This is to certify that the

thesis entitled

Maximal Commutative Subalgebras
of n by n Matrices Over a Field

presented by

Young Kwon Song

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

William C. Brown
Major professor

Date April 18, 1996

MAXIMAL COMMUTATIVE SUBALGEBRAS OF n BY n MATRICES OVER A FIELD

By

Young Kwon Song

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1996

ABSTRACT

MAXIMAL COMMUTATIVE SUBALGEBRAS OF n BY n MATRICES OVER A FIELD

BY

Young Kwon Song

The existence of R.C. Courter's counterexample to M. Gerstenhaber's conjecture suggests some interesting questions about the isomorphism classes of local algebras in the ring of 14 by 14 matrices. It was conjectured for a long time that Courter's example is unique up to isomorphism.

In Chapter 2, we will show that the class of maximal, local, commutative algebras which are isomorphic to $\mathcal{B} \ltimes N^2$ has only one isomorphism class. Next, we will show the class of pairs (R, V) which are (σ, τ)-isomorphic to $(\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$ has only one isomorphism class.

In Chapter 3, we will construct a new algebra $\mathcal{S}$ which is maximal, local, commutative, index of Jacobson radical 3, and dimension 13. We will use $\mathcal{S}$ to show the (B, N)-construction depends on the field k. The algebra $\mathcal{S}$ is not a (B, N)-construction if k is the real numbers and is a (B, N)-construction if k is an algebraically closed field. Finally, we will answer the above conjecture by showing the algebra $\mathcal{S}$ is not isomorphic to the Courter's algebra.

DEDICATION

To my parents, brother, sisters, loving wife, and pretty daughter, Christina.

ACKNOWLEDGEMENTS

I would like to give heartfelt thanks to my advisor, Professor William C. Brown for his helpful guidance, encouragement, patience, and understanding. I would also like to thank Professors Wei-Eihn Kuan, Christel Rotthaus, Charles Seebeck, and Bernd Ulrich for their helpful advice.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF SYMBOLS

LIST OF SYMBOLS (continued)

k^n	$M_{1\times n}(k)$	5
$\mathcal{H}$	$Hom_k(V,V)$	5
$\underline{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_n\}$	the canonical basis of k^{14}	5
$\Gamma : \mathcal{H} \rightarrow T_n$	matrix representation	5
V_R	right R-module V	5
$i(J(R))$	index of Jacobson radical of R	7
$B \ltimes M$	idealization of M	7
Ω	$\{(R, J, k) \in \mathcal{M}_{14}(k) \mid dim_k R = 13,\ i(J) = 3\}$	8
$M\mathcal{B}(4)$	the category of all finitely generated, faithful, $\mathcal{B}$-modules of dimension 4	8
$\Omega_{\mathcal{B}}$	$\{(R, J, k) \in \Omega \mid R \cong \mathcal{B} \ltimes N^2$ for some $N \in M\mathcal{B}(4)\}$	8
ΩC_1	$\{(R, V) \in \mathcal{M}\mathbf{X} \mid (\mathbf{R}, \mathbf{V}) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes \mathbf{N^2}, \mathcal{B}^2 \ltimes \mathbf{N})$ for some $N \in M\mathcal{B}(4)\}$	8
$[R]$	the k-algebra isomorphism class containing R	8
$[(R, V)]$	the (σ, τ)-isomorphism class containing (R, V)	8
$(0) :_V J$	$\{v \in V \mid vJ = 0\}$	10
$L(\alpha_1, \ldots, \alpha_n)$	the linear span of $\alpha_1, \ldots, \alpha_n$	10
$Soc(R)$	$Ann_R(J)$	11
$\mu_{\mathcal{B}}(N)$	the minimal number of generator of $\mathcal{B}$-module N	18
$\mathcal{S}$	the k-algebra defined on Equation (104)	48

Chapter 1

Notation and History

In this thesis, k will denote an arbitrary field. We will let $\mathbb{N}$ denote the natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$. If $m, n \in \mathbb{N}$, then $M_{m\times n}(k)$ will denote the set of all $m \times n$ matrices with entries in k.

If $m = n$, then we will abbreviate $M_{m\times n}(k)$ by T_n. We will assume $n \geq 2$ throughout this thesis. An associative ring R will be called a k-algebra if R is a k-vector space and $a(rr') = (ar)r' = r(ar')$ for all $a \in k$ and $r, r' \in R$. In this thesis, all k-algebras will be assumed to contain a (multiplicative) identity $1 \neq 0$. In particular, if R is a k-algebra, then $dim_k(R) \geq 1$. T_n is an example of k-algebra. A k-subspace R_0 of a k-algebra R will be called a k-subalgebra of R if R_0 is closed under multiplication from R and R_0 contains the identity of R. We will assume all k-algebra homomorphisms take the identity to identity.

Let R be a commutative, k-subalgebra of T_n. Thus, $xy = yx$ for all $x, y \in R$. R is called a maximal, commutative, k-subalgebra of T_n if R satisfies the following property : If R' is a commutative, k-subalgebra of T_n and $R \subseteq R'$, then $R = R'$. Thus, a maximal, commutative, k-subalgebra of T_n is a maximal element with respect to inclusion in the set of all maximal, commutative, k-subalgebras of T_n. We will let $\mathcal{M}_n(k)$ denote the set of all maximal, commutative, k-subalgebras of T_n.

Thus, if $C_{T_n}(S) = \{A \in T_n \mid As = sA, \text{ for all } s \in S\}$ is the centralizer of a set S in T_n, then a commutative, k-subalgebra R of T_n is maximal if and only if $C_{T_n}(R) = R$.

Maximal, commutative, k-subalgebras of T_n come in many different shapes and sizes. Here are a few examples.

Example 1: Let $p, q \in \mathbb{N}$ such that $\mid p - q \mid \leq 1$. Set $n = p + q$. Let

$$R = \left\{ \begin{pmatrix} xI_p & Z \\ O_{q\times p} & xI_q \end{pmatrix} \in T_n \mid x \in k, Z \in M_{p\times q}(k) \right\}. \tag{1}$$

In Equation (1), I_p denotes the identity matrix of size p by p and $O_{q\times p}$ denotes the zero matrix of size q by p. Then, R is a commutative, k-subalgebra of T_n and it is easy to check that $C_{T_n}(R) = R$. Thus, $R \in \mathcal{M}_n(k)$. We will call R a Schur algebra of size n.

Throughout this thesis, we will denote the Schur algebra of size 4 by $\mathcal{B}$. Thus,

$$\mathcal{B} = \left\{ \begin{pmatrix} x & 0 & a & b \\ 0 & x & c & d \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \mid x, a, b, c, d \in k \right\}. \tag{2}$$

□

Example 2: Let $R = k[D]$, where $D \in T_n$ is a nonderogatory matrix. Since D is nonderogatory, the characteristic polynomial c_D of D and the minimal polynomial m_D of D are the same. Consequently, $dim_k(k[D]) = deg(m_D) = deg(c_D) = n$. Suppose $D' \in C_{T_n}(R)$. Then, $DD' = D'D$ and by [7: Theorem 2], $n = dim_k(k[D]) \leq dim_k(k[D, D']) \leq n$. Thus, $k[D, D'] = k[D]$ and in particular, $D' \in k[D] = R$. We conclude $R \in \mathcal{M}_n(k)$. □

Example 3: Let

$$R = \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & a_1 \end{pmatrix} \in T_n \mid a_i \in k, i = 1, 2, \ldots, n \right\}. \tag{3}$$

Then, R is clearly a commutative k-subalgebra of T_n, and $C_{T_n}(R) = R$. Thus, $R \in \mathcal{M}_n(k)$. □

Example 4 (R.C. Courter): Let $\mathcal{C} = kI_{14} \bigoplus J \subseteq T_{14}$, where J is the set of all matrices of the following form:

$$\left(\begin{array}{cc|cccccccccc|cc}
\multicolumn{2}{c|}{O_{2\times 2}} & \multicolumn{10}{c|}{O_{2\times 10}} & \multicolumn{2}{c}{O_{2\times 2}} \\ \hline
x_{11} & O & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
O & x_{11} & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
x_{12} & O & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
O & x_{12} & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
x_{21} & O & \multicolumn{10}{c|}{O_{10\times 10}} & \multicolumn{2}{c}{O_{10\times 2}} \\
O & x_{21} & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
x_{22} & O & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
O & x_{22} & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
z_{11} & z_{12} & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\
z_{21} & z_{22} & \multicolumn{10}{c|}{} & \multicolumn{2}{c}{} \\ \hline
y_{11} & y_{12} & z_{11} & z_{12} & z_{21} & z_{22} & O & O & O & O & x_{11} & x_{12} & \multicolumn{2}{c}{O_{2\times 2}} \\
y_{21} & y_{22} & O & O & O & O & z_{11} & z_{12} & z_{21} & z_{22} & x_{21} & x_{22} & \multicolumn{2}{c}{}
\end{array}\right). \tag{4}$$

In Equation (4), x_{ij}, y_{ij}, and $z_{ij} \in k$. In [6], R.C.Courter showed $\mathcal{C} \in \mathcal{M}_{14}(k)$. Notice $dim_k(\mathcal{C}) = 13$. We will call $\mathcal{C}$ Courter's algebra. □

Recall that a commutative ring R is called a local ring if R has precisely one maximal ideal. If R is a local ring, then the Jacobson radical $J(R)$ of R is the unique maximal of R. $R/J(R)$ is called the residue class field of R. We shall use the notation (R, J, k) to indicate that R is a local ring with maximal ideal J and residue class field k. For example, the Schur algebras discussed in Example 2 are all local rings with maximal ideal $J = \left\{ \begin{pmatrix} O_{p\times p} & Z \\ O_{q\times p} & O_{q\times q} \end{pmatrix} \mid Z \in M_{p\times q}(k) \right\}$ and residue class field k.

If M is a module over a commutative ring R, then M is called a faithful, R-module if $Ann_R(M) = (0)$. Here, $Ann_R(M) = \{r \in R \mid rM = (0)\}$ is the annihilator of M. For example, $k^4 = M_{1\times 4}(k)$ is a faithful, $\mathcal{B}$-module via right multiplication.

Let $\mathbf{X}$ denote the category whose objects are ordered pairs (G, H), where G is a finite dimensional, local, commutative, k-algebra and H is a finitely generated, faithful, G-module. If $(G, H), (G', H')$ are two objects in $\mathbf{X}$, then a morphism from (G, H) to (G', H') is an ordered pair (σ, τ), where $\sigma : G \longrightarrow G'$ is a k-algebra homomorphism, $\tau : H \longrightarrow H'$ is a k-vector space homomorphism and $\tau(hg) = \tau(h)\sigma(g)$ for all $h \in H$ and $g \in G$. We will use the notation $(\sigma, \tau) : (G, H) \longrightarrow (G', H')$ to indicate the morphism (σ, τ) from (G, H) to (G', H'). We call a morphism $(\sigma, \tau) : (G, H) \longrightarrow (G', H')$ an isomorphism if σ is a k-algebra isomorphism and τ is a k-vector space isomorphism. In this case we will use the notation $(G, H) \cong_{(\sigma,\tau)} (G', H')$. The reader can easily check that (σ, τ) is an isomorphism if and only if (σ, τ) is an isomorphism in the category $\mathbf{X}$.

Let $(G, H) \in \mathbf{X}$. We denote the set of G-module homomorphisms from H to H by $Hom_G(H, H)$. Since G is a commutative ring, each $g \in G$ determines a G-module endomorphism μ_g of H given by $\mu_g(h) = hg$ for $h \in H$. We then have a map $\varphi : G \longrightarrow Hom_G(H, H)$ given by $\varphi(g) = \mu_g$. Note that φ is a k-algebra homomorphism. The map φ is called the regular representation of G given by H. We say the G-module has a small endomorphism ring if $Hom_G(H, H) \cong G$ via the regular representation. Let $\mathcal{M}\mathbf{X}$ denote the full subcategory of $\mathbf{X}$ whose objects are those $(G, H) \in \mathbf{X}$ for which the G-module H has a small endomorphism ring. If $(G, H) \in \mathbf{X}$, then $(G, H) \in \mathcal{M}\mathbf{X}$ if and only if $Hom_G(H, H) \cong G$ via the regular representation. Consider the following examples.

Example 5: Let G be a local, commutative, k-algebra with $dim_k(G) = n$.

Then, G is a finitely generated, faithful, G-module. Notice that $Hom_G(G, G) \cong G$ via the regular representation. Thus, G has a small endomorphism ring G, and $(G, G) \in \mathcal{M}\mathbf{X}$. □

Example 6: If R is a commutative, k-subalgebra of T_n, then $V = k^n = M_{1\times n}(k)$ is a finitely generated, faithful, (right) R-module via the usual matrix multiplication. In particular, if $V = k^{14}$ and $R = \mathcal{C}$, Courter's algebra described in Example 4, then $(\mathcal{C}, V) \in \mathbf{X}$. Since $\mathcal{C} \in \mathcal{M}_{14}(k)$, it follows from [4, Proposition 1] that $(\mathcal{C}, V) \in \mathcal{M}\mathbf{X}$. □

In Example 6, we begin to see the connection between algebras in $\mathcal{M}_n(k)$ and objects in $\mathcal{M}\mathbf{X}$. Let R be a k-subalgebra of T_n and set $V = k^n$. Then, V is a finitely generated, faithful, R-module with scalar multiplication given by vr for $v \in V, r \in R$. Set $\mathcal{H} = Hom_k(V, V)$. Let $\mu : T_n \longrightarrow \mathcal{H}$ be the representation given by $\mu(A)(v) = vA$. Notice that μ is an anti k-algebra isomorphism of T_n onto $\mathcal{H}$. For each $i = 1, 2, \ldots, n$, set $\varepsilon_i = (0, \ldots, 1, \ldots, 0) \in V$. We will call $\underline{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ the canonical basis of V. Let $\Gamma : \mathcal{H} \longrightarrow T_n$ denote the matrix representation of elements of $\mathcal{H}$ via $\underline{\varepsilon}$. Thus, if $f \in \mathcal{H}$ and $f(\varepsilon_i) = \sum_{j=1}^{n} a_{ij}\varepsilon_j, i = 1, \ldots, n$, then $\Gamma(f) = (a_{ij})$. Γ is an anti k-algebra isomorphism of $\mathcal{H}$ onto T_n. The reader can easily check that $\Gamma\mu = 1_{T_n}$, the identity map of T_n. We have now constructed the following sequence of k-algebras and anti k-algebra isomorphisms.

$$T_n \xrightarrow{\mu} \mathcal{H} \xrightarrow{\Gamma} T_n \quad \text{with} \quad \Gamma\mu = 1_{T_n}. \tag{5}$$

For any commutative, k-subalgebra $R \subseteq T_n$, $\mu(C_{T_n}(R)) = C_{\mathcal{H}}(\mu(R))$. Here, $C_{\mathcal{H}}(\mu(R))$ is the centralizer in $\mathcal{H}$ of $\mu(R)$. Likewise, $\Gamma(C_{\mathcal{H}}(\mu(R))) = C_{Tn}(\Gamma\mu(R)) = C_{T_n}(R)$. Now, suppose $R \in \mathcal{M}_n(k)$. Then, $R = C_{T_n}(R)$ and $C_{\mathcal{H}}(\mu(R)) = Hom_R(V, V)$. Since R is commutative, $\mu : R \longrightarrow \mu(R)$ is a homomorphism. Hence, the map

$$R = C_{T_n}(R) \xrightarrow{\mu} \mu(R) = \mu(C_{T_n}(R)) = C_{\mathcal{H}}(\mu(R)) = Hom_R(V, V) \tag{6}$$

given in Equation (6) is a k-algebra isomorphism. This map is just the regular representation of R afforded by V. Therefore, V_R (the right R-module V) has a small endomorphism ring. Thus, if R is local and $R \in \mathcal{M}_n(k)$, then $(R, V) \in \mathcal{M}\mathbf{X}$.

Conversely, suppose R is a commutative, k-subalgebra of T_n such that V_R has a small endomorphism ring. Then, $C_{\mathcal{H}}(\mu(R)) = \mu(R)$ and $C_{T_n}(R) = \Gamma(C_{\mathcal{H}}(\mu(R))) = \Gamma\mu(R) = R$. Thus, $R \in \mathcal{M}_n(k)$. Hence, if $(R, V) \in \mathcal{M}\mathbf{X}$, then R is a local and $R \in \mathcal{M}_n(k)$. In summary, if R is a local, k-subalgebra of T_n, then $R \in \mathcal{M}_n(k)$ if and only if $(R, V) \in \mathcal{M}\mathbf{X}$.

Isomorphism classes in the category $\mathcal{M}\mathbf{X}$ correspond to isomorphism classes of local, k-algebras in $\mathcal{M}_n(k)$. Local algebras in $\mathcal{M}_p(k)$ with $p \leq n$ are the fundamental building blocks of algebras in general in $\mathcal{M}_n(k)$. To see this, let $R \in \mathcal{M}_n(k)$. Since $dim_k(R) < \infty$, R is an artinian ring. It follows from [9: Theorem 3, p205] that $R = \oplus_{i=1}^{\ell} R_i$, a finite direct sum of artinian, local rings R_i, $i = 1, \ldots, \ell$. Since R contains the identity, $V = VR = \oplus_{i=1}^{\ell} VR_i$. Set $V_i = VR_i, i = 1, \ldots, \ell$. Then, $V = \oplus_{i=1}^{\ell} V_i$. and each V_i is a finitely generated, faithful, R_i-module. Notice $R \cong Hom_R(V, V) \cong \prod_{i,j=1}^{\ell} Hom_R(V_i, V_j) = \prod_{i=1}^{\ell} Hom_{Ri}(V_i, V_i)$. It follows that $R_i \cong Hom_{Ri}(V_i, V_i)$. Hence, $R_i \in \mathcal{M}_{n_i}(k)$, where $n_i = dim_k(V_i)$, $i = 1, \ldots, \ell$. Thus, $R_i \in \mathcal{M}_n(k)$ can be decomposed into local, maximal, commutative subalgebras of smaller dimensions. Thus, it suffices to study maximal, commutative subalgebras which are local to understand the structure of maximal, commutative subalgebras in general. We will use the notation $(R, J(R), k) \in \mathcal{M}_n(k)$ to denote a local, commutative, k-algebra $R \in \mathcal{M}_n(k)$ which has $J(R)$ as its Jacobson radical and k as its residue class field. If R is clear, then we will use J instead of $J(R)$.

In [7], M.Gerstenhaber conjectured that $dim_k(R) \geq n$ for any $R \in \mathcal{M}_n(k)$. In [6], R.C. Courter constructed an algebra $\mathcal{C} \in \mathcal{M}_{14}(k)$ which is local, $dim_k(\mathcal{C}) = 13$, and

$i(J(\mathcal{C})) = 3$. Here, $i(J(\mathcal{C}))$ is the index of nilpotency of the ideal $J(\mathcal{C})$. Courter's counterexample to Gerstenhaber's conjecture is minimal with respect to both n and $i(= i(J(R)))$. In [8], T.J. Laffey showed that $dim_k(R) \geq n$ for $R \in \mathcal{M}_n(k)$ if $n \leq 13$. Thus, $n = 14$ is the smallest integer for which $dim_k(R)$ can be less than n. In [6], R.C. Courter showed that $i(J(R)) \leq 2$ implies that $dim_k(R) \geq n$ for any $R \in \mathcal{M}_n(k)$. Thus, $i = 3$ is the smallest index of nilpotency for which $dim_k(R)$ can be less than n for $R \in \mathcal{M}_n(k)$. The existence of Courter's example in $\mathcal{M}_{14}(k)$ suggests some interesting questions about the isomorphism classes of local algebras in $\mathcal{M}_{14}(k)$. For example, one could ask if Courter's example is unique up to isomorphism. To be more specific, is $(\mathcal{C}, V)$ unique up to (σ, τ)-isomorphism in $\mathcal{M}\mathbf{X}$? It turns out the Courter's example is not unique and we will construct another example in this thesis. Let B be a commutative ring and M a right B-module. The direct sum $B \oplus M$ of the B-modules B and M can be given the structure of a commutative ring by defining multiplication in the following way.

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, m_2 b_1 + m_1 b_2), b_i \in B, m_i \in M, i = 1, 2. \tag{7}$$

The commutative ring thus defined is called the idealization of M and will be denoted by $B \ltimes M$.

Suppose $R \in \mathcal{M}_n(k)$. We say R is a (B, N)-construction if R is k-isomorphic to $B \ltimes N^\ell$ for some $(B, N) \in \mathbf{X}$ and $\ell \in \mathbb{N}$. Here, N^ℓ denotes the direct sum of ℓ copies of B-module N. The B-module $B^\ell \oplus N$ is a $B \ltimes N^\ell$-module with scalar multiplication defined as follows.

$$(b_1, \ldots, b_\ell, n)(b, n_1, \ldots, n_\ell) = (b_1 b, \ldots, b_\ell b, nb + \sum_{i=1}^{\ell} n_i b_i). \tag{8}$$

It is easy to check that $B^\ell \oplus N$ is a finitely generated, faithful, $B \ltimes N^\ell$-module. In [3: Theorem 2], W.C. Brown and F.W. Call showed that the $B \ltimes N^\ell$-module $B^\ell \oplus N$

has a small endomorphism ring. Thus, $(B \ltimes N^{\ell}, B^{\ell} \oplus N) \in \mathcal{M}\mathbf{X}$ for all $(B, N) \in \mathbf{X}$. We call $(G, H) \in \mathbf{X}$ a C_1-construction if $(G, H) \cong_{(\sigma,\tau)} (B \ltimes N^{\ell}, B^{\ell} \oplus N)$ for some $(B, N) \in \mathbf{X}$ and $\ell \in \mathbb{N}$. In [3], W.C.Brown and F.W.Call showed that Courter's algebra $\mathcal{C}$ is a (B, N)-construction. In [4], W.C.Brown proved that $(\mathcal{C},V)$ is a C_1-construction. In fact, $(\mathcal{C},V) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$, where $\mathcal{B}$ is the Schur algebra of size 4, $N = k^4$, and $V = k^{14}$.

Let $\Omega = \{(R, J, k) \in \mathcal{M}_{14}(k) \mid dim_k(R) = 13, i(J) = 3\}$. Example 4 shows that $\mathcal{C} \in \Omega$. We are interested in how many algebras in Ω are (B, N)-constructions. To be more specific, we are interested in how many algebras in Ω are $(\mathcal{B},N)$-constructions, where $\mathcal{B}$ is the Schur algebra of size 4 defined in Equation (2). To this end, let $M\mathcal{B}(4)$ denote the class of all faithful, $\mathcal{B}$-modules of vector space dimension 4. Let $\Omega_{\mathcal{B}} = \{(R, J, k) \in \Omega \mid R \cong \mathcal{B} \ltimes N^2$ for some $N \in M\mathcal{B}(4)\}$. We have noted that $\mathcal{C} \in \Omega_{\mathcal{B}}$. Let $\Omega C_1 = \{(R, V) \in \mathcal{M}\mathbf{X} \mid (R, V) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$ for some $N \in M\mathcal{B}(4)\}$. We have noted that $(\mathcal{C},k^{14}) \in \Omega C_1$.

In Chapter 2, we will prove that if $R, R' \in \Omega_{\mathcal{B}}$, then $R \cong R' \cong \mathcal{C}$ as k-algebras. Thus, $\Omega_{\mathcal{B}}$ has only one k-algebra isomorphism class $[\mathcal{C}]$. If $(R, V), (R', V) \in \Omega C_1$, then we will show that $(R, V) \cong_{(\sigma,\tau)} (R', V) \cong_{(\sigma_1,\tau_1)} (\mathcal{C}, k^{14})$. Thus, ΩC_1 has only one (σ, τ)-isomorphism class $[(\mathcal{C},k^{14})]$.

In Chapter 3, we will study the following question: If $(R, J, k) \in \Omega$, is R a (B, N)-construction and is (R, k^{14}) a C_1-construction? If $k = \mathbb{R}$, the real numbers, then we will construct a k-algebra $(\mathcal{S}, J, k) \in \Omega$ that is not a (B, N)-construction. Furthermore, we will prove that $(\mathcal{S}, k^{14})$ is not a C_1-construction in Chapter 3. Finally, we will prove that $\mathcal{S}$ is not k-algebra isomorphic to $\mathcal{C}$. and conclude that Ω has at least two k-algebra isomorphism classes $[\mathcal{S}]$ and $[\mathcal{C}]$.

Chapter 2

Uniqueness of Algebras in Ω_β and ΩC_1

2.1 Algebras in Ω

In this section, we will prove two important theorems about the algebras in set Ω. Let $R \in \Omega$. By replacing R with a suitable k-algebra isomorphic copy, we can assume the elements in J have a particularly simple form. Let R_1 and R_2 be k-algebras in T_n. If $R_1 = P^{-1}R_2P$ for some $P \in GL(n, k)$, then we say R_1 and R_2 are conjugate.

Lemma 2.1: *Let $R_1 \in \mathcal{M}_n(k)$ and let R_2 be a commutative, k-subalgebra of T_n. If R_1 and R_2 are conjugate, then $R_2 \in \mathcal{M}_n(k)$.*

Proof: Since R_2 is commutative, it is enough to show that $C_{T_n}(R_2) \subseteq R_2$. Let $r \in C_{T_n}(R_2)$. Since R_2 is conjugate to R_1, $R_2 = P^{-1}R_1P$ for some $P \in GL(n, k)$. Hence, $r(P^{-1}r_1P) \in (P^{-1}r_1P)r$ for all $r_1 \in R_1$. Therefore, $(PrP^{-1})r_1 = r_1(PrP^{-1})$. Thus, $PrP^{-1} \in C_{T_n}(R_1) = R_1$. Hence, $r \in P^{-1}R_1P = R_2$. □

Lemma 2.2: *Let $R_1 \in \mathcal{M}_n(k)$ and let R_2 be a commutative, k-subalgebra of T_n. If $R_2 = R_1^T$ (transpose of R_1), then $R_2 \in \mathcal{M}_n(k)$.*

Proof: Again it is enough to show that $C_{T_n}(R_2) \subseteq R_2$. Let $r \in C_{T_n}(R_2)$. since $R_2 = R_1^T$, $rr_1^T = r_1^Tr$ for all $r_1 \in R_1$. Then, $r_1r^T = r^Tr_1$ and hence $r^T \in C_{T_n}(R_1) = R_1$. Thus, $r \in R_1^T = R_2$. □

Theorem 2.3: *Let $(R, J, k) \in \Omega$. Then, there exists $(R_1, J(R_1), k) \in \Omega$ such that R and R_1 are conjugate and each element $r \in J(R_1)$ is a matrix of the following form*

$$\begin{pmatrix} O_2 & O & O \\ A & O_{10} & O \\ C & B & O_2 \end{pmatrix}. \tag{9}$$

Here, O_n denotes the zero matrix of size n by n, $A \in M_{10\times 2}(k), B \in M_{2\times 10}(k)$, and $C \in T_2$.

Proof: Let $V = k^{14}$, $V_1 = (0) :_V J = \{v \in V \mid vJ = (0)\}, V_2 = (0) :_V J^2$, $p = dim_k(V_1), q = dim_k(V_2/V_1)$, and $\ell = 14 - p - q$. Since $i(J) = 3$, $VJ^2 \subseteq V_1 \subseteq V_2$. Suppose $p = 0$. Then, $V_1 = (0)$ and consequently, $VJ^2 = (0)$. Since V is a faithful, R-module, $J^2 = (0)$. This is impossible since $i(J) = 3$. Thus, $p \geq 1$. Suppose $q = 0$. Then, $VJ \subseteq V_2 = V_1$ and again $VJ^2 = (0)$. This is a contradiction. Hence, $q \geq 1$. In [8: p 203], T.J. Laffey showed that

$$dim_k(R) \geq \frac{q(p+\ell)}{1+p\ell} + 1 + p\ell. \tag{10}$$

Let

$$f(\ell, p, q) = \frac{q(p+\ell)}{1+p\ell} + 1 + p\ell = \frac{14(p+\ell) + (p^2-1)(\ell^2-1)}{1+p\ell}. \tag{11}$$

Since $dim_k(R) = 13, f(\ell, p, q) \leq 13$. An easy computation shows $\ell = p = 2$ and $q = 10$ are the only positive integers satisfying the inequality in (10). Since $dim_k(V_1) = p = 2, V = L(\alpha_1, \alpha_2)$, i.e. V is a linear span of two linearly independent vectors α_1, α_2. Similarly, $V_2 = L(\alpha_1, \ldots, \alpha_{12})$ and $V = L(\alpha_1, \ldots, \alpha_{14})$ for some k-basis $\Delta = \{\alpha_1, \ldots, \alpha_{14}\}$ of V. Then, for any $\alpha \in V, \alpha = \sigma_{i=1}^{14} \alpha_i x_i$ for some $x_i \in k, i = 1, \ldots, 14$. If $r \in J$, then $\alpha_i r = 0$ for $i = 1, 2, \alpha_i r \in V_1$ for $i = 1, \ldots, 12$, and $\alpha_i r \in V_2$ for $i = 13, 14$. Thus, each $r \in J$ has the following matrix representation with respect to the basis Δ.

$$\Gamma_\Delta(R) = \begin{pmatrix} O_2 & O & O \\ A(r) & O_{10} & O \\ C(r) & B(r) & O_2 \end{pmatrix}. \tag{12}$$

Here, $A(r) \in M_{10\times 2}(k), B(r) \in M_{2\times 10}(k)$, and $C(r) \in T_2$.

Let $R_1 = \Gamma_\Delta(R)$. The matrix representation of each $r \in R$ with respect to $\underline{\varepsilon}$ is r and with respect to Δ is $\Gamma_\Delta(R)$. Hence, there exists $P \in GL(14, k)$ such that $P^{-1}rP = \Gamma_\Delta(r)$ for all $r \in R$. Thus, $P^{-1}RP = R_1$. Since $R \cong R_1$ as k-algebras, $dim_k(R_1) = 13$ and $i(J(R_1)) = 3$. By Lemma 2.1, $R_1 \in \mathcal{M}_{14}(k)$. Thus, $(R_1, J(R_1), k) \in \Omega$ and the elements in $J(R_1)$ have the form given in (9). □

For an algebra $R \in \Omega$, Theorem 2.3 has the following interpretation. Any algebra conjugate to R lies in the isomorphism class $[R]$ of R. Hence, in studying $[R]$, we can assume the elements in $J(R)$ are described as in Equation (9). We will use those ideas to study the socle of an algebra R in Ω.

Let R be a commutative, k-algebra with Jacobson radical J and $dim_k(R) < \infty$. The socle of R, $Soc(R)$, is the annihilator of J. Thus, $Soc(R) = Ann_R(J) = \{r \in R \mid rJ = (0)\}$. The following Lemma is obvious from the definition.

Lemma 2.4: *Let R and R_1 be finite dimensional, commutative, k-algebras. If $R \cong R_1$ as k-algebras, then $Soc(R) \cong Soc(R_1)$.* □

Theorem 2.5: *Suppose $(R, J, k) \in \Omega$. Then, $dim_k(Soc(R)) = 4$. Furthermore, R is conjugate to an $(R_1, J(R_1), k) \in \Omega$ such that each element of $Soc(R_1)$ has the following form.*

$$(13) \qquad r = \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ C(r) & O & O_2 \end{pmatrix}.$$

Proof: Using Theorem 2.3, we may assume that each $r \in J$ has the form in Equation (9). Let $V = k^{14}$. Since $i(J) = 3$, we have the following strict containments.

$$(14) \qquad (0) < (0) :_{VJ} J < VJ < V.$$

Let $W_1 = (0) :_{VJ} J$, $W_2 = VJ$, $\mu = dim_k(W_1)$, $\lambda = dim_k(W_2/W_1)$, and $\nu = dim_k(V/W_2)$. Then, $\mu + \lambda + \nu = 14$. Let

$$\check{E}_{ij} = \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ E_{ij} & O & O_2 \end{pmatrix}, \quad 1 \leq i, j \leq 2. \tag{15}$$

In (15), E_{ij} is the i, j-th matrix unit of T_2. Equation (9) implies $\check{E}_{ij} r = 0 = r\check{E}_{ij}$ for all $r \in J(R)$ and $1 \leq i, j \leq 2$. Thus, $\check{E}_{ij} \in R$. Clearly, $\check{E}_{ij} \in Soc(R)$ for $1 \leq i, j \leq 2$ and hence, $dim_k(Soc(R)) \geq 4$.

Since $\varepsilon_1 = \varepsilon_{13}\check{E}_{11}$ and $\varepsilon_2 = \varepsilon_{14}\check{E}_{12}$, $\varepsilon_i \in W_2$ for $i = 1, 2$. Since $\varepsilon_1 J = \varepsilon_2 J = (0)$, $\varepsilon_i \in W_1, i = 1, 2$. Thus, $\mu \geq 2$. We had seen from Theorem 2.3 that $dim_k((0) :_V J) = 2$. Since $W_1 \subseteq (0) :_V J, \mu \leq 2$. Therefore, $\mu = 2$. The strict inclusions in Equation (14) imply that $\lambda \geq 1, \nu \geq 1$. Since $\{\varepsilon_1, \varepsilon_2\}$ is a k-vector space basis of W_1, we can extend it to bases of W_2 and V. Let $\{\beta_1, \ldots, \beta_\lambda, \varepsilon_1, \varepsilon_2\}$ be a basis of W_2 and $\Delta = \{\gamma_1, \ldots, \gamma_\nu, \beta_1, \ldots, \beta_\lambda, \varepsilon_1, \varepsilon_2\}$ be a basis of V. If $r \in J$, then $\varepsilon_i r = 0$ for $i = 1, \ldots, \nu$. Thus, we have the following matrix representation of $r \in J$ with respect to the basis Δ.

$$\Gamma_\Delta(r) = \begin{pmatrix} O_\nu & A(r) & C(r) \\ O & O_\lambda & B(r) \\ O & O & O_2 \end{pmatrix}. \tag{16}$$

Let $R_1 = \Gamma_\Delta(r)$. Since $R_1 = P^{-1}RP$ for some $P \in GL(14, k), R_1 \cong R$ as k-algebras. Thus, by Lemma 2.1, $R_1 \in \Omega$. Let $R_2 = R_1^T$, the transpose of R_2. Then, each element $r \in J(R_2)$ is of the form

$$r = \begin{pmatrix} O_\nu & O & O \\ A_1(r) & O_\lambda & O \\ C_1(r) & B_1(r) & O_2 \end{pmatrix}. \tag{17}$$

Here, $A_1(r) = A(r)^T, B_1(r) = B(r)^T$, and $C_1(r) = C(r)^T$. Since R_1 is commutative, $R_1 \cong R_2$ as k-algebras. By Lemma 2.2, $(R_2, J(R_2), k) \in \Omega$.

Let $r \in Soc(R_2)$. From (17), $\varepsilon_i \in (0) :_V J(R_2)$ for $i = 1, \ldots, \nu$. Thus, $dim_k((0) :_V J(R_2)) \geq \nu$. Since $(R_2, J(R_2), k) \in \Omega$, the proof of Theorem 2.3 implies that $dim_k((0) :_V J(R_2)) = 2$. Thus, $\nu \leq 2$.

Let $r \in Soc(R)$. Then, $\varepsilon_i r = 0$ for $i = 1, 2, \beta_i r = 0$ for $i = 1, \ldots, \lambda$ and $\gamma_i r \in W_1$ for $i = 1, \ldots, \nu$. Thus, we have the following matrix representation of $r \in Soc(R)$ with respect to the basis Δ

$$\Gamma_\Delta(r) = \begin{pmatrix} O_\nu & O & C(r) \\ O & O_\lambda & O \\ O & O & O_2 \end{pmatrix}. \tag{18}$$

By Lemma 2.4, $4 \leq dim_k(Soc(R)) = dim_k(Soc(R_1)) \leq 2\nu \leq 4$.. Therefore, $dim_k(Soc(R)) = 4$ and each element $r \in J$ has the form in Equation (13). □

Putting Theorem 2.3 and 2.5 together, we can always assume that a specific representative R of an isomorphism class $[R]$ has the following form. Every element $r \in J(R)$ can be written in the form

$$r = \begin{pmatrix} O_2 & O & O \\ A & O_{10} & O \\ C & B & O_2 \end{pmatrix}. \tag{19}$$

Furthermore, the socle of R is the set of all matrices of the form

$$Soc(R) = \left\{ \begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ C & O & O_2 \end{pmatrix} \mid C \in T_2 \right\}. \tag{20}$$

2.2 Classification of isomorphism classes in $\Omega_{\mathcal{B}}$ and ΩC_1

The reader will recall that $\Omega_{\mathcal{B}} = \{(R, J, k) \in \Omega \mid R \cong \mathcal{B} \ltimes N^2$ for some $N \in M\mathcal{B}(4)\}$, where $\mathcal{B}$ is the Schur algebra defined in Equation (2). We had noticed that the

Courter's example $(\mathcal{C}, J(\mathcal{C}), k) \in \Omega_{\mathcal{B}}$. In [3: Example 5], Brown and Call showed that $\mathcal{C} \cong \mathcal{B} \ltimes (k^4)^2$. This leads to a natural question about the role k^4 is playing in this example. One can ask whether other finitely generated, faithful, $\mathcal{B}$-modules $N \in M\mathcal{B}(4)$ give algebras $\mathcal{B} \ltimes N^2$ which determine other isomorphism classes in $\Omega_{\mathcal{B}}$? In this section, we will show $\Omega_{\mathcal{B}}$ has only one isomorphism class $[\mathcal{C}]$. Thus, varing N in $M\mathcal{B}(4)$ yields no new isomorphism classes in Ω.

The reader will also recall that $\Omega C_1 = \{(R, V) \in \mathcal{M}\mathbf{X} |\ (R, V) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$ for some $N \in M\mathcal{B}(4)\ \}$. In this section, we will show that the set ΩC_1 has only one (σ, τ)-isomorphism class $[(\mathcal{C}, k^{14})]$.

The questions above make sense because $M\mathcal{B}(4)$ has at least two isomorphism classes. To see this, we first need a $\mathcal{B}$-module presentation of k^4. We will denote the i, j-th matrix unit of T_4 by E_{ij}. Notice that $E_{ij} \in \mathcal{B}$ if $i = 1, 2, j = 3, 4$.

Lemma 2.6: *Let*

$$A = \begin{pmatrix} E_{23} & E_{24} & E_{13} & E_{14} & O & O \\ -E_{13} & -E_{14} & O & O & E_{23} & E_{24} \end{pmatrix} \in M_{2\times 6}(\mathcal{B}). \tag{21}$$

Then, $\mathcal{B}^2/CS(A) \in M\mathcal{B}(4)$.

Proof: Obviously, $\mathcal{B}^2/CS(A)$ is a finitely generated, $\mathcal{B}$-module. Since $dim_k(\mathcal{B}^2) = 10$ and $dim_k(CS(A)) = 6, dim_k(\mathcal{B}^2/CS(A)) = 4$. Suppose $r \in Ann_{\mathcal{B}}(\mathcal{B}^2/CS(A))$. Then, $r\begin{pmatrix} I_4 \\ O \end{pmatrix}, r\begin{pmatrix} O \\ I_4 \end{pmatrix} \in CS(A)$. Thus, $\begin{pmatrix} r \\ O \end{pmatrix}, \begin{pmatrix} O \\ r \end{pmatrix} \in CS(A)$ which implies that for some $x_i, y_j \in \mathcal{B}$, $1 \leq i, j \leq 6$

$$\begin{aligned} r &= x_1E_{23} + x_2E_{24} + x_3E_{13} + x_4E_{14} \\ 0 &= -x_1E_{13} - x_2E_{14} + x_5E_{23} + x_6E_{24} \\ 0 &= y_1E_{23} + y_2E_{24} + y_3E_{13} + y_4E_{14} \\ r &= -y_1E_{13} - y_2E_{14} + y_5E_{23} + y_6E_{24} \end{aligned} \tag{22}$$

Since $J(\mathcal{B})^2 = (0)$, we can assume $x_i, y_j \in k = kI_4$ for $1 \leq i, j \leq 6$. The second and third equations in (22) imply $x_1, x_2, x_5, x_6, y_1, y_2, y_3, y_4$ are all zero. Thus, $r = x_3E_{13} + x_4E_{14} = y_5E_{23} + y_6E_{24}$. Therefore, $r = 0$. Hence, $Ann_{\mathcal{B}}(\mathcal{B}^2/CS(A)) = (0)$ and $\mathcal{B}^2/CS(A)$ is a faithful, $\mathcal{B}$-module. □

Lemma 2.7: *Let A be the matrix in Equation (21). Then $\mathcal{B}^2/CS(A)$ is $\mathcal{B}$-module isomorphic to k^4.*

Proof: Let $f : \mathcal{B}^2 \longrightarrow k^4$ be the map defined by $f\begin{pmatrix} x \\ y \end{pmatrix} = \varepsilon_2 x + \varepsilon_1 y$. Here, $\varepsilon_1 = (1,0,0,0)$ and $\varepsilon_2 = (0,1,0,0)$. Then, f is a surjective, $\mathcal{B}$-module homomorphism. If $\begin{pmatrix} z \\ w \end{pmatrix} \in ker\ f$, then $z = a_1I_4 + a_2E_{13} + a_3E_{14} + a_4E_{23} + a_5E_{24}$ and $w = b_1I_4 + b_2E_{13} + b_3E_{14} + b_4E_{23} + b_5E_{24}$ for some $a_i, b_i \in k, i = 1, \ldots, 5$. Since $f\begin{pmatrix} z \\ w \end{pmatrix} = \varepsilon_2 z + \varepsilon_1 w = 0, a_1 = b_1 = 0, b_2 = -a_4$, and $b_3 = -a_5$.

Thus,

$$\begin{aligned}\begin{pmatrix} z \\ w \end{pmatrix} &= a_2\begin{pmatrix} E_{13} \\ O \end{pmatrix} + a_3\begin{pmatrix} E_{14} \\ O \end{pmatrix} + a_4\begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix} \\ &+ a_5\begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} + b_4\begin{pmatrix} O \\ E_{23} \end{pmatrix} + b_5\begin{pmatrix} O \\ E_{24} \end{pmatrix}\end{aligned} \tag{23}$$

Hence, $\begin{pmatrix} z \\ w \end{pmatrix} \in CS(A)$. It is easy to check that $CS(A) \subseteq kerf$. Therefore, $CS(A) = kerf$. Hence, $\mathcal{B}^2/CS(A) \cong k^4$ as $\mathcal{B}$-modules. □

We can now construct a faithful, $\mathcal{B}$-module of dimension 4 which is not isomorphic to k^4 as $\mathcal{B}$-modules.

Theorem 2.8: *Let*

$$C = \begin{pmatrix} E_{13} & E_{14} & E_{23} & E_{24} & O & O \\ E_{24} & E_{23} & O & O & E_{13} & E_{14} \end{pmatrix} \in M_{2\times 6}(\mathcal{B}). \tag{24}$$

Then, $\mathcal{B}^2/CS(C) \in M\mathcal{B}(4)$ and $\mathcal{B}^2/CS(C)$ is not $\mathcal{B}$-module isomorphic to k^4.

Proof: Obviously, $\mathcal{B}^2/CS(C)$ is a finitely generated, $\mathcal{B}$-module. Since $dim_k(\mathcal{B}^2) = 10$ and $dim_k(CS(C)) = 6$, $dim_k(\mathcal{B}^2/CS(C)) = 4$. Suppose $r \in Ann_{\mathcal{B}}(\mathcal{B}^2/CS(C))$. Then, $\begin{pmatrix} r \\ O \end{pmatrix}, \begin{pmatrix} O \\ r \end{pmatrix} \in CS(C)$ which implies that for some $x_i, y_j \in \mathcal{B}$, $1 \leq i, j \leq 6$

$$
\begin{aligned}
r &= x_1E_{13} + x_2E_{14} + x_3E_{23} + x_4E_{24} \\
0 &= x_1E_{24} + x_2E_{23} + x_5E_{13} + x_6E_{14} \\
0 &= y_1E_{13} + y_2E_{14} + y_3E_{23} + y_4E_{24} \\
r &= y_1E_{24} + y_2E_{23} + y_5E_{13} + y_6E_{14}
\end{aligned} \tag{25}
$$

Since $J(\mathcal{B})^2 = (0)$, we can assume $x_i, y_j \in k = kI_4$ for $1 \leq i, j \leq 6$. The second and third equations in (25) imply $x_1, x_2, x_5, x_6, y_1, y_2, y_3, y_4$ are all zero. Thus, $r = x_3E_{23}+x_4E_{24} = y_5E_{13}+y_6E_{14}$. Therefore, $r = 0$. Hence, $Ann_{\mathcal{B}}(\mathcal{B}^2/CS(C)) = (0)$ and $\mathcal{B}^2/CS(C) \in M\mathcal{B}(4)$.

Suppose $\mathcal{B}^2/CS(C)$ is $\mathcal{B}$-module isomorphic to k^4. Then, there exists a $\mathcal{B}$-module isomorphism $g : \mathcal{B}^2/CS(C) \longrightarrow k^4$. Let $\beta_1 = \begin{pmatrix} I_4 \\ O \end{pmatrix}^- = \begin{pmatrix} I_4 \\ O \end{pmatrix} + CS(C) \in \mathcal{B}^2/CS(C)$. and $\beta_2 = \begin{pmatrix} O \\ I_4 \end{pmatrix}^-$. Then, $\mathcal{B}^2/CS(C) = \beta_1\mathcal{B}+\beta_2\mathcal{B}$. Since $k^4 = \varepsilon_1\mathcal{B} + \varepsilon_2\mathcal{B}, g(\beta_1) = \varepsilon_1x_1 + \varepsilon_2y_1$ and $g(\beta_2) = \varepsilon_1x_2 + \varepsilon_2y_2$ for some $x_i, y_i \in \mathcal{B}$, $i = 1, 2$. Notice that x_1 or y_1 is unit. To see this, suppose $x_1, y_1 \in J(\mathcal{B})$. Then, $g(\beta_1) = \varepsilon_1x_1 + \varepsilon_2y_1 \in k^4J(\mathcal{B})$. The inclusions

$$
k^4 = g(\beta_1)\mathcal{B} + g(\beta_2)\mathcal{B} \subseteq k^4J(\mathcal{B}) + g(\beta_2)J(\mathcal{B}) \subseteq k^4 \tag{26}
$$

imply $k^4 = k^4J(\mathcal{B}) + g(\beta_2)J(\mathcal{B})$. By Nakayama's Lemma, $k^4 = g(\beta_2)J(\mathcal{B})$. This implies $\mathcal{B}$ is isomorphic to k^4 as $\mathcal{B}$-modules and hence $dim_k(\mathcal{B})= 4$. Since $dim_k(\mathcal{B})= 5$, this is impossible. Hence, x_1 or y_1 is unit in $\mathcal{B}$. Similarly, x_2 or y_2 is unit.

Let A be the matrix given in Equation (21) and let f be the $\mathcal{B}$-module homomorphism given in the proof of Lemma 2.7. If $\begin{pmatrix} z \\ w \end{pmatrix} \in CS(C)$, then

$$\begin{aligned} f\begin{pmatrix} y_1z+y_2w \\ x_1z+x_2w \end{pmatrix} &= \varepsilon_1(x_1z+x_2w)+\varepsilon_2(y_1z+y_2w) \\ &= (\varepsilon_1x_1+\varepsilon_2y_1)z+(\varepsilon_1x_2+\varepsilon_2y_2)w \\ &= g(\beta_1)z+g(\beta_2)w \\ &= g(\beta_1z+\beta_2w) \\ &= g(0)=0. \end{aligned} \tag{27}$$

Thus,

$$\begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} y_1z+y_2w \\ x_1z+x_2w \end{pmatrix} \in ker f = CS(A). \tag{28}$$

Now, there are two cases to consider.

Case 1: Suppose x_1 is a unit. Since $\begin{pmatrix} E_{13} \\ E_{24} \end{pmatrix} \in CS(C)$, $\begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} E_{13} \\ E_{24} \end{pmatrix}$ $\in CS(C)$ by the Equation (28). Hence,

$$\begin{aligned} \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} E_{13} \\ E_{24} \end{pmatrix} &= a_1\begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix} + a_2\begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} + a_3\begin{pmatrix} E_{13} \\ O \end{pmatrix} \\ &= a_4\begin{pmatrix} E_{14} \\ O \end{pmatrix} + a_5\begin{pmatrix} O \\ E_{23} \end{pmatrix} + a_6\begin{pmatrix} O \\ E_{24} \end{pmatrix}. \end{aligned} \tag{29}$$

for some $a_i \in k, 1 \le i \le 6$ (See the comments after Equation (22)). Thus,

$$\begin{aligned} y_1E_{13}+y_2E_{24} &= a_1E_{23}+a_2E_{24}+a_3E_{13}+a_4E_{14} \\ x_1E_{13}+x_2E_{24} &= -a_1E_{13}-a_2E_{14}+a_5E_{23}+a_6E_{24}. \end{aligned} \tag{30}$$

Let $x_1 = t_1I_4 + s_1$ with $t_1 \in k$ and $s_1 \in J(\mathcal{B})$. The first equation in (30) then implies $a_1 = a_4 = 0$. The second equation in (30) then implies $t_1 = 0$. Thus, $x_1 \in J(\mathcal{B})$. Since we are assuming x_1 is a unit, this is impossible.

Case 2: Suppose y_1 is a unit. Since $\begin{pmatrix} E_{23} \\ O \end{pmatrix} \in CS(C)$, $\begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} E_{23} \\ O \end{pmatrix}$ $\in CS(C)$ by the Equation (28). Hence,

$$\begin{aligned} \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}\begin{pmatrix} E_{23} \\ O \end{pmatrix} &= b_1\begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix} + b_2\begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} + b_3\begin{pmatrix} E_{13} \\ O \end{pmatrix} \\ &+b_4\begin{pmatrix} E_{14} \\ O \end{pmatrix} + b_5\begin{pmatrix} O \\ E_{23} \end{pmatrix} + b_6\begin{pmatrix} O \\ E_{24} \end{pmatrix}. \end{aligned} \tag{31}$$

for some $b_i \in k, 1 \leq i \leq 6$. Thus,

$$\begin{aligned} y_1 E_{23} &= b_1 E_{23} + b_2 E_{24} + b_3 E_{13} + b_4 E_{14} \\ x_1 E_{23} &= -b_1 E_{13} - b_2 E_{14} + b_5 E_{23} + b_6 E_{24}. \end{aligned} \tag{32}$$

The second equation in (32) implies $b_1 = 0$ and the first equation in (32) implies $y_1 \in J(\mathcal{B})$. This is impossible. We conclude there is no $\mathcal{B}$-module isomorphism g between $\mathcal{B}^2/CS(C)$ and k^4. □

Thus, $M\mathcal{B}(4)$ has at least two isomorphism classes $[\mathcal{B}^2/CS(A)]$ and $[\mathcal{B}^2/CS(C)]$. But as we will see, the idealizations of these modules are k-algebra isomorphic.

To classify the isomorphism classes in the sets $\Omega_{\mathcal{B}}$ and ΩC_1, we need Theorem 2.9. We will denote the minimal number of generators of $\mathcal{B}$-module N by $\mu_{\mathcal{B}}(N)$.

Theorem 2.9: *Let $N \in M\mathcal{B}(4)$. Then, $\mu_{\mathcal{B}}(N) = 2$.*

Proof : Since $dim_k(N) = 4, 1 \leq \mu_{\mathcal{B}}(N) \leq 4$. Suppose $\mu_{\mathcal{B}}(N) = 1$. Then, $N = \alpha\mathcal{B}$ for some $\alpha \in N$. Let $f : \mathcal{B} \longrightarrow N$ be a map defined by $f(b) = \alpha b$ for $b \in \mathcal{B}$. Then, f is a $\mathcal{B}$-module epimorphism. If $b \in kerf$, then $\alpha b = 0$. Thus, $b \in Ann_{\mathcal{B}}(\alpha) = Ann_{\mathcal{B}}(\alpha\mathcal{B})$. Since N is a faithful, $\mathcal{B}$-module, $Ann_{\mathcal{B}}(\alpha\mathcal{B}) = (0)$. Therefore, $b = 0$ and hence f is a $\mathcal{B}$-module isomorphism. Thus, $5 = dim_k(\mathcal{B}) = dim_k(\alpha\mathcal{B}) = 4$. This is impossible. Hence, $2 \leq \mu_{\mathcal{B}}(N) \leq 4$.

Suppose $\mu_{\mathcal{B}}(N) = 4$. By Nakayama's Lemma, $\mu_{\mathcal{B}}(N) = dim_k(N/NJ(\mathcal{B}))$. Therefore, $dim_k(NJ(\mathcal{B})) = 0$. Thus, $NJ(\mathcal{B}) = (0)$. Since N is a faithful, $\mathcal{B}$-module, we conclude $J(\mathcal{B}) = (0)$. This is impossible.

Suppose $\mu_{\mathcal{B}}(N) = 3$. Then, $N = \alpha_1\mathcal{B}+\alpha_2\mathcal{B} + \alpha_3\mathcal{B}$ for some $\alpha_i, i = 1, 2, 3$. After relabeling the α_i's if need be, we can assume $\alpha_1, \alpha_2, \alpha_3$ satisfy precisely one of the following four conditions :

Case 1: $\alpha_i J(\mathcal{B}) = (0)$ for $i = 1, 2, 3$.

Case 2: $\alpha_i J(\mathcal{B}) = (0)$ for $i = 1, 2$ and $\alpha_3 J(\mathcal{B}) \neq (0)$.

Case 3: $\alpha_i J(\mathcal{B}) = (0)$ and $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 2, 3$.

Case 4: $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 1, 2, 3$.

We will show all four cases lead to a contradiction.

Case 1: Suppose $\alpha_i J(\mathcal{B}) = (0)$ for all $i = 1, 2, 3$. Then, $NJ(\mathcal{B}) = (0)$. Since N is a faithful, $\mathcal{B}$-module, $J(\mathcal{B}) = (0)$. This is impossible.

Case 2: Suppose $\alpha_i J(\mathcal{B}) = (0)$ for all $i = 1, 2$ and $\alpha_3 J(\mathcal{B}) \neq (0)$. Suppose $\alpha_3 b = 0$ for some $b \in \mathcal{B}$. If b is a unit, then $\alpha_3 = 0$. This is impossible. Thus, $b \in J(\mathcal{B})$. Hence, $b \in Ann_{\mathcal{B}}(N)$. Since N is a faithful, $\mathcal{B}$-module, we conclude $b = 0$. Thus, $Ann_{\mathcal{B}}(\alpha_3) = (0)$ and hence $\mathcal{B} \cong \alpha_3\mathcal{B} \subseteq N$ as $\mathcal{B}$-modules. Since $dim_k(\mathcal{B}) = 5$, this is impossible.

Case 3: Suppose $\alpha_1 J(\mathcal{B}) = (0)$ and $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 2, 3$. Since $\left\{\beta_1 = \begin{pmatrix} I_4 \\ O \\ O \end{pmatrix}, \beta_2 = \begin{pmatrix} O \\ I_4 \\ O \end{pmatrix}, \beta_3 = \begin{pmatrix} O \\ O \\ I_4 \end{pmatrix}\right\}$ is a free $\mathcal{B}$-module basis of $\mathcal{B}^3$, the map $\varphi : \mathcal{B}^3 \longrightarrow N$ defined by $\varphi(\sum_{i=1}^3 \beta_i b_i) = \sum_{i=1}^3 \alpha_i b_i, b_i \in \mathcal{B}, i = 1, 2, 3$ is a well defined $\mathcal{B}$-module epimorphism. Thus, $\mathcal{B}^3/ker\varphi \cong N$ as $\mathcal{B}$-modules. Since $dim_k(\mathcal{B}^3) = 15$ and $dim_k(N) = 4, dim_k(ker\varphi) = 11$. Hence, $ker\varphi$ has the following form.

$$ker\varphi = \sum_{i=1}^{11} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \mathcal{B}, \quad x_i, y_i, z_i \in \mathcal{B}, i = 1, \ldots, 11. \tag{33}$$

Furthermore, if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in ker\varphi$, then x, y, z are not units in $\mathcal{B}$. For example, suppose x is a unit in $\mathcal{B}$. Since $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in ker\varphi$, $\alpha_1 = (-1/x)(\alpha_2 y + \alpha_3 z)$. Thus, $\mu_{\mathcal{B}}(N) < 3$ which is impossible.

Since $J(\mathcal{B})^2 = (0), ker\varphi$ can be written in the following form.

$$ker\varphi = \oplus_{i=1}^{11} k \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}. \tag{34}$$

Here, $x_i, y_i, z_i \in J(\mathcal{B}), i = 1, \ldots, 11$. Since $\alpha_1 J(\mathcal{B}) = (0), (\beta_1 + ker\varphi)J(\mathcal{B}) = (0)$ in $\mathcal{B}^3/ker\varphi$. Thus, $\begin{pmatrix} J(\mathcal{B}) \\ O \\ O \end{pmatrix}$. Since $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 2, 3$, $1 \leq dim_k(Ann_{\mathcal{B}}(\alpha_i)) < 4$ for $i = 2, 3$. Therefore, we have the following six subcases to consider.

Subcase 1: $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 2, 3$

Subcase 2: $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$

Subcase 3: $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 2, 3$

Subcase 4: $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$

Subcase 5: $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$

Subcase 6: $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 3$ for $i = 2, 3$

We will show all six subcases lead to a contradiction.

Subcase 1: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 2, 3$. Let $Ann_{\mathcal{B}}(\alpha_i) = ks_i, s_i \in J(\mathcal{B}), i = 2, 3$. Then, $\begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix} \in ker\varphi$. Since $\alpha_1 J(\mathcal{B}) = (0)$, $\begin{pmatrix} J(\mathcal{B}) \\ O \\ O \end{pmatrix} \subseteq$

$ker\varphi$. Let

$$\left\{\begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \right. \tag{35}$$
$$\left. \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}\right\}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B})$ for $i = 1, \ldots, 5, x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, \ldots, 5$. Thus,

$$\left\{\delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \right.$$
$$\delta_5 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \tag{36}$$
$$\left. \delta_9 = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_4 \\ z_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_5 \\ z_5 \end{pmatrix}\right\}$$

is a basis of $ker\varphi$. Therefore, $ker\varphi$ can be written in the following form

$$ker\varphi = \begin{pmatrix} J \\ O \\ O \end{pmatrix} \oplus k \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix} \oplus k \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix} \oplus \sum_{i=1}^{5} k \begin{pmatrix} O \\ y_i \\ z_i \end{pmatrix}. \tag{37}$$

Since $dim_k(J(\mathcal{B})) = 4, \{s_2, y_1, \ldots, y_5\}$ is a linearly dependent set. Thus, there exist $d, c_1, \ldots, c_5 \in k$ not all zero such that $ds_2 + c_1y_1 + \cdots + c_5y_5 = 0$. If $c_i = 0$ for all $i = 1, \ldots, 5$, then $d \neq 0$ and $ds_2 = 0$. This implies $s_2 = 0$. This is impossible since $\begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}$ is a basis vector of $ker\varphi$. Hence, some c_i is not zero. We can assume $c_5 \neq 0$. Thus, $y_5 \in L(s_2, y_1, \ldots, y_4)$. We can repeat this proof on $s_2, y_1, \ldots, y_4$ and assume $y_4 \in L(s_2, y_1, y_2, y_3)$. Hence, we may assume $y_4, y_5 \in L(s_2, y_1, y_2, y_3)$.

Therefore, $y_4 = ds_2 + c_1y_1 + c_2y_2 + c_3y_3$ for some $d, c_1, c_2, c_3 \in k$. If $d\delta_5 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 - \delta_{10} = 0$, then $\{\delta_5, \delta_7, \delta_8, \delta_9, \delta_{10}\}$ is linearly dependent which is impossible. Thus, $d\delta_5 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 - \delta_{10} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = ts_3$ for some $t \in k$, then $d\delta_5 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 - \delta_{10} - t\delta_6 = 0$ and $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}\}$ is linearly dependent which is impossible. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi \backslash k\delta_6$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 2$. This is a contradiction.

Subcase 2: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_3$ for some $s_i \in J(\mathcal{B}), i = 1, 2, 3$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \right\} \tag{38}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B})$ for $i = 1, \ldots, 4, x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, \ldots, 4$. Thus,

$$\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \right.$$
$$\delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \tag{39}$$
$$\left. \delta_9 = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_4 \\ z_4 \end{pmatrix} \right\}$$

is a basis of $ker\varphi$ Since $dim_k(J(\mathcal{B})) = 4, \{s_3, z_1, \ldots, z_4\}$ is a linearly dependent set.

Thus, there exist $d, c_1, \dots, c_4 \in k$ not all zero such that $ds_3 + c_1 z_1 + \cdots + c_4 z_4 = 0$. If $c_i = 0$ for all $i = 1, \dots, 4$, then $d \neq 0$ and $ds_3 = 0$. This implies $s_3 = 0$. This is impossible since $\begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}$ is a basis vector of $ker\varphi$. Hence, some c_i is not zero. We can assume $c_4 \neq 0$. Thus, $z_4 = ds_3 + c_1 z_1 + c_2 z_2 + c_3 z_3$ for some $d, c_1, c_2, c_3 \in k$. If $d\delta_7 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} = 0$, then $\{\delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d\delta_7 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ y \\ O \end{pmatrix}$ with $y \neq 0$ in $J(\mathcal{B})$. If $y = t_1 s_1 + t_2 s_2$ for some $t_1, t_2 \in k$, then $d\delta_7 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} - t_1\delta_5 - t_2\delta_6 = 0$ and $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $\begin{pmatrix} O \\ y \\ O \end{pmatrix} \in ker\varphi \backslash k\delta_5 + k\delta_6$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_2)) \geq 3$. This is a contradiction.

Subcase 3: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 2, 3$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_3 + ks_4$ for some $s_i \in J(\mathcal{B}), i = 1, 2, 3, 4$. Let

$$\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right\} \tag{40}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B})$ for $i = 1, 2, 3$, $x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, 2, 3$. Thus,

$$\left\{\delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix},\right.$$
$$\delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \tag{41}$$
$$\left.\delta_9 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix}\right\}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, y_1, y_2, y_3\}$ is a linearly dependent set. Thus, there exist $d_1, d_2, c_1, c_2, c_3 \in k$ not all zero such that $d_1s_1 + d_2s_2 + c_1y_1 + c_2y_2 + c_3y_3 = 0$. If $c_i = 0$ for all $i = 1, 2, 3$, then $d_1s_1 + d_2s_2 = 0$. Since s_1, s_2 are linearly independent vectors in $J(\mathcal{B}), d_1 = d_2 = 0$. This is impossible. Thus, $c_i \neq 0$ for some $1 \leq i \leq 3$. We can assume $c_3 \neq 0$. Hence, $y_3 = d_1s_1 + d_2s_2 + c_1y_1 + c_2y_2$ for some $d_1, d_2, c_1, c_2 \in k$. If $d_1\delta_5 + d_2\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = 0$, then $\{\delta_5, \delta_6, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d_1\delta_5 + d_2\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = t_3s_3 + t_4s_4$ for some $t_3, t_4 \in k$, then $d_1\delta_5 + d_2\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t_3\delta_7 - t_4\delta_8 = 0$. This is a contradiction since the vectors in Equation (41) form a basis of $ker\varphi$. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi \backslash k\delta_7 + k\delta_8$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 3$ and this is a contradiction.

Subcase 4: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2 + ks_3$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_4$ for some $s_i \in J(\mathcal{B}), i = 1, 2, 3, 4$.

Let

$$\left\{\begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \right. \tag{42}$$
$$\left. \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}\right\}$$

be a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4$ and $x_i \in J(\mathcal{B}), x_i \in L(E_{13}, E_{14}, E_{23}, E_{24})$ for $i = 1, 2, 3$. Thus,

$$\left\{\delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \right. \tag{43}$$
$$\delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix},$$
$$\left. \delta_9 = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix} \delta_{10} = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_3 \\ z_3 \end{pmatrix}\right\}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, s_3, y_1, y_2, y_3\}$ is a linearly dependent set. Thus, there exist $d_1, d_2, d_3, c_1, c_2, c_3 \in k$ not all zero such that $d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1 + c_2y_2 + c_3y_3 = 0$. If $c_i = 0$ for all $i = 1, 2, 3$, then $d_1s_1 + d_2s_2 + d_3s_3 = 0$ Since s_1, s_2, s_3 are linearly independent vectors in $J(\mathcal{B}), d_1 = d_2 = d_3 = 0$. This is impossible. Thus, $c_i \neq 0$ for some i. We can assume $c_3 \neq 0$. Hence, $y_3 = d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1 + c_2y_2$ for some $d_1, d_2, d_3, c_1, c_2 \in k$. If $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = 0$, then $\{\delta_5, \delta_6, \delta_7, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = ts_4$ for some $t \in k$, then $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t\delta_8 = 0$. This is impossible since $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi \backslash k\delta_8$.

Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 2$ and this is a contradiction.

Subcase 5: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$. Then, $Ann_{\mathcal{B}}(\alpha_2) = ks_1 + ks_2 + ks_3$ and $Ann_{\mathcal{B}}(\alpha_3) = ks_4 + ks_5$ for some $s_i \in J(\mathcal{B})$, $i = 1, 2, 3, 4, 5$. Let

$$\begin{aligned} &\left\{ \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \right. \\ &\left. \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\} \end{aligned} \tag{44}$$

be a basis of $ker\varphi$. Since $J(\mathcal{B}) = L(E_{13}, E_{14}, E_{23}, E_{24})$ and $x_1, x_2 \in J(\mathcal{B}), x_1, x_2 \in L(E_{13}, E_{14}, E_{23}, E_{24})$. Thus,

$$\begin{aligned} &\left\{ \delta_1 = \begin{pmatrix} E_{13} \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} E_{14} \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} E_{23} \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} E_{24} \\ O \\ O \end{pmatrix}, \right. \\ &\delta_5 = \begin{pmatrix} O \\ s_1 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix}, \\ &\left. \delta_9 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ y_1 \\ z_1 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ y_2 \\ z_2 \end{pmatrix} \right\} \end{aligned} \tag{45}$$

is a basis of $ker\varphi$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, s_3, y_1, y_2\}$ is a linearly dependent set. Thus, there exist $d_1, d_2, d_3, c_1, c_2 \in k$ not all zero such that $d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1 + c_2y_2 = 0$. If $c_1 = c_2 = 0$, then $d_1s_1 + d_2s_2 + d_3s_3 = 0$. Since s_1, s_2, s_3 are linearly independent vectors in $J(\mathcal{B}), d_1 = d_2 = d_3 = 0$. This is impossible. Thus, $c_i \neq 0$ for some i. We can assume $c_2 \neq 0$. Hence, $y_2 = d_1s_1 + d_2s_2 + d_3s_3 + c_1y_1$ for some $d_1, d_2, d_3, c_1 \in k$. If $d_1\delta_5 + d_2\delta_6 + d_3\delta_7 + c_1\delta_{10} - \delta_{11} = 0$, then $\{\delta_5, \delta_6, \delta_7, \delta_{10}, \delta_{11}\}$ is linearly dependent which is impossible. Thus, $d_1\delta_5+d_2\delta_6+d_3\delta_7+c_1\delta_{10}-\delta_{11} = \begin{pmatrix} O \\ O \\ z \end{pmatrix}$ with $z \neq 0$ in $J(\mathcal{B})$. If $z = t_4s_4+t_5s_5$ for some $t_4, t_5 \in k$, then $d_1\delta_5+d_2\delta_6+d_3\delta_7+c_1\delta_{10}-\delta_{11}-t_4\delta_8-t_5\delta_9 = 0$. This is again impossible since $\{\delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent. Thus, $\begin{pmatrix} O \\ O \\ z \end{pmatrix} \in ker\varphi\backslash k\delta_8+k\delta_9$. Therefore, $dim_k(Ann_{\mathcal{B}}(\alpha_3)) \geq 3$ which is a contradiction.

Subcase 6: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 3$ for $i = 2, 3$. Note that

$$\begin{aligned} dim_k(Ann_{\mathcal{B}}(\alpha_2)) + dim_k k(Ann_{\mathcal{B}}(\alpha_3)) &= dim_k(Ann_{\mathcal{B}}(\alpha_2) + Ann_{\mathcal{B}}(\alpha_3)) \\ &\quad +dim_k(Ann_{\mathcal{B}}(\alpha_2) \cap Ann_{\mathcal{B}}(\alpha_3)). \end{aligned} \tag{46}$$

Since $dim_k(Ann_{\mathcal{B}}(\alpha_2) + Ann_{\mathcal{B}}(\alpha_3)) \leq dim_k(J(\mathcal{B})) = 4$, Equation (46) implies $dim_k(Ann_{\mathcal{B}}(\alpha_2) \cap Ann_{\mathcal{B}}(\alpha_3)) \geq 2$ Thus, there is $0 \neq b \in Ann_{\mathcal{B}}(\alpha_2) \cap Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction. We have now shown any of the subcases in Case 3 lead to a contradiction. Hence, Case 3 is impossible.

Case 4: Suppose $\alpha_i J(\mathcal{B}) \neq (0)$ for $i = 1, 2, 3$. Let $n_i = dim_k(Ann_{\mathcal{B}}(\alpha_i))$. By relabeling the $\alpha_i's$ if need be, there are ten subcases to consider.

Subcase 1: Suppose $n_i = 1$ for $i = 1, 2, 3$

Subcase 2: Suppose $n_1 = 2, n_2 = n_3 = 1$

Subcase 3: Suppose $n_1 = n_2 = 2, n_3 = 1$

Subcase 4: Suppose $n_i = 2$ for $i = 1, 2, 3$

Subcase 5: Suppose $n_1 = 3, n_2 = n_3 = 1$

Subcase 6: Suppose $n_1 = 3, n_2 = 2, n_3 = 1$

Subcase 7: Suppose $n_1 = 3, n_2 = n_3 = 2$

Subcase 8: Suppose $n_1 = n_2 = 3, n_3 = 1$

Subcase 9: Suppose $n_1 = n_2 = 3, n_3 = 2$

Subcase 10: Suppose $n_i = 3$ for $i = 1, 2, 3$.

A proof similar to that given in Case 3 will show that Subcase 1 through Subcase 9 are impossible. The reader can consult the Appendix for the details. Subcase 10 is also impossible. To see this, let V be a vector space and suppose $W_i, i = 1, 2, 3$ are subspaces of V. Suppose $dim_k(V) = n$. Then, we have the following equation which can be found in [2 : Cor.2.15, p13].

$$\begin{aligned} dim_k(W_1 \cap W_2 \cap W_3) &= n - \textstyle\sum_{i=1}^{3}(n - dim_k(W_i)) + \{(n - dim_k(W_1 + W_2)) \\ &\quad +(n - dim_k((W_1 \cap W_2) + W_3))\}. \end{aligned} \tag{47}$$

Suppose $V = \mathcal{B}$ and $W_i = Ann_{\mathcal{B}}(\alpha_i), i = 1, 2, 3$. Then, Equation (47) implies $dim_k(W_1 \cap W_2 \cap W_3) = 9 - dim_k(W_1 + W_2) - dim_k((W_1 \cap W_2) + W_3)$. Since $dim_k(W_1 + W_2) \le 4$ and $dim_k((W_1 \cap W_2) + W_3) \le 4$, we have $dim_k(W_1 \cap W_2 \cap W_3) \ge 1$. Thus, there exists $0 \ne b \in W_1 \cap W_2 \cap W_3$. Since $W_i = Ann_{\mathcal{B}}(\alpha_i), i = 1, 2, 3, \alpha_i b = 0$ for $i = 1, 2, 3$. Thus, $b \in Ann_{\mathcal{B}}(N) = (0)$ which is a contradiction.

Therefore, all four cases are impossible. Hence, we conclude $\mu_{\mathcal{B}}(N) = 2$. □

We can now show that there is only one isomorphism class in $\Omega_{\mathcal{B}}$.

Theorem 2.10: *Let A be the matrix in Equation (21) and let $Q = \mathcal{B}^2/CS(A)$. Then, $\mathcal{B} \ltimes Q^2 \cong \mathcal{B} \ltimes N^2$ as k-algebras for any $N \in M\mathcal{B}(4)$.*

Proof: Let $N \in M\mathcal{B}(4)$. Then, $\mu_{\mathcal{B}}(N) = 2$ by Theorem 2.9. Thus, $N = \alpha_1\mathcal{B} + \alpha_2\mathcal{B}$ for some $\alpha_i \in N, i = 1, 2$. Since $\left\{\gamma_1 = \begin{pmatrix} I_4 \\ O \end{pmatrix}, \gamma_2 = \begin{pmatrix} O \\ I_4 \end{pmatrix}\right\}$ is a free $\mathcal{B}$-module basis of $\mathcal{B}^2$, the map $\psi : \mathcal{B}^2 \longrightarrow N$ (defined by $\psi(\sum_{i=1}^{2} \gamma_i b_i) = \sum_{i=1}^{2} \alpha_i b_i, b_i \in \mathcal{B}, i = 1, 2)$ is a well defined surjective, $\mathcal{B}$-module homomorphism. Hence, $\mathcal{B}^2/ker\psi \cong N$ as $\mathcal{B}$-modules. Since $dim_k(\mathcal{B}^2) = 10$ and $dim_k(N) = 4, dim_k(ker\psi) = 6$. Thus, $ker\psi$

has the following form.

$$(48) \qquad ker\psi = \sum_{i=1}^{6} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \mathcal{B}.$$

Here, $x_i, y_i \in B, i = 1, \ldots, 6$. Furthermore, if $\begin{pmatrix} z \\ w \end{pmatrix} \in ker\psi$, then z and w are not units in $\mathcal{B}$. For example, if z is a unit, then $\alpha_1 z + \alpha_2 w = 0$ implies that $N = \alpha_2\mathcal{B}$. Thus, $\mu_{\mathcal{B}}(N) = 1$ and this is impossible. Therefore, $ker\psi$ has the following form.

$$(49) \qquad ker\psi = \oplus_{i=1}^{6} k \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

Here, $x_i, y_i \in J(B), i = 1, \ldots, 6$. To exhibit an isomorphism between $\mathcal{B} \ltimes Q^2$ and $\mathcal{B} \ltimes N^2$, we need to choose a good basis of $ker\psi$. We may assume $ker\psi$ has the following form.

$$(50) \quad ker\psi = k\begin{pmatrix} E_{ij} \\ y_1 \end{pmatrix} \oplus k\begin{pmatrix} E_{pq} \\ y_2 \end{pmatrix} \oplus k\begin{pmatrix} E_{mn} \\ y_3 \end{pmatrix} \oplus k\begin{pmatrix} E_{uv} \\ y_4 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_5 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_6 \end{pmatrix}.$$

Here, $y_i \in J(\mathcal{B}), 1 \leq i \leq 6$ and the ordered pairs $(i,j), (p,q), (m,n)$, and (u,v) are just $(1,3), (1,4), (2,3), (2,4)$ in some order. To see this, we proceed as follows. Since $dim_k(\mathcal{B}) = 4, \{x_1, \ldots, x_6\}$ in Equation (49) is a linearly dependent set. Thus, by replacing the $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$'s, $i = 1, \ldots, 6$ by suitable linear combination if need be, we may assume $x_5 = x_6 = 0$. It now follows that $\{x_1, x_2, x_3, x_4\}$ is a linearly independent set. For, if $\{x_1, x_2, x_3, x_4\}$ is a linearly dependent, then by the same argument, we may assume $x_4 = 0$ in (49). Then, $ker\psi$ has the following form.

$$(51) \quad ker\psi = k\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \oplus k\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \oplus k\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_4 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_5 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_6 \end{pmatrix}.$$

Since $dim_k(J(\mathcal{B})) = 4, \{y_1, \ldots, y_6\}$ is a linearly dependent set. Thus, by the same argument above, we may assume $y_1 = y_2 = (0)$. Therefore, we have

$$(52) \quad ker\psi = k\begin{pmatrix} x_1 \\ O \end{pmatrix} \oplus k\begin{pmatrix} x_2 \\ O \end{pmatrix} \oplus k\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_4 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_5 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_6 \end{pmatrix}.$$

Now, let $W_1 = kx_1 + kx_2$ and let $W_2 = ky_4 + ky_5 + ky_6$. Then, $dim_k(W_1) = 2, dim_k(W_2) = 3$, and $dim_k(W_1 + W_2) \leq 4$. Notice that

$$(53) \qquad dim_k(W_1 + W_2) + dim_k(W_1 \cap W_2) = dim_k(W_1) + dim_k(W_2).$$

Therefore, $dim_k(W_1 \cap W_2) \geq 1$. This implies that there exists $0 \neq b \in J(\mathcal{B})$ such that $b \in W_1 \cap W_2$. Thus, $\begin{pmatrix} b \\ O \end{pmatrix}, \begin{pmatrix} O \\ b \end{pmatrix} \in ker\psi$. Since $\psi\begin{pmatrix} b \\ O \end{pmatrix} = \alpha_1 b$ and $\psi\begin{pmatrix} O \\ b \end{pmatrix} = \alpha_2 b$, $\alpha_1 b = \alpha_2 b = 0$. Since $N = \alpha_1\mathcal{B}+\alpha_2\mathcal{B}$, $b \in Ann_{\mathcal{B}}(N) = (0)$. This is impossible. Thus, $\{x_1, x_2, x_3, x_4\}$ is a linearly independent set in $J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, we may assume $\{x_1, x_2, x_3, x_4\} = \{E_{13}, E_{14}, E_{23}, E_{24}\}$. Hence, $ker\psi$ can be written as Equation (50).

We next show we can rearrange the six basis vectors given in (50) so that

$$(54) \quad ker\psi = k\begin{pmatrix} z_5 \\ z_1 \end{pmatrix} \oplus k\begin{pmatrix} z_6 \\ z_2 \end{pmatrix} \oplus k\begin{pmatrix} E_{ab} \\ z_3 \end{pmatrix} \oplus k\begin{pmatrix} E_{cd} \\ z_4 \end{pmatrix} \oplus k\begin{pmatrix} O \\ z_5 \end{pmatrix} \oplus k\begin{pmatrix} O \\ z_6 \end{pmatrix}.$$

Here, $z_1, \ldots, z_6 \in J(\mathcal{B}), (a, b), (c, d)$ are distinct ordered pairs in $\{(1,3), (1,4), (2,3), (2,4)\}$. To see this, we proceed as follows. In Equation (50), $y_5, y_6 \in J(\mathcal{B})$. Thus, $y_5 = a_1E_{13} + a_2E_{14} + a_3E_{23} + a_4E_{24}$ and $y_6 = b_1E_{13} + b_2E_{14} + b_3E_{23} + b_4E_{24}$ for some $a_i, b_i \in k, 1 \leq i \leq 4$. Since $y_5 \neq 0, a_i \neq 0$ for some i. We can assume $a_1 \neq 0$. By replacing $\begin{pmatrix} O \\ y_6 \end{pmatrix}$ by $\begin{pmatrix} O \\ y_6 \end{pmatrix} - b_1a_1^{-1}\begin{pmatrix} O \\ y_5 \end{pmatrix}$ if need be, we can assume $b_1 = 0$. Hence, $y_6 = b_2E_{14} + b_3E_{23} + b_4E_{24}$. Since $y_6 \neq 0$, some $b_i \neq 0$ for $i = 2, 3, 4$. We can assume $b_2 \neq 0$. By replacing $\begin{pmatrix} O \\ y_5 \end{pmatrix}$ by $\begin{pmatrix} O \\ y_5 \end{pmatrix} - a_2b_2^{-1}\begin{pmatrix} O \\ y_6 \end{pmatrix} \neq 0$, we can assume $y_5 = a_1E_{13} + a_3E_{23} + a_4E_{24}$. Thus, $y_5 = a_1E_{13} + a_3E_{23} + a_4E_{24}$ and $y_6 = b_2E_{14} + b_3E_{23} + b_4E_{24}$ with $a_1 \neq 0$ and $b_2 \neq 0$. The ordered pair (1,3) is one of four ordered pairs appearing in (50). We can assume $(1,3) = (i,j)$. Since $a_1 \neq 0$, we can write $ker\psi$ as follows.

$$(55) \quad ker\psi = k\begin{pmatrix} y_5 \\ y_7 \end{pmatrix} \oplus k\begin{pmatrix} E_{pq} \\ y_2 \end{pmatrix} \oplus k\begin{pmatrix} E_{mn} \\ y_3 \end{pmatrix} \oplus k\begin{pmatrix} E_{uv} \\ y_4 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_5 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_6 \end{pmatrix}.$$

Here, $y_7 = a_1y_1 + a_3y_3 + a_4y_4$. The ordered pair (1,4) is one of three ordered pairs appearing in (55). We can assume $(1,4) = (p,q)$. Since $b_2 \neq 0$, we can write $ker\psi$ as follows.

$$(56)\ ker\psi = k\begin{pmatrix} y_5 \\ y_7 \end{pmatrix} \oplus k\begin{pmatrix} y_6 \\ y_8 \end{pmatrix} \oplus k\begin{pmatrix} E_{mn} \\ y_3 \end{pmatrix} \oplus k\begin{pmatrix} E_{uv} \\ y_4 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_5 \end{pmatrix} \oplus k\begin{pmatrix} O \\ y_6 \end{pmatrix}.$$

Here, $y_8 = b_2y_2 + b_3y_3 + b_4y_4$. Setting $z_1 = y_7, z_2 = y_8, z_3 = y_3, z_4 = y_4, z_5 = y_5, z_6 = y_6, (a,b) = (m,n)$, and $(c,d) = (u,v)$, we have (54). Note that, $z_1 \neq 0$. For, if not, then $\begin{pmatrix} y_5 \\ O \end{pmatrix}, \begin{pmatrix} O \\ y_5 \end{pmatrix} \in ker\psi$. This implies $y_5 \in Ann_{\mathcal{B}}(N) = (0)$ which is impossible. By the same argument, $z_2 \neq 0$. Furthermore, $\{z_1, z_2, z_5, z_6\}$ is linearly independent. For, if not, then there exist $t_i \in k, i = 1,2,3,4$, not all zero such that $t_1z_1 + t_2z_2 + t_3z_5 + t_4z_6 = 0$. Thus,

$$(57)\qquad \begin{pmatrix} t_1z_5 + t_2z_6 \\ O \end{pmatrix} = t_1\begin{pmatrix} z_5 \\ z_1 \end{pmatrix} + t_2\begin{pmatrix} z_6 \\ z_2 \end{pmatrix} + t_3\begin{pmatrix} O \\ z_5 \end{pmatrix} + t_4\begin{pmatrix} O \\ z_6 \end{pmatrix} \in ker\psi.$$

Suppose $t_1 = t_2 = 0$. Then, (57) implies $t_3 = t_4 = 0$. This is impossible. Thus, $t_1 \neq 0$ or $t_2 \neq 0$ and hence $t_1z_5 + t_2z_6 \neq 0$. Equation (57) implies $t_1z_5 + t_2z_6 \in Ann_{\mathcal{B}}(N) = (0)$ which is impossible. Therefore, $\{z_1, z_2, z_5, z_6\}$ is linearly independent. Thus, a basis for $ker\psi$ can be given as in (54) with $\{z_1, z_2, z_5, z_6\}$ is linearly independent.

Now, we are ready to define an isomorphism between $\mathcal{B}\ltimes Q^2$ and $\mathcal{B}\ltimes(\mathcal{B}^2/ker\psi)^2$. For simplicity, we will denote cosets $\begin{pmatrix} x \\ y \end{pmatrix} + CS(A)$ in Q by $\begin{pmatrix} x \\ y \end{pmatrix}^-$. We will write elements in $\mathcal{B}\ltimes Q^2$ as orders triples (b, q_1, q_2). Here, $b \in \mathcal{B}, q_1, q_2 \in Q$. It is easy to check that the following 13 elements form a k-vector space basis of $\mathcal{B}\ltimes Q^2$.

$$
(58)\quad
\begin{aligned}
\beta_1 &= (I_4, O, O), \beta_2 = (E_{13}, O, O), \beta_3 = (E_{14}, O, O)\\
\beta_4 &= (E_{23}, O, O), \beta_5 = (E_{24}, O, O), \beta_6 = (O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-, O)\\
\beta_7 &= (O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-, O), \beta_8 = (O, \begin{pmatrix} E_{23} \\ O \end{pmatrix}^-, O), \beta_9 = (O, \begin{pmatrix} E_{24} \\ O \end{pmatrix}^-, O)\\
\beta_{10} &= (O, O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-), \beta_{11} = (O, O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-), \beta_{12} = (O, O, \begin{pmatrix} E_{23} \\ O \end{pmatrix}^-)\\
\beta_{13} &= (O, O, \begin{pmatrix} E_{24} \\ O \end{pmatrix}^-).
\end{aligned}
$$

We will denote cosets $\begin{pmatrix} x \\ y \end{pmatrix} + ker\psi$ in $\mathcal{B}^2/ker\psi$ by $\begin{pmatrix} x \\ y \end{pmatrix}^-$. It is easy to check that the following 13 elements form a k-vector space basis of $\mathcal{B}\ltimes(\mathcal{B}^2/ker\psi)^2$.

$$
(59)\quad
\begin{aligned}
\delta_1 &= (I_4, O, O), \delta_2 = (-z_1, O, O), \delta_3 = (-z_2, O, O)\\
\delta_4 &= (z_5, O, O), \delta_5 = (z_6, O, O), \delta_6 = (O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-, O)\\
\delta_7 &= (O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-, O), \delta_8 = (O, \begin{pmatrix} z_5 \\ O \end{pmatrix}^-, O), \delta_9 = (O, \begin{pmatrix} z_6 \\ O \end{pmatrix}^-, O)\\
\delta_{10} &= (O, O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-), \delta_{11} = (O, O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-), \delta_{12} = (O, O, \begin{pmatrix} z_5 \\ O \end{pmatrix}^-)\\
\delta_{13} &= (O, O, \begin{pmatrix} z_6 \\ O \end{pmatrix}^-).
\end{aligned}
$$

Define a map $\sigma : \mathcal{B}\ltimes Q^2 \longrightarrow \mathcal{B} \ltimes (\mathcal{B}^2/ker\psi)^2$ by

$$
(60)\qquad \sigma(\sum_{i=1}^{13} t_i\beta_i) = \sum_{i=1}^{13} t_i\delta_i, \quad t_i \in k, i = 1, \dots, 13.
$$

Then, obviously, σ is a k-vector space isomorphism. Notice that

$$
(61)\quad
\begin{aligned}
\beta_i\beta_j &= 0 = \delta_i\delta_j \text{ for } 2 \le i \le 5, \quad j = 2,3,4,5,8,9,12,13\\
\beta_i\beta_j &= 0 = \delta_i\delta_j \text{ for } 6 \le i,j \le 13\\
\beta_i^2 &= 0 = \delta_i^2 \text{ for } 2 \le i \le 13.
\end{aligned}
$$

Furthermore, $\sigma(\beta_1\beta_i) = \sigma(\beta_i) = \delta_i = \delta_1\delta_i = \sigma(\beta_1)\sigma(\beta_i)$ for $i = 1, \ldots, 13$. Thus, to show σ is a k-algebra isomorphism it remains to show the following.

$$
\begin{aligned}
\sigma(\beta_i\beta_6) &= \sigma(\beta_i)\sigma(\beta_6),\ \sigma(\beta_i\beta_7) = \sigma(\beta_i)\sigma(\beta_7) \\
\sigma(\beta_i\beta_{10}) &= \sigma(\beta_i)\sigma(\beta_{10}),\ \sigma(\beta_i\beta_{11}) = \sigma(\beta_i)\sigma(\beta_{11}) \text{ for } i = 2,3,4,5.
\end{aligned} \tag{62}
$$

Notice that the third and fourth equations in (62) are actually the same as the first and second equations in (62) but in the third slot. Thus, we will finish the proof by verifying the first and second equations in (62).

(63)

$$
\begin{aligned}
\sigma(\beta_2\beta_6) &= \sigma(O, \begin{pmatrix} E_{13} \\ O \end{pmatrix}^-, O) = \sigma(O,O,O) = (O,O,O) = (O, \begin{pmatrix} -z_1 \\ O \end{pmatrix}^-, O) \\
&= (-z_1, O, O)(O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-, O) = \delta_2\delta_6 = \sigma(\beta_2)\sigma(\beta_6) \\
\sigma(\beta_2\beta_7) &= \sigma(O, \begin{pmatrix} O \\ E_{13} \end{pmatrix}^-, O) = \sigma(O, \begin{pmatrix} E_{23} \\ O \end{pmatrix}^-, O) = (O, \begin{pmatrix} z_5 \\ O \end{pmatrix}^-, O) \\
&= (O, \begin{pmatrix} O \\ -z_1 \end{pmatrix}^-, O) = (-z_1, O, O)(O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-, O) = \delta_2\delta_7 = \sigma(\beta_2)\sigma(\beta_7) \\
\sigma(\beta_3\beta_6) &= \sigma(O, \begin{pmatrix} E_{14} \\ O \end{pmatrix}^-, O) = \sigma(O,O,O) = (O,O,O) = (O, \begin{pmatrix} -z_1 \\ O \end{pmatrix}^-, O) \\
&= (-z_2, O, O)(O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-, O) = \delta_3\delta_6 = \sigma(\beta_3)\sigma(\beta_6) \\
\sigma(\beta_3\beta_7) &= \sigma(O, \begin{pmatrix} O \\ E_{14} \end{pmatrix}^-, O) = \sigma(O, \begin{pmatrix} E_{24} \\ O \end{pmatrix}^-, O) = (O, \begin{pmatrix} z_6 \\ O \end{pmatrix}^-, O) \\
&= (O, \begin{pmatrix} O \\ -z_2 \end{pmatrix}^-, O) = (-z_2, O, O)(O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-, O) = \delta_3\delta_7 = \sigma(\beta_3)\sigma(\beta_7) \\
\sigma(\beta_4\beta_6) &= \sigma(O, \begin{pmatrix} E_{23} \\ O \end{pmatrix}^-, O) = (O, \begin{pmatrix} z_5 \\ O \end{pmatrix}^-, O) \\
&= (z_5, O, O)(O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-, O) = \delta_4\delta_6 = \sigma(\beta_4)\sigma(\beta_6) \\
\sigma(\beta_4\beta_7) &= \sigma(O, \begin{pmatrix} O \\ E_{23} \end{pmatrix}^-, O) = \sigma(O,O,O) = (O,O,O) = (O, \begin{pmatrix} O \\ z_5 \end{pmatrix}^-, O) \\
&= (z_5, O, O)(O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-, O) = \delta_4\delta_7 = \sigma(\beta_4)\sigma(\beta_7)
\end{aligned}
$$

(63)

$$
\begin{aligned}
\sigma(\beta_5\beta_6) &= \sigma(O, \begin{pmatrix} E_{24} \\ O \end{pmatrix}^{-}, O) = (O, \begin{pmatrix} z_6 \\ O \end{pmatrix}^{-}, O) \\
&= (z_6, O, O)(O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^{-}, O) = \delta_5\delta_6 = \sigma(\beta_5)\sigma(\beta_6) \\
\sigma(\beta_5\beta_7) &= \sigma(O, \begin{pmatrix} O \\ E_{24} \end{pmatrix}^{-}, O) = \sigma(O, O, O) = (O, O, O) = (O, \begin{pmatrix} O \\ z_6 \end{pmatrix}^{-}, O) \\
&= (z_6, O, O)(O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^{-}, O) = \delta_5\delta_7 = \sigma(\beta_5)\sigma(\beta_7).
\end{aligned}
$$

Thus, $\sigma(\beta_i\beta_j) = \sigma(\beta_i)\sigma(\beta_j), 1 \leq i, j \leq 13$ and hence σ is a k-algebra isomorphism. Therefore, $\mathcal{B} \ltimes Q^2 \cong \mathcal{B} \ltimes (\mathcal{B}^2/ker\psi)^2$.

Notice that $\mathcal{B}^2/ker\psi \cong N$ as $\mathcal{B}$-modules. Let $f : \mathcal{B}^2/ker\psi \longrightarrow N$ be a $\mathcal{B}$-module isomorphism. Then, the map $\sigma' : \mathcal{B} \ltimes (\mathcal{B}^2/ker\psi)^2 \longrightarrow \mathcal{B} \ltimes N^2$ defined by $\sigma'(b, n_1, n_2) = (b, f(n_1), f(n_2))$ is a k-algebra isomorphism. Thus, $\mathcal{B}\ltimes Q^2 \cong \mathcal{B}\ltimes N^2$ as k-algebras. $\square$

In [3], Brown and Call showed that $\mathcal{C} \cong \mathcal{B}\ltimes(k^4)^2$. Thus, by Theorem 2.10, $\mathcal{C} \cong \mathcal{B}\ltimes N^2$ for any $N \in M\mathcal{B}(4)$. This implies there is only one isomorphism class $[\mathcal{C}]$ in $\Omega_{\mathcal{B}}$.

Recall $(B, N) \in \mathbf{X}$ implies $(\mathcal{B}\ltimes N^\ell, \mathcal{B}^\ell \oplus N) \in \mathcal{M}\mathbf{X}$. To classify the isomorphism classes in ΩC_1, we need the following lemma.

Lemma 2.11: *Let $(B, N), (B, M) \in \mathbf{X}$. Suppose $N \cong M$ as B-modules. Then, $(B \ltimes N^\ell, B^\ell \oplus N)$ and $(B \ltimes M^\ell, B^\ell \oplus M)$ are (σ, τ)-isomorphic.*

Proof: Let $f : N \longrightarrow M$ be a B-module isomorphism. Define a map $\sigma : B\ltimes N^\ell \longrightarrow B \ltimes M^\ell$ by $\sigma(b, n_1, \ldots, n_\ell) = (b, f(n_1), \ldots, f(n_\ell))$ for $b \in B, n_i \in N, i = 1, \ldots, \ell$. It is easy to show σ is a k-vector space isomorphism. If $(b, n_1, \ldots, n_\ell), (b', n'_1, \ldots, n'_\ell) \in$

$B \ltimes N^\ell$, then

(64)

$$\begin{aligned}
\sigma((b, n_1, \ldots, n_\ell)(b', n_1', \ldots, n_\ell')) &= \sigma(bb', n_1'b + n_1b', \ldots, n_\ell'b + n_\ell b') \\
&= (bb', f(n_1'b + n_1b'), \ldots, f(n_\ell'b + n_\ell b')) \\
&= (bb', f(n_1')b + f(n_1)b', \ldots, f(n_\ell')b + f(n_\ell)b') \\
&= (b, f(n_1), \ldots, f(n_\ell))(b', f(n_1'), \ldots, f(n_\ell')) \\
&= \sigma(b, n_1, \ldots, n_\ell)\sigma(b', n_1', \ldots, n_\ell').
\end{aligned}$$

Thus, σ is a k-algebra isomorphism. If we define a map $\tau : B^\ell \oplus N \to B^\ell \oplus M$ by $\tau(b_1, \ldots, b_\ell, n) = (b_1, \ldots, b_\ell, f(n))$. Then, τ is a k-vector space isomorphism. Obviously,

$$\tau((b_1, \ldots, b_\ell, n)(b, n_1, \ldots, n_\ell)) = \tau(b_1, \ldots, b_\ell, n)\sigma(b, n_1, \ldots, n_\ell). \tag{65}$$

Thus, $(B \ltimes N^\ell, B^\ell \oplus N) \cong_{(\sigma,\tau)} (B \ltimes M^\ell, B^\ell \oplus M)$. □

Now, we are ready to classify the isomorphism classes in ΩC_1.

Theorem 2.12: *Let Q and N be as in Theorem 2.10. Then, $(\mathcal{B} \ltimes Q^2, \mathcal{B}^2 \oplus Q) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$.*

Proof: Let

$$\begin{aligned}
\xi_1 &= (I_4, O, O),\ \xi_2 = (E_{13}, O, O),\ \xi_3 = (E_{14}, O, O) \\
\xi_4 &= (E_{23}, O, O),\ \xi_5 = (E_{24}, O, O),\ \xi_6 = (O, I_4, O) \\
\xi_7 &= (O, E_{13}, O),\ \xi_8 = (O, E_{14}, O),\ \xi_9 = (O, E_{23}, O) \\
\xi_{10} &= (O, E_{24}, O),\ \xi_{11} = (O, O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-),\ \xi_{12} = (O, O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-) \\
\xi_{13} &= (O, O, \begin{pmatrix} E_{23} \\ O \end{pmatrix}^-),\ \xi_{14} = (O, O, \begin{pmatrix} E_{24} \\ O \end{pmatrix}^-).
\end{aligned} \tag{66}$$

Then, $\{\xi_1, \ldots, \xi_{14}\}$ is a k-vector space basis of $\mathcal{B}^2 \oplus Q$ and let

(67)
$$\begin{aligned}
\eta_1 &= (I_4, O, O),\ \eta_2 = (-z_1, O, O),\ \eta_3 = (-z_2, O, O) \\
\eta_4 &= (z_5, O, O),\ \eta_5 = (z_6, O, O),\ \eta_6 = (O, I_4, O) \\
\eta_7 &= (O, -z_1, O),\ \eta_8 = (O, -z_2, O),\ \eta_9 = (O, z_5, O) \\
\eta_{10} &= (O, z_6, O),\ \eta_{11} = (O, O, \begin{pmatrix} I_4 \\ O \end{pmatrix}^-),\ \eta_{12} = (O, O, \begin{pmatrix} O \\ I_4 \end{pmatrix}^-) \\
\eta_{13} &= (O, O, \begin{pmatrix} z_5 \\ O \end{pmatrix}^-),\ \eta_{14} = (O, O, \begin{pmatrix} z_6 \\ O \end{pmatrix}^-).
\end{aligned}$$

Then, $\{\eta_1, \ldots, \eta_{14}\}$ is a k-vector space basis of $\mathcal{B}^2 \oplus (\mathcal{B}^2/ker\psi)$, where ψ is a map in Theorem 2.10. Let $\tau : \mathcal{B}^2 \oplus Q \longrightarrow \mathcal{B}^2 \oplus (\mathcal{B}^2/ker\psi)$ be the map defined by

(68)
$$\tau(\sum_{i=1}^{14} t_i \xi_i) = \sum_{i=1}^{14} t_i \eta_i, \quad t_i \in k, 1 \le i \le 14.$$

Then, τ is a k-vector space isomorphism.

Let σ be the k-algebra isomorphism in Theorem 2.10. Let $b_i, n_i \in \mathcal{B}$ for $i = 1, 2, 3, j = 1, 2, 3, 4, 5, 6$. Then,

(69)
$$\begin{aligned}
b_i &= r_i I_4 + a_i E_{13} + b_i E_{14} + c_i E_{23} + d_i E_{24} \\
n_j &= s_j I_4 + p_j E_{13} + q_j E_{14} + u_j E_{23} + v_j E_{24}
\end{aligned}$$

for some $r_i, a_i, b_i, c_i, d_i, s_j, p_j, q_j, u_j, v_j \in k, i = 1, 2, 3, j = 1, 2, 3, 4, 5, 6$. Since $\begin{pmatrix} E_{13} \\ O \end{pmatrix}$, $\begin{pmatrix} E_{14} \\ O \end{pmatrix}, \begin{pmatrix} O \\ E_{23} \end{pmatrix}, \begin{pmatrix} O \\ E_{24} \end{pmatrix} \in CS(A)$,

(70)
$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^- = \begin{pmatrix} s_1 I_4 + p_1 E_{13} + q_1 E_{14} + u_1 E_{23} + v_1 E_{24} \\ s_2 I_4 + p_2 E_{13} + q_2 E_{14} + u_2 E_{23} + v_2 E_{24} \end{pmatrix}^- = \begin{pmatrix} s_1 I_4 + u_1 E_{23} + v_1 E_{24} \\ s_2 I_4 + p_2 E_{13} + q_2 E_{14} \end{pmatrix}^-.$$

Since $\begin{pmatrix} E_{23} \\ -E_{13} \end{pmatrix}, \begin{pmatrix} E_{24} \\ -E_{14} \end{pmatrix} \in CS(A)$, Equation (70) becomes

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^- = \begin{pmatrix} s_1 I_4 + (u_1 + p_2)E_{23} + (v_1 + q_2)E_{24} \\ s_2 I_4 \end{pmatrix}^- . \tag{71}$$

By the same reasons, we have

$$\begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^- = \begin{pmatrix} s_3 I_4 + (u_3 + p_4)E_{23} + (v_3 + q_4)E_{24} \\ s_4 I_4 \end{pmatrix}^- . \tag{72}$$

and

$$\begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^- = \begin{pmatrix} s_5 I_4 + (u_5 + p_6)E_{23} + (v_5 + q_6)E_{24} \\ s_6 I_4 \end{pmatrix}^- . \tag{73}$$

Notice that

$$\begin{aligned} \tau(b_1, b_2, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-) &= \tau(b_1, 0, 0) + \tau(0, b_2, 0) + \tau(0, 0, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-) \\ &= (r_1 I_4 - a_1 z_1 - b_1 z_2 + c_1 z_5 + d_1 z_6, 0, 0) \\ &\quad +(0, r_2 I_4 - a_2 z_1 - b_2 z_2 + c_2 z_5 + d_2 z_6, 0) \\ &\quad +(0, 0, \begin{pmatrix} s_1 I_4 + (u_1 + p_2)z_5 + (v_1 + q_2)z_6 \\ s_2 I_4 \end{pmatrix}^-). \end{aligned} \tag{74}$$

Let

$$\begin{aligned} L_1 &= r_1 I_4 - a_1 z_1 - b_1 z_2 + c_1 z_5 + d_1 z_6 \\ L_2 &= r_2 I_4 - a_2 z_1 - b_2 z_2 + c_2 z_5 + d_2 z_6 \\ L_3 &= \begin{pmatrix} s_1 I_4 + (u_1 + p_2)z_5 + (v_1 + q_2)z_6 \\ s_2 I_4 \end{pmatrix}^- . \end{aligned} \tag{75}$$

Then, $\tau(b_1, b_2, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-) = (L_1, L_2, L_3)$. Notice that

(76)

$$\begin{aligned}\sigma(b_3, \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-, \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-) &= \sigma(b_3, 0, 0) + \sigma(0, \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-, 0) + \sigma(0, 0, \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-) \\ &= (r_3 I_4 - a_3 z_1 - b_3 z_2 + c_3 z_5 + d_3 z_6, 0, 0) \\ &+ (0, \begin{pmatrix} s_3 I_4 + (u_3 + p_4) z_5 + (v_3 + q_4) z_6 \\ s_4 I_4 \end{pmatrix}^-, 0) \\ &+ (0, 0, \begin{pmatrix} s_5 I_4 + (u_5 + p_6) z_5 + (v_5 + q_6) z_6 \\ s_6 I_4 \end{pmatrix}^-).\end{aligned}$$

Let

$$\begin{aligned} M_1 &= r_3 I_4 - a_3 z_1 - b_3 z_2 + c_3 z_5 + d_3 z_6 \\ M_2 &= \begin{pmatrix} s_3 I_4 + (u_3 + p_4) z_5 + (v_3 + q_4) z_6 \\ s_4 I_4 \end{pmatrix}^- \\ M_3 &= \begin{pmatrix} s_5 I_4 + (u_5 + p_6) z_5 + (v_5 + q_6) z_6 \\ s_6 I_4 \end{pmatrix}^-. \end{aligned} \tag{77}$$

Then, $\sigma(b_3, \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-, \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-) = (M_1, M_2, M_3)$. Thus,

$$\begin{aligned}\tau(b_1, b_2, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-)\sigma(b_3, \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-, \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-) &= (L_1, L_2, L_3)(M_1, M_2, M_3) \\ &= (L_1 M_1, L_2 M_1, L_3 M_1 + M_2 L_1 + M_3 L_2).\end{aligned} \tag{78}$$

Since $y_i \in J(\mathcal{B})$ for $i = 5, 6, 7, 8$, we have

(79)

$$
\begin{aligned}
L_1M_1 &= r_1(r_3I_4 - a_3z_1 - b_3z_2 + c_3z_5 + d_3z_6) - r_3(a_1z_1 + b_1z_2 - c_1z_5 - d_1z_6) \\
&= r_1r_3I_4 - (r_1a_3 + r_3a_1)z_1 - (r_1b_3 + r_3b_1)z_2 + (r_1c_3 + r_3c_1)z_5 + (r_1d_3 + r_3d_1)z_6
\end{aligned}
$$

$$
\begin{aligned}
L_2M_1 &= r_2(r_3I_4 - a_3z_1 - b_3z_2 + c_3z_5 + d_3z_6) - r_3(a_2z_1 + b_2z_2 - c_2z_5 - d_2z_6) \\
&= r_2r_3I_4 - (r_2a_3 + r_3a_2)z_1 - (r_2b_3 + r_3b_2)z_2 + (r_2c_3 + r_3c_2)z_5 + (r_2d_3 + r_3d_2)z_6
\end{aligned}
$$

$$
\begin{aligned}
L_3M_1 &= \begin{pmatrix} s_1(r_3I_4 - a_3z_1 - b_3z_2 + c_3z_5 + d_3z_6) + r_3((u_1 + p_2)z_5 + (v_1 + q_2)z_6) \\ s_2(r_3I_4 - a_3z_1 - b_3z_2 + c_3z_5 + d_3z_6) \end{pmatrix}^- \\
&= \begin{pmatrix} s_1r_3I_4 + (r_3u_1 + r_3p_2 + s_1c_3 + s_2a_3)z_5 + (r_3v_1 + r_3q_2 + s_1d_3 + s_2b_3)z_6 \\ s_2r_3I_4 \end{pmatrix}^- \\
M_2L_1 &= \begin{pmatrix} r_1(s_3I_4 + (u_3 + p_4)z_5 + (v_3 + q_4)z_6) - s_3(a_1z_1 + b_1z_2 - c_1z_5 - d_1z_6) \\ s_4(r_1I_4 - a_1z_1 - b_1z_2 + c_1z_5 + d_1z_6) \end{pmatrix}^- \\
&= \begin{pmatrix} r_1s_3I_4 + (r_1u_3 + r_1p_4 + s_3c_1 + s_4a_1)z_5 + (r_1v_3 + r_1q_4 + s_3d_1 + s_4b_1)z_6 \\ s_4r_1I_4 \end{pmatrix}^- \\
M_3L_2 &= \begin{pmatrix} r_2(s_5I_4 + (u_5 + p_6)z_5 + (v_5 + q_6)z_6) - s_5(a_2z_1 + b_2z_2 - c_2z_5 - d_2z_6) \\ s_6(r_2I_4 - a_2z_1 - b_2z_2 + c_2z_5 + d_2z_6) \end{pmatrix}^- \\
&= \begin{pmatrix} r_2s_5I_4 + (r_2u_5 + r_2p_6 + s_5c_2 + s_6a_2)z_5 + (r_2v_5 + r_2q_6 + s_5d_2 + s_6b_2)z_6 \\ s_6r_2I_4 \end{pmatrix}^-
\end{aligned}
$$

On the other hand,

(80)

$$
(b_1, b_2, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-)(b_3, \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-, \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-) = (b_1b_3, b_2b_3, \begin{pmatrix} n_1b_3 \\ n_2b_3 \end{pmatrix}^- + \begin{pmatrix} n_3b_1 \\ n_4b_1 \end{pmatrix}^- + \begin{pmatrix} n_5b_2 \\ n_6b_2 \end{pmatrix}^-).
$$

Equation (71),(72),(73) imply that

(81)

$$
\begin{aligned}
b_1b_3 &= r_1(r_3I_4 + a_3E_{13} + b_3E_{14} + c_3E_{23} + d_3E_{24}) + r_3(a_1E_{13} + b_1E_{14} + c_1E_{23} + d_1E_{24}) \\
&= r_1r_3I_4 + (r_1a_3 + r_3a_1)E_{13} + (r_1b_3 + r_3b_1)E_{14} + (r_1c_3 + r_3c_1)E_{23} + (r_1d_3 + r_3d_1)E_{24}
\end{aligned}
$$

$$
\begin{aligned}
b_2b_3 &= r_2(r_3I_4 + a_3E_{13} + b_3E_{14} + c_3E_{23} + d_3E_{24}) + r_3(a_2E_{13} + b_2E_{14} + c_2E_{23} + d_2E_{24}) \\
&= r_2r_3I_4 + (r_2a_3 + r_3a_2)E_{13} + (r_2b_3 + r_3b_2)E_{14} + (r_2c_3 + r_3c_2)E_{23} + (r_2d_3 + r_3d_2)E_{24}
\end{aligned}
$$

(81)

$$\begin{pmatrix} n_1b_3 \\ n_2b_3 \end{pmatrix}^- = \begin{pmatrix} s_1(r_3I_4 + a_3E_{13} + b_3E_{14} + c_3E_{23} + d_3E_{24}) + r_3((u_1+p_2)E_{23} + (v_1+q_2)E_{24}) \\ s_2(r_3I_4 + a_3E_{13} + b_3E_{14} + c_3E_{23} + d_3E_{24}) \end{pmatrix}^-$$
$$= \begin{pmatrix} s_1r_3I_4 + (s_1c_3 + r_3u_1 + r_3p_2 + s_2a_3)E_{23} + (s_1d_3 + r_3v_1 + r_3q_2 + s_2b_3)E_{24} \\ s_2r_3I_4 \end{pmatrix}^-$$

$$\begin{pmatrix} n_3b_1 \\ n_4b_1 \end{pmatrix}^- = \begin{pmatrix} s_3(r_1I_4 + a_1E_{13} + b_1E_{14} + c_1E_{23} + d_1E_{24}) + r_1((u_3+p_4)E_{23} + (v_3+q_4)E_{24}) \\ s_4(r_1I_4 + a_1E_{13} + b_1E_{14} + c_1E_{23} + d_1E_{24}) \end{pmatrix}^-$$
$$= \begin{pmatrix} s_3r_1I_4 + (s_3c_1 + r_1u_3 + r_1p_4 + s_4a_1)E_{23} + (s_3d_1 + r_1v_3 + r_1q_4 + s_4b_1)E_{24} \\ s_4r_1I_4 \end{pmatrix}^-$$

$$\begin{pmatrix} n_5b_2 \\ n_6b_2 \end{pmatrix}^- = \begin{pmatrix} s_5(r_2I_4 + a_2E_{13} + b_2E_{14} + c_2E_{23} + d_2E_{24}) + r_2((u_5+p_6)E_{23} + (v_5+q_6)E_{24}) \\ s_6(r_2I_4 + a_2E_{13} + b_2E_{14} + c_2E_{23} + d_2E_{24}) \end{pmatrix}^-$$
$$= \begin{pmatrix} s_5r_2I_4 + (s_5c_2 + r_2u_5 + r_2p_6 + s_6a_2)E_{23} + (s_5d_2 + r_2v_5 + r_2q_6 + s_6b_2)E_{24} \\ s_6r_2I_4 \end{pmatrix}^- .$$

From Equation (80), we have

(82)

$$\tau((b_1, b_2, \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-)(b_3, \begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-, \begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-)) = \tau(b_1b_3, 0, 0) + \tau(0, b_2b_3, 0)$$
$$+\tau(0,0,\begin{pmatrix} n_1b_3 \\ n_2b_3 \end{pmatrix}^-) + \tau(0,0,\begin{pmatrix} n_3b_1 \\ n_4b_1 \end{pmatrix}^-)$$
$$+\tau(0,0,\begin{pmatrix} n_5b_2 \\ n_6b_2 \end{pmatrix}^-).$$

Since

(83)

$$\tau(b_1b_3, 0, 0) = (r_1r_3I_4 - (r_1a_3 + r_3a_1)z_1 - (r_1b_3 + r_3b_1)z_2 + (r_1c_3 + r_3c_1)z_5 + (r_1d_3 + r_3d_1)z_6, 0, 0)$$

$$\tau(o, b_2b_3, 0) = (0, r_2r_3I_4 - (r_2a_3 + r_3a_2)z_1 - (r_2b_3 + r_3b_2)z_2 + (r_2c_3 + r_3c_2)z_5 + (r_2d_3 + r_3d_2)z_6, 0)$$

$$\tau(0,0,\begin{pmatrix} n_1b_3 \\ n_2b_3 \end{pmatrix}^-) = (0,0,\begin{pmatrix} s_1r_3I_4 + (s_1c_3 + r_3u_1 + r_3p_2 + s_2a_3)z_5 + (s_1d_3 + r_3v_1 + r_3q_2 + s_2b_3)z_6 \\ s_2r_3I_4 \end{pmatrix}^-)$$

$$\tau(0,0,\begin{pmatrix} n_3b_1 \\ n_4b_1 \end{pmatrix}^-) = (0,0,\begin{pmatrix} s_3r_1I_4 + (s_3c_1 + r_1u_3 + r_1p_4 + s_4a_1)z_5 + (s_3d_1 + r_1v_3 + r_1q_4 + s_4b_1)z_6 \\ s_4r_1I_4 \end{pmatrix}^-)$$

$$\tau(0,0,\begin{pmatrix} n_5b_2 \\ n_6b_2 \end{pmatrix}^-) = (0,0,\begin{pmatrix} s_5r_2I_4 + (s_5c_2 + r_2u_5 + r_2p_6 + s_6a_2)z_5 + (s_5d_2 + r_2v_5 + r_2q_6 + s_6b_2)z_6 \\ s_6r_2I_4 \end{pmatrix}^-).$$

Therefore, from Equation (78) to (83),

(84)

$$\tau((b_1,b_2,\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-)(b_3,\begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-,\begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-))=\tau(b_1,b_2,\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^-)\sigma(b_3,\begin{pmatrix} n_3 \\ n_4 \end{pmatrix}^-,\begin{pmatrix} n_5 \\ n_6 \end{pmatrix}^-).$$

Thus,

$$(85)\qquad (\mathcal{B} \ltimes Q^2, \mathcal{B}^2 \oplus Q) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes (\mathcal{B}^2/ker\psi)^2, \mathcal{B}^2 \oplus (\mathcal{B}^2/ker\psi)).$$

Since $\mathcal{B}^2/ker\psi \cong N$ as $\mathcal{B}$-modules, $(\mathcal{B} \ltimes (\mathcal{B}^2/ker\psi)^2, \mathcal{B}^2 \oplus (\mathcal{B}^2/ker\psi)) \cong_{(\sigma',\tau')}$ $(\mathcal{B}\ltimes N^2, \mathcal{B}^2\oplus N)$ by Lemma 2.11. Therefore, $(\mathcal{B}\ltimes Q^2, \mathcal{B}^2\oplus Q) \cong_{(\sigma_1,\tau_1)} (\mathcal{B}\ltimes N^2, \mathcal{B}^2\oplus N)$, where $\sigma_1 = \sigma'\sigma$ and $\tau_1 = \tau'\tau$. □

We have now proven the following assertion. If $N \in M\mathcal{B}(4)$, then $(\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N) \cong_{(\sigma,\tau)} (\mathcal{B} \ltimes (k^4)^2, \mathcal{B}^2 \oplus k^4) \cong_{(\sigma',\tau')} (\mathcal{C}, k^{14})$. Thus, ΩC_1 has exactly one (σ,τ)-isomorphism class $[(\mathcal{C}, k^{14})]$.

Chapter 3

Nonuniqueness of Algebras in Ω

3.1 Construction of New Algebra $\mathcal{S}$ in Ω

It has been conjectured for a long time that the set $\Omega = \{(R, J, k) \in \mathcal{M}_{14}(k) |\ dim_k R = 13 \text{ and } i(J) = 3\}$ has only one isomorphism class $[\mathcal{C}]$. It turns out the isomorphism class $[\mathcal{C}]$ is not unique. In this section, we will construct a new k-algebra $(\mathcal{S}, J, k) \in \Omega$ such that $[\mathcal{S}] \neq [\mathcal{C}]$.

If $(R, J, k) \in \Omega$, then by Theorem 2.3, we may assume every $r \in J$ has the form in (9). From Theorem 2.5, we may assume every $r \in Soc(R)$ has the form in (13). We can then write $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$, where

$$(86) \qquad \lambda_i = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ O & B_i & O_2 \end{pmatrix}, \quad i = 1, \ldots, 8.$$

Conversely, suppose R is a commutative, k-subalgebra of T_{14} of the form $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$, where $dim_k R = 13$ and $\lambda_1, \ldots, \lambda_8$ have the form given in Equation (86). (We are not assuming R is maximal). Then, R is a local ring with Jacobson radical given by $J = (\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$ and residue class field k. We will give a necessary and sufficient condition on the A_i's and B_i's which will imply $R \in \Omega$.

For a matrix $A \in M_{m\times n}(k)$, we will let $ker\ A = \{u \in M_{1\times m}(k) | uA = 0\}$ and $NS(A) = \{v \in M_{n\times 1}(k) | Av = 0\}$.

Theorem 3.1: *Let $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ be a commutative, k-subalgebra of T_{14}. We assume $dim_k R = 13$ and each λ_i has the form given in Equation* (86)*. Suppose $\bigcap_{i=1}^{8} ker(A_i) = (0)$ and $\bigcap_{i=1}^{8} NS(B_i) = (0)$. If $r \in C_{T_{14}}(R)$, then r has the following form.*

$$r = \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix} + aI_{14}, \quad a \in k. \tag{87}$$

Proof: Let $r = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} \in C_{T_{14}}(R)$. Here, $X_1, X_9 \in T_2$ and $X_5 \in T_{10}$.

Then, $r\check{E}_{ij} = \check{E}_{ij}\ r$ and

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}\begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O_2 \end{pmatrix} = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O_2 \end{pmatrix}\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} \tag{88}$$

for all $i = 1, \ldots, 8$. Thus, we have the following equations.

$$\begin{array}{llll} \text{(a)} & X_2A_i + X_3W = 0 & \text{(e)} & A_iX_3 = 0 \\ \text{(b)} & X_3B_i = 0 & \text{(f)} & X_8A_i + X_9W = WX_1 + B_iX_4 \\ \text{(c)} & X_5A_i + X_6W = A_iX_1 & \text{(g)} & X_9B_i = WX_2 + B_iX_5 \\ \text{(d)} & X_6B_i = A_iX_2 & \text{(h)} & WX_3 + B_iX_6 = 0. \end{array} \tag{89}$$

These equations hold for all $i = 1, \ldots, 8$ and all $W \in T_2$. We also have the equations obtained by replacing A_i and B_i in (a) through (h) with the zero matrix. Since $X_3W = 0$ for all $W \in T_2$, we have $X_3 = 0$. Then, (a) implies $X_2A_i = 0$ for all $i = 1, \ldots, 8$. Thus, $X_2 \in \bigcap_{i=1}^{8} ker(A_i) = (0)$. Hence, $X_2 = 0$. Equation (h) implies $B_iX_6 = 0$ for all $i = 1, \ldots, 8$. Thus, $X_6 \in \bigcap_{i=1}^{8} NS(B_i) = (0)$. Hence, $X_6 = 0$. Since $X_9W = WX_1$ for all $W \in T_2$, we have the following equations.

$$X_9E_{11} = E_{11}X_1 \qquad X_9E_{12} = E_{12}X_1$$
$$X_9E_{21} = E_{21}X_1 \qquad X_9E_{22} = E_{22}X_1. \tag{90}$$

Let $X_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $X_9 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Here, $a_{ij}, b_{ij} \in k, \quad i,j = 1,2$. Then, (90) implies $a_{11} = a_{22} = b_{11} = b_{22}$ and $a_{12} = a_{21} = b_{12} = b_{21} = 0$. Thus, $X_1 = X_9 = a_{11}I_2$. In (c), let $W = 0$. Then, $X_5A_i = A_iX_1 = A_i(a_{11}I_2) = a_{11}A_i$. Hence, $(X_5 - a_{11}I_{10})A_i = 0$, for all $i = 1, \ldots, 8$. Thus, $X_5 - a_{11}I_{10} \in \bigcap_{i=1}^{8} ker(A_i) = (0)$ which implies $X_5 = a_{11}I_{10}$. Therefore, r has the form in (86). □

Let $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ be as in Theorem 3.1. Theorem 3.1 implies any $r \in C_{T_{14}}(R)$ has the form given in (87). Notice that all matrices of the form

$$\begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ Z & O & O_2 \end{pmatrix} \text{ and } aI_{14} \tag{91}$$

are elements in $C_{T_{14}}(R)$. In the next theorem, we characterize those P's and Q's for which $r \in C_{T_{14}}(R)$.

Theorem 3.2: *Let $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ be the k-subalgebra in Theorem 3.1. Let $r = \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix} + aI_{14} \in T_{14}$. Then, $r \in C_{T_{14}}(R)$ if and only if $\begin{pmatrix} (Row_1Q)^T \\ Col_1P \\ (Row_2Q)^T \\ Col_2P \end{pmatrix} \in NS(\Lambda)$. Here, Row_iQ is the i-th row of Q, Col_iP is the i-th column of P, and $\Lambda \in M_{32\times 40}(k)$ is the following matrix.*

$$(92) \qquad \Lambda = \begin{bmatrix} \begin{pmatrix} (Col_1A_1)^T & -Row_1B_1 & O & O \\ (Col_2A_1)^T & O & O & -Row_1B_1 \\ O & -Row_2B_1 & (Col_1A_1)^T & O \\ O & O & (Col_2A_1)^T & -Row_2B_1 \end{pmatrix} \\ \vdots \\ \vdots \\ \vdots \\ \begin{pmatrix} (Col_1A_8)^T & -Row_1B_8 & O & O \\ (Col_2A_8)^T & O & O & -Row_1B_8 \\ O & -Row_2B_8 & (Col_1A_8)^T & O \\ O & O & (Col_2A_8)^T & -Row_2B_8 \end{pmatrix} \end{bmatrix}$$

Proof: Suppose $r \in C_{T_{14}}(R)$. Then, for all $i = 1, \dots, 8$,

$$(93) \quad \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix} \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O_2 \end{pmatrix} = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ W & B_i & O_2 \end{pmatrix} \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ Z & Q & O_2 \end{pmatrix}.$$

Therefore, $QA_i = B_iP$ for $i = 1, \dots, 8$. Let

$$(94) \quad \begin{aligned} A_i &= \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ \vdots & \vdots \\ a_{101}^{(i)} & a_{102}^{(i)} \end{pmatrix} \quad B_i = \begin{pmatrix} b_{11}^{(i)} & \cdots & b_{110}^{(i)} \\ b_{21}^{(i)} & \cdots & b_{210}^{(i)} \end{pmatrix}, \text{ for } i = 1, \dots, 8 \\ P &= \begin{pmatrix} p_{11} & p_{12} \\ \vdots & \vdots \\ p_{101} & p_{102} \end{pmatrix} \quad Q = \begin{pmatrix} q_{11} & \cdots & q_{110} \\ q_{21} & \cdots & q_{210} \end{pmatrix}. \end{aligned}$$

Here, $a_{mn}^{(i)}$, $b_{mn}^{(i)}$, p_{mn}, $q_{mn} \in k$. Since $QA_i = B_iP$ for all $i = 1, \dots, 8$, we have

$$(95) \quad \begin{aligned} &\textstyle\sum_{j=1}^{10} q_{1j}a_{j1}^{(i)} - \sum_{j=1}^{10} b_{1j}^{(i)}p_{j1} = 0, \qquad \sum_{j=1}^{10} q_{1j}a_{j2}^{(i)} - \sum_{j=1}^{10} b_{1j}^{(i)}p_{j2} = 0 \\ &\textstyle\sum_{j=1}^{10} q_{2j}a_{j1}^{(i)} - \sum_{j=1}^{10} b_{2j}^{(i)}p_{j1} = 0, \qquad \sum_{j=1}^{10} q_{2j}a_{j2}^{(i)} - \sum_{j=1}^{10} b_{2j}^{(i)}p_{j2} = 0. \end{aligned}$$

It is easy to check (95) is equivalent to

$$\Lambda \begin{pmatrix} (Row_1 Q)^T \\ Col_1 P \\ (Row_2 Q)^T \\ Col_2 P \end{pmatrix} = 0. \tag{96}$$

Conversely, if P and Q satisfy Equation (96), then $QA_i = B_i P$ for all $i = 1, \ldots, 8$. Hence, by Equation (93), $r \in C_{T_{14}}(R)$.

Theorem 3.3: *Let* $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ *be a commutative, k-subalgebra of* T_{14}. *We assume* $dim_k R = 13$ *and each* λ_i *has the form given in* (86). *Then, the following two statements are equivalent.*

(a) $R \in \mathcal{M}_{14}(k)$

(b) $\bigcap_{i=1}^{8} ker(A_i) = (0)$, $\bigcap_{i=1}^{8} NS(B_i) = (0)$, *and* $rank(\Lambda) = 32$.

In Theorem 3.3, Λ is the 32×40 matrix given in (92).

Proof: (a) $\Rightarrow$ (b) Let $u = (u_1, \ldots, u_{10}) \in \bigcap_{i=1}^{8} ker(A_i)$. Then,

$$\begin{pmatrix} O_2 & O & O \\ O & O_{10} & O \\ O & \binom{u}{o} & O_2 \end{pmatrix} \in Soc(R). \tag{97}$$

Theorem 2.5 implies $dim_k\ Soc(R) = 4$. The elements $\check{E}_{ij}$, $i, j = 1, 2$ are clearly in $Soc(R)$. Hence, $Soc(R) = L(\check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$. Thus, $u = 0$ and hence $\bigcap_{i=1}^{8} ker(A_i) = (0)$.

Let $v = (v_1, \ldots, v_{10})^T \in \bigcap_{i=1}^{8} NS(B_i)$. Then,

$$\begin{pmatrix} O_2 & O & O \\ (vo) & O_{10} & O \\ O & O & O_2 \end{pmatrix} \in Soc(R). \tag{98}$$

Since $Soc(R) = L(\check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$, $v = 0$. Therefore, $\bigcap_{i=1}^{8} NS(B_i) = (0)$.

Let

$$(99) \qquad \alpha_i = \begin{pmatrix} (Row_1 B_i)^T \\ Col_1 A_i \\ (Row_2 B_i)^T \\ Col_2 A_i \end{pmatrix}, \quad i = 1, \dots, 8.$$

Since $\lambda_i \in R = C_{T_{14}}(R)$, $\alpha_i \in NS(\Lambda)$ by Theorem 3.2. Since $\lambda_1, \dots, \lambda_8$ are linearly independent, $\alpha_1, \dots, \alpha_8$ are linearly independent. Hence, $dim_k NS(\Lambda) \geq 8$. Let $w \in NS(\Lambda)$. Since $w \in M_{40\times 1}(k)$, we can write w as follows.

$$(100) \qquad w = \begin{pmatrix} (Row_1 Q)^T \\ Col_1 P \\ (Row_2 Q)^T \\ Col_2 P \end{pmatrix}$$

for some $P \in M_{10\times 2}(k)$ and $Q \in M_{2\times 10}(k)$. Let

$$(101) \qquad r = \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ O & Q & O_2 \end{pmatrix}.$$

Then, by Theorem 3.2, $r \in C_{T_{14}}(R) = R$. Thus, $r = c_1\lambda_1 + \cdots + c_8\lambda_8$ for some $c_i \in k$, $i = 1, \dots, 8$. Hence, $w = c_1\alpha_1 + \cdots + c_8\lambda_8$. Therefore, $dim_k NS(\Lambda) \leq 8$ and hence $dim_k NS(\Lambda) = 8$. We conclude $rk(\Lambda) = 32$.

(b) $\Rightarrow$ (a) Since $rank(\Lambda) = 32$, $dim_k NS(\Lambda) = 8$. Let α_i, $i = 1, \dots, 8$ be the vectors defined by (99). Since $dim_k R = 13$, $\lambda_1, \dots, \lambda_8$ are linearly independent over k. It easily follows that $\alpha_1, \dots, \alpha_8$ are linearly independent over k. Thus, $\{\alpha_1, \dots, \alpha_8\}$ is a basis of $NS(\Lambda)$. If $r \in C_{T_{14}}(R)$, then Theorem 3.1 implies r has the form given in (87). Thus, by Theorem 3.2,

$$(102) \qquad \begin{pmatrix} (Row_1 Q)^T \\ Col_1 P \\ (Row_2 Q)^T \\ Col_2 P \end{pmatrix} \in NS(\Lambda).$$

This implies

$$(103) \qquad \begin{pmatrix} O_2 & O & O \\ P & O_{10} & O \\ O & Q & O_2 \end{pmatrix} \in L(\lambda_1, \ldots, \lambda_8).$$

Therefore, $r \in R$ and hence $C_{T_{14}}(R) = R$. We conclude $R \in \mathcal{M}_{14}(k)$. $\square$

Thus, we can easily check whether a k-subalgebra $R = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ of the type given in Theorem 3.3 is in Ω.

Now, we will construct a new k-algebra $(\mathcal{S}, J, k) \in \Omega$ with the following matrices. Let

$$(104) \qquad \delta_i = \begin{pmatrix} O_2 & O & O \\ P_i & O_{10} & O \\ O & Q_i & O_2 \end{pmatrix}, \quad i = 1, \ldots, 8.$$

Here,

$$P_1 = \begin{pmatrix} I_2 \\ O_2 \\ O_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} O_2 \\ I_2 \\ O_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} O_2 \\ O_2 \\ I_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad P_4 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ I_2 \\ O_2 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{11} \end{pmatrix}, \quad P_6 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{12} \end{pmatrix}, \quad P_7 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{21} \end{pmatrix}, \quad P_8 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{22} \end{pmatrix},$$

and

$$Q_1 = (I_2\ O_2\ O_2\ O_2\ E_{11}), \quad Q_2 = (O_2\ I_2\ O_2\ O_2\ E_{12})$$

$$Q_3 = (O_2\ O_2\ I_2\ O_2\ E_{21}), \quad Q_4 = (O_2\ O_2\ O_2\ I_2\ E_{22})$$

$$Q_5 = (E_{11}\ O_2\ E_{21}\ O_2\ O_2), \quad Q_6 = (E_{12}\ O_2\ E_{22}\ O_2\ O_2)$$

$$Q_7 = (O_2\ E_{11}\ O_2\ E_{21}\ O_2), \quad Q_8 = (O_2\ E_{12}\ O_2\ E_{22}\ O_2).$$

Throughout the rest of this thesis, we will let $\mathcal{S} = k[\delta_1, \ldots, \delta_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ with $\delta_1, \ldots, \delta_8$ given by (104). Notice that $\mathcal{S}$ is a k-subalgebra of T_{14}. The multiplication table for $J(\mathcal{S})$ is as follows:

Table 1: Multiplications of δ_i's

	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8
δ_1	$\check{E}_{11} + \check{E}_{22}$	O	O	O	$\check{E}_{11}$	$\check{E}_{12}$	O	O
δ_2	O	$\check{E}_{11} + \check{E}_{22}$	O	O	O	O	$\check{E}_{11}$	$\check{E}_{12}$
δ_3	O	O	$\check{E}_{11} + \check{E}_{22}$	O	$\check{E}_{21}$	$\check{E}_{22}$	O	O
δ_4	O	O	O	$\check{E}_{11} + \check{E}_{22}$	O	O	$\check{E}_{21}$	$\check{E}_{22}$
δ_5	$\check{E}_{11}$	O	$\check{E}_{21}$	O	O	O	O	O
δ_6	$\check{E}_{12}$	O	$\check{E}_{22}$	O	O	O	O	O
δ_7	O	$\check{E}_{11}$	O	$\check{E}_{21}$	O	O	O	O
δ_8	O	$\check{E}_{12}$	O	$\check{E}_{22}$	O	O	O	O

We don't include the multiplications for $\check{E}_{ij}$'s since $\check{E}_{ij}J(\mathcal{S}) = (0)$ for all $i, j = 1, 2$.

Theorem 3.4: *Let $\mathcal{S} = k[\delta_1, \ldots, \delta_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ be the k-subalgebra of T_{14} defined by the equations in* (104). *Then,*

(a) $\mathcal{S} \in \mathcal{M}_{14}(k)$

(b) $(\mathcal{S}, J, k) \in \Omega$

Proof: (a) It is easy to check that $\mathcal{S}$ is a local, commutative, k-subalgebra of T_{14} with $dim_k \mathcal{S} = 13$. Obviously, $\{P_1, \ldots, P_8\}$ is linearly independent. Furthermore, $\bigcap_{i=1}^{8} ker(P_i) = (0)$ and $\bigcap_{i=1}^{8} NS(Q_i) = (0)$. Let

$$\Lambda = \begin{bmatrix} \begin{pmatrix} (Col_1P_1)^T & -Row_1Q_1 & O & O \\ (Col_2P_1)^T & O & O & -Row_1Q_1 \\ O & -Row_2Q_1 & (Col_1P_1)^T & O \\ O & O & (Col_2P_1)^T & -Row_2Q_1 \end{pmatrix} \\ \vdots \\ \vdots \\ \vdots \\ \begin{pmatrix} (Col_1P_8)^T & -Row_1Q_8 & O & O \\ (Col_2P_8)^T & O & O & -Row_1Q_8 \\ O & -Row_2Q_8 & (Col_1P_8)^T & O \\ O & O & (Col_2P_8)^T & -Row_2Q_8 \end{pmatrix} \end{bmatrix}. \tag{105}$$

Then, $\Lambda \in M_{32\times 40}(k)$ and $rank(\Lambda) = 32$. Thus, by Theorem 3.3, $\mathcal{S} \in \mathcal{M}_{14}(k)$.

(b) We can easily check that $dim_k\mathcal{S} = 13$ and $i(J) = 3$. Thus, $(\mathcal{S}, J, k) \in \Omega$ by (a). □

In Theorem 3.4, we constructed a new k-algebra $(\mathcal{S}, J, k) \in \Omega$. In the next section, we will show $[\mathcal{S}] \neq [\mathcal{C}]$. Hence, $\mathcal{S}$ determines a new isomorphism class in Ω.

3.2 The Algebra $\mathcal{S}$

In this section, we will prove the k-algebra $\mathcal{S}$ constructed in Theorem 3.4 is not a (B, N)-construction if $k = \mathbb{R}$ and is a (B, N)-construction if k is an algebraically closed field. We will prove that $\mathcal{S}$ is not k-algebra isomorphic to $\mathcal{C}$. Therefore, we can conclude that Ω has at least two k-algebra isomorphism classes $[\mathcal{S}]$ and $[\mathcal{C}]$. It also follows that $(\mathcal{S}, k^{14})$ is not (σ, τ)-isomorphic to $(\mathcal{C}, k^{14})$. Furthermore, we will prove $(\mathcal{S}, k^{14})$ is not a C_1-construction.

Theorem 3.5: *Suppose $k = \mathbb{R}$. Then, $\mathcal{S}$ is not a (B, N)-construction.*

Proof: Suppose $\mathcal{S}$ is a (B,N)-construction. Then, by [3: Theorem 4], $\mathcal{S}$ contains an ideal I which satisfies the following two properties.

(106)

(a) $Ann_{\mathcal{S}}(I) = I$

(b) $0 \longrightarrow I \longrightarrow \mathcal{S} \xrightarrow{\nu} \mathcal{S}/I \longrightarrow 0$ splits as k-algebras, i.e., there exists a k-algebra homomorphism $\nu\theta = 1_{\mathcal{S}/I}$.

Since $\check{E}_{ij}I = 0$ for $i,j = 1,2$, $\check{E}_{ij} \in I$, $i,j = 1,2$ by (a). Notice that $\delta_1 \notin I$. Otherwise, $\delta_1^2 = 0$ by (a). Since $\delta_1^2 = \check{E}_{11} + \check{E}_{22}$, this is impossible. Thus, $\delta_1 \notin I$. Let $\theta : \mathcal{S}/I \longrightarrow \mathcal{S}$ be a splitting map. Then, $\theta(\delta_1 + I) = \delta_1 + r$, where $r \in I$. Since θ is a k-algebra homomorphism, we have

$$\begin{aligned} \delta_1^2 + 2\delta_1 r &= \delta_1^2 + 2\delta_1 r + r^2 \\ &= (\delta_1 + r)^2 = (\theta(\delta_1 + I))^2 \\ &= \theta((\delta_1 + I)^2) = \theta(\check{E}_{11} + \check{E}_{22} + I) \\ &= \theta(0 + I) = 0. \end{aligned} \tag{107}$$

Let $r = \sum_{i=1}^{8} t_i\delta_i + \sum_{j,\ell=1}^{2} s_{j\ell}\check{E}_{j\ell}$, $t_i, s_{j\ell} \in \mathbb{R}$. Then, (107) implies

$$(1 + 2t_1 + 2t_5)\check{E}_{11} + 2t_6\check{E}_{12} + (1 + 2t_1)\check{E}_{22} = 0. \tag{108}$$

Thus, $t_1 = -\frac{1}{2}$, $t_5 = t_6 = 0$. Hence, we have

$$r = -\frac{1}{2}\delta_1 + t_2\delta_2 + t_3\delta_3 + t_4\delta_4 + t_7\delta_7 + t_8\delta_8 + \sum_{j,\ell=1}^{2} s_{j\ell}\check{E}_{j\ell}. \tag{109}$$

Since $r \in I$, $r^2 = 0$ by (a). Thus,

$$\begin{aligned} &(\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_2t_7)\check{E}_{11} + 2t_2t_8\check{E}_{12} + 2t_4t_7\check{E}_{21} \\ &+(\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_4t_8)\check{E}_{22} = 0. \end{aligned} \tag{110}$$

Therefore, we have the following four equations.

$$(111)\qquad \begin{aligned} &t_2t_8 = 0\\ &t_4t_7 = 0\\ &\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_2t_7 = 0\\ &\tfrac{1}{4} + t_2^2 + t_3^2 + t_4^2 + 2t_4t_8 = 0. \end{aligned}$$

We will show that there is no real solution of the equations given in (111). Since $t_2t_8 = 0,\ t_2 = 0$ or $t_8 = 0$. Thus, we have the following two cases to consider.

Case 1: $t_2 = 0$

Case 2: $t_8 = 0$

We will show both cases lead to a contradiction.

Case 1: Suppose $t_2 = 0$. Then, from the third equation in (111), we have $\frac{1}{4} + t_3^2 + t_4^2 = 0$. This is impossible since $t_3, t_4 \in \mathbb{R}$.

Case 2: Suppose $t_8 = 0$. Then, the fourth equation in (111) implies $\frac{1}{4} + t_2^2 + t_3^2 + t_4^2 = 0$. This is again impossible since $t_2, t_3, t_4 \in \mathbb{R}$.

Thus, the equations in (111) have no real solutions. This implies that there is no $r \in I$ such that $\theta(\delta_1 + I) = \delta_1 + r$. Thus, there is no splitting map of the exact sequence given in (106). Therefore, $\mathcal{S}$ is not a (B, N)-construction. □

It was conjectured that every $R \in \mathcal{M}_{14}(k)$ is a (B, N)-construction. Theorem 3.5 implies this conjecture depends on k. If $k = \mathbb{R}$, then $\mathcal{S}$ is not a (B, N)-construction. If $k = \mathbb{C}$ (complex numbers), then $\mathcal{S}$ is a (B, N)-construction. More generally, we prove $\mathcal{S} \in \mathcal{M}_{14}(k)$ is a (B, N)-construction if k is an algebraically closed field.

Theorem 3.6 *Suppose k is an algebraically closed field. Then, $\mathcal{S}$ is a (B, N)-construction.*

Proof: Since k is an algebraically closed field, the polynomial $f(x) = x^2 + 1 \in k[x]$ has a root i. Set $\alpha_1 = \delta_1 - i\delta_2$, $\alpha_2 = \delta_3 - i\delta_4$, $\alpha_3 = \delta_5 - i\delta_7$, and $\alpha_4 = \delta_6 - i\delta_8$. Then, $\alpha_1\delta_5 = \check{E}_{11}$, $\alpha_1\delta_6 = \check{E}_{12}$, $\alpha_2\delta_5 = \check{E}_{21}$, $\alpha_2\delta_6 = \check{E}_{22}$. Thus, the ideal I generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ contains $\check{E}_{mn}$ for all $m, n = 1, 2$. It is easy to check $I = L(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22})$. Thus, $dim_k(I) = 8$.

Let $\beta \in Ann_{\mathcal{S}}(I)$. Then, $\beta = \sum_{n=1}^{8} t_n\delta_n + \sum_{m,n=1}^{2} s_{mn}\check{E}_{mn}$ for some $t_n, s_{mn} \in k$. Since $\alpha_n \in I$ for all $n = 1, 2, 3, 4$, $\alpha_n\beta = 0$ for all $n = 1, 2, 3, 4$. From $\alpha_1\beta = \alpha_2\beta = 0$, we have

$$
\begin{aligned}
&(t_1 - it_2 + t_5 - it_7)\check{E}_{11} + (t_6 - it_8)\check{E}_{12} + (t_1 - it_2)\check{E}_{22} = 0 \\
&(t_3 - it_4)\check{E}_{11} + (t_5 - it_7)\check{E}_{21} + (t_3 - it_4 + t_6 - it_8)\check{E}_{22} = 0.
\end{aligned} \tag{112}
$$

Equation (112) implies

$$
\begin{aligned}
t_1 - it_2 + t_5 - it_7 &= 0 \\
t_3 - it_4 + t_6 - it_8 &= 0 \\
t_6 - it_8 &= 0 \\
t_1 - it_2 &= 0 \\
t_3 - it_4 &= 0 \\
t_5 - it_7 &= 0
\end{aligned} \tag{113}
$$

Thus, we have $t_1 = it_2, t_3 = it_4, t_5 = it_7$, and $t_6 = it_8$. Hence,

$$
\begin{aligned}
\beta &= it_2\delta_1 + t_2\delta_2 + it_4\delta_3 + t_4\delta_4 + it_7\delta_5 + it_8\delta_6 \\
&\quad + t_7\delta_7 + t_8\delta_8 + \textstyle\sum_{m,n=1}^{2} s_{mn}\check{E}_{mn} \\
&= it_2\alpha_1 + it_4\alpha_2 + it_7\alpha_3 + it_8\alpha_4 + \textstyle\sum_{n,n=1}^{2} s_{mn}\check{E}_{mn}.
\end{aligned} \tag{114}
$$

Therefore, $\beta \in I$ and hence $Ann_{\mathcal{S}}(I) \subseteq I$. Since $I^2 = 0$, $I \subseteq Ann_{\mathcal{S}}(I)$. Thus, $Ann_{\mathcal{S}}(I) = I$.

Notice that $\Delta = \{I_{14} + I, \delta_1 + I, \delta_3 + I, \delta_5 + I, \delta_6 + I\}$ is a k-vector space basis of $\mathcal{S}/I$. Since $dim_k(I) = 8$ and $dim_k(\mathcal{S}) = 13$, we have $dim_k(\mathcal{S}/I) = 5$. Since $i\alpha_n \in I$

for all $n = 1, 2, 3, 4$, we have

$$\begin{aligned} \delta_2 + I &= -i\delta_1 + I \\ \delta_4 + I &= -i\delta_3 + I \\ \delta_7 + I &= -i\delta_5 + I \\ \delta_8 + I &= -i\delta_6 + I. \end{aligned} \tag{115}$$

Let θ be the k-vector space homomorphism from $\mathcal{S}/I$ to $\mathcal{S}$ defined as follows:

$$\begin{aligned} &\theta(I_{14} + I) = I_{14} \\ &\theta(\delta_1 + I) = \tfrac{1}{2}\delta_1 + \tfrac{1}{2}i\delta_2 \\ &\theta(\delta_3 + I) = \tfrac{1}{2}\delta_3 + \tfrac{1}{2}i\delta_4 \\ &\theta(\delta_5 + I) = \tfrac{1}{2}\delta_5 + \tfrac{1}{2}i\delta_7 \\ &\theta(\delta_6 + I) = \tfrac{1}{2}\delta_6 + \tfrac{1}{2}i\delta_8. \end{aligned} \tag{116}$$

Then,

$$\begin{aligned} \theta(\delta_2 + I) = \theta(-i\delta_1 + I) = -i\theta(\delta_1 + I) = \tfrac{1}{2}\delta_2 - \tfrac{1}{2}i\delta_1 \\ \theta(\delta_4 + I) = \theta(-i\delta_3 + I) = -i\theta(\delta_3 + I) = \tfrac{1}{2}\delta_4 - \tfrac{1}{2}i\delta_3 \\ \theta(\delta_7 + I) = \theta(-i\delta_5 + I) = -i\theta(\delta_5 + I) = \tfrac{1}{2}\delta_7 - \tfrac{1}{2}i\delta_5 \\ \theta(\delta_8 + I) = \theta(-i\delta_6 + I) = -i\theta(\delta_6 + I) = \tfrac{1}{2}\delta_8 - \tfrac{1}{2}i\delta_6 \end{aligned} \tag{117}$$

Furthermore, θ is a k-algebra homomorphism. To see this, we proceed as follows. Let $\gamma, \gamma' \in \mathcal{S}/I$. Then, $\gamma = (tI_{14} + a) + I$ and $\gamma' = (t'I_{14} + a') + I$ for some $t, t' \in k$ and $a, a' \in J(\mathcal{S})$. Note that

$$\begin{aligned} \theta(\gamma\gamma') &= \theta(((tI_{14} + a) + I)((t'I_{14} + a') + I)) \\ &= \theta(tt'I_{14} + ta' + at' + aa' + I) \\ &= \theta(tt'I_{14} + ta' + at' + I) \\ &= \theta(tt'I_{14} + I) + \theta(ta' + I) + \theta(at' + I) \\ &= tt'\theta(I_{14} + I) + t\theta(a' + I) + t'\theta(a + I) \\ &= tt'I_{14} + t\theta(a' + I) + t'\theta(a + I). \end{aligned} \tag{118}$$

and

$$\begin{aligned}\theta(\gamma)\theta(\gamma') &= \theta((tI_{14}+a)+I)\theta((t'I_{14}+a')+I)\\ &= (\theta(tI_{14}+I)+\theta(a+I))(\theta(t'I_{14}+I)+\theta(a'+I))\\ &= (tI_{14}+\theta(a+I))(t'I_{14}+\theta(a'+I))\\ &= tt'I_{14}+t\theta(a'+I)+t'\theta(a+I)+\theta(a+I)\theta(a'+I).\end{aligned}$$

Thus, it remains to show $\theta(a+I)\theta(a'+I)=0$. Let $a=\sum_{n=1}^{8}u_n\delta_n+\sum_{m,n=1}^{2}v_{mn}\check{E}_{mn}$ and $a'=\sum_{n=1}^{8}u'_n\delta_n+\sum_{m,n=1}^{2}v'_{mn}\check{E}_{mn}$, $u_n, u'_n, v_{mn}, v'_{mn}\in k$. Then, we have

$$\begin{aligned}a+I &= (u_1-iu_2)\delta_1+(u_3-iu_4)\delta_3+(u_5-iu_7)\delta_5+(u_6-iu_8)\delta_6+I\\ a'+I &= (u'_1-iu'_2)\delta_1+(u'_3-iu'_4)\delta_3+(u'_5-iu'_7)\delta_5+(u'_6-iu'_8)\delta_6+I\end{aligned} \tag{119}$$

Therefore, by (116)

$$\begin{aligned}\theta(a+I) &= \tfrac{1}{2}(u_1-iu_2)(\delta_1+i\delta_2)+\tfrac{1}{2}(u_3-iu_4)(\delta_3+i\delta_4)\\ &\quad+\tfrac{1}{2}(u_5-iu_7)(\delta_5+i\delta_7)+\tfrac{1}{2}(u_6-iu_8)(\delta_6+i\delta_8)\\ \theta(a'+I) &= \tfrac{1}{2}(u'_1-iu'_2)(\delta_1+i\delta_2)+\tfrac{1}{2}(u'_3-iu'_4)(\delta_3+i\delta_4)\\ &\quad+\tfrac{1}{2}(u'_5-iu'_7)(\delta_5+i\delta_7)+\tfrac{1}{2}(u'_6-iu'_8)(\delta_6+i\delta_8).\end{aligned} \tag{120}$$

By using Table 1, we have

$$\begin{aligned}&\theta(a+I)\theta(a'+I)=\tfrac{1}{4}((u_1-iu_2)(\delta_1+i\delta_2)+(u_3-iu_4)(\delta_3+i\delta_4)\\ &+(u_5-iu_7)(\delta_5+i\delta_7)+(u_6-iu_8)(\delta_6+i\delta_8))((u'_1-iu'_2)(\delta_1+i\delta_2)\\ &+(u'_3-iu'_4)(\delta_3+i\delta_4)+(u'_5-iu_7)(\delta_5+i\delta_7)+(u'_6-iu'_8)(\delta_6+i\delta_8))\\ &=0\end{aligned} \tag{121}$$

Recall $\nu : \mathcal{S}\longrightarrow\mathcal{S}/I$ is the natural homomorphism defined by $v(r)=r+I$ for $r\in\mathcal{S}$. Then,

$$\begin{aligned}v\theta(I_{14}+I) &= v(I_{14})=I_{14}+I\\ v\theta(\delta_1+I) &= v(\tfrac{1}{2}\delta_1+\tfrac{1}{2}i\delta_2)=\tfrac{1}{2}\delta_1+\tfrac{1}{2}i\delta_2+I\\ &= (\tfrac{1}{2}\delta_1+\tfrac{1}{2}i\delta_2)+(\tfrac{1}{2}\delta_1-\tfrac{1}{2}i\delta_2)+I\\ &= \delta_1+I\end{aligned} \tag{122}$$

$$\begin{aligned} v\theta(\delta_3+I) &= v(\tfrac{1}{2}\delta_3+\tfrac{1}{2}i\delta_4) = \tfrac{1}{2}\delta_3+\tfrac{1}{2}i\delta_4+I \\ &= (\tfrac{1}{2}\delta_3+\tfrac{1}{2}i\delta_4)+(\tfrac{1}{2}\delta_3-\tfrac{1}{2}i\delta_4)+I \\ &= \delta_3+I \end{aligned}$$

$$\begin{aligned} v\theta(\delta_5+I) &= v(\tfrac{1}{2}\delta_5+\tfrac{1}{2}i\delta_7) = \tfrac{1}{2}\delta_5+\tfrac{1}{2}i\delta_7+I \\ &= (\tfrac{1}{2}\delta_5+\tfrac{1}{2}i\delta_7)+(\tfrac{1}{2}\delta_5-\tfrac{1}{2}i\delta_7)+I \\ &= \delta_5+I \end{aligned}$$

$$\begin{aligned} v\theta(\delta_6+I) &= v(\tfrac{1}{2}\delta_6+\tfrac{1}{2}i\delta_8) = \tfrac{1}{2}\delta_6+\tfrac{1}{2}i\delta_8+I \\ &= (\tfrac{1}{2}\delta_6+\tfrac{1}{2}i\delta_8)+(\tfrac{1}{2}\delta_6-\tfrac{1}{2}i\delta_8)+I \\ &= \delta_6+I. \end{aligned}$$

Thus, it is easy to check $\nu\theta(r+I) = r+I$ for all $r \in \mathcal{S}$. This implies the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}/I \longrightarrow 0 \tag{123}$$

splits as k-algebras. Therefore, the ideal I of $\mathcal{S}$ satisfies the two conditions in [3:Theorem 4] and $\mathcal{S}$ is a (B,N)-construction. □

Theorem 3.5 and 3.6 show that the question: "When is $(R,J,k) \in \Omega$ a (B,N)-construction?" depends on the field k. From Theorem 3.6, one could conjecture that every $(R,J,k) \in \Omega$ is a (B,N)-construction if k is an algebraically closed field. At present, this conjecture is still opened.

Next we show $\mathcal{S}$ is not k-algebra isomorphic to $\mathcal{C}$. In what follows, we will need a multiplication table for $\mathcal{C}$. Let

$$\lambda_i = \begin{pmatrix} O_2 & O & O \\ A_i & O_{10} & O \\ O & B_i & O_2 \end{pmatrix}, \quad i = 1,\ldots,8. \tag{124}$$

Here,

$$A_1 = \begin{pmatrix} I_2 \\ O_2 \\ O_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} O_2 \\ I_2 \\ O_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} O_2 \\ O_2 \\ I_2 \\ O_2 \\ O_2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ I_2 \\ O_2 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{11} \end{pmatrix}, \quad A_6 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{12} \end{pmatrix}, \quad A_7 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{21} \end{pmatrix}, \quad A_8 = \begin{pmatrix} O_2 \\ O_2 \\ O_2 \\ O_2 \\ E_{22} \end{pmatrix},$$

and

$$B_1 = (O_2\ O_2\ O_2\ O_2\ E_{11}), \quad B_2 = (O_2\ O_2\ O_2\ O_2\ E_{12})$$

$$B_3 = (O_2\ O_2\ O_2\ O_2\ E_{21}), \quad B_4 = (O_2\ O_2\ O_2\ O_2\ E_{22})$$

$$B_5 = (E_{11}\ O_2\ E_{21}\ O_2\ O_2), \quad B_6 = (E_{12}\ O_2\ E_{22}\ O_2\ O_2)$$

$$B_7 = (O_2\ E_{11}\ O_2\ E_{21}\ O_2), \quad B_8 = (O_2\ E_{12}\ O_2\ E_{22}\ O_2).$$

Then, $\mathcal{C} = k[\lambda_1, \ldots, \lambda_8, \check{E}_{11}, \check{E}_{12}, \check{E}_{21}, \check{E}_{22}]$ and the multiplication table for $J(\mathcal{C})$ is as follows:

Table 2: Multiplications of λ_i's

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
λ_1	O	O	O	O	$\check{E}_{11}$	$\check{E}_{12}$	O	O
λ_2	O	O	O	O	O	O	$\check{E}_{11}$	$\check{E}_{12}$
λ_3	O	O	O	O	$\check{E}_{21}$	$\check{E}_{22}$	O	O
λ_4	O	O	O	O	O	O	$\check{E}_{21}$	$\check{E}_{22}$
λ_5	$\check{E}_{11}$	O	$\check{E}_{21}$	O	O	O	O	O
λ_6	$\check{E}_{12}$	O	$\check{E}_{22}$	O	O	O	O	O
λ_7	O	$\check{E}_{11}$	O	$\check{E}_{21}$	O	O	O	O
λ_8	O	$\check{E}_{12}$	O	$\check{E}_{22}$	O	O	O	O

We don't include the multiplications for $\check{E}_{ij}$'s since $\check{E}_{ij}J(\mathcal{C}) = (0)$ for all $i, j = 1, 2$.

We will now prove that the set $\Omega = \{(R, J, k) \in \mathcal{M}_{14}(k) | \dim_k R = 13, i(J) = 3\}$ has at least two isomorphism classes $[\mathcal{S}]$ and $[\mathcal{C}]$.

Theorem 3.7: *Let $(\mathcal{S}, J, k) \in \Omega$ be the k-algebra given in Theorem 3.4. Then, $\mathcal{S}$ is not k-algebra isomorphic to the Courter's algebra $\mathcal{C}$.*

Proof: Let $X, \ldots, X$ be indeterminates over k and set $A = k[X_1, \ldots, X_8]$. Let I be the following ideal (in A):

$$\begin{aligned} I = \ & (X_1^2, X_2^2, X_3^2, X_4^2, X_5^2, X_6^2, X_7^2, X_8^2, X_1X_2, X_1X_3, X_1X_4, X_2X_3, \\ & X_2X_4, X_3X_4, X_5X_6, X_5X_7, X_5X_8, X_6X_7, X_6X_8, X_7X_8, X_1X_7, \\ & X_1X_8, X_2X_5, X_2X_6, X_3X_7, X_3X_8, X_4X_5, X_4X_6, X_1X_5 - X_2X_7, \\ & X_1X_6 - X_2X_8, X_3X_5 - X_4X_7, X_3X_6 - X_4X_8). \end{aligned} \tag{125}$$

Let $\pi : A \longrightarrow \mathcal{C}$ be the map defined by $\pi(X_i) = \lambda_i$ for all $i = 1, \ldots, 8$. From Table 2, it is easy to check π is a surjective, k-algebra homomorphism. Table 2 also implies $I \subseteq ker\pi$. Thus, the map $\bar{\pi} : A/I \longrightarrow \mathcal{C}$ defined by $\bar{\pi}(f + I) = \pi(f)$ is a well-defined k-algebra epimorphism. Let $m = (X_1, \ldots, X_8)$. Then, $m^3 \subseteq I$ and hence $\{1 + I, X_1 + I, \ldots, X_8 + I, X_1X_5 + I, X_1X_6 + I, X_3X_5 + I, X_3X_6 + I\}$ is a k-vector space basis of A/I. Thus, $dim_k(A/I) = 13$. Since $\dim_k \mathcal{C} = 13$, $\bar{\pi}$ is a k-algebra isomorphism. Thus, $\mathcal{C} \cong A/I$ as k-algebras.

Let L be the ideal of A defined as follows:

(126)

$$\begin{aligned} L = \ & (X_5^2, X_6^2, X_7^2, X_8^2, X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4, X_5X_6, \\ & X_5X_7, X_5X_8, X_6X_7, X_6X_8, X_7X_8, X_1X_7, X_1X_8, X_2X_5, X_2X_6, X_3X_7, X_3X_8, \\ & X_4X_5, X_4X_6, X_1^2 - X_2^2, X_1^2 - X_3^2, X_1^2 - X_4^2, X_1X_5 - X_2X_7, X_1X_6 - X_2X_8, \\ & X_3X_5 - X_4X_7, X_3X_6 - X_4X_8, X_1^2 - X_1X_5 - X_3X_6). \end{aligned}$$

Let $\pi_1 : A \longrightarrow \mathcal{S}$ be a map defined by $\pi_1(X_i) = \delta_i$ for all $i = 1, \ldots, 8$. Then, π is a surjective, k-algebra homomorphism. Using Table 1, $L \subseteq ker\pi$. Hence, the map

$\bar{\pi}_1 : A/L \longrightarrow \mathcal{S}$ defined by $\bar{\pi}_1(g+L) = \pi_1(g)$ is a well-defined k-algebra epimorphism. Since $m^3 \subseteq L$, $\{1+L, X_1+L, \ldots, X_8+L, X_1^2+L, X_1X_5+L, X_1X_6+L, X_3X_5+L\}$ is a k-vector space basis of A/L. Thus, $dim_k(A/L) = 13$. Since $\dim_k \mathcal{S} = 13$, $\bar{\pi}_1$ is a k-algebra isomorphism. Hence $\mathcal{S} \cong A/L$ as k-algebras.

Suppose $\mathcal{S}$ is k-algebra isomorphic to $\mathcal{C}$. Since $\mathcal{S} \cong A/L$ and $\mathcal{C} \cong A/I$, there is a k-algebra isomorphism $\varphi : A/I \longrightarrow A/L$. Notice that $A/I, A/L$ are standard graded rings. Since $J(\mathcal{C})^3 = (0)$ and $J(\mathcal{S})^3 = (0)$, we have $A/I = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 = k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2$ and $A/L = \mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \mathcal{S}_2 = k \oplus \mathcal{S}_1 \oplus \mathcal{S}_2$. Here, $\mathcal{C}_n$ and $\mathcal{S}_n$ are the n-th homogeneous components of A/I and A/L, respectively. Since $\mathcal{C}$ and $\mathcal{S}$ are local rings, $J(A/I) = m/I$ and $J(A/L) = m/L$. Since φ is a k-algebra isomorphism, $\varphi(m/I) = \varphi(J(A/I)) = J(A/L) = m/L$. Thus, $\varphi((m/I)^n) = (m/L)^n$ for all $n > 0$. We can now define a map $\psi : gr_{m/I}(A/I) \longrightarrow gr_{m/L}(A/L)$ given by $\psi(\alpha + (m/I)^{n+1}) = \varphi(\alpha) + (m/L)^{n+1}$ for all $\alpha \in (m/I)^n$. Since $\varphi((m/I)^n) = (m/L)^n$ for all $n > 0$, ψ is well-defined, k-vector space homomorphism. Since φ is surjective, ψ is surjective. Hence, ψ is a k-vector space isomorphism.

Next we will show ψ is a k-algebra homomorphism. Let $\alpha_1 \in (m/I)^n$ and let $\alpha_2 \in (m/I)^\ell$. Then,

(127)

$$\begin{aligned}
\psi((\alpha_1 + (m/I)^{n+1})(\alpha_2 + (m/I)^{\ell+1})) &= \psi(\alpha_1\alpha_2 + (m/I)^{n+\ell+1}) \\
&= \varphi(\alpha_1\alpha_2) + (m/L)^{n+\ell+1} \\
&= \varphi(\alpha_1)\varphi(\alpha_2) + (m/L)^{n+\ell+1} \\
&= (\varphi(\alpha_1) + (m/L)^{n+1})(\varphi(\alpha_2) + (m/L)^{\ell+1}) \\
&= \psi(\alpha_1 + (m/I)^{n+1})\psi(\alpha_2 + (m/I)^{\ell+1}).
\end{aligned}$$

Thus, ψ is a k-algebra isomorphism.

Let $\rho_1 : k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \longrightarrow gr_{m/I}(A/I)$ be a map defined by $\rho_1(\alpha_n) = \alpha_n + (m/I)^{n+1}$ for all $\alpha_n \in \mathcal{C}_n$, $n = 0, 1, 2$. Then, it is easy to check that ρ_1 is a k-vector space isomorphism. Let $\alpha_n, \beta_n \in \mathcal{C}_n$. Then,

(128)

$$\begin{aligned}
\rho_1((\alpha_0+\alpha_1+\alpha_2)(\beta_0+\beta_1+\beta_2)) &= \rho_1(\alpha_0\beta_0+\alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\beta_2+\alpha_2\beta_0+\alpha_1\beta_1)\\
&= \rho_1(\alpha_0\beta_0)+\rho_1(\alpha_0\beta_1+\alpha_1\beta_0)+\rho_1(\alpha_0\beta_2+\alpha_2\beta_0+\alpha_1\beta_1)\\
&= \alpha_0\beta_0+(\alpha_0\beta_1+\alpha_1\beta_0+(m/I)^2)+(\alpha_0\beta_2+\alpha_2\beta_0+\alpha_1\beta_1\\
&\quad +(m/I)^3))\\
&= (\alpha_o+(\alpha_1+(m/I)^2)+(\alpha_2+(m/I)^3))(\beta_0+(\beta_1+(m/I)^2)\\
&\quad +(\beta_2+(m/I)^3))\\
&= \rho_1(\alpha_0+\alpha_1+\alpha_2)\rho_1(\beta_0+\beta_1+\beta_2).
\end{aligned}$$

Thus, ρ_1 is a k-algebra isomorphism.

Let $\rho_2 : k\oplus\mathcal{S}_1\oplus\mathcal{S}_2 \longrightarrow gr_{m/L}(A/L)$ be a map defined by $\rho_2(\gamma_n) = \gamma_n + (m/L)^{n+1}$ for all $\gamma_n \in \mathcal{S}_n$, $n = 0,1,2$. Then, by the same arguments above, ρ_2 is a k-algebra isomorphism.

Let $\sigma = \rho_2^{-1}\circ\psi\circ\rho_1 : k\oplus\mathcal{C}_1\oplus\mathcal{C}_2 \longrightarrow k\oplus\mathcal{S}_1\oplus\mathcal{S}_2$. Then, σ is a k-algebra isomorphism with $\sigma(\mathcal{C}_n) = \mathcal{S}_n$ for all $n = 0,1,2$. Thus, σ is a k-algebra isomorphism which is homogeneous of degree 0. Therefore, we may assume $\varphi : A/I \to A/L$ is a k-algebra isomorphism which is homogeneous of degree 0. Since $\varphi(\mathcal{C}_1) = \mathcal{S}_1$, $\varphi(X_i + I) = \sum_{j=1}^{8} a_{ij}X_j + L$ for some $a_{ij} \in k$, $i,j = 1,\ldots,8$. Here, $det(a_{ij}) \neq 0$. Let $\eta : A \longrightarrow A$ be a map defined by $\eta(X_i) = \sum_{j=1}^{8} a_{ij}X_j$ for all $i = 1,\ldots,8$. Then, by [10, Corollary 2, p137], η is an automorphism.

Let $\theta_0 : A \longrightarrow A/I$ and $\theta_1 : A \longrightarrow A/L$ be natural homomorphisms. Then the following diagram commutes.

(129)
$$\begin{array}{ccc}
A & \xrightarrow{\theta_0} & A/I\\
\eta\downarrow & & \downarrow\varphi\\
A & \xrightarrow{\theta_1} & A/L
\end{array}$$

To see this, we proceed as follows. The four maps in (129) are homogeneous of degree 0. Hence, it suffices to show that $\varphi\theta_0(f) = \theta_1\eta(f)$ for any homogeneous form f in A. Let $f(x_1, \ldots, x_n)$ be of d-form in A. Then,

$$
\begin{aligned}
\varphi(\theta_0(f(x_1,\ldots,x_n))) &= \varphi(f(x_1+I,\ldots,x_n+I)) \\
&= f(\varphi(x_1+I),\ldots,\varphi(x_n+I)) \\
&= f(\textstyle\sum_{j=1}^{8} a_{1j}x_j + L,\ldots,\sum_{j=1}^{8} a_{8j}x_j + L) \\
&= f(\textstyle\sum_{j=1}^{8} a_{1j}x_j,\ldots,\sum_{j=1}^{8} a_{8j}x_j) + L \\
&= f(\eta(x_1),\ldots,\eta(x_8)) + L \\
&= \eta(f(x_1,\ldots,x_8)) + L \\
&= \theta_1\eta(f(x_1,\ldots,x_n)).
\end{aligned}
\tag{130}
$$

Thus, the diagram in (129) commutes. Hence, φ can be lifted to an automorphism η which is homogeneous of degree 0.

Notice A/I is an A-module via θ_0, $r(a+I) = \theta_0(r)(a+I) = ra + I$ for $r, a \in A$. Also, A/L is an A-module via $\theta_1\eta$, $rb = \theta_1(\eta(r))b$ for $r \in A$, $b \in A/L$. If $r \in A$ and $a \in A/I$, then

$$\varphi(ra) = \varphi(\theta_0(r)a) = \varphi(\theta_0(r))\varphi(a) = \theta_1(\eta(r))\varphi(a) = r\varphi(a). \tag{131}$$

Thus, φ is an A-module isomorphism homogeneous of degree 0.

The minimal free resolution of the A-modules A/I and A/L are as follows:

(132)

$$0 \to A^4 \to A^{27} \to A^{92} \to A^{204} \to A^{296} \to A^{266} \to A^{136} \to A^{32} \to A \to A/I \to 0$$

$$0 \to A^4 \to A^{26} \to A^{87} \to A^{197} \to A^{293} \to A^{266} \to A^{136} \to A^{32} \to A \to A/L \to 0.$$

These resolutions were computed using Macaulay. Notice the resolutions have different betti numbers. By [5: Proposition 1.5.16], this is impossible. Thus, A/I is not k-algebra isomorphic to A/L. We conclude that S is not k-algebra isomorphic to $\mathcal{C}$. □

By Theorem 3.7, we conclude that the set $\Omega = \{(R,J,k) \in \mathcal{M}_{14}(k) | dim_k R = 13,\ i(J) = 3\}$ has at least two isomorphism classes $[\mathcal{C}]$ and $[\mathcal{S}]$.

Since $\mathcal{S}$ is not k-algebra isomorphic to $\mathcal{C}$, $(\mathcal{S}, V_1)$ is not (σ,τ)-isomorphic to $(\mathcal{C}, V_2)$ for any finitely generated, faithful, $\mathcal{S}$-module V_1 and for any finitely generated, faithful $\mathcal{C}$-module V_2.

In our last theorem in this thesis, we will prove that $(\mathcal{S}, k^{14})$ is not a C_1-construction. Let $\mathcal{B}$ be the Schur algebra of size 4 given in (2). Let B be a k-algebra which is k-algebra isomorphic to $\mathcal{B}$. Suppose $f : \mathcal{B} \to B$ is a k-algebra isomorphism. Let N be a finitely generated, faithful, B-module. Then, N is a finitely generated, faithful, $\mathcal{B}$-module via f. Hence, we can form the pairs $(\mathcal{B} \ltimes N^{\ell}, \mathcal{B}^{\ell} \oplus N)$ and $(B \ltimes N^{\ell}, B^{\ell} \oplus N)$ in $\mathcal{M}X$.

Theorem 3.8: *With the notation given above, suppose $dim_k(N) = 4$. Then, f induces a (σ,τ)-isomorphism $(\sigma,\tau) : (\mathcal{B} \ltimes N^{\ell}, \mathcal{B}^{\ell} \oplus N) \to (B \ltimes N^{\ell}, B^{\ell} \oplus N)$.*

Proof: Recall N is a $\mathcal{B}$-module via $nb = nf(b)$. Let $\sigma : \mathcal{B} \ltimes N^{\ell} \to B \ltimes N^{\ell}$ be the map defined by

(133)
$$\sigma(b, n_1, \ldots, n_\ell) = (f(b), n_1, \ldots, n_\ell).$$

Then, it is easy to check σ is a k-vector space isomorphism. Let $(b, n_1, \ldots, n_\ell)$, $(c, m_1, \ldots, m_\ell) \in \mathcal{B} \ltimes N^{\ell}$. Then,

(134)
$$\begin{aligned}
\sigma((b, n_1, \ldots, n_\ell)\,(c, m_1, \ldots, m_\ell)) &= \sigma(bc, m_1 b + n_1 c, \ldots, m_\ell b + n_\ell c) \\
&= (f(bc), m_1 b + n_1 c, \ldots, m_\ell b + n_\ell c) \\
&= (f(b)f(c), m_1 f(b) + n_1 f(c), \ldots, m_\ell f(b) + n_\ell f(c)) \\
&= (f(b), n_1, \ldots, n_\ell)(f(c), m_1, \ldots, m_\ell) \\
&= \sigma(b, n_1, \ldots, n_\ell)\sigma(c, m_1, \ldots, m_\ell).
\end{aligned}$$

Thus, σ is a k-algebra isomorphism.

Let $\tau : \mathcal{B}^\ell \oplus N \to B^\ell \oplus N$ be a map defined by

$$\tau(b_1, \ldots, b_\ell, n) = (f(b_1), \ldots, f(b_\ell), n). \tag{135}$$

Then, it can be easily checked that τ is a k-vector space isomorphism. Let $(b, n_1, \ldots, n_\ell) \in \mathcal{B} \ltimes N^\ell$ and let $(b_1, \ldots, b_\ell, n) \in \mathcal{B}^\ell \oplus N$. Then,

$$\begin{aligned} \tau((b_1, \ldots, b_\ell, n)\,(b, n_1, \ldots, n_\ell)) &= \tau(b_1 b, \ldots, b_\ell b, nb + \textstyle\sum_{i=1}^{\ell} n_i b_i) \\ &= (f(b_1)f(b), \ldots, f(b_\ell)f(b), nf(b) + \textstyle\sum_{i=1}^{\ell} n_i f(b_i)) \\ &= (f(b_1), \ldots, f(b_\ell), n)(f(b), n_1, \ldots, n_\ell) \\ &= \tau(b_1, \ldots, b_\ell, n)\sigma(b, n_1, \ldots, n_\ell). \end{aligned} \tag{136}$$

Thus, we conclude $(\mathcal{B} \ltimes N^\ell, \mathcal{B}^\ell \oplus N) \cong_{(\sigma,\tau)} (B \ltimes N^\ell, B^\ell \oplus N)$. □

We can now prove that $(\mathcal{S}, k^{14})$ is not C_1-construction by using the result in Theorem 3.8.

Theorem 3.9: *Let $(\mathcal{S}, J, k) \in \Omega$ be the k-algebra constructed in Theorem 3.4. Then, $(\mathcal{S}, k^{14})$ is not a C_1-construction.*

Proof: Suppose $(\mathcal{S}, k^{14})$ is a C_1-construction. Then, $(\mathcal{S}, k^{14})$ is (σ, τ)-isomorphic to $(B \ltimes N^\ell, B^\ell \oplus N)$ for some $(B, N) \in X$ and $\ell \in \mathbb{N}$. Let $d = dim_k(B)$ and $n = dim_k(N)$. Since $\mathcal{S}$ is k-algebra isomorphic to $B \ltimes N^\ell$ and k^{14} is k-vector space isomorphic to $B^\ell \oplus N$, $dim_k(\mathcal{S}) = dim_k(B \ltimes N^\ell)$ and $dim_k(k^{14}) = dim_k(B^\ell \oplus N)$. Thus, we have

$$\begin{aligned} 13 &= d + \ell n \\ 14 &= \ell d + n. \end{aligned} \tag{137}$$

The only solution $(d, n, \ell) \in \mathbb{N}^3$ for Equation (136) is $d = 5, n = 4, \ell = 2$. Thus, $(\mathcal{S}, k^{14}) \cong_{(\sigma,\tau)} (B \ltimes N^2, B^2 \oplus N)$.

Notice that $J(B \ltimes N^2) = J(B) \ltimes N^2$. From this, it easily follows that $i(J(B \ltimes N^2)) = i(J(B)) + 1$. Since $\mathcal{S}$ is k-algebra isomorphic to $B \ltimes N^2$ and

$i(J(\mathcal{S})) = 3, i(J(B)) = 2$. Since $dim_k(B) = 5$ and $i(J(B)) = 2$, B is k-algebra isomorphic to $\mathcal{B}$. Theorem 3.8 implies $(B \ltimes N^2, B^2 \oplus N) \cong_{(\sigma_1,\tau_1)} (\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$. Thus, $(\mathcal{S}, k^{14}) \cong_{(\sigma_2,\tau_2)} (\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N)$, where $\sigma_2 = \sigma_1 \circ \sigma$ and $\tau_2 = \tau_1 \circ \tau$. By [4, Proposition 1], $(\mathcal{B} \ltimes (k^4)^2, \mathcal{B}^2 \oplus k^4) \cong_{(\sigma_3,\tau_3)} (\mathcal{C}, k^{14})$ and by Theorem 2.12, $(\mathcal{B} \ltimes N^2, \mathcal{B}^2 \oplus N) \cong_{(\sigma_4,\tau_4)} (\mathcal{B} \ltimes (k^4)^2, \mathcal{B}^2 \oplus k^4)$. Thus, $(\mathcal{S}, k^{14}) \cong_{(\sigma',\tau')} (\mathcal{C}, k^{14})$, where $\sigma' = \sigma_3 \circ \sigma_4 \circ \sigma_2$ and $\tau' = \tau_3 \circ \tau_4 \circ \tau_2$. Therefore, $\mathcal{S}$ is k-algebra isomorphic to $\mathcal{C}$. This is impossible by Theorem 3.7. We can conclude that $(\mathcal{S}, k^{14})$ is not C_1-construction.

□

Theorem 3.9 implies that if $(R, J, k) \in \Omega$, then we can not conclude (R, k^{14}) is a C_1-construction.

Appendix

Appendix

We will prove Subcase 1 through 9 in Case 4 of the proof of Theorem 2.9.

Subcase 1: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 1, 2, 3$. Let $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = ks_i, s_i \in J(\mathcal{B}), i = 1, 2, 3$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix} \in ker\psi$. Let

$$
\begin{aligned}
&\left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_4 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \right. \\
&\quad \delta_5 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \\
&\quad \left. \delta_9 = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_7 \\ y_7 \\ z_7 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_8 \\ y_8 \\ z_8 \end{pmatrix} \right\}
\end{aligned}
\tag{138}
$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ is linearly dependent. Thus, there exist $d, c_i \in k, i = 1, \ldots, 8$, not all zero such that $ds_1 + \sum_{i=1}^{8} c_i x_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 8$, then $d \neq 0$ and $ds_1 = 0$. This implies $s_1 = 0$. This is impossible. Hence, some c_i is not zero. We can assume $c_8 \neq 0$. Thus, $x_8 \in L(s_1, x_1, \ldots, x_7)$. We can repeat this argument four times and assume $x_4, x_5, x_6, x_7, x_8 \in L(s_1, x_1, x_2, x_3)$. Therefore, $x_4 = ds_1 + c_1x_1 + c_2x_2 + c_3x_3$ for some $d, c_1, c_2, c_3 \in k$. Since $\{\delta_1, \delta_4, \delta_5, \delta_6, \delta_7\}$ is linearly independent, $d\delta_1 + c_1\delta_4 + c_2\delta_5 + c_3\delta_6 - \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_7\}$ is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_2 + k\delta_3$. Thus,

$$\left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_4 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_7 \\ y_7 \\ z_7 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_8 \\ y_8 \\ z_8 \end{pmatrix} \end{aligned} \right\} \tag{139}$$

is a basis of $ker\varphi$. We can repeat this argument four times and assume

$$\left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} O \\ s_2 \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ O \\ s_3 \end{pmatrix}, \delta_4 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_4 \\ v_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_5 \\ v_5 \end{pmatrix} \end{aligned} \right\} \tag{140}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4, \{s_2, u_1, \ldots, u_5\}$ is linearly dependent. Thus, there exist $d, c_i \in k, i = 1, \ldots, 5$, not all zero such that $ds_2 + \sum_{i=1}^{5} c_i u_i = 0$. We can assume $c_5 \neq 0$. Since $d\delta_2 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 + c_4\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d, c_i \in k, i = 1, 2, 3, 4, v \in J(\mathcal{B})$ and $\{\delta_2, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = ts_3$ for some $t \in k$, then $d\delta_2 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 + c_4\delta_{10} - \delta_{11} - t\delta_3 = 0$. This is impossible. Thus, $v \notin ks_3$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_3$. This implies $\begin{pmatrix} O \\ O \\ v \end{pmatrix}, \delta_3 \in Ann_{\mathcal{B}}(\alpha_3)$.

This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$.

Subcase 2: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_1)) = 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 2, 3$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2, Ann_{\mathcal{B}}(\alpha_2) = ks_3$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_4, s_i \in J(\mathcal{B}), i = 1, 2, 3, 4$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix} \in ker\varphi$. Let

$$
\begin{aligned}
&\left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix} \right. \\
&\quad \delta_5 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \\
&\quad \left. \delta_9 = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_7 \\ y_7 \\ z_7 \end{pmatrix} \right\}
\end{aligned} \tag{141}
$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, \dots, 7$, not all zero such that $d_1s_1 + d_2s_2 + \sum_{i=1}^{7} c_ix_i = 0$. If $c_i = 0$ for all $i = 1, \dots, 7$, then $d_1s_1 + d_2s_2 = 0$. This implies $d_1 = d_2 = 0$. This is impossible. Hence, some c_i is not zero. We can assume $c_7 \neq 0$. Hence, $x_7 \in L(s_1, s_2, x_1, \dots, x_6)$. We can repeat this argument four times and assume $x_3, x_4, x_5, x_6, x_7 \in L(s_1, s_2, x_1, x_2)$. Therefore, $x_3 = d_1s_1 + d_2s_2 + c_1x_1 + c_2x_2$ for some $d_i, c_i \in k, i = 1, 2$. Since $\{\delta_1, \delta_2, \delta_5, \delta_6, \delta_7\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + c_1\delta_5 + c_2\delta_6 - \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \dots, \delta_7\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_3 + k\delta_4$. Thus,

$$(142)\qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix} \\ \delta_5 = \begin{pmatrix} x_1 \\ y_2 \\ z_2 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_2 \\ y_3 \\ z_3 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_7 \\ y_7 \\ z_7 \end{pmatrix} \end{array} \right\}$$

is a basis of $ker\varphi$. We can repeat this argument four times and assume

$$(143)\qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ O \\ s_4 \end{pmatrix} \\ \delta_5 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_4 \\ v_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_5 \\ v_5 \end{pmatrix} \end{array} \right\}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4, \{s_3, u_1, \ldots, u_5\}$ is linearly dependent. Thus, there exist $d, c_i \in k, i = 1, \ldots, 5$, not all zero such that $ds_3 + \sum_{i=1}^{5} c_i u_i = 0$. We may assume $c_5 \neq 0$. Since $d\delta_3 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 + c_4\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d, c_i \in k, i = 1, 2, 3, 4, v \in J(\mathcal{B})$ and $\{\delta_3, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = ts_4$ for some $t \in k$, then $d\delta_3 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 + c_4\delta_{10} - \delta_{11} - t\delta_4 = 0$. This is impossible. Thus, $v \notin ks_4$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_4$. This implies $\begin{pmatrix} O \\ O \\ v \end{pmatrix}, \delta_4 \in Ann_{\mathcal{B}}(\alpha_3)$.

This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$.

Subcase 3: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 1, 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2$, $Ann_{\mathcal{B}}(\alpha_2) = ks_3 + ks_4$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_5$, $s_i \in J(\mathcal{B}), i = 1, 2, 3, 4, 5$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix} \in ker\varphi$. Let

$$\begin{aligned} (144) \qquad & \left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \right. \\ & \delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \\ & \left. \delta_9 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix} \right\} \end{aligned}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, x_1, x_2, x_3, x_4, x_5, x_6, \}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, \ldots, 6$, not all zero such that $d_1 s_1 + d_2 s_2 + \sum_{i=1}^{6} c_i x_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 6$, then $d_1 s_1 + d_2 s_2 = 0$. This implies $d_1 = d_2 = 0$. This is impossible. Thus,$c_i \neq 0$ for some i. We can assume $c_6 \neq 0$. Hence, $x_6 \in L(s_1, s_2, x_1, \ldots, x_5)$. We can repeat this argument three times and assume $x_3, x_4, x_5, x_6 \in L(s_1, s_2, x_1, x_2)$. Therefore, $x_3 = d_1 s_1 + d_2 s_2 + c_1 x_1 + c_2 x_2$ for some $d_i, c_i \in k, i = 1, 2$. Since $\{\delta_1, \delta_2, \delta_6, \delta_7, \delta_8\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + c_1\delta_6 + c_2\delta_7 - \delta_8 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_8\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_3 + k\delta_4 + k\delta_5$. Thus,

$$(145) \qquad \left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix} \end{aligned} \right\}$$

is a basis of $ker\varphi$. We can repeat this argument three times and assume

$$(146) \qquad \left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_4 \\ v_4 \end{pmatrix} \end{aligned} \right\}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4$, $\{s_3, s_4, u_1, \dots, u_4\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, \dots, 4$, not all zero such that $d_1 s_3 + d_2 s_4 + \sum_{i=1}^{4} c_i u_i = 0$. We may assume $c_4 \neq 0$. Since $d_1\delta_3 + d_2\delta_4 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_1, d_2, c_i \in k, i = 1,2,3,4, v \in J(\mathcal{B})$ and $\{\delta_3, \delta_4, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = ts_5$ for some $t \in k$, then $d_1\delta_3 + d_2\delta_4 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} - t\delta_5 = 0$. This is impossible. Thus, $v \notin ks_5$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_5$. This implies

$\begin{pmatrix} O \\ O \\ v \end{pmatrix}$, $\delta_5 \in Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$.

Subcase 4: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 1, 2, 3$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2$, $Ann_{\mathcal{B}}(\alpha_2) = ks_3 + ks_4$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_5 + ks_6$, $s_i \in J(\mathcal{B}), i = 1, 2, 3, 4, 5, 6$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix} \in ker\varphi$. Let

$$\begin{aligned}
&\left\{\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}\right. \\
&\delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\
&\left.\delta_9 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}\right\}
\end{aligned} \tag{147}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, x_1, x_2, x_3, x_4, x_5\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, \ldots, 5$, not all zero such that $d_1s_1 + d_2s_2 + \sum_{i=1}^{5} c_ix_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 5$, then $d_1s_1 + d_2s_2 = 0$. This implies $d_1 = d_2 = 0$. This is impossible. Thus,$c_i \neq 0$ sor some i. We can assume $c_5 \neq 0$. Hence, $x_5 \in L(s_1, s_2, x_1, \ldots, x_4)$. We can repeat this argument two times and assume $x_3, x_4, x_5 \in L(s_1, s_2, x_1, x_2)$. Therefore, $x_3 = d_1s_1 + d_2s_2 + c_1x_1 + c_2x_2$ for some $d_i, c_i \in k, i = 1, 2$. Since $\{\delta_1, \delta_2, \delta_7, \delta_8, \delta_9\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + c_1\delta_7 + c_2\delta_8 - \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_9\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_3 + k\delta_4 + k\delta_5 + k\delta_6$. Thus,

$$\begin{aligned}
&\left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \right. \\
(148) \qquad &\delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\
&\left. \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \right\}
\end{aligned}$$

is a basis of $ker\varphi$. We can repeat this argument two times and assume

$$\begin{aligned}
&\left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} O \\ s_3 \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \right. \\
(149) \qquad &\delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\
&\left. \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix} \right\}
\end{aligned}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4$, $\{s_3, s_4, u_1, \ldots, u_3\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, 2, 3$, not all zero such that $d_1 s_3 + d_2 s_4 + \sum_{i=1}^{3} c_i u_i = 0$. We may assume $c_3 \neq 0$. Since $d_1\delta_3 + d_2\delta_4 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_i, c_i \in k, i = 1, 2, v \in J(\mathcal{B})$ and $\{\delta_3, \delta_4, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = t_1 s_5 + t_2 s_6$ for some $t_1, t_2 \in k$, then $d_1\delta_3 + d_2\delta_4 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t_1\delta_5 - t_2\delta_6 = 0$. This is impossible. Thus, $v \notin ks_5 + ks_6$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_5 + k\delta_6$. This implies $\begin{pmatrix} O \\ O \\ v \end{pmatrix}$, $\delta_5, \delta_6 \in Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$.

Subcase 5: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_1)) = 3$ and $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 1$ for $i = 2, 3$ Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2 + ks_3$, $Ann_{\mathcal{B}}(\alpha_2) = ks_4$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_5$, $s_i \in J(\mathcal{B}), i = 1, 2, 3, 4, 5,$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \in ker\varphi$. Let

$$(150) \qquad \begin{gathered} \left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \right. \\ \delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \\ \left. \delta_9 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix} \right\} \end{gathered}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, x_1, x_2, x_3, x_4, x_5, x_6\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, \ldots, 6$, not all zero such that $d_1 s_1 + d_2 s_2 + \sum_{i=1}^{6} c_i x_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 6$, then $d_1 s_1 + d_2 s_2 + d_3 s_3 = 0$. This implies $d_1 = d_2 = d_3 = 0$. This is impossible. Thus,$c_i \neq 0$ for some i. We can assume $c_6 \neq 0$. Hence, $x_6 \in L(s_1, s_2, x_1, \ldots, x_5)$. We can repeat this argument four times and assume $x_2, x_3, x_4, x_5, x_6 \in L(s_1, s_2, s_3, x_1)$. Therefore, $x_2 = d_1 s_1 + d_2 s_2 + d_3 s_3 + c x_1$ for some $d_i, c \in k, i = 1, 2, 3$. Since $\{\delta_1, \delta_2, \delta_3, \delta_6, \delta_7\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + c\delta_6 - \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_7\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_4 + k\delta_5$. Thus,

$$
(151) \qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix} \end{array} \right\}
$$

is a basis of $ker\varphi$. We can repeat this argument four times and assume

$$
(152) \qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ O \\ s_5 \end{pmatrix}, \delta_6 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_4 \\ v_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_5 \\ v_5 \end{pmatrix} \end{array} \right\}
$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4, \{s_4, u_1, \ldots, u_5\}$ is linearly dependent. Thus, there exist $d_1, c_i \in k, i = 1, \ldots, 5$, not all zero such that $d_1 s_4 + \sum_{i=1}^5 c_i u_i = 0$. We may assume $c_5 \neq 0$. Since $d_1\delta_4 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 + c_4\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_1, c_i \in k, i = 1, 2, 3, 4, v \in J(\mathcal{B})$ and $\{\delta_4, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = ts_5$ for some $t \in k$, then $d_1\delta_4 + c_1\delta_7 + c_2\delta_8 + c_3\delta_9 + c_4\delta_{10} - \delta_{11} - t\delta_5 = 0$. This is impossible. Thus, $v \notin ks_5$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_5$. This implies $\begin{pmatrix} O \\ O \\ v \end{pmatrix}, \delta_5 \in Ann_{\mathcal{B}}(\alpha_3)$.

This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$.

Subcase 6: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_1)) = 3, dim_k(Ann_{\mathcal{B}}(\alpha_2)) = 2$, and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2 + ks_3, Ann_{\mathcal{B}}(\alpha_2) = ks_4 + ks_5$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_6, s_i \in J(\mathcal{B})$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix} \in$ $ker\varphi$. Let

$$\begin{aligned} &\left\{ \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \right. \\ &\delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \\ &\left. \delta_9 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \right\} \end{aligned} \tag{153}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4, \{s_1, s_2, s_3, x_1, x_2, x_3, x_4, x_5\}$ is linearly dependent. Thus, there exist $d_1, d_2, d_3, c_i \in k, i = 1, \ldots, 5$, not all zero such that $d_1 s_1 + d_2 s_2 + d_3 s_3 + \sum_{i=1}^{5} c_i x_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 5$, then $d_1 s_1 + d_2 s_2 + d_3 s_3 = 0$. This implies $d_1 = d_2 = d_3 = 0$. This is impossible. Thus,$c_i \neq 0$ sor some i. We can assume $c_5 \neq 0$. Hence, $x_5 \in L(s_1, s_2, s_3, x_1, \ldots, x_4)$. We can repeat this argument three times and assume $x_2, x_3, x_4, x_5 \in L(s_1, s_2, s_3, x_1)$. Therefore, $x_2 = d_1 s_1 + d_2 s_2 + d_3 s_3 + c x_1$ for some $d_i, \acute{c} \in k, i = 1, 2, 3$. Since $\{\delta_1, \delta_2, \delta_3, \delta_7, \delta_8\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + c\delta_7 - \delta_8 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_8\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_4 + k\delta_5 + k\delta_6$. Thus,

$$(154) \qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \end{array} \right\}$$

is a basis of $ker\varphi$. We can repeat this argument three times and assume

$$(155) \qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_4 \\ v_4 \end{pmatrix} \end{array} \right\}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4, \{s_4, s_5, u_1, \dots, u_4\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, \dots, 4$, not all zero such that $d_1 s_4 + d_2 s_5 + \sum_{i=1}^{4} c_i u_i = 0$. We may assume $c_4 \neq 0$. Since $d_1\delta_4 + d_2\delta_5 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_1, d_2, c_i \in k, i = 1, 2, 3, v \in J(\mathcal{B})$ and $\{\delta_4, \delta_5, \delta_8, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = ts_6$ for some $t \in k$, then $d_1\delta_4 + d_2\delta_5 + c_1\delta_8 + c_2\delta_9 + c_3\delta_{10} - \delta_{11} - t\delta_6 = 0$. This is impossible. Thus, $v \notin ks_6$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_6$. This implies

$\begin{pmatrix} O \\ O \\ v \end{pmatrix}, \delta_6 \in Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$.

Subcase 7: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_1)) = 3, dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 2$ for $i = 2, 3$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2 + ks_3, Ann_{\mathcal{B}}(\alpha_2) = ks_4 + ks_5$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_6 + ks_7, s_i \in J(\mathcal{B})$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix} \in ker\varphi$. Let

$$\begin{aligned} \Bigg\{ \delta_1 &= \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 &= \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\ \delta_9 &= \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \Bigg\} \end{aligned} \tag{156}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B})$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, s_3, x_1, x_2, x_3, x_4\}$ is linearly dependent. Thus, there exist $d_1, d_2, d_3, c_i \in k, i = 1, \ldots, 4$, not all zero such that $d_1 s_1 + d_2 s_2 + d_3 s_3 + \sum_{i=1}^{4} c_i x_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 4$, then $d_1 s_1 + d_2 s_2 + d_3 s_3 = 0$. This implies $d_1 = d_2 = d_3 = 0$. This is impossible. Thus,$c_i \neq 0$ for some i. We can assume $c_4 \neq 0$. Hence, $x_4 \in L(s_1, s_2, s_3, x_1, x_2, x_3)$. We can repeat this argument two times and assume $x_2, x_3, x_4 \in L(s_1, s_2, s_3, x_1)$. Therefore, $x_2 = d_1 s_1 + d_2 s_2 + d_3 s_3 + c x_1$ for some $d_i, c \in k, i = 1, 2, 3$. Since $\{\delta_1, \delta_2, \delta_3, \delta_8, \delta_9\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + c\delta_8 - \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_9\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_4 + k\delta_5 + k\delta_6 + k\delta_7$. Thus,

$$\left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \end{array} \right\} \tag{157}$$

is a basis of $ker\varphi$. We can repeat this argument two times and assume

$$\left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ O \\ s_6 \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix} \end{array} \right\} \tag{158}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4, \{s_4, s_5, u_1, u_2, u_3\}$ is linearly dependent. Thus, there exist $d_1, d_2, c_i \in k, i = 1, 2, 3$, not all zero such that $d_1 s_4 + d_2 s_5 + \sum_{i=1}^{3} c_i u_i = 0$. We may assume $c_3 \neq 0$. Since $d_1\delta_4 + d_2\delta_5 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_i, c_i \in k, i = 1, 2, v \in J(\mathcal{B})$ and $\{\delta_4, \delta_5, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = t_1 s_6 + t_2 s_7$ for some $t_1, t_2 \in k$, then $d_1\delta_4 + d_2\delta_5 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t_1\delta_6 - t_2\delta_7 = 0$. This is impossible. Thus, $v \notin ks_6 + ks_7$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_6 + k\delta_7$. This implies $\begin{pmatrix} O \\ O \\ v \end{pmatrix}$, $\delta_6, \delta_7 \in Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$.

Subcase 8: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 3$ for $i = 1, 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2 + ks_3, Ann_{\mathcal{B}}(\alpha_2) = ks_4 + ks_5 + ks_6$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_7, s_i \in J(\mathcal{B})$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix} \in$ $ker\varphi$. Let

$$\left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \end{aligned} \right\} \tag{159}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B}), i = 1, 2, 3, 4$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, s_3, x_1, x_2, x_3, x_4\}$ is linearly dependent. Thus, there exist $d_1, d_2, d_3, c_i \in k, i = 1, \ldots, 4$, not all zero such that $d_1s_1 + d_2s_2 + d_3s_3 + \sum_{i=1}^4 c_ix_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 4$, then $d_1s_1+d_2s_2+d_3s_3 = 0$. This implies $d_1 = d_2 = d_3 = 0$. This is impossible. Thus,$c_i \neq 0$ for some i. We can assume $c_4 \neq 0$. Hence, $x_4 \in L(s_1, s_2, s_3, x_1, x_2, x_3)$. We can repeat this argument two times and assume $x_2, x_3, x_4 \in L(s_1, s_2, s_3, x_1)$. Therefore, $x_2 = d_1s_1 + d_2s_2 + d_3s_3 + cx_1$ for some $d_i, c \in k, i = 1, 2, 3$. Since $\{\delta_1, \delta_2, \delta_3, \delta_8, \delta_9\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + c\delta_8 - \delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_9\}$

is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_4 + k\delta_5 + k\delta_6 + k\delta_7$. Thus,

$$
\left\{
\begin{aligned}
&\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\
&\delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&\delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}
\end{aligned}
\right\}
\tag{160}
$$

is a basis of $ker\varphi$. We can repeat this argument two times and assume

$$
\left\{
\begin{aligned}
&\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\
&\delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \\
&\delta_9 = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_3 \\ v_3 \end{pmatrix}
\end{aligned}
\right\}
\tag{161}
$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4, \{s_4, s_5, s_6, u_1, u_2, u_3\}$ is linearly dependent. Thus, there exist $d_i, c_i \in k, i = 1, 2, 3$, not all zero such that $d_1 s_4 + d_2 s_5 + d_3 s_6 + \sum_{i=1}^{3} c_i u_i = 0$. We may assume $c_3 \neq 0$. Since $d_1\delta_4 + d_2\delta_5 + d_3\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_i, c_1, c_2 \in k, i = 1, 2, 3, v \in J(\mathcal{B})$ and $\{\delta_4, \delta_5, \delta_6, \delta_9, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = ts_7$ for some $t \in k$, then $d_1\delta_4 + d_2\delta_5 + d_3\delta_6 + c_1\delta_9 + c_2\delta_{10} - \delta_{11} - t\delta_7 = 0$. This is impossible. Thus, $v \notin ks_7$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_7$. This implies

$\begin{pmatrix} O \\ O \\ v \end{pmatrix}, \delta_7 \in Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 1$.

Subcase 9: Suppose $dim_k(Ann_{\mathcal{B}}(\alpha_i)) = 3$ for $i = 1, 2$ and $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$. Let $Ann_{\mathcal{B}}(\alpha_1) = ks_1 + ks_2 + ks_3, Ann_{\mathcal{B}}(\alpha_2) = ks_4 + ks_5 + ks_6$, and $Ann_{\mathcal{B}}(\alpha_3) = ks_7 + ks_8, s_i \in J(\mathcal{B})$. Then, $\begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \begin{pmatrix} O \\ O \\ s_8 \end{pmatrix} \in ker\varphi$. Let

$$(162) \qquad \left\{ \begin{array}{l} \delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ \delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_8 \end{pmatrix} \\ \delta_9 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \end{array} \right\}$$

be a basis of $ker\varphi$. Here, $x_i, y_i, z_i \in J(\mathcal{B}), i = 1, 2, 3$. Since $dim_k(J(\mathcal{B})) = 4$, $\{s_1, s_2, s_3, x_1, x_2, x_3\}$ is linearly dependent. Thus, there exist $d_i, c_i \in k, i = 1, 2, 3$, not all zero such that $d_1s_1 + d_2s_2 + d_3s_3 + \sum_{i=1}^{3} c_ix_i = 0$. If $c_i = 0$ for all $i = 1, \ldots, 4$, then $d_1s_1 + d_2s_2 + d_3s_3 = 0$. This implies $d_1 = d_2 = d_3 = 0$. This is impossible. Thus,$c_i \neq 0$ for some i. We can assume $c_3 \neq 0$. Hence, $x_3 \in L(s_1, s_2, s_3, x_1, x_2)$. We can repeat this argument and assume $x_2, x_3 \in L(s_1, s_2, s_3, x_1)$. Therefore, $x_2 = d_1s_1 + d_2s_2 + d_3s_3 + cx_1$ for some $d_i, c \in k, i = 1, 2, 3$. Since $\{\delta_1, \delta_2, \delta_3, \delta_9, \delta_{10}\}$ is linearly independent, $d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + c\delta_9 - \delta_{10} = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \neq 0$. Since $\{\delta_1, \ldots, \delta_9\}$ is linearly independent, $\begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix} \notin k\delta_4 + k\delta_5 + k\delta_6 + k\delta_7 + k\delta_8$. Thus,

$$\left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_8 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{11} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \end{aligned} \right\} \tag{163}$$

is a basis of $ker\varphi$. We can repeat this argument and assume

$$\left\{ \begin{aligned} &\delta_1 = \begin{pmatrix} s_1 \\ O \\ O \end{pmatrix}, \delta_2 = \begin{pmatrix} s_2 \\ O \\ O \end{pmatrix}, \delta_3 = \begin{pmatrix} s_3 \\ O \\ O \end{pmatrix}, \delta_4 = \begin{pmatrix} O \\ s_4 \\ O \end{pmatrix} \\ &\delta_5 = \begin{pmatrix} O \\ s_5 \\ O \end{pmatrix}, \delta_6 = \begin{pmatrix} O \\ s_6 \\ O \end{pmatrix}, \delta_7 = \begin{pmatrix} O \\ O \\ s_7 \end{pmatrix}, \delta_8 = \begin{pmatrix} O \\ O \\ s_8 \end{pmatrix} \\ &\delta_9 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \delta_{10} = \begin{pmatrix} O \\ u_1 \\ v_1 \end{pmatrix}, \delta_{11} = \begin{pmatrix} O \\ u_2 \\ v_2 \end{pmatrix} \end{aligned} \right\} \tag{164}$$

is a basis of $ker\varphi$.

Since $dim_k(J(\mathcal{B})) = 4$, $\{s_4, s_5, s_6, u_1, u_2\}$ is linearly dependent. Thus, there exist $d_i, c_1, c_2 \in k, i = 1, 2, 3$, not all zero such that $d_1 s_4 + d_2 s_5 + d_3 s_6 + c_1 u_1 + c_2 u_2 = 0$. We may assume $c_2 \neq 0$. Since $d_1\delta_4 + d_2\delta_5 + d_3\delta_6 + c\delta_{10} - \delta_{11} = \begin{pmatrix} O \\ O \\ v \end{pmatrix}$ for some $d_i, c \in k, i = 1, 2, 3, v \in J(\mathcal{B})$ and $\{\delta_4, \delta_5, \delta_6, \delta_{10}, \delta_{11}\}$ is linearly independent, $v \neq 0$. If $v = t_1 s_7 + t_2 s_8$ for some $t_1, t_2 \in k$, then $d_1\delta_4 + d_2\delta_5 + d_3\delta_6 + c\delta_{10} - \delta_{11} - t_1\delta_7 - t_2\delta_8 = 0$.

This is impossible. Thus, $v \notin ks_7 + ks_8$. Therefore, $\begin{pmatrix} O \\ O \\ v \end{pmatrix} \in ker\varphi \backslash k\delta_7 + k\delta_8$. This implies $\begin{pmatrix} O \\ O \\ v \end{pmatrix}, \delta_7, \delta_8 \in Ann_{\mathcal{B}}(\alpha_3)$. This is a contradiction since $dim_k(Ann_{\mathcal{B}}(\alpha_3)) = 2$.

Bibliography

Bibliography

[1] D. Bayer and M. Stillman, *Macaulay*, A Computer Algebra System for Computing in Algebraic Geometry and Commutative Algebra, 1990.

[2] W.C. Brown, *A Second Course in Linear Algebra*, John Wiley and Sons, New York, 1988.

[3] W.C. Brown and F.W. Call, *Maximal Commutative Subalgebras of $n \times n$ Matrices*, Communications in Algebra, 21(12), 4439-4460, 1993.

[4] W.C. Brown, Two Constructions of Maximal Commutative Subalgebras of $n \times n$ Matrices, Communications in Algebra, 22(10),4051-4066,1994.

[5] Winfried Bruns and Jurgen Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, New York, 1993.

[6] R.C. Courter, *The Dimension of Maximal Commutative Subalgebras of K_n*, Duke Mathematical J. 32, 225-232, 1965.

[7] M. Gerstenhaber, *On Dominance and Varieties of Commuting Matrices*, Annals of Mathematics , (2)73, 324-348, 1961.

[8] T.J. Laffey, *The Minimal Dimension Of Maximal Commutative Subalgebras of Full Matrix Algebras*, Linear Algebra and Its Applications, 71, 199-212, 1985.

[9] O. Zariski and P. Samuel, *Commutative Algebra*, Vol 1, Springer-Verlag, New York, 1958.

[10] O. Zariski and P. Samuel, *Commutative Algebra*, Vol 2, Springer-Verlag, New York, 1958.