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The Moduli Space of Special Lagrangian Submanifolds

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ABSTRACT

The Moduli Space of Special Lagrangian Submanifolds

By

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In this thesis we study the deformation theory of special Lagrangian submanifolds and the singularities of the local moduli space. We show that the moduli space of all infinitesimal special Lagrangian deformations of a smooth, compact, orientable special Lagrangian L in a symplectic manifold with non-integrable almost complex structure is a smooth manifold and its dimension is equal to the dimension of $\mathcal{H}^1(L)$, the space of harmonic 1-forms on L.

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CHAPTER 1

Introduction

The purpose of this chapter is to develop some motivation for the study of special Lagrangian submanifolds and to give some insight into the geometry involved. We will first give the basic definitions in calibrated geometries and study the complex submanifolds of Kähler manifolds as an example. We will then discuss the general concept of special Lagrangian geometries with possible applications.

1.1 Calibrated Geometries

Given a real valued function $f(x) : \mathbf{R} \to \mathbf{R}$ we can describe the local minimum points of f by finding the critical points $(x|\ f'(x) = 0)$ and applying some derivative tests. By comparing the local minimum points we can then describe the global minimum points.

Let X^n be a Riemannian manifold. We can also study a similar problem in the space of immersions of M into X where f is now defined as a volume functional.

Given $\alpha \in H_k(X,\mathbf{Z})$ define the set

$\mathcal{H} = \{$M: compact, oriented submanifolds of $X|\ [M] = \alpha\}$ and the volume functional

$V : \mathcal{H} \to \mathbf{R}$ such that $V(M) = \int_M dvol_M$.

Our goal in this chapter is to study the global minimum points of this functional. The motivation behind this is that a calibrated submanifold is volume minimizing in its homology class. Therefore in order to understand the geometry of this special class of submanifolds we should first understand the geometry of $V(M)$ and in particular the global minimum points of V.

1.1.1 Complex Submanifolds of Kähler Manifolds

In this section we study the global minimum points of the volume functional of a Kähler manifold and show that a complex submanifold has least volume in its homology class by proving the *Wirtinger's Inequality* and the *Federer's Argument.*

First we will give some definitions.

DEFINITION **1.1** : *A symplectic structure on a manifold X^{2n} is a nondegenerate closed 2-form $\omega \in \Omega^2(X)$. A smooth manifold X^{2n} with a symplectic structure is called a symplectic manifold. (e.g $\mathbf{R}^{2n}, \mathbf{S}^2$)*

Nondegeneracy means that each tangent space (T_qX, ω_q) at any point $q \in X$ is a symplectic vector space.

REMARK 1.1 : The symplectic manifold X is necessarily of even dimension because the n-fold wedge product $\omega \wedge \omega \wedge \ldots \wedge \omega$ never vanishes. This also implies that X is orientable.

DEFINITION 1.2 : *An almost complex structure on a* $2n$ *dimensional real manifold is a complex structure* J *on the tangent bundle* TX.

DEFINITION 1.3 : *An almost complex structure* J *on a symplectic manifold is called integrable if and only if it is covariant constant with respect to the Levi-Civita connection of the associated metric* g_J.

DEFINITION 1.4 : *A Kähler manifold is a symplectic manifold* (X, ω) *with an integrable almost complex structure* J. *(e.g* $\mathbf{R}^{2n}, \mathbf{CP}^n$)

Recall that our goal is to study the volume minimizing submanifolds of a Kähler manifold. For this we need two important tools, namely the *Wirtinger's Inequality* and the *Federer's Argument.*

Theorem 1.1 : *(Wirtinger's Inequality)*

Let X^{2n} *be a Kähler manifold and* M^{2m} *be a real, oriented submanifold of* X. *Then for all* $p \in M$,

$$\frac{\omega^m}{m!}|_{T_pM} \leq dvol|_{T_pM}$$

with equality if and only if T_pM *is a complex subspace of* T_pX.

Proof: For any two unit vectors $Y, W \in T_pX$, we can show that

$\omega(Y,W)^2 = < Y, JW >^2$ (compatibility of ω and the metric)

$\leq |Y|^2 |JW|^2$ (by Cauchy-Schwartz inequality)

$= |Y|^2 |W|^2 = 1$ (J preserves the length of the vectors)

and the inequality above will be an equality iff $Y = \pm JW$, in other words iff Y and W span a complex space.

By linear algebra we can show that there exists an oriented, orthonormal basis $e_1, ... e_{2m}$ of $T_p(M)$ such that ω can be written as follows:

$\omega = \sum_{i=1}^{m} \lambda_i \omega_{2i-1} \wedge \omega_{2i}$

where $\lambda_i = \omega(e_{2i-1}, e_{2i})$ for $i = 1, .., m$ and $\omega_1, ..., \omega_{2m}$ are the dual one forms to $e_1, ... e_{2m}$.

With a simple calculation one can show that

$\omega^m = (m!) \lambda_1 ... \lambda_m \omega_1 \wedge ... \wedge \omega_{2m}$.

$|\omega^m| = (m!) \omega_1 \wedge ... \wedge \omega_{2m}$ iff $|\lambda_1 ... \lambda_m| = 1$.

Since we showed earlier that $\omega(Y,W)^2 \leq 1$ and this implies that $|\lambda_i| \leq 1$ for each i. Therefore, $|\lambda_1 ... \lambda_m| = 1$ iff $\lambda_i^2 = \omega(e_{2i-1}, e_{2i})^2 = 1$ for all $i = 1, .., m$. This is equivalent to saying that $e_{2i-1} = \pm J e_{2i}$ for all $i = 1, .., m$ in other words T_pM is a complex space. For more details see [8].

Theorem 1.2 : *(Federer's Argument)*

Let X^{2n} be a Kähler manifold. Let $\Phi : M^{2m} \to X$ be a compact complex submanifold. Then $vol(\Phi) \leq vol(\overline{\Phi})$ where $\overline{\Phi} : \overline{M}^{2m} \to X$ is any real 2m-dimensional submanifold homologous to M rel boundary. (with equality if and only if $\overline{M}^{2m}$ is also complex.)

Proof: Let $\Phi : M \to X$ be a compact complex submanifold and $\overline{\Phi} : \overline{M} \to X$ is any real $2m$-dimensional submanifold homologous to M rel boundary.

Take $\beta : W^{2m+1} \to X$ as the collection of simplices mapping to X such that $M - \overline{M} = \partial W$.

We will first show that $\int_M \Phi^* \omega^m = \int_{\overline{M}} \overline{\Phi}^* \omega^m$.

Since w is closed $dw^m = 0$. Also the pullback and exterior derivative commutes with each other. So we get

$$0 = \int_W \beta^* d\omega^m = \int_W d\beta^* \omega^m$$

$$= \int_{\partial W} \beta^* \omega^m \text{ (by Stoke's theorem).}$$

$$= \int_{M-\overline{M}} \beta^* \omega^m = \int_M \beta^* \omega^m - \int_{\overline{M}} \beta^* \omega^m$$

Since the restrictions of β^* to M and $\overline{M}$ are Φ^* and $\overline{\Phi}^*$, respectively, we get

$$= \int_M \Phi^* \omega^m - \int_{\overline{M}} \overline{\Phi}^* \omega^m = 0$$

Now, we can get the Federer's Argument as follows:

$$vol(M) = \int_M dvol_M = \frac{1}{m!}\int_M \Phi^*\omega^m = \frac{1}{m!}\int_{\overline{M}} \overline{\Phi}^*\omega^m$$
$$\leq \int_{\overline{M}} dvol_{\overline{M}} \text{ (by Wirtinger's Inequality)}$$
$$= vol(\overline{M}).$$

(with equality iff $\overline{M}$ is also complex).

1.1.2 Calibrations

In 1982, Harvey and Lawson extended the fact that complex submanifolds of a Kähler manifold are volume minimizing in their homology classes to the more general context of *calibrated submanifolds.* In their paper they introduced four new examples of calibrated geometries. The first is the special Lagrangian calibration which is a real n form defined on a $2n$ dimensional manifold with holonomy contained in $SU(n)$. The other three are associative, coassociative and Cayley calibrations which occur in specific dimensions. Most of the definitions used here can be found in [7].

Let X be a Riemannian manifold and ϕ be a p-form on X. At each point $x \in X$, the *comass* of ϕ_x is defined as follows:

$||\phi||_x = Sup\{< \phi_x, \xi_x >: \xi_x \text{ is a unit simple p-vector at } x\}$

DEFINITION 1.5 : *A smooth p-form ϕ on a Riemannian manifold X is called a calibration if*

i) ϕ is comass 1.

ii) $d\phi = 0$.

(X, ϕ) is called a calibrated manifold.

Let ϕ be a smooth p-form of comass 1 on X. We will denote the collection of oriented p-planes at $x \in X$ by $G(p, T_x X)$. We can identify this set with the vector space of p-vectors at x. Then we can define $\mathcal{G}(\phi)$ as follows:

$$\mathcal{G}(\phi) = \{ \ \xi_x \in G(p, T_x X) | < \phi, \xi_x >= 1 \}.$$

DEFINITION 1.6 : *A p-dimensional submanifold $S \subset X$ is called a $\mathcal{G}(\phi)$ submanifold if $T_q S \in \mathcal{G}(\phi)$ for all $q \in S$.*

One can also define the calibration as follows: Note that these two definitions are equivalent.

DEFINITION 1.7 : *A calibration is a closed p-form ϕ on a Riemannian manifold X^n such that ϕ restricts to each tangent p-plane of X^n to be less than or equal to the volume form of that p-plane.*

DEFINITION 1.8 : *The submanifolds of X^n for which the p-form ϕ restricts to be equal to the Riemmanian volume form are said to be calibrated by the form ϕ.*

We will use the term *calibrated geometry* for the ambient manifold X, the calibration ϕ, and the collection of submanifolds calibrated by ϕ. Recall that in the previous section we showed that complex submanifolds of a Kähler manifold are volume minimizing in their homology classes, so if ω denotes the Kähler form and if $\phi_p = \frac{\omega^p}{p!}$ then ϕ_p is the calibration and the collection of complex submanifolds are the submanifolds calibrated by ϕ_p.

Next we will give an example:

EXAMPLE **1.1** *:* Take $\omega = dx$ in $\mathbf{R^2}$. We will find the calibrated geometries associated to $\omega = dx$.

The comass of $\omega =$ Sup $\{\omega(e) : |e| = 1, e$ is a vector in $\mathbf{R^2}\}$

We can write e in terms of the basis : $e = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ such that $|e| = \sqrt{a^2 + b^2} = 1$.

Then we get $\omega(e) = a \Rightarrow$ Comass$(\omega) =$ Sup $\{a : |a| \leq 1\}$.

From $\sqrt{a^2 + b^2} = 1$ we see that if comass$= 1$ then $e = \frac{\partial}{\partial x}$.

Therefore the associated 1-dimensional calibrated submanifolds of $\mathbf{R^2}$ will be straight lines parallel to the x-axis.

In section 1.1.1 we showed that the complex submanifolds are volume minimizing in Kähler manifolds by proving Wirtinger's Inequality and the Federer's Argument. One can easily obtain similar properties for calibrated submanifolds:

Let (X, ϕ) be a calibrated manifold.

1) If S is a compact oriented p-dimensional submanifold of X,

$$\int_S \phi \leq Vol(S)$$

with equality if and only if S is a $\mathcal{G}(\phi)$ submanifold.

2) Let S be a $\mathcal{G}(\phi)$ submanifold. Let $\overline{S}$ be any submanifold homologous to S then

$$vol(S) \leq vol(\overline{S})$$

with equality if and only if $\overline{S}$ is also a $\mathcal{G}(\phi)$ submanifold.

These properties imply that the calibrated submanifolds are volume minimizing in their homology classes.

1.1.3 Special Lagrangian Geometries

In this section we will introduce the Special Lagrangian Geometries for $\mathbf{C}^n$. In section 1.2 we will extend these to Calabi-Yau manifolds. Recall that to construct a calibrated geometry we first need a calibration. In our case it is the special Lagrangian calibration which is a real n form defined on a $2n$ dimensional manifold with holonomy contained in $SU(n)$.

We will take the standard coordinates $z_1 = x_1 + ix_{n+1}, ..., z_n = x_n + ix_{2n}$ on $\mathbf{C}^n$. We will also assume that $\mathbf{C}^n$ is equipped with a complex volume form $dz = dz_1 \wedge ... \wedge dz_n$. Here we will examine the real n-form

$$\mu = Re(dz) = \frac{1}{2}(dz + \overline{dz}).$$

REMARK **1.2** : With a simple calculation one can show that this form is closed i.e $d(\mu) = 0$. Note that μ has constant coefficients on $\mathbf{C}^n \cong \mathbf{R}^{2n}$. Also we can show that Re(dz) has comass 1. For more details see [7].

DEFINITION **1.9** : *The real n-form $\mu = Re(dz) = Re(dz_1 \wedge ... \wedge dz_n)$, is the special Lagrangian calibration on $\mathbf{C}^n$.*

DEFINITION **1.10** : *A real n-plane in $\mathbf{C}^n$ is called special Lagrangian if the form $Re(dz)$ restricts to be the volume form.*

This is equivalent to saying that the restriction of Im(dz), the imaginary part of (dz) and the symplectic form ω is zero on a special Lagrangian plane. (See [7])

DEFINITION **1.11** : *An oriented n dimensional submanifold S of $\mathbf{C}^n$ is called a special Lagrangian submanifold if T_qS is a special Lagrangian n plane for all $q \in S$.*

In addition to introducing the special Lagrangian calibration, Harvey-Lawson also discussed several examples in their paper [7]. We will conclude this section with one of these examples.

EXAMPLE **1.2** : Let M_c denote the locus of the equations:

$|z_j|^2 - |z_1|^2 = c_j$, $j = 2, ..., n$ and

$Rez_1...z_n = c_1$ if n is even, or $Imz_1...z_n = c_1$ if n is odd.

Then M_c is a special Lagrangian submanifold of $\mathbf{C}^n$.

1.2 Calabi-Yau Manifolds and Strominger-Yau-Zaslow Conjecture

In this section we will define the special Lagrangian calibration for Calabi-Yau manifolds.

We will first explain why we study the special Lagrangian submanifolds of a Calabi-Yau manifold. In section 1.1.2 we mentioned that the notion of special Lagrangian submanifold was first introduced by Harvey and Lawson [7] as an example of calibrated geometries. In their paper, special Lagrangian submanifolds were studied as volume minimizing submanifolds in a homology class. In the last few years, developments in mirror symmetry and especially the Strominger-Yau-Zaslow conjecture have focused attention on special Lagrangian submanifolds again. Strominger, Yau and Zaslow [13] proposed a geometric construction of mirror manifolds via special Lagrangian tori fibrations. According to their proposal, there should be a close connection between the deformation theory of special Lagrangian submanifolds and the mirror symmetry. Recently, special Lagrangian submanifolds have been studied by several other authors [1], [3], [4], [6], [9], [11].

DEFINITION **1.12** : *A Calabi-Yau manifold X is a Kähler manifold of complex dimension n with a covariant constant holomorphic n-form. (equivalently it is a Riemannian manifold with holonomy contained in $SU(n)$).*

One other equivalent definition for Calabi-Yau manifolds is that they are Kähler manifolds with first Chern class $c_1 = 0$

Calabi-Yau manifolds are equipped with a Kähler 2-form ω, an almost complex structure J which is tamed by ω, the compatible Riemannian metric g and a nowhere vanishing holomorphic $(n,0)$-form $\xi{=}\mu + i\beta$, where μ and β are real valued n-forms.

DEFINITION **1.13** : *An n-dimensional submanifold $L \subseteq X$ is special Lagrangian if L is Lagrangian (i.e. $\omega|_L \equiv 0$) and $Im(\xi)$ restricts to zero on L. Equivalently, $Re(\xi)$ restricts to be the volume form on L with respect to the induced metric.*

We can ask the following question: Do special Lagrangian submanifolds play any role in the construction of the mirror pair ? The answer is not known yet but Strominger, Yau and Zaslow [13] proposed a geometric construction of mirror manifolds via special Lagrangian tori fibrations. Roughly, they argued that any 3-dimensional Calabi-Yau manifold X with a mirror pair $\overline{X}$ should contain a SL 3-torus.

One can state the conjecture in the simplest way as follows:

Strominger-Yau-Zaslow Conjecture: *If X and $\overline{X}$ are mirror pairs of Calabi-Yau 3-folds, then there should exist a special Lagrangian 3-torus fibration $f : X \to B$ (with some singular fibres), such that $\overline{X}$ is obtained by finding some suitable compactification of the dual of this fibration.*

1.3 Organization

The thesis is organized as follows. In chapter 2 we review the deformation theory of special Lagrangian submanifolds in a Calabi-Yau manifold and explain R.C.McLean's result. We will also extend this result to symplectic manifolds with non-integrable almost complex structure.

CHAPTER 2

Deformations of Special Lagrangian Submanifolds

In this chapter we will prove that the moduli space of all infinitesimal deformations of a smooth compact special Lagrangian submanifold L in a symplectic manifold X within the class of special Lagrangian submanifolds is a smooth manifold of dimension $b_1(L)$, the first Betti number of L.

2.1 Deformations in a Calabi-Yau Manifold

In [10], McLean proved the following theorem which says that the moduli space of nearby submanifolds of a smooth compact special Lagrangian submanifold L in a Calabi-Yau manifold X is a smooth manifold and its dimension is equal to the dimension of $\mathcal{H}^1(L)$, the space of harmonic 1-forms on L. In what follows, X will denote a $2n$-dimensional Calabi-Yau manifold with a Kähler 2-form ω and a nowhere vanishing holomorphic $(n, 0)$-form $\xi = \mu + i\beta$, where μ and β are real valued n-forms.

Theorem 2.1 : *The moduli space of all infinitesimal deformations of a smooth, compact, orientable special Lagrangian submanifold L in a Calabi-Yau manifold X within the class of special Lagrangian submanifolds is a smooth manifold of dimension equal to dim($\mathcal{H}^1(L)$).*

REMARK **2.1** : R.C.McLean's theorem is a tool to show the existence of non-explicit examples. It says that given one compact special Lagrangian submanifold L, there is a local finite dimensional moduli space of deformations whose dimension is equal to the first Betti number $b_1(L)$. Hence starting with a set of real points (special Lagrangian submanifold) in a suitable Calabi-Yau and deforming one can assert the existence of compact special Lagrangian submanifolds.

Proof of Theorem: For a small normal vector field V we define the deformation map as follows,

$$F : \Gamma(N(L)) \rightarrow \Omega^2(L) \bigoplus \Omega^n(L)$$

$$F(V) = ((\exp_V)^*(-\omega), (\exp_V)^*(Im(\xi))$$

The deformation map F is the restriction of $-\omega$ and $Im(\xi)$ to L_V and then pulled back to L via $(\exp_V)^*$. Here $N(L)$ denotes the normal bundle of L, $\Gamma(N(L))$ the space of sections of the normal bundle, and $\Omega^2(L)$, $\Omega^n(L)$ denote the differential 2-forms and n-forms, respectively. Also, $\exp_V$ represents the exponential map which gives a diffeomorphism of L onto its image L_V in a neighborhood of 0.

Recall that the normal bundle $N(L)$ of a special Lagrangian submanifold is isomorphic to the cotangent bundle $T^*(L)$. Thus, we have a natural identification of normal vector fields to L with differential 1-forms on L.

Furthermore, since L is compact we can identify these normal vector fields with nearby submanifolds. Under these identifications, it is then easy to see that the kernel of F will correspond to the special Lagrangian deformations.

We compute the linearization of F at 0,

$$dF(0) : \Gamma(N(L)) \to \Omega^2(L) \bigoplus \Omega^n(L)$$

where

$$dF(0)(V) = \frac{\partial}{\partial t} F(tV)|_{t=0} = \frac{\partial}{\partial t}[\exp^*_{tV}(-\omega), \exp^*_{tV}(\beta)]|_{t=0}$$

$=[-(\mathcal{L}_V\omega)|_L, (\mathcal{L}_V\beta)|_L]$ where $\mathcal{L}_V$ denotes the Lie derivative.

Using the Cartan Formula, we get:

$$= (-(i_V d\omega + d(i_V\omega))|_L, (i_V d\beta + d(i_V\beta))|_L)$$

$$= (-d(i_V\omega)|_L, (d(i_V\beta))|_L) \ (\omega \text{ and } \beta \text{ are closed forms})$$

$$= (dv, d * v)$$

where i_V represents the interior derivative and v is the dual 1-form to the vector field JV with respect to the induced metric. For the details of local calculations of $d(i_V\omega)$ and $d(i_V\beta)$ see [10].

Hence $dF(0)(V) = (dv, d * v) = (dv, *d^*v)$.

Next, we need to show that $dF(0)(V) = (dv, d*v) = (dv, *d^*v)$ is onto. McLean showed this by proving that F is a map from $\Gamma(N(L))$ to exact 2-forms and exact n-forms. The proof goes as follows:

The image of F lies in the closed 2-forms and closed n-forms, since F is the pullback of the closed forms ω and β. By replacing V with tV we can see that $exp : L \to X$ is homotopic to the inclusion $i : L \to X$ and since we work with closed forms exp_V^* and i^* give the same map in cohomology. Then $[exp_V^*(\beta)] = [i^*(\beta)] = [\beta|_L] = 0$ and $[exp_V^*(\omega)] = [i^*(\omega)] = [\omega|_L] = 0$ since L is special Lagrangian. So the forms in the image of F are cohomologous to zero and they are exact forms.

Now, one can easily show that for any given exact 2-form a and exact n-form b we can solve for v that satisfies the equation $dv = a$ and $d{*}v = b$. Hence $dF(0)(V)$ is surjective and after completing the space of differential forms with appropriate norms and using the Banach space implicit function theorem and elliptic regularity we can conclude that $F^{-1}(0,0)$ is a smooth manifold with tangent space at 0 equal to $\mathcal{H}^1(L)$, [10].

2.2 A New Interpretation of Special Lagrangian Submanifolds

In this section our aim is to improve R.C.McLean's result which is explained in section 2.1 to symplectic manifolds. Precisely, we want to show that the moduli space of all infinitesimal special Lagrangian deformations of L in a symplectic manifold with non-integrable almost complex structure is also a smooth manifold of dimension $b_1(L)$. We will prove this by extending the parameter space of special Lagrangian deformations, in other words by using a modified definition of special Lagrangian submanifolds.

First, we will explain why we need to change the classical definition of special Lagrangian submanifolds in terms of the calibrated form. Recall that McLean showed the surjectivity of the linearized operator in the Calabi-Yau case by a cohomology argument. This is possible because in the Calabi-Yau case the complex $(n,0)$-form ξ is closed but when we try to extend this result to symplectic manifolds with non-integrable almost complex structure (i.e ξ is no longer closed) we cannot use the same cohomology argument. So we have to seek some other ways to prove the surjectivity. One way is to change the deformation map slightly and that's the main reason for us to use a modified definition of special Lagrangian submanifolds.

Recall that a Lagrangian submanifold L of a Calabi-Yau manifold is special Lagrangian if $Im(\xi)|_L \equiv 0$, where ξ is a nowhere vanishing, closed, complex $(n,0)$-form. In our case, we will drop the assumption that ξ is closed (i.e $d\xi \neq 0$) and introduce a new parameter θ for the deformations. Then the condition $Im(\xi)|_L \equiv 0$ will be replaced by $Im(e^{i\theta}\xi)|_L \equiv 0$ in the definition of special Lagrangian submanifolds.

In what follows, X will denote a $2n$-dimensional symplectic manifold with symplectic 2-form ω, an almost complex structure J which is tamed by ω, the compatible Riemannian metric g and a nowhere vanishing complex valued $(n,0)$-form $\xi = \mu + i\beta$, where μ and β are real valued n-forms. We say ξ is *normalized* if the following condition holds:

$$(-1)^{n(n-1)/2}(i/2)^n \xi \wedge \bar{\xi} = \omega^n/n!$$

So far, all the studies have focused on Calabi-Yau manifolds where this complex form is closed, but for our purposes we need a globally defined $(n, 0)$-form which is not closed on X. There are many non-closed forms on a manifold, and one can easily construct them; for example by multiplying a given closed form with a non-constant function. The fact that ξ is not closed implies that the associated almost complex structure J on the tangent bundle TX is non-integrable. In fact, the integrability of the almost complex structure determined by ξ can be replaced by a weaker condition than $d\xi = 0$ ([4]).

In special Lagrangian calibrations, there is an additional term $e^{i\theta}$, where for each fixed angle θ we have a corresponding form $e^{i\theta}\xi$ and its associated geometry. θ is called the *phase factor* of the calibration and this in fact will be our new parameter in the deformation of special Lagrangian submanifolds. In order to enlarge our parameter space we will allow θ to vary along the deformations. We will also assume that the initial value of θ is 0 to avoid the appearance of additional constants.

Taking the new parameter θ into consideration, one can slightly modify the definition of a special Lagrangian submanifold in a symplectic manifold.

DEFINITION **2.1** : *An n-dimensional submanifold $L \subseteq X$ is special Lagrangian if L is Lagrangian (i.e. $\omega|_L \equiv 0$) and $Im(e^{i\theta}\xi)$ restricts to zero on L, for some $\theta \in \mathbf{R}$. Equivalently, $Re(e^{i\theta}\xi)$ restricts to be the volume form on L with respect to the induced metric.*

2.3 Deformation Theory in a Symplectic Manifold

Under the given assumptions, we are ready to state our theorem:

Theorem 2.2 : *The moduli space of all infinitesimal deformations of a smooth, compact, orientable special Lagrangian submanifold L in a symplectic manifold X within the class of special Lagrangian submanifolds is a smooth manifold of dimension $b_1(L)$, the first Betti number of L.*

REMARK **2.2** : In [10], McLean proved the same theorem for Calabi-Yau manifolds, i.e, for $d\xi = 0$. This is the case where the almost complex structure is integrable.

Proof of Theorem. For a small vector field V and a scalar $\theta \in \mathbf{R}$, we define the deformation map as follows,

$$F : \Gamma(N(L)) \times \mathbf{R} \rightarrow \Omega^2(L) \bigoplus \Omega^n(L)$$

$$F(V, \theta) = ((\exp_V)^*(-\omega), (\exp_V)^*(Im(e^{i\theta}\xi))$$

The deformation map F is the restriction of $-\omega$ and $Im(e^{i\theta}\xi)$ to L_V and then pulled back to L via $(\exp_V)^*$ as in [13]. Here $N(L)$ denotes the normal bundle of L, $\Gamma(N(L))$ the space of sections of the normal bundle, and $\Omega^2(L)$, $\Omega^n(L)$ denote the differential 2-forms and n-forms, respectively. Also, $\exp_V$ represents the exponential map which gives a diffeomorphism of L onto its image L_V in a neighbourhood of 0.

Recall that the normal bundle $N(L)$ of a special Lagrangian submanifold is isomorphic to the cotangent bundle $T^*(L)$. Thus, we have a natural identification of normal vector fields to L with differential 1-forms on L.

Furthermore, since L is compact we can identify these normal vector fields with nearby submanifolds. Under these identifications, it is then easy to see that the kernel of F will correspond to the special Lagrangian deformations.

We compute the linearization of F at (0,0),

$$dF(0,0) : \Gamma(N(L))\times\mathbf{R}\rightarrow \Omega^2(L)\bigoplus\Omega^n(L)$$

where

$$dF(0,0)(V,\theta) = \frac{\partial}{\partial t}F(tV,s\theta)|_{t=0,s=0} + \frac{\partial}{\partial s}F(tV,s\theta)|_{t=0,s=0}$$

Therefore,

$$\frac{\partial}{\partial t}F(tV,s\theta)|_{t=0,s=0} + \frac{\partial}{\partial s}F(tV,s\theta)|_{t=0,s=0}$$

$$= \frac{\partial}{\partial t}[\exp^*_{tV}(-\omega), \exp^*_{tV}(Im((\cos(s\theta) + i\sin(s\theta))(\mu + i\beta))]|_{t=0,s=0}$$

$$+\frac{\partial}{\partial s}[\exp^*_{tV}(-\omega), \exp^*_{tV}(Im((\cos(s\theta) + i\sin(s\theta))(\mu + i\beta))]|_{t=0,s=0}$$

$$=[-(\mathcal{L}_V\omega)|_L, (\mathcal{L}_V\mu)|_L \cdot \sin(s\theta)|_{s=0} + (\mathcal{L}_V\beta)|_L \cdot \cos(s\theta)|_{s=0}$$

$$+ ((\exp^*_{tV}\mu)\cdot\cos(s\theta)\cdot\theta - (\exp^*_{tV}\beta)\cdot\sin(s\theta)\cdot\theta)|_{t=0,s=0}]$$

$$= [-(\mathcal{L}_V\omega)|_L, \mathcal{L}_V\beta|_L\cdot\cos(s\theta)|_{s=0} + ((\exp^*_{tV}\mu)\cdot\cos(s\theta)\cdot\theta)|_{t=0,s=0}]$$

Here $\mathcal{L}_V$ represents the Lie derivative and one should notice that $\exp^*_{tV}\mu|_{t=0}$ is just the restriction of μ to L which is equal to 1 by our assumption that the initial value of θ is 0.

Also, on a compact manifold L, top dimensional constant valued forms correspond to $\mathcal{H}^n(L)$, the space of harmonic n-forms on L and there is a natural identification between the reals and harmonic n-forms. Therefore, $\theta = \theta.dvol$ will play the role of a harmonic n-form in our calculations.

Using the Cartan Formula, we get:

$$= (-(i_V d\omega + d(i_V\omega))|_L, (i_V d\beta + d(i_V\beta))|_L + \theta)$$

$$= (-d(i_V\omega)|_L, (i_V d\beta + d(i_V\beta))|_L + \theta)$$

$$= (dv, \zeta + d * v + \theta), \text{ where } \zeta = i_V(d\beta)|_L$$

Here i_V represents the interior derivative and v is the dual 1-form to the vector field V with respect to the induced metric. For the details of local calculations of $d(i_V\omega)$ and $d(i_V\beta)$ see [10].

Hence $dF(0,0)(V,\theta) = (dv, \zeta + d * v + \theta)$.

Let $x_1, x_2, ..., x_n$ and $x_1, x_2, ..., x_{2n}$ be the local coordinates on L and X, respectively. Then for any given normal vector field $V = (V_1 \frac{\partial}{\partial x_{n+1}}, ..., V_n \frac{\partial}{\partial x_{2n}})$ to L we can show that

$\zeta = i_V(d\beta)|_L = -n(V_1 \cdot g_1 + ... + V_n \cdot g_n)$dvol where g_i $(0 < i \leq n)$ are combinations of coefficient functions in the connection-one forms.

One can also decompose the n-form $\zeta = da + d^*b + h_2$ by using Hodge Theory and because ζ is a top dimensional form on L, ζ will be closed and the equation becomes $dF(0,0)(V,\theta) = (dv, da + d * v + h_2 + \theta)$ for some $(n-1)$-form a and harmonic n-form h_2.

The harmonic projection for $\zeta = -n(V_1.g_1 + ... + V_n.g_n)\text{dvol}$ is $(\int_L -n(V_1.g_1 + ... + V_n.g_n)\text{dvol})\text{dvol}$ and therefore one can show that $da = -n(V_1.g_1 + ... + V_n.g_n)\text{dvol}+(n\int_L (V_1.g_1 + ... + V_n.g_n)\text{dvol})\text{dvol}$ and $h_2 = (-n\int_L (V_1.g_1 + ... + V_n.g_n)\text{dvol})\text{dvol}$.

REMARK **2.3** : One should note that the differential forms a and h_2 both depend on V and therefore should be explored carefully in order to understand the deformations of special Lagrangian submanifolds.

After completing the space of differential forms with appropriate norms, we can consider F as a smooth map from $C^{1,\alpha}(\Omega^1(L))\times\mathbf{R}$ to $C^{0,\alpha}(\Omega^2(L))$ and $C^{0,\alpha}(\Omega^n(L))$, where

$$C^{k,\alpha}(\Omega) = \{f \in C^k(\Omega)|\ [D^\gamma f]_{\alpha,\Omega} < \infty, |\gamma|\leq k\} \text{ and}$$

$$[f]_{\alpha,\Omega} = \underset{x,y\in\Omega,\ x\neq y}{Sup} \frac{dist(f(x),f(y))}{(dist(x,y))^\alpha} \text{ in } \Omega.$$

The Implicit Function Theorem says that $F^{-1}(0,0)$ is a manifold and its tangent space at $(0,0)$ can be identified with the kernel of dF.

$(dv)\bigoplus(\zeta + d * v + \theta) = (0,0)$ implies
$dv = 0$ and $\zeta + d * v + \theta = da + d * v + h_2 + \theta = 0$.

The space of harmonic n-forms $\mathcal{H}^n(L)$, and the space of exact n-forms $d\Omega^{n-1}(L)$, on L are orthogonal vector spaces by Hodge Theory. Therefore, $dv = 0$ and $da+d*v+h_2+\theta = 0$ is equivalent to $dv = 0$ and $d*v+da = 0$ and $h_2+\theta = 0$.

One can see that the special Lagrangian deformations (the kernel of dF) can be identified with the 1-forms on L which satisfy the following equations:

(i) $dv = 0$

(ii) $d * (v + \kappa(v)) = 0$

(iii) $h_2 + \theta = 0$.

Here, $\kappa(v)$ is a linear functional that depends on v and h_2 is the harmonic part of ζ which also depends on v. These equations can be formulated in a slightly different way in terms of decompositions of v and $*a$.

If $v = dp + d^*q + h_1$ and $*a = dm + d^*n + h_3$ then we have

(i) $dd^*q = 0$

(ii) $\Delta(p \pm m) = 0$

(iii) $h_2 + \theta = 0$.

This formulation of the solutions will help us to prove the surjectivity of the linearized operator without using $\kappa(v)$.

REMARK **2.4** : When $\theta = C$, the infinitesimal deformations of θ give no additional special Lagrangian deformations simply because there cannot be two different harmonic representatives in the same cohomology class. Therefore, one can obtain McLean's result by fixing $\theta = C$ along the deformations for some constant C and since $d\beta|_L = 0$ in the integrable case, $da = 0$ and $h_2 = 0$. Hence the deformations correspond to 1-forms which satisfy the equations $dv = 0$ and $d * v + \theta = 0$.

Next, we need to show that the deformation theory of special Lagrangian submanifolds is unobstructed. In order to use the implicit function theorem, we need to show that the linearized operator is surjective at $(0,0)$.

Recall that the deformation map,

$F : \Gamma(N(L)) \times \mathbf{R} \to \Omega^2(L) \bigoplus \Omega^n(L)$

is defined as follows:

$F(V,\theta) = ((\exp_V)^*(-\omega), (\exp_V)^*(Im(e^{i\theta}\xi)).$

Even though $Im(e^{i\theta}\xi)$ is not closed on the ambient manifold X, the restriction of this differential form is a top dimensional form on L, and therefore it will be closed on L. On the other hand, ω is the symplectic form which is by definition closed on X. Therefore, the image of the deformation map F lies in the closed 2-forms and closed n-forms.

At this point we will investigate the surjectivity for ω and $Im(e^{i\theta}\xi)$ separately.

We have the following diagrams for $dF = dF_1 \bigoplus dF_2$ with natural projection maps $proj_1$ and $proj_2$:

$dF_1 : \Gamma(N(L)) \xrightarrow{d} \Omega^2(L) \xrightarrow{proj_1} d\Omega^1(L)$

and,

$dF_2 : \Gamma(N(L)) \times \mathbf{R} \xrightarrow{d*(1+\kappa)+\theta} \Omega^n(L) \xrightarrow{proj_2} d\Omega^{n-1}(L) \bigoplus \mathcal{H}^n(L)$

We will show that the maps dF_1 and dF_2 are onto $d\Omega^1(L)$ and $d\Omega^{n-1}(L) \bigoplus \mathcal{H}^n(L)$, respectively.

Therefore, for any given exact 2-form x and closed n-form $y = u + z$ in the image of the deformation map (here u is the exact part and z is the harmonic part of y), we need to show that there exists a 1-form v and a constant θ that satisfy the equations,

(i) $dv = x$

(ii) $d * (v + \kappa(v)) = u$

(iii)$h_2 + \theta = z$.

alternatively, we can solve the following equations for p, q and θ.

(i) $dd^* q = x$

(ii) $\Delta(p \pm m) = *u$ (Here, the star operator $*$ is defined on L)

(iii) $h_2 + \theta = z$.

For (i), since x is an exact 2-form we can write $x = d(dr + d^* s + \text{harmonic form})$ by Hodge Theory. Then one can solve (i) for q by setting $q = s$.

For (ii), since $\Delta m = d^* dm = d^* * a = *d * *a = \pm * da$,

$\Delta(p \pm m) = \Delta p \pm \Delta m = \Delta p \pm *da$ (here a depends on p)

$= \Delta p \pm (-n(V_1.g_1 + ... + V_n.g_n) + (n\int_L (V_1.g_1 + ... + V_n.g_n)\text{dvol})) = *u$

Since $V = (V_1, ..., V_n)$ is the dual vector field of the one form $v = dp + d^* q + h_1$ we can write the equation above as

$\Delta p \pm (-n(v \cdot g) + (n\int_L (v \cdot g)\text{dvol})) = *u$

$= \Delta p \pm (-n(dp + d^* q + h_1) \cdot g + (n\int_L (dp + d^* q + h_1) \cdot g\text{dvol})) = *u$

where $v \cdot g$ represents the action of the one form v on the vector field $g = (g_1, .., g_n)$

and $n\int_L (dp + d^*q + h_1 \cdot g)\text{dvol}$ is the harmonic projection of $-n(dp + d^*q + h_1) \cdot g$.

Then we get

$$\Delta p \pm n(-(dp \cdot g) + \int_L dp \cdot g\text{dvol}) = *u \mp n[-(d^*q + h_1) \cdot g + \int_L (d^*q + h_1) \cdot g\text{dvol}].$$

For simplicity we put $*u \mp n[-(d^*q + h_1) \cdot g + \int_L (d^*q + h_1) \cdot g\text{dvol}] = h$. Since $\int_L * u = 0$ and $\int_L (d^*q + h_1) \cdot g\text{dvol}$ is the harmonic projection of $(d^*q + h_1) \cdot g$, we get $\int_L h = 0$.

Since L is a compact manifold without boundary, by Leibniz Rule, $\int_L dp \cdot g \text{ dvol} = -\int_L p \cdot divg \text{ dvol}$ and the equation becomes

$$\Delta p \pm n(-(dp \cdot g) - \int_L p \cdot divg \text{ dvol}) = h.$$

Then by adding and subtracting p from the equation

$$(\Delta - Id)p = [\pm n(-(dp \cdot g) - \int_L p \cdot divg \text{ dvol}) - p + h] \text{ and}$$

$$p = (\Delta - Id)^{-1}[.....]p + \overline{h} = \mathcal{K}(p) + \overline{h}, \text{ where } \overline{h} = (\Delta - Id)^{-1}h.$$

and since $||(\Delta - Id)^{-1}\int_L p \cdot divg||_{L^1_1} \leq C|\int_L p \cdot divg| \leq C||p||_{L^2}$, $\mathcal{K}(p)$ is a compact operator which takes bounded sets in L^2 to bounded sets in L^1_1. Also note that we assumed here $1 \notin spec\Delta$, and if this is not the case then we can modify the above argument by adding and subtracting λp, $\lambda \notin spec\Delta$ from the equation.

Next we will show that the set of solutions of the equation $\Delta p \pm n(-(dp \cdot g) - \int_L p \cdot divg \text{ dvol}) = 0$ is constant functions and therefore of dimension 1. Note that this set of solutions also satisfy the equation $(Id - \mathcal{K})(p) = 0$

Note that $\int_L p \cdot divg$ dvol is a constant which depends on p. We denote this as $C(p)$. At maximum values of p, Δp will be negative which implies that $C(p) \leq 0$ and at minimum values of p, Δp will be positive which implies that $C(p) \geq 0$ so $C(p)$ should be zero. Then the maximum principle holds for the equation $\Delta p \pm n(-(dp \cdot g)) = 0$ and since L is a compact manifold without boundary the solutions of this equation are constant functions. Hence the dimension of the kernel of $(Id - \mathcal{K})$ is one.

Next we find the kernel of $(Id - \mathcal{K}^*)$.

$$\int_L (\Delta p \pm n(-(dp \cdot g) - \textstyle\int_L p \cdot divg))q(y)dy$$
$$= \int_L p\Delta q(y) \pm n \int_L -(dp \cdot g)q(y)dy - n \int_L (\textstyle\int_L p \cdot divg)q(y)dy$$
$$= \int_L p(y)\Delta q \pm n \int_L +(pdiv(g \cdot q)(y)dy - n \int_L p(x) \cdot divg(x) \textstyle\int_L q(y)dydx$$
$$= \int_L p(y)\Delta q \pm n \int_L +(pdiv(g \cdot q)(y)dy - n \int_L p(y) \cdot divg(y) \textstyle\int_L q(x)dxdy$$
$$= \int_L p(y)(\Delta q \pm n(+div(g \cdot q) - divg \textstyle\int_L q(x)dx)dy$$

Since we assumed that $1 \notin spec\Delta$, $\dim \ker(Id - \mathcal{K}^*)(\Delta - Id) = \dim \ker\,(Id - \mathcal{K}^*)$ and the kernel of $(Id - \mathcal{K}^*)$ is equivalent to the solution space of the equation

$$\Delta q \pm n(+div(g \cdot q) - divg \textstyle\int_L q(x)dx) = 0$$

By Fredholm Alternative, [2] the dimension of this kernel is 1 and one can check that a constant function $q = 1$ satisfies this equation, therefore the kernel consists of constant functions. Moreover these functions satisfy the compatibility condition $\int h.q = 0$.

Then by Fredholm Alternative we can conclude the existence of solutions of the equation

$$\Delta p \pm (-n(V_1.g_1 + ... + V_n.g_n) + (n\int_L (V_1.g_1 + ... + V_n.g_n)\text{dvol})) = *u$$

(iii) is straightforward.

The only thing remaining is to show that the image of the deformation map F_1 lies in $d\Omega^1(L)$ and the image of F_2 lies in $d\Omega^{n-1}(L) \bigoplus \mathcal{H}^n(L)$.

For ω, we can follow the same argument as in [13]. Since $\exp_V : L \to X$ is homotopic to the inclusion $i : L \to X$, $\exp_V^*$ and i^* induce the same map in cohomology. Thus, $[\exp_V^*(\omega)] = [i^*(\omega)] = [\omega|_L] = 0$. So the forms in the image of F is cohomologous to zero. This is equivalent to saying that they are exact forms.

For $Im(e^{i\theta}\xi)$, we cannot follow the same process, because it is not a closed form on the ambient manifold X and therefore does not represent a cohomology class. But by our construction of our deformation map, it is obvious that the image lies in $d\Omega^{n-1}(L) \bigoplus \mathcal{H}^n(L)$.

One can find the dimension of this manifold by comparing the operators $d + *d^*(v)$ and $d + *d^*(v + \kappa(v))$. Since $\zeta = i_V(d\beta)|_L = -n(V_1.g_1 + ... + V_n.g_n)\text{dvol}$ it is easy to see that the extra term $*d^*(\kappa(v))$ contains no derivatives of v and this implies that the linearized operators $d + *d^*(v)$ and $d + *d^*(v + \kappa(v))$ have the same leading term. Also it is known that the index of an elliptic operator is stable under lower order perturbations. Since the dimension of the kernel of $d + *d^*$ is $b_1(L) + 1$ and the dimension of its cokernel is 1 as a map from $\Gamma(N(L)) \times \mathbf{R} \to$

$d\Omega^1(L) \bigoplus d\Omega^{n-1}(L) \bigoplus \mathcal{H}^n(L)$, we can conclude that both the index of $d + *d^*(v)$ and $d + *d^*(v + \kappa(v))$ are equal to $b_1(L)$. Hence the dimension of tangent space of special Lagrangian deformations in a symplectic manifold is also $b_1(L)$, the first Betti number of L.

Therefore, dF is surjective at $(0, 0)$ and by infinite dimensional version of the implicit function theorem and elliptic regularity, the moduli space of all infinitesimal deformations of L within the class of special Lagrangian submanifolds is a smooth manifold and has dimension $b_1(L)$.

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