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QUADRATIC REPRESENTATIONS FOR GROUPS OF LIE TYPE OVER FIELDS OF CHARACTERISTIC TWO

By

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ABSTRACT

QUADRATIC REPRESENTATIONS FOR GROUPS OF LIE TYPE OVER FIELDS OF CHARACTERISTIC TWO

By

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Let K be a field of characteristic two, G be a group of Lie type defined over K, and let V be an irreducible KG-module. By a theorem due to Steinberg we know that $V \cong \bigotimes_{i \in I} V_i$, where I is an arbitrary index set and each V_i is an algebraic conjugate of a restricted KG-module. Now suppose that G contains a fours-group which acts quadratically on V. We determine then that $|I| \leq 2$. Moreover, by using the weight structure of the modules and information about the parabolic subgroups of G we determine which restricted modules are possible when $|I| = 2$ and, with some restrictions on A, when $|I| = 1$. In all cases the restricted modules are fundamental modules, and in many cases the majority of these are ruled out.

Dedicated to the memory of my grandfather, Frederick A. Dostal

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CHAPTER 1

Introduction

Let V be a vector space over a field K. Then a subgroup $A \leq G \leq GL_K(V)$ is said to act quadratically on V if $[V, A, A] = 0$. V is called a quadratic representation for G.

In [1], Michael Aschbacher mentions the following question about finite groups G with $O_2(G) = 1$ and faithful $GF(2)G$-modules V: " Do there exist 4-subgroups A of G acting quadratically on V; that is, with $[V, A, A] = 0$? Determine the triples (G, V, A) with this property."

In this paper we attempt to answer this question when G is a group of Lie type defined over a field with even characteristic.

Considerations of quadratic action where first made by John Thompson in [15]. There he classified finite irreducible subgroups of $GL_K(V)$ generated by quadratically acting elements for fields K with $char(K) = p \geq 5$. He determined that for $p \geq 5$, the groups satisfying the above conditions are groups of Lie type defined over a field of characteristic p. Ho Chat-Yin solved a similar problem with a few restrictions added for the case of $p = 3$ in [7]. Completing the picture when p is odd, A.A. Premet and I.D. Suprunenko classified the irreducible quadratic representations of groups of Lie type over fields of odd characteristics in [10].

Quadratic $GF(2)$-representations are addressed in [1], [8], and [9] by Michael

Aschbacher, Ulrich Meierfrankenfeld, and Gernot Stroth. Of course, a different set of criteria is needed here since every involution acts quadratically on a $GF(2)$-representation. Consequently, quadratically acting fours-groups were considered instead. The alternating groups, the sporadic groups, and groups of Lie type over fields with odd characteristic containing quadratically acting fours-groups were considered by Ulrich Meierfrankenfeld and Gernot Stroth in [8] and [9]. There it was determined which of the above groups admitted quadratic representations and they indicated these representations.

As was mentioned, in this paper we address the question for groups of Lie type defined over fields of even characteristic. This situation was examined by Gernot Stroth in [13] under assumptions which were essentially equivalent to the assumption that if A is a quadratically acting fours group, then A intersects nontrivially, but is not contained in, a root subgroup. This restriction is indicative of the fact that some restraints on the types of fours-groups that should be considered are necessary. For example, a fours-group contained in a root subgroup would tend to act quadratically on too many representations to make classifying them worthwhile. Towards that end we make the following definition:

Definition 1 *Let V be a vector space over a field K with characteristic two and suppose that a and b are commuting involutions in $GL_K(V)$. We say that $A = \langle a, b\rangle$ acts linearly dependently on V if there exists $\gamma \in K$ such that $[v, a] = \gamma[v, b]$ for all $v \in V$.*

Clearly this is a strong restriction on V and A. In fact, when K is a field of even characteristic and G is a Chevalley group over K, we are able classify the irreducible KG-modules for which G contains a linearly dependently acting fours-group A. We record these in the following theorem. (The ordering used for the root systems is given in Figure 3.1.)

Theorem 1 *If K is a field of even characteristic, G is a Chevalley group over K, V is an irreducible KG-module, and A is fours-subgroup of G which acts linearly dependently on V, then $|K| \geq |A| \geq 4$ and up to algebraic conjugates of V one of the following holds:*

1. *$G \cong A_l(K)$ and V is a fundamental module.*
2. *$G \cong B_l(K)$ and one of the following is true:*
   (a) *r_i is a long root and V is the natural or spin module, or*
   (b) *r_i is a short root and V is a fundamental module.*
3. *$G \cong D_l(K)$ and V is the natural or a half-spin module.*
4. *$G \cong E_6(K)$ and $V = V(\lambda_1)$ or $V(\lambda_6)$.*
5. *$G \cong E_7(K)$ and $V = V(\lambda_7)$.*
6. *$G \cong F_4(K)$ and one of the following is true:*
   (a) *r_i is a long root and $V = V(\lambda_4)$, or*
   (b) *r_i is a short root and $V = V(\lambda_1)$.*
7. *$G \cong G_2(K)$, r_i is short, and $V = V(\lambda_2)$, the natural module.*

We accomplish the proof of this theorem by finding a conjugate of A which is contained in a minimal parabolic subgroup of G and which acts nontrivially and linearly dependently on all the chief factors of V for this parabolic. We note then that the constant $\gamma \in K$ associated with this linearly dependent action on V must remain constant for all these chief factors. Using this we are able to eliminate all but a very few possible values for the weight of V since the weight of V determines the weight of the various chief factors for the parabolic subgroups.

The list of modules obtained in the theorem proves useful because of the following lemma which is central in this paper. To see its significance, recall that by The Steinberg Tensor Product Theorem if V is any irreducible KG-module (K and G as above), then $V \cong \bigotimes_{i \in I} V_i$, where I is some indexing set and V_i is an algebraic conjugate of a restricted module for G.

Lemma 1 *Let G be a Chevalley group over K, $char(K) = 2$. Let $V = \bigotimes_{i \in I} V_i$ be a KG-module with each V_i nontrivial and irreducible. Also let $A \leq G$ be an abelian two-subgroup, $|A| \geq 4$, which acts quadratically on V. Then $|I| \leq 2$. If $|I| = 2$, then A acts linearly dependently on each V_i. Moreover, if A acts linearly dependently on V, then $|I| = 1$.*

The proof of the lemma follows from a few very straight forward calculations.

Once we have the above list of potential modules, we are then able to determine the structure of fours-groups which may act linearly dependently. To do this, in most of the cases, we find subgroups $X \cong Sl_2(K)$ and $Y \leq C_G(X)$ such that $A \leq X \times Y$. A theorem due to Steinberg in [12] states that every chief factor of V for $X \times Y$ is isomorphic to one of the form $V_1 \otimes V_2$ where V_1 is an irreducible KX-module and V_2 is an irreducible KY-module. However, the lemma above implies then that A must act trivially on one of these two and so often we are able to conclude that $A \leq X$ and then in a root subgroup of G. We summarize this result in the following corollary.

Corollary 1 *If K is a field of even characteristic, G is a Chevalley group over K, V is an irreducible KG-module, and A is fours-subgroup of G which acts linearly dependently on V, then one of the following holds:*

1. *If $G \cong A_l(K)$ and if V is not an algebraic conjugate of the natural module for G, then A is contained in a root subgroup.*

2. *If $G \cong B_l(K)$ and if V is not an algebraic conjugate of the natural module for G, then one of the following is true:*

(a) *V is not an algebraic conjugate of the spin module for G and A is contained in a short root subgroup of G.*

(b) *$l \geq 3$, V is an algebraic conjugate of the spin module for G and A is contained in a root subgroup of G.*

3. *If $G \cong D_l(K)$ and V is not an algebraic conjugate of the natural module for G, then A is contained in the product of three commuting root subgroups of G.*

4. *If $G \cong E_6(K)$ or $E_7(K)$, then A is contained in a root subgroup of G.*

5. *If $G \cong F_4(K)$, then one of the following is true:*

   (a) *V is an algebraic conjugate of $V(\lambda_1)$ and A is contained in a long root subgroup of G.*

   (b) *V is an algebraic conjugate of $V(\lambda_4)$ and A is contained in a short root subgroup of G.*

6. *If $G \cong G_2(K)$, then A is contained in a short root subgroup of G.*

After the following definition, we are almost in a position to state the main result of the paper.

Definition 2 *Let G be a group of Lie-type and $A \leq G$ a fours-group.*

1. *We say that A is a linearly dependent fours-group in G if one of the following hold:*

   (a) *$G \cong A_l(K)$ or $D_l(K)$ and A acts linearly dependently on the natural module for G.*

   (b) *$G \cong B_2(K)$ and A act linearly dependently on either the natural or the spin module for G.*

(c) $G \cong B_l(K)$, $l \geq 3$ *and either* A *acts linearly dependently on the natural module for* G *or there exists a rank two connected parabolic subgroup, say* P, *of* G *such that* A *is contained in the Levi complement of* P *and is linearly dependent there.*

(d) $G \cong E_6(K)$, $E_7(K)$, *or* $E_8(K)$ *and there exists a proper connected parabolic subgroup* M_J *such that* $A^g \leq L_J$ *and is linearly dependent there for some* $g \in G$.

(e) $G \cong F_4(K)$ *or* $G_2(K)$ *and* A *is contained in a root subgroup of* G.

(f) $G \cong G^\iota(K)$ *is a twisted Chevalley group and* A *is linearly dependent when considered as a subgroup of* $G(K)$.

2. *We say that* A *is a linearly independent fours-group if* A *is not a linearly dependent fours-group.*

It follows from the corollary above that except for $E_8(K)$, a fours-group is linearly dependent if and only if it acts linearly dependently on some irreducible KG-module.

Moreover, it follows from the lemma above that if A is linearly independent and if it acts quadratically on V, then V is an algebraic conjugate of a restricted module for G.

To prove the main result of the paper, first we show that if $G \cong A_2(K)$ or $B_2(K)$ and if A is linearly independent and acts quadratically on V, then V must be an algebraic conjugate of a fundamental module for G. Then, in an inductive manner, we show that if G is an arbitrary Chevalley group and $A \leq G$ is linearly independent, then there exists a rank two connected parabolic subgroup, say P, such that $AO_2(P)/O_2(P)$ remains a linearly independent subgroup of $O^{2'}(P)/O_2(P)$. By the above, it follows that all the nontrivial chief factors of V for the Levi complement of P must be fundamental modules. This fact is enough to eliminate the majority of the possible values of the weight of V, leaving us with the desired list.

We now state the main result.

Theorem 2 *Let $G = G(K)$ be a Chevalley group defined over K, $char(K) = 2$ and let V a nontrivial, irreducible KG-module. If there exists a linearly independent fours-group A which act quadratically on V, then up to algebraic conjugacy of V, one of the following is true:*

1. *$G \cong A_l(K)$ and V is a fundamental module.*

2. *$G \cong B_l(K)$ and V is a fundamental module.*

3. *$G \cong D_l(K)$ and V is the natural or a half-spin module.*

4. *$G \cong E_6(K)$ and $V = V(\lambda_1)$ or $V(\lambda_6)$.*

5. *$G \cong E_7(K)$ and $V = V(\lambda_7)$.*

6. *$G \cong F_4(K)$ and $V = V(\lambda_1)$ or $V(\lambda_4)$.*

7. *$G \cong G_2(K)$ and $V = V(\lambda_2)$.*

Corollary 2 *Let k be the algebraic closure of K and let $G_0 = G^{\iota}(K) \leq G(k)$ be a twisted Chevalley group. Let V be a nontrivial, irreducible KG_0-module and suppose that G_0 contains a linearly independent fours-group which acts quadratically on V. Then V is obtained from the restriction to G_0 of a rational representation V' of $G(k)$, where V' is one of the representation from the conclusion of Theorem 2.*

The corollary follows easily from the main theorem and a result of Steinberg.

In the next two chapters we record the notation used throughout the paper. The results due to Steinberg mentioned above are recorded in Chapter 3 as Theorems 4.13 and 4.14, and can be found in [12].

CHAPTER 2

Setup, Notation, etc.

Throughout this paper we are concerned with determining the structure of certain modules, involutions, and subgroups of groups of Lie type. For the Chevalley groups which correspond to classical groups we use the structure of the natural module for the group to obtain this information. For the exceptional Chevalley groups, on the other hand, we utilize the (B, N)-structure of the group. Consequently, in this chapter we record notation and a few basic facts and important theorems concerning the weight structure of certain modules.

Except where noted otherwise we use the following set of abbreviations which was, for the most part, adapted from [4]. Let

K be a field with $char(K) = 2$,

G be a group of Lie type defined over K,

V be an non-trivial, irreducible GK-module,

A be a fours-group, $A \leq G$, which acts quadratically on V.

Suppose now that G is a Chevalley group. Within the semi-simple Lie algebra L corresponding to G, we let

Φ be a root system of L,

Π be a fundamental root system, $\Pi \subseteq \Phi$,

Φ^+ be the set of positive roots in Φ,

Φ_J be the root subsystem of Φ generated by $J \subseteq \Pi$,

$\{X_\alpha, H_r \mid \alpha \in \Phi, r \in \Pi\}$ be a Chevalley basis for L,

W be the Weyl group of Φ,

w_r be the reflection in the hyperplane orthogonal to the root r,

W_J be the subgroup of W generated by $\{w_r \mid r \in J \subseteq \Pi\}$,

$\{\lambda_i\}$ be a set of fundamental weights corresponding to Π.

Thus, $\langle \lambda_i, r_j \rangle = \delta_{ij}$.

Within G itself we let

X_r be the root subgroup of G corresponding to $r \in \Phi$,

H be the diagonal subgroup of G,

U be the unipotent subgroup of G generated by the positive root subgroups,

B be the Borel subgroup of G with $B = UH$,

N be the monomial subgroup of G with $N/H \cong W$,

N_J be the inverse image in N of W_J in W and $J \subseteq \Pi$.

We define our notation for the parabolic subgroups of G as follows: Let $J \subseteq \Pi$. Then define

$P_J = \langle B, X_{-r} \mid r \in J \rangle$,

$M_J = \langle B, X_{-r} \mid r \notin J \rangle$,

$L_J = \langle X_{\pm r} \mid r \notin J \rangle$, the Levi complement of M_J,

$U_J = \langle X_r \mid r \in J \cap \Phi_+ \rangle$,

$Q_J = O_2(M_J)$.

Thus, $M_J = Q_J L_J H$.

Note that $M_J = P_{\Pi \setminus J}$.

Lastly, if $r, s \in \Phi$ with $r \neq \pm s$, then we denote the root subsystem of Φ generated by r and s by $\langle \pm r, \pm s \rangle$. That is, $\langle \pm r, \pm s \rangle = (\mathbb{Z}r + \mathbb{Z}s) \cap \Phi$.

Furthermore, we say that $\langle \pm r, \pm s \rangle$ has type

A_2(long) if both r and s are long roots and $\langle \pm r, \pm s \rangle$ is a root system of type A_2.

A_2(short) if both r and s are short roots and $\langle \pm r, \pm s \rangle$ is a root system of type A_2.

B_2 if $\langle \pm r, \pm s \rangle$ is a root system of type B_2.

$L \perp L$ if both roots are long roots and are perpendicular to each other.

$L \perp S$ if one root is a long root and the other is a short root and they are perpendicular.

$S \perp S$ if both roots are short and perpendicular.

Notice that if $w \in W$, then $\langle \pm r, \pm s \rangle^w$ has the same type as $\langle \pm r, \pm s \rangle$.

One last word about some of the terminology used in this paper. When discussing the classical groups we will often mention "the natural module." By this we are referring to the vector space relative to which G is most often defined as a subgroup of non-singular linear transformations with determinant equal to one, preserving some particular form and which, relative to the ordering of the root systems given in the next section, is denoted by $V(\lambda_1)$. Moreover, if $G \cong B_l(K)$, then $V(\lambda_l)$ is identified with "the spin module for G" while if $G \cong D_l(K)$, then $V(\lambda_{l-1})$ and $V(\lambda_l)$ are identified with the "half-spin modules for G."

CHAPTER 3

Root Systems

Basic facts involving root systems, parabolic subgroups, and the Levi decomposition will be used repeatedly throughout this paper, especially when discussing the exceptional groups. Consequently, we need a list of some of the roots in Φ. We have based the the labeling of the roots in each root system on the labeling of the Dynkin Diagrams given in Figure 3.1 below.

For the classical groups we have given an explicit description of the root systems of type A_l, B_l, D_l and G_2 below.

Figure 3.2 shows the roots in system of rank two expressed as integral combinations of fundamental roots.

For the systems of type E_6, E_7, E_8, and F_4, on the other hand, we have explicitly listed a subset of the root system. These are given in Tables 3.1, 3.2, 3.3, and 3.4.

In Figure 3.5 we have given a list of the highest long and short roots in each root system.

The Classical Root Systems

Type A_l

Let $e_0, e_1, \ldots, e_l$ be an orthonormal basis of a Euclidean space with dimension $l+1$. For $1 \leq i \leq l$, let $r_i = e_{i-1} - e_i$. Then the set $\{r_i \mid 1 \leq i \leq l\}$ is a fundamental system of type A_l and the set $\{e_i - e_j \mid i \neq j, 0 \leq i, j \leq l\}$ is the full set of roots.

Type B_l

Let $e_1, e_2, \ldots, e_l$ be an orthonormal basis of a Euclidean space with dimension l. For $1 \leq i \leq l-1$, let $r_i = e_l - e_{l+1}$ and let $r_l = e_l$. Then the set $\{r_i \mid 1 \leq i \leq l\}$ is a fundamental system of type B_l and the set $\{\pm e_i \pm e_j, \pm e_i \mid i \neq j, 0 \leq i, j \leq l\}$ is the full set of roots.

Type D_l

Let $e_1, e_2, \ldots, e_l$ be as above. Let $r_i = e_i - e_{i+1}$ for $1 \leq i \leq l-1$ and let $r_l = e_{l-1} + e_l$. Then the set $\{r_i \mid 1 \leq i \leq l\}$ is a fundamental system of type D_l and the set $\{\pm e_i \pm e_j \mid i \neq j, 0 \leq i, j \leq l\}$ is the full set of roots.

Figure 3.1. Labeling of the Dynkin Diagrams

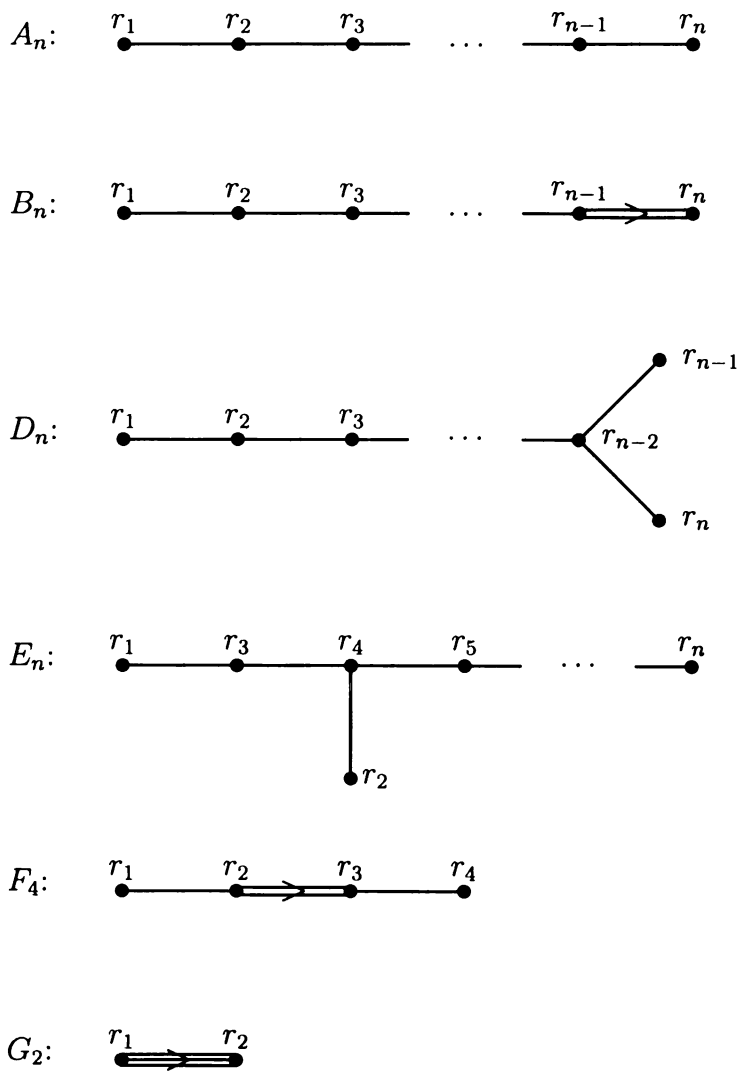

Figure 3.2. The Rank 2 Indecomposable Root Systems

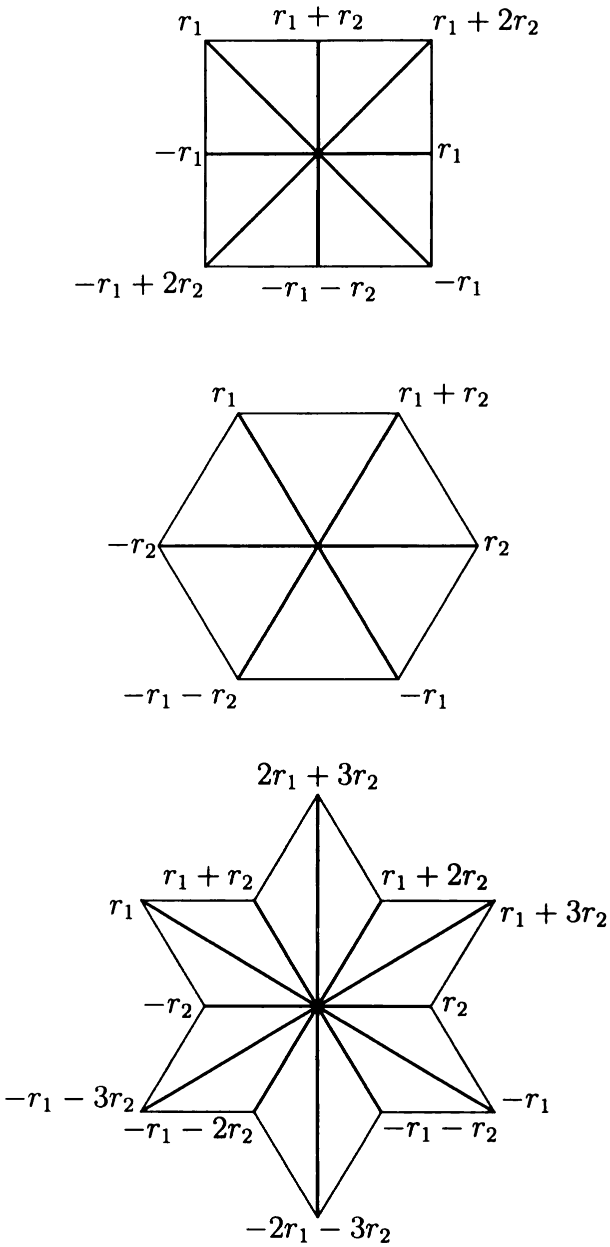

Table 3.1. The roots of E_6

Roots in Φ^+ for Φ of type E_6, where if $r \in \Phi^+$ and if $r = \sum_{i=1}^{6} n_i r_i$ then we represent r in the table as $r = n_1 n_3 \underset{n_2}{n_4} n_5 n_6$. We have ranked them approximately according to their height.

$r_1 = 10\underset{0}{0}00$	$r_2 = 00\underset{1}{0}00$	$r_3 = 01\underset{0}{0}00$	$r_4 = 00\underset{0}{1}00$
$r_5 = 00\underset{0}{0}10$	$r_6 = 00\underset{0}{0}01$	$r_7 = 01\underset{0}{1}00$	$r_8 = 00\underset{0}{1}10$
$r_9 = 01\underset{0}{1}10$	$r_{10} = 00\underset{1}{1}00$	$r_{11} = 01\underset{1}{1}00$	$r_{12} = 00\underset{1}{1}10$
$r_{13} = 01\underset{1}{1}10$	$r_{14} = 01\underset{1}{2}10$	$r_{15} = 11\underset{0}{0}00$	$r_{16} = 11\underset{0}{1}00$
$r_{17} = 11\underset{1}{1}00$	$r_{18} = 11\underset{0}{1}10$	$r_{19} = 11\underset{1}{1}10$	$r_{20} = 11\underset{1}{2}10$
$r_{21} = 12\underset{1}{2}10$	$r_{22} = 00\underset{0}{0}11$	$r_{23} = 00\underset{0}{1}11$	$r_{24} = 00\underset{1}{1}11$
$r_{25} = 01\underset{0}{1}11$	$r_{26} = 01\underset{1}{1}11$	$r_{27} = 01\underset{1}{2}11$	$r_{28} = 01\underset{1}{2}21$
$r_{29} = 11\underset{0}{1}11$	$r_{30} = 11\underset{1}{1}11$	$r_{31} = 11\underset{1}{2}11$	$r_{32} = 12\underset{1}{2}11$
$r_{33} = 11\underset{1}{2}21$	$r_{34} = 12\underset{1}{2}21$	$r_{35} = 12\underset{1}{3}21$	$r_{36} = 12\underset{2}{3}21$

Table 3.2. The roots of E_7

Roots in Φ^+ for Φ of type E_7 which are either fundamental roots or for which the coefficient of r_7 is nonzero. Again, if $r \in \Phi^+$ and if $r = \sum_{i=1}^{7} n_i r_i$, then we represent r in the table as $r = n_1 n_3 \underset{n_2}{n_4} n_5 n_6 n_7$. We have ranked them approximately according to their height.

$r_1 = 10\underset{0}{0}000$	$r_2 = 00\underset{1}{0}000$	$r_3 = 01\underset{0}{0}000$	$r_4 = 00\underset{0}{1}000$
$r_5 = 00\underset{0}{0}100$	$r_6 = 00\underset{0}{0}010$	$r_7 = 00\underset{0}{0}001$	$r_8 = 00\underset{0}{0}011$
$r_9 = 00\underset{0}{0}111$	$r_{10} = 00\underset{0}{1}111$	$r_{11} = 00\underset{1}{1}111$	$r_{12} = 01\underset{0}{1}111$
$r_{13} = 01\underset{1}{1}111$	$r_{14} = 01\underset{1}{2}111$	$r_{15} = 01\underset{1}{2}211$	$r_{16} = 01\underset{1}{2}221$
$r_{17} = 11\underset{0}{1}111$	$r_{18} = 11\underset{1}{1}111$	$r_{19} = 11\underset{1}{2}111$	$r_{20} = 11\underset{1}{2}211$
$r_{21} = 12\underset{1}{2}111$	$r_{22} = 12\underset{1}{2}211$	$r_{23} = 11\underset{1}{2}221$	$r_{24} = 12\underset{1}{2}221$
$r_{25} = 12\underset{1}{3}211$	$r_{26} = 12\underset{1}{3}221$	$r_{27} = 12\underset{1}{3}321$	$r_{28} = 12\underset{2}{3}211$
$r_{29} = 12\underset{2}{3}221$	$r_{30} = 12\underset{2}{3}321$	$r_{31} = 12\underset{2}{4}321$	$r_{32} = 13\underset{2}{4}321$
$r_{33} = 23\underset{2}{4}321$			

Table 3.3. The roots of E_8

Roots in Φ^+ of type E_8 which are either fundamental roots or for which the coefficient of r_8 is nonzero. Once more, if $r \in \Phi^+$ and if $r = \sum_{i=1}^{8} n_i r_i$, then we denote r in the table as $r = \begin{array}{c} n_1 n_3 n_4 n_5 n_6 n_7 n_8 \\ n_2 \end{array}$.

$r_1 = \begin{array}{c}1000000\\0\end{array}$	$r_2 = \begin{array}{c}0000000\\1\end{array}$	$r_3 = \begin{array}{c}0100000\\0\end{array}$	$r_4 = \begin{array}{c}0010000\\0\end{array}$
$r_5 = \begin{array}{c}0001000\\0\end{array}$	$r_6 = \begin{array}{c}0000100\\0\end{array}$	$r_7 = \begin{array}{c}0000010\\0\end{array}$	$r_8 = \begin{array}{c}0000001\\0\end{array}$

Roots for which $n_1 = 0$ and $n_8 \neq 0$.

$r_9 = \begin{array}{c}0000011\\0\end{array}$	$r_{10} = \begin{array}{c}0000111\\0\end{array}$	$r_{11} = \begin{array}{c}0001111\\0\end{array}$	$r_{12} = \begin{array}{c}0011111\\0\end{array}$
$r_{13} = \begin{array}{c}0011111\\1\end{array}$	$r_{14} = \begin{array}{c}0111111\\0\end{array}$	$r_{15} = \begin{array}{c}0011111\\1\end{array}$	$r_{16} = \begin{array}{c}0111111\\1\end{array}$
$r_{17} = \begin{array}{c}0121111\\1\end{array}$	$r_{18} = \begin{array}{c}0122111\\1\end{array}$	$r_{19} = \begin{array}{c}0122211\\1\end{array}$	$r_{20} = \begin{array}{c}0122221\\1\end{array}$

Roots for which $n_1 = 1$ and $n_8 \neq 0$.

$r_{21} = \begin{array}{c}1111111\\0\end{array}$	$r_{22} = \begin{array}{c}1111111\\1\end{array}$	$r_{23} = \begin{array}{c}1121111\\1\end{array}$	$r_{24} = \begin{array}{c}1221111\\1\end{array}$
$r_{25} = \begin{array}{c}1122111\\1\end{array}$	$r_{26} = \begin{array}{c}1222111\\1\end{array}$	$r_{27} = \begin{array}{c}1122211\\1\end{array}$	$r_{28} = \begin{array}{c}1222211\\1\end{array}$
$r_{29} = \begin{array}{c}1232111\\1\end{array}$	$r_{30} = \begin{array}{c}1232211\\1\end{array}$	$r_{31} = \begin{array}{c}1232111\\2\end{array}$	$r_{32} = \begin{array}{c}1233211\\1\end{array}$
$r_{33} = \begin{array}{c}1232211\\2\end{array}$	$r_{34} = \begin{array}{c}1233211\\2\end{array}$	$r_{35} = \begin{array}{c}1242111\\2\end{array}$	$r_{36} = \begin{array}{c}1343211\\2\end{array}$
$r_{37} = \begin{array}{c}1122221\\1\end{array}$	$r_{38} = \begin{array}{c}1222221\\1\end{array}$	$r_{39} = \begin{array}{c}1232221\\1\end{array}$	$r_{40} = \begin{array}{c}1232221\\2\end{array}$
$r_{41} = \begin{array}{c}1233221\\1\end{array}$	$r_{42} = \begin{array}{c}1233221\\1\end{array}$	$r_{43} = \begin{array}{c}1233321\\1\end{array}$	$r_{44} = \begin{array}{c}1233321\\2\end{array}$
$r_{45} = \begin{array}{c}1243221\\2\end{array}$	$r_{46} = \begin{array}{c}1343221\\2\end{array}$	$r_{47} = \begin{array}{c}1243321\\2\end{array}$	$r_{48} = \begin{array}{c}1343321\\2\end{array}$
$r_{49} = \begin{array}{c}1244321\\2\end{array}$	$r_{50} = \begin{array}{c}1344321\\2\end{array}$	$r_{51} = \begin{array}{c}1354321\\2\end{array}$	$r_{52} = \begin{array}{c}1354321\\3\end{array}$

Roots for which $n_1 = 2$ and $n_8 \neq 0$.

$r_{53} = \begin{array}{c}2343211\\2\end{array}$	$r_{54} = \begin{array}{c}2343221\\2\end{array}$	$r_{55} = \begin{array}{c}2343321\\2\end{array}$	$r_{56} = \begin{array}{c}2344321\\2\end{array}$
$r_{57} = \begin{array}{c}2354321\\2\end{array}$	$r_{58} = \begin{array}{c}2454321\\2\end{array}$	$r_{59} = \begin{array}{c}2354321\\3\end{array}$	$r_{60} = \begin{array}{c}2454321\\3\end{array}$
$r_{61} = \begin{array}{c}2464321\\3\end{array}$	$r_{62} = \begin{array}{c}2465321\\3\end{array}$	$r_{63} = \begin{array}{c}2465421\\3\end{array}$	$r_{64} = \begin{array}{c}2465431\\3\end{array}$
$r_{65} = \begin{array}{c}2465432\\3\end{array}$			

Table 3.4. The roots of F_4

Roots in Φ^+ for Φ of type F_4:

$r_1 = 1000$	$r_2 = 0100$	$r_3 = 0010$	$r_4 = 0001$
$r_5 = 1100$	$r_6 = 0120$	$r_7 = 0110$	$r_8 = 0011$
$r_9 = 1120$	$r_{10} = 0122$	$r_{11} = 1110$	$r_{12} = 0111$
$r_{13} = 1220$		$r_{14} = 1111$	$r_{15} = 0121$
$r_{16} = 1122$		$r_{17} = 1121$	
$r_{18} = 1222$		$r_{19} = 1221$	
$r_{20} = 1242$		$r_{21} = 1231$	
$r_{22} = 1342$		$r_{23} = 1232$	
$r_{24} = 2342$			

Notice that the first two columns contain the long roots while the second two contain the shorts roots.

Table 3.5. The highest long and short roots in Φ

If we follow the preceding convention for describing roots as the sum of fundamental roots, then the following is a list of the highest short and long roots in Φ.

Φ	**Highest Short Root**	**Highest Long Root**
A_l		$111\ldots1$
B_l	$111\ldots1$	$122\ldots2$
D_l	$111\ldots1^1_1$	$122\ldots2^1_1$
E_6		$12\underset{1}{3}21$
E_7		$23\underset{2}{4}321$
E_8		$24\underset{3}{6}5432$
F_4	1232	2342
G_2	12	23

CHAPTER 4

General Lemmas

We now prove a number of general lemmas which will be useful later.

Lemma 4.1 *Let $r \in \Phi^+$.*

1. *If r is either a short root or all the roots in Φ have the same length, then r can be written as the sum of fundamental roots $r = r_{i_1} + r_{i_2} + \ldots + r_{i_k}$ in such a way that for each $l \leq k$, $r' = r_{i_1} + r_{i_2} + \ldots + r_{i_l} \in \Phi$, and $|r| = |r'|$.*

2. *If r is a long root in Φ, then r can be written as the integral linear combination of fundamental roots*

$$r = \frac{|r|^2}{|r_{i_1}|^2} r_{i_1} + \frac{|r|^2}{|r_{i_2}|^2} r_{i_2} + \ldots + \frac{|r|^2}{|r_{i_k}|^2} r_{i_k}$$

in such a way that for each $l \leq k$,

$$r' = \frac{|r|^2}{|r_{i_1}|^2} r_{i_1} + \frac{|r|^2}{|r_{i_2}|^2} r_{i_2} + \ldots + \frac{|r|^2}{|r_{i_l}|^2} r_{i_l} \in \Phi,$$

and $|r| = |r'|$.

Proof: Suppose $r = \sum_{r_j \in \Pi} n_j r_j$. We induct on $\sum_{r_j \in \Pi} n_j$ (the height of r). If $\sum_{r_j \in \Pi} n_j = 1$, then we are clearly done, so assume that $\sum_{r_j \in \Pi} n_j > 1$. Since $(r, r) > 0$,

$n_j > 0$ for all n_j, and $(r,r) = \sum_{r_j \in \Pi} n_j(r,r_j)$, it follows that for some $r_i \in \Pi$, $(r,r_i) > 0$. As $\langle \pm r, \pm r_i \rangle$ is a root subsystem of type A_2, B_2, or G_2, one can easily check that if $|r| \leq |r_j|$ for all $r_j \in \Pi$, then $w_{r_i}(r) = r - r_i$. Similarly, if $|r| > |r_j|$ for some $r_j \in \Pi$, then one sees that $\frac{|r|^2}{|r_i|^2} \in \mathbb{Z}$, and $w_{r_i}(r) = r - \frac{|r|^2}{|r_i|^2} r_i$. The lemma follows by induction applied to $w_{r_i}(r)$.

Lemma 4.2 *There is a unique conjugacy class (possibly empty) of roots subsystems of type A_2(long), A_2(short), B_2, $L \perp S$, and $S \perp S$ in Φ under W. Moreover, if $\Phi \neq B_l$ or D_l, then there is also a unique conjugacy class of type $L \perp L$. If $\Phi = B_l$ with $l \geq 2$ or D_l with $l \geq 5$, then there are two conjugacy classes of type $L \perp L$. If $\Phi = D_4$, there are three conjugacy classes.*

Proof: Let r, s, α, and $\beta \in \Phi$ and suppose that $\langle \pm r, \pm s \rangle$ and $\langle \pm\alpha, \pm\beta \rangle$ have the same type. We assume that $|r| = |\alpha|$. Then there is a $w \in W$ such that $\alpha^w = r$, and so $\langle \pm\alpha, \pm\beta \rangle^w = \langle \pm r, \pm\beta^w \rangle$. In particular, it suffices to show that $Stab_W(r)$ has the indicated number of orbits on the set

$$\{\gamma \in \Phi \mid \langle \pm r, \pm\gamma \rangle \text{ has the same type as } \langle \pm r, \pm s \rangle\}.$$

If r is a long root and $\Phi \neq D_4$, then the result follows from [5], Lemma 4.2 and Propositions 4.2, 6.5, and 6.16 (possibly from [3] also).

Note that the two conjugacy classes in B_l of type $L \perp L$ are a result of the fact that sometimes the root subsystem is contained in a larger subsystem of type B_2, and sometimes it is not. For example, $\langle \pm(e_1 - e_2), \pm(e_1 + e_2) \rangle \subseteq \langle \pm(e_1 - e_2), \pm e_2 \rangle$, which has type B_2. On the other hand $\langle \pm(e_1 - e_2), \pm(e_3 - e_4) \rangle$ is not contained in any such subsystem. We will call the first conjugacy class $(L \perp L)_1$ and the second $(L \perp L)_2$.

Also, if $\Phi = D_l$ with $l \geq 5$, then the two conjugacy classes have representatives $\{\pm(e_i - e_j), \pm(e_i + e_j)\}$ and $\{\pm(e_i + e_j), \pm(e_k + e_m)\}$ where $\{i,j\} \cap \{m,k\} = \emptyset$. Similarly, we will call the first conjugacy class $(L \perp L)_1$ and the second $(L \perp L)_2$.

Now suppose that $\Phi = D_4$ and $\langle \pm r, \pm s\rangle$ has type $L \perp L$. Then one can easily check that the conjugacy classes are the following:

$$\{\pm(e_i - e_j), \pm(e_i + e_j)\}$$

$$\{\pm(e_i \pm e_j), \pm(e_k \pm e_l) \mid \{i,j\} \cap \{k,l\} = \emptyset\}$$

$$\{\pm(e_i \pm e_j), \pm(e_k \mp e_l) \mid \{i,j\} \cap \{k,l\} = \emptyset\}$$

We will call these conjugacy classes $(L \perp L)_1$, $(L \perp L)_2$, and $(L \perp L)_3$, respectively.

Lastly, suppose that r is a short root. If $\Phi = B_l$, then as each two short roots are perpendicular and are contained in a root subsystem of type B_2, and as all the root systems of type B_2 are conjugate, the result follows. If $\Phi = F_4$, then the result follows because of the graph automorphism that switches long and short roots while preserving angles between them.

Corollary 4.3 *Let r, s, α, and $\beta \in \Phi$ and suppose that $\langle \pm r, \pm s\rangle$ has type B_2, $L \perp L$, $L \perp S$, or $S \perp S$. Moreover, suppose that $\langle \pm r, \pm s\rangle$ and $\langle \pm\alpha, \pm\beta\rangle$ are in the same conjugacy class of Φ under W and that $(r,s) = (\alpha, \beta)$. Then there exists $w \in W$ such that $\{r,s\}^w = \{\alpha, \beta\}$.*

Proof: As this is true for each set of roots $\{\gamma, \delta\} \leq \langle \pm\alpha, \pm\beta\rangle$ and as there exists $w' \in W$ such that $\{r,s\}^{w'} \subseteq \langle \pm\alpha, \pm\beta\rangle$, the claim is clear.

Lemma 4.4 *Let G be a Chevalley group and suppose $1 \neq M \leq U \in Syl_2(G)$. Let P_1 and P_2 be parabolic subgroups of G containing U such that $G = \langle P_1, P_2\rangle$. Then there exists $g \in G$ and $i \in \{1,2\}$ such that $M^g \leq P_i$ and $M^g \not\subseteq O_2(P_i)$.*

Proof: Let $X = \{R \leq U \mid$ for all $g \in G$, $R^g \leq P_i$ implies that $R^g \subseteq O_2(P_i)\}$ and let $Y = \langle X\rangle \leq G$. Choose $R \in X$. Since $R \leq U \leq P_1$, it follows that $R \leq O_2(P_1)$. Thus $O_2(P_1) \triangleleft P_1$ implies that $\langle R^{P_1}\rangle \leq O_2(P_1) \leq U$. But note that $\langle R^{P_1}\rangle \leq U$ implies that $\langle R^{P_1}\rangle \in X$, by definition of X. Hence $P_1 \leq N_G(Y)$. Similarly, $P_2 \leq N_G(Y)$. So $G = \langle P_1, P_2\rangle$ implies that $Y \lhd G$. Thus, $Y \leq U$ implies that $Y \leq O_2(G) = 1$, proving the claim.

At various points in the paper we will have occasion to explicitly write an involution as the product of elements of root subgroups. Towards that end we include the following lemmas.

Lemma 4.5 *Let $r \in \Phi$, $\Phi \neq G_2$, and $g \in G \setminus C_G(X_r)$. Then $\langle X_r, X_r^g\rangle \leq G$ is either a 2-group or is conjugate in G to $\langle X_r, X_{-r}\rangle \cong SL_2(K)$. More specifically, it is isomorphic to either $Sl_2(K)$, a 2-Sylow subgroup of $Sl_3(K)$, or $K \times K$ (considered as an additive group).*

Proof: Since all roots of the same length are conjugate under W, we can assume without loss that if r is a long root, then it is the highest long root or that if it is a short root, then it is the highest short root. In either case, we have that $B \leq N_g(X_r)$. So write $g = b_1 n b_2$ with $b_1, b_2 \in B$, $n \in N$. Then $\langle X_r, X_r^g\rangle = \langle X_r, X_r^{b_1 n b_2}\rangle = \langle X_r, X_r^n\rangle^{b_2}$. An inspection of the various root systems now yields the result.

Lemma 4.6 *Let G be a Chevalley group, $G \not\cong G_2(K)$, and let $\langle a, b\rangle = A$ be a fours-group in G. If there exists a root subgroup, say X, such that $a \notin C_G(X)$ but such that $b \in C_G(X)$, then $\langle A^{C_G(b)}\rangle$ either contains a nontrivial element of a root subgroup or an element which is conjugate to an element of the form $x_\alpha(1)x_\beta(1)$ where $\alpha, \beta \in \Phi$, $|\alpha| = |\beta|$, and $\alpha \perp \beta$.*

Proof: Let $W = \langle X, X^a\rangle$. By Lemma 4.5, W is either isomorphic to $Sl_2(K)$, a Sylow 2-subgroup of $L_3(K)$, or $K \times K$. We consider each, case by case.

Case 1: $W \cong Sl_2(K)$.

As $Y = \langle [X, a]\rangle$ is normal in $\langle X, a\rangle$, Y is normal in W. Moreover, because Y contains at least one involution, Y is not contained in the center of W. Hence $Y = W$. Thus because $Y \leq \langle A^{C_G(b)} \rangle$, we can choose an element x in a root subgroup of W which satisfies the claim of the lemma.

Case 2: W is isomorphic to a Sylow 2-subgroup of $L_3(K)$.

Again, $Y = \langle [X, a] \rangle$ is normal in W. Moreover, because $Z(W) = [X^a, X]$ we see that $Y \not\leq Z(W)$. Hence, $Z(W) \leq Y$ and so as $Z(W)$ is a root subgroup the claim is also satisfied in this case.

Case 3: $W \cong K \times K$, viewed as an additive group.

Without loss we can assume that $X = X_r$, where r is the highest long or short root in Φ so that $B \leq N_G(X)$. If we write $a = b_1 n b_2$, with $b_1, b_2 \in B$ and $n \in N$, then $X^a X = X^{nb_2} X = (X^n X)^{b_2}$ which is clearly conjugate to $X^n X$. Thus, we can choose an element $1 \neq x \in [X, a] \leq \langle A^{C_G(b)} \rangle$ which is conjugate to an element in $X_\alpha X_\beta$ for some roots $\alpha, \beta \in \Phi$, $|\alpha| = |\beta|$ and such that $\alpha + \beta$ is not a root.

Now suppose that α and β are not perpendicular. Then as $\alpha + \beta$ is not a root, $\alpha - \beta$ is a root, and as $X_\alpha^\sharp X_\beta^\sharp \leq X_\beta^{X_{\alpha-\beta}}$, the claim follows.

Lemma 4.7 *Let X_r be a root subgroup in G, $X = \langle X_r, X_{-r} \rangle$ and suppose that $N_G(X_r) = M_i$. Then $O^{2'}(N_G(X)) = L_i \times X$ and $O^{2'}(C_G(X)) = L_i$.*

Proof: Let $g \in N_G(X)$. Since X is doubly transitive on its Sylow 2-subgroups, there exists $x \in X$ such that $X_r^{gx} = X_r$ and $X_{-r}^{gx} = X_{-r}$. That is, $gx \in N_G(X_r) \cap N_G(X_{-r}) = L_i H$, and so $g \in L_i H X$, proving the lemma.

Lemma 4.8 *Let $a \in G$ be an involution, $G \not\cong G_2(K)$ and let r denote the highest long root in Φ. If applicable, let s denote the highest short root in Φ. Also let $N(X_r) = M_i$ and $N_G(X_s) = M_j$. Then there exists a conjugate a' of a such that either $\langle X_{-r}^{a'}, X_{-r} \rangle = \langle X_{\pm r} \rangle$ and $a' \in X_r L_i$ or $\langle X_{-s}^{a'}, X_{-s} \rangle = \langle X_{\pm s} \rangle$ and $a' \in X_s L_j$.*

Proof: By Lemma 4.4 applied several times to $\langle a \rangle$, there is a minimal parabolic P_k and a conjugate, say a'', of a such that $a'' \in U \backslash O_2(P_k)$. As $\langle X_{-r_k}, X_{-r_k}^{a''} \rangle$ is clearly not a 2-group, it follows from Lemma 4.5 that it is conjugate to $\langle X_{r_k}, X_{-r_k}\rangle \cong SL_2(q)$. Moreover, since by Lemma 4.2 $\langle X_{r_k}, X_{-r_k}\rangle$ is conjugate to $\langle X_r, X_{-r}\rangle$ if r_k is a long root, or $\langle X_s, X_{-s}\rangle$ if r_k is a short root, it follows that for some conjugate, a', of a, we have either $\langle X_{-r}^{a'}, X_{-r}\rangle = \langle X_r, X_{-r}\rangle$ or $\langle X_{-s}^{a'}, X_{-s}\rangle = \langle X_s, X_{-s}\rangle$. The lemma now follows by Lemma 4.7.

Suppose that all the roots in Φ have the same length. We choose roots $s_1, s_2, \ldots, s_n \in \Phi$ as follows: Let s_1 be the highest weight root in Φ and suppose that $N_G(X_{r_1}) = M$. Let L be the Levi subgroup of M and let $\Phi_1 \subseteq \Phi$ be the root subsystem corresponding to L. If G is not an orthogonal group, then Φ_1 is a connected root subsystem. In this case, choose s_2 to be the highest root in Φ_1. On the other hand, if G is an orthogonal group, then $\Phi_1 = \Phi_2 \cup \Phi_3$ where both Φ_2 and Φ_3 are connected root subsystems and Φ_3 has type A_2. In this case choose s_2 to be the highest root in Φ_2 and s_3 to be the unique positive root in Φ_3. We continue the selection of the roots by considering s_2 as we considered s_1 above, until the Levi complement of normalizer of X_{r_n} is trivial.

Lemma 4.9 *If all the roots in Φ have the same length, then every involution in G is conjugate to one of the form $x_{s_{i_1}}(1)x_{s_{i_2}}(1)\cdots x_{s_{i_j}}(1)$ for some $\{i_1, i_2, \cdots, i_j\} \subseteq \{1, 2, \cdots, n\}$.*

Proof: This follows from Lemma 4.8 and our choice of $s_1, s_2, \ldots s_n$.

Lemma 4.10 *Suppose V is a vector space over some field K. Moreover, suppose that $V = \bigcup_{i=1}^{n} V_i$, where each V_i is a proper subspace of V. Then $|K| \leq n$.*

Proof: Without loss, we assume that n above is minimal. Thus $V_j \not\subseteq \bigcup_{\substack{i=1\\ i\neq j}}^{n} V_i$ for each $1 \leq j \leq n$. Choose $x \in V_1 \setminus (\bigcup_{i=2}^{n} V_i)$ and $y \in V_2 \setminus (V_1 \cup \bigcup_{i=3}^{n} V_i)$ and choose λ_1 and $\lambda_2 \in K$, with $\lambda_1 \neq \lambda_2$. If there exist an i such that $\lambda_1 x + y$ and $\lambda_2 x + y \in V_i$, then we see that both x and y are in V_i, contradicting our choice of x and y. Hence, $n \geq |\{\lambda x + y \mid \lambda \in K\}| = |K|$, proving the lemma.

We will now record a few results involving the weight structure of a KG-module. All of the theorems cited below may be found in [12].

Let G be a universal Chevalley group defined over a field K (arbitrary), L be the associated Lie Algebra, and let λ be a weight such that $\langle \lambda, r\rangle \in \mathbb{Z}^+ \cup \{0\}$ for all $r \in \Phi^+$. By [12], Theorem 39(e) (page 209), there exists a unique irreducible rational KG-module, say V, for which λ is the highest weight. We will now present a brief description of the construction of V given there.

Let λ also represent the corresponding weight on L, so $\lambda(H_r) = \langle \lambda, r\rangle \in \mathbb{Z}^+ \cup \{0\}$, for all $r \in \Phi^+$. By [12], Theorem 3(e) (page 14), there exists an irreducible L-module, say (ρ, V'), with λ as the highest weight. Let v^+ denote a nonzero highest weight vector in V'. Let $\mathcal{U}$ be the universal enveloping algebra of L and then let $\mathcal{U}_\mathbb{Z}$ be the subalgebra of $\mathcal{U}$ generated by $\{X_r^m/m! \mid r \in \Phi\}$. By [12], Theorem 2, Corollary 1 (page 17), there exists a lattice M contained in V' which is invariant under $\mathcal{U}_\mathbb{Z}$. In fact, $M = \mathcal{U}_\mathbb{Z} v^+$.

Now define $V^K = M \otimes_\mathbb{Z} K$ and let G' be the Chevalley group constructed as an automorphism group inside of $\mathcal{U}_\mathbb{Z}$. Then by [12], Theorem 7, Corollary 1, there is a rational homomorphism $\phi : G \to G'$ such that $\phi(x_r(t)) = x'_r(t)$ for all $r \in \Phi$ and $t \in K$. This give us a representation of G on V'. This representation, however, need not be irreducible, but it does contain v^+ which has weight λ. Let V'' be the smallest submodule of V' containing v^+, and let V''' be the maximal proper submodule of V''. Then V''/V''' is the required KG-module.

Now suppose that $A \leq G$ is a fours-group and $V(\lambda)$ is an irreducible KG-module with highest weight λ upon which A acts quadratically. Suppose that k is the algebraic closure of K. Let G_k denote the Chevalley group derived from the same representation as G, only over k instead of K. Similarly, denote V_k', V_k'', and V_k''' in the construction given above. Then $V' \leq V_k'$, $V'' \leq V_k''$, and $V''' \leq V_k'''$. Therefore, if we identify G as a subgroup of G_k, then it follows that A acts quadratically on V_k''/V_k''' as well.

Thus we have proven the following proposition.

Proposition 4.11 *Let k be the algebraic closure of K and let V be an irreducible quadratic KG-module. Then the kG-module gotten by extending V remains a quadratic module for G with the same weight structure as V.*

Lemma 4.12 *Let Φ be a connected root system containing only one root length, r be the highest root in Φ and let G be the associated Chevalley group defined over K. Also suppose that $M_i = N_G(X_r)$ and that P is a connected parabolic containing both U and $\langle X_{\pm r_i}\rangle$. Then $X = \langle X_r^P\rangle$ is the unique irreducible KP-module in $Q_i = O_2(P)$. Moreover, X has weight λ_i.*

Proof: Let Y be a P-module in Q. Because U is a 2-group acting on Y, $C_Y(U) \neq 1$. Thus, because $Z(U) = X_r$ and because H acts irreducibly on X_r, it follows that $X_r \leq Y$ and so $X \leq Y$.

Also, $M_i = N_G(X_r)$ implies that if r_j is a fundamental root and if $\langle X_{\pm r_j}\rangle \not\leq C_G(X_r)$, then $j = i$. And as $x_r(t)^{h_{r_i}(\lambda)} = x_r(\lambda t)$, it follows that X is isomorphic, as a KP-module, to $V(\lambda_i)$.

Lastly, the following two theorems due to Robert Steinberg are used throughout this paper. They are stated for an arbitrary field with nonzero characteristic equal to p.

Theorem 4.13 (The Steinberg Tensor Product Theorem) *Assume that G is a universal Chevalley group with $|\Pi| = l$. Let $\mathcal{R}$ be the set of p^l irreducible rational representations of G for which the highest weight λ satisfies $0 \leq \langle\lambda, r_i\rangle \leq p-1$ for all $r_i \in \Pi$. Then every irreducible rational representation of G can be written uniquely*

$\bigotimes_{j=0}^{\infty} \rho_j \circ Fr^j$ *where* $\rho_j \in \mathcal{R}$ *and* Fr *denotes the Frobenius map which replaces the matrix entries of the elements of* G *with their* p*th power.*

This is Theorem 41 (page 217) of [12].

Theorem 4.14 *Let* G *be a finite universal Chevalley group or one of its twisted analogues constructed as the set of fixed points of an automorphism of the form* $x_r(t) \to x_{\rho r}(\pm t^{q(r)})$. *Then the* $\prod_{r\in\Pi} q(r)$ *irreducible rational representations of the including algebraic group (got by extending the base field to its algebraic closure) for which the highest weights satisfy* $0 \le \langle \lambda, r\rangle \le q(r)-1$, *for all* $r \in \Pi$, *remain irreducible and distinct on restriction to* G *and form a complete set.*

This is Theorem 43 (page 230) of [12].

CHAPTER 5

Linear Dependence

Definition 5.1 (Linearly Dependent Action Of A Group On A Module) *Let G be a group, K an arbitrary field, and V a KG-module. We say that G acts linearly dependently on V if for each $a, b \in G$ there exists $\lambda \in K$ (possibly zero) such that either $[v, a] = \lambda[v, b]$ for all $v \in V$, or $[v, b] = \lambda[v, a]$ for all $v \in V$.*

Proposition 5.2 *Suppose V is a vector space over K, K a field of characteristic two, and $a, b \in GL_K(V)$ are involutions. Then $\langle a, b\rangle$ acts linearly dependently on V if and only if a and b normalize the same subspaces of V.*

Proof: Suppose that there exists $\lambda \in K$ such that $[v, b] = \lambda[v, a]$ for all $v \in V$. As $1 \neq b \in GL_K(V)$, $\lambda \neq 0$. Now let $W \leq V$ be a subspace and suppose that $W^a = W$. Choose $x \in W$. Then $[x, b] = \lambda[x, a]$ implies that $x^b = \lambda x^a + (\lambda + 1)x \in W$ and so b normalizes W as well.

So suppose now that a and b normalize the same subspaces of V. Choose $x \in V \setminus C_V(a)$. Then as a normalizes the 1-dimensional subspace $K(x^a + x)$, b does as well. Thus, because b is an involution, $(x^a + x)^b = x^a + x$. Hence $v^{ab} + v^a + v^b + v = 0$ for all $v \in V$. Similarly, $v^{ba} + v^a + v^b + v = 0$ for all $v \in V$, and so $v^{ab} = v^{ba}$, for all $v \in V$. Now choose $y \in V$. Because a normalizes $K\langle y, y^a\rangle$, there exists scalars λ and $\mu \in K$ such that $y^b = \lambda y^a + \mu y$. Hence $y^{ab} = \mu y^a + \lambda y$. Thus because

$y^a + y^b + y = y^{ab} = \mu y^a + \lambda y$ implies that $y^b = (\mu + 1)y^a + (\lambda + 1)y$, it follows that $\mu = \lambda + 1$, and so $y^b = \lambda y^a + (\lambda + 1)y$. Therefore, $[v, b] = \lambda[v, a]$ for all $v \in V$.

Remark: Suppose that $dim_K(V) = k < \infty$ and that a and $b \in GL_K(V)$ are involutions such $\langle a, b\rangle$ acts linearly dependently on V. Then there exists a scalar $0 \neq \lambda \in K$ and a basis for V relative to which a and b have the following matrix form:

$$a = \begin{pmatrix} I_l & 0 & 0 \\ 0 & I_{k-2l} & 0 \\ I_l & 0 & I_l \end{pmatrix}, \quad b = \begin{pmatrix} I_l & 0 & 0 \\ 0 & I_{k-2l} & 0 \\ \lambda I_l & 0 & I_l \end{pmatrix},$$

where $l = dim_K([V, a])$.

Lemma 5.3 *Let A be a 2-group, $|A| \geq 4$; K a field with $char(K) = 2$; $V = \bigotimes_{i\in I} V_i$, with each V_i a KA-module, such that $[V, A, A] = 0$; also let $J = \{i \in I \mid [V_i, A] \neq 0\}$. Then the following are true:*

1. *If $|J| \geq 2$, then A acts linearly dependently on each V_i.*

2. *If $a, b \in A$ and if there is an $j \in J$ and $\lambda \in K$ such that $[v, a] = \lambda[v, b]$, for all $v \in V_j$, then for all $i \in I$ and $v \in V_i$, $[v, a] = \lambda[v, b]$.*

3. *If $|J| \geq 3$, or if A acts linearly dependently on V and $|J| \geq 2$, then $\lambda = 0$ or 1. That is, there exists a subgroup $H < A$ with $[A : H] = 2$ such that $[V_i, H] = 0$, for all $i \in I$.*

Proof: Assume that $|J| \geq 2$, and choose $a, b \in A^\sharp$ with $[V_j, b] \neq 0$ for some $j \in J$. Note that if $i \in I \setminus J$, then by definition of J, $[v, c] = 0$ for all $v \in V_i$ and for all $c \in A$, and so of course $[v, a] = \lambda[v, b]$ for all $\lambda \in K, v \in V_i$. Now choose $i \in J \setminus \{j\}$. Without loss, assume that $i = 1$ and $j = 2$. We wish to consider the action of A on $V' = V_1 \otimes V_2$, so set $I' = I \setminus \{1, 2\}$ and let $V'' = \bigotimes_{i\in I'} V_i$ so that $V = V' \otimes V''$. Choose $0 \neq w \in C_{V''}(A)$. Then $[V, A, A] = 0$ implies that $[V' \otimes w, A, A] = 0$, and so $[V', A, A] \otimes w = 0$. Thus $w \neq 0$ implies that $[V', A, A] = 0$. In particular, for all $v \in V_1, w \in V_2$, we have $[(v, w), a, b] = 0$. That is,

$$(v^{ab}, w^{ab}) + (v^a, w^a) + (v^b, w^b) + (v, w) = 0$$

for all $v \in V_1$ and $w \in V_2$.

Similarly we can show that $[V_n, A, A] = 0$, $n = 1$ or 2; so that $v^{ab} = v^a + v^b + v$ for all $v \in V_1$ or V_2. Substituting this into the equation above, we see that

$$(v^a + v^b + v, w^a + w^b + w) + (v^a, w^a) + (v^b, w^b) + (v, w) = 0$$

which is equivalent to

$$(v^a, w^b + w) + (v^b, w^a + w) + (v, w^a + w^b) = 0$$

for all $v \in V_1$ and $w \in V_2$. In particular, as $w^a + w^b = (w^a + w) + (w^b + w)$, we get that

$$(v^a + v, w^b + w) = (v^b + v, w^a + w)$$

for all $v \in V_1$ and $w \in V_2$.

Because of our choice of V_2, we may choose $w \in V_2$ such that $w^b + w \neq 0$.

Case 1: $[V_1, a] \neq 0$.

Choose $v \in V_1$ such that $v^a + v \neq 0$. As $(v^a + v, w^b + w) \neq 0$ implies that $(v^b + v, w^a + w) \neq 0$ as well, we see that there exists $\lambda \in K^*$ such that $[v, a] = \lambda[v, b]$ and $[w, a] = \lambda[w, b]$. Since our choice of $v \in V_1 \setminus C_{V_1}(a)$ was arbitrary, it follows that $[v, a] = \lambda[v, b]$, for all $v \in V_1 \setminus C_{V_1}(a)$. Now choose $v \in C_{V_1}(a)$. Then $(v^a + v, w^b + w) = 0$ implies that $(v^b + v, w^a + w) = 0$; so $w^a + w \neq 0$ yields $v^b + v = 0$. Thus $[v, a] = \lambda[v, b]$ for all $v \in V_1$. Similarly, $[w, a] = \lambda[w, b]$ for all $w \in V_2$.

Case 2: $[V_1, a] = 0$.

First suppose that $[V_2, a] \neq 0$. Then choose $c \in A$ with $[V_1, c] \neq 0$. By Case 1 applied to $\{a, c\}$, there exists $\mu \in K^*$ such that $[v, c] = \mu[v, a]$ for all $v \in V_1$ or V_2, contrary to $[V_1, a] = 0$ and $[V_1, c] \neq 0$. Thus $[V_2, a] = 0$, and so $[v, a] = 0[v, b]$ for all $v \in V_1$ or V_2, proving the first part of the lemma.

Note that we have just proven that for all $a \in A$, $[V_i, a] \neq 0$ for some $i \in J$ if and only if $[V_j, a] \neq 0$ for every $j \in J$. That is, $C_A(V_i) = C_A(V_j)$ for all $i, j \in J$. Let $H = C_A(V_j)$ for some $j \in J$. Then A/H acts faithfully on each $V_i, i \in J$. To finish the lemma, we may assume that $|A| \geq 2$ and that A is faithful on each V_i, $i \in J$.

Suppose now that $|J| \geq 3$ and choose $a, b \in A^\sharp$. Without loss assume that $\{1, 2, 3\} \subseteq J$. Then as A acts quadratically on $(V_1 \otimes V_2) \otimes V_3$, and as $[V_3, a] \neq 0$, it follows from the above that there exists $\lambda \in K^*$ such that $[v, b] = \lambda[v, a]$ for all $v \in V_1 \otimes V_2$ or V_3.

Thus by considering $V_1 \otimes V_2$ in place of V, to prove part (3) of the lemma we may assume that A acts linearly dependently on V and that $\{1, 2\} \subseteq J$. Moreover, we assume that $|A| > 2$. Choose $a, b \in A^\sharp$ with $a \neq b$. As before, let $V' = V_1 \otimes V_2$ and $V'' = \bigotimes_{k \in I \setminus \{1,2\}} V_k$ so that $V = V' \otimes V''$. Note that A acts linearly dependently on V'. Thus, there exist λ and $\mu \in K^*$ such that $[v, b] = \lambda[v, a]$ for all $v \in V'$, and $[v, b] = \mu[v, a]$ for all $v \in V_1$ or V_2.

Because A is faithful on both V_1 and V_2, for $n = 1$ and 2 we choose $v_n \in V_n$ such that $v_n^a \neq v_n$. Let $z_n = [v_n, a]$, so $v_n^a = z_n + v_n$. Then, $[(v_1, v_2), b] = \lambda[(v_1, v_2), a] = \lambda(v_1^a, v_2^a) + \lambda(v_1, v_2) = \lambda(z_1, v_2) + \lambda(v_1, z_2) + \lambda(z_1, z_2)$. However, we also have that $[v_n, b] = \mu(v_n^a + v_n) = \mu z_n$, and so $v_n^b = \mu z_n + v_n$. In particular, we see that $[(v_1, v_2), b] = (v_1^b, v_2^b) + (v_1, v_2) = \mu(z_1, v_2) + \mu(v_1, z_2) + \mu^2(z_1, z_2)$. Hence,

$$\lambda(z_1, v_2) + \lambda(v_1, z_2) + \lambda(z_1, z_2) = \mu(z_1, v_2) + \mu(v_1, z_2) + \mu^2(z_1, z_2),$$

and so

$$(z_1, (\lambda + \mu)v_2 + (\lambda + \mu^2)z_2) = (v_1, (\lambda + \mu)z_2).$$

Moreover, as $[v_n, a] = z_n$, and $|a| = 2^m$ for some $m \in \mathbb{Z}$, v_n and z_n are linearly independent. Thus it follows that $\lambda + \mu = 0 = \lambda + \mu^2$, and so $\mu \neq 0$ implies that $\mu = \lambda = 1$. But then $[v, b] = \mu[v, a]$ for all $v \in V_1$ and V_2 implies that $v^a = v^b$ for all $v \in V_1$ and V_2. Therefore, since A acts faithfully, we get that $a = b$, contradicting our choice of a and b and completing the proof of the lemma.

For the remainder of this section, we adopt the following hypothesis: Let G be a Chevalley group over K, as before we assume that $char(K) = 2$. Let $V = \bigotimes_{i \in I} V_i$ be a KG-module with each V_i nontrivial and irreducible, I some index set; $A \leq G$ an elementary abelian 2-subgroup with $|A| \geq 4$ such that A acts linearly dependently on V.

Lemma 5.4 *A acts faithfully on each V_i and $|I| = 1$*

Proof: Suppose there is an $i \in I$ and $1 \neq a \in A$ such that $[V_i, a] = 0$. Then $\langle a^G \rangle \leq C_G(V_i)$. Let $M = G/C_G(V_i)$. Since the $char(K) = 2$, $Z(G)$ has odd order and so $C_G(V_i) \not\leq Z(G)$. In particular, we see that M is a 2-group. By definition, $C_M(V_i) = 1$. However, as $|M| = 2^k$, for some $k \in \mathbb{Z}$, $C_M(V_i) \neq 1$ unless $M = 1$. Consequently, G must centralize V_i, a contradiction. Therefore, A acts faithfully on each V_i. It follows from the second part of lemma 5.3 that $|I| = 1$.

Remark: Because A acts faithfully on V_i, it follow that if $a, b \in A^\sharp$ and $\lambda \in K$ such that $[v, a] = \lambda[v, b]$ for all $v \in V_i$, then $\lambda \neq 0$. Moreover, if $c \in A$ such that $[v, c] = \mu[v, b]$ for all $v \in V_i$, then $\lambda = \mu$ if and only if $a = c$.

Lemma 5.5 *Let $1 \neq a, b \in A$. Then $C_G(a) = C_G(b)$.*

Proof: Let $g \in C_G(a)$. Since A acts linearly dependently on V, there exist $\lambda \in K^\sharp$ such that $[v, a] = \lambda[v, b]$, for all $v \in V$. Hence $\lambda(v^{bg} + v^b + v^g + v) = \lambda[v, b, g] =$

$[v,a,g] = [v,g,a] = \lambda[v,g,b] = \lambda(v^{gb} + v^b + v^g + v)$. Thus, $v^{bg} = v^{gb}$ for all $v \in V$. Therefore $g \in C_G(b)$ too, since G acts faithfully on V.

Lemma 5.6 *Let G and V be as above. If $r = \sum_{r_j \in \Pi} n_j r_j \in \Phi$, then $h_r(\lambda)$ and $\prod_{r_j \in \Pi} h_{r_j}\left(\lambda^{n_j \frac{(r_j,r_j)}{(r,r)}}\right)$ act identically on V.*

Proof: By [12], Lemma 19 (page 27), $h_r(\lambda)$ acts as multiplication by $\lambda^{\langle \mu, r\rangle}$ on the weight space V_μ of V, where, $\langle \mu, r\rangle = \mu(H_r)$ and $H_r = \frac{2r}{(r,r)}$. Thus since $r = \sum_{r_j \in \Pi} n_j r_j$,

$$H_r = \sum_{r_j \in \Pi} n_j \frac{2r_j}{(r,r)} = \sum_{r_j \in \Pi} n_j \frac{(r_j,r_j)}{(r,r)} H_{r_j}.$$

So because μ is a linear functional, it follows that $h_r(\lambda)$ and

$$\prod_{r_j \in \Pi} h_{r_j}\left(\lambda^{n_j \frac{(r_j,r_j)}{(r,r)}}\right)$$

act identically on V_μ, and hence on all of V since V is the sum of its weight spaces.

Lemma 5.7 *Let $S \in Syl_2(G)$ with $A \leq S$. Let P_1 and P_2 be parabolic subgroups of G containing S such that $G = \langle P_1, P_2\rangle$. Then there exists $g \in G$ and $i \in \{1,2\}$ such that $A^g \leq P_i$ with $A^g \cap O_2(P_i) = 1$.*

Proof: By Lemma 4.4 there exists $g \in G$ and $i \in \{1,2\}$ such that $A^g \leq P_i$, but $A^g \not\leq O_2(P_i)$. Suppose $1 \neq a \in A^g \cap O_2(P_i)$. Let Y be a chief factor in V for P_i. Then $a \in O_2(P_i)$ implies that $[Y,a] = 0$. Let $1 \neq b \in A^g$. Then by the above remark, there exists $\lambda \in K^\sharp$ such that $[v,a] = \lambda[v,b]$ for all $v \in Y$. Thus $[Y,b] = 0$ as well and so we see that $[Y, A^g] = 0$ for every chief factor Y of V for P_i. However, this is equivalent to $A^g \leq O_2(P_i)$, contrary to the above and proving the lemma.

Remark: It follows from Lemma 5.7 and induction that there exists an $r_i \in \Pi$ and $g \in G$ such that $A^g \leq P_{r_i}$ and $A^g \cap O_2(P_{r_i}) = 1$. Without loss we can assume that

$A \leq P_{r_i}$ and $A \cap O_2(P_{r_i}) = 1$ since if A acts linearly dependently on V, then so do all of its conjugates.

Theorem 5.8 *If G, V, A, and $r_i \in \Pi$ are as above, then $|K| \geq |A| \geq 4$ and up to algebraic conjugates of V one of the following hold:*

1. *$G \cong A_l(K)$ and V is a fundamental module.*
2. *$G \cong B_l(K)$ and one of the following is true:*
   (a) *r_i is a long root and V is the natural or spin module, or*
   (b) *r_i is a short root and V is a fundamental module.*
3. *$G \cong D_l(K)$ and V is the natural or a half-spin module.*
4. *$G \cong E_6(K)$ and $V = V(\lambda_1)$ or $V(\lambda_6)$.*
5. *$G \cong E_7(K)$ and $V = V(\lambda_7)$.*
6. *$G \cong F_4(K)$ and one of the following is true:*
   (a) *r_i is a long root and $V = V(\lambda_4)$, or*
   (b) *r_i is a short root and $V = V(\lambda_1)$.*
7. *$G \cong G_2(K)$, r_i is short, and $V = V(\lambda_2)$, the natural module.*

Proof: It follows from 4.13 and Lemma 5.4 that $V \cong V(\lambda)^{\varsigma}$ where $V(\lambda)$ is a restricted module for G and $\varsigma \in Aut(K)$. We can assume without loss that V is a restricted module.

Let $L = \langle X_{\pm r_i} \rangle \cong Sl_2(K)$ and let Y be a non-trivial chief factor in V for P_{r_i}. Also let $\hat{A} \subseteq L$ be such that $\hat{A}O_2(P_{r_i}) = AO_2(P_{r_i})$. Since $[Y, O_2(P_{r_i})] = 0$, $\hat{A}$ acts linearly dependently on Y. Let V_n denote the natural module for $SL_2(K)$. So as Y is also a chief factor for L and since V_n is the unique non-trivial restricted module for L, it follows as above from Lemma 5.4 that $Y \cong V_n^\sigma$, for some $\sigma \in Aut(K)$.

Let $\Upsilon = \{\sigma \in Aut(K) \mid V_n^\sigma \cong Y$ where Y is a chief factor of V for $L\}$. Choose $\sigma \in \Upsilon$, $a \in A^\sharp$, and Y a chief factor of V for L such that $Y \cong V_n^\sigma$. Then for all $b \in A$, there exists $\lambda_b \in K$ such that $[v,b] = \lambda_b[v,a]$ for all $v \in Y$. Note that by the remark following Lemma 5.4, $\lambda_b = \lambda_c$ if and only if $b = c$. Thus if we let $K(A)_\sigma = \{\lambda_b \mid b \in A\}$, then $|K(A)_\sigma| = |A|$. Thus as $K(A)_\sigma \leq K$, we see that $|K| \geq |A| \geq 4$.

Now choose $\gamma \in \Upsilon$ and $Y' \cong V_n^\gamma$ a chief factor of V for L. Then for each $v \in Y'$, $[v,b] = \lambda_b^{\sigma^{-1}\gamma}[v,a]$. In particular, $\lambda_b = \lambda_b^{\sigma^{-1}\gamma}$, and so we see that

$$\lambda^\sigma = \lambda^\gamma \quad \text{for all} \quad \lambda \in K(A)_\sigma, \;\; \gamma \in \Upsilon \tag{5.1}$$

Now let $0 \neq v \in C_V(U)$, the highest weight space of V. Suppose $r \in \Phi$ with $|r| = |r_i|$ such that $h_r(\lambda)v \neq v$ for some $\lambda \in K$. Let $L' = \langle X_{\pm r}\rangle$. Since $char(K) = 2$ and since $|r| = |r_i|$, there exists $w \in W$ such that $x_{r_i}(t)^w = x_r(t)$ for all $t \in K$, and $L^w = L'$. Since $h_r(\lambda)v \neq v$, there is a non-trivial chief factor Y/X of V for $P_{r_i}^w$ such that $v \in Y \setminus X$. But then $(Y/X)^{w^{-1}}$ is a non-trivial chief factor for L and so, as above, $(Y/X)^{w^{-1}} \cong V_n^\sigma$ as KL-modules for some $\sigma \in \Upsilon$. Let $\overline{v} = v + X$. Then $[\overline{v}^{w^{-1}}, X_{r_i}] = [\overline{v}, X_r]^{w^{-1}} = 0$ since $X_r \leq U$. Hence $\overline{v}^{w^{-1}} \in C_{(Y/X)^{w^{-1}}}(X_{r_i})$ implies that $\overline{v}^{w^{-1}}$ is in the highest weight space of $(Y/X)^{w^{-1}}$. Thus, $h_{r_i}(\lambda)\overline{v}^{w^{-1}} = \lambda^\sigma \overline{v}^{w^{-1}}$. In particular, since $h_r(\lambda) = h_{r_i}(\lambda)^w$ we see that $h_r(\lambda)\overline{v} = \lambda^\sigma \overline{v}$. Therefore,

$$h_r(\lambda)v = \lambda^\sigma v \quad \text{for all} \quad \lambda \in K. \tag{5.2}$$

Let $J = \{r_j \in \Pi \mid h_{r_j}(\lambda)v \neq v \text{ for some } \lambda \in K\}$. Since V is a restricted module, it follows from Lemma 5.6 that

$$h_r(\lambda)v = \prod_{r_j \in J} h_{r_j}\left(\lambda^{n_j \frac{(r_j,r_j)}{(r,r)}}\right)v = \lambda^{\psi}v, \tag{5.3}$$

where $\psi = \sum_{r_j \in J} n_j \frac{(r_j,r_j)}{(r,r)}$.

If $|r| = |r_i|$ and if $h_r(\lambda)v \neq v$, we define $\sigma_r(\lambda) = \lambda^{\psi}$ for all $\lambda \in K$. It then follows from equations (5.2) and (5.3) above that $\sigma_r \in Aut(K)$. In particular, if we choose another $s \in \Phi$ with $|s| = |r_i|$ such that $h_s(\lambda)v \neq v$ for some λ and $s = \sum_{r_j \in \Pi} s_j r_j$, then it follows from equations (5.1) and (5.2) that for all $\lambda \in K(A)_{\sigma_r}$

$$\lambda^{\sum_{r_j \in \Pi} s_j \frac{(r_j,r_j)}{(r_i,r_i)}} = \lambda^{\sum_{r_j \in \Pi} n_j \frac{(r_j,r_j)}{(r_i,r_i)}} \tag{5.4}$$

For convenience of notation, let $p_j = \frac{(r_j,r_j)}{(r_i,r_i)} = \frac{|r_j|^2}{|r_i|^2}$ for all $r_j \in \Pi$.

Now, choose $r \in \Phi^+$ with $|r| = |r_i|$ and such that if $r = \sum_{r_j \in \Pi} n_j r_j$, then $\sum_{r_j \in J} n_j$ is maximal among all such roots.

Case 1: Suppose $|r_i| \leq |r_j|$ for all $r_j \in \Pi$.

We claim that either $\sum_{j \in J} n_j = 1$ or (7.) from the statement of the Theorem is true (i.e. $G \cong G_2(K)$, etc.). So, suppose not. Then by Lemma 4.1 applied to r, we see that there are roots $s, s' \in \Phi^+$ with $|s| = |s'| = |r_i|$ such that if $s = \sum_{r_j \in \Pi} s_j r_j$ and $s' = \sum_{r_j \in \Pi} s'_j r_j$, then $\sum_{j \in J} s_j = 2$ and $\sum_{j \in J} s'_j = 1$, and such that if $k \in J$ with $s'_k \neq 0$, then $s_k \neq 0$ either. Hence, it follows from equation 5.3 that there exists $j, k \in J$ (not necessarily distinct) such that $h_s(\lambda)v = \lambda^{p_k + p_j}v$ and $h_{s'}(\lambda)v = \lambda^{p_k}v$. Thus, by equation 5.4, $\lambda^{p_k + p_j} = \lambda^{p_k}$ for all $\lambda \in K(A)_{\sigma_s}$ and so $\lambda^{p_j} = 1$ for all $\lambda \in K(A)_{\sigma_s}$. However, as $|r_j| \geq |r_i|$ we see that $p_j = \frac{|r_j|^2}{|r_i|^2}$ =1, 2, or 3. But if p_j =1 or 2, we get a contradiction to $|K(A)_{\sigma_s}| = |A| \geq 4$.

To finish this case, we need only show that $G \cong G_2(q)$ and $1 \in J$ is not a possibility, so suppose it is. First, assume that $\{1,2\} = J$. Then if we let $s = r_2$ and $s' = r_1 + 2r_2$, we see that $h_s(\lambda)v = \lambda v$ while $h_{s'}(\lambda)v = \lambda^5 v$, and so $\lambda^5 = \lambda$ for all $\lambda \in K(A)_{\sigma_s}$, yielding a contradiction similar to the one above. Thus we can assume that $J = \{1\}$. So, if $r = r_1 + r_2$, then $\sigma_r \in Aut(K)$, but $\sigma_r(\lambda) = \lambda^3$, for all λ, a contradiction since by Proposition 4.11 we can assume that K is algebraically closed. Hence if $G = G_2(K)$, then $J = \{2\}$.

Case 2: Suppose $|r_i| > |r_j|$ for some $r_j \in \Pi$.

Then we claim that $\sum_{j \in J} n_j \frac{|r|^2}{|r_j|^2} = 1$. So, suppose not. By Lemma 4.1 there are roots s and $s' \in \Phi$ with $|s| = |s'| = |r_i|$, such that if $s = \sum_{r_j \in \Pi} s_j r_j$ and if $s' = \sum_{r_j \in \pi} s'_j r_j$, then $\sum_{j \in J} s_j \frac{|r_j|^2}{|r|^2} = 2$ and $\sum_{j \in J} s'_j \frac{|r_j|^2}{|r|^2} = 1$. It follows from equation 5.3 that $h_s(\lambda)v = \lambda^2 v$ and $h_{s'}(\lambda)v = \lambda v$. And so, as above, we get that $\lambda^2 = \lambda$ for all $\lambda \in K(A)_{\sigma_s}$, contrary to $|A| \geq 4$.

An inspection of the various roots of maximal height in each root system, as recorded in Table 3.5 now yields the result, proving the theorem.

CHAPTER 6

Linearly Dependently Acting Fours-groups

In this chapter we prove the following corollary to Theorem 5.8:

Corollary 6.1 *Suppose G, A, and V are as in Theorem 5.8. Then the following are true:*

1. *If $G \cong A_l(K)$ and if V is not an algebraic conjugate of the natural module for G, then A is contained in a root subgroup.*
2. *If $G \cong B_l(K)$ and if V is not an algebraic conjugate of the natural module for G, then one of the following is true:*
   (a) V is not an algebraic conjugate of the spin module for G and A is contained in a short root subgroup of G.
   (b) $l \geq 3$, V is an algebraic conjugate of the spin module for G and A is contained in a root subgroup of G.
3. *If $G \cong D_l(K)$ and V is not an algebraic conjugate of the natural module for G, then A is contained in the product of two commuting root subgroups of G.*
4. *If $G \cong E_6(K)$ or $E_7(K)$, then A is contained in a root subgroup of G.*

5. *If $G \cong F_4(K)$, then one of the following is true:*

   (a) *V is an algebraic conjugate of $V(\lambda_1)$ and A is contained in a long root subgroup of G.*

   (b) *V is an algebraic conjugate of $V(\lambda_4)$ and A is contained in a short root subgroup of G.*

6. *If $G \cong G_2(K)$, then A is contained in a short root subgroup of G.*

The proof of this corollary will follow from the several lemmas to follow. Consequently, throughout the chapter we will assume that G, A, and V are as in Theorem 5.8. The information on certain parabolic subgroups used below can be found in [5] or can be easily verified computationally.

Assume first that $G \not\cong G_2(K)$. We will handle $G \cong G_2(K)$ separately at the end of the chapter.

Choose $1 \neq a \in A$. By Lemma 4.8 we may assume there exists a root, say $t \in \Phi$, such that $X = \langle X_{-t}^{a}, X_{-t} \rangle = \langle X_{\pm t} \rangle \cong Sl_2(K)$ and $[X_t, a] = 1$.

Lemma 6.2 *If $G \cong A_l(K)$, $D_l(K)$, $E_n(K)$, or $F_4(K)$ or if $G \cong B_l(K)$ and t is a short root, then $A \leq X \times C_G(X)$.*

Proof:

Case 1: $G \cong A_l(K)$.

Without loss we may assume that $t = r_1$. Thus $N_G(X) = (X \times C_G(X))H = L_2H$. By [11], $Y = C_V(Q_2)$ is a nontrivial irreducible KM_2-module. Moreover, because M_2 is a maximal parabolic subgroup of G and because Y is a proper subspace of V, it follows that $M_2 = N_G(Y)$. Thus $A \leq M_2$ since $a \in N_G(Y)$ implies that $A \leq N_G(Y)$, by Proposition 5.2.

Now let $Q_2^- = \langle X_{-r} \mid X_r \in Q_2 \rangle$. Then $M_2^- = L_2 H Q_2^-$ is also a maximal parabolic subgroup of G containing a. By an argument identical to one used above, $A \leq M_2^-$ as well. Hence, $A \leq O^{2'}(M_2 \cap M_2^-) = L_2$.

Case 2: $G \ncong A_l(K)$ or $F_4(K)$.

Then as $a \in C_G(X_t)$, $A \leq C_G(X_t)$ as well by Lemma 5.5. Now let $Y = C_V(X)$. Because $a \in N_G(Y)$, $A \leq N_G(Y)$ too, by Proposition 5.2. Thus if $A \nleq X \times C_G(X)$, then $O_2(N_G(X_t)) \cap (N_G(Y) \backslash X_t) \neq 1$. Hence because $O_2(N_G(X_t))/X_t$ is an irreducible module for the Levi compliment of $N_G(X_t)$, it follow that $G = \langle N_G(X_t), X_{-t} \rangle \leq N_G(Y)$, a contradiction.

Case 3: $G \cong F_4(K)$.

Because of the graph automorphism of F_4, we may assume without loss that t is a short root. Moreover, we may assume that t is the highest short root in Φ, namely $t = r_{23}$. As in Case 2, we have that $A \leq M_4 \cap N_G(Y)$, where $M_4 = N_G(X_t)$ and $Y = C_V(X)$. However, although Q_4/X_t is an indecomposable module for M_4, it is not irreducible. Rather, M_4 acts irreducibly on X_t, S/X_t, and Q_4/S where $S = Z(Q_4) = \langle X_{r_{10}}, X_{r_{16}}, X_{r_{18}}, X_{r_{20}}, X_{r_{22}}, X_{r_{23}}, X_{r_{24}} \rangle$. Thus, by an argument similar to the one used in Case 2, we may assume that $A \leq \langle L_4, S \rangle$.

Now let $s_1 = -r_{24}$, $s_2 = r_1$, $s_3 = r_2$, and $s_4 = r_3$. Then one can easily check that $\Phi' = \langle \pm s_1, \pm s_2, \pm s_3, \pm s_4 \rangle$ is a root subsystem of type B_4 and that $A \leq \langle L_4, S \rangle \leq \langle X_s \mid s \in \Phi' \rangle \cong B_4(K)$. Thus, by Case 2, we get that $A \leq X \times C_G(X)$, proving the lemma.

Lemma 6.3 *If G is as in Lemma 6.2, then Corollary 6.1 holds.*

Proof: We assume that if t is a long root, then it is the highest long root and if t is a short root, then it is the highest short root. Let $J \subseteq \Pi$ such that $N_G(X_t) = M_J$. Thus we have $A \leq X \times L_J$ by Lemmas 4.7 and 6.2.

Now let λ be the weight of V and assume that $\lambda \notin \mathbb{Z}\lambda_j$ for some $r_j \in J$. (Recall that by Theorem 5.8, λ is a fundamental weight.) By Lemma 5.6 neither X nor L_J centralize $C_V(U)$; so choose a non-trivial chief factor Y/W of V for $X \times L_J$ such that $C_V(U) \subseteq Y \setminus W$. Then by the corollary to Lemma 68 in [12], $Y/W \cong V_1 \otimes V_2$ as a $K(X \times L_J)$-module, where V_1 is a nontrivial irreducible KX-module and V_2 is a nontrivial irreducible KL_J-module. Suppose that $[V_2, A] \neq 0$. Then by Lemma 5.3 there exists $1 \neq c \in A$ such that $[V_1, c] = [V_2, c] = 0$. That is, c normalizes every subspace of V_1. However, this contradicts Proposition 5.2 since the rest of A clearly does not. Hence, $[V_2, A] = 0$.

Now, if M_J is a connected parabolic subgroup, then $[V_2, A] = 0$ implies that $A \leq X$ and so, in particular, $A \leq X_t$.

Thus we may assume that $G \cong D_l(K)$. Let $\overline{A} \leq L_J$ be such that $AX/X = \overline{A}X/X$ in $N_G(X)/X$. Then $L_J \cong Sl_2(K) \times D_{l-2}(K)$ if $l \geq 5$ and $L_J \cong Sl_2(K) \times Sl_2(K) \times Sl_2(K)$ if $l = 4$. Then as $\lambda \in \mathbb{Z}\lambda_{l-1}$ or $\mathbb{Z}\lambda_l$ by assumption, $\overline{A}$ must be contained in the $Sl_2(K)$ factor in the first case or in the product of at most two $Sl_2(K)$ factors in the second case.

Now assume that $G \cong D_4(K)$ and suppose that $\lambda \in \mathbb{Z}\lambda_4$. Then by the above, $\overline{A} \leq \langle X_{r_1} \rangle \times \langle X_{r_3} \rangle \times \langle X_t \rangle$. However, by Lemma 4.2, $\langle \pm r_1, \pm r_3 \rangle$ is conjugate to $\langle \pm(e_1 + e_4), \pm(e_2 + e_3) \rangle = \langle \pm(r_1 + r_2 + r_4), \pm(r_2 + r_3 + r_4) \rangle$ and so we see that either $\overline{A} \leq \langle X_{r_1} \rangle \times \langle X_t \rangle$, or $\overline{A} \leq \langle X_{r_3} \rangle \times \langle X_t \rangle$. In either case, however, we get that A is contained in the product of two root subgroups. Similarly if $\lambda \in \mathbb{Z}\lambda_3$ or, in fact, $\lambda \in \mathbb{Z}\lambda_1$.

Lemma 6.4 *If $G \cong B_l(K)$, $l \geq 3$, and if $\lambda \in \mathbb{Z}\lambda_l$, then A is contained in a root subgroup of G.*

Proof: Without loss we may assume that t is the highest long root in Φ. Then $N_G(X) \leq M_2$. As in the lemma above, we have $A \leq N_G(C_V(X)) \cap M_2$.

Let s be the highest short root in Φ. Also let $\Phi_0 = \{r \in \Phi \mid \text{if} \quad r = \Sigma_{r_i \in \Pi} n_i r_i,$ then $\quad n_1 = 0\}$ and let $Q_0 = \langle X_r \mid r \in \Phi_0^+ \rangle \cap Q_2$. Note that Φ_0 is a root subsystem of type B_{l-1} and that $t - s$ is the longest short root there. Thus we get that L_{12} acts irreducibly on Q_0/X_{t-s}.

Similarly, L_{12} acts irreducibly on $(Q_1 \cap Q_2)/\langle X_t, X_s \rangle$. Thus as conjugation by elements in $\langle X_{\pm r_1} \rangle$ interchanges elements in Q_0 with elements in $Q_1 \cap Q_2$. It follows that L_2 acts irreducibly on Q_2/Z where $Z = \langle X_t, X_s, X_{t-s} \rangle$.

In particular, as in Lemma 6.2, we get that $A \leq ZL_2$. However, note that $\Phi' = \{\pm t, \pm s, \pm(t-s), \pm r_1\}$ is a root subsystem of type B_2. So, if we let $M = \langle X_r \mid r \in \Phi' \rangle$, then $ZL_2 = M \times L_{12} \cong B_2(K) \times B_{l-2}(K)$. Therefore, since both of the above factors act nontrivially on $C_V(U)$, it follows as above that $A \leq M$.

Lastly, since $B_2(K)$ has three conjugacy classes of involutions with representatives $x_t(1)$, $x_s(1)$, and $x_t(1)x_s(1)$, we may assume that A contains one of these. However, note that if we let c denote this element and if c is one of the two latter types of involutions, then $\langle X_{-s}^c, X_{-s} \rangle$ is certainly not a 2−group and so we may apply Lemma 6.2. Thus we have $c = x_r(1)$, but then as $Z(C_M(c)) = X_r$, we get that $A \leq X_r$, by Lemma 5.5.

Corollary 6.5 *Let $G \cong B_2(K)$ and A be as in Theorem 5.8. Also let $V = V(\lambda_1)$ denote the natural module for G. If A acts linearly dependently on both the natural and the spin module for G, then A is contained in a root subgroup of G. Moreover, if A acts linearly dependently on either the natural or the spin module for G, then $dim_K([V, A]) \leq 2$.*

Proof: Let ϕ be the graph automorphism of G. Then as

$$\begin{aligned}\phi: \quad & x_{r_1}(t) \mapsto x_{r_2}(t)\\ & x_{r_1+r_2}(t) \mapsto x_{r_1+2r_2}(t)\\ & x_{r_2}(t) \mapsto x_{r_1}(t^2)\\ & x_{r_1+2r_2}(t) \mapsto x_{r_1+r_2}(t^2),\end{aligned}$$

it follows that $V^{\phi^{-1}} = V(\lambda_2)$.

Now assume that A acts linearly dependently on both $V(\lambda_1)$ and $V(\lambda_2)$. Then it follows from the above that both A and A^{ϕ} must act linearly dependently on V. However, because G contains three conjugacy classes of involutions with representatives

$$x_{r_1}(1), \quad x_{r_2}(1), \quad \text{and} \quad x_{r_1+r_2}(1)x_{r_1+2r_2}(1),$$

we may assume that A contains one of these. Thus because A acts linearly dependently on V we see that A must be contained in one of the following sets: X_{r_1}, X_{r_2}, or $\{x_{r_1+r_2}(t)x_{r_1+2r_2}(t) \mid t \in K\}$. However, the last of the above sets is not normalized by ϕ, and so can not contain A. Therefore A must be contained in a root subgroup, proving the first part of the corollary.

The second part of the corollary now follows since, by the above, either A or A^{ϕ} acts linearly dependently on V. Thus, A must be contained in either one of the three aforementioned sets or in the set $\{x_{r_1+r_2}(t^2)x_{r_1+2r_2}(t) \mid \lambda \in K\}$ and so, in particular, the elements of A are conjugate under H. Hence if $1 \neq a, b \in A$ with $b^h = a$ for some $h \in H$, then $[V,a] = [V,b^h] = [V,b]^h = [V,b]$ since H acts as the group of diagonal matrices on V. Therefore, $[V,a] = [V,A]$ for all $1 \neq a \in A$.

Lemma 6.6 *Let $G \cong G_2(q)$, A, and V be as in Theorem 5.8. Then A is contained in a short root subgroup.*

Proof: By [14], (8.1) G has two conjugacy classes of involutions with representatives $t = x_{2a+b}(1)$ and $z = x_{3a+2b}(1)$. Moreover, it follows from [14], (3.3), (8.5), and the commutator relations developed in [14] that $Z(C_G(t)) = X_{2a+b}$ and $Z(C_G(z)) = X_{3a+2b}$.

Without loss we can assume that either t or z is an element of A. If $t \in A$, then by 5.5 and the above we see that A is contained in a short root subgroup and we are done. So assume now that $z \in A$. Then again by 5.5 and the above we see that A is contained in a long root subgroup. However, this implies that a conjugate of A is contained in $P_{r_1} \setminus O_2(P_{r_1})$, contradicting Theorem 5.8 and proving this lemma and Theorem 6.1.

CHAPTER 7

A Result Concerning Root Systems and Weights

Throughout this chapter, assume that G is a Chevalley group, $G \not\cong G_2(K)$, and Φ is the root system associated with G. As usual, let Π be a fundamental root system in Φ.

Definition 7.1 *Let λ be a weight of an irreducible module for G. Define*

$$\lambda^\perp = \{r \in \Phi \mid \langle \lambda, r \rangle = 0\}.$$

Let J be a rank two root subsystem of Φ. Recall that in Lemma 4.2 we determined the orbit of J in Φ under W. We will now prove the following lemma in which we determine all weights λ such that $\lambda^\perp \cap J^w \neq \emptyset$ for all $w \in W$.

Lemma 7.2 *Let $J \subseteq \Phi$ be a rank 2 root subsystem. If λ is a weight such that $\lambda^\perp \cap J^w \neq \emptyset$, for all $w \in W$, then λ is an integral multiple of one of the fundamental weights listed in the table below.*

Φ	J	**Possible Weights**
A_l	$A_2(long)$	λ_i *where* $1 \leq i \leq n$
	$L \perp L$	λ_1 *or* λ_l
B_l	$A_2(long)$	λ_1 *or* λ_l
	B_2	λ_i *where* $1 \leq i \leq l$
	$(L \perp L)_1$	λ_1
	$(L \perp L)_2$	λ_l
	$L \perp S$	λ_1
	$S \perp S$	λ_1
D_4	$A_2(long)$	λ_1, λ_3, *or* λ_4
	$(L \perp L)_1$	λ_3 *or* λ_4
	$(L \perp L)_2$	λ_1 *or* λ_3
	$(L \perp L)_3$	λ_1 *or* λ_4
D_l, $l \geq 5$	$A_2(long)$	λ_1, λ_{l-1}, *or* λ_l
	$(L \perp L)_1$	λ_1
	$(L \perp L)_2$	λ_{l-1} *or* λ_l
E_6	$A_2(long)$	λ_1 *or* λ_6
	$L \perp L$	*None are possible*
E_7	$A_2(long)$	λ_7
	$L \perp L$	*None are possible*
E_8	$A_2(long)$	*None are possible*
	$L \perp L$	*None are possible*

Φ	J	**Possible Weights**
	$A_2(long)$	λ_4
	$A_2(short)$	λ_1
	B_2	λ_1 *or* λ_4
F_4	$L \perp L$	*None are possible*
	$L \perp S$	*None are possible*
	$S \perp S$	*None are possible*

This lemma will follow from the next several lemmas.

Lemma 7.3 *Let $J \subseteq \Phi$, with $J \cong A_2$ or B_2. Choose $r_j \in \Pi$. Also,*

1. *if $J \cong A_2(long)$ or B_2, let $r = \sum_{r_i \in \Pi} n_i r_i$ be the highest long root in Φ, or*

2. *if $J \cong A_2(short)$, let $r = \sum_{r_i \in \Pi} n_i r_i$ be the highest short root in Φ.*

If $n_j = 1$ or if $J \cong B_2$ and $n_j \leq 2$, then $K(\lambda_j)_{J^w} \neq \emptyset$ for all $w \in W$.

Proof: Let $w \in W$. Choose $s, t \in J^w$ to be positive roots such that $|s| = |t|$ and $s - t \in J^w$. Note that if $J \cong B_2$, then s and t must both be short roots. Suppose that $s = \sum_{r_i \in \Pi} s_i r_i$ and $t = \sum_{r_i \in \Pi} t_i r_i$. If either s_j or t_j is zero, then $K(\lambda_j)_{J^w} \neq \emptyset$, so assume that $s_j, t_j \neq 0$.

Suppose first that $n_j = 1$. Then as r was chosen to have maximal height, $0 < s_j, t_j \leq n_j$ and so $s_j = t_j = 1$. Hence $\langle \lambda_j, s - t \rangle = 0$, and so $K(\lambda_j)_{J^w} \neq \emptyset$.

Now suppose that $J \cong B_2$ and $n_j \leq 2$. Then because $s + t$ is a long root in J^w and because r was chosen to have maximal height, we see that $s_j + t_j \leq 2$. Consequently, as above, $s_j = t_j = 1$ and so $K(\lambda_j)_{J^w} \neq \emptyset$.

Hence we have found a sufficient condition, when J is connected, for a fundamental weight to be included in the table above. We will show now that these are in fact the only weights that should be included in the table.

Definition 7.4 *For a weight* λ *and* $r = \sum_{r_i \in \Pi} n_i r_i \in \Phi$, *define*

$$S_\lambda(r) = \sum_{r_i \in \Pi \setminus \lambda^\perp} n_i |r_i|^2 / |r|^2.$$

Lemma 7.5 *Suppose that* J *is a connected rank 2 root subsystem in* Φ *and that* λ *is a weight such that* $\lambda^\perp \cap J^w \neq \emptyset$, *for all* $w \in W$. *If* J *has type* A_2*(long), let* r *be the highest long root in* Φ. *Also, if* J *has type* A_2*(short) or* B_2, *let* r *be the highest short root in* Φ. *If* $J \cong B_2$, *then* $S_\lambda(r) \leq 2$. *Otherwise,* $S_\lambda(r) = 1$.

Proof: Suppose not.

Case 1: Assume that all the roots in Φ have the same length and so $J \cong A_2$.
By Lemma 4.1, there exists $r' \in \Phi$ and $r_i \in \Pi$ with $r' - r_i \in \Phi$, $S_\lambda(r') = 2$, and $S_\lambda(r' - r_i) = 1$. Thus, if $J' = \langle r', r_i \rangle$, then $J' \cong A_2$ implies that there exists $w \in W$ such that $J^w = J'$. However, $\lambda^\perp \cap J' = \emptyset$, contrary to the hypothesis.

Case 2: Assume that $\Phi = B_l$ and $J \cong A_2$.
Since $S_\lambda(r) > 1$, $\lambda \notin \mathbb{Z}\lambda_l$. Then, as in Case 1, there exists long roots $r' \in \Phi$ and $r_i \in \Pi$ such that $r' - r_i \in \Phi$ and such that $S_\lambda(r') = 2$ and $S_\lambda(r_i) = 1$ which brings us to the same contradiction as above.

Case 3: Assume that $\Phi = B_l$ and $J \cong B_2$.
As above, $\lambda \notin \mathbb{Z}\lambda_l$ and so by Lemma 4.1 there exists a short root $r' \in \Phi$ and a long root $r_i \in \Pi$ such that $S_\lambda(r') > S_\lambda(r' - r_i) > 0$. Now, since $J' = \langle \pm r', \pm r_i \rangle = \pm\{r', r_i, r' - r_i, r' + r_i\}$, we see that $\lambda^\perp \cap J' = \emptyset$, contrary to the hypothesis.

Case 4: Assume that $\Phi = F_4$.
Here it seems easiest to simply give explicit sets of roots which generate connected rank 2 root subsystems and which eliminate all but the desired values of λ.

1. Suppose that J has type A_2(long)

   Let $J' = \pm\{r_{13}, r_{16}, r_{24}\} = \pm\{1220, 1122, 2342\}$. Thus, if $\lambda^\perp \cap J' \neq \emptyset$, then

$\lambda \in \mathbb{Z}\lambda_4$ and so $S_\lambda(r) = 1$.

2. Suppose that J has type A_2(short)

   Let $J' = \pm\{r_{12}, r_{17}, r_{23}\} = \pm\{0111, 1121, 1232\}$. Thus, as above, $\lambda \in \mathbb{Z}\lambda_1$.

3. Suppose that J has type B_2

   Let $J' = \pm\{r_{10}, r_{11}, r_{23}, r_{24}\} = \pm\{0122, 1110, 1232, 2342\}$. Thus, as in the two cases above, $\lambda \in \mathbb{Z}\lambda_1$ or $\mathbb{Z}\lambda_4$.

Remark: It follows from Lemma 7.3 and 7.5 that if J is a connected rank 2 root subsystem and λ is a weight such that $\lambda^\perp \cap J^w \neq \emptyset$, for all $w \in W$, then λ is an integral multiple of one of the weights in Lemma 7.2.

Lemma 7.6 *Suppose that $\Phi \neq B_l$ or D_l and let J be a root subsystem of type $L \perp L$, $L \perp S$, or $S \perp S$. Then λ is a weight such that $\lambda^\perp \cap J^w \neq \emptyset$ for all $w \in W$ if and only if $\Phi = A_l$ and $\lambda \in \mathbb{Z}\lambda_1$ or $\mathbb{Z}\lambda_l$.*

Proof: Because of the graph automorphism of F_4, we may assume that J contains a long root. Thus without loss we may assume that J contains the highest long root, say r. As $\langle \lambda, r \rangle \neq 0$ it follows that every root with the appropriate length in $r^\perp$ is in $\lambda^\perp$.

Now choose $r_i \in \Pi \backslash \lambda^\perp$. Then $r_i \notin r^\perp$ and so $|r_i| = |r|$. As above we see that every root with the appropriate length in $r_i^\perp$ is then also in $\lambda^\perp$. However, unless $\Phi = A_l$, $r^\perp$ is a maximal root subsystem in Φ, in which case $\Phi = \langle r_i^\perp, r^\perp \rangle$, a contradiction. Hence $\Phi = A_l$ and $\lambda \in \mathbb{Z}(\lambda_1 + \lambda_l)$. If $\langle \lambda, r_1 \rangle \neq 0$, then $\langle r_2, r_3, \ldots, r_l \rangle = \langle r_1^\perp, r^\perp \rangle \leq \lambda^\perp$ and so $\lambda \in \mathbb{Z}\lambda_1$. Similarly, if $\langle \lambda, r_l \rangle \neq 0$, then $\lambda \in \mathbb{Z}\lambda_l$.

Moreover, we notice that if $r \in \Phi^+$ such that $\langle r, \lambda_1 \rangle \neq 0$, then $r = r_1 + r' = e_0 - e_j$ for some root r' and $1 \leq j \leq l$. Thus if r and s are roots such that $\langle r, \lambda_1 \rangle \neq 0 \neq \langle s, \lambda_1 \rangle$, then $r \not\perp s$. In particular, we see that λ_1 can not be eliminated from the list of possible weights. Similarly, neither can λ_l.

As in Lemma 7.5 it again seems easiest for $\Phi = B_l$ or D_l to explicitly list a set of roots which will eliminate all but the desired weights.

Assume that $\Phi = B_l$.

1. Suppose first that J has type $(L \perp L)_1$.

   Let $r = r_1 + r_2 + \cdots + r_{l-2} + 2r_{l-1} + 2r_l = e_1 + e_{l-1}$ and $s = r_2 + r_3 + \cdots + r_{l-1} + 2r_l = e_2 + e_l$. $J' = \langle r, s \rangle$ has type $(L \perp L)_1$ and so if $K(\lambda)_{J'} \neq \emptyset$, then $\lambda \in \mathbb{Z}\lambda_1$.

   Similarly as in Lemma 7.6, we note that $\langle r, \lambda_1 \rangle \neq 0$ if and only if $r = \pm e_1 \pm e_j$ for some j. Thus, as above, we see that λ_1 can not be eliminated from the list of possible weights.

2. Suppose now that J has type $(L \perp L)_2$.

   Let $r = r_1 + r_2 + \cdots + r_{l-2} + r_{l-1} = e_1 - e_l$ and $s = r_1 + r_2 + \cdots + r_{l-1} + 2r_l = e_1 + e_l$. $J' = \langle r, s \rangle$ has type $(L \perp L)_2$ and so if $\lambda^{\perp} \cap J' \neq \emptyset$, then $\lambda \in \mathbb{Z}\lambda_l$.

   Moreover, we note that as one of the roots, say r must be of the form $e_i - e_j$, it follows that if $r = \sum_{k=1}^{l} n_k r_k$, then $n_l = 0$. Hence λ_l may not be deleted from the list of possible roots.

3. Suppose that J has type $(L \perp S)$.

   Let $r = r_2 + r_3 + \cdots r_l = e_2$ and let $s = r_1 + r_2 + \cdots r_{l-1} + 2r_l = e_1 + e_l$. Thus, $\lambda \in \mathbb{Z}\lambda_1$ and, by the above, λ_1 can not be eliminated.

4. Suppose that J has type $(S \perp S)$.

   Let $r = r_1 + r_2 + \cdots r_l = e_1$ and let $s = r_2 + r_3 + \cdots r_l = e + 2$. As above, we see that only λ_1 has the desired property. Also because the only short root r such that $\langle \lambda_1, r \rangle \neq 0$ is e_1, it follows that λ_1 may not be eliminated.

Now assume that $\Phi = D_l$, $l \geq 5$.

1. Suppose that J has type $(L \perp L)_1$.

   Let $r = r_1 + r_2 + \cdots r_l = e_1 + e_{l-1}$ and let $s = r_2 + \cdots r_{l-2} + 2r_{l-2} + r_{l-1} + r_l = e_2 + e_{l-2}$. Then $\lambda \in \mathbb{Z}\lambda_1$ which, as above, can not be eliminated.

2. Suppose that J has type $(L \perp L)_2$.

   Let $r = r_1 + r_2 + \cdots + r_{l-1} = e_1 - e_l$ and let $s = r_1 + r_2 + \cdots + r_{l-2} + r_l = e_1 + e_l$. Thus $\lambda \in \mathbb{Z}\lambda_{l-1}$ or $\mathbb{Z}\lambda_l$.

   Now, as one of the roots, say r, must be of the form $e_i - e_j$, then if $r = \sum_{k=1}^{l} n_k r_k$, $n_l = 0$. Thus λ_l can not be eliminated from our list. Similarly if r is of the form $e_i + e_j$, $r = \sum_{k=1}^{l} n_k r_k$ and if $n_{l-1} \neq 0$, then $\max\{i, j\} \leq l$. Thus $s = e_i - e_j = \sum_{k=1}^{l} m_k r_k$ with $m_{l-1} = 0$, and so λ_{l-1} can not be eliminated. Note that the preceding statement also follows because of the graph automorphism of Φ.

Lastly assume that $\Phi \cong D_4$.

Here the result follows from inspection of the various conjugacy classes which are given explicitly in Lemma 4.2.

Lemma 7.2 now follows from the preceding arguments.

CHAPTER 8

Linear Independence

Definition 8.1 *Let G be a group of Lie-type and $A \leq G$ a fours-group.*

1. *We say that A is a linearly dependent fours-group in G if one of the following holds:*

   (a) $G \cong A_l(K)$ or $D_l(K)$ and A acts linearly dependently on the natural module for G.

   (b) $G \cong B_2(K)$ and A act linearly dependently on either the natural or the spin module for G.

   (c) $G \cong B_l(K)$, $l \geq 3$ and either A acts linearly dependently on the natural module for G or there exists a rank two connected parabolic subgroup, say P, of G such that A is contained in the Levi complement of P and is linearly dependent there.

   (d) $G \cong E_6(K)$, $E_7(K)$, or $E_8(K)$ and there exists a proper connected parabolic subgroup M_J such that $A^g \leq L_J$ and is linearly dependent there for some $g \in G$.

   (e) $G \cong F_4(K)$ or $G_2(K)$ and A is contained in a root subgroup of G.

   (f) $G \cong G^i(K)$ is a twisted Chevalley group and A is linearly dependent when considered as a subgroup of $G(K)$.

2. *We say that A is a linearly independent fours-group if A is not a linearly dependent fours-group.*

Proposition 8.2 *Let G, A, and V be as in Theorem 5.8. Then A is a linearly dependent fours-group.*

Proof: This follows from Corollary 6.1 and the definition above.

We now state the main theorem of the paper.

Theorem 8.3 *Let $G = G(K)$ be a Chevalley group defined over K, char$(K) = 2$ and let V a nontrivial, irreducible KG-module. If there exists a linearly independent fours-group A which act quadratically on V, then up to algebraic conjugacy of V, one of the following is true:*

1. *$G \cong A_l(K)$ and V is a fundamental module.*
2. *$G \cong B_l(K)$ and V is a fundamental module.*
3. *$G \cong D_l(K)$ and V is the natural or a half-spin module.*
4. *$G \cong E_6(K)$ and $V = V(\lambda_1)$ or $V(\lambda_6)$.*
5. *$G \cong E_7(K)$ and $V = V(\lambda_7)$.*
6. *$G \cong F_4(K)$ and $V = V(\lambda_1)$ or $V(\lambda_4)$.*
7. *$G \cong G_2(K)$ and $V = V(\lambda_2)$.*

Corollary 8.4 *Let k be the algebraic closure of K and let $G_0 = G^i(K) \leq G(k)$ be a twisted Chevalley group. Let V be a nontrivial, irreducible KG_0-module and suppose that G_0 contains a linearly independent fours-group which acts quadratically on V. Then V is obtained from the restriction to G_0 of a rational representation V' of $G(k)$, where V' is an arbitrary representation from the conclusion of Theorem 8.3.*

Proof: The corollary follows directly from Theorem 4.14 and Theorem 8.3.

The proof of Theorem 8.3 will follow from the next several lemmas and sections.

Lemma 8.5 *V is an algebraic conjugate of a restricted module for G.*

Proof: Suppose not. Then, by Theorems 4.13 and 4.14, V is the tensor product of two or more algebraic conjugates of restricted modules. However, A acts linearly dependently on each those restricted modules by Lemma 5.3, and so by Proposition 8.2, A is a linearly dependent fours-group, contrary to the hypothesis of the Theorem.

Remark: It follows from Lemma 8.5 that we may assume that V is an restricted module.

Lemma 8.6 *Suppose $G = A_2(K)$. Then V is isomorphic to the natural module for G.*

Proof: Let V_N denote the natural module for G. Because a is an involution and V_N is 3-dimensional, we may assume that a is in a root subgroup, say $X_{r_1+r_2}$. Also, as $C_G(X_{r_1+r_2}) \leq B$ we have that $A \leq U$. In particular, $U = \langle X_{r_1}, X_{r_2}, X_{r_1+r_2}\rangle$ and $[X_{r_1}, X_{r_2}] = X_{r_1+r_2}$ implies that either $A \leq O_2(P_{r_1})$ or $O_2(P_{r_2})$. Without loss, we assume that $A \leq O_2(P_{r_1})$, and $b \in O_2(P_{r_1}) \setminus X_{r_1+r_2}$. Then for all $g \in N_{r_2}$ we have $a^g \in U \setminus O_2(P_{r_1})$ and $b^g \in P_{r_2} \setminus U$.

Let $X = C_V(O_2(P_{r_1}))$ and note that because X is U-invariant and because $C_V(U)$ is 1-dimensional it follows that $C_V(U) \leq X$. If $C_U(X) \not\leq O_2(P_{r_1})$, then all of $\langle C_U(X)^{P_{r_1}}\rangle = O^{2'}(P_{r_1})$ acts trivially on $C_V(U)$ in which case $\lambda = \lambda_{r_2}$, and we are done. So assume that $C_U(X) < O_2(P_{r_1})$, but $C_V(U) \neq X$. Then, for each $g \in N_{r_2}$, $[X, a^g] \neq 0$. Moreover, $[X, a^g] = [X, O_2(P_{r_1})a^g]$ is invariant under U and so $C_V(U) \leq [X, a^g]$. In particular, we see that $C_V(U)$ is centralized by $\langle U, b^g\rangle$ for all $g \in N_{r_2}$. Hence, $\lambda = \lambda_{r_1}$ and we are done.

Lemma 8.7 *Let $G = B_2(K)$. Then V is an algebraic conjugate of either the natural or the spin module.*

Proof: Suppose not. Then it follows from the definition of linearly independent, Theorem 5.8 and from Theorem 4.13 that V is an algebraic conjugate of $V(\lambda_1 + \lambda_2)$. Recall that r_1 is a long root and r_2 is a short root. Let ϕ be the graph automorphism of G induced by the graph automorphism of B_2. Then as

$$\begin{aligned} \phi: \quad & x_{r_1}(t) \mapsto x_{r_2}(t) \\ & x_{r_2}(t) \mapsto x_{r_1}(t^2) \end{aligned}$$

we see that $V(\lambda_1 + \lambda_2)^{\phi^{-1}}$ is isomorphic to $V(\lambda_1) \otimes V(\lambda_2)^2$ as a KG-module. Thus V is isomorphic to $V_{\lambda_1} \otimes V_{\lambda_2}$. However by Lemma 5.3, this implies that A must act linearly dependently on both the natural and the spin module for G, contrary to A being a linearly independent fours-group, proving the lemma.

Definition 8.8 *Suppose that λ is a weight for G and $w \in W$. Then we define λ^w to be the weight defined by $\lambda^w(H_r) = \lambda(H_{w^{-1}(r)})$.*

Lemma 8.9 *Let G be a Chevalley group and $V = V(\lambda)$ be a nontrivial irreducible KG-module with highest weight λ. Also let P_J be a parabolic subgroup of G. Then for all $w \in W$, λ^w is conjugate under W_J to the highest weight of some chief factor of V for P_J. That is, each orbit of W_J on $\{\lambda^w \mid w \in W\}$ corresponds to a chief factor.*

Proof: Let $M_K = N_G(V_\lambda)$. For convenience of notation, in this lemma only we let $L_J = \langle X_{\pm r} \mid r \in J \rangle$ and $Q_J = O_2(P_J)$ instead of the usual definition of L_J and Q_J.

Note that because $\Phi_{J^{w^{-1}}} \cap \Phi^+$ is a positive root system for $J^{w^{-1}}$, $(\Phi_{J^{w^{-1}}} \cap \Phi^+)^w = \Phi_J \cap (\Phi^+)^w$ is a positive root system in Φ_J. Thus we get that $L_J \cap M_K^w$ is a parabolic subgroup of L_J. Hence there exists $w' \in W_J$ such that $P = Q_J H (L_J \cap M_K^w)^{w'}$ is a parabolic subgroup of G with $B \leq P$. So now let X/Y be a chief factor of V for P_J

such that $(V_{\lambda^{ww'}} + Y/Y) \cap (X/Y) \neq 0$. Then

$$(V_{\lambda^{ww'}} + Y)^B \leq (V_{\lambda^{ww'}} + Y)^{Q_J H (L_J \cap M_K^w)^{w'}} = (V_{\lambda^{ww'}} + Y)^{L_J \cap M_K^w)^{w'}} = V_{\lambda^{ww'}} + Y,$$

since L_J normalizes Y and $M_K^{ww'}$ normalizes $V_{\lambda^{ww'}}$. Thus, it follows that $\lambda^{ww'}$ is the highest weight of X/Y.

Proposition 8.10 *If G, V, and A are as in Theorem 8.3 and if there exists a connected rank two parabolic subgroup, say P_J, of G such that $A \subseteq O^{2'}(P_J) \setminus O_2(P_J)$ and such that $AO_2(P_J)/O_2(P_J)$ is a linearly independent fours-group in $O^{2'}(P_J)/O_2(P_J)$, then $\lambda^{\perp} \cap J^w \neq \emptyset$, for all $w \in W$, where λ is the highest weight in V*

Proof: It follows from Lemmas 8.6 and 8.7 that if γ is the highest weight of any chief factor of V for P_J, then $\gamma^{\perp} \cap J \neq \emptyset$. Thus, by Lemma 8.9, $(\lambda^w)^{\perp} \cap J \neq \emptyset$, for all $w \in W$. However, this is clearly equivalent to the condition that $\lambda^{\perp} \cap J^w \neq \emptyset$, for all $w \in W$.

Remark: Therefore we see that to prove Theorem 8.3 for $G \not\cong G_2(K)$, it is sufficient to show that there exists a connected rank two parabolic subgroup which contains a linearly independent fours-group. We will do exactly that in the next sections.

CHAPTER 9

The Classical Groups

Lemma 9.1 *Let $G = A_n(K)$ and let A be a fours-group in G. Let V be the natural module for G and suppose that A acts linearly independently on V. Then there is a 3-dimensional subspace $X \leq V$ normalized by A upon which A acts linearly independently.*

Proof: Suppose not.

Case 1: There exists $v \in V$ such that $\langle v^A \rangle$ is 4-dimensional.

Then v, v^a, v^b, and v^{ab} are linearly independent vectors in V. Let $X = \langle v^a + v, v^b + v, v^{ab} + v \rangle$. Then X is a 3-dimensional subspace normalized by A, and as $v^a + v \in N_V(a) \setminus N_V(b)$, we see that $N_X(a) \neq N_X(b)$, and so A act linearly independently on X by Lemma 5.2.

Case 2: There exists $v \in V$ such that $\langle v^A \rangle$ is 3-dimensional.

Then $X = \langle v^A \rangle$ satisfies the claim of the lemma since if $[v, b] \in K[v, a]$, then $v^b \in \langle v, v^a \rangle$. However, $v^b \in \langle v, v^a \rangle$ implies that $\langle v^A \rangle$ is only 2-dimensional, contrary to the assumption.

Case 3: $dim_K(\langle v^A \rangle) \leq 2$ for all $v \in V$.

Choose $v \notin C_V(a)$. Then v and v^a are linearly independent and so span $\langle v^A \rangle$. Thus, there exists λ and $\gamma \in K$ such that $v^b = \lambda v + \gamma v^a$. Then $v^{ab} = \lambda v^a + \gamma v$ and

$v = \lambda v^b + \gamma v^{ab}$. Substituting, we get $v = \lambda(\lambda v + \gamma v^a) + \gamma(\lambda v^a + \gamma v) = (\lambda^2 + \gamma^2)v$. So, $\lambda^2 + \gamma^2 = 1$ and so $\lambda = \gamma + 1$. Thus $v^b = (\gamma + 1)v + \gamma v^a$, and so $[v, b] = \gamma[v, a]$.

Suppose now that $C_V(a) \not\subseteq C_V(b)$ and choose $w \in C_V(a) \setminus C_V(b)$. Then $v + w \notin C_V(a)$ implies, by the above, that there exists $\alpha \in K$ such that

$$\gamma[v, a] + [w, b] = [v, b] + [w, b] = [v + w, b] = \alpha[v + w, a] = \alpha[v, a],$$

since $w \in C_V(a)$. Hence, $0 \neq [w, b] \in K[v, a]$. In particular, $X = \langle v, w, v^a + v\rangle$ is an A-invariant subspace of V. Moreover, $w \in C_V(a) \setminus C_V(b)$ implies that $C_X(a) \neq C_X(b)$, and so A acts linearly independently on X, again contrary to our assumptions.

Thus we may now assume that $dim_K(\langle v^A \rangle) \leq 2$ for all $v \in V$ and that $C_V(a) = C_V(b)$.

Choose $v, w \notin C_V(a)$. Then there exists $\gamma, \mu \in K$ such that $[v, b] = \gamma[v, a]$ and $[w, b] = \mu[w, a]$. We claim that $\gamma = \mu$. Let $W = K\langle v, w\rangle$, the subspace generated by v and w.

Suppose that $W \cap C_V(a) \neq 0$. Then there exists $\alpha, \beta \in K^\sharp$ such that $\alpha v + \beta w \in C_V(a)$. Since W is only 2-dimensional and as $W \not\subseteq C_V(a)$, it follows that $C_W(a) = K(\alpha v + \beta w)$. Thus $0 = [\alpha v + \beta w, b] = \alpha\gamma[v, a] + \beta\mu[w, a] = [\alpha\gamma v + \beta\mu w, b]$ implies that $\gamma = \mu$.

Thus we may assume that $W \cap C_V(a) = 0$. In particular, there exists $\eta \in K^\sharp$ such that $[v + w, b] = \eta[v + w, a]$. But then

$$\eta[v, a] + \eta[w, a] = \eta[v + w, a] = [v + w, b] = [v, b] + [w, b] = \gamma[v, a] + \mu[w, a]$$

from which it follows that $[(\eta + \gamma)v + (\eta + \mu)w, a] = 0$. Hence, $\gamma = \eta = \mu$. Therefore A acts linearly dependently on V contrary to our initial hypothesis.

Lemma 9.2 *Let $G \cong B_l(K)$ and suppose that A contains a transvection, but that A is not contained in a short root subgroup. Then there exists a rank 2 connected parabolic subgroup of type B_2 which contains a linearly independent fours-group which also acts quadratically on V*

Proof: Let r and s be the highest long and short roots in Φ, respectively. We may, without loss, assume that $1 \neq a \in A \cap X_s$, but that $A \not\leq X_s$. Because $char(K) = 2$ we have that $N_G(X_s) = M_1$. Moreover, because Q_1/X_s is the unique minimal normal subgroup in M_1/X_s, we get that there exist $1 \neq x \in X_r \cap \langle A^{C_G(a)} \rangle$. Therefore, as the subgroup $A' = \langle a, x \rangle$ acts quadratically on V and as $\langle \pm r, \pm s \rangle$ has type B_2, the lemma is proven.

Lemma 9.3 *Let $G = B_l(K)$ with $l \geq 3$ and $|K| > 6$. Let $A \leq G$ be a fours-group and suppose that A contains no transvections and acts linearly independently on $(V, (\cdot,\cdot))$, the natural module for G. Then A acts linearly independently on either some singular 3-space in V or on $x^\perp/Kx$ for some $x \in C_V(A)$.*

Proof: Suppose not. First we note that if $C_V(A)$ is nonsingular we may choose $x, y \in C_V(A)$ with $(x, y) \neq 0$ and let $W = \langle x, y \rangle$. Then as $V = W \oplus W^\perp$ and $W^\perp \cong x^\perp/Kx$, we get a contradiction. Hence $C_V(A)$ is singular.

The rest of the proof of the lemma will follow from several claims. Choose $a, b \in A$ such that $A = \langle a, b \rangle$.

Claim 1: $[V, A, A] = 0$.

Proof: Suppose not. Then choose $v \in V$ with $[v, a, b] \neq 0$. If c is any involution, then because $dim([W, c]) \leq 1/2dim(W)$ for any c-invariant subspaces W of V, it follows that $X = \langle v^A \rangle$ is 4-dimensional. In particular, $[X, A]$ is a 3-dimensional A-invariant subspace upon which A acts linearly independently. Now if there is a $z \in C_V(A) \setminus [X, A]$, then $[X, A] \leq [V, A] = C_V(A)^\perp \leq z^\perp$ implies that $[X, a] + Kz/Kz \leq z^\perp/Kz$, contrary to the above.

On the other hand, if $C_V(A) \leq [X, A] \leq X$, then $C_V(A) \leq C_X(A)$. But then $dim(C_V(a)) \geq l$ and $dim(C_V(A)) \geq l/2$ imply that $l/2 \leq 1$ and so $l \leq 2$, a contradiction. Therefore $[V, A, A] = 0$.

Let $\Omega = \bigcup\{C_V(c) \mid 1 \neq c \in A\}$.

Claim 2: If $x \in V \setminus \Omega$, then there exists $0 \neq \lambda_x \in K$ such that $[x, a] = \lambda_x[x, b]$.

Proof: Let $U = [Kx, A] = [Kx, a] + [Kx, b]$ and suppose that U is 2-dimensional. Note that $U \leq C_V(A)$ implies that U is singular. Moreover, note that if $x \in U^\perp$, then $Kx+U$ is a 3-dimensional singular subspace upon which A acts linearly independently, contrary to our assumption. Thus we may assume that $x \notin U^\perp$. Then $x^\perp \cap U$ is 1-dimensional. Moreover, since $C_V(A) = [V, A]^\perp$ is at least 3-dimensional, we get that $dim(x^\perp \cap C_V(A)) \geq 2$. In particular, $x^\perp \cap C_V(A)) \not\leq U$. Let $z \in (x^\perp \cap C_V(A)) \setminus U$. $z \in C_V(A)$ implies that $C_V(A)^\perp \leq z^\perp$; so we have $U \leq [V, A] \leq C_V(A)^\perp \leq z^\perp$. Hence $Kx + U + Kz/Kz$ is a 3-space in $z^\perp/Kz$ upon which A acts linearly independently, contrary to our assumption. Thus, U is 1-dimensional and so for each $x \in V \setminus \Omega$ there exists $0 \neq \lambda_x \in K$ such that $[x, a] = \lambda_x[x, b]$.

Claim 3: If $x, y \in V$ such that $(Kx + Ky) \cap \Omega = 0$, then $\lambda_x = \lambda_y$.

Proof: As $[x, a] + [y, b] = \lambda_{x+y}([x, b] + [y, b])$ and as $[x, a] + [y, a] = \lambda_x[x, b] + \lambda_y[y, b]$, we see that $[(\lambda_x + \lambda_{x+y})x + (\lambda_y + \lambda_{x+y})y, b] = 0$. Thus $(Kx + Ky) \cap \Omega = 0$ implies that $\lambda_x = \lambda_{x+y} = \lambda_y$.

Claim 4: If $x, y \in V \setminus \Omega$, then there exists $z \in V$ such that $(Kx + Kz) \cap \Omega = 0$ and $(Kz + Ky) \cap \Omega = 0$.

Proof: Suppose not. Thus for all $z \in V$, either $Kz \leq Kx + \Omega$ or $Kz \leq Ky + \Omega$. That is, $V \leq \{Kx + \Omega\} \cup \{Ky + \Omega\}$. Hence,

$$\begin{aligned} V \leq\ & \{Kx + C_V(a)\} \cup \{Kx + C_V(a)\} \cup \{Kx + C_V(b)\} \cup \{Kx + C_V(ab)\} \\ & \cup \{Ky + C_V(a)\} \cup \{Ky + C_V(b)\} \cup \{Ky + C_V(ab)\} \end{aligned}$$

Because A does not contain any transvections, we get that $dim(C_V(c)) < 2l - 1$ for all $1 \neq c \in A$. Hence V is the union of 6 proper subspaces, contrary to $|K| > 6$, by Lemma 4.10. Therefore, $\lambda_x = \lambda_y$ for all $x, y \in V \setminus \Omega$ and so also for all $x, y \in \langle V \setminus \Omega \rangle$.

Claim 5: $V = \langle V \setminus \Omega \rangle$.

Proof: If not, then let $X = \langle V \setminus \Omega \rangle$. Then V is the union of 4 proper subspaces, namely, X, $C_V(a)$ $C_V(b)$, and $C_V(ab)$ again a contradiction. Therefore, A acts linearly dependently on V, contrary to our initial hypothesis and proving the lemma.

Lemma 9.4 *Let $G = O_{2n}(K)$, $n \geq 4$, K an algebraically closed field with $char(K) = 2$, and let A be a fours-group in G. Let $(V, q, (\cdot,\cdot))$ be the natural module for G and suppose that A acts linearly independently on V. Then A acts linearly independently on either some singular 3-dimensional subspace of V or on $x^{\perp}/Kx$ for some $x \in C_V(A)$.*

Proof: Suppose not. The proof of the lemma will follow from several claims.

Claim 1: Suppose that $v \in V$ such that $[v, A, A] \neq 0$ and let $X = \langle v^A \rangle$. Then there exists $z \in C_V(A) \setminus [X, A]$ with $q(z) = 0$.

Proof: Note that X is 4-dimensional and $[X, A]$ is a 3-dimensional A-invariant subspace upon which A acts linearly independently. Now we note that if U is any singular subspace of V, then U contains an isotropic hyperplane. This is because if we choose $u \in U$ such that $q(u) \neq 0$, then for all $w \in U$ there exists $\gamma \in K$ such that $q(w + \gamma u) = q(w) + \gamma^2 q(u) = 0$, since K is algebraically closed. Thus if the dimension of $C_V(A)$ is greater than or equal to 3, then the claim is clearly true as $C_{[X,A]}(A) = \langle v + v^a + v^b + v^{ab} \rangle$ is 1-dimensional. Similarly the claim is true if there exists $w \in V$ with $\langle w + w^a + w^b + w^{ab} \rangle \neq \langle v + v^a + v^b + vab \rangle$ since $u + u^a + u^b + u^{ab}$ is a singular vector for all $u \in V$.

Thus, we may assume that $dim(C_V(A)) \leq 2$ and $dim([V, A, A]) = 1$. Now note that if $1 \neq c \in A$, then $|c| = 2$ implies that $dim(C_W(c)) \geq \frac{1}{2} dim(W)$ for all c-invariant subspaces W of V. Hence $2 \geq dim(C_V(A)) = dim(C_{C_V(a)}(b)) \geq \frac{1}{2}dim(C_V(a)) \geq \frac{1}{2}n$ implies that $n = 4$, and $dim(C_V(a)) = 4$ too. Thus $dim([V,a]) = 4$. But $[V,a,b]$ 1-dimensional implies that $C_{[V,a]}(b)$ is 3-dimensional. However, $C_{[V,a]}(b) \leq C_V(A)$, contrary to the above.

Claim 2: $[V,A,A] = 0$.

Proof: Suppose not. Choose $[v,A,A] \neq 0$ and let $\langle X^A \rangle$. Choose $z \in C_V(A) \setminus [X,A]$ with $q(z) = 0$. Then $[X,A] \leq [V,A] = C_V(A)^{\perp} \leq z^{\perp}$ implies that $[X,A] + Kz/Kz$ is a 3-space in $z^{\perp}/Kz$ upon which A acts linearly independently, contrary to the above.

Let $\Omega = \bigcup \{C_V(c) \mid 1 \neq c \in A\}$.

Claim 3: Let $x \in V \setminus \Omega$. Then there exists $0 \neq \lambda_x \in K$ such that $[x,a] = \lambda_x[x,b]$.

Proof: Let $U = [Kx,A] = [Kx,a] + [Kx,b]$ and suppose U is 2-dimensional. First suppose that $x \perp U$. We note that U is singular because $q(x + x^a) = q(x) + q(x^a) + (x,x^a) = (x,x^a) = (x,x+x^a) = 0$ since $x \perp U$. Now suppose that for each $y \in C_V(A)$ and for all $\lambda \in K$, $q(x+\lambda y) \neq 0$. Then K algebraically closed and $q(x+\lambda y) = q(x) + \lambda^2 q(y) + \lambda(x,y)$ implies that $x \in C_V(A)^{\perp}$ and that $q(y) = 0$ for all $y \in C_V(A)$. Thus because $[V,A,A] = 0$ implies that $dim(C_V(A)) \geq n$ we get that $C_V(A)$ is a maximal singular subspace of V. But then $C_V(A)^{\perp} = C_V(A)$, contrary to $x \in C_V(A)^{\perp} \setminus C_V(A)$. Hence we choose $z \in x + C_V(A)$ with $q(z) = 0$. However, $C_V(A) \perp [V,A]$ then implies that $Kz + U$ is a singular 3-space upon which A acts linearly independently, contrary to our assumption.

Hence we can assume that $x \not\perp U$. Then $x^{\perp} \cap U$ is 1-dimensional. Moreover, because $[V,A,A] = 0$ and $n \geq 4$ implies that $C_V(A) = [V,A]^{\perp}$ is at least 4-dimensional, we get that $x^{\perp} \cap C_V(A)$ is at least 3-dimensional. Hence there exists $z \in (x^{\perp} \cap C_V(A)) \setminus U$ with $q(z) = 0$. But then $Kx + U + Kz/Kz$ is a 3-dimensional space in $z^{\perp}/Kz$ upon which A acts linearly independently, contrary to the above. Therefore, U is 1-dimensional and so there exists $\lambda_x \in K$ such that $[x,a] = \lambda_x[x,b]$.

The remainder of the proof follows exactly as in the proof of Lemma 9.3.

CHAPTER 10

$E_n(K)$

Choose $a, b \in A$ such that $\langle a, b\rangle = A$.

Lemma 10.1 *Let $G = E_n(K)$, $n = 6, 7,$ or 8, and suppose X_r is a root subgroup of G with $|A \cap X_r| = 2$. Then there exists a connected rank two parabolic subgroup of G which contains a linearly independent quadratically acting fours-group.*

Proof: Suppose that $b \in X_r$ and let $X = \langle A^{C_G(b)}\rangle$. As $O_2(N_G(X_r))/X_r$ is the unique minimal subgroup of $C_G(X_r)/X_r$, it follows that $O_2(N_G(X_r)) \leq X$. Thus we can choose a root subgroup $X_s \leq O_2(N_G(X_r))$ such that r and s are contained in a root subsystem of type A_2 and choose an involution $x \in X_s$. $A' = \langle x, b\rangle$ is then the required fours-group.

Proposition 10.2 *If there exists a root subgroup, say X, such that $a \notin C_G(X)$ but such that $b \in C_G(X)$, then either there exists a connected rank two parabolic subgroup of G which contains a linearly independent quadratically acting fours-group or else we may assume that there exists roots $\alpha, \beta \in \Phi$, with $|\alpha| = |\beta|$ and $\alpha \perp \beta$ such that $a = x_\alpha(1)x_\beta(1)$.*

Proof: This follows directly from Lemmas 4.6 and 10.1.

To complete the proof of Theorem 8.3 for the case when $G \cong E_n(K)$, it suffices to assume that $a = x_\alpha(1)x_\beta(1) \in A$, where α and β are as in Proposition 10.2.

We now investigate the structure of $\langle A^{C_G(a)}\rangle$. In particular, we shall prove that if a and b centralize different root subgroups, then $\langle A^{C_G(a)}\rangle$ contains a fours-group which intersects nontrivially, but is not contained in, a root subgroup and which also acts quadratically on V.

Before we can do this, however, we first determine $C_G(a)$. Towards this end we make a helpful, albeit nonobvious, choice for α and β as follows:
for $G = E_6(K)$ we choose $\alpha = r_{32}$ and $\beta = r_{33}$,
for $G = E_7(K)$ we choose $\alpha = r_{27}$ and $\beta = r_{29}$,
for $G = E_8(K)$ we choose $\alpha = r_{58}$ and $\beta = r_{59}$.
Recall that by Lemma 4.2 all sets of roots $\{\alpha, \beta \in \Phi \mid \alpha \perp \beta\}$ are conjugate under W.

The following lemmas, Lemmas 10.3, 10.4, and 10.5, were adapted from [5] where they appeared in a more general context.

Lemma 10.3 *Let $G = E_6(K)$. Then $C_G(a) \leq M_{1,6}$ and $O^{2'}(C_G(a)) = U_0L_0$ where $U_0 = O_2(M_{1,6}) = Q_1Q_6$ and $L_0 \cong Sp_6(K)$. Moreover, $U_0' = Z(U_0) = Q_1 \cap Q_6$ is isomorphic to the natural module for $L_{1,6} \cong D_4(K) \cong SO_8^+(K)$ with root elements corresponding to isotropic vectors and $a = x_{r_{32}}(1)x_{r_{33}}(1)$ corresponding to an anisotropic vector.*

Proof: Since Q_1 and Q_6 are abelian and since $X_{r_{32}}, X_{r_{33}} \leq Q_1 \cap Q_6$, it is clear that $U_0 = Q_1Q_6 \leq C_G(a)$. Now let

$$L_0 = \langle X_{r_2}, X_{r_4}, x_{r_3}(\lambda)x_{r_5}(\lambda), w_2, w_4, w_3w_5 \mid \lambda \in K\rangle.$$

Then $L_0 \leq L_{1,4}$ and, using Table 3.1 and the Chevalley Commutator Formula, one sees that $L_0 \leq C_G(a)$ and that $L_{1,4} \not\leq C_G(a)$. Moreover, $L_{1,4} \cong D_4(K)$ and L_0 can be obtained as the set of fixed points of the graph automorphism of order 2 of L. Hence,

$L_0 \cong B_3(K) \cong Sp_6(K)$.

Let $X = U_0L_0$. Then $U_0 = O_2(X)$. We claim that M_1 and M_6 are the only parabolic subgroups of G which contain X. So suppose that $X \leq M = M_i^g$, for some $g \in G$. As X involves an $Sp_6(K)$, we see that $i = 1$ or 6. Thus because $O_2(M)$ is abelian and $O_2(X) = U_0$ is not, it follows that X is contained in a proper parabolic subgroup of M. Hence $X \leq M_i^g \cap M_j^{gk}$, where $\{i,j\} = \{1,6\}$ and $k \in M_i^g$. That is, $X \leq (M_1 \cap M_6)^{hg}$ for some $h \in M_i$. In particular, $O_2(X) \leq O_2((M_1 \cap M_6)^{hg})$. But $O_2(X) = O_2(M_1 \cap M_6)$ and so $hg \in N_G(O_2(M_1 \cap M_6)) = M_1 \cap M_6$. Thus $M_i^{hg} = M_i^g = M_i$. Therefore, the only parabolic subgroups which can contain X are M_1 and M_6. On the other hand its easy to check that neither X_{-r_1} nor X_{-r_6} centralize a. Thus, $C_G(a) \leq M_1 \cap M_6 = M_{1,6}$.

Now again using Table 3.1 and the Chevalley Commutator Formula, one can check that $U_0' = Q_1 \cap Q_6 = Z(U_0) = \langle X_{r_i} \mid 29 \leq i \leq 36\rangle$. In particular, $X_{r_{36}} \in U_0'$ where r_{36} is the highest root in Φ. Let $Y = \langle X_{r_{36}}^{L_{1,6}}\rangle$. By Lemma 4.12, Y is the unique irreducible $L_{1,6}$ module in U_0 and has weight λ_2. Thus Y must be the natural module for $L_{1,6} \cong SO_8^+(K)$ and hence is eight dimensional. On the other hand, $X_{r_{36}} \leq Z(U_0)$ implies that $Y \leq Z(U_0)$ which is also eight dimensional. Thus $U_0' = Z(U_0)$ is isomorphic to the natural module for $L_{1,6}$.

Moreover, if $S \in Syl_2(SO_8^+(K))$ and V_N is the natural module, then $C_{V_N}(S)$ is an isotropic 1-dimensional subspace. Hence $X_{r_{36}}$ corresponds to an isotropic vector and therefore because all the root elements in U_0' are conjugate they all correspond to isotropic vectors too. In particular, since a is not conjugate to a root element by assumption, a corresponds to an anisotropic vector. Hence, because the centralizer of an anisotropic vector in $SO_8^+(K)$ is isomorphic to $Sp_6(K)$, $L_0 = C_{L_{1,6}}(a)$. Therefore, $O^{2'}(C_G(a)) = U_0L_0$.

Lemma 10.4 *Let $G = E_7(K)$. Then $C_G(a) \leq M_6$ and $O^{2'}(C_G(a)) = U_0L_0$ where $U_0 = Q_6 = O_2(M_6)$ and $L_0 \cong Sp_8(K) \times SL_2(K)$. Moreover, $U_0' = Z(U_0)$ is isomorphic* *to the natural module for $L_{6,7} \cong D_5(K) \cong SO_{10}^+(K)$ with root elements corresponding to isotropic vectors and $a = x_{r_{27}}(1)x_{r_{29}}(1)$ corresponding to an anisotropic vector.*

Proof: Recall that $L_6 \cong D_5(K) \times A_1(K) \cong SO_{10}^+(K) \times SL_2(K)$. Let L_0 be the direct product of the $SL_2(K)$ factor of L_6 with the fixed point group of the graph automorphism of the $D_5(K)$ factor. Thus,

$$\begin{aligned} L_0 &= \langle X_{r_1}, X_{r_3}, X_{r_4}, x_{r_2}(\lambda)x_{r_5}(\lambda), w_1, w_3, w_4, w_2w_5 \mid \lambda \in K\rangle \times \langle X_{\pm r_7}\rangle \\ &= Sp_8(K) \times SL_2(K). \end{aligned}$$

Let $U_0 = Q_6$. Using Table 3.2 and the Chevalley Commutator Formula, one can easily check that $U_0L_0 \leq C_G(a)$, but that $L_6 \not\leq C_G(a)$. Let $X = U_0L_0$. We claim that M_6 is the unique maximal parabolic subgroup containing X. Suppose that $X \leq M = M_i^g$ for some maximal parabolic subgroup M and $g \in G$. Because X involves an $Sp_8(K) \times SL_2(K)$, it follows that $i = 6$ or 7.

Suppose $i = 7$. Then as $O_2(M_7)$ is abelian while $O_2(X) = Q_6$ is not. It follows that X must be contained in a proper parabolic subgroup of M of the form $M_7^g \cap M_6^{gh}$, where $h \in M_7^g$. However, by comparing orders, we see that $Sp_8(K) \times SL_2(K) \not\leq M_7^g \cap M_6^{gh}$.

Thus $i = 6$. Now if $O_2(X) = U_0 \not\leq O_2(M_6^g)$, then X must be contained in a proper parabolic subgroup of M_6. Thus $X \leq M_6^g \cap M_7^{gh}$, for some $h \in M_6^g$. But this yields the same contradiction as above. Hence, $O_2(X) = U_0 \leq O_2(M_6^g)$. That is, since $U_0 = O_2(M_6)$, we have that $O_2(M_6) \leq O_2(M_6^g)$ and hence $g \in M_6$. Therefore, M_6 is the unique maximal parabolic of G containing X, and thus $C_G(a) \leq M_6$ as well.

Now using Table 3.2 and the Chevalley Commutator Formula, one can easily check that

$$U_0' = Z(Q_6) = \langle X_{r_i} \mid i = 23, 24, 26, 27, 29, 30, 31, 32, 33\rangle.$$

In particular, because $X_{r_{33}} \leq U_0'$, where r_{33} is the highest root in Φ, it follows from Lemma 4.12 that $Y = \langle X_{r_{33}}^{L_{6,7}} \rangle$ is isomorphic to the natural module for $L_{6,7} \cong SO_{10}^+(K)$. Thus as U_0' is ten dimensional, we see that $Y = U_0'$. As in the case of $E_6(K)$, because the centralizer of a Sylow subgroup of $SO_{10}^+(K)$ in the natural module is an isotropic vector, we get that all the root elements of U_0' correspond to isotropic vectors and so a must correspond to an anisotropic vector.

Moreover, because the centralizer of an anisotropic vector in $SO_{10}^+(K)$ is isomorphic to $Sp_8(K)$, we see that $L_0 = C_{L_6}(a)$. Therefore $O^{2'}(C_G(a)) = U_0L_0$.

Lemma 10.5 *Let $G = E_8(K)$. Then $C_G(a) \leq M_1$ and $O^{2'}(C_G(a)) = U_0L_0$ where $U_0 = O_2(M_1) = Q_1$ and $L_0 \cong Sp_{12}(K)$. Moreover, $U_0' = Z(U_0)$ is isomorphic to the natural module for $L_1 \cong D_7(K) \cong SO_{14}^+(K)$ with root elements corresponding to isotropic vectors and $a = x_{r_{58}}(1)x_{r_{59}}(1)$ corresponding to an anisotropic vector.*

Proof: Recall that $L_1 \cong D_7(K)$. Let L_0 be the set of fixed points of the graph automorphism of L_1. Thus

$$\begin{aligned} L_0 &= \langle x_{r_2}(\lambda)x_{r_3}(\lambda), X_{r_4}, X_{r_5}, X_{r_6}, X_{r_7}, X_{r_8}, \\ &\quad\; w_2w_3, w_4, w_5, w_6, w_7, w_8 \mid \lambda \in K \rangle \\ &\cong Sp_{12}(K). \end{aligned}$$

Also let $U_0 = Q_1$. Using Table 3.3 and the Chevalley Commutator Formula, one can easily check that $X = U_0L_0 \leq C_G(a)$ but $L_1 \not\leq C_G(a)$.

We claim that M_1 is the unique maximal parabolic subgroup which contains X. So suppose that $X \leq M = M_i^g$ for some maximal parabolic subgroup M and $g \in G$. Because $X/U_0 \cong Sp_{12}(K)$, it follows by comparing the orders that $i = 1$ or 8. Now if $U_0 \not\leq O_2(M)$, then we get that $X \leq M_i^g \cap M_j^{gh}$ with $\{i, j\} = \{1, 8\}$ for some $h \in M_i^g$. However this is clearly impossible. Thus we can assume that $U_0 \leq O_2(M)$.

Hence because $|Q_8'| < |Q_1'| = |U_0'|$, $U_0 \not\leq Q_8$. Therefore $i = 1$, and as in the previous cases, we can conclude that M_1 is the unique maximal parabolic subgroup of G which contains X.

Now using Table 3.3 and the Chevalley Commutator Formula, one can easily check that

$$\begin{aligned} U_0' &= Z(Q_1) \\ &= \langle X_{r_i} \mid \quad \text{if} \quad r_i = \sum_{j=1}^{8} n_j r_j, \quad \text{then} \quad n_1 = 2\rangle. \end{aligned}$$

In particular, because $X_{r_{65}} \leq U_0'$, where r_{65} is the highest root in Φ, it follows from Lemma 4.12 that $Y = \langle X_{r_{65}}^{L_1}\rangle$ is isomorphic to the natural module for $L_1 \cong SO_{14}^+(K)$. Thus as U_0' has dimension 14, $Y = U_0'$. As in the case of $E_6(K)$, because the centralizer of a Sylow subgroup of $SO_{14}^+(K)$ in the natural module is an isotropic vector, we get that all the root elements of U_0' correspond to isotropic vectors and so a must correspond to an anisotropic vector. Moreover, as the centralize of an anisotropic vector in $SO_{10}^+(K)$ is isomorphic to $Sp_{10}(K)$, we see that $L_0 = C_{L_1}(a)$. Therefore $O^{2'}(C_G(a)) = U_0 L_0$.

Lemma 10.6 *If a and b centralize different subgroups, then G contains a quadratically action fours-group A' and a root subgroup X such that $|A' \cap X| = 2$. In particular, V is one of the modules in Lemma 7.2.*

Proof: Suppose that $a = x_\alpha(1)x_\beta(1)$. Let $Z = \{x_\alpha(\lambda)x_\beta(\lambda) \mid \lambda \in K\}$. One can see from the description of $C_G(a)$ that $Z \leq Z(C_G(a))$ and that all the elements of Z are conjugate in H.

Case 1: $b \in Z$.

Then $L_0U_0 \leq C_G(b)$ implies that $O^{2'}(C_G(a)) \leq O^{2'}(C_G(b))$. But then because a and b are conjugate to each other, $O^{2'}(C_G(a)) = O^{2'}(C_G(b))$, contrary to the hypothesis.

Case 2: $b \in U_0' \setminus Z$.

Recall that U_0' has the structure of a natural module for $SO_{2m}^{+}(K)$, for some m and that L_0 is the centralizer of an anisotropic vector in U_0'. For $n = 6$ or 8, let $M = L_0$. For $n = 7$, recall that $L_0 \cong Sp_8(K) \times SL_2(K)$ and let M be the $Sp_8(K)$ factor. In all cases then, $M \cong Sp_{2m-2}(K)$. Now, when U_0' is viewed as an MK-module, it contains MK-submodules X and Y with $X \leq Y$ such that X is 1-dimensional and is spanned by an isotropic vector, Y/X is $(2m-2)$-dimensional, and U_0'/Y is 1-dimensional. it follows that $\langle b^M + Z/Z\rangle$ is isomorphic to the natural module for M. In particular, $\langle b^{L_0}\rangle$ must contain an isotropic vector. Therefore, because isotropic vectors correspond to root elements in U_0', $\langle A^{C_G(a)}\rangle$ must contain a root element.

Case 3: $b \in U_0 \setminus U_0'$.

Because $U_0' = Z(U_0)$ and because U_0 is generated by root subgroups, it follows that there is a root subgroup, $X \leq U_0$ such that $[X, b] \neq 1$. We claim that $[X, b] \cap Z = 1$.

Consider the set of all triples (R, S, x) where R and S are commuting root subgroups in G and x is an involution in R^*S^* which is not contained in any root subgroup. By Lemmas 4.5 and 4.6 there are perpendicular roots $r, s \in \Phi^+$ such that RS is conjugate to X_rX_s. Thus, it follows from Lemma 4.2 that all such triples are conjugate in G. In particular, for a fixed involution x, we get that $C_G(x)$ is conjugate on the set of all pairs of root subgroups R and S such that $x \in R^*S^*$. Therefore, since U_0' is invariant under $C_G(a)$, it follows that if R and S are root subgroups such that $a \in R^*S^*$, then $R, S \leq U_0'$. That is, $[R, b] \cap Z \neq 1$ only if $R \leq U_0'$. Consequently, $[X, b] \cap Z = 1$. We choose $x \in [X, b]$ with $|x| = 2$ and let $A' = \langle a, x\rangle$. By Case 2, $\langle A'^{C_G(a)}\rangle$ contains a root element.

Case 4: $G = E_6(K)$ or $E_8(K)$ and $b \notin U_0$, or $G = E_7(K)$ and $b \in M_{6,7} \setminus U_0$.

Suppose that $[b, U_0] \leq U_0' = Z(U_0)$. Then by the 3-Subgroup Lemma, $[b, U_0'] = 1$, contrary to U_0' being a faithful module for L_0 if $G = E_6(K)$ or $E_8(K)$ and for $L_{6,7}$ if $G = E_7(K)$. Hence we can choose a root subgroup $X \leq U_0 \setminus U_0'$ and $x \in [b, X]$ such that $x \in U_0 \setminus U_0'$ and $|x| = 2$. We then apply Case 3 to $A' = \langle a, x \rangle$.

Case 5: $G = E_7(K)$ and $b \notin M_{6,7}$.

We can assume that $b \in U$ and so we see that $X = \langle X_{-r_7}, X_{-r_7}^b \rangle$ is not a 2-group. Thus, by Lemma 4.5 $X \cong SL_2(K)$ and so we can choose a root element x in $\langle A^{C_G(a)} \rangle$. Then $A' = \langle x, a \rangle$ is a quadratically acting fours-group containing a root element.

Lemma 10.7 *Let $G = E_n(K)$, $n = 6, 7,$ or 8, and suppose that a and b centralize the same root subgroups in G. Then A is linearly dependent.*

Proof: Suppose there exists a root subgroup X_r such that A centralizes $\langle X_r, X_{-r} \rangle$. If $N_G(X_r) = M_i$, then by Lemma 4.7, $A \leq L_i$. The result then follows by induction on the root systems since the statement is clearly true for $A_l(K)$ and $D_l(K)$.

First we consider $G = E_6(K)$. By Lemma 4.9 we may assume that

$$a = x_{r_{36}}(1)x_{r_{29}}(1)x_{r_9}(1)x_{r_4}(1).$$

Now let $\Phi' = \langle \pm r_2, \pm r_4, \pm r_9, \pm r_{29} \rangle$. Then one can easily check that $\{r_4, r_9, r_{29}, r_{36}\} \subseteq \Phi'$ and that Φ' has type D_4. Thus, without loss, we may assume that $a \in D_4(K)$. We claim that every involution in $D_4(K)$ centralizes a root subgroup and its corresponding negative root subgroup.

Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be fundamental root system for D_4 with highest root $r = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ and assume that a' is an involution in $D_4(K)$. By Lemma 4.9 we may assume that $a' = x_r(1)x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$. However, if we let

$$g = x_{\alpha_2+\alpha_3}(1)x_{\alpha_1+\alpha_2}(1)x_{\alpha_2+\alpha_4}(1)x_{\alpha_1+\alpha_2+\alpha_3}(1),$$

then $a^g = x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_5}(1)$ which centralizes $\langle X_r, X_{-r}\rangle$ and so the claim follows, proving the lemma for $E_6(K)$.

Next we consider $G = E_7(K)$. As above, by Lemma 4.9, we may assume that $a = x_{r_{33}}(1)x_{r_{16}}(1)x_{r_6}(1)a'$ where a' is an involution in $L_{1,6,7} \cong D_4(K)$. However, as we saw above, every involution in $D_4(K)$ centralizes a root subgroup and its corresponding negative root subgroup, so the lemma is also proven for $E_7(K)$.

Lastly, if $G = E_8(K)$, then because $O^{2'}(C_G(\langle X_{r_{65}}, X_{-r_{65}}\rangle)) = L_8 \cong E_7(K)$, the lemma follows by the above.

CHAPTER 11

$F_4(K)$

Choose $a, b \in A$ such that $\langle a, b\rangle = A$. Also let Φ be a root system of type F_4. Recall from Table 3.4 that r_{23} and r_{24} are the highest short and long roots in Φ, respectively. For ease of notation, let $s = r_{23}$ and $r = r_{24}$.

Lemma 11.1 *Let $G = F_4(K)$ and suppose there exists a root subgroup X of G such that $|A \cap X| = 2$. Then there exists a connected rank 2 parabolic subgroup of G of type B_2 which contains a linearly independent fours-group which also acts quadratically on V.*

Proof: Suppose that $1 \neq a \in A \cap X$. Because of the graph automorphism of G, we may assume that $X = X_r$. Then because $Z(O_2(N_G(X_r)))$ is the unique minimal normal subgroup of $N_G(X_r)$ and because $X_s \leq Z(O_2(N_G(X_r)))$, we have that $X_s \leq \langle A^{C_G(a)}\rangle$. Hence if we chose an involution $x \in X_s$, then $A' = \langle x, b\rangle$ is the fours-group claimed above.

Lemma 11.2 *Let $G = F_4(K)$ and suppose that a and b centralize different root subgroups in G. Then there exists a connected rank 2 parabolic subgroup of G of type B_2 which contains a linearly independent fours-group which also acts quadratically on V.*

Proof: First suppose that A contains a nontrivial element of a root subgroup. As above, we can assume that $1 \neq a \in A \cap X_r$. Thus, because $C_G(a) = O^{2'}(M_1)$ and because $Z(O^{2'}(M_1)) = X_r$, it follows that $|A \cap X_r| = 2$, and so the lemma follows from Lemma 11.1.

Hence, by Lemma 4.6 we may assume that $a = x_\alpha(1)x_\beta(1)$ where $\alpha, \beta \in \Phi$, $|\alpha| = |\beta|$ and α is perpendicular to β. Because there exists unique conjugacy classes of type $L \perp L$ and $S \perp S$ in Φ, there exists a root subsystem, say $J \subseteq \Phi$, of type B_2 such that $\alpha, \beta \in J$. Moreover, because there exists a unique conjugacy class with type B_2, we may assume that $J = \langle \pm r, \pm s \rangle$. Lastly, because $B_2(K)$ has three conjugacy classes of involutions with representatives $x_r(1)$, $x_s(1)$, and $x_s(1)x_r(1)$, we may assume that $a = x_s(1)x_r(1)$.

Note that because $a \in Z(U)$, $C_G(a) \leq P_J$ for some $J \subseteq \Phi$. One can easily check then that $C_G(a) = O^{2'}(M_{1,4})$.

Suppose that $b \in X_r^* X_s^*$. Then as H is transitive on $X_r^* X_s^*$, there exists $h \in H$ such that $a^h = b$. But then $C_G(b) = C_G(a)^h = C_G(a)$, contrary to the assumptions of the lemma since $C_G(a) = O^{2'}(M_{1,4})$ is generated by root subgroups.

Thus we may assume that $b \notin X_r X_s$. Because $Z(Q_1) \cap Z(Q_4) = X_r X_s$, it follows that either $b \notin Z(Q_1)$ or $Z(Q_4)$. Also because of the graph automorphism of G which interchanges Q_1 and Q_4 we can assume without loss that $b \notin Z(Q_4)$. Then $\langle b^{Q_4} \rangle \cap Q_4 \not\leq Z(Q_4)$. Thus as $Q_4' = X_s$, we can choose $1 \neq x \in [\langle b^{Q_4} \rangle, Q_4] \cap X_s \leq \langle A^{C_G(a)} \rangle$. $A' = \langle a, x \rangle$ is then the required fours-group.

Lemma 11.3 *Let $G = F_4(K)$ and suppose that a and b centralize the same root subgroups. Then either V is as claimed in Theorem 8.3, or A is contained in a root subgroup of G and hence is linearly dependent.*

Proof: Suppose that V has weight λ. Let ϕ denote the graph automorphism of G.

By [6] G has four conjugacy classes of involutions with representatives $t = x_r(1)$, $u = x_s(1)$, $tu = x_r(1)x_s(1)$, and $v = x_{r_{19}}(1)x_{r_{20}}(1)$.

Now, if $t \in A$, then because $C_G(t) = O^{2'}(M_1)$ is generated by root subgroups and because $Z(O^{2'}(M_1)) = X_r$, it follows that $A \leq X_r$.

Similarly if $u \in A$, then $A \leq X_s$.

If $tu \in A$, then because $Z(C_G(tu)) = X_rX_s$, by the above, we get that $A \leq X_rX_s$. Let $J = \langle \pm r, \pm s \rangle$. Then J has type B_2. Because A is not contained in a root subgroup, it follows from Corollary 6.5 and Lemma 8.7 that every nontrivial chief factor of V for $\langle X_{\pm r}, X_{\pm s} \rangle$ must either be the tensor product of algebraic conjugates of the natural module or the tensor product of algebraic conjugates of the spin module. In particular, $\lambda^{\perp} \cap J^w \neq \emptyset$ for all $w \in W$. Thus, by Lemma 7.2, λ in an integral multiple of either λ_1 or λ_4. Moreover, because A is not contained in a root subgroup of G, it follows from Theorem 6.1 that V must be a restricted module. Therefore V is an algebraic conjugate of either $V(\lambda_1)$ or $V(\lambda_4)$.

Thus we may assume that $v \in A$. By [6], $C_G(v)$ is generated by root subgroups and $Z(C_G(v)) = X_{r_{19}}X_{r_{20}}$. Thus $A \leq X_{r_{19}}X_{r_{20}}$. Suppose that $b = x_{r_{19}}(\gamma_1)x_{r_{20}}(\gamma_2)$. Because $a^{\phi} = a$ and $b^{\phi} = x_{r_{19}}(\gamma_2)x_{r_{20}}(\gamma_1^2)$ we can assume that $\gamma_1 \neq \gamma_2$. Now as r_{19} is a short root and r_{20} is a long root and because $r_{19} \perp r_{20}$, it follows from Lemma 4.2 that there exists $w \in W$ such that $\{r_{19}, r_{20}\}^w = \{r_1, r_3\}$. Hence $A^w \leq L_4 \cong Sp_6(K)$, the Levi compliment of M_4. Note that because $\gamma_1 \neq \gamma_2$ we can assume that A^w does not act linearly dependently on the natural module for $Sp_6(K)$. Moreover, if there exists a rank two parabolic subgroup of type B_2 such that A is contained in the Levi complement and is linearly dependent there, then the result follows from the above cases. Thus we may assume that A^w is linearly independent in L_4. Therefore, it follows from Lemma 9.3 that there exist a rank 2 connected root subsystem, say J, such that $\lambda^{\perp} \cap J^{w'} \neq \emptyset$ for all $w' \in W$. By Lemma 7.2, λ is an integral multiple of either λ_1 or λ_4. Moreover, because A is not contained in a root subgroup, it follows as above that V is an algebraic conjugate of either $V(\lambda_1)$ or $V(\lambda_4)$, as claimed.

CHAPTER 12

$G_2(K)$

Lastly we consider $G_2(K)$. Because of the unique structure of its associated root system it is easiest to do a small amount of computation rather than develop an approach similar to that used in the previous cases.

Lemma 12.1 *Let $G = G_2(q)$. If A is not contained in a root subgroup of G, then V is an algebraic conjugate of the natural module for G.*

Proof: Assume that A is not contained in a root subgroup of G. We claim that, without loss, we may assume that $Q_1 = O_2(M_1) \leq \langle A^{C_G(a)} \rangle$ for some $a \in A$. By [14] G has two conjugacy classes of involutions with representatives $x_{2r_1+3r_2}(1)$ and $x_{r_1+2r_2}(1)$.

Case 1: $a = x_{2r_1+3r_2}(1) \in A$.

Since $a \in Z(U)$, $C_G(a) \leq P_J$ for some $J \subseteq \Phi$. One can easily check then that $C_G(a) = O^{2'}(M_1) = Q_1L_1$, where $L_1 = \langle X_{r_2}, X_{-r_2} \rangle \cong Sl_2(K)$. It follows from the commutator relations that $\overline{Q}_1 = Q_1/X_{2r_1+3r_2}$ is abelian, and so may be regarded as a 4-dimensional KL_1-module with multiplication by a scalar, say γ, defined as conjugation by $h_{2r_1+3r_2}(\gamma)$. Moreover, because $\overline{X}_{r_1+3r_2} \leq C_{\overline{Q}_r}(X_{r_2})$, $X_{r_2} \in Syl_2(L_1)$ and because $x_{r_1+3r_2}(\gamma)^{h_{r_2}(\lambda)} = x_{r_1+3r_2}(\lambda^3\gamma)$, we see that $\langle \overline{X}_{r_1+3r_2}^{L_1} \rangle$ contains a KL_1-submodule isomorphic to $V_n \otimes V_n^2$, where V_n is the natural module for $SL_2(K)$. Since

$|K| > 2$, this submodule is irreducible and because it is four dimensional, it must be all of $\overline{Q}_1$. In particular, since $A \not\leq X_{2r_1+3r_2}$, $Q_1 \leq \langle A^{C_G(a)} \rangle$.

Case 2: $a = x_{r_1+2r_2}(1) \in A$

Then, again by [14], $C_G(a) = U_2 L_2$, where $L_2 = \langle X_{\pm r_1} \rangle$, $U_2 = X_{r_1+2r_2} \times X_{r_1+3r_2} \times X_{2r_1+3r_2}$, and L_2 acts irreducibly on $Q = X_{r_1+3r_2} \times X_{2r_1+3r_2}$. So, $A \not\leq X_{r_1+2r_2}$ implies that $Q \leq \langle A^{C_G(a)} \rangle$. In particular, we can choose an element, say x, of a long root subgroup in Q and apply Case 1 to the fours-group $A' = \langle a, x \rangle$, proving our claim.

Thus we have $[V, X_{2r_1+3r_2}, Q_1] = 0$. Let $g = w_{r_1+3r_2}$. Then $X^g_{2r_1+3r_2} = X_{r_1}$, implies that $[V, X_{r_1}, Q_1^g] = 0$ too.

Let $V_0 = C_V(O_2(P_{r_1}))$. By [11], V_0 is an irreducible module for $\langle X_{r_1}, X_{-r_1} \rangle$. So, $W = [V_0, X_{r_1}]$ is centralized by both Q_1^g and $O_2(P_{r_1})$. However, $\langle Q_1^g, O_2(P_{r_1}) \rangle = G$. Thus $W = 0$, and so also $[V_0, O^{2'}(P_{r_1})] = 0$. Therefore, as V was assumed to be a restricted module, we see that $\lambda = \lambda_2$. That is, V is the natural module for G.

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