R} \times L)$. Similarly, for a finite interval $[a, b] \subset \mathbf{R}$ we can define the space ${}^x\mathcal{M}_k^y([a, b] \times L)$ to be a space of connections over the cylinder $[a-1, b+1] \times L$ having small energy on the subcylinders $[a-1, a] \times L$ and $[b, b+1] \times L$ containing the points x and y.

The noncompactness of U_k^m is a direct consequence of noncompactness of the spaces $\mathcal{M}_{k_Y}(Y)$ and $\mathcal{M}_k(N)$, so it is sufficient to describe the compactification of these factor spaces. Recall that in order to glue back these factor spaces we need to work with spaces of connections having small energy on the fixed cylinder R. We shall assume this throughout this paragraph. Let us focus first on the case of compactifying the space $\mathcal{M}_{k_Y}(Y)$. The boundary component corresponding to the splitting at one region can be described as the fibered product:

$$\mathcal{M}_{k_Y}^y(Y, m_i) \times_{\partial} {}^y\mathcal{M}_{k_i}^x([a-1, b+1] \times L, [m_i, m]) / (SO(3)_y \times SO(3)_x) \tag{5.22}$$

The first Pontrjagin class of the $SO(3)_x$ action defines the point class $-4\mu(x)$. The total space of this $SO(3)_x$–fibration needs to be glued to the space $\mathcal{M}_{k_N}^x(N, m)$ to get the compactified set U_k^m. The time translation parameter of the space

$$\mathcal{M}_{k_i}([a-1, b+1] \times L, [m, m'])$$

is the collar parameter of the fibered space $\mathcal{M}_k(Y, m)$. Denote by $\mathcal{M}'_{k_i}([a-1, b+1] \times L) = \mathcal{M}_{k_i}([a-1, b+1] \times L)/\mathbf{R}$ and let

$$p : \mathcal{M}_{k_-}^y(Y, m_i) \times_{\partial} \mathcal{M}_{k_i}'^{,y}(T, [m_i, m]) / SO(3)_y \to$$

$$\to \mathcal{M}_{k_-}(Y, m_i) \times \mathcal{M}'_{k_i}([a-1, b+1] \times L, [m_i, m])$$

be the projection. We define the compactification of $\mathcal{M}_{k_Y}(Y,m)$ as

$$\mathcal{M}_{k_Y}(Y,m) \bigcup_p \left[\mathcal{M}_{k_-}(Y, m_i) \times \mathcal{M}'_{k_i}([a-1, b+1] \times L, [m_i, m])\right]$$

Notice that the space that we have added in this compactification has codimension at least 2 in $\mathcal{M}^x_{k_Y}(Y,m)$. (One dimension is "lost" while dividing by time translations, and we also "lose" the gluing parameter between Y and the cylinder T). Thus we are in the situation described at the begining of this section, which allows us to define the compactified space $\mathcal{M}_{k_Y}(Y,m)$, or equivalently the relative class $\mathrm{H}^{top}(\mathcal{M}_{k_Y}(Y,m), \partial(\mathcal{M}_{k_Y}(Y,m))\ )$. We shall use the same symbol $\mathcal{M}_{k_Y}(Y,m)$ for the compactified space. We define the classes $\mu(x_i)$ on this compactification as pullbacks of the corresponding classes on the product $\mathcal{M}_{k_-}(Y, m_i) \times \mathcal{M}'_{k_i}([a-1, b+1] \times L, [m_i, m])$. Notice that with the class $\mu(y)$ for y in the splitting region, pulling back from $\mathcal{M}^y_{k_-}(Y, m_i)$ or $\mathcal{M}'^{,y}_{k_i}([a-1, b+1] \times L, [m_i, m])$ gives the same result.

While applying the same construction to the moduli space $\mathcal{M}_k(N,m)$ we get a similar splitting of that moduli space into factor spaces $\mathcal{M}^y_{k_N}(N,m')$ and the moduli space on the nearest cylinder $\mathcal{M}^y_{k_T}([a-1, b+1] \times L, [m', m])$. In this situation $\mathcal{M}^y_{k_N}(N,m')$ is not an $SO(3)$ fibration because of the presence of reducible connections. Thus we shall apply the construction similar to (5.8), relating the classes $\mu(\sigma)$ and $\mu(x_i)$ to the Pontrjagin class of the basepoint fibration based at x, which is at the end of N. This basepoint fibration based at x allows us to define the extension of the boundary value map $\partial : \mathcal{M}^x_{k_Y}(Y,m) \to \mathcal{R}(L)$, and thus the gluing of the compactified spaces $\mathcal{M}^x_{k_Y}(Y)$ and $\mathcal{M}^x_k(N)$. As a result we get a compactification of the set U^m_k. In general it may happen that the fibered product defining the boundary of $\mathcal{M}_k(Y)$ has a bigger number of factors. In this case we apply inductively the above procedure for constructing the compactification of an unbased moduli space. This construction is similar to the Floer compactification of the moduli spaces on the cylinder.

Now we want to ask the question if the relations between $\mu(\sigma)$ and the point classes proven in Corollaries 5.12 and 5.17 hold on the sets C_i after compactifying the moduli spaces $\mathcal{M}_k(N,m)$ and $\mathcal{M}_{k_Y}(Y,m)$. With the definition of the compactified $\mu(\sigma)$ as above, the answer is positive only when there is only one reducible connection A_i is in the the set $\mathcal{M}_{k_1}(N, m_1)$. (We used the relation between *this* connection and the Pontrjagin class of the basepoint fibration based at x to define the compactified class $\mu(\sigma)$.) Otherwise the cohomology class

$$\mu(\sigma) - \pi^*\Sigma^s(\sigma) - \langle c_1(\mathbf{L})\ ,\ \sigma\rangle \cdot h_\partial \tag{5.23}$$

whose evaluation inside the set C_i is zero, has nontrivial support in the neighborhoods of the other reducible connections A_j, $j \neq i$.

5.5 The relative Mayer–Vietoris sequence

So far we have been dealing only with local computations in a single open set U_k^m coming from covering the part of the moduli space $\mathcal{M}_k(X)$ that contains the intersection of $V_z \cap \bigcap_{j=1}^{n} V_\sigma$. Now we want to show how to combine these local computations into a global one.

Assume first for simplicity that the whole moduli space M may be covered by just two open sets U_1 and U_2. As the moduli spaces of connections on bundles over $\mathbf{R} \times L$ allow time translations, then from (5.22) we see that $U_1 \cap U_2$ can be written as $(-2,2) \times V'$, where V' is codimension 1 submanifold of M. Let us put $V_1 = U_1 \setminus (-1,2) \times V'$ and $V_2 = U_2 \setminus (-2,1) \times V'$ (thus V_i's are deformation retracts of U_i's, but they do not cover M). Then the cohomology group $\mathrm{H}^*(M)$ is a part of the relative Mayer–Vietoris sequence:

$$\begin{array}{ccccccc} \rightarrow & \mathrm{H}^*(M,\ V_1 \cup V_2) & \rightarrow \mathrm{H}^*(M,\ V_1) & \oplus & \mathrm{H}^*(M,\ V_2) & \rightarrow \mathrm{H}^*(M,\emptyset) & \rightarrow \\ & \| & \| & & \| & & \\ & \mathrm{H}^*(\Sigma V') & \mathrm{H}^*(U_2,\partial U_2) & & \mathrm{H}^*(U_1,\partial U_1) & & \end{array} \tag{5.24}$$

It follows from this sequence, that any top dimensional class in $\mathrm{H}^{top}(M)$ is equal to the sum of two relative classes in $\mathrm{H}^*(U_1, \partial U_1)$ and in $\mathrm{H}^*(U_2, \partial U_2)$ modulo a "correction term" from $\mathrm{H}^*(\Sigma V')$.

In general, when the set M is covered by open sets $U_1, U_2, \ldots, U_r$, let us denote the complements of these sets by $V_i = M \backslash U_i$. For any subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, r\}$ let $V_{i_1,i_2,\ldots,i_k} = V_{i_1} \cup \cdots \cup V_{i_k}$, and $V^{i_1,i_2,\ldots,i_k} = V_{i_1} \cap \cdots \cap V_{i_k}$. Then we have the following generalization of the relative Mayer–Vietoris sequence:

Theorem 5.25 *Let M be a manifold containing the submanifolds $V_1, \ldots, V_r$, not necessarily covering M. Then for every class $\mu \in \mathrm{H}^{top}(M, V_{1,2,\ldots,r})$ there exist classes $\mu_{i_1,\ldots,i_k} \in \mathrm{H}^{top}(M, V^{i_1,\ldots,i_k})$ for $\{i_1, i_2, \ldots i_k\} \subset \{1, 2, \ldots, r\}$, such that:*

$$\langle \mu\ ,\ [M, V_{1,2,\ldots,r}] \rangle = \sum_k (-1)^{k+1} \sum_{\{i_1,i_2,\ldots i_k\}} \langle \mu_{i_1,i_2,\ldots i_k}\ ,\ [M, V^{i_1,i_2,\ldots i_k}] \rangle$$

Proof: We shall prove this theorem by induction on the number of sets V_i. For $r = 2$ the theorem follows from the relative Mayer–Vietoris Theorem described above. Assume thus that this theorem is valid for all $n < r$. The sequence (5.24) for $M_1 = V_1$ and $M_2 = V_{2,3,\ldots,r}$ is:

$$\mathrm{H}^{top}(M, V_1 \cup V_{2,3,\ldots,r}) \rightarrow \mathrm{H}^{top}(M, V_1) \oplus \mathrm{H}^{top}(M, V_{2,3,\ldots,r}) \rightarrow \mathrm{H}^{top}(M, V_{1,2,\ldots,r}) \rightarrow 0 \tag{5.26}$$

Hence, using the inductive assumption, there exist classes $\mu_{i_1,\ldots,i_s} \in \mathrm{H}^{top}(M, V_1 \cup V_{2,3,\ldots,r})$, where $\{i_1, \ldots, i_s\} \subset \{2, 3, \ldots, r\}$, a class $\mu_1 \in \mathrm{H}(M,\ V_1)$, and a class $\alpha \in \mathrm{H}^{top}(M,\ V_1 \cup V_{2,3,\ldots,r})$ such that

$$\mu = \mu_1 + \sum_k (-1)^{k+1} \sum_{\substack{\{i_1,i_2,\ldots i_k\} \subset \\ \subset \{2,3,\ldots,r\}}} \mu_{i_1,i_2,\ldots i_k} - \alpha$$

By excision we have: $\mathrm{H}^*(M, V_1 \cup V_{2,3,\ldots,r}) \simeq \mathrm{H}^*(U_1, \overline{V}_{2,3,\ldots,r})$, where $U_1 = M \setminus V_1$, and $\overline{V}_i = U_1 \cap V_i$. We shall use the same symbol $\alpha \in \mathrm{H}^*(U_1, \overline{V}_{2,3,\ldots,r})$ for the image of $\alpha \in \mathrm{H}^*(M, V_1 \cup V_{2,3,\ldots,r})$ under the excision isomorphism. Using again the inductive assumption for the set U_1 containing sets $\overline{V}_i$ we get

$$\alpha = \sum_k (-1)^{k+1} \sum_{\substack{\{i_1,i_2,\ldots i_k\}\subset \\ \subset\{2,3,\ldots,r\}}} \alpha_{i_1,i_2,\ldots i_k}$$

where $\alpha_{i_1,i_2,\ldots i_k} \in \mathrm{H}^{top}(U_1, \overline{V})$. By excision $\mathrm{H}^{top}(U_1, \overline{V}) \simeq \mathrm{H}^{top}(M, V^{1,i_1,i_2,\ldots,i_k})$, thus $\alpha_{i_1,i_2,\ldots i_k}$ correspond to $\mu_{1,i_1,i_2,\ldots i_k}$, which concludes the proof. □

Corollary 5.27 *When the sets $U_1, U_2, \ldots, U_r$ cover the space M, then every class $\mu \in \mathrm{H}^{top}(M)$ is equal to a combination of the classes $\mu_{i_1,i_2,\ldots,i_k} \in \mathrm{H}^{top}(\overline{U}_{i_1,i_2,\ldots,i_k})$, where $\overline{U}_{i_1,i_2,\ldots,i_k}$ is a compactification of the intersection $U_{i_1,i_2,\ldots,i_k}$ described in previous paragraph.*

Proof: When the sets $\{U_i\}_{i=1}^r$ cover M, then $V_{1,2,\ldots,r} = \emptyset$. Thus from Theorem 5.25 it is sufficient to prove that $\mathrm{H}^{top}(M, V^{i_1,i_2,\ldots,i_k}) \simeq \mathrm{H}^{top}(\overline{U}_{i_1,i_2,\ldots,i_k})$. The excision isomorphism gives us $\mathrm{H}^*(M, V^{i_1,i_2,\ldots,i_k}) \simeq \mathrm{H}^*(U_{i_1,i_2,\ldots,i_k}, \partial)$. Recall that the compactified sets $\overline{U}$ are of the form $U \underset{\phi}{\cup} K$, where K is a set of codimension at least 2. Let us denote $C_K = \partial U \underset{\phi}{\cup} K$. Then we have another excision isomorphism: $\mathrm{H}^*(U_{i_1,i_2,\ldots,i_k}, \partial) \simeq \mathrm{H}^*(\overline{U}, C_K)$. As the set C_K deformation retracts onto K, which is of codimension at least 2, then we have the last isomophism $\mathrm{H}^{top}(\overline{U}, C_K) \simeq \mathrm{H}^{top}(\overline{U})$ from the exact sequence of the pair $(\overline{U}, C_K)$. □

Let Φ denote the set of these classes $\phi \in \mathrm{H}^{top}(\mathcal{M}_k(X))$ that are equal to a linear combination of terms of the form $a_{\alpha,\alpha_1,\ldots,\alpha_s}\mu(\sigma)^\alpha \cdot \mu(x_1)^{\alpha_1} \cdots \mu(x_s)^{\alpha_s}$. The specific compactification of the classes $\mu(\sigma)$ and $\mu(x_i)$ that we defined before allows us for every $\phi \in \Phi$ to define a "restriction map" $r_{i_1,i_2,\ldots,i_k} : \mathrm{H}^{top}(\mathcal{M}_k(X)) \to \mathrm{H}^{top}(\overline{U}_{i_1,i_2,\ldots,i_k})$ such that ϕ is a combination of the classes $r_{i_1,i_2,\ldots,i_k}(\phi)$ in the sense of Lemma 5.27. The following lemma will allow us to avoid dealing with the classes $r_{i_1,i_2,\ldots,i_k}(\phi)$ for $k > 1$:

Lemma 5.28 *Let M and $V_1, \ldots, V_r$ be as in Theorem 5.25. Let $\mu \in \mathrm{H}^{top}(M)$ be a cohomology class such that $\mu_i = 0 \in \mathrm{H}^{top}(M, V_i)$ for every $i = 1, 2, \ldots, r$. Then $\mu = 0 \in \mathrm{H}^{top}(M, V_{1,2,\ldots,r})$.*

Proof: This is proved by an induction on the number of sets V_i as in (5.25). For $r = 2$ the statement follows from the relative Mayer–Vietoris sequence. The inductive assumption implies that $\mu \in \mathrm{H}^{top}(M, V_{2,3,\ldots,r})$ is zero. Thus the sequence (5.26) gives us the lemma. □

6 The proof of Theorem 1.3

The proof of Theorem 1.3 is based on the idea of replacing the classes $\mu(\sigma)$ with some appropriate point classes $\mu(x_i)$ by using relations described in Lemmas 5.12 and 5.16. In other words we would like to find a linear combination of the form:

$$\mu(\sigma)^n + \sum_{\alpha<n} a_{\alpha,\alpha_1,\ldots,\alpha_s}\mu(\sigma)^\alpha \cdot \mu(x_1)^{\alpha_1} \cdots \mu(x_s)^{\alpha_s} \tag{6.1}$$

that would evaluate as zero on every set U_i. In the above formula, as well as in later notation, we shall often skip the factor $\mu(z)$ that appears in every term. Then from Corollary 5.27 it follows that (6.1) is zero on the whole of $\mathcal{M}_k(X)$, thus giving the main result.

We shall begin by defining a form that will evaluate trivially on a part of the diagram $\mathcal{J}$ defined in (4.6). Let $\{U_i\}_{i=1}^s$ be the open sets of the cover corresponding to the first column of $\mathcal{J}$.

Theorem 6.2 *Let $\sigma\cdot\sigma = -p$, and let s be the number of elements in $\chi(L(p,1)\,)$. Let $n > 2s$, and set $r = \frac{n-2s}{2}$ when n is even, and $r = \frac{n-2s+1}{2}$ when n is odd. There is an $a = (a_1, a_2, \ldots, a_s)$ such that $\phi(a)$ defined by:*

$$\phi(a) = \mu(\sigma)^n - a_s\mu(\sigma)^{2s}\mu(x_s)^r + a_{s-1}\mu(\sigma)^{2s-2}\mu(x_{s-1})^{r+1} + \cdots + a_1\mu(\sigma)^2\mu(x_1)^{r+s-1}$$

when n is even, and

$$\phi(a) = \mu(\sigma)^n - a_s\mu(\sigma)^{2s-1}\mu(x_s)^r + a_{s-1}\mu(\sigma)^{2s-3}\mu(x_{s-1})^{r+1} + \cdots + a_1\mu(\sigma)^1\mu(x_1)^{r+s-1}$$

when n is odd, evaluates trivially on the set $\bigcup_{1\le i\le s} U_i$.

Proof: We shall give the proof in the case when n is even. The proof in the other case is analogous. For convenience let us set $p(x_i) = -4\mu(x_i)$. Using Corollary 5.12 for the set U_s we get the relation: $\mu(\sigma)^2 = s^2 \cdot p(x_i)$ for $i \le s$. Thus, in U_s we get:

$$\phi(a) = \left(s^{2r} + a_s + \frac{1}{s^2}a_{s-1} + \cdots + \frac{1}{(s^2)^{s-1}}a_1\right)\mu(\sigma)^{2s}p(x_s)^r$$

While considering the other sets U_i we have to use the following lemma, which is a direct consequence of Lemma 5.13:

Lemma 6.3 *Let $i > k$, and let the numbers α, β, and γ be such that $\gamma + \alpha \ge \dim \mathcal{M}_{\frac{k^2}{p}}(N,k)$. Then in the set U_k we have:*

$$\mu(\sigma)^\gamma\mu(x_k)^\alpha\mu(x_i)^\beta = \mu(\sigma)^\gamma\mu(x_k)^{\alpha+\beta}$$

From this Lemma and the relation $\mu(\sigma)^2 = k^2 p(x_i)$ for $i \leq k$ we get in U_k:

$$\begin{aligned}
\phi(a) &= \mu(\sigma)^{2k}\left((k^2)^{r+s-k}p(x_k)^{r+s-k} + (k^2)^{s-k}a_s \cdot p(x_k)^{s-k}p(x_s)^r + \cdots\right. \\
&+ \left.\frac{1}{(k^2)^{k-1}}p(x_k)^{r+s-k}a_1\right) \\
&= \mu(\sigma)^{2k}p(x_k)^{r+s-k}\left((k^2)^{r+s-k} + (k^2)^{s-k}a_s + \cdots + \frac{1}{(k^2)^{k-1}}a_1\right)
\end{aligned}$$

Thus we get a system of s equations with s variables, whose matrix is the Vandermonde matrix:

$$W_s = \begin{pmatrix} (s^2)^{s-1} & \dots & (s^2) & 1 \\ \dots & \dots & \dots & \dots \\ (2^2)^{s-1} & \dots & (2^2) & 1 \\ 1 & \dots & 1 & 1 \end{pmatrix} \tag{6.4}$$

whose determinant is $\prod_{1\leq i<j\leq s}(j^2 - i^2)$. In particular, this determinant is nonzero, thus we can always find the numbers $a_1, \dots, a_s$. $\square$

In order to remove the assumption $n > 2s$ from the above theorem we need to define the relative invariants, that have been introduced in [FS2].

Definition 6.5 *Let Y is a 4-manifold with a cylindrical end and $\mathcal{M}_{k_Y}(Y,m)$ a compactified moduli space. Assume that m is nontrivial, and that c denotes the first Chern class of the S^1 fibration $\partial^{-1}(pt)$. Then define the relative Donaldson invariant $D_Y[m] : \mathbf{A}(Y) \to \mathbf{R}$ by:*

$$D_Y[m](z \cdot c^\beta) = \langle \mu(z) \cdot c^\beta \;,\; \mathcal{M}_{k_Y}(Y,m)\rangle$$

When $m = \frac{p}{2}$ we set $D_Y[m](z) = D_{X,\sigma}(z)$ to be a twisted $SO(3)$ invariant. When $m = 0$ we set $D_Y[m](z) = D_X(z)$, the usual invariant.

The class w_2 of that twisted invariant must evaluate as zero on Y and on N, thus it is a unique $\mathbf{Z}_2$ class that comes from $\mathrm{H}^1(L)$ in the cohomology sequence:

$$\to \mathrm{H}^1(L) \to \mathrm{H}^2(X) \to \mathrm{H}^2(Y) \oplus \mathrm{H}^2(N) \to$$

Since the $\mathbf{Z}_2$ reduction of the class σ satisfies these conditions, we have $w_2 = \sigma$.

Lemma 6.6 *Let $n = 2k \leq 2s$ (where s is the number of elements in $\chi(L)$). Then there exist coefficients $a_1, \dots, a_{k-1}$ such that the form*

$$\phi_k = \mu(\sigma)^{2k} - a_{k-1}\mu(\sigma)^{2k-2}p(x_{k-1}) - \cdots - a_1\mu(\sigma)^2 p(x_1)^{k-1}$$

evaluates on the sum of the sets $\cup_{i=1}^{k} U_i$ as $(2k)! D_Y[k](z)$ in the case when k is a trivial element of $\chi(L)$, and evaluates as $\frac{(2k)!}{2} D_Y[k](z \cdot c)$ otherwise.

Proof: Let $W(k-1)$ denote the $k \times (k-1)$ matrix

$$\begin{pmatrix} -((k-1)^2)^{k-1} & ((k-1)^2)^{k-2} & \dots & (k-1)^2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

and let $W(k-1)^j$ denote the $(k-1) \times (k-1)$ matrix obtained from $W(k-1)$ by deleting j^{th} column from the right. Then according to Theorem 6.2 for $a_j = \frac{\det W(k-1)^j}{\det W_{k-1}}$ the form ϕ_k evaluates trivially on the sets U_i for $1 \leq i < k$. So it remains to evaluate ϕ_k on the set U_k:

$$\begin{aligned} \phi_k|_{U_k} &= -(k^2)^k + a_{k-1}(k^2)^{k-1} + \dots + a_1 k^2 = \\ &= \frac{k^2}{\det W_{k-1}} \Big[-(k^2)^{k-1} \cdot \det W_{k-1} + a_{k-1}(k^2)^{k-2} \det W(k-1)^{k-1} + \cdots \\ &+ a_1 \det W(k-1)^1 \Big] = \\ &= k^2 \frac{\det W_k}{\det W_{k-1}} = \prod_{0 \leq i \leq k-1} (k^2 - i^2) = \frac{(2k)!}{2} \end{aligned} \tag{6.7}$$

When k is a trivial element of $\chi(L)$ and U_k corresponds to the set with the biggest amount of energy on the N-side that covers the intersection of divisors V_σ, then the bundle Q apearing in the Lemma 5.13 is $SO(3) \to S^2 = SO(3)/S^1$. As the first Chern class of this bundle is equal to 2, then we have an additional factor of 2 in the formula for $\phi_k|_{U_k}$ in this case. □

To avoid dealing with separate statements for two different cases $n > 2s$ and $n \leq 2s$ we shall introduce the following notation:

Definition 6.8 *Define the Donaldson invariants needed for evaluation $D(z\sigma^n)$:*

$$D_\perp(z\sigma^{2m} x_{m+2d}^{k-2d-m}) = \begin{cases} D(z\sigma^{2m} x_{m+2d}^{k-2d-m}) & \text{if } k-2d-m > 0. \\ D[m](c \cdot z) & \text{if } k-2d-m = 0. \\ 0 & \text{if } k-2d-m < 0. \end{cases}$$

when $n = 2k$, and

$$D_\perp(z\sigma^{2m-1} x_{m+2d}^{k-2d-m}) = \begin{cases} D(z\sigma^{2m-1} x_{m+2d}^{k-2d-m}) & \text{if } k-2d-m > 0. \\ D[m](z) & \text{if } k-2d-m = 0. \\ 0 & \text{if } k-2d-m < 0. \end{cases}$$

in the case $n = 2k+1$.

The particular subscript of x_{m+2d} in the above formulas is taken from the fact that in the $(d+1)^{\text{st}}$ column of the diagram $\mathcal{J}$ the element with boundary value m lies in the level x_{m+2d}. Now we are ready to prove the theorem, from which (1.3) will follow:

Theorem 6.9 *Let X be a smooth manifold containing an embedded sphere S^2 with self-intersection $\sigma^2 = -p$. Let s be the number of elements in $\chi(L(p,1))$, and let*

$z \in \mathbf{A}(X)$ be a cohomology class such that $z \cdot \sigma = 0$. Then there exist constants a_j^r such that:

$$D(\sigma^{2k} z) = \sum_{m=1}^{s} \sum_{d=0} a_m^d D_{\perp}(\sigma^{2m} x^{k-2d-m} z) \tag{6.10}$$

when $n = 2k$, and:

$$D(\sigma^{2k+1} z) = \sum_{m=1}^{s} \sum_{d=0} a_m^d D_{\perp}(\sigma^{2m-1} x^{k-2d-m} z) \tag{6.11}$$

when $n = 2k + 1$.

Proof: We first concentrate on the case when $n = 2k$. Let $U_1, U_2, \dots, U_s$ be the open sets that correspond to the first column of the diagram $\mathcal{J}$ (if there are fewer than s sets in the first column, thanks to the definition of $D_{\perp}$ we can assume that we completed that list by empty sets). Theorem 6.2 and Lemma 6.6 imply that there are constants $a_1^0, a_2^0, \dots, a_s^0$ such that

$$\phi^0 = \sum_{m=1}^{s} a_m^0 D_{\perp}(z\sigma^{2m} x^{k-m})$$

evaluates trivially on the first column of $\mathcal{J}$. Let $\mathcal{J}^{(1)}$ be a diagram obtained from $\mathcal{J}$ by erasing its first column. For every vertex of $\mathcal{J}^{(1)}$, like in (4.6) we can assign an open set in $\mathcal{M}_{k-1}(X)$ (the moduli space of connections on X, with *one charge less* than the space that we started with.) Repeating the argument from Theorem 6.2 we can show that there exists a polynomial

$$\begin{aligned} \phi^1 &= a_s^1 \mu(\sigma)^{2s} \mu(x_{s+2})^{r-1} + a_{s-1}^1 \mu(\sigma)^{2s-2} \mu(x_{s+1})^r + \cdots + a_1^1 \mu(\sigma)^2 \mu(x_3)^{r+s-2} \\ &+ a_0^1 \mu(x_2)^{r+s-1} \end{aligned}$$

such that $\phi^0 - \phi^1$ evaluates trivially on the first column of $\mathcal{J}^{(1)}$. The difference $\phi^0 - \phi^1$ should be understood in the following way: as the diagram $\mathcal{J}^{(1)}$ has been obtained from $\mathcal{J}$ by erasing some of its elements, then we can say that there is an embedding $\mathcal{J}^{(1)} \hookrightarrow \mathcal{J}$. Then ϕ^1 we evaluate on the open set corresponding to a vertex of $\mathcal{J}^{(1)}$, whereas the ϕ^0 we evaluate on the corresponding set of $\mathcal{J}$. Repeating Theorem 6.2 for consecutive diagrams $\mathcal{J}^{(j)}$ we prove the theorem. □

Proof of Theorem 1.3: Lemma 6.6 with Theorem 6.9 implies that the relative Donaldson invariants $D_Y[k]$ can be expressed in terms of $D(\sigma^{2k})$ and $D(\sigma^{2j} x^{k-j})$ for $j < k$. Thus all relative invariants in 6.10 can be replaced with invariants defined on X, which completes the proof. □

7 Examples of computations.

The theory that we have developed gives more than just Theorem 1.3. By using the results of this thesis we can compute the actual coefficients in some of the formulas given by the (1.3). Below we give some examples of such computations.

Example 7.1 Let σ be the cohomology class of an embedded 2–sphere with self–intersection -4. Although Theorem 1.3 does not cover the case of $D(z\sigma^4)$, the only relative Donaldson invariant that appears in Theorem 6.9 is a twisted $SO(3)$ invariant with $w_2 = \sigma$, that is why we get the formula of the form:

$$D(z\sigma^4) = A \cdot D(z) + B \cdot D(z\sigma^2 x) + C \cdot D_\sigma(z)$$

We have already seen that the diagram $\mathcal{J}$ in this situation is

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & \\ 2 & 0 \end{tikzcd}$$

and let us denote the corresponding open sets by $U^1_{1/4}$, U^2_1 and U^0_1. Then from Lemma 6.6 we have that $\phi^0 = \mu(\sigma)^4 - \mu(\sigma)^2 p(x_1)$ evaluates trivially on $U^1_{1/4}$. Since the coefficient B is obtained from the term $\mu(\sigma)^2 p(x_1) = -4\mu(\sigma)^2\mu(x_1)$, then we have $B = -4$. The coefficient C we obtain by evaluating ϕ^0 on U^2_1. The $\dim \mathcal{M}_1(N,2) = 5$, thus the neighborhood of the reducible connection is modelled on $\mathbf{C}^3/S^1 = \mathbf{C}P^2$. From Lemma 5.12 it follows that $U^2_1 = P(SO(3)^C \times \gamma_2^C)$, where $\gamma_n \to \mathbf{C}P^n$ is a line bundle with $c_1(\gamma_n) = [\mathbf{C}P^1]$. If h is the generator of the fiber $\mathbf{C}P^1$ of P, then we have:

$$\mu(\sigma) = 2h; \qquad\qquad p(x_1) = h^2$$

Thus in U^2_1

$$\mu(\sigma)^4 - \mu(\sigma)^2 p(x_1) = \pm c([SO(3)/S^1]) \cdot (2^4 - 2^2) \cdot D_\sigma(z) = \pm 24 \cdot D_\sigma(z)$$

Here $c = 2[S^2]$ is the first Chern class of the fibration $SO(3) \to SO(3)/S^1$.

In order to determine the sign in the above equation we have to refer to the orientations of moduli spaces defined in [D2]. By using a complex structure on the elliptic complex

$$\Omega^0(TX \otimes L) \to \Omega^1(TX \otimes L) \to \Omega^2_+(TX \otimes L)$$

where $L \to X$ is a complex line bundle, one can assign an orientation $o(L)$ to a reducible connection respecting the splitting $L \oplus \bar{L}$ in the case of $SU(2)$ connections, or respecting the splitting $L \oplus \mathbf{R}$ in the case of $SO(3)$ connection. The Example 4.3 from [D2] worked out for an arbitrary reducible bundle $L \oplus \bar{L}$ gives:

Claim 7.2 *The orientation $o(L)$ of a moduli space in the neighborhood of reducible connection on a bundle $E = L \oplus \bar{L}$ is $-\bar{n}\wedge$* (standard orientation on $\mathbf{C}P^n$), *where $\bar{n}$ is the normal pointing away from the reduction. This orientation compares to the standard one with a sign $(-1)^{c_1(L)^2}$.*

For the connections over the space N we have to change the sign in the above Claim, due to the fact that N has the orientation *opposite* to the complex one. The decomposition $X = Y\#N$ yields a splitting of arbitrary line bundle L into

$L \cong L_Y \# L_N$, where $c_1(L) = c_1(L_N)$ on N. The neighborhood of reducible connection on $L_N \oplus \overline{L}_N$ is described as $SO(3) \underset{S^1}{\times} \mathbf{C}^n$ has the orientation $o(L)$. An arbitrary connection may be related to a reducible one through additions and subtractions of instantons. The set U_1^2 is a neighborhood of the reducible connection on $L_N \oplus \overline{L}_N$ with $c_1(L_N) = 2[S^2] \in \mathrm{H}^2(N)$. The associated $SO(3)$ connection is defined on $L_N^2 \oplus \mathbf{R}$. Thus as a bundle L on the whole X we can take the line bundle with $c_1(L) = -PD[S^2] = -\sigma$, since $-\sigma[S^2] = c_1(L_N^2) = 4$. Thus the complex orientation on U_1^2 compares with the standard one with the sign $(-1)^{\sigma^2} = (-1)^4$, giving the "+" sign at $D_\sigma(z)$.

There are no S^1 reducible connections inside the set U_1^0 but there is a trivial one. Following the discussion in [Y], the restriction of the base–point fibration to the link of the trivial connection in $\mathcal{M}_1(N,0)$ is $Fr_+(N) \to N$, where $Fr_+(N)$ is the associated $SO(3)$–principal bundle of $\Lambda_+^2 TN$. Then $U_1^0 = P_G(SO(3)^G \times Fr_+(N)^G)$, and following Corollary 5.16 we have in this set:

$$\mu(\sigma) = \pi^*(\sigma); \qquad\qquad p(x_1) = -4\pi^*(x_1) + H$$

where H is a generator of the fiber $\Sigma SO(3)$ of $P_G(SO(3)^G \times Fr_+(N)^G)$. Thus $\phi^0 = -H[\Sigma SO(3)] \cdot \sigma^2[N] = -2 \cdot (-4) = 8$. The additional "−" sign has to be added again due to the anti–complex orientation of N. Putting all three terms together we get

Theorem 7.3 *When σ is a class of S^2 with self-intersection -4, then the following formula holds:*

$$D(z\sigma^4) = \quad -8 \cdot D(z) + 24 \cdot D_\sigma(z) - 4 \cdot D(z\sigma^2 x) \tag{7.4}$$

7.1 Second stratum connections over $\overline{CP^2}$

The next example requires the description of the reducible connections in the second stratum of the moduli space. The following presentation is based on [Y, Oz]. Let ω be a reducible connection on the bundle $L \oplus \bar{L}$ over X with $c_L = c_1(L)$. Assume that n is such that the neighborhood of $\omega \in \mathcal{M}_{c_L^2}(X)$ is modeled on $\mathbf{C}^n/S^1$. Then an open neighborhood of $\{\omega\} \times X$ in the compactified moduli space $\mathcal{M}_{c_L^2+1}(X)$ is homeomorphic to the bundle:

$$\mathbf{C}^n \underset{S^1_{ST}}{\times} \left(P_L \underset{X}{\times} (Fr_+ X \underset{SO(3)}{\times} cSO(3)\,) \Big/ U(1) \right) \tag{7.5}$$

By the symbol $P \underset{X}{\times} Q$ we have denoted the fibered product of two fibrations P and Q over X. (the pullback of Q over P.) Let P_L denote the principal S^1 bundle associated to L, on which $U(1)$ acts in a natural way from the right. The $U(1)$ action on $cSO(3)$ is a *square* of the right action of $S^1 \hookrightarrow SO(3)$ via the map

$$\exp(i\theta) \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

For any complex line bundle L we have $L \underset{\tilde{S}^1}{\times} SO(3) \cong L^2 \underset{S^1}{\times} SO(3)$, where the $\tilde{S}^1$ action on $SO(3)$ is the square of the S^1 action. Thus in what will follow we shall replace the fibration P_L with P_{L^2} getting the standard $U(1) = S^1$ actions on all bundles appearing in the formula. The stabilizer S^1_{ST} acts in the standard way on $\mathbf{C}^n$ and on the left on the space $P_L \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,) \big/ U(1)$. Thus in the neighborhood of $\{\omega\} \times X$ the bundle $M \to M/S^1$ used in Theorem 5.5 is:

$$\begin{aligned} M \;&=\; S\left(\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,)/S^1)\right) \to \\ &\to P\left(\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,)/S^1)\right) \end{aligned} \tag{7.6}$$

where the projection is the quotient by the S^1_{ST} action. In the above formula we have used the symbol "S" even though the fiber of the boundary of

$$\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,)/S^1)$$

is not a sphere, but $S^{2n-1} * SO(3)$, a space that is double covered by S^{2n+3}. For later computations we need to evaluate the powers of the first Chern class c_M of the fibration $M \to M/S^1_{\mathrm{ST}}$ on M/S^1.

When the $SO(3)$ fibration Fr_+X lifts to an $SU(2)$ fibration $\tilde{F}r_+X$ (i.e. when w_2 of the manifold X vanishes), then, following [Oz], we have an S^1 equivariant double cover map:

$$P_L \underset{X}{\times} (\tilde{F}r_+X \underset{SU(2)}{\times} cSU(2)\,)/S^1 \to P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,)/S^1$$

where S^1 acts in the standard way on $SU(2)$. The fibration

$$P_L \underset{X}{\times} (\tilde{F}r_+X \underset{SU(2)}{\times} cSU(2)\,)/S^1$$

can be *orientation reversing* identified with the sphere bundle of the complex vector bundle $L^{-1} \otimes \Lambda^2_{+,\mathbf{C}}$ with the complex orientation. For the whole space M/S^1 we have the following:

Lemma 7.7 (Compare Proposition 2 in [Oz]) *If X is a four manifold with $w_2 = 0$, then there is a fiberwise covering map p of degree 2^n*

$$p : P\left(\mathbf{C}^n \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})\right) \to P\left(\mathbf{C}^n \times (P_{L^2} \underset{X}{\times} (Fr_+X \underset{SO(3)}{\times} cSO(3)\,)/S^1)\right)$$

Moreover $p^*(c_M) = -2h_P$, where h_P is the positive generator of the fiber $\mathbf{C}P^{n+1}$ of $P\left(\mathbf{C}^n \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})\right) \to X$.

Proof: In order to make the lift an S^1 equivariant we need to lift the S^1 action on P_{L^2} and on each fator of $\mathbf{C}^n = c(\underbrace{S^1 * \cdots * S^1}_{n \text{ times}})$. This gives us a 2^{n+1} cover of the fibrations before projectivization. Dividing by the S^1 action "cancels" one factor of 2.

On the fibers the map p is

$$\begin{array}{ccc} S^{2n+3} & \longrightarrow & S^{2n-1} * SO(3) \\ \downarrow & & \downarrow \\ \mathbf{C}P^{n+1} & \xrightarrow{p} & (S^{2n-1} * SO(3))/S^1 \end{array}$$

By using technics similar to the desingularization of the moduli space (5.6) and Theorem 5.13 we get:

$$\langle c_M^{n+1}, (S^{2n-1} * SO(3))/S^1\rangle = \langle c_M^{n+1}, P(S^{2^C} \times SO(3)^C)\rangle = c_{SO(3)}[S^2] = 2$$

Hence

$$\langle p^*(c_M^{n+1}), \mathbf{C}P^{n+1}\rangle = 2^n \langle c_M^{n+1}, (S^{2n-1} * SO(3))/S^1\rangle = 2^{n+1}\langle \gamma^{n+1}, \mathbf{C}P^{n+1}\rangle$$

giving the second conclusion modulo the sign. The minus in this formula is due to the fact that the identification of 2^n cover space with $P(\mathbf{C}^n \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}}))$ is orientation reversing. □

Now we need to adjust the above results for the case of the space $N \cong \mathbf{C}P^n/\mathbf{Z}_p$.

7.2 Second stratum connections over the orbifold N

Let $\tau : \overline{\mathbf{C}P^2} \to N$ denote the quotient by $\mathbf{Z}_p$ action defined by $\tau[z_0 : z_1 : z_2] = [\exp(\frac{2\pi i}{p})z_0 : z_1 : z_2]$. This action has a fixed sphere $\{z_0 = 0\}$ and one fixed point $\{z_1 = z_2 = 0\}$. The quotient space N is not a manifold, having singularities at the fixed points of τ. Near the fixed point sphere N is diffeomorphic to $-p$ degree disc bundle over S^2, thus the singularity there can be resolved. The neighborhood of the fixed point is modelled on the open set in $\mathbf{R}^n$ divided by $\mathbf{Z}_p$ action with a single fixed point at the origin. Such spaces are called *orbifolds.* We will need the following properties of N:

Lemma 7.8 *Let $\sigma \in \mathrm{H}^2(N)$ be a cohomology class Poincare dual to the sphere with self-intersection $-p$. Also let $x \in \mathrm{H}^4(N)$ denote the generator of this group. Then we have*

$$\tau^*(\sigma) = p\gamma; \qquad \tau^*(x) = p\gamma^2$$

where $\gamma \in \mathrm{H}^2(\overline{\mathbf{C}P^2})$ is a generator of this group.

Proof: For the generator $x \in \mathrm{H}^4(N)$ we have $\tau^*(x) = p\gamma^2$, thus proving second equality. Also we have:

$$\langle \tau^*(\sigma^2), \overline{\mathbf{C}P^2} \rangle = p \cdot \langle \sigma^2, N \rangle = -p^2 = \langle (\pm p\gamma)^2, \overline{\mathbf{C}P^2} \rangle$$

Since $PD(\sigma)$ is a fixed by the action of τ, then we have the "+" sign in the first equality. $\square$

Over the orbifold N the line bundles L and the bundle $\Lambda^2_+(N)$ are defined as $\mathbf{Z}_p$ quotients of the corresponding bundles over $\overline{\mathbf{C}P^2}$. Thus if h_N denotes the class obtained by quotiening the generator h of the fiber $\mathbf{C}P^2$ of $P(\mathbf{C} \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})) \to \overline{\mathbf{C}P^2}$, then we have:

$$\langle h_N^3 \pi^*(\sigma), P(\mathbf{C} \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})) \rangle = \langle h_N \pi^*(\sigma), N \rangle = \frac{1}{p} \langle hp\gamma, \overline{\mathbf{C}P^2} \rangle = \langle h\gamma, \overline{\mathbf{C}P^2} \rangle \quad (7.9)$$

and

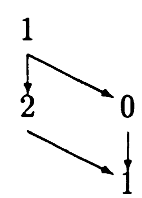
$$\langle h_N^2 \pi^*(x), P(\mathbf{C} \oplus (\bar{L} \otimes \Lambda^2_{+,\mathbf{C}})) \rangle = \langle \pi^*(x), N \rangle = \langle \gamma^2, \overline{\mathbf{C}P^2} \rangle \quad (7.10)$$

In particular this answer does not depend on the intersection number p. Now we are ready to come back to the computations of next examples.

Example 7.11 For the σ with the self-intersection -4, same as in the previous example, we want to increase the number of the classes σ appearing in the Donaldson invariant. According to Theorem 1.3 we expect the formula of the form:

$$D(z\sigma^6) = a_1 D(z\sigma^2 x^2) + a_2 D(z\sigma^4 x) + a_3 D(zx) + a_4 D(z\sigma^2) \quad (7.12)$$

Each term in the above formula corresponds to different open set represented on the diagram:



Solving the system of equations described in Theorem 6.2 gives us that the form

$$\phi^0 = \mu(\sigma)^6 - 5\mu(\sigma)^4 p(x_2) - (-4)\mu(\sigma)^2 p(x_1)^2$$

evaluates trivially on the sets $U^1_{1/4}$ and U^2_1. This shows that the coefficient $a_1 = -4 \cdot (-4)^2 = -64$, and $a_2 = 5 \cdot (-4) = -20$. Again it remains to evaluate ϕ^0 on U^0_1 and $U^1_{5/4}$. Similarly like in the previous example, in U^0_1 we have

$$\phi^0 = 4 \cdot \sigma^2[N] \cdot H[\Sigma SO(3)] \cdot p(x_1)[\mathcal{M}_k(Y,0)] = 4 \cdot (-4) \cdot 2 \cdot (-4) D(zx) = 128 D(zx)$$

This gives us the coefficient at $D(zx)$ with the sign determined like in the previous example. Since the graph $\mathcal{J}$ terminates on one level below the set U_1^0, then we also get $\phi^1 = -32p(x_2)$. In $U_{5/4}^1$ we have:

$$\mu(\sigma) = \pi^*(\sigma) - h; \qquad\qquad p(x_i) = -4\pi^*(x_i) + h^2$$

Thus:

$$\begin{aligned}
\phi^0 &= (\pi^*(\sigma) - h)^6 - 5\,(\pi^*(\sigma) - h)^4 \cdot \left(-4\pi^*(x_i) + h^2\right) + 4\,(\pi^*(\sigma) - h)^2 \cdot \\
&\quad \cdot \left(-4\pi^*(x_i) + h^2\right)^2 \\
&= h^6 - 6h^5\pi^*(\sigma) + 15h^4(\pi^*(\sigma))^2 - 5(h^4 - 4h^3\pi^*(\sigma) + 6h^2(\pi^*(\sigma))^2) \cdot \\
&\quad \cdot (-4\pi^*(x_i) + h^2) + 4\left((\pi^*(\sigma))^2 - 2\pi^*(\sigma)h + h^2\right) \cdot (h^4 - 8h^2\pi^*(x_i)) \\
&= h^5\pi^*(\sigma)\,(-6 + 20 - 8) + h^4\pi^*(x_i)\left(15\sigma^2 + 20 - 30\sigma^2 + 4\sigma^2 - 32\right) \quad (7.13) \\
&= 6h^5\pi^*(\sigma) + 32h^4\pi^*(x_i)
\end{aligned}$$

The set $U_{5/4}^1$ is $P(\mathcal{Q}_Y^C \times M^C)$, where M is the neighborhood of the connection $\omega \times N$ described in (7.6). Thus

$$\langle h^5\pi^*(\sigma), U_{5/4}^1\rangle = \langle c_Q, \mathcal{Q}_Y\rangle \cdot \langle h^3\pi^*(\sigma), M\rangle = D(\sigma^2) \cdot -\frac{1}{2} \cdot (-2)^3 \cdot \langle c_{F_r}\gamma, \overline{CP^2}\rangle$$

Here $c_{F_r} = c_1(\bar{L} \otimes \Lambda^2_{+,C}) = -2\gamma$ and the extra minus in the front of the whole formula is due to the fact that the identification of the 2^n cover of M and $\bar{L} \otimes \Lambda^2_{+,C}$ was orientation reversing. Thus the contribution of the term $6h^5\pi^*(\sigma)$ in $U_{5/4}^1$ is $48D(\sigma^2)$. Similar computations give $\langle h^4\pi^*(x), U_{5/4}^1\rangle = -2D(\sigma^2)$. The contribution of ϕ^1 in $U_{1/4}^1$ is equal to $D(\sigma^2)$, thus adding these terms together we get:

$$D(z\sigma^6) = -64D(z\sigma^2x^2) - 20D(z\sigma^4x) - 128D(zx) - 48D(z\sigma^2)$$

Example 7.14 As the other way of generalizing the above considerations we shall prove that for $\sigma \cdot \sigma = -6$ we have

$$D(z\sigma^6) = -192D(zx) - 64D(z\sigma^2x^2) - 20D(z\sigma^4x) - 108D(z\sigma^2) + 720D_\sigma(z)$$

The graph $\mathcal{J}$ for this situation is

$$\begin{tikzcd}
1 \arrow[d] \arrow[dr] & \\
2 \arrow[d] \arrow[dr] & 0 \arrow[d] \\
3 & 1
\end{tikzcd}$$

and as before the form $\phi^0 = \mu(\sigma)^6 - 5\mu(\sigma)^4 p(x_2) - (-4)\mu(\sigma)^2 p(x_1)^2$ evaluates trivially on the sets $U_{1/6}^1$ and $U_{4/6}^2$, giving the coefficients at $D(z\sigma^2x^2)$ and at $D(z\sigma^4x)$. In $U_{3/2}^3$ we have:

$$\phi^0 = h[S^2] \cdot (3^6 - 5 \cdot 3^4 + 4 \cdot 3^3) = 2(729 - 405 + 36) = 720$$

which gives the constant at $D_\sigma(z)$ and $\phi^1 = -48p(x_2)$. Repeating the reasoning from the previous example we get that $\phi^0 = 4\cdot(-6)\cdot 2\cdot(-4)D(zx) = 192D(zx)$ in U_1^0.

From (7.13) in $U^1_{7/6}$ we have:

$$\phi^0 = 6h^5\pi^*(\sigma) + 54h^4\pi^*(x_i)$$

As the pull–back over $\overline{CP^2}$ of the link of the reducible connection in $U^1_{7/6}$ does not depend on the self–intersection p of σ, we can repeat the computations from the previous example, obtaining:

$$\langle\phi^0, U^1_{7/6}\rangle + \langle\phi^1, U^1_{1/6}\rangle = (6\cdot 8 + 54\cdot(-2) - 48)\cdot D(\sigma^2) = -108D(\sigma^2)$$

□

BIBLIOGRAPHY

BIBLIOGRAPHY

[Au] David M. Austin, *SO(3) Instantons on* $L(p,q) \times \mathbf{R}$, J.Diff. Geom. **32** (1990), 383–413.

[B] Raoul Bott, "Lectures on K(X)", W. A. Benjamin, New York, 1969.

[D1] S. Donaldson, *Polynomial invariants for smooth 4-manifolds*, Topology **29** (1990), 257–315.

[D2] S. Donaldson, *The orientation of Yang–Mills moduli spaces and 4–manifold topology*, J. Diff. Geom. **26** (1987) 397–428.

[DK] S. Donaldson and P. Kronheimer, "The Geometry of Four Manifolds," Oxford Mathematical Monographs, Oxford University Press, Oxford, 1990.

[FL] R. Fintushel and T. Lawson, *Compactness of moduli spaces for orbifold instantons*, Topology and its Applications 23 (1986) 305 - 312.

[FS1] R. Fintushel and R. Stern, *Surgery in cusp neighborhoods and the geography of irreducible 4-manifolds*, Inventiones Mathematicae **117** (1994), 455–523.

[FS2] R. Fintushel and R. Stern, *Donaldson invariants of 4- manifolds with simple type*, to appear in J. Diff. Geom.

[FS3] R. Fintushel and R. Stern, *The blowup formula for Donaldson invariants* , to appear in Annals of Math.

[FM 1] R. Friedman and J. Morgan, *On the diffeomorphism type of certain algebraic surfaces. I*, J. Diff. Geom. **27** (1988), 297-369.

[FM 2] R. Friedman and J. Morgan, "Smooth Four–Manifolds and Complex Surfaces", Springer–Verlag, 1994.

[KoM] D. Kotschick and J. W. Morgan, $SO(3)$ *Invariants for 4- manifolds with* $b_2^+ = 1$, *II*, J. Diff. Geom. **39** (1994), 433–456.

[MM] J. Morgan and T. Mrówka, *On the gluing theorem for instantons on manifolds containing long neck*, preprint.

[MMR] John W. Morgan, Tomasz Mrówka and Daniel Ruberman, "The L^2 Moduli Space and Vanishing Theorem for Donaldson Polynomial Invariants", International Press, 1994

[M] Tomasz Mrówka, *A local Mayer-Vietoris principle for Yang-Mills moduli spaces*, PhD thesis, Univ. of California, Berkeley.

[Oz] Peter Ozsváth, *Some blowup formulas for $SU(2)$ Donaldson Polynomials*, J. Diff. Geom **40** (1994), 411–447.

[P] R. S. Palais, "Foundations of Global Non-linear Analysis", W.A. Benjamin Inc, 1968.

[T1] C. Taubes, *Donaldson - Floer theory for nontrivial circle bundles*, in preparation.

[T2] Clifford H. Taubes, "L^2 Moduli Spaces on 4-manifolds with Cylindrical Ends", International Press, 1994

[Y] HongJie Yang, *Transition functions and a blow–up formula of Donaldson polynomials*, PhD thesis, Columbia University.