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Deletion-Contraction Techniques for the Chromatic Symmetric Function of a Graph

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ABSTRACT

Deletion-Contraction Techniques for the Chromatic Symmetric Function of a Graph

By

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Recently, R. P. Stanley defined and studied a symmetric function, X_G, which generalizes the chromatic polynomial of a graph, G. This generalization has both advantages and disadvantages. The main advantage is that it gives us more information about the colorings of G than the chromatic polynomial. However, one disadvantage is that this new symmetric function does not satisfy a deletion-contraction recurrence similar to the one for the chromatic polynomial.

In this thesis, we define a similar graph invariant called Y_G. This invariant is defined using noncommutative variables, and from it we can recover X_G by allowing the variables to commute. This new invariant is also a symmetric function. More importantly, by using noncommutative variables we will be able to obtain a deletion-contraction recurrence for Y_G. We may then obtain some of Stanley's results for X_G in a uniform manner by using induction. In addition, this will allow us to make some progress on the **3+1** Conjecture of Stanley and Stembridge.

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INTRODUCTION

As early as 1912, Whitney [19] began to study graph colorings from a mathematical point of view. Today the theory of graph coloring has many applications to both scheduling problems and efficient network design. [11]. Here we will use symmetric functions to enumerate graph colorings. While this section contains much of the background leading up to our study, we will try to introduce notation and definitions as they are needed throughout the text, rather than all at once. We will generally follow Stanley [15, 14] for combinatorial notation or anything specifically related to the symmetric function of a graph, X_G, and MacDonald [10] for symmetric functions in general.

To begin, let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$, where the edges consist of unordered pairs of the vertices. We mention here that if the edge set consisted of *ordered* pairs of vertices we would have had a graph with directed edges, referred to as a *digraph*. A v_1–v_n *walk* in a graph is a sequence of vertices, $v_1, v_2, \dots, v_n$ such that $v_{i-1}v_i$ is an edge for all $2 \leq i \leq n$. A graph, G, is *connected* if there is a u, v walk for every pair of vertices, u and v in $V(G)$. The *connected components* of G are just the maximal connected subgraphs of G. Finally, H is a *spanning subgraph* of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. In our study we will actually consider *multigraphs*, in which multiple edges and loops are allowed. The other definitions above extend in the natural way to multigraphs.

Since our main interest here is in coloring graphs, we define a *coloring* of G to be

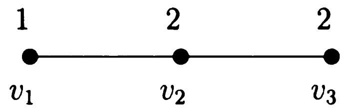

Figure 1. A coloring (not proper) of P_3.

Figure 2. Two proper colorings of P_3

a map $\alpha : V(G) \longrightarrow C$, where C is the *color set.* In particular, a *proper coloring* of G is a coloring such that no two adjacent vertices are the same color, i.e., $\alpha(v_i) \neq \alpha(v_j)$ if $v_i v_j$ is an edge of the graph. For an example, we show a coloring for the path on three vertices, P_3, which is not a proper coloring in Figure 1 and two proper colorings for P_3 in Figure 2.

Whitney's object of study was the *chromatic polynomial* of a graph, $\mathcal{X}_G(n)$, which is defined to be the number of ways to properly color G using the color set $C = \{1, 2, \ldots, n\} \stackrel{\text{def}}{=} [n]$. For P_3, since there are n ways to color v_1 from a set of n colors, and $n-1$ ways to color each of the remaining vertices, we see that $\mathcal{X}_{P_3}(n) = n(n-1)^2$. It is somewhat surprising that $\mathcal{X}_G(n)$ is always going to be a polynomial in n. One easy way to see this is to use induction along with the *Deletion-Contraction Lemma*, which we will now discuss.

Given a graph G and an edge $e \in E(G)$, we can define the graph $G - e$ to be the graph G with the edge e deleted from its edge set. The *contraction of* G *by* e, G/e, is obtained from G by contracting e (in the topological sense) to a single vertex. Given these definitions, the *Deletion-Contraction Lemma* states that

$$\mathcal{X}_G(n) = \mathcal{X}_{G-e}(n) - \mathcal{X}_{G/e}(n).$$

This gives us a recursive way to compute the chromatic polynomial of a graph, as well as to establish various properties of $\mathcal{X}_G(n)$ by induction. Two of Whitney's results that can be proven using this method are stated here.

Theorem 1 *[19] For a finite graph, G,*

$$\mathcal{X}_G(n) = \sum_{S \subseteq E(G)} (-1)^{|S|} n^{c(S)},$$

where $c(S)$ is the number of connected components of the spanning subgraph of G with edge set S, which by abuse of notation we just denote by S. ■

As an illustration, we will use this theorem to again calculate $\mathcal{X}_{P_3}(n)$. If we let the edge set of P_3 be $\{e_1, e_2\}$, where $e_1 = v_1v_2$ and $e_2 = v_2v_3$, then we can make the following table.

$S \subseteq E(G)$	$(-1)^{\|S\|}$	$n^{c(S)}$
ϕ	1	n^3
e_1	-1	n^2
e_2	-1	n^2
e_1, e_2	1	n^1

This shows us that according to the Theorem, $\mathcal{X}_{P_3}(n) = n^3 - 2n^2 + n = n(n-1)^2$, which agrees with our previous calculation.

The other theorem of Whitney's in which we will be interested is the one known as the *Broken Circuit Theorem.* A *cycle* or *circuit* is a closed walk with distinct vertices and edges, $v_1, v_2, \ldots, v_m, v_1$, for $m \geq 1$. If we fix a total order on $E(G)$, a *broken circuit* is a circuit with its largest edge (with respect to the total order) removed. Let the *broken circuit complex* B_G of G denote the set of all $S \subseteq E(G)$ which do *not* contain a broken circuit in our fixed ordering on the edges. The Broken Circuit Theorem then asserts:

Theorem 2 *[19] For any finite graph, G, on d vertices we have*

$$\mathcal{X}_G(n) = \sum_{S \in B_G} (-1)^{|S|} n^{d-|S|}.$$

■

If we again calculate $\mathcal{X}_{P_3}(n)$ using this theorem, we will come out with exactly what we had before, only with n^3 and n^1 reversing positions in the table, since P_3 contains no circuits and hence no broken circuits. As a less trivial example, we will use this theorem to verify that the chromatic polynomial for K_3, the complete graph on 3 vertices is indeed given by $n(n-1)(n-2)$, which can be obtained by noticing that there are n ways to color the first vertex, $n-1$ colors left available for the second vertex, and $n-2$ colors allowed for the last vertex. We label $E(K_3) = \{e_1, e_2, e_3\}$, where the fixed order on the edges is the obvious one induced by the subscripts.

Since the only circuit in K_3 is $\{e_1, e_2, e_3\}$, the only broken circuit will be $\{e_1, e_2\}$. This gives us the following table, where we notice that here $d = 3$.

$S \in B_G$	$(-1)^{\|S\|}$	$n^{d-\|S\|}$
ϕ	1	n^3
e_1	-1	n^2
e_2	-1	n^2
e_3	-1	n^2
e_1, e_3	1	n^1
e_2, e_3	1	n^1

This gives us $\mathcal{X}_{K_3} = n^3 - 3n^2 + 2n$, which again agrees with the previous calculation.

Following these early results, some of the more interesting applications are those of Zaslavsky in [20, 21, 22]. In that series of papers he introduces the notion of colorings for certain generalizations of graphs called signed graphs. These colorings have very nice connections to characteristic polynomials of certain types of hyperplane arrangements. A related result by Zaslavsky and Greene [7] concerns the sinks of acyclic orientations for G. An *orientation* of G is a digraph D obtained by assigning a unique direction to each edge of G. An orientation is *acyclic* if it has no directed cycles. We also define a *sink* of D to be a vertex $v \in V(D)$ such that $\overrightarrow{vx} \notin E(D)$ for all $x \in V(D)$. Also, for notational convenience we adopt the convention that

$$\mathcal{X}_G(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k.$$

Theorem 3 *([7] Theorem 7.3) Let v_0 be any vertex of G. The number of acyclic orientations of G with a unique sink at v_0 is $|a_1|$.* ∎

This theorem is related to one of Stanley, which states:

Theorem 4 *[13] The number of acyclic orientations of G is $\sum_i |a_i|$.* ∎

All of these theorems are actually specializations of results which can be obtained from Stanley's symmetric function generalization of the chromatic polynomial. The first three theorems listed previously can all easily be derived from the recurrence relation for the chromatic polynomial. However, this symmetric function does not satisfy any similar deletion-contraction recursion, which eliminates induction as a tool for these proofs. In what follows we will extend the Stanley's definition by using symmetric functions in *noncommutative* variables. This setting will allow us to establish a recurrence and again allow induction as a valid approach to our proofs.

CHAPTER 1

Preliminaries

1.1 Symmetric Functions

Here we will review the basic facts about symmetric functions in commuting variables. Our development will closely mirror that found in Sagan's book [12]. The interested reader should consult either MacDonald [10] or Sagan [12] for a more comprehensive discussion.

We will begin with the monomial symmetric functions. Let $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$ be a countably infinite set of commutative variables, and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be an integer partition of n, denoted $\lambda \vdash n$, where the λ_i form a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{k} \lambda_i = n$. If we allow r_i to be the number of parts of λ equal to i, then we may also express $\lambda = (1^{r_1}, 2^{r_2}, \dots, n^{r_n})$ as an alternate notation. The *monomial symmetric function corresponding to* λ is given by

$$m_\lambda = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},$$

where the sum is over all *distinct* monomials having exponents $\lambda_1, \dots, \lambda_k$. As an example we can see that

$$m_{(2,1)} = x_1^2x_2 + x_1^2x_3 + \cdots + x_2^2x_1 + x_2^2x_3 + \cdots + x_3^2x_1 + \cdots .$$

We then define the *ring of symmetric functions* as the vector space over $\mathbb{C}$ spanned by the monomial symmetric functions. It is an elementary fact that the monomial symmetric functions are actually linearly independent over $\mathbb{C}$ and so form a basis for the vector space of symmetric functions. It is important to note here that while we will usually consider the symmetric functions as a vector space in this thesis, the fact that they form a closed set under multiplication also makes the symmetric functions a ring.

While it is clear from our development that the monomial symmetric functions form a standard basis for the symmetric functions, there are other nice bases for this vector space which are routinely used. These include the elementary, power sum, and complete homogeneous symmetric functions as well as the Schur functions. We will define the power sum symmetric functions and the elementary symmetric functions here, as they will be relevant to the rest of this thesis. For a description of the complete homogeneous symmetric functions and the Schur functions, please see either [10], [12], or [4]

The r^{th} *power sum symmetric function* is

$$p_r = m_{(r)} = \sum_{i \geq 1} x_i^r,$$

and the r^{th} *elementary symmetric function* is

$$e_r = m_{(1^r)} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

While these are seemingly natural definitions based on the monomial symmetric functions, they obviously do not form bases since they are not even indexed by integer partitions. Hence we must extend these definitions to p_λ and e_λ where $\lambda \vdash n$. We will do this multiplicatively, by defining

$$p_{(\lambda_1,\lambda_2,\dots,\lambda_k)} \stackrel{\text{def}}{=} p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_k} \quad \text{and} \quad e_{(\lambda_1,\lambda_2,\dots,\lambda_k)} \stackrel{\text{def}}{=} e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}.$$

These are the *power sum* and *elementary symmetric functions*, respectively. They also form bases for the space of symmetric functions. To illustrate these definitions, two simple examples are computed. For the integer partition $(2,1)$ we have

$$\begin{aligned} p_{(2,1)} \quad = p_2p_1 \quad &= (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) \\ &= x_1^3 + x_2^3 + \cdots + x_1^2x_2 + x_2^2x_1 + \cdots \\ &= m_{(3)} + m_{(2,1)} \end{aligned}$$

and

$$\begin{aligned} e_{(2,1)} \quad = e_2e_1 \quad &= (x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\ &= x_1^2x_2 + x_2^2x_1 + \cdots + 3x_1x_2x_3 + \cdots \\ &= m_{(2,1)} + 3m_{(1,1,1)}. \end{aligned}$$

These functions are referred to as symmetric functions for the following reason. Given a permutation in the symmetric group on n elements, $\delta \in \mathcal{S}_n$, we define a natural action on the set of functions in $\mathbb{C}[\mathbf{x}]$ by

$$\delta f(x_1, x_2, x_3, \dots) = f(x_{\delta(1)}, x_{\delta(2)}, x_{\delta(3)}, \dots).$$

It should be clear that, for any permutation δ, the elements in our space of symmetric functions will remain invariant under this action.

Given this information about the space of symmetric functions, we now have sufficient background to introduce Stanley's chromatic symmetric function.

1.2 The Chromatic Symmetric Function, X_G

In "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph" [15] (see also [16]), R. P. Stanley introduced a symmetric function, X_G, which generalizes the chromatic polynomial associated with a labeled graph on d vertices.

Definition 1.2.1 *Let G have vertex set $V(G) = \{v_1, v_2, \ldots, v_d\}$. We define*

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} \ldots x_{\kappa(v_d)},$$

where the sum ranges over all proper colorings, $\kappa : V(G) \to \mathbb{P}$, and $\mathbb{P}$ is the set of positive integers.

Note that X_G is homogeneous of degree $d = |V(G)|$, where $|*|$ denotes cardinality. We also notice that if G has loops this sum is empty, giving $X_G = 0$. To illustrate this definition, we will compute the chromatic symmetric function for our standard example of the path on three vertices, P_3. We can see that any proper coloring of this graph will have one of two possible types: the coloring could have v_1 and v_3 one color with v_2 a different color, or it could have all three vertices different colors. Since there are 6 different ways to color the three vertices with the same set of three different colors, we obtain

$$X_{P_3} = x_1^2 x_2 + x_2^2 x_1 + \cdots + 6x_1x_2x_3 + 6x_1x_2x_4 + \cdots$$

It should be clear from the definition that X_G is a symmetric function, since any permutation of the subscripts simply permutes the colors and doesn't affect the *set* of colorings. We can also see it more explicitly in this case, since the previous expression clearly shows that

$$X_{P_3} = m_{(2,1)} + 6m_{(1,1,1)}.$$

For $\lambda \vdash n$ having r_i parts of size i, we can also use the notation $\lambda = (1^{r_1}, 2^{r_2}, \dots, n^{r_n})$, and define $|\lambda| = r_1! \cdots r_n!$. We say that a partition of the vertex set of G is *stable* if no block of the partition contains adjacent vertices. Then it is not hard to see [15] that $X_G = \sum_\lambda a_\lambda |\lambda| m_\lambda$, where a_λ is the number of stable partitions whose block sizes correspond exactly to the parts of λ. We are also easily able to see that for the disjoint union of two graphs, $G = H \uplus I$, we have $X_G = X_H X_I$.

We can verify that this symmetric function is a generalization of the chromatic polynomial, $\mathcal{X}_G(n)$, since setting $x_1 = x_2 = \dots = x_n = 1$ and $x_i = 0$ for all $i > n$ in X_G, denote by $X_G(1^n)$, yields $\mathcal{X}_G(n)$. To see this, note that this substitution will produce a term equal to 1 for each monomial in X_G which comes from a proper coloring of the graph using the first n colors, and a term equal to zero for each monomial arising from a proper coloring which uses a color not in $[n]$. Hence the sum of all these monomial terms after this substitution will just be the number of proper colorings of G which only use the first n colors. This is precisely $\mathcal{X}_G(n)$.

Once we are assured that this is a generalization of the chromatic polynomial, one might expect that previous results about the chromatic polynomial should also generalize. It is also natural to study the expansion of this chromatic symmetric function in terms of the different symmetric function bases. The calculation of X_G for various specific graphs is also of interest. Stanley pursues all of these lines of inquiry in his paper.

Several of Stanley's results for X_G are extensions of Whitney's [19] theorems for the chromatic polynomial. For example, Stanley's symmetric function extension of Theorem 1 utilizes the power sum symmetric functions.

Theorem 1.2.2 *[15, Theorem 2.5] Let G be a finite graph of order d. We have*

$$X_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(S)},$$

where $\lambda(S) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is the integer partition of d with λ_i being the number of vertices in the i^{th} component of S. ∎

We can see that this result directly implies Whitney's first theorem by noticing that $p_r(1^n) = n$ for any r, and so $p_\lambda(1^n) = n^{l(\lambda)}$, where $l(\lambda)$ is the number of parts of λ. Hence $p_{\lambda(S)}(1^n) = n^{c(S)}$, completing the reduction.

Not surprisingly, Stanley also has a generalization of Theorem 2.

Theorem 1.2.3 *[15, Theorem 2.9] For any finite graph G, we have*

$$X_G = \sum_{S \in B_G} (-1)^{|S|} p_{\lambda(S)}.$$

∎

In this thesis, we will be studying an analogue of Stanley's chromatic symmetric function X_G, called Y_G, which is defined using noncommutative variables. We wish to consider this analogue because we know that many results for the chromatic polynomial can be proven easily using induction and the deletion-contraction recurrence. Unfortunately, Stanley's symmetric function has no such deletion-contraction property, which deprives him of induction as a tool for his proofs.

To see where the problem lies, note that X_G is homogeneous of degree d, while $X_{G/e}$ is homogeneous of degree $d-1$. In order to find a recurrence, we would need to add another variable to each monomial in $X_{G/e}$. But which variable? In the proof of the deletion-contraction rule for the chromatic polynomial, we have proper colorings of G/e corresponding to colorings of $G-e$ with u and v the same color, where $e = uv$. However, while $X_{G/e}$ gives us more information about the colorings of G/e than the chromatic polynomial, it does not give us the explicit information we need to fix the problem: namely, what color was assigned to the vertex obtained by contracting uv. We lost that information when we allowed the variables to commute.

To correct this difficulty, in Chapter 2 we introduce an analogue of X_G which is a symmetric function in *noncommutative* variables. That is, for any multigraph G with vertices labeled $v_1, v_2, \ldots, v_d$ *in a fixed order* we define the analogue of X_G as

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)}$$

where again the sum is over all proper colorings of G, but the x_i are now *noncommuting* variables.

The reason for using *noncommuting* variables is so that we can keep track of the color which κ assigns to each vertex. Since we still have the homogeneity problem in $Y_{G/e}$, we define an operation on the non-commutative symmetric functions which will allow us to use deletion-contraction techniques for computing Y_G. In this chapter we also provide some basic expansions of Y_G which closely resemble Whitney's and Stanley's theorems.

In Chapter 3 we will further explore the interrelationships between chromatic polynomials, chromatic symmetric functions, acyclic orientations and sinks. There is an interesting connection between Theorem 2 and the result of Green and Zaslavsky, Theorem 3. From Theorem 2 we can interpret the coefficients of the chromatic polynomial as the number of sets $S \in B_G$ of a certain size. From Theorem 3, we see the coefficient of n in $\mathcal{X}_G(n)$ is the number of acyclic orientations of G with a unique sink at any fixed vertex of G. It follows that the number of acyclic orientations of G with a unique sink at the fixed vertex is the same as the number of sets $S \in B_G$ with $|S| = d - 1$. Elementary graph theory tells us that this is also be the number of spanning trees of G which contain no broken circuits. We will provide a bijective proof of this fact by modifying an algorithm due to Blass and Sagan [1]. Finally, we will also extract some of the information that the non-commutative chromatic symmetric function can give us about acyclic orientations and sinks.

In Chapter 4 we will consider a conjecture about the coefficients of X_G, when it is expanded in terms of the elementary symmetric function basis. We will make some progress here on the $(\mathbf{3}+\mathbf{1})$-free conjecture of Stanley and Stembridge, proving it in some special cases.

We finish in Chapter 5 with some other partial results about acyclic orientations and sinks, as they relate to the $(\mathbf{3}+\mathbf{1})$-free conjecture. We conclude with some open problems, as well as a few ideas on how they might be approached using our techniques. Before we begin, however, we will need to discuss symmetric functions in noncommuting variables and our analogue of X_G in that setting. This is the focus of our next chapter.

CHAPTER 2

The Noncommutative Case

2.1 Symmetric Functions in Noncommuting Variables

We begin with some background on symmetric functions in noncommuting variables. Much of this follows from the work of Doubilet [4], although he does not explicitly mention these functions in his work. These noncommutative symmetric functions will be indexed by *set* partitions, which form a lattice under a certain partial order, which we will define here.

A ***lattice*** is a poset (partially ordered set) $\mathcal{L}$ such that every pair $x, y \in \mathcal{L}$ has a least upper bound (or join) denoted by $x \vee y$ and a greatest lower bound (or meet) denoted $x \wedge y$. Any finite lattice has a unique minimal element denoted by $\hat{0}$ and a unique maximal element denoted by $\hat{1}$. We let Π_d denote the set partitions of $\{1, 2, \ldots, d\} = [d]$. This forms the *set partition lattice*, where the partial order is defined as follows. If $\sigma = A_1/A_2/\cdots/A_k$ and $\tau = B_1/B_2/\cdots/B_m$, then $\sigma \leq \tau$ if and only if for all $1 \leq i \leq k$ there exists some j with $1 \leq j \leq m$ such that $A_i \subseteq B_j$. That this partial order on Π_d actually forms a lattice is an elementary result. As an example, we have included the Hasse diagram for Π_4 in Figure 2.1. Given a poset,

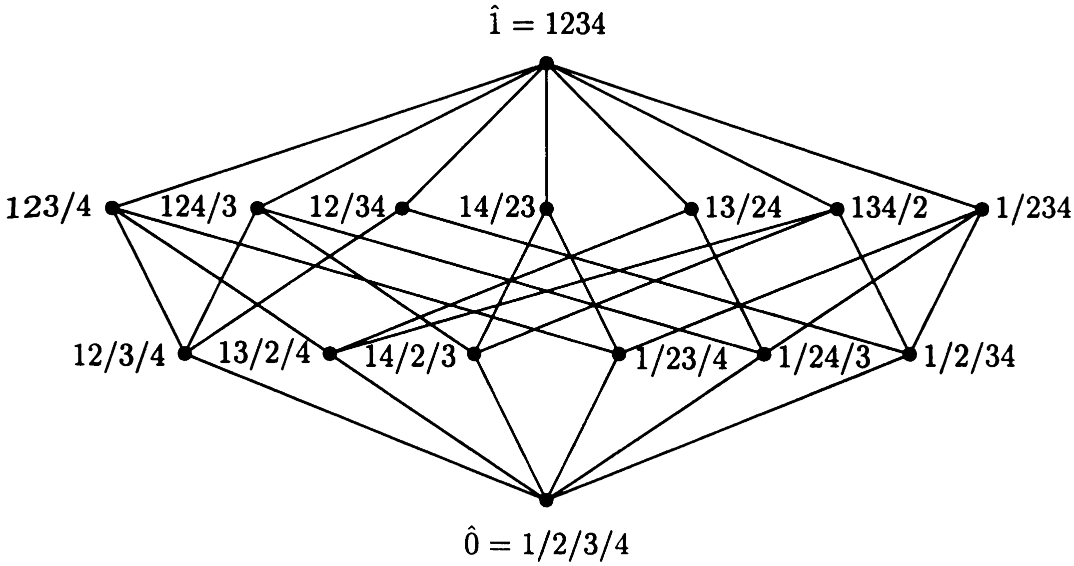

Figure 2.1. The partition lattice Π_4.

P, we also recursively define the *Möbius function*, μ, of P on intervals $[x, y]$ in P by

$$\mu(x,x) = 1 \quad \text{and} \quad \mu(x,y) = -\sum_{x \leq z < y} \mu(x,z) \text{ for all } x, y \in P.$$

It can be shown that for σ, $\tau \in \Pi_d$ with $\sigma = A_1/A_2/\cdots/A_k$ and $\tau = B_1/B_2/\cdots/B_m$, and $\sigma \leq \tau$, then

$$\mu(\sigma,\tau) = \prod_{n=1}^{m} (-1)^{a_n - 1}(a_n - 1)!,$$

where a_n is the number of blocks of σ contained in B_n.

Since we will be using both integer and set partitions in this thesis, our convention will be that λ and μ generally denote integer partitions, while σ, τ, and π are usually set partitions.

If $\pi \in \Pi_d$ we define $\lambda(\pi)$ to be the integer partition of d whose parts are the block sizes of π. So if $\pi = B_1/B_2/\ldots/B_k$, where the block sizes are in weakly decreasing order, we have $\lambda(\pi) = (|B_1|, |B_2|, \ldots, |B_k|)$. Further, if $\lambda(\pi) = (1^{r_1}, 2^{r_2}, \ldots, d^{r_d})$, we define

$$|\pi| = r_1!r_2!\cdots r_d!$$

and

$$\pi! = 1!^{r_1}2!^{r_2}\cdots d!^{r_d}.$$

We can now introduce the vector space for the noncommutative symmetric functions. Let $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$ be a set of *noncommuting* variables. We define the *noncommutative monomial symmetric functions*, m_π, by:

$$m_\pi = \sum_{i_1,i_2,\ldots,i_d} x_{i_1}x_{i_2}\cdots x_{i_d}, \tag{2.1}$$

where the sum is over all multisets (collections in which repetitions are allowed) $\{\{i_1, i_2, \ldots, i_d\}\}$ of the positive integers $\mathbb{P}$ such that $i_j = i_k$ if and only if j and k are in the same block of π. For example,

$$m_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + \cdots$$

is the monomial symmetric function in noncommuting variables corresponding to the partition $\pi = 13/24$.

We notice, from [4] that letting the x_i commute transforms m_π into $|\pi|m_{\lambda(\pi)}$. The noncommutative monomial symmetric functions, $\{m_\pi : \pi \in \Pi_d, d \in \mathbb{N}\}$, are linearly independent over $\mathbb{C}$, and we call their span the set of *noncommutative symmetric functions.* (Note that this is different from the noncommutative symmetric functions studied in [6])

Another useful basis will be the *noncommutative power sum symmetric functions* given by

$$p_\pi \stackrel{\text{def}}{=} \sum_{\sigma \geq \pi} m_\sigma = \sum_{i_1, i_2, \ldots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \tag{2.2}$$

where the second sum is over all multisets $\{\{i_1, i_2, \ldots, i_d\}\}$ of $\mathbb{P}$ such that $i_j = i_k$ if j and k are both in the same block of π. In a similar manner we will define the *noncommutative elementary symmetric function* basis elements by

$$e_\pi = \sum_{\sigma : \sigma \wedge \pi = \hat{0}} m_\sigma = \sum_{i_1, i_2, \ldots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \tag{2.3}$$

where the second sum is over all multisets $\{\{i_1, i_2, \ldots, i_d\}\}$ of $\mathbb{P}$ such that $i_j \neq i_k$ if j and k are both in the same block of π. With these definitions one may derive the formulae found in the appendix of Doubilet's paper [4] which show

$$m_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) p_\sigma$$

and

$$m_\pi = \sum_{\tau \geq \pi} \frac{\mu(\pi, \tau)}{\mu(\hat{0}, \tau)} \sum_{\sigma \leq \tau} \mu(\sigma, \tau) e_\sigma. \tag{2.4}$$

This verifies that these are actually bases for the noncommutative symmetric functions.

As an illustration of these definitions, we see that

$$\begin{aligned} p_{13/24} &= x_1 x_2 x_1 x_2 + x_1 x_3 x_1 x_3 + \cdots + x_1^4 + x_2^4 + \cdots \\ &= m_{13/24} + m_{1234} \end{aligned}$$

and that

$$\begin{aligned} e_{13/24} &= x_1^2x_2^2 + \cdots + x_1x_2^2x_1 + \cdots + x_1^2x_2x_3 + \cdots \\ &\quad + x_1x_2^2x_3 + \cdots + x_1x_2x_3^2 + \cdots + x_1x_2x_3x_1 + \cdots + x_1x_2x_3x_4 \cdots \\ &= m_{12/34} + m_{14/23} + m_{12/3/4} + m_{1/23/4} + m_{1/2/34} + m_{14/2/3} + m_{1/2/3/4}. \end{aligned}$$

Allowing the variables to commute transforms p_π into $p_{\lambda(\pi)}$ and e_π into $\pi! e_{\lambda(\pi)}$. It should be clear that these noncommutative symmetric functions are symmetric in the usual sense, i.e., they are invariant under the previously defined symmetric group action on the variables. However, it will be useful to define a new action of the symmetric group on the noncommutative symmetric functions which permutes the *positions* of the variables. For $\delta \in \mathcal{S}_d$, we define

$$\delta \circ m_\pi \stackrel{\text{def}}{=} m_{\delta(\pi)},$$

where the action of $\delta \in S_d$ on a set partition of $[d]$ is the obvious one acting on the elements of the blocks. It follows that for any δ this action induces a vector space isomorphism, since it merely produces a permutation of the basis elements. Alternatively we can consider this action to be defined on the monomials so that

$$\delta \circ (x_{i_1} x_{i_2} \cdots x_{i_k}) \stackrel{\text{def}}{=} x_{i_{\delta^{-1}(1)}} x_{i_{\delta^{-1}(2)}} \cdots x_{i_{\delta^{-1}(k)}}$$

and extend linearly.

Utilizing the first characterization of this action, it follows straight from definitions (2.2) and (2.3) that $\delta \circ p_\pi = p_{\delta(\pi)}$ and $\delta \circ e_\pi = e_{\delta(\pi)}$.

2.2 Development and Results for Y_G

We begin this section by reviewing the definition of Y_G.

Definition 2.2.1 *For any multigraph G with vertices labeled $v_1, v_2, \ldots, v_d$* in a fixed order, *define*

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where again the sum is over all proper colorings of G, but the x_i are now noncommuting *variables.*

As an example, we can calculate

$$\begin{aligned} Y_{P_3} &= x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + \cdots + x_1x_2x_3 + x_1x_3x_2 + \cdots + x_3x_2x_1 + \cdots \\ &= m_{13/2} + m_{1/2/3}. \end{aligned}$$

We again mention that if G has loops then this sum is empty and we would have $Y_G = 0$. Furthermore, Y_G depends not only on G, but also on the *labeling* of its vertices. In this section we verify that Y_G does indeed satisfy a deletion-contraction recurrence and use this to prove some results about the expansion of Y_G in certain bases for the noncommutative symmetric functions. In order to get such a recurrence, it is necessary to have a distinguished edge.

We want to be able to uniformly choose such an edge, and so we will also define an action of the symmetric group on a graph. For all $\delta \in \mathcal{S}_d$ we let δ act on the vertices of G by $\delta(v_i) = v_{\delta(i)}$. This creates the action on graphs given by $\delta(G) = H$, where H is just a relabeling of G. Finally, any partition P of $V(G)$ induces a set partition $\pi(P)$ of $[d]$ corresponding to the subscripts of the vertices.

Proposition 2.2.2 (Relabeling Proposition) *For any finite multigraph G, we have*

$$\delta \circ Y_G = Y_{\delta(G)},$$

where the vertex order $v_1, v_2, \ldots, v_d$ is used in both Y_G and $Y_{\delta(G)}$.

Proof. Recall that a *stable partition* of the vertex set of G is a set partition of $V(G)$ such that no two vertices in the same block of the partition are adjacent. (If G has a loop, there are none.) It should be clear from the definitions that

$$Y_G = \sum_P m_{\pi(P)} \tag{2.5}$$

where the sum is over all stable partitions, P, of $V(G)$.

If we have two different labelings of G, say G and H, we choose $\delta \in S_d$ such that $\delta(G) = H$. We note that the action of δ produces a bijection between the stable partitions of G and H. Utilizing the above characterization (2.5) of Y_G and denoting the stable partitions of G and H by P_G and P_H respectively, we have

$$Y_H = \sum_{P_H} m_{\pi(P_H)} = \sum_{P_G} m_{\delta(\pi(P_G))} = \sum_{P_G} \delta \circ m_{\pi(P_G)} = \delta \circ \sum_{P_G} m_{\pi(P_G)} = \delta \circ Y_G. \qquad \blacksquare$$

We now turn our attention to deletion-contraction techniques. Using the Relabeling Proposition allows us, without loss of generality, to choose a labeling of G so our distinguished edge is $e = v_{d-1}v_d$. It is this edge for which we will derive the deletion-contraction recurrence for Y_G.

Definition 2.2.3 *We define an operation called* induction, $\uparrow$, *on the monomial*

$x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}$, *by*

$$(x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}})\uparrow \;=\; x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}^2$$

and extend this operation linearly.

Note that this function takes a noncommutative symmetric function which is homogeneous of degree $d-1$ to one which is homogeneous of degree d. Context will make it clear whether the word induction refers to this operation or to the proof technique.

This definition will only be used for deletion-contraction on the edge $e = v_d v_{d-1}$. We can extend induction to any edge $e = v_k v_l$ as follows. For $k < l$, define an operation $\uparrow_k^l$ on noncommutative symmetric functions which simply repeats the variable x_k in the l^{th} position. That is, for a monomial $x_{i_1}\cdots x_{i_k}\cdots x_{i_{d-1}}$, define

$$(x_{i_1}\cdots x_{i_k}\cdots x_{i_{l-1}}x_{i_l}\cdots x_{i_{d-1}})\uparrow_k^l \;\stackrel{\text{def}}{=}\; x_{i_1}\cdots x_{i_k}\cdots x_{i_{l-1}}x_{i_k}x_{i_l}\cdots x_{i_{d-1}}$$

and extend linearly.

At this point it will be useful to adopt the convention that provided G has an edge which is not a loop, we choose a labeling such that $e = v_{d-1}v_d$. We also note here that if there is no such edge, then

$$Y_G = \begin{cases} e_{1/2/\cdots/d} = p_{1/2/\cdots/d} & \text{if } G = \overline{K_d} \\ 0 & \text{if } G \text{ has a loop.} \end{cases} \tag{2.6}$$

We mention here that K_d is the complete graph on d vertices, i.e., the graph (not multigraph) on d vertices having all possible edges. Its edge complement is $\overline{K_d}$, the completely disconnected graph which has d vertices and no edges. In order to allow for multiple edges and loops, we note that contracting a multiple edge will form a loop, and adopt the convention that to contract a loop, we simply delete it.

Proposition 2.2.4 (Deletion-Contraction Proposition) *For $e = v_{d-1}v_d$, we have $Y_G = Y_{G-e} - Y_{G/e}\uparrow$, where the contraction of $e = v_{d-1}v_d$ is labeled v_{d-1}.*

Proof. The proof is very similar to that for the deletion-contraction property of $\mathcal{X}_G$. We consider proper colorings of $G - e$. They can be split disjointly into two types:

1. Proper colorings of $G - e$ with vertices v_{d-1} and v_d different colors;

2. Proper colorings of $G - e$ with vertices v_{d-1} and v_d the same color.

Those of the first type clearly correspond to proper colorings of G. If κ is a coloring of $G - e$ of the second type then, since the vertices v_{d-1} and v_d are the same color, we have

$$x_{\kappa(v_1)}x_{\kappa(v_2)}\cdots x_{\kappa(v_{d-1})}x_{\kappa(v_d)} = (x_{\kappa(v_1)}x_{\kappa(v_2)}\cdots x_{\kappa(v_{d-1})})\uparrow = x_{\tilde{\kappa}}\uparrow$$

where $\tilde{\kappa}$ is a proper coloring of G/e, and the vertex obtained from the contraction is labeled v_{d-1}. Thus we have $Y_{G-e} = Y_G + Y_{G/e}\uparrow$. Rearranging the terms gives $Y_G = Y_{G-e} - Y_{G/e}\uparrow$. ∎

We note that if e is a repeated edge, then the proper colorings of $G - e$ are exactly the same as those of G. The fact that there are no proper colorings of the second type corresponds to the fact that G/e has a loop, and so it has no proper colorings. Also note that since we allow multiple edges when contracting e we have

$$|E(G-e)| = |E(G/e)| = |E(G)| - 1.$$

The Deletion-Contraction Proposition for Y_G gives us a tool for using induction. For $\pi \in \Pi_{d-1}$ we let $\pi + (d) \in \Pi_d$ denote the partition π with d inserted into the block containing $d-1$. From equations (2.1) and (2.2) it is easy to see that

$$m_\pi\!\uparrow = m_{\pi+(d)} \qquad \text{and} \qquad p_\pi\!\uparrow = p_{\pi+(d)}.$$

With this notation we can now provide an example of the deletion-contraction proposition for P_3, where the vertices are labeled sequentially, and the distinguished edge is $e = v_2v_3$.

$$Y_{P_3} = Y_{P_2 \uplus \{v_3\}} - Y_{P_2}\!\uparrow .$$

It is not difficult to compute

$$\begin{aligned} Y_{P_2\uplus\{v_3\}} &= m_{1/2/3} + m_{1/23} + m_{13/2}, \\ Y_{P_2} &= m_{1/2}, \\ Y_{P_2}\!\uparrow &= m_{1/23}. \end{aligned}$$

This gives us

$$\begin{aligned} Y_{P_3} &= m_{1/2/3} + m_{1/23} + m_{13/2} - m_{13/2} \\ &= m_{1/2/3} + m_{13/2}, \end{aligned}$$

which agrees with our previous calculation.

We may use this recurrence to provide noncommutative analogues of the previously cited results of Stanley, where the proofs now follow from induction.

Theorem 2.2.5 *For any finite labeled graph G,*

$$Y_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\pi(S)},$$

where $\pi(S)$ denotes the partition of $\{1, 2, \ldots, d\}$ associated with the partition of $V(G)$

into the connected components of S.

Proof. We induct on the number of non-loops in $E(G)$. If $E(G)$ consists only of n loops, for $n \geq 0$, then for all $S \subseteq E(G)$, we will have $\pi(S) = 1/2/\cdots/d$. This shows us that

$$\sum_{S \subseteq E} (-1)^{|S|} p_{\pi(S)} = \sum_{S \subseteq E} (-1)^{|S|} p_{1/2/\ldots/d} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i p_{1/2/\ldots/d} = \begin{cases} p_{1/2/\ldots/d} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

This agrees with equation (2.6).

Now, if G has edges which are not loops, we use the Relabeling Proposition to obtain a labeling for G with $e = v_{d-1}v_d$. From the Deletion-Contraction Proposition we know that $Y_G = Y_{G-e} - Y_{G/e}\uparrow$ and that both $G - e$ and G/e have one fewer edge than G. This allows us to apply induction to Y_{G-e} and $Y_{G/e}$, obtaining

$$Y_G = \sum_{S \subseteq E(G-e)} (-1)^{|S|} p_{\pi(S)} - \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow .$$

It should be clear that

$$\sum_{S \subseteq E(G-e)} (-1)^{|S|} p_{\pi(S)} = \sum_{\substack{S \subseteq E(G) \\ e \notin S}} (-1)^{|S|} p_{\pi(S)}.$$

Hence it suffices to show that if $e \in S$,

$$- \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow = \sum_{\substack{S \subseteq E(G) \\ e \in S}} (-1)^{|S|} p_{\pi(S)}.$$

To do so we define a map $\Theta : \{\tilde{S} \subseteq E(G/e)\} \to \{S \subseteq E(G) : e \in S\}$ by

$$\Theta(\tilde{S}) = S, \text{ where } S = \tilde{S} \cup e.$$

Then Θ is a bijection, since we allow multiple edges to occur when we contract e to v_{d-1}. We know that $|\tilde{S}|+1 = |S|$ and $\pi(S) = \pi(\tilde{S}) + (d)$, giving $p_{\pi(S)} = p_{\pi(\tilde{S})}\uparrow$. Thus we have

$$\begin{aligned} -\sum_{\tilde{S}\subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow &= \sum_{\tilde{S}\subseteq E(G/e)} (-1)^{|\tilde{S}|+1} p_{\pi(\tilde{S})}\uparrow \\ &= \sum_{\substack{S\subseteq E(G)\\ e\in S}} (-1)^{|S|} p_{\pi(S)}. \end{aligned}$$

This completes the proof. ∎

By letting the x_i commute, we then obtain Theorem 1.2.2 as a corollary. There are also other results which we may obtain by this method, such as Stanley's generalization of Whitney's Broken Circuit Theorem. Before we prove this, however, we will need the following lemma, which appeared in [1]. For the sake of completeness, we include a proof here.

Lemma 2.2.6 *For any non-loop e, there is a bijection between B_G and $B_{G-e} \cup B_{G/e}$ given by*

$$S \longrightarrow \begin{cases} \tilde{S} = S - e \in B_{G/e} & \text{if } e \in S \\ \tilde{S} = S \in B_{G-e} & \text{if } e \notin S, \end{cases}$$

where we take e to be the first edge of G in the total order on the edges .

Proof. It is enough to show that this map is well-defined, and that it has a well-defined inverse.

To show that it is well-defined, we let $S \in B_G$ with $e \in S$. Notice that e is neither a loop nor a multiple edge: if e were a loop, then ϕ is a broken circuit, and so $e \notin S$. If e and e' have the same endpoints, then by the assumption that e is minimal, e is a broken circuit, and so again $e \notin S$.

If $e \notin S$ it is clear that $S \in B_{G-e}$, since any broken circuit of $G-e$ is also a broken circuit of G.

If $e \in S$, we let $S = \{e, e_2, \dots, e_k\}$ be listed in increasing order, according to the total order, $\leq$, on the edges. Then $\tilde{S} = \{e_2, \dots, e_k\}$ does not contain any broken circuits of G/e, for if S' was a broken circuit of G/e contained in $\tilde{S}$, then $S' \bigcup \{e_m\}$ would be a circuit of G/e for some e_m larger than any element of S'. But then $S' \bigcup \{e_m, e\}$ would contain a circuit of G. So $S' \bigcup \{e\}$ would contain a broken circuit of G, since $e \leq e_m$. This contradicts $S \in B_G$. To construct the inverse, we simply map

$$\tilde{S} \longrightarrow \begin{cases} \tilde{S} & \text{if } \tilde{S} \in B_{G-e} \\ \tilde{S} \bigcup \{e\} & \text{if } \tilde{S} \in B_{G/e}. \end{cases}$$

It is clear that this is the inverse, provided again that the map is well-defined. An argument similar to the one given above shows that this is indeed the case. ∎

We can now obtain a characterization of Y_G in terms of the broken circuit complex of G for any fixed total ordering on the edges.

Theorem 2.2.7 *We have*

$$Y_G = \sum_{S \in B_G} (-1)^{|S|} p_{\pi(S)},$$

where again $\pi(S)$ denotes the partition of $\{1, 2, \dots, d\}$ with blocks corresponding to the connected components of S. ∎

Proof. We again induct on the number of non-loops in $E(G)$. If the edge set consists only of n loops, it should be clear that for $n > 0$ we will have every edge being a circuit, and so the empty set is a broken circuit. Thus we have

$$Y_G = \begin{cases} \sum_{S\in\phi}(-1)^{|S|}p_{\pi(S)} = 0 & \text{if } n > 0, \\ \sum_{S\in\{\phi\}}(-1)^{|S|}p_{\pi(S)} = p_{1/2/\ldots/d} & \text{if } n = 0. \end{cases}$$

This again matches equation (2.6).

For $n > 0$ and e a non-loop, we again consider $Y_G = Y_{G-e} - Y_{G/e}\uparrow$, where $G-e$ and G/e both have one less edge than G, and so induction applies. From the preceding lemma and arguments as in Proposition 2.2.5, we have

$$\sum_{\substack{S\in B_G \\ e\notin S}} (-1)^S p_{\pi(S)} = \sum_{\tilde{S}\in B_{G-e}} (-1)^S p_{\pi(S)}$$

and

$$\sum_{\substack{S\subseteq E(G) \\ e\in S}} (-1)^{|S|} p_{\pi(S)} = -\sum_{\tilde{S}\in B_{G/e}} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow,$$

which gives the result. ∎

CHAPTER 3

Orientations and Sinks

3.1 Acyclic Orientations

As we have seen in the introduction, there are some interesting results which relate the chromatic polynomial of a graph to the number of acyclic orientations of the graph and to the sinks of these acyclic orientations. We begin here by proving Theorem 3. While the result is not new, we do offer a new proof here more in keeping with the spirit of the other proofs in this thesis. We denote the set of acyclic orientations of G by $\mathcal{A}(G)$, and the set of acyclic orientations of G with a unique sink at v_0 by $\mathcal{A}(G, v_0)$. We also recall our notation that $\mathcal{X}_G(n) = a_0 + a_1 n + \cdots + a_k n^k$.

Lemma 3.1.1 *For any fixed vertex v_0, and any edge $e = uv_0, u \neq v_0$, the map*

$$D \longrightarrow \begin{cases} D - e \in \mathcal{A}(G - e, v_0) & \text{if } D - e \in \mathcal{A}(G - e, v_0) \\ D/e \in \mathcal{A}(G/e, v_0) & \text{if } D - e \notin \mathcal{A}(G - e, v_0), \end{cases}$$

is a bijection between $\mathcal{A}(G, v_0)$ and $\mathcal{A}(G - e, v_0) \uplus \mathcal{A}(G/e, v_0)$, where the vertex of G/e formed by contracting e is labeled v_0.

Proof. We must first prove that this map is well-defined, by showing that in both cases we actually obtain an acyclic orientation with unique sink at v_0. This is clear in the first case by definition. In the second, where $D - e \notin \mathcal{A}(G - e, v_0)$, it must be true that $D - e$ has sinks both at u and at v_0 (since deleting a directed edge of D will not change the acyclic property of the orientation, nor can it cause us to lose the sink at v_0). So the orientation D/e will be in $\mathcal{A}(G/e, v_0)$: since u and v_0 were the only sinks in $D - uv_0$ the contraction must have a unique sink at v_0, and there will be no new cycles formed. Hence this map is well-defined.

To see that this is actually a bijection, we need only exhibit the inverse. This is obtained by simply orienting all edges of G as in $D - uv_0$ or D/uv_0 as appropriate, and then adding in the oriented edge $\overrightarrow{uv_0}$. It should be clear that this map is also well-defined. ∎

Lemma 3.1.2 *If G is connected, then any $D \in |\mathcal{A}(G)|$ has at least one sink.*

Proof. While this is a well-known graph theory result, we prove it here for completeness, by way of contradiction. Consider the finite set of directed walks in $\mathcal{A}(G)$ given by

$$S = \{v_{i_1} \to v_{i_2} \to \ldots \to v_{i_k} : v_{i_l} \in V(G) \text{ for } 1 \leq l \leq k, \text{ and } k \leq |V(G)| + 1\}.$$

Clearly $S \neq \phi$, since for any vertex $v \in G$, the trivial walk given by v will be an element of S. So we may consider a walk, W, in S with maximum k. If $k = |V(G)|+1$, then W contains a cycle, contradicting $D \in \mathcal{A}(G)$. If $k \leq |V(G)|$, then we claim that v_{i_k} will be a sink of D. If it is not a sink, then there is a directed edge $e = \overrightarrow{v_{i_k}w}$, for some $w \in E(G)$. Adding this directed edge to W will again give us a walk in S. But this contradicts our choice of W. Hence v_{i_k} must be a sink of D, completing the proof. ∎

As an immediate corollary we have the following result.

Corollary 3.1.3 *For any $D \in |\mathcal{A}(G)|$, the number of sinks is greater than or equal to the number of components of G.* ■

Using Lemma 3.1.1 and Corollary 3.1.3, we may now prove Theorem 3 by showing that the boundary conditions and recurrence relations for $|a_1|$ and the number of acyclic orientations with a unique sink at a fixed vertex, v_0, are the same.

Theorem 3.1.4 *[7] For any fixed vertex v_0, the number of acyclic orientations of G with a unique sink at v_0 is $|a_1|$.*

Proof. We prove this theorem by showing that the boundary conditions and recurrence relations for $|a_1|$ and $|\mathcal{A}(G, v_0)|$ both match. Our recurrence will only be valid when there is a non-loop incident with v_0, and so our boundary condition will occur in the case where only loops are incident with v_0. If $d = 1$, then

$$\mathcal{X}_G(n) = \begin{cases} n & \text{if } G = K_1, \\ 0 & \text{if } G \text{ has loops.} \end{cases}$$

So in this case,

$$|a_1| = \left\{ \begin{array}{ll} 1 & \text{if } G = K_1, \\ 0 & \text{if } G \text{ has loops} \end{array} \right\} = |\mathcal{A}(G, v_0)|.$$

If $d > 1$, then having only loops incident with v_0 is equivalent to having at least two components in G. In this case we see from Theorem 1 that $|a_1| = 0$ and from Corollary 3.1.3 that $|\mathcal{A}(G, v_0)| = 0$ as well. Thus the boundary conditions match.

Now if there is a non-loop e incident with v_0, we can see that $|a_1|$ is the sum of the absolute values of the coefficients of n from $\mathcal{X}_{G-e}$ and $\mathcal{X}_{G/e}$ since the signs on the coefficients of the chromatic polynomial alternate. Hence it follows from Lemma 3.1.1 that the recurrence relations are also the same and so the theorem is proven. ■

Stanley has a stronger version of this result.

Theorem 3.1.5 *[15] If $X_G = \sum_\lambda c_\lambda e_\lambda$, then the number of acyclic orientations of G with j sinks is given by* $\sum_{l(\lambda)=j} c_\lambda$. ■

We can prove an analogue of this this theorem in the noncommutative setting by using techniques similar to his, have not been able to do so using induction. We can inductively demonstrate the weaker versions which follow.

Theorem 3.1.6 *Let $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$. Then for any fixed vertex, v_0, the number of acyclic orientations of G with a unique sink at v_0 is $(d-1)!c_{[d]}$.*

Proof. We again induct on the number of non-loops in G. In the base case, if all the edges of G are loops, then

$$Y_G = \begin{cases} e_{1/2/\ldots/d} & \text{if } G \text{ has no edges} \\ 0 & \text{if } G \text{ has loops.} \end{cases}$$

So

$$c_{[d]} = \left\{ \begin{array}{ll} 1 & \text{if } G = K_1 \\ 0 & \text{if } d > 1 \text{ or } G \text{ has loops} \end{array} \right\} = |\mathcal{A}(G, v_0)|.$$

If G has non-loops, then by the Relabeling Proposition , we may let $e = v_{d-1}v_d$. We know that $Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow$. Since we will only be interested in the leading coefficient, let

$$Y_G = ae_{[d]} + \sum_{\sigma < [d]} a_\sigma e_\sigma,$$

$$Y_{G\backslash e} = be_{[d]} + \sum_{\sigma < [d]} b_\sigma e_\sigma,$$

and

$$Y_{G/e} = ce_{[d-1]} + \sum_{\sigma<[d-1]} c_\sigma e_\sigma$$

where $\leq$ is the partial order on set partitions. By induction and Lemma 3.1.1, it is enough to show that $(d-1)!a = (d-1)!b + (d-2)!c$. For $\pi \in \Pi_{d-1}$ we recall that $\pi + (d) \in \Pi_d$ is obtained by inserting d into the block of π which contains $d-1$. It was noted before that for $\pi \in \Pi_{d-1}$, we have $p_\pi \uparrow = p_{\pi+(d)}$. We utilize the change of basis formulae from equations (2.2) and (2.4) to obtain

$$e_\pi \uparrow = \sum_{\sigma \leq \pi} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d))} \sum_{\tau \leq \sigma + (d)} \mu(\tau, \sigma + (d)) e_\tau. \tag{3.1}$$

With this formula, we compute the coefficient of $e_{[d]}$ from $Y_{G/e}\uparrow$. The only term which contributes comes from $ce_{[d-1]}\uparrow$, which gives us

$$\begin{aligned}
ce_{[d-1]}\uparrow &= c \sum_{\sigma \in \Pi_{d-1}} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d))} \sum_{\tau \leq \sigma + (d)} \mu(\tau, \sigma + (d)) e_\tau \\
&= c \frac{\mu(\hat{0}, [d-1])}{\mu(\hat{0}, [d])} e_{[d]} + \sum_{\tau < [d]} d_\tau e_\tau \\
&= \frac{-c}{d-1} e_{[d]} + \sum_{\tau < [d]} d_\tau e_\tau
\end{aligned}$$

Thus, from $Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow$ we have that

$$\begin{aligned}
(d-1)!a &= (d-1)!b + (d-1)! \frac{c}{d-1} \\
&= (d-1)!b + (d-2)!c,
\end{aligned}$$

which completes the proof. ■

The following corollaries follow easily from this result.

Corollary 3.1.7 *If* $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$, *then the number of acyclic orientations of* G *with one sink is* $d!c_{[d]}$.

Proof. From the proof above it follows that the number of acyclic orientations of G with a unique sink at v is independent of the choice of v. Since there are d vertices, the above proposition implies that the total number of acyclic orientations of G with only one sink is $d!c_{[d]}$. ■

Corollary 3.1.8 *If* $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$, *then the number of acyclic orientations of* G *with one source is* $d!c_{[d]}$. ■

This should be obvious, since reversing an acyclic orientation with a unique sink at a given vertex produces an acyclic orientation with a unique source at that vertex, and vice-versa.

Also following immediately from Theorem 3.1.4 and Proposition 3.1.6, we have

Corollary 3.1.9 *If* $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$, *and* $\mathcal{X}_G(n) = a_0 + a_1 n + \cdots + a_k n^k$ *then*

$$(d-1)!c_{[d]} = |a_1|.$$

■

3.2 The Modified Blass-Sagan Algorithm

From Theorem 3.1.4 we know that the number of acyclic orientations of G with a unique sink at v_0 is given by $|a_1|$. From Whitney's Theorem, we also see that $|a_1|$ is the number of sets, $S \in B_G$ with $|S| = d - 1$. We would like to prove that these two quantities are equal directly without using the chromatic polynomial. To do so, we introduce the notation that for any arc $a = \overrightarrow{wu}$, the oppositely oriented arc is denoted $a' = \overrightarrow{uw}$. We also say that to *unorient* an arc, a, in a digraph we will just add the oppositely oriented arc a'. Since we are interested in acyclic digraphs, it is necessary to adopt the convention that a digraph is acyclic if it has no cycles of length ≥ 3. With this convention, unorienting an arc will not necessarily produce a cycle. Also for any acyclic digraph D, we will need to let $c(D)$ be the *contraction of* D, which simply contracts all *unoriented* arcs of D. If D has no unoriented arcs, then it is clear that $D = c(D)$. We notice that $c(D)$ is still acyclic and has no unoriented arcs.

Corollary 3.2.1 *For any fixed vertex $v_0 \in V(G)$, the number of acyclic orientations of G with a unique sink at v_0 is the same as the number of sets, $S \in B_G$ with $|S| = d - 1$.*

Proof. Here we present a bijective proof of this result. We shall do this by means of modifying an algorithm first introduced by Blass and Sagan in [1]. This modified algorithm will examine the arcs of an acyclic orientation of G one at a time, and either delete the arc, or unorient it.

We now present the algorithm. Let us fix an orientation of G, which we will refer to as the *normal orientation* of G, and also choose a fixed vertex v_0 of G. The algorithm will accept any acyclic orientation D of G which has a unique sink at v_0, and consider each arc in turn, using the total order on the edges which defines the broken circuits. At the stage when an arc $a = \overrightarrow{wu}$ is being considered, the algorithm will delete the arc $a = \overrightarrow{wu}$ if either:

I) $D + a'$ has a cycle, or

II) $c(D) - a$ has only one sink, *and* a is *not* normally oriented.

Otherwise, the algorithm will unorient a. For an example of how this algorithm works, see Figure 3.1. The steps of the algorithm in this figure are labeled by either **I**, **II**,

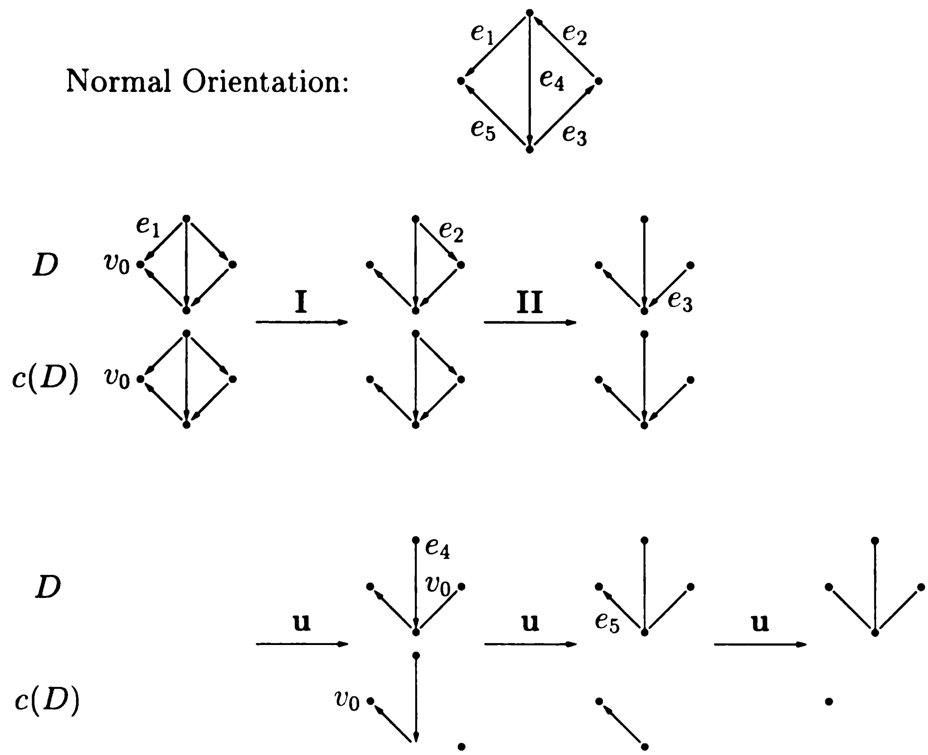

Figure 3.1. An example of the Algorithm

or **u**, indicating if the algorithm deleted the edge for reason I or II, or unoriented it.

To show that this algorithm actually does produce a bijection, we shall first introduce a sequence of sets, $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_q$ such that $\mathcal{D}_0$ is the set of all acyclic orientations of G with a unique sink at v_0 and $\mathcal{D}_q$ (where $q = |E(G)|$) is the set of all $S \in B_G$ with $|S| = d - 1$. Equivalently, $\mathcal{D}_q$ is the set of all spanning trees, T, of G such that $E(T)$ contains no broken circuits.

We will show that the kth step in the algorithm produces a bijection, $\mathcal{A}_k$, from $\mathcal{D}_{k-1}$ to $\mathcal{D}_k$, where $\mathcal{D}_k$ is defined as the set of all spanning subdigraphs D of G satisfying the following conditions:

(a) Each of the first k edges of G is either present in D (as an unoriented edge) or absent from D, but each of the remaining $q - k$ edges is present in D in exactly one orientation.

(b) D is acyclic.

(c) D has a $x \to v_0$ path for every $x \in V(D)$.

(d) The unoriented part of D contains no broken circuit.

From these conditions, it should be clear that $\mathcal{D}_0$ is indeed the set of acyclic orientations of G with a unique sink at v_0, since any finite weakly connected acyclic digraph has at least one sink. It is also clear that any element of $\mathcal{D}_q$ will be an acyclic, connected graph, which implies that the elements of $\mathcal{D}_q$ must be trees with exactly $d-1$ edges. So provided the algorithm produces a bijection at each step, we will produce the desired bijection between acyclic orientations of G with a unique sink at v_0, and edge sets of size $d-1$ which contain no broken circuits.

We should also note here that conditions (b) and (c) together imply that $c(D)$ must have a unique sink which occurs at the vertex identified with v_0. That this is the only possible sink of $c(D)$ is clear from condition (c). We also know that v_0 must be a sink of $c(D)$, since if it is not, then there is a vertex u and arc $a = \overrightarrow{v_0 u}$ in $c(D)$. But from condition (c) there would have to be a $u \to v_0$ path in D. This contradicts (b), which asserts that D is acyclic.

To show that the algorithm does indeed produce a bijection at each step, we use the following three lemmas. We also use the notational convention that a digraph in $\mathcal{D}_k$ will be denoted by D_k.

Lemma 3.2.2 *$\mathcal{A}_k$ maps $\mathcal{D}_{k-1}$ into $\mathcal{D}_k$.*

Proof. We need only prove that each of the properties (a)-(d) listed above is satisfied after the algorithm is applied at the kth step. We proceed to prove each one in turn.

(a) Since at the kth step the algorithm will either delete or unorient the kth edge, this is clear.

(b) Since any edge which would form a cycle if unoriented will be deleted by the algorithm, this also is clear.

(c) Since unorienting an arc can never destroy an $x \to v_0$ path, we need only consider the case where the algorithm deletes an arc. In fact, if the arc $a = \overrightarrow{wu}$ in D_{k-1} was deleted, we need only show that there is still a $w \to v_0$ path.

Now, if the arc $a = \overrightarrow{wu}$ in D_{k-1} was deleted for the first reason, then we must have had another (different) $w \to u$ path in D_{k-1}. Since there was a $u \to v_0$ path in D_{k-1}, (in fact, one which didn't use the arc a) we can then extend our other $w \to u$ path into a walk containing a $w \to v_0$ path in D_k.

If the arc $a = \overrightarrow{wu}$ in D_{k-1} was deleted for the second reason, again we need only consider the possibility that for the vertex w, there is no $w \to v_0$ path in D_k. But then there is no *oriented* arc $\overrightarrow{wu'}$ with $u \neq u'$, since otherwise all $u' \to v_0$ paths must also use the arc a, as there are no $w \to v_0$ paths in D_k. Thus D_{k-1} would have a cycle containing w. Contracting all unoriented arcs from w and repeating this argument as necessary, we see that w would then be a sink of $c(D_{k-1}) - a$, which contradicts our reason for deleting a.

(d) Suppose for the sake of contradiction that the unoriented part of D_k contains a broken circuit, $C - x$, where x is the greatest element of the cycle C. Since the unoriented part of D_{k-1} didn't contain any broken circuits, and since the only difference between D_{k-1} and D_k is at the kth edge a, we see that a must be unoriented in D_k and that $a \in C - x$. But then x is greater than a, and so x is present in D_k in one of its orientations. But all the other edges in C are also present and unoriented. Hence, C forms a cycle in D_k, contradicting the previously verified fact that D_k is acyclic. ■

Lemma 3.2.3 *A_k is one-to-one.*

Proof. Suppose D_1 and D_2 are two distinct elements of $\mathcal{D}_{k-1}$ which are both mapped to D by the algorithm. Since the algorithm only affects the kth edge, we note that D_1 and D_2 (and consequently $c(D_1)$ and $c(D_2)$) must only differ in the arc a. Without loss of generality, we may assume that a has an abnormal orientation in D_2 and a normal orientation in D_1.

We note that D was not obtained from D_1 and D_2 by the deletion of the arc a: we know that a could not have been deleted from either D_1 or D_2 for forming a cycle, as the other would then have contained that cycle. Also, if a was deleted by the algorithm from both D_1 and D_2 for the second reason, then a was abnormally oriented in both D_1 and D_2, which is not possible.

If a was not deleted by the algorithm, then $c(D_2) - a$ has an additional sink. So if $a = \overrightarrow{wu}$ is the arc in $c(D_2)$, then $w \neq v_0$ since v_0 is a sink, and so w must be the additional sink in $c(D_2) - a$. But this means that w was already a sink in $c(D_1)$, contradicting $D_1 \in \mathcal{D}_{k-1}$. ∎

Lemma 3.2.4 *A_k maps $\mathcal{D}_{k-1}$ onto $\mathcal{D}_k$.*

Proof. Given $D_k \in \mathcal{D}_k$ we must construct $D_{k-1} \in \mathcal{D}_{k-1}$ which maps onto it. Hence for any digraph, $D_k \in \mathcal{D}_k$, we must construct a digraph D_{k-1} and verify that the algorithm does indeed map D_{k-1} onto D_k, and that D_{k-1} satisfies properties (a)-(d). For all of the following cases, it will be immediate that the D_{k-1} we construct will satisfy properties (a), (b), and (d), so we will only show the verification of property (c). Let a be the kth edge of G. There are two cases.

The first case is when $a \notin D_k$. If there exists a unique orientation of a in which D_k would remain acyclic, we give a that orientation in D_{k-1}. If both orientations of a would preserve the acyclicity of D_k, then we choose the abnormal orientation for a in D_{k-1}. We note that at least one of the orientations of a must preserve acyclicity, since otherwise a completes two different cycles in D_{k-1}. These two cycles together would contain a cycle in D_k, which is a contradiction.

That the algorithm maps the digraph D_{k-1} obtained in the previous paragraph to D_k is obvious when only one orientation of a produces an acyclic orientation of D_{k-1}. However, if both produce acyclic orientations, we need to check that $c(D_{k-1}) - a$ has a unique sink at v_0. This is true, since it is easy to see that $c(D_{k-1}) - a = c(D_{k-1} - a) = c(D_k)$. To verify that $c(D_{k-1})$ constructed above still satisfies property (c), we note that adding an arc cannot destroy any existing paths.

For the other case, we suppose that a is present in D_k as an unoriented edge e, and so neither orientation can produce a cycle in D_{k-1} . We note that there must be at least one orientation of $e = wu$ such that there remains an $x \to v_0$ path for every $x \in D_{k-1}$. If all $x \to v_0$ paths p use the arc $a = \overrightarrow{wu}$ for some x, and if all $y \to v_0$ paths q use $a' = \overrightarrow{uw}$, then the $x \to w$ portion of P together with the $w \to v_0$ portion of Q contains an $x \to v_0$ path avoiding a, which contradicts our assumption about x.

If there is a unique orientation of $e = wu$ so that there remains an $x \to v_0$ path for every $x \in D_{k-1}$ we choose that one to maintain property (c) for D_{k-1}, say $a = \overrightarrow{wu}$. Using the same argument we used to prove the second case of (c) in Lemma 3.2.2, it is easy to verify that the algorithm will take the D_{k-1} so constructed and map it to D_k by unorienting a since $c(D_{k-1}) - a$ has an additional sink at w .

In the subcase where $e = uw$ is present in D_k as an unoriented edge and we would still retain property (c) with either orientation of e, we will consider the digraph D_{k-1} obtained from D by giving e the normal orientation, say $a = \overrightarrow{wu}$. It is clear that the algorithm maps D_{k-1} to D_k, since $D_{k-1} + a' = D_k$ is acyclic and a has the normal orientation. ∎

CHAPTER 4

Results on e-positivity

4.1 Inducing e_π

We now turn our attention to the expansion of Y_G in terms of the elementary symmetric function basis. We recall that for any fixed $\pi \in \Pi_d$ we use $\pi + (d+1)$ to denote the partition of $[d+1]$ formed by inserting the element $(d+1)$ into the block of π which contains d. We will denote the block of π which contains d by B_π. We also let $\pi/d+1$ be the partition of $[d+1]$ formed by adding the block $\{d+1\}$ to π.

In order obtain information about the coefficients for the expansion of Y_G in non-commutative elementary symmetric functions using our deletion-contraction results, it is necessary for us to understand the coefficients arising in $e_\pi\uparrow$. We have seen that the expression for $e_\pi\uparrow$ is rather complicated (see equation (3.1)). However, if the terms in the expression of $e_\pi\uparrow$ are grouped properly, the coefficients in many of the groups will sum to zero. To see that such a grouping should exist, we use the following lemma.

Lemma 4.1.1 *For $\pi \in \Pi_d$, let $e_\pi\uparrow = \sum_{\tau \in \Pi_{d+1}} c_\tau e_\tau$. Then*

$$\sum_{\substack{\tau\in\Pi_{d+1}\\ \lambda(\tau)\text{fixed}}} c_\tau = \begin{cases} 1/|B_\pi| & \textit{if } \lambda(\tau)=\lambda(\pi/d+1),\\ -1/|B_\pi| & \textit{if } \lambda(\tau)=\lambda(\pi+(d+1)),\\ 0 & \textit{else.}\end{cases}$$

Proof. Let $\pi = B_1/B_2/\ldots/B_\pi/\ldots/B_k \in \Pi_n$. We also let b_i denote the size of the block B_i. We may now consider the graphs on $d+1$ vertices given by

$$G_\pi = K_{B_1} \uplus K_{B_2} \uplus \cdots \uplus (K_{B_\pi}+e) \uplus \cdots \uplus K_{B_k},$$

where K_{B_i} is the complete graph on b_i vertices labeled with the elements of B_i, and where $K_{B_\pi}+e$ is the complete graph on $b_\pi \stackrel{\text{def}}{=} b$ vertices labeled with the elements of B_π which also contains an additional vertex labeled $d+1$, and an additional edge from d to $d+1$.

By the recurrence relation for Y_G, we can see that $Y_{G_\pi} = Y_{G_\pi - e} - Y_{G_\pi/e}\uparrow$. Equivalently, this gives us that $Y_{G_\pi/e}\uparrow = Y_{G_\pi-e} - Y_{G_\pi}$. It is easy to see that $Y_{G_\pi/e} = e_\pi$, and so this gives us that $e_\pi\uparrow = Y_{G_\pi-e} - Y_{G_\pi}$. If we let C be the operator which allows the variables to commute, we get $C(e_\pi\uparrow) = X_{G_\pi-e} - X_{G_\pi}$.

We now proceed to calculate $X_{G_\pi-e}$ and X_{G_π}. Since it is easy to see that $G_\pi - e = K_{B_1} \uplus K_{B_2} \uplus \cdots \uplus K_{B_\pi} \uplus K_{\{d+1\}} \uplus \cdots \uplus K_{B_k}$, we may use the product rule for X_G to show that $X_{G_\pi-e} = \pi! e_{(\lambda(\pi),1)}$. To calculate X_{G_π} we again will use the product rule, together with the fact that

$$X_{K_{B_\pi}} = \sum_{\lambda\vdash d+1} a_\lambda |\lambda| m_\lambda,$$

where a_λ is the number of stable partitions of the vertex set of K_{B_π} of type λ. Letting $\mu = \lambda(B_1/B_2/\cdots/\hat{B}_\pi/\cdots/B_k)$, we can easily count the number of stable partitions of K_{B_π} to obtain the equation

$$X_{G_\pi} = \mu! e_\mu \left[(b-1)(b-1)! m_{(1^{b-1},2)} + (b+1)! m_{(1^{b+1})} \right].$$

We may then use the fact that $m_{(1^{b-1},2)} = e_{(b,1)} - (b+1)e_{(b+1)}$ and $m_{(1^{b+1})} = e_{(b+1)}$, combining this with the previous results to obtain:

$$C(e_\pi\uparrow) = \pi! e_{(\lambda(\pi),1)} - \mu! e_\mu \left[(b-1)(b-1)! \left(e_{(b,1)} - (b+1)e_{(b+1)} \right) + (b+1)! e_{(b+1)} \right].$$

Following simplification, this leads to

$$C(e_\pi\uparrow) = \frac{(\pi/d+1)!}{b} e_{(\lambda(\pi),1)} - \frac{(\pi+(d+1))!}{b} e_{\lambda(\pi+(d+1))}. \tag{4.1}$$

This essentially completes the proof, as we know that if $e_\pi\uparrow = \sum_{\tau\in\Pi_{d+1}} c_\tau e_\tau$, then

$$C(e_\pi\uparrow) = \sum_{\tau\in\Pi_{n+1}} c_\tau C(e_\tau).$$

Hence we can see that

$$\left(\sum_{\substack{\tau\in\Pi_{d+1} \\ \lambda(\tau) \text{ fixed}}} c_\tau \right) \tau!$$

is the coefficient of $e_{\lambda(\tau)}$ in $C(e_\pi\uparrow)$. Putting this together with equation (4.1) gives the result. ■

This lemma show us that the coefficients of various terms from $e_\pi\uparrow$ can be combined in a nice way and even indicates exactly how to do so. We need to sum together the coefficients from set partitions which are of the same type (as integer partitions), and whose block containing d have the same size. If we do this, then almost all these amalgamated coefficients will drop out. We need to know, however, if there is a pattern to these combinations which will allow us to repeatedly use deletion-contraction techniques. We see that the contributing coefficients of $e_\pi\uparrow$ will have type $\lambda(\pi + (d+1))$ or $\lambda(\pi/d+1)$. If we want to be able to repeat this process, though, it will be necessary to know the size of the block of $e_\pi\uparrow$ which contains $d+1$. We want all those terms of $e_\pi\uparrow$ which do not have type $\lambda(\pi+(d+1))$ or $\lambda(\pi/d+1)$, *and* whose block containing $d+1$ is not the same size as the block containing $d+1$ in $\pi+(d+1)$ or $\pi/d+1$ to drop out. With one more bit of notation, we may show this is indeed the way the coefficients will behave. Let $P(\alpha) = P(\alpha_1, \alpha_2, \ldots, \alpha_l)$ be the set of all partitions of $[d+1]$ which are less than or equal to $\pi + (d+1)$, have blocks of size $\alpha_1, \alpha_2, \ldots, \alpha_l$, and for which $d+1$ is in a block of size α_1. The proper grouping of the terms of $e_\pi\uparrow$ is given by the following lemma.

Lemma 4.1.2 *If* $e_\pi \uparrow \; = \sum_{\tau\in\Pi_{d+1}} c_\tau e_\tau$, *then* $c_\tau = 0$ *unless* $\tau \leq \pi + (d+1)$, *and for any* $\alpha \vdash (d+1)$, *we have*

$$\sum_{\tau\in P(\alpha)} c_\tau = \begin{cases} 1/|B_\pi| & \textit{if } P(\alpha) = \{\pi/d+1\}, \\ -1/|B_\pi| & \textit{if } P(\alpha) = \{\pi+(d+1)\}, \\ 0 & \textit{else.} \end{cases}$$

Proof. Fix $\pi \in \Pi_d$. By equation (3.1)

$$e_\pi\uparrow = \sum_{\sigma\leq\pi} \frac{\mu(\hat{0},\sigma)}{\mu(\hat{0},\sigma+(d+1))} \sum_{\tau\leq\sigma+(d+1)} \mu(\tau,\sigma+(d+1))e_\tau.$$

Hence we may express

$$e_\pi\uparrow = \sum_{\tau\leq\pi+(d+1)} c_\tau e_\tau,$$

where for any fixed $\tau \leq \pi + (d+1)$ we have

$$c_\tau = \sum_{\substack{\sigma\leq\pi \\ \sigma+(d+1)\geq\tau}} \frac{-1}{|B_\sigma|}\mu\left(\tau,\sigma+(d+1)\right). \tag{4.2}$$

We first note that if $\tau = \pi/d+1 \in P(\alpha)$, then $|P(\alpha)| = 1$ and we have the interval $[\tau, \pi + (d+1)] \cong \Pi_2$. A simple direct computation shows that $c_{\pi/d+1} = 1/|B_\pi|$. Similarly, if $\tau = \pi + (d+1) \in P(\alpha)$, then again $|P(\alpha)| = 1$ and we can easily compute $c_{\pi+(d+1)} = -1/|B_\pi|$.

We now fix $\tau = B_1/B_2/\cdots/B_{q+2}/\cdots/B_l \in P(\alpha)$ such that $|B_i| = \alpha_i$ for all $1 \leq i \leq l$. For $q \geq -1$, we let $B_1, B_2, \cdots, B_{q+2}$ be the blocks of τ which are contained in $B_{\pi+(d+1)}$. For notational convenience, we will also let $|B_{\pi+(d+1)}| = m+1$, where $m \geq 1$. Finally let β denote the partition obtained from τ by merging the blocks of τ which contain d and $d+1$, allowing $\beta = \tau$ if d and $d+1$ are in the same block of τ. Replacing $\sigma + (d+1)$ by $\sigma \in \Pi_{d+1}$ in equation (4.2), we see that

$$c_\tau = \sum_{\beta \leq \sigma \leq \pi+(d+1)} \frac{-1}{|B_\sigma| - 1} \mu(\tau, \sigma).$$

Now for any $B \subseteq [d+1]$ we will consider the sets

$$L(B) = \{\sigma \in \Pi_{d+1} : \{d, d+1\} \subseteq B \in \sigma, \text{ where } \beta \leq \sigma \leq \pi + (d+1)\}.$$

The nonempty $L(B)$ partition the interval $[\beta, \pi + (d+1)]$ according to the content of the block containing $\{d, d+1\}$ and so we may express

$$c_\tau = \sum_B \frac{-1}{|B| - 1} \sum_{\sigma \in L(B)} \mu(\tau, \sigma).$$

To compute the inner sum, we need to consider the following 2 cases.

Case 1) For some $k > q+2$, B_k is strictly contained in a block of $\pi + (d+1)$. In this case, we see that each non-empty $L(B)$ forms a non-trivial cross-section of a product of partition lattices, and so for this case

$$\sum_{\sigma \in L(B)} \mu(\tau, \sigma) = 0.$$

Thus partitions in this case will not contribute to $\sum\limits_{\tau \in P(\alpha)} c_\tau$.

Case 2) For all $k > q+2$, B_k is a block of $\pi + (d+1)$. In this case, we have $q \geq 0$, since otherwise we must have $\tau = \pi + (d+1)$ which we have already considered. Then we can show

$$\frac{1}{|B|-1} \sum_{\sigma \in L(B)} \mu(\tau, \sigma) = \begin{cases} \frac{(-1)^{q+1}(q+1)!}{m} & \text{if } B = B_{\pi+(d+1)} \\ \frac{(-1)^q q!}{m-\alpha_i} & \text{if } B = B_{\pi+(d+1)} \setminus B_i, \quad 2 \leq i \leq q+2 \\ 0 & \text{else.} \end{cases} \tag{4.3}$$

It is easy to see that if $B = B_{\pi+(d+1)}$ then $L(B) = \{\pi + (d+1)\}$ and so that part is clear. Also, if $B = B_{\pi+(d+1)} \setminus B_i$ for some $2 \leq i \leq q+2$, then we have $L(B) \cong \Pi_1$ and so $\sum_{\sigma \in L(B)} \mu(\tau, \sigma) = (-1)^q q!$. Otherwise, $L(B)$ again forms a non-trivial cross-section of a product of partition lattices, and again gives us no net contribution to the sum.

We notice that since $\{d, d+1\} \subset B$, the second case in (4.3) will only occur if $d \in B_j$ for $j \neq i$. So adding up all these contributions gives

$$c_\tau = (-1)^q q! \left(\sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \alpha_i} - \frac{q+1}{m} \right).$$

In order to compute the sum over all $\tau \in P(\alpha)$, it will be convenient to consider all possible choices for the block of τ containing d. So for $\tau \leq \pi + (d+1)$ and

$1 \leq j \leq q+2$, let

$$P(\alpha, j) = \{(B_1, B_2, \ldots, B_l) \mid B_1/B_2/\ldots/B_l \in P(\alpha), d \in B_j\}.$$

The sequence $(B_1, B_2, \ldots, B_l)$ is called an *ordered* set partition. Also define

$$\delta_j = \begin{cases} \alpha_j - 1 & \text{if } j = 1 \\ \alpha_j & \text{else,} \end{cases}$$

so

$$|P(\alpha, j)| = \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}}.$$

Thus we can see that

$$\sum_{\tau \in P(\alpha, j)} c_\tau = \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}} (-1)^q q! \left(\sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \alpha_i} - \frac{q+1}{m} \right).$$

To obtain the sum over all $\tau \in P(\alpha)$ we need to sum over all $P(\alpha, j)$ for $1 \leq j \leq q+2$. However, for $1 \leq r \leq m+1$, if we let k_r be the number of blocks $B_i, 1 \leq i \leq q+2$ which have size r, then in the sum over all $P(\alpha, j)$, each $\tau \in P(\alpha)$ appears $\Pi_{r=1}^{m+1} k_r!$ times. Combining all this information, we see that

$$\sum_{\tau \in P(\alpha)} c_\tau = \frac{(-1)^q q!}{\prod_{r=1}^{m+1} k_r!} \sum_{j=1}^{q+2} \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}} \left(\sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \delta_i} - \frac{q+1}{m} \right).$$

Hence it suffices to show that

$$\sum_{j=1}^{q+2} \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}} \left(\sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \delta_i} - \frac{q+1}{m} \right) = 0.$$

Using the multinomial recurrence,

$$\sum_{j=1}^{q+2} \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}} = \binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}},$$

we need only show that

$$\sum_{j=1}^{q+2} \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}} \sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \delta_i} = \frac{q+1}{m} \binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}}.$$

However, we may express

$$\begin{aligned}
\sum_{j=1}^{q+2} \binom{m-1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}} \sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \delta_i} &= \sum_{j=1}^{q+2} \frac{\binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}} \delta_j}{m} \sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \delta_i} \\
&= \frac{\binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}}}{m} \sum_{j=1}^{q+2} \sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{\delta_j}{m - \delta_i} \\
&= \frac{\binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}}}{m} \sum_{i=2}^{q+2} \frac{1}{m - \delta_i} \sum_{\substack{j=1 \\ j \neq i}}^{q+2} \delta_j \\
&= \frac{\binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}}}{m} \sum_{i=2}^{q+2} \frac{1}{m - \delta_i} (m - \delta_i) \\
&= \frac{q+1}{m} \binom{m}{\delta_1, \ldots, \delta_j, \ldots, \delta_{q+2}}.
\end{aligned}$$

■

4.2 Some e-positivity Results

We wish to use this result to prove some positivity theorems about Y_G's expansion in the elementary symmetric function basis. If the coefficients of the elementary symmetric functions in this expansion are all non-negative, then we say that Y_G is *e-positive*. Unfortunately, even for some of the simplest graphs, Y_G is usually not $e-$positive. The only graphs which are obviously $e-$positive are the complete graphs on n vertices, for which we have $Y_{K_n} = e_{[n]}$, and their complements, which yield $Y_{\overline{K}_n} = e_{1/2/\cdots/n}$. Even paths, with the vertices labeled sequentially, are not e–positive, for we can compute that $Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$. However, in this example we can notice that while Y_{P_3} is not e–positive, if we group the terms according to their type and the size of the block which contains 3, the sum of these coefficients will be non-negative.

This observation along with the proof of the previous lemma inspires us to define equivalence classes reflecting the sets $P(\alpha)$. That is, if the block of σ containing i is $B_{\sigma,i}$ and the block of τ containing i is $B_{\tau,i}$, we define

$$\sigma \equiv_i \tau \text{ iff } \lambda(\sigma) = \lambda(\tau) \text{ and } |B_{\sigma,i}| = |B_{\tau,i}|.$$

In a similar manner, we can define

$$e_\sigma \equiv_i e_\tau \text{ iff } \sigma \equiv_i \tau$$

and let (τ) and $e_{(\tau)}$ denote the equivalence classes of τ and e_τ respectively. We can take formal sums of these equivalence classes and write expressions such as

$$\sum_{\sigma\in\Pi_d} c_\sigma e_\sigma \equiv_i \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)} e_{(\tau)} \text{ where } c_{(\tau)} = \sum_{\sigma\in(\tau)} c_\sigma.$$

We will refer to this equivalence relation as *congruence modulo i*.

Using this notation, we have $Y_{P_3} \equiv_3 \frac{1}{2}e_{(12/3)} + \frac{1}{2}e_{(123)}$, since $e_{13/2} \equiv_3 e_{1/23}$. We will say that a labeled graph G (and similarly Y_G) is (e)*–positive* if $c_{(\tau)}$ is non-negative for all $(\tau) \subseteq \Pi_d$ under some labeling of G and suitably chosen congruence. We notice that the expression of Y_G for a labeled graph G may have all non-negative amalgamated coefficients for congruence modulo i, but not for congruence modulo j, making the definition for (e)-positivity *seem* to depend on the labeling of the graph. However,

the (e)-positivity of a graph is in fact *not* dependent on the labeling of the graph. If a different labeling for an (e)-positive graph is chosen, then we need only change the congruence class appropriately, to again see the (e)-positivity. This should be clear from the Relabeling Proposition. That is, if $\gamma(G) = H$ for some γ such that $\gamma(i) = j$, and $Y_G \equiv_i \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)} e_{(\tau)}$, then $Y_H \equiv_j \sum_{(\tau)\subseteq\Pi_d} c_{(\gamma(\tau))} e_{(\gamma(\tau))}$. So if G is (e)-positive modulo i, then H is seen to be (e)-positive modulo j. Hence relabeling a graph does not change its (e)-positivity. We now turn our attention to showing that paths, cycles, and complete graphs with one edge deleted are all (e)-positive. We begin with a few more preliminary results about this congruence relation and how it affects our induction of e_π.

We note that in the proof of Lemma 4.1.2, the roles played by the elements d and $d+1$ are essentially interchangeable. That is, if we let $\tilde{P(\alpha)}$ be the set of all partitions of $[d+1]$ which are less than or equal to $\pi+(d+1)$, have blocks of size $\alpha_1, \ldots, \alpha_l$ and for which d is in a block of size α_1, and let $\tilde{\pi} \in \Pi_d$ be the partition π with d replaced by $d+1$, then the same proof will show that

$$\sum_{\tau \in \tilde{P(\alpha)}} c_\tau = \begin{cases} 1/|B_\pi| & \text{if } \tilde{P(\alpha)} = \{\tilde{\pi}/d\}, \\ -1/|B_\pi| & \text{if } \tilde{P(\alpha)} = \{\tilde{\pi}+(d)\}, \\ 0 & \text{else.} \end{cases}$$

Note that here $\tilde{\pi}+(d)$ is the partition obtained from $\tilde{\pi}$ by inserting the element d into the block of $\tilde{\pi}$ containing $d+1$. This allows us to state a corollary in terms of the congruence relationship just defined.

Corollary 4.2.1 *For $\pi \in \Pi_d$,*

$$e_\pi\!\uparrow \equiv_{d+1} \frac{1}{b} e_{(\pi/d+1)} - \frac{1}{b} e_{(\pi+(d+1))}$$

and

$$e_\pi \uparrow \equiv_d \frac{1}{b} e_{(\tilde{\pi}/d)} - \frac{1}{b} e_{(\tilde{\pi}+(d))},$$

where $b = |B_\pi|$. ■

The next lemma simply verifies that the induction operation respects the congruence relation and should be clear without proof.

Lemma 4.2.2 *If* $\gamma \equiv_d \tau$, *then* $e_\gamma \uparrow \equiv_{d+1} e_\tau$. ■

From this we can extend induction to congruence classes in a well-defined manner:

$$e_{(\pi)} \uparrow = \sum_{(\tau) \subseteq \Pi_{d+1}} c_{(\tau)} e_{(\tau)} \quad \text{if and only if} \quad e_\pi \uparrow = \sum_{\tau \in \Pi_{d+1}} c_\tau e_\tau.$$

In order to use induction to prove the (e)-positivity of a graph G, we will usually try to delete a set of edges which will isolate either a single vertex or a complete graph from G in hope of obtaining a simpler (e)-positive graph. In order to see how this procedure will affect Y_G, we use the following lemma.

Lemma 4.2.3 *Given a graph, G on d vertices define* $H = G \uplus K_m$. *If* $Y_G = \sum_{\sigma \in \Pi_d} c_\sigma e_\sigma$, *then* $Y_H = \sum_{\sigma \in \Pi_d} c_\sigma e_{\sigma/d+1,d+2,\ldots,d+m}$.

Proof. From previous statements it follows that

$$\begin{aligned} Y_H &= Y_G e_{[m]} \\ &= \sum_{\sigma \in \Pi_d} c_\sigma e_\sigma e_{[m]} \\ &= \sum_{\sigma \in \Pi_d} c_\sigma e_{\sigma/d+1,d+2,\ldots,d+m}. \end{aligned}$$

The result above naturally suggests that we use the notation G/v_{d+1} for the graph $G \uplus \{v_{d+1}\}$. We are now in a position to prove the (e)-positivity of paths.

Proposition 4.2.4 *For all $d \geq 1$, Y_{P_d} is $(e)-$positive.*

Proof. We proceed by induction, having labeled P_d so that $E(P_d) = \{v_1v_2, v_2v_3, \ldots, v_{d-1}v_d\}$. If $d = 1$, then we have $Y_{P_1} = e_1$ and the proposition is clearly true.

So we assume by induction that

$$Y_{P_d} \equiv_d \sum_{(\tau)\in\Pi_d} c_{(\tau)}e_{(\tau)},$$

where $c_{(\tau)} \geq 0$ for all $(\tau) \in \Pi_d$. From our deletion-contraction recurrence, Corollary 4.2.1 and Lemma 4.2.3, we see that using $e = v_dv_{d+1}$,

$$\begin{aligned} Y_{P_{d+1}} &= Y_{P_d/v_{d+1}} - Y_{P_d}\uparrow \\ &\equiv_{d+1} \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)}e_{(\tau/d+1)} - \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)}e_{(\tau)}\uparrow \\ &\equiv_{d+1} \sum_{(\tau)\subseteq\Pi_d} \left(1 - \frac{1}{|B_\tau|}\right) c_{(\tau)}e_{(\tau/d+1)} + \sum_{(\tau)\subseteq\Pi_d} \frac{c_{(\tau)}}{|B_\tau|}e_{(\tau+(d+1))}. \end{aligned}$$

Since we know that $c_{(\tau)} \geq 0$, and $|B_\tau| \geq 1$ for all τ, this completes the induction step and the proof. ■

In the commutative context we will say that the symmetric function X_G is *e-positive* if all the coefficients in the expansion of the elementary symmetric functions are non-negative. It is easy to see that we can use the (e)-positivity result for Y_{P_d} and specialize it to show the e-positivity of X_{P_d}.

Corollary 4.2.5 *X_{P_d} is $e-$positive.* ■

It is then natural to ask if cycles will also be $(e)-$positive, for when we delete an edge of a cycle we obtain a path. While it is true that cycles will be $(e)-$positive, a stronger relationship exists between paths and cycles. In fact, it turns out that the coefficients in the (e)-expansion of paths will appear again as the coefficients in the (e)-expansion of cycles with only a slight modification. For labeling purposes, however, we will need the following lemma which follows easily from the Relabeling Proposition.

Lemma 4.2.6 *If $\gamma \in \mathcal{S}_d$ fixes d, then $Y_{\gamma(G)} \equiv_d Y_G$.*

Proposition 4.2.7 *For all $d \geq 2$, if*

$$Y_{P_d} \equiv_d \sum c_{(\tau)} e_{(\tau)}, \text{ then } Y_{C_{d+1}} \equiv_{d+1} \sum c_{(\tau)} e_{(\tau+(d+1))},$$

where we have labeled the graphs so $E(P_d) = \{v_1v_2, v_2v_3, \ldots, v_{d-1}v_d\}$ and $E(C_{d+1}) = \{v_1v_2, v_2v_3, \ldots, v_{d-1}v_d, v_dv_{d+1}, v_{d+1}v_1\}$.

Proof. We proceed by induction on d. If $d = 2$, then $Y_{P_2} = e_{[2]}$ and $Y_{C_3} = e_{[3]}$, and so the proposition holds for $d = 2$.

For the induction step, we assume that

$$Y_{P_{d-1}} \equiv_{d-1} \sum c_{(\tau)} e_{(\tau)}$$

and also that

$$Y_{C_d} \equiv_d \sum c_{(\tau)} e_{(\tau+(d))}.$$

We notice that if $e = v_dv_{d+1}$, then $C_{d+1} - e$ does not have the standard labeling for paths. But if we let $\gamma = (d+1)(1,d)(2,d-1)\cdots(\lfloor \frac{d+1}{2} \rfloor, \lceil \frac{d+1}{2} \rceil)$ then we can use the deletion-contraction recurrence to get

$$Y_{C_{d+1}} = Y_{\gamma(P_{d+1})} - Y_{C_d}\uparrow.$$

From our opening observation, since $d+1$ is a fixed point for γ, this allows us to deduce that

$$Y_{C_{d+1}} \equiv_{d+1} Y_{P_{d+1}} - Y_{C_d}\uparrow .$$

In the proof of Proposition 4.2.4 we saw that

$$Y_{P_{d+1}} = Y_{P_d/v_{d+1}} - Y_{P_d}\uparrow .$$

Combining these two equations gives

$$Y_{C_{d+1}} \equiv_{d+1} Y_{P_d/v_{d+1}} - Y_{P_d}\uparrow - Y_{C_d}\uparrow . \tag{4.4}$$

The demonstration of Proposition 4.2.4 also showed us that

$$Y_{P_d} \equiv_d \sum_{(\tau)} \left(\left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d)} + \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d))} \right) . \tag{4.5}$$

Applying Corollary 4.2.1 and Lemma 4.2.3 yields

$$\begin{aligned} Y_{P_d}\uparrow \equiv_{d+1} \quad & \textstyle\sum_{(\tau)} \left[\left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d/d+1)} - \left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d,d+1)} \right. \\ + \quad & \left. \frac{c_{(\tau)}}{|B_\tau|(|B_\tau|+1)} e_{(\tau+(d)/d+1)} - \frac{c_{(\tau)}}{|B_\tau|(|B_\tau|+1)} e_{(\tau+(d)+(d+1))} \right] \end{aligned}$$

and

$$Y_{P_d/v_{d+1}} \equiv_{d+1} \sum_{(\tau)} \left(c_{(\tau)} + \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau+(d)/d+1)} - \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau/d,d+1)}$$

respectively. By the induction hypothesis,

$$\begin{aligned} Y_{C_d}\uparrow \; & \equiv_{d+1} \sum_{(\tau)} c_{(\tau)} e_{(\tau+(d))}\uparrow \\ & \equiv_{d+1} \sum_{(\tau)} \left(\frac{c_{(\tau)}}{|B_\tau|+1} e_{(\tau+(d)/d+1)} - \frac{c_{(\tau)}}{|B_\tau|+1} e_{(\tau+(d)+(d+1))} \right) . \end{aligned}$$

Plugging these expressions for $Y_{P_d/d+1}$, $Y_{P_d}\uparrow$, and $Y_{C_d}\uparrow$ into equation (4.4), grouping the terms according to type, and simplifying gives

$$Y_{C_{d+1}} \equiv_{d+1} \sum_{(\tau)} \left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d,d+1)} + \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d)+(d+1))}.$$

This corresponds to the expression in equation (4.5) for Y_{P_d} in exactly the desired manner, and so we are done.

■

Since we know that $Y_{P_{d-1}}$ is (e)−positive, for all $d \geq 3$, we have an immediate corollary.

Proposition 4.2.8 *For all $d \geq 3$, Y_{C_d} is (e)−positive.* ■

If we allow the variables to commute again, we can specialize to X_{C_d}.

Corollary 4.2.9 *For all $d \geq 1$, X_{C_d} is e−positive.* ■

We are also able to use our recurrence to show the (e)-positivity of complete graphs with one edge removed.

Proposition 4.2.10 *For $d \geq 2$, if $e = v_{d-1}v_d$ then*

$$Y_{K_d - e} \equiv_d \frac{d-2}{d-1} e_{([d])} + \frac{1}{d-1} e_{([d-1]/d)}.$$

Proof. Consider the complete graph K_d and apply deletion-contraction to the edge $e = v_{d-1}v_d$. Together with Corollary 4.2.1 this will give us

$$\begin{aligned} e_{[d]} &= Y_{K_d} \\ &= Y_{K_d-e} - Y_{K_{d-1}}\uparrow \\ &= Y_{K_d-e} - e_{[d-1]}\uparrow \\ &\equiv_d Y_{K_d-e} - \frac{1}{d-1} e_{([d-1]/d)} + \frac{1}{d-1} e_{([d])}. \end{aligned}$$

Simplifying gives the result. ■

This also immediately specializes.

Corollary 4.2.11 *For $d \geq 2$,*

$$X_{K_d-e} = d(d-2)(d-2)!e_d + (d-2)!e_{(d-1,1)}.$$

■

4.3 The (3+1)-free Conjecture

Let **a+b** be the poset which is a disjoint union of an a-element chain and a b-element chain. The poset P is said to be **(a+b)**-*free* if it contains no induced subposet isomorphic to **a+b**. Let $G(P)$ denote the graph obtained from P where the edges of $G(P)$ are $\{uv : u \text{ and } v \text{ are incomparable as elements of } P\}$. Then the **(3+1)**-free Conjecture of Stanley and Stembridge [17] may be stated.

If P is **(3+1)**-free, then $X_{G(P)}$ is e-positive.

A subset of the **(3+1)**-free graphs is the class of *indifference graphs.* They are characterized [16] as having vertices and edges

$$V = [n], \text{ and } E = \{uv : u, v \text{ belong to some } I \in \mathcal{C}\},$$

where $\mathcal{C}$ is a collection of intervals $[i, j] = \{i, i+1, \ldots, j\} \subseteq [n]$. We note that without loss of generality, we can assume no interval in the collection is properly contained in any other. These graphs are both $(\mathbf{3+1})$-free and $(\mathbf{2+2})$-free.

Indifference graphs have a nice structure suitable for our deletion-contraction techniques. That is, if G is an indifference graph on n vertices with at least one edge, then it is not hard to see that there is an edge ($e = v_h v_n$ where $[h, n]$ is the "last" interval in the collection) for which both $G - e$ and G/e are indifference graphs. Thus we may inductively assume the (e)-positivity of indifference graphs with fewer edges than G, and use deletion-contraction to obtain indifference graphs with fewer edges. Unfortunately, the relationship between the coefficients in the (e)-expansion of Y_{G-e} and $Y_{G/e}\uparrow$ is not entirely clear, so we have not been able to make this method work on all indifference graphs. However, we are able to resolve a special case. For any composition (ordered integer partition) of n, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let $\tilde{\lambda_i} = \sum_{j \leq i} \lambda_j$. A $K_\lambda - chain$ is the indifference graph using the collection of intervals $\{[1, \tilde{\lambda_1}], [\tilde{\lambda_1}, \tilde{\lambda_2}], \ldots, [\tilde{\lambda_{k-1}}, \tilde{\lambda_k}]\}$. This is just a string of complete graphs, whose sizes are given by the parts of λ, which are attached to one another sequentially at single vertices. We notice that the K_λ-chain for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ can be obtained from the K_τ-chain for $\tau = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ by attaching the graph K_{λ_k} to its last vertex.

We will be able to handle this type of attachment for any graph G with vertices $\{v_1, v_2, \ldots, v_d\}$. Hence, we define $G + K_m$ to be the graph with

$$V(G + K_m) = V(G) \cup \{v_{d+1}, \ldots, v_{d+m-1}\}$$

and

$$E(G + K_m) = E(G) \cup \{e = v_i v_j : i, j \in [d, d + m + 1]\}.$$

Using deletion-contraction techniques, we are able to obtain information about the (e)-expansion of $G + K_m$ in terms of the coefficients in the (e)-expansion of G. However, we will also need some more notation. For $\pi \in \Pi_d$, we let $\pi + i$ denote the partition given by π with the additional i elements $d + 1, d + 2, \ldots, d + i$ added to B_π. This is in contrast to $\pi + (i)$, which denotes the partition given by π with the element i inserted into B_π. Finally, we follow Stanley in denoting the falling factorial by

$$\langle m \rangle_i \stackrel{\text{def}}{=} m(m-1)\cdots(m-i+1)$$

and the rising factorial by

$$(b)_i \stackrel{\text{def}}{=} b(b+1)\cdots(b+i-1).$$

We begin with a short study of the behavior of $Y_{G+K_m}\uparrow_d^{d+j}$ by proving the following lemma.

Lemma 4.3.1 *If* $1 \leq j < k \leq m$, *then* $Y_{G+K_m}\uparrow_d^{d+j} \equiv_d Y_{G+K_m}\uparrow_d^{d+k}$.

Proof. For any partition, π of $[d + m - 1]$,, define $\delta_{d+j}(\pi)$ to be the partition σ obtained by inserting the element $d + j$ into the block of π containing d, and adding one to each element of π which is at least $d + j$. It should be easy to see that $m_\pi\uparrow_d^{d+j} = m_{\delta_{d+j}(\pi)}$. Similarly, $m_\pi\uparrow_d^{d+k} = m_{\delta_{d+k}(\pi)}$. Now consider the permutation $\gamma = (d+j, d+k, d+k-1, d+k-2, \ldots, d+j+1)$. We can see that $\gamma \circ (m_\pi\uparrow_d^{d+j}) = m_\pi\uparrow_d^{d+k}$. This implies that $\gamma \circ (Y_{G+K_m}\uparrow_d^{d+j}) = Y_{G+K_m}\uparrow_d^{d+k}$. Noticing that d is a fixed point of γ, and so Lemma 4.2.6 applies will complete the proof. ∎

Lemma 4.3.2 *If G is a graph on d vertices with*

$$Y_G \equiv_d \sum_{(\pi)\subseteq \Pi_d} c_{(\pi)} e_{(\pi)},$$

then

$$Y_{G+K_m}\uparrow_d^{d+m} \equiv_{d+m} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)} \langle m-1\rangle_i \left[e_{(\pi+i/d+i+1,\ldots,d+m)} - e_{(\pi+i+(d+m)/d+i+1,\ldots,d+m-1)}\right]}{(b)_{i+1}},$$

where $b = |B_\pi|$.

Proof. We prove the lemma by induction on m. The case $m = 1$ is merely a restatement of Corollary 4.2.1. So we may assume this lemma is true for $Y_{G+K_m}\uparrow_d^{d+m}$, and proceed to prove it for $Y_{G+K_{m+1}}\uparrow_d^{d+m+1}$.

From Lemma 4.3.1, it follows that for $1 \leq j \leq m$, we have

$$Y_{G+K_m}\uparrow_d^{d+j}\uparrow_d^{d+m+1} \equiv_{d+m+1} Y_{G+K_m}\uparrow_d^{d+m}\uparrow_d^{d+m+1}.$$

Now, from $G+K_{m+1}$ we may delete the edge set $\{v_d v_{d+j} : 1 \leq j \leq m\}$ and combine all the terms $Y_G\uparrow_d^{d+j}\uparrow_d^{d+m+1}$ for $1 \leq j \leq m$ to obtain

$$\begin{aligned} Y_{G+K_{m+1}}\uparrow_d^{d+m+1} &\equiv_{d+m+1} Y_{G\uplus K_m}\uparrow_d^{d+m+1} - mY_{G+K_m}\uparrow_d^{d+m}\uparrow_d^{d+m+1} \\ &\equiv_{d+m+1} Y_{G\uplus K_m}\uparrow_d^{d+m+1} - mY_{G+K_m}\uparrow_d^{d+m}\uparrow_{d+m}^{d+m+1}. \end{aligned}$$

From this point on, we need only concern ourselves with the clerical details, making sure that everything matches up properly. We can see from Lemma 4.1.2, Lemma 4.2.3 and the original hypothesis on Y_G, that

$$Y_{G\uplus K_m}\uparrow_d^{d+m+1} \equiv_{d+m+1} \sum_{(\pi)} \frac{c_{(\pi)}}{b} \left(e_{(\pi_1)} - e_{(\pi_2)}\right). \tag{4.6}$$

where

$$\pi_1 = \pi/d+1,\dots,d+m/d+m+1,$$

$$\pi_2 = \pi+(d+m+1)/d+1,\dots,d+m.$$

Similarly, the induction hypothesis shows

$$mY_{G+K_m}\uparrow_d^{d+m}\uparrow_{d+m}^{d+m+1} \equiv_{d+m+1} \sum_{(\pi)}\sum_{i=0}^{m-1} \frac{c_{(\pi)} m\langle m-1\rangle_i}{(b)_{i+1}} \left(\frac{e_{(\pi_3)}-e_{(\pi_4)}}{m-i} - \frac{e_{(\pi_5)}-e_{(\pi_6)}}{b+i+1}\right) \tag{4.7}$$

where

$$\pi_3 = \pi+i/d+i+1,\dots,d+m/d+m+1,$$

$$\pi_4 = \pi+i/d+i+1,\dots,d+m+1,$$

$$\pi_5 = \pi+i+(d+m)/d+i+1,\dots,d+m-1/d+m+1,$$

$$\pi_6 = \pi+i+(d+m)+(d+m+1)/d+i+1,\dots,d+m-1.$$

Simplifying the terms and combining both previous equations (4.6) and (4.7) gives

$$Y_{G+K_{m+1}}\uparrow_d^{d+m+1} \equiv_{d+m+1}$$

$$\sum_{(\pi)} c_{(\pi)} \left(\frac{e_{(\pi_1)}-e_{(\pi_2)}}{b} - \sum_{i=0}^{m-1} \frac{\left(e_{(\pi_3)}-e_{(\pi_4)}\right)\langle m\rangle_i}{(b)_{i+1}} + \sum_{i=0}^{m-1} \frac{\left(e_{(\pi_5)}-e_{(\pi_6)}\right)\langle m\rangle_{i+1}}{(b)_{i+2}}\right).$$

Note that modulo $d+m+1$ we have

$(\pi_5) = (\pi+i+1/d+i+2,\dots,d+m/d+m+1)$ and

$(\pi_6) = (\pi+i+1+(d+m+1)/d+i+2,\dots,d+m).$

So by shifting indices and simplifying, we obtain

$$Y_{G+K_{m+1}}\uparrow_d^{d+m+1} \equiv_{d+m+1}$$

$$\sum_{(\pi)} \sum_{i=0}^{m} \frac{c_{(\pi)} \langle m \rangle_i \left[e_{(\pi+i/d+i+1,\dots,d+m+1)} - e_{(\pi+i+(d+m+1)/d+i+1,\dots,d+m)} \right]}{(b)_{i+1}},$$

which completes the induction step and the proof. ∎

This lemma is useful because it helps us to find an explicit formula for $Y_{G+K_{m+1}}$ in terms of Y_G. Once we have the formula, it will be easy to verify that if G is (e)-positive, then so is $G + K_{m+1}$. For the induction step in establishing this explicit formula, we will need the following observation.

Lemma 4.3.3 *For any graph G on d vertices, and $(i,j) \in S_d$, then*

$$Y_G \uparrow_i^{d+1} \equiv_{d+1} Y_{(i,j)(G)} \uparrow_j^{d+1} .$$

Proof: For any $\pi \in \Pi_d$, we see directly from the definitions for induction and the symmetric group action that

$$(i,j) \circ (m_\pi \uparrow_i^{d+1}) = m_{(i,j)\pi} \uparrow_j^{d+1} .$$

From this it follows that

$$(i,j) \circ (Y_G \uparrow_i^{d+1}) = Y_{(i,j)(G)} \uparrow_j^{d+1} .$$

Since $d+1$ is a fixed point of (i,j), Lemma 4.2.6 gives the result. ∎

We may now give the formula for $Y_{G+K_{m+1}}$ in terms of Y_G.

Lemma 4.3.4 *If $m \geq 1$, and*

$$Y_G \equiv_d \sum_{(\pi)\subseteq\Pi_d} c_{(\pi)} e_{(\pi)},$$

then

$$Y_{G+K_{m+1}} \equiv_{d+m} \sum_{(\pi)\subseteq\Pi_d} \sum_{i=0}^{m-1} \frac{c_{(\pi)}\langle m-1\rangle_i}{(b)_{i+1}} \left[(b-m+i)e_{(\hat{\pi})} + (i+1)e_{(\overline{\pi})}\right]$$

where $b = |B_\pi|$, and

$\hat{\pi} = \pi + i/d+i+1, \dots, d+m,$

$\overline{\pi} = \pi + i + (d+m)/d+i+1, \dots, d+m-1.$

Proof. We induct on m. If $m = 1$, then $Y_{G+K_2} = Y_{G\uplus K_1} - Y_G{\uparrow}_d^{d+1}$. This shows that

$$Y_{G+K_2} \equiv_{d+1} \sum_{(\pi)} \left(\frac{c_{(\pi)}(b-1)}{b} e_{(\pi/d+1)} + \frac{c_{(\pi)}}{b} e_{(\pi+(d+1))}\right),$$

which verifies the base case.

To begin the induction step, we repeatedly utilize the deletion-contraction recurrence to delete the edges $v_{d+i}v_{d+m+1}$ for $0 \leq i \leq m$, and obtain

$$Y_{G+K_{m+2}} \equiv_{d+m+1} Y_{G+K_{m+1}\uplus v_{d+m+1}} - mY_{G+K_{m+1}}{\uparrow}_{d+m}^{d+m+1} - Y_{G+K_{m+1}}{\uparrow}_d^{d+m+1}. \quad (4.8)$$

Note that we are able to combine all the terms from $Y_{G+K_{m+1}}{\uparrow}_{d+i}^{d+m+1}$ together by the previous observation, Lemma 4.3.3 since for all $1 \leq i \leq m$ there is a permutation satisfying the correct conditions.

We now expand each of the terms in equation (4.8). For the first, using Lemma 4.2.3,

$$Y_{G+K_{m+1}\uplus v_{d+m+1}} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)}\langle m-1\rangle_i}{(b)_{i+1}} \left[(b-m+i)e_{(\pi_1)} + (i+1)e_{(\pi_2)}\right],$$

where

$\pi_1 = \pi + i/d + i + 1, \ldots, d + m/d + m + 1,$

$\pi_2 = \pi + i + (d + m)/d + i + 1, \ldots, d + m - 1/d + m + 1,$ and

For the second term, using Corollary 4.2.1, we have

$$mY_{G+K_{m+1}} \uparrow_{d+m}^{d+m+1} \equiv_{d+m+1}$$

$$\sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)} \langle m \rangle_{i+1}}{(b)_{i+1}} \left[\frac{b - m + i}{m - i} \left(e_{(\pi_1)} - e_{(\pi_3)} \right) + \frac{i+1}{b+i+1} \left(e_{(\pi_2)} - e_{(\pi_4)} \right) \right],$$

where

$\pi_3 = \pi + i/d + i + 1, \ldots, d + m + 1,$ and

$\pi_4 = \pi + i + (d + m) + (d + m + 1)/d + i + 1, \ldots, d + m - 1.$

And finally, using Lemma 4.3.2,

$$Y_{G+K_{m+1}} \uparrow_d^{d+m+1} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m} \frac{c_{(\pi)} \langle m \rangle_i}{(b)_{i+1}} \left(e_{(\pi_3)} - e_{(\pi_5)} \right)$$

where

$\pi_5 = \pi + i + (d + m + 1)/d + i + 1, \ldots, d + m.$

Grouping the terms appropriately and shifting indices where needed we have massive cancellation, which yields

$$Y_{G+K_{m+2}} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m} \frac{c_{(\pi)} \langle m \rangle_i \left[(b - (m+1) + i) e_{(\pi_3)} + (i+1) e_{(\pi_5)} \right]}{(b)_{i+1}}.$$

This completes the induction step and the proof. ∎

Examining this lemma, we can see that in $Y_{G+K_{m+1}}$ we have the same sign on all the coefficients as we had in Y_G, with the possible exception of the terms where

$b < m - i$. But it is easy to see that in this case we have

$$e_{(\pi+i/d+i+1,\ldots,d+m)} \equiv_{d+m} e_{(\pi+m-i-k-1+(d+m)/d+m-i-k,\ldots,d+m-1)}.$$

This means that in the expression for $Y_{G+K_{m+1}}$ as a sum over congruence classes modulo $d+m$, we can combine the coefficients on these terms. And so upon simplification, the coefficient on $e_{(\pi+i/d+i+1,\ldots,d+m)}$ will be:

$$\left(\frac{(k-m+i)\langle m-1\rangle_i}{(k)_{i+1}} + \frac{(m-i-k)\langle m-1\rangle_{m-i-k-1}}{(k)_{m-i-k}}\right) c_{(\pi)},$$

where $c_{(\pi)}$ is the coefficient on $e_{(\pi)}$ in Y_G.

Adding these fractions by finding a common denominator, we see that this coefficient is actually zero. corresponding This gives us the next result.

Theorem 4.3.5 *If Y_G is (e)-positive, then Y_{G+K_m} is also (e)-positive.* ■

Notice that Proposition 4.2.4 follows easily from Theorem 4.3.5 and induction, since for paths $P_{m+1} = P_m + K_2$. As a more general result we have the following Corollary.

Corollary 4.3.6 *If G is a K_λ-chain, then Y_G is (e)-positive. Hence, X_G is also e-positive.* ■

We can also describe another class of (e)-positive graphs. We define a *diamond* to be the indifference graph on the collection of intervals $\{[1,3],[2,4]\}$. So a diamond consists of two K_3's sharing a common edge. Then the following holds.

Theorem 4.3.7 *Let D be a diamond. If G is (e)-positive, then so is $G+D$.*

Proof. The proof of this result is analogous to the proof for the case of $G + K_m$. We note that if G is a graph on d vertices, then we can think of $G + D$ as

G with the additional vertices $v_{d+1}, v_{d+2}, v_{d+3}$, and new edges constructed from the intervals $\{[d, d+2], [d+1, d+3]\}$. If we now delete and contract the edges $v_{d+1}v_{d+3}$ and $v_{d+2}v_{d+3}$, we see (using the appropriate symmetry) that

$$Y_{G+D} \equiv_{d+3} Y_{G+K_3 \uplus \{d+3\}} - 2Y_{G+K_3 \uparrow_{d+2}^{d+3}}. \tag{4.9}$$

By Lemma 4.3.4, if $Y_G \equiv_d \sum_{(\pi)} c_{(\pi)} e_{(\pi)}$ then

$$Y_{G+K_3} \equiv_{d+2} \sum_{(\pi)} c_{(\pi)} \left[\frac{b-2}{b} e_{(\pi_1)} + \frac{b-1}{b(b+1)} e_{(\pi_2)} + \frac{1}{b} e_{(\pi_3)} + \frac{2}{b(b+1)} e_{(\pi_4)} \right],$$

where

$\pi_1 = \pi/d+1, d+2,$

$\pi_2 = \pi + (d+1)/d+2,$

$\pi_3 = \pi + (d+2)/d+1,$

$\pi_4 = \pi + (d+1) + (d+2),$

$b = |B_\pi|.$

So by Corollary 4.2.1,

$$\begin{aligned} Y_{G+K_3} \uparrow_{d+2}^{d+3} \equiv_{d+3} & \sum_{(\pi)} c_{(\pi)} \left[\left(\frac{b-2}{2b} \right) \left(e_{(\tilde{\pi}_1)} - e_{(\pi_1')} \right) + \frac{b-1}{(b)_2} \left(e_{(\tilde{\pi}_2)} - e_{(\pi_2')} \right) \right. \\ & \left. + \frac{1}{(b_2)} \left(e_{(\tilde{\pi}_3)} - e_{(\pi_3')} \right) + \frac{2}{(b)_3} \left(e_{(\tilde{\pi}_4)} - e_{(\pi_4')} \right) \right] \end{aligned}$$

where $\tilde{\pi}_i = \pi_i/d+3$, and $\pi_i' = \pi_i + (d+3)$.

Using Lemma 4.2.3 and plugging these equations into (4.9), we have

$$Y_{G+D} \equiv_{d+3} \sum_{(\pi)} c_{(\pi)} \left[\frac{2e_{(\tilde{\pi}_4)}}{(b+1)_2} + \frac{(b-2)e_{(\pi_1')}}{b} + \frac{2(b-1)e_{(\pi_2')}}{(b)_2} + \frac{2e_{(\pi_3')}}{(b)_2} + \frac{4e_{(\pi_4')}}{(b)_3} \right].$$

It should be clear that all these coefficients will be non-negative, provided that the case $b = 1$ works out. But if $b = 1$, then we have $\pi_1' \equiv_{d+3} \pi_3'$. Hence the coefficient on $e_{(\pi_1')}$ is $\frac{b-2}{b} + \frac{2}{(b)_2}$, which is zero for $b = 1$. ■

CHAPTER 5

Open Problems and Conjectures

5.1 Partitioning Acyclic Orientations

In this chapter we conclude with a few open problems, and some philosophical ramblings on how one might attempt to answer them using our inductive machinery. Some of these problems were first discussed in Stanley's paper [15], and remain open at this point. We begin with an approach to the (**3**+**1**)-free conjecture suggested by Timothy Chow [3].

In the commutative setting, we know that if $X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$, then

$$\sum_{l(\lambda)=j} c_\lambda = \text{ the number of acyclic orientations of } G \text{ with } j \text{ sinks.}$$

This suggests that to show X_G is e-positive, we could partition the set of acyclic orientations of G with j sinks into subsets so that each subset has cardinality c_λ. For each such orientation, Tim's idea is to partition the vertices into $V(G) = \uplus_{i=1}^{j} V_i$ so that $|V_i| = \lambda_i$ and each subset V_i contains exactly one sink of the orientation, labeled v_i. This vertex partition is formed as follows. We begin by placing each v_i into V_i. For any other vertex, u, we place u in V_i if there are paths originating from u which terminate at the sinks $v_i, v_k, \ldots, v_m$ and $i = \min\{i, k, \ldots, m\}$.

This procedure works very nicely on paths, but to understand why it works, we need the following Corollary to Theorem 4.2.4. We notice first that for any representative τ of (τ), it is well-defined to take $(\tau)! = \tau!$.

Corollary 5.1.1 *If* $Y_{P_d} = \sum_{\tau\in\Pi_d} c_\tau e_\tau \equiv_d \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)} e_{(\tau)}$, *then*

$$\tau! c_{(\tau)} = |B_\tau| \prod_{B\in\tau-B_\tau} (|B|-1)$$

for every $(\tau) \subseteq \Pi_d$.

Proof. The proof is by induction (what else?). For our base case, $Y_{P_1} = e_{[1]}$ is enough to show the result holds. If we assume the induction hypothesis for P_d, then from equation (4.4, we have

$$Y_{P_{d+1}} \equiv_{d+1} \sum_{(\tau)} c_{(\tau)} \left(\frac{|B_\tau|-1}{|B_\tau|}\right) e_{(\tau/d+1)} + \sum_{(\tau)} \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d+1))}.$$

Examining the coefficients in this expression based on our induction hypothesis and then simplifying, we see that

$$(\tau/d+1)! c_{(\tau)} \frac{|B_\tau|-1}{|B_\tau|} = c_\tau |B_\tau| \prod_{B\in\tau-B_\tau} (|B|-1) \frac{|B_\tau|-1}{|B_\tau|} = c_\tau \prod_{B\in\tau} (|B|-1)$$

and

$$(\tau+(d+1))! \frac{c_{(\tau)}}{|B_\tau|} = (|B_\tau|+1)|B_\tau| \prod_{B\in\tau-B_\tau} (|B|-1) \frac{c_\tau}{|B_\tau|} = c_\tau (|B_\tau|+1) \prod_{B\in\tau-B_\tau} (|B|-1).$$

This completes the induction step and the proof. ∎

Note that this expression for $c_{(\tau)}$ automatically shows that it is non-negative, but we can also use it to verify that Chow's algorithm works for P_d. From the congruence relationship, we see that (τ) can be expressed as $(1, 2, \ldots, b_1/b_1+1, b_1+$

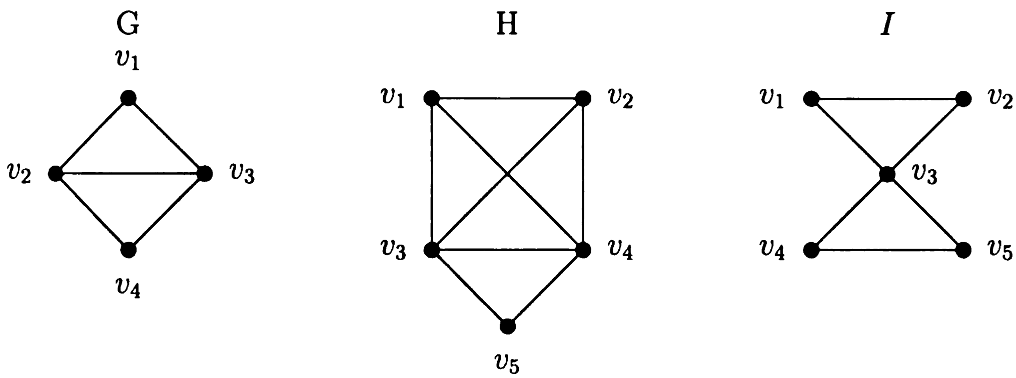

Figure 5.1. The graphs G, H, and I

$2, \ldots, b_2/\ldots/b_{k-1}+1, \ldots, d)$. Given a partition, we need to see that there are exactly $|B_\tau| \prod_{B \in \tau - B_\tau}(|B|-1)$ orientations which the algorithm will map to (τ). (Note that all orientations of a path must be acyclic.) We notice that v_{b_1} could not be a sink for any such orientation of P_d. If it were, then the existence of $\overrightarrow{v_{b_1+1}v_{b_1}}$ would force the algorithm to put both v_{b_1+1} and v_{b_1} into the same block, which was not the case. Similarly $v_{b_2}, \ldots, v_{b_{k-1}}$ are also not candidates for sinks of such an orientation, but v_d is. This gives us $|B_1|-1$ possibilities for the first sink, $|B_2|-1$ for the second, etc., until we reach the last sink, for which we have $|B_\tau|$ choices. It is not hard to see that each of these possibilities actually does give us an appropriate orientation for the paths, and verifies the algorithm works in this case.

In order to see the algorithm's failure to hold in general, as well as show the promise for the existence of a modification, we consider the graphs G, H, and I given in Figure 5.1.

For both the graphs G and H , the algorithm produces exactly the correct subset partitions.

$$X_G = 16e_4 + 2e_{(3,1)}, \quad \text{while} \quad Y_G \equiv_4 \frac{2}{3}e_{([4])} + \frac{1}{3}e_{(123/4)}.$$

The partitions we obtain from the algorithm for G are [4] which occurs 16 times and 123/4, which occurs twice.

$$X_H = 60e_5 + 12e_{(4,1)}, \quad \text{while} \quad Y_H \equiv_5 \frac{1}{2}e_{([5])} + \frac{1}{2}e_{(1234/5)}.$$

The partitions we obtain for H are [5] which occurs 60 times and 1234/5, which occurs 12 times. We notice that not only do we have the correct number of partitions of a given type, but we even have the partitions obtained matching the congruence classes. That is, for G the algorithm only produces the partition 123/4 having type (3,1), just as it only produces the partition 1234/5 for H. When we do the calculations for I, however, the algorithm doesn't produce the desired results. That is,

$$X_I = 20e_5 + 12e_{(4,1)} + 4e_{(3,2)} \quad \text{while} \quad Y_I \equiv_5 \frac{1}{6}e_{([5])} + \frac{1}{3}e_{(1/2345)} + \frac{1}{6}e_{(1234/5)} + \frac{1}{3}e_{(123/45)}.$$

In this case the algorithm produces [5] 20 times as expected, however the partition 1/2345 is produced 4 times, 1234/5 occurs 4 times, and 123/45 occurs 8 times. This is not the desired outcome, for we want 12 of the type $(4,1)$ and only 4 of the type $(3,2)$. However, if we look closely at the coefficients in Y_I we see that by taking $\pi!c_{(\pi)}$ we should get the numbers 20, 8, 4 and 4 just as we have, except these numbers are not associated with the desired partitions. While this shows that there is indeed an intimate relationship between the coefficients of Y_I and this algorithm, it is not clear how to modify this algorithm to produce the correct associated partitions.

5.2 X_G and Trees

We will end with a note on one of the more interesting unsolved problems posed by Stanley. We feel that our method has some promise in approaching this problem, but have been unable to make it work so far.

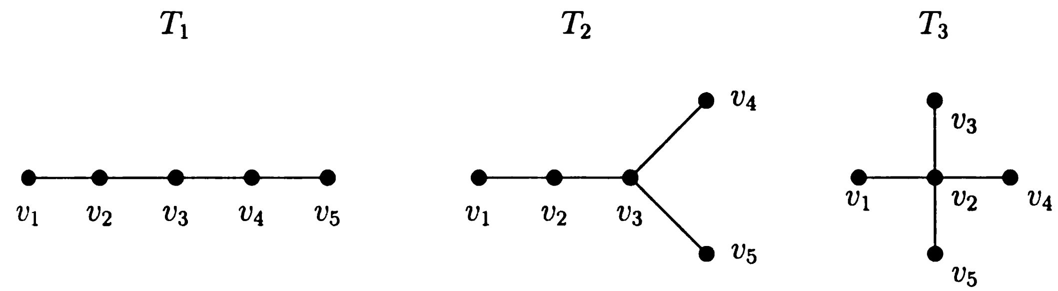

Figure 5.2. The trees T_1, T_2, and T_3.

We begin by noticing that if T is a tree on d vertices, we have $\mathcal{X}_T(n) = n(n-1)^{d-1}$. Since X_G is a generalization of the chromatic polynomial, it might be reasonable to suppose that it also is constant on trees with d vertices. This is far from the case! In fact, it has been verified up to $d = 9$ [3] that X_G will distinguish non-isomorphic trees. This leads to the following question posed by Stanley, which I will give as a conjecure.

Conjecture 5.2.1 *[15] X_G distinguishes non-isomorphic trees.*

To illustrate how we believe this problem may be approached using our deletion-contraction techniques, we will consider the case $d = 5$, where the three different trees labeled as in Figure 5.2. For these trees, we have the symmetric functions:

$$X_{T_1} = p_5 - 2p_{(4,1)} - 2p_{(3,2)} + 3p_{(3,1^2)} + 3p_{(2^2,1)} - 4p_{(2,1^3)} + p_{(1^5)}$$

$$Y_{T_1} \equiv_5 p_{(12345)} - p_{(1/2345)} - p_{(1234/5)} - p_{(12/345)} - p_{(123/45)} + p_{(1/2/345)}$$
$$+2p_{(123/4/5)} + 2p_{(1/23/45)} + p_{(12/34/5)} - p_{(1/2/3/45)} - 3p_{(12/3/4/5)} + p_{(1/2/3/4/5)}$$

$$X_{T_2} = p_5 - 3p_{(4,1)} - p_{(3,2)} + 4p_{(3,1^2)} + 2p_{(2^2,1)} - 4p_{(2,1^3)} + p_{(1^5)}$$

$$Y_{T_2} \equiv_5 p_{(12345)} - 2p_{(1/2345)} - p_{(1234/5)} - p_{(12/345)} + 2p_{(1/2/345)} + 2p_{(123/4/5)}$$
$$+p_{(12/34/5)} + p_{(1/23/45)} - 3p_{(12/3/4/5)} - p_{(1/2/3/45)} + p_{(1/2/3/4/5)}$$

$$X_{T_3} = p_5 - 4p_{(4,1)} + 6p_{(3,1^2)} - 4p_{(2,1^3)} + p_{(1^5)}$$

$$Y_{T_3} \equiv_5 p_{(12345)} - 3p_{(1/2345)} - p_{(1234/5)} + 3p_{(123/4/5)} + 3p_{(1/2/345)}$$
$$-3p_{(12/3/4/5)} - p_{(1/2/3/45)} + p_{(1/2/3/4/5)}.$$

We can see here that the X_{T_i} are distinct, as are the Y_{T_i}, with respect to the congruence relation. It is much more surprising that the X_{T_i} are distinct, however, since it is easy to show the following proposition.

Proposition 5.2.2 *For any graph G on d vertices with no loops or multiple edges, Y_G is unique.*

Proof. We know from equation (2.5) that $Y_G = \sum_P m_{\pi(P)}$ for the stable partitions P. Construct the graph H with vertex set $V(G) = \{v_1, v_2, \dots, v_d\}$ and edge set $E(H) = \{v_i v_j |$ there exists a $\pi(P)$ such that i, j are in the same block of $\pi(P)\}$. Since $\pi(P)$ comes from a stable partition P of G, v_i and v_j are in the same block of some $\pi(P)$ if and only if there is no edge $v_i v_j$ in G. Hence the graph H constructed is the (edge) complement of G and so we can recover G from H. ∎.

From this proposition it is trivially true that Y_G distinguishes trees. It does not, however, follow that Y_G will distinguish trees with respect to equivalence classes. It is easy to see that this will be the case up to $d = 8$, since it is true for X_G, but we need to go further. It seems reasonable to expect to prove this using our deletion-contraction techniques, since trees are reconstructible from their leaf-deleted subgraphs [9]. We proceed in the following manner.

If $T_1 \not\cong T_2$ then by the reconstructibility of trees there must exist labelings of these trees so that v_d is a leaf of T_1, $\tilde{v}_d$ is a leaf of T_2 and $T_1 - v_d \not\cong T_2 - \tilde{v}_d$. By induction we will have $Y_{T_1 - v_d} \not\equiv_{d-1} Y_{T_2 - \tilde{v}_d}$, and consequently, $Y_{T_1 - v_d} \not\equiv_d Y_{T_2 - \tilde{v}_d}$. Hence, our recurrence gives us

$$Y_{T_1} = Y_{T_1 - v_d / v_d} - Y_{T_1 - v_d}\uparrow$$
$$Y_{T_2} = Y_{T_2 - \tilde{v}_d / \tilde{v}_d} - Y_{T_2 - \tilde{v}_d}\uparrow .$$

We believe that from this relationship, we should be able to show that with respect to the congruence classes, Y_G distinguishes non-isomorphic trees. However, even though in the expansion of $Y_{T_1} = Y_{T_1 - v_d / v_d} - Y_{T_1 - v_d}\uparrow$ we have terms from both $Y_{T_1 - v_d / v_d}$ and $Y_{T_1 - v_d}\uparrow$ which do not match those from $Y_{T_2 - \tilde{v}_d / \tilde{v}_d}$ and $Y_{T_2 - \tilde{v}_d}\uparrow$, we have been unable to prove that the differences do not cancel out. Further, even if we were able to show that with respect to congruence classes $Y_{T_1} \not\equiv_d Y_{T_2}$, this does not necessarily imply that $X_{T_1} \neq X_{T_2}$. We hope to overcome these difficulties in future research.

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