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COMPLEXES OF CURVES AND MAPPING CLASS GROUPS

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ABSTRACT

COMPLEXES OF CURVES AND MAPPING CLASS GROUPS

By

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This thesis is composed of two parts.

In the first part of this thesis, we study the complexes of curves on orientable surfaces of small genus in order to better understand the mapping class groups of such surfaces. Our main result is that the group of automorphisms of the complex of curves of a surface is isomorphic to the extended mapping class group of the surface, if the surface is a sphere with at least five punctures or is a tori with at least three punctures. As an application we prove that any isomorphism between two finite index subgroups of the extended mapping class group is induced by an inner automorphism of the extended mapping class group. We conclude that the outer automorphism group of a finite index subgroup of the extended mapping class group is finite.

The second part concerns closed nonorientable surfaces. Namely, we compute the first homology of the mapping class group of such a surface. It turns out that this group is cyclic of order two if the genus of the surface is at least seven. We also show that in this case the the subgroup of the mapping class group generated by Dehn twists is perfect. As an algebraic application, we conclude that the group of isometries of a vector space of dimension $n \geq 7$ over the finite field of order two equipped with the symmetric bilinear form $\langle , \rangle$ defined by $\langle v_i, v_j \rangle = \delta_{ij}$ on a basis $\{v_1, v_2, \ldots, v_n\}$ is perfect.

To my parents, brothers, sisters and to my wife

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Chapter 0

Introduction

The complex of curves was introduced by Harvey [11] to define a certain boundary to the Teichmüller space. Later, this complex is used for many authors in the study of the algebraic structures of the (extended) mapping class groups of surfaces. In particular, Ivanov [16] proved that the canonical map from the extended mapping class group $\mathcal{M}_S^*$ of a surface S to the group of automorphisms of the complex of curves $C(S)$ is onto, provided the genus of the surface S is at least two. In fact, this natural map is an isomorphism if in addition, the surface S is not a closed surface of genus two. Using this result, he proved that any isomorphism between two finite index subgroups of the (extended) mapping class group is induced by some inner automorphism of $\mathcal{M}_S^*$, concluding that the outer automorphism group of a subgroup of $\mathcal{M}_S^*$ of finite index is finite. This is a generalization of the fact that the outer automorphism group of the (extended) mapping class group is finite (cf. [15], [22]). He also gave a new proof of Royden's Theorem saying that the isometries of the Teichmüller space are all induced by diffeomorphisms of the surface.

In this work, we prove that for a surface S the natural map from $\mathcal{M}_S^*$ to the group of automorphisms of the complex of curves $C(S)$ is an isomorphism if S is either a sphere with at least five punctures or a torus with at least three punctures. After proving this result, the result of Ivanov about the finite index subgroups of mapping class groups follows for the surfaces we consider. Since this proof has not been written yet, we give it here. We also prove that (with one exception) the topological type of the surfaces of genus at most one are determined by their complexes of curves.

In Chapter 1, we give the relevant definitions and preliminary information on orientable surfaces. It is well-known to the experts that the map $\mathcal{M}_S^* \to \text{Aut } C(S)$ is injective if S is not a torus with two punctures or a closed surface of genus two. We give a proof of this fact in this section.

We prove our main results about the mapping class group and the automorphisms of the complex of curves in Chapter 2. The proofs for the case of the punctured sphere and for torus are given separately. Although the main ideas of our proofs are the same as that of Ivanov's, the proof in the punctured sphere case differs substantially from the other cases.

We prove our results on nonorientable surfaces in Chapter 3. In this chapter, we first give the necessary background on the mapping class groups of nonorientable surfaces and the vector spaces over the finite field of order two. Relationship between these two concepts comes from a theorem of McCarthy and Pinkall [23], Theorem 2. Then, we compute the first homology group of the mapping class group of a nonorientable surface. It turns out that if the genus of the surface is at least seven, then the first homology group of the mapping class group is cyclic of order two. Another result in this chapter is that the subgroup of the mapping class group generated by the Dehn twists about 2-sided simple closed curves is perfect. We then conclude that the group of isometries of a vector space V over the finite field $\mathbf{F}_2$ with a symmetric bilinear form $\langle \, , \rangle$ defined on a basis $\{v_1, v_2, \ldots, v_n\}$ by $\langle v_i, v_j \rangle = \delta_{ij}$ is perfect if the dimension of V is at least seven.

Chapter 1

Complexes of Curves and Extended Mapping Class Groups

In this chapter, we study the complex of curves of an orientable surface and the action of the extended mapping class group on this complex. The necessary definitions and preliminary informations are given.

1.1 Introduction and Preliminaries

1.1.1 Circles on surfaces

Let S be a connected orientable surface without boundary. Although a simple closed curve on S is an embedding $S^1 \to S$, by a 'simple closed curve' on S (or 'circle' on S) we will mean the image of the map under consideration.

A circle on S is said to be *nontrivial* if it neither bounds a disc nor bounds a disc with one puncture. We denote by $\mathcal{S}$ the set of isotopy classes of nontrivial circles. Circles will be denoted by lower-case letters a, b, c, etc., and their isotopy classes by the Greek letters α, β, γ, etc.

Let C be a collection of pairwise disjoint circles on S. The surface obtained from S by cutting along C is denoted by S_C.

A nontrivial circle a on S is called *k-separating* if the surface S_a is disconnected and one of its components is a disc with k punctures. If S_a is connected we call a a *nonseparating* circle. The isotopy class of a circle is called *nonseparating* (or *k-separating*) if the circle is nonseparating (or k-separating). Note that on a sphere with n punctures, a k-separating circle is also $(n-k)$-separating. In this case, we may take $k \leq \frac{n}{2}$.

Two circles a, b on a surface S are called *topologically equivalent* if there exists a homeomorphism $F: S \to S$ such that $F(a) = b$. It is clear from the classification of surfaces that two circles on a surface are topologically equivalent if and only if they are either both nonseparating or both k-separating for some k.

Let $\alpha, \beta \in \mathcal{S}$. The *geometric intersection number* $i(\alpha, \beta)$ of α and β is the infimum of the cardinality of $a \cap b$, where $a \in \alpha$ and $b \in \beta$. By the geometric intersection of two circles we will mean the geometric intersection number of their classes.

1.1.2 The complex $C(S)$

Let V be a nonempty set. An *abstract simplicial complex* K with vertices V is a collection of nonempty finite subsets of V satisfying the following two conditions:

1) If $x \in V$ then $\{x\} \in K$.

2) If $\sigma \in K$ and if $\sigma' \subset \sigma$ is a nonempty subset of V, then $\sigma' \in K$.

In the second case, σ' is called a *face* of σ. The *dimension* $\dim \sigma$ of a simplex σ is $\text{card}\, \sigma - 1$, where $\text{card}\, \sigma$ is the cardinality of σ. A simplex σ is called a *q-simplex* if $\dim \sigma = q$. The supremum of the dimensions of the simplices of K is called the *dimension* of K, denoted by $\dim K$.

Let K be an abstract simplicial complex and L a subcomplex of K, i.e., L is itself a simplicial complex and $L \subset K$. L is said to be a *full subcomplex* if whenever a set of vertices of L is a simplex in K, it is also a simplex in L.

The *complex of curves* $C(S)$ on an orientable surface S is the abstract simplicial complex with vertex set $\mathcal{S}$ such that a set of vertices $\{\alpha_0, \alpha_1, \ldots, \alpha_q\}$ forms a q-simplex if and only if $\alpha_0, \alpha_1, \ldots, \alpha_q$ have representatives which are pairwise disjoint.

The complex of curves can also be defined for a surface with boundary in a similar manner. In this case a circle is nontrivial if it does not bound a disc with at most one puncture or it is not parallel to a boundary component of S. Clearly, the complex of curves of a surface of genus g with b boundary components and with n punctures, and that of a surface of genus g with $n+b$ punctures are isomorphic. Therefore, we will consider only punctured surfaces. We will think of punctures on the surface S as distinguished points.

An alternative definition of the complex $C(S)$ can be given as follows. The set of vertices is the same as before. Now $\{\alpha_0, \alpha_1, \ldots, \alpha_q\}$ forms a q-simplex if and only if $i(\alpha_i, \alpha_j) = 0$ for all i, j.

If the Euler characteristic $\chi(S)$ of S is negative, then S can be endowed with a hyperbolic metric, and there exists a unique geodesic in the isotopy class of each nontrivial circle (cf. [1], [4]). A simple closed geodesic is automatically nontrivial. The geometric intersection number of two classes are realized by the unique geodesics they contain. In this case, a third definition of $C(S)$ can be given: the vertices are the simple closed geodesics on S and a set of vertices forms a simplex if and only if the geodesics in the set are pairwise disjoint. This definition of the complex of curves is independent of the choice of the metric chosen in the sense that different metrics give rise to isomorphic complexes.

If S is a sphere with at most three punctures, then there are no nontrivial circles on S. Hence $C(S) = \emptyset$. If S is a closed torus and a is a nontrivial circle on S, then the surface S_a is an annulus. Therefore, any nontrivial circle on S other than a either is isotopic to a or intersect a. If S is not a sphere with at most three punctures or a closed torus, then the Euler characteristic of S is negative and the number of nontrivial simple closed curves on S is $3g - 3 + n$, where g is the genus of S and n is the number of punctures on S. Hence the dimension of $C(S)$ is $3g - 4 + n$.

Suppose that dim $C(S) \geq 1$. *A pentagon* in the complex $C(S)$ is an ordered 5-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, defined up to cyclic permutations, of vertices of $C(S)$ such that $i(\alpha_j, \alpha_{j+1}) = 0$ for $j = 1, 2, 3, 4, 5$ and $i(\alpha_j, \alpha_k) \neq 0$ otherwise ($\alpha_6 = \alpha_1$).

Let α be a vertex of $C(S)$. The *link* $L(\alpha)$ of α is the full subcomplex of $C(S)$ whose vertex set is the set of vertices of $C(S)$ which form an edge together with α, i.e., the vertex set of $L(\alpha)$ is

$$\{\beta : \beta \neq \alpha \text{ and } i(\alpha, \beta) = 0\}.$$

Given a subcomplex L of $C(S)$, we define the *dual link* L^d of L to be the graph whose vertices are the vertices of L, and whose edges are those pairs of vertices which do not form an edge in L. We denote the dual link of $L(\alpha)$ by $L^d(\alpha)$.

1.2 The complex $B(S)$ and ideal triangulations

Let S be a surface with $n \geq 1$ punctures. The simplicial complex $B(S)$ is defined as follows. The vertices of $B(S)$ are the isotopy classes of nontrivial embedded arcs on S joining punctures. (By definition, an arc is called nontrivial if it is not deformable to a puncture.) A set of vertices forms a simplex if and only if the vertices in the set have representatives which are pairwise disjoint. A simple Euler characteristic argument shows that dim $B(S)$ is finite and all maximal simplices have the same dimension. As in the definition of a circle, by an *arc* we mean the image of $(0,1)$ under a map $g: [0,1] \to S$ such that $g(0)$ and $g(1)$ are punctures of S (thinking the punctures on S as distinguished points). Therefore we do not include endpoints into arcs. We will denote arcs by the letters a', b', c' etc, and their isotopy class by α', β', γ' etc.

The geometric intersection number of isotopy classes of two arcs is defined similarly to that of circles: if α', β' are two vertices of $B(S)$, then $i(\alpha', \beta')$ is the infimum of the cardinality of $a' \cap b'$, where $a' \in \alpha', b' \in \beta'$. Then an alternative definition of $B(S)$ can be given. The vertex set is the same and a set of vertices form a simplex if and only if any two classes of arcs in the set have geometric intersection number zero.

Similarly, for a vertex α of $C(S)$ and a vertex β' of $B(S)$, we can define the geometric intersection number $i(\alpha, \beta')$ as the infimum of the cardinality of $a \cap b'$, where $a \in \alpha$ and $b' \in \beta'$. We define the geometric intersection number of two arcs or an arc and a circle in the obvious way, similar to that of two circles.

The following lemma is proved in [10], Expose 2, III, and it will be used throughout this work.

Lemma 1.1 *Let S be sphere with three punctures. Then*

(i) *up to isotopy there exists a unique nontrivial embedded arc joining a puncture P to itself, or P to another puncture Q.*

(ii) *any circle on S can be deformed to a puncture.*

Now suppose that S is a sphere with at least five punctures or a torus with at least three punctures for the next three paragraphs. If a is 2-separating circle on S, by Lemma 1.1 there exists up to isotopy a unique nontrivial embedded arc a' on the twice-punctured disc component of S_a joining two punctures. Conversely, an arc a' joining two different punctures of S determines uniquely a 2-separating circle up to isotopy, namely the boundary of a regular neighborhood of the arc and its endpoints (as punctures). This gives a one-to-one correspondence between the set of 2-separating isotopy classes in S and the set of isotopy classes of embedded arcs joining different punctures.

If a and b are two 2-separating circles on S, and α and β are their isotopy classes, such that the corresponding arcs a' and b' can be chosen disjoint with exactly one common endpoint P (see Figure 1.1 (a)), then we say that a and b constitute a *simple pair of circles* and denote it by $\langle a; b\rangle$. Similarly, $\langle a'; b'\rangle$ is called a *simple pair of arcs*. We call P the *center* of the simple pair. We also call $\langle \alpha; \beta\rangle$ and $\langle \alpha'; \beta'\rangle$ simple pairs.

Let $a_1', a_2', \ldots, a_k'$ be embedded pairwise disjoint arcs, P_{i-1} and P_i the endpoints of a_i', with $P_i \neq P_j$ for $i \neq j$, $0 \leq i, j \leq k$. Thus $\langle a_i'; a_{i+1}'\rangle$ is a simple pair of arcs with center P_i for each $1 \leq i \leq k-1$. Let $a_1, a_2, \ldots, a_k$ be the corresponding circles. These circles are well-defined up to isotopy by Lemma 1.1. We call $a_1', a_2', \ldots, a_k'$ (respectively, $a_1, a_2, \ldots, a_k$) a *chain of arcs* (respectively, a *chain of circles*) and denote it by $\langle a_1'; a_2'; \ldots; a_k'\rangle$ (respectively, $\langle a_1; a_2; \ldots; a_k\rangle$) (see Figure 1.1 (b)).

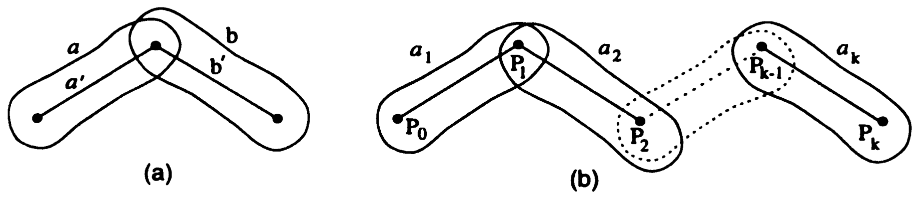


Figure 1.1: A simple pair and a chain.

If the Euler characteristic $\chi(S)$ of S is at most -2, then it follows from Theorem 3.3 of [14] that the geometric realization $|B(S)|$ of $B(S)$ is connected.

An *ideal triangulation* of S is a triangulation whose vertex set is the set of punctures of S, in the sense that vertices of a triangle can coincide, as can a pair of edges. The importance of $B(S)$ comes from its close connection with ideal triangulations of the surface S. Note that the set of isotopy classes of the edges of an ideal triangulation form a codimension-zero simplex in $B(S)$. Conversely, a set of nonintersecting (embedded) representatives of a codimension-zero simplex determines an ideal triangulation of S.

A *'good' ideal triangle* is by definition a set $\{a', b', c'\}$ of nontrivial embedded disjoint arcs such that

(i) a', b' and c' join, say, P_1 to P_2, P_2 to P_3 and P_3 to P_1, respectively, for three different punctures P_1, P_2 and P_3,

(ii) $a' \cup b' \cup c'$ bound a disc in S.

Figure 1.2: Triangles in an ideal triangulation

The following theorem is proved by Hatcher [11].

Theorem 1.2 *Let S be a connected orientable surface with at least one puncture. Then for any two codimension-zero simplices σ and σ' of $B(S)$ there exists a sequence of codimension-zero simplices $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ such that $\sigma_{i-1} \cap \sigma_i$ is a codimension-one simplex for each i, $1 \leq i \leq k$.*

1.3 The extended mapping class group and its actions

Let S be an orientable surface. Let Diff S be the group of self-diffeomorphisms of S and $\mathrm{Diff}_0\ S$ the normal subgroup of Diff S consisting of diffeomorphisms isotopic to the identity. The *extended mapping class group* of S is the group Diff $S/\text{Diff}_0\ S$ and we denote it by $\mathcal{M}_S^*$. Alternatively, one can define the extended mapping class group as the quotient Homeo S by the normal subgroup $\text{Homeo}_0\ S$, where Homeo S is the group of homeomorphisms of S and $\text{Homeo}_0\ S$ is the subgroup consisting of homeomorphisms isotopic to the identity. It is well-known that these two definitions agree.

In the above definition, if we take only orientation preserving diffeomorphisms or homeomorphisms, the group obtained is called the *mapping class group* of S and usually denoted by $\mathcal{M}_S$.

We define an action of $\mathcal{M}_S^*$ on $C(S)$ as follows. For a mapping class f and an $\alpha \in \mathcal{S}$, choose representatives $F \in f$ and $a \in \alpha$. Define $f(\alpha)$ to be the class of the circle $F(a)$. If F and G are two isotopic diffeomorphisms of S, then $F(a)$ is isotopic to $G(a)$. If a and b are two isotopic circles, $F(a)$ and $F(b)$ are isotopic. It follows that $\mathcal{M}_S^*$ has a well-defined action on $\mathcal{S}$. Let $\alpha, \beta \in \mathcal{S}$ be such that $i(\alpha, \beta) = 0$. Choose $a \in \alpha$, $b \in \beta$ with $a \cap b = \emptyset$. If $f \in \mathcal{M}_S^*$ and if $F \in f$, then $F(a)$ and $F(b)$ are disjoint. Therefore $i(f(\alpha), f(\beta)) = 0$. This implies that f is, in fact, a simplicial map $C(S) \to C(S)$. Clearly, it is injective, and surjective. Hence it is an automorphism of $C(S)$. Therefore, there exists a natural map $\mathcal{M}_S^* \to \text{Aut } C(S)$.

Now suppose that S has at least one puncture. For $f \in \mathcal{M}_S^*$ and $\alpha' \in B(S)$, choose $F \in f$ and $a' \in \alpha'$. Define $f(\alpha')$ to be the class of the arc $F(a')$. Again, it is easy to see that this gives a well-defined action and $\mathcal{M}_S^*$ acts on $B(S)$ by automorphisms.

Theorem 1.3 *Let S be a connected oriented surface. If S is not a sphere with at most four punctures or a torus with at most two punctures or a closed surface of genus two, then the natural map $\mathcal{M}_S^* \to \text{Aut } C(S)$ is injective.*

Proof: Suppose that a is a 2-separating circle on S with the isotopy class α. Denote by D the twice-punctured disc component of S_a. By interchanging two punctures on D leaving the boundary component pointwise fixed, we get a homeomorphism of D. Extension of this homeomorphism to $S\backslash D$ by the identity gives a homeomorphism of S. Let $t_\alpha^{\frac{1}{2}}$ be the isotopy class of this homeomorphism. Note that we can choose the twisting of the punctures on D so that $(t_\alpha^{\frac{1}{2}})^2 = t_\alpha$, where t_α is the right Dehn twist about α. It seems right to call $t_\alpha^{\frac{1}{2}}$ a 'half twist' about α. It is clear from the definition that $f\, t_\alpha^{\frac{1}{2}}\, f^{-1} = t_{f(\alpha)}^{\frac{1}{2}}$ if f is orientation preserving and $f\, t_\alpha^{\frac{1}{2}}\, f^{-1} = (t_{f(\alpha)}^{\frac{1}{2}})^{-1}$ if f is orientation reversing.

We claim that an orientation reversing mapping class cannot act as the identity on the complex $C(S)$. Suppose on the contrary that $f \in \mathcal{M}_S^*$ is orientation reversing and $f(\alpha) = \alpha$ for all vertices α of $C(S)$. First, suppose that S is a punctured sphere. By hypothesis, S has at least five punctures. Let γ be a 2-separating vertex of $C(S)$, $c \in \gamma$ and $F \in f$ such that $F(c) = c$. Orient c arbitrarily. Since the components of S_c are not homeomorphic to each other, F preserves each component of S_c and reverses the orientation of c. Thus F permutes the punctures on the twice-punctured disc component of S_c. Considering another circle d such that $\langle c; d\rangle$ is a simple pair, we see that F, in fact, fixes each puncture. Now let a and b be two 2-separating circles intersecting as in Figure 1.3 (a). Clearly, we can choose an $F \in f$ such that $F(a) = a$ and $F(b) = b$. Let us orient a and b arbitrarily. Then $F(a) = a^{-1}$ and $F(b) = b^{-1}$ as oriented circles. Since F fixes each puncture, it follows that $F(a_{[X,Y]}) = a_{[Y,X]}$, where $a_{[X,Y]}$ represent the segment of the oriented circle a from X to Y by following the orientation of a. In particular, $F(X) = Y$. On the other hand, $F(b_{[X,Z]}) = b_{[Z,X]}$, and hence $F(X) = Z$. By this contradiction f cannot be orientation reversing.

Suppose now that the genus of S is at least one. We can consider S in $\mathbf{R}^3$ in such a way that it is invariant under the reflection ρ across XY plane (see Figure 1.3 (b), (c)). Let us denote the isotopy class of ρ by ϱ. Then $h = \varrho f$ is orientation preserving.

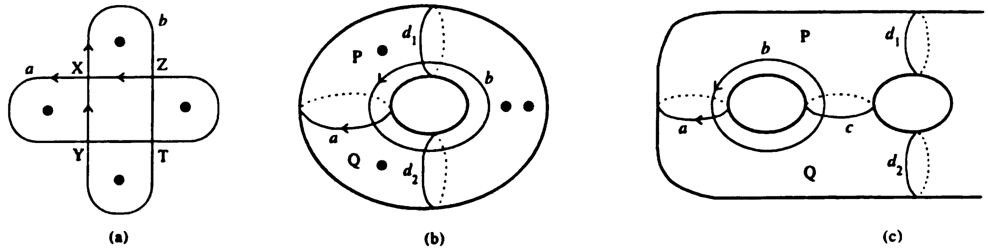


Figure 1.3: Various configurations.

For any vertex α of $C(S)$, we have

$$h(\alpha) = \varrho(f(\alpha)) = \varrho(\alpha).$$

Consider the circles a, b, d_1, d_2 (and c if the genus of S is greater than one) illustrated in the figure. Let P denote the once-punctured annulus bounded by a and d_1 if S is a punctured torus or the sphere with three holes bounded by a, c and d_1 otherwise. Since S is not a torus with at most two punctures or a closed surface of genus two, d_1 and d_2 are not isotopic to each other. Let Q denote $\rho(P)$. It is clear that there exists a homeomorphism $H \in h$ such that $H(a) = \rho(a) = a$, $H(d_1) = d_2$ (and $H(c) = c$ if the genus of S is greater than one) as unoriented circles. Then $H(P) = Q$. Since H is orientation reversing, H reverses the orientation of a, i.e., $H(a) = a^{-1}$. Again since H is orientation reversing, we must have $H(b) = b^{-1}$. Then $H_*([a]) = -[a]$ and $H_*([b]) = -[b]$, where H_* is the automorphism of $H_1(S, \mathbf{Z})$ induced by H and $[a]$ is the homology class of a. It follows that

$$[H(t_a(b))] = H_*([t_a(b)]) = H_*([a] + [b]) = -[a] - [b].$$

On the other hand

$$[\rho(t_a(b))] = \rho_*([t_a(b)]) = \rho_*([a] + [b]) = [a] - [b].$$

Note that $t_a(b)$ is a circle. If two oriented circles c_1 and c_2 are isotopic as unoriented circles, then $[c_1] = \pm[c_2]$. Therefore, $H(t_a(b))$ is not isotopic to $\rho(t_a(b))$ or $\rho((t_a(b))^{-1})$.

As a consequence, if α and β denote the isotopy classes of a and b, respectively, then $t_\alpha(\beta)$ is, of course, a vertex of $C(S)$ and $h(t_\alpha(\beta)) \neq \varrho(t_\alpha(\beta))$. This contradiction proves the claim.

Suppose now that the mapping class f acts trivially on $C(S)$, i.e., $f(\alpha) = \alpha$ for all α. Then $f \in \mathcal{M}_S$. Hence $f\, t_\alpha^{\frac{1}{2}} f^{-1} = t_{f(\alpha)}^{\frac{1}{2}} = t_\alpha^{\frac{1}{2}}$ if α is a 2-separating vertex of $C(S)$ and $f\, t_\beta f^{-1} = t_{f(\beta)} = t_\beta$ for any vertex β of $C(S)$. That is, $f\, t_\alpha^{\frac{1}{2}} = t_\alpha^{\frac{1}{2}} f$ and $f\, t_\beta = t_\beta f$. If S is a punctured sphere, since the mapping class group $\mathcal{M}_S$ is generated by the 'half twists' for a number of 2-separating vertices [2], Theorem 4.5, we conclude that $f \in C(\mathcal{M}_S)$, where $C(\mathcal{M}_S)$ is the center of $\mathcal{M}_S$. On the other hand, $C(\mathcal{M}_S)$ is trivial [17]. Hence $f = 1$. It is well-known that if if the genus of S is positive, the subgroup of $\mathcal{M}_S$ fixing each puncture is generated by the Dehn twists about (nonseparating) vertices of $C(S)$ (a proof of this may be found in [17]). From this it is not difficult to conclude that $\mathcal{M}_S$ is generated by the Dehn twists about vertices of $C(S)$ and the 'half twists' about 2-separating vertices. Therefore $f \in C(\mathcal{M}_S)$. But $C(\mathcal{M}_S)$ is trivial [17]. Hence $f = 1$. The proof of the theorem is now complete. □

Chapter 2

Automorphisms of $C(S)$ and $\mathcal{M}_S^*$

2.1 Introduction

The main purpose of this chapter is to prove that if S is a sphere with at least five punctures or a torus with at least three punctures, then the natural map $\mathcal{M}_S^* \to$ Aut $C(S)$, described in Section 1.3, is an isomorphism. They are stated as Theorem 2.12 and Theorem 2.18. We have already proved in Chapter 1 that this map is injective.

Section 2.2 discusses the proof of Theorem 2.12. Clearly, every mapping class takes a simple pair of circles to a simple pair of circles. So, if automorphisms of $C(S)$ are induced by mapping classes, then the image of any simple pair under an automorphism must be a simple pair. This is the starting point. We first prove Theorem 2.2, which enables us to recognize simple pairs of circles in the complex $C(S)$. After proving that Aut $C(S)$ preserves the topological type of vertices of $C(S)$, we conclude that simple pairs are preserved under the action of Aut $C(S)$. Next, we define actions of Aut $C(S)$ on the punctures of S and on the set of vertices of $B(S)$. This gives rise to a natural injective homomorphism Aut $C(S) \to$ Aut $B(S)$. Then, we show that any element of Aut $B(S)$ induced by an element of Aut $C(S)$ agrees with a mapping class on a codimension-zero simplex, and then on all of $B(S)$. Finally, this mapping class agrees with the original element of Aut $C(S)$ on $C(S)$.

The main results of Section 2.3 are Theorem 2.18 and Theorem 2.19. Instead of simple pairs, we work with the pairs of circles intersecting transversally at one point. The first step is to recognize these configurations in the complex $C(S)$. The difficulty here is to show that the group Aut $C(S)$ preserves the topological type of vertices of $C(S)$. We are able to overcome this difficulty when S has three punctures. Then induction on the number of punctures proves Theorem 2.18. In the proof we use Theorem 2.16, which Ivanov proved, but unfortunately, has not been published yet. We give our own proof of it in the case of a torus with at least three punctures. This proof is basically the same as that given in Section 2.2; we only point out the differences. Theorem 2.19 asserts that the surfaces of genus at most one are determined by their complexes of curves.

In Section 2.4, we give an application of the main results of Section 2.2 and Section 2.3. Namely, we prove that if S is a sphere with at least five punctures or a torus with at least three punctures, then any isomorphism between two subgroups of $\mathcal{M}_S^*$ of finite index is the restriction of an inner automorphism of $\mathcal{M}_S^*$. This implies that two subgroups of finite index are isomorphic if and only if they are conjugate, and that the outer automorphism group of a subgroup of finite index is finite.

2.2 Punctured Spheres

In this section S will, unless otherwise stated, denote a sphere with n punctures, and n will be greater than or equal to five.

2.2.1 Simple pairs in $C(S)$

Lemma 2.1 *Let S be a sphere with $n \geq 5$ punctures and σ be a codimension-zero simplex of $C(S)$. Then at least two vertices of σ are 2-separating.*

Proof: Recall that if R is a sphere with four punctures, then $\dim C(R) = 0$. Also note that any nontrivial circle on R is 2-separating.

We now prove the lemma by induction on n. If S is a sphere with five punctures, then since $\dim C(S) = 1$, (hence card $\sigma = 2$) and since every nontrivial circle on S is 2-separating, we are done.

Let $n \geq 6$. If all vertices in σ are 2-separating, then we are done since card $\sigma \geq 3$. So suppose σ contains a k-separating vertex α for some $3 \leq k \leq \frac{n}{2}$. Choose a circle $a \in \alpha$. Then the surface S_a is a disjoint union of S'_a, a disc with k punctures, and S''_a, a disc with $n-k$ punctures. Note that $C(S'_a)$ and $C(S''_a)$ are isomorphic to the complexes of curves of spheres with $k+1$ and $n-k+1$ punctures, respectively. Also $\sigma - \{\alpha\}$ is the union of two simplices σ' and σ'', where σ' and σ'' are codimension-zero simplices of the complexes $C(S'_a)$ and $C(S''_a)$, respectively.

If $k = 3$ and $n = 6$ then $C(S'_a)$ and $C(S''_a)$ are isomorphic to the complex of curves of a sphere with four punctures. Hence $\dim \sigma' = \dim \sigma'' = 0$. Then the vertex of σ' and that of σ'' are 2-separating on S.

If $k = 3$ and $n \geq 7$ then the only vertex of σ' is 2-separating on S. Also, since $\dim C(S''_a) = n-k+1 \geq 5$, by the induction hypothesis there exist at least two vertices in σ'' which are 2-separating on S''_a. Then one of these vertices is 2-separating on S.

If $k \geq 4$ then $\dim C(S'_a) = k+1 \geq 5$ and $\dim C(S''_a) = n-k+1 \geq 5$. By induction hypothesis, σ' contains at least two vertices which are 2-separating on S'_a. One of these vertices must be 2-separating on S. Similarly, one of the vertices of σ''

must be 2-separating on S. Therefore, we are done. □

Since any diffeomorphism of S takes a simple pair of circles to a simple pair of circles, any automorphism of $C(S)$ must take a simple pair of vertices to a simple pair in order to have a hope that it is induced by some diffeomorphism of S. The following theorem enables us to recognize simple pairs in the complex $C(S)$.

Theorem 2.2 *Let α and β be two 2-separating vertices of $C(S)$. Then $\langle\alpha;\beta\rangle$ is a simple pair if and only if there exist vertices $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{n-2}$ of $C(S)$ satisfying the following conditions.*

(i) *$(\gamma_1, \gamma_2, \alpha, \gamma_3, \beta)$ is a pentagon in $C(S)$,*

(ii) *γ_1 and γ_{n-2} are 2-separating, γ_2 is 3-separating, and γ_k and γ_{n-k} are k-separating for $3 \leq k \leq \frac{n}{2}$,*

(iii) *$\{\alpha, \gamma_3, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$, $\{\alpha, \gamma_2, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$, $\{\beta, \gamma_3, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$ and $\{\gamma_1, \gamma_2, \gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$ are codimension-zero simplices.*

Proof: The 'only if' part of the proof is very easy. Let $a \in \alpha$ and $b \in \beta$ such that $\langle a;b\rangle$ is a simple pair. It is clear that any two simple pairs of circles are topologically equivalent, i.e., if $\langle a_1;b_1\rangle$ is any other simple pair, then there exists a homeomorphism $F: S \to S$ such that $\langle F(a_1);F(b_1)\rangle = \langle a;b\rangle$. Hence we can assume that a and b are the circles illustrated in Figure 2.1. The figure represents the case $n = 8$. In the figure, we think of the sphere as the one point compactification of the plane. Then the isotopy classes γ_i of the circles c_i satisfy (i)-(iii).

Now we prove the converse. Assume that conditions (i)-(iii) above hold. For a k-separating circle c on S with $2 \leq k \leq \frac{n}{2}$, let S_c' and S_c'' denote the connected components of S_c having k and $n-k$ punctures, respectively. In the case of $k = \frac{n}{2}$

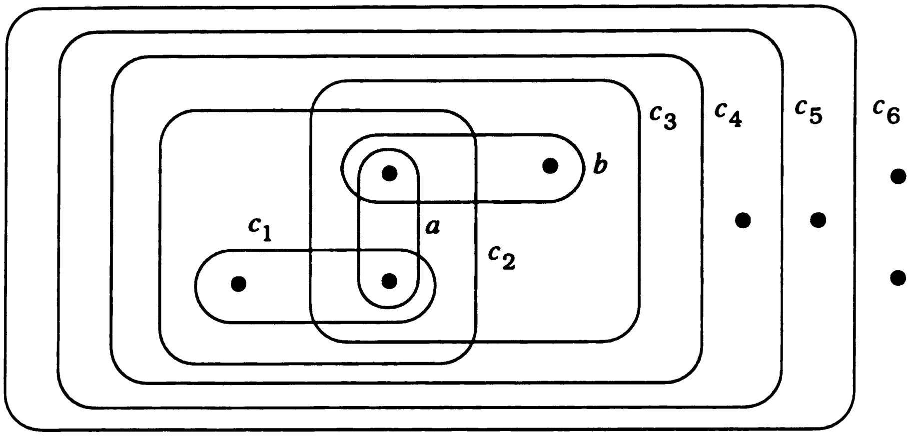


Figure 2.1: A configuration of circles on a punctured sphere.

either of them may represent either component. Let a, b and c_i be the representatives of α, β and γ_i, respectively, intersecting each other minimally.

We claim that a and b lie on a thrice-punctured disc bounded by c_3. By (i), a and b intersect transversally at least once, because the number of points in the intersection $a \cap b$ is the geometric intersection number $i(\alpha, \beta)$ of α and β, which is nonzero. Also $a \cup b$ does not intersect c_3. Hence they lie on the same component of S_{c_3}. For $n = 5$ or $n = 6$ since c_3 is 3-separating, each component of S_{c_3} is either a disc with two punctures or a disc with three punctures. Since there is no nontrivial circle on a disc with two punctures, the claim is obvious. So suppose that $n \geq 7$. Let $C = c_3 \cup c_4 \cup \ldots \cup c_{n-3}$ and consider the surface S_C. We first prove that S_C is a union of two discs with three punctures, whose boundaries are c_3 and c_{n-3}, and a number of annuli with one puncture. Since every circle on S is separating, the number of components of S_C is $n - 4$. Also, for any circle d on S, $\chi(S) = \chi(S'_d) + \chi(S''_d)$. From this it follows that

$$2 - n = \chi(S) = \sum \chi(R)$$

where R runs over the components of S_C. Since all c_i are non-isotopic, $\chi(R)$ is negative for all R. Hence either there is only one R with $\chi(R) = -3$, or there are precisely two components with $\chi(R) = -2$, and the rest of the components have Euler characteristic -1. If there exists a component R with $\chi(R) = -3$, then R is a disc with a number of holes and punctures, the total number of holes and punctures on the disc R being four. We then can find two circles d_1 and d_2 on R such that at most one of d_1 and d_2 is 2-separating on S. Then $\{\delta_1, \delta_2, \gamma_3, \gamma_4, \ldots, \gamma_{n-3}\}$ is a maximal simplex of $C(S)$ containing at most one 2-separating vertex, where δ_i is the class of d_i for $i = 1, 2$. This is a contradiction to Lemma 2.1. Thus S_C has two components with Euler characteristic -2, namely S'_{c_3} and $S'_{c_{n-3}}$.

Since $a \cap b$ is nonempty and since $\{\alpha, \gamma_3, \ldots, \gamma_{n-3}\}$ and $\{\beta, \gamma_3, \ldots, \gamma_{n-3}\}$ are simplices of $C(S)$, $a \cup b$ lies either on S'_{c_3} or on $S'_{c_{n-3}}$, both of which are discs with three punctures. Now let ∂ be the boundary of the thrice-punctured disc on which a and b lie. Since c_2 is a 3-separating circle intersecting b and is not isotopic to ∂, it follows that c_2 and ∂ intersect nontrivially. As $i(\gamma_2, \gamma_{n-3}) = 0$, we must have $\partial = c_3$. This proves the claim.

The condition (iii) implies that the circles a, b, c_1, c_2 and c_3 all lie on $S_{C'}$, where $C' = c_4 \cup c_5 \cup \ldots \cup c_{n-2}$. (If $n = 5$ then c_2 and c_3 are 2-separating circles disjoint from a. Hence the corresponding arcs c_2' and c_3' have a common endpoint. In this case, we take C' to be a trivial simple closed curve deformable to this puncture.) By arguing as above, one can see that the surface $S_{C'}$ is the disjoint union of a number of surfaces of Euler characteristic -1 and a disc D with four punctures, with boundary c_4, after changing the roles of c_4 and c_{n-4} if necessary (see Figure 2.2).

Keeping the correspondence between 2-separating vertices and the arcs in mind, suppose that the endpoints of a' are P and Q. Then c_3 separates P, Q and another puncture, say, R from the forth puncture T on D. Up to a homeomorphism of S, the picture of a', c_3 and c_4 are as illustrated Figure 2.2 (a).

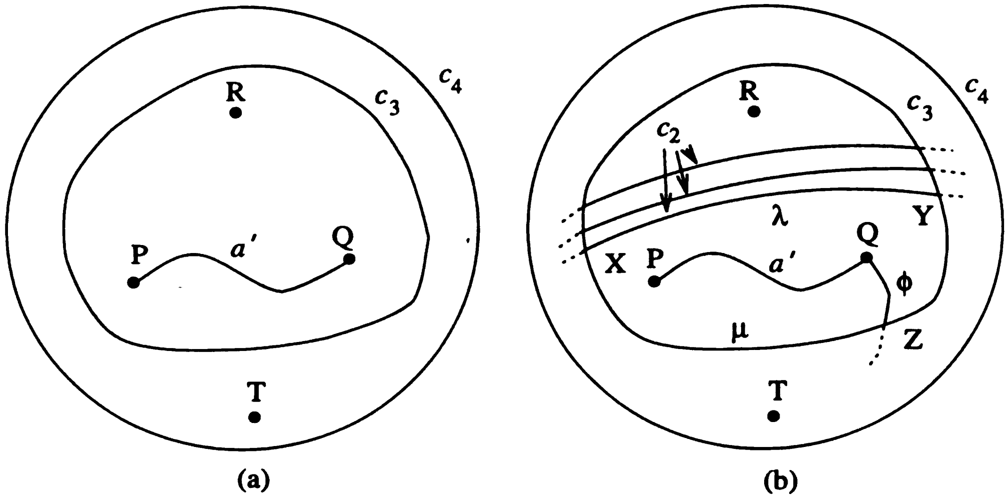


Figure 2.2: The disc D with four punctures.

Since $i(\alpha, \gamma_2) = 0$ and $i(\gamma_2, \gamma_3) \neq 0$, each component of $c_2 \cap S'_{c_3}$ is an arc connecting two points on c_3, and isotopic to each other by an isotopy of S'_{c_3} leaving the endpoints of the arc on c_3 by Lemma 1.1. (Here S'_{c_3} denotes the thrice-punctured disc component of S which does not contain the circle c_4). Let λ be one of these arcs and let $\lambda \cap c_3 = \{X, Y\}$. Then a component μ of $c_3 - \{X, Y\}$ and λ bound a disc D' with two punctures P and Q. Note that P and Q are on S'_{c_2}, which is a thrice-punctured disc.

Since b' is on S'_{c_3}, its endpoints are among P, Q and R. So one of them must be P, by changing the roles of P and Q if necessary. If the other endpoint of b' is Q, then the endpoints of c'_1 must be T and R since c_1 is disjoint from b. This and $c_1 \cap c_2 = \emptyset$ imply that R and T are on S'_{c_2}, too, since at least one of R and T is on S'_{c_2}. This is a contradiction because the four punctures P, Q, R, T cannot be on S'_{c_2} all together. By this contradiction the endpoints of b' are P and R, and that of c'_1 are Q and T.

Since $c_1 \cap c_2 = \emptyset$, c'_1 does not intersect λ and hence it intersects μ. Let Z be the first point where c'_1 meets μ starting from Q. Denote the segment of c'_1 between Q and Z by ϕ. Up to an isotopy of D' leaving the endpoint at Q fixed and keeping the other endpoint on $\lambda \cup \mu$, such an arc is unique by Lemma 1.1. So ϕ can be chosen so that it does not intersect a'.

Finally, by cutting S'_{c_3} along a regular neighborhood of $\phi \cup \{Q\}$ we get a disc with two punctures P and R. The arc b' must lie on this disc since it meets neither c'_1 nor c_3. Again, up to an isotopy of this disc there is only one arc joining the punctures P and R, which can be chosen disjoint from a'. This finishes the proof. □

Lemma 2.3 *The group* Aut $C(S)$ *preserves the topological type of the vertices of* $C(S)$.

Proof: Note that all vertices of $C(S)$ are separating. For a vertex α of $C(S)$, the dual link $L^d(\alpha)$ is connected if and only if α is 2-separating. This implies that each automorphism of $C(S)$ permutes the set of 2-separating vertices.

If α is a k-separating vertex for some $2 < k \leq \frac{n}{2}$, then the dual link $L^d(\alpha)$ has exactly two connected components. Let us denote these components by $L^d_0(\alpha)$ and $L^d_1(\alpha)$. The vertices of these components are the isotopy classes of the circles on the two components of S_a, where $a \in \alpha$. We then define two full subcomplexes $L_0(\alpha)$ and $L_1(\alpha)$ of the link $L(\alpha)$ of α as follows: The vertices of $L_j(\alpha)$ are those of $L^d_j(\alpha)$ for $j = 0, 1$. We can choose $L^d_0(\alpha)$ and $L^d_1(\alpha)$ so that $L_0(\alpha)$ and $L_1(\alpha)$ are isomorphic to the complexes of curves of, respectively, the k-punctured and the $(n-k)$-punctured disc components of S_a. Hence dim $L_0(\alpha) = k - 3$ and dim $L_1(\alpha) = n - k - 3$.

Let α be a k-separating vertex for some $2 < k \leq \frac{n}{2}$ and f be an automorphism of $C(S)$. It is clear that f induces an isomorphism from the disjoint union $L_0(\alpha) \cup L_1(\alpha)$ to the disjoint union $L_0(f(\alpha)) \cup L_1(f(\alpha))$. Since dim $L_0(\alpha) \leq$ dim $L_1(\alpha)$, it follows that dim $f(L_0(\alpha)) \leq$ dim $f(L_1(\alpha))$ and hence $f(L_0(\alpha)) = L_0(f(\alpha))$. (In the case $k = \frac{n}{2}$ we may change the role of $L_0(f(\alpha))$ and $L_1(f(\alpha))$ if necessary.) Since α is k-separating, dim $L_0(\alpha) =$ dim $L_0(f(\alpha)) = k - 3$. Since $L_0(f(\alpha))$ is isomorphic to the complexes of curves of one of the components of $S_{f(a)}$ for $f(a) \in f(\alpha)$, the circle $f(a)$ must be k-separating. □

Corollary 2.4 *Let f be an automorphism of $C(S)$. If $\langle \alpha; \beta \rangle$ (and hence $\langle \alpha'; \beta' \rangle$) is a simple pair, then so is $\langle f(\alpha); f(\beta) \rangle$ (and hence $\langle f(\alpha'); (\beta') \rangle$). Similarly, the image of a chain in $C(S)$ under f is also a chain.*

Proof: It is clear that the conditions (i) and (iii) of Theorem 2.2 are invariant under the automorphisms of $C(S)$. The fact that the condition (ii) is invariant under Aut $C(S)$ is proved in Lemma 2.3 above.

The second part of the corollary follows easily from the first part. □

2.2.2 The map Aut $C(S) \to$ Aut $B(S)$

In this subsection we show that every automorphism of $C(S)$ gives rise to an automorphism of $B(S)$ in a natural way. In fact, we will have an injective homomorphism Aut $C(S) \to$ Aut $B(S)$.

The first step is to define an action of Aut $C(S)$ on the punctures of S. We define this action as follows. For $f \in$ Aut $C(S)$ and for a puncture P of S, take any simple pair $\langle \alpha'; \beta' \rangle$ with center P, and define $f(P)$ to be the center of the simple pair $\langle f(\alpha'); f(\beta') \rangle$. Note that by the one-to-one correspondence between the set of 2-separating vertices of $C(S)$ and the set of those vertices of $B(S)$ which join different punctures, Aut $C(S)$ has a well-defined action on the latter set.

Lemma 2.5 *The definition of the action of* Aut $C(S)$ *on the punctures of S is independent of the choice of the simple pair.*

Proof: Let $a' \in \alpha'$ and $b' \in \beta'$ be such that a' and b' are disjoint, i.e., $\langle a'; b' \rangle$ is a simple pair with center P. Let $f(a')$ and $f(b')$ be the disjoint representatives of $f(\alpha')$ and $f(\beta')$, respectively, and $\tilde{P}$ the center of $\langle f(a'); f(b') \rangle$.

We show first that if c' is an arc joining P to some other puncture, with the class γ', and if $f(c')$ is a representative of $f(\gamma')$ intersecting $f(a')$ and $f(b')$ minimally, then one of the endpoints of $f(c')$ is $\tilde{P}$. The proof of this is by induction on $i = i(\alpha', \gamma') + i(\beta', \gamma')$. Let us denote by P_1, P_2 and P_3 the other endpoints of a', b' and c', respectively.

As the first step of the induction, suppose that $i = 0$. There are two cases to consider.

Case 1: If $\langle \alpha'; \gamma' \rangle$ and $\langle \beta'; \gamma' \rangle$ are simple pairs also (i.e., if P_3 is different from P_1 and P_2), then there is a fourth arc d' such that any two arcs in the set $\{a', b', c', d'\}$ constitute a simple pair with center P (see Figure 2.3 (a)). This is because there are at least five punctures on S. Then any two arcs in $\{f(a'), f(b'), f(c'), f(d')\}$ constitute a simple pair, where $f(d')$ is a representative of the image of the class of d' under f. An easy argument shows that all four arcs $f(a'), f(b'), f(c')$ and $f(d')$ must have a common endpoint, which must be $\tilde{P}$.

Case 2: If the endpoints of a' and c' are the same, (i.e., $P_1 = P_3$), then there exist a puncture Q different from P, P_1 and P_2, and an arc d' joining Q and P not intersecting any of the three arcs a', b', c' (see Figure 2.3 (b)). By an application of Case 1 to $\{a', b', d'\}$ and then to $\{b', d', c'\}$, we see that one of the endpoints of $f(c')$ is $\tilde{P}$. Notice that the center of $\langle f(\beta'); f(\gamma') \rangle$ must be $\tilde{P}$.

For the case $i > 0$, let us orient all three arcs from P to P_j, for each $j = 1, 2, 3$. Let X be the first point where c' meets $a' \cup b'$. Without loss of generality we can assume that X is on a'. Let d' be the arc consisting of the segment of c' from P to X

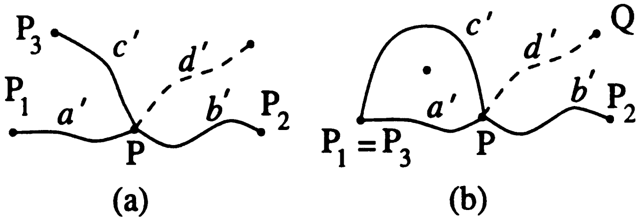


Figure 2.3: Construction of d' if $i = 0$.

and that of a' from X to P_1 (see Figure 2.4). Then $i(\alpha', \delta') + i(\beta', \delta') = 0$, $\langle \delta'; \beta' \rangle$ is a simple pair, and $i(\delta', \gamma') + i(\beta', \gamma') < i(\alpha', \gamma') + i(\beta', \gamma')$, where δ' is the class of d'. By induction, one of the endpoints of $f(c')$ is the center of the simple pair $\langle f(\delta'); f(\beta') \rangle$, which is $\tilde{P}$ by Case 2.

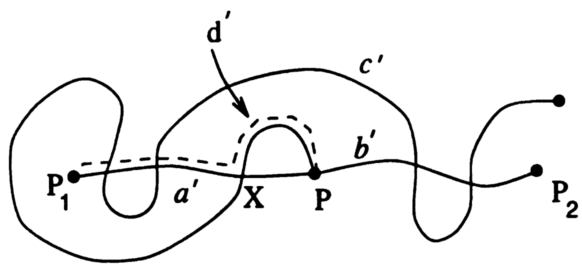


Figure 2.4: Construction of d' if $i > 0$.

If $\langle \gamma'; \delta' \rangle$ is another simple pair with center P, and $c' \in \gamma'$ and $d' \in \delta'$ intersect each other as well as a' and b' minimally, then by applying the argument above first to $\{a', b', c'\}$ and then to $\{a', b', d'\}$ we see that $\tilde{P}$ is an endpoint of both $f(c')$ and $f(d')$, which must be the center of the simple pair $\langle f(\gamma'); f(\delta') \rangle$. Hence the action of Aut $C(S)$ on the set of punctures of S is well-defined. □

For a punctured surface R, let us denote by $\mathcal{P}(R)$ the set of punctures of R.

Lemma 2.6 *Let $f \in$ Aut $C(S)$, α a k-separating vertex of $C(S)$, and $a \in \alpha$. If S'_a and S''_a denote the k-punctured and $(n-k)$-punctured disc components of S_a, then $f(\mathcal{P}(S'_a)) = \mathcal{P}(S'_{f(a)})$, and hence $f(\mathcal{P}(S''_a)) = \mathcal{P}(S''_{f(a)})$. In the case of $k = \frac{n}{2}$, we may change the roles of S'_a and S''_a if necessary.*

Proof: Two punctures P and Q are on the same connected component of S_a if and only if P and Q can be joined by an arc disjoint from a, and P and Q can be joined by an arc disjoint from a if and only if $f(P)$ and $f(Q)$ can be joined by an arc disjoint from $f(a)$. Now the proof of the lemma follows. □

We can now define an action of Aut $C(S)$ on the vertices of $B(S)$. Let $f \in$ Aut $C(S)$, α' a vertex of $B(S)$ and let $a' \in \alpha'$. If a' is joining two different punctures, then $f(\alpha')$ is already defined by the correspondence with the 2-separating vertices of $C(S)$ and the action of Aut $C(S)$ on $C(S)$. That is, $f(\alpha')$ is the isotopy class of the arc, which is unique up to isotopy, joining two punctures on the twice-punctured disc component of $S_{f(a)}$ for $f(a) \in f(\alpha)$. Suppose now that the arc a' is joining a puncture P to itself. Let a_1 and a_2 be the boundary components of a regular neighborhood of $a' \cup \{P\}$ and α_1 and α_2 be their classes. Since a' is not deformable to P, at most one of a_1 and a_2 is trivial. If a_1 is trivial then a_2 is 2-separating, and by Lemma 2.3 $f(\alpha_2)$ is 2-separating. Hence for a (circle) representative $f(a_2)$ of $f(\alpha_2)$, one of the components, say $S'_{f(a_2)}$, of $S_{f(a_2)}$ is a twice-punctured disc, and one of the punctures on $S'_{f(a_2)}$ is $f(P)$ by Lemma 2.6. Define $f(\alpha')$ to be the class of a nontrivial simple arc on $S'_{f(a_2)}$ joining $f(P)$ to itself. Such an arc is unique up to isotopy by Lemma 1.1.

In the case that neither a_1 nor a_2 is trivial, we claim that $f(a_1)$ and $f(a_2)$ bound a once-punctured annulus with only one puncture $f(P)$. Here $f(a_i)$ is a representative of $f(\alpha_i)$. For the proof of this, suppose that the set of punctures on S'_{a_1} and S''_{a_2} are $\mathcal{P}(S'_{a_1}) = \{P_1, \ldots, P_k\}$ and $\mathcal{P}(S''_{a_2}) = \{Q_1, \ldots, Q_{n-k-1}\}$, respectively. Then $P_i \neq Q_j$ for all i, j. By Lemma 3.5, $\mathcal{P}(S'_{f(a_1)}) = \{f(P_1), \ldots, f(P_k)\}$ and $\mathcal{P}(S''_{f(a_2)}) = \{f(Q_1), \ldots, f(Q_{n-k-1})\}$. It follows that, since $f(a_1)$ and $f(a_2)$ are disjoint and nonisotopic, they must bound an annulus with only one puncture $f(P)$.

Then $f(\alpha')$ is defined to be the isotopy class of the unique arc (up to isotopy) on this annulus joining $f(P)$ to itself.

Lemma 2.7 *Let f be an automorphism of $C(S)$ and α' and β' be two distinct vertices of $B(S)$ such that $i(\alpha', \beta') = 0$. Then $i(f(\alpha'), f(\beta')) = 0$. Hence every automorphism of $C(S)$ induces an automorphism of $B(S)$.*

Proof: Let a' and b' be two disjoint representatives of α' and β', respectively. There are seven cases to consider (see Figure 2.5). In the figure, we assume that the arc on the left is a' and the one on the right is b'.

If a' (resp. b') is joining two different punctures, let us denote by α (resp. β) the 2-separating vertex of $C(S)$ corresponding to α' (resp. β'), and by a (resp. b) a representative of α (resp. β).

If a' (resp. b') is connecting a puncture P to itself, let us denote by a_1 and a_2 (resp. b_1 and b_2) the boundary components of a regular neighborhood of $a' \cup \{P\}$ (resp. $b' \cup \{P\}$). Note that we use the classes of a_i (resp. b_i) to define $f(\alpha')$ (resp. $f(\beta')$). We also denote representatives of $f(\alpha)$, $f(\alpha')$ by $f(a)$, $f(a')$ etc.

If a_1 or b_1 is trivial, we work with them by considering the trivial simple closed curve deformable to a puncture to be that puncture.

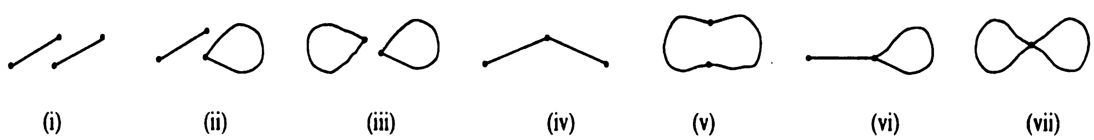


Figure 2.5: Seven possible cases for a' and b'.

We examine each of the seven cases illustrated above.

(i) Here a and b are disjoint, so $i(\alpha, \beta) = 0$. Since f is an automorphism of $C(S)$, this implies that $i(f(\alpha), f(\beta)) = 0$. Another way of saying this is that $f(\alpha)$ and $f(\beta)$ have distinct disjoint representatives $f(a)$ and $f(b)$. Hence a' and b' are disjoint.

(ii) On the annulus determined by $f(b_1)$ and $f(b_2)$ there is only one puncture. Since f is an automorphism of $C(S)$, the 2-separating circles $f(a), f(b_1)$ and $f(b_2)$ are all distinct and no two are isotopic. Therefore $f(a)$ cannot lie on this annulus. Since $f(b')$ is on the annulus, we are done.

(iii) The once-punctured annuli determined by a_1 and a_2, and b_1 and b_2 are disjoint. Since a_1, a_2, b_1 and b_2 are pairwise disjoint, so are $f(a_1), f(a_2), f(b_1)$ and $f(b_2)$. So the annuli determined by $f(a_1)$ and $f(a_2)$, and $f(b_1)$ and $f(b_2)$ are disjoint.

(iv) Corollary 2.4.

(v) Suppose the endpoints of a' and b' are P and Q. Let R be any puncture other than P and Q. Then there exist arcs c' and d' and a 3-separating circle e satisfying the following conditions:

(1) c' and d' are disjoint from $a' \cup b'$,

(2) c' joins P and R and d' joins Q and R,

(3) $b' \cup c' \cup d'$ bounds a disc on S, and

(4) b', c' and b' lie on a thrice-punctured disc component of S_e.

Note that any arc joining P and Q which is disjoint from c' and d' is isotopic to an arc disjoint from b'. (Note that any arc isotopic to b' can be isotoped to an arc disjoint from b'.) Then $f(b') \cup f(c') \cup f(d')$ lies on a thrice-punctured disc component of $S_{f(e)}$ and bounds a disc on S. The latter follows from a similar argument given in the first step of the induction in the proof of Lemma 2.5. Since $\langle a'; c'\rangle$ and $\langle b'; c'\rangle$ are simple pairs, so are $\langle f(a'); f(c')\rangle$ and $\langle f(b'); f(c')\rangle$. In particular, $f(a')$ can be chosen disjoint from $f(c')$ and $f(d')$. Hence it can also be chosen disjoint from $f(b')$.

(vi) Suppose that a' is connecting P to Q and that b' is connecting P to itself. We can assume that b_2 is disjoint from a'. Clearly, there exists an arc c' not meeting

a', b' and b_1, joining P to some other puncture, say, R (see Figure 2.6).

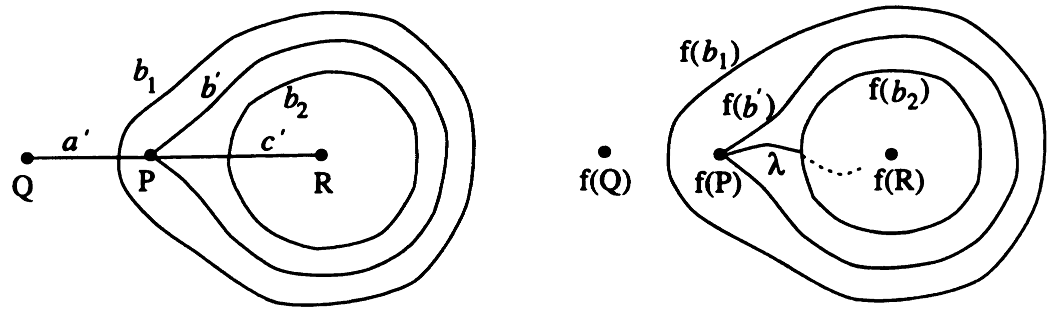


Figure 2.6: Case (vi).

Then $f(c')$ connects $f(P)$ to $f(R)$ and meets $f(b_2)$, but not $f(b_1)$ or $f(a')$. Let λ be the segment of $f(c')$ lying on the once-punctured annulus determined by $f(b_1)$ and $f(b_2)$, and connecting $f(P)$ to $f(b_1)$. Note that the intersection of $f(a')$ with the this annulus is a collection of arcs joining a point on $f(b_1)$ either with another point on $f(b_1)$ or with $f(P)$. Since $f(a')$ is disjoint from λ, the intersection of $f(a')$ and this annulus consists of only one arc connecting $f(P)$ to some point on $f(b_1)$. Then $f(b')$, which is an arc on this annulus joining $f(P)$ to itself, can be chosen so that it does not intersect $f(a')$.

(vii) Let P be the common endpoints of a' and b'. We can assume that a_1 does not meet b_1 and b_2. Let $P_1, \ldots, P_k$ be the punctures on the component of S_{a_1} which does not contain $a' \cup b'$. Choose k arcs $c'_0, c'_1, \ldots, c'_{k-1}$ such that c'_i joins P_i to P_{i+1} for $i = 0, 1, \ldots, k-1$, where $P_0 = P$, and $\{\alpha', \beta', \gamma'_0, \gamma'_1, \ldots, \gamma'_{k-1}\}$ is a simplex of $B(S)$. Then the arc $f(c_i)$ joins $f(P_i)$ and $f(P_{i+1})$, and $\{f(\alpha'), f(\gamma'_0), f(\gamma'_1), \ldots, f(\gamma'_{k-1})\}$ and $\{f(\beta'), f(\gamma'_0), f(\gamma'_1), \ldots, f(\gamma'_{k-1})\}$ are simplices of $B(S)$ by (i), (ii), (iv) and (vi). Since any arc joining P to itself which is disjoint from $c'_0, c'_1, \ldots, c'_{k-1}$ is isotopic to an arc disjoint from a', any arc joining $f(P)$ to itself disjoint from $f(c'_0), f(c'_1), \ldots, f(c'_{k-1})$ is isotopic to an arc disjoint from $f(a')$. Therefore, $f(b')$ can be chosen disjoint from $f(a')$. This completes the proof. $\square$

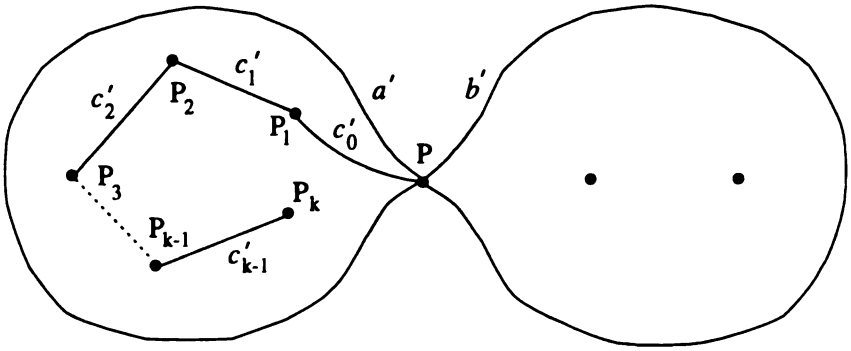


Figure 2.7: Case (vii).

Proposition 2.8 *The group* Aut $C(S)$ *is naturally isomorphic to a subgroup of the group* Aut $B(S)$.

Proof: By Lemma 2.7 every element of Aut $C(S)$ induces an element of Aut $B(S)$. Clearly, the map taking an element of Aut $C(S)$ to the induced element of Aut $B(S)$ is a homomorphism. It is not very difficult to show that if an automorphism of $C(S)$ induces the identity automorphism of $B(S)$, then it is, in fact, the identity. □

2.2.3 Automorphisms of $C(S)$ and $\mathcal{M}_S^*$

In this subsection, S will denote a sphere with at least five punctures.

Lemma 2.9 *Let f and g be two automorphisms of $B(S)$. If they agree on a codimension-zero simplex, then they agree on all of $B(S)$.*

Proof: Let σ be a codimension-zero simplex of $B(S)$. Suppose that f is equal to g on σ. If σ' is another codimension-zero simplex, then by Theorem 1.2 there exist codimension-zero simplices $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ such that $\sigma_{i-1} \cap \sigma_i$ is a codimension-one simplex for each i. Since any codimension-one simplex is a face of either one or two codimension-zero simplices, if two automorphisms $B(S)$ agree on σ_{i-1} then they agree on σ_i. Clearly, this implies that f must be equal to g on σ'. Since every simplex of $B(S)$ is a face of a codimension-zero simplex, we are done. □

Recall the correspondence between the codimension-zero simplices of $B(S)$ and the isotopy classes of ideal triangulations of S. Since all maximal simplices in $B(S)$ have the same dimension, we have a well-defined action of the group Aut $B(S)$ on the codimension-zero simplices.

Lemma 2.10 *Let $f \in$ Aut $C(S)$, $\Delta = \{a', b', c'\}$ be a 'good' ideal triangle on S and let α', β' and γ' be the isotopy classes of a', b' and c', respectively. Then $\{\alpha', \beta', \gamma'\}$, and hence also $\{f(\alpha'), f(\beta'), f(\gamma')\}$, is a 2-simplex in $B(S)$. If $f(\Delta) = \{f(a'), f(b'), f(c')\}$ is a realization of the latter simplex, then it is a 'good' ideal triangle on S.*

Proof: Recall that a 'good' ideal triangle has three different vertices and $\langle \alpha'; \beta' \rangle$, $\langle \beta'; \gamma' \rangle$ and $\langle \alpha'; \gamma' \rangle$ are simple pairs. It follows that $f(\Delta)$ is an ideal triangle all of whose vertices are different. Let P and Q be any two punctures different from the vertices of $f(\Delta)$. Since $f^{-1}(P)$ and $f^{-1}(Q)$ can be joined by an arc d' not intersecting any of the edges of Δ, P and Q can be joined by an arc $f(d')$. Since $f(d')$ has the geometric intersection number zero with each arc in $f(\Delta)$, it can be chosen so that it does not intersect any of them. Thus the proof of the lemma is complete. □

Lemma 2.11 *Let $f \in$ Aut $C(S)$. Suppose that there exists a homeomorphism $F : S \to S$ such that $[F]$ and f are equal as automorphisms of $B(S)$, where $[F]$ denotes the isotopy class of F. Then $[F] = f$ as automorphisms of $C(S)$.*

Proof: Let α be a k-separating vertex of $C(S)$, $a \in \alpha$ and let $\mathcal{P}(S'_a) = \{P_1, \ldots, P_k\}$, $\mathcal{P}(S''_a) = \{Q_1, \ldots, Q_{n-k}\}$. Choose two arbitrary chains $\langle b'_1; \ldots; b'_{k-1} \rangle$ and $\langle c'_1; \ldots; c'_{n-k-1} \rangle$ such that b'_i and c'_j are disjoint from a for i and j. Then $\langle f(b'_1); \ldots; f(b'_{k-1}) \rangle$ and

$\langle f(c'_1);\ldots;f(c'_{n-k-1})\rangle$ are two chains such that $f(b'_i)$ and $f(c'_j)$ are disjoint from $f(a)$ for all i,j. Since $[F](\beta'_i) = f(\beta'_i)$, $[F](\gamma'_j) = f(\gamma'_j)$ by assumption, and since there is only one (k-separating) circle up to isotopy disjoint from the chains $\langle f(b'_1);\ldots;f(b'_{k-1})\rangle$ and $\langle f(c'_1);\ldots;f(c'_{n-k-1})\rangle$, namely $f(a) \in f(\alpha)$, we must have $[F](\alpha) = f(\alpha)$. This proves the lemma. □

We are now ready to prove the main result of Section 2.2.

Theorem 2.12 *Let S be a sphere with at least five punctures. Then the natural map $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is an isomorphism.*

Proof: Injectiveness of the map $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ was proved in Theorem 1.3. Let us now show that it is onto. Let $f \in \operatorname{Aut} C(S)$ and let C' be an arbitrary ideal triangulation of S such that each triangle has three different vertices, i.e., a 'good' triangulation. Existence of such a triangulation is clear. Let σ be the isotopy class of C'. Then σ is a codimension-zero simplex of $B(S)$. By Lemma 2.10, 'good' ideal triangles are mapped to 'good' ideal triangles by f, and it is a well-known fact that f can be realized by a homeomorphism on each such triangle. Since each edge of C' is an edge of exactly two 'good' ideal triangles, the homeomorphisms of these triangles give rise to a homeomorphism F of S. If $[F]$ denotes the isotopy class of F, then f agrees with $[F]$ on σ. By Lemma 2.9, they agree on $B(S)$. Finally, by Lemma 2.11, $[F]$ is equal to f on $C(S)$. □

2.3 Punctured Tori

In this section, unless otherwise stated, S denotes a torus with n punctures.

2.3.1 Automorphisms of $C(S)$ and $\mathcal{M}_S^*$

The following theorem enables us to recognize whether or not two vertices of $C(S)$ have geometric intersection number one, by looking at the complex $C(S)$.

Theorem 2.13 *Let S be a torus with at least two punctures, and let α and β be two vertices of $C(S)$. Then the geometric intersection number $i(\alpha, \beta) = 1$ if and only if there exist three vertices γ_1, γ_2 and γ_3 of $C(S)$ such that*

(i) *$(\gamma_1, \alpha, \gamma_2, \beta, \gamma_3)$ is a pentagon in $C(S)$, and*

(ii) *α, β and γ_3 are nonseparating, and γ_1 and γ_2 are n-separating.*

Proof: Let us first prove the 'only if' clause of the theorem. Clearly, $i(\alpha, \beta) = 1$ implies that α and β are both nonseparating. Thus there exist $a \in \alpha$ and $b \in \beta$ such that a and b intersect transversally at only one point. It is well known that if c and d are any other pair of circles intersecting transversally at only one point, then there exists a homeomorphism $F: S \to S$ such that $F(a) = F(c)$ and $F(b) = F(d)$. Hence we can assume that a and b are the standard circles in Figure 2.8 (a). The existence of the other circles whose isotopy classes satisfy (i) and (ii) is now obvious from Figure 2.8 (b).

For the converse, let $a \in \alpha, b \in \beta$ and $c_i \in \gamma_i$ intersect each other minimally pairwise. Since a is nonseparating, the surface S_a is an annulus with n punctures and two boundary components q_1 and q_2. Then S is a quotient space of S_a. Let $p : S_a \to S$ be the quotient map, so $p(q_1) = p(q_2) = a$. Up to a homeomorphism of S_a

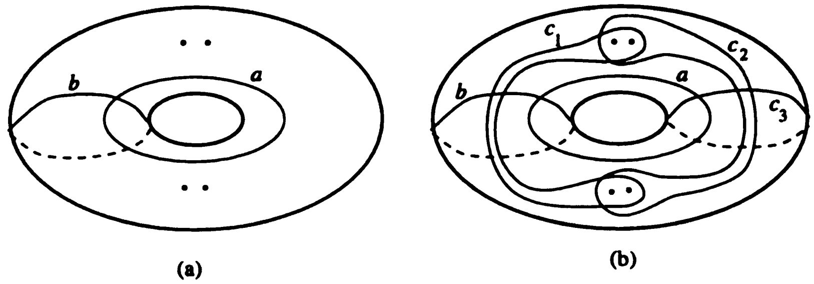


Figure 2.8: A pentagon in $C(S)$.

preserving q_1 and q_2, we can assume that the picture of $p^{-1}(c_2)$ in S_a is as in Figure 2.9 (a). In the figures, c_i represents $p^{-1}(c_i)$.

Let $i(\alpha, \beta) = m$, which is the cardinality of $a \cap b$. Since α and β are not connected by an edge in the pentagon (and hence in $C(S)$), this geometric intersection number m must be positive.

We now consider the components of the preimage $p^{-1}(b)$ of b, which is a collection of arcs. Since $i(\beta, \gamma_2) = 0$, the components of $p^{-1}(b)$ lie on a disc with two holes whose boundary components are q_1, q_2 and c_2, and they do not intersect c_2. Thus it follows from Lemma 1.1 that each arc in $p^{-1}(b)$ joins either a point on q_1 to a point on q_2, or, two points on q_1, or, two points on q_2. Let m_{11}, m_{22} m_{12} be the number of these components joining q_1 to q_1, q_2 to q_2 and q_1 to q_2, respectively. Then $m = m_{12} + 2m_{11} = m_{12} + 2m_{22}$ and hence $m_{11} = m_{22}$. On the other hand, if λ is an embedded arc on S_a connecting two points on q_1 such that λ is not isotopic to a segment of q_1 and does not intersect c_2, then every embedded arc connecting two points on q_2 which is disjoint from λ must be trivial, i.e., isotopic to a segment of q_2. Therefore m_{11} and m_{22} must be zero, so each component of $p^{-1}(b)$ connects a point on q_1 to a point on q_2. Therefore the picture of these arcs on S_a is as in Figure 2.9 (b).

Let us now orient q_1 and q_2 so that the induced orientations of $p(q_1)$ and $p(q_2)$ agree in S, and let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_m$ be the consecutive intersection

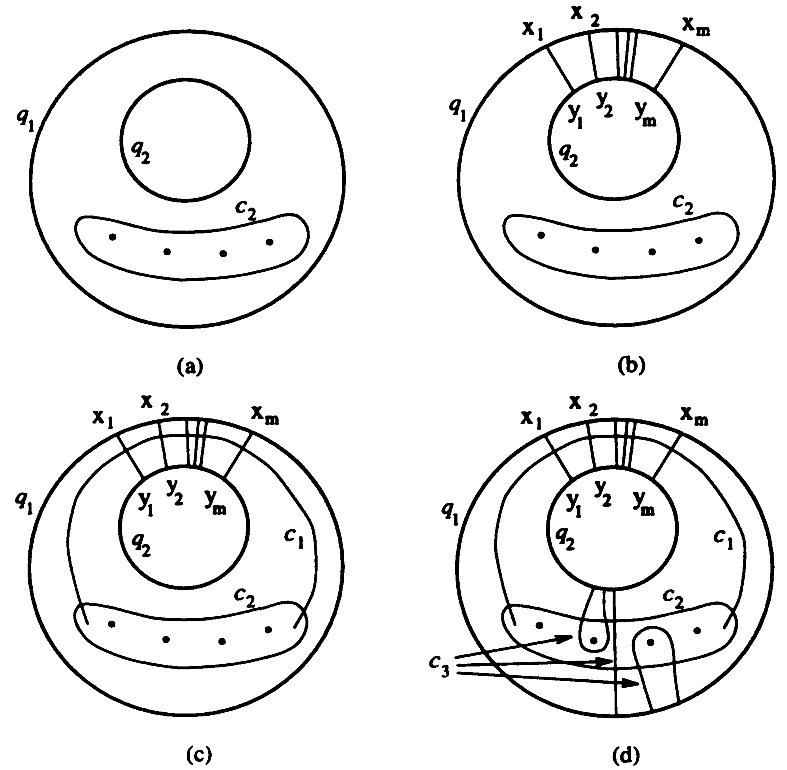


Figure 2.9: The surface S_a.

points of $p^{-1}(b)$ with q_1 and q_2, respectively, such that X_i is joined with Y_i by some arc in $p^{-1}(b)$ for each $i = 1, 2, \ldots, m$. It is clear that there exists a k, $0 \leq k < m$, such that $p(X_i) = p(Y_{i+k})$ for each i. By convention we set $Y_{m+i} = Y_i$.

Since $i(\beta, \gamma_1) \neq 0$ and $i(\alpha, \gamma_1) = 0$, the preimage of c_1, also denoted by c_1 in the figure, intersects every component of $p^{-1}(b)$ (see Figure 2.9 (c)). As $i(\gamma_3, \gamma_1) = i(\gamma_3, \beta) = 0$, each component of $p^{-1}(c_3)$, intersects q_1 only in the open interval $]X_m, X_1[$ and q_2 in $]Y_m, Y_1[$. Therefore $p(]X_m, X_1[) = p(]Y_m, Y_1[)$ in S, i.e., $p(X_1) = p(Y_1)$ and hence $k = 0$. Finally, $p(b)$ is a connected curve only if m is equal to 1. This finishes the proof of the theorem. □

Lemma 2.14 *Let $n \geq 3$. Let α, β and γ be distinct vertices of $C(S)$. If α is non-separating, β is n-separating and γ is separating, and $i(\alpha, \beta) = i(\beta, \gamma) = 0$, then $i(\alpha, \gamma) = 0$.*

Proof: Let a, b and c be representatives of α, β and γ in minimal position. The nonseparating circle a and the separating circle c are, respectively, nonseparating and separating on the surface S_b, the surface obtained from S by cutting along b. But nonseparating and separating circles on S_b lie on different components (see Figure 2.10). □

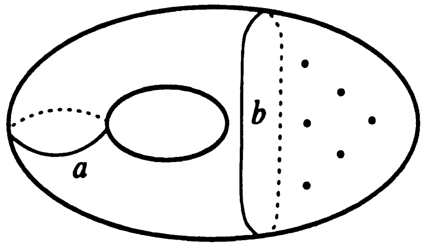


Figure 2.10: A nonseparating and an n-separating circle.

Lemma 2.15 *Let $n \geq 2$ and let S and S' denote a torus with n punctures and a sphere with $n+3$ punctures, respectively. If every automorphism of the complex $C(S)$ is induced by some self-homeomorphism of S, then $C(S)$ and $C(S')$ are not isomorphic.*

Proof: Suppose that φ is an isomorphism from $C(S)$ to $C(S')$. Then φ induces a group isomorphism $\varphi_* : \text{Aut } C(S) \to \text{Aut } C(S')$, defined by $\varphi_*(f) = \varphi f \varphi^{-1}$ for each $f \in \text{Aut } C(S)$. This implies that

$$\text{Aut } C(S') = \{\varphi f \varphi^{-1} : f \in \text{Aut } C(S)\}.$$

We now show that this is impossible. Note that for a vertex α of $C(S)$, the dual link $L^d(\alpha)$ of α is connected if and only if α is either nonseparating or 2-separating, and for a vertex β of $C(S')$, the dual link $L^d(\beta)$ of β is connected if and only if β is 2-separating. From this it follows that the image of the union of the set of nonseparating vertices and the set of 2-separating vertices of $C(S)$ is precisely the set of 2-separating vertices of $C(S')$.

Let α be a nonseparating vertex of $C(S)$, and choose a 2-separating vertex β of $C(S)$ such that $i(\alpha,\beta)=0$, i.e., α and β are joined by an edge in $C(S)$. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are two 2-separating vertices of $C(S')$ and are joined by an edge in $C(S')$. Let c and d be representatives of $\varphi(\alpha)$ and $\varphi(\beta)$, respectively. Then c and d are two disjoint 2-separating circles on S'. By the classification of surfaces there exists a homeomorphism G of S' such that $G(c)=G(d)$. Hence $g(\varphi(\alpha))=\varphi(\beta)$, where g is the isotopy class of G. Then the automorphism $\varphi_*^{-1}(g)=\varphi^{-1}g\varphi$ of $C(S)$ takes the nonseparating vertex α to the separating vertex β. This is impossible since every automorphism of $C(S)$ is induced by a homeomorphism of S, by hypothesis. Hence we have the lemma. □

We need the following theorem of Ivanov [16]. The proof has not been published yet, so for completeness we give a proof of it in the next section for tori with at least three punctures. The proof we give is basically the same as the one we give for punctured spheres.

Theorem 2.16 *Let S be an orientable surface of genus at least one. Suppose that $f\in$ Aut $C(S)$ and α and β two vertices of $C(S)$ with $i(\alpha,\beta)=1$ imply that $i(f(\alpha),f(\beta))=1$. Then every element of* Aut $C(S)$ *is induced by some homeomorphism of S.*

Lemma 2.17 *Let $n\geq 3$. If S is a torus with n punctures and S' is a sphere with $n+3$ punctures, then $C(S)$ and $C(S')$ are not isomorphic.*

We prove this lemma together with the main theorem of this section.

Theorem 2.18 *Let S be a torus with $n\geq 3$ punctures. Then the natural map $\mathcal{M}_S^*\to$* Aut $C(S)$ *is an isomorphism.*

Proofs of Lemma 2.17 *and Theorem* 2.18*:* As we have mentioned in the proof of Lemma 2.15, for a vertex α of $C(S)$, $L^d(\alpha)$ is connected if and only if α is either nonseparating or 2-separating. Hence every element of the group Aut $C(S)$ maps a nonseparating vertex either to a nonseparating vertex or to a 2-separating one, and a k-separating vertex to a k'-separating one for $k, k' \geq 3$.

Let α be a k-separating vertex of $C(S)$ with $k \geq 3$ and let $a \in \alpha$. Let us denote by $S_a^{(0)}$ and $S_a^{(1)}$ the components of S_a of genus zero and of genus one, respectively. The graph $L^d(\alpha)$ has exactly two connected components, say, $L_0^d(\alpha)$ and $L_1^d(\alpha)$. The vertices of these components correspond to the isotopy classes of circles on the connected components of S_a. We can choose $L_0^d(\alpha)$ and $L_1^d(\alpha)$ so that the vertices of $L_i^d(\alpha)$ are the isotopy classes of circles on $S_a^{(i)}$. We then define two full subcomplexes $L_0(\alpha)$ and $L_1(\alpha)$ of $C(S)$ as follows. The set of vertices of $L_i(\alpha)$ are those of $L_i^d(\alpha)$. Then $L_i(\alpha)$ is isomorphic to $C(S_a^{(i)})$.

If f is an automorphism of $C(S)$ and α is a k-separating vertex with $k \geq 3$, then f restricts to an isomorphism from $L^d(\alpha) = L_0^d(\alpha) \cup L_1^d(\alpha)$ to $L^d(f(\alpha)) = L_0^d(f(\alpha)) \cup L_1^d(f(\alpha))$. Since $L_j^d(\alpha)$ and $L_j^d(f(\alpha))$ are connected components, we get $f(L_0^d(\alpha)) = L_r^d(f(\alpha))$ and $f(L_1^d(\alpha)) = L_{1-r}^d(f(\alpha))$ for some $r = 0$ or $r = 1$. Then $f(L_0(\alpha)) = L_r(f(\alpha))$ and $f(L_1(\alpha)) = L_{1-r}(f(\alpha))$. Since dim $L_0(\alpha) =$ dim $C(S_a^{(0)}) = k-3$ and dim $L_1(\alpha) =$ dim $C(S_a^{(1)}) = n-k$, dim $L_r(\alpha) = k-3$ and dim $L_{1-r}(\alpha) = n-k$. From this it is easy to conclude that if $r = 0$ (resp. r=1), then $f(\alpha)$ is k-separating (resp. $(n-k+3)$-separating). The proofs now proceed simultaneously by induction on n.

Suppose that $n = 3$. Let $f \in$ Aut $C(S)$ and let α and β be two vertices of $C(S)$ with $i(\alpha, \beta) = 1$. By Theorem 2.13, there exist vertices γ_1, γ_2 and γ_3 of $C(S)$ such that $(\gamma_1, \alpha, \gamma_2, \beta, \gamma_3)$ is a pentagon, γ_1 and γ_2 are 3-separating, and γ_3 is 2-

separating. Then $(f(\gamma_1), f(\alpha), f(\gamma_2), f(\beta), f(\gamma_3))$ is a pentagon in $C(S)$. From the discussion given in the preceeding paragraph, it follows that $f(\gamma_1)$ and $f(\gamma_2)$ are 3-separating. Note that any two distinct nonisotopic 3-separating circles on S, a torus with three punctures, must intersect. Therefore none of the vertices $f(\alpha), f(\beta)$ and $f(\gamma_3)$ can be 3-separating. By the same argument, any two distinct nonisotopic 2-separating circles on S must intersect. We conclude that one of the vertices $f(\beta)$ and $f(\gamma_3)$, say $f(\beta)$, is nonseparating. By applying Lemma 2.14 twice to the pentagon $(f(\gamma_1), f(\alpha), f(\gamma_2), f(\beta), f(\gamma_3))$, we see first that $f(\alpha)$, and then $f(\gamma_3)$, is nonseparating, i.e., $f(\alpha), f(\beta), f(\gamma_1), f(\gamma_2)$ and $f(\gamma_3)$ satisfy the conditions (i) and (ii) of Theorem 2.13. Hence $i(f(\alpha), f(\beta)) = 1$. This is true for any automorphism of $C(S)$. Therefore, by Theorem 2.16 every automorphism of $C(S)$ is induced by some homeomorphism of S. Using this and Lemma 2.15, we see that $C(S)$ and $C(S')$ are not isomorphic if S' is a sphere with six punctures.

Now suppose that $n \geq 4$. Let α and β be nonseparating and 2-separating vertices of $C(S)$ and let $a \in \alpha$ and $b \in \beta$. Then $L(\alpha)$ is isomorphic to the complex of curves on S_a, a sphere with $n+2$ punctures, and $L(\beta)$ is isomorphic to the complex of curves on $S_b^{(1)}$, a torus with $n-1$ punctures. By the induction hypothesis, every automorphism of $C(S_a)$ is induced by a self-homeomorphism of S_a. Then by Lemma 2.15, $C(S_a)$ is not isomorphic to $C(S_b^{(1)}) = C(S_b)$, i.e., $L(\alpha)$ is not isomorphic to $L(\beta)$. It follows that if $f \in \text{Aut } C(S)$ then $f(\alpha)$ cannot be 2-separating, i.e., nonseparating vertices are preserved under the action of Aut $C(S)$.

To show that n-separating vertices are also preserved under the action of Aut $C(S)$, we assume the converse. Suppose that there exist $f \in \text{Aut } C(S)$ and an n-separating vertex α of $C(S)$ such that $f(\alpha) = \beta$ is not n-separating. Then β is 3-separating by the discussion given above. The automorphism f restricts to an isomorphism from the disjoint union $L_0(\alpha) \cup L_1(\alpha)$ to the disjoint union $L_0(\beta) \cup L_1(\beta)$. Since $L_1(\alpha)$

and $L_0(\beta)$ are discrete, and since $L_0(\alpha)$ and $L_1(\beta)$ are, for instance, connected, we must have $f(L_1(\alpha)) = L_0(\beta)$. But this means that f takes the nonseparating vertices in the link of α to separating vertices, a contradiction. Thus the automorphisms of $C(S)$ preserve the set of n-separating vertices as well.

Now it follows from Theorem 2.13 that if $f \in \text{Aut } C(S)$ and if $i(\alpha, \beta) = 1$, then $i(f(\alpha), f(\beta)) = 1$. Then Theorem 2.16 implies that automorphisms of $C(S)$ are induced by self-homeomorphisms of S. This completes the proof of Theorem 2.18. Again, that $C(S)$ and $C(S')$ are not isomorphic if S' is a sphere with $n+3$ punctures follows from Lemma 2.15. So the proof of Lemma 2.17 is complete, too. □

As a corollary to Lemma 2.17 we can state the following theorem.

Theorem 2.19 *Let S be a sphere with at least five punctures, or a torus with at least three punctures. Let S' be a connected orientable surface of genus at most one. In the case that S is a sphere with five punctures, suppose, in addition, that S' is not a torus with two punctures. If $C(S)$ and $C(S')$ are isomorphic, then S and S' are homeomorphic.*

Proof: If S and S' are either both punctured spheres or both punctured tori, and if $C(S)$ and $C(S')$ are isomorphic, then their dimensions, hence the number of punctures on S and S', are equal. By the classification of surfaces, these two surfaces are diffeomorphic.

Let S be a sphere with at least five punctures and let S' be torus with at least three punctures. Certainly, if the dimensions of $C(S)$ and $C(S')$ are not equal, then S and S' are not homeomorphic. But if their dimensions are the same, then S has three more punctures than S'. In this case $C(S)$ and $C(S')$ are not isomorphic by Lemma 2.17. □

2.3.2 Proof of Theorem 2.16 for punctured tori

The purpose of this section is to give a proof of Theorem 2.16 for tori with $n \geq 3$ punctures. So let S be a torus with at least three punctures, and assume that f is an automorphism of $C(S)$ and α and β two vertices of $C(S)$ with $i(\alpha, \beta) = 1$ imply that $i(f(\alpha), f(\beta)) = 1$. The idea of the proof is that of Section 2.2. Recall that for a 2-separating vertex α of $C(S)$, the corresponding vertex of $B(S)$ is denoted by α'.

Lemma 2.20 (i) *Let f be an automorphism of $C(S)$ and let α_1 and α_2 be two non-separating vertices of $C(S)$ such that a_1 and a_2 bound an annulus with one puncture for $a_i \in \alpha_i$. Then $f(a_1) \in f(\alpha_1)$ and $f(a_2) \in f(\alpha_2)$ bound an annulus with one puncture.*

(ii) *Let f be an automorphism of $C(S)$ and let α and β be nonseparating and 2-separating vertices of $C(S)$. If $i(\alpha, \beta') = 1$, then $i(f(\alpha), f(\beta')) = 1$.*

(iii) *The group* Aut $C(S)$ *preserves the topological type of the vertices of $C(S)$.*

Proof: (i) Note that for a nonseparating vertex α of $C(S)$, there exists a vertex β such that $i(\alpha, \beta) = 1$. By assumption, $i(g(\alpha), g(\beta)) = 1$ for $g \in$ Aut $C(S)$. Hence $g(\alpha)$ is nonseparating. Therefore Aut $C(S)$ preserves nonseparating vertices.

Now let α and β be two nonseparating vertices of $C(S)$ such that $\{\alpha, \beta\}$ is a 1-simplex of $C(S)$. We define the link $L(\alpha, \beta)$ of $\{\alpha, \beta\}$ to be the full subcomplex of $C(S)$ with the vertex set

$$\{\gamma \in C(S) : \gamma \neq \alpha, \gamma \neq \beta, i(\gamma, \alpha) = i(\gamma, \beta) = 0\}.$$

In fact, $L(\alpha, \beta) = L(\alpha) \cap L(\beta)$ and $L^d(\alpha, \beta) = L^d(\alpha) \cap L^d(\beta)$.

Let $a \in \alpha$ and $b \in \beta$ be disjoint representatives. Then the surface $S_{a \cup b}$ has two connected components. The vertices of $L(\alpha, \beta)$ is the isotopy classes of nontrivial circles on these two components. That is, $L(\alpha,\beta)$ is isomorphic to the complex of curves $C(S_{a\cup b})$.

Since the circles on different components do not intersect, if two vertices of $L^d(\alpha,\beta)$ form an edge then their representatives can be isotoped to circles on the same connected components of $S_{a\cup b}$. It follows that $L^d(\alpha,\beta)$ is connected if and only if the complex of curves of one of the connected components of $S_{a\cup b}$ is empty. That is, $L^d(\alpha,\beta)$ is connected if and only if one of the components of $S_{a\cup b}$ is a once-punctured annulus.

Since $g(L^d(\alpha,\beta)) = L^d(g(\alpha),g(\beta))$ for any automorphism g of $C(S)$, the proof of (i) follows.

(ii) We can find nonseparating vertices $\alpha_0,\alpha_1,\ldots,\alpha_n$ of $C(S)$ such that $\alpha_1 = \alpha$, $i(\alpha_1,\beta') = i(\alpha_0,\alpha_i) = 1$, $1 \le i \le n$, and all of the unmentioned intersection numbers are zero (see Figure 2.11 (a)). Then by using the assumption of Theorem 2.16, and part (i), we see that the configuration formed by minimally intersecting representatives $f(a_j)$ of $f(\alpha_j)$ is homeomorphic to the one formed by a_j. (Such a homeomorphism is constructed in [15] and [22].) Hence we can assume that $f(\alpha_j) = \alpha_j$ for all j. Then up to isotopy there exists a unique 2-separating circle, which must be b, disjoint from every a_j for $j \ne 1$. Hence the conclusion follows.

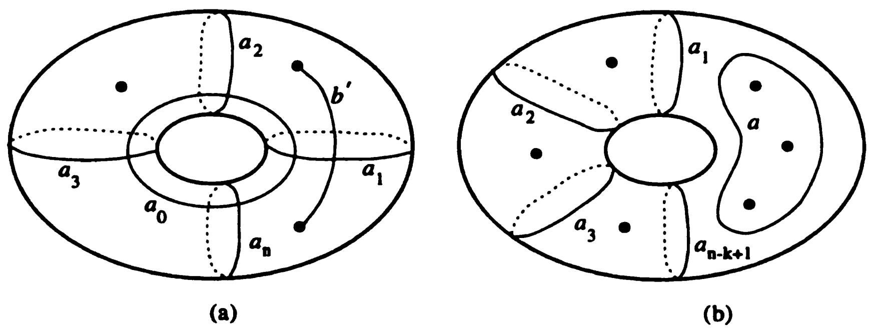


Figure 2.11: Some configurations of circles and arcs.

(iii) The fact that Aut $C(S)$ preserves the nonseparating vertices is proved in (i). Let α be a k-separating vertex of $C(S)$ with $2 \leq k \leq n$. We complete the proof by induction on k.

Let $k = 2$. The dual link of a vertex of $C(S)$ is connected if and only if the vertex is either nonseparating or 2-separating. Since Aut $C(S)$ preserves the nonseparating vertices, it must preserve the 2-separating ones as well.

Suppose now that $k \geq 3$ and that the assertion is true for all l-separating vertices with $l < k$. This implies that $f(\alpha)$ is at least k-separating. Note that there exist nonseparating vertices $\alpha_1, \alpha_2, \ldots, \alpha_{n-k+1}$ such that $\{\alpha, \alpha_1, \alpha_2, \ldots, \alpha_{n-k+1}\}$ is a simplex of $C(S)$, and that a_j and a_{j+1} bound an annulus with one puncture for each $1 \leq j \leq n-k$, where $a_j \in \alpha_j$ (see Figure 2.11 (b)). By part (i), $f(a_j)$ and $f(a_{j+1})$ bound an annulus with one puncture. It follows that the surface obtained by cutting S along the circles $f(a_1), f(a_2), \ldots, f(a_{n-k+1})$ is a disjoint union of $n-k$ once-punctured annuli and an annulus A with k punctures. Since the circle $f(a)$ is on the annulus A, it can be at most k-separating. □

By the correspondence between the set of 2-separating vertices of $C(S)$ and the set of isotopy classes of arcs joining different punctures on S, there is a well-defined action of Aut $C(S)$ on the latter set.

The next step is to prove that Aut $C(S)$ takes simple pair of circles to simple pair of circles.

Lemma 2.21 *If $\langle \alpha; \beta \rangle$ is a simple pair, then so is $\langle f(\alpha); f(\beta) \rangle$ for any automorphism f of $C(S)$.*

Proof: Note that any simple pair $\langle \alpha; \beta \rangle$ is determined uniquely by the existence of vertices $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that

(a) each γ_i is nonseparating, $0 \leq i \leq n$,

(b) $i(\gamma_0, \gamma_i) = 1$ and $i(\gamma_i, \gamma_j) = 0$, $1 \leq i, j \leq n$,

(c) $i(\alpha', \gamma_1) = i(\beta', \gamma_2) = 1$, and

(d) all the other unmentioned intersection numbers of α_i and α, or β are zero (see Figure 2.12 (a)).

Since Aut $C(S)$ preserves the conditions (a)-(d) and 2-separating circles, the simple pairs are preserved by Aut $C(S)$. □

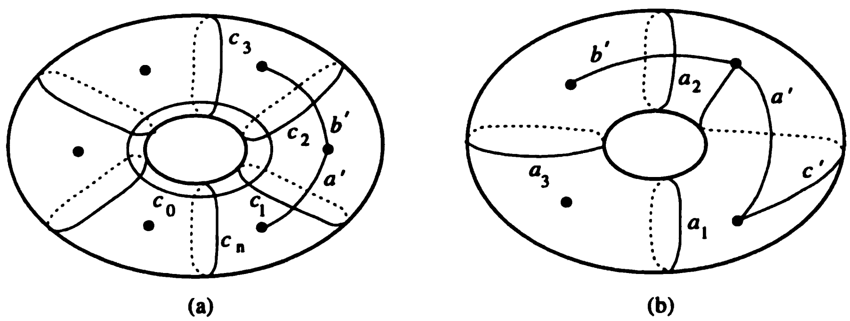


Figure 2.12: Existence of various configurations.

By the correspondence with 2-separating circles and the arcs connecting different punctures, the group Aut $C(S)$ preserves the simple pairs of arcs, too.

The action of Aut $C(S)$ on the punctures of S is defined in the same way as before. For $f \in$ Aut $C(S)$ and a puncture P on S, choose a simple pair $\langle \alpha'; \beta' \rangle$ with center P and define $f(P)$ to be the center of the simple pair $\langle f(\alpha'); f(\beta') \rangle$.

Lemma 2.22 *The definition of the action of* Aut $C(S)$ *on the set of punctures of* S *is independent of the choice of the simple pair.*

Proof: We use the technique and the notation of Lemma 2.5. We induct on $i = i(\alpha', \gamma') + i(\beta', \gamma')$.

If S has at least five punctures, the proof of Lemma 2.5 works here, too. Hence we may assume that S is a torus with three or four punctures.

Suppose that $i = 0$. There are two cases to consider.

Case 1: If P_1, P_2 and P_3 are all different, then S has four punctures and any two circles in the set $\{a', b', c'\}$ constitute a simple pair. Let d be the boundary of a regular neighborhood of the union $a' \cup b' \cup c' \cup \{P, P_1, P_2, P_3\}$. Clearly, d is 4-separating. There exists a nonseparating circle, say, e such that $i(e, a') = i(e, b') = 0$ and $i(e, c') = 1$. Then $f(d)$ is 4-separating and the arcs $f(a'), f(b')$ and $f(c')$ lie on the four-punctured disc component of $S_{f(d)}$. Let us denote the endpoints of $f(a')$ and $f(b')$ by $\tilde{P}, \tilde{P}_1, \tilde{P}_2$. Since any two circles in the set $\{f(a'), f(b'), f(c')\}$ form a simple pair, either one of the endpoints of $f(c')$ is $\tilde{P}$, or else the endpoints of $f(c')$ are $\tilde{P}_1$ and $\tilde{P}_2$. Since there exists a nonseparating circle, namely $f(e)$, intersecting $f(c')$ and not intersecting $f(a') \cup f(b')$, the latter case cannot hold.

Case 2: Suppose that P_3 is one of P_1 and P_2. Without loss of generality we may assume that $P_3 = P_1$. Now the proof splits into two subcases.

Subcase 1: First suppose that the boundaries of a regular neighborhood of $a' \cup c' \cup \{P, P_1\}$ are nonseparating. If S has three (resp. four) punctures, then there exist two (resp. three) nonseparating circles a_1, a_2 (resp. a_1, a_2, a_3) disjoint from a' and c' such that $S_{a_1 \cup a_2}$ (resp. $S_{a_1 \cup a_2 \cup a_3}$) is a disjoint union of an annulus with two punctures P, P_1, and one (resp. two) once-punctured annulus (resp. annuli) (see Figure 2.12 (b)). Then only one of the components of $S_{f(a_1) \cup f(a_2)}$ (resp. $S_{f(a_1) \cup f(a_2) \cup f(a_3)}$) is a twice-punctured annulus, on which the arcs $f(a'), f(c')$ lie. Hence the endpoints of $f(c')$ are those of $f(a')$.

Subcase 2: Now suppose that the boundaries of a regular neighborhood of $a' \cup c' \cup \{P, P_1\}$ are separating. It is clear that there exists an arc d' joining P to P_1 disjoint from $a' \cup b' \cup c'$ such that the components of the boundaries of a regular neighborhood of $a' \cup d' \cup \{P, P_1\}$ are nonseparating. Now we apply Subcase 1 to $\{a', d', b'\}$ to see that one of the endpoints of $f(d')$ is the center of the simple pair $\langle f(a'); f(b')\rangle$. Note that $\langle f(d'); f(b')\rangle$ is a simple pair with center that of $\langle f(a'); f(b')\rangle$. We apply Subcase 1 to $\{d', c', b'\}$ again to conclude that one of the endpoints of $f(c')$ is the center of the simple pair $\langle f(d'); f(b')\rangle$.

The proof now proceeds as the proof of Lemma 2.5. □

We now define an action of Aut $C(S)$ on the vertices of the complex $B(S)$, as in the punctured sphere case in Section 2. This action of Aut $C(S)$ on the isotopy classes of arcs joining different punctures is defined by the correspondence with the isotopy classes of 2-separating circles.

Let us define the action on the classes of arcs joining a puncture to itself. For this purpose let $f \in$ Aut $C(S)$ and let α' be a vertex of $B(S)$ such that $a' \in \alpha'$ is connecting a puncture P to itself. Consider the circles a_1 and a_2 which are the components of the boundary of a regular neighborhood of $a' \cup \{P\}$. Then either both of a_i are nonseparating, or a_1 is k-separating and a_2 is $(k+1)$-separating for some $k \geq 1$. (By convention, a 1-separating circle is the puncture to which the trivial simple closed curve is deformable.) If a_1 and a_2 are both nonseparating, then they bound an annulus with one puncture. By Lemma 2.20, $f(a_1)$ and $f(a_2)$ are nonseparating and bound an annulus with one puncture. Clearly, this puncture must be $f(P)$. We define $f(\alpha')$ to be the isotopy class of the unique (up to isotopy) nontrivial embedded arc on this annulus joining $f(P)$ to itself. If a_1 is k-separating and a_2 is $(k+1)$-separating, then $f(a_1)$ is k-separating and $f(a_2)$ is $(k+1)$-separating. Using the technique of Lemma 2.20 (iii), it is not hard to see that $f(a_1)$ and $f(a_2)$ bound an annulus with one puncture, $f(P)$. Then $f(\alpha')$ is defined as the isotopy class of any nontrivial embedded arc on this annulus joining $f(P)$ to itself.

Lemma 2.23 *Every automorphism of the complex $C(S)$ induces an automorphism of $B(S)$.*

Proof: Let $f \in \text{Aut } C(S)$ and let α' and β' be two distinct vertices of $B(S)$ with $i(\alpha', \beta') = 0$. It suffices to show that $i(f(\alpha'), f(\beta')) = 0$. As in the proof of Lemma 2.7, there are seven cases to consider (see Figure 2.5). If a' (resp. b') joins a puncture to itself, let a_1, a_2 (resp. b_1, b_2) be the circles used to define $f(\alpha')$ (resp. $f(\beta')$).

(i),(ii),(iii) The proof is the same as that of Lemma 2.7 (i),(ii),(iii).

(iv) Lemma 2.21.

(v) Let P and Q be the endpoints of a' and b'. There are two possible cases. The boundary components a, b of a regular neighborhood of $a' \cup b' \cup \{P, Q\}$ are either both nonseparating or both separating (see Figure 2.13). In either case, a and b bound an annulus with two punctures on which a' and b' lie. Moreover, there exists a circle c such that $i(\gamma, \alpha') \neq 0$ and $i(\gamma, \beta') = 0$, where γ is the isotopy class of c. Let α and β be the isotopy classes a and b respectively. By looking at the link and the dual link of $\{\alpha, \beta\}$ as in the proof of Lemma 2.20 (i), it is easy to see that $f(a)$ and $f(b)$ bound an annulus with two punctures, where $f(a) \in f(\alpha)$ and $f(b) \in f(\beta)$. Clearly, the arcs $f(a')$ and $f(b')$ lie on this annulus. Since $i(f(\gamma), f(\alpha')) \neq 0$, a segment of $f(c)$ connects $f(a)$ to $f(b)$. Then any arc disjoint from this segment can be isotoped to an arc disjoint from $f(a')$.

(vi) The argument in the proof of Lemma 2.7 works when b_1 and b_2 are nonseparating circles as well.

(vii) Suppose that a' and b' join the puncture P to itself. If a_1 and a_2, or b_1 and b_2 are separating, then the arguments given in the proof of part (vii) of Lemma

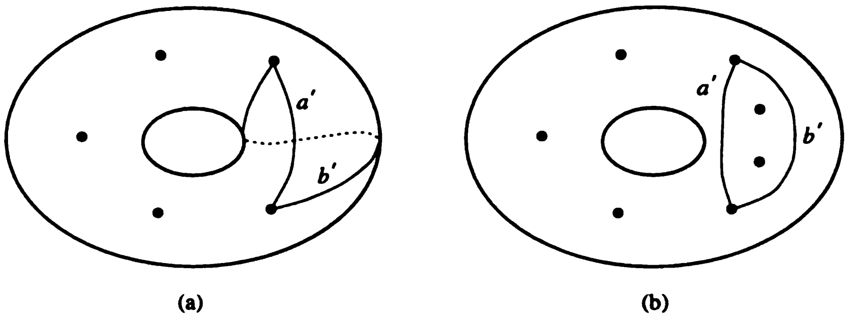


Figure 2.13: Two possible cases for (v).

2.7 finish the proof. So suppose that a_1, a_2, b_1 and b_2 are all nonseparating. Then $i(\beta', \alpha_1) = i(\beta', \alpha_2) = 1$, or $i(\beta', \alpha_1) = 0$ and $i(\beta', \alpha_2) = 2$ (by changing the roles of a_1 and a_2 if necessary). By composing f with a mapping class we can assume that $f(\alpha_1) = \alpha_1, f(\alpha_2) = \alpha_2$ and hence $f(\alpha') = \alpha'$.

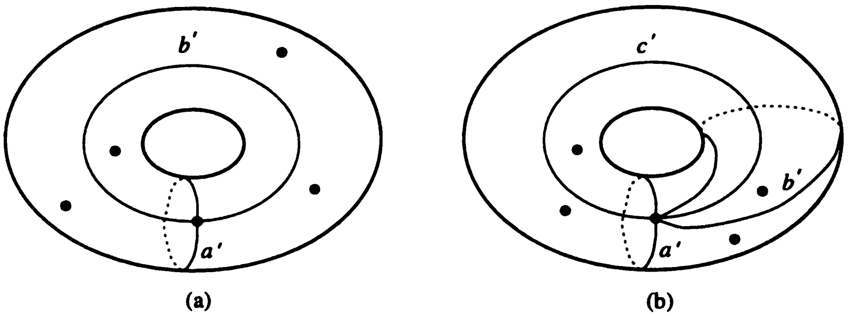


Figure 2.14: The arcs a' and b'.

If $i(\beta', \alpha_1) = i(\beta', \alpha_2) = 1$ (see Figure 2.14 (a)), then by Lemma 2.20 (ii) we have $i(f(\beta'), \alpha_1) = i(f(\beta'), f(\alpha_1)) = 1$ and $i(f(\beta'), \alpha_2) = i(f(\beta'), f(\alpha_2)) = 1$. Choose $f(b') \in f(\beta')$ so that it intersects a_1 and a_2 only once. The intersection of $f(b')$ with the once-punctured annulus determined by a_1 and a_2 has two components, one joining P to a_1 and the other joining P to a_2. Now it is obvious that these components can be isotoped so that they do not intersect a'. Therefore, $i(f(\alpha'), f(\beta')) = 0$.

Suppose now that $i(\beta', \alpha_1) = 0$ and $i(\beta', \alpha_2) = 2$ (see Figure 2.14 (b)). Clearly, there exists a nontrivial embedded arc c' joining P with itself such that it is disjoint from a' and b', and $i(\gamma', \alpha_1) = i(\gamma', \alpha_2) = 1$ and $i(\beta', \gamma_1) = i(\beta', \gamma_2) = 1$, where γ'

is the isotopy class of c'. Then $i(f(\gamma'), f(\alpha')) = i(f(\gamma'), \alpha') = 0$, $i(f(\gamma'), f(\beta')) = 0$ and $i(f(\gamma'), \alpha_1) = i(f(\gamma'), \alpha_2) = 1$. Choose a representative $f(c')$ of $f(\gamma')$ intersecting a', a_1 and a_2 minimally. A representative $f(b')$ of $f(\beta')$ can be chosen so that it is disjoint from a_1 (as $i(f(\beta'), \alpha_1) = 0$) and $f(c')$. But then $f(b')$ can be isotoped to a curve disjoint from a'. This completes the proof of the lemma. □

Corollary 2.24 Aut $C(S)$ *is naturally isomorphic to a subgroup of* Aut $B(S)$.

The proof of Theorem 2.16 for tori with $n \geq 3$ punctures proceeds now as in Section 2.2.3. In fact, Lemmas 2.9, 2.10 and 2.11 hold for punctured tori, too. Then the proof of Theorem 2.12 finishes the proof of Theorem 2.16.

2.4 Subgroups of $\mathcal{M}_S^*$

In this section, S denotes an oriented surface. The purpose of this section is to prove Theorem 2.28 as an application of our main results in Sections 2.2 and 2.3.

If α is a vertex of $C(S)$, we denote by t_α the right Dehn twist about α. It is well-known that for $f \in \mathcal{M}_S^*$, $f\,t_\alpha\,f^{-1} = t_{f(\alpha)}$ if f is orientation preserving and $f\,t_\alpha\,f^{-1} = t_{f(\alpha)}^{-1}$ if f is orientation reversing. An immediate consequence of the definition of Dehn twists is

Theorem 2.25 *Let α and β be two vertices of $C(S)$ and let N, M be two nonzero integers. Then $t_\alpha^N = t_\beta^M$ if and only if $\alpha = \beta$ and $N = M$.*

The following relations between Dehn twists are well known. We will use this theorem to prove our results in this section and in the next chapter. A proof of the theorem may be found in [15] or [22].

Theorem 2.26 *Let α and β be two vertices of $C(S)$ and let N, M be two nonzero integers. Then*

(i) $i(\alpha,\beta) = 0$ *if and only if* $t_\alpha^N t_\beta^M = t_\beta^M t_\alpha^N$.

(ii) (braid relations) $i(\alpha,\beta) = 1$ *if and only if* $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.

Let G be a group and let $f \in G$. We denote by $C_G(f)$ the centralizer of f in G, i.e.,

$$C_G(f) = \{g \in G : gf = fg\}$$

We denote the center of G by $C(G)$.

Let $m \geq 3$ be an integer. Let Γ be a subgroup of finite index of the kernel of the natural homomorphism $\mathcal{M}_S^* \to \text{Aut } H_1(S, \mathbf{Z}_m)$. Then clearly Γ is of finite index in $\mathcal{M}_S^*$.

Theorem 2.27 *Let S be a sphere with at least five punctures or a torus with at least three punctures. An element $f \in \Gamma$ is a power of a Dehn twist if and only if*

(i) $C(C_\Gamma(f))$ *is isomorphic to* $\mathbf{Z}$, *and*

(ii) $C(C_\Gamma(f))$ *is not isomorphic to* $C_\Gamma(f)$.

This theorem is proved in the same manner as Theorem (2.3) of [15]. We now ready to prove

Theorem 2.28 *Let S be a sphere with at least five punctures or a torus with at least three punctures. Let G_1 and G_2 be two subgroups of $\mathcal{M}_S^*$ of finite index. Then any isomorphism $G_1 \to G_2$ is induced by some inner automorphism of $\mathcal{M}_S^*$. In particular, two subgroups of $\mathcal{M}_S^*$ of finite index are isomorphic if and only if they are conjugate. Also, if G is a subgroup of $\mathcal{M}_S^*$ of finite index, then the outer automorphism group* Out G *of G is finite.*

Proof: Let $\Phi : G_1 \to G_2$ be an isomorphism. Consider the subgroup $\Phi^{-1}(G_2 \cap \Gamma) \cap \Gamma$ of $\mathcal{M}_S^*$ of finite index. Let Γ_1 be a finite index subgroup of $\Phi^{-1}(G_2 \cap \Gamma) \cap \Gamma$ and let $\Gamma_2 = \Phi(\Gamma_1)$. Then, Γ_1 and Γ_2 are of finite index in $\mathcal{M}_S^*$ and Φ restricts to an isomorphism from Γ_1 to Γ_2. Clearly, $\Phi(C(C_{\Gamma_1}(f))) = C(C_{\Gamma_2}(\Phi(f)))$ and $\Phi(C_{\Gamma_1}(f)) = C_{\Gamma_2}(\Phi(f))$. It follows from Theorem 2.27 that Φ takes sufficiently high powers of Dehn twists to powers of Dehn twists. More precisely, for a vertex α of $C(S)$, since the index of Γ_1 is finite, there exists a nonzero integer N such that $t_\alpha^N \in \Gamma_1$. If $t_\alpha^N \in \Gamma_1$ then $\Phi(t_\alpha^N) = t_\beta^M$ for some vertex β. If $t_\alpha^{N_1}, t_\alpha^{N_2} \in \Gamma_1$ and $\Phi(t_\alpha^{N_i}) = t_{\beta_i}^{M_i}$, then

$$t_{\beta_1}^{M_1 N_2} = \Phi(t_\alpha^{N_1})^{N_2} = \Phi(t_\alpha^{N_2})^{N_1} = t_{\beta_2}^{M_2 N_1}.$$

It follows from Theorem 2.25 that $\beta_1 = \beta_2$. That is, we have a well-defined map φ from the vertex set of $C(S)$ to itself, defined by the equation $\Phi(t_\alpha^N) = t_{\varphi(\alpha)}^M$, independent of the choice of the powers involved.

Next, we show that φ is an automorphism $C(S) \to C(S)$. Obviously, we also have a map φ^{-1} from the vertex set of $C(S)$ to itself induced by Φ^{-1}. Now for any α of $C(S)$

$$t_\alpha^N = \Phi^{-1}(\Phi(t_\alpha^N)) = \Phi^{-1}(t_{\varphi(\alpha)}^M) = t_{\varphi^{-1}(\varphi(\alpha))}^T$$

and

$$t_\alpha^N = \Phi(\Phi^{-1}(t_\alpha^N)) = \Phi(t_{\varphi^{-1}(\alpha)}^K) = t_{\varphi(\varphi^{-1}(\alpha))}^L$$

for some appropriate integers N, M, T, K, L. Then again by Theorem 2.25, we have $\varphi^{-1}(\varphi(\alpha)) = \alpha$ and $\varphi(\varphi^{-1}(\alpha)) = \alpha$, so φ is a bijection.

Let α, β be two vertices of $C(S)$ with $i(\alpha, \beta) = 0$, then $t_\alpha^{N_1}$ commutes with $t_\beta^{N_2}$, and hence $t_{\varphi(\alpha)}^{M_1}$ commutes with $t_{\varphi(\beta)}^{M_2}$. It follows from Theorem 2.26 that $i(\varphi(\alpha), \varphi(\beta)) = 0$, i.e., φ is an automorphism of the complex of curves $C(S)$.

Now, the automorphism $\varphi : C(S) \to C(S)$ is induced by a mapping class f of S, (i.e., $\varphi(\alpha) = f(\alpha)$), by Theorem 2.12 if S is a punctured sphere, or by Theorem 2.18

if S is a punctured torus. Then

$$\Phi(t_\alpha^N) = t_{\varphi(\alpha)}^M = t_{f(\alpha)}^M = f\, t_\alpha^{\pm M}\, f^{-1}.$$

Let $g \in G_1$. If α is a vertex of $C(S)$, then for appropriate integers N, M and K,

$$\Phi(g\, t_\alpha^N\, g^{-1}) = \Phi(g)\,\Phi(t_\alpha^N)\,\Phi(g^{-1}) = \Phi(g)\, t_{\varphi(\alpha)}^M\, \Phi(g)^{-1} = \Phi(g)\, t_{f(\alpha)}^M\, \Phi(g)^{-1} = t_{\Phi(g)(f(\alpha))}^{\pm M}$$

and

$$\Phi(g\, t_\alpha^N\, g^{-1}) = \Phi(t_{g(\alpha)}^{\pm N}) = t_{\varphi(g(\alpha))}^K = t_{f(g(\alpha))}^K.$$

Then $\Phi(g)(f(\alpha)) = f(g(\alpha))$, or, equivalently, $(\Phi(g)f)(\alpha) = (fg)(\alpha)$, and thus $(fg)^{-1}\Phi(g)f$ is in the kernel of the map $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$, which is trivial by Theorem 1.3. So $\Phi(g)f = fg$, i.e., $\Phi(g) = fgf^{-1}$.

The second conclusion follows easily from the first.

Let G be a subgroup of $\mathcal{M}_S^*$ of finite index. Let us denote by $N_{\mathcal{M}_S^*}(G)$ the normalizer of G in $\mathcal{M}_S^*$. That is,

$$N_{\mathcal{M}_S^*}(G) = \{f \in \mathcal{M}_S^* : f\,G\,f^{-1} \subset G\}.$$

For each $f \in N_{\mathcal{M}_S^*}(G)$, the inner automorphism I_f of $\mathcal{M}_S^*$ maps G to itself. This gives a homomorphism $\psi : N_{\mathcal{M}_S^*}(G) \to \operatorname{Aut} G$ defined by $\psi(f) = I_f$. This homomorphism is surjective by the first part of the theorem. Hence we have a surjective homomorphism $\overline{\psi} : N_{\mathcal{M}_S^*}(G) \to \operatorname{Out} G$. Clearly, G is in the kernel of $\overline{\psi}$. Consequently, the order of Out G is

$$[\operatorname{Out} G : 1] = [N_{\mathcal{M}_S^*}(G) : \ker \overline{\psi}] \leq [N_{\mathcal{M}_S^*}(G) : G] \leq [\mathcal{M}_S^* : G]$$

which is finite by assumption. □

Remark Except, possibly, for closed surfaces of genus two, the conclusion of Theorem 2.28 is, in fact, true for *all* surfaces of genus ≥ 2. A sketch of this is given in [16], and the proof of Theorem 2.28 above is almost identical to that in [16].

Chapter 3

First Homology Groups of Mapping Class Groups of Nonorientable Surfaces

3.1 Introduction

In this chapter, we calculate the first homology groups of the mapping class groups of closed nonorientable surfaces. It turns out that they are isomorphic to $\mathbf{Z}_2$ if the genus of the surface is at least seven. Another result is that the subgroup of the mapping class group generated by Dehn twists about 2-sided circles has trivial first homology group, again, if the genus of the surface is at least seven. As an algebraic application, we deduce Corollary 3.15.

The first homology groups of the mapping class groups of closed orientable surfaces are well-known. Let F be a closed orientable surface of genus g. If F is a sphere ($g = 0$) then $\mathcal{M}_F$ is trivial, and hence $H_1(\mathcal{M}_F) = 0$. If $g \geq 3$ then $H_1(\mathcal{M}_F)$ is again trivial. This result is due to Powell [26]. The group $H_1(\mathcal{M}_F)$ is $\mathbf{Z}_{10}$ if $g = 2$, proved by Mumford [24], and $\mathbf{Z}_{12}$ if $g = 1$.

If $g = 0$ or $g \geq 3$, then it follows from $H_1(\mathcal{M}_F) = 0$ and the fact that $\mathcal{M}_F$ is of index two in $\mathcal{M}_F^*$ that the first homology group $H_1(\mathcal{M}_F^*)$ is $\mathbf{Z}_2$. Also, $H_1(\mathcal{M}_F^*)$

is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ if $g = 1, 2$. Since this is not so well-known as the other results stated above, we give a proof at the end of the present chapter. The case $g = 1$ is, probably, well-known to algebraists. The case $g = 2$ is proved at the end of (5.5) in [15].

In Section 3.2, we give the preliminary information on nonorientable surfaces. We state various results regarding the mapping class groups of nonorientable surfaces, and prove some lemmas which will be used in the proofs of the main results.

In Section 3.3, we discuss vector spaces over the finite field $\mathbf{F}_2$ of order 2. We are mainly interested in the group of isometries of such a vector space with a symmetric bilinear form. This subject is related to the mapping class group of nonorientable surfaces via a theorem of McCarthy and Pinkall.

We prove our main results of this chapter in the last section, Section 3.4. In the proofs, we use known relations between Dehn twists to get an upper bound on the order of the first homology groups. A lower bound is obtained by finding homomorphisms onto abelian groups.

3.2 Mapping Class Groups of Nonorientable Surfaces

Let S be a closed connected nonorientable surface of genus g. For nonorientable surfaces, the genus is defined to be the number of real projective planes in a connected sum decomposition. The *mapping class group* $\mathcal{M}_S^*$ is defined as the group of isotopy classes of self-homeomorphisms of S. Let $\mathcal{M}_S$ be the subgroup of $\mathcal{M}_S^*$ generated by twists about 2-sided circles. It turns out that this subgroup is the analogue of $\mathcal{M}_F$ in the orientable case.

It is known that the mapping class group of a projective plane is trivial and that of a Klein bottle is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (cf. [19]). In addition, the subgroup of the mapping class group of a Klein bottle generated by the Dehn twists about 2-sided circles is $\mathbf{Z}_2$. Therefore, in the rest of the chapter we restrict ourself to surfaces of genus at least three.

Recall that for a simple closed curve (a *circle*) a on a surface S we denote by S_a the surface obtained from S by cutting along a. Two circles a and b on S are *topologically equivalent* if there exists a homeomorphism $H : S \to S$ such that $H(a) = b$.

Lickorish [19] proved that the group $\mathcal{M}_S^*$ is generated by the isotopy classes of Dehn twists about two-sided nontrivial circles and that of so-called Y-homeomorphisms (crosscap slides). He also showed that the isotopy classes of Dehn twists about two-sided nontrivial circles do not generate $\mathcal{M}_S^*$. In fact, the subgroup $\mathcal{M}_S$ generated by the isotopy classes of Dehn twists about 2-sided circles is of index 2 in $\mathcal{M}_S^*$ (cf. [21]). Later, Chillingworth [8] found a finite set of generators. Using the results of Humphries [13], one can eliminate some of Chillingworth's generators of $\mathcal{M}_S^*$.

Let us describe the Y-homeomorphisms of nonorientable surfaces. Consider a Möbius band M with one hole. Let us attach a Möbius band M' to M along the boundary of the hole. The resulting surface K is a Klein bottle with one hole. By moving M' once along the core of M we get a homeomorphism of K which is the identity on the boundary of K (see Figure 3.1). If K is embedded in a surface S, we can extend this homeomorphism by the identity to a homeomorphism of S. This homeomorphism is called a Y-*homeomorphism* or a *crosscap slide*. It is clear from the definition that the square of a Y-homeomorphism is a Dehn twist about the boundary of the Klein bottle K with hole. We note that Y-homeomorphisms act as the identity on the $\mathbf{Z}_2$-homology.

Let S be a closed connected nonorientable surface of genus g. In the rest of this chapter, we will mainly use the models given in Figure 3.2. If g is even, we will also

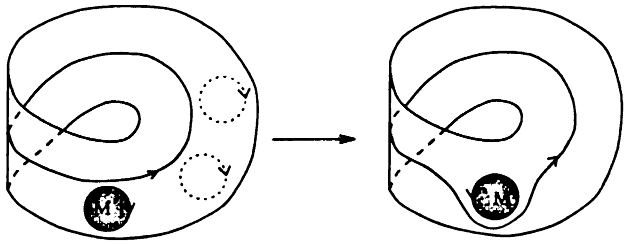


Figure 3.1: A Y-homeomorphism.

use the model for S given in Figure 3.3. In the figures, the interiors of the shaded discs are to be removed and the antipodal points on each boundary component identified.

For a 2-sided circle a on the nonorientable surface S, let us denote by t_a one of the two Dehn twists about a. Consider the models in Figure 3.2. Let d be the boundary of the shaded part if g is odd and $d = b_{r+1}$ if g is even. Clearly, S_d is orientable. After orienting S_d arbitrarily, we can assume that the twists t_a are actually the right twists on S_d for $a = a_i, b_i, c_i$.

Theorem 3.1 (Chillingworth, [8]) *Let S be a closed connected nonorientable surface of genus g. Then $\mathcal{M}_S^*$ is generated by*

$$\{t_{a_i}, t_{b_i}, t_{c_j}, z : 1 \leq i \leq r, 1 \leq j \leq r-1\} \text{ if } g = 2r+1,$$

$$\{t_{a_i}, t_{b_i}, t_{c_i}, t_{b_{r+1}}, z : 1 \leq i \leq r\} \text{ if } g = 2r+2,$$

where z is the isotopy class of a Y-homeomorphism such that z^2 is a twist about e if g is odd and about f if g is even.

Theorem 3.2 (Chillingworth, [8]) *Let S be a closed connected nonorientable surface of genus g. Then $\mathcal{M}_S$ is generated by*

$$\{t_{a_i}, t_{b_i}, t_{c_j}, t_e, t_{e_1} : 1 \leq i \leq r, 1 \leq j \leq r-1\} \text{ if } g = 2r+1,$$

and

$$\{t_{a_i}, t_{b_i}, t_{c_i}, t_{b_{r+1}}, t_f, t_{f_1}, t_{f_2} : 1 \leq i \leq r\} \text{ if } g = 2r+2$$

where e_1, f_1, f_2 *are the circles in Figure* 3.4.

The generators of $\mathcal{M}_S$ can also be described by using the oriented double cover of S (cf. [4]).

Theorem 3.3 (McCarthy-Pinkall, [23]) *Let* S *be a closed nonorientable surface. If* L *is an automorphism of* $H_1(S, \mathbf{Z}_2)$ *which preserves the* $\mathbf{Z}_2$*-valued intersection pairing, then* L *is induced by a diffeomorphism which is a product of Dehn twists.*

In other words the natural map

$$\mathcal{M}_S \to \text{Iso } H_1(S, \mathbf{Z}_2)$$

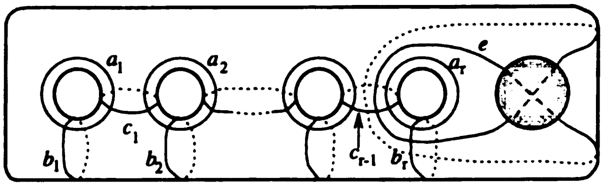
is onto, where Iso $H_1(S, \mathbf{Z}_2)$ is the group of isomorphisms of $H_1(S, \mathbf{Z}_2)$ preserving the $\mathbf{Z}_2$-valued intersection pairing.

Let us recall the lantern relation discovered by Dehn [9] and rediscovered by Johnson [18]. Let S_0 be a sphere with four holes with boundary components d_0, d_1, d_2 and d_3. For $1 \leq i < j \leq 3$, let d_{ij} denote a circle encircling d_i and d_j as in Figure 3.5 (a). If we consider the diffeomorphisms of S_0 fixing ∂S_0, and if we embed S_0 in a surface R, then we can consider the right twists t_{d_i} and $t_{d_{ij}}$ as twists in R by extending them by the identity to the complement of S_0. Then we have

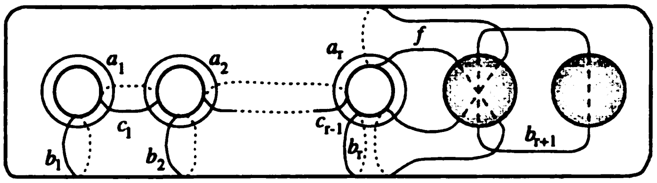
Lemma 3.4 *In this situation* $t_{d_0} t_{d_1} t_{d_2} t_{d_3} = t_{d_{12}} t_{d_{13}} t_{d_{23}}$.

Lemma 3.5 *Consider the torus with two holes in Figure* 3.5(*b*) *as embedded in a surface. Then*

$$t_{q_1} t_{q_2} = (t_a t_c t_b)^2 \, (t_b t_a t_c)^2 = (t_a t_c t_b)^4.$$

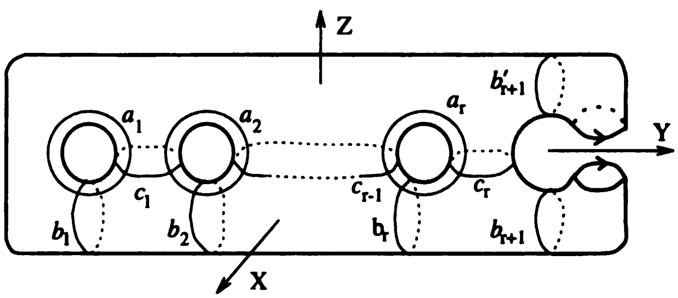


g=2r+1



g=2r+2

Figure 3.2: Models for nonorientable surfaces.



g=2r+2

Figure 3.3: Another model if g is even.

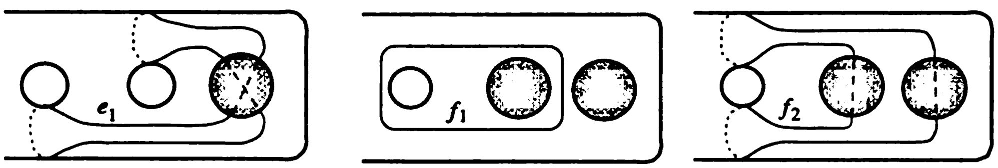


Figure 3.4: The circles e_1, f_1 and f_2.

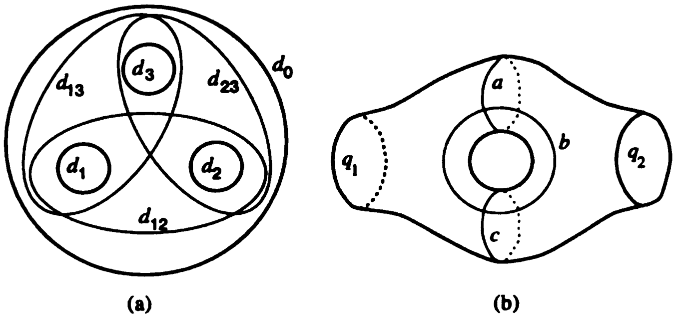


Figure 3.5: Sphere with four holes S_0 and torus with two holes.

Proof: By Lemma 3 of [20], $t_{q_1}t_{q_2} = (t_a t_c t_b)^2 (t_b t_a t_c)^2$. The equality $(t_b t_a t_c)^2 = (t_a t_c t_b)^2$ follows from Theorem 2.26:

$$(t_b t_a t_c)^2 = t_b t_a t_c t_b t_a t_c = t_b t_a t_c t_b t_c t_a = t_b t_a t_b t_c t_b t_a = t_a t_b t_a t_c t_b t_a$$

$$= t_a t_b t_c t_a t_b t_a = t_a t_b t_c t_b t_a t_b = t_a t_c t_b t_c t_a t_b = t_a t_c t_b t_a t_c t_b = (t_a t_c t_b)^2. \ \square$$

Lemma 3.6 *Let S be a closed connected nonorientable surface. Then there exists a mapping class ϱ such that $\varrho t_{a_1} \varrho^{-1} = t_{a_1}^{-1}$.*

Proof: Consider the models given in Figure 3.3. Let d be the boundary of the shaded part if g is odd and $d = b_{r+1}$ if g is even. Then S is a quotient of the surface S_d. Let us embed S_d in $\mathbf{R}^3$ in such a way that it is invariant under the reflection $\tilde{\rho}$ across the XY-plane and $\tilde{\rho}(d) = d$. Then $\tilde{\rho}$ induces a homeomorphism ρ of S. Let ϱ be the isotopy class of ρ. Since $\tilde{\rho}(a_1) = a_1$ and since $\tilde{\rho}$ is orientation reversing, it follows that $\varrho t_{a_1} \varrho^{-1} = t_{a_1}^{-1}$. $\square$

Lemma 3.7 *Let S be a closed connected nonorientable surface of even genus $g \geq 4$. Suppose that a is a separating circle on S such that one of the components of S_a is a Klein bottle with one hole, and the other component is also nonorientable. Then there exists a mapping class ϱ such that $\varrho t_a \varrho^{-1} = t_a^{-1}$.*

Proof: Let $g = 2r$. Consider the sphere $\mathbf{S}^2$ as the one point compactification of $\mathbf{R}^2$. Remove from $\mathbf{S}^2$ the interiors of g discs of radius $\frac{1}{4}$ with centers $(\pm 1, 0), (\pm 2, 0), \ldots,$ $(\pm r, 0)$. Let $\mathbf{S}(g)$ be the resulting surface. Let $\tilde{a}$ be the circle of radius $\frac{3}{2}$ with center $(0,0)$ and ρ be the reflection across Y-axis. Then $\rho(\tilde{a}) = \tilde{a}$. Let us realize S as the quotient of $\mathbf{S}(g)$ obtained by identifying the antipodal points on each boundary component. We may assume that a is the image of $\tilde{a}$. The homeomorphism ρ descends to a homeomorphism of S. Denote the isotopy class of this homeomorphism of S by ϱ. Since ρ is orientation reversing, we have $\varrho t_a \varrho^{-1} = t_a^{-1}$. □

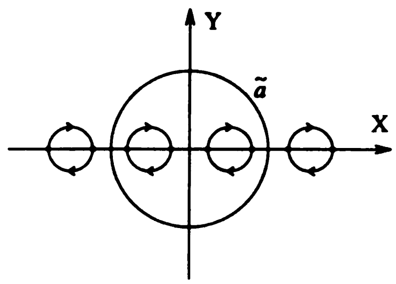


Figure 3.6: Surface $\mathbf{S}(g)$.

Lemma 3.8 *Let G be a group and H a normal subgroup of G. Then $[H,H]$ is a normal subgroup of G*

Proof: The proof of the lemma follows immediately from the following fact:

$$g^{-1}[h_1, h_2]g = [g^{-1}h_1 g, g^{-1}h_2 g]. \ \square$$

3.3 Groups of isometries of vector spaces over the field of order 2

Let V be a vector space space of dimension m over $\mathbf{F}_2$, the finite field of order 2, let $\{v_1, v_2, \ldots, v_m\}$ be a basis of V, and let $\langle\,,\rangle$ be the symmetric bilinear form on V defined by $\langle v_i, v_j\rangle = \delta_{ij}$. Let us denote by Iso V the group of isometries of V.

If m is odd, say $m = 2n+1$, then the restriction of $\langle\, , \rangle$ to the $2n$-dimensional subspace

$$W = \{v \in V : \langle v, v\rangle = 0\}$$

is nondegenerate, and hence it is a symplectic form. Recall that a symmetric bilinear form $\langle\, , \rangle$ on a vector space W over a field of characteristic 2 is called *symplectic* if it is nondegenerate and $\langle w, w\rangle = 0$ for all $w \in W$. Any isometry of V induces an isometry of the symplectic space $(W, \langle\, , \rangle)$, and any isometry h of W can be extended uniquely to an isometry of V by requiring that

$$h(\sum_{i=1}^{2n+1} v_i) = \sum_{i=1}^{2n+1} v_i.$$

From this observation it follows easily that there is an isomorphism between Iso V and Iso W. The group Iso W is called the *symplectic group* and is denoted by $\mathrm{Sp}(2n, 2)$. It is known that $\mathrm{Sp}(2n, 2)$ is perfect for $n \geq 3$ (see, for instance, [27]).

A proof of the next lemma may be found in [25], Chapter 3, or in [27], Chapter 8.

Lemma 3.9 *The order of* $\mathrm{Sp}(2n, 2)$ *is* $2^{n^2} \prod_{i=1}^{n} (2^{2i} - 1)$.

Lemma 3.10 *The group* $\mathrm{Sp}(4, 2)$ *is isomorphic to* Σ_6*, and* $\mathrm{Sp}(2, 2)$ *is isomorphic to* Σ_3*, where* Σ_r *is the symmetric group on* r *letters.*

Proof: A proof of the first assertion may be found in [25], Chapter 3, or in [27], Chapter 8, so let us prove the second. Let W be a vector space over $\mathbf{F}_2$ of dimension 2 with a basis $\{w_1, w_2\}$ and let $\langle\, , \rangle$ be defined by $\langle w_i, w_j\rangle = 1 + \delta_{ij}$. Then $(W, \langle\, , \rangle)$ is a symplectic space and hence Iso W is isomorphic to $\mathrm{Sp}(2, 2)$. Since any permutation of $\{w_1, w_2, w_1 + w_2\}$ induces an isometry, Σ_3 is isomorphic to a subgroup of Iso W. Also, since the orders of Σ_3 and Iso W are both 6, they are isomorphic. □

Lemma 3.11 *Suppose that $m = 2n$. Let us consider V as a subspace of a vector space V' over $\mathbf{F}_2$ of dimension $2n+1$ with a basis $\{v_1, \dots, v_{2n}, v_{2n+1}\}$. Extend the form $\langle\,,\rangle$ to V' by $\langle v_i, v_{2n+1}\rangle = \delta_{i,2n+1}$. Then* Iso V *is isomorphic to the stabilizer of v_{2n+1} in* Iso V' *and the order* [Iso $V : 1$] *of* Iso V *is*

$$[\text{Iso } V : 1] = 2^{n^2} \prod_{i=1}^{n-1} (2^{2i} - 1).$$

Proof: We define a map

$$\psi\colon \text{Iso } V \to \text{Stab}_{\text{Iso } V'}(v_{2n+1})$$

by

$$\psi(h)(v) = \begin{cases} h(v) & \text{if } v \in V \\ v_{2n+1} & \text{if } v = v_{2n+1} \end{cases}$$

for $h \in$ Iso V, where $\text{Stab}_{\text{Iso } V'}(v_{2n+1})$ is the stabilizer of v_{2n+1} under the canonical action of Iso V'. If $v \in V$ then

$$\langle \psi(h)(v), \psi(h)(v_{2n+1})\rangle = \langle h(v), v_{2n+1}\rangle = 0 = \langle v, v_{2n+1}\rangle.$$

As a consequence of this, $\psi(h)$, the extension of h to V', is an isometry fixing v_{2n+1}. Also, for any $h \in \text{Stab}_{\text{Iso } V'}(v_{2n+1})$

$$h(V) = h(\{v_{2n+1}\}^{\perp}) = \{v_{2n+1}\}^{\perp} = V.$$

Therefore the map ψ is onto. The injectiveness of ψ is clear. Hence, the first assertion follows.

Now we calculate card $\text{Orb}(v_{2n+1})$, the cardinality of the orbit of v_{2n+1} under the canonical action of Iso V'. We claim that $\text{Orb}(v_{2n+1})$ is the set of sums of odd numbers of basis elements, except for $\sum_{i=1}^{2n+1} v_i$. It is obvious that for any $k < n$ there exists an $h \in$ Iso V' such that $h(\sum_{j=1}^{2k+1} v_{i_j}) = \sum_{i=1}^{2k+1} v_i$ if $i_j \neq i_{j'}$ for $j \neq j'$.

Therefore we only need to show that for any $k < n$ there exists an $h \in \text{Iso } V'$ such that $h(v_{2n+1}) = \sum_{i=1}^{2k+1} v_i$. This is true if and only if $\sum_{i=1}^{2k+1} v_i$ can be completed to an orthonormal basis of V'. For this purpose let $k < n$, and let

$$v'_{2n+1} = \sum_{i=1}^{2k+1} v_i$$

and

$$v'_i = \begin{cases} v_i + v_{2k+2} + v'_{2n+1} & \text{if } 1 \leq i \leq 2k+1 \\ v_{i+1} & \text{if } 2k+2 \leq i \leq 2n. \end{cases}$$

It is easy to check that $\{v'_1, v'_2, \ldots, v'_{2n+1}\}$ is an orthonormal basis of V'. Since $\langle v, v \rangle = 1$ if and only if v is an odd number of sums of basis elements and since $h(\sum_{i=1}^{2n+1} v_i) = \sum_{i=1}^{2n+1} v_i$ for all $h \in \text{Iso } V'$, the claim follows.

Now it follows that card $\text{Orb}(v_{2n+1}) = 2^{2n} - 1$.

Recall that for any finite group G acting on a finite set X,

$$[G:1] = [\text{Stab}_G(x):1] \, \text{card} \, \text{Orb}(x)$$

for $x \in X$, where $[G:1]$ is the order of G. Hence we have

$$[\text{Iso } V' : 1] = [\text{Iso } V : 1](2^{2n} - 1).$$

Now, since Iso V' is isomorphic to $\text{Sp}(2n, 2)$, the conclusion follows from Lemma 3.9.

□

The next lemma is stated in [27] as Exercise 1 on page 174.

Lemma 3.12 *Suppose that $m = 2n$. Then there is a surjective homomorphism from* Iso V *to* $\text{Sp}(2n-2, 2)$.

Proof: Let $W = \{v \in V : \langle v, v \rangle = 0\}$. If $w_i = v_1 + v_{i+1}$ then $\{w_1, w_2, \ldots, w_{2n-1}\}$ is a basis of W. Clearly, $W^\perp = \{0, v_1 + v_2 + \ldots + v_{2n}\}$ and $W^\perp \subset W$. Then $W/W^\perp$ with the induced form is a symplectic space of dimension $2n-2$. Clearly, Iso $W/W^{\perp}$ is isomorphic to $\mathrm{Sp}(2n-2,2)$.

Let $h \in$ Iso V. Then $h(W) = W$. That is, h induces an isometry of W, and $h(W^{\perp}) = W^{\perp}$. This implies that h induces an isometry of $W/W^{\perp}$. It follows that there is a homomorphism from Iso V to Iso $W/W^{\perp}$. We show that this homomorphism is surjective.

Let $\overline{h}$ be an isometry of $W/W^{\perp}$. Then

$$\overline{h}(w_i + W^{\perp}) = w_i' + W^{\perp}$$

for some $w_i' \in W$. If $w_1' + w_2' + \ldots + w_{2n-1}' = 0$ then by replacing w_{2n-1}' with $w_{2n-1}' + v_1 + v_2 + \ldots + v_{2n}$ we may assume that $w_1' + w_2' + \ldots + w_{2n-1}' \neq 0$. The set $\{w_1', w_2', \ldots, w_{2n-1}'\}$ is a basis of W and

$$\begin{aligned} \langle w_i', w_j' \rangle &= \langle w_i' + W^{\perp}, w_j' + W^{\perp} \rangle \\ &= \langle \overline{h}(w_i + W^{\perp}), \overline{h}(w_j + W^{\perp}) \rangle \\ &= \langle w_i + W^{\perp}, w_j + W^{\perp} \rangle \\ &= \langle w_i, w_j \rangle. \end{aligned}$$

Then $h(w_i) = w_i'$ defines an isometry of W which induces $\overline{h}$ on $W/W^{\perp}$.

It remains to prove that isometries of W can be extended to V. So let h be an isometry of W. Consider the subspace U of W generated by $\{w_1, w_2, \ldots, w_{2n-2}\}$. Then

$$U^{\perp} = \{0, v_{2n}, v_1 + v_2 + \ldots + v_{2n-1},, v_1 + v_2 + \ldots + v_{2n}\}.$$

Hence the restriction of $\langle\,,\rangle$ to U is nondegenerate. Since h is an isometry between U and $h(U)$, the restriction of $\langle\,,\rangle$ to $h(U)$ is also nondegenerate. This implies that $V = h(U) \oplus h(U)^{\perp}$. In particular, $h(U)^{\perp}$ is not a subspace of W. Let us choose

$v \in h(U)^{\perp} \backslash W$. Then $\langle v, v \rangle = 1$ and $\langle v, h(v_1 + v_i) \rangle = 0$ for $1 \leq i \leq 2n - 1$. If $\langle v, h(v_i + v_{2n}) \rangle = 0$ for some $1 \leq i \leq 2n-1$, we would conclude, since W is also generated by

$$\{h(v_1 + v_2), h(v_1 + v_3), \ldots, h(v_1 + v_{2n-1}), h(v_i + v_{2n})\},$$

that $v \in W^{\perp} \subset W$, a contradiction. Therefore, $\langle v, h(v_i + v_{2n}) \rangle = 1$ for all $1 \leq i \leq 2n - 1$. Hence $\langle v, h(v_i + v_{2n}) \rangle = 1 + \delta_{2n,i}$. Now define $k : V \to V$ by the formula $k(v_i) = v + h(v_i + v_{2n})$. Then

$$\begin{aligned}
\langle k(v_i), k(v_j) \rangle &= \langle v + h(v_i + v_{2n}), v + h(v_j + v_{2n}) \rangle \\
&= \langle v, v \rangle + \langle h(v_i + v_{2n}), v \rangle + \langle v, h(v_j + v_{2n}) \rangle + \langle h(v_i + v_{2n}), h(v_j + v_{2n}) \rangle \\
&= 1 + 1 + \delta_{2n,j} + 1 + \delta_{2n,i} + \langle v_i + v_{2n}, v_j + v_{2n} \rangle \\
&= \delta_{2n,j} + 1 + \delta_{2n,i} + \langle v_i, v_j \rangle + \langle v_i, v_{2n} \rangle + \langle v_{2n}, v_j \rangle + \langle v_{2n}, v_{2n} \rangle \\
&= \delta_{2n,j} + 1 + \delta_{2n,i} + \delta_{ij} + \delta_{2n,i} + \delta_{2n,j} + 1 \\
&= \delta_{ij}.
\end{aligned}$$

Clearly, k defines an isometry of V which is an extension of h. □

3.4 The first homology groups

We are now ready to prove the main results of this chapter. Recall that for a group G, $H_1(G) = G/[G, G]$. We will use multiplicative notation for the operation in $H_1(G)$.

Theorem 3.13 *Let S be a closed connected nonorientable surface of genus g. Then $H_1(\mathcal{M}_S^*)$ is equal to*

(i) 0 *if* $g = 1$,

(ii) $\mathbf{Z_2} \oplus \mathbf{Z_2}$ *if* $g = 2, 3, 5, 6$,

(iii) $\mathbf{Z_2} \oplus \mathbf{Z_2} \oplus \mathbf{Z_2}$ *if* $g = 4$,

(iv) $\mathbf{Z_2}$ *if* $g \geq 7$.

Proof: If $g \geq 2$, then the subgroup $\mathcal{M}_S$ is of index 2 in $\mathcal{M}_S^*$ (cf. [21]). Hence, it is a normal subgroup. Then the quotient $\mathcal{M}_S^*/\mathcal{M}_S$ is of order 2, and hence is abelian. Clearly, this implies that the index of the commutator subgroup $[\mathcal{M}_S^*, \mathcal{M}_S^*]$ is at least 2 in $\mathcal{M}_S^*$, i.e., the order of $H_1(\mathcal{M}_S^*)$ is at least 2. It is well-known that if a group G is generated by X, then $G/[G,G]$ is generated by $\overline{X} = \{x[G,G] : x \in X\}$. For $h \in \mathcal{M}_S^*$ let us denote by $\overline{h}$ the class of h in $\mathcal{M}_S^*/[\mathcal{M}_S^*, \mathcal{M}_S^*]$.

If $g = 1$, then $\mathcal{M}_S^*$ is trivial. Hence so is $H_1(\mathcal{M}_S^*)$.

If $g = 2$, then $\mathcal{M}_S^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ (cf. [19]). So $H_1(\mathcal{M}_S^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

If $g = 3$, Birman and Chillingworth [4] proved that $\mathcal{M}_S^*$ admits the presentation:

$$\langle a, b, y : aba = bab, yay^{-1} = a^{-1}, yby^{-1} = b^{-1}, y^2 = 1, (aba)^4 = 1 \rangle.$$

It follows that $\mathcal{M}_S^*/[\mathcal{M}_S^*, \mathcal{M}_S^*]$ has the presentation:

$$\mathcal{M}_S^*/[\mathcal{M}_S^*, \mathcal{M}_S^*] = \langle \overline{a}, \overline{y} : \overline{a}^2 = \overline{y}^2 = 1, \overline{a}\,\overline{y} = \overline{y}\,\overline{a} \rangle.$$

Therefore $H_1(\mathcal{M}_S^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. This can also be deduced from the arguments below.

We now assume that $g \geq 4$. The rest of the proof splits into two cases.

Case 1: g is odd. Let $g = 2r + 1$. From the braid relations it follows that $\overline{t}_{a_i} = \overline{t}_{b_j} = \overline{t}_{c_k}$ in $H_1(\mathcal{M}_S^*)$. Hence, it follows from Theorem 3.1 that $H_1(\mathcal{M}_S^*)$ is generated by $\{\overline{t}_{a_1}, \overline{z}\}$.

Lemma 3.6 implies that $\overline{t}_{a_1}^2 = 1$. We now show that $\overline{z}^2 = 1$.

Notice that e is a separating circle and one of the components of S_e is a Klein bottle with one hole. There exist circles $q_1, q_2, \ldots, q_r = e$ on S such that q_1 bounds a

Möbius band, and that q_i and q_{i+1} bound a torus with two holes for $i = 1, 2, \ldots, r-1$ (see Figure 3.7). Then by Lemma 3.5

$$t_{q_i} t_{q_{i+1}} = (t_{a'_i} t_{c'_i} t_{b'_i})^4$$

for some nonseparating 2-sided circles a'_i, b'_i and c'_i each of whose complement is nonorientable. Hence, a'_i, b'_i, c'_i are topologically equivalent to a_1. Therefore, each of the Dehn twists $t_{a'_i}, t_{b'_i}$ and $t_{c'_i}$ is conjugate to either t_{a_1} or $t_{a_1}^{-1}$. Since the conjugate elements induces the same element in the abelianization and since $\bar{t}_{a_1} = \bar{t}_{a_1}^{-1}$, we have $\bar{t}_{a'_i} = \bar{t}_{b'_i} = \bar{t}_{c'_i} = \bar{t}_{a_1}$. This implies that

$$\bar{t}_{q_i} \bar{t}_{q_{i+1}} = (\bar{t}_{a'_i} \bar{t}_{c'_i} \bar{t}_{b'_i})^4 = 1$$

Since the Dehn twist about a 2-sided circle which bounds a Möbius band is isotopic to the identity, $t_{q_1} = 1$. Now it follows that $\bar{t}_{q_i} = 1$ for all i. Therefore $\bar{z}^2 = \bar{t}_{q_r} = 1$.

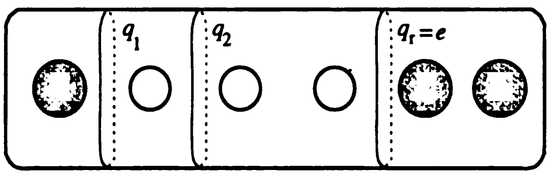


Figure 3.7: Circles $q_1, q_2, \ldots, q_r$.

From these discussions it follows that the generators of $H_1(\mathcal{M}_S^*)$ satisfy the relations $\bar{t}_{a_1}^2 = 1, \bar{z}^2 = 1$ and the obvious commutation relation. Hence $H_1(\mathcal{M}_S^*)$ is either $\mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, as its order is at least 2.

Let $g = 5$. In view of Section 3.3, Iso $H_1(S, \mathbf{Z}_2)$ is isomorphic to Sp(4, 2), where $H_1(S, \mathbf{Z}_2)$ is considered with the $\mathbf{Z}_2$-valued intersection pairing. Let

$$\varphi : \mathcal{M}_S^* \to \text{Iso } H_1(S, \mathbf{Z}_2) \cong \text{Sp}(4, 2)$$

be the canonical map. By Theorem 3.3 the restriction of φ to $\mathcal{M}_S$ is onto. By Lemma 3.10, Sp(4, 2) is isomorphic to Σ_6, the symmetric group on six letters. Since the commutator subgroup of Σ_6 is of index 2 in Σ_6, the commutator subgroup of Sp(4, 2) is of index 2. In particular, there is a homomorphism from Sp(4, 2) onto $\mathbf{Z}_2$. Composition of this homomorphism with φ gives a homomorphism $\overline{\varphi}$ from $\mathcal{M}_S^*$ onto $\mathbf{Z}_2$. Since the restriction of $\overline{\varphi}$ to $\mathcal{M}_S$ is also onto, $\ker\overline{\varphi} \cap \mathcal{M}_S$ is normal of index 4 in $\mathcal{M}_S^*$. Recall that for any prime number p, any group of order p^2 is abelian. Hence the group $\mathcal{M}_S^*/\ker\overline{\varphi} \cap \mathcal{M}_S$ is abelian. It follows that the index of the commutator subgroup of $\mathcal{M}_S^*$ is at least 4. Therefore $H_1(\mathcal{M}_S^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

If $g \geq 7$ then the sphere with four holes of Figure 3.5 (a) can be embedded in S so that $d_0 = b_1, d_1 = c_1, d_2 = c_2, d_3 = b_3$. Then each d_{ij} is 2-sided nonseparating with a nonorientable complement (see Figure 3.8). Since a_1 intersects d_{12} and d_{13} only once and since a_2 intersects d_{23} once, it follows from the braid relations that $\overline{t}_{d_i} = \overline{t}_{d_{ij}} = \overline{t}_{a_1}$. From the lantern relation we have

$$\overline{t}_{d_0}\overline{t}_{d_1}\overline{t}_{d_2}\overline{t}_{d_3} = \overline{t}_{d_{12}}\overline{t}_{d_{13}}\overline{t}_{d_{23}}$$

and hence $\overline{t}_{a_1}^4 = \overline{t}_{a_1}^3$. Thus $\overline{t}_{a_1} = 1$. Therefore $H_1(\mathcal{M}_S^*) = \mathbf{Z}_2$.

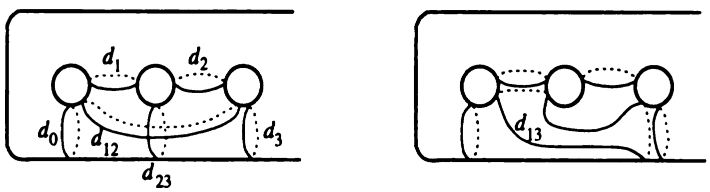


Figure 3.8: Embedding of S_0.

Case 2: g is even. Let $g = 2r + 2$. It follows from the braid relations that $\overline{t}_{a_i} = \overline{t}_{b_j} = \overline{t}_{c_k}$ for $1 \leq i, j, k \leq r$. Hence, $H_1(\mathcal{M}_S^*)$ is generated by $\{\overline{t}_{a_1}, \overline{t}_{b_{r+1}}, \overline{z}\}$ by Theorem 3.1.

By Lemma 3.6, $\varrho t_{a_1}\varrho^{-1} = t_{a_1}^{-1}$ for some mapping class ϱ. It follows that $\overline{t}_{a_1}^2 = 1$.

We now claim that $\overline{t}_{b_{r+1}}^2 = 1$. Let us consider the model for S given in Figure 3.3. Let $\tilde{\rho} : S_d \to S_d$ be the reflection across the XY-plane, where d is a circle isotopic to b_{r+1}. The homeomorphism $\tilde{\rho}$ induces a homeomorphism ρ. Since $\tilde{\rho}$ is orientation reversing, since $\tilde{\rho}(b_{r+1}) = b'_{r+1}$ on S_d and since b_{r+1} is isotopic to b'_{r+1} on S, we have

$$\varrho t_{b_{r+1}} \varrho^{-1} = t^{-1}_{b'_{r+1}} = t^{-1}_{b_{r+1}},$$

where ϱ is the isotopy class of ρ. The claim now follows.

The next step is to prove that $\overline{t}_f^2 = \overline{t}_{f_1}^2 = \overline{t}_{f_2}^2 = 1$. Consider the generators of $\mathcal{M}_S$ given by Theorem 3.2. Note that f is a separating circle such that the surface S_f is a disjoint union of a Klein bottle with one hole, and a nonorientable surface. It follows from Lemma 3.7 that $\overline{t}_f^2 = 1$. Since f_1 intersects b_r at only one point, from the braid relations we have $\overline{t}_{f_1} = \overline{t}_{b_r}$. That is, $\overline{t}_{f_1}^2 = 1$. Also, since f_2 is 2-sided nonseparating and since S_{f_2} is orientable, f_2 is topologically equivalent to b_{r+1}. In another words, t_{f_2} is conjugate to $t^{\pm 1}_{b_{r+1}}$. This implies that $\overline{t}_{f_2}^2 = \overline{t}^{\pm 2}_{b_{r+1}} = 1$.

It is clear now from Theorem 3.2 that if $h \in \mathcal{M}_S$ then $\overline{h}^2 = 1$.

Now let q be a separating circle on S such that one of the components of S_q is a Klein bottle K with one hole and the other is an orientable surface. Let z_0 be the isotopy class of a Y-homeomorphism supported in K such that $z_0^2 = t_q$. We now show that $\overline{t}_q = 1$. On the orientable component of S_q, there are circles $q_0, q_1, q_2, \ldots, q_r = q$ such that q_0 bounds a disc, and q_{i-1} and q_i bound a torus with two holes for each $i = 1, 2, \ldots, r$. Then by Lemma 3.5, $t_{q_{i-1}} t_{q_i}$ is the fourth power of a product of isotopy classes of twists about circles which are topologically equivalent to a_1. This implies that $\overline{t}_{q_{i-1}} \overline{t}_{q_i} = 1$. Since $\overline{t}_{q_0} = 1$, it follows that $\overline{t}_q = \overline{z}_0^2 = 1$.

Finally, we show that $\overline{z}^2 = 1$. As indicated in [19], the automorphism of $H_1(S, \mathbf{R})$ induced by z_0 has determinant -1. However, Dehn twists about 2-sided circles induce automorphisms with determinant $+1$. Therefore, $z_0 \notin \mathcal{M}_S$. Since $\mathcal{M}_S$ is of index two in $\mathcal{M}_S^*$ and since $z \notin \mathcal{M}_S$, $z_0 = hz$ for some $h \in \mathcal{M}_S$. Hence,

$$1 = \overline{z}_0^2 = \overline{h}^2 \overline{z}^2 = \overline{z}^2.$$

By putting the results of the last six paragraphs together, we conclude that the generators of $H_1(\mathcal{M}_S^*)$ satisfy $\bar{t}_{a_1}^2 = \bar{t}_{b_{r+1}}^2 = \bar{z}^2 = 1$ and the commutation relations. Therefore, $H_1(\mathcal{M}_S^*)$ is $\mathbf{Z}_2, \mathbf{Z}_2 \oplus \mathbf{Z}_2$, or $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Let us consider the case $g = 4$ first. Consider $H_1(S, \mathbf{Z}_2)$ with the $\mathbf{Z}_2$-valued intersection pairing and let

$$\varphi\colon \mathcal{M}_S^* \to \text{Iso } H_1(S, \mathbf{Z}_2)$$

be the natural homomorphism. The restriction of φ to $\mathcal{M}_S$ is onto by Theorem 3.3.

It follows from Lemma 3.11 that the order of Iso $H_1(S, \mathbf{Z}_2)$ is 48. Since any permutation of an orthonormal basis of $H_1(S, \mathbf{Z}_2)$ induces an isometry of $H_1(S, \mathbf{Z}_2)$, Iso $H_1(S, \mathbf{Z}_2)$ has a subgroup isomorphic to the symmetric group Σ_4. Let us identify this subgroup with Σ_4. Since Σ_4 is of index 2 in Iso $H_1(S, \mathbf{Z}_2)$, it is normal. Recall that the commutator subgroup $[\Sigma_4, \Sigma_4]$ of Σ_4 is the alternating group, which is of index 2 in Σ_4, and hence of index 4 in Iso $H_1(S, \mathbf{Z}_2)$. By Lemma 3.8, $[\Sigma_4, \Sigma_4]$ is normal in Iso $H_1(S, \mathbf{Z}_2)$. Note that any group of order 4 is abelian. Therefore, Iso $H_1(S, \mathbf{Z}_2)/[\Sigma_4, \Sigma_4]$ is abelian. It follows that the commutator subgroup of Iso $H_1(S, \mathbf{Z}_2)$ is equal to $[\Sigma_4, \Sigma_4]$. In particular, there is a homomorphism from Iso $H_1(S, \mathbf{Z}_2)$ onto an abelian group G of order 4.

Let $\overline{\varphi}$ be the composition of φ with the quotient map from Iso $H_1(S, \mathbf{Z}_2)$ to G. Then $[\mathcal{M}_S^*, \mathcal{M}_S^*] \subset \ker \overline{\varphi}$. Since $[\mathcal{M}_S^*, \mathcal{M}_S^*] \subset \mathcal{M}_S$, we have $[\mathcal{M}_S^*, \mathcal{M}_S^*] \subset \ker \overline{\varphi} \cap \mathcal{M}_S$, and since the restriction of $\overline{\varphi}$ to $\mathcal{M}_S$ is onto, $\ker \overline{\varphi} \cap \mathcal{M}_S$ is of index 8 in $\mathcal{M}_S^*$. Therefore, the order of $H_1(\mathcal{M}_S^*)$ is at least 8. Consequently, $H_1(\mathcal{M}_S^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Suppose now that $g \geq 6$. The sphere with four holes of Figure 3.5 (a) can be embedded in S in such a way that $d_0 = b_{r-1}, d_1 = c_{r-1}, d_2 = c_r, d_3 = b_{r+1}$. Then each d_{ij} is 2-sided nonseparating with a nonorientable complement. Since $d_1, d_2, d_3, d_{12}, d_{13}$ and d_{23} are all topologically equivalent to a_1, and $\bar{t}_{a_1} = \bar{t}_{a_1}^{-1}$, it follows that $\bar{t}_{d_{ij}} = \bar{t}_{d_k} =$

$\bar{t}_{a_1}$ for $1 \le i < j \le 3$ and $1 \le k \le 3$. Hence, as a consequence of the lantern relation we have

$$\bar{t}_{d_0}\bar{t}_{d_1}\bar{t}_{d_2}\bar{t}_{d_3} = \bar{t}_{d_{12}}\bar{t}_{d_{13}}\bar{t}_{d_{23}},$$

which implies that $\bar{t}_{b_{r+1}} = 1$. Hence $H_1(\mathcal{M}_S^*)$ is $\mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Let $g = 6$. Consider $H_1(S, \mathbf{Z}_2)$ with the $\mathbf{Z}_2$-valued intersection pairing again. Since $H_1(S, \mathbf{Z}_2)$ is isomorphic to $\mathbf{Z}_2^6$, by Lemma 3.12 we have a surjective homomorphism from Iso $H_1(S, \mathbf{Z}_2)$ to Sp(4, 2). Since Sp(4, 2) is isomorphic to Σ_6, there is a natural map from Sp(4, 2) onto $\mathbf{Z}_2$. Therefore, the composition of the following natural maps is onto:

$$\mathcal{M}_S^* \to \text{Iso } H_1(S, \mathbf{Z}_2) \to \text{Sp}(4,2) \to \mathbf{Z}_2.$$

In fact, even the restriction of $\overline{\varphi}$ to $\mathcal{M}_S$ is onto by Theorem 3.3. Therefore $\ker \overline{\varphi} \cap \mathcal{M}_S$ is normal and of index 4 in $\mathcal{M}_S^*$. Clearly, this implies that the order of $H_1(\mathcal{M}_S^*)$ is at least 4. It follows that $H_1(\mathcal{M}_S^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

If $g \ge 8$ then, as in the case of g odd, the lantern relation implies that $\bar{t}_{a_1} = 1$, because the sphere with four holes of Figure 3.5 (a) can be embedded in S such that $d_0 = b_1, d_1 = c_1, d_2 = c_2$ and $d_3 = b_3$. Hence, the order of $H_1(\mathcal{M}_S^*)$ is at most 2, and thus exactly 2. The proof of Theorem 3.13 is now complete. □

Theorem 3.14 *Let S be a closed connected nonorientable surface of genus g. Then $H_1(\mathcal{M}_S)$ is equal to*

(i) $\mathbf{Z}_2$ *if* $g = 2$,

(ii) 0 *if* $g = 1$ *or* $g \ge 7$.

Proof: If $g = 1$ then $\mathcal{M}_S$ is trivial. If $g = 2$ then it is easy to conclude from $\mathcal{M}_S^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ that $\mathcal{M}_S = \mathbf{Z}_2$. Hence, the proof of the theorem follows in these two cases.

Suppose now that $g \geq 7$. Let us denote by $\bar{t}_a$ the class of t_a in the quotient $\mathcal{M}_S/[\mathcal{M}_S, \mathcal{M}_S]$. Consider the generators of $\mathcal{M}_S$ given by Theorem 3.2.

If g is odd, then for $1 \leq i \leq r$ the relations $\bar{t}_{a_i} = \bar{t}_{b_i} = \bar{t}_{c_i} = 1$ follow from the braid relations and the lantern relations. In the proof of Theorem 3.13, to prove the triviality of $\bar{t}_e = \bar{z}^2$ in $H_1(\mathcal{M}_S^*)$, we used only Dehn twists. The same argument shows that $\bar{t}_e = 1$ in $H_1(\mathcal{M}_S)$. Since e_1 intersects a_r once, the braid relations imply that $\bar{t}_{e_1} = \bar{t}_{a_r} = 1$. This implies the theorem for g odd.

Suppose g is even. Again, for $1 \leq i \leq r$ the equalities $\bar{t}_{a_i} = \bar{t}_{b_i} = \bar{t}_{c_i} = 1$ follow from the braid relations and lantern relation as in the proof of Theorem 3.13. The sphere with four holes of Figure 3.5 (a) can be embedded in S so that $d_0 = b_{r-1}, d_1 = c_{r-1}, d_2 = c_r, d_3 = b_{r+1}$. By the lantern relation, $\bar{t}_{b_{r+1}} = 1$. Since S_f is a disjoint union of a Klein bottle with one hole and a nonorientable surface of genus ≥ 6, the sphere with four holes of Figure 3.5 (a) can be embedded in S in such a way that $d_0 = f$ and the circles $d_1, d_2, d_3, d_{12}, d_{13}, d_{23}$ are all 2-sided nonseparating with nonorientable complements (see Figure 3.9). Now by the lantern relation

$$t_f t_{d_1} t_{d_2} t_{d_3} = t_{d_{12}} t_{d_{13}} t_{d_{23}}.$$

It follows that $\bar{t}_f = 1$. Since f_1 intersects b_r once, we have $\bar{t}_{f_1} = 1$ by the braid relations. The circle f_2 is nonseparating and its complement is orientable. Hence it is topologically equivalent to b_{r+1}. By the same reason that $\bar{t}_{b_{r+1}} = 1$, we have $\bar{t}_{f_2} = 1$. This completes the proof of the theorem. □

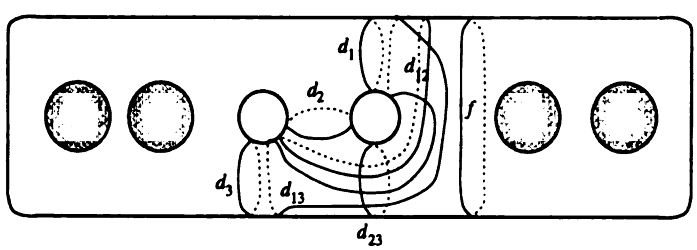


Figure 3.9: Embedding of S_0.

Remark: If $g = 3, 4, 5, 6$, the problem of the computation of $H_1(\mathcal{M}_S)$ remains open.

Our theorems support a little surprising similarity between the subgroups $\mathcal{M}_S$ and $\mathcal{M}_F$ of $\mathcal{M}_S^*$ and $\mathcal{M}_F^*$ respectively: they are both of index 2 and perfect for sufficiently high genera.

Corollary 3.15 *Let V be a vector space over the finite field $\mathbf{F}_2$ of order 2 with a basis $\{v_1, v_2, \dots, v_n\}$ and let $\langle\,,\rangle$ be the symmetric bilinear form defined by $\langle v_i, v_j\rangle = \delta_{ij}$. If $n \geq 7$ then the group $\operatorname{Iso} V$ is perfect.*

Proof: This is a consequence of Theorem 3.3 and Theorem 3.14. □

Remark: If the dimension n of V is odd, then $\operatorname{Iso} V$ is the symplectic group $\operatorname{Sp}(n-1, 2)$. This group is known to be perfect if $n \geq 7$ (cf. [25], [27]).

Theorem 3.16 *Let F be a closed connected orientable surface of genus g. Then $H_1(\mathcal{M}_F^*)$ is equal to*

(i) $\mathbf{Z}_2$ *if $g = 0$ or $g \geq 3$,*

(ii) $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ *if $g = 1$ or $g = 2$.*

Proof: If $g = 0$ then $\mathcal{M}_F^*$ is equal to $\mathbf{Z}_2$ and hence $H_1(\mathcal{M}_F^*) = \mathbf{Z}_2$. For $g \geq 3$, since $H_1(\mathcal{M}_F) = 0$, $[\mathcal{M}_F, \mathcal{M}_F] = \mathcal{M}_F$. Since $\mathcal{M}_F$ is of index 2 in $\mathcal{M}_F^*$, it follows that $[\mathcal{M}_F^*, \mathcal{M}_F^*] = \mathcal{M}_F$. Hence $H_1(\mathcal{M}_F^*) = \mathbf{Z}_2$.

We now prove (ii). Let us realize F in $\mathbf{R}^3$ invariant under the reflections across coordinate planes, as in Figure 3.10. Recall that $\mathcal{M}_F$ is generated by $\{t_{A_1}, t_{A_2}\}$ if $g = 1$ and by $\{t_{A_1}, t_{A_2}, t_{A_3}, t_{A_4}, t_{A_5}\}$ if $g = 2$ (cf. [20]). Then these generators, together with

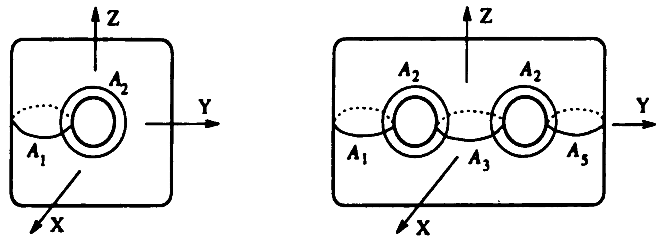


Figure 3.10: Orientable surfaces of genus one and two.

the isotopy class of an orientation reversing homeomorphism, generate $\mathcal{M}^*_F$. Let ϱ be the isotopy class of the reflection ρ across the XY-plane. Then, $\mathcal{M}^*_F$ is generated by $\{t_{A_1}, t_{A_2}, \varrho\}$ if $g = 1$ and by $\{t_{A_1}, t_{A_2}, t_{A_3}, t_{A_4}, t_{A_5}, \varrho\}$ if $g = 2$. If $g = 1$ then it follows from Theorem 2.26 that $\bar{t}_{A_1} = \bar{t}_{A_2}$. Similarly, $\bar{t}_{A_1} = \bar{t}_{A_2} = \bar{t}_{A_3} = \bar{t}_{A_4} = \bar{t}_{A_5}$ if $g = 2$. This implies that $H_1(\mathcal{M}^*_F)$ is generated by $\{\bar{t}_{A_1}, \bar{\varrho}\}$. Since ρ is orientation reversing and $\rho(A_1) = A_1$, we have $\varrho t_{A_1} \varrho^{-1} = t_{A_1}^{-1}$. Hence $\bar{t}_{A_1}^2 = 1$. Also, since $\rho^2 = 1$, $\bar{\varrho}^2 = 1$. Since $\mathcal{M}^*_F$ has a normal subgroup of index 2, namely $\mathcal{M}_F$, the first homology group $H_1(\mathcal{M}^*_F)$ cannot be 0. Therefore, it is either $\mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

We now prove that the order of $H_1(\mathcal{M}^*_F)$ is at least 4. Consider $H_1(F, \mathbf{Z}_2)$ with the $\mathbf{Z}_2$-valued intersection pairing. Note that the group Iso $H_1(F, \mathbf{Z}_2)$ is isomorphic to $\mathrm{Sp}(2,2) \cong \Sigma_3$ if $g = 1$ and $\mathrm{Sp}(4,2) \cong \Sigma_6$ if $g = 2$, i.e., Iso $H_1(F, \mathbf{Z}_2) \cong \Sigma_{3g}$ for $g = 1, 2$. Consider the natural homomorphism

$$\varphi : \mathcal{M}^*_F \to \text{Iso } H_1(F, \mathbf{Z}_2) \cong \Sigma_{3g}.$$

It is well-known that the restriction of φ to $\mathcal{M}_F$ is onto. Let $\bar{\varphi}$ be the composition of φ with the quotient map from Σ_{3g} onto $\Sigma_{3g}/[\Sigma_{3g}, \Sigma_{3g}] \cong \mathbf{Z}_2$. Then $\ker \bar{\varphi} \cap \mathcal{M}_F$ is a normal subgroup of $\mathcal{M}^*_F$ of index 4. Therefore, the order of $H_1(\mathcal{M}^*_F)$ is at least 4. This implies the theorem. □

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