This is to certify that the

dissertation entitled

SUBPOSETS OF
THE BOOLEAN ALGEBRA

presented by

Ping Zhang

has been accepted towards fulfillment
of the requirements for

Doctor Doctor degree in Philosophy Philosophy

Wei-Eihn Kuan
Major professor
Wei-Eihn Kuan

Date Aug. 1, 1995
August 1, 1995

SUBPOSETS OF THE BOOLEAN ALGEBRA

By

Ping Zhang

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1994

ABSTRACT

SUBPOSETS OF THE BOOLEAN ALGEBRA

BY

Ping Zhang

This work deals with two important subposets of the Boolean algebra.

The first subposet $Q_{n:k}$ is called truncated Boolean algebra, which consists of all subsets, whose cardinality is at least k together with the empty set. We first compute its Möbius function in various ways. Since $Q_{n:k}$ can be consudered as the intersection lattice of the k-equal subspace arrangements, we then compute its charateristic polynomial, $\chi(Q_{n:k}, t)$, by different methods. As a result, we obtain two different expressions for $\chi(Q_{n:k}, t)$. One of them has a nice form in the terms of the basis $(t-1)^i, i \geq 0$, for the polynomial ring.

The second subposet $Q_{n|k}$ is called the k-divisible Boolean algebra, which consists of all subsets whose cardinality is divisible by k together with the whole set. The generalized Euler number $E_{n|k}$ is the absolute value of Möbius function of $Q_{n|k}$. So $E_{n|k}$ counts the number of permutations of an n-set with all the descents in the position m, where m is divisible by k. The well known classic Euler number is a special case when $k = 2$. we study the arithmetic properties of the generalized Euler numbers and their q-analogs. We derive two different expressions for their recursions and obtain their divisibilities. We also provide new proofs of previousely known results already in the literature.

DEDICATION

To my parents

ACKNOWLEDGEMENTS

I would like to take this opportunity to thank my advisor Professor Bruce Sagan. However, I simply do not know how to express my gratitude in words. This work is a very small return for much given.

Special thanks to my mathematical uncle, Professor Edgar Palmer, for his encouragement and kindness. I owe him a great deal. I would also like to thank my doctoral committee, Bruce Sagan, Ronald Fintushel, Wei-Eihn Kuan, Edgar Palmer and Richard Phillips, for their time and support. Finally, I would like to thank, my mathematical brother, Chris Weiss, Ms. Dorothy Manderscheid, the head of math library, and my friends, Steven Plemmons and Laurie Church, for their generous help any time I asked.

Contents

Introduction

The history of partially ordered sets, or posets, and lattices begins in the nineteenth century. The subject was systematically developed in the 1930's, first in G. D. Birkhoff's work [4] and then by others. Although the Möbius function originated in several forms related to number theory, geometries, algebra, topology and combinatorics, the first version of the Möbius Inverse Theorem for posets was due to L. Weisner [44] in 1935. Shortly after Weisner, P. Hall independently rediscovered this theorem [27]. In 1939, M. Ward was able to generalized the Möbius Inverse Theorem [43]. Then in 1964 Rota began the first systematic study the Möbius functions of the posets within combinatorics [32]. He also established the connection between the Möbius function and the efficient enumeration of objects represented by posets. The combinatorial properties of the Möbius function provide a great deal of information regarding the structure of posets and related enumerative problems.

The characteristic polynomial of a lattice was also first considered by G. D. Birkhoff [5], and has been called the Birkhoff polynomial [39] and the Poincaré polynomial [13]. Since the characteristic polynomial of a lattice is the generating function for the Möbius function, much has been done to exploit the combinatorial and algebraic properties of this polynomial. For example, Stanley has produced a factorization theorem for the modular elements in a finite geometric lattice [38] and has also shown that the characteristic polynomial of a supersolvable lattice has only nonnegative integer roots [39]. In fact, much recent work has been related to finding conditions under which this polynomial has only integral roots. Raising interest in this topic stems, in fact, from the fact that a polynomial with real roots has a log concave coefficient sequence. If, in addition, the coefficients are positive items, they are unimodal [37].

The topic of hyperplane arrangements has developed rapidly in recent years. The combinatorial implications of this subject arise from a simple fact: An affine hyperplane cuts $\mathbf{R}^n$ into two connected regions. By introducing several hyperplanes, $\mathbf{R}^n$ can be partitioned into a number of bounded and unbounded regions. The problem of counting these regions dated back to to the mid-1800's. However, no satisfactory explanation or general formulas were produced until 1975 when Zaslavsky [45] first used the Möbius function of the intersection lattice $\mathcal{L}(\mathcal{A})$ (defined in Chapter 2) to enumerate the regions of the complement of a hyperplane arrangement. Zaslavsky's results illustrate the important roles played by intersection lattices, their Möbius functions, and their characteristic polynomials. Zaslavsky also established the theory of signed graphs and exploited the connection between the chromatic polynomials of these graphs and the characteristic polynomials of certain arrangements [47, 48]. More recently Blass and Sagan were able to generalize one of Zaslavsky's fundamental theorems [9] by demonstrating that both of these polynomials count a set of lattice points in $\mathbf{Z}^n$, This gives a surprising relationship between these two polynomials and the Ehrhart polynomial of a polytope [40, 9]. We will use this result and many others to computer Möbius functions and characteristic polynomials for various subsposets of the Boolean algebra.

The history of the Euler numbers can be traced all the way back to the eighteenth century. They posses many interesting number-theoretic properties that can also be interpreted in various combinatorial ways. D. André [1] showed that the coefficient of $x^n/n!$ in sec x + tan x, or the Euler number E_n, is the number of alternating permutations $a_1a_2\cdots a_n$ of $\{1,2,\cdots,n\}$, where alternating means $a_1 < a_2 > a_3 <$

$a_4 > \cdots$. This result was extended by Carlitz [11] to generalized Euler numbers, $E_{n|k}$, which count the permutations $a_1a_2\cdots a_n$ of $\{1, 2, \cdots, n\}$ such that $a_i > a_{i+1}$ if and only if i is divisible by k. In particular, the ordinary Euler number is the case when $k = 2$. Furthermore, Stanley [36] used a q-analog of the Euler number, to generalize this result. He has shown that they count the same permutations by weight, where the weight of a permutation with i inversions is q^i.

The divisibility properties of these numbers have also received much attention over the years. It is well-known that E_{2n+1} is divisible by 2^n and that $(n+1)E_{2n+1}$ is divisible by 2^{2n} but by no higher power of two [12, p.259]. The divisibility properties of Stanley's q-Euler numbers were studied by G. Andrews, I. Gessel, G. Viennot and D. Foata [2, 26, 19]. The odd integer $(n+1)E_{2n+1}/2^{2n}$ is called a Genocchi number. D. Dumont, G. Viennot and J. Francon have given nice combinatorial interpretations for the Genocchi numbers and the Euler numbers [16, 20]. Moreover, a formal power series extension of these numbers has also been investigated by Ranrianarivony, J. Zeng, D. Dumont [33, 18] and others. We will generalize some of these results to the number $E_{n|k}$ in this thesis.

Chapter 1

Definitions and Notation

In this section, we set up some definitions and notation. We will follow Stanley [40] as much as possible. Any terms not defined can be found described in Stanley's book.

A *partially ordered set*, or poset, P is a set together with a binary relation $\leq$ satisfying the following three axioms:

1. For all $x \in P, x \leq x$. (reflexivity)

2. If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)

3. If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

We use $x < y$ to mean $x \leq y$ and $x \neq y$, $y \geq x$ to mean $x \leq y$. We say $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable*. The element y covers x ($x \prec y$) if $x < z \leq y$ implies $z = y$.

We say that P has a minimal element $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $x \geq \hat{0}$ for all $x \in P$. Similarly, P has a maximal element $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$.

A *subposet* Q of P is a subset Q of P and a partial order of Q such that for any

$x, y \in Q$, we have $x \leq y$ in Q if and only if $x \leq y$ in P. A special type of subposet of P is the (closed) interval $[x, y] = \{z \in P : \quad x \leq z \leq y\}$, defined whenever $x \leq y$.

Two posets P, Q are *isomorphic* if there exists an order-preserving bijection $\eta : P \to Q$ whose inverse is also order-preserving; that is,

$$x \leq y \text{ in } P \quad \text{if and only if} \quad \eta(x) \leq \eta(y) \text{ in } Q.$$

A *chain* (or totally ordered set) C in P is a subset so that every two elements are comparable. So if the elements of C are $\{x_0, x_1, \ldots, x_k\}$ with $x_i \leq x_j$ when $i \leq j$, we can write C as

$$C : \quad x_0 < x_1 < \cdots < x_k.$$

The *length* of this chain is k. We say this chain C is *saturated* if we can write C as

$$C : \quad x_0 \prec x_1 \prec \cdots \prec x_k.$$

A *maximal chain* in a poset P is a saturated chain from a minimal element in P to a maximal element in P.

We say P is a *graded poset of rank n* if every maximal chain in P has the same length n. In this case, there is a unique rank function $\rho : P \to \{0, 1, \ldots, n\}$ such that

$$\rho(y) = \begin{cases} 0 & \text{if } x \text{ is a minimal element in } P \\ \rho(x) + 1 & \text{if } x \prec y. \end{cases}$$

Given a poset P, then the *Möbuis function* μ of P is defined recursively on intervals $[x, y]$ in P by

$$\mu(x, x) = 1 \text{ and } \mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) \text{ for all } x, y \in P.$$

If there is a possibility of confusion, we use μ_p to denote the Möbuis function of a poset P. Suppose that P is a finite poset that has a unique minimal element $\hat{0}$. For brevity, we let $\mu(x)$ denote $\mu(\hat{0}, x)$ and $\mu(P)$ denote $\mu_P(\hat{0}, \hat{1})$. The Möbius function of P can be equivalently defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -\sum_{y<x} \mu(y) & \text{if } x > \hat{0}. \end{cases} \tag{1.1}$$

It is well-known that if $\eta : P \to Q$ is an isomorphism between two finite posets P and Q, then $\mu(x) = \mu(\eta(x))$ for all $x \in P$. Also, the Möbuis function of the interval $[x, y]$ of P equals the restriction to $[x, y]$ of the Möbuis function of P.

The following result is fundamental [44] and a proof can be found in [32].

Theorem 1.0.1 (Möbius Inversion Theorem) *Let V be a vector space over a field K. Let P be a finite poset with $\hat{0}$. If f and $g : P \to V$ satisfying condition that $f(x) = \sum_{y \geq x} g(y)$ for all x in P, then $g(\hat{0}) = \sum_{y \geq \hat{0}} \mu(y) f(y)$.*

Let P be a finite graded poset with $\hat{0}$ and rank n. The *characteristic polynomial* of P is

$$\chi(P, t) = \sum_{x \in P} \mu(x)\ t^{n-\rho(x)}. \tag{1.2}$$

One uses the corank of x, rather than its rank, as the exponent on t so that the polynomial will be monic. Since the characteristic polynomial is just the generating function for the Möbius function, it is of fundamental importance.

A *lattice* is a poset $\mathcal{L}$ for which every pair $x, y \in \mathcal{L}$, has a least upper bound (or join) $x \vee y$ and a greatest lower bound (or meet) $x \wedge y$. Clearly, all finite lattices have a $\hat{0}$ and a $\hat{1}$.

For $n \in \mathbf{N}$, let Q_n be the poset of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion; that is,

$$x \leq y \text{ in } Q_n \text{ if and only if } x \subseteq y \text{ as sets.}$$

Then $x \wedge y = x \cap y$ and $x \vee y = x \cup y$ for all $x, y \in Q_n$. Hence Q_n forms a graded lattice with the minimal element $\hat{0} = \emptyset$, the empty set, and the maximal element $\hat{1} = \{1, 2, \ldots, n\}$. The rank function ρ of Q_n is $\rho(x) = |x|$, where $|\cdot|$ denotes the cardinality of x. Then Q_n is called the *Boolean algebra* of order n. It is well-known [40, 3.8.3] that

Theorem 1.0.2 *The Möbuis function of Q_n is*

$$\mu(x) = (-1)^{|x|}$$

for all $x \in Q_n$.

By the definition, the characteristic polynomial of Q_n is

$$\chi(Q_n, t) = \sum_{x \in Q_n} (-1)^{|x|} t^{n-|x|} = \sum_{|x|=k \geq 0} (-1)^k \binom{n}{k} t^{n-k} = (t-1)^n. \tag{1.3}$$

In this work, we study two classes of the subposets of the Boolean algebra Q_n: the truncated Boolean algebra and the k-divisible Boolean algebra. We denote these two subposets by $Q_{n:k}$ and $Q_{n|k}$, respectively.

We begin in Chapter 2 with the study of the truncated Boolean algebra. In the spirit of J. W. Moon [30], we first compute the möbius function of $Q_{n:k}$ in as many ways as possible. We then derive two forms of the characteristic polynomial of $Q_{n:k}$. In particular, after a review of the definitions and some premilinary materials related to subspace arrangements, we use a lattice point counting method due to Blass and Sagan to get this result.

In Chapter 3, we study the k-divisible Boolean algebra. We start by determining the Möbius function of $Q_{n|k}$. We then define the corresponding generalized Euler number to be the absolute value of this Möbius function and study its combinatorial properties. We derive two different recursions for these numbers. By using these recursions, we extend some well-known facts about Euler numbers (the case $k = 2$) to generalized Euler numbers. In Section 3, we introduce the q-Euler numbers defined by Stanley [40]. We establish combinatorially an explicit expression of the recursion for the q-Euler numbers. Using this recursion, we are able to obtain some nice divisibility properties of the q-Euler numbers and then generalize two q-divisibility theorems of Andrews and Gessel. Along the way, we also provide a few different proofs for known results already in the literature.

This work ends with some comments on related results, open questions and various conjectures.

Chapter 2

The Truncated Boolean Algebra

2.1 The Möbius Function of the Truncated Boolean Algebra

For k fixed, $1 \leq k \leq n$, let $Q_{n:k}$ be the set of all $S \subseteq \{1, 2, \ldots, n\}$ such that $|S| \geq k$ or $|S| = 0$. Ordering $Q_{n:k}$ by inclusion, we see that $Q_{n:k}$ is a poset. In fact, $Q_{n:k}$ is the subposet obtained from Q_n by eliminating all elements that have ranks $1, 2, \ldots, k-1$. For this reason, $Q_{n:k}$ is called the *truncated Boolean algebra.*

The Möbius function of $Q_{n:k}$ is well known. We first state the Möbius value $\mu(x)$ for elements from $Q_{n:k}$. Then we give various new algebraic and combinatorial proofs of this result. Our proofs illustrate various standard combinatorial techniques.

Theorem 2.1.1 *The Möbius function of $Q_{n:k}$ is*

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ (-1)^{m-k+1}\binom{m-1}{k-1} & \text{if } x > \hat{0} \text{ and } |x| = m \quad \text{where } k \leq m \leq n. \end{cases} \tag{2.1}$$

First, we give an algebraic proof of Theorem 2.1.1 simply using the definition of the Möbius function and the following lemma.

Lemma 2.1.2 *Let $P(x)$ be a polynomial of degree at most n with real coefficients. If $P(x)$ has more than n distinct real roots, then $P(x) \equiv 0$.*

First proof of theorem 2.1.1 : By the definition of the Möbius function of a poset, $\mu(\hat{0}) = 1$. Let $x \in Q_{n:k}$ with $|x| = k + i$ where $0 \le i \le n - k$. We proceed by induction on i. For $i = 0$, the result is trivial since $\mu(x) = -1$ if $|x| = k$. Assume that the result holds for all $0 \le i \le I$. Now let $x \in Q_{n:k}$ with $|x| = k + I + 1$. It is easy to see that

$$\begin{aligned} & |\{y \in Q_{n:k} :\ \hat{0} < y < x \text{ and } |y| = k + i \text{ where } 0 \le i \le I\ \}| \\ = & \binom{k+I+1}{k+i} = \binom{k+I+1}{I-i+1}. \end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned} \mu(x) &= -\sum_{\hat{0} \le y < x} \mu(y) = -1 - \sum_{i=0}^{I} \sum_{\substack{\hat{0} < y < x \\ |y| = k+i}} \mu(y) \\ &= -1 - \sum_{i=0}^{I} \binom{k+I+1}{I-i+1} (-1)^{i+1} \binom{k+i-1}{i}. \end{aligned}$$

Let

$$P(k) = (-1)^I \binom{k+I}{I+1} - \left\{ -1 - \sum_{i=0}^{I} \binom{k+I+1}{I-i+1} (-1)^{i+1} \binom{k+i-1}{i} \right\}.$$

It is enough to show that $P(k) \equiv 0$. First note

$$\binom{k+I}{I+1} = \frac{(k+I)(k+I-1)\cdots(k+1)k}{(I+1)\,!} \tag{2.2}$$

and

$$\binom{k+I+1}{I-i+1}\binom{k+i-1}{i} = \frac{(k+I+1)(k+I)\cdots(k+i+1)(k+i-1)\cdots(k+1)k}{(I-i+1)\,!\,i\,!} \tag{2.3}$$

are polynomials in k of degree $I + 1$. Hence if $P(k) \not\equiv 0$, then it is a polynomial of degree at most $I + 1$. Moreover, by equation (2.2), $\binom{-j+I}{I+1} = 0$ for all $0 \le j \le I$ and $\binom{-(I+1)+I}{I+1} = \binom{-1}{I+1} = (-1)^{I+1}$. Also by equation (2.3),

$$\sum_{i=0}^{I} (-1)^{i+1} \binom{-j+I+1}{I-i+1} \binom{-j+i-1}{i} = (-1)^{j+1} \binom{-j+I+1}{I-j+1} \binom{-j+j-1}{j} = -1$$

for all $0 \leq j \leq I$ and

$$\sum_{i=0}^{I}(-1)^{i+1}\binom{-(I+1)+I+1}{I-i+1}\binom{-(I+1)+i-1}{i} = 0.$$

It follows that $P(-j) = 0$ for all $0 \leq j \leq I+1$. By theorem 2.1.2, we have $P(k) \equiv 0$ and the result follows. ∎

The second proof of theorem 2.1.1 involves certain properties of Möbius functions of Eulerian posets which were originally studied by Stanley [41, Proposition 2.2].

A finite graded poset P with $\hat{0}$ and $\hat{1}$ is *Eulerian* if $\mu(x,y) = (-1)^{l(x,y)}$ for all $x \leq y$ in P, where $l(x,y)$ is the length of interval $[x,y]$ in P; that is, the length of a maximal chain in $[x,y]$. It is clear that the Boolean algebra Q_n is Eulerian. The following theorem is due to Stanley [40, p.137].

Theorem 2.1.3 (Stanley) *Let P be Eulerian of rank n, and let Q be any subposet of P containing $\hat{0}$ and $\hat{1}$. Set $\bar{Q} = (P \setminus Q) \cup \{\hat{0}, \hat{1}\}$. Then*

$$\mu(Q) = (-1)^{n-1}\mu(\bar{Q}).$$

The second proof of theorem 2.1.1: By the definition of Möbius functions of posets, $\mu_{Q_{n:k}}(\hat{0}) = 1$. Let $x \in Q_{n:k}$ with $|x| = m$, where $k \leq m \leq n$. Since the interval $[\hat{0}, x]$ in $Q_{n:k}$ is isomorphic to the poset $Q_{m:k}$, we can apply Theorem 2.1.3 to Q_m and $Q_{m:k}$. Let $\hat{0}$ and $\hat{1}$ denote the minimal and the maximal element of Q_m, respectively. It follows that $\bar{Q}_{m:k} = (Q_m \setminus Q_{m:k}) \bigcup \{\hat{0}, \hat{1}\}$ and then by Theorem 2.1.3,

$$\mu(Q_{m:k}) = (-1)^{m-1}\mu(\bar{Q}_{m:k}).$$

Moreover, if $y \in \bar{Q}_{m:k}, y \neq \hat{1}$, then $\mu_{\bar{Q}_{m:k}}(y) = \mu_{Q_m}(y) = (-1)^{|y|}$. It follows that

$$\mu(\bar{Q}_{m:k}) \quad = \quad -\sum_{y<\hat{1}} \mu_{\bar{Q}_{m:k}}(y)$$

$$
\begin{aligned}
&= -\sum_{j=0}^{k-1} |\{y \in \bar{Q}_{m:k} : \ |y| = j \ \}| \ (-1)^j \\
&= -\sum_{j=0}^{k-1}(-1)^j\binom{m}{j} = (-1)^k\binom{m-1}{k-1}.
\end{aligned}
$$

The last equality follows from identity (1.5) in [22, p.1]. Finally we have

$$
\begin{aligned}
\mu_{Q_{n:k}}(x) &= \mu(Q_{m:k}) \\
&= (-1)^{m-1}(-1)^k\binom{m-1}{k-1} \\
&= (-1)^{m-k+1}\binom{m-1}{k-1}
\end{aligned}
$$

as desired. ∎

The third proof of theorem 2.1.1 uses Stanley's results characterizing the Möbius function of a rank-selected poset in terms of the number of permutations with given descent set [42, Proposition 14.1]. A few preliminaries are required for this proof.

Let P be a finite graded poset with $\hat{0} \neq \hat{1}$ and $n = \rho(\hat{1})$ where ρ is P's rank function. Let $S \subseteq \{1, \ldots, n-1\}$ and then the corresponding *S-rank-selected subposet* of P is

$$P_S = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$$

with the same partial order as P. Now define $\alpha(P, S) = \alpha(S)$ to be the number of maximal $\hat{0} - \hat{1}$ chains of P_S. Then define $\beta(P, S) = \beta(S)$ by

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\alpha(T).$$

If μ_S denotes the Möbius function of the P_S, by proposition 14.1 in [42], we have

$$\beta(S) = (-1)^{|S|-1}\mu_S(\hat{0}, \hat{1}). \tag{2.4}$$

Let $\mathcal{C}(P)$ be the set of all pairs (x, y) of elements of P for which y covers x. A function $\lambda : \mathcal{C}(P) \to \mathbf{Z}$ is called an *R-labeling* of P if, for every interval $[x, y]$ of P, there is a unique saturated chain $x = x_0 < x_1 < \cdots < x_l = y$ satisfying

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{l-1}, x_l). \tag{2.5}$$

A poset P possessing an R-labeling λ is call an R-poset and the chain $x = x_0 < x_1 < \cdots < x_l = y$ satisfying (2.5) is called the increasing chain from x to y.

R. P. Stanley has shown [35, Theorem 3.1] the following theorem and a proof can be find in [40, Theorem 3.13.2].

Theorem 2.1.4 (Stanley) *Suppose that P is an R-poset with $\hat{0} \neq \hat{1}$ and $\rho(P) = n$. Let λ be an R-labeling of P, and let $S \subseteq \{1, 2, \ldots, n-1\}$. Then $\beta(P, S)$ equals the number of maximal chains $M : \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ of P for which the sequence $\lambda(M) := (\lambda(x_0, x_1), \ldots, \lambda(x_{n-1}, x_n)$ has descent set S; that is, for which*

$$\mathrm{Des}(\lambda(M)) := \{i : \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})\} = S.$$

Let $P = Q_n$ the Boolean algebra of rank n and $S \subseteq \{1, 2, \ldots, n-1\}$. If $H \prec T$, then let $\lambda(H, T)$ be the unique element of $T \setminus H$. So for any interval $[x, y]$ in Q_n, there is a unique increasing chain $x = x_0 < x_1 < \cdots < x_l = y$ defined by letting the sole element $x_i - x_{i-1}$ consist of the least integer contained in $y - x_{i-1}$. Hence λ is an R-labeling of Q_n and $\beta(Q_n, S)$ is the number of maximal $\hat{0} - \hat{1}$ chains M in Q_n such that $\mathrm{Des}(\lambda(M)) = S$.

Let $\mathcal{S}_n$ be the set of all permutations of $\{1, 2, ..., n\}$. We define the *descent set of* π to be

$$\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1} \text{ and } 1 \leq i \leq n-1 \ \}.$$

Note that for each maximal $\hat{0} - \hat{1}$ chain $M : x_0 < x_1 < \cdots < x_n$ in Q_n with $\mathrm{Des}(\lambda(M)) = S$, the sequence $\lambda(M)$ determines a permutation

$$\pi = \lambda(x_1, x_0)\lambda(x_2, x_1) \cdots \lambda(x_n, x_{n-1})$$

with $\mathrm{Des}(\pi) = S$. Conversely, for each $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ with $\mathrm{Des}(\pi) = S$, we see that π determines a maximal $\hat{0} - \hat{1}$ chain

$$M : \emptyset < \{\pi_1\} < \{\pi_1, \pi_2\} < \cdots < \{\pi_1, \pi_2, \cdots \pi_n\}$$

with $\mathrm{Des}(\lambda(M)) = S$. It is easy to check that it is a one to one and onto correspondence between the set of all maximal $\hat{0} - \hat{1}$ chains M in Q_n with $\mathrm{Des}(\lambda(M)) = S$ and the set of all permutations $\pi \in \mathcal{S}_n$ with $\mathrm{Des}(\pi) = S$. Hence $\beta(Q_n, S) = \beta_n(S)$ is the number of permutations of $\{1, 2, \ldots, n\}$ with descent set S.

Now we are in the position to give another proof of Theorem 2.1.1 using the idea of the R-labeling of a poset.

The third proof of theorem 2.1.1: Let $P = Q_n$ and $S = \{k, k+1, \ldots, n-1\}$. Then $P_S = Q_{n:k}$ and

$$\mu(Q_{n:k}) = (-1)^{|S|-1}\beta_n(S) = (-1)^{n-k-1}\beta_n(S)$$

where $\beta_n(S)$ is the total number of the permutations of $\{1, 2, \ldots, n\}$ with the descent set S. So it is enough to show that

$$\beta_n(S) = \binom{n-1}{k-1}.$$

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of $\{1, 2, \ldots, n\}$ with $\mathrm{Des}(\pi) = S$. Then π is built up as follows:

$$\pi_1 < \pi_2 < \cdots < \pi_{k-1} < \pi_k > \pi_{k+1} > \pi_{k+2} > \cdots > \pi_n.$$

Hence $\pi_k = n$. Since $\pi_i \neq n$ for all $1 \leq i \leq k-1$, there are $\binom{n-1}{k-1}$ ways to choose $\pi_1 < \pi_2 < \cdots < \pi_{k-1}$. Having chosen $\pi_1 < \pi_2 < \cdots < \pi_{k-1}$, since there is only one way to order the rest numbers, we have only one way to choose $\pi_k > \pi_{k+1} > \cdots > \pi_n$. It follows that $\beta_n(S) = \binom{n-1}{k-1}$.

Let $x \in Q_{n:k}$ with $|x| = m$. Since the interval $[\hat{0}, x]$ in $Q_{n:k}$ is isomorphic to $Q_{m:k}$, $\mu(x) = \mu(Q_{m:k}) = (-1)^{m-k-1}\binom{m-1}{k-1}$ as desired. ∎

2.2 Subspace Arrangements and the Intersection Lattices

A *central subspace arrangement* $\mathcal{A} = \{K_1, K_2, \ldots, K_m\}$ in the Euclidean space $\mathbf{R}^n$ is a finite collection of linear subspaces K_i of $\mathbf{R}^n$. Then $\mathcal{A}$ is a *hyperplane arrangement* if $\operatorname{codim} K_i = 1$ for all i.

The *intersection lattice* of a subspace arrangement, $\mathcal{L} = \mathcal{L}(\mathcal{A})$, is the poset of nonempty intersections of these subspaces ordered by reverse inclusion; that is,

$$x \leq y \quad \text{if and only if} \quad y \subseteq x.$$

Thus in $\mathcal{L}$, $\hat{0}$ corresponds to $\mathbf{R}^n$ and $\hat{1}$ corresponds to $\cap_{K\in\mathcal{A}} K$. Given two arrangements $\mathcal{A}$ and $\mathcal{B}$, we say $\mathcal{A}$ is *embedded* in $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{L}(\mathcal{B})$. The characteristic polynomial of $\mathcal{L}$ is

$$\chi(\mathcal{L}, t) = \sum_{x\in\mathcal{L}} \mu(x)\, t^{\dim(x)}. \tag{2.6}$$

Note that this charateristic polynomial differs from the one in definition (1.2) since $\mathcal{L}$ may not be graded, and even if it is, then the dimension and corank of x may not be the same. However, using the dimension often gives more interesting polynomials.

One of the most important combinatorial invariants of an arrangement is its characteristic polynomial. Zaslavsky has related the characteristic polynomials of certain arrangements to the chromatic polynomials of signed graphs [47, 48]. Blass and Sagan [9] have generalized one of Zaslavskly's results by showing that these two polynomials both count a certain set of lattice points in $\mathbf{Z}^n$, which provides us an efficient way to compute characteristic polynomials (see Theorem 2.7).

let $\mathbf{R}^n = \{(x_1, x_2, \cdots, x_n)\}$ be the Euclidean space of dimension n. For each

$i, 1 \leq i \leq n$, let H_i be the hyperplane $x_i = 0$ and

$$\mathcal{Q}_n = \{H_1, H_2, \cdots, H_n\}.$$

This hyperplane arrangement is called the *coordinate hyperplane arrangement* and the intersection lattice of $\mathcal{Q}_n$ is lattice isomorphic to the Boolean algebra Q_n. For this reason, we often write Q_n in place of $\mathcal{L}(\mathcal{Q}_n)$.

For fixed k, $1 \leq k \leq n$, let $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ and

$$K_I = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}.$$

Let $\mathcal{Q}_{n:k}$ be the set of all such K_I's so that $\mathcal{Q}_{n:k}$ is a subspace arrangement. It is easy to check that $\mathcal{L}(\mathcal{Q}_{n:k})$ is a lattice and it is lattice isomorphic to the truncated Boolean algebra $Q_{n:k}$. Also for this reason, we simply denote $\mathcal{L}(\mathcal{Q}_{n:k})$ by $Q_{n:k}$. Since $\mathcal{Q}_{n:k} \subseteq \mathcal{L}(\mathcal{Q}_n)$, we see that it can be considered as a subspace arrangement embedded in the coordinate hyperplane arrangement $\mathcal{Q}_n$.

2.3 The Characteristic Polynomial of the Truncated Boolean Algebra

We now consider the characteristic polynomial of $Q_{n:k}$. In this section, we will derive two forms of the characteristic polynomial of $Q_{n:k}$ in various ways, which involve the Blass-Sagan Theorem [9], hypergeometric series, generating functions, binomial coefficient identities, induction, etc.

Theorem 2.3.1 *The characteristic polynomial of $Q_{n:k}$ has the following two forms:*

$$\chi(Q_{n:k}, t) = \sum_{i=0}^{k-1} \binom{n}{i} (t-1)^{n-i}, \tag{2.7}$$

$$\chi(Q_{n:k}, t) = (t-1)^{n-k+1} \sum_{i=0}^{k-1} \binom{n-k+i}{i} t^{k-i-1}. \tag{2.8}$$

It is clear that $\chi(Q_{n:1},t) = \chi(Q_n,t)$. Also observe that if $n < k$, then $Q_{n:k} = \hat{0}$ and $\chi(Q_{n:k},t) = \mu(\hat{0})t^{n-0} = t^n$. Moreover, if $n < k$, then

$$\sum_{i=0}^{k-1} \binom{n}{i}(t-1)^{n-i} = \sum_{i=0}^{n} \binom{n}{i}(t-1)^{n-i} = t^n,$$

also

$$\begin{aligned} & (t-1)^{n-k+1} \sum_{i=0}^{k-1} \binom{n-k+i}{i} t^{k-i-1} \\ = \quad & (t-1)^{n-k+1} t^{k-1} \sum_{i=0}^{k-n-1} \binom{k-n-1}{i} \left(\frac{-1}{t}\right)^i \\ = \quad & (t-1)^{n-k+1} t^{k-1} \left(1 - \frac{1}{t}\right)^{k-n-1} = t^n. \end{aligned}$$

Much work has been devoted to finding conditions under which the characteristic polynomial of a lattice has only integral roots. This is true for Q_n but not in general for $Q_{n:k}$, $k \geq 2$. However, Theorem 2.3.1 shows that $\chi(Q_{n:k},t)$ at least factors partially over $\mathbf{Z}$, in particular that it is divisible by $(t-1)^{n-k+1} = \chi(Q_{n-k+1},t)$.

Furthermore, Equation (2.7) shows that one gets a nicer form for the coefficients of $\chi(Q_{n:k},t)$ when it is expanded using the basis $1, t-1, (t-1)^2, \cdots$ for the polynomial ring, rather than using the usual basis $1, t, t^2, \cdots$.

Since the truncated Boolean algebra can be considered as a subspace arrangement embedded in the coordinate hyperplane arrangement, we first give a combinatorial proof of Equation (2.7) using the Blass-Sagan interpretation of certain characteristic polynomials as counting a set of lattice points in $\mathbf{Z}^n$ [9].

Theorem 2.3.2 (Blass-Sagan) *Let*

$$\mathcal{B}_n = \{(x_i = \pm x_j), (x_k = 0)\}_{1 \leq i \leq j \leq n, 1 \leq k \leq n}$$

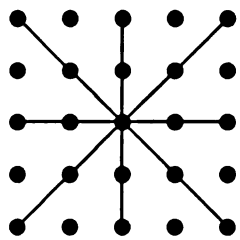
and let $\mathcal{A}$ be a subspace arrangement such that $\mathcal{A} \subseteq \mathcal{L}(\mathcal{B}_n)$. For $t = 2s+1$, define $[-s,s] = \{-s, -(s-1), \ldots, -1, 0, 1, \ldots, s\}$ and $C_t = [-s,s]^n$. Then $\chi(\mathcal{L}(\mathcal{A}),t) = |C_t \setminus \mathcal{A}|$.

The significance of Theorem 2.3.2 is that it provides us an efficient way to determine certain characteristic polynomials without even computing any Möbius functions. Before proving this theorem, we would like to give the readers an example to show intuitively what is going on. Consider $\mathcal{B}_2$ in $\mathbf{R}^2$ and C_5 in $\mathbf{Z}^2$. It is well-known [28] that

$$\chi(\mathcal{L}(\mathcal{B}_n),t) = (t-1)(t-3)\cdots(t-2n+1).$$

So $\chi(\mathcal{L}(\mathcal{B}_2),5) = (5-1)(5-3) = 8$.

On the other hand, let $\mathcal{A} = \mathcal{B}_2 = \{(x_1 = \pm x_2), (x_1 = 0), (x_2 = 0)\}$.

$Q_5 =$ 

We see that $|C_5 \setminus \mathcal{B}_2| = 8$ and then $|C_5 \setminus \mathcal{B}_2| = \chi(\mathcal{L}(\mathcal{B}_2),5)$.

The following proof is due to Blass and Sagan [9]. I am including their proof here for completeness.

Proof (of Theorem 2.3.2): For $X \in \mathcal{L}(\mathcal{A})$, let

$$f(X) = |X \cap C_t| \quad \text{and} \quad g(X) = |(X \cap C_t) \setminus (\cup_{Y>X} Y \cap C_t)|.$$

Given $X \in \mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B}_n)$, there is a one to one correspondence between $X \cap C_t$ and the cube of side t in $\mathbf{Z}^{\dim(X)}$ centered at 0. It follows that $f(X) = |X \cap C_t| = t^{\dim(X)}$. Clearly $f(X) = \sum_{Y \geq X} g(Y)$. So by Möbius Inversion Theorem 1.0.1

$$|C_t \setminus \mathcal{A}| = g(\mathbf{R}^n) = \sum_Y \mu(Y) f(Y) = \sum_Y \mu(Y) t^{\dim(Y)} = \chi(\mathcal{L}(\mathcal{A}),t)$$

and this ends the proof. ∎

A combinatorial proof of Equation (2.7): Since $\mathcal{Q}_{n:k} \subseteq \mathcal{L}(\mathcal{B}_n)$, by the Blass-Sagan Theorem, it is enough to show that

$$|C_t \setminus \mathcal{Q}_{n:k}| = \sum_{i=0}^{k-1} \binom{n}{i} (t-1)^{n-i}. \tag{2.9}$$

Note that $C_t \setminus Q_{n:k}$ consists of all $x = (x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n$ where $-s \leq x_i \leq s$ for all i and the number of zeros in $(x_1, x_2, \ldots, x_n)$ is at most $k-1$. This observation enables us to partition $C_t \setminus Q_{n:k}$ into k parts. For fixed i, $0 \leq i \leq k-1$, let

$$S_i = \{x = (x_1, x_2, \ldots, x_n) \in C_t \setminus Q_{n:k} : \text{the number of zeros in } (x_1, x_2, \ldots, x_n) \text{ is } i\ \}.$$

Then

$$\begin{aligned} |S_i| &= (\text{the number of ways to choose } i \text{ elements} \\ &\quad \text{from } \{x_1, x_2, \ldots, x_n\} \text{ as zero coordinates of } x) \\ &\quad \cdot(\text{the number of ways to choose the remaining } n-i \\ &\quad \text{non-zero coordinates of } x \text{ from } [-s, s] \setminus \{0\}) \\ &= \binom{n}{i} (t-1)^{n-i}. \end{aligned}$$

It is clear that $\{S_i\}_{0 \leq i \leq k-1}$ are mutually disjoint and

$$C_t \setminus Q_{n:k} = \bigcup_{i=0}^{k-1} S_i.$$

Hence

$$|C_t \setminus Q_{n:k}| = \sum_{i=0}^{k-1} |S_i| = \sum_{i=0}^{k-1} \binom{n}{i} (t-1)^{n-i}$$

as desired. ∎

Next, we will give an algebraic proof of Equation (2.7) which only involves the use of binomial identities and the definition of the characteristic polynomial of a poset.

An algebraic proof of Equation (2.7) : We start from the right hand side of Equation (2.7):

$$\sum_{i=0}^{k-1} \binom{n}{i} (t-1)^{n-i}$$

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} t^j (-1)^{n-i-j}$$

$$= \sum_{j=0}^{n-k+1} \left\{ \sum_{i=0}^{k-1} \frac{n!}{i!(n-i-j)!} (-1)^{n-i-j} \right\} \frac{t^j}{j!}$$

$$+ \sum_{j=n-k+2}^{n} \left\{ \sum_{i=0}^{n-j} \frac{n!}{i!\,(n-i-j)!} (-1)^{n-j-i} \right\} \frac{t^j}{j!}$$

$$= \sum_{j=0}^{n-k} (-1)^{n-j} \left\{ \sum_{i=0}^{k-1} (-1)^i \frac{(n-j)!}{i!(n-i-j)!} \right\} n(n-1)...(n-j+1) \frac{t^j}{j!}$$

$$+ \sum_{j=n-k+2}^{n-1} \left\{ \sum_{i=0}^{n-j} \frac{(n-j)!}{i!\,(n-i-j)!} (-1)^{n-j-i} \right\} n(n-1)...(n-j+1) \frac{t^j}{j!} + t^n$$

(the $j = n-k+1$ term in the first sum is the expansion of $(1-1)^{k-1} = 0$ and the $j = n$ term in the second sum is t^n)

$$= t^n + \sum_{j=0}^{n-k} (-1)^{n-j} \left\{ (-1)^{k-1} \binom{n-j-1}{k-1} \right\} n(n-1)...(n-j+1) \frac{t^j}{j!}$$

(applying identity (1.5) in [22, p.1] to each term in the first sum and the expansion of $(1-1)^{n-j} = 0$ to each term in the second one)

$$= t^n + \sum_{j=0}^{n-k} (-1)^{n-j-k+1} \frac{n(n-1)...(n-j+1)(n-j-1)!}{(k-1)!\,(n-j-k)!} \frac{t^j}{j!}$$

$$= \quad t^n + \sum_{p=0}^{n-k}(-1)^{p+1}\ \frac{n(n-1)...(k+p+1)(k+p-1)!}{(k-1)!\ p!}\ \frac{t^{n-k-p}}{(n-k-p)!}$$

(let $n-j-k=p$)

$$= \quad t^n + \sum_{p=0}^{n-k}(-1)^{p+1}\binom{n}{k+p}\frac{(k+p-1)...(k+1)k}{p!}\ t^{n-k-p}$$

$$= \quad t^n + \sum_{p=0}^{n-k}\binom{n}{k+p}(-1)^{p+1}\binom{k+p-1}{p}t^{n-k-p}.$$

On the other hand, the left hand side of Equation (2.7) is

$$\chi(Q_{n:k}, t)$$

$$= \quad \sum_{x\in Q_{n:k}} \mu(x)t^{\dim(x)}$$

$$= \quad \mu(\hat{0})t^n + \sum_{j=0}^{n-k}\quad \sum_{\dim(x)=j} \mu(x)t^j$$

$$= \quad t^n + \sum_{j=0}^{n-k}\binom{n}{j}(-1)^{n-k-j+1}\binom{n-j-1}{k-1}t^j$$

$\Big($ using Theorem 2.1.1 and $|\{x \in Q_{n:k}:\ \dim(x) = j\}| = \binom{n}{j}$ $\Big)$

$$= \quad t^n + \sum_{p=0}^{n-k}\binom{n}{k+p}(-1)^{p+1}\binom{k+p-1}{p}t^{n-k-p}$$

(let $n-j-k=p$).

Hence we have shown algebraically that Equation (2.7) holds. ∎

As for the proofs of Equation (2.8), we first give it a combinatorial proof using the Blass-Sagan Theorem; that is, count the number of elements in $C_t \setminus Q_{n:k}$ as we did in the combinatorial proof of Equation (2.7) except that in this case we partition $C_t \setminus Q_{n:k}$ in a different way.

A combinatorial proof of Equation (2.8) : For non-negative integers t and s with $t = 2s + 1$,we let $E = [-s, s]$ and $D = E \setminus \{0\}$. It is clear that $|E| = 2s + 1 = t$ and $|D| = 2s = t - 1$. Also we let $A \times B$ denote the product of two sets A and B and A^m denote the product of m copies A's where m is a non-negative integer.

By Theorem 2.3.2, it is enough to show that

$$|C_t \setminus Q_{n:k}| = (t-1)^{n-k+1} \sum_{i=0}^{k-1} \binom{n-k+i}{i} t^{k-i-1}. \tag{2.10}$$

First we give a outline of this proof. To prove equation (2.10), we construct a partition $\{A_0,\ A_1, \ldots\ A_{k-1}\}$ of $C_t \setminus Q_{n:k}$ such that each A_i is a disjoint union of $\binom{n-k+i}{i}$ sets, say $A_{i,1},\ A_{i,2}, \ldots, A_{i,\binom{n-k+i}{i}}$ with $|A_{i,j}| = t^{k-i-1}(t-1)^{n-k+1}$, where $0 \le i \le k-1$ and $1 \le j \le \binom{n-k+i}{i}$. In this way, we have

$$C_t \setminus Q_{n:k} = \biguplus_{i=0}^{k-1} A_i = \biguplus_{i=0}^{k-1} \left\{ \biguplus_{j=1}^{\binom{n-k+i}{i}} A_{i,j} \right\} \tag{2.11}$$

where $\biguplus$ denotes the disjoint union of sets. Then

$$\begin{aligned} |C_t \setminus Q_{n:k}| &= \sum_{i=0}^{k-1} |A_i| = \sum_{i=0}^{k-1} \left\{ \sum_{j=1}^{\binom{n-k+i}{i}} |A_{i,j}| \right\} \\ &= \sum_{i=0}^{k-1} \sum_{j=1}^{\binom{n-k+i}{i}} (t-1)^{n-k+1} t^{k-i-1} \\ &= (t-1)^{n-k+1} \sum_{i=0}^{k-1} \binom{n-k+i}{i} t^{k-i-1}. \end{aligned}$$

Now we start a detailed proof of equation (2.10) by constructing $\{A_{i,j}\}_{1 \le j \le \binom{n-k+i}{i}}$ for each i. Keep in mind that $C_t \setminus Q_{n:k}$ consists of all $x = (x_1, x_2, \ldots, x_n)$, such that $\{x_1, x_2, \ldots, x_n\} \subseteq [-s, s]$ and the number of zeros in $(x_1, x_2, \ldots, x_n)$ is at most $k-1$.

For fixed i, $0 \leq i \leq k-1$, take products of $n-k$ copies of D's and i copies $\{0\}$'s in all possible ways. Since the number of all such products is the number of ways of placing i 0-component's in the $n-k+i$ possible positions, we get a collection of $\binom{n-k+i}{i}$ sets, say $\{B_{i,j}\}_{1\leq j\leq \binom{n-k+i}{i}}$. Note that all the $B_{i,j}$ are mutually disjoint subsets in $\mathbf{R}^{n-k+i}$ and

$$|B_{i,j}| = |D^{n-k}||\{0\}| = (t-1)^{n-k}$$

which is independent of i and j. Moreover, if $x = (x_1, x_2, \ldots, x_{n-k+i}) \in B_{i,j}$, then $\{x_1, x_2, \ldots, x_{n-k+i}\} \subseteq [-s, s]$ and the number of zeros in $(x_1, x_2, \ldots, x_{n-k+i})$ is i.

Now for fixed i and j, $0 \leq i \leq k-1$ and $1 \leq j \leq \binom{n-k+i}{i}$, let $A_{i,j} = E^{k-i-1} \times D \times B_{i,j}$. It is clear that

$$|A_{i,j}| = |E^{k-i-1}||D||B_{i,j}| = t^{k-i-1}(t-1)^{n-k+1}$$

which is independent of j. Since all $B_{i,j}$'s are mutually disjoint subsets of $\mathbf{R}^{n-k+i}$, we see that all $A_{i,j}$'s are also mutually disjoint subsets of $\mathbf{R}^n$. Moreover, if $x = (x_1, x_2, \ldots, x_n) \in A_{i,j}$, then $\{x_1, x_2, \ldots, x_n\} \subseteq [-s, s]$ and the number of zeros in $(x_1, x_2, \ldots, x_n)$ is at most $(k-i-1)+i = k-1$. Hence $A_{i,j} \subseteq C_t \setminus Q_{n:k}$.

Now we define

$$A_i = \biguplus_{j=1}^{\binom{n-k+i}{i}} A_{i,j} \quad \text{for all } 0 \leq i \leq k-1$$

and then

$$|A_i| = \sum_{j=1}^{\binom{n-k+i}{i}} |A_{i,j}| = \binom{n-k+i}{i}|A_{i,1}| = \binom{n-k+i}{i} t^{k-i-1}(t-1)^{n-k+1}. \qquad (2.12)$$

We claim that $\{A_i\}_{0\leq i\leq k-1}$ are mutually disjoint. To prove this claim, we proceed by contradiction. Suppose not and let $x \in A_i \cap A_h$ for some $0 \leq i < h \leq k-1$. Then $x \in A_{i,j} \cap A_{h,l}$ for some $1 \leq j \leq \binom{n-k+i}{i}$ and $1 \leq l \leq \binom{n-k+h}{h}$. It follows that

(i) the number of zeros in $(x_{k-i}, \ldots, x_n)$ is i since $x \in A_i$, and $x_{k-i} \neq 0$;

(ii) the number of zeros in $(x_{k-h+1}, \ldots, x_n)$ is h since $x \in A_h$.

But the fact $|\{x_{k-h+1}, x_{k-h+2}, \ldots, x_{k-i-1}\}| = h - (i+1)$ implies, by (ii), that the number of zeros in $(x_{k-i}, \ldots, x_n)$ is at least $i+1$ which contradicts (i). Hence all A_i's are mutually disjoint.

Finally, we claim that $C_t \setminus Q_{n:k} = \bigcup_{i=0}^{k-1} A_i$. Since $A_{i,j} \subseteq C_t \setminus Q_{n:k}$, it follows that

$$A_i = \bigcup_{j=1}^{\binom{n-k+i}{i}} A_{i,j} \subseteq C_t \setminus Q_{n:k} \quad \text{and then} \quad \bigcup_{i=0}^{k-1} A_i \subseteq C_t \setminus Q_{n:k}.$$

On the other hand, let $x = (x_1, x_2, \ldots, x_n) \in C_t \setminus Q_{n:k}$. Since the number of zeros in $(x_1, x_2, \ldots x_n)$ is at most $k-1$, the number of non-zeros in $(x_1, x_2, \ldots, x_n)$ is at least $n-k+1$. Let x_m be the $(n-k+1)$-st non-zero coordinate in $\{x_1, x_2, \ldots, x_n\}$ counting from the right. Moreover, m is at most k since $|\{x_{m+1}, x_{m+2}, \ldots, x_n\}| = n-m$ is at least $n-k$ by the definition of x_m. If $k - m = i$, then we claim that $x \in A_i$. By the definition of x_m, the number of zeros in $(x_{m+1}, \ldots, x_n)$ is $k - m = i$. It follows that $(x_{m+1}, x_{m+2}, \ldots, x_n) \in B_{i,j}$ for some $1 \leq j \leq \binom{n-k+i}{i}$. Since $x_m \neq 0$ and $k - m = i$, we have $(x_1, x_2, \ldots, x_{m-1}, x_m) \in E^{m-1} \times D = E^{k-i-1} \times D$. So $x = (x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n) \in E^{k-i-1} \times D \times B_{i,j} = A_{i,j}$. This proves that $C_t \subseteq \bigcup_{i=0}^{k-1} A_i$.

Hence $C_t \setminus Q_{n:k} = \biguplus_{i=0}^{k-1} A_i$ and $|C_t \setminus Q_{n:k}| = \sum_{i=1}^{k-1} |A_i|$. Now the result follows from equation (2.12). ■

Next, we set up some definitions for the algebraic proofs of Equation (2.8).

The two most common types of generating functions are ordinary generating functions and exponential generating functions. The *ordinary generating function* of a sequence $\{a_n\}_{n\geq 0}$ is the formal power series

$$\sum_{n\geq 0} a_n x^n,$$

while the *exponential generating function* of $\{a_n\}_{n\geq 0}$ is the formal power series

$$\sum_{n\geq 0} a_n x^n/n!.$$

For information about generating functions, see Stanley's book [40]

If $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$ are constants, then we can form the *hypergeometric series*

$$ {}_pF_q\left[\begin{matrix} a_1, & a_2, & \ldots, & a_p \\ & b_1, & \ldots, & b_q \end{matrix} \middle| \, x \right] := \sum_{k\geq 0} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{x^k}{k!} \tag{2.13}$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is a rising factorial with respect to k. For information about such functions, see the books of Bailey [3] or Slater [34]. We also need the following result known as the Chu-Vandermonde Theorem [34, p.28].

Theorem 2.3.3 (Chu-Vandermonde) *If n is a positive integer, then*

$$ {}_2F_1\left[\begin{matrix} -n, & a \\ & -n+b \end{matrix} \middle| \, 1 \right] = \frac{(1-b+a)_n}{(1-b)_n} \tag{2.14}$$

Proof. See [34, p.28]. ■

Now we are ready to give algebraic proofs of Equation (2.8) by hypergeometric series and generating functions, respectively. We assume that Equation (2.7) is given. First factor out $(t-1)^{n-k+1}$ from $\sum_{i=0}^{k-1}\binom{n}{i}(t-1)^{n-i}$, and then expand $(t-1)^{k-i-1}$. We find

$$\begin{aligned}\sum_{i=0}^{k-1}\binom{n}{i}(t-1)^{k-i-1} &= \sum_{i=0}^{k-1}\binom{n}{i}\left\{\sum_{j=0}^{k-i-1}\binom{k-i-1}{j}t^j(-1)^{k-i-1-j}\right\} \\ &= \sum_{j=0}^{k-1}\left\{\sum_{i=0}^{k-j-1}\binom{n}{i}\binom{k-i-1}{j}(-1)^{k-i-1-j}\right\}t^j.\end{aligned}$$

The coefficient of t^j is

$$\sum_{i=0}^{k-j-1}\binom{n}{i}\binom{k-i-1}{j}(-1)^{k-i-1-j}.$$

Let $k-j-1=h\geq 0$. We see that the coefficient of $t^j=t^{k-h-1}$ is

$$(-1)^h\sum_{i=0}^{h}(-1)^i\binom{n}{i}\binom{k-i-1}{h-i}.$$

Now we let

$$g(h)=\sum_{i=0}^{h}(-1)^i\binom{n}{i}\binom{k-i-1}{h-i}=\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}\binom{k-i-1}{h-i}.$$

It remains to show that

$$g(h)=(-1)^h\binom{n-k+h}{h}\qquad\text{for all } h\geq 0. \tag{2.15}$$

1. We first prove (2.15) by using hypergeometric series. To do this, we need first to express the binomial coefficients in terms of rising factorials:

$$\binom{n}{i}=(-1)^i\frac{(-n)_i}{i!}\quad\text{and}\quad\binom{k-i-1}{h-i}=\frac{(-1)^h\,(-k+1)_h\,(-h)_i}{h\,!\,(-k+1)_i}.$$

Then we see that

$$\begin{aligned}
g(h) &= \sum_{i=0}^{h}(-1)^i\binom{n}{i}\binom{k-i-1}{h-i}\\
&= \frac{(-1)^h\,(-k+1)_h}{h!}\left\{\sum_{i=0}^{\infty}\frac{(-n)_i(-h)_i}{(-k+1)_i}\frac{1}{i!}\right\}\\
&= \frac{(-1)^h(-k+1)_h}{h!}\left\{{}_2F_1\left[\begin{matrix}-n,\ -h\\ -k+1\end{matrix}\,\middle|\,1\right]\right\}
\end{aligned}$$

(by the definition of hypergeometric series)

$$= \frac{(-1)^h(-k+1)_h}{h!}\frac{(-n+k-h)_n}{(-n+k)_n}$$

(using equation (2.14) with $a=-h$ and $b=n-k+1$)

$$= \frac{(-1)^h(n-k+1)_h}{h!}=(-1)^h\binom{n-k+h}{h}.\qquad\blacksquare$$

2. Next we prove (2.15) using generating functions. Consider the generating function $G(x)$ of $\{g(h)\}_{h\geq 0}$.

$$\begin{aligned}
G(x) &= \sum_{h=0}^{\infty} g(h)x^h \\
&= \sum_{i=0}^{\infty}(-1)^i\binom{n}{i}\left\{\sum_{h=i}^{\infty}\binom{k-i-1}{h-i}x^h\right\} \\
&= \sum_{i=0}^{\infty}(-1)^i x^i\binom{n}{i}\left\{\sum_{p=0}^{\infty}\binom{k-i-1}{p}x^p\right\} \\
&= \sum_{i=0}^{\infty}(-1)^i\, x^i\binom{n}{i}(1+x)^{k-i-1} \\
&= (1+x)^{k-1}\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}\left(\frac{x}{1+x}\right)^i \\
&= (1+x)^{k-1}\left(1-\frac{x}{1+x}\right)^n \\
&= \frac{1}{(1+x)^{n-k+1}} = \sum_{h=0}^{\infty}\binom{n-k+h}{h}(-1)^h x^h.
\end{aligned}$$

It follows that $g(h) = (-1)^h\binom{n-k+h}{h}$ for all $h \geq 0$. Then the coefficient of t^{k-h-1} is $(-1)^h g(h) = \binom{n-k+h}{h}$ as desired. ∎

We conclude this section by giving the exponential generating function of the characteristic polynomial of $Q_{n:k}$.

Theorem 2.3.4 *The exponential generating function of* $\{\chi(Q_{n:k}, t)\}_{n\geq 0}$ *is*

$$G_k(x,t) = \left\{\sum_{i=0}^{k-1}\frac{x^i}{i!}\right\}e^{x(t-1)} \tag{2.16}$$

where k is a fixed non-negative integer.

Proof. By the definition, we have

$$G_k(x,t) = \sum_{n=0}^{\infty}\left\{\sum_{i=0}^{k-1}\binom{n}{i}(t-1)^{n-i}\right\}\frac{x^n}{n!}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} \sum_{n=i}^{\infty} \binom{n}{i} (t-1)^{n-i} \frac{x^n}{n!} \\
&= \sum_{i=0}^{k-1} \left\{ \sum_{n=i}^{\infty} \frac{[(t-1)x]^{n-i}}{(n-i)!} \right\} \frac{x^i}{i!} \\
&= e^{x(t-1)} \left\{ \sum_{i=0}^{k-1} \frac{x^i}{i!} \right\}
\end{aligned}
$$

as desired. ∎

Chapter 3

The k-divisible Boolean Algebra

Let n and k be positive integers with $1 \leq k \leq n$. Define $Q_{n|k}$ be the set of all subsets $x \subseteq \{1, 2, ..., n\}$ such that k divides $|x|$ or $|x| = n$. Ordering $Q_{n|k}$ by inclusion, we see that $Q_{n|k}$ is a subposet obtained from Q_n by eliminating all elements that have ranks not divisible by k, except $\{1, 2, \ldots, n\}$ if $k \not| \, n$. So $Q_{n|k}$ is called the *k-divisible Boolean algebra*. It is clear that $Q_{n|1} = Q_n$.

3.1 The Möbius Function of the k-divisible Boolean Algebra

The main theorem in this section concerns the Möbius function of $Q_{n|k}$. To state this theorem, we let $\lceil \ \rceil$ and $\lfloor \ \rfloor$ denote the ceiling and floor functions (round up and round down), respectively. Clearly, if $x \in Q_{n|k}$, then either $|x| = mk$ for $0 \leq m \leq \left\lfloor \frac{n}{k} \right\rfloor$, or $|x| = n$.

Theorem 3.1.1 *The Möbius function of $Q_{n|k}$ is*

$$\mu(x) = \begin{cases} \displaystyle\sum_{r=0}^{m} (-1)^r \sum_{\substack{j_1+j_2+\cdots+j_r=m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1k, j_2k, \ldots, j_rk}, & \text{if } |x| = mk\ ; \\ \\ \displaystyle\sum_{r=0}^{\lfloor \frac{n}{k} \rfloor} (-1)^{r+1} \sum_{h=r}^{\lfloor \frac{n}{k} \rfloor} \sum_{\substack{j_1+j_2+\cdots+j_r=h \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{n}{n-hk, j_1k, j_2k, \ldots, j_rk}, & \text{if } |x| = n \text{ and } k \nmid n. \end{cases} \tag{3.1}$$

We will give both algebraic and combinatorial proofs for Theorem 3.1.1. Our algebraic proof of this theorem simply uses the definition of the Möbius function, induction, binomial identities, etc.

An algebraic proof of Theorem 3.1.1: For any $x \in Q_{n|k}$, we see that either $|x| = mk$ for some non-negative m, or $|x| = n$ with $k \nmid n$. We will consider these two cases separately.

Case 1: Suppose that $|x| = mk$ and induct on m. If $m = 0$, then $x = \hat{0}$ and $\mu(\hat{0}) = 1$ by the definition. Also $(-1)^0(0!/p) = 1$ where p is an empty product. So the result holds when $m = 0$. Next, by the definition,

$$\mu(x) = -\sum_{\substack{y < x \\ y \in Q_{n|k}}} \mu(y)$$

$$= -\sum_{h=0}^{m-1} |\{y \in Q_{n|k}: |y| = hk \text{ and } y < x\}|\ \mu(y)$$

$$= -\sum_{h=0}^{m-1} \binom{mk}{hk} \sum_{r=0}^{h} (-1)^r \sum_{\substack{j_1+j_2+\cdots+j_r=h \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{hk}{j_1k,\ j_2k,\ \ldots,\ j_rk}$$

(using the induction hyperthesis)

$$= \sum_{r=0}^{m-1} (-1)^{r+1} \sum_{h=r}^{m-1} \sum_{\substack{j_1+j_2+\cdots+j_r=h \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{(m-h)k,\ j_1k,\ j_2k, \ldots,\ j_rk}$$

$$= \sum_{r=0}^{m-1} (-1)^{r+1} \sum_{\substack{j_1+j_2+\cdots+j_r+j_{r+1}=m \\ j_i \geq 1 \\ 1 \leq i \leq r+1}} \binom{mk}{j_1k,\ j_2k,\ \ldots, j_rk,\ j_{r+1}k}$$

(putting $j_{r+1} = m - h$)

$$= \sum_{r=1}^{m} (-1)^r \sum_{\substack{j_1+j_2+\cdots+j_r=m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1k,\ j_2k,\ \ldots,\ j_rk}$$

(shifting the index r by 1)

$$= \sum_{r=0}^{m} (-1)^r \sum_{\substack{j_1+j_2+\cdots+j_r=m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1k,\ j_2k,\ \ldots,\ j_rk}.$$

Note that we get the last equality by adding the $r = 0$ term into the sum. We can do so since if $r = 0$ and $m > 0$, then

$$\sum_{\substack{j_1 + j_2 + \cdots + j_r = m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1 k,\ j_2 k,\ \ldots,\ j_r k}$$

is an empty sum and hence is 0.

Case 2: Suppose that $|x| = n$ with $k \nmid n$. By the definition,

$$\mu(\hat{1}) = - \sum_{\substack{y < \hat{1} \\ y \in Q_{n|k}}} \mu(y)$$

$$= - \sum_{h=0}^{\lfloor \frac{n}{k} \rfloor} |\{y \in Q_{n|k} : |y| = hk\}| \ \mu(y)$$

$$= - \sum_{h=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n}{hk} \sum_{r=0}^{h} (-1)^r \sum_{\substack{j_1 + j_2 + \cdots + j_r = h \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{hk}{j_1 k,\ j_2 k, \ldots,\ j_r k}.$$

(applying case 1 to each y)

$$= \sum_{r=0}^{\lfloor \frac{n}{k} \rfloor} (-1)^{r+1} \sum_{h=r}^{\lfloor \frac{n}{k} \rfloor} \sum_{\substack{j_1 + j_2 + \cdots + j_r = h \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{n}{n - hk, j_1 k,\ j_2 k, \ldots,\ j_r k}$$

as desired. This ends our algebraic proof. ∎

Our combinatorial proof of Theorem 3.1.1 employs an important way of computing the Möbius function due to Phillip Hall [27].

Theorem 3.1.2 (Phillip Hall) *Let P be a finite poset with $\hat{0}$ and $\hat{1}$. Let $x \in P$ and c_r be the number of chains $\hat{0} = x_0 < x_1 < \cdots < x_r = x$ of length r between $\hat{0}$ and x. Thus $c_0 = 0$ and $c_1 = 1$. Then*

$$\mu(x) = c_0 - c_1 + c_2 - c_3 + \cdots.$$

A proof of this theorem can be found in [40, p.119].

The significance of Theorem 3.1.2 is that it shows that $\mu(P)$ can be interpreted as an Euler characteristic and thus links Möbius inversion with algebraic topology.

First combinatorial proof of Theorem 3.1.1: First consider the $|x| = mk$ case. Since a maximal $\hat{0} - x$ chain in $Q_{n|k}$ has length m, it follows from Theorem 3.1.2 that

$$\mu(x) = \sum_{r=0}^{m}(-1)^r c_r \tag{3.2}$$

where c_r is the number of chains $\hat{0} = x_0 < x_1 < \cdots < x_r = x$ of length r in $Q_{n|k}$. So it is enough to show that

$$c_r = \sum_{\substack{j_1 + j_2 + \cdots + j_r = m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1 k,\ j_2 k,\ \ldots,\ j_r k} \quad \text{for all } 0 \leq r \leq m. \tag{3.3}$$

First assume that $r = 0$. In this case, if $m = 0$, then $c_0 = 1$ and the sum is $0!/p = 1$, where p is an empty product. If $m > 0$, then $c_0 = 0$ and the sum is 0 since it is an empty sum. So the result holds when $r = 0$.

Next assume that $r > 0$. Let $C : \hat{0} = x_0 < x_1 < \cdots < x_{r-1} < x_r = x$ be a $\hat{0} - x$ chain with fixed $|x_i| = h_i k$ for all i. Clearly, $0 = h_0 < h_1 < \cdots < h_{r-1} < h_r = m$ and the number of such chains C with fixed h_i's is

$$\prod_{i=1}^{r}(\text{the number of ways to choose } x_i)$$

$$= \prod_{i=1}^{r} \begin{pmatrix} \text{the number of ways to choose } h_i k - h_{i-1} k \\ \text{elements from a } (mk - h_{i-1}k)\text{-element set} \end{pmatrix}$$

$$= \prod_{i=1}^{r} \binom{mk - h_{i-1}k}{h_i k - h_{i-1}k} = \binom{mk}{j_1 k,\ j_2 k,\ \ldots,\ j_{r-1}k,\ j_r k},$$

where we assume that $j_i = h_i - h_{i-1} > 0$ for all i. Then Equation (3.3) follows by taking sum over all possible $0 = j_0 < j_1 < \cdots < j_r \leq m$ such that $j_1 + j_2 + \cdots + j_r = m$ and all $j_i > 0$.

Now suppose $|x| = n$ with $k \not| \, n$. Then $c_0 = 0$ and a maximal $\hat{0} - \hat{1}$ chain has length $\lceil \frac{n}{k} \rceil$. By Theorem 3.1.2 ,

$$\mu(\hat{1}) = \sum_{r=1}^{\lceil \frac{n}{k} \rceil} (-1)^r c_r = \sum_{r=0}^{\lfloor \frac{n}{k} \rfloor} (-1)^{r+1} c_{r+1},$$

where c_{r+1} is the number of $\hat{0} - \hat{1}$ chains of length $r+1$. It is enough to show that

$$c_{r+1} = \sum_{h=r}^{\lfloor \frac{n}{k} \rfloor} \sum_{\substack{j_1 + j_2 + \cdots + j_r = h \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{n}{n - hk,\ j_1 k,\ j_2 k, \ldots, j_r k} \qquad \text{for all } r\ .$$

Following the same lines as in the $|x| = mk$ case, we let $C : \hat{0} = x_0 < x_1 < \cdots < x_{r-1} < x_r < x_{r+1} = \hat{1}$ be a $\hat{0} - \hat{1}$ chain in $Q_{n|k}$ with fixed $|x_i| = h_i k$ for all i. Then the number of such chains C is

$$\prod_{i=1}^{r} (\text{the number of ways to choose } x_i) \ = \ \binom{n}{j_1 k,\ j_2 k, \ldots, j_{r-1}k,\ j_r k,\ n - h_r k}$$

where each $j_i = h_i - h_{i-1} > 0$, $r \leq h_r \leq \lfloor \frac{n}{k} \rfloor$ and $j_1 + j_2 + \cdots + j_r = h_r$. Now the result follows by substituting h_r by h and taking sum over all possible such $j_1, \cdots, j_r$ and h. ■

The second combinatorial proof of Theorem 3.1.1 uses one of Stanley's results which characterizes the Möbius function of a rank-selected poset in terms of the number of permutations with a certain descent set [42, Proposition 14.1].

The Second Combinatorial Proof of Theorem 3.1.1: Let

$$S = \{k, 2k, \ldots, \left\lfloor \frac{n}{k} \right\rfloor k\} \subseteq \{1, 2, \ldots, n-1\}.$$

Then $Q_{n|k}$ is an S-rank-selected subposet of Q_n. By Theorem 2.1.4, we see that

$$\mu(Q_{n|k}) = (-1)^{|S|-1}\beta_n(S).$$

where $\beta_n(S)$ is the total number of the permutations of $\{1, 2, \ldots, n\}$ with the descent set S. So it remains to find $\beta_n(S)$.

Let $\alpha_n(S)$ be the number of permutations $\pi \in S_n$ with $\mathrm{Des}(\pi) \subseteq S$. Then

$$\alpha_n(S) = \sum_{T \subseteq S} \beta_n(T). \tag{3.4}$$

By the Principle of Inclusion-Exclusion [32], we have

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha_n(T).$$

If $|T| = 0$, then $\alpha_n(T) = 1$. If $|T| = r \geq 1$, then let

$$T = \{1 \leq h_1 k < h_2 k < \cdots < h_r k \leq n-1\} \subseteq S.$$

To obtain a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ satisfying $\mathrm{Des}(\pi) \subseteq S$, first choose $\pi_1 < \pi_2 < \cdots < \pi_{h_1 k}$ in $\binom{n}{h_1 k}$ ways. Then choose $\pi_{h_1 k+1} < \pi_{h_1 k+2} < \cdots < \pi_{h_2 k}$ in $\binom{n-h_1 k}{h_2 k - h_1 k}$ ways and so on. From this we obtain

$$\begin{aligned} \alpha_n(T) &= \binom{n}{h_1 k}\binom{n-h_1 k}{h_2 k - h_1 k}\cdots\binom{n-h_{r-1}k}{h_r k - h_{r-1}k}\binom{n-h_r k}{n-h_r k} \\ &= \binom{n}{h_1 k,\ (h_2-h_1)k, \cdots, (h_r - h_{r-1})k,\ n-h_r k}. \end{aligned}$$

There are two different cases here. So we will consider them sparetely.

Case 1: If $k|n$, then let $n = mk$. Since the $m = 0$ case is trivial, we may assume that m is a positive integer. Then $S = \{k, 2k, \ldots, (m-1)k\}$ and $|S| = m-1$. So we

have

$$\alpha_n(T) = \binom{mk}{h_1k,\ (h_2-h_1)k, \cdots,\ (h_r-h_{r-1})k,\ (m-h_r)k}.$$

Let $j_1 = h_1, j_i = h_i - h_{i-1}$ where $2 \leq i \leq r \leq m-1$, and $j_{r+1} = m - h_r$. Then all $j_i > 0$ and $j_1 + j_2 + \cdots + j_{r+1} = m$. By Equation (3.4), we see that

$$\beta_n(S) = (-1)^{m-1} + \sum_{r=1}^{m-1}(-1)^{m-1-r} \sum_{\substack{j_1+j_2+\cdots+j_{r+1} = m \\ j_i \geq 1 \\ 1 \leq i \leq r+1}} \binom{mk}{j_1k,\ j_2k, \ldots, j_rk,\ j_{r+1}k}.$$

Then it follows that

$$\begin{aligned}
\mu(Q_{n|k}) &= (-1)^{(m-1)-1}\beta_n(S) \\
&= -1 + \sum_{r=1}^{m-1}(-1)^{r+1} \sum_{\substack{j_1+j_2+\cdots+j_r = m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1k,\ j_2k, \ldots, j_rk,\ j_{r+1}k} \\
&= \sum_{r=1}^{m}(-1)^r \sum_{\substack{j_1+j_2+\cdots+j_{r+1} = m \\ j_i \geq 1 \\ 1 \leq i \leq r+1}} \binom{mk}{j_1k,\ j_2k, \ldots,\ j_rk}
\end{aligned}$$

(shifting the index r by 1 and then putting -1 as the $r = 1$ term)

$$= \sum_{r=0}^{m}(-1)^r \sum_{\substack{j_1+j_2+\cdots+j_r = m \\ j_i \geq 1 \\ 1 \leq i \leq r}} \binom{mk}{j_1k,\ j_2k, \ldots, j_rk}$$

(adding the $r = 0$ term that is 0 if $m > 0$).

So the result holds in case 1.

Case 2: Suppose $k \not| n$. If $0 < n < k$, then $\mu(Q_{n|k}) = 1$. Also, the sum has only $r = 0$ term which equals to 1. So we are done. Now suppose $n > k$. Let $m = \left\lfloor \frac{n}{k} \right\rfloor$ and then $S = \{k, 2k, \ldots, m\} \subseteq \{1, 2, \ldots, n-1\}$. If $T \subseteq S$, then

$$\alpha_n(T) = \binom{n}{j_1k,\ j_2k,\ \ldots,\ j_rk,\ n - j_rk}$$

where $j_1 + j_2 + \cdots + j_r = h_r$ and $r \leq h_r \leq m$. It follows that

$$\beta_n(S) = \sum_{r=1}^{m}(-1)^{m-1-r}\sum_{h_r=r}^{m}\sum_{\substack{j_1+j_2+\cdots+j_r=h_r\\ j_i\geq 1\\ 1\leq i\leq r}}\binom{n}{j_1k,\ j_2k,\ldots,j_rk,\ n-h_rk}.$$

Changing the notation h_r to h, we have

$$\mu(Q_{n|k}) = \sum_{r=0}^{m}(-1)^{r+1}\sum_{h=r}^{m}\sum_{\substack{j_1+j_2+\cdots+j_r=h\\ j_i\geq 1\\ 1\leq i\leq r}}\binom{n}{j_1k,\ j_2k,\ldots,j_rk,\ n-hk}.$$

We add the $r = 0$ term into the sum to include the $m = 0$ case.

Let $x \in Q_{n|k}$ with $|x| = mk$. Since the interval $[\hat{0}, x]$ in $Q_{n|k}$ is isomorphic $Q_{(mk)|k}$, $\mu(x) = \mu(Q_{(mk)|k})$ and then the result follows from case 1. ∎

3.2 Old Numbers and New Numbers

Let $\mu(x)$ be the Möbius function of $Q_{n|k}$. We define

$$|\mu(\hat{1})| = E_{n|k}$$

where $|\cdot|$ denotes the absolute value of a real number.

By Theorem 2.1.4, we see that $E_{n|k}$ is the number of all permutations of $\{1, 2, \cdots, n\}$ with the descent set $\{k, 2k, \ldots, \lfloor \frac{n}{k} \rfloor k\}$. In particular, if $k = 2$, then $E_{n|2}$ is the number of alternating permutations in $\mathcal{S}_n$. The number $E_{n|2}$ is well known as an *Euler number.* So we call $E_{n|k}$ a *generalized Euler number* for $k > 2$. Define

$$E'_{n|k} = (-1)^{\lfloor \frac{n}{k} \rfloor} E_{n|k}.$$

If $k > 2$, then $E'_{n|k}$ is called a *generalized signed Euler number* and $E'_{n|2} = (-1)^{\lfloor \frac{n}{2} \rfloor} E_{n|2}$ is a *signed Euler number.*

The Euler numbers are classical numbers and have been thoroughly studied in the past century. So we consider them as old numbers. By comparison, the generalized Euler numbers will be our new numbers.

The following results are well-known [12, p.48] for the old Euler numbers .

(1). The exponential generating functions of the $E_{n|2}$ are

$$\sum_{n \geq 0} E_{(2n)|2} \frac{x^{2n}}{(2n)!} = \sec x,$$

$$\sum_{n \geq 0} E_{(2n+1)|2} \frac{x^{2n+1}}{(2n+1)!} = \tan x,$$

$$\sum_{n \geq 0} E'_{2n} \frac{x^{2n}}{(2n)!} = \operatorname{sech}(x).$$

For this reason, $E_{(2n)|2}$ and $E_{(2n+1)|2}$ are sometimes called a *secant number* and a *tangent number,* respectively. Naturally, $E'_{(2n)|2}$ and $E'_{(2n+1)|2}$ will be called a *signed secant number* and a *signed tangent number*, respectively.

(2). The number $E_{n|2}$ satisfies the following recursion: for $n \geq 1$,

$$E_{(2n)|2} = \sum_{m=1}^{n} \binom{2n-1}{2m-1} E_{(2m-1)|2} E_{(2n-2m)|2} + E_{(2n-1)|2};$$

$$E_{(2n+1)|2} = \sum_{m=1}^{n} \binom{2n}{2m-1} E_{(2m-1)|2} E_{(2n-2m+1)|2}$$

with the boundary conditions $E_{i|2} = 1$ for all $0 \le i \le 2$.

(3). The number $E'_{(2n)|2}$ satisfies the following formula [10]:

$$(E+1)^m + (E-1)^m = 0$$

where $E^m = E_m$, such that $E_0 = 1$ and

$$E_m = E'_{(m)|2}, \qquad \text{if } m \text{ is even;}$$

$$E_m = 0, \qquad \text{if } m \text{ is odd .}$$

(4). $(n+1)E_{(2n+1)|2} = 2^{2n}|G_{2n+2}|$, where G_{2n+2} is called a *Genocchi number*, which is an odd integer. Thus it is clear that the tangent number $E_{(2n+1)|2}$ is divisible by 2^n.

It is worthwhile to mention that

Lemma 3.2.1 (**3**) *is equivalent to the following condition:*

(3'). *The number* $E'_{(2n)|2}$ *satisfies the following recursion:*

$$\sum_{j\ge 0}^{n} \binom{2n}{2j} E'_{(2j)|2} = 0 \quad \textit{and} \quad E'_0 = 1. \tag{3.5}$$

Proof. By the binomial formula, we see that

$$\begin{aligned}
& (E+1)^m + (E-1)^m \\
= & \sum_{j\ge 0}^{m} \binom{m}{j} E^{m-j}[1+(-1)^j] \\
= & \sum_{\substack{j \ge 0 \\ j \text{ even}}}^{m} 2\binom{m}{j} E_{m-j} = \sum_{j\ge 0}^{\lfloor \frac{m}{2} \rfloor} 2\binom{2\lfloor \frac{m}{2} \rfloor}{2j} E_{2(\lfloor \frac{m}{2} \rfloor - j)}
\end{aligned}$$

$$= \sum_{j\geq 0}^{n} 2\binom{2n}{2j} E'_{(2n-2j)|2} \quad \left(\text{putting } n = \left\lfloor \tfrac{m}{2} \right\rfloor\right)$$

$$= \sum_{j\geq 0}^{n} 2\binom{2n}{2j} E'_{(2j)|2}.$$

It follows that (**3**) and (**3'**) are equivalent. ∎

Our goal here is to extend these well known results for $E_{n|2}$ to $E_{n|k}$ for $k > 2$. We first consider the exponential generating functions in (**1**). Stanley [36] has shown the following result.

Theorem 3.2.2 *Let n and k be positive integers with $k \leq n$. Then the exponential generating function of $E'_{n|k}$ is:*

$$\sum_{n\geq 0} E'_{n|k} \frac{x^n}{n!} = \frac{1 + \sum_{i=1}^{k-1} \sum_{n\geq 0} x^{nk+i}/(nk+i)!}{\sum_{n\geq 0} x^{nk}/(nk)!}. \tag{3.6}$$

Alternatively,, for $1 \leq i \leq k-1$,

$$\sum_{n\geq 0} E'_{(nk+i)|k} \frac{x^{nk+i}}{(nk+i)!} = \frac{\sum_{n\geq 0} x^{nk+i}/(nk+i)!}{\sum_{n\geq 0} x^{kn}/(kn)!},$$

$$\sum_{n\geq 0} E'_{(nk)|k} \frac{x^{kn}}{(kn)!} = \frac{1}{\sum_{n\geq 0} x^{nk}/(nk)!}.$$

Proof. See [40, Proposition 3.16.4]. ∎

By substituting x^k by $-x^k$ in the exponential generating function for $E'_{n|k}$ in Theorem 3.2.2, one can easily see that

Theorem 3.2.3 *For each fixed $k \geq 2$, the exponential generating function of $E_{n|k}$ is*

$$\sum_{n\geq 0} E_{n|k} \frac{x^n}{n!} = \frac{1 + \sum_{i=1}^{k-1} \sum_{n\geq 0} (-1)^n x^{nk+i}/(nk+i)!}{\sum_{n\geq 0} (-1)^n x^{nk}/(nk)!}. \tag{3.7}$$

Also, for $1 \leq i \leq k-1$,

$$\sum_{n\geq 0} E_{(nk+i)|k} \frac{x^{nk+i}}{(nk+i)!} = \frac{\sum_{n\geq 0} (-1)^n x^{nk+i}/(nk+i)!}{\sum_{n\geq 0} (-1)^n x^{kn}/(kn)!},$$

$$\sum_{n\geq 0} E_{(nk)|k} \frac{x^{kn}}{(kn)!} = \frac{1}{\sum_{n\geq 0}(-1)^n x^{nk}/(nk)!}.$$

Remark: If $k = 2$, we get the same exponential generating function of $E_{n|2}$ and $E'_{n|2}$ in (**1**):

$$\sum_{n\geq 0} E_{(2n+1)|2} \frac{x^{2n+1}}{(2n+1)!} = \frac{\sum_{n\geq 0}(-1)^n x^{2n+1}/(2n+1)!}{\sum_{n\geq 0}(-1)^n x^{2n}/(2n)!} = \frac{\sin(x)}{\cos(x)} = \tan(x),$$

$$\sum_{n\geq 0} E_{(2n)|2} \frac{x^{2n}}{(2n)!} = \frac{1}{\sum_{n\geq 0}(-1)^n x^{2n}/(2n)!} = \frac{1}{\cos(x)} = \sec(x),$$

$$\sum_{n\geq 0} E'_{(2n)|2} \frac{x^{2n}}{(2n)!} = \frac{1}{\sum_{n\geq 0} x^{2n}/(2n)!} = \frac{1}{\mathrm{ch}(x)} = \mathrm{sech}(x).$$

Next we give our first recursion for $E_{n|k}$, which generalizes the recursion for $E_{n|2}$ in (**2**).

Theorem 3.2.4 *The number $E_{n|k}$ satisfies the recursion*

$$E_{(n+1)|k} = \sum_{m=1}^{\lfloor n/k \rfloor} \binom{n}{km-1} E_{(km-1)|k} E_{(n-km+1)|k} + b(k \not| \, n) E_{n|k}. \tag{3.8}$$

where $b(\cdot)$ is the boolean function defined by

$$b(c) = \begin{cases} 1 & \textit{if the condition } c \textit{ is true,} \\ 0 & \textit{if the condition } c \textit{ is not true.} \end{cases} \tag{3.9}$$

The boundary conditions are $E_{n|k} = 1$ for all $0 \leq n \leq k$.

Remark: If $k = 2$, we have the same recursions for $E_{n|k}$ in (**2**).

As for the proof of this theorem, we leave it to the next section, since Theorem 3.2.4 is just a simple corollary of Theorem 3.3.2.

Our second recursion for $E'_{n|k}$ generalizes the recursion for $E'_{n|2}$ in (**3**) or in (**3'**).

Theorem 3.2.5 *Suppose that $n \geq 1$ and $1 \leq i \leq k-1$. Then the number $E_{n|k}$ satisfies the following recursion:*

$$(1) \quad \sum_{j\geq 0}^{n}(-1)^j \binom{nk}{kj} E_{(kj)|k} = 0;$$

$$(2) \quad \sum_{j=0}^{n}(-1)^j \binom{kn+i}{kj} E_{(kj)|k} + (-1)^{n+1} E_{(kn+i)|k} = 0$$

with the boundary conditions $E_{i|k} = 1$ for all $0 \leq i \leq k-1$.

By the definition, we see that suppose $n \geq 1$ and $1 \leq i \leq k-1$. Then $E'_{n|k}$ satisfies the following recursion:

$$(1') \quad \sum_{j\geq 0}^{n} \binom{nk}{kj} E'_{(kj)|k} = 0;$$

$$(2') \quad \sum_{j=0}^{n} \binom{kn+i}{kj} E'_{(kj)|k} = E'_{(kn+i)|k}$$

with the boundary conditions $E'_{i|k} = 1$ for all $0 \leq i \leq k-1$.

Remark: If $k = 2$, we have the same recursion for $E'_{n|2}$ in (**3**') and (**3**).

We will give Theorem 3.2.5 an algebraic proof as well as a combinatorial proof. Our algebraic proof uses exponential generating functions.

An Algebraic Proof of Theorem 3.2.5 To prove (1), it is equivalent to show that

$$E_{(kn)|k} = (-1)^{n+1} \sum_{j\geq 0}^{n-1}(-1)^j \binom{kn}{kj} E_{(kj)|k}. \tag{3.10}$$

We have

$$\sum_{n\geq 1}\left[(-1)^{n+1}\sum_{j\geq 0}^{n-1}(-1)^j \binom{kn}{kj} E_{(kj)|k}\right] \frac{x^{kn}}{(kn)!} + 1$$

$$= \quad -\sum_{n\geq 1}\left\{\sum_{j\geq 0}(-1)^{n-j}\frac{(kn)!}{[k(n-j)]!(kj)!}\ E_{(kj)|k}\frac{x^{k(n-j)}\ x^{kj}}{(kn)!}\right\} + 1$$

$$
\begin{aligned}
&= -\sum_{j\geq 0}\left[\sum_{n-j\geq 0}(-1)^{n-j}\frac{x^{k(n-j)}}{[k(n-j)]!}-1\right]E_{(kj)|k}\frac{x^{kj}}{(kj)!}+1 \\
&= -\sum_{j\geq 0}\left\{\left[\sum_{j\geq 0}E_{(kj)|k}\frac{x^{kj}}{(kj)!}\right]^{-1}-1\right\}E_{(kj)|k}\frac{x^{kj}}{(kj)!}+1 \\
&= -1+\sum_{j\geq 0}E_{(kj)|k}\frac{x^{kj}}{(kj)!}+1=\sum_{j\geq 0}E_{(kj)|k}\frac{x^{kj}}{(kj)!}.
\end{aligned}
$$

Equating the coefficients of $\frac{x^{kj}}{(kj)!}$, the result follows.

To prove (2), it is equivalent to show that

$$
E_{(kn+i)|k}=(-1)^n\sum_{j=0}^{n}(-1)^j\binom{kn+i}{kj}E_{(kj)|k} \quad \text{for all } i\ .
$$

By Theorem 3.2.3, we have

$$
\begin{aligned}
&\sum_{n\geq 0}E_{(nk+i)|k}\frac{x^{nk+i}}{(nk+i)!} \\
&= \left[\sum_{n\geq 0}(-1)^n\frac{x^{nk+i}}{(nk+i)!}\right]\left[\sum_{n\geq 0}E_{(nk)|k}x^{kn}/(kn)!\right] \\
&= \sum_{n\geq 0}(-1)^n\left[\sum_{j=0}^{n}(-1)^j\binom{kn+i}{kj}E_{(kj)|k}\right]\frac{x^{nk+i}}{(nk+i)!}
\end{aligned}
$$

Equating the coefficients of $\frac{x^{nk+i}}{(nk+i)!}$, the result follows. ■

We now give a combinatorial proof of Theorem 3.2.5. To do this, we need some preliminary materials related to signed sets and the involutions of such sets.

A *signed set* S is a set with a function $\epsilon : S \to \{+1,-1\}$. An *involution* f on S is a function $f : S \to S$ such that

$$
f^2 = id, \quad \text{where } id \text{ is the identity map of } S\ .
$$

A function$f$ on a signed set S is *sign-reversing* if

$$
\text{sign}\ \{f(a)\} \ = -\text{sign}\ \{f(a)\} \text{ for all } a \in S\ .
$$

A point $a \in S$ is called a *fixed point* of f if $f(a) = a$. The set of all fixed points of f in S is denoted by F_f; that is,

$$F_f = \{a \in S : f(a) = a\}.$$

Then the following theorem will play a crucial role in our proof [40, 2.6].

Theorem 3.2.6 *Let S be a signed set with a sign function ϵ. If f is a sign-reversing involution on S with the fixed point set F_f. Then*

$$\sum_{a\in S} \epsilon(a) = \sum_{a\in F_f} \epsilon(a) = |F_f|. \tag{3.11}$$

A combinatorial Proof of Theorem 3.2.5 : First, we construct a signed set S as follows: we let π_0 represent the 0 permutation for convenience of notation.

$$S(0) = \{\pi_0\},$$

$$S(n+1)$$

$$= \begin{cases} \{\pi = \pi_1\pi_2\cdots\pi_{kn+i} \in S_{kn+i} : \ \mathrm{Des}(\pi) = \{k, 2k, \cdots, kn\}\}, & i \neq 0; \\ \emptyset, & i = 0. \end{cases}$$

Let $1 \leq j \leq n$ and $0 \leq i \leq k-1$. For fix i and j, let J be any kj-subset of $\{1, 2, \cdots, kn+i\}$ and $S(J)$ be the set of all permutations of J with descent set $\{k, 2k, \cdots, (j-1)k\}$. Then let

$$S(j) = \biguplus_J S(J), \quad \text{where } J \text{ runs over all } kj\text{-subset of } \{1, 2, \cdots, kn+i\}.$$

By the definition of $E_{n|k}$, one can easily see that

$$|S(0)| = 1,$$

$$|S(j)| = \binom{kn+i}{kj} E_{(kj)|k}, \quad \text{if } 1 \leq j \leq n \text{ and } 0 \leq i \leq k-1,$$

$$|S(n+1)| = \begin{cases} E_{(nk+i)|k}, & \text{if } i \neq 0; \\ 0, & \text{if } i = 0\ . \end{cases}$$

we now contruct a signed set S by

$$S = \biguplus_{j=0}^{n+1} S_j$$

with a signed function $\epsilon : S \to \{+1, -1\}$ defined by

$$\epsilon(\pi) = (-1)^j \quad \text{if} \quad \pi \in S_j.$$

Secondly, we define a sign reversing involution f on S as follows: if $\pi \in S$, then $\pi \in S_j$ for some $0 \leq j \leq n+1$. there are three cases:

Case 1: If $j = 0$, then $\pi = \pi_0$. Define

$$f(\pi_0) = 123 \cdots k.$$

Case 2: If $j = n+1$ and $i > 0$ and then $\pi = \pi_1 \pi_2 \cdots \pi_{kn+i}$. Define

$$f(\pi_1 \pi_2 \cdots \pi_{kn+i}) = \pi_1 \pi_2 \cdots \pi_{kn}.$$

Case 3: If $0 < j < n+1$, then $\pi = \pi_1 \pi_2 \cdots \pi_{kj}$. we let

$$\{1, 2, \ldots, kn+i\} \setminus \{\pi_1, \pi_2, \ldots, \pi_{kj}\} = \{a_1 < a_2 < \cdots\}$$

and define

$$f(\pi) = \begin{cases} \pi_1 \pi_2 \cdots \pi_{kj} a_1 a_2 \cdots a_k \in S_{j+1}, & \text{if } a_1 < \pi_{kj} \text{ and } 1 \leq j < n \text{ ;} \\ \pi_1 \pi_2 \cdots \pi_{kn} a_1 a_2 \cdots a_i \in S_{n+1}, & \text{if } a_1 < \pi_{kn} \text{ and } j = n; \\ & \\ \pi_1 \pi_2 \cdots \pi_{kj-k} \in S_{j-1}, & \text{if } a_1 > \pi_{kj} \text{ and } 1 < j \leq n. \\ \pi_0 & \text{if } a_1 > \pi_k j \text{ and } j = 1 \end{cases} \tag{3.12}$$

Remark: Note that f is a well defined map on S. Moreover, if $\pi \in S_j$, then either $f(\pi) \in S_{j+1}$ or $f(\pi) \in S_{j-1}$. It follows that

1. f is sign reversing.

2. f has no fixed point; that is, $F_f = \emptyset$.

It remains to show that f is an involution on S.

1. If $\pi \in S(0)$, then

$$f^2(\pi_0) = f(12\cdots k) = \pi_0,$$

since $\{1,2,\ldots,nk+i\} \setminus \{1,2,\ldots,k\} = \{k+1 < k+2 < \cdots\}$ and $k < k+1$.

2. If $\pi \in S(n+1)$, then

$$\begin{aligned} f^2(\pi_1\pi_2\cdots\pi_{nk+i}) &= f(\pi_1\pi_2\cdots\pi_{kn}) \quad \text{(by the definition)} \\ &= \pi_1\pi_2\cdots\pi_{kn+i} \quad (\text{since } kn \in \mathrm{Des}(\pi)\ ). \end{aligned}$$

3. If $\pi \in S(j)$ for some $0 < j < n+1$, by a routine procedure, one can easily see that $f^2(\pi) = \pi$.

So we see that $f^2 = id$.

By Theorem 3.2.6 and the remark, we see that, on one hand,

$$\sum_{\pi \in S} \epsilon(\pi) = \sum_{\pi \in F_f} = |F_f| = 0.$$

On the other hand,

$$\begin{aligned} \sum_{\pi \in S} \epsilon(\pi) &= \sum_{j=0}^{n+1} |S_j|(-1)^j \\ &= \begin{cases} \sum_{j=0}^{n}(-1)^j \binom{kn+i}{kj}\ E_{(kj)|k} + (-1)^{n+1} E_{(kj+i)|k}, & \text{if } i \neq 0; \\ \sum_{j=0}^{n}(-1)^j \binom{kn}{kj}\ E_{(kj)|k} & \text{if } i = 0. \end{cases} \end{aligned}$$

Combining these two equations, the result follows. ■

So for, we have generalized the well-known results (**1**), (**2**), (**3**) and (**3'**) about the old numbers. It remains to generalize their divisibility properties in (**4**). We leave it to the next two sections.

3.3 The Generalized q-Euler numbers

We start this section by introducing the concept of a q-analog of a mathematical object, which will play an important role in the rest of this work.

A q-analog of a mathematical object $\mathcal{O}$ ($\mathcal{O}$ could be a definition, number, theorem or an identity) is an object $\mathcal{O}(q)$ depending upon a parameter q such that

$$\lim_{q\to 1} \mathcal{O}(q) = \mathcal{O}.$$

Note that q-analogs are not necessarily unique. There may be several equally valid q-analogs for the same object.

Let $k, j \in \mathbf{N}$ and q be an indeterminate. Define q-analogs of k by

$$[k]_j = 1 + q^j + q^{2j} + q^{3j} + \cdots + q^{(k-1)j}.$$

In particular,

$$[k]_1 = [k] = 1 + q + q^2 + \cdots + q^{k-1}.$$

Further, define a q-analog of the factorial by

$$[k]! = [k][k-1][k-2]...[2][1].$$

We can then define q-binomial coefficients by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}.$$

Clearly, when $q = 1$, $[k] = [k]_j = k, [k]! = k!$ and $\begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}$.

It is well-known that any q-binomial coefficient is a polynomial in q.

R. P. Stanley [40, p148] defines a natural q-analog $E_{n|k}(q)$ of $E_{n|k}$ as follows:

$$\sum_{n\geq 0} E_{(kn+i)|k}(q)\frac{x^{kn+i}}{(q)_{kn+i}} = \frac{\sum_{n\geq 0}(-1)^n x^{nk+i}/(q)_{nk+i}}{\sum_{n\geq 0}(-1)^n x^{nk}/(q)_{nk}} \tag{3.13}$$

for $1 \leq i \leq k-1$ and

$$\sum_{n\geq 0} E_{(kn)|k}(q)\frac{x^{kn}}{(q)_{kn}} = \frac{1}{\sum_{n\geq 0}(-1)^n x^{nk}/(q)_{nk}} \tag{3.14}$$

where $(q)_m = (1-q)(1-q^2)...(1-q^m) = (1-q)^m[m]!$ for any positive integer m.

We call $E_{n|k}(q)$ a generalized q-Euler number. In particular, $E_{n|2}(q)$ is an ordinary q-Euler number and a natural q-tangent number is given by

$$\sum_{n\geq 0} E_{(2n+1)|2}(q)\frac{x^{2n+1}}{(q)_{2n+1}}$$

$$= \frac{\sum_{n\geq 0}(-1)^n x^{2n+1}/(q)_{2n+1}}{\sum_{n\geq 0}(-1)^n x^{2n}/(q)_{2n}} = \frac{\sin_q(x)}{\cos_q(x)} = \tan_q(x).$$

Also, it is clearly that $E_{n|k}(1) = E_{n|k}$.

The main theorem in this section is concerned with a recursion for $E_{n|k}(q)$, from which we derive the first recursion for $E_{n|k}$. Once we have established this theorem, we can more efficiently study the divisibility properties of $E_{n|k}$. First, the following preliminaries are needed.

Let $x_1, x_2, ..., x_n$ be a set of variables. Then the *elementary symmetric function* of degree r in n variables is the sum of all square-free monomials of degree r in $x_1, x_2, ..., x_n$. More formally, for all $r \geq 0$ and $n \geq 0$,

$$e_r(x_1, x_2, \ldots, x_n) = \sum_{1\leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} ... x_{i_r}. \tag{3.15}$$

Let $x_i = q^{i-1}$ for all $1 \leq i \leq n$, where q is an indeterminate. Then

$$e_r(1, q, q^2, \ldots, q^{n-1}) = q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} \tag{3.16}$$

where $\begin{bmatrix} n \\ r \end{bmatrix}$ is the q-binomial coefficient [29, p.19]. Suppose that $\{P_1, P_2\}$ is a partition of $\{1, 2, ..., n\}$. For $a \in P_1$ and $b \in P_2$, we call a pair (a, b) an *inversion of*

$\{P_1, P_2\}$ *caused by* a if $a > b$. We define

$$I(P_1, P_2) = |\{(a, b) : a \in P_1,\, b \in P_2 \text{ and } a > b\}|.$$

Lemma 3.3.1 *Let* $\{P_1, P_2\}$ *be a partition of* $\{1, 2, ..., n\}$ *with* $|P_1| = r$. *Then*

$$\sum_{\{P_1, P_2\}} q^{I(P_1, P_2)} = \begin{bmatrix} n \\ r \end{bmatrix}. \tag{3.17}$$

Proof. This lemma is well known. But I am including a proof for completeness.

Let $P_1 = \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1, 2, ..., n\}$ and $P_2 = \{1, 2, ..., n\} \setminus P_1$. It is clear that i_j causes $i_j - j$ inversions for all $1 \leq j \leq r$. Then

$$I(P_1, P_2) = \sum_{j=1}^{r} (i_j - j) = \sum_{j=1}^{r} (i_j - 1) - \binom{r}{2}. \tag{3.18}$$

By using (3.16), we get

$$\begin{aligned} \sum_{\{P_1, P_2\}} q^{I(P_1, P_2)} &= q^{-\binom{r}{2}} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} q^{\sum_{j=1}^{r} (i_j - 1)} \\ &= q^{-\binom{r}{2}}\, e_r(1, q, q^2, \ldots, q^{n-1}) \\ &= q^{-\binom{r}{2}}\, q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix} \end{aligned}$$

as desired. ∎

If $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathcal{S}_n$, then define an *inversion of a permutation* π to be a pair (π_i, π_j) where $i < j$ and $\pi_i > \pi_j$. We let the *number of inversions of* π to be

$$\text{inv}(\pi) = |\{(\pi_i, \pi_j) :\ i < j \text{ and } \pi_i > \pi_j\ \}|.$$

Recall that we defined the descent set of π by

$$\text{Des}(\pi) = \{i : \pi_i > \pi_{i+1} \text{ and } 1 \leq i \leq n - 1\ \}.$$

Theorem 3.3.2 *Let n and k be any non-negative integers. Then $E_{n|k}(q)$ satisfies the following recursion:*

$$E_{(n+1)|k}(q) = \sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} \begin{bmatrix} n \\ km-1 \end{bmatrix} q^{n-km+1} E_{(km-1)|k}(q) E_{(n-km+1)|k}(q) + b(k \not| n) E_{n|k}(q) \quad (3.19)$$

where $b(\cdot)$ is the boolean function defined by (3.9).

Proof. R. P. Stanley has shown [40, Proposition 3.16.4] that

$$E_{n|k}(q) = \sum_{\pi} q^{\mathrm{inv}(\pi)} \quad (3.20)$$

where the sum is over all $\pi \in \mathcal{S}_n$ with the descent set $\mathrm{Des}(\pi) = \{k, 2k, ..., \lfloor \frac{n-1}{k} \rfloor k\}$. So it is enough to show that $E_{n|k}(q)$ defined by equation (3.20) satisfies (3.19).

Let

$$\mathcal{E}_{n+1} = \{\pi : \ \pi \in \mathcal{S}_{n+1} \text{ and } \mathrm{Des}(\pi) = \{k, 2k, ..., \lfloor \tfrac{n}{k} \rfloor k\} \ \}.$$

If $\pi = \pi_1\pi_2...\pi_{n+1} \in \mathcal{E}_{n+1}$, then either $n+1 = \pi_{km}$ for some $1 \le m \le \lfloor \frac{n}{k} \rfloor$ or $n+1 = \pi_{n+1}$ where $k \not| n$. One easily sees that $n+1 = \pi_{n+1}$ can happen if and only if $k \not| n$. Now define

$$\mathcal{E}^m_{n+1} = \{\pi : \ \pi \in \mathcal{E}_{n+1} \text{ and } \pi_{km} = n+1\}$$

for $1 \le m \le \lfloor \frac{n}{k} \rfloor$ and

$$\mathcal{E}^0_{n+1} = \{\pi : \pi \in \mathcal{E}_{n+1} \text{ and } \pi_{n+1} = n+1 \ \}.$$

It is clear that $\mathcal{E}_{n+1} = \uplus_{m=0}^{\lfloor \frac{n}{k} \rfloor} \mathcal{E}^m_{n+1}$. Note that k divides n if and only if $\mathcal{E}^0_{n+1} = \emptyset$. Hence

$$E_{(n+1)|k}(q) = \sum_{\pi \in \mathcal{E}_{n+1}} q^{\mathrm{inv}(\pi)} = \sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{\pi \in \mathcal{E}^m_{n+1}} q^{\mathrm{inv}(\pi)} + b(k \not| n) \sum_{\pi \in \mathcal{E}^0_{n+1}} q^{\mathrm{inv}(\pi)}. \quad (3.21)$$

Now consider the first summation in equation (3.21). For fixed m, $1 \leq m \leq \left\lfloor \frac{n}{k} \right\rfloor$, and write

$$\pi = \underbrace{\pi_1\pi_2...\pi_{km-1}}_{\tau}(n+1)\underbrace{\pi_{km+1}...\pi_{n+1}}_{\sigma} \in \mathcal{E}^m_{n+1}$$

where $\pi_{km} = n+1$. Then

$$\text{inv}(\pi) = \text{inv}(\tau) + \text{inv}(\sigma) + (n - km + 1) + i(\tau, \sigma)$$

where

$$n - km + 1 = |\{(n+1, \pi_j) : km + 1 \leq j \leq n+1\}|$$

and

$$i(\tau, \sigma) = |\{(\pi_i, \pi_j) : \pi_i > \pi_j, 1 \leq i \leq km - 1 \text{ and } km + 1 \leq j \leq n+1\}|.$$

Suppose $\{P_1, P_2\}$ is a partition of $\{1, 2, ..., n\}$ with $|P_1| = km - 1$. Define $\mathcal{E}(P_1)$ to be the set of all permutations of P_1 with the descent set $\{k, 2k, \ldots, (m-1)k\}$. Then there is a one-to-one order-preserving correspondence between $\mathcal{E}(P_1)$ and $\mathcal{E}_{km-1}$; and Similarly for $\mathcal{E}(P_2)$ and $\mathcal{E}_{n-km+1}$. For any $\tau = \tau_1\tau_2...\tau_{km-1} \in \mathcal{E}(P_1)$ and any $\sigma = \sigma_1\sigma_2...\sigma_{n-km+1} \in \mathcal{E}(P_2)$,

$$\begin{aligned} i(\tau, \sigma) &= |\{(\tau_i, \sigma_j) : \ \tau_i > \sigma_j, \ 1 \leq i \leq km - 1 \text{ and } 1 \leq j \leq n - km + 1 \ \}| \\ &= |\{(a, b) : a \in P_1, b \in P_2 \text{ and } a > b\}| \\ &= I(P_1, P_2). \end{aligned}$$

For any partition $\{P_1, P_2\}$ of $\{1, 2, ..., n\}$ with $|P_1| = km - 1$,

$$\sum_{\tau \in \mathcal{E}(P_1) \sigma \in \mathcal{E}(P_2)} q^{\text{inv}(\tau)+\text{inv}(\sigma)}$$

$$= \sum_{\tau \in \mathcal{E}_{km-1} \sigma \in \mathcal{E}_{n-km+1}} q^{\text{inv}(\tau)+\text{inv}(\sigma)}$$

$$= \left\{ \sum_{\tau \in \mathcal{E}_{km-1}} q^{i(\tau)} \right\} \left\{ \sum_{\sigma \in \mathcal{E}_{n-km+1}} q^{i(\sigma)} \right\}$$

$$= E_{(km-1)|k}(q) E_{(n-km+1)|k}(q).$$

If r is replaced by $km - 1$ in Lemma 3.3.1, then

$$\sum_{\pi \in \mathcal{E}^m_{n+1}} q^{\mathrm{inv}(\pi)} = \sum_{(P_1,P_2)} \sum_{\substack{\tau \in \mathcal{E}(P_1) \\ \sigma \in \mathcal{E}(P_2)}} q^{\mathrm{inv}(\tau)+\mathrm{inv}(\sigma)+n-km+1+I(P_1,P_2)}$$

$$= \begin{bmatrix} n \\ km-1 \end{bmatrix} q^{n-km+1} E_{(km-1)|k}(q) E_{(n-km+1)|k}(q).$$

It follows that

$$\sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{\pi \in \mathcal{E}^m_{n+1}} q^{\mathrm{inv}(\pi)} = \sum_{m=1}^{\lfloor \frac{n}{k} \rfloor} \begin{bmatrix} n \\ km-1 \end{bmatrix} q^{n-km+1} E_{(km-1)|k}(q) E_{(n-km+1)|k}(q).$$

So we are done if k divides n. Suppose that $k \not| \; n$. Then there is an extra term $\sum_{\pi \in \mathcal{E}^0_{n+1}} q^{\mathrm{inv}(\pi)}$ in equation (3.21). Note that the correspondence

$$\pi = \pi_1 \pi_2 \ldots \pi_n (n+1) \Longleftrightarrow \tilde{\pi} = \pi_1 \pi_2 \ldots \pi_n$$

between $\mathcal{E}^0_{n+1}$ and $\mathcal{E}_n$ is one-to-one and order-preserving. Hence

$$\sum_{\pi \in \mathcal{E}^0_{n+1}} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{E}_n} q^{\mathrm{inv}(\pi)} = E_{n|k}(q).$$

This completes the proof of Theorem 3.3.2. ∎

Remark: Observe that when $q = 1$, we have the recursion for $E_{n|k}$ as in Theorem 3.2.4

Theorem 3.3.2 and Theorem 3.2.4 are very useful in further studying the divisibility properties of $E_{n|k}$ and $E_{n|k}(q)$, which will be seen in the next section. To demonstrate this point, we will end this section by a direct application of Theorem 3.2.4.

We know that $(n+1)E_{(2n+1)|2} = 2^{2n}|G_{2n+2}|$ where G_{2n+2} is the Genocchi number. It can be verified using the exponential generating functions of $E_{(2n+1)|2}$ and $|G_{2n+2}|$ as follows: it is well-known [12, p.48 - p.49] that.

$$G(t) = \frac{2t}{e^t+1} = t + \sum_{n\geq 0} |G_{2n+2}| \frac{t^{2n+2}}{(2n+2)!}$$

and

$$T(t) = \frac{2}{e^{2t}+1} = 1 + \sum_{n\geq 0} E_{(2n+1)|2} \frac{t^{2n+1}}{(2n+1)!}.$$

Let $2t = s$ and then $t = s/2$. On one hand,

$$\begin{aligned} t\,T(t) &= \frac{2t}{e^{2t}+1} \\ &= t + \sum_{n\geq 0} E_{(2n+1)|2}(2n+2)\frac{t^{2n+2}}{(2n+2)!} \\ &= 1/2\left\{ s + \sum_{n\geq 0} \frac{E_{(2n+1)|2}(n+1)}{2^{2n}} \frac{s^{2n+2}}{(2n+2)!} \right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} t\,T(t) &= 1/2\left\{ \frac{2(2t)}{e^{2t}+1} \right\} \\ &= 1/2\left\{ \frac{2s}{e^{s}+1} \right\} = 1/2\ G(s) \\ &= 1/2\left\{ s + \sum_{n\leq 0} |G_{2n+2}| \frac{s^{2n+2}}{(2n+2)!} \right\}. \end{aligned}$$

It follows that $(n+1)E_{(2n+1)|2} = 2^{2n}|G_{2n+2}|$.

Since Genocchi numbers are odd, we see that following theorem is true.

Theorem 3.3.3 *Let n be a non-negative integer. Then*

1. 2^{2n} *divides* $(n+1)E_{(2n+1)|2}$;

2. $(n+1)E_{(2n+1)|2}/2^{2n}$ *is odd.*

Proof. Using the recursion for $E_{n|k}$ when $k = 2$, we can now give a direct proof of this theorem without using their generating functions. For brevity, we first define

$$E^*_{(2n+1)|2} = (n+1)E_{(2n+1)|2}.$$

We first prove (1): Induct on n. The case $n = 0$ is trivial. Suppose $n > 0$ and then

$$\begin{aligned} E^*_{(2n+1)|2} &= \sum_{m=1}^{n} (n+1)\binom{2n}{2m-1} E_{(2m-1)|2}E_{(2n-2m+1)|2} \\ &= \sum_{m=1}^{n} \frac{(n+1)\binom{2n}{2m-1}}{m(n-m+1)} E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2} \\ &= \frac{2}{2n+1}\sum_{m=1}^{n}\binom{2n+2}{2m} E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2}. \end{aligned}$$

It follows that

$$(2n+1)E^*_{(2n+1)|2} = 2\sum_{m=1}^{n}\binom{2n+2}{2m} E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2}.$$

By the induction, we have

$$2^{2n-2} \text{ divides } E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2}$$

where $1 \le m \le n$.

Since $2n+1$ is odd, it is enough to show that

$$\sum_{m=1}^{n}\binom{2n+2}{2m} E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2} \tag{3.22}$$

is divisible by 2^{2n-1}. Note that for fixed m, $1 \le m \le n$, we see that

$$\binom{2n+2}{2m} = \binom{2n+2}{2n+2-2m} = \binom{2n+2}{2(n-m+1)}.$$

Moreover, since $n-(n-m+1)+1=m$, $2(n-m+1)-1=2n-2m+1$ and $2n-2(n-m+1)+1=2m-1$, Also since $1\le m\le n$ implies that $1\le n-m+1\le n$, it follows that the m-th term equals to the $(n-m+1)$-th term in sum (3.22). It follows that sum (3.22) is

$$2\sum_{m=1}^{\lfloor\frac{n}{2}\rfloor}\binom{2n+2}{2m}E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2}$$

$$+b\ (n \text{ is odd})\binom{2n+2}{2\lceil\frac{n}{2}\rceil}\left\lceil\frac{n}{2}\right\rceil E_{(2\lceil\frac{n}{2}\rceil-1)|2}\left(n-\left\lceil\frac{n}{2}\right\rceil+1\right)E_{(2n-2\lceil\frac{n}{2}\rceil+1)|2}$$

where $b(\cdot)$ is the boolean function. By the induction,

$$2^{2n-1}\ \text{ divides } 2\binom{2n+2}{2m}E^*_{(2m-1)|2}E^*_{(2n-2m+1)|2}\text{ where } 1\le m\le\left\lfloor\tfrac{n}{2}\right\rfloor.$$

So we are done if n is even. If n is odd, there is an extra term. Now let $n=2l+1$ for some non-negative integer l and then

$$\binom{2n+2}{2\lceil\frac{n}{2}\rceil}=\binom{2(2l+1)+2}{2(l+1)}=\binom{2(2l+2)}{2l+2}$$

which is divisible by 2, since that the number of carries in adding $2l+2$ to $(4l+4)-(2l+2)=2l+2$ in 2-ary arithmetic is at least 1. Hence the extra term is also divisible by 2^{2n-1} and the result follows.

We next prove (2): Induct on n. The $n=0$ case is trivial. Now suppose $n>0$ and consider

$$\begin{aligned}\frac{E^*_{(2n+1)|2}}{2^{2n}} &= \sum_{m=1}^{n}\frac{n+1}{2^2m(n-m+1)}\binom{2n}{2m-1}\left\{\frac{E^*_{(2m-1)|2}}{2^{2(m-1)}}\right\}\left\{\frac{(E^*_{(2(n-m)+1)|2}}{2^{2(n-m)}}\right\}\\ &= \frac{1}{2(2n+1)}\sum_{m=1}^{n}\binom{2(n+1)}{2m}\left\{\frac{E^*_{(2m-1)|2}}{2^{2(m-1)}}\right\}\left\{\frac{E^*_{(2(n-m)+1)|2}}{2^{2(n-m)}}\right\}.\end{aligned}$$

Let

$$O_m=\left\{\frac{E^*_{(2m-1)|2}}{2^{2(m-1)}}\right\}\left\{\frac{E^*_{(2(n-m)+1)|2}}{2^{2(n-m)}}\right\}$$

and then

$$(2n+1)\frac{E^*_{(2n+1)|2}}{2^{2n}} = \frac{1}{2}\sum_{m=1}^{n}\binom{2(n+1)}{2m}O_m.$$

By the induction, O_m is an odd number for all $1 \le m \le n$.

Since $2n+1$ is odd, it is enough to show that

$$V(n) = \frac{1}{2}\sum_{m=1}^{n}\binom{2(n+1)}{2m}O_m$$

is odd for all $n \ge 1$. Note that, by the induction,

$$O_m - 1 \equiv 0 \pmod 2 \quad \text{for all } 1 \le m \le n.$$

By [22, (1.91)], we see that

$$\sum_{m=0}^{n+1}\binom{2n+2}{2m} = 2^{2(n+1)-1} = 2^{2n+1}.$$

It follows that

$$\frac{1}{2}\left\{\sum_{m=1}^{n}\binom{2n+2}{2m}\right\} = \frac{1}{2}(2^{2n+1}-2) = 2^{2n}-1$$

is odd. Now it is enough to show that

$$V(n) - (2^{2n}-1) = \sum_{m=1}^{n}\frac{\binom{2n+2}{2m}}{2}(O_m - 1)$$

is even, since then $V(n)$ must be odd and the result follows.

Note that $O_m = O_{n-m+1}$ for all $1 \le m \le n$. It follows that

$$V(n) - (2^{2n}-1)$$

$$= \ 2\sum_{m=1}^{\lfloor \frac{n}{2}\rfloor}\frac{\binom{2n+2}{2m}}{2}(O_m - 1) + \delta\,(n \text{ is odd })\frac{\binom{2n+2}{2\lceil \frac{n}{2}\rceil}}{2}\left(O_{\lceil \frac{n}{2}\rceil} - 1\right).$$

It is clear that for $1 \le m \le \lfloor \frac{n}{2}\rfloor$, $\binom{2n+2}{2m}(0_m - 1)$ is even since each $(0_m - 1)$ is even and $\binom{2n+2}{2m}$ is an integer. Moreover, we have the extra term in the odd-n case is also even since $\frac{\binom{2n+2}{2\lceil \frac{n}{2}\rceil}}{2}$ is an integer and $O_{\lceil \frac{n}{2}\rceil} - 1$ is even. Hence $V(n) - (2^{2n}-1)$ is even and then $V(n)$ is odd. ∎

3.4 The Divisibility of the Generalized q-Euler Numbers

G. E. Andrews and I. Gessel have shown [2, Theorem 1 and Theorem 2] that

1. $E_{(2n+1)|2}(q)$ is divisible by $(1+q)(1+q^2)...(1+q^n) = [2][2]_2[2]_3...[2]_n$.

2. $E_{(2n+1)|2}(q)$ is divisible by $(1+q)^n = [2]^n$.

It follows that 2^n divides $E_{(2n+1)|2}$ by letting $q = 1$.

We adopt the techniques which G. E. Andrews and I. Gessel have used in their work and extend their results to $E_{(nk+i)|k}(q)$ for all prime $k \geq 2$ and $1 \leq i \leq k-1$.

First, we study some related divisibility properties of certain binomial coefficients and q-binomial coefficients .

Lemma 3.4.1 *Let k be prime. Then $\binom{kN+i}{km-1}$ is divisible by k for all $0 \leq i \leq k-2$.*

Althought Lemma 3.4.1 is just a special case of Theorem 3.4.3 and Theorem 3.4.4, we include a proof here to show a special method used in proving such a problem.

Proof. By Kummer's Theorem [15, p.270, item 71], it is enough to show that the number of carries when adding $km-1$ to $kN+i-(km-1)$ in base k is greater than or equal to 1. Now $kN+i-(km-1) = k(N-k)+(i+1)$ where $1 \leq (i+1) \leq k-1$ and $km-1 = k(m-1)+(k-1)$. So these two numbers have one's digit $i+1$ and $k-1$ in base k. Since $(i+1)+(k-1) \geq k$, we have a carry out of the one's digit and are done. ∎

In general, q-analogs of these binomial coefficients have the similar divisibility. To show it , we first need the following lemma [24].

Lemma 3.4.2 *If ρ is a primitive kth roots of unity, then ρ is a simple root of $1-q^M$ if and only if $k|M$.*

Remark: Note that Lemma 3.4.2 requires the condition that k is prime. Since our proofs are based on Lemma 3.4.2, the condition that k is prime can not be omitted. Also, it is easy to find a counter example to show that this condition is nesessary.

Lemma 3.4.3 *Let prime $k \geq 2$ and $0 \leq i \leq k-2$. For any nonnegative integers n and m, the expression*

$$\begin{bmatrix} kn+i \\ km-1 \end{bmatrix} \frac{[k][k]_2...[k]_{m-1}}{[k]_n[k]_{n-1}...[k]_{n-m+1}}$$

is a polynomial in q. Clearly, when $q=1$, we have Lemma 3.4.1.

Proof. The expression in question is a rational function and the roots of the denominator are roots of unity. To prove Lemma 3.4.3, we need only show that each zero of the denominator appears with at least as large multiplicity in the numerator as in the denominator.

We know that that $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$ is a polynomial in q. By Lemma 3.4.2, for each $1 \leq j \leq km-1$, j must divide at least $\left\lfloor \frac{km-1}{j} \right\rfloor$ of the numbers $kn+i, kn+i-1, ..., k(n-1)+1, k(n-1), k(n-1)-1, ..., kn-km+i+2$ (otherwise this q-binomial coefficient would not be a polynomial). Now

$$\begin{bmatrix} kn+i \\ km-1 \end{bmatrix} = \frac{(1-q^{nk+i})(1-q^{kn+i-1})...(1-q^{kn-km+i+2})}{(1-q^{km-1})(1-q^{km-2})...(1-q^2)(1-q)}. \tag{3.23}$$

Since $[k]_j = (1-q^{jk})/(1-q^j)$, we have

$$[k][k]_2...[k]_{m-1} = \frac{(1-q^k)(1-q^{2k})...(1-q^{k(m-1)})}{(1-q)(1-q^2)...(1-q^{m-1})} \tag{3.24}$$

and

$$[k]_n[k]_{n-1}...[k]_{n-m+1} = \frac{(1-q^{nk})(1-q^{k(n-1)})(1-q^{k(n-2)})...(1-q^{k(n-m+1)})}{(1-q^n)(1-q^{n-1})...(1-q^{n-m+1})}. \tag{3.25}$$

As a result, we have

$$\frac{[k][k]_2 \ldots [k]_{m-1}}{[k]_n [k]_{n-1} \ldots [k]_{n-m+1}}$$

$$= \frac{(1-q^k)(1-q^{2k})\ldots(1-q^{k(m-1)})(1-q^n)(1-q^{n-1})\ldots(1-q^{n-m+1})}{(1-q^{nk})(1-q^{k(n-1)})\ldots(1-q^{k(n-m+1)})(1-q)(1-q^2)\ldots(1-q^{m-1})}$$

Now we get

$$\begin{bmatrix} kn+i \\ km-1 \end{bmatrix} \frac{[k][k]_2 \ldots [k]_{m-1}}{[k]_n [k]_{n-1} \ldots [k]_{n-m+1}} = \frac{Q_1(q)}{Q_2(q)}, \tag{3.26}$$

where $Q_1(q)$ and $Q_2(q)$ are as follows:

$$\begin{aligned}
Q_1(q) &= (1-q^{kn+i})(1-q^{kn+i-1}) \\
&\ldots(1-q^{kn+1})(1-q^n)(1-q^{kn-1}) \\
&\ldots(1-q^{k(n-1)+1})(1-q^{n-1})(1-q^{k(n-1)-1}) \\
&\ldots \\
&\ldots(1-q^{k(n-m+1)+1})(1-q^{n-m+1})(1-q^{k(n-m+1)-1}) \\
&\cdot(1-q^{k(n-m+1)-2})\ldots(1-q^{k(n-m)+i+2}),
\end{aligned}$$

$$\begin{aligned}
Q_2(q) &= (1-q^{km-1})(1-q^{km-2}) \\
&\ldots(1-q^{k(m-1)+1})(1-q^{m-1})(1-q^{k(m-1)-1}) \\
&\ldots(1-q^{k(m-2)+1})(1-q^{(m-2)})(1-q^{k(m-2)-1}) \\
&\ldots \\
&\ldots(1-q^{k+1})(1-q)(1-q^{k-1}) \\
&\cdot(1-q^{k-2})\ldots(1-q^2)(1-q).
\end{aligned}$$

Note that $Q_1(q)/Q_2(q)$ is almost same as $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$ except that the factor

$$\frac{(1-q^{kn})(1-q^{k(n-1)})\ldots(1-q^{k(n-m+1)})}{(1-q^{k(m-1)})(1-q^{k(m-2)})\ldots(1-q^k)}$$

in $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$ is replaced by

$$\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-m+1})}{(1-q)(1-q^2)...(1-q^{m-1})}.$$

More precisely, $Q_1(q)/Q_2(q)$ is same as $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$ except that each kN exponent in the numerator and denominator has been divided by k. Suppose that $1 \leq j \leq km-1$ and $j|kN$. If $k \not| j$, then $j|kN$ implies that $j|N$. If $k|j$, then $\frac{j}{k}|N$. Hence we still have that each zero of the denominator of $Q_1(q)/Q_2(q)$ appears with at least as large multiplicity in the numerator as in the denominator. Thus the divisibility properties previously described are preserved since the only change dose not affect whether a denominator exponent divides a numerator exponent. ∎

Lemma 3.4.4 *Let $k \geq 2$ be aprime and $0 \leq i \leq k-2$. For any nonnegative integers n and m,$[k]$ is a factor of* $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$.

Clearly, when $q = 1$, we also have Lemma 3.4.1.

Proof. For all non-negative integers n and m, we see that

$$\begin{bmatrix} kn+i \\ km-1 \end{bmatrix} = \frac{(1-q^{nk+i})(1-q^{kn+i-1})...(1-q^{k(n-m+1)})...(1-q^{kn-km+i+2})}{(1-q^{km-1})(1-q^{km-2})...(1-q^{k(m-1)})...(1-q^2)(1-q)}. \quad (3.27)$$

By Lemma 3.4.2, for any integer M, $1-q^M$ has one and only one factor $[k]$ if and only if M is divisible by k. There are $n-(n-m) = m$ factors of $[k]$ in the numerator and $m-1$ factors of $[k]$ in the denominator of $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$. As a result, there is one factor $[k]$ in $\begin{bmatrix} kn+i \\ km-1 \end{bmatrix}$. ∎

Now we are in the position to prove the divisibility properties of the generalized q-Euler numbers.

Theorem 3.4.5 *Let k be any prime and $1 \le i \le k-1$. We have $E_{(nk+i)|k}(q)$ is divisible by $[k][k]_2[k]_3\ldots[k]_n$.*

Remark: When $k = 2$, we have the Andrews and Gessel's first result about the divisibility of the q-tangent numbers in [2].

Proof. Induct on n. For $n = 0$, the result is trivial. Suppose the result is true up to but not including N. By Theorem 3.3.2, we have two cases.

Case 1: If $i = 1$, then

$$E_{(kN+1)|k}(q)$$

$$= \sum_{m=1}^{N} \begin{bmatrix} kN \\ km-1 \end{bmatrix} q^{kN-km+1} E_{(km-1)|k}(q) E_{(kN-km+1)|k}(q),$$

Case 2: If $2 \le i \le k-1$, then

$$E_{(kN+i)|k}(q)$$

$$= \sum_{m=1}^{N} \begin{bmatrix} kN+(i-1) \\ km-1 \end{bmatrix} q^{kN-km+1} E_{(km-1)|k}(q) E_{(kN-km+i)|k}(q) + E_{(kN+i-1)|k}(q)$$

We first consider case 1. By the induction hypothesis, for all $1 \le m \le N$,

$$E_{(km-1)|k}(q) = [k][k]_2[k]_3\ldots[k]_{m-1} P_1(q)$$

and

$$E_{(kN-km+1)|k}(q) = [k][k]_2[k]_3\ldots[k]_{N-m} P_2(q)$$

where $P_1(q)$ and $P_2(q)$ are different polynomials in q. Then

$$\begin{bmatrix} kN \\ km-1 \end{bmatrix} E_{(km-1)|k}(q) E_{(kN-km+i)|k}(q)$$

$$= \begin{bmatrix} kN \\ km-1 \end{bmatrix} \frac{[k][k]_2\ldots[k]_{m-1}}{[k]_N[k]_{N-1}\ldots[k]_{N-m+1}} [k][k]_2[k]_3\ldots[k]_N P_1(q) P_2(q).$$

By Lemma 3.4.3,

$$\begin{bmatrix} kN \\ km-1 \end{bmatrix} \frac{[k][k]_2 ... [k]_{m-1}}{[k]_N [k]_{N-1} ... [k]_{N-m+1}}$$

is a polynomial in q. We have that $[k][k]_2[k]_3...[k]_N$ is a factor of

$$\begin{bmatrix} kN \\ km-1 \end{bmatrix} E_{(km-1)|k}(q) E_{(kN-km+1)|k}(q)$$

for all $1 \leq m \leq N$ and then a factor of

$$\sum_{m=1}^{N} \begin{bmatrix} kN \\ km-1 \end{bmatrix} q^{kN-km+1} E_{(km-1)|k}(q) E_{(kN-km+i)|k}(q).$$

So the result holds in case 1.

For case 2, we see that $[k][k]_2[k]_3...[k]_N$ divides the extra term by an induction on i, where $i = 1$ is done in case 1. So the result also holds in case 2. ■

Theorem 3.4.6 *Let k be prime and $1 \leq i \leq k-1$. $E_{(kn+i)|k}(q)$ is divisible by $[k]^n$.*

Remark: When $k = 2$, we have the Andrews and Gessal's second result about the divisibility of q-tangent numbers in [2].

Proof. Induct on n. For $n = 0$, the result is trivial. Suppose the result is true up to but not including N. By Theorem 3.3.2, we consider the two same cases as in Theorem 3.4.5.

Case 1: If $i = 1$, then

$$E_{(kN+1)|k}(q)$$

$$= \sum_{m=1}^{N} \begin{bmatrix} kN \\ km-1 \end{bmatrix} q^{kN-km+1} E_{(km-1)|k}(q) E_{(kN-km+1)|k}(q)$$

Case 2: If $2 \leq i \leq k-1$, then

$$E_{(kN+i)|k}(q)$$

$$= \sum_{m=1}^{N} \begin{bmatrix} kN+(i-1) \\ km-1 \end{bmatrix} q^{kN-km+i} E_{(km-1)|k}(q) E_{(kN-km+i)|k}(q) + E_{(kN+i-1)|k}(q).$$

We first consider case 1. By the induction hypothesis, for all $1 \leq m \leq N$, then

$$E_{(km-1)|k}(q) = [k]^{m-1} H_1(q)$$

and

$$E_{(kN-km+1)|k}(q) = [k]^{N-m} H_2(q)$$

where $H_1(q)$ and $H_2(q)$ are different polynomials in q. By Lemma 3.4.4, $[k]$ is a factor of $\begin{bmatrix} kN \\ km-1 \end{bmatrix}$. We have that $[k]^{N-m+m-1+1} = [k]^N$ is a factor of

$$\begin{bmatrix} kN \\ km-1 \end{bmatrix} E_{(km-1)|k}(q) E_{(kN-km+1)|k}(q),$$

for all $1 \leq m \leq N$ and then a factor of

$$\sum_{m=1}^{N} \begin{bmatrix} kN \\ km-1 \end{bmatrix} q^{kN-km+1} E_{(km-1)|k}(q) E_{(kN-km+1)|k}(q).$$

So the result holds in case 1.

For case 2, we see that $[k]^N$ divides the extra term by an induction on i, where $i = 1$ is done in case 1. So the result also holds in the case 2. ■

Observe that if put $q = 1$ in Theorem 3.4.5 and Theorem 3.4.6, we have the divisibility properties of $E_{n|k}$ as follows.

Corollary 3.4.7 *If k is prime, then k^n divides $E_{(kn+i)|k}$ for all $1 \leq i \leq k-1$. In particular, when $k = 2$, we have 2^n divides $E_{(2n+1)|2}$.*

Chapter 4

Open Problems and Conjectures

We know that $(n+1)E_{(2n+1)|2}$ is divisible by 2^{2n} but by no higher power of two. More precisely, we have

$$(n+1)E_{(2n+1)|2} = 2^{2n}G_{2n+2} \tag{4.1}$$

where the Genocchi number G_{2n+2} is odd. It is then natural to ask the following question:

Problem: For a fixed prime k, are there two simple functions of n and k, say $a_{n,k}$ and $b_{n,k}$, such that

$$a_{n,k}E_{n|k} = k^{b_{n,k}}G_{n|k} \tag{4.2}$$

with $G_{n|k}$ being an integer that is not divisible by k?

In our study, we see that for each $i, 1 \le i \le k-1$,

$$\frac{(n+1)E_{(nk+i)|k}}{k^{\left\lfloor \frac{kn+i-1}{k-1} \right\rfloor}} \tag{4.3}$$

is an integer, but in general this number and k are not relatively prime. For example: if $k = 5$ and $n = 9$, then $(9+1)E_{(9\cdot5+4)|5}$ has factor 5^{14}, while $\left\lfloor \frac{49-1}{5-1} \right\rfloor = 12$. So the number $(9+1)E_{(9\cdot5+4)|5}/5^{12}$ has factor 5^2. But we believe that $E_{(3n+i)|3}$ behaves exactly like the ordinary Euler numbers.

Conjecture 4.0.8 *For $i = 1$ or 2, we have*

$$(n+1)E_{(3n+i)|3} = 3^{\left\lfloor \frac{3n+i-1}{2} \right\rfloor} G_{(3n+i)|3}$$

such that $G_{(3n+i)|3}$ is an integer that is not divisible by 3.

Note that if $k = 2$, expression (4.3) agrees with the Genocchi number.

The signed Euler number $E'_{n|2}$ is defined by

$$E'_{n|2} = (-1)^{\lfloor \frac{n}{2} \rfloor} E_{n|2}$$

where $\lfloor\ \rfloor$ is the round down function. L. Carlitz [10] found a congruence for $E'_{n|2}$:

$$E'_{(2n)|2} \equiv \begin{cases} 0 \pmod{p^e} & (p \equiv 1 \pmod 4)\ ) \\ 2 \pmod{p^e} & (p \equiv 3 \pmod 4)\ ) \end{cases} \tag{4.4}$$

where p is an odd prime such that $(p-1)p^{e-1}|2n$.

We expect that the generalized Euler number and the signed Euler number have the similar congruence properties.

Bibliography

[1] D. André, Developpements de sec x et de tang x, *C.R. Acad. Sci. Paris* **88** (1879), 965–967.

[2] G. Andrews and I. Gessel, Divisibility properties of the q-tangent numbers, *Proc. Amer. Math. Soc.* **68** (1978), 380–384.

[3] W. N. Bailey, "Generalized Hypergeometric Series," Cambridge University Press, Cambridge, 1935.

[4] G. Birkhoff, " Lattice theory", 3rd ed., Amer. Math. Soc. Providence, RI, 1967.

[5] G. Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. of Math., II ser.* **14** (1913), 42–46.

[6] A. Björner and L. Lovász, Linear decision trees, subspace arrangements and Möbius functions, to appear.

[7] A. Björner, M. Las Vergnas, B. Bturmfels, N, white and G. M. Ziegler, "Oriented Matroids," Cambridge University Press, 1992.

[8] A. Björner, Subspace arrangements, to appear.

[9] A. Blass and B. Sagan, Characteristic, chromatic and Ehrhart polynomials, in preparation.

[10] L. Carlitz, A note on the Euler numbers and polynomials, *Nagoya Math. J* **7** (1954), 35–43.

[11] L. Carlitz, Permutations with prescribed pattern, *Math. Nachr.* **58** (1973), 31–53.

[12] L. Comtet, "Advanced Combinatorics," D. Reidel Pub. Co., Dordrecht, 1974.

[13] H. H. Crapo, The Tutte polynomial, *Aequationes Math.* **3** (1969), 211–229.

[14] H. H. Crapo and G. -C. Rota, On the foundation of combinatorial theory: Combinatoria; geometries, M.I.T. Press 1970.

[15] Leonard Eugene Dickson, "History of the Theory of Number," Chelsea Publishing Company, New York, vol. I, (1966).

[16] D. Dumont, Interpretations combinatorire des nombers de Genocchi, *Duke Math. J.* **41** (1974), 305–318.

[17] D. Dumont, Sur une conjecture de gandhi concernant les nomres de genocchi, *Discrete Math.* **4** (1972), 321–327.

[18] D. Dumont and J. Zeng, A note on the Euler and Genocchi nubers, Actes 24^4 Séminarire Lotharingien, p.87-97.

[19] D. Foata, Further divisibility properties of the q-tangent numbers, *Proc. Amer. Math. Soc.* **81** (1981), 143–148.

[20] J. Francon and G. Viennot, Permutations selon leurs pics, creux, doubles montées et double descentes, numbers d'euler et nombres de Gnocchi, *Discrete Math.* **28** (1979), 21–35.

[21] I. Gessel, Congruence for Bell and Tangent numbers *Fibonacci Quarterly* **19** (1981), 137–144.

[22] H. W. Gould, "Combinatorial Identities," revised edition, Morgantown Printing and Binding Co., Morgantown, W. Va., 1972.

[23] H. W. Gould, The q-Stirling numbers of the first and second kinds, *Duke Math. J.* **28** (1961), 281–289.

[24] L. C. Grove "Algebra" New York, London, 1983.

[25] B. Grübaum,"Arrangements of hyperplanes," *Proc. Second Lousiana Conf. on Combinatoriacs and Graph Theory*, Baton Rouge, (1971), 41-106.

[26] I. Gessel and G. Viennot, Generalized Tangent and Genocchi Numbers, preprint.

[27] P. Hall, "The Eulerian function of a group," *Quart. J. Math. Oxford Ser.* **38** (1936), 134-151.

[28] T. Józefiak and B. E. Sagan, Basic derivations for subarrangements of Coxeter arrangements, *J. Algebraic Combin.* **2** (1993), 291–320.

[29] I. G. Macdonald ," Symmetric Functions and Hall Polynomials," Clarendon Press, Oxford, 1979.

[30] J. W. Moon, "Counting Labeled Trees," Canad. Math. Monographs, Vol. 1, Canad. Math. Congress, Montréal, P.Q., 1970.

[31] P. Orlik and H. Terao, "Arrangements of Hyperplanes," Grundlehren 300, Springer-Verlag, New York, NY, 1992.

[32] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.

[33] A. Ranrianarivony and J. Zeng, Sur une extension des nomberes d'Euler et les records des permutations alternantes, preprint

[34] L. J. Slater, "Generalized Hypergeometric Functions," Cambridge University Press, Cambridge, 1966.

[35] R. P. Stanley, Finite lattices and Jordan-Hölder sets, *Alg. Univ.* **4** (1974), 361-371.

[36] R. P. Stanley, Binomial posets, Möbuis inversion, and permutation enumeration, *J. Combin. Theory Ser. A* **20** (1976), 336-356.

[37] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in "Graph Theory and Its Applications: East and West," Ann. NY Acad. Sci. **576** (1989), 500–535.

[38] R. P. Stanley, Modular elements of geometric lattices, *Algebra Universalis* **1** (1971), 214–217.

[39] R. P. Stanley, Supersolvable lattices, *Algebra Universalis* **2** (1972), 197–217.

[40] R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986.

[41] R. P. Stanley, Some aspects of groups acting on finite posets, *J. Combin. Theory Ser. A* **32** (1982), 132-161.

[42] R. P. Stanley, "Ordered Structures and Partitions," Ph. D. thesis, Harvard University, Cambride, MA, 1971.

[43] M. Ward, The algebra of lattice functions, *Duke Math. J.*, **5** (1939), 357–371.

[44] L. Weisner, " Abstract theory of the inversion of finite series," *Trans. Amer. Math. Soc.* **2** (1935), 474–484.

[45] T. Zaslavsky, "Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes," Memoirs Amer. Math. Soc., No. 154, Amer. Math. Soc., Providence, RI, 1975.

[46] T. Zaslavsky, The geometry of root systems and signed graphs, *Amer. Math. Monthly* **88** (1981), 88–105.

[47] T. Zaslavsky, Signed graphs, *Discrete Applied Math.* **4** (1982), 47–74.

[48] T. Zaslavsky, Signed graph coloring, *Discrete Math.* **39** (1982), 215–228.

[49] T. Zaslavsky, Chromatic invariants of signed graphs, *Discrete Math.* **42** (1982), 287–312.