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SOME AMALGAMS IN CHARACTERISTIC 3
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Panagiotis Papadopoulos

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By

Panagiotis Papadopoulos

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ABSTRACT

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Co_1 has parabolic subgroups of the shape $3^6 2M_{12}$ and $3^{1+4}Sp_4(3).2$

Lyons' simple group Ly has parabolics of the form $3^5(M_{11} \times 2)$ and $3^{2+4}A_5D_8$. We will characterize these parabolics and similarly the ones found in subgroups of Co_1 using the amalgam method introduced by Goldschmidt.

Dedicated

to the memory of Amanda

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Contents

1 Introduction

Let G be a finite group, p a prime, S a Sylow p-subgroup of G and $B = N_G(S)$. A **parabolic subgroup** of G is a proper subgroup of G which contains a conjugate of B. Consider the set $\mathcal{J}$ of parabolic subgroups of G ordered by inclusion; then $\mathcal{J}$ becomes a partially ordered set called **the parabolic geometry** of G. In the case where G is a finite simple group of Lie type in characteristic p, $\mathcal{J}$ is the usual building given by Lie theory. The parabolic geometry may be viewed as a generalization of the concept of a building to an arbitrary group. In recent years the parabolic geometry (in particular for p=2) has been used to study, construct, characterize and prove uniqueness of many of the sporadic finite simple groups. The parabolic geometries (again for p=2) also play an important role in the ongoing revision of the classification of the finite simple groups, in particular in the so called quasi-thin and uniqueness cases.

While parabolic subgroups have most intensively been studied for p=2, many interesting examples exist (besides the groups of Lie type) for arbitrary primes. In [RS], Ronan and Stroth determined all the minimal parabolic geometries for all the 26 sporadic groups and all primes. One of the most interesting series of examples arises for the prime p=3 and G being the first Conway group Co_1.

Co_1 has parabolic subgroups of the shape $3^6 2M_{12}$ and $3^{1+4}Sp_4(3).2$ (we will explain this notation later).

Lyons' simple group Ly has parabolics of the form $3^5(M_{11} \times 2)$ and $3^{2+4}A_5D_8$. These parabolics have been used by M. Aschbacher and Y. Segev to prove the uniqueness of the Ly. It is the goal of this paper to characterize these parabolics and similarly the ones found in subgroups of Co_1.

Let B_G be the largest normal subgroup of G contained in B; then B_G is contained in all the parabolic subgroups of G and thus acts trivially on the parabolic geometries and so the parabolic geometry carries out information only about G/B_G. Also when B is contained in a unique maximal parabolic subgroup of G, the parabolic geometry becomes disconnected.

So let us assume that $B_G = 1$ and that P_1 and P_2 are parabolic subgroups of G containing B, with $G = \langle P_1, P_2 \rangle$, for example two different maximal parabolic subgroups of G. Then we see that G, P_1 and P_2 fulfill the following statement:

(A_1) P_1 and P_2 are finite subgroups of G.

(A_2) $G = \langle P_1, P_2 \rangle$.

(A_3) Let $S \in Syl_p(P_1 \cap P_2)$ and $B = N_{P_1 \cap P_2}(S)$; then $B = N_{P_i}(S)$, i=1, 2. In particular $S \in Syl_p(P_i)$, i=1, 2.

(A_4) No non-trivial normal subgroup of G is contained in B.

If (G, P_1, P_2) fulfill $(A_1) - (A_4)$, we say that G is an **amalgamated product of P_1 and P_2.**

We remark that we allow G to be infinite in this definition in order to cover the case where $G = P_1 *_B P_2$, the free amalgamated product of P_1 and P_2 over B (see [S] for a precise definition). Notice that if (G, P_1, P_2) is an amalgamated product then also $(P_1 *_B P_2, P_1, P_2)$ is an amalgamated product in our sense.

To any amalgamated product (G, P_1, P_2) we can associate a graph Γ whose vertices are the cosets of P_1 and P_2 in G and two cosets are adjacent if they are distinct and have non-empty intersection. We remark that if $B = N_G(S)$ then the graph Γ can be embedded into the parabolic geometry of G.

The amalgamated method introduced by Goldschmidt [G] and refined by Stellmacher [St], Delgado [DS] and Timmesfeld [T] uses Γ as a tool to define important subgroups of B and as a book-keeping device to determine relations between the subgroups. This method has proven very succesful in determining the structure of P_1 and P_2 assuming the action of P_1 and P_2 on their neighbours $\Delta(P_1)$ and $\Delta(P_2)$ respectively in the graph Γ is given.

Let us assume for simplicity that $P_1 \cap P_2 = B$ (which will always be true for $G = P_1 *_B P_2$). Let $Q_i = O_p(P_i)$, $L_i = O^{p'}(P_i) = \langle S^{P_i} \rangle$ and $P_i^{(1)}/Q_i = C_{G_i/Q_i}(L_i/Q_i)$. Then it is easy to see that $P_i^{(1)}$ is precisely the kernel of the action of P_i on $\Delta(P_i)$ and L_i acts transitively on $\Delta(P_i)$. Hence the group L_i/Q_i carries most of the information about the action of P_i on $\Delta(P_i)$ and we then refer to the pair $(L_1/Q_1, L_2/Q_2)$ as the type of the amalgamated product (G, P_1, P_2).

The main task of the amalgam method can now be described as determining (P_1, P_2) from the type $(L_1/Q_1, L_2/Q_2)$. For example, the main part of Goldschmidt's paper [G] determines the structure of (P_1, P_2) of type (Sym(3),Sym(3)) for p=2.

For the remainder of this paper we will work under the following hypothesis:

(P) (G, P_1, P_2) is an amalgamated product of type (Θ, Ψ) for p=3 so that:

(P_0) $\Theta \cong PSL_2(9)$, M_{11}, M_{12} or $2 \cdot M_{12}$,

(P_1) $\Psi \cong PSL_2(3)$, $SL_2(3)$, A_5, $2 \cdot A_5$, $2^4 A_5$, $2_-^{1+4} A_5$, $PSL_2(9)$, $SL_2(9)$, $Sp_4(3)$ or $PSp_4(3)$,

(P_2) $C_{P_i}(O_3(P_i)) \leq O_3(P_i)$ for i=1, 2.

Before we state the main theorem recall the following standard definitions:
For a finite group X and a prime number p,

$O_p(X)$ is the largest normal p-subgroup of X;

$O^p(X)$ is the smallest normal subgroup of X such that $G/O^p(X)$ is a p-group, or, equivalently, the group generated by all p' elements;

$O^{p'}(X)$ is the smallest normal subgroup of X such that its index in X is not divisible by p, or, equivalently, the group generated by all Sylow p-subgroups of X.

Now introduce the following notation:
$G \sim 3^{d_1 + \cdots + d_n} H$ means that there exists a normal series

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n \leq G,$$

so that for i=1, 2, $\cdots$, n, H_i/H_{i-1} are elementary abelian minimal normal subgroups of G/H_{i-1} with $|H_i/H_{i-1}| = 3^{d_i}$ and $G/H_n \cong H$.

Also, by $G \sim 2{\cdot}H$ we mean that $G/Z(G) \cong H$, $|Z(G)| = 2$ and $Z(H) \leq H'$.

We are now able to state our main result.

THEOREM P: Under hypothesis P the possible pairs (L_1, L_2) are as follows:

(i) $(3^4PSL_2(9), 3^{1+4}2{\cdot}A_5)$,

(ii) $(3^4PSL_2(9), 3^{1+4}2_-^{1+4}A_5)$,

(iii) $(3^62{\cdot}M_{12}, 3^{1+1+1+2+2+1}SL_2(3))$,

(iv) $(3^62{\cdot}M_{12}, 3^{1+4}Sp_4(3))$,

(v) $(3^5M_{11}, 3^{1+1+2+2}SL_2(3))$,

(vi) $(3^5M_{11}, 3^{1+4}SL_2(9))$,

(vii) $(3^4PSL_2(9), 3^{1+2+2}SL_2(3))$,

(viii) $(3^5M_{11}, 3^{1+1+4}2{\cdot}A_5)$,

(ix) $(3^6PSL_2(9), 3^{1+1+4}SL_2(9))$.

Note that the examples for (i)-(ix) can be found in $G \cong McL$, Co_2, Co_1, Co_1, Suz, Co_3, $U_4(3)$, Ly and $PSp_4(9)$ respectively. We will see later that all the cases occur when b=1 and $[Z_\alpha, Z_{\alpha'}] = 1$ where the notation in this remark will become apparent momentarily.

2 Properties of Θ, Ψ and their modules

In this section we will list some of the properties of the groups Θ and Ψ and their modules.

A non-abelian p-group P (p a prime) is **called extra-special** if

$$|\Phi(P)| = |Z(P)| = p.$$

There are two extra-special groups of order 2^5; one, denoted by 2_+^{1+4}, contains an elementary abelian subgroup of order 8 and the other one, denoted by 2_-^{1+4}, does not.

It is a well-known fact (see for example [Go; 5.5.2]) that

$$2_-^{1+4} \cong Q_8 * D_8,$$

where * denotes the central product.

A **Steiner system S(l, m, n)** is a pair $(\Omega, \mathcal{B})$, where Ω is a set of size n, $\mathcal{B}$ is a set of subsets of size m called blocks and such that every subset of size l in Ω lies in a unique member of $\mathcal{B}$.

By [W], there exists a unique, up to isomorphism, Steiner system of type S(5, 6, 12). Let $\mathcal{S}$=S(5, 6, 12). Define then the **Mathieu group on 12 points** to be the group

$$M_{12} = Aut(\mathcal{S}) = \{\pi \in Sym(12) | B^{\pi} \textit{ is a block for all blocks } B\}.$$

Define M_{11} to be the stabilizer of a point in M_{12}. Then M_{11} is 4-transitive on eleven points and its corresponding Steiner system is S(4, 5, 11).

Lemma 2.1 *(a) M_{12} is sharply 5-transitive on 12 points, i.e., M_{12} is 5-transitive on 12 points and the stabilizer of any five points in M_{12} is the identity group.*

(b) $|M_{12}| = 12{\cdot}11{\cdot}10{\cdot}9{\cdot}8 = 2^6{\cdot}3^3{\cdot}5{\cdot}11$.

(c) The normalizer of a Sylow 3-subgroup of M_{12} has orbits of lengths 3 and 9 and therefore if an involution acts on these it has a fixed point.

(d) M_{12} has two classes of involutions, say D_1 and D_2. Moreover

$x \in D_1$ *if and only if x fixes a point*

if and only if x fixes four points

if and only if x belongs to a normalizer of a Sylow 3-subgroup of M_{12}

if and only if x lifts to an involution in $2{\cdot}M_{12}$.

Proof: [A] and [Gr].

Notation 2.2 To avoid repetitions we will use the following notation throughout:

$X \cong (2)H$ means that either $X \cong 2{\cdot}H$ or $X \cong H$. Similarly, $X \cong 2^{(1+)4}A_5$ will denote a group X such that $X/O_2(X) \cong A_5$, $O_2(X)/O_2(X)'$ is the even permutation module on five letters for A_5 and $|O_2(X)'|$=1 or 2 respectively.

Definition 2.3 *Let X be a finite group. Slightly abusing the standard definition we will say that X is 3-stable provided that the following condition holds: If V is an irreducible GF(3)X-module and* $A \leq X$ *is such that* $[V, A, A] = 1$ *then* $[V, A] = 1$.

Lemma 2.4 *Let Y be a finite group. Then:*

(a) The following statement is equivalent to Y being 3-stable: let V be any GF(3)-module and $A \leq Y$ *with* $[V, A, A] = 1$. *Then* $AC_Y(V)/C_Y(V) \leq O_3(Y/C_Y(V))$.

(b) Y is 3-stable if and only if $Y/O_3(Y)$ *is 3-stable.*

(c) If every element of order 3 in Y lies in a perfect simple 3-stable subgroup of Y then Y is 3-stable.

Proof: (a) Suppose first that Y is 3-stable and let V and A as in the statement. Let W be any compposition factor for Y and V. Then $[W, A, A] = 1$ and so by definition of 3-stable, $[W, A] = 1$. Hence $AC_Y(V)/C_Y(V) \leq O_3(Y/C_Y(V))$.

Suppose next that the statement holds and let V be any irreducible GF(3)Y-module. Then $O_3(Y/C_Y(V)) = 1$ and so by the statement $A \leq C_Y(V)$ and Y is 3-stable.

(b) It is clear, since $O_3(Y)$ acts trivially on every irreducible GF(3)Y-module.

(c) Suppose that V is a GF(3)Y-module and $a \in V$ with $[V, a, a] = 1$. Then $a^3 = 1$ and we may assume that $|a| = 3$. Then $a \in X \leq Y$, where X is perfect and 3-stable. Then $[W, a, a] = 1$ for any composition factor W for X on V and so $[W, a] = 1$ and since $X = \langle a^X \rangle$, X is simple and $[W, X] = 1$ we have $[V, \underbrace{X, \ldots, X}_{(n-times)}] = 1$ for some n and since X is perfect, $[V, X] = 1$ and $[V, a] = 1$.

Remark 2.5 It follows directly from [Go; p.111] that $PSL_2(3)$, A_5 and $PSL_2(9)$ are all 3-stable. It is also easy to see that any element of order 3 in M_{11} or $(2)M_{12}$ lies in a subgroup A_5 of these groups and since A_5 is 3-stable, 2.4 implies that so are M_{11} and $(2)M_{12}$. Finally, 2^4A_5 is 3-stable as it contains A_5 which in turn it contains a Sylow 3-subgroup of 2^4A_5.

Definition 2.6 *A GF(3)X-module V is called an FF-module for X if $C_X(V) = 1$ and if there exists a non-identity 3-subgroup A of X such that $|V|/|C_V(A)| \leq |A|$.*

Lemma 2.7 *If X has an irreducible FF-module then X is not 3-stable.*

Proof: It follows from Thompson's Replacement Theorem, see [Go; 8.2.4].

Lemma 2.8 *Let $X \cong \Theta$, $PSL_2(3)$, $(2)A_5$, 2^4A_5, $2^{1+4}_-A_5$ or $PSp_4(3)$. Then X does not have an FF-module.*

Proof: The proof for Θ, $PSL_2(3)$, A_5 and 2^4A_5 follows from 2.5 and 2.7 and the proof for $PSp_4(3)$ can be found in [M]. So we only worry about the cases $2{\cdot}A_5$ and $2^{1+4}_-A_5$, namely the cases where $X/O_2(X) \cong A_5$.

Let V be a faithful irreducible GF(3)X-module. Let A be a non-trivial 3-subgroup of X and suppose that $|V/C_V(A)| \leq |A|$. We want a contradiction. First, $|A| = 3$ since $|X|_3 = 3$ where $|X|_3$ denotes the 3-part of X. Second, since there exists an element of order 5 in X and since 5 does not divide

We can choose $d_1, d_2 \in L_\delta$ (where $X \cong L_\delta/Q_\delta$) of order 3 such that $D := \langle d_1, d_2 \rangle$ has a quotient A_5. Since $C_V(D) = C_V(d_1) \cap C_V(d_2)$ has codimension less than or equal to two in V and since $GL_2(3)$ is solvable, D acts trivially on V, a contradiction since $[V, d_1] \neq 0$. Hence X does not have an FF-module.

Definition 2.9 *(a) Let $X \cong Sp_4(3)$. A faithful GF(3)X-module W is called a natural $Sp_4(3)$-module for X, if W carries the structure of a 4-dimensional symplectic space over GF(3) which is invariant under the action of X.*

(b) Let $X \cong SL_2(3^k)$ and W a faithful GF(3)X-module. Then W is called a natural $SL_2(3^k)$-module for X if W carries the structure of a 2-dimensional vector space over $GF(3^k)$ invariant under the action of X.

It is worth mentioning at this point that

$$A_4 \cong PSL_2(3) \text{ and } 2{\cdot}A_4 \cong SL_2(3),$$

$$A_5 \cong PSL_2(5) \text{ and } 2{\cdot}A_5 \cong SL_2(5),$$

$$A_6 \cong PSL_2(9) \text{ and } 2{\cdot}A_6 \cong SL_2(9),$$

and

$$U_4(2) \cong PSp_4(3) \text{ and } 2{\cdot}U_4(2) \cong Sp_4(3).$$

Remark 2.10 (i) $SL_2(3)$ has a unique faithful irreducible GF(3)-module; moreover, this module is an FF-module and its order is 3^2.

(ii) $PSL_2(9)$ has four irreducible GF(3)-modules; their dimensions are: 1, 4, 6 and 9.

(iii) Let $X = SL_2(3)$, $SL_2(9)$ or $Sp_4(3)$ and let V be an FF-module. Then

$$V = [V, Z(X)] \oplus C_V(X)$$

and [V,Z(X)] is a natural $SL_2(3)$, $SL_2(9)$ or $Sp_4(3)$-module respectively. [M; p.469 and 470]

(iv) M_{11} has two irreducible modules of dimension less than or equal to 8; moreover, both have dimension five and they are dual to each other. [J]

(v) $2{\cdot}M_{12}$ has a unique non-trivial irreducible GF(3)-module of dimension less than 10; moreover this module has dimension six and is faithful; in particular, M_{12} does not have any non-trivial module of dimension less than 10. [J]

Lemma 2.11 *Let $G \cong (P)SL_2(3)$, $(P)SL_2(9)$, M_{11} or $(2)M_{12}$. Then G has no automorphism of order 2 centralizing a Sylow 3-subgroup.*

Proof: Well-known, see for example [A].

Lemma 2.12 *Let X be any of our groups Θ or Ψ, $S_1 \in Syl_3(X)$ and $B_1 = N_X(S_1)$. Then B_1 is irreducible on $Z(S_1)$; in particular B_1 is irreducible on S_1 for $X \cong (P)SL_2(3)$, $(2)A_5$, 2^4A_5, $2^{(1+)4}A_5$, $(P)SL_2(9)$ or M_{11}.*

Proof: If $|X|_3 = 3$ then $|S_1| = 3$ and the lemma holds trivially.

$$If\ X \cong (2)M_{12}\ or\ (P)Sp_4(3)\ then\ |Z(S_1)| = 3$$

and

$$if\ X \cong PSL_2(9)(\cong A_6)\ then\ S_1 = <(123),(456)>\ and\ B_1 = S_1<(1425)(36)>$$

and the lemma holds for $PSL_2(9)$ and so also for $SL_2(9)$. Since $PSL_2(9) \leq M_{11}$ and $|M_{11}|_3 = |PSL_2(9)|_3$ it also holds for M_{11}.

Lemma 2.13 *Let $H \in \Theta$, $T \in Syl_3(H)$, $t \in Aut(H)$ with $|t| = 2$, $[N_H(T), t] \leq T$ and $T^t = T$. Then t is an inner automorphism.*

Proof: Suppose first that Z(H)=1. View H as a subgroup of Aut(H).

Suppose $H = PSL_2(9)$. By 2.12, $N_H(T)$ is irreducible on H and so t either inverts T or centralizes T. Now the same is true for any involution z in $N_H(T)$. Hence by 2.11, both t and z invert T and so, again by 2.11, tT=zT and $t \in H$. If $H = M_{11}$, Aut(H)=H and we are done. If $H = M_{12}$, $N_H(T)/T \cong C_2 \times C_2$ and $N_{Aut(T)}(T)/T \cong D_8$ (see [A]); hence no element in $N_{Aut(T)}(T) \setminus Inn(H)$ centralizes $N_H(T)/T$.

If $Z(H) \neq 1$ then by the previous case, t induces an inner automorphism s^* on H/Z(H). Pick s in H with $sZ(H) = s^*$. Then $[s^{-1}t, H] \leq Z(H)$ so $[s^{-1}t, H, H] = 1$. Since $H' = H$, the 3-subgroup lemma now implies $[s^{-1}t, H] = 1$.

Lemma 2.14 *Let $H = PSL_2(9)$, M_{11} or $(2)M_{12}$ and $T \leq R \in Syl_3(H)$ with $|T| = 9$. Then:*

(a) $H = <R, R^g>$ for some $g \in H$.

(b) If $H = (2)M_{12}$ then $H = <T, R'^g> = <T, T^g> = <R, R'^g>$ for some $g \in H$.

Proof: Note first that

(∗) $PSL_2(9) \cong A_6 = <(123),(125)(346)>$

and in particular (a) holds for $H = PSL_2(9)$.

Clearly the statement for M_{12} implies the statement for $2{\cdot}M_{12}$ and we may assume now that $H = M_{11}$ or M_{12}.

Let $(\Omega, \mathcal{B})$ be a Steiner system of type S(4, 5, 11) and S(5, 6, 12) respectively with $H = Aut(\Omega, \mathcal{B})$. Let $D \leq T$ with $|D| = 3$ if $H = M_{11}$ and $D = R'$ if $H = M_{12}$. Then, in any case, $D \leq T$ and D normalizes a block $B \in \mathcal{B}$. Hence $N_H(B) \cong$Sym(5) or Sym(6) respectively. In the M_{12} case, $N_H(\{B, \Omega \setminus B\})(\cong Aut(A_6))$ interchanges the two conjugacy classes of elements of order 3 in $N_H(B)$. Hence, using

$$A_5 = <(123),(345)> \textit{ and } (*) \textit{ respectively},$$

$$N_H(B)' = <D, D^g> \textit{ for some } g \in H.$$

It is easy to see that $N_H(\{B, \Omega \setminus B\})$ is the unique maximal subgroup of H containing $N_H(B)'$ (see for example [A]). Since $R \not\leq N_H(\{B, \Omega \setminus B\})$ (by Lagrange's Theorem), (a) is proved.

Also, (b) holds unless $T \leq N_H(B)$. So assume $H = M_{12}$ and $T \leq N_H(B)$. Then T has four orbits of length 3 on Ω. Let X be a set of size two in Ω normalized by D. Then $T \not\leq N_H(X)$, $N_H(X) \cong Aut(A_6)$ and $N_H(X)' = <D, D^g>$ for some $g \in N_H(X)$ by (∗). $H = <T, D^g>$ and the lemma is proved.

Lemma 2.15 *The normalizer of a Sylow 3-subgroup is maximal in $SL_2(3)$ and in $SL_2(9)$; for $Sp_4(3)$ the maximal overgroups of a normalizer of a Sylow 3-subgroup are $N(E_1)$ and $N(E_2)$ where E_i, i=1 or 2 is the i-dimensional singular subspace of W normalized by the Sylow 3-subgroups (W any natural GF(3)-module for $Sp_4(3)$).*

Proof: [C; 8.3.2 and 11.3.2].

3 Properties of the graph Γ

In this section we will define a graph Γ and we will list some of its properties.

Definition 3.1 *Let* $\Gamma = \{P_i x | x \in G, i = 1,2\}$. *From now on, small Greek letters will always denote elements of* Γ. *Make* Γ *into a graph by defining* α *to be adjacent to* β *if and only if* $\alpha \neq \beta$ *and* $\alpha \cap \beta \neq \emptyset$. *Then* G *operates on* Γ *by right multiplication.*

For $\delta \in \Gamma$ *let* $G_\delta = Stab_G(\delta)$, $G_\delta^{(n)}$ = *largest normal subgroup of* G_δ *fixing all vertices of distance at most* n *from* δ *and* $\Delta(\delta)$ *the set of all vertices adjacent to* δ.

Lemma 3.2 *Let i=1, 2. Then:*

(a) $G_{P_i x} = P_i^x$,

(b) The edge-stabilizers in G *are conjugate to* B,

(c) Let $\delta_i = P_i$. *Then* $\Delta(\delta_i) \cong P_i/B$ *as a* G_{δ_i}*-set; in particular,* G_{δ_i} *is transitive on* $\Delta(\delta_i)$,

(d) Let (δ, λ) *be an edge; then* $G = <G_\delta, G_\lambda>$,

(e) G *acts faithfully on* Γ,

(f) Γ *is connected.*

Proof: (a), (b) and (d) follow directly from the definitions; we will now prove the rest of the claims.

(c) [DS; 2.1(c)].

(e) Let $g \in G$ be such that $\gamma^g = \gamma$ for all $\gamma \in \Gamma$. Then $P_i g = P_i$ and therefore $g \in P_i$. Also, if $h \in G$ then $\gamma^{g^h} = \gamma$ for all $\gamma \in \Gamma$. Hence $<g^G> \leq B$ and claim is proved by (A_4).

(f) Let Γ_0 be the connected component of Γ containing P_1. Then also $P_2 \in \Gamma_0$. By (a), $<P_1, P_2> \leq N_G(\Gamma_0)$ and (A_2) implies $\Gamma = \Gamma_0$.

Notation 3.3 Let d(,) denote the usual distance on the graph Γ.

For $\delta \in \Gamma$ and $i \geq 1$,

$\Delta^{(i)}(\delta) = \{\lambda \in \Gamma / d(\delta, \lambda) \leq i\}$,

$Q_\delta = O_3(G_\delta)$,

$Z_\delta = <\Omega_1 Z(T)/T \in Syl_3(G_\delta)>$,

$V_\delta = <Z_\lambda / \lambda \in \Delta(\delta)>$,

$b_\delta = min_{\delta' \in \Gamma}\{d(\delta, \delta')/Z_\delta \not\leq G_{\delta'}^{(1)}\}$,

$b = min_{\delta' \in \Gamma}\{b_{\delta'}\}$,

$G_{\delta\lambda} = G_\delta \cap G_\lambda$ and $Q_{\delta\lambda} = Q_\delta \cap Q_\lambda$ if $\delta \in \Delta(\lambda)$.

A pair of vertices (δ, δ') such that $Z_\delta \not\leq G_{\delta'}^{(1)}$ and $d(\delta, \delta') = b$ is called a critical pair.

The bounding of the parameter b which we just introduced, will allow us to deduce a considerable amount of information about P_1 and P_2.

Lemma 3.4 *(a) G acts edge- but not vertex-transitively on* Γ,

(b) G_δ *is finite,*

(c) $C_{G_\delta}(Q_\delta) \subseteq Q_\delta$,

(d) If α *is adjacent to* β *then* $Syl_3(G_\alpha \cap G_\beta) \subseteq Syl_3(G_\alpha) \cap Syl_3(G_\beta)$.

Proof: [DS; p.73].

Remark 3.5 Notice that as G acts edge-transitively, $b = min\{b_\alpha, b_\beta\}$ for any pair of adjacent vertices α, β. Thus, we are allowed to choose α, β such that $b_\alpha = b \leq b_\beta$ and $\{G_\alpha, G_\beta\} = \{P_1, P_2\}$. In particular, $G_\alpha \cap G_\beta = B$ and $S \in Syl_3(G_\alpha) \cap Syl_3(G_\beta)$.

Let $\alpha' \in \Gamma$ such that $d(\alpha, \alpha') = b$ and $Z_\alpha \not\leq G_{\alpha'}^{(1)}$. Let p be a path of length b from α to α'. We label the vertices of p by

$$p = (\alpha, \alpha + 1, \ldots, \alpha + b) = (\alpha' - b, \cdots, \alpha' - 1, \alpha'),$$

i.e. α+i (respectively α'−i) is the unique vertex in p with d(α,α+i)=i (respectively d(α' -i)=i). Furthermore, from 3.2 (c) we may assume that

$$\beta = \alpha + 1 \; if \; b \geq 1.$$

Note also that if $Q_\delta = Q_\lambda$ for some $\delta \in \Delta(\lambda)$ then $Q_\delta \trianglelefteq \langle G_\delta, G_\lambda \rangle = G$, a contradiction. Hence

$$Q_\delta \neq Q_\lambda \ \forall \delta \in \Delta(\lambda).$$

Lemma 3.6 *Let (δ, λ) be an edge and N a subgroup of $G_{\delta,\lambda}$ such that $N_{G_\mu}(N)$ acts transitively on $\Delta(\mu)$ for $\mu \in \{\delta, \lambda\}$. Then N=1.*

Proof: See [DS; (3.2)].

Lemma 3.7 *For $\delta \in \Gamma$,*

(a) $Q_\delta \leq G_\delta^{(1)}$,

(b) $Z_\delta \leq Z(Q_\delta) \cap V_\delta$; in particular, $b \geq 1$,

(c) $Z_{\alpha'} \leq G_\alpha$ and $[Z_\alpha, Z_{\alpha'}] \leq Z_\alpha \cap Z_{\alpha'}$,

(d) $Z_\alpha \neq \Omega_1 Z(T)$, $T \in Syl_3(G_\alpha)$,

(e) If $S \in Syl_3(B)$ and $\Omega_1(Z(S))$ is centralized by a subgroup R of G_β which acts transitively on $\Delta(\beta)$ then $Z(L_\alpha) = 1$,

(f) $G_\delta^{(1)}/Q_\delta$ is a p'-group.

Proof: (a) Let $\lambda \in \Delta(\delta)$ and $T_0 \in Syl_3(G_{\delta\lambda})$. Then

$$Q_\delta = O_3(G_\delta) \leq T_0 \leq G_\lambda.$$

Hence $Q_\delta \in G_\lambda \ \forall \lambda \in \Delta(\delta)$ and therefore $Q_\delta \leq G_\delta^{(1)}$.

(b) $Z_\delta \leq V_\delta$ is immediate from the definitions. Now show that $Z_\delta \leq Z(Q_\delta)$. Since $C_{G_\delta}(Q_\delta) \leq Q_\delta$ it is enough to show $Z_\delta \leq C_{G_\delta}(Q_\delta)$. Let T_0 be a Sylow 3-subgroup of G_δ containing Q_δ. Then $Q_\delta = Q_\delta^g \leq T_0^g$ for every $g \in G_\delta$. Hence $Q_\delta \leq T$ for every $T \in Syl_3(G_\delta)$. As T centralizes Z(T) so does Q_δ for every $T \in Syl_3(G_\delta)$. Thus, by the definition of Z_δ, $[Z_\delta, Q_\delta] = 1$ and $Z_\delta \leq C_{G_\delta}(Q_\delta)$. In particular, by (a) $Z_\alpha \leq G_\alpha^{(1)}$ and so $\alpha \neq \alpha'$ and $b \geq 1$.

(c) Minimality of b gives $Z_{\alpha'} \leq G_{\alpha+1}^{(1)} \leq G_\alpha$. In particular, $Z_{\alpha'}$ normalizes Z_α. Hence $[Z_\alpha, Z_{\alpha'}] \leq Z_\alpha$. Similarly, $[Z_\alpha, Z_{\alpha'}] \leq Z_{\alpha'}$.

(d) Assume that $Z_\alpha = \Omega_1 Z(T)$ for some $T \in Syl_3(G_\alpha)$. Hence $S = T^g$ for some $g \in G_\alpha$ and therefore

$$\Omega_1(Z(S)) = \Omega_1(Z(T^g)) = (\Omega_1(Z(T)))^g = Z_\alpha$$

where the last equality holds because $Z_\alpha \trianglelefteq G_\alpha$. Thus $Z_\alpha \leq Z_\beta$. Since $Z_\alpha \nleq G_{\alpha'}^{(1)}$ we get that $Z_\beta \nleq G_{\alpha'}^{(1)}$ and since $d(\alpha', \alpha) = b - 1$, we get a contradiction to minimality of b.

(e) Let R be a subgroup of G_β that acts transitively on $\Delta(\beta)$ and such that $\Omega_1(Z(S))$ is centralized by R. Then

$$\Omega_1(Z(L_\alpha)) \leq C_{G_\alpha}(Q_\alpha) \leq Q_\alpha \leq S.$$

Hence $\Omega_1(Z(L_\alpha)) \leq \Omega_1(Z(S))$ and therefore $\Omega_1(Z(L_\alpha))$ is also centralized by R. Thus, by 3.6, $\Omega_1(Z(L_\alpha)) = 1 = Z(L_\alpha)$.

(f) Without loss of generality, $\delta \in \{\alpha, \beta\}$. Since $B = N_{G_\delta}(S)$ we get

$$O_3(B) \in Syl_3(B);$$

hence $B/O_3(B)$ is a $3'$-group. Let $Q = O_3(G_\delta^{(1)})$. Then, as $G_\delta^{(1)} \leq B$, $G_\delta^{(1)}/Q$ is a $3'$-group. Now Q is a normal characteristic subgroup of $G_\delta^{(1)}$ which is normal in G_δ and so $Q \trianglelefteq G_\delta$. But Q is a 3-group. Thus $Q \leq O_3(G_\delta) \trianglelefteq G_\delta^{(1)}$ and we conclude $Q = O_3(G_\delta)$. Thus $G_\delta^{(1)}/O_3(G_\delta)$ is a $3'$-group.

Remark 3.8 (i) By 3.7(b), $Z_\alpha \nleq Q_{\alpha'}$.

(ii) $Z_\delta \leq G_\gamma \ \forall \gamma \in \Delta^{(1)}(\delta)$, $B = G_{\alpha\beta}$, $Z_\alpha \leq B$, $Z_\beta \leq B$.

(iii) Also $Syl_3(B) \subseteq Syl_3(P_1) \cap Syl_3(P_2)$.

(iv) Frattini argument gives $L'_\delta S = L_\delta$ and for $\mu \in \Delta(\delta)$, $G_\delta = L_\delta G_{\delta\mu}$.

Lemma 3.9 *(1)* $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$,

(2) $V_\delta \trianglelefteq G_\delta \ \forall \delta \in \Gamma$,

(3) Z_α *normalizes* $V_{\alpha'}$.

Proof: (1) $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] \leq [Z_\alpha \cap Z_{\alpha'}, Z_{\alpha'}] \leq [Z_{\alpha'}, Z_{\alpha'}] = 1$.

(2) For $\varepsilon \in \Delta(\delta)$ and $g \in G_\delta$, $Z_\varepsilon^g = Z_{\varepsilon^g} \leq V_\delta$ as $d(\varepsilon^g, \delta) = d(\varepsilon^g, \delta^g) = d(\varepsilon, \delta) = 1$.

(3) It follows from $V_{\alpha'} \trianglelefteq G_{\alpha'}$ and $Z_\alpha \leq G_{\alpha'}$.

Lemma 3.10 *If $b>2$ then V_β is abelian.*

Proof: Let $\lambda, \varepsilon \in \Delta(\beta) \cup \{\beta\}$; then $d(\lambda, \varepsilon) \leq 2 < b$ so $Z_\lambda \leq Q_\varepsilon$ and

$$[Z_\lambda, Z_\varepsilon] \leq [Q_\varepsilon, Z_\varepsilon] = 1.$$

Hence $[V_\beta, V_\beta] = 1$.

Lemma 3.11 *If $Z_\delta \leq Z(L_\delta)$ then $Z_\delta \leq Z_\lambda$ $\forall \lambda \in \Delta(\delta)$.*

Proof: Let $T \in Syl_3(G_{\delta\lambda})$, $\lambda \in \Delta(\delta)$. By Frattini argument we now have that

$$\Omega_1 Z(T) \trianglelefteq L_\delta N_{G_\delta}(T) = G_\delta.$$

Hence,

$$Z_\delta = <\Omega_1 Z(T)^{G_\delta}> = \Omega_1 Z(T) \leq Z_\lambda.$$

Corollary 3.12 *(a) $Z_\alpha \not\leq Z(L_\alpha)$,*

(b) If $Z_{\alpha'} \leq Z(L_{\alpha'})$ then α is not conjugate to α'.

Proof: (a) follows from 3.11 as $Z_\alpha \not\leq Z_\beta$. For (b) notice that if α were conjugate to α' then, since by (a) $Z_\alpha \not\leq Z(L_\alpha)$, we get $Z_{\alpha'} \not\leq Z(L_{\alpha'})$ and we are done.

Corollary 3.13 $Z_\alpha \cap Q_{\alpha'} \neq C_{Z_\alpha}(Z_{\alpha'})$ *if and only if* $Z_{\alpha'} \leq Z(L_{\alpha'})$.

Proof: Assume first that $Z_\alpha \cap Q_{\alpha'} \neq C_{Z_\alpha}(Z_{\alpha'})$. By 3.15(c), $Z_{\alpha'} \leq Q_\alpha$, so $[Z_\alpha, Z_{\alpha'}] = 1$ and $Z_{\alpha'} \leq Z(L_{\alpha'})$.

Conversely, let $Z_{\alpha'} \leq Z(L_{\alpha'})$. Then, since $Z_\alpha \leq L_{\alpha'}$ and since $Z_\alpha \not\leq Q_{\alpha'}$,

$$Z_\alpha \cap Q_{\alpha'} \neq C_{Z_\alpha}(Z_{\alpha'}).$$

Lemma 3.14 *Let $\delta \in \{\alpha, \beta\}$.*

(a) Let A be a 3-subgroup of G_δ with $A \not\leq Q_\delta$. Then

$$O^3(L_\delta) \leq \langle A^{L_\delta} \rangle \text{ and } L_\delta = \langle A^{L_\delta} \rangle Q_\delta.$$

(b) Let $N \trianglelefteq G_\delta$ with 3 dividing $|NQ_\delta/Q_\delta|$; then $NG_{\alpha\beta} = G_\delta$.

Proof: Let $\overline{L_\delta} = L_\delta/Q_\delta$. Since $\overline{L_\delta} = \Theta$ or Ψ, $\overline{L_\delta}/O_2(\overline{L_\delta})$ is simple in all possible cases. Since $A \not\leq Q_\delta$ and A is a 3-group, $\overline{A} \not\leq O_2(\overline{L_\delta})$. We conclude that $\langle \overline{A}^{\overline{L_\delta}} \rangle O_2(\overline{L_\delta}) = \overline{L_\delta}$. Thus $\overline{L_\delta}/\langle \overline{A}^{\overline{L_\delta}} \rangle$ is a 3′-group. Since $\overline{L_\delta}$ is generated by 3-elements, $\overline{L_\delta} = \langle \overline{A}^{\overline{L_\delta}} \rangle$ and so

$$\langle A^{L_\delta} \rangle Q_\delta = L_\delta.$$

In particular, $L_\delta/\langle A^{L_\delta} \rangle$ is a 3-group and (a) is proven.

(b) By (a) applied to a Sylow 3-subgroup of N, $L_\delta \leq NQ_\delta$, so

$$G_\delta = L_\delta G_{\alpha\beta} = (NQ_\delta)G_{\alpha\beta} = NG_{\alpha\beta}.$$

Remark 3.15 (a) By 3.12(a), $Z_\alpha \not\leq Z(L_\alpha)$ and so by 3.14(a) $C_{G_\alpha}(Z_\alpha)/Q_\alpha$ is a 3′-group.

(b) If $Z_{\alpha'} \leq Q_\alpha$ then $[Z_\alpha, Z_{\alpha'}] = 1$ and so

$$Z_\alpha = C_{Z_\alpha}(Z_{\alpha'}).$$

Hence $Z_\alpha \cap Q_{\alpha'} \neq Z_\alpha$ and $Z_\alpha \cap Q_{\alpha'} \neq C_{Z_\alpha}(Z_{\alpha'})$.

(c) If $Z_{\alpha'} \not\leq Q_\alpha$ then by (a) $C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha \cap Q_{\alpha'}$ and since we have a complete symmetry between α and α' in this case, we get that $C_{Z_{\alpha'}}(Z_\alpha) = Z_{\alpha'} \cap Q_\alpha$.

Definition 3.16 *(a) $\overline{L_\delta} = L_\delta/O_3(L_\delta)$.*

(b) Let K be a complement for S in B and

$$K_\alpha = K \cap L_\alpha \text{ and } K_\beta = K \cap L_\beta.$$

(c) Let $\delta\in\{\alpha,\beta\}$. Let t_δ be an element of order 2 in K_δ with $t_\delta Q_\delta/Q_\delta\in Z(L_\delta/Q_\delta)$ if L_δ/Q_δ is isomorphic to one of the groups $SL_2(3)$, $2{\cdot}A_5$, $2_-^{1+4}A_5$, $SL_2(9)$, $2{\cdot}M_{12}$, $Sp_4(3)$; otherwise let $t_\delta = 1$.

Corollary 3.17 *Let $\delta\in\Delta(\lambda)$ and $t_\delta \neq 1$. If $L_\lambda/Q_\lambda \cong (P)SL_2(3)$, $(P)SL_2(9)$, M_{11} or $(2)M_{12}$ then t_δ does not centralize S/Q_λ.*

Proof: Suppose t_δ centralizes S/Q_λ. Then 2.11 implies that t_δ centralizes L_λ/Q_λ. By Frattini argument $L_\lambda = C_{L_\lambda}(t_\delta)Q_\lambda$. Similarly $L_\delta = C_{L_\delta}(t_\delta)Q_\delta$ and by 3.6 we get $t_\delta = 1$, a contradiction.

Lemma 3.18 *Q_δ is not contained in Q_λ for any pair of adjacent vertices δ and λ.*

Proof: Without loss of generality assume $\{\delta,\lambda\} = \{\alpha,\beta\}$. Assume that Q_δ is contained in Q_λ. By 3.5 Q_δ is properly contained in Q_λ. Hence Q_λ/Q_δ is normal in B and as $Q_\lambda \neq S$ we get that B is not irreducible on S/Q_δ. Hence by 2.12, $L_\delta/Q_\delta \not\cong PSL_2(3)$, $SL_2(3)$, 2^4A_5, $2_-^{1+4}A_5$, A_5, $2{\cdot}A_5$, $PSL_2(9)$, $2PSL_2(9)$ or M_{11} and therefore

$$L_\delta/Q_\delta \cong (P)Sp_4(3) \text{ or } (2)M_{12}.$$

For $\gamma\in\Gamma$ let $\mathcal{M}(Q_\gamma)$ be the set of all maximal with respect to order abelian subgroups of Q_γ and $J(Q_\gamma) = <A/A\in \mathcal{M}(Q_\gamma)>$, namely . $J(Q_\gamma)$ is the Thompson subgroup of Q_γ.

If $J(Q_\lambda) \leq Q_\delta$ then clearly $\mathcal{M}(Q_\delta)$=$\mathcal{M}(Q_\lambda)$ and so

$$1 \neq J(Q_\lambda) = J(Q_\delta) \trianglelefteq <G_\lambda, G_\delta>,$$

a contradiction to (A_4). Hence,

$$J(Q_\lambda) \not\leq Q_\delta$$

and there exists $A\in \mathcal{M}(Q_\lambda)$ with $A \not\leq Q_\delta$.

Now notice that $<\Omega_1 Z(Q_\lambda)^{G_\delta}> \trianglelefteq G_\delta$ and therefore $<\Omega_1 Z(Q_\lambda)^{G_\delta}>$ can not be normal in G_λ. Let $Z_\delta^* = <\Omega_1 Z(Q_\lambda)^{G_\delta}>$. Since Q_δ is contained in Q_λ we get that $\Omega_1 Z(Q_\lambda) \leq C_{G_\delta}(Q_\delta) \leq Q_\delta$ and therefore $Z_\delta^* \leq Z(Q_\delta)$. Hence

$$Z_\delta^* \cap A = Z_\delta^* \cap A \cap Q_\delta.$$

Let $X = C_{G_\delta}(Z_\delta^*)$.

Suppose $C_A(Z_\delta^*) \not\leq Q_\delta$. Then 3.14 gives $XG_{\delta\lambda} = G_\delta$. Since $\Omega_1 Z(Q_\lambda)$ is normalized by G_λ and therefore by $G_{\delta\lambda}$ as well and since it is also normalized by X, we get that $\Omega_1 Z(Q_\lambda)$ is normalized by G_δ and G_λ, a contradiction. Hence $A \cap Q_\delta = C_A(Z_\delta^*)$. Also, since

$$Z_\delta^* \cap A = Z_\delta^* \cap A \cap Q_\delta$$

we now get

$$Z_\delta^* \cap A = Z_\delta^* \cap C_A(Z_\delta^*).$$

$Z_\delta^*(A \cap Q_\delta)$ is abelian. Hence $|Z_\delta^* C_A(Z_\delta^*)| \leq |A|$. Then we have

$$|A| \geq |Z_\delta^* C_A(Z_\delta^*)| = |Z_\delta^*||C_A(Z_\delta^*)|/|Z_\delta^* \cap C_A(Z_\delta^*)| =$$

$$|Z_\delta^*||C_A(Z_\delta^*)|/|Z_\delta^* \cap A|.$$

Hence $|Z_\delta^*|/|Z_\delta^* \cap A| \leq |A|/|C_A(Z_\delta^*)|$ and so

$$|Z_\delta^*/C_{Z_\delta^*}(A)| \leq |Z_\delta^*/Z_\delta^* \cap A| \leq |A|/|C_A(Z_\delta^*)| = |AQ_\delta|/|Q_\delta|.$$

Thus Z_δ^* is an FF-module for $L_\delta/C_{L_\delta}(Z_\delta^*)$ and 2.8 gives

$$L_\delta/Q_\delta \cong Sp_4(3).$$

Now t_δ centralizes S/Q_δ and as $Q_\delta < Q_\lambda$ we get that t_δ centralizes S/Q_λ a contradiction by 3.17 since $L_\lambda/Q_\lambda \in \Theta$.

Remark 3.19 If S$\neq Q_\alpha Q_\beta$ then B is irreducible neither on S/Q_α nor on S/Q_β. Thus by 2.12

$$\{L_\alpha/Q_\alpha, L_\beta/Q_\beta\} \cong \{(P)Sp_4(3), (2)M_{12}\}.$$

Hence either

$$S = Q_\alpha Q_\beta$$

or

$$S \neq Q_\alpha Q_\beta \textit{ and } \{L_\alpha/Q_\alpha, L_\beta/Q_\beta\} \cong \{(P)Sp_4(3), (2)M_{12})\}).$$

Lemma 3.20 *Any V_δ has a non-central chief factor for L_δ.*

Proof: Let $\lambda \in \Delta(\delta)$. If V_δ has no non-central chief factor then $O^3(L_\delta)$ centralizes Z_λ and therefore $Z_\lambda \trianglelefteq <L_\delta, L_\lambda>$, a contradiction to 3.6.

4 The case $Z_{\alpha'} \not\leq Q_\alpha$

In this section we work under the hypothesis $Z_{\alpha'} \not\leq Q_\alpha$. Notice that under this hypothesis, we have a complete symmetry between α and α', so $Z_{\alpha'} \not\leq Z(L_{\alpha'})$.

Lemma 4.1 *(a) $Z_\alpha \cap Q_{\alpha'} = C_{Z_\alpha}(Z_{\alpha'})$; in particular b is even,*

(b) $Z_{\alpha'} \cap Q_\alpha = C_{Z_{\alpha'}}(Z_\alpha)$,

(c) $O^{3'}(C_{G_\alpha}(Z_\alpha)) = Q_\alpha$,

(d) $O^{3'}(C_{G_{\alpha-1,\alpha}}(Z_{\alpha-1})) = Q_{\alpha-1}$ and

(e) $O^{3'}(C_{G_\alpha}(Z_{\alpha-1}Z_\alpha)) = Q_\alpha \cap Q_{\alpha-1}$.

Proof: (a) follows from $Z_{\alpha'} \not\leq Z(L_{\alpha'})$ and 3.13; (b) follows from (a) and symmetry between α and α'; (c) follows from 3.15; (d) is an immediate consequence of (c) and 3.7(a); (c) and (d) imply (e).

Definition 4.2 *$\varepsilon = 1$ if $Z_\beta \neq \Omega_1 Z(S)$ and $\varepsilon = 2$ if $Z_\beta = \Omega_1 Z(S)$.*

The main result in this section will be the following

Proposition 4.3 *$b = 2$ and $\varepsilon = 2$.*

Lemma 4.4 *(a) $L_\alpha/Q_\alpha \cong L_{\alpha'}/Q_{\alpha'} \in \{SL_2(3),\ SL_2(9),\ Sp_4(3)\}$.*

(b) Z_α is an FF-module for L_α/Q_α, $Z_\alpha = [Z_\alpha, L_\alpha] \oplus \Omega_1 Z(L_\alpha)$ and $[Z_\alpha, L_\alpha]$ is the unique natural $SL_2(3)$, $SL_2(9)$ or $Sp_4(3)$ module for L_α/Q_α respectively.

Proof: (a) Since $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$ and $[Z_\alpha, Z_{\alpha'}] \neq 1$ and as $Z_\alpha \nleq Q_{\alpha'}$ we get that $L_{\alpha'}/Q_{\alpha'}$ cannot be 3-stable. Similarly L_α/Q_α is not 3-stable. Hence

$$L_\alpha/Q_\alpha \cong L_{\alpha'}/Q_{\alpha'} \cong SL_2(3),\ 2{\cdot}A_5,\ SL_2(9),\ 2^{1+4}_{-}A_5 \text{ or } (P)Sp_4(3).$$

I want to exclude the possibility of $2{\cdot}A_5$, $2^{1+4}_{-}A_5$ and $PSp_4(3)$. Without loss of generality we may assume that

$$|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leq |Z_{\alpha'} Q_\alpha/Q_\alpha|.$$

Let $V = Z_\alpha$ and $A = Z_{\alpha'}Q_\alpha/Q_\alpha$. Then

$$|V/C_V(A)| = |Z_\alpha/C_{Z_\alpha}(Z_{\alpha'})| = |Z_\alpha/Z_\alpha \cap Q_{\alpha'}| =$$

$$|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leq |Z_{\alpha'}Q_\alpha/Q_\alpha| = |A|.$$

Therefore Z_α is an FF-module for L_α/Q_α. Since $2{\cdot}A_5$, $PSp_4(3)$ and $2^{1+4}_{-}A_5$ do not have an FF-module we conclude that

$$L_\alpha/Q_\alpha \cong L_{\alpha'}/Q_{\alpha'} \in \{SL_2(3),\ SL_2(9),\ Sp_4(3)\}.$$

(b) follows from 2.10. □

By 4.4, L_α fixes some symplectic form on Z_α with $\Omega_1 Z(L_\alpha)$ in its radical. In what follows "$\perp$" and "singular" is meant with respect to that form on Z_α (or also on $Z_{\alpha'}$).

Lemma 4.5 *Let $X \leq G_{\alpha'}$. Then $C_{Z_{\alpha'}}(X)^\perp = [Z_{\alpha'}, X] + \Omega_1 Z(L_{\alpha'})$.*

Proof: [As; 22.1].

Definition 4.6 *For $L_\alpha/Q_\alpha \cong SL_2(3)$ or $SL_2(9)$ let*

$$\Lambda(\alpha,\alpha') = \Lambda = \Delta(\alpha) \setminus \{\beta\}$$

and for $L_\alpha/Q_\alpha \cong Sp_4(3)$ let $E_{\alpha\beta}$ be the 2-dimensional singular subspace of Z_α normalized by S and define $\Lambda(\alpha,\alpha') = \Lambda$ by

$$\Lambda = \{\alpha-1\in\Delta(\alpha)/Z_{\alpha-1} \not\leq [Z_\alpha, Z_{\alpha'}] \text{ and } E_{\alpha-1,\alpha}\cap[Z_\alpha, Z_{\alpha'}] = 1\}$$

4.7 $\Lambda \neq \emptyset$.

Proof: For $L_\alpha/Q_\alpha \cong SL_2(3)$ or $SL_2(9)$, this is clear. So suppose $L_\alpha/Q_\alpha \cong Sp_4(3)$ and pick a singular 2-space in Z_α whose intersection with $[Z_\alpha, Z_{\alpha'}]$ is 1 and is not perpendicular to $[Z_\alpha, Z_{\alpha'}]$. Call this space E and pick any 1-space in E which is not perpendicular to $[Z_\alpha, Z_{\alpha'}]$, say W. Then $N_{G_\alpha}(W)\cap N_{G_\alpha}(E)$ is the normalizer of a Sylow 3-subgroup of G_α and so there exists $\alpha-1\in\Delta(\alpha)$ with $Z_{\alpha-1} = W$ and $E_{\alpha-1,\alpha} = E$. Then $\alpha-1\in\Lambda$.

Lemma 4.8 *If $L_\alpha/Q_\alpha \cong SL_2(3)$ or $SL_2(9)$ then*
$Z_\beta = C_{Z_\alpha}(Z_{\alpha'}) = [Z_\alpha, Z_{\alpha'}] + \Omega_1 Z(L_\alpha) = [Z_\alpha, Q_\beta] + \Omega_1 Z(L_\alpha) = C_{Z_\alpha}(Q_\beta) = C_{Z_\alpha}(S)$.

Proof: By 4.10, $[Z_\alpha, L_\alpha]$ is 2-dimensional over GF(q), where q=3 or 9 respectively. Hence $[Z_\alpha, L_\alpha]$, $C_{Z_\alpha}(Z_{\alpha'})$, $[Z_\alpha, Z_{\alpha'}]$, $[Z_\alpha, Q_\beta]$ and $C_{[Z_\alpha,L_\alpha]}(Q_\beta)$ are all 1-dimensional over GF(q). Moreover, $[Z_\beta, Q_\beta] = 1 = [Z_\beta, Z_{\alpha'}]$ and the lemma follows.

Lemma 4.9 *Let $\alpha-1\in\Lambda$. Then $\langle G_{\alpha-1,\alpha}, Z_{\alpha'}\rangle = G_\alpha$.*

Proof: If $L_\alpha/Q_\alpha \cong SL_2(3)$ or $SL_2(9)$, Lemma 4.8 implies

$$[Z_{\alpha-1}, Z_{\alpha'}] \neq 1$$

and so $Z_{\alpha'} \not\leq G_{\alpha-1,\alpha}$. By 2.15, $G_{\alpha-1,\alpha}$ is maximal in G_α and so $\langle G_{\alpha-1,\alpha}, Z_{\alpha'}\rangle = G_\alpha$.

So suppose $L_\alpha/Q_\alpha \cong Sp_4(3)$ and $\langle G_{\alpha-1,\alpha}, Z_{\alpha'}\rangle \neq G_\alpha$. By 2.15, $\langle G_{\alpha-1,\alpha}, Z_{\alpha'}\rangle$ normalizes $Z_{\alpha-1}$ or $E := E_{\alpha-1,\alpha}$.

If $Z_{\alpha'}$ normalizes $Z_{\alpha-1}$ then $[Z_{\alpha-1}, Z_{\alpha'}] = 1$ and

$$Z_{\alpha-1} \leq C_{Z_\alpha}(Z_{\alpha-1}) = [Z_\alpha, Z_{\alpha'}]^\perp,$$

a contradiction. Thus $Z_{\alpha'}$ does not normalize $Z_{\alpha-1}$. Hence $Z_{\alpha'}$ normalizes E and

$$[Z_{\alpha-1}, Z_{\alpha'}] \leq [E, Z_{\alpha'}] \leq E \cap [Z_\alpha, Z_{\alpha'}] = 1$$

Therefore $[Z_{\alpha-1}, Z_{\alpha'}] = 1$, a contradiction since $Z_{\alpha'}$ does not normalize $Z_{\alpha-1}$.

Lemma 4.10 $\varepsilon = 2$. *In particular* Z_α *is a natural* $SL_2(3)$, $SL_2(9)$ *or* $Sp_4(3)$*-module and* $Z_\beta \leq Z_\alpha$.

Proof: Suppose $\varepsilon = 1$. Let $\alpha - 1 \in \Lambda$.

If $Z_{\alpha-1} \not\leq Q_{\alpha'-1}$ then $(\alpha-1, \alpha'-1)$ has the same properties as (α, α'), which can't happen as the vertices alternate in terms of 3-stability. Hence

$$Z_{\alpha-1} \leq Q_{\alpha'-1} \leq G_{\alpha'-1}^{(1)} \leq G_{\alpha'}$$

and

$$[Z_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha, Z_{\alpha'} \cap Q_\alpha] \leq [G_{\alpha'}, Z_{\alpha'}, Z_{\alpha'}] = 1.$$

Now, 3-stability of $G_{\alpha-1}$ implies $[Z_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] = 1$ which gives

$$C_{Z_{\alpha'}}(Z_\alpha) = Z_{\alpha'} \cap Q_\alpha \leq C_{Z_{\alpha'}}(Z_{\alpha-1}).$$

Hence

$$C_{Z_{\alpha'}}(Z_{\alpha-1})^\perp \leq C_{Z_{\alpha'}}(Z_\alpha)^\perp$$

and by 4.5,

$$[Z_{\alpha'}, Z_{\alpha-1}] \leq [Z_{\alpha'}, Z_\alpha].$$

Hence $Z_{\alpha-1}Z_\alpha$ is normalized by $Z_{\alpha'}$ and by $G_{\alpha-1} \cap G_\alpha$ we get by choice of $\alpha - 1$ that $Z_{\alpha-1}Z_\alpha \trianglelefteq G_\alpha$ and therefore

$$C_{G_\alpha}(Z_{\alpha-1}Z_\alpha) \trianglelefteq G_\alpha.$$

By 4.1(e) now $O^{3'}(C_{G_\alpha}(Z_{\alpha-1}Z_\alpha)) = Q_\alpha \cap Q_{\alpha-1}$ and so we conclude that

$$Q_{\alpha-1} \cap Q_\alpha \trianglelefteq G_\alpha \textit{ and } Q_\beta \cap Q_\alpha \trianglelefteq G_\alpha.$$

Let $L = <Q_\beta^{G_\alpha}>$. As $[Q_\beta, Q_\alpha] \leq Q_\beta \cap Q_\alpha \trianglelefteq G_\alpha$, $[L, Q_\alpha] \leq Q_\alpha \cap Q_\beta \leq Q_\beta$. Recall the definition of t_α (see 3.16) now. Since $Q_\beta \not\leq Q_\alpha$, 3.14(a) implies that $t_\alpha \in O^3(L_\alpha) \leq L$. Hence

$$[t_\alpha, Q_\alpha] \leq Q_\alpha \cap Q_\beta \leq <t_\alpha>(Q_\alpha \cap Q_\beta) \trianglelefteq <t_\alpha>Q_\alpha$$

and

$$O^2(<t_\alpha>(Q_\alpha \cap Q_\beta)) \leq <t_\alpha>(Q_\alpha \cap Q_\beta).$$

Thus

$$[t_\alpha^S] \leq <t_\alpha S> \cap Q_\alpha \leq O^2(<t_\alpha>Q_\alpha) \cap Q_\alpha \leq$$

$$(<t_\alpha>(Q_\alpha \cap Q_\beta)) \cap Q_\alpha \leq Q_\alpha \cap Q_\beta \leq Q_\beta.$$

Hence t_α centralizes S/Q_β, a contradiction by 3.17. Thus $\varepsilon = 2$. So

$$Z_\beta = \Omega_1 Z(L_\beta) = \Omega_1 Z(S)$$

and by 3.7(e), $\Omega_1 Z(L_\alpha) = 1$. So the last statement of the lemma follows from 4.4(b).

Notation 4.11 $X_\alpha = \Omega_1 Z(Q_\alpha)$,

$\bar{b}$ =min$\{d(\alpha, \delta) / X_\alpha \not\leq Q_\delta\}$.

Lemma 4.12 $\bar{b}$=b.

Proof: $Z_\alpha \not\leq Q_{\alpha'}$ and $Z_\alpha \leq X_\alpha$ give $X_\alpha \not\leq Q_{\alpha'}$. Hence $\bar{b} \leq b$. Suppose $\bar{b}<b$ and choose $d(\alpha, \delta) = \bar{b}$ so that $X_\alpha \not\leq Q_\delta$. Since $\bar{b}<b$ we get $V_\delta \leq G_\alpha$ which implies

$$[V_\delta, X_\alpha, X_\alpha] = 1.$$

If δ is not conjugate to α then G_δ is not 3-stable and 3.20 gives $X_\alpha \leq Q_\delta$, a contradiction. Hence δ is conjugate to α and so $\bar{b} \leq b$ implies $Z_\delta \leq Q_\alpha$ and $[Z_\delta, X_\alpha] = 1$, a contradiction to 4.1. Therefore the claim is proved.

4.13 $X_\alpha = Z_\alpha$.

Proof: $d(\alpha, \alpha' - 1) < b = \bar{b}$ gives $X_\alpha \leq Q_{\alpha'-1} \leq G^{(1)}_{\alpha'-1} \leq G_{\alpha'}$ which gives

$$[X_\alpha, Z_\alpha \cap Q_{\alpha'}] = 1.$$

Hence by 4.1(a), $[X_\alpha, C_{Z_\alpha}(Z_{\alpha'})] = 1$ and by 4.5, $[X_\alpha, Z_{\alpha'}] \leq Z_\alpha$. By 3.14(a),

$$L_\alpha = <Z_{\alpha'}^{L_\alpha}> Q_\alpha$$

and therefore $[X_\alpha, L_\alpha] \leq Z_\alpha = [Z_\alpha, t_\alpha]$.

Now: $X_\alpha = C_{X_\alpha}(t_\alpha) \oplus [X_\alpha, t_\alpha]$; but the first summand is normalized by L_α and the second is $[X_\alpha, L_\alpha] = Z_\alpha$. Hence

$$C_{X_\alpha}(t_\alpha) \leq C_{X_\alpha}(L_\alpha) \leq \Omega_1 Z(L_\alpha) = 1$$

which implies $X_\alpha = Z_\alpha$.

Remark 4.14 The following are equivalent:

(i) $Z_{\alpha-1} \not\leq [Z_\alpha, Z_{\alpha'}]$;

(ii) $C_{Z_{\alpha-1}}(Z_{\alpha'}) = 1$;

Define now Y^*_β and Y_β by

$$Y^*_\beta / Z_\beta = <C_{Z_\delta / Z_\beta}(Q_\beta) / \delta \in \Delta(\beta)>$$

and

$$Y_\beta = C_{Z_\alpha}(O^3(L_\beta)).$$

Note that $[Y^*_\beta, Q_\beta] \leq Z_\beta$.

4.15 *If* $b>2$ *then* $Y^*_\beta \leq Z_\alpha$.

Proof: Let $\alpha - 1 \in \Lambda$. Since $Y^*_{\alpha-1} \leq V_{\alpha-1} \leq G_{\alpha'-2}$ by minimality of b we have $[Y^*_{\alpha-1}, Z_{\alpha'-2}] \leq Z_{\alpha'-2}$. Now $Z_{\alpha'-2}$ is centralized by $Z_{\alpha'}$ since $b>2$ and therefore

$[Y^*_{\alpha-1}, Z_{\alpha'-2}] \leq [Y^*_{\alpha-1}, Q_{\alpha-1}] \leq Z_{\alpha-1}$ and $[Y^*_{\alpha-1}, Z_{\alpha'-2}] \leq C_{Z_{\alpha-1}}(Z_{\alpha'}) = 1$. But then $Y^*_{\alpha-1} \leq Q_{\alpha'-2} \leq G_{\alpha'-1}$ and since $b>2$ implies that $V_{\alpha'-1}$ is abelian (see 3.10), we get $[Y^*_{\alpha-1}, V_{\alpha'-1}, V_{\alpha'-1}] \leq [G_{\alpha'-1}, V_{\alpha'-1}, V_{\alpha'-1}] \leq (V_{\alpha'-1})' = 1$ and

$$[Y^*_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha, Z_{\alpha'} \cap Q_\alpha] = 1.$$

Look at $<V_{\alpha-1}^{V_{\alpha'-1}}>$ now. For $\delta \in \Delta(\alpha-1)$ and $g \in V_{\alpha'-1} \leq G_\alpha$ we have

$$d(\delta^g, \alpha) = d(\delta^g, \alpha^g) = d(\delta, \alpha) \leq 2.$$

Hence $d(\delta^g, \alpha-1) \leq 3$ and since $b \geq 4$ we get that $Z_{\delta^g} \leq Q_\alpha \cap Q_{\alpha-1}$ and

$$<V_{\alpha-1}^{V_{\alpha'-1}}> \leq Q_\alpha \cap Q_{\alpha-1}.$$

Therefore

$$[Y^*_{\alpha-1}, <V_{\alpha-1}^{V_{\alpha'-1}}>] \leq [Y^*_{\alpha-1}, Q_{\alpha-1}] \leq Z_{\alpha-1}.$$

Hence

$$[V_{\alpha'-1}, Y^*_{\alpha-1}] \leq [V_{\alpha'-1}, V_{\alpha-1}] \leq <V_{\alpha-1}^{V_{\alpha'-1}}>.$$

Thus

$$[V_{\alpha'-1}, Y^*_{\alpha-1}, Y^*_{\alpha-1}] \leq Z_{\alpha-1} \cap V_{\alpha'-1} \leq C_{Z_{\alpha-1}}(Z_{\alpha'}) = 1$$

and 3-stability of $L_{\alpha'-1}/Q_{\alpha'-1}$ gives $[V_{\alpha'-1}, Y^*_{\alpha-1}] = 1$ whence

$$Y^*_{\alpha-1} \leq Q_{\alpha'-1} \leq G_{\alpha'}.$$

Now, if $Z_\alpha \cap Q_\alpha \not\leq Q_{\alpha'-1}$ then since $Z_{\alpha'} \cap Q_\alpha$ is quadratic on $Y^*_{\alpha-1} \leq Q_{\alpha'-2}$, we get $[Y^*_{\alpha-1}, O^3(L_{\alpha-1})] = 1$ by 3-stability. Hence

$$C_{Z_\alpha}(Q_{\alpha-1}/Z_{\alpha-1}) \trianglelefteq O^3(L_{\alpha-1})G_{\alpha,\alpha-1} = G_{\alpha-1}$$

so

$$Y^*_{\alpha-1} = C_{Z_\alpha}(Q_{\alpha-1}/Z_{\alpha-1}) \leq Z_\alpha$$

and claim is proved. Hence, assume now that

$$Z_{\alpha'} \cap Q_\alpha \le Q_{\alpha'-1}.$$

Then $[Y^*_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \le Z_{\alpha-1} \cap V_{\alpha'-1} \le C_{Z_{\alpha-1}}(Z_{\alpha'}) = 1$ which by 4.5 implies $[Y^*_{\alpha-1}, Z_{\alpha'}] \le Z_\alpha$. Hence, $Y^*_{\alpha-1}Z_\alpha \trianglelefteq <G_{\alpha,\alpha-1}, Z_{\alpha'}> = G_\alpha$. Therefore $[Y^*_{\alpha-1}, Q_\alpha] \trianglelefteq G_\alpha$.

If $[Y^*_{\alpha-1}, Q_\alpha] \ne 1$ then $C_{[Y^*_{\alpha-1}, Q_\alpha]}(Q_\alpha) \ne 1$ and since $Z_\alpha = \Omega_1 Z(Q_\alpha)$ and Z_α is irreducible, $Z_\alpha \le [Y^*_{\alpha-1}, Q_\alpha]$. So $Y^*_{\alpha-1} = Y^*_{\alpha-1}Z_\alpha \trianglelefteq G_\alpha$ and $Y^*_{\alpha-1} \trianglelefteq <G_{\alpha-1}, G_\alpha>$, a contradiction. Hence $[Y^*_{\alpha-1}, Q_\alpha] = 1$ and $Y^*_{\alpha-1} \le Z_\alpha$ by 4.13.

Corollary 4.16 *If $b>2$ then $L_\alpha/Q_\alpha \cong Sp_4(3)$.*

Proof: Suppose $L_\alpha/Q_\alpha \cong SL_2(3)$ or $SL_2(9)$. Then, by 4.8,

$$C_{Z_\alpha/Z_\beta}(Q_\beta/Z_\beta) = Z_\alpha.$$

Therefore,

$$Z_\alpha \le Y^*_\beta \le Z_\alpha$$

whence

$$Z_\alpha = Y^*_\beta \trianglelefteq <G_\alpha, G_\beta>,$$

a contradiction.

Remark 4.17 Suppose $b>2$.

(1) Since $Y^*_\beta \le Z_\alpha$, $[Y^*_\beta, <Q_\alpha^{G_\beta}>] = 1$ and so $Y^*_\beta \le Y_\beta$ by 3.14. In particular, $Y_\beta \ne Z_\beta$ and since $G_{\alpha,\beta}$ normalizes Y_β, $E_{\alpha\beta} \le Y_\beta$. Hence

$$Y_\beta^\perp \le E_{\alpha\beta} \le Y_\beta,$$

where $Y_\beta^\perp$ is the perp of Y_β in Z_α. So $G_\beta = O^3(L_\beta)G_{\alpha\beta}$ normalizes $Y_\beta^\perp$ and $Y_\beta^\perp$ does only depend on β and not on α. Moreover, $Y_\beta^\perp = Z_\beta$ if $|Y_\beta| = 3^3$ and $Y_\beta^\perp = Y_\beta = E_{\alpha\beta}$ if $|Y_\beta| = 3^2$.

Let $\alpha-1\in\Lambda$. Note that if $Y_{\alpha-1}^{\perp}=Z_{\alpha-1}$ then $|C_{Y_{\alpha-1}}^{\perp}(Z_{\alpha'})|=1$ and if $Y_{\alpha-1}^{\perp}=Y_{\alpha-1}$ then $|C_{Y_{\alpha-1}}^{\perp}(Z_{\alpha'})|\leq 3$.

(2) By 4.5, $[Z_\lambda, C_{Q_\beta}(Y_\beta)]\leq Y_\beta^{\perp}\ \forall\lambda\in\Delta(\beta)$ and hence

$$[V_\beta, C_{Q_\beta}(Y_\beta)]\leq Y_\beta^{\perp} \textit{ and } [V_\beta, Q_\beta\cap Q_\alpha]\leq Y_\beta^{\perp}.$$

4.18 *Suppose $b>2$. Then $Z_\beta=C_{Z_\alpha}(Q_\beta)$.*

Proof: We have $C_{Z_\alpha}(Q_\beta)\leq Y_\beta^*\leq Y_\beta$ so $C_{Z_\alpha}(Q_\beta)$ is centralized by $O^3(L_\beta)$ and by Q_β. By 3.14(a) and $Q_\alpha\not\leq Q_\beta$, $L_\beta=Q_\beta O^3(L_\beta)Q_\alpha$ and hence L_β centralizes $C_{Z_\alpha}(Q_\beta)$. So $Z_\beta\leq C_{Z_\alpha}(Q_\beta)\leq\Omega_1 Z(S)\leq Z_\beta$ and the claim is proved.

4.19 *If $b>2$ then for all $\alpha-1\in\Lambda, V_{\alpha-1}\not\leq Q_{\alpha'-2}$.*

Proof: Suppose we can pick $\alpha-1\in\Lambda$ such that

$$V_{\alpha-1}\leq Q_{\alpha'-2}.$$

Then $V_{\alpha-1}\leq Q_{\alpha'-2}\leq G_{\alpha'-1}$ and since $b>2$ we get that $V_{\alpha'-1}$ is abelian and therefore $[V_{\alpha-1}, V_{\alpha'-1}, V_{\alpha'-1}]\leq (V_{\alpha'-1})'=1$.

In particular, $V_{\alpha'-1}\cap Q_\alpha\leq Q_{\alpha-1}$ by 3-stability and 3.20. Hence by 4.17(2),

$$[V_{\alpha-1}, V_{\alpha'-1}\cap Q_\alpha]\leq Y_{\alpha-1}^{\perp}.$$

Since $b>2$ and b is even, $b\geq 4$. Let $\delta\in\Delta(\alpha-1)$ and $g\in V_{\alpha'-1}\leq G_\alpha$. Then $d(\delta^g,\alpha)=d(\delta,\alpha)=2$ and so $d(\delta^g,\alpha-1)\leq 3$. Hence, by minimality of b,

$$Z_\delta^g\leq Q_{\alpha-1}\cap Q_\alpha.$$

Thus $<V_{\alpha-1}^{V_{\alpha'-1}}>\leq Q_\alpha\cap Q_{\alpha-1}$ and therefore we get using 4.17(2) that

$$R:=[V_{\alpha'-1}, V_{\alpha-1}, V_{\alpha-1}]\leq V_{\alpha'-1}\cap[<V_{\alpha-1}^{V_{\alpha'-1}}>, V_{\alpha-1}]\leq$$

$$V_{\alpha'-1}\cap\ Y_{\alpha-1}^{\perp}\leq C_{Y_{\alpha-1}^{\perp}}(Z_{\alpha'}).$$

Hence by 4.17(1) we get $|R| = 1$ or 3.

Suppose $V_{\alpha-1} \leq Q_{\alpha'-1}$. By 4.9,

$$<G_{\alpha-1,\alpha}, Z_{\alpha'}> = G_\alpha$$

and by (A_4), $V_{\alpha-1}$ is not normal in G_α. So $Z_{\alpha'}$ does not normalize $V_{\alpha-1}$ and since $[Z_\alpha, Z_{\alpha'}][V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \leq V_{\alpha-1}$ we get that

$$[V_{\alpha-1}, Z_{\alpha'}] \not\leq [Z_\alpha, Z_{\alpha'}][V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha].$$

Let $W = [Z_\alpha, Z_{\alpha'}][V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha]$.

Since $W = [Z_\alpha, Z_{\alpha'}][V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \leq V_{\alpha-1} \cap Z_{\alpha'} \leq C_{Z_{\alpha'}}(V_{\alpha-1})$ we get

$$W \leq [Z_{\alpha'}, V_{\alpha-1}] = C_{Z_{\alpha'}}(V_{\alpha-1})^\perp \leq W^\perp.$$

If $|W| \geq 3^2$ then $|W^\perp| \leq 3^2$ and so $[V_{\alpha-1}, Z_{\alpha'}] \leq W$, a contradiction. Hence

$$|[Z_\alpha, Z_{\alpha'}][V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha]| = 3.$$

If $[V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] = 1$ then by 4.5 $[V_{\alpha-1}, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}]$ a contradiction and therefore $[V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \neq 1$; since $[Z_\alpha, Z_{\alpha'}] \neq 1$ we get $[Z_\alpha, Z_{\alpha'}] = [V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha]$. But $[V_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \leq Y_{\alpha-1}^\perp$ gives $[Z_\alpha, Z_{\alpha'}] \leq Y_{\alpha-1}^\perp \leq Z_{\alpha-1}^\perp$ a contradiction to the choice of $\alpha - 1 \in \Lambda$ as $[Z_\alpha, Z_{\alpha'}] \not\leq Z_{\alpha-1}^\perp$.

Hence $V_{\alpha-1} \not\leq Q_{\alpha'-1}$.

Suppose $|R| = 1$. Then 3-stability gives $V_{\alpha-1} \leq Q_{\alpha'-1}$, a contradiction. Therefore $|R| = 3$ and so $|Y_{\alpha-1}^\perp| = 3^2$ and $|Y_{\alpha-1}| = 3^2$.

If $V_{\alpha'-1}/Z_{\alpha'-1}$ has more than one non-central chief factor for $L_{\alpha'-1}$, say X_2/X_1 and X_4/X_3 with

$$X_1 < X_2 < X_3 < X_4$$

then, since we can not have a quadratic action on noncentral chief factors and since $V_{\alpha-1} \not\leq Q_{\alpha'-1}$, we get $[X_2, V_{\alpha-1}, V_{\alpha-1}] \not\subseteq X_1$ (3-stability applied to X_2/X_1). So

$$[X_2, V_{\alpha-1}, V_{\alpha-1}] \neq 1.$$

Therefore

$$R = [X_2, V_{\alpha-1}, V_{\alpha-1}]$$

and hence

$$[X_4, V_{\alpha-1}, V_{\alpha-1}] \leq R \leq X_2 \leq X_3$$

contradicting 3-stability on the chief factor X_4/X_3. So, $V_{\alpha'-1}/Z_{\alpha'-1}$ has only one noncentral chief factor.

Suppose now $[V_\beta, Q_\beta, O^3(L_\beta)] \leq Z_\beta$. Then $[Z_\alpha, Q_\beta]Z_\beta \trianglelefteq G_\beta$. Now by 4.18 we have $Z_\beta^\perp = C_{Z_\alpha}(Q_\beta)^\perp = [Z_\alpha, Q_\beta]$. Also $[Z_\alpha, Q_\beta]Z_\beta$ is centralized by $<Q_\alpha^{G_\beta}>$ and as $O^3(L_\beta) \leq <Q_\alpha^{G_\beta}>$ by 3.14(a) we get that $[Z_\alpha, Q_\beta]$ is centralized by $O^3(L_\beta)$. Thus $Z_\beta^\perp = [Z_\alpha, Q_\beta] \leq Y_\beta$. But $|Z_\beta^\perp| = 3^3$ and, as seen above, $|Y_\beta| = 3^2$ a contradiction. Therefore

$$[V_\beta, Q_\beta, O^3(L_\beta)] \not\leq Z_\beta.$$

So there is a noncentral chief factor in $[V_\beta, Q_\beta]$. Thus

$$[V_\beta, O^3(L_\beta)] \leq [V_\beta, Q_\beta]Z_\beta$$

(otherwise we get another noncentral chief factor, but we should only have one). Hence $Z_\alpha[V_\beta, Q_\beta] \trianglelefteq G_\beta$ and therefore $V_\beta = Z_\alpha[V_\beta, Q_\beta]$ which implies

$$V_\beta/Z_\alpha = [V_\beta, Q_\beta]Z_\alpha/Z_\alpha = [V_\beta/Z_\alpha, Q_\beta].$$

So we have a 3-group (Q_β) acting on a 3-group (V_β/Z_α) such that

$$V_\beta/Z_\alpha = [V_\beta/Z_\alpha, Q_\beta].$$

Hence $V_\beta/Z_\alpha = 1$ and $V_\beta = Z_\alpha \trianglelefteq <G_\alpha, G_\beta>$, a contradiction.

Lemma 4.20 *b=2.*

Proof: Assume that b>2. Suppose

$$|Y_\beta| = 3^3 \text{ or } |[Z_\alpha, Z_{\alpha'}]| = 3^2;$$

then

$$Z_\beta = Y_\beta^\perp \textit{ or } Y_\beta = E_{\alpha\beta} \textit{ and } [Z_\alpha, Z_{\alpha'}] = C_{Z_\alpha}(Z_{\alpha'}).$$

Let $\alpha - 1 \in \Lambda$. Then, by the definition of Λ in both cases,

$$Y_{\alpha-1}^\perp \cap C_{Z_\alpha}(Z_{\alpha'}) = 1.$$

By 4.17(2),

$$[V_{\alpha-1}, Z_{\alpha'-2}] \leq Y_{\alpha-1}^\perp \cap C_{Z_\alpha}(Z_{\alpha'}) = 1,$$

contradicting 4.19.

So we can assume that

$$|Y_\beta| = 3^2 \textit{ and } |[Z_\alpha, Z_{\alpha'}]| = 3.$$

Note that $[Y_\beta, Z_{\alpha'}] = 1$ and so $Y_\beta \leq [Z_\alpha, Z_{\alpha'}]^\perp$.

Look at $C_{Z_\alpha}(Z_{\alpha'}) \setminus Y_\beta = Z_\alpha \cap Q_{\alpha'} \setminus Y_\beta = [Z_\alpha, Z_{\alpha\prime}]^\perp \setminus Y_\beta$. Pick 1-spaces

$$E_1, E_2, E_3 \leq [Z_\alpha, Z_{\alpha'}]^\perp$$

so that they generate everything, (note that $[Z_\alpha, Z_{\alpha'}]^\perp$ is a 3-dimensional vector space) i.e. $|E_i| = 3$, $E_i \leq [Z_\alpha, Z_{\alpha'}]^\perp$, i=1, 2, 3 and $E_1 \cdot E_2 \cdot E_3 = [Z_\alpha, Z_{\alpha'}]^\perp$. Moreover, pick the above E_i's so that $E_i \not\leq Y_\beta$. Choose $\beta_i \in \Lambda$ in such a way that Z_{β_i} is perpendicular to E_i but not to $[Z_\alpha, Z_{\alpha'}]$. Also, choose the β_i's in such a way that $Z_{\beta_i} E_i = Y_{\beta_i}$ (which implies that $C_{Y_{\beta_i}}(Z_{\alpha'}) = E_i$). Then, by 4.17(2) applied to β_i in place of β, $[V_{\beta_i}, Z_{\alpha'-2}] \leq Y_{\beta_i}^\perp = Y_{\beta_i}$ and since $b>2$, $[V_{\beta_i}, Z_{\alpha'-2}] \leq C_{Y_{\beta_i}}(Z_{\alpha'}) = E_i$. By 4.17(2) applied to $\alpha' - 3$ in place of β, $[V_{\beta_i}, Z_{\alpha'-2}] \leq Y_{\alpha'-3}$. So,

$$Z_\alpha \cap Q_{\alpha'} = <E_i / i \in I> \subseteq Y_{\alpha'-3}.$$

But $|Z_\alpha \cap Q_{\alpha'}| = 3^3$ and $|Y_{\alpha'-3}| = 3^2$, a contradiction.

Proof of the Proposition: It follows from 4.10 and 4.20.

5 The case b=2

In this section we assume that $Z_{\alpha'} \not\leq Q_\alpha$. Recall from the previous section that

$$L_\alpha/Q_\alpha \cong SL_2(3),\ SL_2(9)\ or\ Sp_4(3),$$

$$Z_\alpha\ is\ a\ natural\ SL_2(3),\ SL_2(9)\ or\ Sp_4(3)-module,$$

$$L_\beta/Q_\beta \cong PSL_2(9),\ M_{11}\ or\ (2)M_{12},$$

$$\varepsilon = b = 2,\ \alpha' - 1 = \beta,\ Z_\beta = \Omega_1 Z(S)$$

and α is conjugate to α'.

Proposition 5.1 *The hypothesis in this section leads to a contradiction.*

Proof of the Proposition: Since $[t_\alpha, K] \leq Q_\alpha \cap K = 1$ we have $[t_\alpha, K_\beta] = 1$ and the order of t_α is 2. By 2.13, t_α induces an inner automorphism on L_β/Q_β.

By 3.17 t_α does not centralize L_β/Q_β. Also, as t_α is an inner automorphism we can pick $t \in K_\beta$ which acts on the same way on L_β/Q_β i.e. pick $t \in K_\beta$ so that $x_\beta = t_\alpha t$ and x_β centralizes L_β/Q_β.

I now claim that the order of t is 2 as well. By choice of t,

$$|t| = |tQ_\alpha/Q_\alpha|$$

and the image of t in $L_\beta/{<}t_\beta{>}Q_\alpha \cong L_\beta{<}x_\beta{>}/{<}t_\beta, x_\beta{>}Q_\alpha$ is t_α which has order two. Hence the claim holds if $t_\beta = 1$ and so we are done for the cases $PSL_2(9)$, M_{11} or M_{12}. The only problem could appear in $2{\cdot}M_{12}$ since when we lift M_{12} to $2{\cdot}M_{12}$ the order of t could become 4. But this does not happen by 2.1(d). Moreover in any case x_β centralizes L_β/Q_β and the order of x_β is also one or two.

Now t_α acts non-trivially on Z_α which is irreducible for L_α so t_α inverts Z_α. K_α acts on Y_β faithfully and K_β centralizes Y_β so $[K_\alpha, K_\beta] = 1$. We will distinguish two cases.

Case 1: $|Z_\alpha| \leq |Y_\beta|^2$.

K_α acts on Y_β faithfully and K_β centralizes Y_β so $[K_\alpha, K_\beta] = 1$. Since K_α centralizes t and K_α centralizes t_α we get that K_α centralizes x_β. Thus $[x_\beta, K_\alpha] = 1$.

Now define $Y_\beta = C_{Z_\alpha}(O^2(L_\beta))$. Let

$$A = Z_\beta \; if \; L_\alpha/Q_\alpha \cong SL_2(3) \; or \; SL_2(9)$$

and

$$A = E_{\alpha\beta} \; if \; L_\alpha/Q_\alpha \cong Sp_4(3).$$

Since t centralizes Y_β and t_α inverts Y_β, x_β inverts Y_β and so x_β inverts A. This means that if x^*_β is the image of x_β in Aut(A) then $x^*_\beta \in Z(Aut(A))$ and so $[N_{G_\alpha}(A), x^*_\beta]$ centralizes A.

Let $L = N_{L_\alpha}(A)$ and $Q = C_{L_\alpha}(A)$. Since Z_α is a natural $SL_2(3)$, $SL_2(9)$ or $Sp_4(3)$-module, $L/C_L(A) \cong GL_F(A)$ where $F = GF(3)$, $GF(9)$ or $GF(3)$ respectively and L acts irreducibly on A. Since $A = A^\perp$, $[Z_\alpha, Q] \leq A^\perp = A$. Hence $[Z_\alpha, Q, Q] = 1$ and Q is a 3-group. So $Q = O_3(L)$. Now $[L, x_\beta] \leq Q$ and so by Frattini argument $L = C_L(x_\beta)Q$. Hence $C_L(x_\beta)$ acts irreducibly on A and on Z_α/A (which is isomorphic to the dual of A). In particular x_β inverts or centralizes Z_α/A. Since

$$V_\beta = <Z_\alpha^{G_\beta}> = <Z_\alpha^{C_{G_\beta}(x_\beta)}>$$

we conclude that x_β inverts or centralizes V_β/A.

Note that x_β inverts A so if x_β inverts V_β/A, x_β inverts V_β and V_β is abelian, a contradiction to $1 \neq [Z_\alpha, Z_{\alpha'}] \leq V_\beta$.

If x_β centralizes V_β/A then $V_\beta = C_{V_\beta}(Z_\beta)A = C_{V_\beta}(Z_\beta) \times A$. Hence $V'_\beta \leq (C_{V_\beta}(Z_\beta))'$ (as $A \leq \bigcap_{g \in G_\beta} Z_\alpha^g \leq Z(V_\beta)$) and so

$$V'_\beta \cap Z_\beta \leq (C_{V_\beta}(Z_\beta))' \cap A = 1.$$

Hence $C_{V'_\beta}(S) = 1$ and $V'_\beta = 1$, again a contradiction.

Case 2: $|Z_\alpha| > |Y_\beta|^2$.

As $Z_\beta \leq Y_\beta$ and $|Z_\beta^2| = |Z_\alpha|$ for $L_\alpha/Q_\alpha \cong SL_2(3)$ or $SL_2(9)$, this implies that $L_\alpha/Q_\alpha \cong Sp_4(3)$.

Then $|Z_\alpha| = 3^4$, Z_α is a natural $Sp_4(3)$-module for L_α/Q_α and $|Y_\beta| = 3$. Thus $C_{Z_\alpha}(O^3(L_\beta)) = Z_\beta$. Recall the definition of $E_{\alpha\beta}$ in 4.6. Since $|Y_\beta| = 3$ we have that

$$[E_{\alpha\beta}, O^3(L_\beta)] \neq 1.$$

Subcase 1: $E_{\alpha\beta} \not\leq Z(V_\beta)$.

Choose $\alpha' \in \Delta(\beta)$ such that $[E_{\alpha\beta}, Z_{\alpha'}] \neq 1$ (hence (α, α') is a critical pair). On the other hand we have $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$.

Suppose $|[Z_\alpha, Z_{\alpha'}]| = 3$. Then $[Z_\alpha, Z_{\alpha'}] = [E_{\alpha\beta}, Z_{\alpha'}] \leq E_{\alpha\beta}$ and so

$$C_{Z_\alpha}(Z_{\alpha'}) = [Z_\alpha, Z_{\alpha'}]^\perp \geq E_{\alpha\beta}^\perp = E_{\alpha\beta},$$

a contradiction to $[E_{\alpha\beta}, Z_{\alpha'}] = 1$. Hence $|[Z_\alpha, Z_{\alpha'}]| \neq 3$. If $|Z_{\alpha'}Q_\alpha/Q_\alpha| = 3$ then $|Z_{\alpha'}/Z_{\alpha'} \cap Q_\alpha| = |Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha)| \leq 3$ and so $[Z_\alpha, Z_{\alpha'}] = |C_{Z_{\alpha'}}(Z_\alpha)^\perp| = 3$, a contradiction. Thus,

$$|Z_{\alpha'}Q_\alpha/Q_\alpha| \geq 3^2.$$

Since $[Z_\alpha, Z_{\alpha'}, Z_{\alpha'}] = 1$ we have by the choice of α' that

$$|[Z_\alpha, Z_{\alpha'}]| = 3^2 = C_{Z_\alpha}(Z_{\alpha'}) \neq E_{\alpha\beta}.$$

Now Q_β normalizes $Z_{\alpha'}$ and hence it also normalizes $C_{Z_\alpha}(Z_{\alpha'})$. Hence $Q_\beta Q_\alpha$ normalizes $C_{Z_\alpha}(Z_{\alpha'})$. But the only S-invariant subgroup of order 3^2 in Z_α is $E_{\alpha\beta}$. Hence $Q_\alpha Q_\beta \neq S$ which means (recall that from 3.19, $Q_\alpha Q_\beta \neq S$ implies $\{L_\alpha/Q_\alpha, L_\beta/Q_\beta\} \cong \{(P)Sp_4(3), (2)M_{12})\}$) $|Q_\alpha Q_\beta/Q_\beta| < |S/Q_\beta| = 3^3$. Thus $|Q_\alpha Q_\beta/Q_\beta| \leq 3^2$.

Now look at $Q_\alpha \cap Q_\beta$; it centralizes $[Z_\alpha, Z_{\alpha'}] = C_{Z_{\alpha'}}(Z_\alpha)$ and the latter has order 3^2. Hence by 4.5 $[Q_\alpha \cap Q_\beta, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}]^\perp = [Z_\alpha, Z_{\alpha'}]$.

Let $\overline{Q_\alpha} = Q_\alpha/Z_\alpha$. Then $|[\overline{Q_\alpha \cap Q_\beta}, Z_{\alpha'}]| = 1$ and so

$$|\overline{Q_\alpha}/C_{\overline{Q_\alpha}}(Z_{\alpha'})| \leq |Q_\alpha Q_\beta/Q_\beta| \leq 3^2 \leq |Z_{\alpha'}Q_\alpha/Q_\alpha|$$

. So there exists a unique non-central composition factor and it is an FF-module isomorphic to Z_α (uniqueness of the FF-module).

Now $C_{\overline{Q_\alpha}}(Z_{\alpha'}) = \overline{Q_\alpha \cap Q_\beta} \trianglelefteq G_{\alpha\beta}$ but on the other hand $C_{Z_\alpha}(Z_{\alpha'})$ is not normal in $G_{\alpha\beta}$. So we found one FF-module in which the centralizer of $Z_{\alpha'}$ is normal in $G_{\alpha\beta}$ and another FF-module in which the centralizer of $Z_{\alpha'}$ was not normal in $G_{\alpha\beta}$, a contradiction to the uniqueness of FF-modules.

Subcase 2: $E_{\alpha\beta} \leq Z(V_\beta)$.

Define $W_\beta = \langle E_{\alpha\beta}^{G_\beta} \rangle \leq Z(V_\beta)$. In particular, $W_\beta \leq Q_\alpha$ and W_β is abelian. Also W_β has a non-central L_β chief factor since $E_{\alpha\beta}$ is not centralized by $O^3(L_\beta)$. Hence $[W_\beta, O^3(L_\beta)] \neq 1$. Choose again $\alpha - 1 \in \Lambda$.

Let's also note that

$$[W_\beta, Q_\beta] = \langle [E_{\alpha\beta}, Q_\beta]^{G_\beta} \rangle \leq Z_\beta.$$

If $W_{\alpha-1} \not\leq Q_\beta$ then, as $W_\beta \leq Q_\alpha \leq G_{\alpha-1}$, we get

$$[W_\beta, W_{\alpha-1}, W_{\alpha-1}] \leq [W_{\alpha-1}, W_{\alpha-1}] = 1$$

contradicting the 3-stability of β.

Hence $W_{\alpha-1} \leq Q_\beta \leq G_{\alpha'}$. So $W_{\alpha-1}$ normalizes $Z_{\alpha'} \cap Q_\alpha$ and therefore $Z_{\alpha'} \cap Q_\alpha$ is quadratic on $W_{\alpha-1}$. Then 3-stability of $L_{\alpha-1}$ gives $Z_{\alpha'} \cap Q_\alpha \leq Q_{\alpha-1}$. Since $[W_{\alpha-1}, Q_{\alpha-1}] \leq Z_{\alpha-1}$ we now get $[W_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \leq Z_{\alpha-1}$. In particular,

$$[W_{\alpha-1}, E_{\alpha'\beta}] \leq Z_{\alpha-1} \cap Z_\beta = 1.$$

Hence by 4.5 $[Z_{\alpha'}, W_{\alpha-1}] \leq E_{\alpha'\beta}^{\perp} = E_{\alpha'\beta}$. Thus

$$[W_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] \leq Z_{\alpha-1} \cap E_{\alpha'\beta} \leq C_{Z_{\alpha-1}}(Z_{\alpha'}) = 1$$

as $E_{\alpha'\beta} \leq Z(V_\beta)$ and $|Z_{\alpha-1}| = 3$. Hence $[W_{\alpha-1}, Z_{\alpha'} \cap Q_\alpha] = 1$ which implies that

$$[Z_{\alpha'}, W_{\alpha-1}] \leq (Z_{\alpha'} \cap Q_\alpha)^{\perp} = [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha.$$

This means that

$$W_{\alpha-1}Z_\alpha \trianglelefteq <G_{\alpha,\alpha-1}, Z_{\alpha'}> = G_\alpha$$

and therefore

$$[W_{\alpha-1}, Q_\alpha] \trianglelefteq G_\alpha.$$

But $[W_{\alpha-1}, Q_\alpha] \neq 1$ (since if $W_{\alpha-1}$ centralizes Q_α, $Q_\alpha \not\leq Q_{\alpha-1}$ and 3.14 imply $[W_{\alpha-1}, O^3(L_{\alpha-1})] = 1$, a contradiction).

Now since $|Z_{\alpha-1}| = 3$ we get $Z_{\alpha-1} \leq [W_{\alpha-1}, Q_\alpha]$. On the other hand,

$$[W_{\alpha-1}, Q_\alpha] \trianglelefteq G_\alpha$$

and therefore $Z_\alpha \leq W_{\alpha-1}$.

Hence $W_{\alpha-1}Z_\alpha = W_{\alpha-1} \trianglelefteq <G_\alpha, G_{\alpha-1}>$, a contradiction.

6 The case $Z_{\alpha'} \leq Q_\alpha$

In this section we will deal with the case $Z_{\alpha'} \leq Q_\alpha$. We will show that b=1 and start the analysis of the case b=1.

It follows from the hypothesis that there is no symmetry between α and α' any more. Also $[Z_\alpha, Z_{\alpha'}] \leq [Z_\alpha, Q_\alpha] = 1$ gives

$$C_{Z_\alpha}(Z_{\alpha'}) = Z_\alpha.$$

Now notice that $Z_\alpha \cap Q_{\alpha'} \neq Z_\alpha$ (otherwise get $Z_\alpha \leq Q_{\alpha'}$, a contradiction). Hence, $C_{Z_\alpha}(Z_{\alpha'}) \neq Z_\alpha \cap Q_{\alpha'}$ and by 3.13, $Z_{\alpha'} \leq Z(L_{\alpha'})$, α and α' are not conjugate and b is odd. Therefore we have

$$Z_\beta = \Omega_1 Z(L_\beta) \textit{ and } Z_{\alpha'} = \Omega_1 Z(L_{\alpha'}).$$

Lemma 6.1 $[Q_\beta, Z_\alpha, Z_\alpha] = 1$.

Proof: 3.7(b) gives $Q_\beta \leq G_\beta^{(1)} \leq G_\alpha$. Hence, $<Z_\alpha^{Q_\beta}> \leq <Z_\alpha^{G_\alpha}> = Z_\alpha$ which gives $[Q_\beta, Z_\alpha, Z_\alpha] \leq [Z_\alpha, Z_\alpha] = 1$.

Lemma 6.2 $L_{\alpha'}/Q_{\alpha'} \cong L_\beta/Q_\beta \cong (P)Sp_4(3)$, $SL_2(3)$, $SL_2(9)$, $2^{1+4}_{-}A_5$ *or* $2{\cdot}A_5$.

Proof: If $b>1$ then $[V_{\alpha'}, Z_\alpha, Z_\alpha] \le [V_{\alpha'}, V_\beta, V_\beta] \le [V_\beta, V_\beta] = 1$ by 3.10, so, since $Z_\alpha \not\le Q_{\alpha'}$, we conclude that L_β/Q_β is not 3-stable and the claim follows by 2.5.

If b=1, 6.1 and $Z_\alpha \not\le Q_\beta$ again imply that L_β/Q_β is not 3-stable and the claim follows by 2.5.

Notation 6.3 For $\gamma\in\Gamma$ let $D_\gamma = C_{Q_\gamma}(O^3(L_\gamma))$.

Lemma 6.4 $Z(L_\alpha) = D_\alpha = 1$.

Proof: Since $\Omega_1(C_{Q_\alpha}(L_\alpha)) \le Z_\beta \le \Omega_1(C_{Q_\alpha}(S)) \le \Omega_1 Z(S) \le Z(L_\beta)$ we get that $\Omega_1(C_{Q_\alpha}(L_\alpha))$ is centralized by L_α and L_β and therefore $C_{Q_\alpha}(L_\alpha) = 1$. Hence

$$C_{Q_\alpha}(O^3(L_\alpha)) = D_\alpha = 1.$$

Also, $\Omega_1 Z(L_\alpha) \le D_\alpha{=}1$ and therefore $Z(L_\alpha) = 1$.

Lemma 6.5 *Q_β is not abelian.*

Proof: By 3.18, $Q_\beta \not\le Q_\alpha$. So 3- stability of L_α gives $[Z_\alpha, Q_\beta, Q_\beta] \ne 1$. Hence $1 \ne [[Z_\alpha, Q_\beta], Q_\beta] \le Q'_\beta$.

Proposition 6.6 *b=1.*

Proof of the Proposition: Assume that b>1. Since b is odd, $b \ge 3$.

6.6.1 $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$.

Proof: Since b$\ge$ 3, 3.10 implies $[V_\beta, V_\beta] = 1$. Clearly $V_\beta \cap Q_{\alpha'}$ centralizes $Z_{\alpha'}$. Let $\delta\in\Delta(\alpha')$. Since $d(\delta,\beta) \le b$ we get $Z_\delta \le G_\beta$ and since $V_\beta \trianglelefteq G_\beta$ we now have that Z_δ normalizes V_β. Then

$$[Z_\delta, V_\beta \cap Q_{\alpha'}, V_\beta \cap Q_{\alpha'}] \le [Z_\delta, V_\beta, V_\beta] \le [V_\beta, V_\beta] = 1.$$

But L_δ is 3-stable as δ is conjugate to α and $Z_\delta \trianglelefteq G_\delta$ and $V_\beta \cap Q_{\alpha'} \le G_\delta$ (since $d(\alpha',\delta) = 1$ so $V_\beta \cap Q_{\alpha'} \le Q_{\alpha'} \le G^{(1)}_{\alpha'} \le G_\delta$). Therefore, $[Z_\delta, V_\beta \cap Q_{\alpha'}] = 1$.

6.6.2 *$L_\beta/Q_\beta \cong Sp_4(3)$, $SL_2(3)$ or $SL_2(9)$ and V_β has a unique non-central L_β-composition factor; moreover, this composition factor is the natural module for L_β/Q_β.*

Proof: 6.6.1 gives $V_\beta \cap Q_{\alpha'} \leq C_{V_\beta}(V_{\alpha'})$ and by a similar argument we also have that $V_{\alpha'} \cap Q_\beta \leq C_{V_{\alpha'}}(V_\beta)$. Without loss of generality, assume

$$|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leq |V_{\alpha'} Q_\beta/Q_\beta|.$$

Now let X=Y/Z be a non-central chief factor in V_β. As

$$C_Y(V_{\alpha'})Z/Z \leq C_{Y/Z}(V_{\alpha'})$$

we get that

$$|X/C_X(V_{\alpha'})| = |Y/Z/C_{Y/Z}(V_{\alpha'})| \leq$$

$$|Y/Z/C_Y(V_{\alpha'})Z/Z| = |Y/C_Y(V_{\alpha'})Z| \leq$$

$$|Y/C_Y(V_{\alpha'})| = |Y/Y \cap C_{V_\beta}(V_{\alpha'})| =$$

$$|Y \cdot C_{V_\beta}(V_{\alpha'})/C_{V_\beta}(V_{\alpha'})| \leq |V_\beta/C_{V_\beta}(V_{\alpha'})| \leq$$

$$|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leq |V_{\alpha'} Q_\beta/Q_\beta|$$

so X is an FF-module; similarly, the direct sum of the L_β chief factors on V_β is still an FF-module for L_β/Q_β and lemma follows by 2.8.

6.6.3 $[V_\beta, Q_\beta] \leq D_\beta$.

Proof: Assume that $[V_\beta, Q_\beta] \not\leq D_\beta$. Then by 6.6.2, $Z_\alpha[V_\beta, Q_\beta]$ is normalized by $G_{\alpha\beta}O^3(L_\beta) = G_\beta$ and we get that $Z_\alpha[V_\beta, Q_\beta] = V_\beta$. Hence $V_\beta/Z_\alpha = [V_\beta/Z_\alpha, Q_\beta]$. Since Q_β is a 3-group acting on the 3-group V_β/Z_α in the above manner, we conclude that $V_\beta/Z_\alpha = 1$. Therefore $V_\beta = Z_\alpha$, a contradiction. Hence $[V_\beta, Q_\beta] \leq D_\beta$.

Notation 6.6.4 *Let $Q^*_\beta = [Q_\beta, O^3(L_\beta)]$.*

6.6.5 $Q^*_\beta \leq Q_\alpha$.

Proof: By 6.6.3, $[V_\beta, Q^*_\beta] \leq [V_\beta, Q_\beta] \leq D_\beta$. Note that $Q^*_\beta \leq O^3(L_\beta)$ and therefore

$$[V_\beta, Q^*_\beta, Q^*_\beta] \leq [D_\beta, Q^*_\beta] \leq [D_\beta, O^3(L_\beta)] = 1.$$

Hence $[Z_\alpha, Q^*_\beta, Q^*_\beta] = 1$ and 3-stability of L_α gives that $[Z_\alpha, Q^*_\beta] = 1$ whence $Q^*_\beta \leq Q_\alpha$.

6.6.6 *The hypothesis that b>1 gives a contradiction.*

Proof: By 6.6.5, Q^*_β centralizes Z_α and so it centralizes $<Z_\alpha^{G_\beta}> = V_\beta$ as well. Since $[t_\beta, Q_\beta] \leq Q^*_\beta$, t_β is the unique involution in $t_\beta Q_\beta / Q^*_\beta$ and so $t_\beta Q^*_\beta \in Z(L_\beta / Q^*_\beta)$. In particular, L_β normalizes $[V_\beta, t_\beta]$. By 6.6.2, $[V_\beta, t_\beta] \neq 1$ and so $C_{[V_\beta, t_\beta]}(S) \neq 1$. Hence

$$Z_\beta \cap [V_\beta, t_\beta] \neq 1.$$

On the other hand, since by 3.10 V_β is abelian,

$$V_\beta = C_{V_\beta}(t_\beta) \times [V_\beta, t_\beta]$$

and $[Z_\beta, t_\beta] \leq [Z_\beta, L_\beta] = 1$, a contradiction.

Notation 6.7 For $\gamma \in \Gamma$ let F_γ be a normal 3-subgroup of L_γ minimal with respect to the property $F_\gamma \not\leq D_\gamma$.

Remark 6.8 As F_γ is a 3-group we get $F_\gamma \leq Q_\gamma$ and $F'_\gamma \neq F_\gamma$. Also, the definition implies $F_\gamma \neq 1$. Since Q_γ is a 3-group acting on the 3-group F_γ, $F_\gamma \neq [F_\gamma, Q_\gamma]$ and by minimality of F_γ, $[F_\gamma, Q_\gamma] \leq D_\gamma$. Also it is clear from the definitions that $F_\beta = [F_\beta, O^3(L_\beta)] \leq O^3(L_\beta)$ and therefore $[D_\beta, F_\beta] \leq [D_\beta, O^3(L_\beta)] = 1$.

Lemma 6.9 *$F_\beta \not\leq Q_\alpha$ and $D_\beta \leq Q_\alpha$.*

Proof: If $F_\beta \leq Q_\alpha$, $[F_\beta, Z_\alpha] = 1$ and by 3.14 $[F_\beta, O^3(L_\beta)] = 1$, a contradiction.

By 6.8 we have $[D_\beta, F_\beta] = 1$ and since $[Z_\alpha, D_\beta] \leq D_\beta$,

$$[Z_\alpha, D_\beta, F_\beta] = 1$$

and

$$[Z_\alpha, D_\beta, <F_\beta^B>] = 1.$$

Suppose now that $D_\beta \not\leq Q_\alpha$. By 2.12, B is irreducible on $Z(S/Q_\alpha)$ and so

$$Z(S/Q_\alpha) \leq D_\beta Q_\alpha / Q_\alpha \leq L_\beta Q_\alpha / Q_\alpha.$$

Similarly, $Z(S/Q_\alpha) \leq <F_\beta^B> Q_\alpha$. Hence

$$[Z_\alpha, Z(S/Q_\alpha), Z(S/Q_\alpha)] = 1,$$

a contradiction to the 3-stability of L_α. Thus $D_\beta \leq Q_\alpha$.

Lemma 6.10 *Q_α is elementary abelian, $[Q_\alpha, O^3(L_\alpha)]$ is an irreducible L_α-module and $F_\alpha = Z_\alpha = [Q_\alpha, O^3(L_\alpha)]$.*

Proof: Step 1: $F_\alpha \cap F_\beta \not\leq D_\beta$.

Proof of step 1: $<F_\alpha^{F_\beta}> \leq <F_\alpha^{Q_\beta}> \leq <F_\alpha^{L_\alpha}> \leq F_\alpha$. Similarly, $<F_\beta^{F_\alpha}> \leq F_\beta$. Hence

$$[F_\alpha, F_\beta] \leq F_\alpha \cap F_\beta.$$

Assume now that $F_\alpha \cap F_\beta \leq D_\beta$. Then, since

$$F_\beta = [F_\beta, O^3(L_\beta)] \leq O^3(L_\beta)$$

we get $[F_\alpha, F_\beta, F_\beta] \leq [D_\beta, F_\beta] \leq [D_\beta, O^3(L_\beta)] = 1$. But $F_\beta \leq Q_\beta \leq G_\alpha$ and by 6.9 $F_\beta \not\leq Q_\alpha$. Hence $[F_\alpha, F_\beta, F_\beta] = 1$ and 3-stability of L_α gives $[F_\alpha, O^3(L_\alpha)] = 1$, a contradiction to the definition of F_α.

Step 2: $[F_\alpha \cap F_\beta, F_\beta] \trianglelefteq L_\beta$ and $F_\beta' \leq F_\alpha$.

Proof of step 2: $F_\alpha \cap F_\beta \not\leq D_\beta$ implies $<(F_\alpha \cap F_\beta)^{L_\beta}> \not\leq D_\beta$. And since

$$<(F_\alpha \cap F_\beta)^{L_\beta}> \trianglelefteq L_\beta$$

minimality of F_β gives $F_\beta \leq <(F_\alpha \cap F_\beta)^{L_\beta}>$. Clearly the other inclusion is also true so $F_\beta = <(F_\alpha \cap F_\beta)^{L_\beta}>$. Now $F_\beta' \trianglelefteq L_\beta$ and $F_\beta' \neq F_\beta$ so minimality of F_β gives $F_\beta' \leq D_\beta$.

This now means that $F'_\beta = [F_\beta, F_\beta] = \langle [F_\alpha \cap F_\beta, F_\beta]^{L_\beta} \rangle \leq D_\beta = C_{Q_\beta}(O^3(L_\beta))$. Hence $[F_\alpha \cap F_\beta, F_\beta]$ is centralized by $O^3(L_\beta)$. Since $[F_\alpha \cap F_\beta, F_\beta] \trianglelefteq S$ and $O^3(L_\beta)S = L_\beta$ we get $[F_\alpha \cap F_\beta, F_\beta] \trianglelefteq L_\beta$. Thus $F'_\beta = [F_\alpha \cap F_\beta, F_\beta] \leq F_\alpha$.

Step 3: Q_α is elementary abelian and $[Q_\alpha, O^3(L_\alpha)] = F_\alpha$.

Proof of step 3: Since $\langle F_\beta^{Q_\alpha} \rangle \leq \langle F_\beta^{L_\beta} \rangle = F_\beta$ we get

$$[Q_\alpha, F_\beta, F_\beta] \leq F'_\beta \leq F_\alpha.$$

Then 3-stability of α gives $[Q_\alpha/F_\alpha, O^3(L_\alpha)] = 1$ i.e. $O^3(L_\alpha)$ centralizes Q_α/F_α. So, Q_α has a unique non-central chief factor. By the properties of the Frattini group (for example see [Go; p.173]) we get that

$$\Phi(Q_\alpha) \subseteq D_\alpha = 1.$$

Hence Q_α is elementary abelian.

Step 4: $[Q_\alpha, O^3(L_\alpha)]$ is irreducible L_α-module and $F_\alpha = Z_\alpha$.

Proof of step 4: Since $D_\alpha = 1$, Z_α is the unique non-central chief factor for L_α on Q_α; moreover, by Gaschütz' Theorem,

$$Z_\alpha = Z_\beta F_\alpha = \Omega_1 Z(L_\alpha) F_\alpha = F_\alpha.$$

Corollary 6.11 *Note that from 6.9, $D_\beta \leq Q_\alpha$ and so $\Phi(D_\beta) = 1$.*

Corollary 6.12 $C_{G_\alpha}(Q_\alpha) = Q_\alpha$.
In particular, if $X \leq G_\alpha$ then $X \cap Q_\alpha = C_X(Q_\alpha)$.

Proof: By (P_2), $C_{G_\alpha}(Q_\alpha) \leq Q_\alpha$. But as Q_α is abelian we get

$$Q_\alpha \leq C_{G_\alpha}(Q_\alpha)$$

and therefore the claim follows.

Lemma 6.13 *If $L_\beta/Q_\beta \cong (P)Sp_4(3)$ then $L_\alpha/Q_\alpha \cong (2)M_{12}$.*

Proof: Q_α is abelian implies $Q_\alpha Q_\beta/Q_\beta$ is abelian.

If $L_\beta/Q_\beta \cong (P)Sp_4(3)$ then the group S/Q_β is not abelian and we conclude $S \neq Q_\alpha Q_\beta$. But then 3.19 gives $L_\alpha/Q_\alpha \cong (2)M_{12}$.

7 The case b=1 and Θ≅(2)M₁₂

Proposition 7.1 *If $S = Q_\alpha Q_\beta$ then* $(L_\alpha, L_\beta) \sim (3^6 2{\cdot}M_{12}, 3^{1+1+1+2+2+1}SL_2(3))$.

Proof: Suppose that $Q_\alpha Q_\beta = S$.

Then by 6.10 S/Q_β is abelian and therefore

$$L_\beta/Q_\beta \not\cong (P)Sp_4(3).$$

Hence by 6.2

$$L_\beta/Q_\beta \cong SL_2(3), 2{\cdot}A_5, 2_-^{1+4}A_5, SL_2(9).$$

Also from $Q_\alpha Q_\beta = S$ we get

$$[F_\beta Q_\alpha/Q_\alpha, S] = [F_\beta Q_\alpha/Q_\alpha, Q_\beta]$$

and as $[F_\beta, Q_\beta] \leq D_\beta \leq Q_\alpha$ (see 6.8 and 6.9) we conclude that

$$[F_\beta Q_\alpha/Q_\alpha, Q_\beta] = 1.$$

Hence $F_\beta Q_\alpha/Q_\alpha \leq Z(S/Q_\alpha)$.

Since $|S/Q_\alpha| = 3^3$ and S/Q_α is not abelian we get that

$$|Z(S/Q_\alpha)| = 3$$

and therefore $F_\beta Q_\alpha/Q_\alpha = Z(S/Q_\alpha)$. But $F_\beta \leq G_\alpha$ and therefore 6.12 gives

$$|F_\beta/C_{F_\beta}(Q_\alpha)| = 3.$$

In particular F_β/D_β is an FF-module for L_β/Q_β so 2.8 implies that

$$L_\beta/Q_\beta \not\cong 2{\cdot}A_5, 2_-^{1+4}A_5.$$

If $L_\beta/Q_\beta \cong SL_2(9)$, then by 2.10, F_β/D_β is a natural $SL_2(9)$-module, a contradiction to $|F_\beta/C_{F_\beta}(Q_\alpha)| = 3$. So,

$$L_\beta/Q_\beta \cong SL_2(3).$$

Now $[Q_\beta, Q_\alpha, Q_\alpha] = 1$ and $PSL_2(3)$ is 3-stable imply that t_β acts non-trivially on each non-central chief factor of L_β in Q_β and therefore it inverts every non-central chief factor of L_β in Q_β. Also t_β inverts $F_\beta Q_\alpha/Q_\alpha = Z(S/Q_\alpha)$; in particular t_β acts non-trivially on $\overline{S} = S/Q_\alpha$. Then by [Go; p.173], t_β has to act non-trivially on $\overline{S}/\Phi(\overline{S})$. If t_β completely inverts $\overline{S}/\Phi(\overline{S})$ then, since t_β also inverts $\Phi(\overline{S}) = Z(\overline{S})$, it completely inverts $\overline{S}$. Since a fixed point free automorphism of order 2 of a group implies that the group is abelian we get that $\overline{S}$ is abelian, a contradiction. Therefore $|[S, t_\beta]Q_\alpha/Q_\alpha| = 3^2$. Recall that by 6.6.4, $Q_\beta^* = [Q_\beta, O^3(L_\beta)]$. Then, as t_β inverts each of the non-central chief factors we get that

$$|Q_\beta^* Q_\alpha/Q_\alpha| = 3^2$$

and so $|Q_\beta^*/C_{Q_\beta^*}(Q_\alpha)| = 3^2$. Hence L_β has exactly two non-central chief factors in Q_β. Moreover, $\Phi(Q_\beta^*) \le Q_\alpha$ and so $[\Phi(Q_\beta^*), Z_\alpha] = 1$ and $\Phi(Q_\beta^*) \le D_\beta$ by 3.14(a) applied to $<Z_\alpha^{L_\beta}>$. Put $\widetilde{Q_\beta} = Q_\beta/D_\beta$. Since Q_β^* acts trivially on $\widetilde{Q_\beta^*}$ and (see proof of 6.6.6) $t_\beta Q_\beta^*/Q_\beta^* \in Z(L_\beta/Q_\beta^*)$ we have $\widetilde{Q_\beta^*} = C_{\widetilde{Q_\beta^*}}(t_\beta) \times [\widetilde{Q_\beta^*}, t_\beta]$ and L_β normalizes $C_{\widetilde{Q_\beta^*}}(t_\beta)$. Now since t_β inverts all the non-central chief factors in Q_β,

$$C_{\widetilde{Q_\beta^*}}(t_\beta) \le C_{\widetilde{Q_\beta}}(O^3(L_\beta)) = 1.$$

Thus $\widetilde{Q_\beta^*} = [\widetilde{Q_\beta^*}, t_\beta]$ has order 3^4.

Let $E = C_{Q_\beta}(\widetilde{Q_\beta^*})$.

7.1.1 $C_E(t_\beta) \le D_\beta$.

Proof: First notice that $C_{Q_\beta}(t_\beta)$ normalizes $C_E(t_\beta)$. Now if the claim is not true, pick $F \le C_E(t_\beta)$ with $[F, C_{Q_\beta}(t_\beta)] \le D_\beta$ and $F \not\le D_\beta$. Since by Frattini argument $L_\beta = Q_\beta C_{L_\beta}(t_\beta)$, a composition series for L_β in Q_β is also a composition series for $C_{L_\beta}(t_\beta)$ in Q_β and we conclude that $[C_{Q_\beta}(t_\beta), O^3(C_{L_\beta}(t_\beta))] = 1$. Hence FD_β/D_β is centralized by $C_{Q_\beta}(t_\beta)$ (by choice of F), $\widetilde{Q_\beta^*}$ (by choice of E) and $O^3(C_{L_\beta}(t_\beta))$.

As $Q_\beta = Q^*_\beta C_{Q_\beta}(t_\beta)$ and $L_\beta = Q_\beta C_{L_\beta}(t_\beta)$, we conclude

$$O^3(L_\beta) \leq Q^*_\beta C_{Q_\beta}(t_\beta) O^3(C_{L_\beta}(t_\beta)).$$

Hence

$$[F, O^3(L_\beta)] \leq D_\beta$$

and

$$[F, O^3(L_\beta), O^3(L_\beta)] \leq [D_\beta, O^3(L_\beta)] = 1.$$

Now we have a group generated by 3′ elements ($O^3(L_\beta)$) acting quadratically on a 3-group (F); thus

$$[F, O^3(L_\beta)] = 1$$

which implies $F \leq D_\beta$, a contradiction. Hence $C_E(t_\beta) \leq D_\beta$.

7.1.2 $Q_\alpha \cap Q_\beta \leq E$.

Proof: Recall that by 6.10 Q_α is abelian and therefore

$$[Q_\alpha \cap Q_\beta, Q_\alpha \cap Q^*_\beta] = 1$$

which gives $[(Q_\alpha \cap Q_\beta)Q^*_\beta, \widetilde{Q^*_{\alpha\beta}}] = 1$ (where $Q^*_{\alpha\beta} = Q_\alpha \cap Q^*_\beta$). But

$$(Q_\alpha \cap Q_\beta)Q^*_\beta \trianglelefteq G_{\alpha\beta} O^3(L_\beta) = G_\beta;$$

thus $[(Q_\alpha \cap Q_\beta)Q^*_\beta, <\widetilde{Q^*_{\alpha\beta}}^{G_\beta}> = 1$. Since $\widetilde{Q^*_\beta} = [\widetilde{Q^*_\beta}, O^3(L_\beta)] = <[\widetilde{Q^*_\beta}, Q_\alpha]^{G_\beta}>$ and $[Q^*_\beta, Q_\alpha] \leq Q^*_{\alpha\beta}$ we get $<\widetilde{Q^*_{\alpha\beta}}^{G_\beta}> = \widetilde{Q^*_\beta}$ and therefore we get $[(Q_\alpha \cap Q_\beta)\widetilde{Q^*_\beta}, \widetilde{Q^*_\beta}] = 1$. Then $[Q_\alpha \cap Q_\beta, \widetilde{Q^*_\beta}] = 1$ and the claim follows. □

Now $E = C_{Q_\beta}(\widetilde{Q^*_\beta}) = C_E(t_\beta)Q^*_\beta = D_\beta Q^*_\beta$ by 7.1.1 and so by 7.1.2,

$$Q_\alpha \cap Q_\beta \leq D_\beta Q^*_\beta.$$

Since $[\widetilde{Q_\beta}, Q_\alpha, Q_\alpha] = 1$, $[\widetilde{Q_\beta}, Q_\alpha] \leq C_{\widetilde{Q_\beta}}(Q_\alpha)$. As $\widetilde{Q^*_\beta}$ has two non-central chief factors,

$$|\widetilde{Q^*_\beta}/C_{\widetilde{Q^*_\beta}}(Q_\alpha)| \geq 3^2 \; and \; |[\widetilde{Q^*_\beta}, Q_\alpha] \geq 3^2.$$

From $|\widetilde{Q^*_\beta}| = 3^4$ and $[\widetilde{Q^*_\beta}, Q_\alpha] \leq \widetilde{Q_{\beta\alpha}} \leq C_{\widetilde{Q^*_\beta}}(Q_\alpha)$ we conclude $[Q^*_\beta, Q_\alpha]D_\beta = Q_\alpha \cap Q_\beta$ and $|Q_\alpha \cap Q_\beta/D_\beta| = 3^2$. Since $|Q_\alpha/Q_\alpha \cap Q_\beta| = 3$ we finally get that $|Q_\alpha/D_\beta| = 3^3$. Hence $|Q_\alpha/C_{Q_\alpha}(Q^*_\beta)| \leq 3^3$. Now since by 2.14 $L_\alpha = <Q^*_\beta, Q^{*g}_\beta>Q_\alpha$ for some $g \in G_\alpha$, we get $|Q_\alpha/C_{Q_\alpha}(L_\alpha)| \leq 3^6$. By 2.10, only $2{\cdot}M_{12}$ has an irreducible module of dimension less than or equal to six. Moreover, this module is unique and its dimension is actually six. Hence, $L_\alpha/Q_\alpha \cong 2{\cdot}M_{12}$ and $|Q_\alpha| = 3^6$ and therefore we also get that $|D_\beta| = 3^3$, $|S| = 3^9$ and $|Q_\beta| = 3^8$.

It is clear now that since Q_α is an irreducible elementary abelian normal subgroup of L_α of order 3^6, $L_\alpha \sim 3^6 2{\cdot}M_{12}$.

Reviewing, $|D_\beta| = 3^3$ and D_β is central for $O^3(L_\beta)$. Also $|\widetilde{Q^*_\beta}| = 3^4$ and Q^*_β has two composition factors each of dimension 2. Finally, $|Q_\beta| = 3^8$ and so $|Q_\beta/Q^*_\beta| = 3$. Thus, $L_\beta \sim 3^{1+1+1+2+2+1}SL_2(3)$.

Proposition 7.2 *If* $S \neq Q_\alpha Q_\beta$ *then* $(L_\alpha, L_\beta) \sim (3^6 2{\cdot}M_{12}, 3^{1+4}Sp_4(3))$.

Proof: Suppose that $S \neq Q_\alpha Q_\beta$.
Then by 3.19 and 6.13, $L_\alpha/Q_\alpha \cong (2)M_{12}$ and $L_\beta/Q_\beta \cong (P)Sp_4(3)$.

$$Therefore \; |S/Q_\alpha| = 3^3 \; and \; |S/Q_\beta| = 3^4.$$

Hence,

$$|Q_\alpha|/|Q_\beta| = 3.$$

Then

$$|Q_\beta/C_{Q_\beta}(Q_\alpha)| = |Q_\beta/Q_\alpha \cap Q_\beta| = |Q_\alpha Q_\beta/Q_\alpha| \leq$$
$$|Q_\alpha Q_\beta/Q_\beta|.$$

Hence all composition factors for L_β in Q_β are FF-modules for L_β/Q_β. Thus by 2.8, $L_\beta/Q_\beta \sim Sp_4(3)$ and L_β has a unique non-central composition factor in Q_β; moreover, this composition factor is a natural module. In particular, $\Phi(Q_\beta) \leq D_\beta$ and so by 2.10 $[Q_\beta/D_\beta, t_\beta]$ is a natural $Sp_4(3)$-module for L_β/Q_β and $C_{Q_\beta/D_\beta}(t_\beta) = C_{Q_\beta/D_\beta}(L_\beta)$. Hence

$$[C_{Q_\beta}(t_\beta), L_\beta, L_\beta] = 1,\ C_{Q_\beta}(t_\beta) = D_\beta \text{ and } |Q_\beta/D_\beta| = 3^4.$$

Thus, as $t_\beta \in O^3(L_\beta)$,

$$Q_\beta = [Q_\beta, O^3(L_\beta)]D_\beta$$

and since $L_\beta = O^3(L_\beta)Q_\beta$ and $[Q_\beta, O^3(L_\beta)] \leq O^3(L_\beta)$ we get that

$$L_\beta = O^3(L_\beta)[Q_\beta, O^3(L_\beta)]D_\beta = O^3(L_\beta)D_\beta.$$

By 6.11, D_β is elementary abelian and so

$$[D_\beta, L_\beta] = [D_\beta, O^3(L_\beta)D_\beta] = 1.$$

Hence $D_\beta = \Omega_1 Z(L_\beta) = Z_\beta$.

Now $|Q_\alpha/Z_\beta| = 3 \cdot |Q_\beta/Z_\beta| = 3 \cdot |Q_\beta/D_\beta| = 3 \cdot 3^4 = 3^5$.

Pick $Z_\beta < X \leq Q_\alpha$ with $[X, S] \leq Z_\beta$. Then [X,S,S]=1 and so $[X, S'] = 1$. Hence $|Q_\alpha/C_{Q_\alpha}(S')| \leq 3^4$.

By 2.14, $L_\alpha = \langle S, S'^g \rangle$ for some $g \in G$ and so

$$|Q_\alpha/C_{Q_\alpha}(L_\alpha)| \leq 3^5 \cdot 3^4 = 3^9.$$

Since $C_{Q_\alpha}(L_\alpha) = 1$, $|Q_\alpha| \leq 3^9$.

By 6.10, Z_α is the unique non-central chief factor for L_α in Q_α.

From 2.10 now we get that

$$L_\alpha/Q_\alpha \cong 2 \cdot M_{12} \text{ and } |Z_\alpha| = 3^6.$$

Furthermore, $Q_\alpha = C_{Q_\alpha}(t_\alpha) \times [Q_\alpha, t_\alpha]$. But $C_{Q_\alpha}(t_\alpha) \leq D_\alpha = 1$. Since Z_α is the unique non-central chief factor for L_α in Q_α,

$$[Q_\alpha/Z_\alpha, O^3(L_\alpha)] = 1$$

and

$$1 \neq [Q_\alpha, t_\alpha] \leq Z_\alpha.$$

Irreducibility of Z_α yields now that

$$[Q_\alpha, t_\alpha] \leq Z_\alpha.$$

As Q_α is an irreducible elementary abelian normal subgroup of L_α of order 3^6 we now get that $L_\alpha \sim 3^6 2{\cdot}M_{12}$. Also, $|Q_\beta| = |Q_\alpha|/3 = 3^5$ and as $|Q_\beta/D_\beta| = 3^4$, $|D_\beta| = 3$ and so we get $L_\beta \sim 3^{1+4}Sp_4(3)$.

8 The case b=1 and $\Theta \cong PSL_2(9)$ or M_{11}

In this section, $\Theta \cong PSL_2(9)$, M_{11} and b=1. Notice that by 6.13 $\Psi \not\cong (P)Sp_4(3)$ and therefore by 6.2,

$$\Psi \cong SL_2(3),\ 2{\cdot}A_5,\ 2_-^{1+4}A_5,\ SL_2(9).$$

Recall also from 3.19 that $S = Q_\alpha Q_\beta$. Moreover $[Z_\alpha, Z_{\alpha'}] = 1$.

Remark 8.1 Since a Sylow 3-subgroup of Θ is elementary abelian we have

$$\Phi(Q_\beta) \leq Q_\alpha.$$

Similarly $\Phi(Q_\alpha) \leq Q_\beta$.

Lemma 8.2 *If $N \leq S, N \trianglelefteq B, \delta \in \{\alpha, \beta\}$ then $N \leq Q_\delta$ or $NQ_\delta = S$.*

Proof: It follows from irreducibility of B on S/Q_δ (see 2.12).

Corollary 8.3 $S = Z_\alpha Q_\beta$.

Proof: It is an immediate consequence of 8.2.

Lemma 8.4 *Let* $X_\beta = \bigcap_{\delta\in\Delta(\beta)} Q_\delta$. *Then:*

(a) Q_β/X_β *is an irreducible* G_β*-module,*

(b) $[Q_\beta/D_\beta, t_\beta] = Q_\beta/D_\beta$ *and* $C_{Q_\beta/D_\beta}(t_\beta) = 1$,

(c) $C_{Q_\beta}(t_\beta) \leq D_\beta$ *and*

(d) $X_\beta = D_\beta$.

Proof: Let $X_\beta < A \leq Q_\beta$ with $A \trianglelefteq G_\beta$. Then $A \not\leq Q_\alpha$ (since if $A \leq Q_\alpha$ and $\gamma = \beta^g$ with $g \in G_\beta$ then since $A \trianglelefteq G_\beta$ we get

$$A = A^g \leq Q_\beta^g = Q_{\beta^g} = Q_\gamma$$

which gives $A \leq X_\beta$, a contradiction). Hence by 8.2, $AQ_\alpha = S$ and therefore $[Z_\alpha, Q_\beta] \leq [Z_\alpha, A] \leq A$. By 8.1 $Q'_\beta \leq X_\beta$ and so

$$[L_\beta, Q_\beta] = [<Z_\alpha^{G_\beta}>Q_\beta, Q_\beta] \leq A.$$

Let $\widetilde{Q_\beta} = Q_\beta/X_\beta$. Then $\widetilde{Q_\beta}$ is abelian. Now $\widetilde{Q_\beta} = C_{\widetilde{Q_\beta}}(t_\beta) \times [\widetilde{Q_\beta}, t_\beta]$ and both parts are normalized by L_β.

If $C_{\widetilde{Q_\beta}}(t_\beta) \neq 1$, we may assume $A = C_{Q_\beta}(t_\beta)X_\beta$ (since then $A \leq Q_\beta, A \trianglelefteq G_\beta$ and as $C_{\widetilde{Q_\beta}}(t_\beta) \neq 1$ we also have $X_\beta \neq A$). Hence

$$\tilde{A} = C_{\widetilde{Q_\beta}}(t_\beta)$$

and get $[[\widetilde{Q_\beta}, t_\beta], t_\beta] \leq [[L_\beta, Q_\beta], t_\beta] \leq [A, t_\beta, t_\beta] = 1$. Hence (element of order 2 acting on a 3-group) $[\widetilde{Q_\beta}, t_\beta] = 1$ a contradiction to $[\widetilde{Q_\beta}, Q_\alpha, Q_\alpha] = 1$ and the 3-stability of $L_\beta/<t_\beta Q_\beta>$. Therefore $C_{\widetilde{Q_\beta}}(t_\beta) = 1$ and $\widetilde{Q_\beta} = [\widetilde{Q_\beta}, t_\beta] = [\widetilde{Q_\beta}, L_\beta]$. Thus $\widetilde{Q_\beta} \leq [L_\beta, \widetilde{Q_\beta}] \leq \tilde{A}$ which implies $\tilde{A} = \widetilde{Q_\beta}$ and Q_β/X_β is an irreducible G_β-module.

Now by 6.9, $D_\beta \leq Q_\alpha$ and as $D_\beta \trianglelefteq G_\beta$ we get $D_\beta \leq X_\beta$. But

$$[X_\beta, Z_\alpha] \leq [Q_\alpha, Z_\alpha] = 1$$

and $Z_\alpha \not\leq Q_\beta$ give $X_\beta \leq D_\beta$. Hence $X_\beta = D_\beta$.

Lemma 8.5 *There is* $g \in G_\beta$ *such that* $t_\beta \in <Z_\alpha, Z_\alpha^g> Q_\beta$.

Proof: If $\Psi \cong SL_2(3)$ or $2{\cdot}A_5$ it is clear since in these cases

$$L_\beta = <Z_\alpha, Z_\alpha^g> Q_\beta$$

for some $g \in G_\beta$ and $t_\beta \in L_\beta$ by definition. Since inside $SL_2(9)$ we can generate a $2{\cdot}A_5$ this case is also clear. The case $2_-^{1+4}A_5$ is left. Let a, b be two elements in L_β/Q_β of order three and $H = <a,b>$ be such that $2^{1+4}H = 2_-^{1+4}A_5$. The possibilities for H then are $2_-^{1+4}A_5$, $2{\cdot}A_5$ or A_5. In the first two cases $t_\beta Q_\beta \in H$ and we are done and the last case can not happen as A_5 is 3-stable and Z_α acts quadratically on Q_β.

Notation 8.6 $\overline{Q_\gamma} = Q_\gamma / D_\gamma$.

Lemma 8.7 $|\overline{Q_\beta}| = 3^4$.

Proof: By 8.5, pick g $\in G_\beta$ such that

$$t_\beta \in <Z_\alpha, Z_\alpha^g> Q_\beta.$$

Since $|Q_\beta / C_{Q_\beta}(Z_\alpha)| = |Q_\beta Q_\alpha / Q_\alpha| = |S/Q_\alpha| = 3^2$, we get

$$|\overline{Q_\beta} / C_{\overline{Q_\beta}}(t_\beta)| \leq 3^4.$$

By 8.4(b), $C_{\overline{Q_\beta}}(t_\beta) = 1$ and therefore $|\overline{Q_\beta}| \leq 3^4$. Suppose $|\overline{Q_\beta}| < 3^4$. Since 5 does not divide $|GL_3(3)|$ we conclude that

$$L_\beta / Q_\beta \cong SL_2(3).$$

From 8.4(a) and 2.10, $|\overline{Q_\beta}| = 3^2$ and so $|Q_\beta / [Q_\beta, Q_\alpha] D_\beta| \leq 3$. Since $[Q_\beta, Q_\alpha] D_\beta \leq Q_\alpha$, $|Q_\beta Q_\alpha / Q_\alpha| \leq 3$ and $S \neq Q_\alpha Q_\beta$ since $|S/Q_\alpha| = 3^2$, a contradiction. Hence $|\overline{Q_\beta}| = 3^4$.

Lemma 8.8 $|[Z_\alpha, \overline{Q_\beta}]| = |\overline{Q_{\alpha\beta}}| = |\overline{Q_\beta \cap Z_\alpha}| = 9$.

Proof: If $|[Z_\alpha, \overline{Q_\beta}]| = 3$, then, with same argument as before, we get

$$|\overline{Q_\beta}| = |[\overline{Q_\beta}, t_\beta]| \leq 3^2,$$

a contradiction. Hence

$$9 \leq |[Z_\alpha, \overline{Q_\beta}]| \leq |\overline{Q_\beta \cap Z_\alpha}| \leq |\overline{Q_{\alpha\beta}}| \leq 9$$

and lemma is proved.

Lemma 8.9 $D_\beta = Z_\beta$.

Proof: First, show $D_\beta \leq Z_\beta$. Let $L = <Z_\alpha^{G_\beta}>$. Then by 3.14(a), $O^3(L_\beta) \leq L$ and $L_\beta = LQ_\beta$. Since $\overline{Q_\beta}$ is irreducible for G_β we get $[\overline{Q_\beta}, L] = 1$ or $\overline{Q_\beta}$. If $[\overline{Q_\beta}, L] = 1$ then $[Q_\beta, L] \leq D_\beta$ so $[Q_\beta, O^3(L_\beta)] = 1$, a contradiction. Therefore $[\overline{Q_\beta}, L] = \overline{Q_\beta}$ which gives $[Q_\beta, L]D_\beta = Q_\beta$.

Also, as $L \trianglelefteq G_\beta$, we have $Q_\beta \leq N_{G_\beta}(L)$. Hence $[Q_\beta, L] \subseteq L$, $Q_\beta \leq D_\beta L$ and $L_\beta = LD_\beta$. But from 6.11 now, $[D_\beta, D_\beta] \leq \Phi(D_\beta) = 1$. As $D_\beta \leq Q_\alpha$, $[L, D_\beta] = 1$ so D_β and L both centralize D_β. But then, we also get $[D_\beta, L_\beta] = [D_\beta, LD_\beta] = 1$. Thus $D_\beta \leq Z(L_\beta) \leq Z_\beta$. Therefore $D_\beta \leq Z_\beta$.

Since $Z_\beta = \Omega_1 Z(L_\beta) \leq C_{Q_\beta}(O^3(L_\beta) = D_\beta$ the lemma follows.

Lemma 8.10 $Q_\alpha \cap Q_\beta = Z_\alpha \cap Q_\beta$.

Proof: It is enough to show that $Q_\alpha \cap Q_\beta \leq Z_\alpha \cap Q_\beta$. Let $x \in Q_\alpha \cap Q_\beta$. Then $xD_\beta \in Q_\alpha \cap Q_\beta / D_\beta = \overline{Q_{\alpha\beta}} = \overline{Z_\alpha \cap Q_\beta} = Z_\alpha \cap Q_\beta / D_\beta$. Therefore, $xD_\beta = yD_\beta$, where $y \in Z_\alpha \cap Q_\beta$. Then $x = yd$, $d \in D_\beta$. 3.11 gives

$$Z_\beta \leq Z_\alpha.$$

By 8.9, $D_\beta = Z_\beta \leq Z_\alpha$. Therefore $x \in Z_\alpha$ and hence $x \in Z_\alpha \cap Q_\beta$.

Corollary 8.11 $Q_\alpha = Z_\alpha$.

Proof: Since $Q_\alpha \subseteq S = Z_\alpha Q_\beta$ we get $Q_\alpha \subseteq Z_\alpha Q_\beta \cap Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta)$ and hence $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta) = Z_\alpha$.

Lemma 8.12 *(1) $Q_\alpha = Z_\alpha$ is irreducible,*

(2) If $\Theta \cong PSL_2(9)$ and $\Psi \cong SL_2(3)$, $2{\cdot}A_5$ or $2^{1+4}_-A_5$ then $|Z_\alpha| = 3^4$, $|Z_\beta| = 3$ and $|Q_\beta| = 3^5$,

(3) If $\Theta \cong PSL_2(9)$ and $\Psi \cong SL_2(9)$ then $|Z_\alpha| = 3^6$, $|Z_\beta| = 3^2$ and $|Q_\beta| = 3^6$,

(4) If $\Theta \cong M_{11}$ and $\Psi \cong SL_2(3)$, $2{\cdot}A_5$ or $2^{1+4}_-A_5$ then $|Z_\alpha| = 3^5$, $|Q_\beta| = 3^6$ and $|Z_\beta| = 3^2$,

(5) If $\Theta \cong M_{11}$ and $\Psi \cong SL_2(9)$ then then $|Z_\alpha| = 3^5$, $|Q_\beta| = 3^5$ and $|Z_\beta| = 3$.

Proof: 3.11 and 8.9 give $D_\beta = Z_\beta \leq Z_\alpha$. Hence

$$|Z_\alpha/Z_\beta| = |Z_\alpha Q_\beta/Q_\beta||Z_\alpha \cap Q_\beta/Z_\alpha \cap D_\beta| = |Z_\alpha Q_\beta/Q_\beta||Z_\alpha \cap Q_\beta/Z_\beta|.$$

Recall now 8.8 to get $|Z_\alpha \cap Q_\beta/Z_\beta| = 3^2$ and hence

$$|Z_\alpha/Z_\beta| = 3^2|Z_\alpha Q_\beta/Q_\beta| = 3^2|S/Q_\beta|.$$

Since $S/Q_\beta \in Syl_3(\Psi)$ we get that

$$|S/Q_\beta| = 3 \; if \; \Psi \not\cong SL_2(9)$$

and

$$|S/Q_\beta| = 3^2 \; if \; \Psi \cong SL_2(9).$$

Hence if $\Psi \not\cong SL_2(9)$ then $|Z_\alpha/Z_\beta| = 3^3$ and if $\Psi \cong SL_2(9)$ then $|Z_\alpha/Z_\beta| = 3^4$; in particular, $|Z_\alpha/Z_\beta| \leq 3^4$. Since by 2.14 we can generate L_α by two Sylow 3-subgroups we get $|Z_\alpha| \leq 3^8$.

By 6.10, Z_α is irreducible as L_α-module.

Case $\Theta \cong PSL_2(9)$: Then by 2.10 $|Z_\alpha| = 3^4$ or 3^6. Moreover, if $|Z_\alpha| = 3^4$ then $|Z_\beta| = 3$ and therefore $|Z_\alpha/Z_\beta| = 3^3$ and $\Psi \not\cong SL_2(9)$ and if $|Z_\alpha/Z_\beta| = 3^6$ then

$|Z_\beta| = 3^2$ and therefore $|Z_\alpha/Z_\beta| = 3^4$ and $\Psi \cong SL_2(9)$.

Case $\Theta \cong M_{11}$: 2.10 gives that $|Z_\alpha| = 3^5$ and $|Z_\beta| = 3$ or 3^2. If $|Z_\beta| = 3$ then $|Z_\alpha/Z_\beta| = 3^4$ and $\Psi \cong SL_2(9)$ and if $|Z_\beta| = 3^2$ then $|Z_\alpha/Z_\beta| = 3^3$ and $\Psi \ncong SL_2(9)$.

Corollary 8.13 *(1) If* $\Theta \cong PSL_2(9)$ *and* $\Psi \cong SL_2(3)$ *then* $(L_\alpha, L_\beta) \sim (3^4PSL_2(9), 3^{1+2+2}SL_2(3))$.

(2) If $\Theta \cong PSL_2(9)$ *and* $\Psi \cong 2{\cdot}A_5$ *then* $(L_\alpha, L_\beta) \sim (3^4PSL_2(9), 3^{1+4}2{\cdot}A_5)$.

(3) If $\Theta \cong PSL_2(9)$ *and* $\Psi \cong 2_-^{1+4}A_5$ *then* $(L_\alpha, L_\beta) \sim (3^4PSL_2(9), 3^{1+4}2_-^{1+4}A_5)$.

(4) If $\Theta \cong PSL_2(9)$ *and* $\Psi \cong SL_2(9)$ *then* $(L_\alpha, L_\beta) \sim (3^6PSL_2(9), 3^{1+1+4}SL_2(9))$.

(5) If $\Theta \cong M_{11}$ *and* $\Psi \cong SL_2(3)$ *then* $(L_\alpha, L_\beta) \sim (3^5M_{11}, 3^{1+1+2+2}SL_2(3))$.

(6) If $\Theta \cong M_{11}$ *and* $\Psi \cong 2{\cdot}A_5$ *then* $(L_\alpha, L_\beta) \sim (3^5M_{11}, 3^{1+1+4}2{\cdot}A_5)$.

(7) If $\Theta \cong M_{11}$ *and* $\Psi \cong SL_2(9)$ *then* $(L_\alpha, L_\beta) \sim (3^5M_{11}, 3^{1+4}PSL_2(9))$.

Proof: By 8.12, $Q_\alpha = Z_\alpha$ is an irreducible elementary abelian normal subgroup of L_α. Moreover,

$$|Q_\alpha| = 3^4 \ if\ \Theta \cong PSL_2(9)\ and\ \Psi \ncong SL_2(9),$$

$$|Q_\alpha| = 3^6 \ if\ \Theta \cong PSL_2(9)\ and\ \Psi \cong SL_2(9)$$

and

$$|Q_\alpha| = 3^5 \ if\ \Theta \cong M_{11}.$$

Thus, the structure of L_α is as given in the corollary.

Notice now that in all the above cases, D_β is central as by 8.9 we have $D_\beta = Z_\beta$. Moreover in cases (1), (2), (3) and (7), $|D_\beta| = 3$ and for the rest of the cases we have $|D_\beta| = 3^2$.

Finally, in all the cases, $|Q_\beta/Z_\beta| = 3^4$ and hence Q_β/Z_β is an irreducible L_β-module whenever $\Psi \ncong SL_2(3)$ which proves (2), (3), (4), (6) and (7). By 8.4 though, t_β inverts Q_β/Z_β. Since by 2.10 $SL_2(3)$ has a unique faithful irreducible GF(3)-module which is of order 3^2, (1) and (5) follow.

Lemma 8.14 *The case* $(L_\alpha, L_\beta) \sim (3^5 M_{11}, 2_-^{1+4} A_5)$ *is impossible.*

Proof: Let $\overline{L_\beta} = L_\beta/Q_\beta$. Since $O_2(\overline{L_\beta})/<\overline{t_\beta}>$ is the even permutation module, $\overline{S}$ centralizes a group $\overline{D}$ of order 8 in $O_2(\overline{L_\beta})$. It is easy to see that $\overline{D} \cong D_8$. Let D^* be the inverse image of D in L_β. Then $[D^*, S] \leq Q_\beta \leq S$ and so $D \leq N_{G_\beta}(S) = B$. Now recall the definition of K from 3.16 and let $D = K \cap D^*$ and pick $t \in D \setminus <t_\beta>$ with $|t| = 2$. Since t_β inverts $Q_\alpha \cap Q_\beta/Z_\beta$ and $<t_\beta> = [t, D]$, t neither centralizes nor inverts $Q_\alpha \cap Q_\beta/Z_\beta$. Since $|Q_\alpha \cap Q_\beta/Z_\beta| = 3^2$, $|[Q_\alpha \cap Q_\beta/Z_\beta, t]| = 3$. Now $[Z_\beta, t] \leq [Z_\beta, L_\beta] = 1$ and $[Q_\alpha, t] \leq [Q_\alpha, B] \cap [S, D] \leq Q_\alpha \cap Q_\beta$ and we get $|[Q_\alpha, t]| = 3$. Similarly,

$$|[Q_\alpha, t_\beta]| = |[Q_\alpha \cap Q_\beta/Z_\beta, t_\beta]| = 3^2.$$

Since M_{11} has no outer automorphism and only one class of involutions, there exists $g \in L_\alpha$ so that $[t^g t_\beta, L_\alpha] \leq Q_\alpha$. Since Q_α is an irreducible L_α-module, $t^g t_\beta$ centralizes or inverts Q_α. In the first case

$$[Q_\alpha, t^g] = [Q_\alpha, t_\beta]$$

and in the second case

$$[Q_\alpha, t^g] = C_{Q_\alpha}(t_\beta).$$

But $|[Q_\alpha, t^g]| = |[Q_\alpha, t]| = 3$, $|[Q_\alpha, t_\beta]| = 3^2$ and

$$|C_{Q_\alpha}(t_\beta)| = |Q_\alpha|/|[Q_\alpha, t_\beta]| = 3^5/3^2 = 3^3.$$

So, in both cases we obtain a contradiction.

Proof of Theorem P: It follows from 7.1, 7.2, 8.13 and 8.14.□

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