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Some Sharp Estimates Involving Hilbert Transform

By

Stefanie Petermichl

A DISSERTATION

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ABSTRACT

Some Sharp Estimates Involving Hilbert Transform

By

Stefanie Petermichl

We construct the Hilbert transform on $\mathbb{R}$ as an average of dyadic operators, allowing us to translate norm estimates to estimates of dyadic type. We are going to apply this representation to give the sharp dimensional growth of the commutator of the Hilbert transform with matrix multiplication by a *BMO* matrix of size $n \times n$. The bound is a multiple of $\log n$ times the *BMO*-norm of the matrix. Furthermore we will apply this representation to give an elegant proof of the fact that H , as an operator in scalar but weighted L^2 , is bounded by the cube of the classical A_2 -norm of the weight. By the different method of Bellman functions, we show the quadratic bound for H in terms of invariant A_2 and this estimate is sharp.

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CHAPTER 1

Introduction

It has been of interest for a long time to give sharp estimates for the norm of the Hilbert transform and related operators in $L^p(\mathbb{R})$. In the present work we look at the estimates of the Hilbert transform in weighted and/or vector spaces. We restrict ourselves to the case $p = 2$.

It will be of great help to reduce various estimates for H to estimates of dyadic type. We will prove that there is a very nice representation of the Hilbert transform via averaging operators that we will refer to as dyadic shifts.

Let us start with vector problems. Recently some activity has been focused on the area of non-commutative weighted estimates. By non-commutative weighted estimates we understand, for example, the estimate of the singular integral operators T , say, the Hilbert transform H , in the space of vector functions with matrix or operator weights. Let us refer the reader to [16], [17], [8], [18].

The problem of estimating $\|T\|_{L^2(W) \rightarrow L^2(W)}$ is equivalent to estimating $\|W^{1/2}TW^{-1/2}\|_{L^2 \rightarrow L^2}$ which is non-linear in W . But there is a linearized counterpart, which comes down to estimates of commutators of the form $TB - BT$. This problem is linear in B . It is well known that estimates of commutators of matrix multiplication with the Hilbert transform yield estimates of Hankel operators.

In the present work we are going to give the estimate from above in terms of the

dimension n ,

$$\|HB - BH\|_{L^2(\mathbb{C}^n) \rightarrow L^2(\mathbb{C}^n)} \leq C \log n \|B\|_{BMO}. \quad (1.1)$$

We refer to [11]. In [7] it has been proven that there exists an $n \times n$ matrix function B such that

$$\|HB - BH\|_{L^2(\mathbb{C}^n) \rightarrow L^2(\mathbb{C}^n)} \geq C' \log n \|B\|_{BMO} \quad (1.2)$$

(the symbol $\|B\|_{BMO}$ is defined below), which proves sharpness.

The same idea of averaging can be tried for weighted estimates of the Hilbert transform and other CZ operators. Now we are in the scalar but weighted situation. The space considered is $L^2(\omega)$, where ω is in A_2 , the exact class of weights that allows the Hilbert transform to be bounded.

The question for sharp estimates for the Hilbert transform, the square function and a uniform bound for martingales on weighted L^2 spaces in terms of the A_2 constant of the weight has attracted considerable interest in recent years. S. Buckley in [1] proved that the square function is bounded by $\|\omega\|_{A_2}^3$ and that the Hilbert transform is bounded by $\|\omega\|_{A_2}^4$. More recently, in [5] the quadratic bound for the square function has been proven. This bound is sharp. An alternative proof can be found in [19] and [12], where the estimate is found using the lower bound for the square function, which is linear. The linear lower bound for a harmonic version has been proven in [2] and independently for the dyadic version in [12]. We prove the cubic bound for all dyadic shifts and hence the Hilbert transform.

By the different method of Bellman functions we obtained the sharp estimate in terms of invariant A_2 , which is the version of A_2 using Poisson averages instead of box averages. The proof uses a nice ‘duality’ between sharp uniform estimates for dyadic martingales and its continuous analog, the Hilbert transform. The proof includes an alternative to what is known as bilinear imbedding theorem in [10] involving two

weights. We establish slightly simpler conditions on the two weights v and ω for the imbedding. The function to prove the theorem is

$$B(X, Y, x, y, r, s, G, H, M, N, K) =$$

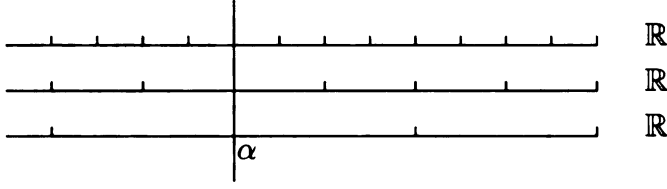
$$5X + 5Y - 2\frac{x^2}{r} - 2\frac{y^2}{s} - \frac{x^2}{r + \frac{G}{Q^4}} - \frac{y^2}{s + \frac{H}{Q^4}} - \frac{x^2}{r + \frac{M}{Q^4}} - \frac{y^2}{s + \frac{N}{Q^4}} - \frac{x^2s - 2xy\frac{K}{Q^2} + y^2r}{rs - \frac{K^2}{Q^4}}$$

CHAPTER 2

The Hilbert Transform and Dyadic Shift Operators

2.1 Definitions

We will be using a variety of dyadic grids in $\mathbb{R}$. The standard dyadic grid is the collection of the following intervals with disjoint interiors : $\{[2^n k, 2^n(k+1)], n, k \in \mathbb{Z}\}$. The point 0 has the significance to be the only point that does not lie in the interior of any of the intervals. We call the standard dyadic grid $\mathcal{D}^{0,1}$ and we denote by $h_J^{0,1}$ the Haar function for $J \in \mathcal{D}^{0,1}$, namely $h_J^{0,1} = 1/\sqrt{|J|} (\chi_{J_-} - \chi_{J_+})$ where J_- is the left half of J and J_+ the right half of J . We obtain a variation of $\mathcal{D}^{0,1}$ by first shifting the starting point 0 to $\alpha \in \mathbb{R}$ and secondly choosing intervals of length $r \cdot 2^n$ for positive r . The resulting grid is called $\mathcal{D}^{\alpha,r}$. The corresponding Haar functions $h^{\alpha,r}$ are chosen so that they are still normalized in L^2 . We often omit the indices α, r in our notations for the Haar functions. The following is an illustration of $\mathcal{D}^{\alpha,r}$.



For $f \in L^2(\mathbb{R})$ we have

$$f(x) = \sum_{I \in \mathcal{D}^{\alpha,r}} (f, h_I) h_I(x).$$

We define a dyadic shift operator $\mathbb{I}\mathbb{I}^{\alpha,r}$:

$$(\mathbb{I}\mathbb{I}^{\alpha,r} f)(x) = \sum_{I \in \mathcal{D}^{\alpha,r}} (f, h_I) (h_{I_-}(x) - h_{I_+}(x)).$$

The symbol ‘ $\mathbb{I}\mathbb{I}$ ’ is a cyrillic letter that reads ‘sha’. Its L^2 operator norm is $\sqrt{2}$ and its representing kernel is

$$K^{\alpha,r}(t, x) = \sum_{I \in \mathcal{D}^{\alpha,r}} h_I(t) (h_{I_-}(x) - h_{I_+}(x)), \quad (2.1)$$

by which we mean that

$$(\mathbb{I}\mathbb{I}^{\alpha,r} f)(x) = \int_{\mathbb{R}} K^{\alpha,r}(t, x) f(t) dt$$

for smooth compactly supported f and x outside the support of f .

2.2 The Representation of the Kernel via Limits of Averages

The kernel of the Hilbert transform is $K(t, x) = \frac{1}{x - t}$. Constant multiples of this function are characterized by the four properties that K only depends on the distance $x - t$, $K \not\equiv 0$, the correct degree of homogeneity and antisymmetry. We will pick the correct averaging process to form such properties from $K^{\alpha,r}$ defined in equation (2.1).

Lemma 2.1 *The convergence of sum (2.1) is uniform for $|x - t| \geq \delta$ for every $\delta > 0$.*

For $x \neq t$ let

$$K(t, x) = \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(t, x) d\alpha \frac{dr}{r}.$$

The limits exist pointwise and the convergence is bounded for $|x - t| \geq \delta$ for every $\delta > 0$. $K(t, x)$ is a nonzero constant multiple of the kernel of the Hilbert transform:

$$K(t, x) = \frac{c_0}{t-x} \text{ for some } c_0 > 0.$$

PROOF.

First note that $\forall \alpha \in \mathbb{R}$ and $\forall r > 0$:

$$\sum_{I \in \mathcal{D}^{\alpha, r}} |h_I(t)(h_{I_-}(x) - h_{I_+}(x))| \leq \frac{2\sqrt{2}}{|t - x|}.$$

In fact, if $|I| < |t - x|$ then $h_I(t)(h_{I_-}(x) - h_{I_+}(x)) = 0$, so picking n_0 minimal such that $r2^{n_0} \geq |t - x|$ we get

$$\begin{aligned} & \sum_{I \in \mathcal{D}^{\alpha, r}} |h_I(t)(h_{I_-}(x) - h_{I_+}(x))| \\ & \leq \sum_{n=n_0}^{\infty} \sum_{\substack{I \in \mathcal{D}^{\alpha, r} \\ |I|=r2^n}} |h_I(t)(h_{I_-}(x) - h_{I_+}(x))| \\ & \leq \sum_{n=n_0}^{\infty} \frac{\sqrt{2}}{r2^n} \\ & \leq \frac{2\sqrt{2}}{|t - x|}. \end{aligned}$$

In particular, the sum converges absolutely and uniformly for $|x - t| \geq \delta$ for every $\delta > 0$. So

$$\begin{aligned} & \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(t, x) d\alpha \\ & = \sum_{n=n_0}^{\infty} \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R \sum_{\substack{I \in \mathcal{D}^{\alpha, r} \\ |I|=r2^n}} h_I(t)(h_{I_-}(x) - h_{I_+}(x)) d\alpha \end{aligned}$$

by dominated convergence. The limit

$$\lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R \sum_{\substack{I \in \mathcal{D}^{\alpha,r} \\ |I|=r2^n}} h_I(t)(h_{I_-}(x) - h_{I_+}(x)) d\alpha$$

exists for all fixed r because shifting a certain grid by $r2^n$ will result in the same intervals of that particular length:

$$\sum_{\substack{I \in \mathcal{D}^{\alpha,r} \\ |I|=r2^n}} h_I(t)(h_{I_-}(x) - h_{I_+}(x)) = \sum_{\substack{I \in \mathcal{D}^{\alpha+r2^n,r} \\ |I|=r2^n}} h_I(t)(h_{I_-}(x) - h_{I_+}(x)).$$

The outer limit in L exists for similar reasons. Here the entire grid repeats itself, i.e. $\mathcal{D}^{\alpha,r} = \mathcal{D}^{\alpha,2^k r}$ for any integer k , so $K^{\alpha,r} = K^{\alpha,2^k r}$. The main point is to show that $K(t, x) = \frac{c_0}{t-x}$. For this purpose it is enough to prove the following properties of $K(t, x)$:

1. Translation invariance, i.e. $K(t, x) = K(t + c, x + c) \quad \forall c \in \mathbb{R}$, so $K(t, x) = K(t - x)$
2. Antisymmetry, i.e. $K(t, x) = -K(-t, -x)$, so $K(x - t) = -K(t - x)$
3. Dilation invariance, i.e. $K(t, x) = \lambda K(\lambda t, \lambda x) \quad \forall \lambda > 0$
4. $K(1) = c_0 > 0$

In order to check the first three properties we observe the following simple relationships between the Haar functions of different dyadic grids for translations, reflections and dilations. For any interval $I \in \mathcal{D}^{\alpha,r}$ there exists an interval of the same length in $\mathcal{D}^{\alpha-c,r}$ so that $h_I^{\alpha,r}(t + c) = h_I^{\alpha-c,r}(t)$. In a similar sense $h_I^{\alpha,r}(-t) = -h_I^{-\alpha,r}(t)$ when changing grids from $\mathcal{D}^{\alpha,r}$ to $\mathcal{D}^{-\alpha,r}$ and $h_I^{\alpha,r}(\lambda t) = \lambda^{-1/2} h_I^{\alpha/\lambda, r/\lambda}(t)$ when changing from $\mathcal{D}^{\alpha,r}$ to $\mathcal{D}^{\alpha/\lambda, r/\lambda}$.

In more detail:

$$\begin{aligned}
& \sum_{\substack{I \in \mathcal{D}^{\alpha,r} \\ |I|=r2^n}} h_I^{\alpha,r}(t+c)(h_{I_-}^{\alpha,r}(x+c) - h_{I_+}^{\alpha,r}(x+c)) \\
&= \sum_{\substack{I \in \mathcal{D}^{\alpha-c,r} \\ |I|=r2^n}} h_I^{\alpha-c,r}(t)(h_{I_-}^{\alpha-c,r}(x) - h_{I_+}^{\alpha-c,r}(x)) \quad \forall c \in \mathbb{R}.
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
& \sum_{\substack{I \in \mathcal{D}^{\alpha,r} \\ |I|=r2^n}} h_I^{\alpha,r}(-t)(h_{I_-}^{\alpha,r}(-x) - h_{I_+}^{\alpha,r}(-x)) \\
&= \sum_{\substack{I \in \mathcal{D}^{-\alpha,r} \\ |I|=r2^n}} -h_I^{-\alpha,r}(t)(-h_{I_+}^{-\alpha,r}(x) + h_{I_-}^{-\alpha,r}(x)) \\
&= - \sum_{\substack{I \in \mathcal{D}^{-\alpha,r} \\ |I|=r2^n}} h_I^{-\alpha,r}(t)(h_{I_-}^{-\alpha,r}(x) - h_{I_+}^{-\alpha,r}(x)).
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
& \sum_{\substack{I \in \mathcal{D}^{\alpha,r} \\ |I|=r2^n}} h_I^{\alpha,r}(\lambda t)(h_{I_-}^{\alpha,r}(\lambda x) - h_{I_+}^{\alpha,r}(\lambda x)) \\
&= \sum_{\substack{I \in \mathcal{D}^{0,r} \\ |I|=r2^n}} h_I^{0,r}(\lambda t - \alpha)(h_{I_-}^{0,r}(\lambda x - \alpha) - h_{I_+}^{0,r}(\lambda x - \alpha)) \\
&= \frac{1}{\lambda} \sum_{\substack{I \in \mathcal{D}^{0,r/\lambda} \\ |I|=r/\lambda 2^n}} h_I^{0,r/\lambda}(t - \alpha/\lambda)(h_{I_-}^{0,r/\lambda}(x - \alpha/\lambda) - h_{I_+}^{0,r/\lambda}(x - \alpha/\lambda)) \\
&= \frac{1}{\lambda} \sum_{\substack{I \in \mathcal{D}^{\alpha/\lambda,r/\lambda} \\ |I|=r/\lambda 2^n}} h_I^{\alpha/\lambda,r/\lambda}(t)(h_{I_-}^{\alpha/\lambda,r/\lambda}(x) - h_{I_+}^{\alpha/\lambda,r/\lambda}(x)) \quad \forall \lambda > 0.
\end{aligned} \tag{2.4}$$

Now we are ready to prove the first three properties.

Proof of translation invariance:

$$\begin{aligned}
& K(t+c, x+c) \\
&= \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(t+c, x+c) d\alpha \frac{dr}{r} \\
&= \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha-c, r}(t, x) d\alpha \frac{dr}{r} \\
&= \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R-c}^{R-c} K^{\alpha, r}(t, x) d\alpha \frac{dr}{r} \\
&= K(t, x)
\end{aligned}$$

by first using (2.2), then a substitution in α and the fact that the modulus of the integrand is bounded by $\frac{2\sqrt{2}}{|x-t|}$.

Proof of antisymmetry:

$$\begin{aligned}
& K(-t, -x) \\
&= \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(-t, -x) d\alpha \frac{dr}{r} \\
&= \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R -K^{-\alpha, r}(t, x) d\alpha \frac{dr}{r} \\
&= -K(t, x)
\end{aligned}$$

by applying (2.3) and a substitution in α .

Proof of dilation invariance:

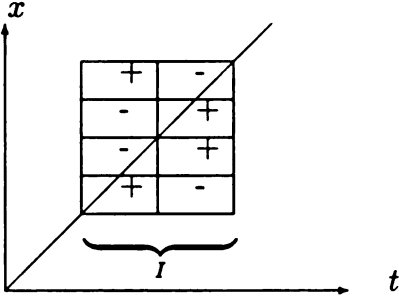
$$\begin{aligned}
K(\lambda t, \lambda x) &= \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(\lambda t, \lambda x) d\alpha \frac{dr}{r} \\
&= \frac{1}{\lambda} \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha/\lambda, r/\lambda}(t, x) d\alpha \frac{dr}{r} \\
&= \frac{1}{\lambda} \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/L}^L \lim_{R \rightarrow \infty} \frac{\lambda}{2R} \int_{-R/\lambda}^{R/\lambda} K^{\alpha, r/\lambda}(t, x) d\alpha \frac{dr}{r} \\
&= \frac{1}{\lambda} \lim_{L \rightarrow \infty} \frac{1}{2 \log L} \int_{1/(L\lambda)}^{L/\lambda} \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(t, x) d\alpha \frac{dr}{r} \\
&= \frac{1}{\lambda} K(t, x)
\end{aligned}$$

by first using (2.4), then a substitution in α , then a substitution in r and the fact that the integrand is bounded by $\frac{2\sqrt{2}}{|x-t|}$.

These three properties prove that the kernel we obtain must be $K(t, x) = \frac{c_0}{t-x}$. For our purposes it is essential to know that $c_0 \neq 0$.

2.2.1 Proof that $K(1) = c_0 > 0$

Let us first illustrate $h_I(t)(h_{I_-}(x) - h_{I_+}(x))$:

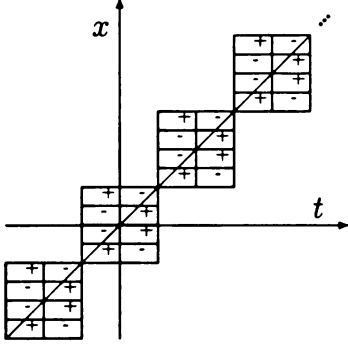


$h_I(t)(h_{I_-}(x) - h_{I_+}(x)) \neq 0$ if and only if the point (t, x) lies in this square. Its value is $\pm \frac{\sqrt{2}}{|I|}$, where the correct sign is indicated inside the smaller rectangles.

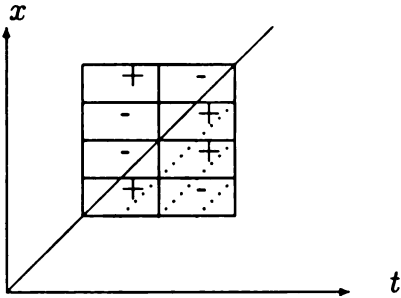
Let us first compute

$$K_n^r(t, x) := \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R \sum_{\substack{I \in \mathcal{D}^{\alpha, r} \\ |I| = r2^n}} h_I(t) (h_{I_-}(x) - h_{I_+}(x)) d\alpha, \quad (2.5)$$

for fixed $r > 0$ and $n \in \mathbb{Z}$ and assuming $t > x$. The picture is the following:



The exact location of the squares along the diagonal is influenced by the starting point α . The picture will repeat for two values of α that differ by an integer multiple of $|I|$. We compute (2.5) in (t, x) by considering the probability that (t, x) lies in any of the squares. Due to the averaging process in α , this is only going to depend on $t - x$. We only need to compute for $|t - x|$ on the dotted lines:



If

$$t - x = 0 \quad \text{then } K_n^r(t, x) = \frac{1}{4} \frac{\sqrt{2}}{|I|} + \frac{1}{4} \left(-\frac{\sqrt{2}}{|I|} \right) + \frac{1}{4} \frac{\sqrt{2}}{|I|} + \frac{1}{4} \left(-\frac{\sqrt{2}}{|I|} \right), \text{ and similarly}$$

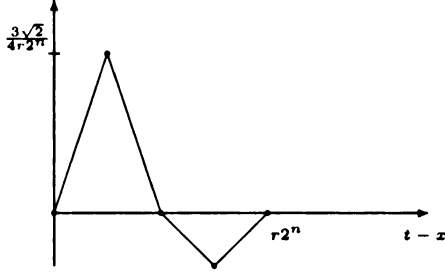
$$t - x = \frac{1}{4}|I| \quad \text{then } K_n^r(t, x) = \frac{3}{4} \frac{\sqrt{2}}{|I|}$$

$$t - x = \frac{1}{2}|I| \quad \text{then } K_n^r(t, x) = 0$$

$$t - x = \frac{3}{4}|I| \quad \text{then } K_n^r(t, x) = -\frac{1}{4} \frac{\sqrt{2}}{|I|}$$

$$t - x \geq |I| \quad \text{then } K_n^r(t, x) = 0.$$

Since $K_n^r(t, x) = K_n^r(t - x)$ is piecewise linear in $t - x$, we obtain the following graph, depending on n and r :



Next we compute

$$K^r(t, x) := \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(t, x) d\alpha. \quad (2.6)$$

First note that

$$K^r(t, x) = \sum_{n \in \mathbb{Z}} \lim_{R \rightarrow \infty} \frac{1}{2R} \int_{-R}^R \sum_{\substack{I \in \mathcal{D}^{\alpha, r} \\ |I| = r2^n}} h_I(t)(h_{I-}(x) - h_{I+}(x)) d\alpha, \quad (2.7)$$

so we compute $K^r(t, x)$ using $K_n^r(t, x)$ defined in equation (2.5) for different values of n and summing over $n \in \mathbb{Z}$. It suffices to compute $K^r(t, x)$ for values $t - x = \frac{3}{4}r2^n$ and $t - x = r2^n$ since the graph is piecewise linear on intervals $[2^{n-1}, \frac{3}{4}r2^n]$ and $[\frac{3}{4}r2^n, 2^n]$ for all n .

For $t - x = \frac{3}{4}r2^n$ we obtain that

$$K^r(\frac{3}{4}r2^n) = -\frac{1}{4} \frac{\sqrt{2}}{r2^n} + \frac{3}{16} \frac{\sqrt{2}}{r2^n} + \frac{9}{64} \frac{\sqrt{2}}{r2^n} (1 + \frac{1}{4} + \frac{1}{16} + \dots) = \frac{\sqrt{2}}{8r2^n}, \quad (2.8)$$

and for $t - x = r2^n$ we get

$$K^r(r2^n) = \frac{3}{16} \frac{\sqrt{2}}{r2^n} (1 + \frac{1}{4} + \frac{1}{16} + \dots) = \frac{\sqrt{2}}{4r2^n}. \quad (2.9)$$

Equations (2.8) and (2.9) imply that

$$\frac{3\sqrt{2}}{32(t-x)} \leq K^r(t-x) \leq \frac{\sqrt{2}}{4(t-x)} \quad \forall r > 0. \quad (2.10)$$

The expression in Lemma 2.1 is obtained from $K^r(t-x)$ by a limit of averages in r , so it is clear from equation (2.10) that $c_0 > 0$. ■

CHAPTER 3

Application to Hankel Operators with Matrix Symbol

3.1 Definitions and Statement

We consider vector valued $L^2 = L^2_{\mathbb{R}}(\mathbb{C}^n)$ for $n \geq 2$, i.e., measurable $f : \mathbb{R} \rightarrow \mathbb{C}^n$ such that $\int_{\mathbb{R}} \langle f, f \rangle_{\mathbb{C}^n} = \|f\|_{L^2_{\mathbb{R}}(\mathbb{C}^n)}^2 < \infty$. We also consider BMO - matrix functions B of size $n \times n$ with ' $B \in BMO$ ' in the following sense. If B is an $n \times n$ matrix function on $\mathbb{R}$, we say that $B \in BMO_c$ if for all $e \in \mathbb{C}^n$, $\|e\|$ denoting its Hilbert space norm,

$$\sup_J \langle \|Be - \langle B \rangle_J e\|^2 \rangle_J \leq a \|e\|^2, \quad (3.1)$$

where the supremum is taken over all intervals in $\mathbb{R}$. The notation $\langle \cdot \rangle_J$ means averaging each entry over J . We say that $B \in BMO_r$ if $B^* \in BMO_c$, meaning that for all $e \in (\mathbb{C}^n)^*$

$$\sup_J \langle \|eB - e \langle B \rangle_J\|^2 \rangle_J \leq a \|e\|^2. \quad (3.2)$$

We say that $B \in BMO$ if $B \in BMO_c \cap BMO_r$. The square root of the best a satisfying both (3.1) and (3.2) will be denoted as $\|B\|_{BMO}$. Note that this definition is 'symmetric', meaning that B and B^* have the same norm.

We are going to prove that the commutator $HB - BH$, H denoting the Hilbert transform, as an operator from $L^2_{\mathbb{R}}(\mathbb{C}^n)$ to itself is bounded by a $\log n$ multiple of $\|B\|_{BMO}$. To be precise,

Theorem 3.1 *There exists $C > 0$ such that for all $B \in BMO$*

$$\|HB - BH\|_{L^2_{\mathbb{R}}(\mathbb{C}^n) \rightarrow L^2_{\mathbb{R}}(\mathbb{C}^n)} \leq C \log n \|B\|_{BMO}.$$

3.2 The Proof

PROOF.

We estimate $\|HB - BH\|$ by relating it to $\|\text{III}B - B\text{III}\|$.

The operators $\text{III}^{\alpha,r}$ are norm bounded by $\sqrt{2}$, so the collection of operators T_R^r , defined by

$$T_R^r : f \mapsto \frac{1}{2R} \int_{-R}^R \text{III}^{\alpha,r} f \, d\alpha$$

are, as averages, also norm bounded by $\sqrt{2}$. By compactness of the closed unit ball in $\mathcal{B}(L^2)$ in the weak operator topology and the fact that the unit ball is metrizable, there is a subsequence $T_{R_k}^r$ such that $T_{R_k}^r$ converges weakly, which means there exists T^r with norm $\leq \sqrt{2}$ such that $\forall f, g \in L^2 : (T_{R_k}^r f, g) \longrightarrow (T^r f, g)$ as $k \rightarrow \infty$. Then consider

$$T_L : f \mapsto \frac{1}{2 \log L} \int_{1/L}^L T^r f \frac{dr}{r}.$$

By the same argument there exists a subsequence T_{L_k} that converges weakly to a bounded operator T .

Let us show that T is represented by the kernel $c_0/(t-x)$. Because of bounded convergence in Lemma 2.1, we have for f, g compactly supported with disjoint supports,

$$\begin{aligned}
(Tf, g) &= \lim_{k \rightarrow \infty} (T_{L_k} f, g) \\
&= \lim_{k \rightarrow \infty} \frac{1}{2 \log(L_k)} \int_{1/L_k}^{L_k} (T^r f, g) \frac{dr}{r} \\
&= \lim_{k \rightarrow \infty} \frac{1}{2 \log(L_k)} \int_{1/L_k}^{L_k} \lim_{l \rightarrow \infty} \frac{1}{2R_l} \int_{-R_l}^{R_l} (\text{III}^{\alpha, r} f, g) d\alpha \frac{dr}{r} \\
&= \left(\int f(t) \lim_{k \rightarrow \infty} \frac{1}{2 \log(L_k)} \int_{1/L_k}^{L_k} \lim_{l \rightarrow \infty} \frac{1}{2R_l} \int_{-R_l}^{R_l} K^{\alpha, r}(t, x) d\alpha \frac{dr}{r} dt, g \right) \\
&= c_0 (Hf, g).
\end{aligned}$$

By [15] (p. 33) there exists a bounded measurable function a such that $Tf(x) = c_0 Hf(x) + a(x)f(x)$ a.e., hence $c_0(HB - BH)f = (TB - BT)f$. By convexity $\|TB - BT\| \leq \sup_{\alpha, r} \|\text{III}^{\alpha, r} B - B \text{III}^{\alpha, r}\|$. So we are left to estimate the commutators with $\text{III}^{\alpha, r}$ uniformly. We show that for all $\alpha \in \mathbb{R}$ and for all $r > 0$

$$\|\text{III}^{\alpha, r} B - B \text{III}^{\alpha, r}\|_{L^2_{\mathbb{R}}(\mathbb{C}^n) \rightarrow L^2_{\mathbb{R}}(\mathbb{C}^n)} \leq C \log n \|B\|_{BMO}. \quad (3.3)$$

In the following α, r will be omitted because all estimates do not depend on the dyadic grid. It suffices to consider $f \in \mathcal{D}(\mathbb{R})$ only, because the estimates do not depend on the support of f . First let us decompose the product Bf , at least for ‘almost compactly supported’ B in the sense that B differs from a compactly supported function only by an additive constant. Note that for any such BMO matrix function, the sum

$$B(x) = \sum_{I \in \mathcal{D}} (B, h_I) h_I(x) \quad (3.4)$$

is meaningful, in the sense that it converges unconditionally to an L^2 function. The reason is that coefficients of constant functions vanish, so (3.4) ignores constants and

treats our B like a compactly supported one, whose entries are in L^2 . For our vector function f we have

$$f(x) = \sum_{I \in \mathcal{D}} (f, h_I) h_I(x). \quad (3.5)$$

By multiplying the sums (3.4) and (3.5) formally one gets

$$Bf = A_B(f) + \Pi_B(f) + R_B(f),$$

where

$$A_B(f) = \sum_{I \in \mathcal{D}} (B, h_I) (f, h_I) h_I^2,$$

$$\Pi_B(f) = \sum_{I \in \mathcal{D}} (B, h_I) \langle f \rangle_I h_I,$$

$$R_B(f) = \sum_{I \in \mathcal{D}} \langle B \rangle_I (f, h_I) h_I.$$

The formal multiplication is meaningful due to our assumptions on f and B . Hence $SB - BS = SA_B - A_BS + S\Pi_B - \Pi_BS + SR_B - R_BS$ and we can estimate the terms separately for our special class of B .

By [9] and [6]

$$\|\Pi_B\|_{L^2_{\mathbf{R}}(\mathbb{C}^n) \rightarrow L^2_{\mathbf{R}}(\mathbb{C}^n)} \leq C \log n \|B\|_{BMO},$$

where C does not depend on n . The proof in [9] is by a Bellman function construction, whereas the independent proof in [6] is by a stopping time procedure.

For the part involving A_B , note that $A_B^* = \Pi_{B^*}$. So

$$\|A_B\| \leq C \log n \|B^*\|_{BMO} = C \log n \|B\|_{BMO},$$

so the parts involving Π_B and A_B are bounded by $\sqrt{2}C \log n \|B\|_{BMO}$.

We estimate the last term as commutator:

$$\begin{aligned}
& \text{III} R_B f - R_B \text{III} f \\
&= \sum_I \langle B \rangle_I (f, h_I) h_{I_-} - \sum_I \langle B \rangle_{I_-} (f, h_I) h_{I_-} + \\
&\quad \sum_I \langle B \rangle_I (f, h_I) h_{I_+} - \sum_I \langle B \rangle_{I_+} (f, h_I) h_{I_+} \\
&= \frac{1}{2} \sum_I (\langle B \rangle_{I_+} - \langle B \rangle_{I_-}) (f, h_I) (h_{I_-} - h_{I_+}).
\end{aligned}$$

So it suffices to show that

$$\|(\langle B \rangle_{I_+} - \langle B \rangle_{I_-})e\|^2 \leq C \|B\|_{BMO}^2 \|e\|^2 \quad \forall e \in \mathbb{C}^n$$

with C independent of n . In fact,

$$\begin{aligned}
& \|(\langle B \rangle_{I_+} - \langle B \rangle_{I_-})e\|^2 \\
&= 4 \|(\langle B \rangle_I - \langle B \rangle_{I_-})e\|^2 \\
&= 4 \left\| \frac{1}{|I_-|} \int_{I_-} (\langle B \rangle_I - B(t))e \, dt \right\|^2 \\
&\leq \frac{4}{|I_-|} \int_{I_-} \|(\langle B \rangle_I - B(t))e\|^2 \, dt \\
&\leq \frac{8}{|I|} \int_I \|(\langle B \rangle_I - B(t))e\|^2 \, dt \\
&\leq 8 \|B\|_{BMO}^2 \|e\|^2.
\end{aligned}$$

Now we pass from our ‘almost compactly supported’ B to general $B \in BMO$. For fixed B we consider the sequence of intervals $I_k = [-k, k]$. We can construct B_k so that $B_k = B$ on I_k , $B_k = \langle B \rangle_{I_k}$ outside the interval concentric with I_k and three times its length and furthermore, $\|B_k\|_{BMO} \leq c \|B\|_{BMO}$ with c independent of k . A suggestion for such a construction can be found in [3] (p.269). So the family of operators $\text{III} B_k - B_k \text{III}$ is uniformly bounded in L^2 , by $C \log n \|B\|_{BMO}$. Hence by a weak compactness argument, the operator $\text{III} B - B \text{III}$ is bounded by the same norm, which gives us the estimate for general B , finishing the proof of inequality (3.3). ■

3.3 Sharpness of Result

In [7] it has been proven that there exists an $n \times n$ matrix function B such that

$$\|HB - BH\|_{L^2(\mathbb{C}^n) \rightarrow L^2(\mathbb{C}^n)} \geq C' \log n \|B\|_{BMO} \quad (3.6)$$

which shows that the estimate in terms of $\log n$ is sharp. We would like to point out that the same averaging technique has been used to obtain the lower bounds for paraproduct operators and hence for Carleson imbedding theorem using the lower bounds for commutators. The proofs can be found in [7]. As a consequence, the $\log n$ upper bounds for paraproducts and Carleson imbedding theorem are sharp.

CHAPTER 4

The Cubic Bound for the Hilbert Transform

4.1 Definitions and Statement

We are now in scalar but weighted situation. We consider weights $\omega \in L^1_{loc}$, where $\omega \in A_2$ with norm

$$\|\omega\|_{A_2} = \sup_I \sqrt{\langle \omega \rangle_I \langle \omega^{-1} \rangle_I},$$

where the supremum runs over all intervals. We also consider a dyadic version, A_2^d , where the supremum runs over dyadic intervals only. So the norm $\|\omega\|_{A_2^d}$ may depend on the choice of the dyadic grid. We are concerned with weighted L^2 spaces, denoted by $L^2_{\mathbb{R}}(\omega)$, containing functions so that $\|f\|_{\omega}^2 := \int_{\mathbb{R}} |f|^2 \omega < \infty$. Our main theorem in this chapter is the following:

Theorem 4.1 $H : L^2_{\mathbb{R}}(\omega) \rightarrow L^2_{\mathbb{R}}(\omega)$ has operator norm $\|H\| \leq c\|\omega\|_{A_2}^3$.

We will reduce the problem to upper and lower bounds of certain square functions, using averaging technique from chapter 2. The square function S is defined by

$$Sf(t) = \sqrt{\int_{\Sigma} |(T_{\varepsilon}f)(t)|^2 d\varepsilon} = \sqrt{\sum_I |(f, h_I)|^2 \frac{\chi_I(t)}{|I|}}$$

where Σ denotes the space $\{-1, 1\}^{\mathcal{D}}$ provided with the natural measure $d\varepsilon$ which assigns equal measure 2^{-k} to every cylindrical subset of $\{-1, 1\}^{\mathcal{D}}$ of length 2^k and T_{ε} is the martingale transform

$$T_{\varepsilon} : f \mapsto \sum_I \varepsilon(I)(f, h_I)h_I$$

associated with the sequence $\varepsilon(I) \in \{-1, 1\}^{\mathcal{D}}$. The following has been proven in [2] (in a harmonic version) and independently in [12]:

Theorem 4.2 *There exists $c > 0$ such that for all $f \in L^2(\omega)$*

$$\|f\|_{\omega} \leq c\|\omega\|_{A_2^d}\|Sf\|_{\omega}.$$

In [5] the sharp upper bound has been proven:

Theorem 4.3 *There exists $c > 0$ such that for all $f \in L^2(\omega)$*

$$\|Sf\|_{\omega} \leq c\|\omega\|_{A_2^d}^2\|f\|_{\omega}.$$

Both [19] and [12] contain a proof of the fact that the quadratic upper bound follows from the linear lower bound, which implies that the lower bound is sharp as well.

4.2 The Proof

We will give a short and elegant proof of Theorem 4.1. It was found together with Sandra Pott.

As seen in Chapter 2, H lies in the closed convex hull of dyadic shift operators. The square function does not ‘see’ the dyadic shift:

Proposition 4.4 $(S\mathbb{I}\mathbb{I}f)(x) = \sqrt{2}(Sf)(x)$ for all x .

PROOF.

$$\begin{aligned}
& S\mathbb{I}\mathbb{I}f(x)^2 \\
&= \int_{\Sigma} |(T_{\varepsilon}\mathbb{I}\mathbb{I}f)(x)|^2 d\varepsilon \\
&= \int_{\Sigma} \left| \sum_I \varepsilon(I)(\mathbb{I}\mathbb{I}f, h_I)h_I(x) \right|^2 d\varepsilon \\
&= \int_{\Sigma} \left| \sum_I (f, h_I)(\varepsilon(I_-)h_{I_-} - \varepsilon(I_+)h_{I_+}) \right|^2 d\varepsilon \\
&\stackrel{(*)}{=} 2 \int_{\Sigma} \left| \sum_I \varepsilon(I)(f, h_I)h_I(x) \right|^2 d\varepsilon \\
&= 2Sf(x)^2
\end{aligned}$$

where $(\star)$ is an effect of the averaging over sequences of signs $\varepsilon(I)$ and the fact that for each fixed x there exists a sequence of signs $\tilde{\varepsilon}(I)$ so that we have for all I : $\sqrt{2}h_I(x) = \tilde{\varepsilon}(I)(\varepsilon(I_-)h_{I_-} - \varepsilon(I_+)h_{I_+})(x)$. ■

Now it is easy to prove Theorem 4.1:

PROOF.

Dyadic shifts with respect to all translates and dilates of the standard dyadic grid have cubic bound, indeed

$$\|\mathbb{I}\mathbb{I}^{\alpha,r}f\|_{\omega} \stackrel{(1)}{\leq} c\|\omega\|_{A_2^{d\alpha,r}}\|S\mathbb{I}\mathbb{I}^{\alpha,r}f\|_{\omega} \stackrel{(2)}{=} c\|\omega\|_{A_2^{d\alpha,r}}\|Sf\|_{\omega} \stackrel{(3)}{\leq} c\|\omega\|_{A_2^{d\alpha,r}}^3\|f\|_{\omega}$$

where (1) is by Theorem 4.2, the lower bound for the square function, (2) by Proposition 4.4, the fact that the square function does not see the dyadic shift and (3) by Corollary 4.3, the upper bound for the square function. By convexity, as before, we obtain the desired bound for the Hilbert transform:

$$\|H\|_{L^2(\omega) \rightarrow L^2(\omega)} \leq c \sup_{\alpha,r} \|\mathbb{I}\mathbb{I}^{\alpha,r}\|_{L^2(\omega) \rightarrow L^2(\omega)} \leq c \sup_{\alpha,r} \|\omega\|_{A_2^{d\alpha,r}}^3 \leq c\|\omega\|_{A_2}^3.$$

■

CHAPTER 5

A Sharp Bound for Weighted Hilbert Transform

5.1 Definitions and Statement

In this section, it is more convenient to work on the unit circle $\mathbb{T}$. We consider the space $L^2_{\mathbb{T}}(\omega)$ where, as before, ω is a positive L^1 function, called a weight. Let m be normalized Lebesgue measure on $\mathbb{T}$. The norm of $f \in L^2_{\mathbb{T}}(\omega)$ is $(\int_{\mathbb{T}} |f|^2 \omega dm)^{1/2}$ and denoted by $\|f\|_{\omega}$. We are, as before, concerned with a special class of weights, called A_2 . We say $\omega \in A_2$ if

$$\|\omega\|_{A_2} := \sup_I \sqrt{\langle \omega \rangle_I \langle \omega^{-1} \rangle_I} < \infty$$

where the supremum is taken over all dyadic subarcs $I \subset \mathbb{T}$. The notation $\langle \omega \rangle_I$ means the average of the function ω over I . We also consider a version of A_2 that is invariant under Möbius transforms, called $A_{2,inv}$. A weight $\omega \in A_{2,inv}$ if

$$\|\omega\|_{A_{2,inv}} := \sup_{z \in \mathbb{D}} \sqrt{\omega(z) \omega^{-1}(z)} < \infty$$

where $\omega(z)$ denotes the harmonic extension of ω , so $\omega(z) = \int \omega(t) P_z(t) dm(t)$, where $P_z(t) = \frac{1-|z|^2}{|1-\bar{z}t|^2}$. Note that in general $\omega^{-1}(z)$ and $\omega(z)^{-1}$ have different meaning.

The first expression means taking the harmonic extension of the reciprocal of ω , the second one means taking the reciprocal of the harmonic extension of ω . Observe that $\omega(z)\omega^{-1}(z) \geq 1$ by Jensen's inequality, so $\|\omega\|_{A_{2,inv}} \geq 1$. In [4] it has been proven that the exact sharp relationship between the two different A_2 norms is as follows:

$$c_1\|\omega\|_{A_2} \leq \|\omega\|_{A_{2,inv}} \leq c_2\|\omega\|_{A_2}^2.$$

In particular, $\omega \in A_2$ if and only if $\omega \in A_{2,inv}$.

In what follows, H stands for Hilbert transform on the circle $\mathbb{T}$. H acts on trigonometric polynomials as follows:

$$H\left(\sum a_k e^{i\theta k}\right) = -i \sum_{k \geq 0} a_k e^{i\theta k} + i \sum_{k < 0} a_k e^{i\theta k}.$$

Let H_0 be the operator $H + iP_0$ where $P_0 : f \mapsto f(0)$.

Theorem 5.1 $H : L^2_{\mathbb{T}}(\omega) \rightarrow L^2_{\mathbb{T}}(\omega)$ has operator norm $\|H\| \leq c\|\omega\|_{A_{2,inv}}^2$.

Note again, that in our notation $\|\omega\|_{A_{2,inv}}^2 = \sup_{z \in \mathbb{D}} \omega(z)\omega^{-1}(z)$.

5.2 Proof

$\|P_0\|_{L^2(\omega) \rightarrow L^2(\omega)} \leq \|\omega\|_{A_{2,inv}}$. Indeed, applying Jensen's inequality we obtain $\|P_0(f)\|_{\omega}^2 = |f(0)|^2 \omega(0) \leq (|f|^2 \omega)(0) \omega^{-1}(0) \omega(0) \leq \|\omega\|_{A_{2,inv}}^2 \|f\|_{\omega}^2$. As $\|H\| \leq \|H_0\| + \|P_0\|$ and $\|\omega\|_{A_{2,inv}} \geq 1$ it suffices to show that $\|H_0\| \leq c\|\omega\|_{A_{2,inv}}^2$. We estimate $\|H_0\|_{L^2_{\mathbb{T}}(\omega) \rightarrow L^2_{\mathbb{T}}(\omega)}$ by duality. Since $(H_0 t f, g/t) = (H_0 f, g)$, it is enough to show that $|(H_0 f, g)| \leq c\|\omega\|_{A_{2,inv}}^2 (\|f\|_{\omega}^2 + \|g\|_{\omega^{-1}}^2)$ for all $f \in L^2_{\mathbb{T}}(\omega)$ and $g \in L^2_{\mathbb{T}}(\omega^{-1})$ (just use $t = \sqrt{\|g\|_{\omega^{-1}}/\|f\|_{\omega}}$ if $f \neq 0$). It suffices to consider real valued and non-negative functions f and g . By polarizing [3] (p. 236) we have

$$\int_{\mathbb{T}} (H_0 f - H_0 f(0))(g - g(0)) dm = \frac{1}{2\pi} \int_{\mathbb{D}} (\nabla H_0 f)(\nabla g) \log \frac{1}{|z|} dA(z).$$

Due to $H_0 f(0) = 0$ the left hand side equals $(H_0 f, g)$. Since $|\nabla H_0 f| = |\nabla f|$ we get

$$|(H_0 f, g)| \leq \frac{1}{2\pi} \int_{\mathbf{D}} |\nabla f| |\nabla g| \log \frac{1}{|z|} dA(z).$$

Note that for f real valued, $|\nabla f| = 2|\partial f/\partial z|$. We will write $f(z)'$ for the holomorphic function $\partial f/\partial z$.

We split the integral into four parts. One can see that

$$\begin{aligned} \int_{\mathbf{D}} |f(z)'| |g(z)'| \log \frac{1}{|z|} dA(z) \leq & \\ & \int_{\mathbf{D}} |f(z)| |g(z)| \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|} dA(z) \\ & + \int_{\mathbf{D}} |f(z)| |g(z)| \left| \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|} dA(z) \\ & + \int_{\mathbf{D}} |f(z)| |g(z)| \left| \frac{\omega(z)'}{\omega(z)} \right| \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \log \frac{1}{|z|} dA(z) \\ & + \int_{\mathbf{D}} |f(z)| |g(z)| \left| \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|} dA(z). \end{aligned}$$

The first integral can be controlled in the same way as done in [9], in fact, the proofs are identical. For the second and third integral we need to proceed in two steps. Again, we want to use the same proof as in [9], but in order to do that we need to have an estimate for a certain Green's potential function involving the weight. But there is a dyadic analog for this estimate, which showed up in the proof for sharp bounds for the dyadic square function in $L^2(\omega)$ and was proven by Bellman function technique (see [5]). We will use the same Bellman function to obtain estimates for the Green's potential. The fourth integral requires what is known as bilinear harmonic imbedding theorem. The appropriate Bellman function was constructed with help from [9]. We give an explicit expression for the function. The imbedding conditions are again certain Green's potentials. We find the appropriate bounds using Bellman functions found in [9] and [19].

Before we start to estimate the four integrals we need the following lemma to relate Laplacians to second differentials:

Lemma 5.2 *If $b(z) = B(h(z))$ where $h = (f_i)_i : \mathbb{C} \rightarrow \mathbb{R}^n$ and $B : \mathbb{R}^n \rightarrow \mathbb{R}$ with B and h sufficiently smooth, then*

$$\Delta b(z) = 4 \left(d^2 B(h(z)) \left(\frac{\partial f_i}{\partial z} \right)_i, \left(\frac{\partial f_i}{\partial z} \right)_i \right) + 4(\nabla B)(h(z)) \left(\frac{\partial^2 f_i}{\partial z \partial \bar{z}} \right)_i \quad (5.1)$$

In particular, if all f_i are harmonic, then

$$\Delta b(z) = 4 \left(d^2 B(h(z)) \left(\frac{\partial f_i}{\partial z} \right)_i, \left(\frac{\partial f_i}{\partial z} \right)_i \right) \quad (5.2)$$

PROOF.

By elementary computation. ■

We will be using the appropriate Bellman functions to bound all integrals. Each variable carries meaning, usually harmonic extensions of functions or Green's potentials for some fixed z . The following variables show up frequently:

$$\begin{aligned} X &= f^2 \omega(z) & Y &= g^2 \omega^{-1}(z) \\ x &= f(z) & y &= g(z) \\ r &= \omega^{-1}(z) & s &= \omega(z) \end{aligned}$$

If we assume f, g to be real and nonnegative, all variables will be nonnegative. Furthermore we have the following natural estimates:

$$1 \leq rs \leq Q^2 \text{ if we write } Q \text{ for } \|\omega\|_{2,inv} \quad (5.3)$$

$$x^2 \leq Xr \text{ and } y^2 \leq Ys \text{ because of Jensen's inequality.} \quad (5.4)$$

These restrictions give a natural domain of our Bellman functions.

5.2.1 The First Integral

Consider the following function of six real variables

$$B(X, x, r, Y, y, s) = X - \frac{x^2}{r} + Y - \frac{y^2}{s}$$

then we get the following size estimate within the natural domain of B :

$$0 \leq B \leq X + Y.$$

and by direct computation of the second differential we get

$$-d^2B = \frac{2x^2}{r} \left| \frac{dx}{x} - \frac{dr}{r} \right|^2 + \frac{2y^2}{s} \left| \frac{dy}{y} - \frac{ds}{s} \right|^2. \quad (5.5)$$

Also consider the function $b : \mathbb{C} \rightarrow \mathbb{R}$

$$b(z) = B(h(z)) = B(f^2\omega(z), f(z), \omega^{-1}(z), g^2\omega^{-1}(z), g(z), \omega(z)),$$

then we obtain the following estimate for $-\Delta b(z)$ using (5.2) and (5.5)

$$\begin{aligned} & -\Delta b(z) \\ &= 8 \frac{|f(z)|^2}{\omega^{-1}(z)} \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right|^2 + 8 \frac{|g(z)|^2}{\omega(z)} \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right|^2 \\ &\geq 16 \frac{|f(z)g(z)|}{\sqrt{\omega^{-1}(z)\omega(z)}} \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \\ &\geq 16 \frac{|f(z)g(z)|}{Q} \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right|. \end{aligned}$$

We use the above estimate for $-\Delta b(z)$ and Green's formula to estimate the first integral:

$$\begin{aligned} & \int_{\mathbf{D}} |f(z)g(z)| \left| \frac{f(z)'}{f(z)} - \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|} dA(z) \\ &\leq c \|\omega\|_{A_{2,inv}} \int_{\mathbf{D}} -\Delta b(z) \log \frac{1}{|z|} dA(z) \\ &= c \|\omega\|_{A_{2,inv}} \left(b(0) - \int_{\mathbf{T}} b dm \right) \\ &\leq c \|\omega\|_{A_{2,inv}} (\|f\|_{\omega}^2 + \|g\|_{\omega^{-1}}^2). \end{aligned}$$

The last step uses that $b \equiv 0$ on $\mathbb{T}$ and that the size estimate $B \leq X + Y$ means $b(0) \leq \|f\|_\omega^2 + \|g\|_{\omega^{-1}}^2$.

5.2.2 The Second and the Third Integral

The second and the third integral are analogous, so let us only prove the estimate for the second one.

We consider the function from [5]

$$B(r, s) = r \left(-\frac{4Q^4}{rs} - rs + 4Q^4 + 1 \right),$$

this function has the following properties:

$$1 \leq rs \leq Q^2 \implies 0 \leq B(r, s) \leq cQ^4 r,$$

$$1 \leq rs \leq Q^2 \implies -d^2 B \geq Cs(dr)^2.$$

Let us also consider the function $b : \mathbb{C} \rightarrow \mathbb{R}$

$$b(z) = B(h(z)) = B(\omega^{-1}(z), \omega(z)),$$

$$0 \leq b(z) \leq cQ^4 \omega^{-1}(z) \text{ and } -\Delta b(z) \geq c\omega(z)|\omega^{-1}(z)'|^2.$$

This function will help us to estimate the following Green's potential:

$$\begin{aligned} & G(|\omega^{-1}'|^2 \omega)(z) \\ &= \int_{\mathbf{D}} \log \frac{1}{|S_z(\xi)|} |\omega^{-1}(\xi)'|^2 \omega(\xi) dA(\xi) \\ &\leq c \int_{\mathbf{D}} -\Delta b(\xi) \log \frac{1}{|S_z(\xi)|} dA(\xi) \\ &\stackrel{(*)}{=} c \int_{\mathbf{D}} -\Delta b(S_{-z}(\xi)) \log \frac{1}{|\xi|} dA(\xi) \\ &= c \left(b(z) - \int_{\mathbf{T}} b dm \right) \\ &\leq cQ^4 \omega^{-1}(z), \end{aligned}$$

where $S_z(\xi) = \frac{\xi-z}{1-\bar{z}\xi}$. In $(\star)$ we just did a change of variables $\xi \mapsto S_{-z}(\xi)$ (note that the symbol Δ carries the variable as well). Hence we proved that

$$G(|\omega^{-1}|^2\omega)(z) \leq c\|\omega\|_{A_{2,\text{inv}}}^4 \omega^{-1}(z)$$

and analogously

$$G(|\omega'|^2\omega^{-1})(z) \leq c\|\omega\|_{A_{2,\text{inv}}}^4 \omega(z).$$

The reader should note the similarity between the estimate for the Green's potential and its dyadic analog found in [5]

$$\frac{1}{|J|} \sum_{I \subset J} |\langle \omega \rangle_{I_+} - \langle \omega \rangle_{I_-}|^2 \langle \omega^{-1} \rangle_I |I| \leq cQ^4 \langle \omega \rangle_J.$$

Functions of similar form as discussed in the proposition below will appear frequently. We take care of their concavity.

Proposition 5.3 *Functions of the form*

$$f(w, x, y, z) = w - \frac{x^2}{y+z} \tag{5.6}$$

with $y > 0$ and $z \geq 0$ are concave.

PROOF. The matrix

$$-d^2 f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{y+z} & \frac{-2x}{(y+z)^2} & \frac{-2x}{(y+z)^2} \\ 0 & \frac{-2x}{(y+z)^2} & \frac{2x^2}{(y+z)^3} & \frac{2x^2}{(y+z)^3} \\ 0 & \frac{-2x}{(y+z)^2} & \frac{2x^2}{(y+z)^3} & \frac{2x^2}{(y+z)^3} \end{pmatrix}$$

is positive semidefinite. ■

Let us introduce a new variable

$$G = G(|\omega^{-1'}|^2 \omega)(z).$$

Now we are ready to steal from [9] the Bellman function used to prove weighted dyadic imbedding theorem. We let

$$B(X, x, r, G, Y, y, s) = X - \frac{x^2}{r + \frac{G}{Q^4}} + Y - \frac{y^2}{s}$$

B is, as a sum of two functions of the form discussed in Proposition 5.3, concave.

Consider

$$b(z) = B(h(z)) = B(f^2 \omega(z), f(z), \omega^{-1}(z), G(|\omega^{-1'}|^2 \omega)(z), g^2 \omega^{-1}(z), \omega(z))$$

We will have to estimate $-\Delta b(z)$ from below. We use equation (5.1) to estimate the part involving X, x, r, G , where the concavity of B allows us to drop the part involving the second differential. We only need to consider partial derivative in the ‘non-harmonic variable’ G . Note that $-\Delta G(|\omega^{-1'}|^2 \omega) = |\omega^{-1'}|^2 \omega$. We use (5.2) and (5.5) for the part involving Y, y, s .

$$\begin{aligned} -\Delta b(z) &\geq Q^{-4} \frac{f(z)^2 (-\Delta G(|\omega^{-1'}|^2 \omega)(z))}{(\omega^{-1}(z) + Q^{-4} G(|\omega^{-1'}|^2 \omega)(z))^2} + 8 \frac{g(z)^2}{\omega(z)} \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right|^2 \\ &\geq cQ^{-4} \frac{f(z)^2 |\omega^{-1}(z)'|^2 \omega(z)}{(\omega^{-1}(z) + Q^{-4} G(|\omega^{-1'}|^2 \omega)(z))^2} + c \frac{g(z)^2}{\omega(z)} \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right|^2 \\ &\geq cQ^{-4} \frac{f(z)^2 |\omega^{-1}(z)'|^2 \omega(z)}{\omega^{-1}(z)^2} + c \frac{g(z)^2}{\omega(z)} \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right|^2 \\ &\geq cQ^{-2} |f(z)g(z)| \left| \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right|. \end{aligned}$$

Now we use Green’s formula, the fact that $b \geq 0$ on $\mathbb{T}$ and $b(0) \leq \|f\|_\omega^2 + \|g\|_{\omega^{-1}}^2$

to estimate the second integral:

$$\begin{aligned}
& \int_{\mathbf{D}} |f(z)| |g(z)| \left| \frac{\omega^{-1}(z)'}{\omega^{-1}(z)} \right| \left| \frac{g(z)'}{g(z)} - \frac{\omega(z)'}{\omega(z)} \right| \log \frac{1}{|z|} dA(z) \\
& \leq c \|\omega\|^2 \int_{\mathbf{D}} -\Delta b(z) \log \frac{1}{|z|} dA(z) \\
& \leq c \|\omega\|^2 (\|f\|_{\omega}^2 + \|g\|_{\omega^{-1}}^2).
\end{aligned}$$

5.2.3 The Fourth Integral

We will apply Lemma 6.1, the harmonic bilinear imbedding theorem whose statement and proof can be found in chapter 6. We apply it for the weights ω and $v = \omega^{-1}$ with $\|\omega\|_{A_{2,inv}} = Q$ and, up to a normalization constant not depending on Q ,

$$\alpha(z) = \frac{|\omega(z)'| |\omega^{-1}(z)|}{\omega(z) \omega^{-1}(z)}.$$

We need to prove the imbedding condition inequalities (6.1), (6.2) and (6.3). We first turn to (6.1):

Consider the function from [19]

$$B(s, r) = s \left(-\frac{4Q^2}{rs} - \frac{rs}{4Q^2} + 4Q^2 + 1 \right),$$

this function has the following properties:

$$1 \leq rs \leq Q^2 \implies 0 \leq B(r, s) \leq cQ^2 s$$

$$1 \leq rs \leq Q^2 \implies -d^2 B \geq Cs \left| \frac{dsdr}{sr} \right|.$$

Let us also consider the function $b : \mathbb{C} \rightarrow \mathbb{R}$

$$b(z) = B(h(z)) = B(\omega(z), \omega^{-1}(z)),$$

then

$$0 \leq b(z) \leq cQ^2 \omega(z) \text{ and}$$

$$-\Delta b(z) \geq 4C\omega(z) \frac{|\omega(z)'||\omega^{-1}(z)'|}{\omega(z)\omega^{-1}(z)} = 4C\omega(z)\alpha(z).$$

This function will help us to estimate the following integral:

$$\begin{aligned} & \int_{\mathbf{D}} \log \frac{1}{|S_z(\xi)|} \alpha(\xi) \omega(\xi) dA(\xi) \\ & \leq \int_{\mathbf{D}} -\Delta b(\xi) \log \frac{1}{|S_z(\xi)|} dA(\xi) \\ & = \int_{\mathbf{D}} -\Delta b(S_{-z}(\xi)) \log \frac{1}{|\xi|} dA(\xi) \\ & = \left(b(z) - \int_{\mathbf{T}} b dm \right) \\ & \leq cQ^2\omega(z), \end{aligned}$$

similarly we obtain

$$\int_{\mathbf{D}} \log \frac{1}{|S_z(\xi)|} \alpha(\xi) \omega^{-1}(\xi) dA(\xi) \leq cQ^2\omega^{-1}(z),$$

which give the desired estimates (6.1) and (6.2).

We are left to show the inequality (6.3), namely that

$$\int_{\mathbf{D}} \log \frac{1}{|S_z(\xi)|} \alpha(\xi) \omega(\xi) \omega^{-1}(\xi) dA(\xi) \leq cQ^2.$$

Consider the function from [9]

$$B(s, r) = 4Q\sqrt{sr} - sr,$$

this function has the following properties:

$$1 \leq rs \leq Q^2 \implies 0 \leq B(r, s) \leq 4Q^2$$

$$1 \leq rs \leq Q^2 \implies -d^2 B \geq c|dsdr|.$$

Let us also consider the function $b : \mathbb{C} \rightarrow \mathbb{R}$

$$b(z) = B(h(z)) = B(\omega(z), \omega^{-1}(z)),$$

then

$$0 \leq b(z) \leq cQ^2 \text{ and}$$

$$-\Delta b(z) \geq 4C\omega(z)\omega^{-1}(z) \frac{|\omega(z)'||\omega^{-1}(z)'|}{\omega(z)\omega^{-1}(z)} = 4C\omega(z)\omega^{-1}(z)\alpha(z).$$

This function will take care of the following integral:

$$\begin{aligned} & \int_{\mathbf{D}} \log \frac{1}{|S_z(\xi)|} \alpha(\xi) \omega(\xi) \omega^{-1}(\xi) dA(\xi) \\ & \leq c \int_{\mathbf{D}} -\Delta b(\xi) \log \frac{1}{|S_z(\xi)|} dA(\xi) \\ & = c \int_{\mathbf{D}} -\Delta b(S_{-z}(\xi)) \log \frac{1}{|\xi|} dA(\xi) \\ & = c \left(b(z) - \int_{\mathbf{T}} b dm \right) \\ & \leq cQ^2, \end{aligned}$$

proving (6.3).

5.3 Shortcut

There is a faster, but less instructive way to obtain the desired result. We use the dyadic analog and deduce the existence of the corresponding Bellman function. In [19] it has been proven that

$$\sum_I |(f, h_I)| |(g, h_I)| \leq c \|\omega\|_{A_2}^2 \|f\|_w \|g\|_{w^{-1}}.$$

By restricting f and g to J and discarding some positive terms on the left, we can deduce the following for all dyadic J

$$\frac{1}{|J|} \sum_{I \subset J} |I| |\langle f \rangle_{I_+} - \langle f \rangle_{I_-}| |\langle g \rangle_{I_+} - \langle g \rangle_{I_-}| \leq c \|\omega\|_{A_2}^2 \sqrt{\langle f^2 \omega \rangle_J \langle g^2 \omega^{-1} \rangle_J}.$$

Again, the reader should note the similarity with our integral:

$$\int_{\mathbf{D}} |f(z)'| |g(z)'| \log \frac{1}{|z|} dA(z).$$

Since the above dyadic estimate holds, there exists a function $B : D \rightarrow \mathbb{R}$ where $D = \{v = (X, Y, s, r, x, y) \in \mathbb{R}_{\geq 0}^6 : 1 \leq sr \leq Q^2\}$ so that

$$v \in D \implies 0 \leq B \leq cQ^2\sqrt{XY} \quad (5.7)$$

and

$$B(v) - \frac{1}{2}(B(v_+) + B(v_-)) \geq c|x_+ - x_-||y_+ - y_-| \quad (5.8)$$

whenever $v, v_+, v_- \in D$ and $v = 1/2(v_+ + v_-)$. Such a function is

$$B(X, Y, s, r, x, y) := \sup \frac{1}{|J|} \sum_{I \subset J} |I| |\langle f \rangle_{I_+} - \langle f \rangle_{I_-}| |\langle g \rangle_{I_+} - \langle g \rangle_{I_-}|,$$

where the supremum runs over functions $f \in L^2(\omega)$, $g \in L^2(\omega^{-1})$ and weights $\omega \in A_2$ with norm $\|\omega\|_{A_2} = Q$ so that

$$\langle f \rangle_J = x, \quad \langle g \rangle_J = y,$$

$$\langle \omega \rangle_J = s, \quad \langle \omega^{-1} \rangle_J = r,$$

$$\langle f^2 \omega \rangle_J = X, \quad \langle g^2 \omega^{-1} \rangle_J = Y.$$

The lower bound in (5.7) is clear by definition of B and the upper bound is just the fact that the dyadic estimate holds. Inequality (5.8) follows by investigating the relationship between the contributions to the supremum that are made by the right and left hand sides of the interval J . In fact, for $v \in D$ and $v_+, v_- \in D$ so that $1/2(v_+ + v_-) = v$ we have that

$$B(v) \geq \sup \frac{1}{|J|} \sum_{I \subset J} |I| |\langle f \rangle_{I_+} - \langle f \rangle_{I_-}| |\langle g \rangle_{I_+} - \langle g \rangle_{I_-}|,$$

where the supremum is taken over the (smaller) set of functions $f \in L^2(\omega)$, $g \in L^2(\omega^{-1})$ and weights $\omega \in A_2$ with norm $\|\omega\|_{A_2} = Q$ so that the averages match:

$$\langle f \rangle_{J_{\pm}} = x_{\pm}, \quad \langle g \rangle_{J_{\pm}} = y_{\pm},$$

$$\langle \omega \rangle_{J_{\pm}} = s_{\pm}, \quad \langle \omega^{-1} \rangle_{J_{\pm}} = r_{\pm},$$

$$\langle f^2 \omega \rangle_{J_{\pm}} = X_{\pm}, \quad \langle g^2 \omega^{-1} \rangle_{J_{\pm}} = Y_{\pm}.$$

By splitting up the sum and realizing that the supremum does not depend on the choice of J we obtain (5.8).

To proceed, we are going to need a smooth version of B satisfying the same estimates. For any compact subset $K \in D$ with $\text{dist}(K, D^c) > \epsilon$, consider dilates $\Phi_{\epsilon}(x) = 1/\epsilon^6 \Phi(x/\epsilon)$ of a smooth bellshaped function Φ supported in the unit ball of $\mathbb{R}^6$. Then the convolution $B_{\epsilon} = B * \Phi_{\epsilon}$ is smooth and satisfies the same estimates than B in the set K with different constants not depending on ϵ . So we have the following size condition on $B_{\epsilon}(v)$ for all $v \in K$.

$$0 \leq B_{\epsilon}(v) \leq cQ^2\sqrt{XY}.$$

The condition (5.8) implies the following estimate for the second differential of B_{ϵ} :

$$-d^2 B_{\epsilon}(v) \geq c|dx||dy|. \tag{5.9}$$

Now let $v(z) = (f^2 \omega(z), g^2 \omega^{-1}(z), \omega(z), \omega^{-1}(z), f(z), g(z))$. The set $\{v(z) : |z| \leq r\}$ is a compact subset of D . Choose ϵ accordingly and consider

$$b_{\epsilon}(z) = B_{\epsilon}(f^2 \omega(z), g^2 \omega^{-1}(z), \omega(z), \omega^{-1}(z), f(z), g(z)).$$

Then applying (5.2) together with (5.9) gives

$$-\Delta b_{\epsilon}(z) \geq c|f(z)'||g(z)'|.$$

Now we are ready to estimate the fourth integral:

$$\begin{aligned}
& \int_{\mathbf{D}} |f(z)'||g(z)'| \log \frac{1}{|z|} dA(z) \\
&= \lim_{r \rightarrow 1} \int_{r\mathbf{D}} |f(z)'||g(z)'| \log \frac{1}{|z|} dA(z) \\
&\leq \lim_{r \rightarrow 1} c \int_{r\mathbf{D}} -\Delta b_\epsilon(z) \log \frac{1}{|z|} dA(z) \\
&= c \left(b_\epsilon(0) - c \lim_{r \rightarrow 1} \int_{r\mathbf{T}} b_\epsilon dm \right) \\
&\leq cQ^2(\|f\|_\omega \|g\|_{\omega^{-1}}).
\end{aligned}$$

5.4 Sharpness of Result

Sharpness can be seen using power weights. We refer the reader to [5] and [13].

CHAPTER 6

Harmonic Bilinear Imbedding

Theorem

6.1 Statement and Proof

Lemma 6.1 *Let $\alpha(z) \geq 0$ and ω, v be two weights so that $1 \leq \omega(z)v(z) \leq Q^2$ for all $z \in \mathbb{D}$ and*

$$\int_{\mathbb{D}} \alpha(\xi) \omega(\xi) \log \frac{1}{|S_z(\xi)|} dA(\xi) \leq Q^2 \omega(z) \quad (6.1)$$

$$\int_{\mathbb{D}} \alpha(\xi) v(\xi) \log \frac{1}{|S_z(\xi)|} dA(\xi) \leq Q^2 v(z) \quad (6.2)$$

and

$$\int_{\mathbb{D}} \alpha(\eta) \omega(\eta) v(\eta) \log \frac{1}{|S_\xi(\eta)|} dA(\eta) \leq Q^2. \quad (6.3)$$

Then for $f, g \geq 0 \in L^2(\mathbb{T})$ we have

$$\int_{\mathbb{D}} \alpha(z) f(z) g(z) \log \frac{1}{|z|} dA(z) \leq cQ^2 \|f\|_{v^{-1}} \|g\|_{\omega^{-1}}.$$

PROOF.

As before, it is more convenient to switch to Young's inequality. It suffices to show that

$$\int_{\mathbb{D}} \alpha(z) f(z) g(z) \log \frac{1}{|z|} dA(z) \leq cQ^2(\|f\|_{v^{-1}}^2 + \|g\|_{\omega^{-1}}^2).$$

Let us consider the following variables:

$$X = f^2 v^{-1}(z) \quad x = f(z) \quad r = v(z)$$

$$Y = g^2 \omega^{-1}(z) \quad y = g(z) \quad s = \omega(z)$$

and the following non-harmonic variables

$$M = \int_{\mathbb{D}} \alpha(\xi) v(\xi) \log \frac{1}{|S_z(\xi)|} \int_{\mathbb{D}} \alpha(\eta) v(\eta) \omega(\eta) \log \frac{1}{|S_\xi(\eta)|} dA(\eta) dA(\xi)$$

$$N = \int_{\mathbb{D}} \alpha(\xi) \omega(\xi) \log \frac{1}{|S_z(\xi)|} \int_{\mathbb{D}} \alpha(\eta) \omega(\eta) v(\eta) \log \frac{1}{|S_\xi(\eta)|} dA(\eta) dA(\xi)$$

$$K = \int_{\mathbb{D}} \alpha(\eta) v(\eta) \omega(\eta) \log \frac{1}{|S_z(\eta)|} dA(\eta)$$

We have the natural estimates:

$$1 \leq rs \leq Q^2$$

$$x^2 \leq Xr \quad \text{and} \quad y^2 \leq Ys \quad \text{by Jensen's inequality}$$

$$M \leq Q^4 r \quad \text{and} \quad N \leq Q^4 s \quad \text{by (6.1), (6.2), (6.3)}$$

$$K \leq Q^2 \quad \text{by (6.3) .}$$

Let us consider the following function of nine (!) real variables:

$$B(X, x, r, Y, y, s, M, N, K) =$$

$$B_1(X, x, r, M)$$

$$+ B_2(Y, y, s, N)$$

$$+ B_3(X, x, r, Y, y, s, K)$$

where

$$B_1(X, x, r, M) = X - \frac{x^2}{r + \frac{M}{Q^4}}$$

$$B_2(Y, y, s, N) = Y - \frac{y^2}{s + \frac{N}{Q^4}}$$

$$B_3(X, x, r, Y, y, s, K) = X + Y - \frac{x^2s - 2xy\frac{K}{Q^2} + y^2r}{rs - \frac{K^2}{Q^4}}$$

and, as before, $b(z), b_1(z), b_2(z), b_3(z)$ the corresponding functions on $\mathbb{D}$. We discuss the properties of B .

Derivative estimates:

$$\frac{\partial B_1}{\partial M} \geq \frac{1}{4Q^4} \frac{x^2}{r^2} \text{ since } M \leq Q^4 r$$

$$\frac{\partial B_2}{\partial N} \geq \frac{1}{4Q^4} \frac{y^2}{s^2} \text{ since } N \leq Q^4 s$$

$$\frac{\partial B_3}{\partial K} \geq \begin{cases} \frac{c}{Q^2} \frac{xy}{rs} & \text{if both } K \leq Q^2 \frac{yr}{4x} \text{ and } K \leq Q^2 \frac{xs}{4y} \\ 0 & \text{else} \end{cases}$$

By exchanging x and y we only need to consider the case $\frac{x^2}{r} \geq \frac{y^2}{s}$ for the derivative estimate of B_3 . Let us point out where B_3 came from. It was taken from an early version of [9], where it was written (up to normalization) in the following form:

$$B_3(X, x, r, Y, y, s, K) = X + Y - \sup_{a \geq 0} \beta(a, X, x, r, Y, y, s, K)$$

where

$$\beta(a, X, x, r, Y, y, s, K) = \frac{x^2}{r + a \frac{K}{Q^2}} + \frac{y^2 a}{as + \frac{K}{Q^2}}.$$

Let us write $\tilde{K}$ for K/Q^2 . It has been shown in an early version of [9] that under the restrictions above

$$\beta(a, X, x, r, Y, y, s, K) \geq \frac{x^2}{r} + \frac{1}{2} \frac{y^2}{s} \quad \text{for } a_0 = \frac{yr}{xs}; \quad (6.4)$$

we will include the proof for the sake of completeness. Let us first observe that

$$\frac{x^2}{r + a\tilde{K}} \geq \frac{x^2}{r} - a\tilde{K} \frac{x^2}{r^2} \quad \text{and} \quad \frac{y^2}{s + a^{-1}\tilde{K}} \geq \frac{y^2}{s} - a^{-1}\tilde{K} \frac{y^2}{s^2}$$

and hence

$$\beta(a, X, x, r, Y, y, s, K) \geq \frac{x^2}{r} + \frac{y^2}{s} - \left(a\tilde{K} \frac{x^2}{r^2} + a^{-1}\tilde{K} \frac{y^2}{s^2} \right). \quad (6.5)$$

The part in parentheses for $a = a_0 = \frac{yr}{xs}$ is

$$\frac{yr}{xs} \tilde{K} \frac{x^2}{r^2} + \frac{xs}{yr} \tilde{K} \frac{y^2}{s^2} = 2 \frac{xy}{rs} \tilde{K} \stackrel{(*)}{\leq} \frac{y^2}{2s}, \quad (6.6)$$

where $(*)$ uses the assumption $\tilde{K} \leq \frac{yr}{4x}$. Now we obtain the required estimate from below for β at a_0 :

$$\beta(a_0, X, x, r, Y, y, s, K) \stackrel{(*)}{\geq} \frac{x^2}{r} + \frac{y^2}{s} - \frac{y^2}{2s} = \frac{x^2}{r} + \frac{1}{2} \frac{y^2}{s}$$

where $(*)$ uses (6.5) for a_0 together with (6.6). So taking supremum in the first variable yields

$$\sup_{a \geq 0} \beta(a, X, x, r, Y, y, s, K) \geq \frac{x^2}{r} + \frac{1}{2} \frac{y^2}{s}.$$

Note that β is continuously differentiable in a for $a > 0$ and that β is close to $\frac{x^2}{r}$ for a near 0 and close to $\frac{y^2}{s}$ for very large a . So β as a function of a attains its maximum in $(0, \infty)$. Testing for critical points yields

$$\frac{\partial \beta}{\partial a} = -\frac{x^2 \tilde{K}}{(r + a\tilde{K})^2} + \frac{y^2 \tilde{K}}{(as + \tilde{K})^2} \quad \text{and} \quad \frac{\partial \beta}{\partial a} = 0 \iff a = \frac{yr - x\tilde{K}}{xs - y\tilde{K}}.$$

By the above it is already clear that $a_m := \frac{yr - x\tilde{K}}{xs - y\tilde{K}} > 0$ and that β attains its maximum at this point. We found our B_3 by letting $a = a_m = \frac{yr - x\tilde{K}}{xs - y\tilde{K}}$. We consider the one parameter family of functions

$$B_3^a(X, x, r, Y, y, s, K) := X + Y - \beta(a, X, x, r, Y, y, s, K).$$

In an early version of [9] the following derivative estimate has been proven:

$$\left. \frac{\partial B_3^a}{\partial \tilde{K}} \right|_{a=a_m} \geq c \frac{xy}{rs} \quad \text{where} \quad a_m = \frac{yr - x\tilde{K}}{xs - y\tilde{K}}, \quad (6.7)$$

but $B_3(X, x, r, Y, y, s, K) = B_3^{a_m}(X, x, r, Y, y, s, K)$ so

$$\frac{\partial B_3}{\partial \tilde{K}} = \left. \frac{\partial B_3^a}{\partial a} \right|_{a=a_m} \cdot \frac{\partial a_m}{\partial \tilde{K}} + \left. \frac{\partial B_3^a}{\partial \tilde{K}} \right|_{a=a_m}.$$

Note that $\left. \frac{\partial B_3^a}{\partial a} \right|_{a=a_m} = -\left. \frac{\partial \beta}{\partial a} \right|_{a=a_m} = 0$ since β attains its maximum in a_m . We have the derivative estimate

$$\frac{\partial B_3}{\partial \tilde{K}} \geq \frac{c}{Q^2} \frac{xy}{rs}.$$

Let us include the proof of (6.7). First observe that according to (6.4) we have that

$$\frac{x^2}{r + a_m \tilde{K}} + \frac{y^2}{s + a_m^{-1} \tilde{K}} \geq \frac{x^2}{r} + \frac{1}{2} \frac{y^2}{s},$$

which implies $\frac{y^2}{s + a_m^{-1} \tilde{K}} \geq \frac{y^2}{2s}$ and hence $s \geq a_m^{-1} \tilde{K}$. But since $\frac{x^2}{r} \geq \frac{y^2}{s}$ inequality (6.4) implies also

$$\frac{x^2}{r + a_m \tilde{K}} + \frac{y^2}{s + a_m^{-1} \tilde{K}} \geq \frac{y^2}{s} + \frac{1}{2} \frac{x^2}{r}$$

and similarly we obtain $r \geq a_m \tilde{K}$.

Now we are ready to show the estimate in (6.7).

$$\frac{\partial B_3^a}{\partial \tilde{K}} = \frac{ax^2}{(r + a\tilde{K})^2} + \frac{a^{-1}y^2}{(s + a^{-1}\tilde{K})^2} \geq 2 \frac{xy}{(r + a\tilde{K})(s + a^{-1}\tilde{K})}$$

now using $r \geq a_m \tilde{K}$ and $s \geq a_m^{-1} \tilde{K}$ we obtain the desired estimate.

Size:

We have the following obvious size estimates for B_i :

$$0 \leq B_1 \leq X \quad 0 \leq B_2 \leq Y \quad 0 \leq B_3 \leq X + Y$$

where $0 \leq B_3$ follows from the fact that $X - \frac{x^2}{r+a\tilde{K}} \geq 0$ and $Y - \frac{y^2}{s+a^{-1}\tilde{K}} \geq 0$ for positive a .

Concavity:

B_1 and B_2 are of the form (5.3) so

$$-d^2 B_1 \geq 0 \quad -d^2 B_2 \geq 0.$$

Functions B_3^a are concave for all parameters a , so B_3 is, as infimum of a family of concave functions, concave, so

$$-d^2 B_3 \geq 0.$$

So B is concave.

We turn to the main estimate. The functions B_1 , B_2 and B_3 will play their main roles in different parts of the unit disk. We divide $\mathbb{D}$ into three parts

$$A_1 = \left\{ z \in \mathbb{D} : K(z) \geq Q^2 \frac{g(z)v(z)}{4f(z)} \right\}$$

$$A_2 = \left\{ z \in \mathbb{D} : K(z) \geq Q^2 \frac{f(z)\omega(z)}{4g(z)} \right\}$$

$$A_3 = \mathbb{D} \setminus (A_1 \cup A_2)$$

If $z \in A_1$, then

$$\begin{aligned} -\Delta b_1(z) &\geq \frac{\partial B_1}{\partial M}(-\Delta M) \\ &\geq \frac{1}{4Q^4} \frac{f(z)^2}{v(z)^2} \alpha(z) v(z) K(z) \\ &\geq \frac{1}{4Q^4} \frac{f(z)^2}{v(z)^2} \alpha(z) v(z) Q^2 \frac{g(z)v(z)}{4f(z)} \\ &= \frac{1}{16Q^2} \alpha(z) f(z) g(z). \end{aligned}$$

and similarly, if $z \in A_2$, then

$$-\Delta b_2(z) \geq \frac{1}{16Q^2} \alpha(z) f(z) g(z).$$

If $z \in A_3$ then

$$\begin{aligned} -\Delta b_3(z) &\geq \frac{\partial B_3}{\partial K}(-\Delta K) \\ &\geq \frac{c}{Q^2} \frac{f(z)g(z)}{\omega(z)v(z)} \alpha(z) \omega(z) v(z) \\ &= \frac{c}{Q^2} \alpha(z) f(z) g(z). \end{aligned}$$

Since $-\Delta b_{1,2,3} \geq 0$ on all of $\mathbb{D}$ we have all together

$$-cQ^2 \Delta b(z) \geq \alpha(z) f(z) g(z)$$

and we are ready to run the Green's formula trick:

$$\begin{aligned} &\int_{\mathbb{D}} \alpha(z) f(z) g(z) \log \frac{1}{|z|} dA(z) \\ &\leq cQ^2 \int_{\mathbb{D}} -\Delta b(z) \log \frac{1}{|z|} dA(z) \\ &= cQ^2 (b(0) - \int_{\mathbf{T}} b dm) \\ &\leq cQ^2 (\|f\|_{v^{-1}}^2 + \|g\|_{\omega^{-1}}^2). \end{aligned}$$

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