This is to certify that the

dissertation entitled

Going Public:
The Development of a Teacher Educator's
Pedagogical Content Knowledge

presented by

Jeffrey Joe Wanko

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Teacher Education

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Date August 24, 2000

GOING PUBLIC:
THE DEVELOPMENT OF A TEACHER EDUCATOR'S PEDAGOGICAL CONTENT KNOWLEDGE

By

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Teacher Education

2000

ABSTRACT

GOING PUBLIC:
THE DEVELOPMENT OF A TEACHER EDUCATOR'S
PEDAGOGICAL CONTENT KNOWLEDGE

By

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When Lee Shulman and his colleagues introduced pedagogical content knowledge (PCK) to the education lexicon in the 1980s, they gave teachers and teacher educators some technical language that could be used for talking about the knowledge needed for work that they do in classrooms, thereby helping to establish teaching as a profession. Since that time, the PCK of classroom teachers has been studied and documented across various content areas. But the PCK of teacher educators has remained a largely unexamined area of research, especially in the providing experiences in helping preservice teachers develop their own PCK. This study examines this issue more fully. Specifically, "Can pedagogical content knowledge be a useful framework for a teacher educator in designing and teaching a mathematics content course for preservice teachers and if so in what ways?"

In this study, I use my own teaching and classroom of prospective elementary teachers as the site for investigation. I examine the ways in which my own PCK as a teacher educator influenced and was influenced by my work with students. Data for the study are provided by my teaching journal, lesson and units plans, student work, and audiotapes of class proceedings.

In conclusion, I present three major findings of this study. First, this study highlights and problematizes Shulman's notion of representation that is used in defining pedagogical content knowledge. In mathematics there are mathematical and empirical representations—classifications which do not map easily onto Shulman's use of representation. This study exposes some of those inherent distinctions and seeks to make Shulman's work more applicable to the field of mathematics. Second, this study describes the importance of task design—a process that is particularly essential in teaching mathematics—and finds that Shulman's notions of PCK and the pedagogical reasoning and action cycle miss or obscure its significance. And third, this study introduces the notion of shared reflection to Shulman's model for pedagogical reasoning and action when it is applied to teacher education. It also finds that the act of going public with one's ideas through shared reflection can be a useful tool for teacher educators in the development of their pedagogical content knowledge.

ACKNOWLEDGEMENTS

Conceptualizing, conducting, analyzing, and writing this research and dissertation have been part of a long and difficult process. Without the help and support of many colleagues, family members, and friends, it would have been next to impossible.

I wish to express my sincerest gratitude to Dr. Sandra K. Wilcox. You have served as a mentor and a guiding force through all of my work as a doctoral student. Your constant assistance and advice have been invaluable. Thank you also to Dr. Glenda Lappan, Dr. Daniel Chazan, and Dr. Helen Featherstone for your help in reviewing my work and providing feedback.

To the staff of the Connected Mathematics Project, I thank you for all of your support over the past six years.

To my parents and the rest of my family, I thank you for encouraging me through my professional life and for providing the support structure that I have always relied on.

And to my wife, Molly, I thank you for coming into my life and for your undying patience and love.

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INTRODUCTION TO THE STUDY

Introduction

The preparation of elementary school teachers of mathematics has traditionally been viewed as having two distinctly separate components—pedagogy and content. Preservice teachers are a part of teacher education programs in which students might take courses on child development, educational psychology, teaching methods, and other matters that are pedagogical in nature. The focus in these courses is learning how to teach—sometimes with respect to specific subject matter, but often from a generalist standpoint. On the other hand, prospective teachers are expected to learn and understand mathematics by taking courses often taught by instructors in a mathematics department. These courses vary in both content and rigor, but invariably the focus remains on learning mathematics.

A problem arises, however, when preservice teachers become practicing teachers and they attempt to cross this divide—when they search for ways to make their experiences in understanding mathematics useful tools for teaching children and, in turn, helping them to understand some more mathematics themselves (Brown & Borko, 1992; Copes, 1996). It is one thing for a teacher to learn some mathematics him- or herself, but it is quite another for a teacher to take that knowledge and apply it to ways for helping students learn and understand some mathematics. Replicating the teachers' experiences is obviously not the answer for a number of reasons: the content being addressed can be quite different, children may approach the mathematics in very different ways, the children's experiences are likely to be quite different from the teacher's and from those of other students, and the teacher has to think about the mathematics from a teaching perspective rather than from a learning perspective. With such a large curriculum of study—in both content and pedagogy—it is not surprising to hear stories of novice teachers having limited knowledge in mathematics. For example, while preservice teachers may get to learn some mathematics in their content courses, that mathematics typically lacks connectedness, a look at multiple perspectives, work beyond studying basic ideas, and attention to what content is taught across grade levels (Ma, 1999). Suffice it to say, the separation of content and pedagogy in a teacher preparation program has often not served preservice teachers well when they begin teaching (Becher, 1989).

An important skill for teachers is the ability to *transform* one's own knowledge into something that can be understood by one's students. Successful teaching is much more than demonstrating one's own knowledge; it involves leading others to know—and believe—things that they didn't before (Hansen, 1995). This happens when a teacher recognizes how students can best experience the acquisition of that knowledge—using various strategies, representations, and problems to move students toward a desired goal. An understanding of the mathematics is central to the transformation process, but it involves much more than processing content—it recognizes that children learn in different ways and that teachers need a multitude of strategies for helping their students to learn mathematics. Thus it is unclear where preservice teachers can and should begin to attend to transforming their mathematical knowledge—as a part of a course on pedagogy or in content.

The transformative process became recognized as an essential element of a teacher's knowledge through the work of Lee Shulman and his colleagues in the mid-1980s. Here I describe the essence of their work as well as how it is applied to my study.

Pedagogical Content Knowledge: A Brief Historical Perspective

Within cognitive psychological research, schema theory developed as a model for ways in which people organize abstract thoughts into various collections and representations of those ideas (Anderson, 1984). When Shulman and others applied schema theory to teaching (Shulman & Grossman, 1988; Wilson, Shulman, & Richert, 1987), they theorized that teachers draw upon seven different domains of knowledge as they teach: knowledge of subject matter, pedagogical content knowledge, knowledge of other content, knowledge of the curriculum, knowledge of learners, knowledge of educational aims, and general pedagogical knowledge.

When pedagogical content knowledge (PCK) was first introduced to the educational community (Shulman, 1986), it was described as a domain of knowledge that was different from both knowledge of the content and general knowledge of teaching. PCK was portrayed as a specialized knowledge for teachers, one which recognizes the subject matter needed for teachers as it differs from that as understood by other content specialists. Shulman's work was actually a continuation of some of the work done by progressive educators (Dewey, 1902) and theorists (Bruner, 1960), but he was the first to label and define PCK as such.[1]

[1] Some have argued that by giving a name to pedagogical content knowledge, Shulman was also working to establish his domain of educational research as a legitimate area of study when compared with other long-standing areas of research.

In analyzing successful teaching, Shulman (1987) described the process as one in which the teacher transforms content knowledge into forms that are pedagogically powerful and adaptive to particular groups of students—ones which draw upon a teacher's pedagogical content knowledge. To this end, he employed the framework of pedagogical reasoning and action, detailing a cycle of comprehension, transformation, instruction, assessment[2], reflection, and new comprehension (Table 1 and Figure 1).

Table 1: A Model of Pedagogical Reasoning and Action

Comprehension
Of purposes, subject matter structures, ideas within and outside the discipline

Transformation
Preparation: critical interpretation and analysis of texts, structuring and segmenting, development of a curricular repertoire, and clarification of purposes
Representation: use of a representational repertoire which includes analogies, metaphors, examples, demonstrations, explanations, etc.
Selection: choice from among an instructional repertoire which includes modes of teaching, organizing, managing, and arranging
Adaptation and Tailoring to Student Characteristics: consideration of conceptions, preconceptions, misconceptions, and difficulties, language, culture, and motivations, social class, gender, age, ability, aptitude, interests, self concepts, and attention

Instruction
Management, presentations, interactions, group work, discipline, humor, questioning, and other aspects of active teaching, discovery or inquiry instruction, and the observable forms of classroom teaching

Assessment
Checking for student understanding during interactive teaching
Testing student understanding at the end of lessons or units
Assessing one's own performance, and adjusting for experiences

Reflection
Reviewing, reconstructing, reenacting and critically analyzing one's own and the class's performance, and grounding explanations in evidence

New Comprehension
Of purposes, subject matter, students, teaching, and self
Consolidation of new understandings, and learning from experiences

[2] Shulman used *evaluation* instead of *assessment*. I have chosen to replace Shulman's terminology with one that is more accepted today to indicate Shulman's original intent. Lesh and Lamon (1992) made this distinction by defining evaluating as "assigning a value to testing" and assessing as "describing a current state—probably with reference to some conceptual, or procedural or developing landmarks" (p. 7).

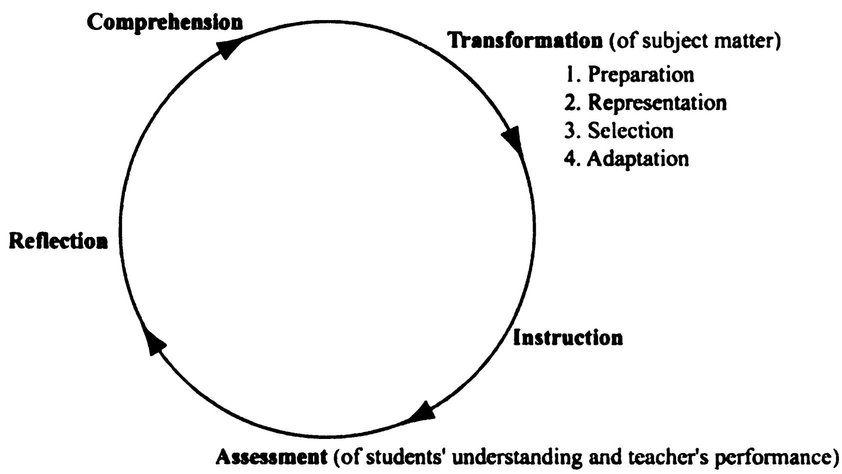

Figure 1: Shulman's Cycle of Pedagogical Reasoning and Action

Each phase of the process is an important one and while the phases can occur in a slightly different sequence, one fact remains constant—the process begins and ends with comprehension. As Shulman states, "To teach is first to understand" (Shulman, 1987, p. 14). Research evidence indicates that pedagogical reasoning is underdeveloped in novice teachers (Brown & Borko, 1992). At any number of places in the pedagogical reasoning and action cycle, novice teachers experience difficulties. The transformation phase of the process is perhaps one of the most critical—for ideas that are comprehended by the teacher must be transformed in some way if they are to be taught to and understood by the students—and one of the most difficult for novice teachers (Geddis & Wood, 1997).

What Do Preservice Teachers Need to Know for Teaching?

The preparation of elementary school teachers needs to address their content knowledge of the subject matter as well as how to teach it. But there also needs to be some focus on preservice teachers' pedagogical content knowledge. The dilemma comes in finding an appropriate place in the teacher preparation program for it. PCK enables teachers to make connections between their knowledge of teaching and their knowledge of subject matter—connections that are critical for teaching effectiveness (Cochran, DeRuiter, & King, 1993). With PCK such an essential part of a teacher's knowledge base, it makes sense that they begin to develop their own PCK even as preservice teachers. It also makes sense that PCK development occurs not only in teaching methods courses, but in content courses as well. Since PCK involves understanding ways to transform one's own understanding of subject matter, a sensible place to learn about this process would be within the context of studying that subject matter itself.

In this study, I suggest an approach to the development of preservice teachers' PCK within a mathematics content course. In Chapter 2, I detail the evolution of my research question and study. Initially, I was interested in what a mathematics content course might look like when the instructor is also attending to issues around teaching the mathematics to elementary school students. As the study progressed, my focus shifted to how, as an instructor, a closer look at my own PCK development is helpful in designing and teaching the course. Little research has been done around the subject of a teacher educator's PCK development (for an example, see Fernández-Balboa & Stiehl, 1995), much less around teaching such a course designed around a content area. In this study, I look at some specific ways in which PCK can be considered an area of study for teacher educators as well as their students within the domain of mathematics—by looking at the role of representations (Chapter 3) and the process of task design (Chapter 4).

This is a study in which the researcher and the subject are one and the same. In the field of qualitative educational research, the first-person perspective has emerged as a relatively new genre and one that is useful in some very specific cases—where the researcher is interested in studying in-depth some element of his or her practice as it pertains to the larger research community.

This study is done from the first-person perspective by necessity as well as by design. Since mathematics content courses are often taught by professors, graduate students, and instructors well-versed in the subject matter but lacking a background in the study of pedagogy, it was a matter of necessity that I had to study my own practice around this question. I was the only mathematics instructor in the department at the time who was interested in the question of looking at what PCK preservice teachers could learn in a content course, much less researching what PCK was important for a teacher of preservice teachers. But there are also issues of design which necessitated that this be a self study. Much of what I was interested in centered around the decisions that a teacher educator makes when attending to issues of PCK. Since I was the course instructor, I have particular insight to the decision making process I used in the design of the course. These decisions were based on my cumulative experiences in classrooms and all of these experience became an important part of my story and my study. As the researcher, I had access to my own experiences in ways that would have been much more difficult—and potentially less enlightening—if I had studied someone else's practice. In Chapter 2, I describe some of these experiences and how they contributed to the development of both the course and the study.

Recently, questions have been raised within the mathematics education community as to how useful the idea of pedagogical content knowledge is when applied specifically to mathematics. In this study, I set out to look at ways in which Shulman's construct of PCK was useful to me in teaching content to preservice teachers. I found places where the model was insufficient in describing aspects of mathematics teaching—places which were revealed as critical when I reflected on my own teaching and when I shared my reflections with my students. These deficiencies are described in the findings of my study in Chapter 5.

CHAPTER 2

METHODOLOGY

Introduction: Designing the Course and Study

In Chapter 1, I described this study as one in which I look more closely at my own teaching. In particular, I look at the decisions I made in teaching a mathematics course for preservice teachers where issues of pedagogical content knowledge are also addressed. This is, then, a qualitative study in the field of "inquiry in teaching," but also one in which I am both researcher and subject. Thus I am using what Lampert called "the perspective of practice" (Lampert, 1998; Ball, 2000) to examine teaching from the inside.

The MTH 202 course that I taught evolved over a three-year period and was based on ideas I learned from teaching previous sections as well as discussions I had about the course design with some of my colleagues in mathematics education. For the purposes of this study, it culminated in the course that I taught in the spring of 1999. In this chapter, I describe some of the aspects of the course in its final iteration and the design of the study I conducted around my teaching of the course.

In this chapter, I also describe the evolution of my study—from one that centered around identifying and including PCK ideas in a course for preservice teachers to one that focused on the development of my own PCK as a teacher educator.

Forming An Initial Question

As a teacher educator, I was both aware of and intrigued by the interplay between teaching mathematics content and teaching pedagogy. There is much to cover in each domain, and the instructors of teacher education and instructors of mathematics can be extremely proprietary of their subject matter. At times, the situation plays itself out as a choice between extremes—forsake the pedagogy to focus solely on the mathematics or give too little attention to the content so that students could learn about teaching. Between these two extremes, however, lies an expansive continuum of possibilities for design of courses in teacher preparation programs.

When given the chance to teach a mathematics course for preservice elementary teachers, I looked forward to creating a content course designed to meet the needs of a specific population of students. Because the students were all future teachers, I saw it as a unique opportunity to incorporate some issues of pedagogy within the study of mathematics. In particular, I wondered what a mathematics content course might look like when the instructor wants to attend to issues that relate to the teaching of that content to K-8 students?

As I considered this question, I realized there were a number of issues that needed to be addressed. For example: What would the mathematics content be of such a course? What pedagogical issues would be addressed and in what ways? How would decisions about the content and design of the course be made and on what experiences would they be based? How could pedagogical concerns be addressed without taking away from the mathematics experiences expected of the course?

Background: Why Did This Interest Me?

My reasons for wanting to design and study my teaching of such a course were threefold. First, it grew out of my experiences teaching teacher education courses and mathematics courses for preservice teachers. I was frustrated by time constraints as well as the convention of separating the learning of subject matter from learning how to teach that subject matter. Second, I drew upon my own experiences teaching children. As a novice middle school teacher, I was often unable to replicate in my classroom for my students the kinds of rich learning experiences I had as a preservice teacher. I felt as if there was something missing in my undergraduate experience in moving from my understanding the mathematics to helping students understand the content. I wanted to find out how to make that transition smoother and more explicit for future teachers. Third, there was a growing recognition in the field that developing a teacher's PCK was critical in the shaping of his or her practice. The vast majority of the research, however, was centered around the development of PCK in teacher education courses. I began to consider whether attention to preservice teachers' PCK development in a mathematics content course might be possible and even desirable.

Learning About and Teaching Preservice Teachers

From the fall of 1994 through the spring of 1999, I taught several different courses for students in the elementary education program at Michigan State University. I began teaching a course for interns[1] and eventually I shifted my focus to teaching a

[1] At Michigan State University, teacher education is a five-year program. After four years of study, students graduate from the university with a baccalaureate degree. The program requires a fifth year internship for certification. During this fifth year, interns are placed in a public school for two semesters with a collaborating teacher and over time take the lead teaching role. As a part of the internship,

content course in the mathematics department for preservice teachers—the site for this research. What follows is a description of how I grew to reconceptualize the content course through my experiences as an instructor in teacher education with a growing awareness of the theory of pedagogical reasoning, and as an instructor in the mathematics department.

Teaching in the Teacher Education Department. For two years, I taught courses in the "Reflection and Inquiry in Teaching Practice" sequence—TE 802 and TE 804 (Table 2). These courses were designed to work in support of the internship placement as a second round of methods courses. In their senior years, students were placed in elementary classrooms for 4 hours per week. As interns, they spent most of a school year in a classroom, working closely with a collaborating teacher and taking increased responsibility for the daily and long-term planning, teaching, and assessment. The course I taught was designed to take up issues that students weren't ready to grapple with as seniors. As they took more of the lead teaching responsibility in the elementary classrooms, the students were faced with needing to think about curriculum and unit planning, learning about what sense their elementary students were making of the subject matter, and making instructional decisions based on assessment of student learning.

Table 2: Senior and Intern Methods Courses with Classroom Placements

Course	Year	Elementary Classroom Placement
TE 401	Senior	2 hours, 2 days/week
TE 802	Intern, 1st semester	Typically 4 days/week
TE 804	Intern, 2nd semester	Typically 4 days/week

students also take two master's level courses each semester. These four courses can count toward the university's Master of Arts in Curriculum and Teaching.

In the first semester course, I taught mathematics methods to groups of about 20 interns at a time for five-week periods, using measurement as the mathematical context for the pedagogical seminar. For the second semester course, I worked with the two dozen interns who selected mathematics as a focus area, and I used geometry as the site for further developing their practice as mathematics teachers. In each course, the mathematics served as a launching point for discussing important pedagogical issues that were critical for interns to be thinking about as novice teachers. For example, interns used their schools' curricular guidelines for their grade level placement to develop plans for teaching geometry and measurement units. They worked with various curriculum materials and made informed decisions about the appropriateness of the materials based on district and state curriculum guidelines and national standards. In addition, interns assessed not only their student's understanding of the content, but they assessed their own knowledge about and teaching of the content as well. These issues were central to the courses and its main goal of developing reflective educators who use inquiry of their own practice to grow as professionals.

I enjoyed working with the interns as they began their teaching careers. But at the same time, I experienced some frustration with not having the opportunity to focus more directly on establishing a strong subject matter base for these novice teachers. By the time they were interns, they were moving beyond the mathematics content and were focusing on applying that knowledge to their teaching. But I was bothered by how weak that content knowledge was—particularly in geometry and measurement.

One of the goals of teaching interns in the teacher education department was to use their classroom placements as sites for studying and developing their teaching practices. This involved helping preservice teachers work through the transformation process—selecting appropriate lessons and adapting those lessons to meet the needs of their students. And yet it is impossible for teachers to move to the transformation phase effectively if comprehension is weak, as was the case with some of the interns I taught. Their lack of mathematical understanding became evident as they struggled to create lesson and unit plans, and they typically had to rely on lessons presented in a textbook which did not draw upon any understanding of what their students knew and didn't know.

This served to underscore the importance of focusing on preservice teachers' understanding of the mathematics content.

Teaching in the Mathematics Department with the Connected Mathematics Project. After teaching in the teacher education department for two years, I shifted my focus to teach a course in the mathematics department. I eventually concentrated my efforts primarily on MTH 202, the geometry course for preservice elementary educators.

In late 1996, I was asked by Glenda Lappan if I would be interested in co-teaching MTH 202 with her in the spring of 1997. Not only was I intrigued by teaching a mathematics content course, but Glenda was using some of the Connected Mathematics Project materials in the course as well and I was interested in seeing what preservice teachers could learn from these materials.

The Connected Mathematics Project (CMP) is a middle school (grades 6-8) mathematics curriculum project funded by the National Science Foundation and was based at Michigan State University. Glenda was one of the authors of the project and I was in my second year as a graduate assistant for the project.[2] Glenda explained that she was planning to use some of the CMP units as a part of the content for the section of MTH 202 that she was teaching.

I was surprised to hear that Glenda wanted to use middle school materials in a course for preservice teachers. These would be twenty-year-olds, not the early teenagers for whom the units were written. I wondered if the students would feel as if they were being taught a watered-down version of the MTH 202 course material and if it would be challenging enough to them. Glenda assured me that the course that we were going to teach would be thought-provoking and mathematically challenging for the students. She also suggested that there were other ways that the CMP materials could be useful in a teacher preparation course. For example, she described a course in which the mathematics would be the focus, but in which the curriculum materials (specifically the teacher editions) would give students a chance to experience some of the complexities of making instructional decisions about what students need to know. She also explained that she viewed CMP as a curriculum that was being written to help teach mathematics to kids, but even more she viewed it as a curriculum to help teach some mathematics to teachers. If that mathematics could be taught to teachers before they entered the teaching profession, then all the better.

As I taught MTH 202 over the next few years, I tried to capitalize on some of these ways in which the materials might be useful and I began to develop a different kind

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[2] The CMP materials had gone through five years of development, writing, field-testing, revision, and subsequent publication. In the spring of 1997, the project focus was shifting from the initial development phase to an implementation phase, but there were still a few eighth grade units to be finalized.

of course, one that kept the mathematics front and center, but also included some rich, embedded pedagogical discussions.

Learning About and Teaching Middle School Students

Before teaching preservice teachers, I graduated with a teaching degree in secondary mathematics education and taught middle school mathematics in southwest Ohio for six years. As a part of my undergraduate program, I took two methods courses in teacher education and over a dozen content courses in mathematics and statistics. All of the content courses were taught by mathematics professors and those of us majoring in education were mixed in with students of other majors. Understandably, these courses focused solely on mathematics content.

The first methods course, "Instructional Procedures in Secondary School Mathematics" (EDT 335), was taught by a professor in mathematics teacher education. The course focused on preparing preservice mathematics teachers for teaching middle school; the second methods course in the sequence concentrated on issues of teaching high school. Although EDT 335 was based in the department of teacher education and was centered around learning to teach mathematics, content was an integral part of our work. Pedagogical ideas were typically framed around mathematical investigations, and the mathematical topics studied—rational numbers, number theory, computation, informal geometry, measurement, and algebra—were linked to some of the core ideas in upper elementary and middle school mathematics curricula.

Although I don't remember it being stated as such, some of the mathematical content of EDT 335 was centered around several big mathematical problems (Figure 2).

These problems were often open-ended tasks designed to build on preservice teachers' existing mathematical knowledge and although I was fairly adept at a wide range of mathematics, I was invariably challenged by the content of these problems.

Construct an organized chart showing all of the proper fractions (one-half, one-third, two-thirds, one-fourth, two-fourths, etc.) through twelfths. Include an exact decimal for each. By studying these and other examples as necessary, develop hypotheses to answer these questions:

- What about a fraction determines the nature of its decimal? [i.e., What determines whether it repeats or terminates, how long its repetend is, how many places it has, when is it a decimal that repeats but not at first, … ?]
- How can we predict the nature of a fraction's decimal?
- What fractions turn out to produce cyclic permutations of the same digits for their decimals?
- In a/b, what is the worst possible decimal?

Figure 2: Sample Problem from EDT 335

In retrospect, I think that all of the problems posed in EDT 335 were designed to push at preservice teachers' mathematics deeper understanding of topics taught in middle school. The problems also provided access to understanding some of the underlying structure of the mathematics beyond what middle schoolers need to know. For example, the problem in Figure 1 asks preservice teachers to investigate patterns of decimal equivalents of fractions. Middle school students typically work with a lot of fractions and decimals. And although middle schoolers do not usually need to understand a great deal behind the patterns that emerge when looking at decimal equivalents for groups of fractions, it can be helpful for the middle school teacher to have an understanding of the structure of our number system that causes those patterns to emerge.

When I started teaching middle school, I wanted to recapture some of the investigative nature of the problems I worked on in EDT 335. For me, the problems were challenging and interesting, and they provided access to exciting mathematical discoveries. I thought that middle schoolers would be able to learn in much the same ways I did from these problems and I wanted to find ways to incorporate some of the problems into the classes I was teaching. But while I understood that many of the problems would need to be altered in ways to make them accessible to middle school students, I found that what I lacked was an understanding of ways to do this.

As a novice teacher, I tried to recreate some of the lessons and experiences that I had as a preservice teacher, only to find that I was unable to initiate the same learning situations that I remembered as a student. For example, I recall working as a preservice teacher on the well-known Locker Problem[3] and having a lively, informative conversation about what we, as students, learned from the problem. However, when I tried to use the problem with my seventh grade students, I was unable to provide the kinds of learning experiences for my students I thought were possible. With my seventh

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[3] The Locker Problem is used in many curricula today, but it is first attributed to George Pólya, often thought of as the father of today's problem solving heuristics. The Locker Problem (Pólya, 1957) lends itself to using a number of problem solving techniques and the solution is surprising until some simple number theory is applied to its understanding.

One version of the Locker Problem is as follows: In a school of 1,000 students there are lockers numbered from 1 to 1,000. One the first day of school, students line up outside of the building with all of the lockers starting out closed. Student #1 goes through and opens every locker. Student #2 goes through and closed every second locker, starting with locker #2. Student #3 goes through and reverses every third locker, starting with locker #3—that is, if a locker is closed, he opens it, if a locker is open, he closes it. Student #4 goes through and reverses every fourth locker, starting with locker #4. Student #5 goes through and reverses every fifth locker, starting with locker #5. This continues until all 1,000 students graders, the problem degenerated into an exercise in looking for patterns[4] and the students failed to make the connections between important mathematical concepts.[5] It was a lack of understanding of this set of necessary prerequisite skills that kept me from moving my students to making an important mathematical discovery.

After thinking about examples like this, I decided that one important piece was missing. When I was an undergraduate student, my professor presented the problem and lead us through a discussion of the mathematics, but we were not given any insight about what he did when presenting the problem or leading the discussion. I viewed his work from a student's limited perspective and I was unable to play the part of the teacher in the lesson when I tried to do so. I gave minimal consideration to the fact that I would be working with seventh graders who had more limited mathematical backgrounds. I did not provide the right mathematical scaffolding for their learning, leaving out some important prior knowledge—like square numbers and factor pairs—that are integral to understanding the solution.

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have gone through the building, reversing their appropriate lockers. After all students are done, which lockers are left open and why?

4 Upon looking at a simpler version of the problem (with 20 to 30 lockers), students quickly found an interesting pattern of open lockers emerging: lockers 1, 4, 9, 16, and 25 are left open. Some students noticed that the open locker numbers increase by consecutive odd numbers ($1 \underline{+3} = 4$, $4 \underline{+5} = 9$, $9 \underline{+7} = 16$, $16 \underline{+9} = 25$, etc.), other students noticed that the lockers that are left open are the square numbers. Either way, students may end up focusing more on the patterns than on the underlying mathematical reason for why these are the lockers that are left open. The patterns are important, but they can actually serve to illuminate the deeper reasons for the patterns and give students access to higher-powered mathematics.

5 After finding the pattern of lockers that are left open, the question of *why* they are the square numbers was raised, challenging the students to access the underlying mathematics. An important idea comes from realizing that each locker is touched by the students whose numbers are the factors of the locker number. For example, locker 20 is touched by students 1, 2, 4, 5, 10, and 20—all of the factors of 20. The lockers that are left open, then, are those that are touched by an *odd* number of students (since the first student to touch the locker opens it, the second one closes it, the third opens it, etc.). Once this is established, students think about which numbers have an odd number of factors. From their work with factor pairs, they recall that the square numbers have an odd number of factors and are therefore the lockers that are left open. With my students, however, I didn't assess for any of the necessary background knowledge, and my students were not ready to talk about factors, factor pairs, or special properties of square numbers.

When planning for teaching a mathematics content course, I recalled the experiences of trying to use problems from EDT 335 in teaching middle school students. Preservice teachers need rich mathematical experiences as students, but I also felt that learning about these problems should also include learning about how to create appropriate learning experiences around these or other challenging problems for elementary and middle school students.

Learning About Sources for Pedagogical Content Knowledge

While I was teaching MTH 202, I became aware of a field of educational research that received a fair amount of attention in the late 1980s and early 1990s which also looked at the ways that pedagogy and subject matter were linked—the study of pedagogical content knowledge.

Shulman writes about the development of a teacher's pedagogical content knowledge (PCK)—an area of knowledge that is actually different from both the content and pedagogy. In fact, PCK is a knowledge base that is truly unique to the teaching profession in that it involves the teacher's developing understanding of the content and the ways in which the teacher transforms his or her understanding of the content to make it understandable to students (Shulman, 1986, 1987).

Pedagogical content knowledge builds on the content knowledge but looks at how it is affected by what the students learn and how they learn it. This is what makes pedagogical content knowledge a unique category of knowledge for teachers—it is one in which the focus begins with what the teacher knows and how that knowledge is transformed into ways in which is it learnable for students. In short, it is for teachers, a "special form of professional understanding" (Shulman, 1987). Educational researchers have come to realize that students learn concepts in a variety of ways and that students' understandings are built upon previous knowledge. Pedagogical content knowledge consists of learning how to represent specific topics in ways that are appropriate for and make sense to the diverse abilities and interests of the students (Borko et al., 1993). In describing the importance of pedagogical content knowledge, Grossman (1991) wrote, "If teachers are to guide students in their journey in to unfamiliar territories, they will need to know the terrain well. Both knowledge of the content and knowledge of the best way to teach that content to students help teachers construct meaningful representations, representations that reflect both the nature of the subject matter and the realities of students' prior knowledge and skills" (p. 203).

The transformation of subject matter for teaching occurs when the teacher reflects on and interprets the subject matter (Cochran, DeRuiter, & King, 1993), adapting it to meet the students' needs and tailoring the material to those students. Gudmundsdottir (1987) recognized this constant restructuring of the subject matter to meet the students' constantly changing needs and Buchmann (1984) discussed this need of a flexible understanding of the subject matter knowledge as crucial to the success of the instruction, thus validating the importance of connected subject matter knowledge for teachers.

For preservice teachers, pedagogical content knowledge is of particular importance. While it may be argued that knowledge of how to teach may best be learned through experience, teacher education programs have long held that meaningful pedagogical knowledge can be learned as a preservice teacher. How that pedagogy can be played out within a specific subject matter is an area ripe for study as perspective teachers learn about the subject matter. Brown and Borko (1992) reported that novice teachers are especially weak in their pedagogical content knowledge and that university courses should make pedagogical content knowledge a priority. Grossman (1988, 1991) also believed that pedagogical content knowledge can and should be taught to preservice teachers, stating that teachers must have pedagogical maps of content as well as an understanding of the subject matter to help students toward new conceptions of mathematics.

I wondered if my MTH 202 course could be a source for looking at preservice teachers' PCK development, for here was a place in which I wanted to introduce pedagogical concerns in a content course. Teachers develop PCK through teaching content to students, but I wondered how PCK could begin to be developed during the preservice period—when teachers' ideas and impressions of the profession are being fostered in a college setting.

Grossman (1990) suggested that there are four possible sources for PCK development—apprenticeship of observation, subject matter knowledge, teacher education, and classroom (teaching) experience (Figure 3). In Grossman's model, the most important and influential source is a teacher's teaching experience (the bottom triangle which appears separately from the other three sources). I, on the other hand, was interested in how teachers develop PCK through their teacher education experiences.

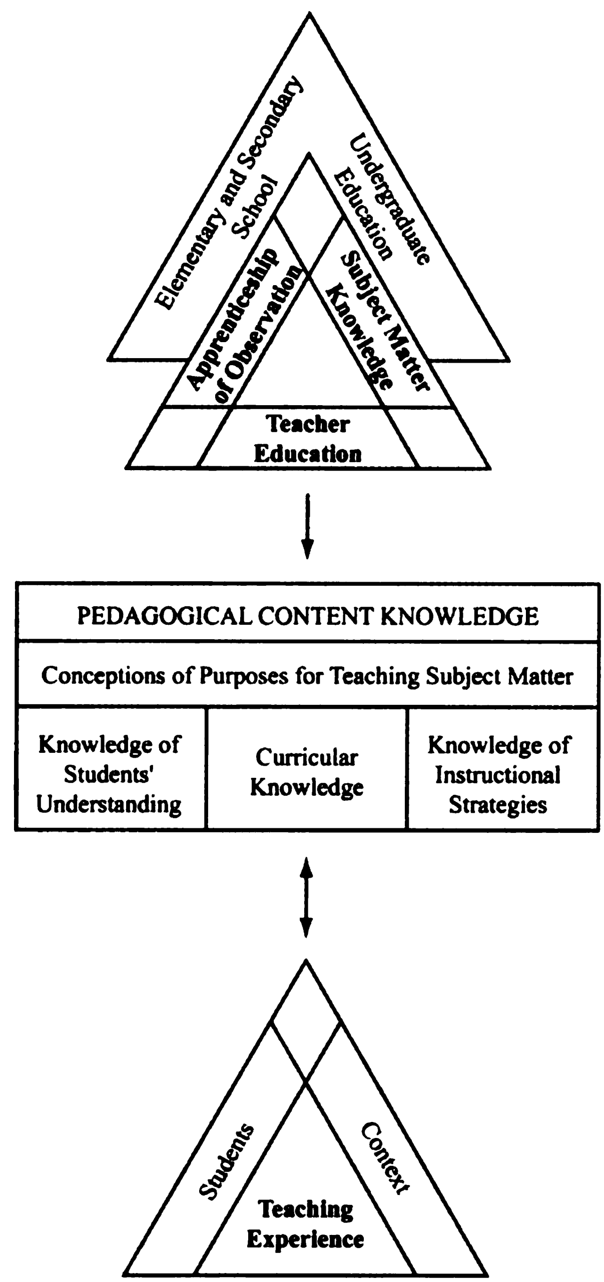

Figure 3: Sources for a Teacher's PCK Development

But most of what Grossman and other researchers refer to as teacher education sources for PCK are methods courses taught in departments of teacher education. My interest lay in considering how PCK might be developed in the context of a content course in which pedagogical ideas were also addressed and studied. What I was beginning to think about was how a mathematics content course—such as MTH

202—might provide a particularly useful site for PCK development, especially when taught by an instructor with a background in teacher education (Figure 4).

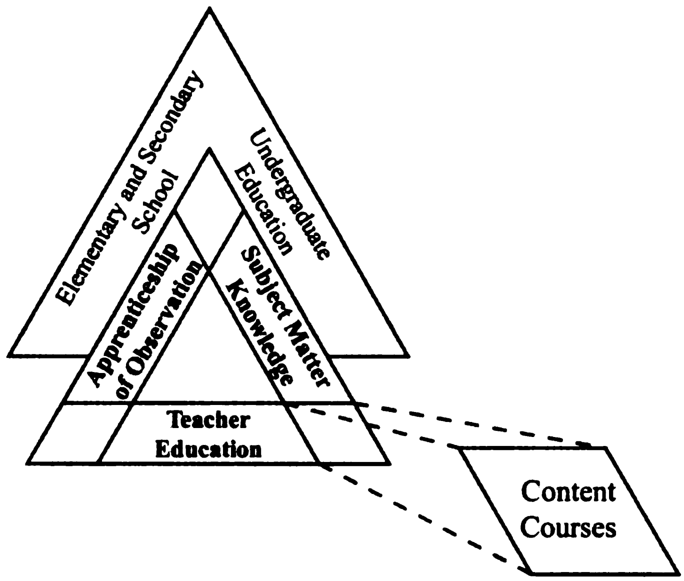

Figure 4: Sources for a Teacher's PCK with Focus on Content

This appeared as a relatively untapped area of research—the study of PCK development in a content course for preservice teachers. It also addressed many of the concerns I experienced working with preservice teachers in trying to find an effective way to combine teaching content with teaching pedagogy and concerns as a middle school teacher working with children in understanding ways to transform meaningful learning experiences for me into ones that carried much the same impact for students.

The Course

When making plans for teaching MTH 202, I was given the opportunity to develop my own curriculum—as long as it focused primarily on geometry and measurement ideas for preservice teachers. An implicit expectation of the mathematics department was that the course needed to remain focused on the mathematics content.[6] The mathematics department chair had full knowledge of my interest in using the CMP materials to teach MTH 202 and he expressed his confidence in my ability to design a course that was appropriate and challenging for students.

Mathematics Content

Good teaching involves an understanding of the major topics in a domain of mathematics and having the ability to make connections between these ideas (Schifter, 1999). To model this thinking for students, I decided to construct my sections of MTH 202 around what I perceived as some of the important mathematical topics in geometry and measurement. By looking at the recommended text for the course and by talking to my colleagues who were also teaching sections of the course, I identified those ideas as perimeter and area, similarity, volume and surface area, the Pythagorean Theorem, symmetry, and transformational geometry. These ideas were chosen based on my work with CMP and what I had observed as critical areas of misconceptions and partial understandings held by preservice teachers.

The first semester that I taught MTH 202, I attempted to cover all of these topics, but I found that I was not able to give adequate time to each one. In each iteration of the course I taught over the next two years, I chose some subset of the mathematical topics listed above, based on the work done by students in previous semesters. These topics aligned with the CMP curriculum as indicated in Table 3.

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[6] This is a salient point, in that I was a doctoral student in teacher education—unlike almost every other graduate student teaching MTH 201 and MTH 202 who were graduate students in mathematics.

Table 3: Important Geometry and Measurement Topics and Alignment in CMP

Mathematical Topic(s)	CMP Unit(s)
Perimeter and Area	*Covering and Surrounding*
Similarity	*Stretching and Shrinking*
Volume and Surface Area	*Filling and Wrapping*
The Pythagorean Theorem	*Looking for Pythagoras*

Model for Learning and Classroom Environment

I wanted students to experience the mathematics content by investigating questions that I posed and that grew out of class discussions. To that end, I worked to create a classroom environment that encouraged group and individual investigations, writing and talking about mathematics, and using manipulatives to enhance learning. I also made use of a teaching model that enabled many of these ideas to be incorporated into the daily classroom routine.

Mathematical Investigations. The mathematics of the course was problem-centered—that is, the mathematics was developed through the introduction of sequenced problems that encouraged students to employ appropriate mathematics and problem solving strategies. This approach differs from other, more traditional models in that the problems provide the impetus for learning and the mathematics is utilized as a tool for solving the problems. This is in contrast to a "traditional" classroom, in which skills are taught and then problem are (sometimes) introduced as applications of the skills.

Students were assigned to small groups (2 or 3 students) and were typically given time to work individually and in their groups on problems. This recognizes that there is often more to be learned working with others, utilizing each individual's respective strengths, than by oneself.

Writing and Talking About Mathematics. Writing has become an important part of every subject matter in schools. Children are often given opportunities to write about what they have learned and the process of writing is recognized as a critical element of the learning process (Rose, 1989). Since the publication of the NCTM Standards (1989), there has been a renewed interest in helping students communicate about mathematics—and writing is one important aspect of communication.

In MTH 202, I regularly asked students to write about the mathematics—directing them to think as much about the *why* behind the answer as the answer itself. On formal assessments, students were asked to explain their answers, an indication that writing about the mathematics was necessary. In addition, students were regularly given writing prompts as a part of their student learning logs. These are described in more detail on page 42.

Closely connected to group investigation and writing is the subject of meaningful student discourse. The NCTM Professional Standards (1991) describe the importance of establishing good classroom discourse, comparing it to a piece of music as it "has themes that pull together to create a whole that has meaning" (NCTM, 1991, p. 35). In establishing good classroom discourse, the teacher plays the pivotal role, establishing norms for discussion, agreement, and disagreement; valuing various opinions and ideas; and conveying messages about who is able to contribute and who has status in the classroom (Cazden, 1988).

Using Manipulatives for Learning. Most of the problems and investigations that were a part of MTH 202 utilized some kind of manipulative to enhance the learning. When mathematical manipulatives became a popular component of many elementary curricula in the 1980s, many teachers embraced using manipulatives as a panacea for alleviating students' difficulties in understanding mathematics. Instead, manipulatives should be used when developmentally appropriate and when they can effectively contribute to the learning process. Manipulatives need to be used when they offer something new and relevant to the learning, not used just for the sake of giving students something to play with. Nor should it be assumed that the use of manipulatives will automatically provide insight to mathematical knowledge (Ball, 1992).

In MTH 202, students investigated problems using a variety of concrete materials. In understanding the concepts of perimeter, area, surface area, and volume, they utilized square grid paper, square dot paper, isometric dot paper, transparency paper, square tiles, cubes, string, clay, and rice. The square dot paper also proved to be particularly useful in understanding notions of similarity and the Pythagorean Theorem. Students also used a variety of tools in working with these manipulatives throughout the course, including scissors, calculators, rulers, and angle rulers. For specific problems, I provided students with labsheets that had drawings, figures, diagrams, and/or data that I found useful.

<u>Instructional Model.</u> In the CMP curriculum, problems and lessons are built around a specific instructional model that I have also found useful in my teaching MTH 202. With this model, lessons are organized around three phases of instruction—the launch, explore, and summarize phases. In the launch phase, the teacher introduces the problem—establishing baseline knowledge for the entire class, introducing new terms or ideas, and issuing the problem's challenge to the students. In the explore phase, students work alone, in pairs, or in small groups on the problem, employing new ideas or utilizing mathematics or strategies from previous lessons. Also during the explore phase, the

teacher works as a facilitator as he or she works between individuals and groups—answering questions, keeping students on-task and on-track, posing questions that can help students get unstuck, and providing additional questions and extra challenges for students. During the final phase, the teacher typically brings the class together again and the class works together to summarize the mathematics that has been learned, the strategies that have been used, and the answer(s) to the problem(s). During this summary phase, the teacher keeps classroom discourse and activity focused on the problem, its solution, and the underlying mathematics.

Through my work with CMP, I had a reasonable amount of experience in designing lessons around this instructional model and in facilitating lessons using it. Many of the other aspects of the course—mathematical investigations, group and individual discovery, writing about mathematics, student discourse, and using manipulatives for learning—are embedded in the instructional model. Students responded well to this model and midway through the course, they were able to describe every aspect of it in detail, before it was ever made explicit to them.

Pedagogical Content and PCK

The knowledge base for teaching is far from being fixed and final. In fact, the systematic study of teaching as an enterprise is relatively new and largely uncharted (Shulman, 1987). The transmission of knowledge to preservice teachers thus becomes a grossly inefficient model for teacher education, and one that is frustrating for novice teachers as they encounter situations for which they feel unprepared. As argued by Fenstermacher (1978, 1986), the goal for teacher education is "not to indoctrinate or train teachers to behave in prescribed ways, but to educate teachers to reason soundly about their teaching as well as to perform skillfully." In this model, the teacher's knowledge base must be flexible and under constant development. Teachers must also be able to use their knowledge base to make reasoned decisions. In this view, teaching becomes a process, rather than an endpoint. A teacher takes an idea and works to place it in his or her existing framework. Then the teacher begins to mold and shape the idea in new ways that will enable his or her students to better understand the idea. In this respect, teaching focuses on the teacher's interaction with ideas and how they can be transformed in ways that students can understand them—more simply, by giving attention to the teacher's pedagogical content knowledge.

When I began the course, I wasn't completely sure what the pedagogical content of the course would be, nor to what extent the course would focus on matters of teaching. In general, I hoped that I could use the mathematical conversations and investigations of the course to address some pedagogical issues, such as assessing student learning and making day-to-day and long-term curriculum decisions.

I was also not exactly clear what I thought pedagogical content knowledge was. When I was planning for the course, I turned to Shulman's definition and description of PCK for some initial guidance:

> Pedagogical content knowledge embodies the aspects of content most germane to its teachability. Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others.... [It] also includes an understanding of what makes the learning of specific concepts easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning. (Shulman, 1986, p. 9)

My initial understanding of PCK centered around the fact that it differed from subject matter knowledge and knowledge about teaching. I viewed it as "more than just a marriage of pedagogy and content. [I]t is another type of knowledge and it is one which teachers must draw upon extensively in order to teach" (Personal Teaching Journal, December 1998). As I taught the course and tried to identify the places in which pedagogical content and PCK were appropriate (see Appendix A), I found that I was mostly focusing on ideas of representation and task development (which provided the basis for the analytic framework in my study), discourse, writing, and teacher professionalism—all of which are part of a knowledge base that is unique to the teaching field.

The Participants

The class met two days a week for an hour and twenty minutes a class session. The class was comprised of thirty-four students—thirty-one females and three males—all majoring in elementary education. One female student dropped the course about two-thirds of the way into the semester due to a family crisis. The other thirty-three students completed the course in its entirety.

Assessment and Grading

Formal assessment of the students consisted of the following: five quizzes (one of which was a group quiz), three tests, responses to ten learning log prompts (see p. 42 and Appendix F), a project and a final cumulative exam. A description of the grading policy that was given to the students is included as Appendix B. Students were regularly assigned homework problems which provided extra practice and further investigation of content that was covered that day in class and which extended the discussions from class.

Influences from Research Literature

The course was focused on students learning meaningful mathematics using a problem-centered curriculum with an emphasis on students' constructing their mathematics, situated cognition, and connections. In addition to the mathematics content, students were exposed to ideas about transforming the content they learned into appropriate mathematics for elementary and middle school students. I approached the teaching of the course primarily as a reflective practitioner—reflecting on my teaching and the students' learning, and involving students in the process.

The current movement of using problems as the framework for a mathematics curriculum can be traced back to John Dewey in *The Child and the Curriculum* and Jerome Bruner in *The Process of Education*—both of whom recognized the ways students learn by applying preexisting knowledge to new problem situations. The value of building a classroom and a curriculum around the investigation of problems has been documented by educators and mathematicians alike (Lo, Gaddis, & Henderson 1996, Schoenfeld, 1996, Fellows, 1992). In describing a mathematics classroom where investigations grow out of interesting problems, Larry Copes advocated that students do "real mathematics"—the kinds of work that mathematicians do (Copes, 1996). In preparing teachers to teach in a problem-centered classroom, he stressed that:

> Teachers should know ways that they can teach mathematical investigations. When the emphasis in on discovering processes rather than reading answers from Tablets in the Sky, teachers need not be able to read those tablets themselves. They need not know all of the answers to engage students in investigating. (Copes 1996 pg. 273)

The bottom-up approach to curriculum design (in which problems form the foundation) was recommended in *Everybody Counts* (National Research Council, 1989) and the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). In the *Standards*, NCTM envisioned "classrooms as places where interesting problems are regularly explored using important mathematical ideas. . . . *What* a student learns depends to a great degree on *how* he or she has learned it" (NCTM, 1989, p. 5). Brown, Collins, and Duguid (1989) took it even further, describing this type of learning as social construction, where the learning outcomes become inseparably associated with the learning process. The current push by some curriculum developers for using "authentic problem situations" recognizes the influence of situated cognition. When students interact with problems that have meaning for them, they learn concepts within problem solving contexts. This helps to create a web of "connected mathematics" for each learner in which the knowledge can be easily accessed in new problem situations. This act of situated cognition also conceives mathematics learning as an "essentially constructive activity instead of an absorptive one" (Schoenfeld, 1992). This theory has yielded the current wave of mathematics literature around constructivism, which focuses education on the learner and his or her efforts to construct meaning around problems (Davis, Maher, & Noddings, 1990; Hiebert & Carpenter, 1992).

The theory of constructivism applies to teaching as well. The teacher's role is to find and adapt tasks which set up learning situations for students. Just as social interactions between students are critical to knowledge construction, so are the interactions between students and the teacher (Koehler & Grouws, 1992). The teacher plays a critical role in what students learn and understand—not only in task design, but in choosing representations, interacting with students, assessing their understanding, etc. Shulman's (1987) description of successful teaching using the model of pedagogical reasoning and action demonstrates how important a role the teacher plays in developing student understanding. One critical phase of Shulman's model is reflection—the process of critically analyzing one's own efforts—as well as those of the students. Yet teachers often see themselves as sources of knowledge—confident in their understanding of the content as well as the pedagogy. Unless challenged, teaching is therefore not a practice that naturally engenders self-reflection (Schön, 1987). But reflection is the very action that enables teachers to learn from their experience and become better teachers (Brown & Borko, 1992).

The Study

In this study, I set out to find ways to construct a course for preservice teachers that covered both mathematics content and pedagogical content knowledge. I was interested in finding ways in which PCK could be an embedded, natural part of the curriculum, much as constructivist teachers want mathematical understanding to grow out of experiences their students have. To increase my understanding of this goal, a focal question and several sub-questions were developed to guide the study and the analysis. The focal research question was:

- What does a mathematics content course look like when the instructor also wants to attend to issues related to teaching that content to K-8 students?

Sub-questions included:

- What do I, as an instructor of my course, do to address what preservice teachers need to know about geometry and measurement to teach it to elementary level students?

- What do preservice teachers need to know about ways of transforming what they know about geometry and measurement for the purposes of teaching it to elementary students and how do I attend to those needs as an instructor?
- In what ways and to what extent is the course contributing to the development of preservice teachers' content knowledge and pedagogical content knowledge?

Studying One's Own Practice

This study is an example of a relatively new field of research—one in which the researcher studies his or her own practice. There is a long tradition of teachers writing about their own practice (e.g., Parker, 1993; Tsuruda, 1994), but the first-person perspective has only become accepted as a viable form of educational research in the past fifteen years (Richardson, 1996). Through the work done by some of the early advocates of first-person educational research (e.g., Lampert, 1986; Lensmire, 1991; Heaton, 1994; Simon, 1995), this method of inquiry has developed into one which is distinctive from other case study research in two critical ways—through design and subject (Ball, 2000).

The first issue of design is critical in that with first-person research, it begins with the identification of an issue and then a context is designed in which the issue can be examined. That is, the context grows out of the study, instead of the other way around. This places the emphasis more on the study itself, for without it, the phenomenon that is being studied wouldn't necessarily exist (Ball, 2000).

It could be argued that the version of MTH 202 I was studying was simply the result of a natural progression of previous sections I taught. To some degree, this is true.

But in many ways, the course was designed not only because of what I learned from teaching other sections, but because I was studying this iteration of the course.

For example, I was aware that geometric formulas were becoming a larger focus of the MTH 202 course each time I taught it. I saw how preservice teachers struggled with formulas and ways to validate them and I wondered if there were some "geometric proofs" which were more helpful, more convincing, and generally more powerful than others in establishing this understanding. I tried various ways to engage students in looking at these geometric proofs and it is clear to me that whether I was conducting a study of my teaching this version of MTH 202 or not, formulas would have continued to play an important role. However, in thinking about what I wanted to study, I began to focus on the development of PCK in preservice teachers. When I saw that Shulman's definition of PCK includes the phrase "the most useful forms of representation of those ideas" (Shulman, 1986), I was struck with the realization that formulas and their geometric proofs are two forms of representation, and that there was the potential for using these representations as the basis for part of my study (in fact, it provided the framework for my analysis in Chapter 3). So because the study was an integral part of the design of the course, the language of "representations" became a regular part of my vocabulary.

Not only did I develop better ways of talking about what I wanted to look at, but I also began to focus more on representations in my teaching journal. For example, in reflecting on a class discussion on the development of formulas for volume and surface area of rectangular solids, I wrote:

> I explained that in middle school, we "don't stop until we get to variables"… and I pushed them to write down a representation for the surface area of that rectangular solid. Three different forms were presented and we were able to have a rich conversation about these forms and their

> equivalence. … I feel a lot better about taking the time to talk about the flat patterns and the development of the surface area formulas than when I did this last year and didn't focus on the flat patterns. They really are a critical step in [my students'] understanding surface area. (Personal Teaching Journal, 3/16/99)

In thinking about the design of the study and reflecting on my teaching, important issues, such as the role that representations play in the development of PCK, were raised and attended to. The course that I taught—and which became an important part of my study—would not have existed in this form had I not been conducting this research.

The second critical difference between first-person research and other forms of case study is the subject and the relationship between the subject and the researcher. Since, in first-person research, these are one and the same, it is apparent that the researcher has access to special insight on his or her subject. Ideas, memories, attitudes, impressions—all of these can be easier to address in a self-study than in even the closest third-person participant observation.

But this also raises a valid concern about maintaining an appropriate distance between researcher and subject. When analyzing data, the researcher needs to stay outside of the experience, remaining an objective observer of events and phenomena. But when the researcher is himself an integral part of these events, that separation can be difficult. On the other hand, distancing himself too far from the events can negate some of the unique insight that can be gained from doing first-person research. It's important to find a middle ground within these two extremes.

In teaching MTH 202, I was constantly struggling with posing good questions. I often had an idea that I was trying to move students toward, much like Martin Simon's "hypothetical learning trajectory" (Simon, 1995), but I experienced difficulties in wording my questions to get at the ideas I wanted to address. This issue of task writing was often on my mind as I prepared for class—as when writing good test questions that would cover all of the important mathematical concepts in a unit but would be concise enough that students could complete in one class period, or when asking questions in the course of a lesson that would move students toward an intended goal without giving away too much of the discovery process. These concerns over the development of good questions, problems, and tasks surfaced frequently in my personal reflections as in the following:

> I think I need to look harder at the questions that I ask and how I can open up the floor for better discussion. It seems like some of my questions are too directed and not open enough ("Anyone know anything about the history of the standard measurement system?") and I need to pay better attention to how I ask these questions. (Personal Teaching Journal, 1/26/99).

Yet many of these concerns that I had would not have been apparent to an outside observer. These were issues that I thought about in practically every stage of course design and development—in planning lessons, in writing assessment questions, in interacting with students in class, through e-mail, and in my office, and in reflecting on the class and my teaching. This theme emerged because the study was about my own teaching and I had much broader insight to the issues I was struggling with as a teacher. The developing attention to task design—an issue that was not an explicit part of my original research design—thus grew to be an important aspect of my study and provided the framework for my analysis in Chapter 4.

I was able to keep an objective perspective on my task analysis because much of the analysis was taking place long after the fact. As I explained to the students in MTH 202, I worked to make sure that the study would not add work to or detract my attention from the course in which they were enrolled. The actual analysis of classroom interactions, teacher reflections, and student work (for the purpose of the study) would not occur until after the course was over. This was to ensure students that they would have my undivided attention during the semester as students—not as participants in a study. It was also my attempt to separate the subject from the researcher.[7]

Informing Students

As this was a study of my own teaching involving college students, I wanted to make sure that they understood the separation between the course and the study I was conducting. And although I was the actual subject of the study, I thought it was important to inform the students throughout the process since they would be participants in the study and their work would be a part of my data. I obtained a preliminary class list and on January 5, 1999—one week before the first class meeting—I sent the students an e-mail to describe the course and the study I was conducting (Appendix C). I wanted students to know how this section of MTH 202 would be different from others and to allow them the opportunity to choose another section if they desired. In the e-mail, I explained that students in the class could choose to be a participant in the study or not. This decision would be made at the end of the course and I would not know who granted permission until after the grades were submitted to the university.

On the first day of class, students received the course syllabus which outlined the mathematics content, expectations, and grading policy. Students also received copies of the Consent for Documentation Form (Appendix D) which described the documentation

[7] Heaton took an even more novel approach when attending to issues of distance (Heaton 1994). In her doctoral dissertation, she referred to herself as Ruth 1, Ruth 2, and Ruth 3—identities that situated her in different time periods with relation to the data (Ruth 1 was the teacher teaching a class, Ruth 2 was the teacher reflecting during the same year but some time later, and Ruth 3 was the teacher looking back 3 years later on her work). Not only did I want to avoid talking about myself in the third person, but since some of my data actually goes back to work I did as an undergraduate student, I wanted to avoid referring to myself with non-positive subscripts.

process I was using and the Informed Consent Form (Appendix E). I explained that the documentation I would be doing (audiotaping, collecting student work, etc.) would be a part of the study, but it was also part of the practice of good teaching. Students were asked to read and sign the Consent for Documentation—either that day or as soon as they wished. I explained that I was giving them a copy of the Informed Consent Form to them on the first day so that they would know what they were going to agree or not agree to at the end of the course.

On the final day of class, students were given another copy of the Informed Consent Form. I again explained the process and that their decisions whether to grant consent would not have any repercussions on their grades. While I was out of the room, students filled out their forms and handed them to one classmate who kept the completed forms until after I submitted the course grades to the university. At that time, I obtained the completed forms from the student and found that all granted permission for their work to be included in the study.

Documentation

There were two types of documentation used for the study—artifacts from the course (data that was a regular part of the course, but collected for the study), and data that was designed and/or collected specifically for the purposes of the study. Artifacts included the lesson and unit plans I made for the course, my teaching journal, the texts that students and I used, and students' written work on both instructional and assessment tasks—including students' "learning logs." For the purposes of the study, I collected audiotapes of all class proceedings, pre- and post-course concept maps showing students' understanding of geometry, and pre- and post-course surveys of students' attitudes toward geometry.

Lesson and Unit Plans. Since this study is centered around my planning and teaching of MTH 202, it was clear that my own thoughts and decisions about the makeup of the course would be important data. Before the course began and as it progressed, I kept written records of what I planned to teach. My plans included outlines and mathematical work that I did to prepare for class (working out homework problems, planning for assessments, investigating ideas that had been raised in class, etc.). These plans always included notes about mathematical content and tasks I planned to use, problems that arose in other students' work that I wanted to use in class discussion, and ideas for homework assignments. At times, the plans contained notes about aspects of PCK I wanted to include in class. The lesson and unit plans were kept in a binder and were helpful in maintaining an ongoing written account of how I envisioned the content of the course.

The unit plans focused on the mathematical topics I identified in geometry and measurement and that I planned to use as the basis for the course. My mathematical goals for the course included in-depth investigation of these mathematical topics (outlined for the semester in Appendix A) as well as:

- Giving students opportunities to read and write about the mathematics and to engage in thoughtful discourse about the mathematical content.
- Making students aware of additional resources that are available to teachers, such as professional journals, the NCTM Standards, Internet resources, and multiple elementary and middle school curricula.

- Highlighting and encouraging students to find connections between different mathematical ideas.
- Connecting the mathematics of the course to other subjects taught in elementary school, such as literature, social studies, and science.
- Identifying preconceptions about what geometry is and giving students a set of positive experiences around the mathematics content of the course.

Teaching Journal. Each day after class, I immediately returned to my office where I sat down for two or three hours, thinking about what had happened in class that day and recording my thoughts into a teaching journal. The teaching journal afforded me the opportunity to reflect on my teaching and what had just transpired in the classroom. When teaching in a middle school or two back-to-back college courses, my teaching schedule often did not afford me the time to reflect—especially this soon after the event—and this resulted in some missed opportunities for me and my students. But when I was able to block out time immediately after teaching MTH 202, I was able to recall important, but small, events which might have been forgotten, I was able to attribute particular insights to students who had made them, and I was able to make better plans on where to pick up for the next class session. The teaching journal was also helpful in that it provided me with a record of my thoughts and ideas which was a rich source of data.

Textbooks. The texts for the course were student editions (SE's) and teacher editions (TE's) of the following CMP units: *Covering and Surrounding* (SE only), *Stretching and Shrinking* (SE and TE), *Filling and Wrapping* (SE and TE), and *Looking for Pythagoras* (SE and TE).

The other instructors of MTH 202 used a college geometry text in their sections (*Geometry: An Investigative Approach* by O'Daffer and Clemens). The O'Daffer text did a nice job presenting problems for preservice teachers to solve and making some connections from the mathematics to teaching the mathematics. But the text fell short in preparing teachers to make curricular decisions—which topics to cover, what topics are appropriate for students at various levels, how to engage students in the mathematics, etc. Russell (1997) suggested that elementary and middle school curriculum materials might be more appropriate for teaching preservice teachers the mathematics content and ways to teach it. Bright (1999) reviewed some current reform curricula with this purpose in mind—of teaching preservice teachers content using student and teacher texts. He suggested that those teacher texts which contained "dialogue boxes"[8] to have the most potential in teaching preservice teachers.

For two years prior to this study, students in my sections of MTH 202 purchased and used various student and teacher editions of geometry and measurement CMP texts. Initially, I had students use SE's for all of the units and TE's for only one of the units. I did this for several reasons. First, I thought that if students had the answer keys to some of the homework questions, it would decrease the amount of time spent in class going over answers to problems. This would leave more time for discussion of the mathematics embedded in the problems, applications of these ideas to teaching elementary school, and investigations of new ideas. Second, I reasoned that since many of the problems and their solutions involved geometric diagrams, it would be easier if students could refer to an

[8] The dialogue boxes appeared in teacher editions and were used to give evidence of student reasoning on problems, strategies that students often utilized, and suggestions for teaching the mathematical content to students.

answer key with figures drawn more accurately than those that I might be able to reproduce in class.

As I taught the course, however, I found that by giving students access to more of the TE's we began to talk about other issues as well. The CMP TE's contain more than just correct answers to student problems. They are developed through interacting with dozens of experienced teachers who used the materials as they were being developed and so they reflect lessons learned by those teachers and suggestions that they made. The TE's contain multiple answers (when appropriate) and various examples of strategies that middle school students may use when working on problems. They also push teachers to understand the content at a much deeper level than that of their students, encouraging them to know more about the *why* behind the mathematics they encounter.

By having access to the TE's, the preservice teachers were being exposed to a number of concepts that are often talked about in teacher education courses but may be difficult to comprehend when not placed in a mathematics context—things like discussions on various strategies that middle school students could use in solving problems and possible ways to engage students in the mathematics that is being studied. As a result of this exposure, students initiated class conversations about lesson planning, connections between mathematical ideas, assessment, curriculum design, and other pedagogical concerns.

For example, a large part of one class session midway through the course centered around my asking the students to "describe the typical mathematics lesson" from MTH 202. The students were writing their own grade-level specific unit plans or lesson plans for a topic in elementary school geometry. In preparing to write their own lessons, I wanted them to think about the normal mathematics class they remembered from their school experiences as compared to what I typically did in MTH 202. Some of the students' responses were:

- "You start by bringing out our background knowledge for a certain mathematical idea."
- "Without telling us what we are going to find, you let us explore first."
- "You use 'guided experimenting'… You set up the situations for us to go in specific directions."
- It's very hands-on. I learn from touching something—to be able to move the tiles or to wrap the yarn around my foot."
- "(After we get back together as a class), you go back and talk about each group's answers and how they are different, (with a goal of) finding the most accurate or the most efficient method."
- "At the end, you tie everything into the big picture, how it all relates to each other."

Some students simply picked up on these ideas after being a student in the class for eight weeks while other students explained that the TE's were helpful in seeing what I was doing when teaching the class. At this point, we talked about phases of instruction and an instructional model (see p. 28) that are explicit in the TE's, and how this model for learning differed from what was experienced in other mathematics classes. Students commented that this conversation served as a helpful analysis of teaching and learning mathematics as they prepared their own lesson plans. I think that fewer students would have been able to participate in this classroom analysis had they not used TE's on a regular basis.

Students' Written Work. I collected and photocopied many pieces of student work. These included quizzes, tests, the final exam, and responses to learning log prompts. All students consented to the use of their work in the study.

This is primarily a study that looks at what I did to plan for and teach a mathematics content course for preservice teachers that gave attention to issues related to teaching that content to children. However, I found it helpful to use evidence of student work in looking at the impact of what I did on what students learned. Students were often encouraged to supply reasons and show work for their answers. This evidence supports not only what they learned, but it also provides support of claims I make about how the design of the course has an impact on the learners.

One of the richest sources of data around student work was the student learning logs. As part of the course, I required each student to supply a small notebook that I called a learning log. The learning logs were used for students to reply to designated prompts throughout the semester (a total of ten different prompts). I regularly collected, read, evaluated and orchestrated class discussions about their responses. The learning log prompts were designed, in part, to give students some experience in writing about mathematics. Many elementary teachers are asking students to keep mathematics journals to record their thoughts, insights, methods, and attitudes. I wanted preservice teachers to have some experience in writing about mathematics as well, just as they might ask their students to do. In addition, the learning log prompts were written in conjunction with the learning goals I had for the class and were designed to:

- Establish a base-level understanding of what students understood about a concept before it was addressed in class. In the first prompt, I asked students to *"write down the first three geometry formulas you think of and tell what they are the formulas for.... Describe some of the pros and cons for using mathematical formulas. What do you think about using formulas with elementary school students?"*
- Address some of their preconceived ideas and attitudes about geometry. It has been my experience that some preservice elementary teachers view geometry only through the lens of their high school geometry course—positively and/or negatively. In one prompt I asked students to *"describe in detail an event from your high school geometry course which is particularly memorable to you. Why do you think this event sticks with you?"*
- Give students a mathematical task in addition to what was done in class. I used this opportunity to pose more open-ended tasks. This gave me some rich data on how students could write about mathematics and on their mathematical thinking. In one prompt I asked, *"When the midpoints of the sides of a quadrilateral are connected in order, another quadrilateral is formed on the inside. ... What kinds of quadrilaterals can be formed on the inside? Which original quadrilaterals produce which types of quadrilaterals on the inside? ..."*
- Raise other pedagogical issues. After the class discussion on how some of the geometry ideas in the course connect to other areas of mathematics (number, algebra, data analysis, etc.), I included a learning log prompt asking students

to *"...give examples of some of the ways in which geometry and measurement connect to other school subjects (possibly including science, social studies, language arts, music, etc.)."*

- Have students reflect on the MTH 202 course. Since I was interested in how the design of the course had an impact on my students' understanding and attitudes, I included one prompt in which I asked them to *"describe in detail one event from your MTH 202 course which is particularly memorable to you. Why do you think this event sticks with you?"*

A complete list of learning log prompts that I used in the course are given in Appendix F.

Audiotapes of Class Proceedings. At the start of each class session, I turned on a small cassette recorder that was placed at the front of the classroom and recorded the entire class session. All students were made aware that this type of data would be collected and they granted consent for their voices to be recorded. The tape recorder was near me and it did a good job picking up my voice throughout. Some parts of student conversations were harder to hear.

Although I did not use the recordings while teaching the course (finding the time to listen to class sessions and review what was said might have been helpful but extremely time-consuming), they provided a relatively thorough record of class proceedings and discussions. I used the teaching journal and the lesson and unit plans to identify critical events in the course and was then able to turn to the audiotapes for a more detailed account which I transcribed.

Other Data. Two additional pieces of data were collected in the course and were designed to help me look at student growth in knowledge and disposition. The first was a set of students' concept maps that were drawn at two different times in the course. After a brief discussion on making concept maps early in the course, I asked students to place the word "geometry" in the center of a blank piece of paper. Students then constructed their own concept maps of how they thought about the content of geometry, naming big and small ideas and connecting them as a representation of their thinking. These maps were collected and redistributed on the last day of class. Students were then given the same task—to construct a concept map of how they thought about the content of geometry—and asked to make those maps on the backside of their initial ones. Students were then asked to compare their two maps with the hope of their seeing how much their understanding of the domain grew.

The second was a set of student responses to an attitude survey about geometry and measurement. As with the concept maps, these surveys were administered at both the beginning and at the end of the course. In the case of the surveys, however, students were not given access to their earlier responses and so could not use them to make a self-assessment about any changes in attitude or disposition.

As the analysis of the data progressed, I didn't find the concept maps very helpful in the study, so they were not used. A few of the student responses to the attitude survey helped shed some light on their dispositions toward learning geometry and thus became a part of the study.

Data Analysis

This was a study of what I did as a teacher educator to attend to issues of pedagogical content knowledge in a content course. To that end, I used coding methods to find themes that surfaced in my teaching of MTH 202—reading my personal accounts of the course in my teaching journal and lesson and units plans, listening to and transcribing episodes of class discussions, and analyzing student work. Data is drawn from thirty class sessions, my writings in preparing for those sessions, and my writings in reflecting after those sessions and after the course was completed.

The Emergence of Two Themes

After the course was completed and all students agreed to be participants in the study, I began the work of analysis by reading—and rereading—my teaching journal. I had almost four months of reflections and observations that were recorded and I wanted to refamiliarize myself with what I wrote over the course of the semester. In particular, I was looking for trends or recurrent themes in my writing—ideas which could potentially develop as themes for analysis.

<u>Formulas and Other Representations.</u> One of the most prevalent themes that emerged came as no surprise to me—that of geometric formulas. As my MTH 202 course evolved over a three-year period, I found myself constantly coming back to issues concerning formulas—understanding where they come from, helping students make sense of them, finding ways to remember them, etc. Many of the problems and investigations I developed and that I used from the CMP curriculum involved areas and perimeters of shapes and volumes and surface areas of solids, topics that often led to

discussions around understanding formulas used to calculate those measures. I recognized early on that formulas would be a possible theme for analysis and I made a conscious effort to pinpoint places in my teaching journal where discussions around formulas occurred.

The importance of understanding representations has not been lost on the mathematics education community. Mathematics has long concerned itself with various categories of representations (e.g., symbolic, spatial, and language) as well as specific types of representations that are particularly useful in mathematics (e.g., graphs, tables, diagrams, and formulas). These mathematical representations are established models in the world of mathematics. Recently, the National Council of Teachers of Mathematics raised mathematics educators' level of consciousness about representations by adding it to its list of essential process standards for school mathematics—a list that now includes problem solving, reasoning and proof, communication, connections, and representations (NCTM, 2000).

Mathematical formulas are one example of a type of representation. A formula is a symbolic representation of a mathematical relationship. For example, $A = s^2$ is a formula for the area of a square. This formula shows—concisely and eloquently—that the area (A) of a square can be determined by multiplying the side (s) by itself, (squaring it). With this type of representation, a great deal can typically be said in a small space. Another example is a pictorial or spatial representation. As I taught different sections of MTH 202 over the years, I became increasingly interested in ways in which various spatial representations could be used to explain or validate the geometric formulas that were a part of the course. I began to wonder if different forms of representation were helpful for students in understanding a concept.

But the word *representation* refers not only to the product (the formula, the picture, etc.) but also to the process—the "act of capturing a mathematical concept or relationship in some form" (NCTM, 2000, p. 67). The research around the role of representations gives credence to both aspects, the process and the product. For not only is it important to look at which types of representations can be used with students and have an impact on student learning, but it is also important to recognize the development of modes of representation as they occur in the classroom.

But representations also play an important role in the development of pedagogical content knowledge as well—although the representations that are mentioned in PCK research literature do not necessarily map easily onto the representations in the mathematics literature. Ball identified "'forms of representation'...[as] the crucial *substance* of pedagogical content knowledge" (Ball, 1988). Borko et al. claimed that making subject matter understandable for students hinges on the teacher's ability to "represent specific topics in ways appropriate to the diverse abilities and interests of learners" (Borko et al., 1993). And in summarizing the literature on PCK, Van Driel et al. suggested that "all scholars agree on Shulman's two key elements—that is, knowledge of representations of subject matter and understanding of specific learning difficulties and student conceptions" (Van Driel, Verloop, & De Vos, 1998). Indeed, representation appears as a critical component of PCK not only in Shulman's introduction of PCK to the field (Shulman, 1986), but also as a key element of the transformation phase in his model for pedagogical reasoning and action (Shulman, 1987).

It seemed to me that the mathematical representations (like spatial, symbolic, and language) could be examples of some of the representations mentioned in the literature on PCK, as in Ball's "substance of pedagogical content knowledge," but that much of the writing about PCK didn't involve physical representations as much as it did representations of knowledge and understanding. With the word *representation* being used in many different ways, I found that it was easy to get confused about its meaning when applying ideas of PCK specifically to learning mathematics—a content area in which representations already play an important role.

I decided to focus some of my analysis around the important mathematical representations as they appeared in my data. Because I identified formulas and their accompanying spatial representations as a potential theme before collecting data, I encountered numerous references to these representations in my unit and lesson plans, my teaching journal, the learning log prompts and assessment questions I wrote as well as the student work done around them, and in the audiotapes of the class proceedings—any source with evidence of my thinking and/or my words. The CMP textbooks (specifically, the TE's) also provided some evidence of identifying various representations and the ways that they can be used in helping students understand a mathematical concept.

A useful method of analysis in thinking about types and roles of representations was in developing and using coding categories. In Chapter 3, I give evidence of my work in forming various categories of representations and how this process revealed a type of representation that I previously disregarded—the representation of mathematical language. I also discuss how the process of shared reflection was an important tool in developing my understanding of the various ways in which representations were used, as

I shared some of struggles and frustrations with preservice teachers and made these conversations a part of my learning about teaching.

Nature of Tasks and Issues of Task Design. After looking more carefully at what I wrote in my lesson plans and teaching journal, I uncovered a second theme. I noticed that one particular problem—one of the learning log prompts—stayed a part of our class discussions for parts of four different sessions. This was considerably longer than the one day I planned to investigate this task. In thinking about why this task embedded itself so deeply in my (if not the class') conscious, I conjectured that if the task were worded differently, it might not create the difficulties that it did for my students and me.

This supposition was gradually generalized to include basic ideas of task design. That is, what goes into the development and construction of a good task—one that does not mislead students but provides an avenue for investigating the problem as the teacher intended.

The NCTM Professional Standards described *worthwhile mathematical tasks* as ones "that are likely to promote the development of students' understandings of concepts and procedures in a way that also fosters their ability to solve problems and to reason and communicate mathematically" (NCTM, 1991, p. 25). While it goes on to explain that tasks may be selected or adapted from other resources, or generated by teachers themselves, it provides little detail on the process that teachers go through in creating meaningful tasks.

Since task design clearly depends on a flexible understanding of the underlying mathematics around which the task is developed as well as identifying ways to help students understand the mathematics themselves, it seems that a teacher education course with an interest in developing preservice teachers' PCK should include some attention to its development.

In my data, I found several sources for data around task development. In my lesson and unit plans and in my teaching journal, I found a number of entries around this particular learning log prompt alone. I also had copies of all of my students' work on this task—data which I used to triangulate my findings and theories on how elements of the task I designed produced unexpected results. In addition to this, I had historical records of this task when it was given to me as a student in a preservice teacher education course. This was helpful in doing a critical analysis of the two versions of the task and looking at the differences in expectations that the tasks make of students. In Chapter 4, I give evidence of my using student work to investigate the design of a task and of my comparing two different versions of the same mathematical task. I also discuss how the process of shared reflection was a useful tool in helping me learn more about the ways in which the task could be revised to better reflect the potential for learning that I was hoping to capture.

In each of these themes—representations and task design—I started by looking at ways in which they exemplified preservice teachers' PCK development. But in thinking about each case, I kept wanting to turn from what my *students* needed to know to what *I* needed to know to teach preservice teachers and what I learned through the use of shared reflection. In light of this, I wondered if this was really a story about developing my own PCK as a teacher educator in the context of attending to students' PCK development.

Revisiting Shulman's Pedagogical Reasoning

As described from the start, this study was built around my wanting to look at what I did to design and teach a mathematics content course that also attended to issues of teaching that content to K-8 students. The students in the course were not intended to be the subjects, but rather participants in a study of my own practice. So while I initially thought the themes that I was developing centered on looking at developing students' PCK, I began to recognize that another important aspect was the development of my own pedagogical content knowledge.

Little had been written about the development of a teacher educator's PCK. Bennett and Carré (1993) applied Shulman's model to a study of a postgraduate teacher education program and Cochran, DeRuiter and King (1993) analyzed a constructivist teacher education program using Shulman's model. But both of these studies were done on a general level, looking at an entire program instead of a specific case. These studies also focused more on looking for PCK in the final end product, rather than on examining the complexities of its development.

Geddis and Wood (1997) described a case study of a teacher educator in which they documented a specific instance of teaching as a transformation of knowledge about mathematics pedagogy for preservice teachers. Although rich in detail, their study looks only at part of the transformation process of teacher education and does not address the process the teacher educator went through himself to develop his own PCK about the subject matter.

It appears that a critical missing element in these studies is understanding the teacher educator's reflection on his or her learning and understanding—about what the students know about the content and pedagogy, and about what the teacher educator knows about the content and the pedagogy. Because the teaching is taking place in a teacher education context, the preservice teachers and the teacher educator can all benefit when this reflection becomes a part of the classroom conversation.

In reflecting on my own experiences as an undergraduate student and a novice middle school teacher, I recalled instances where I struggled to recreate a rich learning situation for my students that I had as a preservice teacher. As with the example of using the locker problem (page 14), I found that I was lacking some important understanding about the problem in making it meaningful for my middle schoolers. Some of my shortcomings centered around an understanding of the mathematics—the necessary prerequisite skills that were needed and the sequencing of problems that could lead to a deeper understanding. Other deficiencies were around the pedagogy—not knowing how to organize my students and classroom to work on the problem, introducing a strategy too early and not giving students an opportunity to construct their own, etc. But after further reflection, I realized that all of these problems fell under the umbrella of my lacking PCK, that is, ways to transform what I understood about the problem as a teacher and making the mathematical knowledge accessible to my students. In the case of the locker problem, I decided that what I was missing was some insight on what my college professor thought about when he chose the task, adapted it for our class, made decisions about introducing representations, and other elements of the pedagogical reasoning process. With some of that pedagogical content knowledge, my experience of using the locker problem with middle school students might have been quite different.

Teaching teachers and giving some attention to their PCK could involve taking an additional step—providing for them some ideas for making curricular and pedagogical decisions. These are skills that are often developed through years of teaching, but novice teachers can also begin to have access to this decision-making process when the teacher educator shares with them the process that he or she went through for using that particular lesson or problem.

In fact, this process makes the pedagogical reasoning and action model explicit. Instead of the teacher educator just working through the phases, the process itself is highlighted and the preservice teachers have exemplary teaching modeled for them. This is one way in which *shared reflection* can be utilized—where the teacher educator shares his or her thinking process with preservice teachers. The subject of shared reflection can be any phase in the pedagogical reasoning and action cycle but by definition, it needs to occur within the teacher educator's reflection cycle. Figure 5 shows an adaptation of my model of Shulman's pedagogical reasoning and action cycle to describe the process of a teacher educator employing shared reflection.

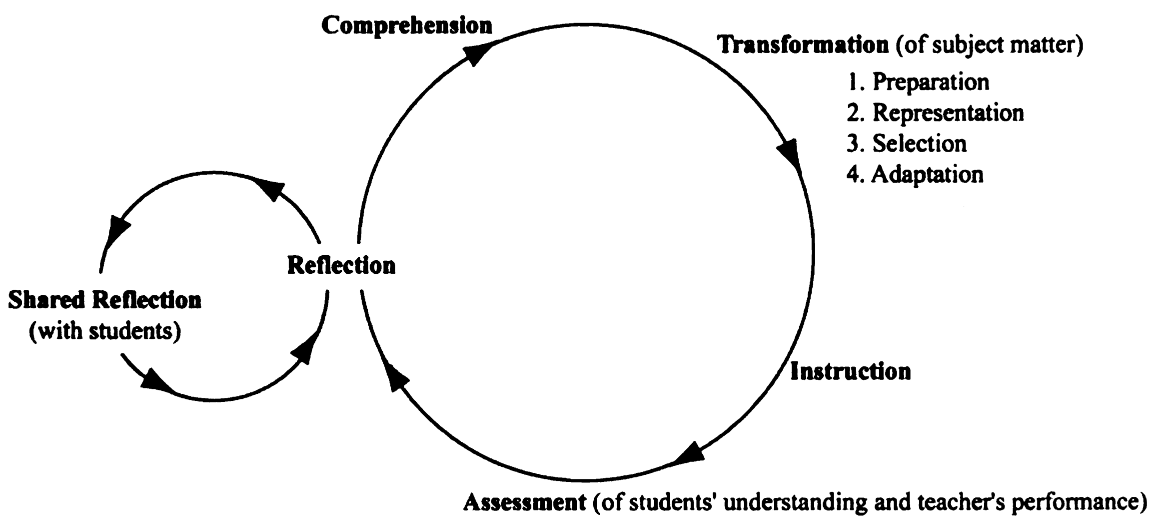

Figure 5: A Teacher Educator's Pedagogical Reasoning and Action

For example, the teacher educator may choose to wait until a complete lesson has been planned, taught, and assessed before he or she begins to share with the preservice teachers some of the decisions that were made around the choice of the problem, the representations that were used, the way in which the lesson was presented, or the assessment has been completed. In other instances, the teacher educator may take the time to address some of the phases either before, during, or after those phases present themselves in the lesson, but still after the teacher educator has reflected on the phase him- or herself. These are decisions that the teacher educator needs to address at all times, deciding whether the students should be let in on the decision-making process or if they should be allowed to "muck about" in the mathematics or the pedagogy for awhile.

Refocusing My Study

Looking back at my data, I found many instances of how I used aspects of shared reflection in helping my students develop their PCK. In each case, the journey begins and ends with my looking at my own PCK of the problem and thinking about how giving attention to the role of representations or task design using shared reflection may have helped students with their PCK development. To that end, I clarified my central question. I was now interested in tackling the issue of: "Can pedagogical content knowledge be a useful framework for a teacher educator in designing and teaching a mathematics content course for preservice teachers and if so in what ways?"

The Story

This study is a story about my efforts to give attention to the pedagogical content knowledge development of preservice teachers through a process of examining my own PCK development as a teacher educator. Although I started the study with an idea of focusing on students' PCK development, it wasn't until midway through the course that I began thinking about focusing on the development of my PCK for teaching prospective teachers and using that experience as a lens for thinking about what the subject matter of PCK might be for preservice teachers. Reflecting on my experiences as an undergraduate student helped me to formulate a model for teacher education in which an emphasis on shared reflection gives preservice teachers some insight to teaching content to K-8 students. This study details my some of my efforts to use shared reflection in lessons about representations and task design.

CHAPTER 3

REPRESENTATIONS

Introduction

This chapter takes the form of three episodes. Each episode describes: (a) a mathematical topic that was covered in the course; (b) the ways in which the mathematics was presented with respect to the development and use of representations; and (c) how shared reflection played an important role in the course.

The three episodes center around trapezoid area, circle area, and volumes of prisms, cylinders, cones, and spheres. The first episode deals with my gaining a deeper understanding of different types of representations—symbolic, spatial, and mathematical language. The second episode deals with my looking at limitations of some representations and the ways in which multiple representations are linked with each other. The third episode deals with my struggling to reconstruct some of my own prior mathematical knowledge and my sharing those struggles with the students. In the third episode, I also look at the use of a blended representation that proved to be a particularly powerful one for preservice teachers.

Throughout each episode, there is also an underlying tension concerning the importance of providing *justification.* Students come to believe that certain mathematical ideas are true. But as described in the NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principals and Standards for School Mathematics* (2000), students need to consider what it takes to prove the validity of these assertions. In a course for preservice teachers, some argue that attention should be given to understanding what it takes to validate a mathematical statement and various methods—or proof schemes—for providing this justification (Harel & Sowder, 1998). The reasoning goes that just as school students are being expected to provide reasoning about their mathematics, so must their future teachers work on understanding what it takes to justify a mathematical assertion.

In looking at justification in this chapter, I use the classifications developed by Harel and Sowder in which proof schemes are described as externally based, empirical, or analytic. Each scheme has several sub-categories (Figure 6) which are helpful for looking at various representations that are used (Sowder & Harel, 1998). In each of the episodes, I examine what proof schemes are implied by the inherent assumptions that were made by me as the teacher educator.

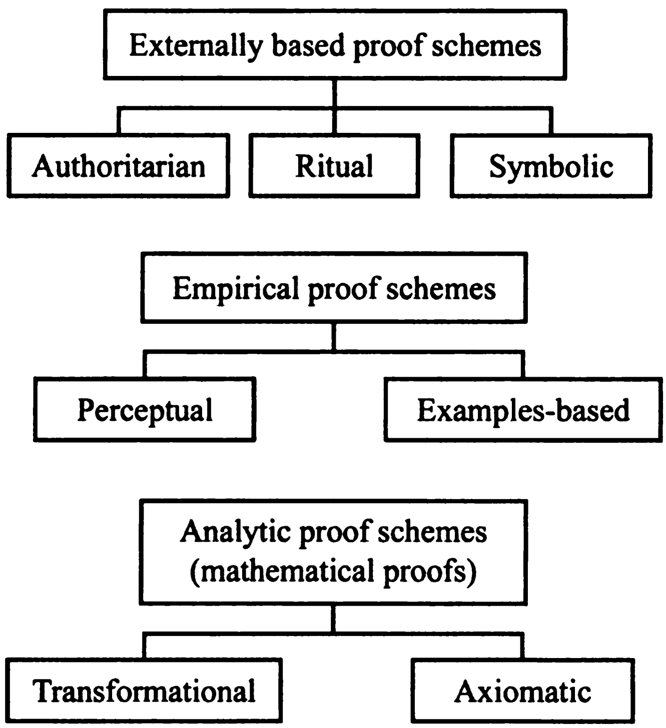

Figure 6: Proof Schemes

Episode 1: Trapezoid Area

Background—Working with Triangles

Students' work with triangles earlier in the course involved investigating ways to generalize the area of a triangle and developing a symbolic formula for triangle area. A series of triangles were presented on grid paper and students were given time to develop strategies for finding area. The goal was to find one or more methods that could be used with as many different triangles as possible. (Two triangles from this labsheet are presented in Figure 7.) Two methods were common: pairing a triangle with a congruent figure or dissecting the triangle and rearranging the pieces to form a new figure.

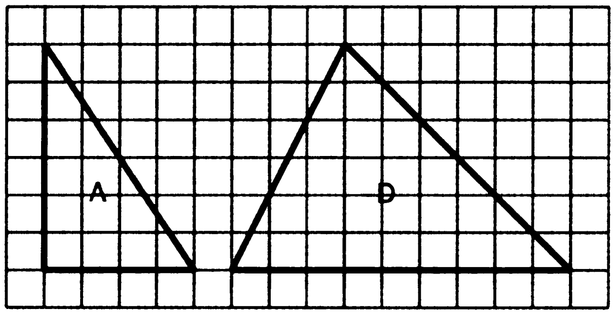

Figure 7: Triangles A and D on Grid Paper

Using Two Copies of a Figure. When investigating a right triangle like A, students often paired it with a congruent copy, rotated 180° and placed so that the hypotenuses matched, creating a rectangle with an area twice that of the original shape (Figure 8). The rectangle's area was easily determined (by counting squares or by using the base multiplied by the height) and the area of the triangle was commonly stated as "half of the area of the rectangle."

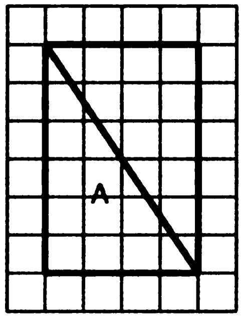

Figure 8: Two Copies of Triangle A

Although non-right triangles, like D, were less likely to be paired with a congruent copy, some students continued the reasoning from the right triangles—using the resulting parallelogram instead of a rectangle (Figure 9).[1]

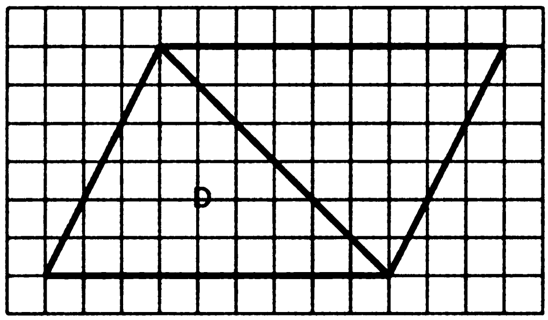

Figure 9: Two Copies of Triangle D

With both right triangles and non-right triangles, students used two copies of the original triangle to make a new shape for which they could find the area. And since two copies had been used, the area of the resulting figure was double that of the original shape. Students then concluded that the area of the original triangle was only $\frac{1}{2}$ the area of the resulting figure, leading to the general formula $A = \frac{1}{2}bh$.

[1] In a previous class, students discovered that a parallelogram has an area equal to a rectangle with the same base and height as those of the parallelogram, leading to the general formula for the area of a parallelogram (rectangular or not) as $A = bh$.

Dissecting and Rearranging. Many of our earlier investigations with area centered on ways to take a shape, subdivide it, and then put the pieces back together in a new shape whose area could be determined. This strategy—which I call dissection and rearrangement—is important in mathematics both historically and as an example of a proof scheme.

Some simple dissections were well-known to the ancient Greeks, but the first formal collection of dissection problems was most likely written by the tenth-century Persian astronomer, Abul Wefa (Gardner, 1969). The problems were typically presented as interesting mathematical diversions, showing for example how three congruent squares can be cut into nine pieces to make one single square. In the early twentieth century, mathematicians (like the puzzle creator Henry Ernest Dudeney) took on geometric dissections with a renewed recreational fervor, working to set records based on the fewest number of cuts needed to dissect a given figure and rearrange the pieces into a specific shape.

In MTH 202, I referred to some of the ways in which methods of dissection and rearrangement were used by mathematicians to determine the areas of unknown regions, like how Leonardo da Vinci calculated the areas of many curved figures by cutting them apart and rearranging the pieces to create figures with known areas. I referred to this method not as a mathematical recreation, but as a viable way to justify a geometric idea. Many students employed this technique in a number of investigations on area, making it a common strategy for looking at the triangles.

When working with right triangles like A, the most typical approach to dissecting and rearranging was to make one cut—horizontal or vertical—and reduce either the height or the base to half its original length. These two pieces could then be formed into a rectangle with half of the height and the same base as the original triangle (Figure 10), or the half of the base and the same height (Figure 11).

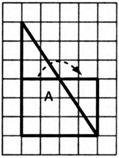

Figure 10: Triangle A as Rectangle with Half of the Height

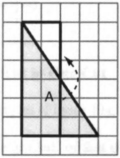

Figure 11: Triangle A as Rectangle with Half of the Base

The notion of making one cut to create two pieces which could be rearranged to make a known shape was used with non-right triangles as well. However with these triangles students discovered that the cut had to be made parallel to the "base" (the side that lay on one of the grid lines)[2]. For example, on Triangle D, one cut could be made parallel to the base that would intersect each of the other sides at their midpoints (Figure 12). The top piece could be rotated 180° and placed on either side to create a non-rectangular parallelogram with a base equal in length to the base of the triangle, but with a height half of the triangle's height.

[2] Because this triangle activity was focused on developing and understanding the basic triangle area formula $A = \frac{1}{2}bh$, all of the triangles presented in class had at least one side drawn on a grid line. The idea of finding the area of a triangle whose endpoints all lie on the intersection of grid lines but none of whose sides lie on grid lines was also presented and was played out in the context of surrounding a shape with a rectangle and subtracting off the areas of the unwanted pieces. It was important that students saw and worked with triangles of many different types, but for the purposes of the lesson on developing a general formula for areas of triangles, only triangles with at least one side on a grid line were considered.

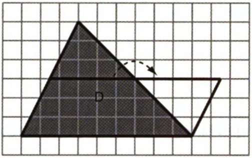

Figure 12: Triangle D with Half of the Height in One Cut

Some students made two cuts with the non-right triangles (Figure 13), still recognizing the fact that the resulting figure would have a height half of that of the original triangle. These students thought of the top triangle as being composed of two smaller right triangles, each of which fit nicely with the larger bottom piece to create a rectangle. These students were likely guided by the existing grid lines to consider this second cut and to create a final shape with all sides lying on the grid lines.

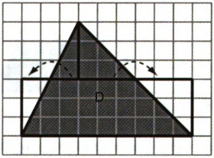

Figure 13: Triangle D with Half of the Height in Two Cuts

In each of the above examples (Figures 5-8), the shape resulting from dissection and rearrangement was a parallelogram—rectangular or not. And each of these resulting parallelograms had one dimension (either the base or the height) the same length as the original triangle. The other dimension was half the length of the original triangle. Students reasoned that the parallelograms they made from the triangles had areas equal to either (a) the original triangle's base multiplied by half of its height, or (b) the original triangle's height multiplied by half of its base. Symbolically, this led to the triangle area formula of $A = \frac{1}{2}bh$.

Area of a Trapezoid—Connecting Symbolic and Spatial Representations

The course was designed so that one subject typically led to the next, helping students to see not only how different mathematical ideas were connected, but also how a strategy that was useful with one topic might be employed with another. For this reason, I chose to follow triangles with some work on trapezoids.[3]

Since many students had used the dissection and rearrangement approach with triangles, I decided that it would be a useful strategy to use with the class when reasoning about the trapezoid formula. When I introduced the trapezoid area formula, I presented the diagram in Figure 14.

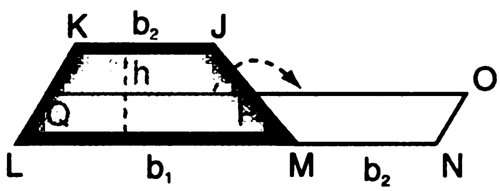

Figure 14: Trapezoid Method A

I expected students to reason through the problem as they had done with other shapes before, using the figure to justify the parts of the formula. In particular, I expected that students would recognize that the resulting figure (OQLN) was a parallelogram with an area equal to the area of the original trapezoid (JKLM). Furthermore, the parallelogram had

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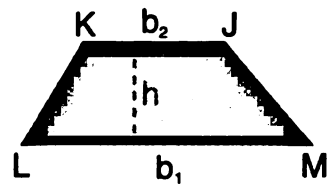
[3] A trapezoid is sometimes defined as a quadrilateral with *at least one pair of parallel sides*. Using this definition, the set of parallelograms (quadrilaterals with two sets of parallel sides) is a subset of the set of trapezoids. I chose to use the more limiting definition of a trapezoid as a quadrilateral with *exactly one pair of parallel sides*. This allowed me to introduce parallelograms first, giving students some knowledge on a figure which was useful when reasoning about the formula for the area of a trapezoid. The alternate definition has parallelograms as a special case of trapezoids, rendering parallelograms much less useful in understanding trapezoids.

a base equal to the sum of the trapezoid's two bases ($b_1 + b_2$) and the height of the resulting parallelogram was equal to half the height of the original trapezoid. Substituting these values into the basic parallelogram area formula $A = bh$, we would get $A = (b_1 + b_2) \times \frac{1}{2}h$ which is equivalent to the more standard form: $A = \frac{1}{2}(b_1 + b_2)h$.

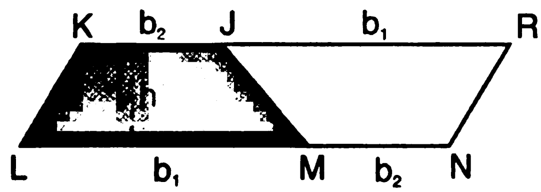
I was intrigued with the idea of using a spatial representation to justify a known area formula. I thought that it was a good way to show preservice teachers that geometric figures and their manipulation were legitimate means of proving a mathematical statement (here, an area formula) and that there were multiple ways in which a figure could be manipulated to yield similar results. I knew that these students, as teachers, would need to be open to different approaches presented by their students and so I decided to present three other methods for reasoning about the trapezoid area formula. I did this in the context of an assessment task (Figure 15), given after students had worked with the trapezoid area formula and after they had been shown the first justification of the area formula using spatial reasoning.

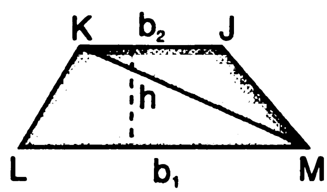
The three spatial representations in Figure 15 had not been discussed in class. I asked students to choose one and explain how it could be used to justify the known formula. All three of the methods were in some way related to the parallelogram and triangle work we had done earlier and I expected students would use this prior knowledge to give good mathematical justifications.

We learned in class that the formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$.



Each of the three diagrams below is a different verification of the formula for the area of a trapezoid. Choose one (by circling it) and explain in detail how that picture shows that the area of the original trapezoid is $\frac{1}{2}(b_1 + b_2)h$.





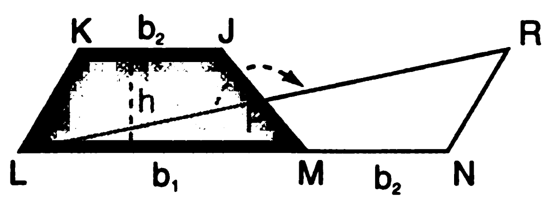


Figure 15: Trapezoid Quiz Task

A majority of the students chose the first spatial representation in the task (reproduced in Figure 16) to write about. Students recalled that we had used two copies of a triangle to create a parallelogram and they followed that same reasoning with the trapezoid. Thus, parallelogram RKLN had an area double that of trapezoid JKLM. They knew that the area of RKLN was $(b_1 + b_2)h$, so they reasoned that the area of JKLM was $\frac{1}{2}(b_1 + b_2)h$.

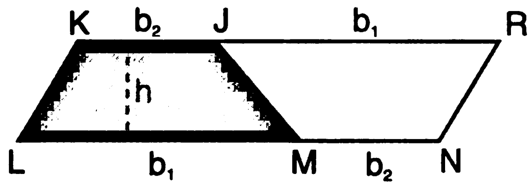


Figure 16: Trapezoid Method B

Students who chose the second representation in the task (Figure 17) saw that the trapezoid was divided into two triangles and that the necessary dimensions of these triangles (base and height) corresponded to known dimensions on the trapezoid. They reasoned that since the area of each triangle could be represented symbolically as one-half of its base length multiplied by the height, the area of the trapezoid could be written as the sum of the areas of the two triangles, or $\frac{1}{2}b_1h + \frac{1}{2}b_2h$. To make this representation look like a trapezoid area formula, they then had to factor and rearrange factors, or as some put it, "combine like terms." Recognizing this, students were able to show that:

$$\begin{aligned} \tfrac{1}{2}b_1h + \tfrac{1}{2}b_2h &= \tfrac{1}{2}h(b_1 + b_2) \\ &= \tfrac{1}{2}(b_1 + b_2)h \end{aligned}$$

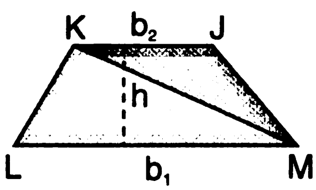


Figure 17: Trapezoid Method C

The students who chose to write about the third spatial representation in the task (Figure 18), also used something they had learned about triangles, but with this diagram they had to recognize that the trapezoid JKLM was dissected and rearranged as triangle

RNL. They observed that the base of the triangle was the sum of the trapezoid's bases ($b_1 + b_2$) and that the height of RNL was the same as the height of JKLM. And with their knowledge of the triangle area formula, students reasoned that the area of RNL (and thus JKLM) was $\frac{1}{2}(b_1 + b_2)h$.

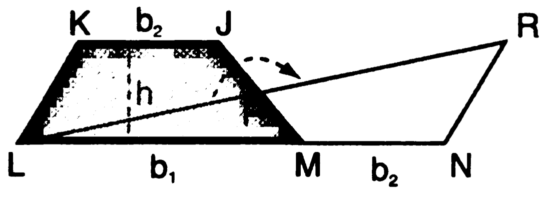


Figure 18: Trapezoid Method D

Area of a Trapezoid—Introduction of Mathematical Language Representation

As I planned for our work with trapezoids, I approached the task of making sense of the symbolic rule for area only through spatial reasoning—primarily through various forms of geometric manipulation. So when I introduced the trapezoid discussion by asking students what they knew about the area of a trapezoid, I thought I would handle any interpretation of the formula with an appropriate spatial representation. What I got, however, was a statement that I was not at all prepared to hear, much less think about.

JW: ...What is the formula for the area of a trapezoid? Anybody remember this? (Drawing picture and labeling it.) This is b_1 this is b_2 and this is h. OK? Anybody remember the formula for the area of a trapezoid? (Pause.)

Suzanne: The average of b_1 and b_2 times the height.

JW: (Repeating for class.) The average of b_1 and b_2 times the height. Another way to think about that... (Writing) Area equals $\frac{1}{2}$ of b_1 plus b_2... the average of base 1 and base 2... and then times the height. OK? And I want you think about *why* that might be true.

In my thinking, the $\frac{1}{2}$ in the formula represented something spatially. The figures for Methods A, B, C, and D were firmly established in my mind and I was planning on having a discussion on where the $\frac{1}{2}$ comes from—the parallelogram with half the height of the original trapezoid, for example. But in the context of the spatial representations that we had been using in class, I struggled to make some sense of how this student's interpretation could be modeled with a drawing. In my attempt to transform the subject matter into something that students could better understand, I had not considered the possibility that another form of representation—mathematical language[4]—might be presented and used by the students. And so I was unprepared to recognize it, much less handle its appearance in class.

It was possible that Suzanne's use of "average of the two bases" was a restatement of a rule she remembered from a high school geometry teacher. Or she may have applied some knowledge from her previous mathematics course for elementary education majors—one with a focus on statistics and probability and in which the concept of averaging was covered. It was also possible that Suzanne adapted a method that we had

[4] My use of "mathematical language" is not meant to imply that the other representations do not involve ways of communicating mathematical ideas. Here, mathematical language indicates the language (words) that are used to communicate ideas in mathematics—the language of mathematics. I previously referred to this as "natural language" to indicate that Suzanne's language employed words that were fairly common in everyday usage. It became clear, however, that the words that Suzanne was using had mathematical meanings that were far from "natural" (see Layzer, 1989).

discussed in class the previous week for estimating the area of an irregular shape on grid paper.[5]

But at the time, I was focused on how a geometric manipulation like that in Method A could justify Suzanne's work, and I couldn't immediately see how the concept of the average of the two base lengths could be represented in this diagram. As time was running short that day, I worked to steer the students back into thinking about a spatial representation with manipulation, and I demonstrated Method A. I essentially forgot about Suzanne's method.

Reflecting on the Use of Mathematical Language, Spatial Manipulation, and Symbolic Representations

While I was caught off guard by Suzanne's statement, it wasn't until the semester was over and I went back over my course records that I began to think more about the implications of her method. While I wish I could say that I immediately recognized how it might influence the way I was thinking about representations, the reality was that I got caught up in some other students' concerns after class and I forgot about Suzanne's introduction of some mathematical language that did not fit with any of the methods I had thought about.

My first impulse was to go back to her statement and think about whether a good spatial representation of the average of the two bases exists. After some trial and error

[5] With this method, students drew a number of horizontal and vertical cross sections on a shape and the average of the horizontal lengths was multiplied by the average of the vertical lengths, giving the area of a rectangle that approximates the area of the original figure. In the case of the trapezoid, there were two different horizontal lengths indicated (the two bases) and one vertical length (the height), so the average of the two bases would make some sense in estimating—but not necessarily finding an exact measurement of—the area of the trapezoid.

and talking with some colleagues, I developed the diagram shown in Figure 19 and the accompanying reasoning.

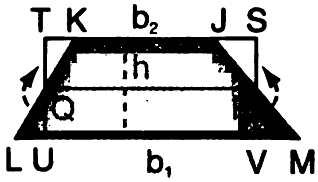


Figure 19: Trapezoid Method E

Q and P are the midpoints of KL and JM respectively. Segment QP is the midsegment[6] of the trapezoid and by its definition, has a length that is the average of the two bases, or $\frac{1}{2}(b_1 + b_2)$. Triangles QUL and PVM are rotated 180° to create the rectangle STUV, which has a base the same length as QP and a height the same as trapezoid JKLM. The area of STUV (and thus, JKLM) is $\frac{1}{2}(b_1 + b_2)h$.

After developing this spatial representation, I felt as if I had finally made some sense of Suzanne's statement in a way that was satisfying to me. While it was readily apparent to me that the mathematical language that she was using—the "average of the two bases multiplied by the height"—was mathematically equivalent to the symbolic representation, I was not interested in recognizing her statement as a legitimate representation until I developed an appropriate spatial manipulation to accompany it.

Reflecting on Mathematical Assumptions

In thinking through this reasoning, however, I realized that a number of mathematical assumptions are embedded in it. In fact, my subsequent analysis of the spatial representations for Methods A, B, C, and D shows that each method carries with it

[6] Some sources refer to the midsegment as the *median line*.

a set of mathematical ideas that would have to be taken for granted if the figures are to be accepted as geometric proofs of the formula. These are not ideas that I had originally considered in presenting these representations, but they grew out of my continued reflection on the methods and my wondering what it would really take to understand the spatial manipulations involved. By investigating these underlying assumptions, I hoped to push myself to reason about my own mathematical knowledge and what it might take to transform that understanding into something I could use in teaching preservice teachers.

I begin by looking at a few general assumptions that are common to all of the methods and then look more carefully at the mathematical assumptions that are specific to each of the methods.

Dissecting and Rearranging. The first assumption comes from this underlying principle—that the area of a shape stays constant when it is cut into pieces and those pieces are moved around. From a psychological perspective, this is Piaget's *conservation of area*, an indicator of the concrete operational stage of child development (Piaget, 1952/1963; Piaget & Inhelder, 1969). While this is not a concern for preservice teachers as students, it is a relevant concern for them to consider as future teachers. They would need to be aware that not every child in the elementary grades would consider this a plausible approach—that some of the original figure's area may be lost when the shape is subdivided.

There is another underlying issue that comes up in most every case of dissection and rearranging—whether the pieces actually fit together as nicely as the drawing suggests. Mathematical issues such as parallelism, congruency, and similarity are embedded within this issue and they are detailed in the discussion of the five methods which follows. This mathematics involved in looking at how pieces fit together when they are rearranged is far from trivial. A classic puzzle first credited to W. W. Rouse Ball in his *Mathematical Recreations and Essays* but popularized by Sam Lloyd in his *Cyclopedia of 5000 Puzzles* illustrates an inherent problem to this issue (Gardner, 1956; De Villiers, 1998). The problem involves taking a square that has been cut into four pieces and rearranging them into a rectangle (Figure 20).

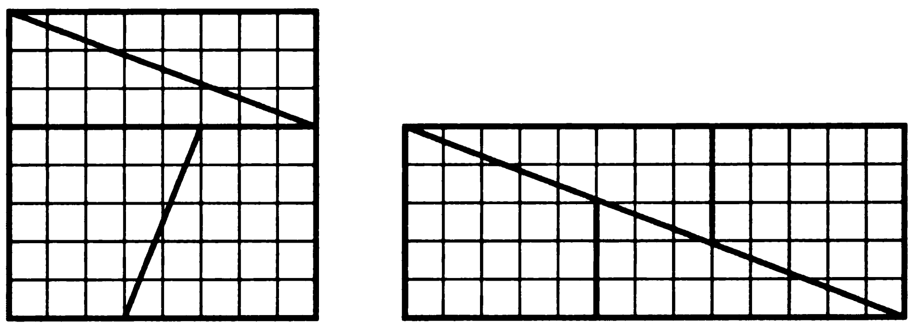


Figure 20: Dissection Puzzle

In looking at these two figures, it would appear that since the two shapes are made up of the same four pieces, they would have the same area, but the square has area 64 and the rectangle has area 65! How could the pieces of one shape be rearranged to create another shape with a different area?

This is a case where a visual observation leads to an incorrect mathematical statement. The assumption is that the four pieces of the square actually do fit together as neatly as they appear, when in actuality, a small gap exists between the pieces when they are used to create the rectangle. This gap has an area of one, thus enabling the four pieces (plus the gap) to have an area of 65.

Preservice teachers made many similar assumptions about the ways pieces fit together and created new figures. In each of these cases, though, their assumptions were correct—although neither I nor the students accounted for the what it would take to prove their assumptions true. My investigation of Method E during the analysis of the data first raised this issue of mathematical assumptions, so I present an investigation of that method first.

Method E. The first issue concerns the midpoints Q and P and whether the segment that connects them is parallel to the bases JK and LM—an important factor in establishing the fact that the resulting figure is truly a parallelogram. To accept this, students need to have an understanding about ratios of segment lengths that are formed by parallel lines. Specifically, we can start by working backwards through the problem.

First, we take Q as the midpoint of KL. The must be a line through Q that is parallel to KJ and LM. This line will intersect JM at some point (say, point A). Since we have three parallel lines (KJ, QA, and LM) with two transversals (KL and JM), the ratio of KQ to QL must be the same as the ratio of JA to AM. Since Q is the midpoint of KL, that ratio is 1:1, which means that the ratio of JA to AM is also 1:1, indicating that A is the midpoint of JM. A is therefore concurrent to the existing point P, so QA and QP are the same line segment. QP is therefore parallel to JK and LM.

The next assumption that must be addressed in this dissection and rearrangement method is that the midsegment QP is indeed the average of the two bases KJ and LM. The definition of a midsegment addresses this concern, but not everyone is familiar with this definition. To understand this idea, students would need to observe that when perpendiculars are dropped from J, K, P, and Q, a number of rectangles are formed (Figure 21). These rectangles are very helpful in setting up a set of equations which, when combined, lead to the desired result.

$$(1)\ QP = KJ + UW + ZV.$$

Then, it must be noted that $\Delta KQX \cong \Delta QLU$, giving LU = QX = UW. A similar argument is made to show VM = YP = ZV. From these two statements, it is shown that U is the midpoint of LW and V is the midpoint of ZM. So:

$$(2)\ UW = \tfrac{1}{2} LW \text{ and } ZV = \tfrac{1}{2} ZM.$$

Combining (1) and the equations in (2), you get

$$QP = KJ + \tfrac{1}{2} LW + \tfrac{1}{2} ZM$$

$$QP = \tfrac{1}{2}[KJ + KJ] + \tfrac{1}{2} LW + \tfrac{1}{2} ZM$$

$$QP = \tfrac{1}{2} KJ + \tfrac{1}{2} KJ + \tfrac{1}{2} LW + \tfrac{1}{2} ZM$$

$$QP = \tfrac{1}{2} KJ + \tfrac{1}{2}[KJ + LW + ZM]$$

$$QP = \tfrac{1}{2} KJ + \tfrac{1}{2} LM$$

$$QP = \tfrac{1}{2}[KJ + LM]$$

The midsegment QP is thus shown to be the average of the two bases KJ and LM.

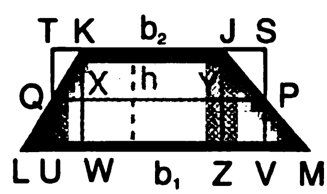


Figure 21: Proof of Trapezoid Method E

All that remains to be shown is whether trapezoid JKLM and rectangle STUV have equivalent areas. This would be true if triangle LUQ could be cut out and moved to fill in the space of triangle KTQ. Consider the fact that JK and LM are parallel. With KL as a transversal, $\angle QLU \cong \angle QKT$ making $\angle QKT$ and $\angle QKJ$ supplementary angles. And since Q is the midpoint of KL, $KQ \cong QL$ and the right triangles KTQ and LUQ are congruent. Similar reasoning can be employed to prove that triangles JSP and MVP are also congruent and so trapezoid JKLM and rectangle STUV have equivalent areas.

Method A. As with Method E, it is important to establish that the midsegment QP is parallel to the bases JK and LM, but there are several additional assumptions that need to be addressed. The first is that in drawing the midsegment QP, the height of PQLM is half the height of JKLM. This is simply another case of the argument that was previously made. If a perpendicular is dropped from J to intersect LM at Z and QP at Y (Figure 22), JZ can also be thought of as a transversal through parallel lines KJ, QP, and LM. The ratio of JY to YZ is also 1:1, so the height of PQLM (which is YZ) is half the height of JKLM (which is JZ).

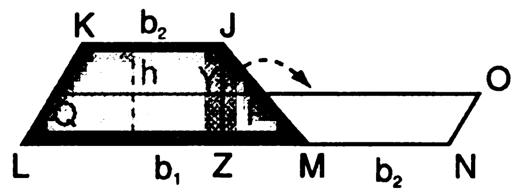


Figure 22: Proof of Trapezoid Method A

The final assumption is that when JKQP is rotated to MNOP, the pieces actually form a parallelogram. The easy part of this is observing that since P is the midpoint of JM, JP = MP and so one piece is not longer than the other when placed together. The more detailed piece is seeing that OQ and LN are indeed straight line segments. Since vertical angles $\angle$JPQ and $\angle$MPO are congruent and since $\angle$JPQ and $\angle$QPM are supplementary, then $\angle$MPO and $\angle$QPM are also supplementary, indicating that OQ is a straight line segment. To show that LN is a straight line segment, students need to observe that with JM as a transversal to JK and ML, the adjacent interior angles $\angle$KJP and ∠PML are supplementary. Thus when JKQP is placed on MNOP, ∠PML and ∠NMP are also supplementary.

We have shown that QO and LN are straight line segments and that they are parallel. To show that OQLN is a parallelogram, all that remains to be shown is that QL and ON are parallel. Consider KL as the transversal to JK and LN. ∠JKQ and ∠QLM are supplementary. When JKQP is rotated to MNOP, K coincides with N, so ∠ONP and ∠QLM are supplementary. This shows that LN is a transversal to parallel line segments QL and ON.

<u>Method B.</u> As with Method A, showing that final figure (here it's RKLN) is indeed a parallelogram is critical. RK and LN can be shown to be a straight line segments by noting that ∠JML ≅ ∠MJR and that ∠JML and ∠MJK are supplementary. Therefore, ∠JML and ∠NMJ are supplementary (as are ∠MJK and ∠RJM). Thus RK and LN are straight line segments and are also parallel. Using an argument similar to the one described in Method A above, it can easily be shown that KL and NR are also parallel, making RKLN a parallelogram.

<u>Method C.</u> By drawing the diagonal KM and subdividing JKLM into two triangles, the only geometrical assumption that needs to be made is that triangles KLM and MJK have the same height. But since JK and LM are parallel, the height remains perpendicular to each of the bases, even when it falls outside of the triangle MJK.

Seeing that the symbolic representation $\frac{1}{2}b_1h + \frac{1}{2}b_2h$ is equivalent to the form $\frac{1}{2}(b_1 + b_2)h$ requires students to understand some symbol manipulation, factoring the original expression as $\frac{1}{2}h(b_1 + b_2)$ and rearranging the factors.

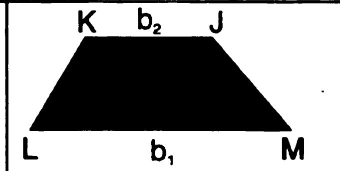
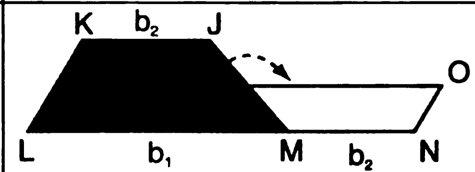
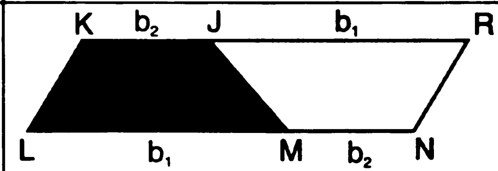
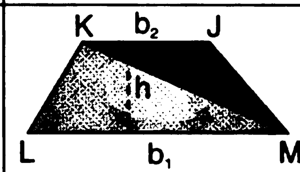
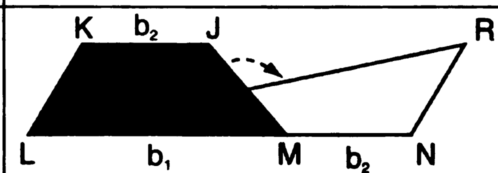
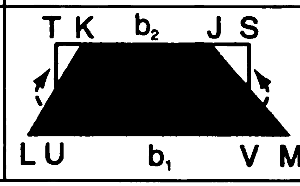
Method D. This dissection and rearrangement method requires students to understand two things—that the pieces fit together as neatly as they appear to make a triangle, and that the height of the triangle is the same as the height of the original trapezoid. To show that RLN is a triangle, it must be understood that the midpoint P creates two congruent segments, JP and MP. Then to show that LN is a straight line segment, students need to observe that with JM as a transversal to JK and ML, the adjacent interior angles ∠KJP and ∠PML are supplementary. Thus when JKLP is placed on MNRP, ∠PML and ∠NMP are also supplementary. And since vertical angles ∠JPL and ∠MPR are congruent and since ∠JPL and ∠LPM are supplementary, then ∠MPL and ∠RPM are also supplementary, indicating that RL is a straight line segment.

The simplest way to show that the height of LNR is equal to the height of JKLM is to observe that R, J, and K are collinear and so the height remains constant.

Comparing Representations through Shared Reflection

With these analyses and a developing appreciation for mathematical language as an additional representation, I decided to investigate the ways in which mathematical language might also be used with the other methods for understanding trapezoid area (Table 4). Not surprisingly, the other four methods didn't seem to lend themselves as nicely to mathematical language representations. In fact, the mathematical language representations that I came up with were either literal translations of the symbolic representations (e.g., "Half of one base multiplied by the height plus half of the other base multiplied by the height" for Method C) or a description of the spatial reasoning (e.g., "The sum of the areas of two triangles").

Table 4: Five Methods for Understanding Area of a Trapezoid with Three Types of Representations

	Mathematical language	Symbolic	Figure with Manipulation – Spatial Reasoning
	(Original trapezoid)		
A	The sum of the two bases multiplied by half of the height. Parallelogram with an area equivalent to that of the original trapezoid.	$(b_1 + b_2) \times \frac{1}{2}h$	
B	Half of the sum of the two bases multiplied by the height, or, the sum of the two bases multiplied by the height, then divided by two. Half of parallelogram with an area double that of the original trapezoid.	$\frac{1}{2}[(b_1 + b_2)h]$ or $\frac{(b_1 + b_2)h}{2}$	
C	Half of one base multiplied by the height plus half of the other base multiplied by the height. The sum of the areas of two triangles.	$\frac{1}{2}b_1h + \frac{1}{2}b_2h$	
D	Half of the sum of the bases multiplied by the height. A triangle with an area equivalent to that of the original trapezoid.	$\frac{1}{2}(b_1 + b_2)h$	
E	The average of the two bases multiplied by the height. A rectangle with an area equivalent to that of the original trapezoid.	$\frac{b_1 + b_2}{2} \times h$ or $[\frac{1}{2}(b_1 + b_2)]h$	

The mathematical language of Method E that Suzanne shared with the class was the one example of language representation that students used regularly. It is fundamentally different from the others in that it employs some language ("average of…") that is neither geometric in origin nor in use. It is a case where mathematical language helps to explain the accompanying spatial reasoning, instead of describing it

Preservice teachers were most at ease using the mathematical language of Method E in remembering and using a way to understand the area of a trapezoid. In subsequent class sessions, the phrase "the average of the two bases multiplied by the height" was almost exclusively used by students—the "one-half" that I had been so concerned about helping them to understand in a geometric sense was folded into the averaging of two lengths. Students seemed to like the mnemonics of the representation and no one was bothered by the fact that we had not come up with an appropriate spatial representation to describe it.

But the fact that this mathematical language emerged as the representation of choice does not indicate that it should be presented as the sole method for preservice teachers' understanding of trapezoid area. In fact, it underscores the importance of helping them to develop multiple representations for understanding. Shulman wrote, "Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice" (Shulman, 1986, p. 9). I was originally focused on helping preservice teachers justify symbolic representations with multiple forms of spatial reasonings, and I was surprised to find that many students were more inclined to focus on a mathematical language representation.

By the same token, these students—as future teachers—need to be prepared to think beyond the use of the one method or representation with which they may be most comfortable, in meeting the varying needs and levels of understanding of their future students. Sharing this experience with preservice teachers is one way in which a teacher educator's attention to his or her own pedagogical content knowledge growth can contribute to the developing knowledge bases of his or her students. In this case, this reflection occurred in retrospect and so was not as enlightening for these students. But I envision using this experience as a central piece of a future course that I or others may teach for teacher educators—one with attention to shared reflections.

The Perceptual Proof Scheme of Dissecting and Rearranging

When preparing for MTH 202, I did not emphasize students' use of justifications. From my past experience in teaching preservice teachers, I knew that many come into MTH 202 with a dislike of geometry—often stemming from their high school experiences in geometry classes where proofs were emphasized. In the attitude survey that was given on the first day of class, I asked students: "What is geometry? What comes to your mind when you hear the word 'geometry'?" One student wrote:

> When I hear the word geometry I think of shapes and measurements. I also think of my high school geometry class and doing endless amounts of proofs!

Another student wrote:

> A bunch of lines and angles where we have to find measures. Also, proofs and theorems to be defined which I do <u>not</u> like.

While many students mentioned the importance of *shape* and *measurement* in geometry, most of the students included some reference to proofs, theorems, or postulates—and these were rarely mentioned in a very positive way. While I thought that justifications were important to consider when making geometric hypotheses, I felt it was more important that students view geometry as more than a study of proofs and theorems, and that the study of geometry can be less daunting than they may remember from high school. This was my main purpose for de-emphasizing proofs in MTH 202.

The first (and only) reference in my teaching journal concerning proofs comes when I was thinking about what content I wanted to teach during the last week of the course. I wrote:

> The only other thing that I had been thinking about is the notion of proof. It's an issue that I have not really addressed with my students, yet it's one that is getting a fair amount of attention right now, especially in conjunction with the ways in which the [Principals and Standards for School Mathematics] is looking at proof and reasoning together across the grade levels. I know that we have a nice geometric proof in *Looking for Pythagoras* and that there are some good algebraic proofs that make a lot of sense with respect to using similar triangles, so I'm not sure how (or if) I will work this in. ... If I only had another three or four weeks with them... (Personal Teaching Journal, 4/20/99)

In my thinking, *proof* was a topic that would be covered—time permitting—at the end of the course and not something that was important to think about in the context of various mathematical situations during the course. And throughout my teaching journal, I only mentioned *justification* as a synonym for *proof*, focusing on the mathematical construct of a formal proof rather than as justifying the reasoning behind one's mathematical thinking.

But I did think about the spatial reasoning as a kind of proof of the symbolic representation. Referring to my learning about the spatial reasoning of Method D, I wrote:

> Actually, this very idea of a new method for me was what had pushed me to consider making the topic of formulas and geometric justifications an emphasis of my course this semester. (Personal Teaching Journal, 1/29/99)

In this way, I was thinking of the dissection and rearrangement method as an example of an *empirical* proof scheme. In these schemes, justification is based solely on the use of examples. More specifically, since the examples were of spatial reasoning, I was relying on a ideas that were highly *perceptual*. In the perceptual proof scheme, conclusions are often based on one or several drawings that are used to argue a mathematical hypothesis. When teaching MTH 202, I was bound by the idea that there had to be a geometric representation (typically using some form of dissection and rearrangement) of a formula. I first classified the five methods of understanding the trapezoid area formula in Table 1 by the spatial representations—thinking that the accompanying figures were the most important representation.

In doing so, I was also giving students the idea that a perceptual proof scheme was a sufficient way to justify mathematical thinking. Students were not challenged to consider the possibility that the perceptions implied by the drawings were erroneous—that the pieces might not fit together as neatly as they appear. In retrospect, there were a number of places in the course where this issue could have been raised—like asking students how they knew that two copies of a trapezoid would make a parallelogram or even whether each of the methods would work on different trapezoids. I have little doubt that preservice teachers need more experiences developing mathematical justifications for observations and hypotheses—to enrich and deepen their own mathematical understanding as well as to understand what it may take in helping kids do the same. However, I decided before the course and throughout the semester that proof and justification would not be a critical element of what I was teaching.

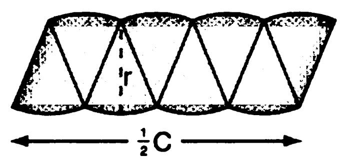
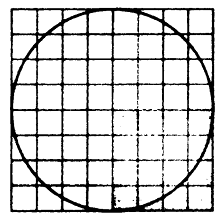
Episode 2: Circle Area

Representations and Their Limitations

When investigating the area of a circle, the representations used have an impact on the ways in which students might make sense of the problem. In addition, each representation bring with it different mathematical assumptions that must be addressed within the context of each method. But not all representations convey the mathematical ideas they symbolize in the same way nor with the same potential for providing access to understanding. And some representations even carry some inherent problems to understanding the underlying mathematics. Preservice teachers need to understand how these representations relate to one another and be aware of the potential difficulties that are inherent with each one (Wilson et al., 1987).

Both of the methods that we used for understanding the formula for the area of a circle (Table 5) bring new problems into the conversation and reveal some of the limitations of relying on the accompanying representations.. The first method requires students to make a mathematical assumption that, while seemingly obvious, truly requires an understanding of integral calculus for its complete justification. Mathematical language plays an important role in the less commonly used second method that the students and I investigated in class. But the introduction of this second method also raises an important, and somewhat problematic idea—that of mathematical precision.

Table 5: Methods for Understanding Area of a Circle

	Mathematical language	Symbolic	Spatial Reasoning
A	An “almost rectangle” with an area equivalent to that of the original circle.	$A = \frac{1}{2} C r$	
B	Pi *radius squares.*	$A = \pi r^2$	

By comparing these two methods and looking at the limitations that are embedded in each one, I found that another case for using multiple representations is in using the strengths of one representation to help cover the weaknesses of another. By juxtaposing the first method with the second, I found that preservice teachers were able to gain some knowledge about the area of a circle that they may not have otherwise had.

<u>Method A.</u> The first method that we looked at is one that is typically used to help middle school students understand the formula for the area of a circle. This method employs the strategy of dissection and rearrangement that was used in many other area examples in the class. With the Piagetian idea of conservation of area firmly in place, the dissection and rearrangement strategy can be a powerful one for preservice teachers. Lengths and angles from one geometric representation remain constant as they reappear in another spatial representation, enabling students to use a rearrangement of pieces that resembles another (already accepted) shape.

With a circle, though, the curved outside of the shape presents a problem. A circle is typically the first shape with a curved edge presented to students and no amount of dissecting and rearranging can transform the curves into the straight lines of triangles and parallelograms.

Historically, this conundrum caused problems for ancient peoples as well. Our best guess as to how people first discovered the formula for the area of a circle is the same way in which I first introduced the concept to my preservice teachers—using a method of dissection and rearrangement and connecting the calculation of the area to that of the circumference. The Babylonians and Chinese calculated the area of a circle using the formula $A = Cd/4 = (C/2)(d/2)$ where C is the circumference and d is the diameter, a formula which was likely derived from a dissection and rearrangement (Katz, 1993). Japanese documents from the 1698 show this method being used and Leonardo da Vinci himself also employed this idea in the 16th century (Beckmann, 1971).

With this method, a circle is cut along any number of radii (Figure 23), and the pieces are rearranged to create a figure that resembles an "almost rectangle"[7] (Figure 24).

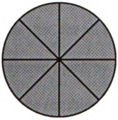


Figure 23: Circle Divided into Sectors

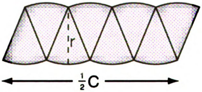


Figure 24: Circle Sectors Rearranged as an "Almost Rectangle"

The reasoning goes that as the circle is cut into more and more smaller sectors, the curved edges of the resulting figure become less pronounced and appear more like a

[7] I refer to this shape as an "almost rectangle" because as the size of the sectors becomes smaller, the shape looks more like a rectangle. Also, it's important to think of the limiting case as a rectangle rather than a parallelogram because only then is the height of the figure equal to the radius of the circle.

straight line, creating a figure that is more rectangular in each iteration. After a while, those curves become so small that the resulting figure might as well be taken as a rectangle and the rectangle area formula (A = base × height) can be used. Here, however, the base length of the rectangle can be thought of as half of the original circle's circumference and the height of the rectangle is equal to the radius of the original circle, yielding the formula $A = \frac{1}{2} C\, r$.[8] Once the constant, π, was accepted, this gave us a formula for determining the circumference of a circle, $C = \pi\, d$ or $C = 2\, \pi\, r$. Substituting this last expression into the circle area formula above, we got $A = \frac{1}{2}(2\, \pi\, r)\, r$ and after some simplification, $A = \pi\, r^2$.

This proved to be a nice way for students to make some sense of how the formula could be derived from a geometric representation of the problem. However, it required at least one leap of faith. I have always been bothered by the mathematical hand-waving that must be done in assuming that arranging smaller and smaller sectors of the circle will result in a figure that, for all intents and purposes, is a rectangle. Granted, the students were never bothered much by it for it seemed to make sense to them that this could be true. And since the resulting formula was the one that students already knew for circle area, then this seemed to be an ideal time to let the ends justify the means.[9]

With this method, the symbolic representation appears to be a powerful one. For students who have dissected and rearranged pieces to make sense of parallelogram,

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[8] Prior to the class session in which this method was presented, we had investigated the circumference of circles and had looked at the development of π as the ratio of a circle's circumference to its diameter, or $\pi = \frac{C}{d}$.

[9] In actuality, some higher level mathematics can easily simplify the problem, employing integral calculus and the theory of limits to the example. But this doesn't typically help those preservice elementary teachers who will never study calculus, and it certainly did not help da Vinci whose work predated the discovery of integral calculus by about a century.

triangle, and trapezoid areas, this strategy makes a great deal of sense. But unlike the polygons, the circle presents a unique twist as the resulting figure does not appear to be a true rectangle. The curved edge of the circle forces students to take an additional step in reasoning about how this method might be useful—noticing that with a greater number of sectors, the resulting shape looks more like a rectangle and assuming that it will eventually be one. For the first time, students are confronted with the possibility that the dissection and rearrangement method cannot completely solve the problem.

Method B. The second method for looking at the area of a circle comes from the *Connected Mathematics Project* (CMP), a middle school mathematics curriculum that was developed at Michigan State from 1991-1997. Having been a part of the writing of many CMP units and having had the opportunity to talk with the authors of the curriculum as well as with a number of teachers who have used the materials, I found that many of the middle school ideas which are presented in CMP are extremely appropriate for preservice elementary teachers. Not only do they give us the opportunity to expand on the preservice teachers' knowledge base, but they provide us with a forum to discuss a number of pedagogical issues and a set of problems to access the pedagogical content knowledge that I am trying to address.

The sixth grade CMP unit that investigates the ideas of perimeter and area is called *Covering and Surrounding*. In the final set of problems in *Covering and Surrounding*, middle school students are exposed to several big ideas around circles—the relationship between the circumference and the diameter of any circle as well as ways to estimate and calculate the area of a circle. The work with circumference results in the students' seeing that the ratio of a circle's circumference to its diameter is a constant (although, due to measurement error, this constant ratio may not initially be apparent) and that this constant ratio is a number with a value a little greater than three. For middle school students, this is often the first time that they are exposed to the irrational number π, or even the notion that numbers exist that cannot be expressed as the ratio of two integers. The topic of irrationality is not intended to be investigated at this point in the curriculum and the subject is only mentioned in describing the existence of π. Preservice teachers, however, already have some knowledge of π and irrational numbers, so I chose to use these problems more as a review and an example of a way that a concept can be introduced in a more investigative way—where students learn through doing and not just rote memorization.

After reviewing the idea of π and methods for estimating areas of shapes drawn on grid paper, we moved to the problem of making sense of the area formula for circles. The question was embedded in an investigation of the relationship between two measures of the circle, just as the circumference and diameter lengths had been compared. Students were given a number of circles drawn on grid paper. Each circle included a square drawn on the radius of the circle (Figure 25). This square was referred to as the *radius square*.

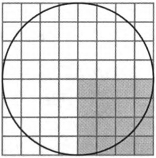

Figure 25: Circle with a Radius Square

Students were asked to find a relationship between the area of a circle's radius square and the (estimated) area of the circle. In *Covering and Surrounding*, this question is posed using language that is consistent with the language of the unit, instructing students to "find out how many radius squares it takes to cover each circle."

At this point, I observed a variety of techniques being used. Some students focused on the language of the question and immediately began covering the circle with radius squares (Figure 26). This helped us to establish an upper limit of the relationship, that four radius squares could *completely* cover the circle, but that there was a fair amount of material left over.

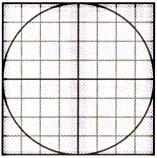

Figure 26: Circle with Four Radius Squares

Other students started dissecting and rearranging radius squares, covering the circle with three radius squares as best as possible (one way is shown in Figure 27) and then cutting the leftover pieces apart and fitting them into the remaining space. This method typically yielded the intriguingly familiar statement that the area of the circle was "a little more than three" radius squares.

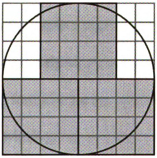

Figure 27: Circle with Three Radius Squares

And other students stuck with their initial numerical estimates of the areas of the circles and compared them with the numerical values for the areas of the corresponding radius squares. Students employed a variety of techniques for estimating the circle areas and these area estimates varied, but they were often within several square units of each other. These students then gave answers that represented the ratio between the circles' (estimated) areas and their corresponding radius square's area, with answers such as 3.1, 3.15, and 3.25.

After all of the students had a chance to work on the problem, I brought the class back together and asked for different students to share their findings. Students began to notice that most of the answers that they came up with for the number of radius squares it would take to cover a circle was "a little more than three" or actual numbers a little greater than 3. I asked where else we had already looked at answers similar to these, and they referred to the work we had done with relating a circle's circumference to its diameter.

At this point I led the students to take that "leap of faith." I pointed out that it would be quite a coincidence that two different problems involving circles would lead us to two different numbers, each one "a little more than three." I went on to state that, in fact, this ratio of the circle's area to the number of radius squares needed to cover the circle is exactly the same ratio as the circle's circumference to its diameter—the irrational number π. I went on to point out how helpful this representation of the area of the circle is in looking at the corresponding formula, for the circle can be thought of as having the same area as "pi radius squares" which gives way to $A = \pi r^2$. Here, the mathematical language of the problem fits with the symbolic representation quite nicely, with the notation for square matching with the geometric picture of the square. The three representations—mathematical language, symbolic, and spatial—coalesce in an almost ideal example of how one representation connects to another.

The method itself can be open to criticism for its apparent lack of mathematical rigor. There is an assumption that, because the work is being done with circles (which had previously resulted in an investigation on the number π), the students' answers for the number of radius squares it takes to cover a circle must also be approximations for π. Students can argue, however, that their answers may actually be approximations for another number—a number close (but not exactly equal) to π. With middle school students, this method alone may be convincing proof that a circle can be covered with "pi radius squares," but for preservice teachers, it is important to understand that there is a level of mathematical rigor that is being glossed over with this method, no matter how helpful the mnemonics of the representations may be.

Reflecting and Shared Reflection on Multiple Representations

In fact, this investigation is helpful in looking at the ways in which these representations provide substance for pedagogical content knowledge. In talking with other instructors of MTH 202, I found that the example using radius squares to cover a circle (Method B) was not used to show why the formula $A = \pi r^2$ makes sense for the area of a circle—in fact, most instructors didn't give any attention to understanding the formula at all. In the course that I developed, representations and understanding how they can be used to make sense of formulas became a central focus of the class. In the case of circles, there are some mathematical assumptions being made about how precise one must be to use this model for understanding the circle area formula. But in a course where pedagogical content knowledge is valued and given some attention, the investigation with radius squares—when paired with the previous one using sectors of a circle—can help to move preservice teachers into thinking about ways to help their students make better sense of the mathematical ideas and to move between the representations.

After both methods were introduced and discussed, I talked to the students about my reasons for presenting both methods. I explained that for the purposes of the class, I felt that neither one was complete enough to stand alone—that as I was thinking about the models, I realized that the representations used in each model had their limitations. I asked the students to explain what they liked and disliked about using each model and what each model helped them to understand about circle area.

About Method A, students explained that they liked the idea of cutting a shape apart and fitting the pieces together to make a new shape that is like one that was already studied. Several also described how using algebraic manipulation was satisfying—transforming what looked like a new formula for the area of a circle ($A = \frac{1}{2} C r$) into a known formula ($A = \pi r^2$) using some simple algebraic substitutions. They liked using strategies that had been utilized with other area problems, making the circle investigation more similar to, than different from, the ones done with other shapes. But students also wondered how elementary school students would be able to make sense of this method, since the symbol manipulation would not be a tool that elementary students would have at their disposal. Several students also expressed some concern about how cutting the circle into only a few sectors would make a shape that had noticeably curved bases but none expressly described any hesitation to accept that a larger number of circle sectors would actually create a rectangle.

With Method B, students described the use of radius squares as "interesting," "totally new," and "something [they] had never thought about before." A number of students liked seeing a method that can be used with middle school and upper elementary school students. Students also explained that they liked how this method reinforces the verbalization of the formula ("pi radius squares" for "pi times the radius squared") and how this method helps to explain the use of "square units in covering something with a curved edge." However, a number of students explained that it was difficult to accept using approximately 3.14 radius squares to cover the circle when their work resulted in a different number.

By centering the discussion around the representations instead of just the outcomes, I wanted students to be more aware of the power behind using different and multiple representations and to be thinking about the ways in which these representations can have an impact on what they contribute to other understanding.

Perceptual and Ritual Proof Schemes

In many ways, the first method for understanding circle area used in MTH 202 was highly perceptual—it involved dissection and rearrangement of pieces to make a figure look like a figure with a known area. It also implied that subsequent iterations (i.e., more sectors) would produce a final figure that more closely resembled a rectangle, leaving students to reason about the argument based on only a small number of examples.

But since it required students to accept that the rearranged pieces form an "almost rectangle," my use of the method also constituted the use of a *ritual* proof scheme.

A ritual proof scheme is one in which students judge "the correctness of an argument solely by the form of the argument rather than by the correctness of the reasoning involved" (Martin & Harel, 1989). A typical example of a ritual proof scheme is a students' reliance—based on prior experience in a traditional geometry class—on the two-column format as the only acceptable method of proof. To these students, mathematical truth lies in the structure that is used in making the argument, rather than in the reasoning itself. With Method A for understanding the area of the circle, students were likely led to believe in the results because the method of subdividing and rearranging pieces was established in earlier examples (parallelograms, triangles, and trapezoids), and so they were willing to accept the method as justification for the results.

My use of Method B does not follow a proof scheme. In that method, I did not ask students to accept the process as absolute mathematical truth, but that the end-product (seeing the area of a circle as "approximately pi radius squares" and relating it to the area formula, $A = \pi r^2$) was the important element. My plans in introducing students to this method centered around their using it more as a mnemonic device for remembering the formula rather than for understanding the irrational number π and for making sense of the method. I placed the emphasis on providing students with multiple representations for the same idea, and not on trying to validate one method with the other. In my mind, I viewed Method A as the justification of the area formula and Method B more as a spatial representation of the symbols that are used.

Episode 3: Volume of Prisms, Cylinders, Cones, and Spheres

Background—Understanding Layers

After completing some additional work in two-dimensional geometry, I decided to push on some of the concepts that we developed in 2-space (like relationships between perimeter and area) and see how students could apply these ideas to 3-space (investigating surface area and volume relationships). The titles of the *Connected Mathematics Project* units that we used (first *Covering and Surrounding* and then *Filling and Wrapping*) were helpful in signaling this shift as well as in attaching some mathematical language to the important mathematical ideas.

In previous times that I taught MTH 202, I spent very little time on the idea of surface area—thinking that it was less important (especially in understanding some of the more common formulas used in solid geometry) and easier to understand. However, I learned that preservice teachers had a fair amount of difficulty in visualizing how various flat patterns (or nets) could be folded to wrap a shape and that these flat patterns were helpful in understanding basic formulas for surface area of rectangular prisms. So this semester, I included several investigations on working with flat patterns—visualizing the resulting solid that would be created and using the flat pattern to determine the surface area as well as seeing how to generalize with a formula. This worked very well with the students this semester, and it helped them to get a better handle on the concept of surface area without spending too much time on it. But with volumes of prisms, I have always used some of the same approaches with preservice teachers that the CMP curriculum has found to be powerful for middle school students—particularly the notion of layers.

When moving from flat figures to solids, it's helpful to fall back upon the tried-and-true method of counting. Kids find counting reassuring, and I have found that preservice teachers do as well. Just as we had first counted squares to determine areas of rectangles, students began by counting cubes to determine the volume of solids. When I asked them to think about ways in which kids might start to find shortcuts for counting a large number of cubes in a rectangular solid, students suggested that they could count the number of cubes in one layer of the shape and multiply by the number of layers. For example, students found that the box that measured 3 cm by 2 cm by 5 cm had 6 cubes in the bottom layer and 5 layers of cubes, for 6×5 or 30 cubes (Figure 28).

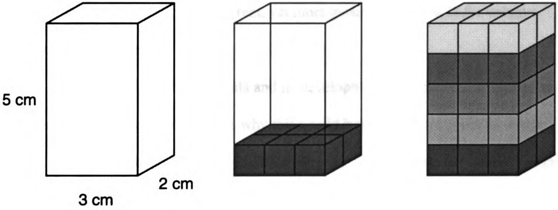

Figure 28: Box with Layers of Cubes

With a rectangular prism, this naturally leads to a generalization that the number of cubes in each layer can be found by multiplying the two dimensions of the base and that the volume can be determined by multiplying the three dimensions of the solid, or $V = l \times w \times h$. This is a formula with which most preservice teachers have some familiarity and as such, they are eager to encounter it and use it. However, it is common to see some students over-generalize and try to use this formula to calculate the volume of any solid (especially ones in which three different dimensions are given).[10]

I found that more preservice teachers were able to avoid this misunderstanding when the concept of layers was used and that language was applied to the situation. And instead of moving to the well-known volume formula, $V = l \times w \times h$ (which is only useful with some prisms), it was more powerful to use the concept of layers which can be applied to all prisms. By using layering, students develop a firmer understanding of volume as the number of cubes in one layer (typically thought of as the bottom layer or the base of the solid) multiplied by the number of layers (or the height of the solid). With this idea, the result is a slightly different, yet more generalized formula for volume: $V =$ area of base $\times$ height.[11]

The power behind this formula and its development is that it can be applied to any prism and it makes sense to students why this would be true. It also makes for a nice application of—and connection to—areas of polygons, and it provides a useful tool in reasoning about cylinder volumes as well. But before I get to the cylinders and how representations provided some interesting avenues for discoveries about spheres and cones with preservice teachers, I first turn to the investigation of volumes of prisms with various bases as an example of my own initial comprehension of knowledge, the beginning of its transformation, and how this was a powerful example of shared reflection for preservice teachers.

[10] Similarly, students who have little understanding of how formulas can be justified with spatial reasoning often use the formula for the area of a rectangle when determining the areas of triangles, trapezoids, and other shapes.

[11] As it is the accepted notation in mathematics, this came to be represented as $V = B \times h$ where the capital B stands for the area of the base (instead of a lower-case b which is used for the length of a side labeled as a base of a parallelogram).

Areas of Regular Polygons—My Comprehension of Subject Matter

In our class work on perimeter and area, we constantly came back to the ideas of how perimeter and area are related. In particular, questions kept coming up about whether a shape with a greater perimeter necessarily indicated a shape with a greater area and students wrestled with these ideas throughout the course. They quickly reasoned that in the case of rectangles with a fixed perimeter, many possible areas could result.[12] Students eventually discovered that the rectangle with a *maximum* area and a fixed perimeter is a square.

But I also wanted them to think about what shape would have a maximum area for a given perimeter if the rectangle constraint were removed from the task—a problem that was not investigated when we were working in two dimensions, so I decided to revisit the issue when working with cylinders.

A problem in *Filling and Wrapping* set the stage for this investigation quite nicely. As part of a larger problem, students were asked to find the volumes of triangular, square, pentagonal, and hexagonal prisms formed by taking identical sheets of paper and folding them into 3, 4, 5, or 6 equal-sized rectangles that would represent the lateral faces of the prisms (two examples are shown in Figure 29).

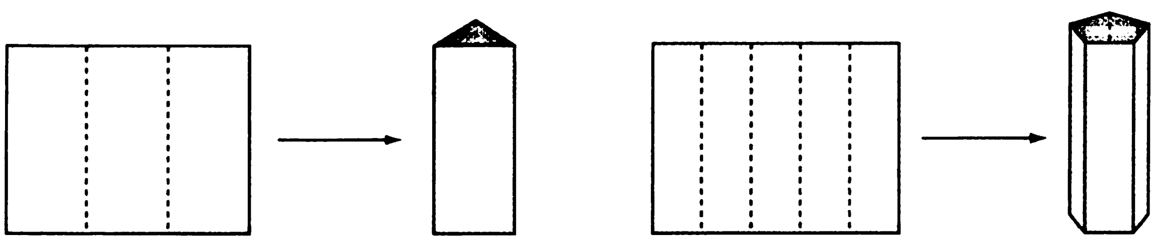

Figure 29: Construction of Triangular and Pentagonal Prisms

[12] For example, rectangles with a fixed perimeter of 12 and whole number lengths include a 1×5, a 2×4, and a 3×3, with areas 5, 8, and 9 respectively.

Students were given this task to work on in class and results were then compiled for the entire group. My primary goal had been to set up the situation where students would see that as the number of lateral sides increased, the volume would also increase. And since the height of the prisms is constant, the problem is actually analogous to the problem of keeping a perimeter of a polygon constant (represented as the base) and maximizing the area.

It turned out that the preservice teachers in my class had very few problems with this concept. Some students quickly reasoned that since the prism bases were looking more and more like a circle, then for a fixed lateral surface area, a cylinder (a "circular prism") would have the greatest volume. From our previous work, these students had possibly realized that a circle would be the maximal two-dimensional shape for a given perimeter and they were ready to apply that knowledge to this new situation. Other students might not have realized this fact prior to this investigation but were able to work back to that conclusion based on their experiences with the cylinders. Either way, it was an important lesson for the class—either in validating a prior belief with an extension problem or in tying together a number of previously-seen but disconnected ideas.

But what caused a great deal of concern for the class was the fact that there was little agreement on what the volumes for the various solids should be (Table 6).

Table 6: Student Responses to Prism Volume Problem

Triangular Prism	Square Prism	Pentagonal Prism	Hexagonal Prism
817 cm^3	1058.4 cm^3	1242 cm^3	1144.8 cm^3
825 cm^3	1053.5 cm^3	933.12 cm^3	1196.5 cm^3
823 cm^3	1043.6 cm^3		2364 cm^3
820 cm^3	1053.5 cm^3		906 cm^3
940.5 cm^3	1043.6 cm^3		
756 cm^3	1078 cm^3		
	1051 cm^3		

Students were encouraged to measure, estimate, and/or calculate the volumes of their prisms in any way they could. Some students took precise measurements of their prisms and did the calculations with a high level of accuracy. Other students did a fair amount of estimation—tracing the base of each prism on grid paper and counting the number of square units in the bottom layer as a part of their calculations. One student suggested subdividing an unknown base shape into other shapes with areas that could be calculated—drawing one diagonal in the pentagon and turning it into a triangle and a trapezoid. Some students worked with what they could recall from their previous mathematics experiences to figure out what the "right" answers were supposed to be. Since all students were working with standard sheets of $8\frac{1}{2}$ in by 11 in paper, I thought that their answers would have less variance, but something was obviously not working right. The class session ended with a discussion about maximizing volume and area, but the students and I were troubled by the messiness of their answers (when Cara shared her answer of 933.12 cm^3 for the volume of the pentagonal prism, she saw how it was smaller than any of the square prism volumes and added "but it's probably off").

As part of my preparation for the next class, I decided that I would take a look at the problem again myself. I decided that for ease of calculation, I would convert the measurements from inches to centimeters (using the conversion 2.54 cm = 1 inch and rounding to the nearest tenth, making the original paper 21.6 cm by 27.9 cm). The square base was the easiest to calculate, giving an approximate volume of 1051 cu cm. Next, I tackled the triangular base (Figure 30). The triangle has a base of 9.3 cm. To determine the height, I recognized the fact that the small triangle ($\triangle ABD$) is a 30-60-90 triangle, and therefore the ratio of the long leg to the short leg is $\sqrt{3}:1$, giving a height of about 8.1 cm. Using this, the volume of the triangular prism would be approximately 814 cu cm.

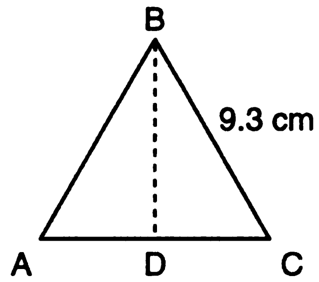

Figure 30: Base of Triangular Prism

Having already tackled the equilateral triangle, I decided to move on to the hexagon, recognizing that some of my equilateral triangle work might be applicable to this problem as well. First I determined that one side of the hexagon would have a length of 4.65 cm (I decided to leave the extra decimal place in since it was exactly half of the equilateral triangle's side length). I drew in the diagonal BE (see Figure 31) and recognized that this creates two congruent "special isosceles trapezoids" (see page 125 in Chapter 4), each of which is made up of three equilateral triangles. Since one of these triangles has a side that is half as long as the triangle in Figure 30, the height of one of these triangles is also half the height of the triangle in Figure 30, or about 4.0 cm. This gives a volume of about 1213 cu cm for the hexagonal prism.

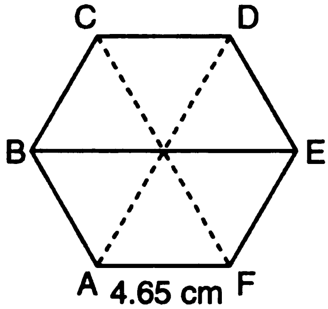

Figure 31: Base of Hexagonal Prism

But when I looked at the pentagonal prism base, I had no idea where to start. I knew that I was looking at a regular pentagon with sides of about 5.58 cm, but I was at a loss for figuring out the area. I recalled doing something like this in high school geometry, but 17 years is a long time and it was not a technique that I remembered using very much since then. I started to doodle around on my paper, to see if anything would come to me. The equilateral triangles in the hexagon had been so helpful, so I decided to subdivide the pentagon into five congruent triangles as well (see Figure 32).

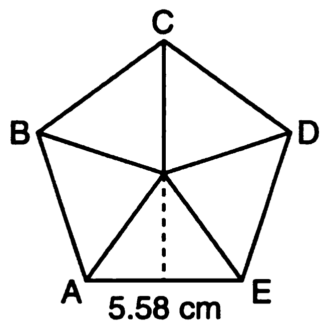

Figure 32: Base of Pentagonal Prism

These triangles were obviously not equilateral, so I had no quick way of figuring out the height of one of them. I remembered that this was something that could be solved with some trigonometry, so I used what I relearned from working with CMP to figure out one of the interior angles of the pentagon[13] and I recalled sohcahtoa[14] and calculated the height of the triangle, only to get a number that was much smaller than I was anticipating. After doing the same process on the hexagon and comparing the result with what I *knew* had to be the answer, I realized that I was doing something wrong—my calculator was in radian mode. It had been almost 15 years since the last time I worked with radians, but here it was again, staring me in the face. I switched to degrees and quickly found the pentagonal prism to have a volume of approximately 1157 cu cm.

None of this mathematics was particularly difficult. In high school, I would not have had any problem pulling these pieces together to solve the problem. But I was clearly rusty in some of my mathematical content knowledge. As I wrote in my teaching journal:

> This exercise has helped me to think about my own mathematical knowledge and how I need to constantly exercise some of those skills that have been dormant for too long. There are some things in higher level mathematics that I haven't done for a long time (like calculus and trigonometry) that are going to be important to me in my future as a mathematics educator that I need to be able to recall more efficiently. I think that by doing the puzzles on the calendar page in the *Mathematics Teacher* and some other recreational mathematics, I can get some of what I need, but also having engaging, mathematical discussions with teachers and others here at the university, we can push each other to think harder about the mathematics that lies beneath what we talk about every day. (Personal Teaching Journal, 3/30/99)

In addition to working with colleagues on additional mathematics tasks however, I have come to realize that my own teaching has also become a site for challenging my mathematical thinking and for revisiting ideas that I encountered some time ago. Through continued reflection on the challenges that arise in the preparation of teaching as well as

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[13] In the CMP unit *Covering and Surrounding*, students learn that the sum of the interior angles of a polygon can be determined by multiplying two less than the number of sides by 180°. This builds from the idea that any triangle has an angle sum of 180° and that the number of triangles (with all vertices on the vertices of the polygon) that a polygon can be divided into is two less than the number of polygonal sides.

[14] Sŏhcahtoa is a mnemonic device introduced in many high school trigonometry classes. It is helpful for remembering the basic trigonometric ratios by relating them to their corresponding angle lengths in a right triangle (sine = opposite/hypotenuse, cosine = adjacent/hypotenuse, tangent = opposite/adjacent).

from the classroom itself, I have been able to expand my own content knowledge. I have recognized that my knowledge base is indeed in a constant state of reconstruction and development through reflection on practice.

Shared Reflection—Modeling Good Teaching Practice

I decided not to stop simply with my own reflection, but to share my experiences with the students. The next day in class, I decided to recreate for them the process I went through in determining the volumes of the four prisms. Utilizing the same mathematics that I made use of, I recreated through the problem and the process again—engaging students once again in the investigation while pointing out where I made mistakes and where I struggled to find an appropriate mathematical approach to solving the problem.

In Shulman's model for pedagogical reasoning I, as a teacher, was moving from what I had originally understood about this problem to a new level of comprehension. This was occurring as I was beginning the transformation process—specifically, I was in the preparation phase of my pedagogical reasoning cycle. I had to think through the mathematics of the problem on my own, just so that I could begin to talk about it with the students.

Drawing on my model which includes shared reflection, I replicated my thinking for the class. These students had all taken several years of mathematics in high school—experiences which were much more recent for them than my high school days were to me. With this background, most of the students were able to follow the logic I used as evidenced by their ability to participate in the discussion and suggest other possible approaches. The accompanying conversation allowed us to cover more mathematics than I thought would be possible with the class and in a short amount of time. By asking the students for their suggestions in tackling my questions and then walking them through my reasoning on this problem, I was able to touch on ideas of similarity, trigonometry, and the Pythagorean Theorem. I demonstrated some of the ways in which these mathematical ideas are connected and how one strategy may have an impact on other approaches.

Even for those students who may not have seen the trigonometry before, the experience of watching and listening to a teacher explain how he struggled with these ideas and refused to give up until an answer had been reached was, I feel, an important one for them. The discussion also continued to demonstrate to preservice teachers the importance of being a reflective practitioner and that their developing knowledge was constantly growing and making new connections. When I thought about how many of the students were able to make the connections between maximizing area with a given perimeter and maximizing volume with a given lateral surface area without my making these connections explicit, I realized that the task of helping preservice teachers recognize and build their content knowledge is a complex one—yet one that I was making some headway on. From my own personal reflections:

> Last year, my students had a hard time extending this idea to why the cylinder has a maximum volume for a given lateral surface area—this year that question was solved before I even asked the question. My students now have such a flexible, connected knowledge of maximizing and minimizing perimeters, areas, surfaces areas, and volumes that they are hypothesizing about new situations with little difficulty. … It's nice to see them making sense of the formulas they had learned but didn't understand and reasoning about new problem situations in a variety of ways. (Personal Teaching Journal, 3/30/99)

Rice and Clay—Selection of Appropriate Tools and Representations

With a useful formula for the volume of a prism in hand and an informal understanding of a cylinder as a "circular prism", students easily tackled the task of finding the volume of a cylinder. This problem, however, only served as a brief prelude to the investigation that was cited by more students than any other as the "most memorable" of the semester. *Filling and Wrapping* contains a numbers of problems that I find extremely impressive when I see middle school students, teachers, and preservice teachers investigate them. But there are two problems which really stand out for me—the development of formulas for the volume of a sphere and a cone.

I had often seen this particular cone investigation before, even as an eighth grader and as a preservice teacher in a methods course. It involves the comparison of a cone and a cylinder with equal heights and congruent bases (Figure 33).

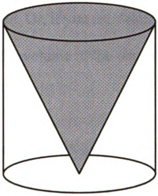


Figure 33: Cone and Corresponding Cylinder

The cone is filled (with rice, popcorn, water, or some other medium) and dumped into the cylinder (Figure 34). The cone is filled and emptied into the cylinder a total of three times.

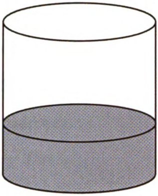


Figure 34: Cylinder Containing Volume of One Corresponding Cone

This demonstrates that the volume of the cone appears to be $\frac{1}{3}$ that of the cylinder. If a numerical value has been determined for the cylinder's volume, then the cone's volume is easily determined. Or, if you are dealing with the cylinder's volume symbolically as $V = \pi r^2 h$, then the volume of the corresponding cone could be expressed as $V = \frac{1}{3}\pi r^2 h$.

But similar to some of our work with circle area, the exactness of the answer comes into question. While it may take a little higher level of acceptance to agree that it takes exactly π radius squares to cover a circle, students must take at least a small leap of faith to buy into the fact that it takes exactly 3 cones to fill the cylinder. Perhaps it takes just a little more than 3 (maybe even π!) cones to fill the cylinder and the rice that the teacher uses fills in the space of the cylinder better than the space of the cone. Or maybe the teacher has chosen a cone and a cylinder of a specific size to make it come out evenly. As a teacher, I always felt that too many questions were raised by this investigation to irrefutably prove the desired relationship, but my middle school students seemed convinced by it and my preservice teachers were always impressed with it (especially since it helped to validate a formula they may have already seen). So I continued to use this problem, not realizing its full potential until I saw how it was embedded in a set of problems in the CMP *Filling and Wrapping* unit.

In some of the recent CMP workshops that are done at Michigan State, a new credo has arisen to help teachers understand the CMP philosophy of curriculum development—*nothing is arbitrary*. That is, there is a reason to the order that problems are presented and they are developed and written the way they are for some specific reasons. The cone/cylinder activity is used in other resources, but in the CMP curriculum the problem is immediately preceded by one that sheds some new light on the task.

After students are introduced to some of the basic vocabulary that they need for the problems (sphere, cone, height of a sphere, radius of the cone base, and radius of the sphere), they are given some clay or modeling dough and are instructed to "make a sphere with a diameter between 2 inches and 3.5 inches." Then, they are given some transparent plastic and instructed to create a cylinder that fits snugly around the sphere, trimming the height of the cylinder so it matches the height of the sphere (Figure 35).

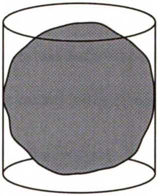


Figure 35: Clay Sphere in a Cylinder

Next, the students are instructed to "flatten the sphere so that it fits snugly in the bottom of the cylinder" and to "mark the height of the flattened sphere on the cylinder" (Figure 36).

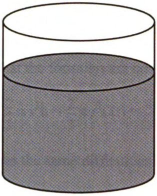


Figure 36: Flattened Sphere in a Cylinder

Following this, students are asked to measure the height of the flattened sphere, the height of the cylinder, and the height of the empty space for the purpose of looking for a relationship between the volume of the sphere and of the cylinder.

At this point most middle school teachers and preservice teachers find that the height of their flattened sphere is about double that of the empty space. This indicates that the original sphere had a volume of about $\frac{2}{3}$ that of the cylinder. This typically becomes even more powerful when data are compared for the entire class and students see that the ratio holds true for whatever size sphere they started with. With what they know about cylinder volume, students can then reason that to find the volume of the sphere, they would take $\frac{2}{3}$ of the volume of the corresponding cylinder. Symbolically, this leads to a sphere volume formula $V = \frac{2}{3}\pi r^2 h$. The only part of this process that students typically find difficult is realizing that the height of the sphere is not given and that since it is the same as the diameter of the sphere, it is double the radius.

For the mathematically-savvy preservice teacher, this formula often comes as a bit of a surprise, for the formula that is usually presented for the volume of a sphere is

$V = \frac{4}{3}\pi r^3$. But with a little symbolic manipulation (and the understanding that with a sphere, $h = 2r$), it can be shown that the formulas are equivalent:

$$V = \frac{2}{3}\pi r^2 h = \frac{2}{3}\pi r^2(2r) = \frac{4}{3}\pi r^3$$

But this problem also raises the same difficulties as the one involving the cone and the cylinder—that some amount of hand-waving must be used to gather an agreement that the 2:3 ratio between the sphere and the cylinder volumes is indeed correct. But here I return to the fact that these two problems are placed one after the other. The students even use the same cylinder that was created by wrapping the plastic around the sphere to investigate the cone and cylinder relationship. It is the combination of these two investigations that lends some additional credence to the outcome. There may still be some question about the exactitude of the relationship, but the fact that two related problems produce such seemingly related results is harder to dismiss.

In these two problems, the interaction of the representations is particularly helpful for understanding the relationships between the volumes. The spatial reasoning that is used here is not initially done with a picture, but with actual objects—the clay sphere, the paper cone of rice, and the plastic cylinder. Students transform these objects to investigate the relationships—creating the cylinder by wrapping the plastic around the sphere and trimming it to fit, flattening the sphere so that it fits tightly into the cylinder, and filling the cone with rice and dumping it repeatedly into the cylinder. These physical transformations themselves become important and powerful ways of building an understanding of the relationship. Students can recall the image of how high the flattened sphere appears in relation to the cylinder, but the physical process of pressing down the clay becomes intricately embedded in their understanding. When Janet poured the rice from the cone on top of the flattened sphere (Figure 37), she observed that the combined volumes of the sphere and cone were equal to the volume of the cylinder. By doing this, she added a new dimension to the spatial reasoning that she was developing—one that demonstrated to her more about the interplay of the various relationships.

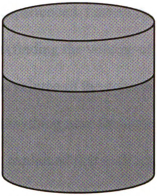


Figure 37: Combined Flattened Sphere and Rice in Cylinder

Using Shared Reflection to Move Learners to Thinking as Future Teachers

As a teacher educator, I observed a growing deepness of my own understanding as well as my students' with these problems. In Shulman's model for pedagogical reasoning, I look primarily at the transformational phase of the teaching process, where the selection of subject matter and the adaptation of lessons results not only in important representations of the subject matter, but also in the accompanying instructional strategies (the ways I set up the tasks, the language, drawings, and models I used to aid in understanding). The choice of these models and the order of their introduction was far from arbitrary—these decisions were made based on the work done around CMP, my presentation of these lessons with practicing teachers in inservice sessions, and my previous use of these investigations with preservice teachers.

But what set this presentation of these problems apart from the other times that I had tried them was my use of shared reflection throughout the lessons. I constantly asked the preservice teachers to change their hats—to move from being a student of mathematics to a future teacher of children. I shared my reflections with the students on the decisions that I had made about teaching these problems and I shared with them the successes and failures that I had experienced. I asked the students to compare the combination of these two lessons (finding the volume of the sphere and the volume of the cone) with using the lesson on the volume of the cone by itself. I asked the students to think about whether they learned anything new about the cone by doing it in conjunction with the sphere activity. Students explained that each lesson helped to validate (in their minds) the other. They described the ways in which combining the two lessons helped them to recall the formulas. In her learning log, Tammy wrote:

> I liked this lesson not necessarily just because we played with clay, but I liked it because of the way it sticks in my head. During this lesson, we learned the formulas for a sphere and a cone. These formulas are very easy for me to remember surprisingly because they are actually tough formulas. The reason I am able to remember them so easily is because in my mind I remember what happened with the rice and clay. I can see how the sphere made of clay filled up two-thirds of the cylinder, and also how the cone of rice filled up one-third of the cylinder. (4/20/99)

I asked the students to think about the kinds of things they learned as students in the class and to reflect on how they might use their experiences as learners when they are teachers. Tammy went on to write:

> This lesson was a favorite of mine, because you found a good way for us to easily remember a tough idea. My goal when I become a teacher is to do the same thing. (4/20/99)

These comments are particularly insightful when considering that Tammy was one of the brightest students in the class. She had a rich mathematical background and was often the first student to recognize a problem's underlying mathematics. In her learning log, Tammy shows that she has been thinking about how powerful these representations might be for her students—explaining that manipulatives involved more than just "play[ing] with clay," but that it makes the mathematics more accessible to students at all ability levels and with various learning styles.

Using a New "Blended" Representation

In preparing to teach these lessons to my preservice teachers, I recalled reading about Archimedes and how he had discovered this fascinating set of volume relationships. I turned to some mathematical history books and found what I was looking for, a succinct visual representation of Archimedes' discovery[15] (Figure 38).

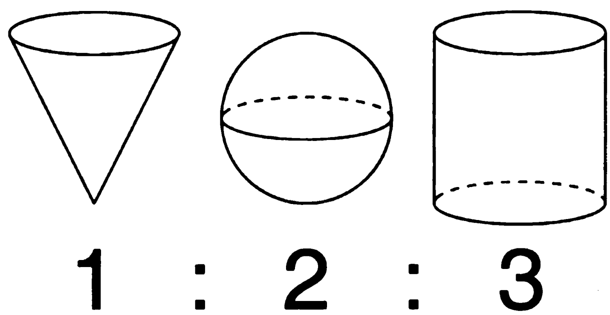


Figure 38: Archimedes' Ratios of Volumes

I decided that while this representation captures much of the beauty of the volume relationships, I wouldn't show it to the students right away. For even though it describes the ratio between the volumes of any two solids quite nicely (just cover up one of the figures and the number beneath it, leaving behind the ratio relating the volumes of the

[15] Archimedes' original representation in of the volume relationships actually involved a cone enclosed in a hemisphere which was enclosed in a cylinder. To discover this relationship, Archimedes clearly had to know the formula for the volume of sphere—a formula that was unknown at the time. In 1906, the Danish philologist John Ludvig Heiberg discovered *The Method*, a lost work by Archimedes in which he described his technique for finding the formula for the volume of a sphere. Archimedes employed principles of mathematics and physics to derive the formula. Today, most mathematicians use calculus to derive the volume formula for a sphere—a tool that was not available to Archimedes as it wasn't discovered until almost 2,000 years after Archimedes' death (Swetz, 1994, pp. 180-181).

two remaining solids), it circumvents the discovery nature of the volume experiment, a powerful lesson for preservice teachers and an experience that I want them to have.

But after the preservice teachers had the opportunity to work on the problems, I drew the three solids and the accompanying compound ratio on the board without an explanation. I asked the students to describe what this representation meant to them. A few students immediately pointed out the ratio 2:3. I asked them to elaborate and they explained that the ratio corresponds with the volumes of the two figures above it, that the sphere has a volume that is $\frac{2}{3}$ the cylinder's volume. Several other students quickly added that the 1:3 ratio is also contained in the representation, indicating that the cone has a volume that is $\frac{1}{3}$ the cylinder's volume. This representation also indicated to the students the additional ratio of 1:2, showing that the sphere's volume is double the cone's volume—a fact that some students discovered in manipulating the rice and clay but a fact that became easier to comprehend with this new representation.

Many students chose to use the representation (Figure 38) in later work—I found it drawn on many tests and quizzes as students worked to recall the relationships—while others relied solely on what they learned from the lessons. Students used this representation to help them reconstruct the formulas for the volume of a cone and a sphere from the formula for the volume of a cylinder. This blended representation is a powerful abbreviation of the spatial, symbolic, and mathematical language representations of the volume relationships and it serves as an ideal example of how important representations can be in the development of preservice teachers' deeper mathematical understanding.

Making Sense of Formulas: An Informal Proof Scheme

As with the circle area, these investigations on the volumes of the sphere and the cone were less about developing an understanding of the mathematical reasoning involved and more about remembering and being able to reconstruct formulas for calculating those volumes. This focus was largely a result of my experience with preservice teachers who had misunderstandings about formulas coming into MTH 202.

On the first day of class I posed a learning log prompt asking students, in part, to "Write down the first three geometry formulas you think of and tell what they are the formulas for." A few were not able to come up with three formulas and many of the other students either stated incorrect formulas or did not write down what the formulas measured. For example, Tina wrote:

$l \times w = \text{area}$ $\quad$ $\frac{1}{2}bh = \text{triangle area}$ $\quad$ $a^2 + b^2 = c^2$

Tina's second formula and description of what it measured were fine, but her first formula lacks the specificity of the shape it measures and she shows no evidence of understanding what the third formula represents or can be used for.

Maryanne's entries were:

$A = 2l \cdot 2w$ $\quad$ $P = 2l + 2w$ $\quad$ perimeter of a square = 4s

Maryanne's third formula was both correct and specific about what it represented, but her second one gave no indication of what kind of shape the formula can be applied to and her first one was incorrect, although it may show Maryanne's generalization of the perimeter formula for a rectangle (an additive process) to an area formula (a multiplicative process).

I found that these students were fairly representative of the preservice teachers I worked with in MTH 202. For this reason, I decided that I would focus the course around helping students remember formulas used in geometry correctly, what the formulas are used for, and how to make sense of the formulas.

The third goal—making sense of formulas—is the only one that alluded to any level of proof schemes. Even then, I truly thought about it as "making sense," and not anything as formal as a kind of proof, much less the all-encompassing "justification." To me, "making sense" implied a less formal level of justification. It indicated that students reasoned about the mathematics of the problem and by making connections to other ideas, using multiple approaches or representations, or thinking about the underlying mathematics of the problem were able to be more confident about their results. I wanted students to feel that the formulas made sense, in much the same way teachers help students gain number sense (e.g., by reasoning that an estimate is in the ballpark and by understanding the magnitude of numbers and the effects that operations have on those numbers). *Geometric formula sense* for preservice teachers can be thought of as a specific type of mathematical PCK that encompasses ideas of using symbolic, spatial, and mathematical language representations to think about the reasonableness of a formula, learning about ways in which various representations help students to reason about formulas in words and in symbols, and pushing at a teacher's own mathematical content knowledge.

Conclusion

In this chapter, I used representation as the organizing tool of analysis in looking at how I attended to issues of PCK while focusing on the development of my own pedagogical content knowledge. Although I began the course focusing on ways to make sense of geometric formulas using spatial reasoning with various manipulations, my understanding of the mathematics and ways to help students deepen their knowledge grew as I expanded my perceptions to include other types of representations that were used in class. The development of a blended representation that encompassed ideas inherent to symbolic, spatial, and mathematical language was a particularly powerful tool for students in understanding relationships between figures and it provided students with a way to remember formulas and relationships that might otherwise be difficult to recall.

I also used Harel and Sowder's proof scheme to investigate further my approach to using justification in teaching MTH 202. The notion of geometric formula sense doesn't exactly classify as a proof scheme, but I found it to be a helpful way to describe the ways in which I encouraged preservice teachers to "make sense" of the formulas they encountered.

CHAPTER 4

TASK DEVELOPMENT

Introduction

Effective teaching involves not only finding and creating tasks that are engaging and mathematically challenging, but also recognizing the complexity of using those tasks in interactions with students (NCTM, 2000). There is a great deal that goes into decisions about the tasks that are chosen—understanding students' abilities and potentials, thinking about how students might interpret the task in the way it appears to them, and deciding whether the task truly reflects the mathematical goals the teacher has for the students. These decisions involve knowing students, knowing mathematics, and knowing how students might best learn the content—aspects of teaching that are difficult for experienced teachers, much less novices.

Yet descriptions about the process of designing tasks give little detail about how these decisions are made. Are there words or phrases that are more useful than others in conveying specific desired mathematical processes? Are there instances when a spatial representation might be helpful and times when it might be misleading? What assumptions are being made when a task introduces terminology that may be new to some students and how might this affect students' performance on the task?

In this chapter, I don't look to establish definitive answers to these and other similar questions about task design. Instead, I examine how thinking about questions like these can help a teacher, or in this case a teacher educator, create tasks that better reflect what he or she intends. This chapter gives a detailed account of the evolution of a task, following it from its creation for MTH 202, examining how students interpreted and answered task, comparing it to another version of the task, and charting its development through several subsequent iterations.

In analyzing student work and the evolution of the task in each of its iterations, one theme that kept bubbling up to the surface was the issue of *generality*. It is not uncommon for teachers to find that students are interacting with a specific case in a problem and not generalizing their results to a larger domain, as the teacher may have expected (Schwartz, 1986; Chazan, 1993; Goldenberg & Cuoco, 1998). Geometry classes are particularly susceptible to the phenomenon, as problems are often posed with an accompanying drawing, intended to exemplify one possibility or case, and students treat hypotheses based on the example as also holding for the more general.

Parzysz made a distinction between the diagrams that indicated cases of the particular and the general when he introduced the words *drawing* and *figure* to the research literature in geometric learning (Parzysz, 1988). A drawing was one specific picture and a figure was meant to represent a larger class of shapes which were related by some shared traits. I use this terminology in describing the task being analyzed in this chapter and to underscore the importance of making students aware of the difference between the two types of diagrams and the tasks that accompany them.

Posing a Task

Following a series of learning log prompts that asked students to write about some of their previous geometry experiences, I posed a question that I thought would link some of the topics we were studying in class and would provide access to some additional geometry topics. Since I was still getting them used to writing in a mathematics class, I posed an open-ended question that would encourage the students to write as much as they could about a problem. I recalled a problem that seemed to meet all of these needs—a problem that was given to me as an undergraduate student in mathematics education that centered around investigating a unique kind of trapezoid, so I recreated the problem as I remembered it.

As students walked into class one day, I distributed slips of paper with a problem and figure (Figure 39).

Learning Log Prompt #4

Here is a *special isosceles trapezoid*. Write down all of the interesting things you can find about this isosceles trapezoid.

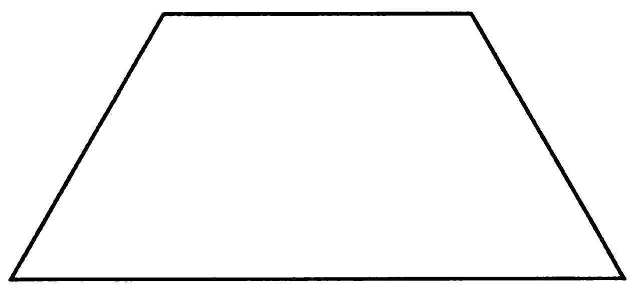


Figure 39: Trapezoid Task Version A

I chose to introduce the term "special isosceles trapezoid" to encourage students to wonder what was special about this figure. In my mind, I envisioned a classification of trapezoids (Figure 40) in which special isosceles trapezoids make up a subset of the set of isosceles trapezoids (trapezoids with congruent non-parallel sides). The special isosceles trapezoids are those in which the two congruent non-parallel sides are also congruent to the smaller base and in which the larger base is twice the smaller base.

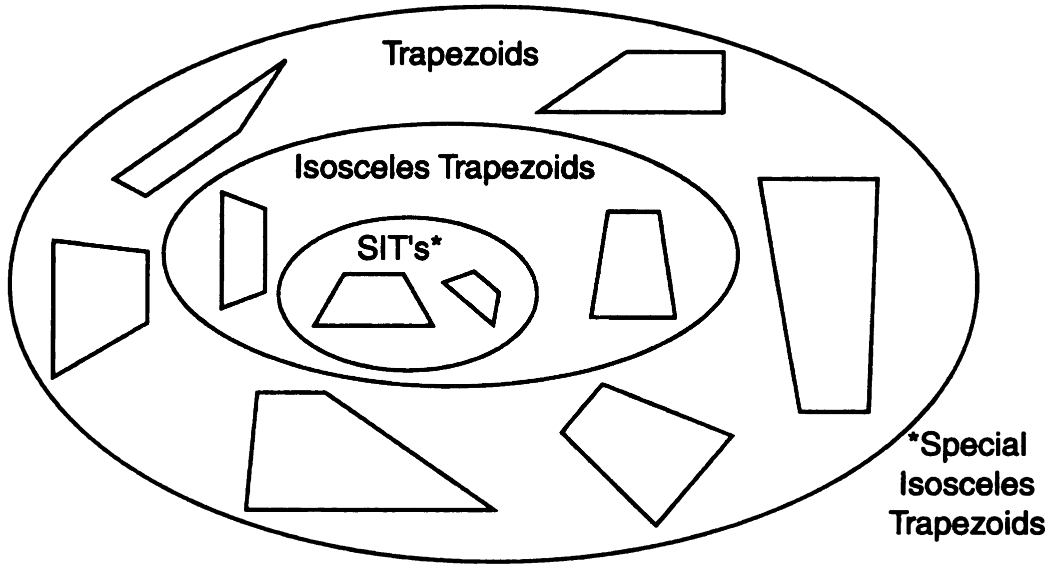


Figure 40: Trapezoid Venn Diagram

At this point, I expected students to at least notice some interesting properties about the figure—that three sides are one length and the fourth side is double that length, that there are two 60° angles and two 120° angles, that the figure has a line of symmetry, etc. I expected students then to branch out and look at other interesting properties of the figure involving diagonals, subdividing and rearranging the shape, segments connecting midpoints of sides, and any other geometric ideas they could think of.

I also planned on using this task to launch a unit centered around similarity. I wanted students to begin looking at what it means for two different shapes to be similar, since the figure I included with the task was meant to be one example of the set of different-sized special isosceles trapezoids. However, the design of the task led most of my students to assume that the only trapezoid that they were supposed to consider was the single one drawn with the prompt and thus gave them a more restricting view of the question than I planned. In much of their work, they treated the diagram as a drawing and not as a figure as I intended.

Using Student Work to Analyze the Design of the Task

When students turned in their learning logs, I was surprised at their limited interpretation of the question I posed—until I reread what I had asked them to do and realized that they had followed my instructions. Instead of looking at special isosceles trapezoids in general, many students focused on the one that they were given and treated it as the one special isosceles trapezoid in existence. For them, what was special was this drawing, not the set of special isosceles trapezoids which was a subset of the set of isosceles trapezoids. I did an inadequate job expressing the question I planned, which made it nearly impossible for students to approach the problem in the same ways that I anticipated.

Four sets of student work are presented here as examples of how preservice teachers answered the question based on their interpretation of the task (Figures 41, 42, 43, and 44). I look at some of the general themes present in the work of the four students—Dana, Rachel, Margie, and Patti—and how the design of the task may have led students to respond in the some of the ways that they did. The samples of student work that are provided also include my comments, corrections, and evaluation of the student's work, giving each student a grade based on a 10-point scale.

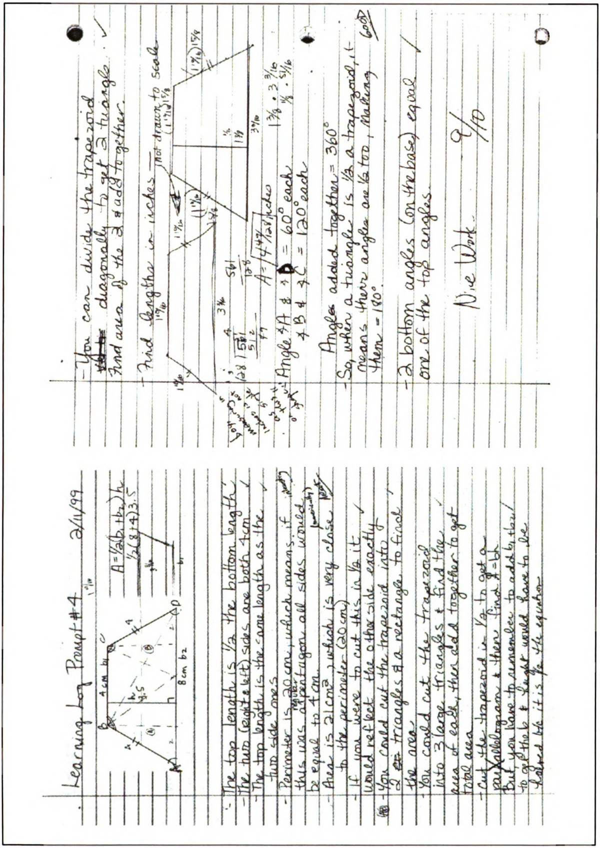


Figure 41: Dana's Work on Trapezoid Task

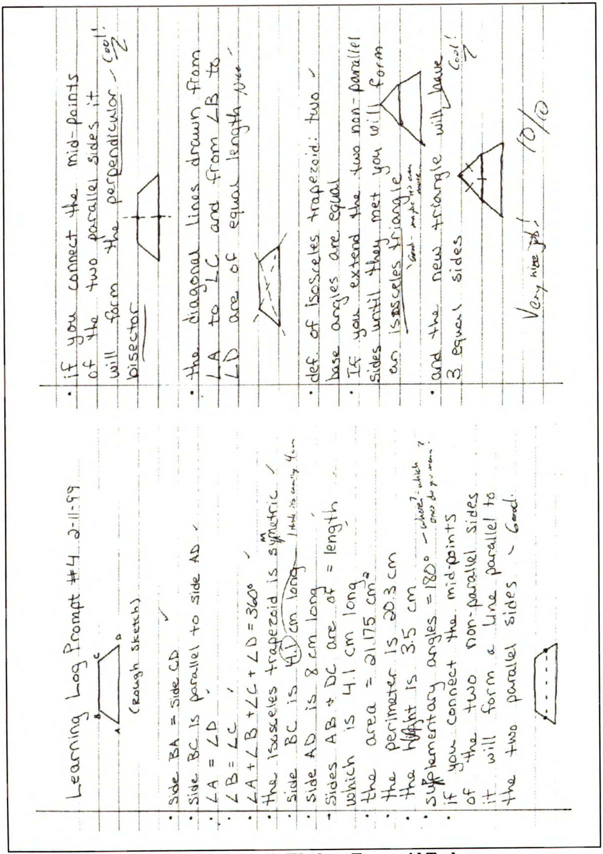


Figure 42: Rachel's Work on Trapezoid Task

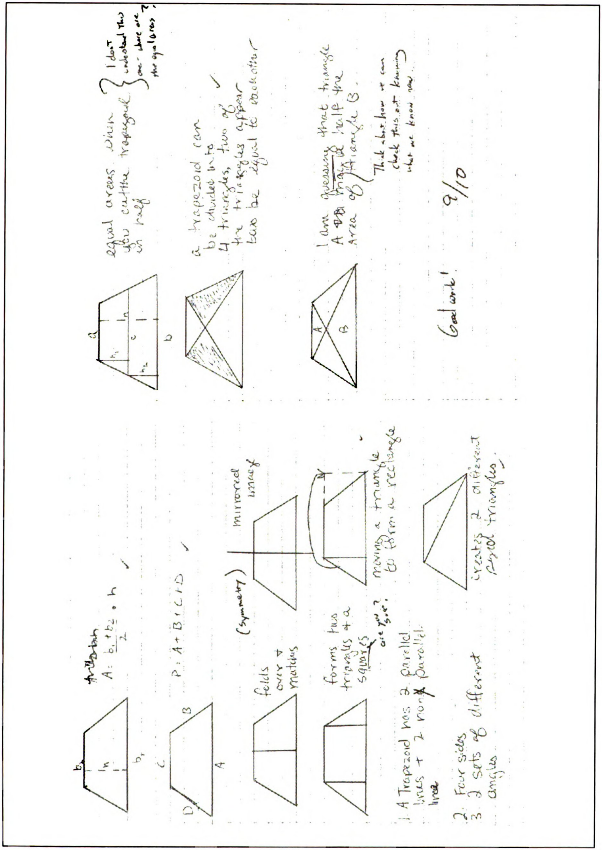
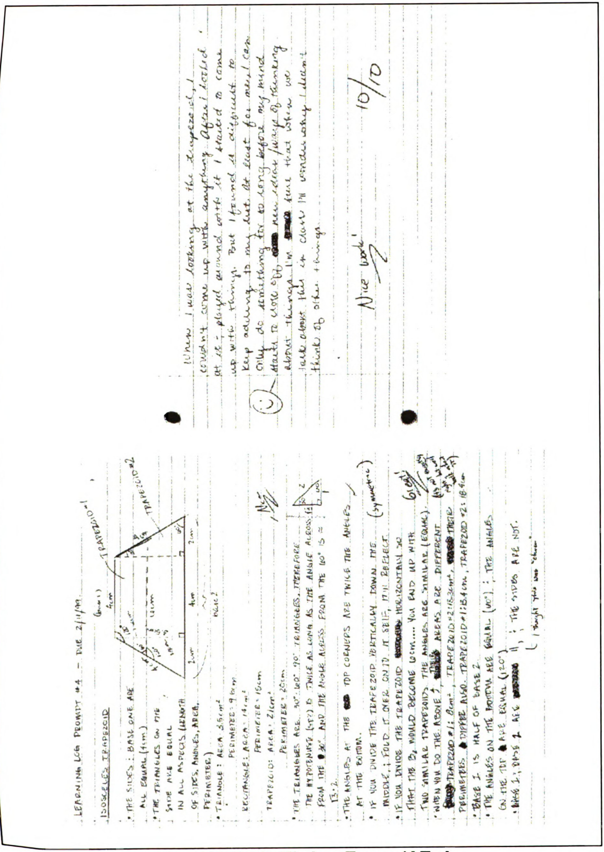


Figure 43: Margie's Work on Trapezoid Task



LEARNING LOG PROMPT #4 – DUE 2/11/99.

ISOSCELES TRAPEZOID

- THE SIDES ; BASE ONE ARE ALL EQUAL (4cm)
- THE TRIANGLES ON THE SIDE ARE EQUAL IN ALL ASPECTS (LENGTH OF SIDES, ANGLES, AREA, PERIMETER)
- TRIANGLE : AREA 3.5cm²
  PERIMETER : 9.6cm
  RECTANGLE : AREA : 14cm²
  PERIMETER : 15cm
  TRAPEZOID : AREA : 21cm²
  PERIMETER : 20cm
- THE TRIANGLES ARE 30°: 60°: 90° TRIANGLES. THEREFORE THE HYPOTENUSE (SP?) IS TWICE AS LONG AS THE ANGLE ACROSS FROM THE 30°, AND THE ANGLE ACROSS FROM THE 60° IS ≈ √3 : 2.
- THE ANGLES AT THE TOP CORNERS ARE TWICE THE ANGLES AT THE BOTTOM.
- IF YOU DIVIDE THE TRAPEZOID VERTICALLY DOWN THE MIDDLE, ; FOLD IT OVER ON TO IT SELF, IT'LL REFLECT. (symmetric)
- IF YOU DIVIDE THE TRAPEZOID HORIZONTALLY SO THAT THE B_2 WOULD BECOME 6cm.... YOU END UP WITH TWO SIMILAR TRAPEZOIDS. THE ANGLES ARE SIMILAR (EQUAL). Great!
- WHEN YOU DO THE ABOVE ↑, THEIR AREAS ARE DIFFERENT TRAPEZOID #1: 8cm², TRAPEZOID #2: 13.3cm², THEIR PERIMETERS DIFFER ALSO. TRAPEZOID #1: 13.4cm, TRAPEZOID #2: 18.4cm (It's not exactly right but we'll talk about it)
- BASE 1 IS HALF OF BASE 2.
- THE ANGLES ON THE BOTTOM ARE EQUAL (60°) ; THE ANGLES ON THE TOP ARE EQUAL (120°)
- BASE 1 ; BASE 2 ARE ∥, ; THE SIDES ARE NOT.
  I thought this was "chosen"

When I was looking at the trapezoid, I couldn't come up with anything. After I looked at it ; played around with it I started to come up with things. But I found it difficult to keep adding to my list. At least for me, I can only do something for so long before my mind starts to close off new ideas / ways of thinking about things. I'm sure that when we talk about this in class I'll wonder why I didn't think of other things.

Nice work!

10/10

Figure 44: Patti's Work on Trapezoid Task

Analyzing Themes in Student Work

The four pieces of student work on this task are fairly representative of what was done by students in the class. All students made at least one kind of error in their work, although I was not assessing students on the accuracy of their findings alone. Almost all students made at least one statement that grew out of an understanding of the definition of trapezoid—these statements could be viewed as trivial. And almost all students applied some ideas that were introduced earlier in the course to this task.

I grouped the responses of these four students into five different categories:

1. Trivial observations
2. Applications of mathematical ideas previously covered in MTH 202
3. Applications of other basic or advanced mathematics
4. Unproven hypotheses
5. Incorrect statements

While there is some overlap between categories (e.g., a statement might be an unproven hypothesis that is also an application of a previously covered idea from MTH 202), I tried to make some distinctions between the kinds of statements I placed in each category.

Trivial Observations. A word of clarification before these are presented: these observations may or may not have seemed trivial to the students who made them. I am classifying them as trivial with respect to the task that was given, to find "all of the interesting things" students could about the figure. The statements that fall into this category are generally a result of the shape being a trapezoid and have nothing to do with the uniqueness of the shape.

Margie made several trivial observations about the special isosceles trapezoid, noting that has four sides, its perimeter can be found by adding up the lengths of the four sides ("P = A + B + C + D"), and its area can be found using a formula that was discussed at length in class ("$A = \frac{b_1 + b_2}{2} \cdot h$"). Unlike the other students, Margie didn't use either of the formulas to determine the trapezoid's perimeter or area, she simply restated a formula that was used in class. Both Margie and Dana listed at least one of the spatial representations for finding the area of the trapezoid, but did not determine the area using that method. And three students (Rachel, Margie, and Dana) stated that the two bases of the figure are parallel—a reiteration of a part of the definition of a trapezoid.

It is possible that each of these trivial observations is a result of some confusion that arose from the wording of the task. Although I wanted students to discover aspects of the figure that made it "special," I instructed the students to find "interesting things" about the figure, a phrase that is somewhat vague and open to interpretation. In addition, I often asked students to make connections back to prior class work. We spent several previous class periods discussing general trapezoids. The trapezoid in the task likely indicated to some students that the task was also an assessment of what they already knew about trapezoids, and so they reproduced some facts that were not unique to the special isosceles trapezoid. Finally, students knew that their responses to the learning log prompt were evaluated as a part of their course grade. By asking for students to find "*all* of the interesting things" they could find, I may have inadvertently indicated to some students that the more things they wrote, the higher their grade would be. These students may have decided to include everything they could come up with about the figure—no matter how trivial.

Applications of Mathematical Ideas Previously Covered in MTH 202. In addition to some of the trivial observations presented in the previous section, there was a set of statements made by students that related to things discussed in class. I classified these statements in a different category from the trivial observations because to me, these observations involved an application (instead of a simple restatement) of an idea that was previously discussed.

Dana, Rachel, and Patti measured the dimensions of the drawing and calculated its perimeter and its area. This involved making some decisions about the trapezoid's measurements—how accurate to be, what measurement system to use, etc. After measuring in centimeters (the drawing was made so that its sides were whole number centimeter lengths), Dana went to the trouble of measuring in inches and calculating its perimeter and area. Two students, Dana and Margie, also applied ideas about subdividing a shape to find its area with Dana listing no fewer than four different ways that this could be done.

As with the trivial observations, it is possible that these applications of previously covered topics drew from my common insistence on connecting back to other ideas. Students may have assumed that I wanted them to apply concepts from the course to this task.

Applications of Other Basic or Advanced Mathematics. I was pleased to see the introduction of ideas that I wasn't planning to cover in MTH 202 or had not yet covered in the course. These ideas included some of what I was hoping to see in students' work—either some ideas of what made the figure "special" or thinking about some of the relationships between the figure's attributes. Some of the more advanced ideas that students gave involving terminology or ideas that are not part of student's daily lives were likely carried over from the their middle school mathematics or high school geometry course.

Dana, Rachel, and Patti observed some relationships between side lengths (three sides were the same, one base was half the length of the other base, etc.) and all four students discussed at least one of the angle relationships (two sets of two different sized angles, larger angles measured twice the smaller angles, etc.). All students also made an observation about the symmetry of the shape, showing how it "folds over and matches" (Margie), that "if you divide the trapezoid vertically down the middle and fold it over onto itself, it'll reflect" (Patti), or that the diagonals are of the same length (Rachel).

Patti recalled some relationships she learned before but I was planning to introduce later in the course—relationships of side length in a 30-60-90 triangle. She subdivided the trapezoid into two 30-60-90 triangles and rectangle and used the relationships she knew to find the exact height of the special isosceles trapezoid. This was a clever approach to use and it showed some nice geometric insight on Patti's part, especially when compared to the other students who took a measurement of the height (3.5 cm) to be exact.

Rachel also made a number of unique observations about the special isosceles trapezoid, finding that "if you connect the midpoints of the two non-parallel sides it will form a line parallel to the two parallel sides" (a restatement of the definition of a midsegment, which was not explicitly discussed in class) and that "if you connect the midpoints of the two parallel sides it will form the perpendicular bisector [to the bases]." Rachel also turned to an outside source to write a definition of an isosceles trapezoid—one in which "two base angles are equal" (assuming that one accepts the fact that the shape is trapezoid).

In responses such as these, students felt free to begin exploring relationships within the figure. This is the type of response I was anticipating when I included the word *interesting* in the design of the task. I wanted students to feel free to experiment with the figure, drawing diagonals, connecting midpoints, and basically wondering, "what if I did this…" when playing with the shape. When designing a task, it's helpful to find some way to express to students when the desired outcome involves trying out something new. Asking students to find "interesting things" about a situation is one possible way, but as was previously discussed, that choice of words may connote other unintended ideas as well.

Unproven Hypotheses. When I posed a learning log prompt asking students to "Describe in detail an event from your high school geometry course which is particularly memorable…", a number of them included a discussion about working with *proofs*. As mentioned in Chapter 3, I wasn't interested in making proof a formal part of the course. However, I did want to find some place in the MTH 202 curriculum to allow for some discussion of issues of justification and I thought that this task might provide that kind of access.

Some of the students' observations about the special isosceles trapezoid were quite interesting and yet they left me wanting to know if students could make a formal or informal argument about whether the statements were true. For example, Rachel stated that "if you extend the two non-parallel sides until they meet, you will form an isosceles triangle…and the new [smaller] triangle [on top of the trapezoid] will have 3 equal sides." Aside from the fact that Rachel could have used to word *equilateral* to describe both the large and small triangle that were created, Rachel's two statements seemed like great examples of hypotheses that preservice teachers might be able to prove.

Margie was the queen of unproved hypotheses—statements that I felt almost begged to be justified in a proof—two-column or otherwise. She wrote that by drawing the two diagonals, the trapezoid "can be divided into 4 triangles—two of the triangles appear to be equal to each other [or congruent]." Although she was not very eloquent in describing her hypothesis, in her accompanying diagram, Margie clearly indicated the two triangles she was talking about (Figure 45).

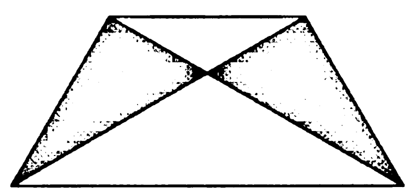


Figure 45: Margie's Special Isosceles Trapezoid 1

In the next statement, Margie looked at the other triangles formed by two diagonals and wrote, "I am guessing that triangle A is half the area of triangle B" (Figure 46). Margie's statement was not correct—the area of triangle A is actually one-fourth the area of triangle B—but I was excited about the kind of statement that she made. I also appreciated that Margie admitted that this observation included some "guessing," just as in her previous one, she noted that the two triangles "appear" to be equal. Unlike most other students, Margie didn't make unfounded claims, she gave unproven hypotheses.

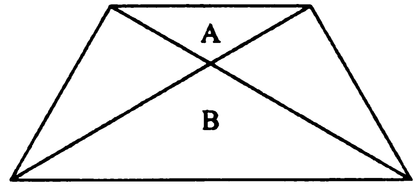


Figure 46: Margie's Special Isosceles Trapezoid 2

This is what initially bothered me about many of the statements that students made. Not only did they offer any justification for why their statements were true, but most students didn't seem to recognize that some of the observations they were making were actually hypotheses until they were proven. Then they could be considered statements of truth.

But to be fair, students had no indication that this was something I was concerned about. We had not really talked about what it means to prove something and I was not ready to tackle that subject yet in class. There was never any expectation made of students to recognize their statements as hypotheses, and nothing in the design of the task would indicate that. I simply asked for "interesting things." While I, personally, found hypotheses to be interesting, there was no reason for me to think that students would do the same.

Incorrect Statements. As previously indicated, some of the students' observations that were listed in other categories also contained incorrect statements. These categories are not mutually exclusive and I coded these statements as incorrect because they specifically indicated to me some mathematical misunderstandings that were relevant to the course.

The first statement was one that was made by Margie concerning the areas of two parts of the figure after it was subdivided. She wrote that you get "equal areas when you cut the trapezoid in half." Without the drawing she provided, I would have thought she was writing about cutting the trapezoid in half along the vertical line of symmetry. Her statement would then have been correct. However, in Margie's drawing that accompanied this statement (Figure 47), she drew the midsegment of the trapezoid—how she was likely thinking about cutting it in half. Since she gave no indication of what shapes she claimed have equal areas, she could be talking about the two trapezoids (which clearly do not have equal areas), or the two triangles at the left-hand side of the figure (which do have equal areas). Since we just finished several weeks of understanding areas of shapes, I was surprised that Margie might still have this misconception about areas of shapes.

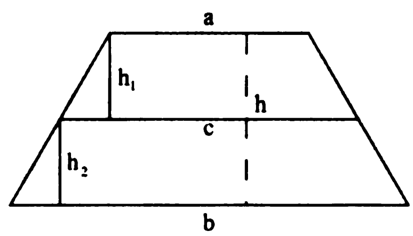


Figure 47: Margie's Special Isosceles Trapezoid 3

An incorrect statement that Patti made also looked at the isosceles trapezoid after the median line was drawn in. In Patti's statement, she claimed that "if you divide the trapezoid horizontally so that the B_2 would become 6 cm, you end up with two similar trapezoids. The angles are similar." Patti's mistake was in using the notion of similarity—an important mathematical topic that we were in the process of investigating when this task was assigned. Before this task, we established the fact that similar shapes have congruent corresponding angles—the idea that Patti seemed to be getting at in saying that the "angles are similar." But while we weren't to the point of understanding about the proportionality of corresponding sides in similar figures, we had established the fact that the similar figures had to have the same basic shape. By looking at her response to this task, Patti may not have completely understood what was meant by two trapezoids that are mathematically similar having the "same shape," since the bottom trapezoid is visibly stretched from side to side when compared to the top trapezoid. Patti had problems with using the word *similar* and/or the idea of similarity in her statement. These misconceptions led to an incorrect statement that a preservice teacher at this level should not have made.

The final example of an incorrect statement was made by Dana as she noticed something interesting about the perimeter and area of the special isosceles trapezoid drawing. Dana wrote, "Area is 21 cm^2, which is very close to the perimeter (20 cm)." In my notes to Dana, I corrected her to make her statement read that they are "numerically very close." I wanted her to realize that while the measurements and calculations she found may be interesting, her statement is not quite mathematically correct. Dana focused on the fact that the numerical parts of the two measurements were close while ignoring the fact that the units were different (cm^2 versus cm).

Her statements should also have raised some questions when in her later observation, she calculated the area to be "4$\boxed{\text{X}}$(square) inches," which is quite different from what she would have found if she calculated the perimeter of the trapezoid when measured in inches (8$\boxed{\text{X}}$, by her measurements). In light of this, her statement about the closeness of the two measurements might read "when measured in centimeters and square centimeters, the perimeter and area are numerically very close."

Dana was mathematically incorrect in the language she used in describing something she noticed about measurements she made of the drawing. We discussed this distinction between units in an earlier class session, and I was surprised to find Dana confusing the issue once again.

Helping Students Learn from Others—Supplementing a Task

After considering the work that some students did on the task as it was posed to them, I decided that there was more that could be learned from investigating the special isosceles trapezoid even further. I felt that other students could benefit from some of the individual observations that were made—both right and wrong—and I looked at the students' work to pull out some of the items I felt would be most beneficial. I decided to create a new labsheet of questions that I pulled from some of the students' observations.

In Shulman's model for pedagogical reasoning I was taking what I learned from assessing students and transforming that knowledge into appropriate questions and new tasks that would prompt more investigation of various ideas in geometry. As part of the transformative process, I was thinking about the original task that I created and what may have caused students to respond to it in the ways that they did. I considered the language that was used, the spatial representation that was presented, and the introduction of the new term *special isosceles trapezoid.* I looked at ways students responded to the learning log prompt and considered questions such as:

- What level of observations should be expected of preservice teachers and how can I encourage students to reach that level?
- How could I get students to consider the properties of the family of special isosceles trapezoids instead of one specific example?
- Should I introduce the term *special isosceles trapezoid* in the task? Does the term mislead students or take away from some of the investigative nature of the task?

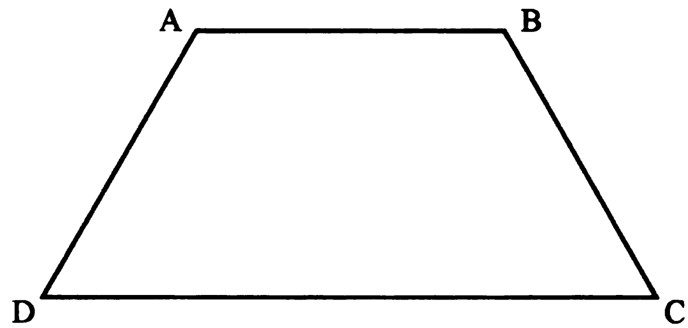
- Should students simply state conjectures or should they also include some proof of their hypotheses? What role should proof play in MTH 202?

Thinking about questions like these helped me decide which individual student observations I would include on the labsheet. I compiled a series of observations and conjectures that were representative of varying types and levels (Figures 48 and 49). I also decided to reformulate their observations as questions. Rather than presenting facts (which may prove true or false), I wrote questions to encourage further inquiry into the task and the observations that students made.

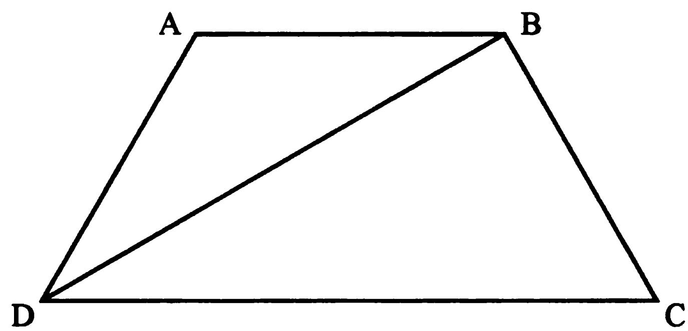
With regards to my PCK development, I was taking what I learned about what students understood about the special isosceles trapezoid, and designed a new level of the task to address the specific misunderstandings and insights that were made by individual students of the class. I was becoming more aware of the ways I designed the initial task that didn't satisfy the goals I had for my students' understanding, and I used their work to help me design a new task comprised of a series of specific questions to reach those goals.

The shared reflection in this stage of the task design involved my giving students some feedback on the work that was done by the class as a whole. I explained that the initial learning log prompt that I wrote was, in some ways, misleading and did not indicate some of what I wanted them to do with the problem. I explained how I decided to revisit some of the more intriguing observations that were made by creating a new labsheet of problems that would address some of the issues I hoped students would observe.

Most people found it helpful to name the vertices of the trapezoid in some way like those shown here. This helps when referring to vertices and line segments later.

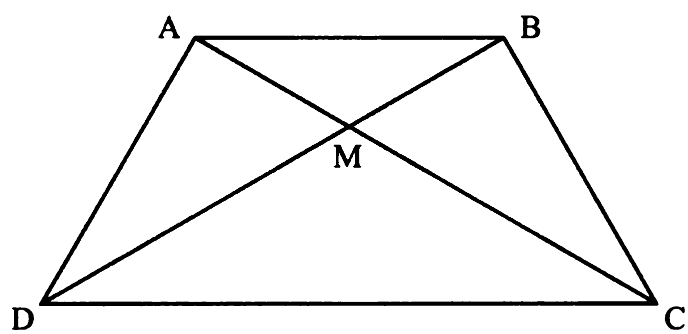


1. By drawing one diagonal as shown, two triangles are created. Describe these two triangles as much as possible. How are their areas related?

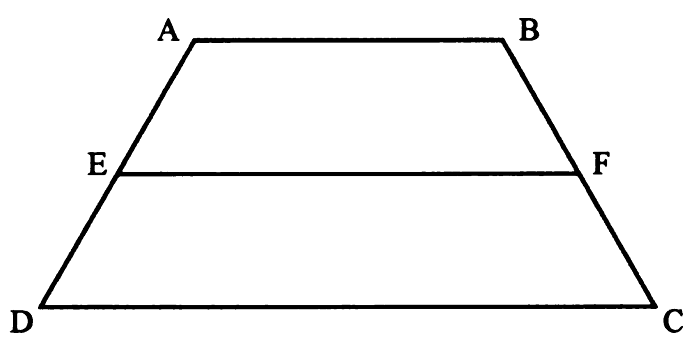


2. By drawing both diagonals as shown, four triangles are created. Describe the relationship between ΔADM and ΔBCM.

3. Describe the relationship between ΔABM and ΔCDM.



4. The line segment that connects the midpoints of the sides is called the midsegment. How is the midsegment length related to the lengths of the two bases?

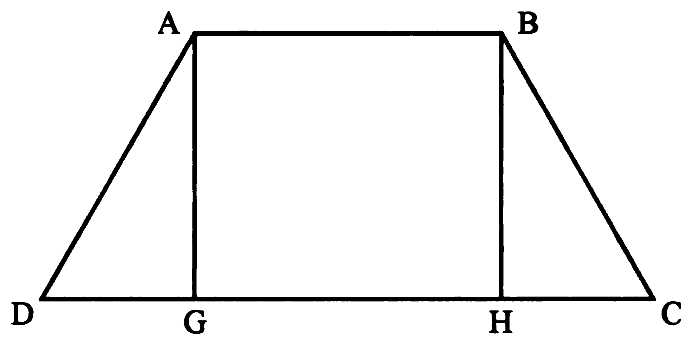


5. One student said that ABFE ~ EFCD. Is this true? Why or why not?

Figure 48: Special Isosceles Trapezoid Labsheet, Page 1

6. If perpendicular line segments are drawn from A and B, what are the three shapes that result?

7. Can you find the length of AG only knowing the lengths of the sides of the original trapezoid?

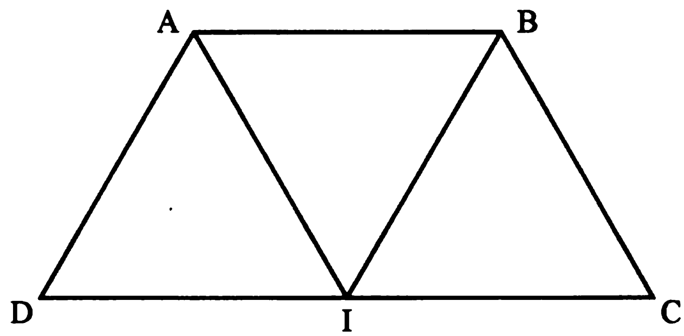


8. One student said that the height is equal to the square root of the sum of the two bases. Is this true? Why or why not?

9. Fold ΔBCH over BH. How does this help to find the area of the original trapezoid?

10. If line segments are drawn from A and B to the midpoint of CD, what three shapes result?

11. If the sides are extended to meet, what shape results?



12. In general, what is the sum of $\angle DAB$ and $\angle ADC$?

13. What is the angle sum of the original trapezoid? Are there any other polygons with the same angle sum?

14. What shape(s) can be made by putting together two copies of the original trapezoid?

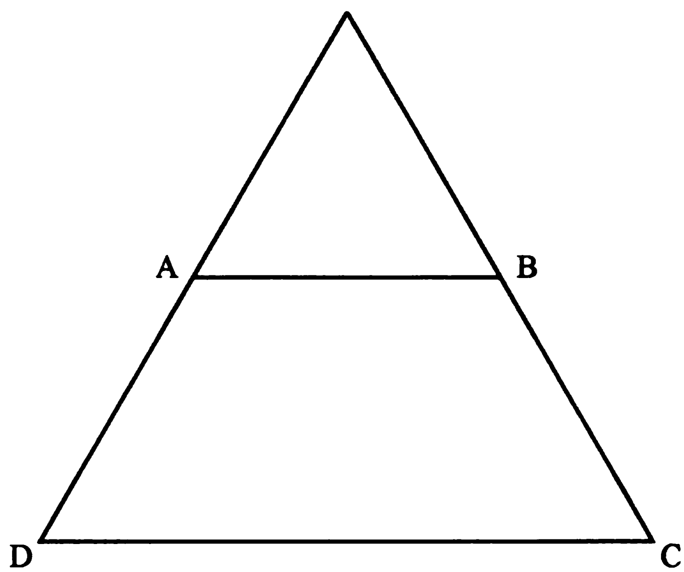


Figure 49: Special Isosceles Trapezoid Labsheet, Page 2

I separated the class into groups of three students each and I distributed the labsheet. I explained that the questions were pulled from students' responses to the learning log prompt. I asked them to work on the questions—some of which were designed to push them to think about aspects of the problem they might not have considered. Students worked on the questions for about 15 minutes and I pulled the class together to go over some of their work.

When I asked students to name one of the questions they thought was interesting or that they wanted to talk about in class, Terry requested question 10.

JW: So what did you find out about question 10?

Terry: It makes three equilateral triangles.

JW: How do you know that they are equilateral?

Terry: I measured the sides and they came out the same. They looked like they would, and when I measured them, I saw that they are.

JW: Is that what you found interesting about this question?

Terry: Yeah, I was surprised by this. I mean, I knew that drawing these two lines [AI and BI] would make three triangles, but I didn't think that they would be equilateral. I figured that they would be isosceles, but I wasn't sure about equilateral.

JW: Why did you think that they would be isosceles triangles?

Terry: Well… I knew that I is the middle of DC, so I figured that the middle triangle [ABI] would be isosceles. Then I figured that the other two triangles would be the same…

JW: As the middle one or as each other?

Terry: Each other. Since there's a line of symmetry down the middle of the trapezoid, I figured that the left and right sides would be the same.

JW: This is good. Well, let me show you something else that's neat about this problem in particular. (Takes out pattern blocks.) A lot of elementary school teachers use pattern blocks to teach different topics in math class. As a couple of you pointed out earlier, there is a pattern block that is the same shape as the special isosceles trapezoid (places red transparent pattern block on overhead). There is also a pattern block that is one of these equilateral triangles (places green transparent pattern block on overhead). By playing with these shapes, kids learn early on that they can place cover the trapezoid pattern block with three of these equilateral triangles. What mathematical ideas does this help you think about teaching elementary students?

Paula: Area.

Patti: Fractions.

What started as a discussion around an observation about the special isosceles trapezoid moved into an introduction to using pattern blocks in teaching elementary school. This is an example of my trying to teach students content, and also trying to move them beyond thinking about the mathematics solely as students but also as how this knowledge pertains to the teaching of elementary students. Terry's initial interest in this property of the special isosceles trapezoid triggered an idea in my head that I wasn't sure

I would cover. I had the transparent pattern blocks with me to use the red pattern block as an example of the special isosceles trapezoid, but when the situation presented itself, I decided to talk with the students about how the pattern blocks are used to teach other mathematical topics as well.

I was pleased with how the questions on the labsheet initiated student conversations and investigations like the one described above. But I was bothered by the fact that I wasn't able to elicit this level of inquiry from all of my students with the learning log prompt I wrote. I knew that I needed to do better at writing questions that indicate the kinds of work I wanted students to do, but I wasn't sure how to go about it on a regular basis. From my personal reflections:

> One other thing that keeps rearing its ugly head in my class is the issue of my asking a different question than what I am trying to get at. This happened on the first test and it happened to a much lesser degree on the most recent quiz and the learning log prompt about the special isosceles trapezoid. On quiz #2, I asked students to compare areas of similar rectangles numerically and pictorially—at least, that's what I thought I had asked. Only four students drew the picture to "validate (their) answer(s)" with the sketch. I knew what I wanted, but I didn't ask it very clearly. (Personal Teaching Journal, 2/18/99).

Comparing to the Original Task

After the course was over, I turned back to my notes from my undergraduate course and found the "special isosceles trapezoid question" as it was originally posed to me (Figure 50) in the EDT 335 methods course.

Make a list of hypotheses about trapezoids that have three congruent sides and a fourth side twice as long as each of the others.

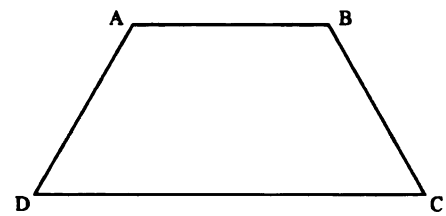


Figure 50: Trapezoid Task Version B

It's not clear to me whether the accompanying figure was a part of the question as it was originally posed or if it is something I added to aide in my understanding as a student. Either way, there is a distinct difference in the information that is given—in the words that are used and the way in which the important information is of a generalized nature.

Each of the following differences highlights how Version A provided students with a *miscue*—a characteristic of the task which can lead students to an incorrect response or at least one that is different from the instructor's intent. A miscue can take the form of a misleading graphic, sentence structure that does not accurately communicate the instructor's meaning, or even an assumption that is not common to both the instructor and the students (Shannon, 1999). Looking at the structure of Version B, I was able to better identify some of the miscues that were a part of Version A of the trapezoid task.

The first noted difference in Version B is the word "hypotheses." Compared to the use of "interesting things" in Version A, the prompt to develop hypotheses indicates to me now (and likely indicated to me as a student) a more rigorous mathematical approach to the task. For while a hypothesis is not necessarily something verifiable, it does connote that the discoveries that are made should be more mathematical in nature. When I asked students to find "interesting things" about the shape, I was opening up the door for my students to observe things like how two of these shapes could fit together to make a regular hexagon. Both ways of wording can lead to some interesting observations and discoveries, but they are likely to be quite different from each other.

When I constructed Version A of the task, I remembered more about the kinds of responses I gave to the task than the task itself. I specifically remembered making a number of hypotheses about the figure and those responses were a part of how I was reconstructing the task in my head. In writing my version of the task, I was trying to create something that might get students to make some of the same kinds of conjectures I did, without recognizing the fact that the students were not necessarily at the same mathematical level I was when I worked on the task. The word "hypothesis" in Version B carried a specific meaning for me—a hypothesis was the part of a proof that needed to be proven. I wanted MTH 202 students to create hypotheses as well, but I knew that the word hadn't been introduced in class. My instruction to find "interesting things" was my attempt to get students to make hypotheses (that could stand up to a geometric proof) without using the word.

The second major difference in the language that is used is the naming of the figure—a "special isosceles trapezoid" in Version A. By giving it this name before students had a chance to investigate its properties, some of the discovery was taken away. The word "isosceles" already implies something about the shape. Students may not have been familiar with this word in this context ("isosceles triangles" are much more common), but some carry over was to be expected. Some students even looked up "isosceles trapezoid" in a dictionary and were able to find out some facts about general isosceles trapezoids.

The inclusion of the word "special" in this case may have been problematic. Students may have interpreted it to imply that this one trapezoid was special, and not the set of isosceles trapezoids which have additional properties. There are other indicators in the question as it was originally worded that compelled me to address the general shape as a student. In that version, the side lengths of the isosceles trapezoid were described in relationship to each other. The shape was described as a trapezoid with "three congruent sides and a fourth side twice as long as each of the others." By seeing the question posed this way, I was given some immediate indication about what made this shape unique and worthy of being investigated in the problem. But by giving students one copy of the shape and asking them to make some discoveries about it, these special qualities were not readily apparent. In fact, some students measured the sides and did not come up with the relationships that were intended—finding the longer base not exactly double the length of any of the other sides due to some measurement error or a distortion of my original drawing due to the effects of a copy machine.

Conclusion

Task design is a complicated process. When done well, it recognizes the prior mathematical background of the students and capitalizes on that background to build deeper mathematical understanding. Good tasks are also designed with a clear vision of what direction the teacher has for the students—making connections between mathematical topics and laying the foundation for future investigations. In Simon's description of planning for effective teaching, he emphasizes the importance of the teacher clearly identifying a *hypothetical learning trajectory* for his or her students, which includes an understanding of the teacher's learning goal, the teacher's plan for learning activities, and the teacher's hypothesis of the learning process (Simon, 1995). All of the issues are also important aspects of good task development.

There is little evidence on ways to support preservice teachers' learning about designing good mathematical tasks. But involving them in the process through a teacher educator's shared reflection is one possible place to start. It would also be helpful for the teacher educator to have a clear understanding of some of the elements of task design—the role that specific words and language in general can play, differences between drawings and figures and what each can contribute to student understanding, and the ways in which written language and geometric representations can be used together to best convey the intended ideas of the task.

CHAPTER 5

IMPLICATIONS AND LIMITATIONS

Implications

There are three major findings in this study in which I examined the growth of my own pedagogical content knowledge. First, this study highlights and problematizes Shulman's notion of representation. In mathematics there are mathematical and empirical representations—classifications which do not map easily onto Shulman's use of representation. This study exposes some of those inherent distinctions and seeks to make Shulman's work more applicable to the field of mathematics. Second, this study describes the importance of task design—a process that is particularly essential in teaching mathematics—and finds that Shulman's notions of PCK and the pedagogical reasoning and action cycle miss or obscure its significance. And third, this study introduces the notion of shared reflection to Shulman's model for pedagogical reasoning and action when it is applied to teacher education. It also finds that the act of going public with one's ideas through shared reflection can be a useful tool for teacher educators in the development of their pedagogical content knowledge.

Complex Ways of Thinking About Representation

In his definition of pedagogical content knowledge and his description of the transformation phase of his pedagogical reasoning and action cycle, Shulman makes liberal use of the word *representation*. As a part of PCK, he includes "the most useful forms of representation of [the most regularly taught topics in one's subject area]" (Shulman, 1986, p. 9). In describing a teacher's transformation of subject matter understanding, Shulman points out that representation "involves thinking through the key ideas in the text or lesson and identifying the alternative ways of representing them to students" (Shulman, 1987, p. 16). In this respect, representation is both a product and a process. These two aspects of representation may be useful to educational researchers and teachers in a number of content areas, including science (Van Driel at al., 1996), geography (Ormrod & Cole, 1996), and physical education (Griffin et al., 1996). In mathematics, however, the word *representation* has another set of meanings that can cloud the issue of pedagogical content knowledge.

There are many forms of representations that have been agreed upon by the mathematics community—such as diagrams, tables, graphs, and formulas. These *mathematical representations* can be powerful embodiments of mathematical ideas that can provide some insight to the mathematics that might not otherwise be possible. Tables of values can highlights patterns which signify whether a relationship is linear, quadratic, or exponential. Symbolic representations and their manipulations can be useful in determining whether two expressions are equivalent and whether one version of a formula may be used in place of another.

But as students struggle with mathematical concepts and try to make sense of new ideas and contexts, they sometimes make use of representations that do not carry the high level of mathematical exactitude as mathematical representations do. However, these *empirical representations* of the mathematics can also be quite powerful tools in helping them make sense of what they are struggling with or to empirically justify an idea that they may have previously seen. For example, most of the preservice teachers in my study entered the class having some knowledge of how the area of a circle could be calculated (various students reported on the first day of class that a formula that could be used was "$A = \pi r^2$," "A = pie * r^2," or "$A = \frac{1}{2}\pi$ (r) squared"). After students used the radius square method of estimating the area of the circle, virtually none of the students made these mistakes again. Although it was not a formal proof of why the formula $A = \pi r^2$ makes mathematical sense, this empirical representation (covering the circle with pi radius squares) helped to make the formula more plausible.

When I began to analyze the data I collected, I had very little appreciation for the various types of representations I would encounter. First, I didn't realize that there were important differences between the mathematical and the empirical representations that I and my students used. These differences became apparent, however, when I found that some of the representations that students found the most powerful (using the radius squares to cover a circle and making and squishing a clay sphere to see how its volume related to a corresponding cylinder) were the ones that were far from rigorous. The empirical representations were never offered as mathematical proofs, but instead as ways for students to develop some informal sense of the formulas and other ideas we were studying.

Second, as a mathematics teacher, I was aware of the use of the word *representation* in mathematics. However, this prior understanding became problematic for me as I encountered the multiple ways in which Shulman and others wrote about representation with respect to pedagogical content knowledge. And even though Shulman describes representation as both a product and a process, he is not referring only to types of mathematical representations—as I initially believed—but to empirical representations as well. As Shulman elaborates, "Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternate forms of representation, some of which derive from research whereas others originate in the wisdom of practice" (Shulman, 1986, p. 9).

If the language of pedagogical content knowledge continues to be applied to mathematics education, more care must be given when the word representation is used and there needs to be an increased awareness of the specialness of mathematics in this regard. In describing PCK generically across the content areas, Shulman introduced some language that confounds the issue when applied specifically to mathematics. With respect to its importance in the transformative process of PCK, representation is a critical element, but the term also masks the complexity behind considering mathematical vs. empirical representations.

Growing Appreciation of Task Design

While the transformational phase of Shulman's model of pedagogical reasoning and action may prove problematic in its overgeneralization of the concept of representation when applied to mathematics, it is also shortsighted in its omission of one of the more critical elements of good mathematics teaching—task design.

A task can be designed with one of two major objectives in mind—as an *opportunity to perform* or as an *opportunity to learn* (Shannon, 1999). When it's intended as an opportunity to perform, the task allows students to demonstrate something that they have already learned, often in a new context. When the task is designed as an opportunity to learn, it is intended to build upon prior classroom experiences for the students' acquisition of new knowledge. Whether a task is constructed as an opportunity to perform, an opportunity to learn, or both, it requires that teachers have a firm understanding of students' content knowledge, the mathematics of the task, and the ways in which the task is designed to elicit student responses. As such, it is helpful to look more carefully at the transformation phase of Shulman's pedagogical reasoning and action cycle to find if and to what degree it can be applied to task development.

As described by Shulman, the transformation phase—in which teachers develop their pedagogical content knowledge through transforming what they know about a subject into ways that their students can understand it—consists of four essential processes: preparation, representation, selection, and adaptation. The preparation phase focuses on reviewing the texts and other materials of instruction and examining the accompanying educational purposes—as determined by the curriculum, the school, or the teacher. The attention is clearly on using the resources that are available and determining whether they fit what the teacher deems as appropriate. In the representation phase, the teacher begins to consider various ways to represent an idea in transforming what he or she knows about the content. The selection phase refers to the teacher's consideration of various instructional methods for presenting an idea to students—including lecture, group work, projects, and discovery learning. And the adaptation phase concerns the ways in which the teacher fits an activity to his or her students' needs, making decisions based on ability, culture, prior knowledge, skills, and other important factors.

Throughout each of these stages, Shulman describes ways in which the teacher makes decisions about what he or she teaches and how the subject matter is presented.

This process is common to good teachers in all subject areas and it is helpful in describing some of the basic commonalities of pedagogy from a general viewpoint. But in attempting to apply the general to a specific case (such as mathematics), some shortcomings begin to appear. Since a number of mathematics educators have described their classrooms and their curricula as problem-based or problem-centered, and since an increased attention is being given to the role that assessment plays in student learning, it is understandable to consider the nature of task development and task design with respect to Shulman's model.

Unfortunately, the model is never very helpful in explaining what the teacher does in designing tasks. Some of the process is hinted at in the preparation and adaptation phases, but never to the degree that is necessary in making detailed decisions about language, diagrams, types of questions (e.g., divergent vs. convergent), and format. In Chapter 4, I described Version A of the trapezoid task I presented to preservice teachers and the difficulties I had in eliciting the kinds of responses I anticipated from this task. After looking at student work, designing an in-class activity with more explicit questions around the task, and looking back on Version B of the task as it was presented to me in my undergraduate studies, I found a number of ways in which Version A was not useful in reaching the learning and process goals I desired. Version A was an example of a task that was replete with miscues—misleading and miscommunicative elements that undermined students' opportunities to perform and to learn. Examples such as this can be helpful for teachers and teacher educators in learning to design tasks which: (a) use vocabulary that is both understandable and unambiguous, (b) recognize diagrams as either drawings (for specific examples) or figures (to represent a larger class of shapes), and (c) are generally helpful in achieving the teacher's learning goals for the students.

This process of instruction, reflection, and new comprehension is clearly outlined in Shulman's model of pedagogical reasoning and action, but the model falls short in capturing the practice of initial task design and the difficulties that are entailed in making important decisions even at the level of word choice and sentence structure. Perhaps what is needed is a mathematics-specific version of Shulman's model of pedagogical reasoning and action, one that addresses the common aspects of good teaching across various subject matters while attending to specific needs—like task design—of the mathematics community.

Shared Reflection: Going Public

The decisions I made about what to teach and why, the struggles I encountered when lessons did not go as planned or students interpreted my questions differently than I intended, the alterations in lessons and tasks based on my past experiences and current student feedback—these were all a regular part of my personal reflection as a teacher. But when I decided that I wanted to attend to issues of teaching along with those of content in MTH 202, I found that I could address these multiple concerns by sharing those reflections with students.

I originally planned to use shared reflection with preservice teachers as a way to focus on the development of their pedagogical content knowledge. With this approach, I would act as an instructor of pedagogical practice—giving examples of ways in which I transformed my understanding of the content in making it learnable for students. But what I didn't consider was the effect that this process of "going public" with my experiences would have on my own PCK development. More than a mode of instruction, shared reflection became another important stage in my growth as a teacher educator.

Instead of just making decisions about what to teach, I was now having to make decisions about what to tell preservice teachers about those pedagogical decisions. For example, when students responded to my initial trapezoid task by making measurements of the drawing that was provided and not realizing that the shape was intended as an example of a larger family of shapes, I went public the next class meeting, explaining that the task was not a good indicator of what I wanted students to learn from the task or what they should use as future teachers. But instead of telling students what I thought was wrong about the task and what I hoped they would do with it, I decided that they might benefit from questions that were a little more direct and exemplary of the ways I wanted them to interact with the problem. I used some of the statements made by the students in the class, framed as questions, to get at some of the issues of similarity, angle and side length relationships, and applications of other mathematical ideas that many students did not use on their own. I thus had over one hundred possible statements that I could have used, but I had to make decisions about which ones would address the underlying mathematics that I wanted to focus on and which ones would best move the students toward their developing notions of similarity.

This process was helpful to me in sorting through what I thought were the critical issues that I wanted to cover. Instead of opening a class discussion on what interesting findings students may have noticed, I worked to focus their attention on several key issues that I felt were a better fit in the learning trajectory I envisioned for the class.

These were issues that I might have taken up through my own personal reflection, but the knowledge that I would have to justify these reasons for a classroom of future teachers forced me to think harder about the decisions I was making and the reasons behind those decisions.

Shared reflection can be a useful tool for the teacher educator who is interested in attending to issues of both content and pedagogy. In particular, it is a powerful device for helping the teacher educator frame issues that are important in the subject matter of pedagogical content knowledge.

Discussion

Helping preservice teachers begin to establish their pedagogical content knowledge is a worthy goal. Practicing teachers often struggle with making mathematics meaningful for their students and some attention to their own PCK can assist in this endeavor. And while its development naturally builds on lessons learned through experience working with kids, PCK can begin to be addressed when teachers are learning about teaching—most notably when they themselves are struggling with content and learning to make connections within the domain of mathematics.

The teacher educator is also a teacher. His or her experience teaching mathematics to prospective teachers also constitutes a site for the study of PCK development. In this study, I set out to examine the ways in which I, as the teacher educator, could use what I learned about my own PCK development with an eye on attending to preservice teachers' PCK. I found that my own PCK was underdeveloped in many ways—in thinking about how various representations might enable students to learn and understand mathematics, in addressing the importance of reasoning and justification for preservice teachers and their future students, in looking at what is involved in designing and implementing a task, and in considering the importance of students' abilities to generalize from a specific case to a larger class. The development of my own PCK was thus both a model for preservice teachers in thinking about their own PCK development and an important factor in making educational decisions about the best avenues to follow in teaching MTH 202. The shared reflection further focused this development for me as I thought about how much to let students in on my decision-making process and to what extent they should be aware of the complexities of teaching.

Pedagogically Useful Mathematical Understanding

In recent years, some members of the mathematics education community have begun to consider the interplay between subject matter knowledge and pedagogical knowledge from a somewhat different perspective. PCK, as it is widely understood, represents a specialized middle ground between the content and the pedagogy. PCK gives attention to the ways in which a teacher makes his or her content knowledge accessible to students, thus recognizing the importance of the mathematics from the perspective of teaching it. In this new perspective, that relationship is somewhat reversed, understanding that the teaching of the mathematics is an important element in the perspective of understanding the content.

Ball and Bass describe this not as a new knowledge base, but an important subdivision of subject matter knowledge, labeling it *pedagogically useful mathematical understanding* (Ball & Bass, in press). In other cases, people have talked about *knowing* *mathematics for teaching* or *applied mathematics for teaching*. Whichever terminology becomes widely-used, the underlying concept is relatively the same—that the important element is the understanding of the mathematics in ways that are significant for teaching it.

For years, mathematicians have talked about what mathematics is important for specific professions. There are calculus courses that are specifically designed for the engineer, geometry and spatial reasoning courses for the architect, and accounting courses for the business major. There are also mathematics courses for the elementary teacher yet, as many have found, the mathematics that is taught in these courses is not regarded as a unique knowledge base for a specialized profession. Some researchers in mathematics education have begun to address this issue by describing the mathematics that teachers (and most specifically, elementary school teachers) need to know in terms of what is pedagogically useful.

Ball and Bass contend that PCK is an important field of study, but that it does not attend to all of the complexities of the practice of teaching. They argue that since teaching often involves addressing more than one mathematical idea at once and, in many cases, within the context of a new problem or novel situation, no amount of PCK can adequately prepare a teacher for all of the circumstances which they may encounter. For this reason, they argue, attention should be focused more on the underlying mathematics that is involved and helping teachers understand it more completely. Unlike Ma (1999) who describes the importance of teachers developing a "profound understanding of fundamental mathematics," Ball and Bass advocate a profound understanding of mathematics that is pedagogically useful.

My understanding of PCK, however, does not employ such a narrow view of the domain. PCK does not downplay the importance of reasoning nor does it advocate that teachers establish a finite repertoire of strategies and answers. Instead, PCK is an attempt to describe the process that teachers employ in making their content knowledge both teachable and learnable. It recognizes that no two teaching situations are alike, and thus it encourages teachers to develop a flexible knowledge of what it takes to teach the mathematics—for example, by understanding ways that multiple representations of the content can be used and understood, and by recognizing the complexities involved in designing a task to meet the needs of the students at a particular time.

Using PCK As the Lens

This, then, raises the question of what a focus on a teacher educator's pedagogical content knowledge development can distinctly contribute to the field of mathematics education. While it has helped to raise a number of issues for me in my teaching, I must admit that attending to my PCK development was probably not the only avenue I could have taken to realize these issues. It seems as if these concerns could also be addressed—and perhaps more effectively—by centering on the content and making decisions about what understanding is pedagogically useful. Ball and Bass (in press) talk about the importance of *decompression*—a teacher's ability to deconstruct his or her own mathematical knowledge to elements which can be flexibly reorganized and adapted to different educational purposes. Since teachers must contend with students' developing knowledge, they must be prepared to help students unpack the pieces which make up that knowledge. To do so, the teacher must be confident and capable of looking at his or her own knowledge in the same way.

It is important to consider how mathematics pedagogy may be both illuminated and confused by using the lens of PCK. First, by giving attention to transformation—the phase of the pedagogical reasoning and action cycle that centers on PCK development—the teacher educator can increase his or her understanding of mathematics. For me, this occurred in a number of places—like learning to distinguish between mathematical and empirical representations and how these forms of representations relate to the ways in which Shulman and others talk about representations, becoming more aware of the ways in which mathematical language can interact with symbolic and spatial representations, and distinguishing between drawings and figures and learning to use them in appropriate ways when developing tasks. These were issues that may have been raised even without focusing on my PCK development, but because I was using PCK as my framework, it gave me some useful language to talk about the process and to relate my personal experiences with those that are a part of the larger educational research community.

Second, it forced me to consider whether Shulman's model of pedagogical reasoning and action was a useful framework specifically for mathematics and the work that teacher educators do in mathematics education. I found that Shulman's categories were helpful, but also problematic when applied to mathematics—in the ways that the multiple uses of representations could easily be confused and in how task development (a critical element in mathematics instruction) is not distinctly attended to. I also found that the pedagogical reasoning and action cycle is a good model even for teacher educators, although it could be augmented with shared reflection to help move preservice teachers to think of themselves not only as students, but teachers as well.

Limitations

As I recognized from the beginning, the fact that this study was done from the first-person perspective provided me with some special insight to the subject in ways that would not have been possible had the study been done by an outside observer. By the same token, there were some limitations raised by the design of the study.

As much as I tried, it was very difficult to completely remove all subjectivity from the analysis of my data. For example, even though I used pseudonyms for my students when writing about their work in Chapter 4, I knew exactly who they were and what my interactions were with them throughout the semester. In most cases, I would have been able to identify their work by looking at the handwriting alone. This might have been avoided had I analyzed the data after a longer time period had elapsed or if another researcher had chosen student work to analyze, rewritten it in his or her own handwriting, or done some of the analysis of the student work him- or herself. This, however, would have taken away from some of the richness of the data afforded by the design of the study. One of the benefits of doing a self-study is proximity between the subject and the researcher. More details can be gleaned, attitudes and feelings may be recalled, and connections between seemingly disparate ideas might be better realized when the researcher has such an intimate relationship with the data. While this proximity can be a detriment to the effectiveness of the research, this limitation can be lessened with careful acknowledgement of the difficulties from the onset of the study.

As a corollary, this mode of qualitative research carries the inherent problem of sample size. As far as sample sizes go, it can't get much less than n = 1. This can be a limitation of the study if implications are made that the subject is representative of a population and that lessons learned from the study are meant to be generalizable to an entire population. This was never the goal of my study. As with most case studies, my intent was to look closely and deeply at a particular event—in this case, my teaching of a mathematics content course for preservice teachers. The event was also a case of the teacher educator thinking hard about what constitutes his own PCK development in the service of furthering the development of his own students' PCK—a particular event that had not been investigated in the research literature prior to this study. It was meant to raise questions concerning the feasibility and implementation of such a course design and not to imply that any other teacher educators use—or should use—the same approach. It would be extremely helpful, however, to learn of some of the ideas learned from this study can have an impact on other teacher educators and their students.

Another limitation around generalizability concerns the mathematical content of the MTH 202 course described in the study. Since the course centered around geometry, certain issues were naturally raised that contributed to this study that might not have been so critical an issue had the course focused on another area of mathematics. Two such examples come to mind: issues of representation and proof. While multiple forms of representation are common in some other areas of mathematics, geometry seems to be one of the most obvious mathematical subjects in which this issue becomes relevant. One of the representations that played the biggest role in my analysis was the spatial representation—a form that, while not unique to geometry, is hard to separate from it.

Likewise, the symbolic representations were an inherent part of geometry, through the use of formulas for area, perimeter, circumference, volume, and surface area. On the other hand, had the study been done in a course on algebra, it is clear that multiple forms of symbolic representations would have played an important role, but not all non-geometric symbolic forms lead to easily-accessible corresponding spatial representations (e.g., polynomials of order higher than 3 and exponentials).[1] The fact that this course was focused on geometry seemed to enable the study of representations better than other topics would have.

Proof falls even more clearly as a particular focus of geometry than does representation. Formal and informal proofs have long been a part of the study of geometry and were clearly candidates for further analysis when considering the mathematics content of MTH 202. But when proofs are considered a part of the broader study of justification or even mathematical reasoning, all mathematical topics are fair game. However, it is clear that the word "proof" carries with it some negative connotations for a number of preservice teachers within the context of geometry. Issues of justification, then, might be easier to address in a course that is centered around a study of probability, number theory, or some other area of mathematics.

[1] Unless you consider graphs of these functions as a form of spatial representation. I tend to think of graphs as a different category of representation in and of itself.

APPENDICES

APPENDIX A

Class Meetings with Mathematical and Pedagogical Content

Date	Task(s)	Mathematical Content	Pedagogical Content PCK
1/12/99	*Comparing and Scaling* 1.3	• Understanding definition of perimeter and area • Comparing shapes with same area/perimeter • Giving basic formulas (generalizations)	
1/14/99	*Comparing and Scaling* 3.1, 4.1	• Investigating constant area, changing perimeter • Investigating constant perimeter, changing area • Seeing that most "square-like" shape has smallest perimeter for given area, largest area for given perimeter (max/min)	• Understanding some of the history behind the NCTM Standards and mathematics reform in the US • Developing and maintaining good classroom discourse
1/19/99	*Comparing and Scaling* 2.1	• Discovering various estimating methods for area of odd shapes	• Establishing good definitions (e.g., formula vs. equation) • Managing classroom data collection
1/21/99	*Comparing and Scaling* 5.1, 6.1, trapezoid formula derivations	• Finding areas of parallelograms and triangles • Understanding how these area formulas make sense • Subdividing shapes into other shapes to find areas • Looking at one specific reasoning of the trapezoid area formula derivation	• Becoming familiar with some of the commonly used (or mentioned) mathematics curricula (e.g., Saxon, UCSMP) • Understanding "geometry" as more than the content of high school geometry course • Looking at multiple strategies for solving a problem • Using concept maps to assess students' baseline understanding of a subject
1/26/99	Measuring in inches and centimeters, (Quiz 1)	• Measuring and reading centimeters and inches on rulers	• Helping students to be more focused in writing about mathematics—differentiatin g between a learning log and a journal
1/28/99	*Comparing and Scaling* 7.2, 7.4	• Understanding formulas for circumference and area of circles (area as pi radius squares)	• Discussing and using etymologies of mathematical terms

Date	Task(s)	Mathematical Content	Pedagogical Content PCK
2/2/99	Circle area formula derivation, *Stretching and Shrinking* 1.1	• Cutting a circle into smaller and smaller wedges and rearranging them to validate area formula • Developing informal understanding of similarity (lengths, angles, and area) • Identifying corresponding parts of similar shapes	• Helping students learn metric prefixes using whimsical situations • Helping students make connections between mathematical ideas and representations • Using manipulatives to help students make sense of some mathematics
2/4/99	Test 1		
2/9/99	*Stretching and Shrinking* 2.1	• Drawing shapes on coordinate grids • Formalizing understanding of similarity (lengths, angles, and general shape) • Using basic algebraic rules to describe a transformation • Understanding scale factor	• Making sense of new student strategies for solving problems • Understanding the purpose of assessment • Constructing good assessments of student understanding
2/11/99	3 rubber bands – show using dot paper, *Stretching and Shrinking* 2.2, 2.3	• Using coordinate geometry to understand how shapes change with two or three rubber bands • Understanding similarity using ratios of corresponding sides • Using algebraic rules to understand more complex transformations of size change and translations	• Developing alternate representations to help students understand a concept
2/16/99	Isosceles trapezoid observations, (Quiz 2)	• Investigating answers to an open-ended question • Looking at interesting relationships between angles, lengths, etc. of a given shape • Developing hypotheses for general cases from one example (good and bad)	• Providing extensions to questions
2/18/99	*Stretching and Shrinking* 3.2, 3.3	• Understanding how area increases as the square of the scale factor	• Dealing with parental misunderstanding and community backlash to mathematics reform • Using mathematical recreations to develop mathematical understanding • Using various tools to measure and to create circles
2/23/99	Basic conversions and symbols used, *Stretching and Shrinking* 4.3, 4.4	• Looking at basic conversions within systems of measurement • Investigating some real-world applications of similarity and scale factor	• Using student work to summarize class discoveries
2/25/99	Test 2		

Date	Task(s)	Mathematical Content	Pedagogical Content PCK
3/2/99	NCTM Standards, *Stretching and Shrinking* 5.1, 5.2, 5.3	• Using similar triangles to solve various real-world problems	• Reading and applying the NCTM Standards to course's content at different grade levels
3/4/99	Analysis of typical class lesson, *Filling and Wrapping* 1.1, 1.2	• Creating flat patterns for unit cubes and other simple boxes	• Analyzing the typical class lesson for this section of MTH 202 and using it as a model for writing lesson plans
3/16/99	Flat pattern review, *Filling and Wrapping* 1.3	• Reviewing concept of flat patterns • Moving between flat patterns and the boxes that are made from them	• Using "guided discovery" in teaching students
3/18/99	Drawing oblique and isometric perspectives, (Quiz 3)	• Drawing oblique and isometric perspective drawings of various boxes and other simple rectangular shapes	• Utilizing various models for creating lesson plans
3/23/99	*Filling and Wrapping* 2.1, 3.1	• Finding various dimensions for boxes with a given volume • Establishing an informal understanding that a cube is a box with the smallest surface area for a given volume • Connecting surface area/volume relationship to perimeter/area relationship	• Understanding various ways to establish connections within mathematics, between mathematics and other subject areas, and between mathematics and the real world
3/25/99	Connections with geometry, *Filling and Wrapping* 3.3	• Validating a volume formula for prisms • Investigating volumes of various prisms with a given lateral surface area, looking for generalizations	• Finding and discussing ways in which geometry content of MTH 202 connects with other content areas • Being a reflective practitioner
3/30/99	Cylinders – *Filling and Wrapping* 4.2, ACE 13, 9	• Finding volumes and surface areas of cylinders • Connecting the concept of volume of cylinder to the maximizing of prism volumes with a given lateral surface area	• Understanding accuracy of measurement and validating student answers • Looking at mathematics problems from different geometric perspectives
4/1/99	Big ideas in *Investigations*, (Quiz 4)	• Identifying big mathematical ideas in *Investigations*	• Reading geometry and measurement units from *Investigations* curriculum in grade level groups
4/6/99	*Filling and Wrapping* 5.1, 5.2	• Discovering relationships between volumes of cylinders, spheres, and cones with congruent radii and heights • Validating volume formulas for spheres and cones	• Altering tasks for multiple purposes • Maintaining professional development through NCTM teaching journals

Date	Task(s)	Mathematical Content	Pedagogical Content PCK
4/8/99	Revision of grade level big ideas, *Filling and Wrapping* 6.1, 6.2	• Applying idea of scale factor to solids • Understanding that the surface area grows as the square of the scale factor and the volume grows as the cube of the scale factor	• Analyzing teacher texts from *Investigations* to identify big ideas in geometry and measurement
4/13/99	Using literature in math class, *Looking for Pythagoras* 2.1	• Developing informal ways to determine areas of shapes drawn on square dot paper • Establishing that the "surround and subtract" method is an efficient strategy that can be used in all cases	• Finding ways in which mathematical ideas from MTH 202 get played out in children's literature
4/15/99	*Looking for Pythagoras* 2.2, 2.3, (Quiz 5)	• Finding all of the different sized squares that can be drawn on 3 x 3 dot paper grids • Establishing the length of the side of a square as the square root of the area • Understanding irrational numbers • Finding the length of a line segment by relating it to the square that has the segment as a side	• Using group quizzes to assess student learning
4/20/99	*Looking for Pythagoras* 3.1	• Using dot paper to draw squares of various sizes on given line segments • Discovering the Pythagorean Theorem using squares on dot paper	• Teaching students elementary notions of reasoning and proof
4/22/99	Test 3		
4/27/99	Pythagorean Triples	• Finding various whole number triples which satisfy the Pythagorean Theorem • Looking for patterns in Pythagorean Triples to try to develop rules or formulas for generating other Pythagorean Triples	• Using concept maps to assess growth in student understanding
4/29/99	Pick's Theorem	• Looking at ways in which mathematical formulas are developed and/or discovered • Finding a generalization for shapes drawn on dot paper and their areas—Pick's Theorem	• Using formulas with elementary students

APPENDIX B

MTH 202 Grading Policy

Homework will be assigned, but not collected or graded. Five quizzes over the homework and classwork will be given. Each quiz will count 30 points for a total of 150 points. In addition, there will be one take-home projects which will require writing and some research and will be worth 100 points. Students will also be required to turn in their learning logs on a regular basis (about 8 to 10 times during the semester). Learning logs will be evaluated and the entire logbook will be worth 100 points at the end of the semester. There will also be three 100 point tests and a 150 point final exam, for a total of 800 points. The final will cover the entire course. Some quizzes and tests may have a group problem solving component.

Grades will be determined at the end of the course on a total point basis. That is, your quiz and test scores, your project grade, your learning log grades, and your final exam scores will be added together to determine your grade. The percentage of the total number of points that you will have accumulated will be calculated. Grades will be determined as follows:

Points	**Percentages**	**Grade**
716 – 800	90 - 100	4.0
676 – 715	85 - 89	3.5
636 – 675	80 - 84	3.0
596 – 635	75 - 79	2.5
556 – 595	70 - 74	2.0
516 – 555	65 - 69	1.5
476 – 515	60 - 64	1.0
0 – 475	0 - 59	0.0

APPENDIX C

Initial E-Mail to Students

You are presently enrolled in the section of MTH 202 that I am teaching during the spring semester. I thought I would let you know a little about the course before the class starts.

MTH 202 is a geometry course for elementary education majors and this will be the fourth semester that I will have taught this course. As for myself, I am a doctoral student in mathematics education. I am at the stage in my work here at MSU where I am beginning on my dissertation and I have chosen this class as the site for my research.

What this means for you is that the course you are taking is not only going to be a learning experience for you, but it will also be a learning experience for me. However, the things that I will be doing in the course will be no different than what I would be doing if I were not using it for my own research. As a student in my class, you will have the opportunity to choose whether you want to be a subject in my data--and you will not have to make that decision until the end of the semester. This way, no one will have to worry that I would be treating students differently based on whether they wanted to be a part of my study.

The fact that this course is a part of my research does not mean that you will be given extra work to do. As I mentioned before, the course will proceed as it would have if it were not the site for my dissertation.

Now, a little about the class. My section will differ from the other sections of MTH 202 in several ways. First of all, it will focus primarily--but not exclusively--on the mathematics. In addition to the geometry that we cover, we will also be talking a lot about what it takes to TEACH geometry to elementary and middle school students. I feel that this is a critical piece for learning the subject matter and I have approached all of my sections of MTH 202 this way.

To help with this, we will be using a series of books different from the one posted in the book stores (by O'Daffer). You will be asked to purchase a series of student and teacher books that are a part of the Connected Mathematics Project--a reform-based curriculum for upper elementary and middle schools. We will use some of the problems in these books to launch our class discussions and I will often assign other more challenging problems to follow them up. I have ordered these books myself and they can be purchased from me (along with a measuring tool called an angle ruler) in my office for a total of around $35 (the price is yet to be solidified).

And that's about it for now. I wanted to take this opportunity to contact you as early as possible to let you know about the basic structure of the course, the books we will be using, and the fact that I will be collecting data on the class for my dissertation. If you have any questions, feel free to contact me either by email or by calling my office at 432-0054.

I look forward to meeting all of you on Tuesday at (ugh) 8:00am.

-Jeff Wanko

APPENDIX D

Documentation of MTH 202

An important part of my work as a teacher involves examining our class discussions closely as well as looking carefully at what you write to me and to yourselves. Having records of our discussions and copies of your work is invaluable to me as a teacher. These help me to shape the instruction of the class to be responsive to you and also help me to continue to develop my own practice as a teacher. For myself, I write journal entries about each of our class sessions that give me a record of my own thoughts and reflections. I often share what happens in our classes and talk with colleagues about teaching. These kinds of activities are an important part of learning in and from my own teaching.

I do these things because I find that they help me to teach well and to continue to develop my teaching. Although I have been teaching for a number of years now, I find that there is always more to learn. These activities that I describe above are also an important facet of being in a professional community and supporting the learning and development of other teachers. They are the same kinds of things that you will be encouraged to do as an intern—audiotaping your lessons, collecting students' written work, writing journal entries about your class sessions, discussing your work with colleagues—and, hopefully, you will have opportunities, resources, and support to do these kinds of things as a practicing teacher. In many ways, this is an opportunity for you to get a sense of how one teacher whom you get to watch closely (me) draws on these kinds of activities and resources to inform and develop his teaching practice.

The materials we make and collect around this course will only be used for the teaching of this course and CANNOT be used for research or presentations without your consent.

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I understand that the records that are being made of my MTH 202 class—the audiotapes and written documents—are part of the teaching practice of my instructor and cannot be used as data for research or as a part of a formal presentation without my consent. These documents will only be used to support the instruction of this course and the professional development of my instructor. I understand that other teacher educators may be viewing these documents for these purposes during the duration of this course.

Signature Date

APPENDIX E

Informed Consent for Study of MTH 202, Section 1

The purpose of this study is to gain a better understanding of the potential of the materials and approaches utilized in your MTH 202 course to support and facilitate the development of your content knowledge as well as your pedagogical content knowledge. Through this research, I hope to learn more about the challenges and benefits of incorporating more innovative materials, pedagogy, and curriculum to mathematics education of future teachers. Because of this, I would like to analyze data collected in your section of MTH 202. The things I would like to use to conduct the research are:

Audiotapes of classes involving the discussions of the mathematics in the course: I, as a researcher, will study the kinds of activities and interactions that occurred during the class, examining what the students and instructor are doing and talking about. Transcripts of some discussions may be made for the purpose of discourse analysis in the study. I will not identify you by name in these transcripts and pseudonyms will be used.

Copies of your writing and work for the course: Another important source of information about what students and instructors do in the course is to look at students' writing and reflections, students' mathematical work, and instructors' comments. I would like to use students' work to study students' experiences and reactions to different class activities and events, and to analyze what different students are exploring and learning at different points in the semester. These formal analyses are for research purposes only. In the study, everything will be labeled with pseudonyms so that you are not identifiable and the analyses that I conduct will not be used in any connection with grades, placements, or other decisions about your development and learning in your teacher education program. You have the option to provide all or any portion of your work for the purpose of this research.

Some self-studies on learning: In TE 150, students are asked to pick one other class in which they are concurrently enrolled and are asked to do a self-study of their own learning in that class. If you are currently taking TE 150 and have chosen your MTH 202 class as the site for your self-study, I invite you to share your self-study with me for the purposes of my own research. This is done entirely on a voluntary basis and I will not ask for nor look at your self-study until after MTH 202 is completed and final grades have been determined.

If you desire further information about this study, you may call Dr. Sandra Wilcox at 355-1741 or Jeffrey Wanko at 432-0054. Results of this research will be shared with you on request.

- -

Permission to participate and permission to use data collected in MTH 202 for research:

I have read the previous statement and agree to participate in this research by giving my permission for Jeffrey Wanko to analyze this information collected during my MTH 202 class: audiotapes of class sessions and photocopies of my class work. I understand that I may provide all or only a portion (as opposed to all) of my work for research purposes, if I so desire. I also understand that if I am also taking TE 150 and have chosen MTH 202 for my self-study, I may also give permission to use my self-study as additional data for research. I understand that my real name will never be used in any written reports of the research and my responses will be treated with confidentiality to anyone outside of the investigators of the project.

Signature: ______________________________________ Date: ________________

APPENDIX F

Learning Log Prompts

Learning Log Prompt #1 _ Due 1/14/99

1. Write down the first three geometry formulas you think of and tell what they are the formulas for (you will not be graded down if there are mistakes, but try to be accurate). Describe some of the pros and cons for using mathematical formulas. What do you think about using formulas with elementary school students?

Learning Log Prompt #2 _ Due 1/21/99

2. Describe in detail an event from your high school geometry course which is particularly memorable to you. Why do you think this event sticks with you?

Learning Log Prompt #3 - Due 2/2/99

3. One measurement attribute is weight. Some units of measurement of weight are pounds, grams, and ounces. Make a list of as many measurement attributes as you can and beneath each attribute list as many different appropriate units as you can (customary, metric, and any others that may apply). Which of these units do you think a third grader should know and understand and why?

Learning Log Prompt #4 - Due 2/11/99

4. Here is a special isosceles trapezoid. Write down all of the interesting things you can find about this isosceles trapezoid (Figure 51).

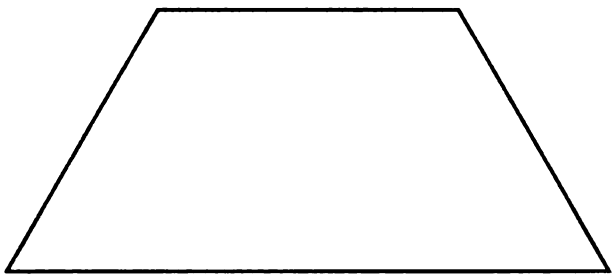

Figure 51: Learning Log Trapezoid

Learning Log Prompt #5 - Due 2/23/99

5. When the midpoints of the sides of a quadrilateral are connected in order, another quadrilateral is formed on the inside. What kinds of quadrilaterals can be formed on the inside? (That is, can the quadrilaterals be squares? rectangles? parallelograms? rhombi? trapezoids? kites? irregular quadrilaterals? etc.) Which original quadrilaterals produce which types of quadrilaterals on the inside? Experiment by drawing many different quadrilaterals and connecting the midpoints of their sides in order. Use these examples to make some general conclusions.

Learning Log Prompt #6 - Due 3/4/99

6. In the draft of the NCTM *Standards 2000*, Standard 3 (Geometry and Spatial Sense) and Standard 4 (Measurement) relate to some of what we have been discussing in MTH 202 this semester. Pick one grade level band (either PreK-2 or 3-5) and describe the things that we have talked about in class that are also recommended to be taught in both Standards 3 and 4 in the Standards 2000.

Learning Log Prompt #7 - Due 3/25/99

7. In the NCTM *Curriculum and Evaluation Standards* (1989), the notion of "mathematical connections" is highlighted. We can think about connections between school mathematics and real world situations, connections between different areas of mathematics, and connections between mathematics and other school subjects. Consider this last approach to connections and give examples of some of the ways in which geometry and measurement connect to other school subjects (possibly including science, social studies, language arts, music, etc.).

Learning Log Prompt #8 - Due 4/13/99

8. In the article "Measuring Up with The Principal's New Clothes" (from *Teaching Children Mathematics*, April 1999, pp. 476-479), a teacher describes how she has taken a children's literature book and developed a mathematics lesson about measurement around it for her third and fourth grade students. Describe another children's literature book or story and the lesson you could develop around it to use in your elementary classroom. Make sure that you identify the grade level and mathematical objective for your lesson, as well as brief description of the lesson you would use (it does not have to be about geometry or measurement, although that connection would be preferable).

Learning Log Prompt #9 - Due 4/20/99

9. Describe in detail one event from your MTH 202 course which is particularly memorable to you. Why do you think this event sticks with you?

Learning Log Prompt # 10 - Due 4/29/99

10. Write down all of the geometry formulas you know and tell what they are the formulas for (do not look these up, do as many as you can from memory and try to be accurate). Describe what you have learned this semester about using formulas and explain some of the power and some of the problems of relying on formulas.

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