HEAVY MESONS AND THEIR EFFECTS ON THE NN AND $\bar{N}N$ ELASTIC SCATTERING AMPLITUDES

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George C. Summerfield

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ABSTRACT

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Body of Abstract

Dispersion relations are developed for the NN and $\bar{N}N$ elastic scattering amplitudes for neutral, scalar nucleons from causality and unitarity. These are evaluated to fourth order in perturbation theory. The fourth order terms are shown to be about ten percent of the pole terms for neutral, scalar nucleons, and low energies, if the coupling constants are not significantly larger than the pion-nucleon pseudoscalar coupling constant. Dispersion relations are developed for the NN and $\bar{N}N$ elastic scattering amplitudes for charged, spinor nucleons. Mesons with zero strangeness are considered and it is shown that there are ten that interact linearly with the nucleon field, conserving J, P, T, and G. The pole terms of these mesons are determined in the NN and $\bar{N}N$ elastic scattering amplitudes, and the neutron-neutron, proton-proton and neutron-proton differential elastic scattering cross sections are determined from these pole terms. This gives the differential cross sections in terms of twelve coupling constants and ten masses.

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CHAPTER I

INTRODUCTION

Presently, no satisfactory theory of the fundamental interaction between nucleons is available. Calculations have been made using phenomenological potentials and meson theories,[1] but none of these can be classified as satisfactory in the sense of giving quantitative predictions of scattering cross sections to any desired accuracy. Any potential theory serves primarily as a parameterization scheme and gives little insight into the fundamental nature of the interaction process. Meson theories attempt to explain the nature of the interaction. However, the role of the meson is still not well determined, since meson theory has yet to give a quantitative result for the short range part of the interaction.

The usual pi-meson theory assumes that the nuclear force arises from a pion field in the same way the electromagnetic force arises from a photon field.[2]

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1. For an extensive survey of the theories of the nuclear interaction, See M.J. Moravcsik and H.P. Noyes, Ann. Rev. Nuclear Sci. 11, 1 (1962).

2. A discussion of the pion theory of nuclear forces can be found in J.D. Jackson, "The Physics of Elementary Particles," Princeton Univ. Press, Princeton, 1958.

The long range nucleon-nucleon force is given by the exchange of a single virtual pion between two nucleons. The pion is virtual in that it does not conserve energy and can only exist for a finite length of time by the Heisenberg uncertainty relation. The pion carries momentum between the two nucleons and, thus, gives rise to a force of finite range equal to the inverse mass of the pion exchanged.[3] The long range interaction predicted by single pion exchange is well verified experimentally.

Since the range of the interaction equals the inverse mass of the particle exchanged, single pion exchange should predict accurately the long range part of the interaction. However, the short range part should depend on the exchange of more than one pion. The effects of the exchange of more than one meson have never been satisfactorily calculated, and, considering the complexity of the problem, it is unlikely that they will be using usual field theoretic techniques.

Since the calculation of the short range nucleon-nucleon interaction has not been made, it is not certain that the pion is fundamental in the interaction. Sakurai[4] suggests that vector mesons heavier than the pion are the fundamental particles in the nuclear interaction, and the pion is a bound state of an antinucleon-

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3. We use units such that $\hbar = c = 1$.

4. J. Sakurai, Ann. Phys. 11, 1 (1960).

nucleon pair under the influence of the heavy meson field.

The existance of heavy mesons has been confirmed by experiment. If these heavy mesons contribute to the nucleon-nucleon interaction, their effects would be felt chiefly at short range, and would not contribute appreciably to the long range part of the interaction. Also, the heavy mesons need not be observed as free particles. They could decay rapidly into pions if their masses are greater than the mass of two pions and their quantum numbers allow the decay. We discuss these points in Chapter VI. With the recent discoveries of the two and three pion resonances,[5] this heavy meson theory of nuclear forces becomes more attractive.

It may be immaterial which of the mesons are considered elementary and which are considered composites of other particles. Supporting this viewpoint, Nishijima[6] has shown the equivalence of elementary and composite particle theories for local, renormalizable field theories.

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5. The resonances are $\rho \to 2\pi$ reported by A.R. Erwin, et al., Phys. Rev. Letters 6, 628 (1961); $\omega \to 3\pi$ reported by B.C. Maglic, et al., Phys. Rev. Letters 7, 178 (1961); $\eta \to 3\pi$ reported by A. Pevsner, et al., Phys. Rev. Letters 7, 421 (1961); and $\zeta \to 2\pi$ reported by R. Barloutaud, et al., Phys. Rev. Letters 8, 114 (19[illegible]).

6. K. Nishijima, Phys. Rev. 111, 995 (1958).

Similar considerations apply to the anti-nucleon-nucleon interaction, with the added complication that annihilation into pions or heavy mesons is possible.

Regardless of whether further progress in understanding the nucleon-nucleon and nucleon-anti-nucleon interactions can be made within the framework of field theory, basic objections have been raised to the use of field theory to explain fundamental processes. As long ago as 1943, Heisenberg[7] suggested that the failure of the concept of the continuum for short time and space intervals makes even the definition of a Hamiltonian impossible. More recently, Landau[8] and Chew[9] expressed the opinion that field theory is incapable of completely explaining elementary phenomena.

Our discussion is based within the framework of dispersion relations, and the form of a dispersion relation is independent of which particles are chosen as elementary. Chew and Frautschi[10] claim that it is possible to use dispersion relations to completely determine all strong interactions. If this claim is

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7. W. Heisenberg, Z. Physik 120, 513 and 673 (1943).

8. L.D. Landau, Proceedings Ninth Ann. Inter, Conf. High Energy Phys., Kiev, 1959.

9. G.F. Chew, "S-Matrix Theory of Strong Interactions," W.A. Benjamin Inc., New York, 1961.

10. G.F. Chew and S.C. Frautschi, Phys. Rev. 123, 1478 (1961).

correct, the choice of elementary particles is immaterial and field theory is unnecessary. However, the claim has not been substantiated.

CHAPTER II

THE BASIS OF DISPERSION RELATIONS.

Even in the absence of a complete theory, dispersion relations can predict some effects of pions and heavy mesons on the nucleon-nucleon and antinucleon-nucleon interactions.

Before discussing the use of dispersion relations for the nuclear interaction, we must describe the interaction. For elastic scattering of equal mass particles, the differential cross section is related to the T-matrix as follows:

$$\sigma_{fi}(\theta,\varphi) = \pi^2 E^2 |\langle i|T|f\rangle|^2 \tag{1}$$

Where E is the center of mass energy of each particle. The T-matrix is related to the S-matrix as follows:

$$\langle i|S|f\rangle = \delta_{if} + (2\pi)^4 i\, \delta^4(P_i - P_f)\langle i|T|f\rangle \tag{2}$$

Where P_i and P_f are the initial and final total four-momenta. The S-matrix is defined in terms of the incident and scattered wave functions in the next chapter. We use the invariant scattering amplitude introduced by Møller.[1] For reactions with two incident and two

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1. C. Møller, Det. Kgl. Danske Vidensk. Selsk., Mat.-Fys. Medd. 23, NR1 (1945).

scattered neutral, spinless particles, the invariant amplitude A is related to the T-matrix as follows:

$$\langle q_1 q_2 | T | q_3 q_4 \rangle = \frac{i}{(16\, q_{o1} q_{o2} q_{o3} q_{o4})^{1/2}} A \tag{3}$$

The initial and final four-momenta are q_1, q_2 and q_3, q_4. Labeling the initial and final particles A,B and C,D, we write the scattering represented by equations (1), (2), and (3) as:

$$A + B \rightarrow C + D \tag{4}$$

The scattering amplitude for this reaction is a function of two independent variables, e.g. the total energy and the scattering angle. However, it is more convenient to choose the invariants s, the square of the total center of mass energy, and t, the negative square of the momentum transfer:

$$\begin{aligned} s &= (q_1 + q_2)^2 \\ t &= (q_1 - q_3)^2 \end{aligned} \tag{5}$$

In the formalism of dispersion relations, A(s,t) is continued into the complex s-plane, while t is kept fixed. The continued function may have singularities in the s-plane. If the location of these singularities can be determined, an integral representation for

$A(s,t)$ can be written using the Cauchy formula.

For example, consider a function $A(s,t)$, not necessarily a scattering amplitude, which has two simple poles and two branch points in the s-plane, as shown in Figure 1. The location of the singularities may depend on the second variable, t. The branch points are at $s_1(t)$ and $s_2(t)$, and the poles are at $s_3(t)$ and $s_4(t)$.

Applying the Cauchy formula for the contour shown in Figure 1, we obtain an integral representation for $A(s,t)$.

$$A(s,t) = \frac{1}{2\pi i} \oint_{C(t)} \frac{A(s',t)}{s'-s}\, ds' \tag{6}$$

Letting the border of the contour extend to infinity and assuming the contribution of the integral around the border vanishes,[2] we are left with only the pole terms and the integrals along the cuts. Defining $A(s+\epsilon, t)$ and $A(s-\epsilon, t)$ as the values of $A(s,t)$ on different sides of the cuts, we rewrite equation (6) for Figure 1 as:

$$A(s,t) = R_3/(s_3-s) + R_4/(s_4-s) + \frac{1}{2\pi i}\int_{s_1}^{\infty + i\,\mathrm{Im}\, s_1} ds' \, \frac{A(s'+\epsilon,t) - A(s'-\epsilon,t)}{s'-s} + \frac{1}{2\pi i}\int_{s_2}^{\mathrm{Re}\, s_2 + i\infty} ds' \, \frac{A(s'+\epsilon,t) - A(s'-\epsilon,t)}{s'-s} \tag{7}$$

2. See Chapter III for a discussion of this point.

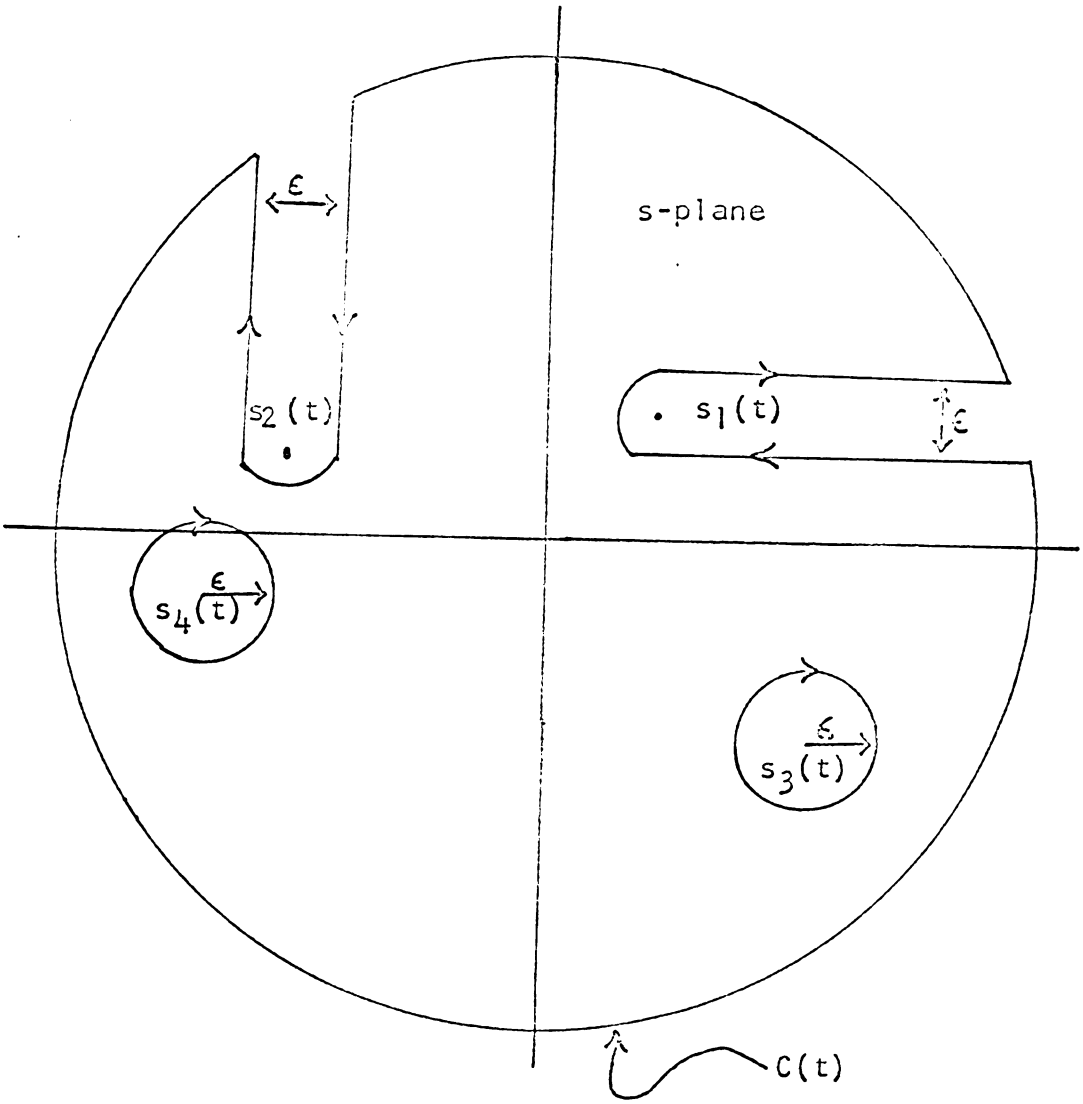

FIGURE I

A Function A(s,t) With Two Poles and Two Branch Points.

R_3 and R_4 are residues, and the numerators of the integrands are the discontinuities of A(s,t) across the cuts:

$$\text{Disc. } A(s,t) = A(s+\epsilon\ , t) - A(s-\epsilon ,t) \quad (8)$$

Double dispersion relations are obtained by applying the same considerations to Disc. A(s,t) as a continued function in the complex t-plane.[3]

Our use of dispersion relations depends entirely on our ability to determine the singularities of A(s,t). The determination of the singularities of the scattering amplitude constitutes the major part of this work. We indicate how the singularities are located and how residues and discontinuities are determined from causality and unitarity in Chapter III. In Chapter IV and V, we discuss the double dispersion relations for neutral, scalar, and charged, spinor nucleons. In Chapter VI, we discuss the properties of pions and heavy mesons and how these mesons affect the dispersion relations.

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3. See Chapter IV.

CHAPTER III

CAUSALITY AND UNITARITY

We now show how to determine the singularities of the scattering amplitude using causality and unitarity. The use of causality has a rather long history in the theory of scattering. In 1926 Kronig[1] and Kramers[2] used causality to write an integral equation relating the dispersive and absorptive parts of the refractive index for the scattering of light.

Causality was used by Karplus and Ruderman,[3] and Goldberger[4] in elementary particle theory. A rigorous proof that causality implies a Kramers-Kronig type of integral equation for the scattering amplitude in field theory was given by Bogoliubov[5] and Symanzik.[6]

The mathematical details of the field theoretic proofs obscure the physical content of the theory.

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1. R. Kronig, J. Opt. Soc. Am. 12, 546 (1926).

2. H.A. Kramers, Atti. cong. intern. fis. Como 2, 545 (1927).

3. R. Karplus and M.A. Ruderman, Phys. Rev. 98, 771 (1955).

4. M.L. Goldberger, Phys. Rev. 97, 508 (1955), and 99, 979 (1955).

5. N.N. Bogoliubov, Report of the International Conf. Theor. Phys., Seattle, 1956.

6. K. Symanzik, Phys. Rev. 105, 743 (1957).

In order to maintain a clear insight into the physical nature of the theory, we avoid the field theoretic analysis. Our discussion is based on physically reasonable arguments using causality and unitarity. This approach sacrifices some of the rigor and completeness of the more formal arguments.

We limit the scattering to two particle initial and final states represented by equation (4), Chapter II.

Following the usual analysis,[7] we consider that the interaction induces a transition between non-interacting stationary states. These are the initial state of particles A and B incoming to the region of interaction with total energy E, and the final state of outgoing particles C and D. The wave functions of these states are $|\psi\, in(E)\rangle$ and $|\psi\, out(E)\rangle$. The scattering is represented by an operator S(E) connecting the "in" and "out" wave functions.

$$|\psi\, out(E)\rangle = S(E)\, |\psi\, in(E)\rangle \qquad (1)$$

The relation between the matrix elements of S(E) and the cross section is given in equation (1), Chapter II.

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7. For example see J. Hamilton, "The Theory of Elementary Particles," Clarendon Press, London, 1959, Page 243.

This is the S-matrix introduced by Wheeler[8] and used by Heisenberg.[9]

The Fourier transforms of the quantities in equation (1) obey the following relation.

$$|\Psi\,\mathrm{out}(t)\rangle = \int_{-\infty}^{\infty} S(t-t')\,|\Psi_{IN}(t')\rangle\,dt' \qquad (2)$$

Equation (2) states that, for an incident wave packet, the outgoing wave at a time t is a linear superposition of contributions from the incoming wave at times t'.

Causality demands that $|\Psi\,\mathrm{in}(t')\rangle$ contributes to $|\Psi\,\mathrm{out}(t)\rangle$ only if t' comes before t. Thus, the scattering operator vanishes for t' greater than t.

$$S(t-t') = 0, \text{ for } t<t'. \qquad (3)$$

We use the Fourier representation of S(E),

$$S(E) = \int_{-\infty}^{\infty} S(\tau)\, e^{iE\tau}\, d\tau \qquad (4)$$

to extend the definition of S(E) into the complex E-plane. Equation (3) limits τ to positive values in equation (4). Thus, if E has a positive imaginary part, the integral has an exponential damping factor and is well

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8. J.A. Wheeler, Phys. Rev. <u>52</u>, 1107 (1937).

9. W. Heisenberg, op. cit.

defined and finite. Then, S(E) is analytic in the upper half E-plane. On the real E axis, there is no exponential damping and S(E) may not be analytic.

We use the results of causality together with unitarity to derive the analytic properties of the scattering amplitude. Unitarity is a statement of the conservation of probability in the scattering. If the initial state is normalized to unity,

$$\langle \Psi_{in(E)} \mid \Psi_{in(E)} \rangle = 1, \tag{5}$$

the final state must also be normalized to unity.

$$\langle \Psi_{out(E)} \mid \Psi_{out(E)} \rangle = 1. \tag{6}$$

Using the definition of S(E) in equation (6), we get the unitarity property.

$$\langle \Psi_{in(E)} \mid S^{\dagger} S \mid \Psi_{in(E)} \rangle = \langle \Psi_{in(E)} \mid \Psi_{in(E)} \rangle = 1. \tag{7}$$

$$S(E)^{\dagger}\, S(E) = 1 \tag{8}$$

S(E) is not the most convenient quantity to use in dispersion theory. A more convenient quantity is the invariant amplitude, related to the $\underset{\sim}{T}$-operator by a function of the energy given in the definition of A (equation (3), Chapter II).

$$A = C(E)\underset{\sim}{T}. \tag{9}$$

$\underset{\sim}{T}$ is related to S by

$$S = 1 - i\underset{\sim}{T}. \tag{10}$$

$\underset{\sim}{T}$ is related to the T-matrix introduced in equation (1), Chapter II by:

$$\langle i|\underset{\sim}{T}|f\rangle = -(2\pi)^4\,\delta^4(P_i - P_f)\langle i|T|f\rangle$$

$\underset{\sim}{T}$ and S obviously have the same singularities. $\underset{\sim}{T}(E)$ is analytic in the upper half E-plane. The unitarity of S leads to the following condition on $\underset{\sim}{T}$.

$$\begin{aligned} S^{\dagger} S &= (1 + i\underset{\sim}{T}^{\dagger})(1 - i\underset{\sim}{T}) = 1, \\ \underset{\sim}{T}^{\dagger}\underset{\sim}{T} &= i(\underset{\sim}{T} - \underset{\sim}{T}^{\dagger}). \end{aligned} \tag{11}$$

We make one further convenient change. Instead of the energy E, we use the square of the energy s as a dynamical variable. Since $\underset{\sim}{T}(E)$ is analytic in the upper half E-plane $\underset{\sim}{T}(s)$ is analytic on the sheet of s corresponding to the upper half E-plane, excluding the real s axis.

$$E = |E| e^{i\varphi}$$

$$s = |E|^2 e^{2i\varphi} \tag{12}$$

$$0 < \varphi < \pi$$

Unless C(E) introduces new singularities,[10] A(s) is also analytic in the complex s-plane, except possibly on the real s axis.

With the analytic properties of A(s) established, we use the Cauchy formula to write an integral representation for A(s).

$$A(s) = \frac{1}{2\pi i} \oint_{CU+CL} \frac{A(s')}{s'-s} ds' \tag{13}$$

The contours CU and CL are the two semi-circles shown in Figure 2. If A(s) vanishes for large s, we can extend the borders of the contours to infinity, and the contributions from the borders vanish, leaving only the integrals along the real axis in equation (13).

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10. C(E) actually removes an undesirable singularity at s=0.

$$A(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} ds' \frac{A(s'+i\epsilon)}{s'-s} + \frac{1}{2\pi i}\int_{\infty}^{-\infty} ds' \frac{A(s'-i\epsilon)}{s'-s}$$

Combining the two integrals gives:

$$A(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} ds' \frac{A(s'+i\epsilon) - A(s'-i\epsilon)}{s'-s} \qquad (14)$$

We assume that:

$$A^*(s) = A(s^*). \qquad (15)$$

This assumption comes from the perturbation theory expressions for A(s). For certain perturbation theory diagrams it does not hold. (See the Appendix). It is true in perturbation theory for nucleon-nucleon and antinucleon-nucleon scattering, and we apply the condition in equation (14).

$$A(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} ds' \frac{A(s'+i\epsilon) - A^*(s'+i\epsilon)}{s'-s}$$

$$A(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} ds' \frac{2i \operatorname{Im} A(s')}{s'-s} \qquad (16)$$

Unitarity (equation (11)) gives an expression for Im A(s).

$$\operatorname{Im} A(s) = C(E) \operatorname{Im} \underset{\sim}{T}(s),$$

$$\operatorname{Im} A(s) = -iC(E)(\underset{\sim}{T} - \underset{\sim}{T}^{\dagger})/2 \qquad (17)$$

$$\operatorname{Im} A(s) = -C(E)\underset{\sim}{T}^{\dagger}\underset{\sim}{T}/2$$

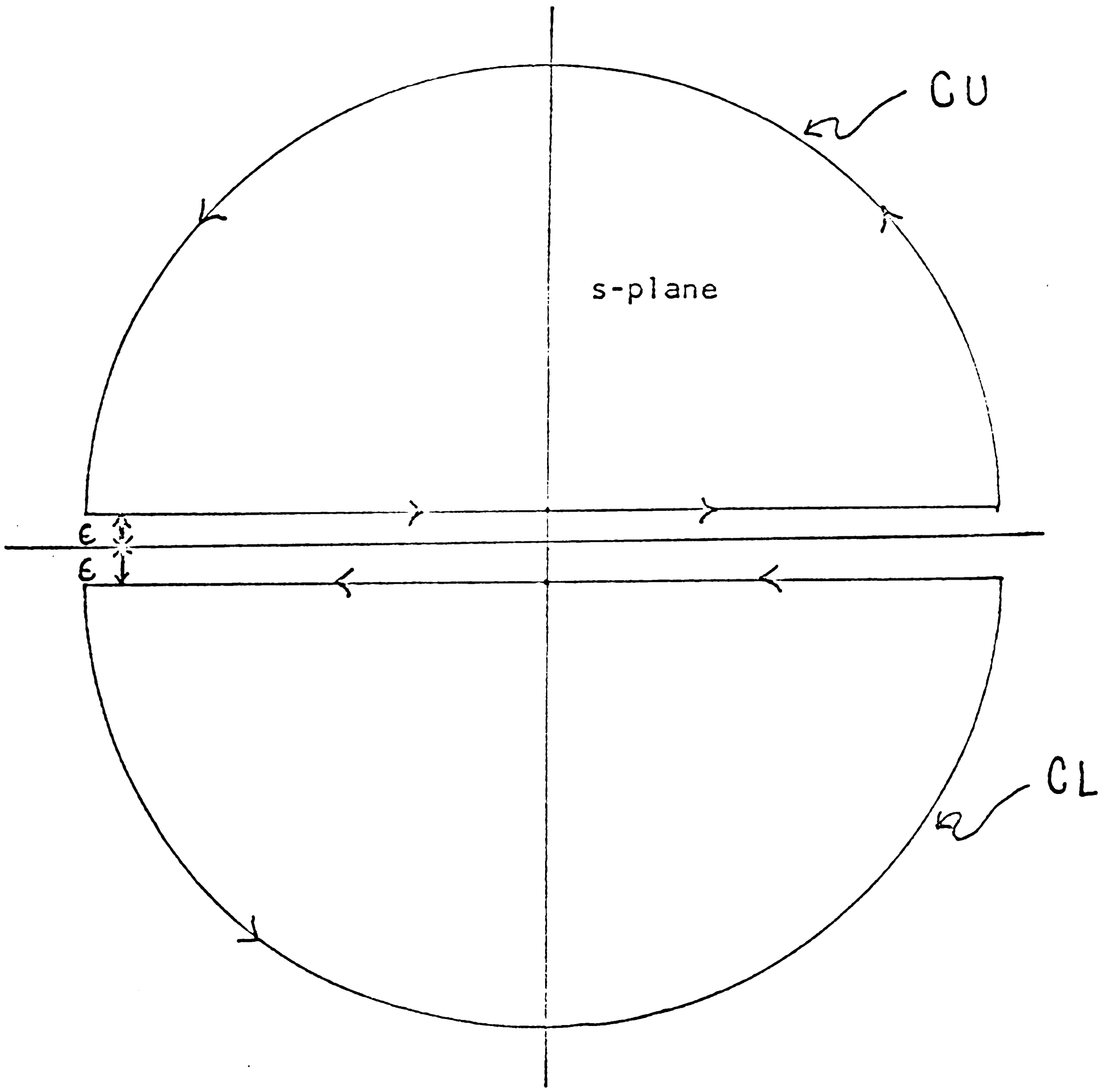

FIGURE 2

Contours for the Invariant Scattering Amplitude.

Taking matrix elements between initial and final states, we get:

$$\mathrm{Im}\,\langle\beta|A|\alpha\rangle = -\frac{C(E)}{2}\langle\beta|\underset{\sim}{T}^{\dagger}\underset{\sim}{T}|\alpha\rangle \qquad (18).$$

Summing over a complete set of intermediate states gives

$$\mathrm{Im}\,\langle\beta|A|\alpha\rangle = -\sum_{n}\frac{C(E)}{2}\langle\beta|\underset{\sim}{T}^{\dagger}|n\rangle\langle n|\underset{\sim}{T}|\alpha\rangle \quad (19)$$

Equation (19) shows that $\mathrm{Im}\,\langle\beta|A|\alpha\rangle$ fails to vanish only at energies for which there is an allowed intermediate state $|n\rangle$, since $\underset{\sim}{T}$ has only energy conserving matrix elements. If $|n\rangle$ is a single particle state with the same quantum numbers as $|\alpha\rangle$ and $|\beta\rangle$, $\mathrm{Im}\,\langle\beta|A|\alpha\rangle$ does not vanish at an energy equal to the mass of the particle in $|n\rangle$. If $|n\rangle$ is a multiparticle state with the same quantum numbers as $|\alpha\rangle$ and $|\beta\rangle$, $\mathrm{Im}\,\langle\beta|A|\alpha\rangle$ does not vanish at all energies greater than the sum of the masses of the particles in $|n\rangle$. Of course, $|n\rangle$ could be either $|\alpha\rangle$ or $|\beta\rangle$.

Using equation (19), we can write equation (16) as:

$$\langle\beta|A|\alpha\rangle = \sum_{M}\frac{R_M}{M^2-s} + \frac{1}{\pi}\int_{\{M_1+M_2+\cdots\}^2}^{\infty} ds'\,\frac{\mathrm{Im}\,\langle\beta|A(s')|\alpha\rangle}{s'-s} \qquad (20)$$

The sum on the pole terms is over all allowed single particle intermediate states. The lower limit on the integral is the square of the mass of the particles in the multiparticle intermediate state with the smallest total mass, and R_M is the residue at the pole.

We now have rules connecting the analytic properties of the scattering amplitude and the allowed intermediate states.

1. If there is an allowed single particle intermediate state, the scattering amplitude has a pole at at energy equal to the mass of the particle.

2. If there is an allowed multiparticle intermediate state, the scattering amplitude has a branch point at an energy equal to the sum of the masses of the particles.

3. The residues at the poles and the discontinuities across the cuts are given by equation (19).

To obtain equation (20), we assumed that A(s) vanished for large real s. If A(s) is constant the borders of the contours CU and CL in equation (13) cannot be extended to infinity. However, if s_o is not equal to any of the M's, equation (13) applies to $A(s)/(s - s_o)$

and the contours can be extended to infinity for this function.

$$\frac{A(s)}{s-s_o} = \frac{1}{2\pi i} \oint_{CU+CL} ds' \frac{A(s')}{(s'-s)(s'-s_o)} + \frac{A(s_o)}{s-s_o} \tag{21}$$

Instead of equation (20), this gives:

$$\frac{\langle\beta|A(s)|\alpha\rangle}{s-s_o} = \frac{\langle\beta|A(s_o)|\alpha\rangle}{s-s_o} + \sum_M \frac{R_M}{(M^2-s)(M^2-s_o)}$$

$$+ \frac{1}{\pi} \int_{\{M_1+M_2+\cdots\}^2}^{\infty} ds' \frac{Im\langle\beta|A(s')|\alpha\rangle}{(s'-s)(s'-s_o)} \tag{22}$$

Multiplying equation (22) by $(s-s_o)$ gives:

$$\langle\beta|A(s)|\alpha\rangle - \langle\beta|A(s_o)|\alpha\rangle = \sum_M R_M\left(\frac{1}{M^2-s} - \frac{1}{M^2-s_o}\right)$$

$$+ \frac{1}{\pi} \int_{\{M_1+M_2+\cdots\}^2}^{\infty} ds' \; Im\langle\beta|A(s')|\alpha\rangle \left(\frac{1}{s'-s} - \frac{1}{s'-s_o}\right) \tag{23}$$

Equation (23) is the subtracted form of the dispersion relation. If the scattering amplitude goes asymptotically as the n^{th} power of s, $n+1$ subtractions will give a valid dispersion relation. Notice that in equation (23) there is one new parameter, $A(s_o)$. Each subtraction adds one new parameter, since no two subtractions can be made at the same value of s_o.

Consider neutron-proton elastic scattering.

$$n + p \rightarrow n + p$$

The deuteron is an allowed intermediate state, giving rise to a pole at $s = M_D^2$. Since the mass of the deuteron is less than the sum of the masses of the neutron and proton, the pole is below the start of the physical region of $s, (\{M_p + M_n\}^2 \leq s < \infty$). The lowest mass multi-particle intermediate state is the neutron-proton state. The branch cut starts at $s = \{M_n + M_p\}^2$ and extends to plus infinity.[11]

Now consider antineutron-proton elastic scattering.

$$\bar{n} + p \rightarrow \bar{n} + p \; .$$

The π^+-meson is an allowed intermediate state, giving a pole at $s = M_\pi^2$. The lowest mass multi-particle state is the two-pion state. The branch cut starts at $s = 4M_\pi^2$ and extends to plus infinity.[11] Here, the cut starts well below the physical threshold, while in neutron-proton scattering the cut started at the physical threshold. This difference makes antinucleon-

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11. There will be higher mass allowed intermediate states and they will contribute to the discontinuity across the cut by equation (19), when s is greater than the square of the sum of the masses in the intermediate state.

nucleon scattering more difficult to analyze than nucleon-nucleon scattering.

The results presented here were shown to be valid under the assumptions that $A^*(s) = A(s^*)$ (equation (15)),[12] and that $C(E)$ introduces no singularities in A beyond those in $\underset{\sim}{T}$. For particular scatterings, of certain particles, these conditions do not hold and there are perturbation theory graphs which have different singularities than those given by rules 1. and 2. However, these rules are valid for the graphs involved in nucleon-nucleon scattering.

Thusfar, we have not justified fully our use of unitarity. Unitarity is well defined for physical values of s, but it is certainly not obvious that unitarity applies for non-physical values of s. In the next Chapter, we show that the scattering amplitude for negative values of s represents a different physical process.[13] For s between zero and the physical threshold, the meaning of unitarity is still not clear and our use of it must be justified. Mandelstam[14] has shown that unitarity can be unambiguously extended to this region consistent with the continuation of $A(s)$.

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12. The failure of $A^*(s) = A(s^*)$ can lead to anomalous thresholds and complex singularities. (See the Appendix).

13. See Chapter IV.

14. S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

CHAPTER IV

NEUTRAL, SCALAR NUCLEONS

1. Kinematics

Before deriving the dispersion relations, we discuss the kinematics of the two body problem. Consider the reaction represented by:

$$A + B \rightarrow C + D. \tag{1}$$

The kinematics are specified by the four-momenta of the initial and final particles. Figure 3 is the diagram of the four-momenta.

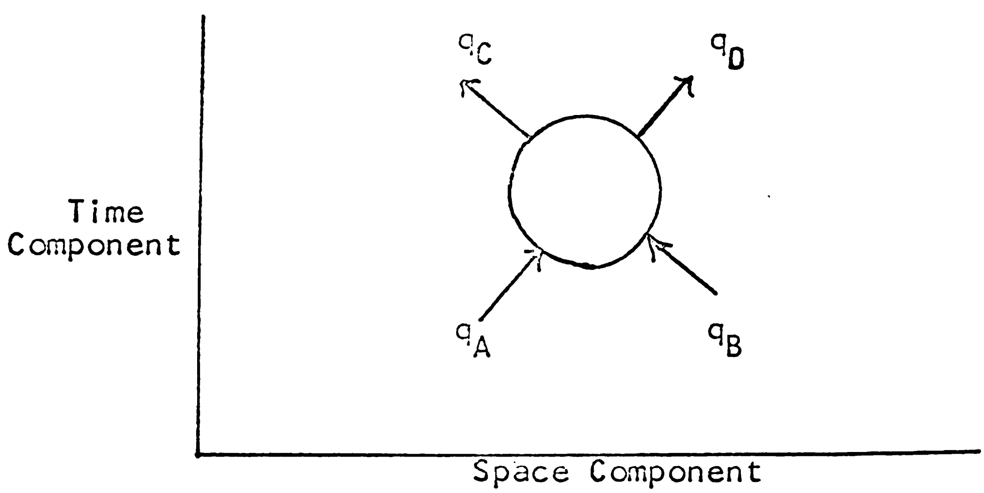

FIGURE 3

Four-Momenta of the Reaction $A + B \rightarrow C + D$.

The metric is defined such that:

$$q^2 = -\vec{q}^{\,2} + E^2 = M^2. \tag{2}$$

The statement of energy and momentum conservation in the reaction is:

$$q_A + q_B = q_C + q_D \;. \tag{3}$$

We choose the same variables as in equation (5), Chapter II and equation (12), Chapter III.

$$\begin{aligned} s &= (q_A + q_B)^2 = (q_C + q_D)^2 \\ t &= (q_A - q_D)^2 = (q_B - q_C)^2 \\ u &= (q_A - q_C)^2 = (q_B - q_D)^2 \;. \end{aligned} \tag{4}$$

In a two body scattering, there are only two independent variables. Therefore, there must be a relation between s,t, and u. Equations (2) and (3) imply that

$$\begin{aligned} s+t+u &= (q_A+q_B)^2 + (q_A - q_D)^2 + (q_A - q_C)^2 \\ s+t+u &= 3M_A{}^2 + M_B{}^2 + M_C{}^2 + M_D{}^2 - 2q_A\cdot(q_C + q_D - q_B) \\ s+t+u &= M_A{}^2 + M_B{}^2 + M_C{}^2 + M_D{}^2 \;. \end{aligned} \tag{5}$$

For nucleon-nucleon and antinucleon-nucleon scattering:

$$M_A = M_B = M_C = M_D = M.$$

We give the connection between s, t, u and the center of mass energy and scattering angle for this case. Kibble[1] has derived the connection between s, t, u and the energy and scattering angle for arbitrary masses, but the results are more complicated.

The center of mass conditions for equal mass particles are:

$$\begin{aligned} |\vec{q}_A| &= |\vec{q}_B| = |\vec{q}_C| = |\vec{q}_D| = |\vec{q}| \\ q_{oA} &= q_{oB} = q_{oC} = q_{oD} = (\vec{q}^{\,2} + M^2)^{1/2} \qquad (6) \\ \vec{q}_A + \vec{q}_B &= \vec{q}_C + \vec{q}_D = 0. \end{aligned}$$

Using equations (6), the following expressions for s, t, and u derive.

$$\begin{aligned} s &= -(\vec{q}_A + \vec{q}_B)^2 + (q_{oA} + q_{oB})^2 = 4(\vec{q}^{\,2} + M^2) \\ t &= -(q_A - q_D)^2 + (q_{oA} - q_{oD})^2 = -2\vec{q}^{\,2}(1-\cos\theta) \qquad (7) \\ u &= -(q_A - q_C)^2 + (q_{oA} - q_{oC})^2 = -2\vec{q}^{\,2}(1+\cos\theta), \end{aligned}$$

where $\vec{q}^{\,2}\cos\theta = \vec{q}_A \cdot \vec{q}_D = -\vec{q}_A \cdot \vec{q}_C$.

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1. T.W.B. Kibble, Phys. Rev. 117, 1159 (1960).

From equations (7), the physical range of s, t, and u is:

$$
\begin{aligned}
4M^2 &\leq s < \infty \\
-s + 4M^2 &\leq t \leq 0 \qquad\qquad (8) \\
-s + 4M^2 &\leq u \leq 0
\end{aligned}
$$

2. Crossing Relations

Consider the three reactions represented by Figure 4. Each reaction is called a channel.

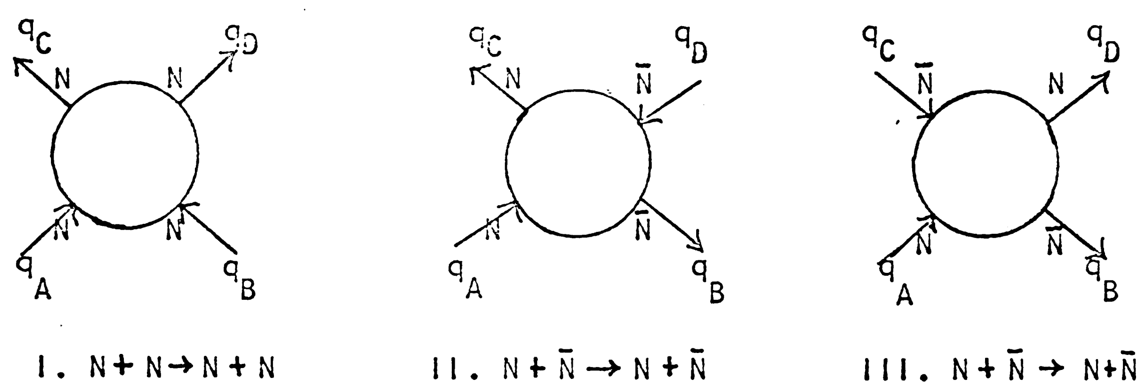

FIGURE 4

Nucleon-Nucleon Channels.

Crossing relations for neutral scalar particles state that the same function of s, t, and u, A(s,t,u), represents the scattering amplitudes of all three of these reactions.[2] (See the Appendix). However, the connections between s, t, and u and the center of mass energies and scattering angles, and the physical regions of s, t, and u are different for each channel. Table 1 shows the connections between s, t, and u and the center of mass energies and scattering angles, and the physical regions of s, t, and u for each channel.

A(s,t,u) represents $A_I(s,t)$ when s, t, and u are in the physical region for channel I, $A_{II}(t,s)$ when s, t, u are in the physical region for channel II, and $A_{III}(u,t)$ when s, t, and u are in the physical region for channel III. A_I, A_{II}, and A_{III} are the physical scattering amplitudes in channels I, II, and III.

---

2. H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 1, 205 (1955), and 6, 319 (1957); and S. Gasiorowicz, Fort. der Phys. 8, 665 (1960); and M.L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. 2, 226 (1957).

TABLE I

Nucleon-Nucleon Channels

CHANNEL		CENTER OF MASS VARIABLES	PHYSICAL REGION OF s, t, and u
I	$N+N\rightarrow N+N$	$s=4(\vec{q}^{\,2}+M^2)$ $t=-2\vec{q}^{\,2}(1-\cos\theta_{I})$ $u=-2\vec{q}^{\,2}(1+\cos\theta_{I})$	$4M^2\leq s<\infty$ $-s+4M^2\leq t\leq 0$ $-s+4M^2\leq u\leq 0$
II	$N+\bar{N}\rightarrow N+\bar{N}$	$s=-2\vec{q}^{\,2}(1-\cos\theta_{II})$ $t=4(\vec{q}^{\,2}+M^2)$ $u=-2\vec{q}^{\,2}(1+\cos\theta_{II})$	$-t+4M^2\leq s\leq 0$ $4M^2\leq t<\infty$ $-t+4M^2\leq u\leq 0$
III	$N+\bar{N}\rightarrow N+\bar{N}$	$s=-2\vec{q}^{\,2}(1+\cos\theta_{III})$ $t=-2\vec{q}^{\,2}(1-\cos\theta_{III})$ $u=4(\vec{q}^{\,2}+M^2)$	$-u+4M^2\leq s\leq 0$ $-u+4M^2\leq t\leq 0$ $4M^2\leq u<\infty$

Since we are going to relate allowed intermediate states to singularities in the scattering amplitude, we show in Table 2 the pertinent, allowed intermediate states for each of the three channels.

TABLE 2

Intermediate States

CHANNEL	ONE PARTICLE INTERMEDIATE	MULTIPARTICLE INTERMEDIATE STATE WITH SMALLEST TOTAL MASS
I	(D) deuteron*	NN -Nucleon-Nucleon
II	M - Meson**	$\pi\pi$ - Pion-Pion**
III	M - Meson**	$\pi\pi$ - Pion-Pion**

*The deuteron state is only present for neutron-proton scattering. We include it in our discussion inside brackets. In the particular cases of neutron-neutron and proton-proton scattering, the contents of the brackets should be taken as zero.

**There are different mesons possible with the quantum numbers of some nucleon-antinucleon state. Only five have been observed and of these the pion has the lowest mass (See Chapter VI).

By the arguments of Chapter III, $A_I(s,t)$ has no singularities in the complex s-plane for t real and negative. Therefore, we can apply equation (16), Chapter III.

$$A_I(s,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} ds' \, \frac{\mathrm{Im}\, A_I(s',t)}{s'-s} \qquad (9)$$

When s' is greater than $4M^2$, $\mathrm{Im}\, A_I(s',t)$ is the imaginary part of the scattering amplitude in channel I. When s' is negative, since $s'+t+u' = 4M^2$, and since t' is negative, u' is greater than $4M^2$. For negative s', $\mathrm{Im}\, A_I(s',t)$ is the imaginary part of the scattering amplitude in channel III. This second statement follows from the crossing relations. Thus, $\mathrm{Im}\, A_I(s',t)$ is the sum of contributions from channels I and III.

$$\mathrm{Im}\, A_I(s',t) = A_1(s',t) + A_3(u',t).$$

A_1 and A_3 are related to the imaginary parts of T-matrix elements by equation (3), Chapter II. For example, A_1 is:

$$A_1(s,t) = (16\, q_{oA} q_{oB} q_{oC} q_{oD})^{1/2} \; \mathrm{Im} \langle q_A q_B | T_I | q_C q_D \rangle ,$$

where the T-matrix is related to the S-matrix by:

$$\langle i|S|f\rangle = \delta_{if} + (2\pi)^4 i \, \delta^4(P_i - P_f) \langle i|T|f\rangle$$

The unitarity of S,

$$\sum_n \langle i|S^\dagger|n\rangle\langle n|S|f\rangle = \delta_{if}$$

gives the following expression for $\mathrm{Im}\langle i|T|f\rangle$.

$$\mathrm{Im}\langle i|T|f\rangle = \frac{(2\pi)^4}{2}\sum_n \delta^4(P_i - P_n)\langle i|T^\dagger|n\rangle\langle n|T|f\rangle$$

Thus, A_1 and A_3 are:

$$A_1(s,t) = \frac{(2\pi)^4}{2}\sum_n (16\, q_{0A}\, q_{0B}\, q_{0C}\, q_{0D})^{\frac{1}{2}}\,\delta^4(q_A+q_B-P_n)\langle q_A q_B|T_1^\dagger|n\rangle\langle n|T_1|q_C q_D\rangle \tag{10}$$

$$A_3(u,t) = \frac{(2\pi)^4}{2}\sum_n (16\, q_{0A}\, q_{0B}\, q_{0C}\, q_{0D})^{\frac{1}{2}}\,\delta^4(q_A+q_C-P_n)\,\langle q_A q_C|T_{III}^\dagger|n\rangle\langle n|T_{III}|q_B q_D\rangle$$

In terms of A_1 and A_3, equation (9) is:

$$A_1(s,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} ds'\,\frac{A_1(s',t)}{s'-s} + \frac{1}{\pi}\int_{-\infty}^{\infty} ds'\,\frac{A_3(u',t)}{s'-s} \tag{11}$$

Equations (10) and Table 3 show that $A_1(s',t)$ is non-zero at s' equal to the square of the deuteron mass, $s' = M_D^2$ (for neutron-proton scattering only), and for s' greater than the square of the mass of two nucleons, $s' \geq 4M^2$; and that $A_3(u',t)$ is non-zero at u' equal to the squares of the masses of the mesons discussed in Chapter VI, $u' = m^2$, and for u' greater than the square of the mass of two pions, $u' \geq 4M_\pi^2$. Since $s' + t + u' = 4M^2$, $A_3(u't)$ fails to vanish for $s' = 4M^2 - m^2 - t$ and $s' \leq 4M^2 - 4M_\pi^2 - t$. Using these results, we write equation (11) as:

$$A_1(s,t) = \left(\frac{R_D}{M_D^2 - s}\right) + \sum_m \frac{R_m}{s-(4M^2-m^2-t)} + \frac{1}{\pi}\int_{4M^2}^{\infty} ds'\,\frac{A_1(s',t)}{s'-s}$$

$$-\frac{1}{\pi}\int_{4M^2-4M_\pi^2-t}^{-\infty} ds'\,\frac{A_3(u',t)}{s'-s} \tag{12}$$

The sum on m extends to all single particle intermediate states. Unitarity (equation (10)) gives the residues R_D and R_m and the discontinuities $A_1(s',t)$ and $A_3(u',t)$. For example, the term $\frac{R_m}{s-(4M^2-m^2-t)}$ comes from the single particle contribution of the meson, m, to equation (10) for $A_3(u',t)$.

$$\frac{R_m}{s-(4M^2-m^2-t)} = \frac{1}{\pi}\int \frac{(2\pi)^4}{2} \sum_{P_m} (16\, q'_{0A}\, q'_{0B}\, q'_{0C}\, q'_{0D})^{\frac{1}{2}}\, \delta^4(q'_A + q'_C - P_m)\ X$$

$$\frac{ds'}{s'-s} \langle q'_A\, q'_C | T^{\dagger} | P_m \rangle \langle P_m | T | q'_B\, q'_D \rangle$$

where

$$s' = (q_A' - q_B')^2$$

$$t = (q_A' - q_D')^2$$

$$u' = (q_A' + q_C')^2$$

Since the sum is over a continuum of intermediate states of four-momentum P_m,

$$\sum_{P_m} = \int \frac{d^4 P_m}{(2\pi)^3}\, \delta(P_m^2 - m^2)$$

and

$$\frac{R_m}{s-(4M^2-m^2-t)} =$$

$$\int \frac{ds'}{s'-s} (16\, q'_{0A}\, q'_{0B}\, q'_{0C}\, q'_{0D})^{\frac{1}{2}}\, \delta(\{q'_A + q'_C\}^2 - m^2) \langle q'_A\, q'_C | T^{\dagger} | (q'_A + q'_C) \rangle \langle (q'_A + q'_C) | T | q'_B\, q'_D \rangle$$

Since $u' = (q_A' + q_C')^2$ and $u' + t + s' = 4M^2$, the s' integration gives:

$$R_m = (16 q_{0A} q_{0B} q_{0C} q_{0D})^{\frac{1}{2}} \langle q_A' q_C' | T^\dagger | (q_A' + q_C') \rangle \langle (q_A' + q_C') | T | q_B' q_D' \rangle$$

where:
$$q_A' + q_C' = q_B' + q_D'$$
$$(q_A' + q_C')^2 = m^2$$
$$(q_A' - q_B')^2 = s$$
$$(q_A' - q_D')^2 = t$$

This expression for R_m is the square of the renormalized meson-nucleon coupling constant.

$$R_m = g_m^2 \; .$$

Similarly, we can obtain a dispersion relation for $A_{II}(t,s)$ in t for s real and negative,

$$A_{II}(t,s) = \sum_m \frac{R_m}{m^2 - t} + \sum_m \frac{R_m}{t - (4M^2 - m^2 - s)} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \, \frac{A_2(t',s)}{t' - t} + \frac{1}{\pi}\int_{-\infty}^{4M^2 - 4M_\pi^2 - s} dt' \, \frac{A_3(u',s)}{t' - t} \quad (13)$$

and for $A_{III}(u,t)$ in u for t real and negative,

$$A_{III}(u,t) = \sum_m \frac{R_m}{m^2 - u} + \left(\frac{R_D}{u - (4M^2 - M_D^2 - t)}\right) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \, \frac{A_3(u',t)}{u' - u} + \frac{1}{\pi}\int_{-\infty}^{-t} \frac{A_1(s',t)}{u' - u} du' \quad (14)$$

Single dispersion relations have been obtained by Goldberger[3], Capps and Takeda[4] for pion-nucleon scattering and by Goldberger, Nambu and Oehme[5] for nucleon-nucleon scattering.

Since we have not considered the asymptotic behavior of the scattering amplitudes, "subtractions", described in Chapter III, may be necessary in equations (12), (13), and (14).

3. Double Dispersion Relations

Single dispersion relations have been used for pion-nucleon forward scattering by Davidon and Goldberger[6] to exclude one of the two sets of phase shifts that fitted the scattering data; and by Haber-Schaim[7] and Gilbert[8] to determine the pion-nucleon coupling constant. It is difficult to extract more information than this from equations (12), (13), and (14), since they do not include the momentum transfer dependence of the scattering amplitude. The momentum transfer dependence is given in double dispersion relations.

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3. M.L. Goldberger, Phys. Rev. 99, 979 (1955).

4. R.H. Capps and G. Takeda, Phys. Rev. 103, 1337 (1956).

5. M.L. Goldberger, Y. Nambu & R. Oehme, Ann. Phys. 2, 226 (1957).

6. W.C. Davidon and M.L. Goldberger, Phys.Rev. 104, 1119 (1956).

7. U. Haber-Schaim, Phys. Rev. 104, 1113 (1956).

8. W. Gilbert, Phys. Rev. 108, 1078 (1957).

Since A(s,t,u) must satisfy equations (12), (13), and (14), it has poles at ($s=M_D^2$), $t=m^2$ and $u=m^2$, and branch points at $s=4M^2$, $t=4M_\pi^2$ and $u=4M_\pi^2$. In equations (12), (13), and (14), one of the variables must be real and negative, so these might not be all the singularities of A(s,t,u).

Mandelstam[9] assumed that the only singularities are those required by the single dispersion relations. Figure 5 shows the location of these singularities in the real s, real t plane. Figure 5 shows only the pion intermediate states for channels II and III, but remember that the intermediate states in these channels should include all the mesons discussed in Chapter VI.

If the only singularities are as shown in Figure 5, A(s,t,u) has singularities only for s real when t is constant and for t real when s is constant. Therefore, equation (16), Chapter III holds for A(s,t,u) in both s and t.

$$A(s,t,u)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}ds'\;\frac{A(s'+i\epsilon,t,u')-A(s'-i\epsilon,t,u')}{s'-s} \qquad (15)$$

$$A(s\pm i\epsilon,t,u)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dt'\;\frac{A(s\pm i\epsilon,t'+i\epsilon,u')-A(s\pm i\epsilon,t'-i\epsilon,u')}{t'-t} \qquad (16)$$

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9. S. Mandelstam, Phys. Rev. 112, 1344 (1958).

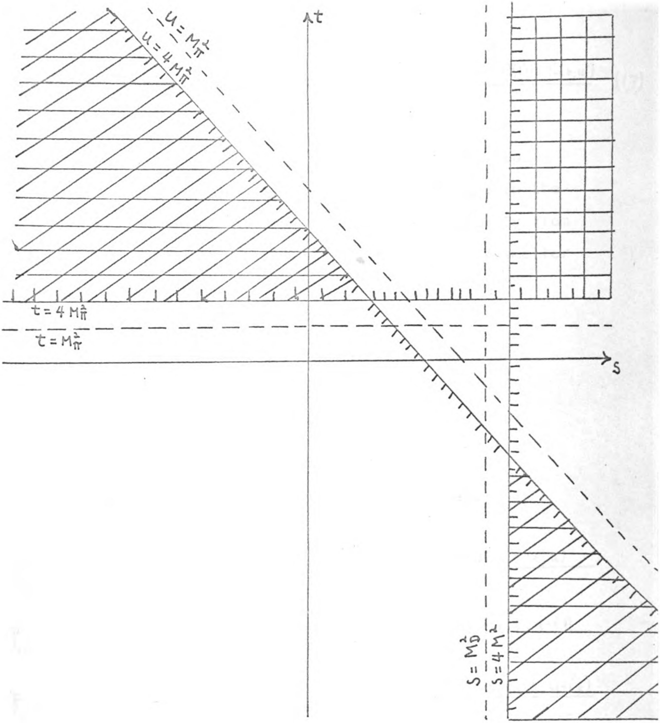

FIGURE 5

Singularities of A(s,t,u)

Poles — — — —

Branch Pts.

Applying equation (16) in equation (15), we get:

$$A(s,t,u) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} dt' \frac{A(s'-i\epsilon,t'+i\epsilon,u') - A(s'-i\epsilon,t'-i\epsilon,u') - A(s'+i\epsilon,t'+i\epsilon,u') + A(s'+i\epsilon,t'-i\epsilon,u')}{(s'-s)(t'-t)} \quad (17)$$

The integrand in equation (17) fails to vanish only for ($s' = M_D{}^2$), $t' = m^2$, $u' = m^2$, and in the three cross hatched regions in Figure 5. Actually from perturbation theory the integrand vanishes in a smaller region than these. (See the Appendix).

$s > 4M^2$, $t > 4M^2$

$s > 4M^2$, $u > 4M^2$

$t > 4M^2$, $u > 4M^2$

Defining the spectral functions as:

$$\rho_{12}(s,t) = \frac{A(s-i\epsilon,t+i\epsilon,u) - A(s-i\epsilon,t-i\epsilon,u) - A(s+i\epsilon,t+i\epsilon,u) + A(s+i\epsilon,t-i\epsilon,u)}{4}$$

$$\rho_{13}(s,u) = \frac{A(s-i\epsilon,t,u+i\epsilon) - A(s-i\epsilon,t,u-i\epsilon) - A(s+i\epsilon,t,u+i\epsilon) + A(s+i\epsilon,t,u-i\epsilon)}{4}$$

$$\rho_{23}(t,u) = \frac{A(s,t-i\epsilon,u+i\epsilon) - A(s,t-i\epsilon,u-i\epsilon) - A(s,t+i\epsilon,u+i\epsilon) + A(s,t+i\epsilon,u-i\epsilon)}{4}$$

we get the Mandelstam representation from equation (17).

$$A(s,t,u) = \left(\frac{R_D}{M_D^2 - s}\right) + \sum_m \frac{g_m^2}{m^2 - t} + \sum_m \frac{g_m^2}{m^2 - u} + \frac{1}{\pi^2}\int_{4M^2}^{\infty} ds' \int_{4M_\pi^2}^{\infty} dt' \frac{\rho_{12}(s',t')}{(s'-s)(t'-t)}$$

$$+ \frac{1}{\pi^2}\int_{4M^2}^{\infty} ds' \int_{4M_\pi^2}^{\infty} du' \frac{\rho_{13}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dt' \int_{4M_\pi^2}^{\infty} du' \frac{\rho_{23}(t',u')}{(t'-t)(u'-u)} \tag{19}$$

The assumptions made in obtaining the Mandelstam representation are too stringent. Eden[10] has shown the validity of the Mandelstam representation for the following conditions.

a. In the real s, real t plane the singularities of A(s,t,u) are those shown in Figure 5.

b. A(s,t,u) is analytic when one of the variables is real and positive and the other two are complex and satisfy $s + t + u = 4M^2$.

The most general conditions for the validity of the Mandelstam representation have not been determined.

4. Determination of the Scattering Amplitude

Attempts have been made to use the complete two dimensional Mandelstam representation and unitarity for pion-pion scattering.[11] However, most applications reduce the Mandelstam representation to a single dispersion relation.

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10. R. J. Eden, Phys. Rev. 120, 1514 (1960).

11. See G.F. Chew, "S-Matrix Theory of Strong Interactions", W.A. Benjamin Inc., New York, 1961.

These single dispersion relations do not fully utilize the implications of the Mandelstam representation, but they do give more information than equations (12), (13), and (14). The difference between single dispersion relations obtained from the Mandelstam representation and equations (12), (13), and (14) lies in the assumption that the only singularities of $A(s,t,u)$ are those required by equations (12), (13), and (14).

Assuming these are the only singularities, we can remove the restrictions that one of the variables be negative. For example, equation (13) is valid in the physical region for channel I, where previously s must have been negative. This is the single dispersion relation that Cini and Fubini[12] use to analize nucleon-nucleon scattering. We sketch their analysis below.

Making the change of variables:

$$u = 4M^2 - s - t$$

$$u' = 4M^2 - s - t'$$

$$-du' = dt'$$

---

12. M. Cini and S. Fubini, Ann. Phys. 10, 352 (1960).

in the second pole term and the second integral in equation (13), we get:

$$A(s,t,u) = \sum_m g_m^2 \left(\frac{1}{m^2 - t} + \frac{1}{m^2 - u} \right) + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \, \frac{A_2(t', s)}{t' - t} + \frac{1}{\pi^2} \int_{4M_\pi^2}^{\infty} du' \, \frac{A_3(u', s)}{u' - u} \qquad (20)$$

Unitarity gives expressions for A_2 and A_3 similar to equation (10).

$$A_2(t,s) = \frac{(2\pi)^4}{2} \sum_n (16\, q_{oA}\, q_{oB}\, q_{oC}\, q_{oD})^{1/2}\, \delta^4(q_A + q_D - P_n) \langle q_A q_D | T^\dagger | n \rangle \langle n | T | q_C q_B \rangle \qquad (21)$$

$$A_3(u,s) = \frac{(2\pi)^4}{2} \sum_n (16\, q_{oA}\, q_{oB}\, q_{oC}\, q_{oD})^{1/2}\, \delta^4(q_A + q_C - P_n) \langle q_A q_C | T^\dagger | n \rangle \langle n | T | q_B q_D \rangle$$

The sums are over multiparticle intermediate states, since single particle intermediate states have been split off in the pole terms. Cini and Fubini consider only the two pion intermediate states and approximate the rest of the sum by a finite series of Legendre polynomials. They do not consider the possibility of heavy mesons.

Ignoring the heavy mesons, this approximation is reasonable, since intermediate states only contribute when the energy is greater than the mass of the particles in the intermediate state. Thus, the two pion state contributes for t or u greater than $4M_\pi^2$, while the three pion state contributes for t or u greater than $9M_\pi^2$.

Noticing the symmetry of equation (20) under the interchange of t and u and applying the Cini-Fubini approximation, we get:

$$A(s,t,u) = \frac{g_\pi^2}{M_\pi^2 - t} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \, \frac{J_{\pi\pi}(t',s)}{t'-t} + (t \leftrightarrow u) \sum_{\ell=0}^{L} \gamma_\ell(s)\, P_\ell(\cos\theta)$$

where s, t, and u are in the physical region for channel I.

$$s = 4(\vec{q}^{\,2} + M^2)$$

$$t = -2\vec{q}^{\,2}(1-\cos\theta)$$

$$u = -2\vec{q}^{\,2}(1+\cos\theta).$$

$J_{\pi\pi}(t',s)$ is the two pion contribution to $A_2(t',s)$.

$$J_{\pi\pi} = \frac{(2\pi)^4}{2} \sum_{P_\pi P_\pi'} (16\, q_{0A}' q_{0B}' q_{0C}' q_{0D}')^{\frac{1}{2}}\, \delta^4(q_A' + q_D' - P_\pi - P_\pi') \langle q_A' q_D' | T^\dagger | P_\pi P_\pi' \rangle \langle P_\pi P_\pi' | T | q_C' q_B' \rangle$$

We express the T-matrix in terms of the scattering amplitude for antinucleon-nucleon annihilation into two pions:

$$\langle q_A q_C | T^\dagger | P_\pi P_\pi' \rangle = \frac{1}{(16\, P_{0\pi} P_{0\pi}'\, q_{0A}\, q_{0D})^{1/2}}\, B(q_A q_C P_\pi P_\pi')$$

B is the scattering amplitude for the annihilation reaction.

$$N + \bar{N} \rightarrow \pi + \pi.$$

These results give the following expression for A(s,t,u).

$$A(s,t,u) = \frac{g_\pi^2}{M_\pi^2 - t} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'-t} \sum_{P_\pi P_\pi'} \frac{(2\pi)^4}{2} \frac{B^*(q_B' q_C' P_\pi P_\pi')\, B(q_A' q_D' P_\pi P_\pi')}{4 P_{0\pi} P_{0\pi}'} \delta^4(q_A + q_D - P_\pi - P_\pi') \tag{22}$$

$$+ (t \leftrightarrow u) + \sum_{\ell=0}^{L} \gamma_\ell(s) P_\ell(\cos\theta)$$

L and γ_ℓ must be determined from experiment. Hopefully, they are small.

Cini and Fubini write the Mandelstam representation for B , bringing in the crossed channels $\pi + N \to \pi + N$. Then they reduce the Mandelstam representation for B to a single dispersion relation and evaluate the single dispersion relation approximately, including only the two pion state in the unitarity expression. Thus, the dispersion relation for B contains the scattering amplitudes for $\pi + \pi \to \pi + \pi$ and $N + \bar{N} \to \pi + \pi$.

Following the Cini-Fubini approach, one must simultaneously determine the scattering amplitudes for three different reactions.

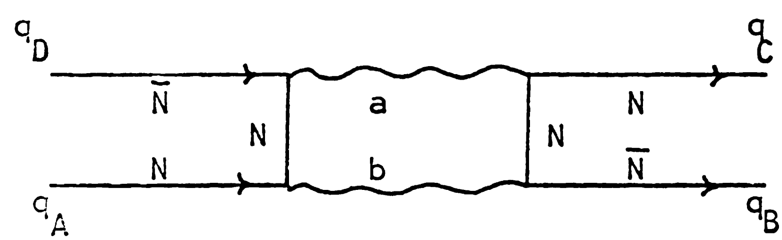
$$\begin{aligned} &\pi + \pi \to \pi + \pi \\ &\pi + N \to \pi + N \\ &N + N \to N + N \end{aligned} \tag{23}$$

Chew and Mandelstam[13] use equation (20) to determine the singularities of the partial wave amplitude in the complex $\vec{q}^{\,2}$ -plane. This method also involves the simultaneous determination of the scattering amplitudes for the reactions in equations (23), unless only pole terms are considered.

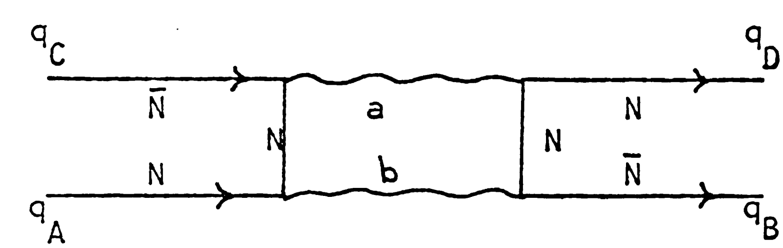
Since we consider the effects of ten mesons, the large number of different reactions to be considered in the methods mentioned above prohibits their use. We deal directly with the Mandelstam representation, but only consider the pole terms explicitly, and attempt to find a convenient parameterization for the integrals.

The spectral functions have been determined in fourth order perturbation theory for neutral scalar nucleons from the diagrams shown below (See the Appendix).

For $\rho_{12}(s,t)$



For $\rho_{13}(s,u)$



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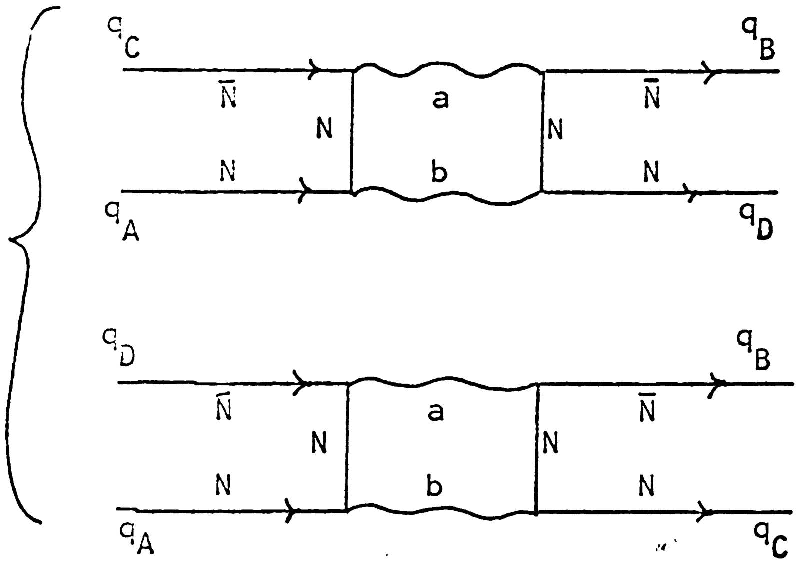
13. G.F. Chew and S. Mandelstam, Phys. Rev. <u>119</u>, 467, (1960).

For $\rho_{23}(t,u)$



a and b are any two mesons.

We express the contributions of these diagrams to the spectral functions in the following way.

$\rho^{ab}(s,t)$ is the contribution of the first diagram to $\rho_{12}(s,t)$.

$\rho^{ab}(s,u)$ is the contribution of the second diagram to $\rho_{13}(s,u)$.

$\rho^{ab}(t,u)$ is the contribution of the third diagram to $\rho_{23}(t,u)$.

$\rho^{ab}(u,t)$ is the contribution of the fourth diagram to $\rho_{23}(t,u)$.

The form of all of the ρ^{ab}'s is the same. Thus, we can write:

$$I_{ab}(x,y) = \int dx' \int dy' \frac{\rho^{ab}(y',x')}{(x'-x)(y'-y)}$$

and substitute the appropriate variables for x and y to obtain the contribution of anyone of the diagrams. An approximate expression for $I_{ab}(x,y)$ is given in the following.

With the spectral functions approximated in fourth order perturbation theory, equation (19) is:

$$A(s,t,u) = \left(\frac{R_D}{M_D^2 - s}\right) + \sum_m g_m^2 \left(\frac{1}{m^2-t} + \frac{1}{m^2-u}\right) + \sum_{ab}\left(I_{ab}(t,s) + I_{ab}(u,s) + I_{ab}(t,u) + I_{ab}(u,t)\right) \tag{24}$$

From the generalized unitarity condition (See Appendix), the spectral function $\rho^{ab}(x,y)$ is in fourth order

$$\rho^{ab}(x,y) = \frac{\left(\frac{f_a f_b}{4}\right)^2}{\left\{(y^2 - 4M^2 y)\left([(m_a + m_b)^2 - x]^2 - 4m_a^2 m_b^2\right)\right\}^{1/2}} \tag{25}$$

$\rho^{ab}(x,y) = 0$, for x or y less than $4M^2$ and $\rho^{ab}(x,y)$ complex.

Using this spectral function, $I_{ab}(x,y)$ has been approximated to within 30% as follows:

$\mathrm{Im}\ I_{ab}(x,y) =$

$$\left(\frac{f_a f_b}{4}\right)^2 \frac{1}{\pi}\left\{\frac{\theta(y-4M^2)}{(cx^2 - Dx + E)^{1/2}} \log\left[\frac{(D^2 - 4CE)^{1/2}}{D - 2CX - 2(C^2X^2 - CDX + CE)^{1/2}}\right] + \frac{\theta(x-4M^2)}{(Ay^2 - By)^{1/2}} \log\left[\frac{B}{B - 2AY - 2(A^2Y^2 - ABY)^{1/2}}\right]\right\} \tag{26}$$

$$\mathrm{Re}\; I_{ab}(x,y) =$$

$$\left(\frac{f_a f_b}{4}\right)^2 \frac{1}{\pi^2} \left\{ \frac{\Theta(y-4M^2)}{(y^2-4M^2y)^{\frac12}(x-\alpha)^{\frac12}(x-\beta)^{\frac12}} \log\left(\frac{4M^2}{2(y^2-4M^2y)^{\frac12}+2y-4M^2}\right) \times \right. \quad (27)$$

$$\left[\log\left(\frac{2\sqrt{(x-\alpha)(x-\beta)}}{2\sqrt{(x-\alpha)(x-\beta)}+2x-(\alpha+\beta)}\right) + \log\left(\frac{\sqrt{(a-\alpha)(a-\beta)}+\sqrt{(x-\alpha)(x-\beta)}}{a-x} + \frac{2x-\alpha-\beta}{2\sqrt{(x-\alpha)(x-\beta)}}\right)\right]$$

$$- \frac{M_0\Theta(y-4M^2)}{2M^2}\left[(d-c)\sqrt{(x-\alpha)(x-\beta)}\log\left(\frac{\sqrt{(a-\alpha)(a-\beta)}+\sqrt{(x-\alpha)(x-\beta)}}{a-x}+\frac{2x-\alpha-\beta}{2\sqrt{(x-\alpha)(x-\beta)}}\right)\left(\frac{2\sqrt{(x-\alpha)(x-\beta)}}{\beta-\alpha}\right)\right.$$

$$+ (x-d)\sqrt{(c-\alpha)(c-\beta)}\log\left(\frac{\sqrt{(a-\alpha)(a-\beta)}+\sqrt{(c-\alpha)(c-\beta)}}{a-c}+\frac{2c-\alpha-\beta}{2\sqrt{(c-\alpha)(c-\beta)}}\right)\left(\frac{2\sqrt{(c-\alpha)(c-\beta)}}{\beta-\alpha}\right)$$

$$\left. + (c-x)\sqrt{(d-\alpha)(d-\beta)}\log\left(\frac{\sqrt{(a-\alpha)(a-\beta)}+\sqrt{(d-\alpha)(d-\beta)}}{a-d}+\frac{2d-\alpha-\beta}{2\sqrt{(d-\alpha)(d-\beta)}}\right)\left(\frac{2\sqrt{(d-\alpha)(d-\beta)}}{\beta-\alpha}\right)\right]$$

$$- \frac{N\,\Theta(-y)}{(y^2-4M^2y)^{\frac12}}\log\left(\frac{4M^2}{4M^2-2y-2(y^2-4M^2y)^{\frac12}}\right)\left[(x-b)\sqrt{(a-\alpha)(a-\beta)}\log\left(\frac{2\sqrt{(a-\alpha)(a-\beta)}+2a-\alpha-\beta}{\beta-\alpha}\right)\right.$$

$$+ (a-x)\sqrt{(b-\alpha)(b-\beta)}\log\left(\frac{2\sqrt{(b-\alpha)(b-\beta)}+2b-\alpha-\beta}{\beta-\alpha}\right) + (b-a)\sqrt{(x-\alpha)(x-\beta)}\,\Theta(-x)\log\left(\frac{2\sqrt{(x-\alpha)(x-\beta)}+2x-\alpha-\beta}{\beta-\alpha}\right)$$

$$\left.\left. + (b-a)\sqrt{(x-\alpha)(x-\beta)}\,\Theta(x-4M^2)\log\left(\frac{2\sqrt{(x-\alpha)(x-\beta)}+2x-\alpha-\beta}{\alpha-\beta}\right)\right]\right\}$$

where

$\alpha = (m_a + m_b)^2$

$\beta = (m_a - m_b)^2$

$A = (X - \alpha)(X - \beta)$

$$B = 4M^2 A + 4 m_a^2 m_b^2 X$$

$$C = (Y^2 - 4M^2 Y)$$

$$D = 2(m_a^2 + m_b^2)\,C + 4 m_a^2 m_b^2 Y$$

$$E = \alpha\beta C$$

$$a = (m_a^2 + m_b^2) + \frac{2m_a^2 m_b^2 Y}{C} + \sqrt{\left(\{m_a^2 + m_b^2\} + \frac{2m_a^2 m_b^2 Y}{C}\right)^2 - \alpha\beta}$$

$$b = a - 2\sqrt{\left(\{m_a^2 + m_b^2\} + \frac{2m_a^2 m_b^2 Y}{C}\right)^2 - \alpha\beta}$$

$$c = (m_a^2 + m_b^2) - \frac{m_a^2 m_b^2}{2M^2} + \sqrt{\left(m_a^2 + m_b^2 + \frac{m_a^2 m_b^2}{2M^2}\right)^2 - \alpha\beta}$$

$$d = c - 2\sqrt{\left(m_a^2 + m_b^2 + \frac{m_a^2 m_b^2}{2M^2}\right)^2 - \alpha\beta}$$

$$M_0 = \frac{1}{c\,d(d-c) + Xd(X-d) + Xc(c-X)}$$

$$N = \frac{1}{ab(b-a) + Xb(X-b) + Xa(a-X)}$$

$\theta(x) = 0$ for $x < 0$

$\theta(x) = 1$ for $x > 0$.

An estimate of the size of $I_{ab}(x,y)$ is given in Table 3 by a comparison of $I_{aa}(t,s)$ and $f_a^2/(m_a^2-t)$ for forward scattering.

TABLE 3

$I_{aa}(t,s)$, for $t = 0$, $m_a = |\vec{q}| = m_\pi$

$\dfrac{f_a^2}{m_a^2 - t} = f_a^2$
$\mathrm{Im}\, I_{aa}(t,s) = 5f_a^4 \times 10^{-4}$
$\mathrm{Re}\, I_{aa}(t,s) = 4f_a^4 \times 10^{-4}$

I_{ab} is the fourth perturbation theory contribution to the scattering amplitude. However, unlike the usual treatment, it is not evaluated in the static approximation, and it is completely covariant.

We see from Table 3 that if the meson coupling constant is not significantly larger than the pion-nucleon pseudoscalar coupling constant ($\frac{g^2}{4\pi} \sim 15$) of the spectral function in fourth order perturbation theory contributes only 10% of the pole term to the scattering amplitude at this energy. The similarities of the fourth order expressions for scalar and spinor nucleons suggest that the fourth order contribution in the charged spinor case will not be more than $\sim$ 10% of the pole term. Thus, either the spectral function is unimportant in the NN problem, which is unlikely, or the fourth order approximation is not very good. Since evaluation of the spectral function to a better approximation is very difficult, we consider only pole terms in what follows. These have a chance of being accurate at energies which are neither too high nor too low (150-300 Mev.).

CHAPTER V

CHARGED SPINOR NUCLEONS

1. The Scattering Amplitude

Since we write the scattering amplitude in terms of nucleon spin and i-spin wave functions, we record the following quantities for future reference. The Dirac spinor for a nucleon or antinucleon with four-momentum q and spin projection σ is[1]:

$$u(q,\sigma,\lambda) = \frac{1}{2}\left[\frac{1+\lambda M/q_3}{1+2\sigma q_3/(q_4^2-q^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \begin{pmatrix} 1+\dfrac{q_3}{2\sigma(q_4^2-q^2)^{\frac{1}{2}}} \\ \dfrac{q_1+iq_2}{2\sigma(q_4^2-q^2)^{\frac{1}{2}}} \\ \dfrac{2\sigma(q_4^2-q^2)^{\frac{1}{2}}+q_3}{\lambda q_4+M} \\ \dfrac{q_1+iq_2}{\lambda q_4+M} \end{pmatrix} \qquad (1)$$

$$\bar{u}(q,\sigma,\lambda) = u^{+}(q,\sigma,\lambda)\,\gamma_4$$

$q^2 = q_4^{\,2} - q_1^{\,2} - q_2^{\,2} - q_3^{\,2} = M^2$. $\sigma = \pm 1/2$ corresponds to spin projections parallel and antiparallel to the three-momentum $\vec{q}$. $\lambda = 1$ corresponds to a nucleon, and $\lambda = -1$ corresponds to an antinucleon.

1. See D.R.Bates,"Quantum Theory", Vol.III,Academic Press, New York, 1962.

In this representation, the spinor operators are:

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad (2)$$

$$\gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\sigma_{\mu\nu} = \frac{1}{i}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right)$$

The nucleon and antinucleon i-spin[2] wave functions are:

$$\begin{aligned} &\text{Proton} && \chi(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\text{Neutron} && \chi(n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\text{Antineutron} && \chi(\bar{n}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\text{Antiproton} && \chi(\bar{p}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned} \qquad (3)$$

In this representation, the i-spin matrices are:

$$\begin{aligned} \tau_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \tau_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \tau_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \qquad (4)$$

For nucleon-nucleon and antinucleon-nucleon elastic scattering, there are ten independent scattering amplitudes, five for i-spin triplet scattering and five for i-spin singlet scattering. This can be seen as follows. Assuming charge independence, we get no singlet-triplet i-spin transitions. There are four states of total angular momentum J, orbital angular momentum ℓ,

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2. The isotopic spin of a nucleon is one-half. See J. Hamilton, op. cit., p. 198.

and spin S, for a particular total i-spin T. The five allowed, independent transitions between these four states are shown in Table 4.

TABLE 4

Allowed Transitions in N-N and $\bar{N}$-N Elastic Scattering

SPIN SINGLET
$J=\ell_i \rightarrow J=\ell_f$
SPIN TRIPLET
$J=\ell_i-1 \rightarrow J=\ell_f-1$
$J=\ell_i \rightarrow J=\ell_f$
$J=\ell_i+1 \rightarrow J=\ell_f+1$

That these are the only independent transitions can be seen from the following arguments. For a particular i-spin state in nucleon-nucleon scattering, the Pauli principle[3] requires that the parity of singlet and triplet spin states is opposite since the total wave function must be antisymmetric under the interchange of the nucleons. Thus, a singlet-triplet spin transition violates parity conservation insofar as T is conserved. In the spin triplet state, transitions from $J = \ell_i$ to $J = \ell_f \pm 1$ and from $J = \ell_i \pm 1$ to $J = \ell_f$ also violate parity conservation. The transition from $J = \ell_i - 1$ to $J = \ell_f + 1$ is equivalent to the transition from $J = \ell_i + 1$ to $J = \ell_f - 1$ by time reversal invariance.

All of the arguments above apply to anti-nucleon-nucleon elastic scattering except the application of the Pauli principle. However, in this case, G-parity and parity conservation imply $(-1)^{S_i + T} = (-1)^{S_f + T}$ and since T is conserved, there are no singlet-triplet spin transitions (See Chapter VI).

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3. Notice that in the case of neutral, scalar nucleons (equations (23), Chapter IV) Bose statistics were applied. This was done to avoid the inconsistancies involved when Fermi statistics are applied to scalar particles. See F. Mandl, "Introduction to Quantum Field Theory", Interscience, New York, 1959, pp. 16 and 48.

Following Goldberger, et al.,[4] we take the ten invariant amplitudes shown below.

Channel I

$$\mathcal{J}_1 = \left[F_1^0(S_1-\tilde{S}_1) + F_2^0(T_1+\tilde{T}_1) + F_3^0(A_1-\tilde{A}_1) + F_4^0(V_1+\tilde{V}_1) + F_5^0(P_1-\tilde{P}_1)\right] P_0$$

$$+ \left[F_1'(S_1-\tilde{S}_1) + F_2'(T_1+\tilde{T}_1) + F_3'(A_1-\tilde{A}_1) + F_4'(V_1+\tilde{V}_1) + F_5'(P_1-\tilde{P}_1)\right] P_1 \qquad (5)$$

where P_0 and P_1 are singlet and triplet i-spin projection operators.

$$P_0 = \frac{1-\vec{\tau}_C\cdot\vec{\tau}_D}{4} \qquad (6)$$

$$P_1 = \frac{3+\vec{\tau}_C\cdot\vec{\tau}_D}{4}$$

Channel II

$$\mathcal{J}_{II} = \left[\bar{F}_1^0(S_{II}-\tilde{S}_{II}) + \bar{F}_2^0(T_{II}+\tilde{T}_{II}) + \bar{F}_3^0(A_{II}-\tilde{A}_{II}) + \bar{F}_4^0(V_{II}+\tilde{V}_{II}) + \bar{F}_5^0(P_{II}-\tilde{P}_{II})\right] \bar{P}_0 \qquad (7)$$

$$+ \left[\bar{F}_1'(S_{II}-\tilde{S}_{II}) + \bar{F}_2'(T_{II}+\tilde{T}_{II}) + \bar{F}_3'(A_{II}-\tilde{A}_{II}) + \bar{F}_4'(V_{II}+\tilde{V}_{II}) + \bar{F}_5'(P_{II}-\tilde{P}_{II})\right] \bar{P}_1$$

where

$$\bar{P}_0 = \frac{1-\vec{\tau}_C\cdot\vec{\tau}_B}{4} \qquad (8)$$

$$\bar{P}_1 = \frac{3+\vec{\tau}_C\cdot\vec{\tau}_B}{4}$$

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4. M.L. Goldberger, M.T. Grisaru, S.W. MacDowell, and D.Y. Wong, Phys. Rev. 120, 2250 (1960). A slightly different amplitude is used by D. Amati, E. Leader, and B. Vitale, Nuovo Cimento 17, 68 (1960).

Channel III

$$\mathcal{J}_{III} = \left[\bar{\bar{F}}_1^{\,o}(S_{III}-\tilde{S}_{III}) + \bar{\bar{F}}_2^{\,o}(T_{III}+\tilde{T}_{III}) + \bar{\bar{F}}_3^{\,o}(A_{III}-\tilde{A}_{III}) + \bar{\bar{F}}_4^{\,o}(V_{III}+\tilde{V}_{III}) + \bar{\bar{F}}_5^{\,o}(P_{III}-\tilde{P}_{III})\right]\bar{\bar{P}}_0$$

(9)

$$\left[\bar{\bar{F}}_1^{\,\prime}(S_{III}-\tilde{S}_{III}) + \bar{\bar{F}}_2^{\,\prime}(T_{III}+\tilde{T}_{III}) + \bar{\bar{F}}_3^{\,\prime}(A_{III}-\tilde{A}_{III}) + \bar{\bar{F}}_4^{\,\prime}(V_{III}+\tilde{V}_{III}) + \bar{\bar{F}}_5^{\,\prime}(P_{III}-\tilde{P}_{III})\right]\bar{\bar{P}}_1$$

where $$\bar{\bar{P}}_0 = \frac{1-\vec{\tau}_D\cdot\vec{\tau}_B}{4} \qquad (10)$$

$$\bar{\bar{P}}_1 = \frac{3+\vec{\tau}_D\cdot\vec{\tau}_B}{4}$$

The operators S_I, etc. are analogous to the β- decay operators.

Channel I, for $S_1\ A_1\ T_1\ V_1\ P_1$

$$\bar{u}(q_D\sigma_D\lambda_D)\,\Omega\, u(q_B\sigma_B\lambda_B)\,\bar{u}(q_c\sigma_c\lambda_c)\,\Omega\, u(q_A\sigma_A\lambda_A)$$

for $\tilde{S}_1\ \tilde{A}_1\ \tilde{T}_1\ \tilde{V}_1\ \tilde{P}_1$

$$\bar{u}(q_c\sigma_c\lambda_c)\,\Omega\, u(q_B\sigma_B\lambda_B)\,\bar{u}(q_D\sigma_D\lambda_D)\,\Omega\, u(q_A\sigma_A\lambda_A) \qquad (11)$$

$$\lambda_A = \lambda_B = \lambda_c = \lambda_D = 1$$

where:

$$\Omega = \begin{cases} 1 & \text{for } S \ \& \ \tilde{S} \\ \gamma_\mu & \text{for } V \ \& \ \tilde{V} \\ \frac{1}{2}\sigma_{\mu\nu} & \text{for } T \ \& \ \tilde{T} \\ i\gamma_5\gamma_\mu & \text{for } A \ \& \ \tilde{A} \\ \gamma_5 & \text{for } P \ \& \ \tilde{P} \end{cases}$$

for example:

$$A_1 = \bar{u}(q_D\sigma_D\lambda_D)\gamma_5\gamma_\mu i\, u(q_B\sigma_B\lambda_B)\,\bar{u}(q_c\sigma_c\lambda_c)\gamma_5\gamma_\mu i\, u(q_A\sigma_A\lambda_A)$$

<u>Channel II</u>, for $S_{II}\ A_{II}\ T_{II}\ V_{II}\ P_{II}$

$$\bar{u}(q_B\sigma_B\lambda_B)\Omega\, u(q_D\sigma_D\lambda_D)\,\bar{u}(q_c\sigma_c\lambda_c)\Omega\, u(q_A\sigma_A\lambda_A)$$

for $\tilde{S}_{II}\ \tilde{A}_{II}\ \tilde{T}_{II}\ \tilde{V}_{II}\ \tilde{P}_{II}$

$$\bar{u}(q_c\sigma_c\lambda_c)\Omega\, u(q_D\sigma_D\lambda_D)\,\bar{u}(q_B\sigma_B\lambda_B)\Omega\, u(q_A\sigma_A\lambda_A) \quad (12)$$

$$\lambda_A = \lambda_c = 1$$

$$\lambda_B = \lambda_D = -1$$

Similarly, we define the operators for Channel III.

S,T,A,V,P are related to $\tilde{S},\tilde{T},\tilde{A},\tilde{V},\tilde{P}$ as follows:

$$\begin{pmatrix}\tilde{S}\\ \tilde{V}\\ \tilde{T}\\ \tilde{A}\\ \tilde{P}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1 & 1\\ 4 & -2 & 0 & 2 & -4\\ 6 & 0 & -2 & 0 & 6\\ 4 & 2 & 0 & -2 & -4\\ 1 & -1 & 1 & -1 & 1\end{pmatrix}\cdot\begin{pmatrix}S\\ V\\ T\\ A\\ P\end{pmatrix} \quad (13)$$

Equation (13) can be verified directly using the definitions of the spinors and spinor operators (equations (1) and (2)). We also record the following relations which result directly from equation (13).

$$\begin{bmatrix} S \\ V \\ T \\ A \\ P \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ -4 & 2 & 0 & -2 & 4 \\ -6 & 0 & 2 & 0 & -6 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} S - \tilde{S} \\ V + \tilde{V} \\ T + \tilde{T} \\ A - \tilde{A} \\ P - \tilde{P} \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} \tilde{S} \\ \tilde{V} \\ \tilde{T} \\ \tilde{A} \\ \tilde{P} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 4 & 2 & 0 & 2 & -4 \\ 6 & 0 & 2 & 0 & 6 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & -1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} S - \tilde{S} \\ V + \tilde{V} \\ T + \tilde{T} \\ A - \tilde{A} \\ P - \tilde{P} \end{bmatrix} \tag{15}$$

2. Crossing Relations

We can determine the behavior of $\mathcal{J}_1(s,t,u)$ under the interchange of t and u from the Pauli principle. From the definition of t and u (equations (7), Chapter IV), this interchange corresponds to the exchange of particles C and D. In Channel I, the wave function must be anti-symmetric under the exchange, and

$$\mathcal{J}_1(s,t,u) = -\mathcal{J}_1(s,u,t) \tag{16}$$

The exchange $C \leftrightarrow D$ gives:

$$
\begin{aligned}
&C \leftrightarrow D \\
&P_0 \leftrightarrow -P_0 \\
&P_1 \leftrightarrow P_1 \\
&S \leftrightarrow \tilde{S} \\
&V \leftrightarrow \tilde{V} \\
&T \leftrightarrow \tilde{T} \\
&A \leftrightarrow \tilde{A} \\
&P \leftrightarrow \tilde{P}
\end{aligned}
\tag{17}
$$

Equations (16) and (17) imply:

$$F_i^j(s,t,u) = (-1)^{i+j}\, F_i^j(s,u,t). \tag{18}$$

The crossing relation between Channels I and II has the following form (See Reference 4).

$$
\begin{aligned}
F_i^j &= \Gamma_{ik} B_{jh} \bar{F}_k^h \\
\bar{F}_i^j &= \Gamma_{ik} B_{jh} F_k^h
\end{aligned}
\tag{19}
$$

where B_{jh} is the i-spin crossing matrix and Γ_{ik} is the spin crossing matrix.

$$\left(B_{jh}\right) = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \tag{20}$$

$$\left(\Gamma_{ik}\right) = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & \Gamma_{45} \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 & 6 & -4 & 4 & -1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & -1 \\ -1 & 6 & 4 & -4 & -1 \end{pmatrix} \tag{21}$$

The crossing relation between Channels I and III can be determined from equations (18) and (19), but it is not necessary to give it explicitly. Equations (18) and (19) contain sufficient information to determine the features of the dispersion relation.

3. Double Dispersion Relations

Assuming each of the F_i^j (s,t,u) has only the singularities required by causality and unitarity, we get the Mandelstam representation (equation (19), Chapter IV) for each $F_i^j(s,t,u)$.

$$F_i^j(s,t,u) = \sum_m R_{mij}\left(\frac{1}{m^2-t} + \frac{(-1)^{i+j}}{m^2-u}\right) + \left(\frac{R_{Dij}}{M_D^2-s}\right)$$

$$+ \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dt' \int_{4M_\pi^2}^{\infty} du' \frac{\rho_{23ij}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\int_{4M^2}^{\infty} ds' \int_{4M_\pi^2}^{\infty} dt' \frac{\rho_{12ij}(s',t')}{(s'-s)(t'-t)} \qquad (22)$$

$$+ \frac{1}{\pi^2}\int_{4M^2}^{\infty} ds' \int_{4M_\pi^2}^{\infty} du' \frac{(-1)^{i+j}\rho_{12ij}(s',u')}{(s'-s)(u'-u)}$$

where $\rho_{23ij}(t',u') = (-1)^{i+j}\rho_{23ij}(u',t')$.

We have applied equation (18) in equation (22).

The residues R_{mij} and R_{Dij} and the spectral functions ρ_{12ij} and ρ_{23ij} can be determined using unitarity, similarly to equations (14), Chapter IV. However, since equation (19) has related

$\rho_{13i\gamma}$ to $\rho_{12i\gamma}$ and the u pole term to the t pole term, unitarity applied in Channels I and II gives all of these residues and spectral functions. This is the reason we do not need explicit crossing relations between Channels I and III.

4. Cross Sections

We now relate the invariant amplitude defined in equation (5) to the cross section. First we define the amplitude for transitions between states of given helicities, following the formalism of Jacob and Wick.[5]

$$\langle \sigma_C \sigma_D | \varphi^i | \sigma_A \sigma_B \rangle$$

The helicity, σ , is the projection of the spin of a nucleon along the direction of motion of the nucleon. φ^i is defined such that the relation between the helicity amplitude and a particular i-spin component of $\mathcal{F}_1$ (s,t,u) is:

$$\frac{4\pi(\vec{q}^2+M^2)^{1/2}}{M^2}\langle \sigma_C \sigma_D | \varphi^i | \sigma_A \sigma_B \rangle = F_1^i(S_1 - \tilde{S}_1) + F_2^i(T_1+\tilde{T}_1) + F_3^i(A_1-\tilde{A}_1) + F_4^i(V_1+\tilde{V}_1) + F_5^i(P_1-\tilde{P}_1) \quad (23)$$

where i = 0,1.

5. M. Jacob & G.C.Wick, Ann.Phys. 7, 404 (1959).

The cross section for scattering in a given i-spin state between states of definite helicities is:

$$\frac{d\sigma^i(\sigma_A\sigma_B \to \sigma_C\sigma_D)}{d\Omega} = \left|\langle\sigma_C\sigma_D|\,\varphi^i\,|\sigma_A\sigma_B\rangle\right|^2 \tag{24}$$

There are no singlet-triplet i-spin transitions. The ten independent helicity amplitudes are[5]

$$\begin{aligned}
\varphi_1^i &= \langle \tfrac12\,\tfrac12|\,\varphi^i\,|\tfrac12\,\tfrac12\rangle = \langle -\tfrac12\,-\tfrac12|\,\varphi^i\,|-\tfrac12\,-\tfrac12\rangle \\
\varphi_2^i &= \langle \tfrac12\,\tfrac12|\,\varphi^i\,|-\tfrac12\,-\tfrac12\rangle = \langle -\tfrac12\,-\tfrac12|\,\varphi^i\,|\tfrac12\,\tfrac12\rangle \\
\varphi_3^i &= \langle \tfrac12\,-\tfrac12|\,\varphi^i\,|\tfrac12\,-\tfrac12\rangle = \langle -\tfrac12\,\tfrac12|\,\varphi^i\,|-\tfrac12\,\tfrac12\rangle \\
\varphi_4^i &= \langle \tfrac12\,-\tfrac12|\,\varphi^i\,|-\tfrac12\,\tfrac12\rangle = \langle -\tfrac12\,\tfrac12|\,\varphi^i\,|\tfrac12\,-\tfrac12\rangle \\
\varphi_5^i &= \langle \tfrac12\,\tfrac12|\,\varphi^i\,|\tfrac12\,-\tfrac12\rangle = -\langle \tfrac12\,-\tfrac12|\,\varphi^i\,|\tfrac12\,\tfrac12\rangle \\
&= \langle -\tfrac12\,-\tfrac12|\,\varphi^i\,|-\tfrac12\,\tfrac12\rangle = -\langle -\tfrac12\,\tfrac12|\,\varphi^i\,|-\tfrac12\,-\tfrac12\rangle \\
&= \langle \tfrac12\,\tfrac12|\,\varphi^i\,|-\tfrac12\,\tfrac12\rangle = -\langle -\tfrac12\,\tfrac12|\,\varphi^i\,|\tfrac12\,\tfrac12\rangle \\
&= \langle -\tfrac12\,-\tfrac12|\,\varphi^i\,|\tfrac12\,-\tfrac12\rangle = -\langle \tfrac12\,-\tfrac12|\,\varphi^i\,|-\tfrac12\,-\tfrac12\rangle
\end{aligned} \tag{25}$$

where $i = 0,1$.

The connection between the F_i^j and the helicity amplitude, φ_k^j, can be determined from equation (23) and the definitions of the F_i^j [4] (equations (5) and (11)).

$$
\begin{aligned}
\varphi_1^i &= \frac{1}{2\pi\sqrt{s}}\left[M^2\left\{F_1^i+(F_2^i+F_4^i)\cos\theta\right\}-\left\{3s/4+\vec{q}^{\,2}\right\}F_3^i\right]\\
\varphi_2^i &= \frac{1}{2\pi\sqrt{s}}\left[-sF_1^i/4+\left\{(s/4+\vec{q}^{\,2})F_2^i+M^2F_4^i\right\}\cos\theta+3M^2F_3^i-\vec{q}^{\,2}F_5^i\right]\\
\varphi_3^i &= \frac{1}{2\pi\sqrt{s}}\left[2M^2F_2^i+sF_4^i/2+\vec{q}^{\,2}(-F_1^i+2F_3^i+F_5^i)\right]\cos^2\theta/2 \qquad (26)\\
\varphi_4^i &= \frac{1}{2\pi\sqrt{s}}\left[2M^2F_2^i+sF_4^i/2-\vec{q}^{\,2}(-F_1^i+2F_3^i+F_5^i)\right]\sin^2\theta/2\\
\varphi_5^i &= -\frac{M}{2\pi}\left(F_2^i+F_4^i\right)\sin\theta
\end{aligned}
$$

where $s = 4(\vec{q}^{\,2}+M^2)$

$t = -2\vec{q}^{\,2}\,(1-\cos\theta)$

$u = -2\vec{q}^{\,2}\,(1+\cos\theta)$

$$\frac{d\sigma(nn)}{d\Omega} = \frac{d\sigma(pp)}{d\Omega} = \frac{1}{8\pi^2}\Big\{\frac{1}{s}\Big|M^2\big[F_1^1 + (F_2^1 + F_4^1)\cos\theta\big] -$$

$$\Big[\frac{3s}{4} + \vec{q}^{\,2}\Big]F_3^1\Big|^2 + \frac{1}{s}\Big|-sF_1^1/4 + \Big[\Big(\frac{s}{4}+\vec{q}^{\,2}\Big)F_2^1 + M^2F_4^1\Big]\cos\theta \qquad (29)$$

$$+ 3M^2F_3^1 - \vec{q}^{\,2}F_5^1\Big|^2 + \frac{1}{s}\Big|2M^2F_2^1 + sF_4^1/2 + \vec{q}^{\,2}(-F_1^1 + 2F_3^1 + F_5^1)\Big|^2\cos^4\theta/2$$

$$+ \frac{1}{s}\Big|2M^2F_2^1 + sF_4^1/2 - \vec{q}^{\,2}(-F_1^1 + 2F_3^1 + F_6^1)\Big|^2\sin^4\theta/2$$

$$+ 4M^2\big|F_2^1 + F_4^1\big|^2\sin^2\theta\Big\}$$

$$\frac{d\sigma(n,p)}{d\Omega} = \frac{1}{32\pi^2}\Big\{\frac{1}{s}\Big|M^2\big[F_1^0 + F_1^1 + (F_2^0 + F_2^1 + F_4^0 + F_4^1)\cos\theta\big] -$$

$$\Big[\frac{3s}{4} + \vec{q}^{\,2}\Big]\big[F_3^0 + F_3^1\big]\Big|^2 + \frac{1}{s}\Big|-\frac{s}{4}\big[F_1^0 + F_1^1\big] + \Big[\Big(\frac{s}{4} + \vec{q}^{\,2}\Big)(F_2^0 + F_2^1) + \qquad (30)$$

$$M^2(F_4^0 + F_4^1)\Big]\cos\theta + 3M^2F_3^1 - \vec{q}^{\,2}F_6^1 + 3M^2F_3^0 - \vec{q}^{\,2}F_5^0\Big|^2 +$$

$$\frac{1}{s}\Big|2M^2(F_2^0 + F_2^1) + \frac{s}{2}(F_4^0 + F_4^1) + \vec{q}^{\,2}(-F_1^0 - F_1^1 + 2F_3^0 + 2F_3^1 + F_5^0 + F_5^1)\Big|^2\cos^4\frac{\theta}{2}$$

$$+ \frac{1}{s}\Big|2M^2(F_2^0 + F_2^1) + \frac{s}{2}(F_4^0 + F_4^1) - \vec{q}^{\,2}(-F_1^0 - F_1^1 + 2F_3^0 + 2F_3^1 + F_5^0 + F_5^1)\Big|^2\sin^4\theta/2$$

$$+ 4M^2\big|F_2^0 + F_2^1 + F_4^0 + F_4^1\big|^2\sin^2\theta\Big\}$$

Neutron-neutron and proton-proton scatterings are in pure i-spin triplet states. The center of mass cross sections for these scatterings with unpolarized incident and target nucleons is the square of the amplitude obtained by averaging over initial and summing over final helicity states.

$$\frac{d\sigma(nn)}{d\Omega} = \frac{d\sigma(pp)}{d\Omega} = \frac{1}{4} \sum_{i=-\frac{1}{2}}^{\frac{1}{2}} \sum_{j=-\frac{1}{2}}^{\frac{1}{2}} \sum_{i'=-\frac{1}{2}}^{\frac{1}{2}} \sum_{j'=-\frac{1}{2}}^{\frac{1}{2}} \left| \langle i j | \varphi' | i' j' \rangle \right|^2$$

Applying equations (25), we get:

$$\frac{d\sigma(nn)}{d\Omega} = \frac{d\sigma(pp)}{d\Omega} = \frac{1}{4} \left\{ 2|\varphi_1^1|^2 + 2|\varphi_2^1|^2 + 2|\varphi_3^1|^2 + 2|\varphi_4^1|^2 + 8|\varphi_5^1|^2 \right\} \tag{27}$$

Neutron-proton scattering is in an equal mixture of i-spin triplet and singlet states and, similarly, the center of mass cross section for unpolarized incident and target nucleons is:

$$\frac{d\sigma(np)}{d\Omega} = \frac{1}{16} \left\{ 2|\varphi_1^0 + \varphi_1^1|^2 + 2|\varphi_2^0 + \varphi_2^1|^2 + 2|\varphi_3^0 + \varphi_3^1|^2 + 2|\varphi_4^0 + \varphi_4^1|^2 + 8|\varphi_5^0 + \varphi_5^1|^2 \right\} \tag{28}$$

Applying equations (26) to equations (27) and (28), we get:

CHAPTER VI

PIONS AND HEAVY MESONS

1. G-parity

The quantum numbers specifying a state of strongly interacting particles with zero baryon number, B, and strangeness, Σ , are the spin, J, the parity, P, the i-spin, T, and the charge conjugation parity of the neutral member of the i-spin multiplet, C. Lee and Yang[1] introduced the operation G, which is the product of charge conjugation and a 180° rotation about the 2-axis in i-spin space.

$$G = C e^{i\pi T_2} \tag{1}$$

All members of an i-spin multiplet with $B = 0$, $\Sigma = 0$ are eigenstates of G with the same G-parity. We specify the G-parity of all charge states instead of C for the neutral state. Also, since G is conserved in strong interactions, it can lead to selection rules.

The i-spin wave function, χ , transforms under $e^{i\pi T_2}$ as follows:

For i-spin zero,

$$e^{i\pi T_2} \chi_0 = \chi_0 \tag{2}$$

---

1. T.D. Lee and C.N. Yang, Nuovo Cimento **3**, 749 (1956).

For i-spin one, since

$$\chi^{+} = \chi_1 + i\chi_2$$
$$\chi^{-} = \chi_1 - i\chi_2$$
$$\chi^{0} = \chi_3$$

and since

$$e^{i\pi T_2}\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix} = \begin{pmatrix}-\chi_1\\ \chi_2\\ -\chi_3\end{pmatrix}$$

then

$$e^{i\pi T_2}\begin{pmatrix}\chi^{+}\\ \chi^{-}\\ \chi^{0}\end{pmatrix} = -\begin{pmatrix}\chi^{-}\\ \chi^{+}\\ \chi^{0}\end{pmatrix}$$

2. G-parity of Baryon-Antibaryon States

For a nucleon-antinucleon state with zero charge, charge conjugation interchanges spacial and spin coordinates of the particle and antiparticle. The nucleon-antinucleon wave function is

$$\Psi_{neut.}(N\bar{N}) = |n\, x_1\sigma_1\rangle|\bar{n}\, x_2\sigma_2\rangle \pm |p\, x_1\sigma_1\rangle|\bar{p}\, x_2\sigma_2\rangle$$

σ and x are spin and space coordinates.

$$C\,\Psi_{neut.}(N\bar{N}) = |\bar{n}\, x_1\sigma_1\rangle|n\, x_2\sigma_2\rangle \pm |\bar{p}\, x_1\sigma_1\rangle|p\, x_2\sigma_2\rangle$$

If a^{+} and b^{+} are antinucleon and nucleon creation operators then:

$$C\,\Psi_{neut.}(N\bar{N}) = \left\{a_n^{+}(x_1\sigma_1)\,b_n^{+}(x_2\sigma_2) \pm a_p^{+}(x_1\sigma_1)\,b_p^{+}(x_2\sigma_2)\right\}|0\rangle$$

Since the fermion operators anticommute

$$C\,\Psi_{neut.}(N\bar{N}) = -\left\{ b_n^\dagger(x_2\sigma_2)\,a_n^\dagger(x_1\sigma_1) \pm b_p^\dagger(x_2\sigma_2)\,a_p^\dagger(x_1\sigma_1) \right\} |0\rangle$$

$$C\,\Psi_{neut.}(N\bar{N}) = -|n\,x_2\sigma_2\rangle|\bar{n}\,x_1\sigma_1\rangle \mp |p\,x_2\sigma_2\rangle|\bar{p}\,x_1\sigma_1\rangle$$

The interchange of the X_i and σ_i gives a factor $(-1)^L\,(-1)^{S+1}$. Thus, we get

$$C\,\Psi_{neut.}(N\bar{N}) = (-1)^{L+1}(-1)^{S+1}\,\Psi_{neut.}(N\bar{N}) \tag{4}$$

L and S are the orbital angular momentum and the total spin. Since the G-parity of all members of an i-spin multiplet is the same, equations (2), (3), and (4) give the G-parity of nucleon-antinucleon states as:

$$G = (-1)^{L+S+T} \tag{5}$$

Equation (4) shows that the parity of an $\bar{N}N$ state is $(-1)^{L+1}$. Thus, $\bar{N}$ and N have opposite intrinsic parity.

Table 5 shows the G-parity of nucleon-antinucleon states with $J < 2$.

TABLE 5

G-Parity of $N\bar{N}$ States With $J < 2$

$N\bar{N}$ State	T	J P	L S	G
$^1S_0$	0	0 -	0 0	+
$^1S_0$	1	0 -	0 0	-
$^3P_0$	0	0 +	1 1	+
$^3P_0$	1	0 +	1 1	-
$^1P_1$	0	1 +	1 0	-
$^1P_1$	1	1 +	1 0	+
$^3P_1$	0	1 +	1 1	+
$^3P_1$	1	1 +	1 1	-
$^3S_1$	0	1 -	0 1	-
$^3S_1$	1	1 -	0 1	+
$^3D_1$	0	1 -	2 1	-
$^3D_1$	1	1 -	2 1	+

For a lambda-antilambda state, the wave function is:

$$\Psi(\Lambda\bar{\Lambda}) = |\Lambda x_1 \sigma_1\rangle |\bar{\Lambda} x_2 \sigma_2\rangle$$

$$C\,\Psi(\Lambda\bar{\Lambda}) = |\bar{\Lambda} x_1 \sigma_1\rangle |\Lambda x_2 \sigma_2\rangle = (-1)^{L+1}(-1)^{S+1}\,\Psi(\Lambda\bar{\Lambda})$$

Thus, the G-parity of a lambda-antilambda state is:

$$G = (-1)^{L+S} \tag{6}$$

since the i-spin of the lambda is zero.

For a sigma-antisigma state with zero charge, the wave function is:

$$\Psi_{neut}(\Sigma\bar{\Sigma}) = a\,|\Sigma^0 x_1 \sigma_1\rangle|\bar{\Sigma}^0 x_2 \sigma_2\rangle + b\,|\Sigma^+ x_1 \sigma_1\rangle|\bar{\Sigma}^+ x_2 \sigma_2\rangle$$

$$+ c\,|\Sigma^- x_1 \sigma_1\rangle|\bar{\Sigma}^- x_2 \sigma_2\rangle$$

where a, b, and c determine the i-spin state.

$$C\,\Psi_{neut.}(\Sigma\bar{\Sigma}) = a\,|\bar{\Sigma}^0 x_1 \sigma_1\rangle|\Sigma^0 x_2 \sigma_2\rangle + b\,|\bar{\Sigma}^+ x_1 \sigma_1\rangle|\Sigma^+ x_2 \sigma_2\rangle$$

$$+ c\,|\bar{\Sigma}^- x_1 \sigma_1\rangle|\Sigma^- x_2 \sigma_2\rangle$$

$$= (-1)^{L+1}(-1)^{S+1}\,\Psi_{neut.}(\Sigma\bar{\Sigma})$$

Thus, the G-parity of a sigma-antisigma state is:

$$G = (-1)^{L+S+T} \tag{7}$$

A complete discussion of the G-parity of baryon-antibaryon states should include $\Sigma\bar{\Lambda}$ and $\bar{\Sigma}\Lambda$ states. However, the G-parity of these states depends on the relative $\Lambda - \Sigma$ parity, which has not been determined.

3. Quantum Numbers of Mesons

A meson with strangeness zero is specified by the quantum numbers T, J, P and G. Table 6 shows the sixteen combinations of these quantum numbers with $J < 2$, $T < 2$. Only ten of the mesons shown in Table 6 have the same quantum numbers as a nucleon-antinucleon state. Notice that equivalence to a nucleon-antinucleon state places the following restrictions on the G-parity.

$$
\begin{array}{ll}
\text{Scalar meson} & G = (-1)^{T} \\
\text{Vector meson} & G = (-1)^{T+1} \\
\text{Axial Vector meson} & G = \pm(-1)^{T} \\
\text{Pseudoscalar meson} & G = (-1)^{T}
\end{array}
\qquad (8)
$$

TABLE 6

Mesons With J<2, T<2

MESON	J^{PG}	T	NN STATE
SCALAR	0^{++}	0	$^3P_0$
	0^{+-}	0	-
	0^{++}	1	-
	0^{+-}	1	$^3P_0$
VECTOR	1^{-+}	0	-
	1^{--}	0	$^3S_1$ & $^3D_1$
	1^{-+}	1	$^3S_1$ & $^3D_1$
	1^{--}	1	-
AXIAL VECTOR	1^{++}	0	$^3P_1$
	1^{+-}	0	$^1P_1$
	1^{++}	1	$^1P_1$
	1^{+-}	1	$^3P_1$
PSEUDO SCALAR	0^{-+}	0	$^1S_0$
	0^{--}	0	-
	0^{-+}	1	-
	0^{--}	1	$^1S_0$

Table 7 shows the meson-nucleon interactions that are linear in the meson field and display strong interaction symmetries (conservation of J, P, G, and T).

TABLE 7

Meson-Nucleon Interactions

MESON	T	INTERACTIONS*		
SCALAR	0	$g_s^0 \bar{N} N \varphi_s$	$[f_s^0 \bar{N} \gamma_\mu N \partial_\mu \varphi_s]$	
	1	$g_s^1 \bar{N} \vec{\tau} N \cdot \vec{\varphi}_s$	$[f_s^1 \bar{N} \gamma_\mu \vec{\tau} N \cdot \partial_\mu \vec{\varphi}_s]$	
VECTOR	0	$[g_V^0 \bar{N} N \partial_\mu \varphi_V^\mu]$	$f_V^0 \bar{N} \gamma_\mu N \varphi_V^\mu$	$t_V^0 \bar{N} \frac{\sigma_{\mu\lambda}}{2} N \frac{(\partial_\mu \varphi_V^\lambda - \partial_\lambda \varphi_V^\mu)}{2}$
	1	$[g_V^1 \bar{N} \vec{\tau} N \cdot \partial_\mu \vec{\varphi}_V^\mu]$	$f_V^1 \bar{N} \gamma_\mu \vec{\tau} N \cdot \vec{\varphi}_V^\mu$	$t_V^1 \bar{N} \frac{\sigma_{\mu\lambda}}{2} \vec{\tau} N \cdot \frac{(\partial_\mu \vec{\varphi}_V^\lambda - \partial_\lambda \vec{\varphi}_V^\mu)}{2}$
AXIAL VECTOR	0	$[g_A^0 \bar{N} \gamma_5 N \partial_\mu \varphi_A^\mu]$	$f_A^0 \bar{N} \gamma_5 \gamma_\mu N \varphi_A^\mu$	$t_A^0 \bar{N} \gamma_5 \frac{\sigma_{\mu\nu}}{2} N \frac{(\partial_\mu \varphi_A^\lambda - \partial_\lambda \varphi_A^\mu)}{2}$
	1	$[g_A^1 \bar{N} \gamma_5 \vec{\tau} N \partial_\mu \vec{\varphi}_A^\mu]$	$f_A^1 \bar{N} \gamma_5 \gamma_\mu \vec{\tau} N \cdot \vec{\varphi}_A^\mu$	$t_A^1 \bar{N} \gamma_5 \frac{\sigma_{\mu\nu}}{2} \vec{\tau} N \cdot \frac{(\partial_\mu \vec{\varphi}_A^\lambda - \partial_\lambda \vec{\varphi}_A^\mu)}{2}$
PSEUDO SCALAR	0	$g_P^0 \bar{N} \gamma_5 N \varphi_P$	$[f_P^0 \bar{N} \gamma_5 \gamma_\mu N \partial_\mu \varphi_P]$	
	1	$g_P^1 \bar{N} \gamma_5 \vec{\tau} N \cdot \vec{\varphi}_P$	$[f_P^1 \bar{N} \gamma_5 \gamma_\mu \vec{\tau} N \cdot \partial_\mu \vec{\varphi}_P]$	

*We do not use the bracketed interactions for reasons explained in the text.

For vector and axial vector mesons, Wentzel[2] shows that $\partial_\mu \varphi^\mu$ must vanish identically to avoid negative energy states. The vector and axial vector interactions for the scalar and pseudoscalar mesons can be transformed as follows:

$$f_s \bar{\Psi}_N \gamma_\mu \Psi_N \partial_\mu \varphi_s = - f_s (\partial_\mu \bar{\Psi}_N \gamma_\mu \Psi_N) \varphi_s \tag{9}$$

$$f_p \bar{\Psi}_N \gamma_5 \gamma_\mu \Psi_N \partial_\mu \varphi_p = - f_p (\partial_\mu \bar{\Psi}_N \gamma_5 \gamma_\mu \Psi_N) \varphi_p \tag{10}$$

Application of Euler's equations:

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial \{\partial_\mu \bar{\Psi}\}} \right\}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial \{\partial_\mu \varphi\}} \right\}$$

gives the same field equations for both interactions in equation (9).

$$(\gamma_\mu \partial_\mu + M) \Psi_N = f_s \gamma_\mu (\partial_\mu \varphi_s) \Psi \tag{11-a}$$

$$(\Box - m_s^2) \varphi_s = - f_s \partial_\mu \bar{\Psi}_N \gamma_\mu \Psi_N \tag{11-b}$$

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2. G. Wentzel, "Quantum Theory of Fields," Interscience, New York, 1949, Chapter 3.

The Euler equations also give the same field equations for both interactions in equation (10).

$$(\gamma_\mu \partial_\mu + M)\Psi_N = f_p \gamma_5 \gamma_\mu (\partial_\mu \varphi_p)\Psi \tag{12-a}$$

$$(\Box - m_p^2)\varphi_p = -f_p \partial_\mu(\bar{\Psi}_N \gamma_5 \gamma_\mu \Psi_N) \tag{12-b}$$

Expanding the right hand side of equation (9), we get:

$$f_s \bar{\Psi}_N \gamma_\mu \Psi_N \partial_\mu \varphi_s = -f_s(\{\partial_\mu \bar{\Psi}_N \gamma_\mu\}\Psi_N + \bar{\Psi}_N\{\gamma_\mu \partial_\mu \Psi_N\})\varphi_s$$

Applying equation (11-a) and the conjugate equation, we get:

$$f_s \bar{\Psi}_N \gamma_\mu \Psi_N \partial_\mu \varphi_s = -f_s(M\bar{\Psi}_N\Psi_N - M\bar{\Psi}_N\Psi_N + 2f_s\bar{\Psi}_N\gamma_\mu\Psi_N\partial_\mu\varphi_s)\varphi_s$$

$$= -2f_s^2 \bar{\Psi}_N \gamma_\mu \Psi_N(\partial_\mu \varphi_s)\varphi_s \tag{13}$$

Equation (13) shows that the vector interaction for scalar mesons is equivalent to an interaction that is quadratic in the meson field.

Expanding the right hand side of equation (10), we get:

$$f_p \bar{\Psi}_N \gamma_5 \gamma_\mu \Psi_N \partial_\mu \varphi_p = -f_p(\{\partial_\mu \bar{\Psi}_N \gamma_5 \gamma_\mu\}\Psi_N + \bar{\Psi}_N\{\gamma_5 \gamma_\mu \partial_\mu \Psi_N\})\varphi_p$$

since $[\gamma_5, \gamma_\mu]_+ = 0$

$$f_p \bar{\Psi}_N \gamma_5 \gamma_\mu \Psi_N \partial_\mu \varphi_p = -f_p(-\{\partial_\mu \bar{\Psi}_N \gamma_\mu\}\gamma_5\Psi_N + \bar{\Psi}_N\gamma_5\{\gamma_\mu \partial_\mu \Psi_N\})\varphi_p$$

Applying equation (12-a) and the conjugate equation, we get:

$$f_p \bar{\Psi}_N \gamma_5 \gamma_\mu \Psi_N \partial_\mu \varphi_p = f_p \left(2M \bar{\Psi}_N \gamma_5 \Psi_N - 2 f_p \bar{\Psi}_N \gamma_\mu \Psi_N \partial_\mu \varphi_p \right) \varphi_p \quad (14)$$

Equation (14) shows that the axial vector interaction for pseudoscalar mesons is equivalent to the pseudoscalar interaction to linear terms in the meson field.

Since we calculate only the pole terms explicitly, and since we eliminate negative energy states, we take:

$$f_s^0 = f_s^1 = f_p^0 = f_p^1 = g_V^0 = g_V^1 = g_A^0 = g_A^1 = 0$$

The G-parity of the mesons in the interactions without brackets in Table 7 is determined by the conservation of G-parity. The G-operation has the following effect on the nucleon fields:

$$G \bar{\Psi}_N \Omega \Psi_N G^{-1} = \omega_c \bar{\Psi}_N \Omega \Psi_N$$

$$G \bar{\Psi}_N \Omega \vec{\tau} \Psi_N G^{-1} = -\omega_c \bar{\Psi}_N \Omega \vec{\tau} \Psi_N \quad (15)$$

where

$$\omega_c = 1 \quad \text{for} \quad \Omega = 1,\ \gamma_5 \gamma_\mu,\ \gamma_5$$

$$\omega_c = -1 \quad \text{for} \quad \Omega = \gamma_\mu,\ \frac{\sigma_{\mu\lambda}}{2},\ \gamma_5 \frac{\sigma_{\mu\lambda}}{2}$$

The mesons have the same G-parity given in equations (15), since G-parity is conserved.

Scalar Meson	$G = (-1)^{T}$	
Vector Meson	$G = (-1)^{T+1}$	
Pseudoscalar Meson	$G = (-1)^{T}$	(16)
Axial Vector Meson	$G = (-1)^{T}$ for $\Omega = \gamma_5 \gamma_\mu$	
	$G = (-1)^{T+1}$ for $\Omega = \gamma_5 \frac{\sigma_{\mu\lambda}}{2}$	

Equations (8) and (16) are the same, as we would expect, since both sets of equations derive from strong interaction symmetries. Notice that the vector meson has two linear interactions, and there are two axial vector mesons.

Five mesons have been observed. Table 8 shows these mesons with their masses and quantum numbers.

TABLE 8

Observed Mesons

MESON	MASS	T	J^{PG}	DECAY
π	140 Mev	1	0^{--}	$\pi^0 \to 2\gamma$ $\pi^\pm \to \mu^\pm + \nu$
ω [3]	780 Mev	0	1^{--}	$\omega \to 3\pi$
ρ [4]	750 Mev	1	1^{-+}	$\rho \to 2\pi$
η [5]	550 Mev	?	?	$\eta^0 \to 3\pi$
(ζ)? [6]	575 Mev	1	?	$\zeta^\pm \to 2\pi$

4. Mesons and the Nucleon-Nucleon Scattering Amplitude.

The ω, ρ, η and ζ mesons have been studied primarily by their decays.[7] However, Lichtenberg[8] and others have considered the effects of mesons with various quantum numbers on the static nuclear potential.

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3. B.C. Maglic, et al., Phys. Rev. Letters 7, 178(1961); & M.L. Stevenson, et al.,Phys. Rev. 125,687 (1962).

4. J.A. Anderson, et al., Phys. Rev. Letters 6,365 (1961); D. Stonehill, et al.,Phys. Rev. Letters 6, 624 (1961); and A.R. Erwin, et al., Phys. Rev. Letters 6, 628 (1961).

5. A. Pevsner, et al., Phys. Rev. Letters 7, 421 (1962); P.L. Bastien, et al., Phys. Rev. Letters 8, 114 (1962); & D.D. Carmony, et al., Phys. Rev. Letters 8, 117 (1962).

6. R. Barloutaud, et al., Phys. Rev. Letters 8, 32 (1962); and B. Sechi Zorn, Phys. Rev. Letters 8, 282 (1962).

7. See for example D.B. Lichtenberg & G.C. Summerfield, Phys. Rev. (to be published).

8. D.B. Lichtenberg, Nuovo Cimento (to be published); D.B. Lichtenberg, J. Kovacs, & H. McManus, Bull. Am. Phys. Soc. 7,55 (1962); and N. Hoshizaki, I. Lin, & S. Machida, Prog. Theor. Phys. 26, 680 (1961).

We determine the mesons' pole terms in the nucleon-nucleon dispersion relation.

The pole terms are given by single particle exchange in perturbation theory. The meson exchange diagrams that give poles in the nucleon-nucleon amplitude are shown in Figure 6.

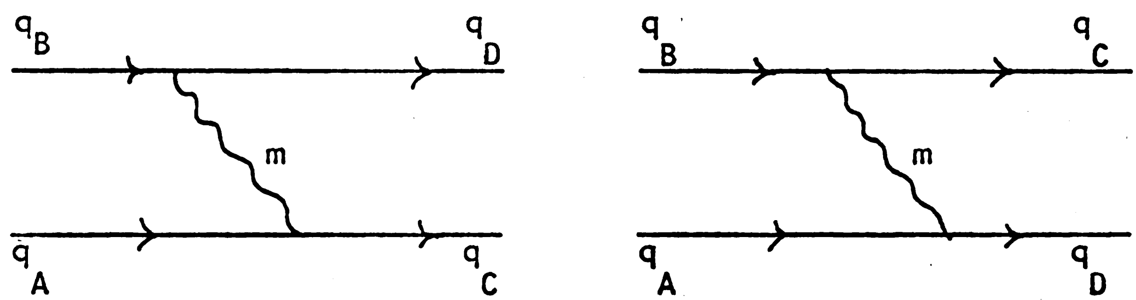

FIGURE 6

Meson Exchange Diagrams

The contributions of these diagrams for the ten mesons that interact linearly with nucleons are given in Table 9.

In equation (22), Chapter V, the pole terms for the ten invariant amplitudes F_i^j are given as:

$$F_i^{j}(s,t,u) = \sum_m R_{mij}\left(\frac{1}{m^2-t} + \frac{(-1)^{i+j}}{m^2-u}\right) \qquad (17)$$

Equations (14) & (15), Chapter V, give the expressions in Table 9 in the representation of equation (17). The R_{mij} are given in terms of the coupling constants and masses in Table 10. Equation (17) and equation (29)

TABLE 9

Meson Pole Terms

MESON			POLE TERM
J^{PG}	T	Mass	
0^{++}	0	M_1	$(g_s^0)^2(P_0+P_1)S_1/(M_1^2-u)+(g_s^0)^2(P_0-P_1)\tilde{S}_1/(M_1^2-t)$
0^{+-}	1	M_2	$(g_s^1)^2(3P_0-P_1)S_1/(M_2^2-u)+(g_s^1)^2(3P_0+P_1)\tilde{S}_1/(M_2^2-t)$
1^{--}	0	M_3	$-(a^0V_1+b^0S_1)(P_0+P_1)/(M_3^2-u)-(a^0\tilde{V}_1+b^0\tilde{S}_1)(P_0-P_1)/(M_3^2-t)$
1^{-+}	1	M_4	$-(a^1V_1+b^1S_1)(3P_0-P_1)/(M_4^2-u)-(a^1\tilde{V}_1+b^1\tilde{S}_1)(3P_0+P_1)/(M_4^2-t)$
1^{++}	0	M_5	$(f_A^0)^2(P_0+P_1)A_1/(M_5^2-u)+(f_A^0)^2(P_0-P_1)\tilde{A}_1/(M_5^2-t)$
1^{+-}	1	M_6	$(f_A^1)^2(3P_0-P_1)A_1/(M_6^2-u)+(f_A^1)^2(3P_0+P_1)\tilde{A}_1/(M_6^2-t)$
1^{+-}	0	M_7	$-\{M_7^2(t_A^0)^2(P_0+P_1)P_I/(M_7^2-u)+M_7^2(t_A^0)^2(P_0-P_1)\tilde{P}_I/(M_7^2-t)\}$
1^{++}	1	M_8	$-\{M_8^2(t_A^1)^2(3P_0-P_1)P_I/(M_8^2-u)+M_8^2(t_A^1)^2(3P_0+P_1)\tilde{P}_I/(M_8^2-t)\}$
0^{-+}	0	M_9	$(g_P^0)^2(P_0+P_1)P_I/(M_9^2-u)+(g_P^0)^2(P_0-P_1)\tilde{P}_I/(M_9^2-t)$
0^{--}	1	M_{10}	$(g_P^1)^2(3P_0-P_1)P_I/(M_{10}^2-u)+(g_P^1)^2(3P_0+P_1)\tilde{P}_I/(M_{10}^2-t)$

$a^0=(f_V^0+2Mt_V^0)^2$; $a^1=(f_V^1+2Mt_V^1)^2$

$b^0=M_3^2(t_V^0)^2-4Mt_V^0(f_V^0+2Mt_V^0)$; $b^1=M_4^2(t_V^1)^2-4Mt_V^1(f_V^1+2Mt_V^1)$

and (30), Chapter V give the differential cross sections.

$$\frac{d\sigma(nn)}{d\Omega} = \frac{d\sigma(pp)}{d\Omega} = \frac{1}{8\pi^2}\Big\{\frac{1}{s}\Big|\sum_m M^2\Big[R_{m11}\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)+(R_{m21}+R_{m41})\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)\cos\theta\Big]- \sum_m\Big[\frac{3s}{4}+\vec{q}^{\,2}\Big]R_{m51}\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)\Big|^2+\frac{1}{s}\Big|\sum_m\Big\{-\frac{s}{4}R_{m11}\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)+ \Big[\Big(\frac{s}{4}+\vec{q}^{\,2}\Big)R_{m21}+M^2R_{m41}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]\cos\theta+\Big[3M^2R_{m31}- \vec{q}^{\,2}R_{m51}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]\Big\}\Big|^2+\frac{1}{s}\Big|\sum_m\Big\{\Big[2M^2R_{m21}+\frac{s}{2}R_{m41}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]+ \vec{q}^{\,2}(-R_{m11}+2R_{m31}+R_{m51})\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)\Big\}\Big|^2\cos^4\frac{\theta}{2}+\frac{1}{s}\Big|\sum_m\Big[2M^2R_{m21}+ \frac{s}{2}R_{m41}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]-\sum_m\vec{q}^{\,2}(-R_{m11}+2R_{m31}+R_{m51})\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)\Big|^2\sin^4\theta/2+ 4M^2\Big|\sum_m(R_{m21}+R_{m41})\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)\Big|^2\sin^2\theta\Big\} \tag{18}$$

$$\frac{d\sigma(np)}{d\Omega} = \frac{1}{32\pi^2}\Big\{\frac{1}{s}\Big|\sum_m M^2\Big[R_{m10}\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)+R_{m11}\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)+\Big[(R_{m20}+R_{m40})\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)+ (R_{m21}+R_{m41})\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)\Big]\cos\theta\Big]-\sum_m\Big[\frac{3s}{4}+\vec{q}^{\,2}\Big]\Big[R_{m30}\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)+R_{m31}\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)\Big]\Big|^2+ \frac{1}{s}\Big|\sum_m\Big\{-\frac{s}{4}R_{m10}\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)-\frac{s}{4}R_{m11}\Big(\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big)+\Big\{\Big(\Big[\frac{s}{4}+\vec{q}^{\,2}\Big]R_{m20}+M^2R_{m40}\Big)\Big(\frac{1}{m^2-t}+ \frac{1}{m^2-u}\Big)+\Big(\Big[\frac{s}{4}+\vec{q}^{\,2}\Big]R_{m21}+M^2R_{m41}\Big)\Big(\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big)\Big\}\cos\theta+\Big[3M^2R_{m30}-\vec{q}^{\,2}R_{m50}\Big]\Big[\frac{1}{m^2-t}- \frac{1}{m^2-u}\Big]+\Big[3M^2R_{m31}-\vec{q}^{\,2}R_{m51}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]\Big\}\Big|^2+\frac{1}{s}\Big|\sum_m\Big\{\Big[2M^2R_{20}+ \frac{s}{2}R_{m40}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]+\Big[2M^2R_{m21}+\frac{s}{2}R_{m41}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]+\vec{q}^{\,2}\Big[-R_{m10}+ 2R_{m30}+R_{m50}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]+\vec{q}^{\,2}\Big[-R_{m11}+2R_{m31}+R_{m51}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]\Big\}\Big|^2\times \cos^4\theta/2+\frac{1}{s}\Big|\sum_m\Big\{\Big[2M^2R_{m20}+\frac{s}{2}R_{m40}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]+\Big[2M^2R_{m21}+ \frac{s}{2}R_{m41}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]-\vec{q}^{\,2}\Big[-R_{m10}+2R_{m30}+R_{m50}\Big]\Big[\frac{1}{m^2-t}-\frac{1}{m^2-u}\Big]- \vec{q}^{\,2}\Big[-R_{m11}+2R_{m31}+R_{m51}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]\Big\}\Big|^2\sin^4\theta/2+ 4M^2\Big|\sum_m\Big\{\Big[R_{m20}+R_{m40}\Big]\Big[\frac{1}{m^2-t}+\frac{1}{m^2-u}\Big]+\Big[R_{m21}+R_{m41}\Big]\Big[\frac{1}{m^2-t}- \frac{1}{m^2-u}\Big]\Big\}\Big|^2\sin^2\theta\Big\} \tag{19}$$

In equations (18) and (19),

$$s = 4(\vec{q}\,^2 + M^2)$$

$$t = -2\vec{q}\,^2 (1-\cos\theta)$$

$$u = -2\vec{q}\,^2 (1+\cos\theta)$$

and the sum on m is taken from M_1 to M_{10}.

Equations (18) and (19) can be used to study the properties of mesons from nucleon-nucleon phenomena; or, when the masses and coupling constants are known, they can be used to determine the NN cross section. Because of the large number of parameters in equations (18) and (19), comparison with experiment must await further clarification of heavy mesons.

We have considered only the pole terms in equation (18) and (19). The effects of heavy mesons should include an evaluation of the spectral function in equation (22), Chapter V. We showed in Chapter IV that fourth order perturbation theory does not give adequate evaluation of the spectral function for scalar particles, and we expect it is not any better for spinor particles. A more complete evaluation than this is beyond the scope of this work.

TABLE 10

Residues, R_{mij}

i	j	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	0	$-\frac{(g_s^0)^2}{2}$	$-\frac{3(g_s^1)^2}{2}$	$-a^0 + b^0/2$	$-3a^1 + 3b^1/2$	0	0	0	0	0	0
2	0	$\frac{(g_s^0)^2}{4}$	$\frac{3(g_s^1)^2}{4}$	$-\frac{b^0}{4}$	$-\frac{3b^1}{4}$	0	0	$-\frac{(M_7 t_A^0)^2}{4}$	$-\frac{3(M_8 t_A^1)^2}{4}$	$\frac{(g_p^0)^2}{4}$	$\frac{3(g_p^1)^2}{4}$
3	0	0	0	$-\frac{a^0}{2}$	$-\frac{3a^1}{2}$	$-\frac{(f_A^0)^2}{2}$	$-\frac{3(f_A^1)^2}{2}$	0	0	0	0
4	0	$\frac{(g_s^0)^2}{4}$	$\frac{3(g_s^1)^2}{4}$	$-a^0/2 - b^0/4$	$-3a^0/2 - 3b^0/4$	$\frac{(f_A^0)^2}{2}$	$\frac{3(f_A^1)^2}{2}$	$\frac{(M_7 t_A^0)^2}{4}$	$\frac{3(M_8 t_A^1)^2}{4}$	$-\frac{(g_p^0)^2}{4}$	$-\frac{3(g_p^1)^2}{4}$
5	0	0	0	$a^0/4$	$+3\frac{a^1}{4}$	0	0	$\frac{(M_7 t_A^0)^2}{2}$	$\frac{3(M_8 t_A^1)^2}{2}$	$-\frac{(g_p^0)^2}{2}$	$-\frac{3(g_p^1)^2}{2}$
1	1	$\frac{(g_s^0)^2}{2}$	$-\frac{(g_s^1)^2}{2}$	$a^0 - b^0/2$	$-a^1 + b^1/2$	0	0	0	0	0	0
2	1	$-\frac{(g_s^0)^2}{4}$	$\frac{(g_s^1)^2}{4}$	$b^0/4$	$-b^1/4$	0	0	$\frac{(M_7 t_A^0)^2}{4}$	$-\frac{(M_8 t_A^1)^2}{4}$	$-\frac{(g_p^0)^2}{4}$	$\frac{(g_p^1)^2}{4}$
3	1	0	0	$a^0/2$	$-\frac{a^1}{2}$	$\frac{(f_A^0)^2}{2}$	$-\frac{(f_A^1)^2}{2}$	0	0	0	0
4	1	$-\frac{(g_s^0)^2}{4}$	$\frac{(g_s^1)^2}{4}$	$a^0/2 + b^0/4$	$-a^1/2 - b^1/4$	$-\frac{(f_A^0)^2}{2}$	$\frac{(f_A^1)^2}{2}$	$-\frac{(M_7 t_A^0)^2}{4}$	$\frac{(M_8 t_A^1)^2}{4}$	$\frac{(g_p^0)^2}{4}$	$-\frac{(g_p^1)^2}{4}$
5	1	0	0	$-a^0/4$	$+\frac{a^1}{4}$	0	0	$-\frac{(M_7 t_A^0)^2}{2}$	$\frac{(M_8 t_A^1)^2}{2}$	$\frac{(g_p^0)^2}{2}$	$-\frac{(g_p^1)^2}{2}$

$a^0 = (f_v^0 + 2M t_v^0)^2$ $\qquad$ $a^1 = (f_v^1 + 2M t_v^1)^2$

$b^0 = \{M_3^2 (t_v^0)^2 - 4M t_v^0 (f_v^0 + 2M t_v^0)\}$ $\qquad$ $b^1 = \{M_4^2 (t_v^1)^2 - 4M t_v^1 (f_v^1 + 2M t_v^1)\}$

APPENDIX

SINGULARITIES IN PERTURBATION THEORY

1. Integral Transforms[1]

Consider a function F(Z) defined in a region, R, of the complex Z-plane by:

$$F(Z) = \int_C g(W,Z)\ dW \tag{1}$$

C is a contour in the W-plane, as shown in Figure 7.

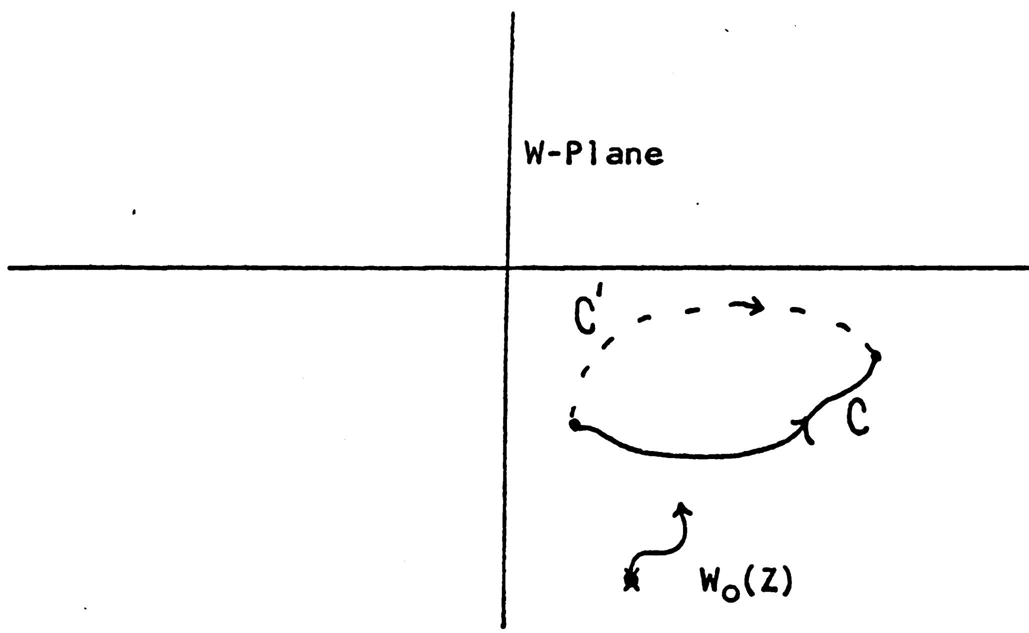

FIGURE 7

Contour Defining F(Z)

1. Our discussion follows that of R.J. Eden, Univ. of Maryland, Phys. Dept. Tech. Report #211(1961), and Jan Tarski, J. Math. Phys. 1, 149 (1960).

F(Z) is well defined and analytic in R, if there are no singularities of g(Z,W) in the neighborhood of C for Z in R.

F(Z) can be analytically continued into any region adjacent to R for which no singularities of g(Z,W) are in the neighborhood of C. Suppose that as a point Z_o is approached from a certain direction a singularity, W_o, of g(Z,W) approaches C. F(Z) can be analytically continued to Z_o by deforming the contour C to C' ahead of the approaching singularity, and without crossing any other singularity.

$$F(Z_o) = \int_{C'} g(Z_o, W)\, dW \qquad (2)$$

Applying the Cauchy theorem before the approaching singularity gets to C, we get:

$$\int_{C} g(Z,W)\, dW = \int_{C'} g(Z,W)\, dW \qquad (3)$$

F(Z) is analytic in the region away from Z_o and in the neighborhood of Z_o, and from equation (3) F(Z) is continuous across the boundary of the two regions.

There are two cases when the prescription for analytically continuing F(Z) breaks down:

1) <u>Pinching Singularities</u>

Two singularities of g(Z,W) approach the same point on C from opposite sides.

2). End Point Singularities

A singularity of g(Z,W) approaches either end point of C.

Polkinghorne and Screaton[2] show that the singularities of continued, multiple integral transforms are given by 1) or 2) applied to each variable of integration.

2. Perturbation Integrals

The general perturbation theory term for the Feynman amplitude of a graph with n internal lines is[3]

$$F_\epsilon = \int d^4K_1 \cdots \int d^4K_\ell \, \frac{B(q_i)}{\prod_{i=1}^{n}(q_i^2 - m_i^2 + i\epsilon)} \qquad (4)$$

ℓ is less than n since the vertex delta functions have been integrated out. The q_i are linearly related to the K_i and the external momenta p_j. $B(q_i)$ is a polynomial for particles with spin and a constant for scalar particles. It does not effect the singularities in either case[3] and we set it equal to one.

$$F_\epsilon = \int d^4K_1 \cdots \int d^4K_\ell \, \frac{1}{\prod_{i=1}^{n}(q_i^2 - m_i^2 + i\epsilon)} \qquad (5)$$

Using the identity,[4]

$$\frac{1}{u_1 \cdots u_n} = C \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n \, \frac{\delta(1 - \sum_1^n \alpha_i)}{\{\sum_1^n \alpha_i u_i\}^n}$$

2. J.C. Polkinghorne & G.R. Screaton, Nuovo Cimento <u>15</u>, 289 and 925 (1960).

3. For example, See J.Hamilton, "The Theory of Elementary Particles," Clarendon Press, London, 1959.

4. J.S.R. Chisholm, Proc. Camb. Phil.Soc.<u>48</u>,300 (1952).

We get:

$$F_\epsilon = C \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n \int d^4k_1 \cdots \int d^4 K_\ell \frac{\delta(1-\Sigma\alpha_i)}{\{\Psi_\epsilon(\alpha, K, p)\}^n} \tag{7}$$

where (8)

$$\Psi_\epsilon(\alpha, K, p) = \sum_1^n \alpha_i (q_i^2 - m_i^2 + i\epsilon)$$

We take the limit $\epsilon \rightarrow 0$, remembering the convention $+i\epsilon$, when defining the physical scattering amplitude. F has possibly the following singularities.

1) End Point Singularities

$$\Psi = 0$$
$$\alpha_i = 0, 1 \tag{9}$$
$$K_i = 0$$

2) Pinching Singularities

$$\Psi = 0$$

and (10)

$$\frac{\partial \Psi}{\partial \alpha_i} = 0$$

$$\frac{\partial \Psi}{\partial K_i} = 0$$

Either end point or pinching singularities must occur in all variables of integration. The form of Ψ is such that:

$$\Psi = \Sigma\, \alpha_i \frac{\partial \Psi}{\partial \alpha_i} \tag{11}$$

The end point conditions are redundant, since $\alpha_i = 1$ implies that $\frac{\partial \Psi}{\partial \alpha_i} = 0$. Now transform the K_i to put Ψ in canonical form.

$$K_i' = K_i + \ell_i \tag{12}$$

where ℓ_i is a linear combination of the $K_j (j \neq i)$ and

$$\Psi = \sum_1^{\ell} C_i(\alpha) K_i'^2 + D(\alpha, p) \tag{13}$$

Then (14)

$$\frac{\partial \Psi}{\partial K_i'} = 2 C_i(\alpha) K_i'$$

The pinching conditions are also redundant, since $K_i' = 0$ implies that $\frac{\partial \Psi}{\partial K_i'} = 0$. Since $\frac{\partial \Psi}{\partial K_i'} = \frac{\partial \Psi}{\partial K_i}$ we get the following necessary conditions for singularities of F.

$$1) \quad \frac{\partial \Psi}{\partial K_i} = 0 \qquad i = 1, \cdots, \ell$$

(15)

$$2) \text{ either } \left.\begin{array}{l} \alpha_j = 0 \\ \text{or} \quad q_j^2 = m_j^2 \end{array}\right\} j = 1, \cdots, n$$

Consider a closed loop within a perturbation graph.

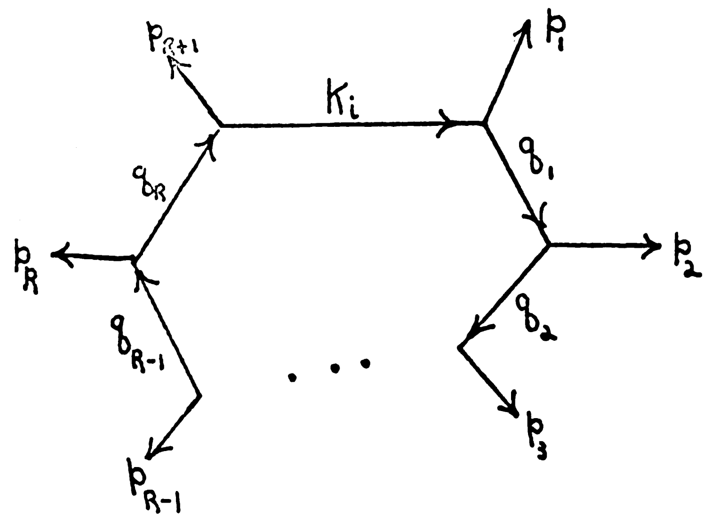

FIGURE 8

Closed Loop Within a Feynman Graph

The p's need not be single lines. In the evaluation of the Feynman amplitude we can choose K_i as shown in Figure 8. The R+1 four-momenta delta functions at the vertices give R equations relating the q_i, K_i, and p_i, and the condition $\sum_1^{R+1} p_i = 0$. Thus we write:

$$
\begin{aligned}
q_1 &= K_i - p_1 \\
q_2 &= K_i - p_1 - p_2 \\
q_R &= K_1 - \sum_1^R p_j
\end{aligned}
\tag{16}
$$

and

$$\frac{\partial \Psi}{\partial K_i} = \frac{\partial}{\partial K_i}\left\{\alpha_i(K_i^2 - m_i^2) + \sum_{j=1}^{R} \alpha_i(q_j^2 - m_j^2)\right\}$$

or (17)

$$\sum \alpha_i q_i = 0$$

Where the sum is taken around a closed loop in the graph. This gives the Landau-Bjorken[5] conditions for the singularities of a Feynman amplitude associated with a Feynman graph.

1) $\sum \alpha_i q_i = 0$, sum taken around closed loops in the graph.

2) $\left.\begin{array}{ll} \text{Either} & \alpha_i = 0 \\ \text{or} & q_i^2 = m_i^2 \end{array}\right\}$ $i = 1, \ldots, n.$

Notice that $\alpha_i = 0$ removes the line i from consideration, leading to a reduced diagram in which the line does not appear.

3. Normal Thresholds

Consider the fourth order diagram shown in Figure 9.

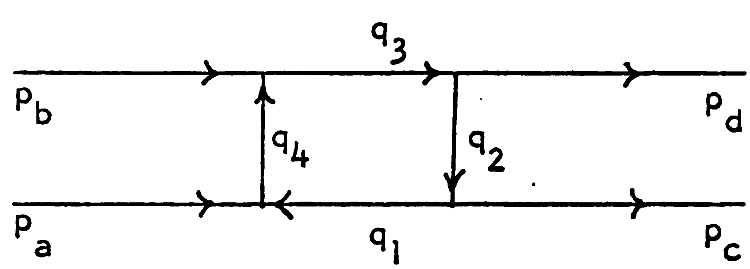

FIGURE 9

Fourth Order Box Graph

5. L.D. Landau, Nuc.Phys. 13, 181 (1959).

The Feynman amplitude of this graph has a singularity for the following conditions:

a) $\alpha_2 = \alpha_4 = 0$

b) $q_1^2 = m_1^2$

$q_3^2 = m_3^2$

c) $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4 = 0$

From conditions a) and c), we get:

$$\alpha_1 q_1 + \alpha_3 q_3 = 0 \tag{18}$$

Taking the scalar product of equation (18) with q_1 and q_3, we get:

$$\alpha_1 q_1^2 + \alpha_3 q_1 \cdot q_3 = 0$$

$$\alpha_1 q_1 \cdot q_3 + \alpha_3 q_3^2 = 0$$

From condition b), we get:

$$\begin{aligned} \alpha_1 m_1^2 + \alpha_3 q_1 \cdot q_3 &= 0 \\ \alpha_1 q_1 \cdot q_3 + \alpha_3 m_3^2 &= 0 \end{aligned} \tag{19}$$

For non zero solutions of these equations,

$$\begin{vmatrix} m_1^2 & q_1 \cdot q_3 \\ q_1 \cdot q_3 & m_3 \end{vmatrix} = 0 \tag{20}$$

$$m_1^2 m_3^2 - (q_1 \cdot q_3)^2 = 0$$

$$q_1 \cdot q_3 = \pm m_1 m_3 \ .$$

The positive solution leads to negative values of α , which are not on the contour.[6] So there is only one singularity from equations (20).

$$q_1 \cdot q_3 = -m_1 m_3$$

Since $s = (p_a + p_b)^2 = (q_1 - q_3)^2$, this singularity is at

$$s = (q_i - q_3)^2 = m_1^2 + m_3^2 - 2q_i q_3 = (m_1 + m_3)^2 \tag{21}$$

4. Anomalous Thresholds

Consider the singularity of the Feynman amplitude for Figure 9 for the following conditions.

a) $\alpha_4 = 0$

b) $q_1^2 = m_1^2$

$q_2^2 = m_2^2.$

$q_3^2 = m_3^2$

c) $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0$

Condition c) gives:

$$\begin{aligned} \alpha_1 m_1^2 + \alpha_2 q_1 \cdot q_2 + \alpha_3 q_1 \cdot q_3 &= 0 \\ \alpha_1 q_1 \cdot q_2 + \alpha_2 m_2^2 + \alpha_3 q_2 \cdot q_3 &= 0 \\ \alpha_1 q_1 \cdot q_3 + \alpha_2 q_2 \cdot q_3 + \alpha_3 m_3^2 &= 0 \end{aligned} \tag{22}$$

Defining $y_{ij} = q_i \cdot q_j / m_i m_j$ we get non zero solutions of equations (22), only if

$$\begin{vmatrix} 1 & y_{12} & y_{13} \\ y_{12} & 1 & y_{23} \\ y_{13} & y_{23} & 1 \end{vmatrix} = 0 \tag{23}$$

$$y_{13} = y_{12} y_{13} \pm \{(1 - y_{12}^2)(1 - y_{23}^2)\}^{1/2}$$

6. This needs further justification by a detailed study of the method of continuation. See References 1,2,or 5.

If none of the particles in Figure 9 can energetically decay into the other two at the same vertex, we get[7]

$$-1 \leq Y_{12} \leq 1$$
$$-1 \leq Y_{23} \leq 1 \qquad (24)$$

Equations (24) and the positive solution of equation (23) give negative values of the α. For positive α the negative solution of (23) must be taken and the following condition must hold.

$$Y_{12}\{1-Y_{23}^2\}^{1/2} + Y_{23}\{1-Y_{12}^2\}^{1/2} < 0 \qquad (25)$$

and $$Y_{13} = Y_{12}Y_{23} - \{(1-Y_{12}^2)(1-Y_{23}^2)\}^{1/2} \qquad (26)$$

In terms of the Y_{ij}, s is

$$s = m_1^2 + m_3^2 - 2\,Y_{13}\,m_1 m_3 .$$

For Y_{13} given by equation (26), $Y_{13} > -1$ and the threshold is below $s = (m_1 + m_3)^2$, in contradiction to the results of Chapter III. This is an anomalous

7. See Reference 1.

threshold, and it will not appear unless equation (25) is satisfied. Expressing equation (25) in terms of the masses of the particles in Figure 9, we get the following results:

1) If $m_3^2 + m_4^2 < M_d^2$ and $m_1^2 + m_4^2 > M_a^2$, there is an anomalous threshold, if

$$M_d^2 - m_3^2 - m_4^2 > m_1^2 + m_4^2 - M_a^2 .$$

2) If $m_3^2 + m_4^2 < M_d^2$ and $m_1^2 + m_4^2 < M_a^2$, there is an anomalous threshold.

3) If neither 1) or 2) are satisfied there is no anomalous threshold.

If the internal masses of Figure 9 are much smaller than the exterior masses singularities occur for complex values of s.

5. Fourth Order Diagrams

Consider the singularities of the amplitude for Figure 9, for

$$q_i^2 = m_i^2 , \qquad i = 1, 2, 3, 4, \qquad (27)$$
$$\sum_1^4 \alpha_i \gamma_i = 0$$

This gives the following condition for singularities:

$$\begin{vmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{12} & 1 & y_{23} & y_{24} \\ y_{13} & y_{23} & 1 & y_{34} \\ y_{14} & y_{24} & y_{34} & 1 \end{vmatrix} = 0 \tag{28}$$

where

$$\begin{aligned}
s &= (p_a+p_b)^2 = (q_1-q_3)^2 = m_1^2+m_3^2-2y_{13}m_1m_3 \\
t &= (p_a-p_c)^2 = (q_2-q_4)^2 = m_2^2+m_4^2-2y_{24}m_2m_4 \\
p_a^2 &= M_a^2 = (q_3-q_4)^2 = m_3^2+m_4^2-2y_{34}m_3m_4 \qquad (29) \\
p_b^2 &= M_b^2 = (q_4-q_1)^2 = m_1^2+m_4^2-2y_{14}m_1m_4 \\
p_c^2 &= M_c^2 = (q_2-q_1)^2 = m_1^2+m_2^2-2y_{12}m_1m_2 \\
p_d^2 &= M_d^2 = (q_3-q_2)^2 = m_2^2+m_3^2-2y_{23}m_2m_3
\end{aligned}$$

For the case $M_a = M_b = M_c = M_d = m_1 = m_3 = M$, equations (28) and (29) give singularities of the Feynman amplitude for s and t such that:

$$\begin{aligned}
&(s-4M^2)\left(t^2-2(m_2^2+m_4^2)t+(m_2^2-m_4^2)^2\right)-4m_2m_4t=0 \\
&s > 4M^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (30) \\
&t > (m_2+m_4)^2
\end{aligned}$$

FIGURE 10

Singularities in Fourth Order Perturbation Theory

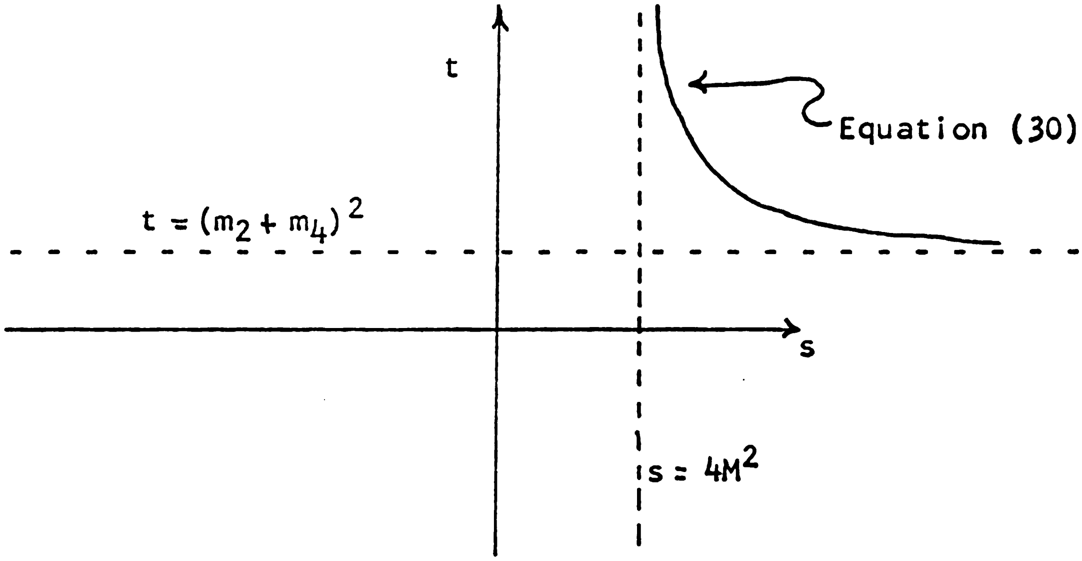

Figure 10 shows all the singularities for Figure 9, in the case $M_a = M_b = M_c = M_d = m_1 = m_3 = M$. The dashed lines are normal thresholds and the solid curve is the solution of equation (30). For this case there are no anomalous thresholds or complex singularities.[8]

6. Generalized Unitarity

Cutkosky[9] has shown that the discontinuity of the Feynman across a cut starting at one of the singularities discussed above is given as follows: In equation (4), for each pinching singularity $1/(q_i^2 - m_i^2)$ is replaced by $2\pi i\, \delta(q_i^2 - m_i^2)$

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8. See Reference 1.

9. R.E. Cutkosky, J. Math. Phys. 1, 429 (1960).

Discontinuity $F =$

$$(2\pi i)^p \int d^4K_1 \cdots d^4K_\ell \, \frac{\delta(q_1^2-m_1^2)\cdots\delta(q_p^2-m_p^2)}{(q_{p+1}^2-m_{p+1}^2)\cdots(q_n^2-m_n^2)} \tag{31}$$

p is the number of pinching singularities. For example, the discontinuity for a normal threshold (equation (20)) is:

Discontinuity $F =$

$$-4\pi^2 \int d^4K \, \frac{\delta(q_2^2-m_2^2)\,\delta(q_4^2-m_4^2)}{(q_1^2-m_1^2)(q_3^2-m_3^2)} \tag{32}$$

The discontinuity for an anomalous threshold (equation (26)) is:

Discontinuity $F =$

$$-8\pi^3 i \int d^4K \, \delta(q_1^2-m_1^2)\,\delta(q_2^2-m_2^2)\,\delta(q_3^2-m_3^2)/(q_4^2-m_4^2) \tag{33}$$

The discontinuity for equation (30) is:

Discontinuity $F =$

$$16\pi^4 \int d^4K \, \delta(q_1^2-m_1^2)\,\delta(q_2^2-m_2^2)\,\delta(q_3^2-m_3^2)\,\delta(q_4^2-m_4^2) \tag{34}$$

$$= \frac{\pi^2}{M^2 m_2 m_4} \begin{vmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{12} & 1 & y_{23} & y_{24} \\ y_{13} & y_{23} & 1 & y_{34} \\ y_{14} & y_{24} & y_{34} & 1 \end{vmatrix}^{-1/2}$$

7. Crossing Relations

Consider the Feynman amplitude for the graph in Figure 9, representing the reaction $A+B \rightarrow C+D$.

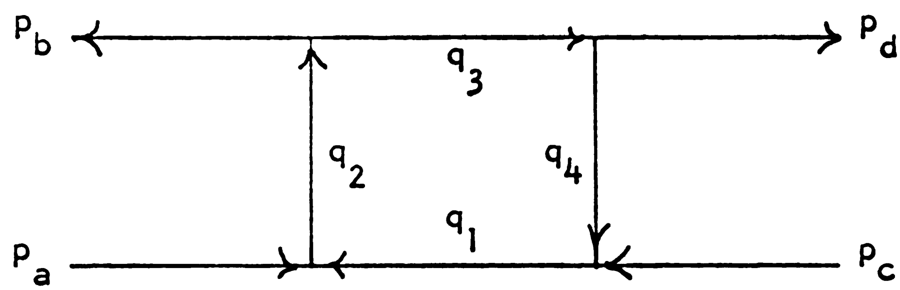
$$F_I = \frac{1}{\delta^4(p_a+p_b-p_c-p_d)} \int d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4 \; \delta^4(p_a+q_1-q_4)\,\delta(p_a+q_4-q_3) \times \tag{35}$$

$$\frac{\delta^4(p_c+q_1-q_2)\,\delta^4(p_d+q_2-q_3)}{\prod_{i=1}^{4}(q_i^2-m_i^2)}$$

Now consider the Feynman amplitude for the graph in Figure 11, representing the reaction $A + \bar{C} \rightarrow \bar{B} + D$.

FIGURE 11

Crossed Box Diagram



$$F_{III} = \frac{1}{\delta^4(p_a+p_c-p_b-p_d)} \int d^4q_1 \cdots d^4q_4 \, \frac{\delta^4(p_a+q_1-q_4)\,\delta^4(-p_b+q_4-q_3)\,\delta^4(-p_c+q_1-q_2)\,\delta^4(p_d+q_2-q_3)}{\prod_{i=1}^{4}(q_i^2-m_i^2)} \tag{36}$$

By inspection of equations (35) and (36) F_{III} can be obtained from F_I by reversing the signs of p_b and p_c. Thus, F_I and F_{III} are the same function of p_a, p_b, p_c, and p_d and consequently they are the same function of s, t, and u. However, the connection between s, t, and u and the center of mass energy and scattering angle must be different for F_I and F_{III}. s, t, and u for F_{III} are:

$$s = (p_A - p_B)^2$$

$$t = (p_A - p_D)^2 \tag{37}$$

$$u = (p_A + p_C)^2$$

The center of mass conditions for equal masses are:

$$|\vec{p}_A| = |\vec{p}_B| = |\vec{p}_C| = |\vec{p}_D| = |\vec{q}|$$

$$p_{oA} = p_{oB} = p_{oC} = p_{oD} = (\vec{q}^2 + M^2)^{1/2} \tag{38}$$

$$\vec{q}_A + \vec{q}_C = \vec{q}_B + \vec{q}_D$$

The equations corresponding to (37) and (38) for F_I are equations (4) and (6), Chapter IV. Equations (37) and (38) give the results shown in Table 1. Lehmann, Symanzik and Zimmermann,[10] and Goldberger, Nambu and Oehme[11] derive crossing relations without reference to perturbation theory.

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10. H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 1, 205 (1955), and 6, 319 (1957).
11. M.L. Goldberger, Y. Nambu, and R. Oehme, Ann. Phys. 2, 226 (1957).