PROPAGATION OF A SOUND PULSE IN A MEDIUM WITH A COMPLEX ELASTIC MODULUS

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By

SALAH IZZAT TAHSIN

A THESIS

Submitted to the School of Graduate Studies of Michigan
State College of Agriculture and Applied Science
in partial fulfillment of the requirements
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AN ABSTRACT

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ABSTRACT

In the first part of this thesis, the physical causality principle along with the assumption that a real (not complex) cause gives rise to a real effect, is used to derive the following properties of the complex propagation constant $k(u)$.

1. It is analytic in the lower half of the complex u-plane.

2. $\lim k(u) = \frac{u}{c}$ as $u \to \infty$.

3. Its real and imaginary parts satisfy the Kronig-Kramers relations.

4. It possesses symmetry properties; i.e.,

$$\overline{k(u)} = -k(-\bar{u}) \quad .$$

where the bar represents the complex conjugate.

The second part of this thesis discusses the propagation of a plane wave pulse in an infinite, homogeneous, and isotropic medium whose elastic modulus is assumed to be represented by a simple form consistent with the relaxation theory of the elastic moduli.

Since the Fourier integral which occurs in this problem cannot be evaluated exactly, two approximate methods are employed to find the shape of the pulse. The precursors of the pulse are shown to be exponential in form, negative for $\omega_0 \ll \frac{\sqrt{a}}{\tau}$ and positive for

$\omega_0 \gg \frac{\sqrt{a}}{\tau}$, where (a) is the amplitude of the relaxing part of the modulus and (τ) is the relaxation time.

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INTRODUCTION

This thesis is divided into two parts. In Part I we shall find some properties of the propagation constant applying the causality principle and the assumption that a real (not complex) cause gives use to a real effect. In Part II we shall discuss the propagation of a plane sound wave in a dispersive medium, characterized by a complex propagation constant.

Before we start the work on this thesis, we would like to give a brief account of what has been done in connection with the principles we are going to use.

The physical causality principle, namely that the effect cannot precede the cause, has been, in various equivalent forms, the reason in establishing some properties of many physical constants. Upon the suggestion of Krõnig (1), Shützer and Tiomno (2) analyzed the relation between causality and the scattering matrix S. They found that, for the causality principle to apply, the analytic function S(k) must have its singularities either in the lower half plane or on the imaginary axis. The same principle is believed to be the deep cause of the properties of Wigner's function (3) R (the reciprocal logarithmic derivative of the wave function). Van Kampen (4) worked on the same relation and found more properties of the S-matrix for the scattering of the electromagnetic field by a fixed center and for nonrelativistic particles. E. Hiedemann and R. D. Spence (5), by applying the causality principle and using function theory methods, obtained the Kronig-Kramers relations (6) between the real and imaginary parts of the complex elastic modulus K. They further found that the singularities of K must lie in the upper half of the complex frequency plane.

Sommerfeld (7) and Brillouin (8) discussed the propagation of light in dispersive media and established the presence of "precursors" or forerunners to the main pulse. They showed that the first precursor travels in any optical medium with the velocity of light in vacuum. The propagation function for light waves used by Brillouin leads to pure resonance phenomena. In our work in Part II we will use a propagation constant for sound waves that lead to pure relaxation phenomena. The mathematical difference between the two phenomena will be discussed at the end of Part I.

PART I

FUNCTION THEORETICAL CONDITIONS ON THE PROPAGATION CONSTANT OF A SOUND WAVE

In this portion we define the propagation constant and present a discussion of its most general properties.

Let a pulse be applied at the plane x = o in an infinite, homogeneous and isotropic medium. The amplitude of the pulse in this plane is given by

$$\begin{aligned} f &= 0 & t &< 0 \\ f &= f(0,t) & t &\geq 0 \end{aligned} \tag{1}$$

where $f(0,t)$ is a real quantity. The frequency spectrum of the pulse is then.

$$\varphi(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(0,t)\, e^{-iut}\, dt \tag{2}$$

where (u) represents the complex frequency $\omega + i\nu$ which is introduced to insure convergence of the Fourier representations used in this thesis and to facilitate a discussion of the problem in terms of the usual methods of the theory of the complex variable.

By application of Cauchy's theorem and Jordan's lemma it can be shown that (1) implies that in the lower half plane $\varphi(u)$ is

analytic and tends uniformly to zero as $u \to \infty$. The statement that $f(0,t)$ is real leads to the requirement that $\overline{\varphi(u)} = \varphi(-\bar{u})$ where the bar denotes the complex conjugate.

The amplitude of the pulse which arrives at the plane $x \neq 0$ can be found by superposing all the plane waves which belong to the various frequencies in the original pulse. These plane waves are of the form $e^{i[ut-k(u)x]}$ where $k(u)$ is the propagation constant appropriate to the frequency (u). Thus the amplitude of the pulse in the plane $x \neq 0$ is given by

$$f(x,t) = \int_{-\infty-i\alpha}^{+\infty-i\alpha} \varphi(u)\, e^{i[ut-k(u)x]}\, du \tag{3}$$

If we set

$$\varphi(u) = \varphi'(u) + i\varphi''(u)$$

then

$$f(0,t) = \int_{-\infty-i\alpha}^{+\infty-i\alpha} [\varphi'(u)\cos ut - \varphi''(u)\sin ut]\, du + i\int_{-\infty-i\alpha}^{+\infty-i\alpha} [\varphi'(u)\sin ut + \varphi''(u)\cos ut]\, du \tag{4}$$

The response must be real since the pulse is real which demands the vanishing of the second integral in the last expression, thus

$$\int_{-\infty-i\alpha}^{+\infty-i\alpha}\left[\varphi'(u)\sin ut+\varphi''(u)\cos ut\right]du = 0 \tag{5}$$

This in turn requires that

$$\varphi'(\omega,\nu) = \varphi'(-\omega,\nu)$$

$$\varphi''(\omega,\nu) = -\varphi''(-\omega,\nu)$$

This is the condition for a real pulse at x = o.

Now let

$$\varphi(u)\,e^{iut} = \psi(u) = \psi'(u) + i\psi''(u)$$

where

$$\psi'(u) = \varphi'(u)\cos ut - \varphi''(u)\sin ut$$

$$\psi''(u) = \varphi'(u)\sin ut + \varphi''(u)\cos ut$$

and $\psi'(\omega,\nu) = \psi'(-\omega,\nu)\ ,\quad \psi''(\omega,\nu) = -\psi''(-\omega,\nu)$

Thus equation (3) becomes

$$f(x,t) = \int_{-\infty-i\alpha}^{+\infty-i\alpha}\psi(u)\,e^{-ik(u)x}\,du$$

$$= \int_{-\infty-i\alpha}^{+\infty-i\alpha}\left[\psi'(u)\cos kx+\psi''(u)\sin kx\right]du + i\int_{-\infty-i\alpha}^{+\infty-i\alpha}\left[\psi''(u)\cos kx-\psi'(u)\sin kx\right]du \tag{6}$$

Let $k(u) = k'(u) - i\,k''(u)$ (7)

Substituting (7) in (6) we obtain

$$f(x,t) = \int_{-\infty-i\alpha}^{+\infty-i\alpha} [\psi' \cos(k'x)\cosh(k''x) + \psi'' \sin(k'x)\cosh(k''x)$$

$$- \psi'' \sin(k'x)\sinh(k''x) - \psi' \cos(k'x)\sinh(k''x)]\,du$$

$$+ i \int_{-\infty-i\alpha}^{+\infty-i\alpha} [\psi' \sin(k'x)\sinh(k''x) - \psi'' \cos(k'x)\sinh(k''x)$$

$$+ \psi'' \cos(k'x)\cosh(k''x) - \psi' \sin(k'x)\cosh(k''x)]\,du$$

Requiring a real response as before leads to the following properties

$$k'(\omega, \nu) = -k'(-\omega, \nu)$$
$$k''(\omega, \nu) = k''(-\omega, \nu)$$

However, if we expand $k(u)$ in a power series

$$k(u) = \frac{u}{c} K(u) = \frac{u}{c}[K'(u) - i K''(u)] \tag{8}$$

where

$$K'(u) = 1 + \frac{b_2}{u^2} + \frac{b_4}{u^4} + \cdots\cdots$$
$$K''(u) = \frac{b_1}{u} + \frac{b_3}{u^3} + \frac{b_5}{u^5} + \cdots\cdots$$

then

$$K'(\omega, \nu) = K'(-\omega, \nu)$$
$$K''(\omega, \nu) = -K''(-\omega, \nu)$$

Hence $$\overline{K(u)} = K(-\bar{u}) .$$

At a certain plane $X_0 \neq 0$ there is a $T \neq 0$ such that the response is zero for $t < T$, then

$$\int_{-\infty - i\alpha}^{+\infty - i\alpha} \varphi(u)\, e^{-ik(u)x_0}\, e^{iut}\, du = 0 \qquad t < T \tag{9}$$

If we make the substitution

$$\mu = t - T$$

the last equation becomes

$$\int_{-\infty - i\alpha}^{+\infty - i\alpha} \varphi(u)\, e^{-ik(u)x_0}\, e^{-iu|\mu| + iuT}\, du = 0$$

Now we set $X_0 = cT$, where (c) is an undetermined constant, to get the integral in the form

$$\int_{-\infty - i\alpha}^{+\infty - i\alpha} \varphi(u)\, e^{iu[1 - \frac{c}{u}k(u)]T}\, e^{-iu|\mu|}\, du = 0 \tag{10}$$

The condition that this integral vanish requires that

(a) $\varphi(u)\, e^{iu[1 - \frac{c}{u}k(u)]T}$ is analytic in the lower half-plane.

$$\text{(b)} \quad \int_C \varphi(u)\, e^{iu[1 - \frac{c}{u}k(u)]T}\, e^{-iu|\mu|}\, du = 0 \tag{11}$$

where C is an infinite semicircle in the lower half plane.

From (b), on the application of Jordan's lemma, we find that integral (10) is satisfied if $\varphi(u)e^{iu[1-\frac{c}{u}k(u)]T}$ is analytic within C and tends to zero uniformly as $u \to \infty$ in C. But we already stated that

$$\varphi(u) \xrightarrow[u\to\infty]{} 0$$

in the same region. Therefore

$$e^{iu[1-\frac{c}{u}k(u)]T} \xrightarrow[u\to\infty]{} \text{finite constant}$$

and

$$iu[1-\tfrac{c}{u}k(u)]T \xrightarrow[u\to\infty]{} iD$$

where D is also a finite constant. This leads us to the result

$$k(u) \xrightarrow[u\to\infty]{} \frac{u}{c} - \frac{D}{cT} \longrightarrow \frac{u}{c} \tag{12}$$

which shows that the undetermined constant (c) is the velocity of propagation as the frequency tends to infinity. By virtue of (8), (12) shows also that

$$K(u) \xrightarrow[u\to\infty]{} 1 \tag{12a}$$

Let

$$e^{iu[1-\frac{c}{u}k(u)]T} = e^{iF(u)} = e^{i[\alpha(\omega,\nu)+i\beta(\omega,\nu)]}$$

$$= e^{-\beta}\cos\alpha + ie^{-\beta}\sin\alpha = \lambda' + i\lambda''$$

Upon the application of the Cauchy-Riemann conditions on λ' and λ'', and after some simple algebra, we obtain

$$\frac{\partial\alpha}{\partial\omega} = \frac{\partial\beta}{\partial\nu} \qquad \text{and} \qquad \frac{\partial\alpha}{\partial\nu} = -\frac{\partial\beta}{\partial\omega} ,$$

which proves the analyticity of $F(u)$. We can also write

$$k(u)\,T = u\,T - F(u)$$

to show that $k(u)$ is analytic, since the sum of analytic functions is analytic.

Making use of relation (8) and following the method of E. Hiedemann and R. D. Spence (5), we obtain the following:

$$\frac{K'(\omega)}{\omega} = \frac{1}{\pi}\int_0^\infty \frac{K''(u)}{u} \cdot \frac{u}{(u^2-\omega^2)}\,du \tag{13a}$$

$$\frac{K''(\omega)}{\omega} = -\frac{1}{\pi}\int_0^\infty \frac{K'(u)}{u} \cdot \frac{u}{(u^2-\omega^2)}\,du \tag{13b}$$

$$K(\infty) = 1 = \frac{1}{\pi}\int_0^\infty \frac{K''(u)}{u}\,du \tag{14}$$

$$K'(u) = K(\infty) + \frac{1}{\pi}\int_0^\infty K''(u)\,\frac{u}{u^2-\omega^2}\,du \tag{15}$$

The above four forms of the Kronig-Kramer relations can be supplemented by the following four forms if we assume that $K(u)$ possesses no singularities on the real frequency axis other than the origin

$$\frac{K'(\omega)}{\omega} = \frac{K(0)}{\omega} + \frac{2}{\pi} P \int_0^\infty \frac{K''(u)}{u} \cdot \frac{\omega}{(u^2-\omega^2)} \, du \tag{16a}$$

$$\frac{K''(\omega)}{\omega} = -\frac{2}{\pi} P \int_0^\infty \frac{K'(u)}{u} \cdot \frac{u}{(u^2-\omega^2)} \, du \tag{16b}$$

$$K(\infty) - K(0) = -\frac{2}{\pi} \int_0^\infty \frac{K'(u)}{u} \, du \tag{17}$$

$$K'(\omega) = K(\infty) + \frac{2}{\pi} P \int_0^\infty K''(u) \frac{u}{(u^2-\omega^2)} \, du \tag{18}$$

where (P) stands for the Cauchy principal value.

Equations (13a), (13b), (14), and (15), as well as the last four relations, make clear the reciprocal dependence of the real and imaginary parts of $K(u)$.

The singularities of $K(u)$ have been shown to lie in the upper half plane. The effect of such singularities on the propagation constant may conveniently be discussed on the basis of whether they lie on or off the positive imaginary axis.

The singularities which lie off the imaginary axis physically represent resonance phenomena such as occurs in dispersion of light waves in optical media and which are almost never present

in sonic media. The singularities which occur on the positive imaginary axis represent relaxation processes and lead to the type of dispersion found in many sonic problems and in electromagnetic waves of long wavelength. Throughout the remainder of this thesis we shall assume that all the singularities of $K(u)$ lie on the positive imaginary axis.

PART II

THE PROPAGATION OF A SOUND PULSE IN A RELAXING MEDIUM

In this portion of the thesis we consider the propagation of a plane wave pulse in a medium characterized by a complex propagation constant which arises from a complex elastic modulus.

We assume, as in Part I, an infinite, homogeneous and isotropic medium. The pulse applied at the plane $X = 0$ is given by

$$f(t) = 0 \qquad t < 0 \; , \; t > \Delta$$

$$f(t) = \sin \omega_0 t \qquad 0 \leq t \leq \Delta \; .$$

where $\omega_0 = \frac{2\pi}{p}$ and $\Delta = \frac{2\pi m}{\omega_0} = mp$,

p is the period and m is an integer.

The Fourier transform of $f(t)$ is

$$\varphi(u) = \int_0^\Delta \sin \omega_0 t \, e^{-iut} \, dt$$

$$= \int_0^\Delta \frac{e^{i(\omega_0 - u)t} - e^{-i(\omega_0 + u)t}}{2i} \, dt \qquad (19)$$

$$= \frac{1}{2i}\left[\frac{e^{2\pi i m} e^{-\frac{2\pi i m}{\omega_0} u} - 1}{i(\omega_0 - u)} + \frac{e^{-2\pi i m} e^{-\frac{2\pi i m}{\omega_0} u} - 1}{i(\omega_0 + u)}\right]$$

$$\varphi(u) = \omega_0 \, \frac{e^{-i\frac{2\pi m}{\omega_0}u} - 1}{u^2 - \omega_0^2}$$

The amplitude of the pulse observed at a point $X \neq 0$ at a time (t) is given by

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(u)\, e^{i[ut - k(u)x]}\, du$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\omega_0}{u^2-\omega_0^2}\left(e^{i[ut - k(u)x]} - e^{i[u(t-\Delta) - k(u)x]}\right) du \tag{20}$$

$$= I_1 + I_2$$

To find an expression for $k(u)$ we assume that the relaxing modulus of elasticity may be represented by

$$M(u) = M(\infty)\left(1 - \frac{a}{1+iu\tau}\right)$$

Where (a) is a constant characteristic of the medium and (τ) is the relaxation time. This is the simplest form of the complex modulus which can be employed to exhibit the effect of relaxation on the propagation of the pulse. More general representations would include the contributions from a spectrum (discrete and/or continuous) of relaxation times. In actual practice it is often found that the previous expression for the modulus suffices over a fairly wide frequency range.

As was shown previously, (c) is the velocity of propagation as (u) tends to infinity, thus we write

$$c = \sqrt{\frac{M(\infty)}{D}} \qquad \text{and} \qquad v = \sqrt{\frac{M(u)}{D}}$$

where D is the density of the medium and (v) is the complex velocity associated with the frequency (u).

By definition the complex propagation is

$$k(u) = \frac{u}{v} = \frac{u}{c} \cdot \frac{1}{\sqrt{1 - \frac{a}{1+iu\tau}}} = \frac{u}{c}\sqrt{\frac{1+iu\tau}{(1-a)+iu\tau}} \tag{21}$$

This form of $k(u)$ satisfies the restrictions which were developed in the first portion of this thesis. From (8) and (21) we get

$$K(u) = \sqrt{\frac{1+iu\tau}{(1-a)+iu\tau}}$$

which is analytic in the lower half-plane. And

$$\lim_{u \to \infty} K(u) = 1 \tag{22}$$

It possesses two branch points on the positive imaginary axis. Furthermore, $\overline{K(u)} = K(-\bar{u})$.

Evaluation of Integral (20)

Making use of (22), the exponent in I_1, equation (20), becomes $iu(t - \frac{x}{c})$ as $u \to \infty$, it has a negative real part,

in the lower half-plane when $t < \frac{x}{c}$

in the upper half-plane when $t > \frac{x}{c}$.

Similar results apply for $iu(\delta - \frac{x}{c})$, $\delta = t - \Delta$.

When $t < \frac{x}{c}$

Both I_1 and I_2 vanish along a path deep below the real axis.

When $\frac{x}{c} < t < \frac{x}{c} + \Delta$

In this case the integration path has to be moved up to the upper half-plane. In so doing, the path encounters the singularities at $u = \pm\omega_0$ and the branch cut and branch points, if any, of the function $k(u)$. These, as was assumed earlier, lie on the positive imaginary axis. Figures 1 and 2 show the paths for I_1 and I_2. The contribution of those parts of the path around the singularities $\pm\omega_0$ are easily found by Cauchy's theorem. For this case we obtain

$$I_1 = e^{-k''(\omega_0)x} \sin\left[\omega_0 t - k'(\omega_0)\right] + \frac{1}{2\pi}\int_{C'} \frac{(-\omega_0)}{u^2 - \omega_0^2} \, e^{i(ut - kx)} \, du \qquad (23)$$

$$I_2 = 0$$

When $t > \frac{x}{c} + \Delta$

In this case I_1 still has the value (23), and I_2 becomes

$$I_2 = -e^{-k''(\omega_0)x} \sin\left[\omega_0 t - k'(\omega_0)x\right] - \frac{1}{2\pi}\int_{C'} \frac{(-\omega_0)}{u^2 - \omega_0^2} \, e^{i(u\delta - kx)} \, du \qquad (24)$$

Thus in this case the main pulse cancels. Only the integrals in (23) and (24) remain. The integral in (23) will be discussed fully later. The integral in (24), however, starts when the main pulse

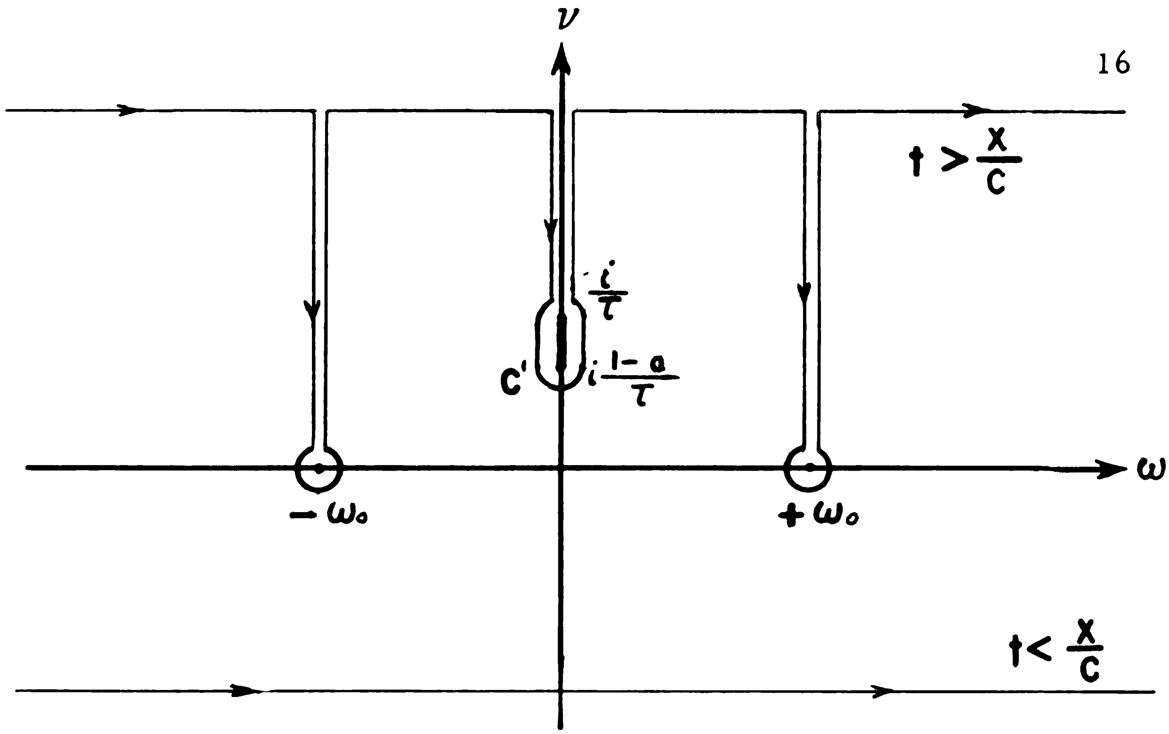

Figure 1. The integration path for I_1 , eq. (20).

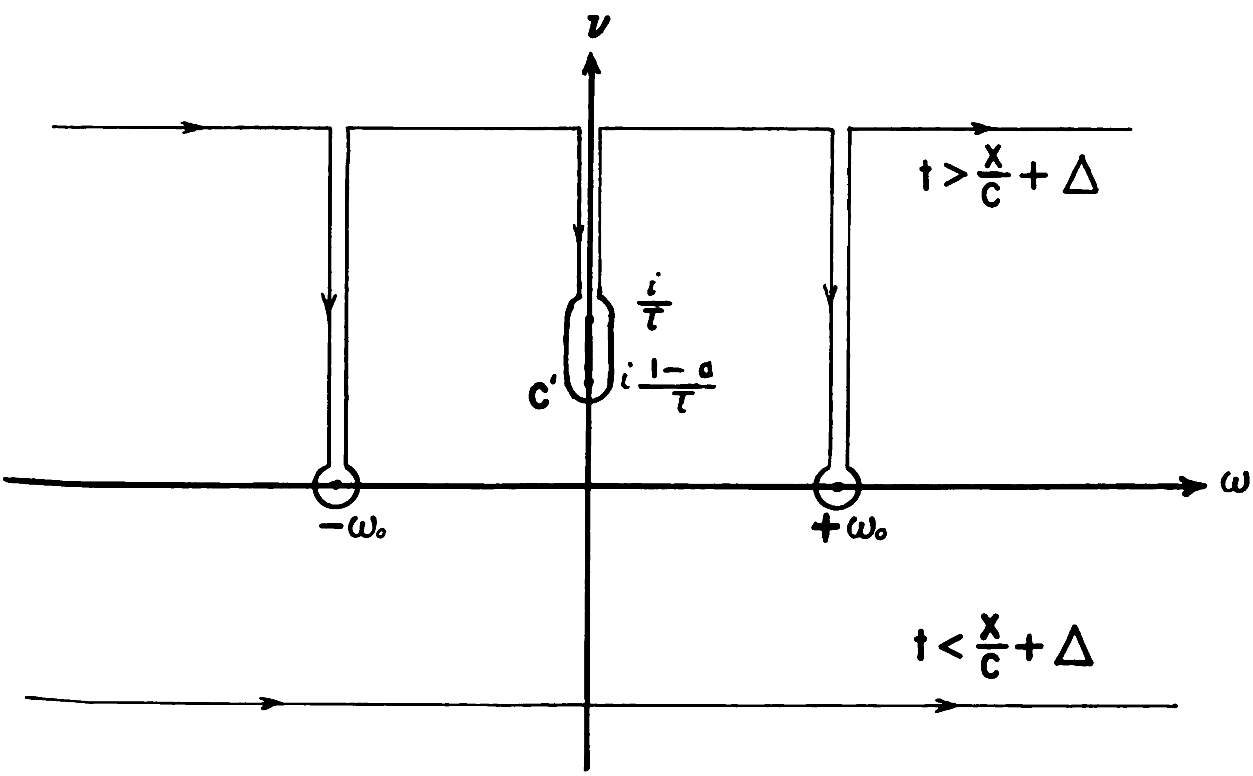

Figure 2. The integration path for I_2 , eq. (20).

vanishes and is of the same form as the integral in (23) except for the sign. Since we are mainly interested in the manner of build-up of the pulse, we shall, in the following, devote our attention to the integral in (23), namely

$$\frac{1}{2\pi}\int_{C'} \frac{(-\omega_0)}{u^2-\omega_0^2} \quad e^{i(ut-kx)} \, du \tag{25}$$

The contour C' can be deformed into other paths, such as the one shown in Figure 3, after reorienting the branch cut.

However, the evaluation will not be carried out along C' or the latter alternative path. Instead two approximate methods will be used. One for $\omega_0 \ll \frac{\sqrt{a}}{\tau}$ and the other for $\omega_0 \gg \frac{\sqrt{a}}{\tau}$.

Case I. $\quad \omega_0 \ll \frac{\sqrt{a}}{\tau}$

We first make the following changes in our variable

$$\xi = u\tau = \rho + i\eta \qquad ,$$

and set

$$ut - kx = \frac{\xi}{\tau}(t - \frac{x}{c}) - (k - \frac{\xi}{c\tau})x = \frac{\xi}{\tau}t^* - k^*x \qquad ,$$

where $\quad t^* = t - \frac{x}{c} \quad$ and $\quad k^* = k - \frac{\xi}{c\tau} = \frac{\xi}{c\tau}\left(\sqrt{\frac{1+i\xi}{(1-\alpha)+i\xi}} - 1\right) \quad .$

Now consider integral (25) in terms of the new variable.

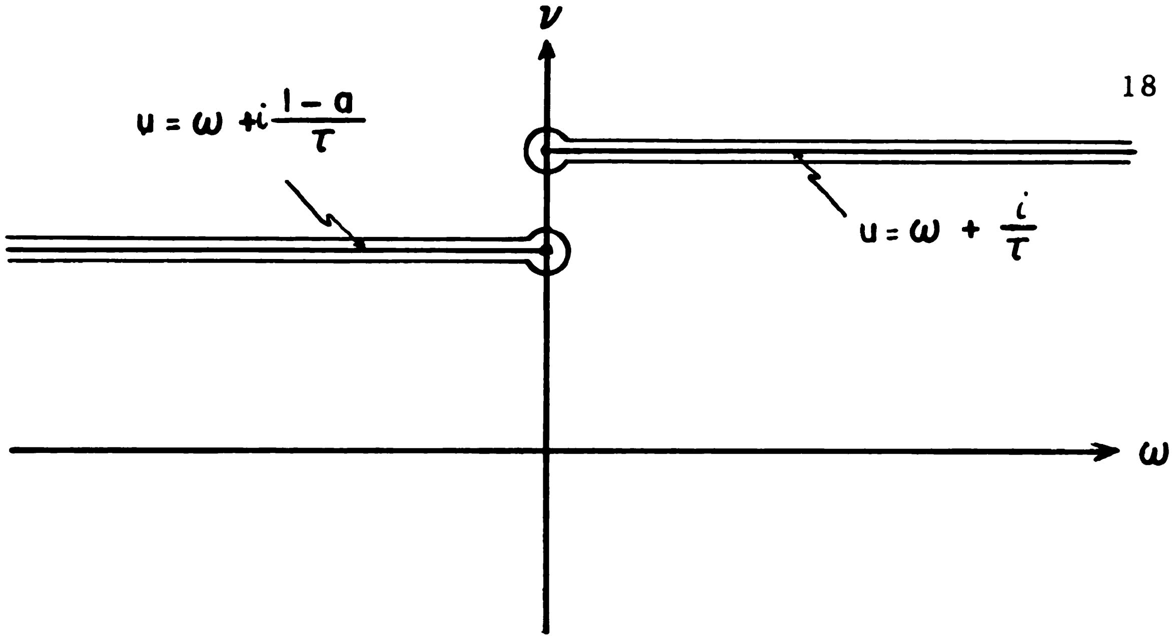

Figure 3. Alternate path for integral (25).

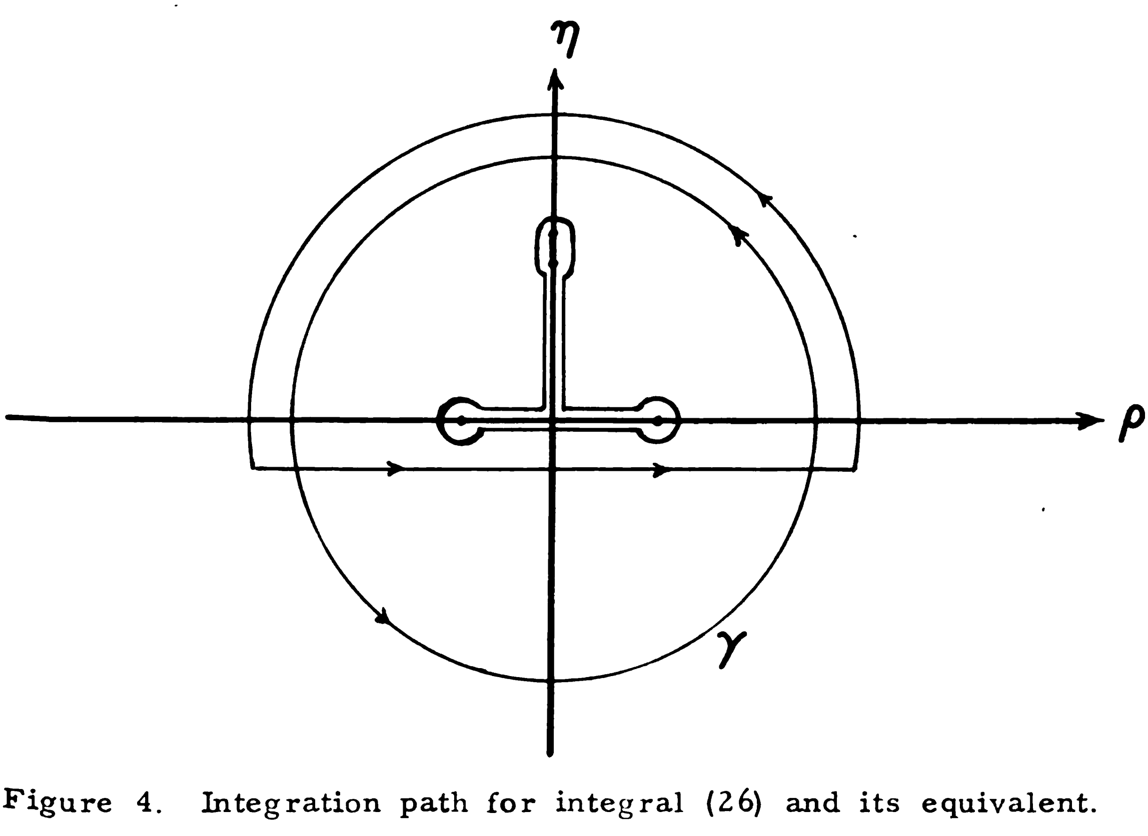

Figure 4. Integration path for integral (26) and its equivalent.

$$f(x,t) = -\frac{1}{2\pi}\int_{\gamma} \frac{\rho_0}{\xi^2-\rho_0^2}\, e^{i\left(\frac{\xi}{\tau}t^* - K^* x\right)}\, d\xi \qquad (26)$$

where γ is the path shown in Figure 4, along with two more equivalent paths. For $\xi \gg 1$, we can write

$$K^* = \frac{\xi}{c\tau}\left(\frac{1}{\sqrt{1-\frac{a}{1+i\xi}}} - 1\right)$$

$$= \frac{\xi}{c\tau}\left(1 + \frac{1}{2}\cdot\frac{a}{1+i\xi} + \frac{3}{8}\cdot\frac{a^2}{(1+i\xi)^2} + \cdots\cdots \quad -1\right)$$

$$= -\frac{ia}{2c\tau} + \frac{1}{2}\left(a - \frac{3}{4}a^2\right)\frac{1}{c\tau^2}\cdot\frac{\tau}{\xi} \quad .$$

Let now

$$\epsilon = \frac{a}{2c\tau} \qquad \text{and} \qquad \alpha = \frac{1}{2c\tau^2}\left(a - \frac{3}{4}a^2\right) \quad ,$$

then

$$f(x,t) = -\frac{\rho_0}{2\pi}\, e^{-\epsilon x}\int_{\gamma} \frac{1}{\xi^2}\, e^{i\left(\frac{\xi}{\tau}t^* - \frac{\alpha x \tau}{\xi}\right)}\, d\xi \quad , \; \rho_0 \ll |\xi| \qquad (27)$$

We write

$$\frac{\xi}{\tau}t^* - \frac{\alpha x\tau}{\xi} = \sqrt{t^*\alpha x}\left(\frac{\xi}{\tau}\sqrt{\frac{t^*}{\alpha x}} - \frac{\tau}{\xi}\sqrt{\frac{\alpha x}{t^*}}\right)$$

Let

$$\frac{\xi}{\tau}\sqrt{\frac{t^*}{\alpha x}} = i\, e^{-i\sigma} \qquad \text{and} \qquad \frac{\tau}{\xi}\sqrt{\frac{\alpha x}{t^*}} = -i\, e^{i\sigma} \qquad (28)$$

then

$$\frac{\xi}{\tau}t^* - \frac{\alpha x\tau}{\xi} = 2i\sqrt{t^*\alpha x}\, \cos\sigma$$

$$d\xi = \tau\sqrt{\frac{\alpha x}{t^*}}\, e^{-i\sigma}\, d\sigma$$

Now on (γ)

$$\xi_0 = R e^{i\varphi}$$

multiplying by $\frac{1}{\tau}\sqrt{\frac{t^*}{\alpha x}}$, we obtain

$$\frac{\xi_0}{\tau}\sqrt{\frac{t^*}{\alpha x}} = \frac{1}{\tau}\sqrt{\frac{t^*}{\alpha x}}\, R e^{i\varphi} = e^{-i(\sigma - \frac{\pi}{2})} \qquad \text{from (28)}$$

with $\varphi = -(\sigma - \frac{\pi}{2})$

$$\frac{1}{\tau}\sqrt{\frac{t^*}{\alpha x}}\, R = 1 \qquad \text{or} \qquad R = \tau\sqrt{\frac{\alpha x}{t^*}}$$

and

$$\frac{1}{\xi_0^2} = -\frac{t^*}{\alpha x \tau^2}\, e^{2i\sigma}$$

Function (27) now becomes

$$f(x,t) = \frac{\rho_0}{\tau}\sqrt{\frac{t^*}{\alpha x}}\, e^{-\epsilon x} \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{i(2i\sqrt{t^*\alpha x}\cos\sigma + \sigma)}\, d\sigma$$

$$= \frac{\rho_0}{\tau}\sqrt{\frac{t^*}{\alpha x}}\, e^{-\epsilon x} \cdot i\, J_1(2i\sqrt{t^*\alpha x}) \qquad (29)$$

For small values of (a) we can write

$$\alpha = \frac{1}{2c\tau^2}\left(a - \frac{3}{4}a^2\right) \simeq \frac{a}{2c\tau^2}$$

then

$$f(x,t) = \rho_0\sqrt{\frac{2}{a}(\theta-1)}\; e^{-\frac{a}{2c\tau}x} \cdot i\, J_1\left(2i\frac{x}{c\tau}\sqrt{\frac{a}{2}(\theta-1)}\right) \qquad (30)$$

where $\theta = \frac{ct}{x}$.

For very small $(\theta-1)$, $i\,J_1(iy) \simeq -\frac{1}{2}y$, so that in this case we can write

$$f(x,t) = -\omega_0 \frac{x}{c}\, e^{-\frac{a}{2c\tau}x} \cdot (\theta-1) \qquad (31)$$

To get an idea of the orders of magnitude, let us take

$a = 0.1$, $c = 1.5 \times 10^{5}$ cm./sec. , $\tau = 10^{-6}$ sec. , $\omega_0 = 10^{5}$ c./sec.

Then for $(\theta - 1)$ of the order of 10^{-2}, $f(x,t)$ is of the order of 10^{-3}. This is a very small magnitude compared to that of the pulse.

The interval of validity of expression (30) is obtained from the condition in (27), i.e.

$$\rho_0 << |\xi| \qquad \text{where} \qquad |\xi| = R = \tau\sqrt{\frac{\alpha x}{t^*}}$$

From which we get the condition

$$\theta - 1 << \frac{a}{2\omega_0^2 \tau^2}$$

To make the expression valid for a large interval, we demand that

$$\omega_0^2 \tau^2 << a$$

And hence the limitation on (ρ_0) which was set for this case, i.e. $\omega_0 << \frac{\sqrt{a}}{\tau}$.

<u>Case II</u> $\omega_0 >> \frac{\sqrt{a}}{\tau}$

For this case we need to use the saddle point method. This method is used because of the difficulty the function $k(u)$ presents in a conventional evaluation. We have already defined the propagation function as

$$k(u) = \frac{u}{c} K(u) = \frac{u}{c}\sqrt{\frac{1 + iu\tau}{(1-u) + iu\tau}} \qquad \text{See (8) and (21).}$$

Thus $k(u)$ has two branch points, one of the type of a pole at $u = i\frac{1-a}{\tau}$ and an ordinary branch point at $u = i\frac{1}{\tau}$.

The first of these points presents an essential singularity of the integrand which renders the evaluation impractical. An approximate nethod (the saddle point method) will be discussed in the following.

The complex u-plane will be surveyed as to the negative domains of the real part of the exponent for varying (θ). Saddle points of the exponent will be located and shown to shift with time. The path of integration will be made to run in the negative domains, passing through saddle points whenever it goes from one negative domain to the neighboring one. The integral will have negligible values along most of the path. It yields an appreciable value only when it goes through a saddle point, where the real part is not too negative. The shifting with time of the saddle points forces the path to intersect the poles and thus give rise to the main pole.

Mapping the complex plane.

The integral to be evaluated is integral (25). We rewrite the integral in terms of the new variable

$$-\frac{1}{2\pi}\int_{c'}\frac{\rho_0}{\xi^2-\rho_0^2}\,e^{\frac{x}{c\tau}\cdot i\xi[\theta-K(\xi)]}\,d\xi \tag{32}$$

Let now

$$v(\xi) = i\,\xi[\theta-K(\xi)] = X(\rho,\eta) + i\,Y(\rho,\eta)$$

where

$$X(\rho,\eta) = -\eta\theta + \eta\sqrt{\frac{r_1}{r_2}}\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) + \rho\sqrt{\frac{r_1}{r_2}}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right) \tag{33a}$$

$$Y(\rho,\eta) = \rho\theta - \rho\sqrt{\frac{r_1}{r_2}}\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) + \eta\sqrt{\frac{r_1}{r_2}}\sin\left(\frac{\varphi_1-\varphi_2}{2}\right) \tag{33b}$$

$$r_1 = \sqrt{\rho^2+(\eta-1)^2} \qquad , \quad r_2 = \sqrt{\rho^2+[\eta-(1-a)]^2}$$

$$\varphi_1 = \tan^{-1}\frac{\eta-1}{\rho} \qquad , \quad \varphi_2 = \tan^{-1}\frac{\eta-(1-a)}{\rho}$$

It was shown earlier that $K(\xi)\xrightarrow[\xi\to\infty]{} 1$, so that for large ξ we can write

$$v(\xi) = i\,\xi\,(\theta-1) = i\rho(\theta-1) - \eta(\theta-1)$$

Then $X = -\eta(\theta-1)$ behaves as follows at infinity

For $\theta < 1$ is negative infinity below the real axis.

is positive infinity above the real axis.

For $\theta = 1$ is zero.

For $\theta > 1$ is positive infinity below the real axis.

is negative infinity above the real axis.

The domains of $X(\rho,\eta)$ in the finite plane are shown in Figures 5 to 8. In the shaded region X is positive; in the unshaded region it is negative. These figures were obtained by plotting $X(\rho,\eta)$ as given in (33a).

Location of the saddle points.

Since the function

$$v(\xi) = X(\rho,\eta) + i\,Y(\rho,\eta)$$

is analytic except at $\xi = ia$ and $\xi = i(1-a)$, $X(\rho,\eta)$ and $Y(\rho,\eta)$ cannot possess finite maxima and minima, they can only possess saddle points, at which

$$\frac{\partial X}{\partial \rho} = \frac{\partial X}{\partial \eta} = \frac{\partial Y}{\partial \rho} = \frac{\partial Y}{\partial \eta} = 0 \qquad ,$$

or where

$$\frac{\partial v}{\partial \xi} = 0$$

Carrying out the differentiation and equating to zero we get the quartic,

$$\xi^4 - i(3b+1)\xi^3 - \left[3b(1+b) + \frac{4b^2-a^2}{4(\theta^2-1)}\right]\xi^2 + i\left[b^2(3+b) - \frac{b(1-b^2)}{\theta^2-1}\right]\xi + \left[b^3 - \frac{b^2(1-b)}{\theta^2-1}\right] = 0 \qquad (34)$$

where $b = 1-a$.

The solutions of this equation represent the loci of the saddle points for a certain (a) and a varying (θ). Since the algebraic solution of this equation for arbitrary (a) and (θ) is unwieldy, it

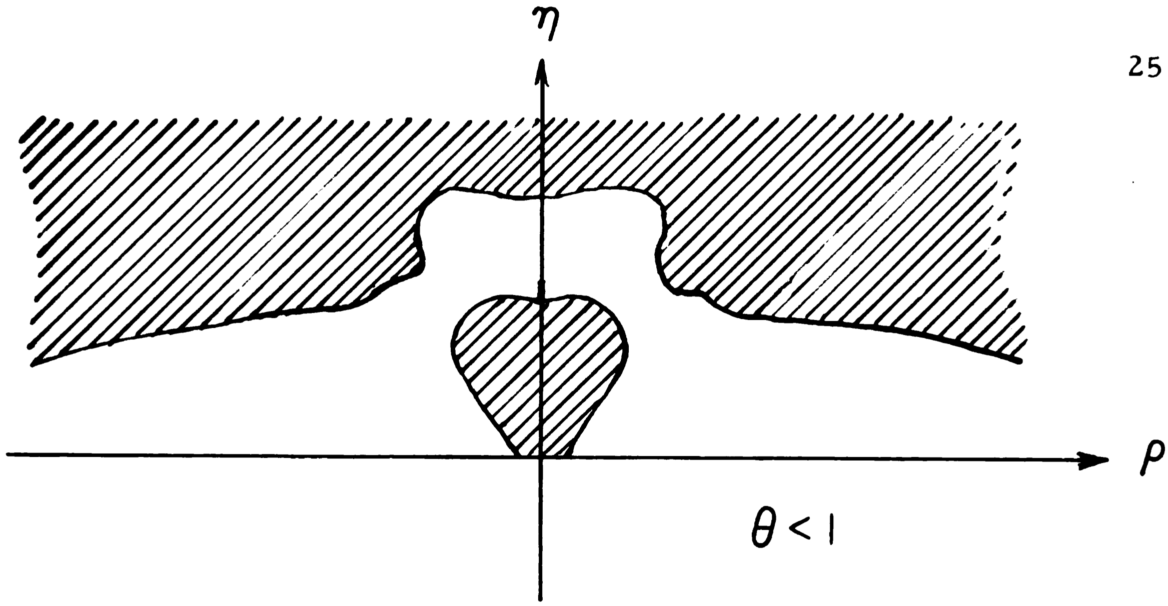

Figure 5. Topographic map of the function $X(\rho,\eta)$ in the complex ξ_0-plane. Shaded area is positive.

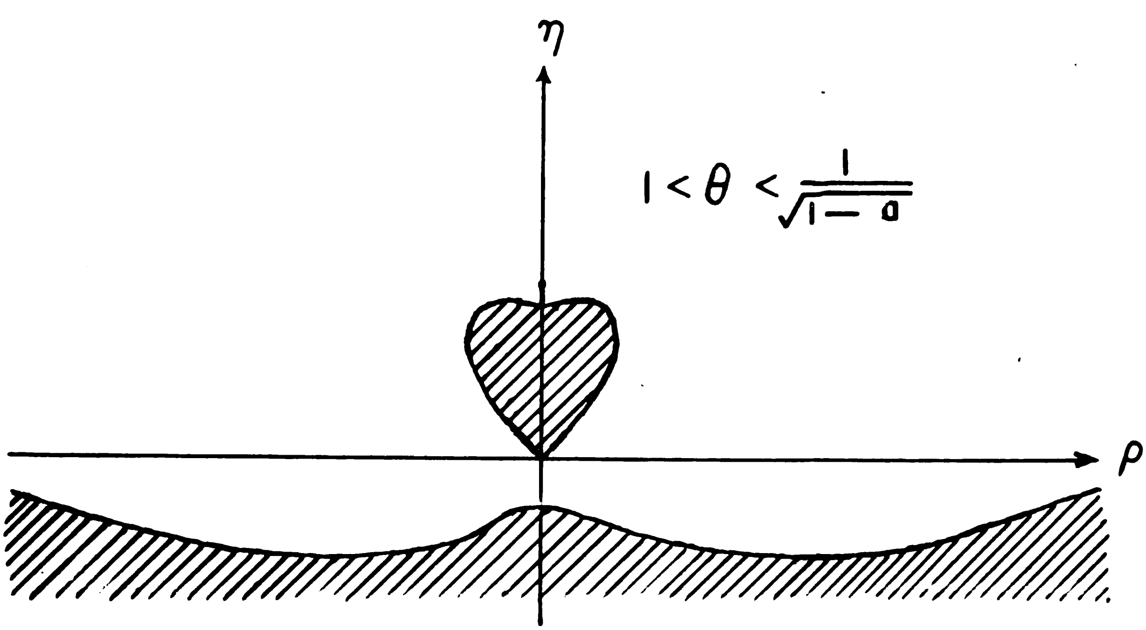

Figure 6.

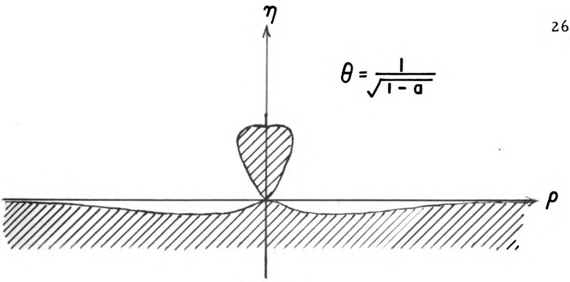

Figure 7.

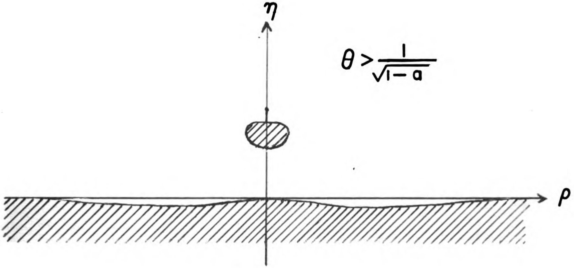

Figure 8.

was solved numerically for a number of values of (a) and (θ) sufficient to obtain a general picture of the location of the saddle points.

The exact solution of the quartic for $a = 0.1$ and varrying (θ) is shown in Figure 9.

Thus two sets of saddle points are obtained, one pair starts at $\xi = \pm\infty$ when $\theta = 1$ and proceeds towards the finite plane with increasing (θ). The other pair is in the finite plane and on the imaginary axis at $\theta = 1$. The two points of the latter pair approach each other with increasing (θ) until they meet in the upper half-plane and fork out as shown.

It is possible to obtain an approximate representation of the locus of the points on the imaginary axis for values of (θ) very near unity. For such values of (θ), the saddle points lie very near the origin where $\xi < 1$. This makes it possible to drop the terms with the third and fourth powers of (ξ). Thus we obtain the following quadratic which proves to be a very good representation of the locus:

$$\left[3b(1+b) + \frac{4b^2 - a^2}{4(\theta^2-1)}\right]\eta^2 - i\left[b^2(3+b) - \frac{b(1-b^2)}{\theta^2-1}\right]\eta - \left[b^3 - \frac{b^2(1-b)}{\theta^2-1}\right] = 0 \qquad (35)$$

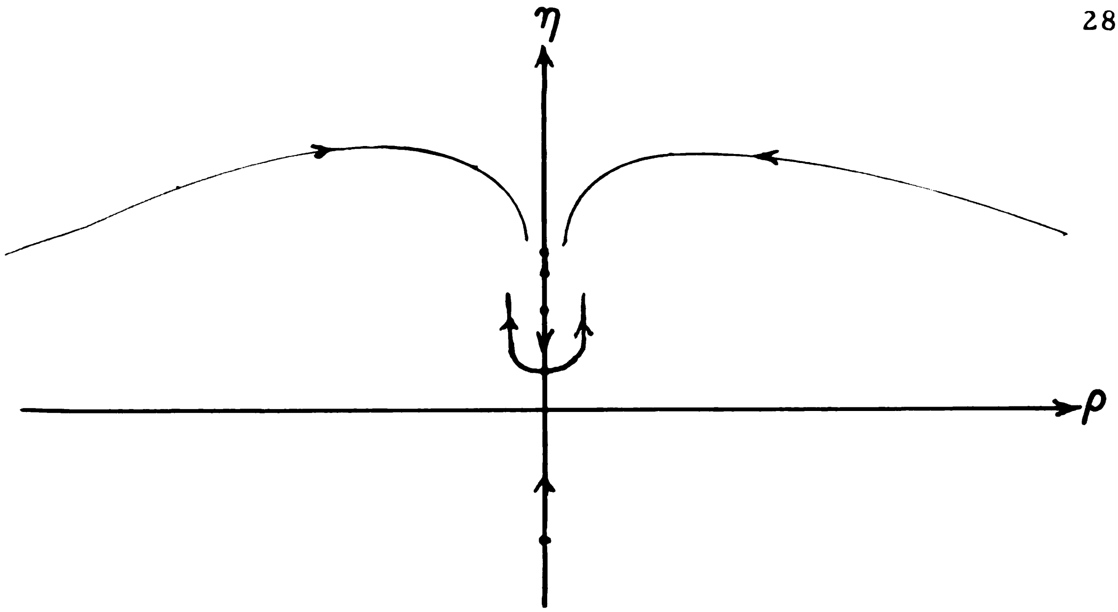

Figure 9. Loci of saddle points for varrying (θ). Arrows show direction of shift with increasing (θ) .

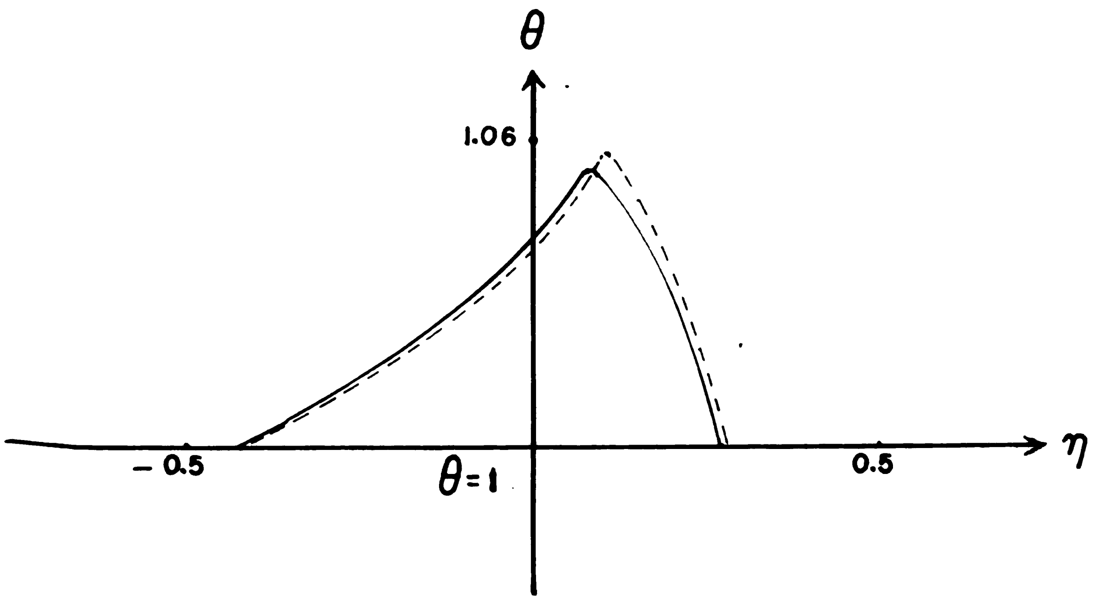

Figure 10. Location of saddle points on the imaginary axis for varying (θ) . The solid curve is obtained from eq. 34 and the broken curve from eq. 35 and both for $a=0.1$.

Figure 10 shows how good the approximate equation is, especially for small values of (θ), which we are most interested in.

We can obtain another good approximate expression for the locus of the other pair of saddle points, if we divied (34) by (35) and drop the remainder after obtaining a quadratic. We will prove later, however, that the contribution of these saddle points will be canceled by the contribution of other parts of the path.

The integration path.

Integral (32) is to be evaluated for $\theta > 1$ and $\rho_o \gg \frac{\sqrt{a}}{t}$.

Figure 11 is the same as Figure 6 with the addition of less negative ridges in the negative domains and the integration path. The integration path is $(-\infty) f a d c d' b f' (+\infty)$, made to pass through saddle points whenever possible.

For $\theta < 1$, the path runs completely below the real axis and gives no contribution. At $\theta = 1$ or $t = \frac{x}{c}$, the path remains in the lower finite plane except at infinity. However, the path has to shift completely to the upper plane at $\theta = \frac{1}{\sqrt{1-a}}$, see Figure 7. The part of the path going around the poles $\pm \rho_o$ moves up before $\theta = \frac{1}{\sqrt{1-a}}$. In so doing, the path intersects the poles and an

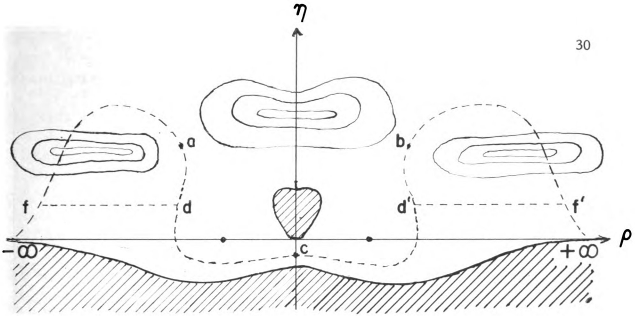

Figure 11. The true integration path for eq. (32). a , b , and c are saddle points.

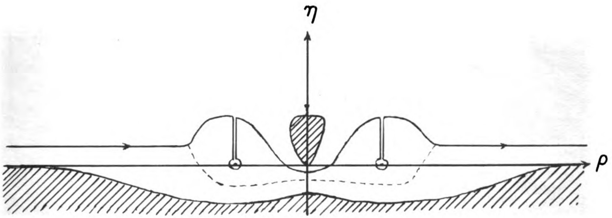

Figure 12. The approximate integration path for eq. (32), before (broken) and after (solid) the arrival of the pulse.

oscillating term is obtained, which builds up very quickly to an appreciable value. We will discuss this point with more detail in a later section.

The total contribution of the path to the integral is obtained by finding the contribution of those parts of the path that are in the immediate neighborhood of the saddle points. However, as will be shown shortly, the contributions of the saddle points (a) and (b), Figure 11, are canceled by the contribution of that part of the path crossing the ridge at the far end of the plane. This leaves the contribution from the vicinity of the point (c) to be calculated.

To prove the above statement, we draw the shortcuts (df) and $(d'f')$, Figure 11. These two shortcuts are equivalent to (fad) and $(d'bf')$ respectively. Let G be the contribution of (df), L the contribution of the saddle point (a), and L' the contribution of the part crossing the ridge between (f) and (a), it follows that

$$G = L + L'$$

We now proceed to show that G is negligible, thus proving that $L \simeq -L'$ and hence the contribution of the saddle point (a) is canceled.

Along (fd) and $(d'f')$, ξ_o is very large, so that we can rewrite integral (32), making use of relation (12) as follows:

$$-\frac{\rho_0}{2\pi}\int_{-\infty}^{\lambda}\frac{1}{\xi^2}\, e^{\frac{x}{c\tau}\cdot i\rho(\theta-1)}\, e^{-\frac{x}{c\tau}\cdot\eta(\theta-1)}\, d\xi \qquad \lambda < \rho_0 < 0$$

The second exponent is a constant, since (η) is constant along (fd). The integral thus has an upper bound of $\frac{1}{\lambda}$, which is very small except when the path approaches the pole. In this case form (32) of the integral must be considered. However, by this time, the contribution of the path in the neighborhood of the pole becomes too prominent, giving rise to the main pulse, and the contribution of the rest of the path becomes negligible. Thus, before the pulse builds up, the contribution G_1 of (fd) is negligible and we are justified in considering the relation $L \simeq -L'$ as valid. Furthermore, G_1 can still be kept negligible after the pulse builds up by taking the path along a large circle around (ρ_0). Figure 12 shows the new path before and after it crosses the poles at $\pm\rho_0$.

Next we proceed to find the contribution of the path in the immediate neighborhood of the saddle point at (c), Figure 11.

We start by expanding the function

$$v(\xi_0) = i\,\xi_0[\theta - K(\xi_0)]$$

in a Taylor series around the saddle point ξ_s obtaining

$$v(\xi_0) = v(\xi_s) - \frac{1}{2}\left|v''(\xi_s)\right|(\xi_0 - \xi_s)^2 \tag{36}$$

where

$$|v''(\xi_s)| = \left| \frac{4(1-a) + a(4-3a)i\xi_s - 3\xi_s^2 - 3i\xi_s^3}{2[(1-a)+i\xi_s]^{5/2}(1+i\xi_s)^{3/2}} \right|$$

the first derivative is zero at the saddle points.

Integral (32) now becomes

$$-\frac{1}{2\pi}\int \frac{\rho_o}{\xi_o^2-\rho_o^2}\, e^{\frac{x}{c\tau}v(\xi_s)} \cdot e^{-\frac{x}{2c\tau}|v''(\xi_s)|(\xi-\xi_s)^2}\, d\xi \tag{37}$$

At the saddle points, on the negative imaginary axis, the path is parallel to the real axis. If now we limit our integration path to a very small portion in the neighborhood of the saddle point, we can write

$$\xi_o = \epsilon + i\eta_s \qquad \text{where} \quad i\eta_s = \xi_{os}$$

so that $\qquad d\xi_o = d\epsilon$

Along such a small path we may consider $\frac{1}{\xi_o^2-\rho_o^2}$ a constant, which is especially true for $\rho_o^2 \gg \xi_o^2$ as we are considering in this case. Taking all constant terms out of the integration sign we get

$$-\frac{\rho_o e^{\frac{x}{c\tau}v(\xi_s)}}{2\pi(\xi_o^2-\rho_o^2)}\int_{-\epsilon}^{+\epsilon} e^{-\frac{x}{2c\tau}|v''(\xi_s)|\epsilon^2}\, d\epsilon \tag{38}$$

$v(\xi_s)$ is always negative and $v''(\xi_{os})$ is of the order of 10^{-1} for $a = 0.1$. If now we choose $x = 1$ cm., $c = 1.5 \times 10^5$ cm./sec. and $\tau = 10^{-7}$ sec. or smaller. The order of magnitude of $\frac{x}{2c\tau}|v''(\xi_{os})|$ is about (3) or larger. This value of the exponent makes most of the contribution come from a very small interval about the saddle point. Thus, extending the limits of integration to infinity does not alter the result appreciably. In so doing we obtain the result

$$- \frac{\rho_0 e^{-\frac{x}{c\tau}|v(\xi_{0s})|}}{2\pi(\xi_{0s}^2 - \rho_0^2)} \sqrt{\frac{2\pi c\tau}{x|v''(\xi_{0s})|}} \qquad (39)$$

For this expression to be valid we should keep $\frac{1}{\xi_{0s}^2 - \rho_0^2}$ as nearly a constant as possible. This can only be done by requiring that

$$\rho_0^2 \gg |\xi_{0s}|^2$$

An expression for the maximum value of $|\xi_{0s}|$ is obtained by setting $(\theta^2 - 1)$ equal to zero in equation (35). The result is simplified considerably if we note that (a) is usually very small compared to unity. After some simplifications we obtain the following expression for $\xi_{0s\,max}$.

$$|\xi_{0s\,max.}| = a + \sqrt{a}$$

Now $\sqrt{a} > a$, so that we can write

$$\rho_0 \gg \sqrt{a} \qquad \text{or} \qquad \omega_0 \gg \frac{\sqrt{a}}{\tau}$$

which is the condition of validity we stated for this case.

Discussion of the solutions (29) and (39).

Expression (31) shows a linear dependence of $f(x,t)$ on (θ) for very small $(\theta-1)$. However, for large $(\theta-1)$ expression (29) decreases exponentially and rapidly with increasing (θ) . This exponential decrease of the function continues until the function loses its validity according to the condition

$$\theta-1 << \frac{a}{2\rho_0^2}$$

obtained earlier. To find the order of magnitude of the function for values of $(\theta-1)$ restricted by the above condition, we assume $a=0.1$, $\rho_0 = 0.1$, $\tau = 10^{-6}$ sec. , $x = 1$ cm. and $c = 1.5 \times 10^5$ cm./sec. . These values require that $\theta-1 << 5$. Let us choose $\theta-1 = 0.05$, then from (29) we get $f(x,t)$ of the order of 10^{-3} which is very small compared to the amplitude of the pulse. We will show later that it is not possible to determine exactly the time of build-up of an observable pulse. However, to be safe in applying expression (29), we should assume a very small $(\theta-1)$ in accord with the above condition. This in turn limits the amplitude of the function to very small values until the arrival of the pulse. It can easily be seen from expression (30) and the condition on $(\theta-1)$ that, within the limits of validity, an increase in (ρ_0) decreases the absolute value of the function.

Expression (39), on the other hand, builds up exponentially from a finite nonzero value at $\theta = 1$. It reaches a maximum at a certain (θ) which is a function of (τ) alone. The amplitude, including that of the maximum, is a function of both (ρ_0) and (τ), it changes directly with (τ) and inversely with (ρ_0).

The difference in sign between the two expressions (29) and (39) is expected because of the factor $\frac{1}{\xi_0^2 - \rho_0^2}$ which changes sign as $|\xi_0|$ assumes values smaller and larger than (ρ_0). Figure 13 shows three plots of (39) versus (θ) for $\rho_0 = 10$ and three different values of (τ).

The contributions of both expressions (29) and (39) are usually referred to as the "precursors" or "forerunners" to the observable pulse, which will be discussed next.

<u>The pulse</u>.

We have shown in the last section that the amplitude of the precursor is very small compared to the pertaining pulse. We also stated that, the moment the integration path crosses a pole, the amplitude builds up to a noticeable value.

Theoretically, the pulse arrives at the plane $x \neq 0$ with a velocity (c), the maximum velocity. This is certainly true of those

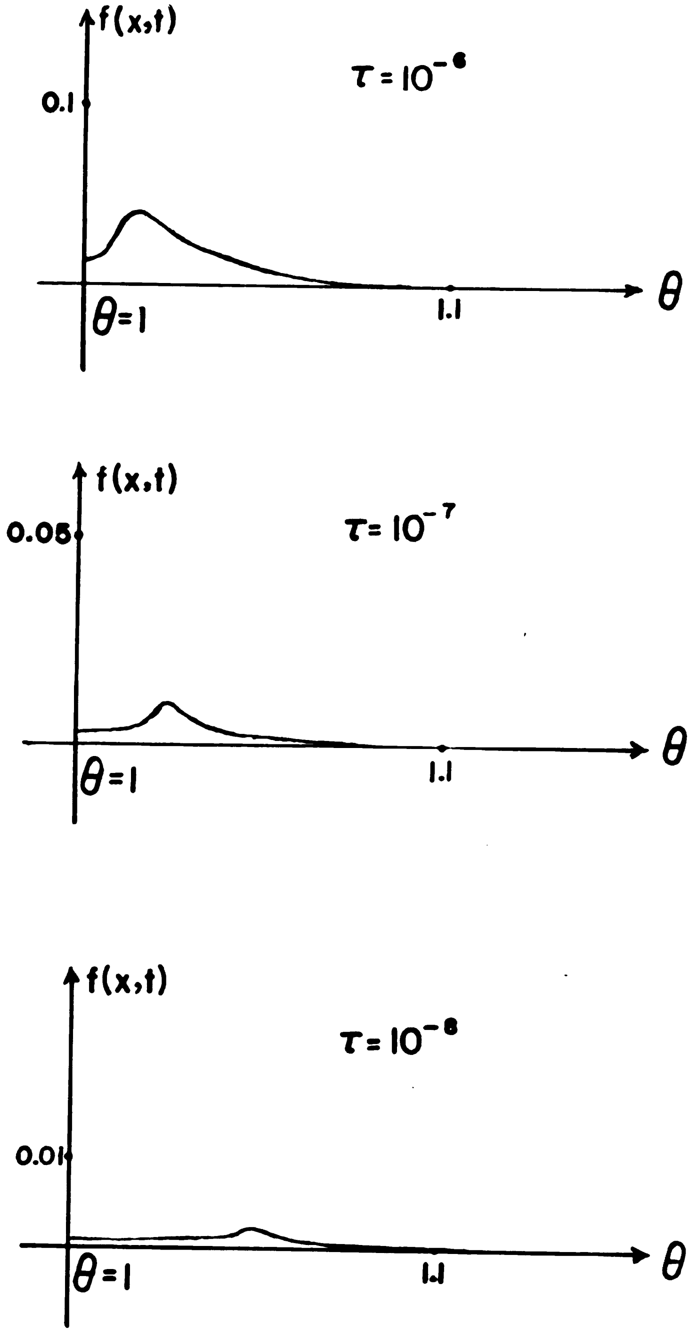

Figure 13. Plots of expression (39) versus (θ) for three values of (τ) and $\rho_o = 10$.

Fourier components of the pulse with the highest frequencies. These, however, do not constitute the bulk of the components constituting the pulse. Most of the components have lower frequencies and hence lower velocities compared to (c) . The whole set of components can be subdivided into groups of different average frequencies which move with different average velocities. The "group velocity" v_g is defined by the relation

$$\frac{1}{v_g} = \frac{d}{d\omega} k'(\omega) = \tau \frac{d}{d\rho} k'(\rho) \tag{40}$$

where $k'(\omega)$ is a function of the real average frequency (ω) and is the real part of $k(\omega)$.

The group of frequency components centered on (ω_o) arrives at the plane $X \neq 0$ at $t = \frac{x}{v_g}$, and since $\theta = \frac{ct}{x}$, at $\theta = \theta_g = \frac{c}{v_g}$. Thus (θ_g) is the ratio of the maximum velocity to the group velocity.

Figure 14 is a plot of (θ_g) as a function of (ρ). This was obtained from (21) according to the definition (40). The figure shows that (θ_g) approaches unity as the frequency tends to infinity. In other words, as the frequency increases, the group velocity approaches the maximum velocity.

It is not possible to find a reasonable definition for the pulse velocity in a medium for reasons that will become clear later. However, it is evident from the past discussion that the pulse

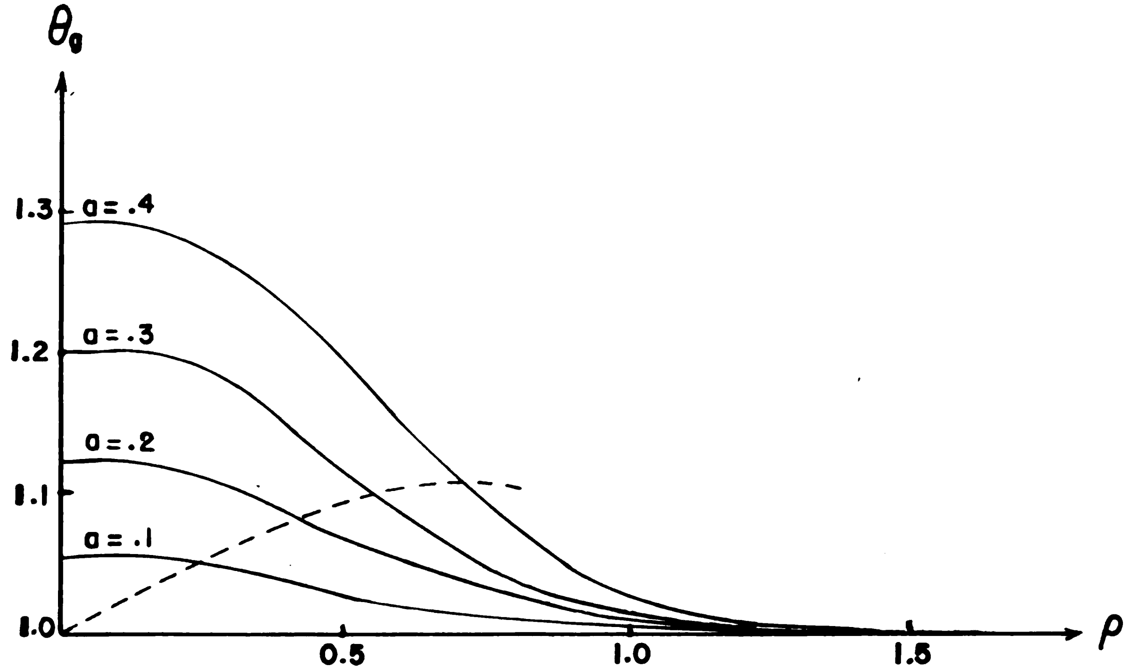

Figure 14. θ_g as a function of (ρ) (solid curves). Broken curve represents (θ_g) as a function of $\rho = \sqrt{a}$.

velocity must not exceed the group velocity. We now make the assumption that the pulse velocity is only slightly smaller than the group velocity. This is especially true for high and low frequencies; it is not so true for frequencies lying in the region of anomaly.

The broken line in Figure 14 represents the limit of validity of both (29) and (39). Expression (29) is valid in a region to the right of the point of intersection of the broken line with the θ_g curve for a certain value of (a). Expression (39), on the other hand, is valid in a region to the left of the points of intersection. The figure yields the following useful information. First, the interval of validity of (29) increases with increasing (a). Second, θ_g is smaller for (39) than it is for (29). Taking notice of our assumption in the preceding paragraph, we conclude that the signal arrives earlier when (39) is used than it does when (29) is used. Usually the constant (a) is of the order of 10^{-2}, which means that θ_g is very small as can be seen from Figure 14. Thus our choice, in a previous section, of $\theta = 1.05$ to find the order of magnitude of (29), was rather generous. Nevertheless, (29) was found to be very small compared to the pulse.

Figures 15 and 16 are schematic representations of what happens at the plane $X \neq 0$, using expressions (29) and (39) respectively. Until $t = \frac{x}{c}$ nothing happens, at $t = \frac{x}{v_g} \approx \frac{x}{v_p}$, ($v_p$ = pulse velocity), the pulse builds up very rapidly to its full amplitude.

Nothing was said, so far, about the decay of the pulse. In equation (20) we had two integrals I_1 and I_2. These two integrals are identical except for their sign and their lower time limits. Δ was picked such that I_2 starts in phase with I_1, but the difference in sign puts them completely out of phase and leads to the destruction of the pulse. Thus I_2 yields the same contributions as I_1, only the former leads to a trailing phenomenon instead of the precursor. This tail phenomenon, not being of any special value, is not shown in Figures 15 and 16. From Figure 14 we can easily establish that, for the same(ω), the pulse arrives earlier the larger (τ) is. Figure 13, on the other hand, shows earlier maxima for larger τ's. Thus we find that the rise of (39) to its maximum value occurs at the same range of (θ_g) for the pulse arrival. In Figure 16 we let the pulse build up from the maximum value of (39).

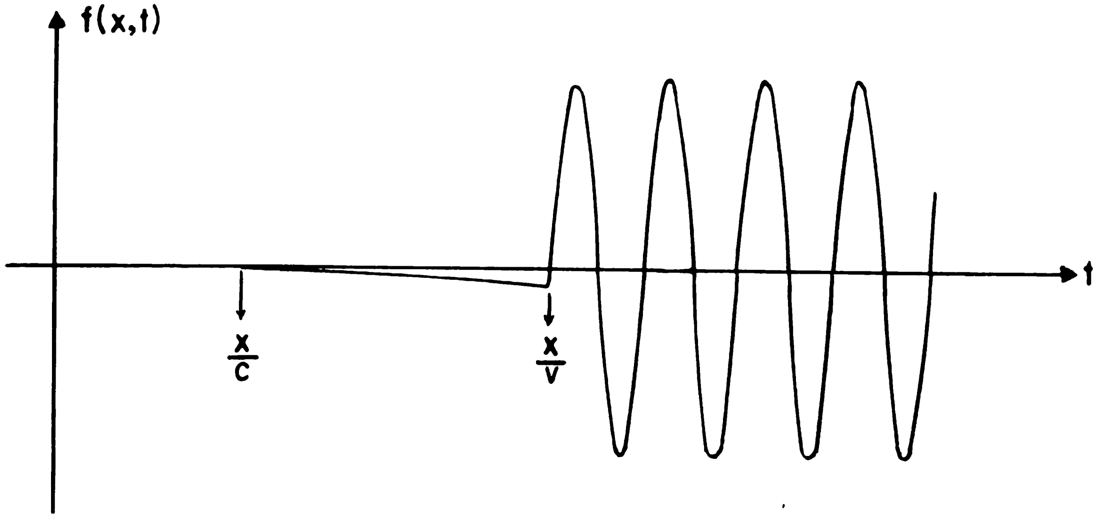

Figure 15. The pulse build-up for $\omega_o \ll \frac{\sqrt{a}}{\tau}$, eq. (29).

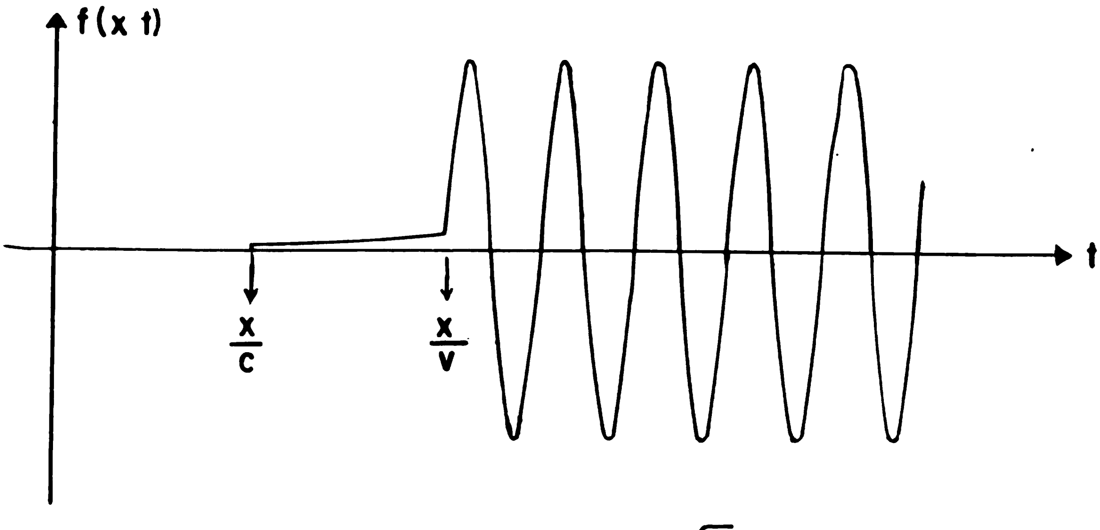

Figure 16. The pulse build-up for $\omega_o \gg \frac{\sqrt{a}}{\tau}$, eq. (39).

CONCLUSION

A pulse traveling in an infinite, homogeneous and isotropic medium, characterized by a complex elastic medium, moves with a maximum velocity (C). This is the velocity of the Fourier component of the pulse with the highest frequency. Other components move with lower velocities, each a function of the pertaining frequency.

Because of the above-stated dispersive property of all media, a pulse arrives distorted at the plane $X \neq 0$. Of particular interest to the observer is the manner in which the pulse builds up to an observable amplitude. Experimental determination of sound velocity in different media is a case where a knowledge of the process of building up of a pulse might prove to be of considerable importance.

We have found that a sonic pulse, in a medium having a relaxation time (τ) builds up exponentially at first and later assumes its oscillatory property.

A detector of infinite sensitivity, at a distance (X) from the source of the pulse, registers the arrival of the pulse at $t = \frac{X}{C}$. As the sensitivity of the detector decreases, the detection of the pulse arrival is delayed. This is one of many effects that make the experimental determination of sound velocity subject to controversies. Such a determination leads to values dependent on the following properties:

1. the dispersive properties of the medium,
2. the Fourier composition of the pulse,
3. the sensitivity of the detecting device.

Further treatment is needed to discuss the behavior of a pulse for the case $\rho \sim \sqrt{a}$ which is not covered by the approximate methods developed in this thesis. However, it appears that, for most of the values of (a) met in practice, the error introduced, in the velocity determination, by the presence of the precursor will not be too large.

Finally, it must be pointed out that the general methods developed here may be applied to the propagation of an electromagnetic pulse in a medium in which the dispersion arises from a relaxation process. In this case, however, the values of (a) will not, in general, be small and the effects of the precursor may be much larger than in the sonic case.

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