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TODDLERS' SYMBOLIZING
AND ITS MATHEMATICAL POTENTIAL

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Helene Alpert Furani

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TODDLERS' SYMBOLIZING

AND ITS MATHEMATICAL POTENTIAL

By

Helene Alpert Furani

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ABSTRACT

TODDLERS' SYMBOLIZING AND ITS MATHEMATICAL POTENTIAL

By

Helene Alpert Furani

This dissertation provides a detailed, ethnographic account of three toddlers' symbolizing. Gathered through participant-observation in home settings, data include the words, gestures and play of three boys between 16 and 22 months of age. The toddlers' activity around naming experience, symbolizing imagined situations and playing with systems of symbols is described and analyzed through a mathematical lens. The toddlers' symbolizing then serves as a springboard for exploring mathematical symbolizing and commonalties between the two. Implications are made with respect to such issues as the action based origins of symbolizing, the 'symbolic continuum,' varieties of symbolic attention and expression, the role of abstraction, the role of playfulness and the impact of individual differences.

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Acknowledgements

I wish to begin by thanking the toddlers of this study for being the joyful and brilliant souls they are, for making me their playmate, for giving me the opportunity to learn so much, and without whom none of this would have been possible. I am also deeply greatful to their parents, Jim Dearing, Joe Eisenmann, Beth Herbel-Eisenmann, Sam Larson, Mary McVee and Jian Zhang for their friendship and for allowing me access to their children, homes and selves and assisting me as co-researchers of their children's minds.*

Next, I wish to thank the faculty and students who have helped me in so many ways to reach this point. In their courses, Bill Rosenthal, Glenda Lappan and John P. (Jack) Smith III all encouraged me to pursue the interests that led to my dissertation research and eventually joined my dissertation committee. I was so very fortunate to have David Pimm join my committee as my dissertation director after the study was underway. He helped to shape the work in key ways that I believe never would have happened without him. Helen Featherstone and Sandra Crespo graciously joined my committee at key junctures and provided valuable input both before and after their formal participation. Helen took on the role of chair once David could no longer officially serve due to his move to the University of Alberta.

I especially value David and Bill's commitment in sticking by me across great distances when I (and also they) left campus. They maintained close contact and helped me manage the colleague-less isolation as I plugged away far away. Deborah Ball was my first dissertation chair, but left my committee following her move to the University of Michigan. She stayed on as long as possible and helped me through my dissertation proposal and at other crucial stages in my graduate career. I value her continued friendship.

* The parents have given me permission to thank them using their real names.

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I appreciate the assistance of all my committee members. They frequently went the extra mile in offering me ideas, suggestions for reading and even mailing me copies of helpful pieces. I hope they will forgive me for drawing liberally on their ideas without always acknowledging their contribution. Most of all, I appreciate their moral support and the confidence they showed in me. They were true teachers in the best sense of the word, warm and caring, friends and guides, supporting me as a person as well as a student. I additionally wish to thank Prentice Starkey at the University of California at Berkeley, who became my mentor for the Spencer Fellows program and whose input was crucial in enabling me to continue traveling down the research path I had begun.

I wish to thank my fellow students, who formed a supportive community that nurtured me as a scholar. I especially wish to thank those participants of the Mathematics Learning Research Group (MLRG) who taught me by letting me see their work and giving feedback on mine: Garnet Hauger, Melissa Dennis, Angela Krebs, Tat-Ming Sze, Beth Herbel-Eisenmann, Candy Baguilat, Dara Sandow, Kyle Ward, Faaiz Gierden and Jan Gormas. I also thank my dear friend and colleague, Fernando Cajas.

I wish to thank my parents, Joan and Hugh Alpert, who provided ever needed moral and material support and encouragement that only parents can give. My mother additionally provided painstaking and expert copy-editing on final drafts. My husband, Khaled Furani, was always by my side and gave assistance in every way possible, including frequently needed boosts to my confidence and even valuable references from time to time. This document is nearly as much his accomplishment as it is mine. My daughter, Mysoon was born in the middle of it all and might be seen by some as having delayed or hindered my progress. To the contrary, her embodiment of love and joy (and not to mention, cuteness) enabled me to persist in lonely New York, so far from my scholarly community. Her smile was often enou

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was often enough to get me started each day. Watching her grow since birth has given me helpful perspectives on the three boys I began studying as toddlers. She may become my next subject as all have anticipated.

I wish to thank the Eternal Spirit that can have no single name, for health, strength and inspiration and allowing me to see this day.

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Chapter I: Introduction

As the title implies, this study is essentially a mathematically informed interpretation of young toddlers' symbolizing. As such, it is unique. Although aspects of symbolizing in young children have been studied by linguists (e.g., various aspects of language learning) and developmental psychologists (e.g., symbolic or pretend play), I have not come across research that places toddlers' symbolic continuum within a single analysis, let alone from a mathematical viewpoint. This is what this study attempts. It carefully examines, explicates and analyzes the symbolizing activity of three young toddlers in its various expressions, with an eye to understanding the underlying cognitive processes and their potential relevance for doing mathematics.

In this introductory chapter, I offer personal assumptions and experiences that have motivated and shaped this study, address preliminary methodological concerns and indicate themes and undercurrents that run throughout. I follow this with a brief overview of the remaining chapters.

At first glance, and even second and third, toddlers and mathematics appear to be an outlandish juxtaposition. Mathematics is a cool, calculating, abstract activity practiced by a 'handful' of adults, while in young children "the blood still runs warm" (Donaldson, 1978, p. 24). Not only is mathematics not for children, but to many adults, it approaches the realm of the inhuman. It belongs to computers and robots and people who think like computers and robots. No wonder countless recoiled in near horror when I answered their cocktail party question with, "I'm a math major."

Although I majored in math as an undergraduate, I came about it rather circuitously, and for a long time, I too shared the common view of mathematics as being somehow inhuman and the world

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and the world divided into 'math people' and 'non-math people,' the latter forming the majority by far. 'Math people' were even further categorized as 'truly math people' and those who managed to fake it. I was someone who could fake it.

I went through school mathematics without truly understanding it, particularly in high school. I began to understand and even enjoy mathematics in college, hence my choice of major, but I suspected that my attending a woman's college had something to do with my success. I only began to believe I could hold my own among the genuine virtuosos, a.k.a. 'math men,' when I did well in a math class at a co-ed university.

On the other hand, my younger brother Chuck was born a 'math person.' 'Two' was one of Chuck's first words. He would toddle around with a plastic, magnetic version of the symbol and proudly name it for anyone in sight. Chuck could count to one hundred and recognize the corresponding symbols before the age of two and a half. In elementary school, he began to calculate batting averages in his head while watching baseball games. He squared the numbers of license plates as we rode around town and read Martin Gardiner books (of mathematical puzzles) for fun. In sixth grade, Chuck was bussed to the junior high school to take algebra and in eleventh grade, to the university for post-calculus courses. He was a star on the county math team.

Compared to Chuck, I was a mere impostor. Not that I minded being 'lesser' in mathematics. I had other interests and talents and never really 'got into' math. My sixth grade teacher 'recognized my abilities' and based on his recommendation, my mother pushed me ahead of my grade. As I said before I got by, but without really understanding. However, once in a while I would really try to puzzle out a problem and eventually manage to understand it. When that happened, I recognized that I was thinking about the problem quite differently from how the teacher or textbook presented it.

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Those experiences planted the kernel of an idea that remained dormant until I entered the classroom 'on the other side of the desk,' but it was there nonetheless and received gradual fortification over the years. Rather than the world being split dichotomously into those near-robots who 'got math' and those who did not, or even along a continuum between these two poles, perhaps people thought about math *differently*; but different ways of thinking were not fostered or even accepted in school.

I fell into teaching by accident just as I had become a mathematics major. Perhaps because it was by accident, I had a terrific first experience. I taught mathematics in a private school populated by home-schooled children. This meant that parents were responsible for all the 'basics,' and I was responsible for anything extra that I chose. I was free from curriculum mandates, standardized tests, and departmental oversight. I was considered the expert; no one questioned what I did.

My personal experiences with math perhaps pre-disposed me to be open-minded about students' ideas, but I also had the fortunate opportunity to receive training from *Project SEED*, a program that seeks to build student self-esteem in disadvantaged school districts by helping students experience success with challenging mathematics. *SEED* promotes student success through such means as classroom discussion, active engagement by all students, bodily participation (through use of gestures for 'agree,' 'disagree,' showing answers with fingers and the like), unconventional symbolism that helps to connect new ideas to ones students already know, step by step development of ideas, fostering of disagreement that must be worked out by argument and proof, deliberate errors by teachers that students must watch for and correct, and the posing of challenging and exciting problems with an attitude of great excitement as well. However, what most influenced me at the time was *SEED's* teaching that students thought logically. Whatever 'errors' students mad

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students made had a logical foundation that was worth uncovering, honoring and building upon in order to reach 'correct' understanding. Honoring children's ways of thinking played a crucial role in enhancing their self-esteem and helping them achieve success in mathematics.

I took this perspective into my first classroom. I looked for the logical ways students thought as I introduced them to interesting mathematics. And logical thinking I found, but I also found something else. In my attempts to uncover students' understandings, I learned that theirs were not only frequently different from my own, experienced view, as might be expected from young learners, but their understanding differed from one another's. The students had different entry points, different ways of making sense, different prior knowledge and different dispositions. And they challenged my preconceptions. There were times the math-enamored 'calculator' missed the mark, while the 'head in the clouds philosopher' suddenly came down to earth and broke the problem wide open. The 'sociable girl' gave the clearest, most thorough explanations, and the 'quiet, least-prepared' (in terms of prior education) boy showed he was fully keeping up by giving correct answers on his weekly 'fun sheets.' Different students took different paths and thought about mathematics differently.

I began to formulate the opinion that far from being inhuman, math was decidedly human and being human had multiple forms of expression. Math connected organically to all my students' minds but did so in various ways. Perhaps society's problem with mathematics lay in an intolerance of this multiplicity, in the uniformity in which mathematics is presented in school and used in the wider society. Mathematics is treated as a monolith, a *fait accompli*, not as something living, changeable and variegated. Children write stories and poems. They make their own drawings, but not their own mathematics.

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My classroom was sadly no exception. Although I was flexible, I was not flexible enough. I did not even imagine students creating their own mathematical terms, symbols and objects of study. The students did a lot of talking while I stood by with a poker face, but I did not truly hear them. I could not hear much beyond what my mathematics training already predisposed me to hear. I knew too little outside the bounds of my education, too little about what my students might actually have been thinking.

This study has been an attempt to overcome this deficiency -- to enable me to actually hear students, to open my mind to multiple ways of thinking mathematically by uncovering alternatives. It has been a step towards understanding cognitive aspects of doing mathematics in their purest, most human, least adulterated form. Naturally, my experiences as a teacher and a learner of mathematics have guided my inquiry throughout. My desire to give voice to varied student thinking in the mathematics classroom has led me to search for differences among the toddlers in this study and begin considering how those differences might inform curriculum and pedagogy.

Young toddlers were the ideal subjects for many reasons. Since they had not yet begun formal schooling of any kind, they were free from its 'indoctrinating' force. While they were beginning to express themselves verbally and physically, they were still at the 'cusp' of language, choosing words and meanings that meshed with their thoughts, rather than having language take a dominating role.[1] Being young, inexperienced, playful and wildly curious about all sorts of things, their activity was still relatively simple, limited in scope,

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[1] This view of language as taking a dominant position as a mediator of thought has gained increasing popularity in recent years, known as the 'Sapir-Whorf hypothesis.' Gee (1999) describes language as "simultaneously *reflect[ing]* reality ('the way things are') and *construct[ing]* (*constru[ing]*) it to be a certain way" (p. 82). He goes on to say, "Different sign systems and different ways of knowing have, in turn, different implications for what is taken as the 'real' world, and what is taken as probable and possible here and now, since it is only through sign systems that we have access to 'reality.'" (p. 83).

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I began this study wanting to learn about 'mathematical thinking.' I was not at all aware that my explorations would lead to a focus on symbolizing. But symbolizing emerged as a dominant activity in the toddlers' lives, from their encounters with language, to pretending or 'symbolic play,' to playing with symbols of a more abstract sort. And symbolizing is such a crucial aspect of doing mathematics, one still largely ignored in mathematics education research and practice.[2]

This study is both empirical and theoretical. I analyze data gathered on three toddlers with particular concerns in mind. I also draw connections to mathematics based on my experiences as a doer of mathematics, a teacher of mathematics and a participant in the mathematics education research community.

This study can be seen as an extensive application of metaphor. I continually ask and try to answer, "In what ways does this activity (of one of the toddlers) resemble a particular mathematical activity?" from an underlying, cognitive perspective. Mathematics is a lens through which I view the toddlers' actions (although I do not necessarily make this explicit). The resulting analysis in turn offers fresh views into doing mathematics.

Metaphor can be understood as crossing domains (Donnelly, 2000), and indeed I crossed domains that appear far-reaching if not outright bizarre. Toddlers and mathematics? Yet, in

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[2] This situation is beginning to change. More researchers have taken up questions of language and mathematics in recent years, e.g., English (1997), but the area is still very sparse. In NCTM's (2000) recent standards document, a new process strand has been added called 'representation.' While NCTM touches eloquently on some of the issues explored here, cognitive aspects and the role of symbols in mathematics itself (i.e., that there is no mathematics -- that is, the cultural artifact taught in schools -- without symbols) are largely missing.

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addition to my own powerful learning from this undertaking, I am encouraged by the words of Poincaré, who said of science:

> [I]t is by unexpected union between its diverse parts that it progresses....Among the chosen combinations the most fertile will often be those formed of elements drawn from domains which are far apart (1982, p. 378, 386).

One may wonder how I could answer the metaphor question as delineated above. Did I see these toddlers write arithmetic problems? Did they solve equations for *x* and *y*? Well, no. Of course I did not actually see the toddlers doing *mathematics* by any current definition of the term, but I did see them involved in symbolizing, in some of the central processes required to engage in mathematics.

Mathematics is often viewed as a collection of objects and tools, and topics of study around those objects and tools: numbers, shapes, formulae, algorithms, measurement, geometry, etc. However, these are cultural products of certain processes of the human mind, processes which are also considered *mathematical* (see e.g., NCTM, 2000). The cultural products taught in school are not the inevitable and only outcomes of mathematical processes, and while I believe the processes are essentially human ways of interacting with the world, the products now subsumed under the title 'mathematics' may not 'fit' all minds. I am here referring to experiences of 'mis-match' between individual ways of thinking about mathematics and those embodied in canonized concepts, symbols and procedures, which I have observed in myself and in students, as well as the knowledge that alternatives can exist.

Researchers in ethnomathematics, the study of mathematics practices in other, frequently non-literate cultures, have catalogued numerous mathematical products that differ from those canonized in the Western mathematics tradition (which also has decidedly non-

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Western roots). Some have even identified cultural processes that give rise to mathematics. Bishop (1988) includes among these processes: counting, locating, measuring, designing, playing and explaining. D'Ambrosio (1994) includes observing, counting, ordering, sorting, measuring and weighing. Ethnomathematics is one area of research that leads to uncovering alternative mathematical processes and products. History of mathematics is another. Research on young children is yet a third fruitful avenue as this study demonstrates.

If learning mathematics were more about developing the processes and less about adopting the products, there could possibly be more widespread success. There might truly arise a 'mathematics for all,' and an empowering mathematics at that, mathematics that would connect more organically to children's minds, enabling them to think in powerful ways.[3]

Regarding the question of toddlers and mathematics, no, I did not see the toddlers make use of mathematical *products* much at all, but they did make heavy use of certain mathematical *processes*, including many that I had never before considered. Of course, the problem remains as to how I could identify mathematical processes without the products attached. This challenge did indeed pose some difficulty, but the toddlers offered me a fruitful solution. As they clearly engaged in symbolizing -- using and creating meaningful words and gestures, playing make-believe, playing games with patterns and symbols as place-holders -- I was able to conduct a deep analysis of their activity and speculate as to its relevance to mathematical symbolizing and to learning mathematics in the classroom.

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[3] By 'mathematics for all' I am referring to what NCTM is currently stating as its 'equity principle': "Equity requires high expectations and worthwhile opportunities for all" (2000, p. 12). While meeting this principle and other stated goals, NCTM talks about processes of doing mathematics (1989) and teaching mathematics (1991). However, NCTM has yet to emphasize processes over products or to consider that through developing the mathematical processes of *all* children, the content of curriculum could itself change.

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When I speak of mathematical processes, I am in many ways referring to 'processes of thought,' which appears to be an ephemeral object of study. Every teacher, every researcher of children would probably love to just (figuratively) open the brain of every child and see what they are thinking and know their thought. Since this is impossible, the only recourse is to infer thought from what children say, do or write. In mathematics education research, children are generally posed tasks, problems to solve, or are interviewed. With toddlers, however, these methods are impossible. Even though they are beginning to use conventional language, they could not possibly explain why they are doing something or what they are thinking. Hence, the demands for inference from outward behavior are even stronger. However, this situation need not pose a problem. Toddlers are incredibly physically active, and through their actions (including actions with words) they express their thoughts.

Donnelly (1998) observed the actions of her sons from their earliest movements and noted how particular ones held their interest to the point of compulsion. As they grew older, her sons' actions manifested themselves in certain patterns of thought, ways of interacting with the world that came out in how they made sense of events, told stories and created fantasy worlds.

> In childhood, the capacity to create metaphors is manifested in play and it is the gesture or action that forms the link between one object and another object or idea. Gradually these metaphor-actions become internalized and become visual or verbal metaphors and form the basis for our conceptual understanding of the world (Donnelly, 2000).

Toddlers' actions can thus be 'read' for meaning as with any text. They offer a relatively 'transparent' view of thought that at later ages becomes internal, requiring a degree of excavation in order to be seen.

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Before I introduce the rest of the chapters in this text, I will summarize some of the primary assumptions and motives that have guided this study. Mathematics is essentially a human sense-making activity, based in human mind and action, which derives its content and character from these foundations. However, as taught in schools today, mathematics is largely rigid and constraining of thought and creativity. A possible means out of that rigidity involves attending to the processes that give rise to mathematics, including to individual differences in these processes, both in research and classroom practice. Rather than primarily enculturating children into the mathematical canon, perhaps children's ways of thinking should additionally guide what happens in mathematics classrooms. Children might then create mathematics that 'fit' *their* minds, thus leading to enhanced self-esteem and true 'mathematical power.'

I see all of these views as resting on a fallibilist mathematics epistemology, one that regards mathematics as any other form of human knowledge, a social historical product that is highly valuable, but does not represent 'truths' about the world outside of human attempts to understand it. By stepping outside of the 'cult of objectivity,' which holds that objective knowledge is possible, multiple and untold forms of thought and expression may be permitted and encouraged.[4]

In Chapter II, I review scholarship from a wide range of fields and carve out my own position on what is involved in learning mathematics, the relationship between thought and language (which naturally includes mathematical language), and the nature of mathematical knowledge. I conclude the chapter with a section that clarifies terms central to the data analysis. Here I define 'symbolizing' and begin to sketch out both the role of symbolizing

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[4] Gill (1993) explains how the 'cult of objectivity' rests on the fallacy that knowledge can be separated from human knowing that must take place within human bodies and actions. "The contention is that only that which is purged of all 'subjective' elements can qualify as genuine knowledge" (p. 50).

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in doing mathematics and particular features of mathematical symbolizing. These issues are naturally taken up in greater depth in the data chapters.

In Chapter III, I describe the study's *Methods*. I include the usual details on what, when, where, why and how of gathering data for the study. I also explicate the study's questions and discuss why and how these questions, primary analytical tools and methods evolved.

Chapters IV, V, and VI are the 'data chapters.' Each is framed around particular aspects of the toddlers' symbolizing activity, which are described, analyzed and illustrated with examples great and small. Each chapter also contains a section on mathematical connections that draws links between the data analysis and aspects of symbolizing in mathematics.

Chapter VII revisits the primary concepts developed in the data chapters and offers further directions for research.

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Chapter II: On Learning, Language and Mathematical Knowledge

One might expect me to begin with Piaget, widely regarded as the "father" and foremost expert on child development. Not only am I dealing with cognition in young children and with mathematical processes, particular foci of Piaget's, but I also studied toddlers in a very similar fashion to how Piaget famously studied his own children. Indeed, I do begin here with Piaget, or more accurately with challenges to his work, and I return to him throughout the chapters in dialogue. However, key aspects of Piaget's work are losing relevance and scholars from alternative perspectives have much insight to offer on human knowledge, learning and development. I discuss their views as well.

Research conducted from the 1970's onward has challenged the construct validity of some of Piaget's most famous experimental tasks (i.e., whether the tasks measure what they are purported to measure, e.g., 'object concept,' 'egocentrism,' 'number conservation,' etc.; see Donaldson, 1978; Hughes, 1986; Starkey, 1992) as well as foundational premises to his theories. These studies and a subsequent widening of the field endorses deviation from Piaget's pursuit of universal, invariant characteristics among children, and the converse investigation of varying manifestations of fundamental cognitive processes. Rather than regard young children as 'egocentric' and 'illogical' (Hughes, 1986), I work to uncover the quite awesome cognitive powers of children.

I review one famous 'finding' of Piaget's by way of example, the 'finding' that before the age of 7 young children cannot 'conserve number.' The Piagetian task used to determine 'conservation of number' involves a researcher asking a child questions. This use of language is one aspect that has called for reexamination. In the words of Donaldson (1978),

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> [Piaget] is sensitive to differences between what language has become for the adult and what language is for the child in the early stages. However, when he himself as an experimenter, uses language, as part of his method for studying children's thinking, he appears to lose sight of the significance of this issue (p. 61).

The standard number conservation task involves an experimenter displaying for a child two linear rows of objects, each of the same number, placed in a one-to-one correspondence. The experimenter then asks the child if there are the same number of objects in each row. If the child agrees, the experimenter lengthens one of the rows by spacing the objects further apart and repeats the question. Children 6 years and under typically respond that the longer row now has more. Children are thereby deemed unable to 'conserve' number because they are apparently unaware that moving objects in space does not change their number. They are believed to rely on visual cues that might indicate 'more' (e.g., a longer row) rather than on logical knowledge about the stability of number through spatial displacement.

Typically, the rows in number conservation tasks contain from 6 to 22 objects each. It would be difficult even for adults to determine the absolute number or compare the numerosities of sets of this size without counting or relying on some other means for judgment, such as one-to-one correspondence, length or density.[1] How do the children in the experiments know that the researchers only moved the objects around and did not somehow surreptitiously add to them? If nothing significant has changed, why then repeat the question?

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[1] Dehaene (1997) reports on studies in which adults were able to quickly and correctly recognize the numerosities of sets of dots up to three without counting. However, both response time and inaccuracies increased linearly with numerosities beyond three. The process by which people (and animals) recognize small quantities immediately, 'apprehend' them without counting, is called 'subitizing.'

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Walkerdine (1988) reports how, "[C]hildren when asked such apparently nonsensical questions as 'Is yellow bigger than green?' would search for objects to justify an answer" (p. 48). The questions experimenters ask in conservation tasks might similarly not make sense to children and yet children provide the answers they think are desired. This possibility fits with findings on number conservation tests discussed by Dehaene (1997) in which:

> the youngest children, who were about two years old, succeeded perfectly in the test....Only the older children failed to conserve [in one of the tasks]. Hence, performance on number conservation tests appears to drop temporarily between two and three years of age (p. 45).

It may be that after age two, children make greater attempts to understand the researchers' intentions and, as a result, mis-interpret what is said. In other words, children may believe the researchers want a different answer than the one first given and that they want the longer row to be identified.

When number conservation tasks are administered differently from the standard, in ways meant to take into consideration potential difficulties, children succeed at greater rates. For example, Hughes (1986) reports on findings in which children as young as three conserved when the quantities involved were small (under four objects). Donaldson (1978) discusses experiments in which a 'naughty teddy' accidentally messed up the situation, rather than a researcher performing deliberate manipulations, and again young children were found to conserve. In experiments conducted by Starkey (1992), rather than compare two rows of objects using sight only and in response to questions, children displaced singular sets of balls themselves into an opaque 'searchbox' and then retrieved them. Balls were secretly removed from the box in the interim. Children from 18 to 48 months of age demonstrated surprise when they could not find the identical number of balls they had placed inside the box and were, therefore, found to 'conserve number.'

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Discrepancies among success rates with various number conservation tasks reveal several important issues: absolute number matters (i.e., a number small enough to perceive absolutely without counting), children's interpretation of the researcher's words and intent matters, and in the words of Donaldson (1978) that the situation make "human sense" matters. Putting balls in a box and then looking for them makes 'human sense' to children; it is a fun 'game,' similar to hide and seek. Comparing long rows of vases and flowers in light of a strange adult's curious movements and questions does not . Such a task is "abstracted from all basic human purposes and feelings and endeavors" (p. 17).

Donaldson (1978) sheds further light on how a child could misinterpret a researcher's questions. She discusses how children first try to 'understand the situation' in which words are uttered, then the people and their intentions and lastly the words. When a child's interpretation of the meaning of a situation and of the words uttered in the situation conflict, the situation might take precedence. Children initially learn word meanings by interpreting situations. "It is possible to figure out what words mean because they occur *together with* certain non-linguistic events" (p. 37). Therefore, although in conservation tasks researchers may have asked children to compare the *number* of objects, the *situations* may have led children to believe they were being asked to compare *length*.

Although the traditional Piagetian number conservation tasks might remain highly 'reliable,' children's success with variations on the tasks suggest that the tasks fail to measure what they purport to measure. In other words, even though young children fail Piaget's tasks, they can 'conserve number.' This fact naturally calls into question Piagetian stage theory and the accompanying view that correct knowledge of number is built up over the early years through 'logico-mathematical experience,' although not completely. The studies cited above show that seven years is not required to construct number knowledge, but they still

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but they still allow for the possibility that 18 months is necessary. However, there is growing evidence that this is not the case; the 'number concept' may be innate.

Another study by Starkey (Starkey, Spelke & Gelman, 1990) found that infants from 6 to 8 months of age could distinguish between the numerosities of 2 and 3 in images and sounds. Dehaene (1997) reports on another study in which 4 day old infants could distinguish quantitatively between 2 and 3 syllable words. These and other studies, most notably in linguistics and animal cognition, point to the possibility of innate cognitive structures that guide attention to certain aspects of experience. People are born with a basic knowledge. Learning begins from there:

> [H]uman cognition...is a collection of specific capabilities which evolved because they contributed in specific ways to survival and reproduction....[C]ognitive development is not a completely general process that equips the child to learn anything at all to which he or she might be exposed; rather, it is a collection of adaptive systems that have evolved to enable children to acquire specific kinds of knowledge that have proved valuable evolutionarily (Sophian, 1995, p. 18).

Regarding mathematics, studies indicate that a number concept and perhaps basic knowledge of addition and subtraction and certain spatial abilities may be innate (see e.g. Wynn, 1992; Hermer & Spelke, 1996). It is upon the human species' innate constructs and processes that mathematics has been constructed. While this nativist view of species-wide knowledge may seem to conflict with a perspective on individual differences, it need not.

There may be some common cognitive inheritance to all people, but this hypothesis does not imply that all people must think alike. For example, all humans may possess the same innate cognitive structures that guide the learning of language, but these structures do not prevent children from constructing the various different grammars that govern different languages (see e.g., Brown, 1973; Hirschfeld & Gelman, 1994). Regarding mathematics, some basic co

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some basic concepts may be present in all children, but where children go from there could vary a great deal. This possibility is evident in the wide range of mathematical practices existent among various cultures as well as the difficulty most children face with much of school mathematics. As Dehaene (1997) puts it,

> Though a few years of education now suffice for a child to learn digital notation, we should not forget that it took centuries to perfect this system before it became child's play. Some mathematical objects now seem very intuitive only because their structure is well adapted to our brain architecture. On the other hand, a great many children find fractions very difficult to learn because their cortical machinery resists such a counter-intuitive concept (p. 7).

The cortical machinery of many children may resist certain concepts, but not necessarily all. Some children might find fractions completely obvious, whereas others find them non-sensical. Some children may have difficulty with the idea that all parts must be equal, or that wholes must be equal when comparing fractions. Others may have no problem with these notions and yet find it more sensible to compare parts to parts rather than parts to wholes, employing a ratio rather than a fraction understanding (e.g., how much pizza left versus how much pizza already eaten, rather than what portion of the original pizza remains). Thus, 'cortical machinery' can differ from child to child.

Thus far, this discussion on nativism has focused on mathematical concepts and objects, but not on processes or ways of thinking. These too might be innate and may also differ in character from person to person. For example, processes of organizing perception, generalizing and symbolizing may be part of the human inheritance that permits the construction of mathematics from basic knowledge. However, the ways in which people engage in these processes can vary, as the three toddlers in this study demonstrate.

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Nativism may also fit well with fallibilist theories of mathematical knowledge since it gives a non-realist explanation for *where* number comes from and why it appears so obvious and intuitive. Number is not in the world, but in the mind. Leading scholars from diverse fields say as much. Here are the concordant and eloquent words of respectively, a mathematics educator, a linguist, a philosopher of language, and a neuro-scientist:

> [W]e see mathematics in the world, namely...we project mathematical forms onto it (Pimm, 1995, p. 33).
>
> One might argue, along Aristotelian lines, that the world is structured in a certain way and that the human mind is able to perceive this structure....A more fruitful approach shifts the main burden of explanation from the structure of the world to the structure of the mind....[O]ur systems of belief are those that the mind, as a biological structure, is designed to construct (Chomsky, 1975, pp. 5, 6, 7).
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> The spirit apprehends itself and its antithesis to the 'objective' world only by bringing certain distinctions inherent in itself into its view of the phenomena and, as it were, injecting them into the phenomena (Cassirer, 1953, p. 178).
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> [Evolution] has biased [the human brain], finally, to project onto physical phenomena an anthropocentric framework that causes all of us to see evidence for design where only evolution and randomness are at work. Is the universe 'written in mathematical language' as Galileo contended? I am inclined to think instead that this is the only language with which we can try to read it (Dehaene, 1997, p. 252).

While nativism addresses a certain basic innate knowledge as well as processes for constructing additional knowledge, it falls far short of offering a full picture of how children learn in a social and physical world. However, other theories offer insight, such as those of social cognition.

Social cognition is also known as 'social constructivism' to emphasize that although society plays a formative role, it does not absolutely determine an individual's development or intelligence. Rather, the social world provides tools and assistance upon which an individual dr

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individual draws in constructing knowledge. Vygotsky is credited with first developing theories of social cognition that others have sampled and expanded.

One of Vygotsky's most influential concepts is the Zone of Proximal Development (ZPD). It indicates that children's potential for learning should be determined by what they are capable of doing with the assistance of 'more knowledgeable others.' "[W]hat children can do with the assistance of others might be in some sense even more indicative of their mental development than what they can do alone" (Vygotsky, 1978, p. 85). Vygotsky stresses that a child's ZPD is not determined only by prior knowledge and experience: Children at the same level of actual development (what they can do independently) can differ widely in regards to their potential development. This notion encourages teaching and for teachers to try out whether a child can learn something new with assistance and appropriate 'scaffolding.'[2] Vygotsky explains that his view differs from that of Piaget, Binet and others who:

> assume that development is always a prerequisite for learning and that if a child's mental functions (intellectual operations) have not matured to the extent that he is capable of learning a particular subject, then no instruction will prove useful. They especially feared premature instruction, the teaching of a subject before the child was ready for it (p. 80).

Vygotsky's view of development is naturally more sensible to me. Children grow and learn but not by moving lock-step through pre-determined stages. Rather, children's learning relies both on what they bring with them (prior knowledge, experiences, interests, faculties) as well as the opportunities they encounter. This view also offers a more helpful guide for education. Each student must be assessed individually and given the chance to show what they know and can do. If a novel task seems too difficult at first, it can be

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2 'Scaffolding' describes a process by which a teacher scales back a problem or a concept to simpler and simpler forms until it reaches a point within a child's ZPD.

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scaled back to a point at which the child can engage. New ideas should not be withheld from children until they are deemed 'ready' as determined by age or some standard test.

In addition to considering the impact other people can have in children's learning, social cognition includes attention to the intelligence embodied in cultural tools such as language, calculators and screwdrivers and the roles they play in individual knowledge (see e.g., Resnick, 1987). This view is often described as 'distributed cognition,' since it regards cognition as enhanced by and distributed among the tools an individual uses. "Intelligence is not a quality of mind alone, but a product of the relation between mental structures and the tools of the intellect provided by culture" (Pea, quoted in Salomon, 1993, p. 112).

Another perspective on social cognition views cognition as not only developed within and aided by social contexts and tools, but actually *situated within* social contexts. Research in situated cognition demonstrates that the knowledge people use in various settings differs; knowledge is built up within and characterized in part by the particular contexts of use (Lave, 1988; Nunes, Schliemann & Carraher, 1993). Situated cognition challenges the notion that knowledge learned in school can be transferred to out-of-school applications. Rather, knowledge must be learned independently in out-of-school settings for it to be viable. This notion fits with many people's experiences of mathematics who successfully solve school math problems (albeit under duress), but cannot make use of mathematics' tools to solve real problems they face in everyday life.

While situated cognition theories are highly plausible, I do not believe they describe the inevitable. If so, institutionalized schooling has no real purpose. Knowledge transfer remains a real problem of education, but perhaps one that can be solved. Regarding mathematics, embracing a fallibilist epistemology could play a role. Rather than school mathematics consisting of learning 'truths,' it could involve acquiring and developing tools for identifyi

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for identifying and solving problems wherever they are found. Attention in schools to developing mathematical processes of thought, rather than only correct conceptual knowledge could additionally contribute, because again people may be able to apply their processes of thinking to numerous diverse situations, even if particular concepts or procedures cannot transfer from context to context.

As an interest in uncovering individual differences motivates this study, theories on social cognition do not actively frame the analysis; rather, they influence it as an undercurrent. Although the analysis focuses on the actions and minds of the toddlers, the role of people, language and cultural artifacts is included. However, again I place emphasis on what the toddlers *do* with that which others provide. One may wonder if a certain event would have occurred had a particular person or artifact not been present. This is a question that cannot be answered directly since no experimental control is possible. In some ways, it seems moot since no children live in a vacuum. Maybe events would not have occurred in exactly the same ways, but relevant events would have occurred nonetheless. However, this question may reflect larger concerns about material, social and emotional resources and the effects poverty or neglect might have on children's intellectual development.

None of my subjects had millionaire parents and none of them were impoverished. All had toys and all had attentive, loving, concerned parents as their primary care-takers. I therefore have no substantial basis for addressing these concerns. Even so, my feeling from observing these three boys (and now a child of my own) is that children will create toys out of everyday objects if none are provided for them, and all will insist on attention when none is given. Real deprivation would arise if children are prevented from playing with things around them or from interacting with caring adults. What this study can contribute is an understanding that people and things do matter, but one cannot control or predict exactly how. Again, with a nod to nativism as well as constructivism, children will attempt m

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To summarize my views on learning, children enter the world with a certain intelligence. While particular faculties form a common inheritance, they can differ in character from child to child. Children construct knowledge of the experiential world by drawing on their own faculties, interests and predilections, but with the assistance and influence of the social environment, one that contains people, language, and objects that are largely cultural artifacts. Influence works the other way as well; children's interests and faculties can affect the shape of their social and physical environment (Hatch & Gardner, 1993). Regarding development, children certainly learn and grow over time, but along no fixed predictable path. Each child is a universe, whose unique mind awaits exploration.

A potent means for exploring a child's mind is to examine outward expressions of thought, the processes and products of symbolizing. However, there is a complex and interdependent relationship between thought and language. Language does not only express thought, but influences it and plays a large role in its construction as well. I explore this complex dynamic next.

Language and thought

One of my motives for choosing to study toddlers was to limit the effects language could have in restraining possibilities for thought. I wished to uncover ways of thinking that may not be possible within the current mathematical 'language game,' ways that could drive the invention of

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invention of new language, language that might better 'fit' children's thought and permit a wider range of expression.[3] As Whorf (1956) puts it:

> Every language and every well-knit technical sublanguage incorporates certain points of view and certain patterned resistances to widely divergent points of view....These resistances not only isolate artificially the particular sciences from each other; they also restrain the scientific spirit as a whole from taking the next great step in development -- a step which entails viewpoints unprecedented in science and a complete severance from traditions (p. 247).

Whorf explains that 'severance from traditions' can only begin with a "re-examination of the linguistic backgrounds of thinking" (p. 247). It is perhaps only by uncovering the relationship between thought and language -- understanding how words and ideas come into being and the ways systems of ideas and systems of signs interrelate -- can language become a true tool in human hands, rather than a force that holds people's lives in illusion. The illusion of Western language involves encoding a "provisional analysis of reality" and regarding it as final.

> The commitment to illusion has been sealed in Western Indo-European language, and the road out of illusion for the West lies through a wider understanding than Western Indo-European alone can give (p. 263).

Whorf intimates that by learning languages outside this tradition and understanding the alternative ways of thinking embodied in other 'language games' a wider view of reality can be had, because the tools for 'seeing' (i.e., language) would have a wider reach. I agree, although another, simultaneous avenue is to allow for a loosening of the strictures of a given language, freeing it from a certain rigidity. Recognizing that language does not mirror reality, stepping out of that illusion is key to approaching language flexibly.

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[3] 'Language game' is a Wittgensteinian concept that will receive additional attention throughout the text.

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Indeed any language can be flexible. For example, there are numerous versions of English (though not necessarily full dialects) with various word usages, meanings, turns of phrase and grammatical forms, although those versions accepted in print are only a small subset of the total. New words come into English constantly from various communities, not the least of which consist of youth and children. Brown (1958) draws on Wittgenstein's notion of 'language game' to explain how this happens.

> From observations of a limited sample of play one learns the rules of this game and becomes a kind of creative participant, extending the game along lines permitted by its structure (p. 107).

However, there are still limits, those given by structure. Even if a whole new system of thought could be invented, it cannot merely be grafted onto an existing system. Hopi cannot be grafted onto English. For one, grammar is set. For example, if prepositions, the part of speech, are expressed in a particular way, they cannot simultaneously be expressed differently. However, differing systems can exist side by side and be honored for the potential world views they offer.

Interestingly, mathematics is already an possible place for allowing this experience to occur. Differing sign systems can and do exist side by side in mathematics. However, the degree of difference may be limited and furthermore, children rarely receive exposure to alternative mathematical systems, let alone participate in constructing them. Changing this practice is important if the West is ever to step out of the illusion Whorf identifies.

Yet, language is not only limiting. It also enables thought to reach levels impossible without it (see Gill, 1993; Osborne, 1999). It is a tool and like other tools, "extend[s] the powers of sight and touch -- but never neutrally" (Pimm, 1995, p. 7). This is perhaps nowhere clearer than in mathematics. Mathematical language enables manipulation of abstra

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abstract objects, independent from the world of things. Symbolic manipulation leads to new symbolic relationships that provide yet additional perspectives for regarding experience. Yet again, it must be remembered that these 'symbolic relationships' can serve as lenses or tools, but cannot represent reality in some neutral way.

The non-neutrality of language is endemic. No language can be neutral, neither can any individual or collective human experience. Yet alternative systems, be they linguistic or mathematical sign systems, can permit a wider range of expression as well as a wider range of experience to the person who masters them; for having new words permits new thoughts and experiences. Extension within a given system -- new words, metaphors, interpretations for existing words, symbolic constructions -- can serve similar purposes. Holding stubbornly onto a singular sign system, keeping it rigid, claiming it 'is' reality, only 'keeps imagination on a short leash,' if not imprisons the mind outright.[4]

Although thought and language are very much intertwined, each influencing and enabling the other, thought can also occur outside language. Even Whorf, who is considered extreme on the issue of language's controlling force, admits this possibility. He discusses experiences of perceiving systems without form, 'extraordinary mental states' and "'nonlexical vistas' that cannot be well put into words" (1956, p. 254). Gill (1993) also maintains that "not all knowledge can be explicated" (p. 53). So there is thought and knowing that can occur outside language that can never be signified within any sign system. Yet, there is also the experience of thinking both in and outside language simultaneously. It is that experience where one reaches for a word, searches images for a

[4] I borrow this expression from Skovsmose (1994) who asks, much as I do, "Does mathematics keep our imagination on a short leash? Does mathematics make us see reality in a distorted way? Could mathematics be interpreted as a language by means of which we not only observe certain structures of reality and ignore others, but also organise reality?" (p. 4).

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metaphor in order to encode a thought not yet encoded. It is out of such experiences that new signs are born.

I earlier quoted Donaldson as saying, “It is possible to figure out what words mean because they occur *together with* certain non-linguistic events” (1978, p. 37). She seems to give primacy to experience. It is out of the need to express experience that language arises. Gattengo (1963) appears to agree when he says, “Only when experiences are meaningful can words appear as convenient short signs for them” (p. 98). This seems to be especially true of the creation of new words, whether the idiosyncratic inventions of toddlers that generally fade from use or the adult namings of things as yet unnamed within a language that eventually enter the lexicon (and simultaneously enter ‘existence’).

However, other scholars I quote in these pages (Pimm, 1995; Dehaene, 1997; Piaget, 1962) also describe the reverse possibility, words giving birth to meaning, signs invoking new experiences. And I maintain this stance as well -- an interconnectedness, give and take, interdependency even, between thought and language. Nevertheless, there appears to be this assumption that experience is somehow basic, primary, with words coming out of experience, after it. Experience is immediate, whereas words are abstract, timeless, at some higher level and distant from experience. Indeed, my ordering of data chapters seems to betray this view as well.

I first discuss ***naming***, processes of generalizing and signifying experience. Then, in ***symbolizing the imagined***, I explore the development of sign systems, signs that relate to one another and birth symbolic creations rather than describe immediate experience. In ***systematizing and playing with symbols***, I examine those instances in which the toddlers leave signification behind on some level and play with abstract form. Thus, I seem to put naming experience as primary and playing with abstract form as somehow more advanced, at a higher

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at a higher level. Indeed, mathematics seems to involve this view. I am not quite sure why, since mathematics rests on all these symbolizing processes. Yet, perhaps it is because the latter process belongs more or less to mathematics alone that mathematics bestows on it ultimate value, while all explicit knowledge and creation, whether in scholarship or art, relies on the other processes as well.[5]

Outside of mathematics, the abstract is also valued over the experiential, often termed 'concrete.' Indeed, this perspective is part and parcel of Piagetian stages and is generally perceived in children's development. Infants clearly experience the world; they then begin to say words. Later they speak in sentences and learn their linguistic system. Later still, they learn algebra. Algebra does not precede words. Or does it? Whorf (1956) talks about no word having an 'exact meaning,' that words only attain 'reference' in context. Communicating thereby relies on a form of algebraic thinking. Words are not fixed, but variable:

> Sentences, not words, are the essence of speech, just as equations and functions, and not bare numbers, are the real meat of mathematics....There is a queer Western notion that the ancients who invented algebra made a great discovery, though the human unconscious has been doing the same sort of thing for eons! (p. 258).

Nevertheless, my own language prevents me from completely shaking the belief that 'abstract' is somehow equivalent to higher, more difficult to attain and therefore more valued. Even so, I would like to leave it an open question. Just as the toddlers do not fall into 'stages' on 'performance' of other activities (e.g., 'stages' of pretend play), they do not fall into 'stages' along some continuum between concrete and abstract thinking, either individually over time or as compared with one another. The analysis shows that all the

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[5] I use 'explicit knowledge' following Gill (1993) to refer to signifiable knowledge.

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toddlers engaged in all of the symbolizing processes explored here, although in different ways and to varying degrees.

Whorf was concerned with issues of time and space, stasis and dynamism, continuity and change and how language influences worldviews in these matters. As he says, "Scientific language, being founded on western Indo-European and not Hopi, sees sometimes actions and forces where there may only be states" (p. 263). Similarly, western Indo-European may see hierarchies and dichotomies where there may only be unordered continua. I will still admit a continuum, rather than a dichotomy, ranging from thinking in experience to abstract thought that moves further and further away from experience (and permits reflection on experience). Yet, every point on this continuum has equal value. All forms of thought are important and perhaps the ability to move back and forth along the continuum is most important of all.

Another dichotomy that parallels the concrete-abstract dichotomy is that between body and mind. Experience is seen as belonging to the body and abstract thought to the mind. Perhaps this dichotomy too is false and should be reconsidered. Thought may very much be done with the body, in the body, even abstract thought. Gill (1993) notes, "We are so 'close to' ourselves in our bodies that we generally fail to appreciate the degree to which our entire existence is conditioned by our embodiment" (p. 42). He emphasizes that, "[K]nowing is a kind of doing...knowledge is not had as much as it is done" (p. 48). Just as humans act within bodies, knowing and thinking are bodily acts as well. Rather than a dichotomy or even a continuum between body and mind, body and mind may be very much intertwined, mind and thought embodied. Language is thus similarly embodied as well. One acts with and experiences language. Language does not stand apart from bodily existence in some distant plane. Although abstract, it does not belong to mind somehow separate from body.

Although this section has focused on language and its relationship to thought, it can be understood as a discussion on mathematics as well for mathematics is frequently referred to as a language or a 'language game' (see e.g., Ernest, 1998). Language and thought cannot be disjoined, neither can one thought from another, despite attempts to compartmentalize them. Thus, I have spoken about mathematics already, in this and the previous section. I was not able to keep all characterizations hidden until now. However, I will now directly address the question as to the substance and nature of mathematics.

What is mathematics?

What is mathematics? 'A bunch of numbers' was the answer once given to me by a pre-service elementary teacher. Seemingly simple at face value, it is not too far from the prevalent view. In his review of the evolution of mathematical concepts, the mathematician Raymond Wilder (1968) explains, "Not only did mathematics begin with number, but number concepts cut through every field of mathematics in some form" (p. x). To Wilder, geometry's inclusion was the result of cultural evolution, perhaps even a chance occurrence:

> Indeed, there can be little doubt that this association of *number*, the most basic element of mathematics, with *line*, one of the most basic elements of geometric form, lay at the heart of the absorption of geometric form into mathematics (p. 88).

Not everyone will agree. Euclid's *Elements* is often viewed as the quintessence of mathematical exposition, an axiomatic system built upon through definition, theorem and proof, and the *Elements* is geometry (Ernest, 1991; Thurston, 1994). Hence, geometry was part and parcel of mathematics from the very beginning, particularly of mathematics as a deductive system.

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Another mathematician, William P. Thurston (1994) discusses how it is difficult even for mathematicians to define what it is they study. He says that for him "the theory of formal patterns" comes closest, but then he goes on to say simply that "[m]athematics includes the natural numbers and plane and solid geometry" (p. 162). While geometry surely warrants a noble place in the history of mathematics -- the Greeks even practiced a geometric algebra (Wilder, 1968) -- it is largely neglected today in school mathematics. Number takes the primary role. Psychologists also view number as the basic element in mathematical knowledge and at times equate it with mathematics. This is evident in Piaget's conservation tasks (including those on continuous *quantity*, e.g., length and volume) and numerous studies on number and arithmetic knowledge in young children, infants and animals (and not on other aspects of mathematical knowing; see e.g., Carey & Gelman, 1991; Gelman & Gallistel, 1978; Starkey, Spelke & Gelman, 1990; Dehaene, 1997; Sophian, 1995; Hughes, 1986; Wynn, 1992).

The central role of number in mathematics likely contributes to the prevalence of a positivist view of mathematical knowledge. Number is an idea present in all cultures and to adults and children alike appears so obvious and real. Even numerous animal species have been found to perceive number! Number must therefore surely exist independent of human perception. This positivist view of number and other parts of mathematics is generally attributed to Plato:

> Platonism is the view that the objects of mathematics have a real, objective existence in some ideal realm....Platonists maintain that the objects and structures of mathematics have a real existence independent of humanity, and that doing mathematics is the process of discovering their pre-existing relationships (Ernest, 1991, p. 29).

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The German philosopher Ernst Cassirer (1953) challenges this characterization of Platonism. He sees Plato's development of the principle of an 'idea' as involving the recognition that concepts are in the mind rather than in the world.

> [T]he Pre-Socratics identified being with a particular existing thing and took it as a fixed point of departure, while [Plato] for the first time recognized it as a *problem*...[In accepting Plato's notion of 'idea'], [t]he fundamental concepts of each science, the instruments with which it propounds its questions and formulates its solutions, are regarded no longer as passive images of something given but as *symbols* created by the intellect itself (pp. 74, 75).

The belief in the ontological existence of mathematical objects might be more appropriately attributed to Plato's student Aristotle. As Hawkins (1993) explains,

> [Aristotle] rejected Plato's view of a world of Ideas known independently of the world of nature. The Ideas were, rather, essential forms of things we find in nature, some of which we earliest learn (p. 15).

Regardless of the philosophical origins of positivism, it remains that it is commonly believed that numbers and other mathematical objects exist, are real and true and furthermore, they describe 'truth.' Doing mathematics involves searching for and discovering 'truths.' Although common, this is a belief that I do not accept and that I cannot accept if I am to regard toddlers with an open mind, if I am to permit them to teach me about mathematical thinking -- to reveal things that may not yet have found their place among mathematical 'truths.'

If mathematics is not absolutely true, if the positivist view is rejected, what will take its place? If not in the world, where then *do* mathematical objects come from? What then *is* mathematics? Answers to these questions that challenge positivist views can be found in what Ernest (1994) describes as a growing 'maverick tradition' of the philosophy of mathematics, in which I am pleased to delve.

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First and foremost, mathematics is a human activity and being human, it is fallible like any human endeavor. Mathematics involves the attempt to understand structures and patterns, be they observed empirically or generated within mathematics itself. Mathematics produces models and explanations that aim to be the best possible accounts of what is observed. Yet as with any explanations, they are and must be partial. It has even been proved mathematically that mathematics can not be unified into a single, coherent, logical system (known as the 'Gödel incompleteness theorem' Thurston, 1994). "Like science, mathematics can advance by making mistakes, correcting them and recorrecting them" (Hersh, 1994, p. 14).

Mathematics rests largely on our language faculty and is often described as a language in itself. Just as no language is capable of mirroring reality, neither can mathematical language. Rather than exist ontologically, mathematical objects exist in human language. They are shared linguistic conventions that reflect social concepts (Hersh, 1994). Hersh (1994) draws upon the linguistic basis of mathematical knowledge in challenging the widespread belief that '2 + 2 = 4' is ultimately true, independent of human language, perception or even existence (as can be seen by his use of pre-human 'brontosauruses' in this example):

> ['T]wo' plays two linguistic roles; it is an adjective and it is a noun. When we talk about two brontosauruses, 'two' is an adjective. When we use 'two' as an adjective, we are using it to talk about objects, usually physical objects. The fact that two brontosauruses plus two brontosauruses equals four brontosauruses is a statement about brontosauruses, not numbers....This is a statement in elementary physics. The noun 'two' on the other hand...is not physically observable. It is some kind of abstract or ideal object.... I am pointing out that these abstract ideal objects are social concepts (p. 18).

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Thus mathematical objects are linguistically coded social concepts which form networks of interrelated social concepts. Hersh even defines mathematics as "[t]he study of the lawful, predictable parts of the social-conceptual world"(p. 19). That mathematical concepts interrelate and form coherent and even predictable structures is a key essence of mathematics' nature. It is the larger structures and theories that give meaning to particular mathematical objects.

> [W]e don't first learn to refer to objects with an 'internal' composition arranged in a structure, we learn a network of interrelated objects. For an object to exist is just for it to exist in a network of interrelated objects, that is, to exist as a posit of our theories...[I]t simply makes no sense to ask what 'three' really is apart from the position of three in the structure of arithmetic, nor does it make sense to ask what 'rabbit' really refers to apart from the theoretic framework encapsulated by English (Tymoczko, 1994, p. 50).

It is in part the coherence and predictability of mathematical structures that conveys the impression that mathematics is true (in addition to the seeming obvious existence of basic mathematical objects). That mathematics can have direct applications in the real world with stunning results further enforces this impression. As Tymoczko puts it,

> We know that natural numbers exist for pretty much the same kind [of] reason that we know that atoms exist, because we know that number theory and atomic theory are true. And we know these are true because they force themselves on us as the best (best? are there even alternatives?) accounts of what is happening around us (p. 52).

While I concur, I wish to emphasize that this sense of truth must remain a tentative and pragmatic one, with the understanding that 'better truths' might one day be found (to atomic as well as number theory). Something can be regarded as true, treated as such because it is useful, not because it is ultimately true. While science has taught me that my perception of red is a function of how my eyes perceive certain wavelengths of light and that no object intrinsically *has* a color, I will still regard, talk about and behave as if a red object rea

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object really *is* red, because this attitude permits me to function. Questioning my eyes or my ears or the ability of my language to describe experience would only lead to insanity. While I know that my senses, my language and my conceptual tools are limited, I must act in daily life as if they do reflect reality. Although for certain purposes (such as research and education) it is important to question and reflect upon what one takes to be 'true,' doing so constantly would be debilitating.

Yet here is an appropriate place to question what are taken to be mathematical truths. Although I have argued that mathematical objects and theories are social concepts generated by a fallible humanity, this fact does not necessarily imply that they could not miraculously at one and the same time be ultimately true. It may be that despite the fallibility of human senses and the limitations of human language, humanity had somehow hit upon 'truth' with its invention of mathematics. It is a collection of amazingly powerful tools after all, but again potency must not be equated with truth.

Just as the fallibility of the human sense of sight can be demonstrated by showing its susceptibility to optical illusions or the existence of wavelengths of light that the eye cannot see, I can argue for mathematics' fallibility in part by pointing out its limitations. First, there are limitations to coherence and completeness even within a given mathematical system. For example, certain numerical values remain undefined, such as '$2 \div 0$.' Second, there are limitations to coherence among mathematical systems as a whole. Axiomatic systems with equally plausible assumptions can exist simultaneously and yet produce results that are in direct contradiction to one another, and all can potentially have practical applications (e.g., Euclidean and non-Euclidean geometries, standard and modular arithmetics). Third, there are limitations to mathematical measuring and modeling of experienced phenomena. One only has to try to rely on a weather forecast to confirm this.

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Bloor (1994) points out another means for demonstrating that mathematics is fallible and consists of sets of social conventions, rather than truths:

> Conventions are shared ways of acting that could in principle be otherwise. They are contingent arrangements, not necessary ones....Demonstrating conventionality therefore involves demonstrating alternative possibilities. Although this necessary condition is easy to state, it isn't always easy to satisfy in practice. For one thing, our imaginations are limited (p. 21).

Indeed imaginations are limited. Having been schooled in mathematics, people are limited to thinking about mathematics in very particular ways (if they have not been terrorized out of thinking mathematically at all). It is difficult to find alternatives within a given society.[6] Alternative mathematical systems (e.g., various geometries) can be seen as a sort of alternative, yet they somehow do not challenge positivist views at their source and are rarely taught in school. Researchers who study the mathematical systems, practices and worldviews in various non-Western cultures have uncovered multiple alternatives to such constructs as base ten numeration, numbers as ideals (i.e., nouns disconnected from experience), division into equal parts, and a segmentable and infinite space-time, to name a few (Pinxten, Van Dooren & Harvey, 1983; Closs, 1986; Zaslavsky, 1973; Lancy, 1983). However, power relations necessarily keep these alternatives from being given serious consideration in schools, and as they are often seen as mere 'proto-mathematics' (Chevallard, 1990), it seems unlikely that 'ethnomathematical' alternatives will challenge convention in the way Bloor (1994) notes is required.

What remains a possibility (save encountering a more powerful mathematics of some non-earthly species) is to find alternatives in our own society, in the 'simple' minds of children rather than the inaccessible minds of mathematicians. If permitted and encouraged to do so,

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[6] Mathematicians may be exempt from this assessment. Having gone through years of 'hoops' in their own education, they have reached a level where they are permitted and encouraged to think about mathematics creatively, to invent mathematics.

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children might invent alternative ways of thinking mathematically and alternative mathematical systems and even objects, ones that also importantly conform to some of the basic tenants of mathematics that give it validity, such as logic, internal consistency and social validation of proof (Thurston, 1994). However, this proposal presents a chicken and egg problem. Finding alternatives is necessary to promote fallibilism. A belief in fallibilism is necessary to permit the development of alternatives. As a result, both fallibilist views and alternative approaches to mathematics must slowly move forward in turn. There has already been some headway on both.

Fallibilism seems to have been embraced by those mathematics educators who show interest in the nature of mathematics. Invention of mathematical alternatives by children has been documented (e.g., 'Sean numbers,' Ball, 1992), despite their rarity and fragility. I hope this study will contribute to the latter by communicating both the possibility and necessity of invention by children.

I began this section with a discussion of number. Number is considered a mathematical 'object,' and yet a number is not an object in the material sense. It is a characteristic of a collection of objects, e.g., two oranges, three bananas, etc. 'Seeing' this characteristic is in part a function of human perception. Noting that number is a characteristic independent of particular objects (e.g., any collection of two discrete object can have 'two-ness') is a function of human abstraction. Turning the characteristic of 'two' into an object that can be operated upon requires symbols, be they iconic or ideographic,[7] and is therefore a function of human symbolizing.

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[7] 'Iconic' refers to symbols that "possess the very structure" of the numbers they stand for, such as tally marks (Hawkins, 1993, p. 14). 'Ideographic' refers to symbols that embody a certain meaning within a particular human language(s), such as '2' but carry no indication of pronunciation (see Hughes, 1986). Symbols can take other forms as well, such as words or gestures.

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Rather than regard mathematics as the study of certain objects, such as number, it can be viewed as a product of particular human processes, such as perceiving, abstracting and symbolizing among others, which are engaged in an effort to understand patterns and structures. Number is a product of these processes, but there are others and there could be others still. Understanding mathematical thinking requires uncovering these and other processes. I have given a very brief illustration here, but throughout the text, I explore in depth processes that give rise to mathematics, particularly those involved in symbolizing.

Thurston (1994) claims the mathematician's role is to "enable *people* to understand and think more clearly and effectively about mathematics" (p. 163). He then offers a list of several "different ways of *thinking about* or *conceiving of* the derivative" as an example of how a particular concept can be understood in multiple ways (p. 163). Thus, Thurston too is emphasizing mathematical *thinking* and the importance of allowing for and encouraging multiple conceptions.

Multiple conceptions is intimately intertwined with multiplicity in language, multiple forms of symbolizing. It is multiplicity in symbolizing's processes and products that I have searched for in the toddlers of this study. Yet before I go on to discuss how I proceeded and what I found, I must explicate some key terms for framing the analysis, some of which I have already used.

Terms and definitions

Since this thesis explores toddlers' symbolizing, it is important that I clarify what I mean by *symbolizing*, *symbol* and a host of other terms. I begin where others begin, with Saussure, the 'father of linguistics.' Walkerdine (1988) explains:

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> For Saussure the sign is constituted by the unifying of the signifier and the signified. This is commonly presented schematically as a 'fraction': signified/signifier (p. 2).

Therefore words are signs as are the signs in sign language. A *sign* fuses together a sound or a gesture -- a *signifier* -- with an object, action, person, idea -- that which the signifier signifies, i.e., a *signified.*

My appropriation of the Saussurian scheme does not carry with it the mathematical interpretation of fraction notation. The signified is not a part of the signifier the way a numerator expresses the part(s) of a denominator's cut-up whole. The mathematical metaphor notwithstanding, the 'fraction' scheme helps to express the unity or fusion occurring within signs that make it so "words don't get in the way" (Pimm 1995, p. 5). When one is able to 'see through' signifiers directly to signifieds, then one is making use of *signs.*

As I am interested in the processes that give rise to signs, that enable toddlers to 'read' and generate signs, I need to be able to look at signs 'unfused.' I find it helpful to separate signifier from signified with distance and to rejoin them with an arrow pointing from signifier to signified, giving: signifier → signified. This change helps to elucidate the pointing function of a signifier (as expressed by the verb 'signify') and to preserve the notion that, in the words of Walkerdine (1988), signifiers have "multiple meanings, articulated within different and sometimes contradictory practices...."(p. 30). Thus, a signifier is not always fused with a particular signified but can point to multiple signifieds. Likewise a given signified can have multiple signifiers pointing to it (just as I am 'Helene,' or 'Mrs. Furani,' or 'mama,' or 'my daughter,' etc.).

Symbol has a particular place in this framework, one derived from mathematics. As Pimm (1995) explains,

> [S]ymbols may in the very least mediate our contact with '[mathematical] objects,' and at times provide the primary experience (p. 4).

Thus in mathematics, signifiers are called 'symbols.' They 'mediate' our experience or in other words 'point' to mathematical objects. So for example, the symbol '5' points to the mathematical object of number five. Signifier to signified has in a sense its mathematical parallel in symbol to object. Yet mathematical objects are not *real* objects. They are not physical or tangible; rather, they are ideas. Mathematical symbols thus get their meanings in large part not through pointing to objects (certainly not to objects outside of mathematics) but through their relationships to other mathematical symbols, through their membership in a symbol system. When using the symbol '5' while doing mathematics, what matters is that '5' is the fifth member of the natural numbers, $5 = 2 + 3$, $5 = 77 - 72$, and so forth. Relationships are what matter.

Resnick (1997) puts this notion another way. He views mathematics as a science of patterns:

> [F]or some time the practice of pure mathematics has reflected the idea that mathematics is concerned with structures [(or patterns)] involving mathematical objects and not with the 'internal' nature of the objects themselves...[Mathematical objects] emerge as positions in these patterns (pp. 201, 226).

Thus, what the symbol '5' points to is largely irrelevant. Relevance lies in its position within a pattern. In mathematics, rather than signifying a signified object, '5' itself becomes an object to be manipulated according to rules and governed by the structures in which it takes its place.

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When I use 'symbol,' I mean to evoke this mathematical connotation. A *symbol* is a signifier somehow severed from its signified. It becomes an object in its own right, perhaps regarded for its own characteristics and forming relationships with other symbols. In other words, if I regard the signifier, 'Helene' as a symbol, I may attend to its sound, its visual image, that it rhymes with 'spleen,' or can take the role of a noun in a sentence, and not to what it signifies (at least for the moment).

Symbols are created by severing the connection of signifiers to signifieds, but they can also be created irrespective of signifieds. They resemble other symbols in form (e.g., marks on paper, gestures) but as yet point to nothing. They have the potential to signify and may do so at some later time, but for the moment they are objects in their own right, ones that may take positions in patterns. For example, this is likely how 'the artist formerly known as Prince' got his new 'name.' He devised a new symbol (one meaningfully constructed out of existing symbols) and then let it signify himself (i.e., he played around with symbols and upon creating one to his satisfaction, gave it a meaning). Dr. Seuss is also well known for inventing new symbols. He invented 'words' that fit rhymes in his stories, then gave them fanciful meanings. He even 'extended' the alphabet with new 'letters' (see, e.g., ***The Lorax, On Beyond Zebra***).

Piaget (1962) defines sign and symbol somewhat differently. For him what is at issue is the degree to which signifier and signified are paired arbitrarily:

> A sign, as conceived by the school of de Saussure is an 'arbitrary' signifier, related to its signified by a social convention and not by any resemblance between them. Such are words, or verbal signs, and mathematical symbols (which are not symbols in the sense in which we use the word here)....The symbol...is a 'motivated' signifier, i.e., there is a resemblance of some kind between it and its signified. A metaphor, for instance is a symbol because there is a relationship between the image used and the object to

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> which it refers, a relationship which is not due to social convention but directly experienced by the mind of the individual (p. 169).

Whether any signs are arbitrary is unclear. Lakoff and Johnson (1980) argue that the expressions we use in everyday talk evoke metaphor (e.g., 'Argument is War,' 'Time is Money'). Brown (1958) explains how words themselves can have metaphorical origins and offers an example:

> Both [the languages of French and English] arrived at their word for the moon by metaphorical means, though the metaphors are constructed on different attributes of the referents -- its luminosity for the French, its periodic cycle for the English (p. 139).

Brown (1958) also cites experiments by Sapir and others in which subjects rated vowels and consonants as expressing qualities such as small and large. "Here speech sounds suggest spatial and visual dimensions, size and brightness" (p. 114). Cassirer (1953) mentions similar investigations, including ones by the mathematician Leibniz, and notes that the tones in certain tonal languages have been found to evoke relative speed and distance (pp. 192, 194).

Upon reflection or investigation or through deliberate questioning in careful experiments, one may come to sense the metaphorical origins of words and expressions or the implied meanings of phonemes, tones, vowels or consonants. Yet when using speech mundanely, one is generally unaware of these connotations even though they may still influence behavior (Whorf, 1956). Discussing the metaphor *foot of the mountain* and its origins in human anatomy, Brown (1958) suggests:

> This metaphor blazed briefly for the person who created it and it lights up again when anyone hears it for the first time, but for most of us it is dead (p. 141).

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For some, the metaphor may never 'light up,' rather be simply accepted for the meaning implied through context. When this happens as with any 'dead metaphor,' the sign is treated as arbitrary.

These perspectives problematize Piaget's distinction between the 'arbitrary sign' and 'motivated symbol.' Nevertheless, the distinction between arbitrary and 'motivated' is still experienced. For example to me, the baseball umpire's gesture for 'you're out' -- an outstretched thumb moved in an arc to rest facing back and away -- resembles the signified action of being kicked out and is therefore 'motivated' in the Piagetian sense, whereas the gesture for 'safe' -- crossed arms outstretched quickly -- appears arbitrary (although another person may 'read' them differently). In this analysis, when I wish to note resemblance between signifier and signified in a sign, I will use Piaget's term in single quotes: 'motivated symbol.'

Vygotsky (1976) employs the Saussurian scheme but with different terms:

> [T]he structure of human perception could be figuratively expressed as a fraction in which the object is the numerator and the meaning the denominator...[When a child pretends] this fraction is inverted and meaning predominates, giving: meaning/object (pp. 546-7).

Thus for Vygotsky an object (or action) carries with it a meaning. Object and meaning are fused. By use of the fraction scheme, Vygotsky seems to equate 'object' with 'signified' and 'meaning' with 'signifier' (although he neither makes mention of Saussure nor uses his terms). 'Meaning' is attributed through words or by use of an object. Thus, Vygotsky says that children can ascribe new meanings to objects by giving them new names (e.g., calling a 'stick' a 'horse') or otherwise attributing to them alternative meanings (e.g., by 'riding' a stick.) Vygotsky implies that meaning is automatically given by language. When

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I call a stick a 'horse,' I am regarding it as a horse and disregarding the usual functions and purposes it has as a stick. I am not merely renaming it 'horse.'

While helpful in elucidating processes involved in pretend play, this formulation has its problems. 'Word' is often juxtaposed rather than equated with 'meaning.' Words can have *different* meanings. Different words can *mean* the same thing. To maintain consistency with the framework thus far, 'meaning' should thereby be equated with 'signified.' Words can point to meanings. Words can also point to objects. 'Meaning' and 'object' then become interchangeable. If so, how can the movement among meanings in relation to objects be incorporated?

Rather than seeing the imaginative signifier of 'horse' replacing the mundane signifier 'stick' to signify the same object, movement can be understood as also happening among signifieds. When I call a stick 'horse,' I imagine it no longer to be a stick. It is now a horse. The object has been transformed through signification in how I see it, how I use it. The word 'horse' still points to the *meaning* 'horse' but in using it to name a stick, the object has undergone a metamorphosis. Its meaning has changed because the signifier pointing to it has changed.

Placing 'meaning' in the 'signified' camp is consistent with Pimm (1995) who starting from his book's title (*Symbols and Meanings in School Mathematics*) juxtaposes symbol with meaning. Pimm discusses how meanings can precede and give rise to symbols or names as well as how:

> [N]ames and other symbols can also bring meanings into being, reversing that sense of antecedent priority. At times, the form of the words can give rise to meaning, to understanding, making links across the symbolic gulf in the reverse direction (p. 4).

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The arrow is here as well. Signifiers may often arise from, but nevertheless point to meanings. Pimm further pairs symbol with 'referent,' which provides for the verb 'refer.' These should also be included in the lexicon, along with 'mean.' The following figure summarizes the terms thus far.

symbol	(no arrow)	
sign	(intrinsic arrow)	
signifier	*signifies*	signified
	refers	referent
	means	meaning
	→	object

A 'sign' exists when the far left and far right columns are fused, a 'symbol' when the left column stands alone, not pointing towards external meaning (even if only momentarily). Signifiers can take numerous forms: spoken words, written words, gestures, photographs, drawings, scale models, numerals, etc.

Down each column the terms are parallel, equivalent in some sense but not synonymous. I have differentiated symbol from sign because these terms are used differently in different contexts. I will use them in particular and specific ways. There will be no such confusion with the other terms. I will use 'refers,' 'meaning,' 'object,' etc. according to common usage.

Still awaiting description is the term 'symbolizing.' 'Symbolizing' evokes a sense of active signification. A word signifies, but a person *symbolizes*. Furthermore, the activity is one of effort and consciousness. Once people gain fluency with a language, they are signifying through the words they use, but not symbolizing. When symbolizing, one is actively involved

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involved in creating and reading symbols, making and giving meaning, turning symbols into signs.

For example, if I had never encountered the word 'dog,' upon encountering it for the first time, it would be a symbol for me -- an utterance or arrangement of marks that failed to point to anything. However, over time, through exposure to the word in different contexts, I would arrive at (i.e., *construct*, although in consonance with encountered usage) some meaning for it. It would then signify. Once 'dog' became fused with its meaning so that I would 'read' it automatically, it would be a sign for me. Thus, 'symbolizing' will refer to all that this process entails, by which symbols are created or met and become signs.

Missing from this definition is the very mathematical notion of symbolizing -- positioning symbols within patterns and creating and exploring relationships among them. 'Symbolizing' will include this activity as well. Thus 'symbolizing' comes to encompass all of these activities with symbols -- turning symbols into signs, turning signs back into symbols, creating symbols as objects in their own right (not signifying as yet) for use in patterns and setting up relationships among symbols. What this process involves -- how it takes shape in toddlers' worlds and how it is involved in doing mathematics -- will be thoroughly explored in the coming chapters.

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Chapter III: Methods

My initial research question was: "What characterizes the mathematical thinking of young toddlers in the domains (flexibly named and defined) of number, pattern, symmetry and space?" I always intended these domains to be tentative, capable of change. I did not anticipate that they would fade into the background and that symbolizing would become both the primary focus of analysis and an axis along which I would examine mathematically relevant thought and action. To reflect this change, I rephrased my research question as:

> What characterizes the symbolizing of young toddlers and what relevance does it hold for understanding mathematical symbolizing and thought?

In many ways, I answered this revised question. In the data chapters, I offer evidence, describe and characterize the symbolizing of the three toddlers, both individually and collectively, with an eye to its mathematical relevance (i.e., mathematics as a lens through which I look at the toddlers' activity in the first place). I also draw explicit connections between the toddlers' symbolizing and ways people engage with mathematics. However, more than characterizations, I feel this study offers conceptual tools for thinking about mathematical symbolizing and for recognizing its importance in doing mathematics and its basis in early activity. These conceptual tools may be additionally useful in uncovering individual differences in ways of symbolizing and thinking mathematically. Given this emphasis on conceptual tools, I rephrase the research question once more:

> What conceptual lenses are effective in analyzing toddlers' symbolizing that simultaneously offer insight into mathematical symbolizing?

Both questions of characterizations and conceptual lenses can and must be answered together. No characterization of cases can be helpful beyond understanding those

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particulars unless they include conceptual tools that may be applied elsewhere. And conceptual tools cannot be deemed useful unless they are demonstrated to be effective and even better developed out of attempts to understand observed phenomena. Thus, in order to answer one question, I must and do answer both. The characterizations may have applicability beyond these pages if, for example, particular descriptions or episodes serve as provocative illustrations or invite further analysis. However, the conceptual tools possess a greater potential for applicability; they could be brought to other studies of learners of all ages and into the classroom to inform ways of assessing student understanding and conducting teaching.

In this chapter, I describe how I went about trying to answer these questions both empirically and analytically, and how my questions, primary conceptual lenses and methods evolved as the study progressed. I begin with a discussion on the evolution of method.

Drawing on other studies

This study relied primarily on methods of participant observation. I entered the lives of the toddlers; I played with them, observed them, and sought additional information from parents and care-givers. However, prior studies on young children's mathematical knowledge (ages 5 months to 5 years), to which I turned for guidance, used very different methods of inquiry (e.g., Starkey, Gelman & Spelke, 1990; Gelman & Gallistel, 1978; Starkey, 1992; Bryant, 1974; Wynn, 1992). In these studies, experimental and clinical methods were omnipresent, as was quantitative analysis with its coding, counting and aggregating. While these studies had much to offer, both in their findings about young children and their sophisticated task design and administration, I knew I did not want to conduct a similar study. Because I believed in the possibility of individual differences in mat

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mathematical thinking, I had difficulty with the aggregation of data. I wished to know what differed among children and why. I also had difficulty with the fact that the researchers' tasks posed questions stemming from their perspectives of mathematics -- adult, Western, schooled perspectives. I wished to learn about unique, individual and decidedly unschooled views.

Therefore, I turned to studies of the mathematics of older children and adults -- ones that aimed at uncovering situated and unschooled mathematical competence (Lave, 1988; Nunes, Schliemann & Carraher, 1993). These studies in situated cognition compared performance on purposeful tasks conducted *in situ* with performance on tasks modeled to mimic real situations and on school-like tasks. These studies began with ethnography and drew a great deal of their results from ethnographic data. Tasks were designed and administered in relation to ethnographic findings. For example, in Lave (1988), subjects were presented similar shopping problems that they had solved in the grocery store, together with real food items. In Nunes *et al.* (1993), Brazilian street sellers were given paper and pencil tasks using the same arithmetic problems they had solved successfully in purposeful transactions. Task data permitted deeper and firmer interpretations than ethnographic data offered alone.

Situated cognition studies appeared as a more applicable general model for gathering data given my research perspectives although I hoped to obtain inspiration and guidance for constructing tasks from the experimental studies. Bowers (1996) explains,

> Any experiment depends upon an (often) implicit and unexplicated 'ethnographic' knowledge that experimenters have acquired through being informal 'participant-observers' in forms of everyday life they believe they share with their subjects (p. 135).

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Rather than follow the traditional experimenter's *modus operandi*, I hoped to acquire *explicit* ethnographic knowledge of subjects through *deliberate* participant observation and let this knowledge guide me in designing and interpreting tasks. I later discovered that Piaget had advocated as much: "Observation must be at once the starting point of all research dealing with child thought and also the final control of the experiments it has inspired" (1929, p. 4).

A pilot study

I undertook a pilot study to try out 'this' method of inquiry. I arranged to sit in on classes in the infant-toddler program at the laboratory pre-school on my university's campus. I visited two separate weekly, hour-long classes where toddlers, who ranged in age from 10 months to close to 2 years, attended with 1 or 2 parents. I visited for several weeks, observed the toddlers at play, although rarely interacted with them, and tried to develop conjectures about their mathematical thinking, which I could then 'test' with tasks.

I managed to develop a conjecture about one of the children: *Many* objects excited him. I constructed a task that I hoped would give some confirmation of that conjecture and also help me determine exactly how *many* were 'many' or would bring on this enthusiastic behavior. The task was a home-made variation of one used in a published experiment (from Starkey, 1992).

I tried the task with five of the toddlers and obtained very indefinite and frustrating results. Discussions with the parents and my supervising professor on the study (Jack Smith), along with my reflections led me to conclude that task administration failed because of a number of factors. For one, the children were not very familiar with me. They were also unfamiliar with the task apparatus and with the activities the task demanded. Although conduct

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conducted in a familiar setting (their 'classroom'), the fixed time frame of the task did not suit all subjects (e.g., some were hungry or tired). Although I directed the toddlers to play with the task in a specific way (with parents' help), they often wanted to play with it in other ways, either in familiar and/or favorite ways they played with other objects and/or in new and novel ways.

I realized that if I wanted to pursue this line of research, I needed to engage in more extensive participant observation. I needed to know the subjects better and have greater opportunity to introduce and modify multiple tasks gradually. For my dissertation research, I decided to use only three subjects and spend time with them in their homes and possibly day care over a number of months. I maintained my intention to design and introduce tasks in line with conjectures that stemmed from observed behavior.

The actual method: abandoning experimental tasks

At the outset of my study, I had two subjects lined up. They were the children of friends and fellow graduate students in education. Since these two boys were already familiar with me, I thought my increased presence could appear relatively 'normal' to them. I was able to observe them at similar ages and they were both only children. When I obtained my third subject, he was also a boy and only child and the son of a former education graduate student. As I expected there to be a good deal of variation among the toddlers as individuals in different environments, it helped that they shared these commonalties.

Although I expected an observation period of 2 to 3 months per toddler, I spent 5 months with my first subject, Jacob, whom I began observing when he was age 16 months. The first 2 months with Jacob went rather slowly. I got to know him better. I played with Jacob and watched him play on his own or with his parents. I learned a good deal of his 'lan

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'language,' as well as ways he played with toys and read books. I participated and co-constructed games with him. However, I had difficulty locating mathematical thinking in his activity. In particular, number, which had been the focus of nearly every experimental study in infant and toddler mathematics, was conspicuously absent. I was hard pressed to come up with a task to try.

I was committed to using familiar objects in tasks and a shape-fitter toy (a cube with three cut-out shapes on each face, each of which corresponded to a unique, three dimensional piece) was a regular component of Jacob's play. Jacob's parents helped him locate the holes for particular shapes and Jacob dexterously dropped them through. Being a shape-fitter, its pieces were of different shapes and came in three colors: yellow, green and blue. I was unsure of the relationship between classifying and mathematics, yet I knew Piagetians thought there was one. I had also noticed how successfully Jacob classified and distinguished vehicles: cars, trucks, motorcycles, bicycles, etc. I wrote in my field notes:

> I would like to try a classification task with Jacob -- sorting his shape sorter blocks, offering just two colors, putting them in jars. I'll leave him a jar to play with next time so that he can get used to the object and not get distracted by its novelty. Then hopefully the week after that, I can encourage him to try to play with jars and blocks in a specific way.

I did just that. On my next visit, I brought in a jar as well as a video camera for the first time. I had always intended to video-tape the toddlers' engagement with tasks as well as other occasions.

I tried to interest Jacob in putting blocks inside the jar and dumping them out. He played that way a bit, but was more interested in putting the lid on the jar and taking it off. He also enjoyed putting my pens inside the jar (which I was using to take notes). I left the jar with

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Jacob so he could 'get used to it' over a period of time. I would try the task I had in mind at a later date.

The later date never arrived. The jar made no additional appearance in my notes. I think I tried to have Jacob play with it on my next visit, although again he had little interest in playing with it in the ways I wanted. I recall his mom telling me she did not let him play with the jar on his own since it was glass. I abandoned the idea of the task with little remorse (certainly none that merited recording).

The task model presented itself as a slow, painstaking and frustrating avenue for inquiry given my questions and age group. Even with my relatively extensive experience with Jacob, tasks seemed to offer no greater hope for insight than they did in my pilot study. I also knew of the long process involved in developing the tasks used in other studies.[1] Fortunately, at the same time that I was despairing of tasks, I caught a glimpse of the potential of ethnography.

Just after my experience with introducing the jar to Jacob, I had the chance to talk about my study with faculty and fellow graduate students at the Mathematics Learning Research Group (MLRG), an informal group that supports graduate student research in mathematics education. Preparing for the forum, I saw that from just two months of observations with a single subject, I was able to pick out episodes from my data that dealt with the mathematical areas of pattern and space. The group found them provocative and rife with possibility.

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[1] I was fortunate to have the opportunity to visit Prentice Starkey, view some of his data and talk with him about his work and my own. I learned that the task I had used as a model for my pilot study involved years of piloting and revision before it was 'veritable' enough for use in the published study (1992).

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I began to believe I would be able to see what I wanted to see through participant observation alone. I acted on that belief and never looked back. I abandoned attempts with formal tasks and have not regretted it.

Hypothesis testing in ethnography

Although I abandoned formal tasks and embraced participant observation with gusto, I did not entirely abandon the idea of forming hypotheses and then making alterations in the setting in order to test them. This was most apparent with Jacob. On the same day I introduced the jar and video camera, I joined Jacob and his mom in playing hide and seek. After playing spontaneously, I decided to play in a patterned way to see if Jacob would pick up the pattern. I did the same on the following visit but used a different pattern. Jacob's father also tried out two additional patterns of hiding when playing hide and seek with Jacob (outside my presence). From all of these 'experiments,' we learned that Jacob failed to attend to any specific pattern but always first looked in the last place of hiding and then randomly searched the other possibilities of the moment.

Another pseudo-task I tried with Jacob involved a peg puzzle. It had nine shapes of vehicles (with pictures) and matching holes. Jacob could complete it very quickly and I wondered how he did it. Did he match shapes or use orientation (e.g., remember that the boat always went in the lower left hand corner)? To answer this question, while Jacob was playing with the puzzle, I surreptitiously turned the frame upside-down. To my surprise, Jacob matched the shapes, even though he failed to reorient them correctly and could not fit them without my help. Orientation of the whole puzzle was irrelevant to his reasoning.

Midway through my data collection of Jacob's activities, he began to pretend in my presence. I took note of this and it became an area of attention. For one, I knew

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developmental psychologists held pretending in high esteem (see e.g., Bretherton, 1984; Weinberger & Starkey, 1994; Piaget, 1962). I thought that a relationship might exist between the complex cognitive activities involved in pretending and the complex cognitive activities of mathematics. When sharing this interest with my advisor, David Pimm, he further encouraged me to explore it. He explained how he viewed the ability to pretend as an essential requirement for doing mathematics -- for setting up assumptions and parameters within which mathematical 'play' can be conducted, behaving 'as if' something were true as with a hypothetical problem, letting variables take on roles or specific values just as a person would in a pretending game.

During my final observation with Jacob, I decided to try push his pretending further than I had seen it go. I initiated novel ways of pretending (to Jacob), such as pretending with invisible 'objects' and making sounds for visible ones (e.g., saying "chugga chugga choo choo" as I moved a train) and took note of Jacob's reactions.

I continued this form of hypothesis testing in less deliberate ways with the next two subjects, Jeremy and George. After the toddlers had been playing in a particular way, at times I suggested or introduced changes to see what they would do with them. I also attempted to extend or incite certain activities for the benefit of the video camera. However, as time wore on, my efforts became more spontaneous. My 'manipulations' had less forethought and were more in keeping with the actual play at hand. They were perhaps even more intended for play and fun than science.

When it came to doing analysis, places where I attempted to test hypotheses proved useful but no less so than spontaneous interactions and points of conflict between the toddlers and me or others. In fact, very fruitful were episodes when I failed to understand the toddlers and they made multiple attempts to teach the 'dumb adult,' as were parental attempts to cc

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correct and rein in their children. Fortunately, many of these encounters were captured on videotape. Videotape proved an invaluable tool since I was able to view episodes again and again and come to understand things that had eluded me in the moment.

And now for specifics

I have offered a narrative of how my method of data collection evolved but have yet to provide some of the nuts and bolts of the resulting method. As I mentioned, I had three subjects, all boys and only children during the time of the study. I observed each subject for roughly two hours per visit, first once weekly and then after some sufficient 'getting to know' period, twice weekly.

Jacob was born on May 26, 1996, and was 16 months old when I began observing him on September 24, 1997. I conducted 21 observations over 5 months, ending on February 26, 1998, when Jacob was 21 months old. I recorded roughly 5 hours of video data. Jacob's parents are Ann, a graduate student in mathematics education and Carl, a graduate student in sports physiology. I knew the whole family before the start of the study.

Jeremy was born on July 28, 1996, and was 17.5 months old when I began my observations on January 16, 1998. I conducted 20 observations over nearly 4 months, ending on May 6, 1998, when Jeremy was 21 months old. I recorded roughly 6 hours of video data. Jeremy's parents are Birgit, a graduate student in literacy education and Lao, an artist. Lao's native language is Chinese and Jeremy is being raised bilingual. I also knew this family before the start of the study.

George was born on September 8, 1996, and was 19.5 months old when I began my observations on April 20, 1998. I conducted 16 observations over 2.5 months, ending on

July 2, 1998, when George was 22 months old. I recorded roughly 5 hours of video data. George's parents are Lynn, a former graduate student in educational administration and John, a faculty member in communication. This family learned about my study through the e-mail grapevine. We met and mutually agreed to George's participation. While I observed Jacob and Jeremy at home exclusively, I also observed George at day care. Patty was his day-care provider. One observation was conducted at home when a babysitter Milady was caring for him.

When with the subjects, I wrote sketchy notes on a small note pad. Once home, I wrote up expanded field notes for most of the observations (73 percent). After an initial period of at least one month for each subject, I introduced a video camera. I switched it on when there were particular things of interest happening, but also just let it run for long periods (20 to 30 minutes). As the study progressed, I became more adept at using the camera and capturing better and more interesting images.

The bulk of the observations were conducted uniquely in that I focused on one subject at a time. Exceptions to this were a 41-day overlap between Jacob and Jeremy and a 16-day overlap between Jeremy and George. This proved to be a good structure. Each child was unique and had his own rhythms, language, activities, environment, toys, etc. Therefore, I needed to concentrate on the children one at time to get into their rhythms, their ways of being in the world. Observing two at once, I felt scattered and could not concentrate deeply on both. Transitions were particularly difficult; the experience was similar to culture shock. It took time to move from the rhythm of one child to another.

Nevertheless, some overlap was good. As an initial, gradual, 'getting to know' period was needed for each subject, having it coincide with an intense finale with the previous subject helped to avoid lag periods and keep up the study's momentum. When I finished with one subject, I had spent enough time with the next that I was soon prepared to 'dig in deep.' Overlap also permitted some inter-subject comparison in the moment, which likely heightened my attention to certain particulars.

The number of months I spent with each subject decreased over time, yet for each I feel I obtained comparably rich and sufficient data. As the method was new to me (that of closely observing toddlers and trying to locate mathematically relevant thinking in their actions), it took some time for me to get to know the method in addition to the first subject. The last subject was the only one I did not know previously and yet curiously, I spent the least amount of time with him. Despite my increasing skills as a researcher, it seems likely that I still would have needed at least as much time with the third subject as the second, but this was not the case.

While I do think I became more effective at gathering data, credit for the reduced time-frame is also due to George and his parents. All were warm and outgoing and took me in rather quickly. John and Lynn actively signed on to the project and were perhaps the most forthcoming of the parents with information about their son, seeing themselves very much as co-researchers. George eagerly adopted me as a playmate, interacting directly with me from my initial visit forward. My presence at his day care as well as at home was additionally helpful. The variety of situations produced a variety of data.

A word on instruments

"An ethnographer is a human instrument...." (Fetterman, 1989, p. 13). This is perhaps more so than in any other form of research. I recall my first advisor, Deborah Ball, discussing how I would have to 'fine-tune' myself as the means for detecting and eliciting that which I wanted to know about the toddlers. This was indeed very much the case, hence my talk of getting into the toddlers' individual rhythms, experiencing 'culture shock' when moving from one subject to the next, and even 'tuning myself' into doing this sort of research in the first place.

It took time to determine where my attention should be, what I should record, what I should ask parents, and how I should spend my time with the toddlers. Although it took time, not all of it took deliberate effort. I felt the families largely involved me in the general flow of their lives and did not change too much by way of everyday routines because of my presence. As for the toddlers, I was an additional playmate or 'audience,' somewhat like a parent, but not quite. I took my cues from the parents and toddlers, standing back, waiting to be invited in, participating in ways they showed me.

All three toddlers were most accepting of my presence. It seemed to make perfect sense to them that an adult would want to come spend time with them, watch their every move, and be available to play at their beck and call. I tried to encourage my role as a playmate although I also knew I was an adult and needed to maintain a certain parity with the parents. If the parents stated a rule, I also needed to enforce it or if they made a statement, I needed to agree, even if for research purposes, I wanted to do the opposite.

However, the toddlers clearly distinguished me from their parents. I may have echoed parental dictates, but I did not initiate prohibitions and the like and the toddlers noticed. For example, Jacob played a game that involved hitting my rear repeatedly (until it hurt and even my research aims could not entice me to continue). He would not have dared try that with his parents or even in their presence (Jacob and I were downstairs). I watched Jeremy play contentedly with pens for some twenty minutes. He acted guiltily the moment his mom walked in the room and indeed she stopped him.

I paid attention to the toddlers at an intensity their parents could not (because they were, of course, meeting other responsibilities). The toddlers knew this too. For example, at George's day care where there were other children, George made sure that I was always paying attention to him, luring me back should another child momentarily attract me.

Informants can also be considered 'instruments' for gathering (and interpreting) data, and the parents in this study were particularly useful informants. Their participation provided data on the toddlers' actions outside my presence and parental interpretations as to the meanings of actions. At times I shared nascent conjectures and parents offered their input. This often meant that they began to pay greater attention to particular behaviors. Parents would share their subsequent observations and interpretations with me. Parents offered spontaneous interpretations of behavior we observed together and others upon my questioning. Parents also alerted me to behaviors that were new or that they found noteworthy.

Jacob and Jeremy both knew me before I started showing up with a note pad and writing in it and both were initially somewhat distracted by my writing. At times the boys would come and take my pen, sometimes even communicating they wanted me to stop using it. At other times, they wanted to write themselves, and I dedicated a final page on each pad that was theirs for doodling upon demand. Even so, their attention to my writing or pad never lasted long and did not interfere with my overall recording of activities. George doodled too, although more rarely than the others, and he never seemed bothered by my writing. A couple times he even returned my pad to me when I had placed it down somewhere. It was as if to George the pad must always accompany me.

The introduction of the video camera met with similar reactions. With each subject, I began with it set up on a tripod. With Jacob and Jeremy, I even left the tripod in their homes so

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they could get used to it. Both of these boys were rather interested in the camera. Jacob seemed to freeze at times when I first turned it on. He also became quite fixated on a flashing red light until I learned how to disable it. Jeremy and Jacob also frequently asked to look through the camera at various points during the observation period, which I let them do, usually first stopping any recording. The tripod became a plaything in itself for Jeremy. He would look through the spaces formed by the supports as if peeking through windows and explore what parts could move. None of this activity interfered with data collection.

I experimented with different microphones for the video camera. I tried a remote transmitter microphone that included a transmitter placed on me or a subject and a receiver connected to the video. This was particularly recommended to me for working with children who may have soft voices. I also tried a sound-grabber microphone. Both were not only unnecessary, but caused difficulties by disconnecting by accident and hence losing sound altogether or by picking up unnecessary and distracting sounds, such as body movements, breathing or sound caused by tripping over a cord. The video camera alone picked up sufficient sound, even at distances when the zoom feature was used. I quickly abandoned the additional microphones.

With Jacob and Jeremy, I noticed that when I went behind the camera I disappeared to some extent. They did not interact with me as they usually did. Thus, for the most part I fixed the camera on the tripod and aimed it in the direction of activity in which I continued to participate. From time to time, I would go to check the camera and make any needed adjustments.

However, with George, the camera's presence seemed to make no difference whatsoever. I was Helene with or without a camera in front of my face. George surely noticed the

camera; the first few days I used it, he said, "cheese," when he initially saw it pointing at him. However, he then seemed to quickly forget it. He talked with me and dragged me into games just as when there was no camera present. As such, the video data of George contains many more hand-held images and I feature less in the picture. The action is viewed more directly as if through my eyes and appears quite natural. The images are also clearer and more precise with appropriate aim and zoom.

A change of attention

I described earlier how my methods shifted from a hybrid of ethnography and experiment to participant observation exclusively. My focus of analysis also underwent transformation although not along the same time frame and for different reasons.

If analysis is taken to include all uses of theory and theoretical constructs throughout the life of a research project, then analysis begins before 'entering the field.' In my initial research question, I delineated categories to guide my attention to events in the field. Even though I wanted to uncover authentic ways of thinking, ones not already assumed to exist by me, and to take a grounded theory approach,[2] I knew I needed to begin somewhere. I wrote:

> One can liken the process I must undertake...to the using of a camera. To take a picture, one must point the camera in a certain direction, focus it (assuming that it is a manual one) and shoot. In this study, the mathematical notions with which I am beginning serve as the 'pointers' -- the directions in which I point the camera. Without having any pointers, I would not know where to begin, where to turn my attention. It would be as if I were never to remove the lens cap. However, pointing the camera in a certain direction does not pre-determine a focus, nor the image that will appear. In this study, the focusing of the camera is a task to be done gradually, gently

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[2] Now that the study has been completed, I find 'grounded theory' to most closely resemble my research approach, with its emphasis on allowing categories to emerge from the data, continual revaluation and revision of categories throughout the life of a project and "constant interplay between data and the researcher's developing conceptualizations" (Pidgeon, 1996, p. 82; see also Glaser & Strauss, 1967).

> and very tentatively, from the ground up, in a sense, and as jointly as possible with the subjects and their settings.

When I began, I did indeed look for number, pattern, symmetry and space, my initial domains for attention. While these can in some sense be considered canonical objects or areas of mathematical study and I was interested in thinking or processes, I had not yet imagined that I could attend to processes without obvious mathematical products attached. It actually took an involved process of evolution for this approach to change.

In my first preliminary analysis of Jacob two months into the study, I examined episodes that I thought related to pattern and space. My next preliminary analysis was conducted nearly three months later, just before I concluded my observations on Jacob. It again involved him exclusively. With that analysis, I developed six categories that were variations on the first four, although symmetry and discrete number were excluded, and pretending was a new addition. My categories had begun to shift from an object to an activity orientation, mostly due to attempts to describe Jacob's activities, although this was not clearly delineated.

My next preliminary analysis was on Jeremy alone, conducted two and a half months into my observations with him. I attempted to apply the Jacob-generated categories to him. I managed to use four of them and tentatively developed an additional one. I now had a total of seven categories.

In meeting with my committee, they noted how my categories were not all of the same type. Some were derived from my original mathematical domains but had been somewhat transformed into descriptions of the toddlers' actions. Others, most notably pretending, described activity of the toddlers without making it clear what I saw as mathematical about that activity. They suggested I make two separate sets of categories, one that described observed activity and another that described mathematical features of activity.

I completed my observations of George, conducted a preliminary analysis on the data for him, reviewed the data and analysis for Jacob and Jeremy and generated two sets of categories as my committee suggested. I had six activity categories (e.g., forms of play) and eight mathematical feature categories, also described in active terms (as processes). 'Symbolizing' appeared for the first time among the latter. At the time, I used it to denote: "inventing and using meaningful symbols (words and gestures) or showing playful flexibility with common language." It was my first formal mention of attending to language.[3]

Committee members had no disagreements with my categories for describing the toddlers' activities, but they found my other set of categories included overlapping elements. I pared it down to three: symbolizing, regularity and shape and space. 'Symbolizing' now also included that which I had considered as mathematical features of pretending, which I had previously described as: abstracting, representing, imagining, and acting 'as if.' Equipped with these new lenses, I returned to my data once more. I intended to conduct an analysis with each of the three categories in turn and for each to form the focus of a data chapter.

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[3] I later discovered that for my second observation of Jacob, I wrote the following interpretation in my notes, referring to a game he had been playing with his mother. "I wonder if Jacob's pushing his legs could be sort of a cue or even a symbol that communicated he wanted to play this game. I suppose one could think about it as a form of mathematical thinking -- using a very specific and concise symbolism to express a more complex notion, but I suppose that all language functions in this sort of way. It may be even more the case for young children as they figure out what sort of symbols will yield what sorts of actions from others, ones that are often unconventional and idiosyncratic. Later with language, it functions so automatically that we forget that words stand for things and we even begin to equate them with the things themselves. This seems that much more true with mathematics." Thus although not obviously intentional or conscious, I seem to have been attuned to issues of language and symbolizing from the start. Perhaps this contributed to my having sufficient data for an analysis of symbolizing, despite only naming it as a focus after completing data collection.

Analyzing and writing: more changes

Although my previous development of categories involved moving back and forth between the data, including being 'in the field,' and concepts, which included discussions with committee members and student colleagues, getting 'down and dirty' into fine-grained analysis was much more intense. It was iterative (it involved moving back and forth between examining 'raw data' and conceptualizing) at a higher frequency by scores.

I decided to begin with 'symbolizing.' I expected it to be the richest and most involved of the categories. I carefully reviewed all the data on each subject in turn. I compiled all video descriptions and field note excerpts that I considered as relating to symbolizing into single documents (one for each toddler). Using the compilations, I wrote analytical memos on each subject's symbolizing.

In order to write the memos, I needed to organize each compilation of data. I placed related data items together, frequently making finer distinctions two or three times over before I could write. I often returned to the original video or field notes to ensure I captured the episodes correctly and in context. This often meant drafting detailed descriptions of key video excerpts. Specific pieces of data usually connected to multiple others and for varying reasons. Multiple pairings provided for multiple interpretations, which I sorted through, keeping some and dropping others, as I worked to achieve a wider consistency. I wrote out my interpretations in the subject-specific memos, pieces that could be considered 'within-case analyses.'

In making sense of related pieces of data, I continually asked myself, "What is going on here? What are the events? What stayed the same and what changed over time? What can account for changes?" To answer these questions, I brought in other data that were similar in some aspects and different in others. I asked, "What is the same and what is different? What can account for similarities and differences?" I also considered data that seemed to counter initial interpretations, which led me to question my assessments. I asked, "Can I truly say this? Can I account for what seems to be incongruous?" Digging deeper, I usually found that I could. This often led to changes and expansions of initial interpretations.

Although I wrote each analytical memo independently, I had a general knowledge of all the data and likely made implicit comparisons among the toddlers even when generating the first memo on Jacob. However, when writing the second and third memos, comparisons became more explicit. I talked about new data in relation to previous descriptions and sometimes referred to specific examples. These points of comparison could be considered 'between-case analyses.' In addition to this comparative aspect, each memo brought new issues into play.

Once I completed all the symbolizing memos, I realized I had a great deal to say about the toddlers' symbolizing alone. What I had learned from the process fascinated me. I had not been at all aware of the potential of 'symbolizing' as an analytical lens applied to my data. I had also been unaware of the extent to which I had data relevant to symbolizing. I began to consider focusing my dissertation on symbolizing and realized I could fold the other analytical categories of regularity and shape and space into symbolizing. Symbolizing would be the umbrella. The symbolic aspects of the other categories were the most interesting anyway. For example, regarding regularity, patterns made use of symbols and rules were expressed symbolically as well. Regularity and symbolizing were very much intertwined. Regarding space, the toddlers established rules about space and played with regularity in space, also through use of signs. Regarding shape, shapes were visual symbols to the toddlers, marks as it were, that they could recognize and compare, not objects endowed with geometric properties.

I thus delineated five data chapters that would capture all the analysis within a symbolizing frame, but would also include relevant aspects drawn from views through the other analytical lenses. Out of the five, two chapters could be written based on the symbolizing memos alone. I decided to begin with those and then return to the data once more using the other two categories as lenses. I did indeed draft those two chapters first. They are now Chapters IV and V of this document.

Writing chapters involved an additional level of analysis. I now had chapter names to organize my focus. They brought different issues to the fore than were apparent in the memos. Writing chapters also involved explicit comparison of the toddlers around the same issues. I could no longer regard each child only independently. Concepts thereby came more to the fore. In the context of chapters, I also greatly expanded the mathematical connections I was making to the toddlers' activity.

Once I had strong drafts of these first two chapters, I turned to the other three. I realized that two of the chapters dealt with overlapping issues and could be combined into one, which became Chapter VI. The remaining chapter was to be called 'Visual Symbols.' I returned to prior, preliminary analyses on shape and sketched out what I thought this chapter might include. I realized that I had little to say, especially in comparison to the extent of the other chapters and nothing too earth-shattering. Furthermore, some of the issues I had originally planned for the chapter, I had already discussed in the drafted chapters. I decided to drop it. I thus had one data chapter still to write that would interrelate regularity and symbolizing. This required another return to all of the data, to view it once more through the regularity lens.

While I intended to look at regularity through a symbolizing lens in writing the chapter, I still wanted a full a compilation of all data I considered independently relevant to 'regularity.' I could later choose to exclude certain data that were irrelevant to the discussion at hand. Thus as with 'symbolizing,' I reviewed all the data and pulled out episodes that related to regularity. However this time, I compiled the data on all three toddlers first, before proceeding to analysis.

In doing the analysis of these data for Chapter VI, I again came up with a set of sub-categories, yet these arose out of a simultaneous within- and between-case analysis. They formed a singular set that involved little overlap and required no sub-division before I began my efforts to write. Because the analysis was for one chapter only, I was able to begin writing the chapter directly from the data, organized by these categories, without generating prior analytical memos. However, not having the memos did make the process more cumbersome. I had to analyze all three toddlers at once and write analysis within the strictures of chapter form, rather than in the more free-flowing form that memos allow. Fortunately, the regularity data was more straightforward; it was easier to categorize and interpret than the 'symbolizing' data.

Some thoughts on the analytical process

There is not much written about conducting analysis nor is it much talked about despite its centrality to the research endeavor. In order to write this methods chapter, I searched literature to help me find ways to talk about what I had done after I had done it. I found practically nothing by way of explicit descriptions of how analysis has been carried out. How to do analysis was also conspicuously absent from my formal training in graduate school. My most helpful insights into the process occurred when more advanced graduate students shared their data and analytical tools at MLRG sessions and asked for assistance from the rest of us.

Despite this partial glimpse into analytical work, I really did not know how to go about analyzing my data. I knew it was important to review data again and again and to come up with categories. That was about it. I do not think I realized I was even doing analysis until I had been doing it for some time.

Perhaps one reason that doing analysis is not much discussed is that it is difficult to talk about in general terms. Particularly in any form of qualitative research, analysis must be very project-specific. How one goes about it is intimately related to what one is trying to find and to what one eventually finds. The shape analysis takes likely changes many times within a given project, as is apparent in my brief description. Nevertheless, I feel there is both the need and possibility to begin to offer descriptions of how analysis is conducted, even if only within the context of specific research projects. Such descriptions can be considered cases just as any other cases, which taken together can lead to the generation of conceptual tools, ways of talking about what is done. Analysis need not remain the mysterious process it currently is. It only need see the light of day.

I have offered a very cursory and reflective description of how I conducted analysis for this project. I would not say it constitutes a 'case' in the way I have suggested, but it could still prove helpful to some nascent researchers. I will also offer a few insights about my experience of the process that may be of interest.

If I could describe in one word the analytical process, I would say 'iterative.' This is hardly a new notion. Pidgeon (1996) talks about the "constant interplay between data and the researcher's developing conceptualizations, a 'flip flop' between ideas and research experience" (p. 82). Yet, I was amazed by how much back and forth there was, as well as the degree to which this movement led to revision of concepts and lenses for viewing the data. Literature also played a key role in iteration. I would pull back from the data to read. Reading would then lead to my seeing new things in the data upon returning to it. New interactions with data changed concepts and new concepts provided new views on the data. Surely knowledge of my data also influenced how I read others' work, what I chose to notice, jot down, how I interpreted their words. Thus, there were three poles among which I moved in no certain order -- data, literature, and concepts I was developing. Each movement from pole to pole influenced what happened in the encounter that followed.

Another experience that struck me was of data 'ballooning.' Time and again I looked at a set of data that I had gathered into a whole and cut it up into sections, delineating separate instances, aspects or categories. I then focused in on one of the pieces. As I did so, it ballooned up, becoming as rich and complex as the original set. So, I cut the piece into smaller pieces and then looked at one of the new pieces closely. The new smaller piece once again ballooned up, turned out to be complex and convoluted as well. Although I perhaps did not need to segment the piece yet a third time to get a handle on it, it took some effort to 'pin it down.' I then had to look at it in light of the wider set it came from and the wider set that contained the set, comparing it both to the larger wholes and to other parts within the wholes. Thus handling data involved juggling between parts and wholes on multiple levels, with parts becoming 'larger and larger' through the process than they had first appeared.

One last aspect worth mentioning was the 'call to write.' After completing preliminary analysis of all the data and narrowing down my major analytical categories to three, I turned my focus to writing. Of course, I did not write immediately, but as I did my final analysis with the new lenses, I had the writing of chapters in mind. Chapters would be my analytical product.

I had originally felt unprepared to turn to writing so soon. I imagined I would continue to do analysis as I had been, generating categories and offering examples and illustrations, but not writing actual chapters until I completed the analysis. I thought that instead I would carefully and methodically go through the data once more, compile it and organize it by category. However, writing helped me to go farther, deeper and closer to the actual analysis I needed for chapters than would have a methodical pass at the data alone. Writing at that juncture proved an essential part of the analytical process.

As I mentioned, I began with 'symbolizing' and needed to write a memo on each toddler in turn before writing a 'symbolizing' chapter. However, I wrote the memos in preparation for writing the chapter. I already had that goal in mind. What were to become chapters changed and changed once more, but writing in full sentences and paragraphs within a common theme forced me to make full sense of all of the relevant data in necessary ways. The depth I achieved in the memos proved crucial in enabling me to write those first chapters and to determine what the chapters might be in the first place.

A few helpful notes for reading

I draw upon two material sources of data, field notes and video tape. I summarize from both these sources and when I do, I speak in the past tense, because the events already occurred. In the summaries, I do not necessarily make mention of the particular sources of the data. When I wish to make my data available to direct inspection and offer a fuller description of events, I either quote directly from my expanded field notes or give extensive descriptions of video footage. Here I use the present tense. This helps to convey the sense that I recorded what I experienced in the moment, whether in writing or on camera.

Regardless of source, I indent these descriptions as one does with extended quotations. However, only the field notes data are true quotations of written text. They embody my perspectives from the field, fixed in time so to speak, whereas video descriptions were drafted for purposes of writing chapters. They remained malleable throughout and naturally highlight aspects of events relative to the analysis at hand. They are not complete, blow by blow descriptions. As the notes offer the possibility of 'real' citations, I simply include excerpts as quotations without noting the source. It can be assumed that unreferenced quotes come from field notes. Since my 'quoting' video data is slightly unorthodox, I deliberately note when 'quotations' are in fact descriptions of video footage.

I took the liberty of editing quotations from field notes in regard to grammar, tense, punctuation, spelling, and typographical errors. I also substituted names for pronouns, pronouns for names and pseudonyms for names. However, the quotations otherwise reflect what I wrote during the time of observation.

I also use quotation marks in ways that may be considered unorthodox. I directly quote from the data (including what a subject said) as well as from other sources. For these quotes, I use the familiar, double quotation marks. However, I also use quotation marks quite frequently to highlight certain words within the text, to emphasize their functioning as signifiers, and to denote terms, the toddlers' invented words, unorthodox usage and colloquialisms. To distinguish between direct quotations and these other 'quotations,' I use single quotation marks for the latter. In light of my use of quotation marks to distinguish signifiers, I distinguish signifieds with *italics*, in addition to using italics in a standard way, e.g., for emphasis.

I provided the 'cast of characters' earlier in this chapter, but list them here for clear and easy reference. The three toddlers are Jacob, Jeremy and George, observed respectively. Jacob's parents are Ann and Carl. Jeremy's parents are Birgit and Lao. George's parents are Lynn and John. Patty was George's day care provider and Milady babysat George during one observation.

Chapter IV: Naming

In her memoir *Thinking in Pictures and Other Reports from My Life with Autism*, Temple Grandin, a highly functional autistic adult describes the foreignness of signs. She opens with, "I think in pictures. Words are like a second language to me" (1995, p. 19). An accomplished designer and engineer, she invents things directly from pictures, without resorting to language. She explains how it is easiest for autistic children to learn nouns, since they signify objects that can be seen, pictured. Even for her as an adult, generalized concepts come from having built up catalogues of specific experiences. She offers an example:

> If I think about Great Danes, the first memory that pops into my head is Dansk, the Great Dane owned by the headmaster at my high school. The next Great Dane I visualize is Helga, who was Dansk's replacement. The next is my aunt's dog in Arizona, and my final image comes from an advertisement for Fitwell seat covers that featured that kind of dog. My memories usually appear in my imagination in strict chronological order, and the images I visualize are always specific. *There is no generic, generalized Great Dane* (p. 28, my emphasis).

Grandin also states that she cannot connect her visual images with mathematical symbols. Perhaps this is due to her way of thinking, which starts from multiple specifics and works towards generalities (which still contains specifics within them). Mathematical signs always refer to general categories, even more so when moving away from using mathematical tools to solve particular real world problems to working with them in their own right, such as when exploring properties of number.

I mention Grandin in order to draw a contrast between her way of thinking and the thinking of Jacob, Jeremy and George, as reflected in the ways they *do* readily learn language, use signs, name their world. In this chapter, I will present and discuss data that show the complex and varied ways the toddlers come to name their world. I do not wish to say that these boys do not have strong visual skills, for they might. Grandin explains how Einstein also thought visually, although in vaguer images than hers, and unlike Grandin he could encode his ideas with mathematical symbols.

Being able to represent mathematical ideas symbolically is crucial to doing mathematics. Skemp (1987) explains, "*Abstracting* is an activity by which we become aware of similarities among our experiences" (p. 11). Language enables us to name these abstractions and thus form concepts. However, more importantly for mathematics, language enables us to bring "primary concepts...together to form concepts of higher order" (p. 15). By first naming these higher concepts and then turning these names into symbols, one can operate upon them independently of their referents. For example, rather than dealing with *three apples*, one can consider the idea of *three* independently of any concrete object to which it can refer, and through it and other numbers explore properties of the number system. "Symbols and the gap, the separation between symbol and object are fundamental to the very possibility of mathematics" (Pimm 1995, p. 109).

That children can and do learn language is obvious. That the processes for learning language also comprise an important component in doing mathematics is perhaps notable, but what makes these processes worth studying? For one, as the data will show, coming to name is hardly straightforward. It is multi-faceted, complex and requires effort and often imagination. However, perhaps more importantly, the data will also show how, as part of learning language, children invent language. They invent symbols and signs.

The linguist Michael Halliday (1977) views children as creating systems of language, ones that bear resemblance and make use of resources available in their mother tongue, yet are unique to them. This capacity to create language is central to the mathematical endeavor. In his study of the evolution of mathematical concepts, the mathematician Raymond Wilder

(1968) gives symbols credit for the "tremendous spurt of mathematical analysis during the seventeenth century" (p. 171). As he explains,

> So long as one was tied to a natural language, the language of common discourse, or even special *words* created expressly for mathematical purposes, mathematical advance was hampered....For on analyzing the mathematical progress at that time, one is struck by how much it actually consisted of the invention of new and powerful symbolic apparatus (p. 171).

Wilder views human beings as uniquely possessing 'symbolic initiative,' the ability to "assign symbols to stand for objects or ideas, set up relationships between them, and operate with them on a conceptual level" (p. 5). They also possess the contrasting ability of 'symbolic reflex,' which enables them to learn and respond to symbols others have created. Both are important to doing mathematics and as Jacob, Jeremy and George will teach us, both are involved in using language.

Symbol adoption

Wilder's term 'symbolic reflex' captures what he likely saw (and criticized) of math teaching: students passively accepting symbols as received wisdom, their making use of them as if automatons, not thinking, merely reacting. Hence he likened use of given symbols to a reflex. Indeed, once symbols of any kind become familiar enough, become ***signs***, then one does respond to them automatically, without pause for reflection. Yet the toddlers were encountering countless symbols for the first (or second, or third) time. Their adoption of conventional signs required active effort to determine meaning and usage. I therefore wish to describe the toddlers' use of *a priori* conventional symbols as 'symbol adoption' rather than 'reflex.'

That symbol adoption and not mere reflex occurs with toddlers is not readily apparent. Toddlers are often popularly viewed as imitating speech and indeed imitation of sound is an important activity for achieving effective usage of oral language. However, learning to use words in purposeful ways could not possibly be achieved through imitation alone. Toddlers make active sense of conventional signs and come to their own meanings for words. This is most apparent from their mistakes. One hears frequent tales of all animals being named, 'dog,' for example. Piaget (1962) provides a similar example from one of his children: "[T]he meaning of a term such as 'bow wow' in the case of J. changed in a few days from dogs to cars and even to men" (p. 220).

Mistakes such as these can be thought of as conflicts between conventional usage of conventional signs and toddlers' own idiosyncratic usage.[1] I will discuss such conflicts in a later section. Although they support the claim of toddlers' active symbol adoption and would be appropriate to discuss here, other more subtle evidence also supports the claim and permits additional facets of symbolizing to be considered.

George seemed to show a particular interest in conventional names. He had his own idiosyncratic way of asking how things were named (to be discussed in the next section) and would spontaneously name objects he saw or pictures in a book. His parents encouraged this activity; they would ask, 'What's this?' of an image (drawing or photo) and George would give a name, several such exchanges occurring in a row. George took note of how different things could be named the same and the same things named differently. The following incident occurred while George was playing in the bathtub.

> George points to an eye on one of the rubber ducks and says, "eye." I agree. George points to each eye on each duck, repeating the word. I ask,

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[1] I use 'idiosyncratic' to denote signs that are particular to the individual toddlers, unconventional, invented. I do not mean the term to carry the connotation that Hughes (1986) ascribes, whereby idiosyncratic signs are unsystematic or that their logic is undetectable.

> "Where are George's eyes?" George touches his own eye. Then he points towards my eyes and says, "eye." I touch my eye and agree. I ask if he sees any more eyes. George looks down at the picture strip in front of him. He points to eye dots on the teddy bear picture and then touches the eyes on a cat picture.

The video camera recorded a similar example, although I failed to understand George's intent at the time.

> George stands by his kid-sized car on the porch and points to its door, saying "see door?" I answer in the affirmative. He then points straight out in front of him and says "door." I do not know what he is talking about and do not respond. George continues to walk forward, away from the car, with pointing arm outstretched saying 'door' repeatedly. I ask George if he wants to go inside. He tells me, "no."

What I failed to see at the time was that George was pointing at the door from the house out to the porch, telling me that it is named 'door' too.

Naming multiple instances of eyes 'eyes' and of doors 'door' reflects more than just adopting appropriate language use. It reveals the establishment of concepts. If all of these different items are called 'eye,' then they all belong together, to the same category, the same concept. Piaget (1962) concurs, noting a dialectical relationship.

> [L]anguage makes possible the construction of concepts, for the relationship is naturally reciprocal and the capacity for constructing conceptual representations is one of the conditions necessary for the acquisition of language (p. 221).

Skemp (1987) appears to agree when he states, "Making an idea conscious seems closely connected to associating it with a symbol" (p. 57). Concepts and language interrelate, but when a child first articulates words, some degree of concept formation may necessarily precede. This possibility is supported by another example from George.

George enjoyed clinking glasses with his parents as they said, 'cheers' at dinner. One day while in the bathtub, George enacted the ritual with his dad. George used two fish-shaped toys that held water (one 'clink' with each) and his dad used an imaginary object (i.e., a closed fist). Then a couple weeks later the following occurred:

> John undresses George and puts him in the bath. George has a little plastic 'cup' in his hand. He reaches towards his bag where his bath toys are kept. Jim isn't sure what he wants and offers a couple things in turn. George tells him "no." When he gets to the 'fish cups,' George says, "yep." John hands them to him and George hands him the other cup he has.

In two different ways, George communicated that he viewed his fish toys as 'cups,' first through use of them in a ritual normally performed with cups and second through direct exchange of one 'cup' for another. George also revealed this view by drinking and pretending to drink from them. He placed the toys in the category 'cup,' and displayed this concept through action prior to his use of the word.[2]

Once George did adopt the word 'cup,' there were additional things to explore about cups and names for cups. On a particular day at day care, the video camera recorded several relevant episodes.

> George is at day care and seated at a picnic table among numerous toy items of food, dishes, cookware, etc. He is happily playing with them, pretending to drink from a baby bottle, eat a piece of toast, etc. George looks at me and says something that sounds like "I getta juice." He gets up from the table, goes over to the toy 'kitchen' and picks up a toy sip-cup (the sort with a lid so liquid will not spill out). He holds it up to show me and

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[2] Although 'cup' was not yet part of George's active vocabulary, he must have surely understood the word as naming a particular object(s). Language comprehension could have played a role in George's concept formation. For example, upon seeing George drink from the 'fish cups,' his parents could have called them that. Although such speculation complicates the relationship between words and concepts, one could still expect use of a word to reflect a more solid understanding or concept than a nonvocal response, e.g., a situation in which George pointed to a cup upon being asked 'where is a cup?' That 'solid understanding' may even be necessary before a toddler ventures to use a word in the first place. Bloom (1998) discusses findings of children's language learning that support this interpretation, which indicate that "much of word learning involves mapping words onto concepts that exist prior to language, and do not support the alternative that such concepts must be created through the shaping powers of words" (p. 218).

> says, "Here juice." Then he says "milk" a number of times and finally "cuk" (for 'cup') and returns to the table.
>
> George continues to explore the items on the table. Then he takes two items, one in each hand. One is a short cup with a handle, like a coffee cup. The other is a larger, tall cup with angled sides, like a tumbler. Holding the first cup up, George declares "cuk," then raising the other he says, "nother cuk." A couple minutes later, George shows me half a toy egg shell and says, "cuk." I show that I don't understand. He says "cuk" three times and then clarifies saying "juice." Then he picks up another cup, one that I had previously recognized as such, and says, "cuk."

George showed how many different items could be called 'cup.' The relevant characteristic appeared to be a shape that permitted the holding of liquid, hence something in the shape of half an egg shell could be called 'cup,' as could a fish-shaped item with a depression capable of holding water. In fact, George later raised up one of his 'fish cups' and declared, "cuk."

Naming and accompanying conceptualization clearly involve categorizing, giving multiple items the same name. However, they also involve giving the same item multiple names, which George did as well; he called a single item 'juice,' 'milk' and 'cup.' Multiple names reflect multiple features, characteristics or uses and demonstrate a richer concept.

Another instance of multiple naming began during my time with George. He started saying his mother's first name, 'Lynn' and delighted in saying it. 'Mama' had been George's name for her. Adults called her 'Lynn.' George's use of 'Lynn' positioned him differently. He was not just her baby. He was like an adult. No wonder he said it with enthusiasm and a little mischief.

A third function of naming (in addition to classifying and offering multiple descriptions) is to draw distinctions. One cannot give everything the same name; otherwise important differences are lost. In an episode caught on video, George made sure that did not happen.

> I film as George plays with his matchbox cars. He lines them all up along the edge of a shelf as he frequently does. I ask if the camera can see how he does that, how he 'lines up his cars.' George quickly tells me, pointing at his vehicles while naming them, that there is also a 'truck' and a 'van,' not only cars.

To George, who has a great interest in vehicles, trucks and vans are different from cars and should not be equated with them, demoted to their level: I was oblivious to key differences.

In the toddlers' worlds, there were conventional gestures in addition to verbal signs or words. According to Armstrong *et al.* (1995), the term 'gesture' has been used to denote a range of meanings, from describing movements that signify or co-occur with speech, sometimes only of the hands and arms, to any movement of the body whether or not communicative, to speech itself also described as 'articulatory gesture.' I use 'gesture' to mean bodily movements that communicate, from those that are very specific and deliberate (resembling the signs of sign language) to more amorphous ones such as those adults use along with speech (waving of hands, pointing, etc.). Specific, deliberate movements that have the potential to communicate, but do not yet signify, should also be considered gestures, 'gestural symbols' in fact.[3] I wish 'gesture' to also include articulations, but ones that do not function in language, articulations that are not phonemes, such as a sigh or a smack of the lips. These will be called 'articulatory gestures.'

George figures as the most adoptive of the toddlers in the gestural realm. His gestures were very adult-like in that they largely accompanied speech and were vague. However, one adopted gesture stood out as deliberate, that of George wiping his hands together to signal

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[3] I think that adult body movements that accompany speech and are mere fidgets as it were, but have no communicative function (except perhaps to say 'I am nervous') should not be considered *gestures*. I do not see them as holding communicative potential.

'all done,' which he often enacted to accompany the words. Yet, he was not that wedded to this meaning. I once misinterpreted it.

> George is at the stove. He hands me invisible 'food' with both hands. Then he wipes his hands together. I ask him if he needs to wash his hands. George goes down the hallway towards the bathroom at the end and returns, then does this once more. On his return he puts his hands out in front of him and says, "hand."

Unlike what occurred with my 'car-van-truck' mistake, George was not troubled by my mistake here. He did not correct me, but instead took up my suggestion that his action indicated his hands were dirty from cooking and he needed to wash them.

In contrast to George, gesture seemed to be Jeremy's preferred mode of communication, even his native tongue (before English and Chinese). Jeremy adopted conventional gestures, using them with great animation: pointing, reaching, raising up his arms, pulling someone by the hand. He gave them conventional meanings, but also expanded their usage. For example, Jeremy pointed to a desk lamp, not because he wanted me to hand it to him, but because he wanted me to turn it on. He reached towards crayons sitting on a shelf, but he also communicated he wanted them to color by first touching a pad of paper several times saying, "ooo, ooo, ooo."

When Jeremy wanted an adult to lift him, it was always for a specific purpose. He made sure to stand by the relevant object before raising his arms. He stood under a tree and raised his arms because he wanted to hang from it. He did the same standing by my camera because he wanted to look through it. He used the same gesture when he was done eating and wanted help getting down from his chair.

Another gesture whose usage Jeremy expanded was the articulatory gesture of a gasp. A gasp is a spontaneous reaction to something surprising and/or scary. When one employs it deliberately rather than automatically, it becomes a gesture.[4] Jeremy gasped when he heard a train pass, an airplane fly overhead, a toilet flush, the telephone ring, water run, the vacuum turn on and a paper clip drop. That Jeremy gasped during mundane and not particularly surprising situations reflects an expanded use of the gesture. He employed it with drama to impart sentiments of surprise or fear to a situation that was not normally surprising or scary. For example, the first time the vacuum turned on, Jeremy gasped and quickly grabbed some small dolls as if to protect them. The second time, Jeremy's gasp was accompanied by an O-shaped mouth and wide eyes.

'Talking' was another articulatory gesture that Jeremy employed. It can be considered conventional in that talking is a conventional way to communicate. Jeremy adopted his form of 'talking' to communicate various purposes that talking serves: to explain, make a request, show concern, etc. Jeremy's 'talking' just happened to have almost no intelligible words.

Jeremy articulated sounds that resembled speech and used accompanying gestures, objects of reference, and change of pitch and form (e.g., whisper or shout), which together conveyed the feeling that he was truly talking. When I first encountered Jeremy's 'talking,' I asked Birgit if he were speaking Chinese. The answer was 'no.' Upon seeing video footage of Jeremy, my research group (MLRG) asked me the same question. In other words, to an observer, Jeremy appeared to be speaking a genuine language, just one the observer did not understand. While Jeremy's 'talking' served functions of genuine

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[4] While I am drawing a distinction between a spontaneous and deliberate symbolic act, it may be that all such signifiers are somewhat learned. A baby's smile might be initially spontaneous, but social reinforcement encourages it to recur and gives it particular meanings. Likewise, while a gasp might be a spontaneous reflex, it might also have its particular meaning due to social interpretation and may not exist or carry the same meaning in all cultures.

language, and he frequently succeeded in communicating those functions, it differed from language in that individual elements failed to signify.[5]

Jeremy used his 'talking' as part of making a request. For example, he 'talked' while reaching for a box of Legos, wanting me to get it for him. He asked a boy for a turn with his basketball by 'talking' and reaching out his arms.

Jeremy also 'talked' to argue. He tried to convince Birgit to let him leave with her by bringing her his coat and 'talking.' After throwing his sand bucket and having it taken away, Jeremy stood by it and 'talked,' asking for it back. He got it.

Jeremy 'talked' to show concern. Once when I was showing Jeremy my video camera, I removed an attached flap. Jeremy 'talked' in an awed voice. He attempted to replace the flap but then removed it and 'talked' again, punctuating his 'speech' with "oh my." While playing with markers, Jeremy got some marks on his fingers. He held them up to show me and looked at them himself. He 'talked' as if with great concern.

Jeremy 'talked' to express satisfaction. One time he was hungry but refused several items Birgit suggested. When she finally offered him a muffin, Jeremy laughed and 'talked' animatedly, clearly happy to get one. He proceeded to down several.

Jeremy 'talked' to 'explain' (or perhaps pretend to explain). For example, Jeremy pointed to pictures on the wall and 'talked' as if explaining to me something about them. While exploring my video camera, Jeremy frequently turned to me and 'talked' as if discussing

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[5] Donaldson (1978) suggests that children might learn the meanings of situations in which speech is used even before having the notion that individual elements of speech, i.e., words have meanings. Jeremy's 'talking' might therefore reflect an understanding that speech is used in particular situations. He employs it when wanting to convey the meaning that such situations are occurring.

what he saw. One time when Jeremy was playing on the see-saw at the playground, he stopped and 'talked' to his mom, gesturing towards the see-saw, seeming to explain what had been going on.[6]

Symbolic adoption includes adoption of conventional words and gestures, but the toddlers pointed out an additional aspect -- adopting convention for its own sake. In other words, verbal and gestural signs generally signify in very clear ways. They name objects and actions and serve purposeful functions. Jacob showed an interest in a verbal sign that failed to signify a particular signified, rather it served a social function. It was the expression, 'thank you.'

Several months into my observations, I noted that Jacob was suddenly saying 'thank you' frequently. I wrote, "I am impressed by how often and consistently Jacob is saying, 'thank you.' He said it after his mom gave him a cookie and even after his grandma wiped his nose." I also wrote, "Ann says Jacob does not always use it appropriately, sometimes saying, 'thank you' when he gives someone something."

'Thank you' is not one of a child's first expressions. Some children (and adults for that matter) never seem to learn it. Parents are frequently seen reminding their children to say it. 'Bye-bye' can be viewed as similar to 'thank you' in that it does not name and yet children regularly learn 'bye-bye' very early. However, to young children, 'bye-bye' could name the action of going away and 'hi' the reverse. In fact, children say 'go bye-bye' to mean just that. So to young toddlers, I believe 'thank you' falls into a particular category of expressions that serve a social function; they take on a particular role in a linguistic social system and do not signify something external to that system. In fact, they are *symbols.*

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[6] Jeremy also 'talked' as part of telling stories, discussed in Chapter V.

The toddlers clearly held a ready interest in symbolic adoption, albeit with their individual attentions and twists, yet they also did not hesitate to take symbolic initiative. While they eagerly adopted signs from the world around them, they also invented their own and taught them to the world.

Symbolic initiative

Jeremy was perhaps the most forward of the toddlers in taking symbolic initiative. He had a number of conventional words in his vocabulary from English and Chinese and he clearly understood both languages; yet when he spoke, he more often used his own signs, many of them gestures. Adults even 'spoke' his language of gestures back to him, when Jeremy ignored their words. Birgit would raise her arms like Jeremy to ask him if he was done eating and wanted to get down. Once when Jeremy was 'riding' Lao like a horse, I used a gesture (opening and closing fists) to tell him to hang on to Lao's shirt after he ignored my verbal pleas. Even so, Jeremy continued to ignore my intent, first mirroring my gesture as I sometimes did with him in a game before finally capitulating and grabbing on.

Jeremy invented numerous gestural signs on the spot in order to communicate in a unique situation. For example one time, Jeremy lifted his arms and rocked his upper body singing "dee, dee, dee..." Birgit asked, "Oh you want the dee, dee music?" She got up and turned on a butterfly mobile by the window. That was indeed what Jeremy wanted.[7] Another time, Jeremy was exploring a music box. On the top were some little glass birds. Jeremy touched one of the bird's beaks. Birgit said, "That's the bird's mouth." Jeremy sniffed and attempted to wiggle his nose. Birgit answered, "I suppose that could be his nose too," thus

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[7] This gesture can be seen as a variation on one Jeremy used for dancing. Pairing it with a particular utterance indicated the unique object of the butterfly mobile.

interpreting Jeremy's sentiments. One one occasion, Jeremy pulled off my mittens and opened the flap on my coat, telling me to take it off and stay a while. In these episodes, had Jeremy used body movements alone, his gestures may not have served as effective signs. Coupled with utterances and references to objects, they did communicate. The first two gestures described above referred to objects and the third to an action, that of taking off my coat.

Gestures are well suited to signifying actions since they too involve movement. Jeremy took advantage of the medium's potential, employing numerous gestures that fit the principle, 'do as I do.' When Jeremy wanted an adult to continue an action she had stopped, he mimicked the action. To ask an adult to continue spinning him around, Jeremy said, "whee, whee..." and either swung his arms side to side, spun around full circle, or shook his head side to side. When I chased Jeremy with wiggling fingers and stopped, he asked me to continue by wiggling his fingers and saying, "ticka, ticka." To ask Birgit to continue bouncing him on her leg, Jeremy jumped up and down saying, "hoo, hoo, hoo..." often astride her leg.

Jeremy also demonstrated to adults the actions he wanted them to take. For example, Jeremy bent his body into a V-shape, hands and feet on the floor. Then he motioned for me to do the same, 'talking' as well. In another incident, Jeremy shook his head side to side and I mirrored him. He did it again and turned to look at Birgit. He waved his arm side to side and towards her saying, "this," telling her to do the same.

Jeremy employed objects in telling adults to 'do as I do.' For example one time, Jeremy handed his mom a toy cow and said, "dink," a way he indicated dancing, but Birgit did not understand. Jeremy picked up another doll and made it 'dance,' showing his mom what he wanted her to do. When Jeremy was coloring, he handed a crayon each to Birgit and me to tell us to color too. After putting a vacuum attachment to his mouth and making a horn sound, Jeremy handed it to me and Birgit in turn, telling us to do the same. When Birgit did not respond right away, Jeremy restated his point by putting his hand to his mouth and making the sound. One time, Lao put on a birthday hat, rocked his head side to side and sang, "dee, dee, dee...." Jeremy then handed the hat to me. I put it on but did not sing. So Jeremy rocked his head and arms side to side singing, "dee, dee, dee...." I listened to him and sang. When I stopped singing, Jeremy reached his arm out towards me and twisted his hand as if turning up the volume on a stereo, telling me to continue.

The gesture of outstretched arm with twisting hand is worth particular mention. Jeremy based it on an action adults used when turning on the stereo. If he wanted the stereo played or the song changed (such as when a song too slow for dancing came on) he used his gesture in the stereo's direction. Although Jeremy's gestures mostly followed the principle, 'do as I do,' they were generally quite malleable as to form; however, this 'play music' gesture was interestingly codified in Jeremy's 'language.' Jeremy used it over and over again. I never saw him use a different signifier for the signified of playing music. As in the example above, he even expanded its usage to situations other than the one from which it was derived. He told me, a person, to sing.

The general malleability of Jeremy's gestural signs is well demonstrated in a videotaped episode I call 'Dancing Dinosaurs.'

> Jeremy has on his arm a very scary looking dinosaur puppet which he makes growl. Birgit introduces a stuffed Barney doll into the scene, who also happens to be a 'dinosaur.' She makes him growl, too. However, Barney must not look very scary to Jeremy. Jeremy grabs his nose and says, "dance" and then "dink." Birgit takes Barney and makes him growl again. Jeremy again tells her to make Barney dance by saying "dance" and "dink," but this time, he also moves his hand as if it were dancing, bouncing it on the table, fingers to wrist. Birgit complies and makes Barney dance. When Barney stops, Jeremy again says "dink" and "dance" and pushes Barney towards Birgit. She again complies.

> Jeremy stops the action by giving Barney a couple 'drinks' from his orange juice and taking one himself. To ask for the dancing to continue, Jeremy sways side to side saying "dink" and then sings, "dee, dee, dee, dee, dee, dee." Barney is losing steam, so Jeremy has to ask him to dance some more. He first says, "dink" but follows it up by bouncing his cup as if it were dancing and singing, "dee, dee, dee, dee." Still another time Jeremy bounces his cup and sings, "dum digga dum digga dum dee dee digga dum dee dee." Jeremy once again asks Barney to dance with "dink" and singing, "dee, dee, dee, dee." Jeremy also responds with identical utterances when I ask whether his dinosaur can dance too. He then makes his dinosaur dance and sings accompaniment. Barney joins in, too, of course.

This episode included eight communications for 'dance.' They involved gestures and utterances, including the conventional word, 'dance,' Jeremy's own word 'dink,' and various portrayals of singing. The gestures mimic dancing in some way -- a dancing hand, cup, or abbreviated version of dancing (swaying side to side). Jeremy was telling his mom 'do as I do,' signifying dancing in multiple ways, even within this single episode.

Jeremy's idiosyncratic word 'dink' appeared a number of times. One might wonder why he used it when he had a conventional word, 'dance.' To answer this question, it is helpful to look at other examples of Jeremy's idiosyncratic verbal signs. Some have appeared above, including, 'dee, dee, dee,' 'whee, whee,' 'hoo, hoo,' and 'ticka, ticka.' All have a sound-like quality to them and appear to be verbal imitations of sounds or *onomatopoeia*. 'Ticka, ticka' likely derived from 'tickle, tickle' but for Jeremy the sign may have had nothing to do with tickling *per se* but rather signified 'the sound' of wiggling fingers.

Jeremy used numerous other examples of onomatopoeia, including: 'ding, ding,' 'moo,' 'vrmmm, vrmmm,' 'yum, yum,' 'yuck,' 'choo-choo,' the Chinese version of 'quack, quack,' and 'piew, piew' for the sound of a gun (which may be a Chinese form of 'bang, bang'). While these are conventional uses of onomatopoeia, he used some unconventionally, letting them name certain objects such as a music box, cow, train or duck, rather than having them merely signify the sounds the objects make. How this sort of naming may have taken place can be intimated from an episode during which Jeremy was exploring items in a desk drawer.

> Jeremy takes out a small cardboard box containing paper clips and some thumbtacks and puts it on top of the desk. He says, "ow," and picks up a thumbtack, holding the sharp end between his thumb and forefinger. He plays with it and another one for a while, using both hands sometimes. I watch carefully to make sure he doesn't hurt himself. As he manipulates them, he says, "ow," periodically. His mom comes in and we manage to get him to put away the thumbtacks and go to the living room.

Jeremy could have been saying 'ow' because he was pretending the thumbtack was hurting him. However, during the episode Jeremy did not signify pain in any other way and being the dramatist that he is, I would have expected faces if not jumps and yelps.[8] A more likely explanation is that Jeremy was naming the thumbtacks 'ow.' He made use of a sound he had heard an adult say when handling a thumbtack.

Jeremy's predilection towards onomatopoeia, in both symbolic adoption and invention leads me to believe that 'dink' employed onomatopoeia as well; it was the *sound* of dancing. Just as other forms of onomatopoeia named the objects that issued them (e.g., 'choo-choo' for *train*), 'dink' named *dancing*.[9]

George also made use of onomatopoeia in symbolic invention. He pretended that he was certain animals by issuing their sounds. George pretended to be a dog by crawling on his hands and knees and saying, "mmh, mmh." George would growl as his parents named

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[8] Furthermore, Jeremy was a daredevil and impervious to pain. He would fall, even from some height and pick himself right up again. I only saw him cry when he bled or hit his head pretty hard. As pain was something to which Jeremy chose not to attend, I doubt he would play out pain in pretending. There were no other incidents in which Jeremy even hinted he might be pretending pain.

[9] This sort of naming, letting the sound something makes stand for the thing itself, is a form of metonymy. The toddlers employed other forms of metonymy in their namings. Metonymy as a systematic means for generating names will be discussed in the 'Mathematical connections' section of this chapter.

him ‘lion,’ ‘tiger’ or ‘bear.’ When they asked for a monkey, George made the sound, ‘eee, eee, eee...’ while putting his hands in his armpits. George also offered animal sounds in response to the question, ‘What sound does a ______ (animal name) make?’ In addition to using animal sounds to conjure up the animals, George also used animal sounds to name; they constituted his first animal names.

For example, he named his rubber ducks, ‘little quack.’ When Lynn asked George to name images on a picture strip in the bath, he gave conventional names such as ‘star,’ ‘house,’ ‘car,’ ‘shoe,’ for those items, but for the cat picture George said, “meow.”

Towards the end of my observations, George had begun to adopt some conventional animal names. They intertwined with onomatopoeia in his naming of animal pictures. He said, ‘owl,’ ‘duck,’ and ‘lion,’ but ‘maaa, maaa’ for a sheep and horse, both ‘how, how, how’ and ‘mmh, mmh, mmh,’ for a dog, ‘meow,’ for a cat, ‘peep, peep’ for chicks, ‘eee, eee, eee’ for a gorilla and ‘loo, loo’ for a rooster. When George wanted to *become* a rooster, he said a full-fledged, dramatic, “loo, loo, loo, loo, loo,” his version of ‘cockadoodledoo.’ ‘Loo, loo’ was an abbreviated form George used to name.

While George derived his idiosyncratic animal names from the sounds the animals made, he had other idiosyncratic verbal signs that appeared completely arbitrary. ‘Nana’ was George’s name for his favorite blanket and ‘gaga’ for ball, any ball, real or image (i.e., picture). I do not believe these were onomatopoeia or somehow otherwise derived from characteristics of the objects themselves or from conventional signs. Rather, they were utterances, symbols, that George likely was able to make consistently back in infancy, which he then assigned to certain beloved objects.

One particularly notable idiosyncratic verbal sign rose around the time of my first visit with George:

> John tells me that lately George has been adding a 'da' suffix to words, most nouns it seems. He isn't sure what it means. Perhaps 'the' or maybe 'my' were his thoughts.

'Da' appeared in my presence that day, however, by itself. George hit a pillow on a chair several times saying, "da, da." I learned that he wanted me to sit there. 'Da' continued to surface throughout the course of my observations and eventually came to take on a particular meaning, at least to adults in George's world.

> George takes out a pop-up toy -- one with four buttons and four doors that pop up with four animals coming up. He opens each one in order and says "da" and points to each animal. I name them as he does so. 'Da' is clearly George's way of saying, 'what's this called?' or 'name this.'

However, on that same day, George also handed me a crayon he found in a toy chest and said, "da." With a pointed finger, he touched several keys on a xylophone, saying "da" at each one. Looking at all recorded instances of 'da,' I now see it as having a more expanded meaning, similar to the word *this*, which could attain particular nuances dependent on context. In the above examples, 'da' could have meant 'look at *this*' and even 'sit on *this.*' George continued to use 'da' in this way. Towards the end of my observations he said "Helene, da" and touched a chair where I was to sit, although it could have here meant '*this* is for Helene.' I had already been sitting down, but in a chair George viewed as belonging to someone else (John in fact). 'Da' could also perhaps have substituted for a specific name. It was a way for George to signify an object although he had no name for it.

Even so, adults all responded to George as if he meant 'name this,' when there was something for which they had a name (e.g., I did not know how to name the keys on the xylophone). When George said, "da," pointing to an illustration in a book, John responded, "That's a red bird, a cardinal perhaps." On another occasion, after lifting up his feet and saying, "shoes," he then pointed to Milady's feet and said, "da." She said, "I have socks." George then pointed to my feet and said, "da." Milady answered, "Helene has bare feet. George has shoes." Although 'da' may have begun as a syllable that made many appearances in various ways, and even playfully (i.e., not necessarily always with intent to communicate something), it came to have a particular meaning within social interaction.[10]

Although George did not appear to invent gestures involving bodily movement, he did use an idiosyncratic articulatory gesture. As with his gestures of body, this gesture accompanied and enhanced the meaning of verbal signs. George's idiosyncratic articulatory gesture was a special strong and deep intonation to his voice that revealed a sense of awe. On my first visit, he used it numerous times while saying "c(r)uck," and "big c(r)uck," meaning *truck*. I asked Lynn about this. She explained that truck received a special enthusiasm reserved for it alone. However, it turns out that she exaggerated somewhat. George later used the same intonation frequently when saying 'lion' and 'big, big lion,' another object of great interest. Then one time the video camera caught George using his intonation in a unique situation.

> George stands in front of a wall of shelves housing many, many mugs (perhaps a hundred or so). George points towards them, although not clearly at any specific one and says, "Dada's cuk," meaning *cup*. Then George names one, "Mama's cuk." Lynn responds, "Mama's cup. Oh, thank you George." We ask if there is a 'Baby's cup' (meaning for George) but he does not respond. Instead, he says, "Dada's cuk," in his special, awed voice. Lynn says, "Oh, Dada's cup's a big cup."

[10] Before George used 'da' as a way to ask for names or to name the unnamed he had another means for doing this. Also on my first observation I wrote, "George hits the truck and John names it, 'red truck,' then he hits the bus and John says, 'yellow bus'."

George's articulatory gesture therefore seemed to evolve into having a clear meaning. As Lynn articulated, it signified that something was *big*. The signified *big* was intrinsic to certain deserving items (truck and lion), but could also be signified in describing other objects.

That George's big voice meant *big* is supported by another piece of evidence. Once when George was naming animals in a book, Lynn pointed out to me that George used a little voice in his naming *little* chicks, 'peep, peep.'

Jacob took symbolic initiative as well. He had idiosyncratic gestures for 'come here,' tugging at my shirt, and 'help me down,' standing on his chair. However, his idiosyncratic verbal signs are particularly worthy of attention. Jacob used 'momo' to signify both *motorcycle* and *lawnmower*. 'Mimi' meant *music*. For his rocking horse, he said, 'hee-hee.' These signifiers all appear to be derived from conventional signs.

Jacob used 'mimi' flexibly in various instances where the conventional signifier 'music' could be used. He said 'mimi' to request that the stereo in the living room or radio in his room be turned on. 'Mimi' also indicated a book that had numerous buttons with illustrations that when pushed made corresponding sounds. (Or it could have indicated specific buttons on the book that played tunes; others played noises, e.g., hammer noise, telephone ring, etc.). 'Mimi' referred to a toy xylophone. 'Mimi' also named various kinds of music, which Jacob began to differentiate. 'Mommy's mimi' was up-tempo dance music, 'Daddy's mimi' was country or easy listening, and 'Jacob's mimi' was children's songs.

Jacob based other idiosyncratic signs on conventional ones. In the following examples, he gave entire expressions idiosyncratic meanings. I was video taping and Jacob showed interest in the camera. I lifted him so he could see through it. The camera was pointed towards Carl on the sofa and I asked Jacob if he could see Daddy. Later that day, Jacob said several times, "want to see Daddy," to ask to look through the camera even though Carl was not in the room. At one point, I contradicted him, saying he couldn't see Daddy, but he could see his basketball hoop. Jacob echoed, "Want to see bee-ball hoop." On my next visit Jacob also used 'want to see Daddy' to mean he wanted to look through the camera, although Carl was not even in the house. I again contradicted Jacob and offered that he could see Pooh instead.

I do not believe that Jacob thought he could really see Carl in the camera each time. 'Want to see Daddy' successfully signified *looking through the camera* in Jacob's first use of the signifier. Jacob thereupon continued to let it signify the same meaning. He used it in subsequent situations, even ones in which the word for word meaning failed to apply.

Another signifier with a similar bent was the expression 'want to build house.' Jacob used it to mean he wanted to play with a set of large foam squares with letter and numeral cutouts. Jacob did not really intend to build a house, although a structure could be built from the squares that bore a mild resemblance to a house. However, other structures could also be built or in fact destroyed. Taking apart a structure actually seemed to be Jacob's preferred activity among those signified by 'want to build house.' However, when destroying, Jacob was often met with resistance from adults who were trying to build as he had literally asked. As in the previous example, 'want to build house' likely arose from Jacob letting a signifier that succeeded in signifying a first instance of a general activity (in this case, playing with the foam squares) signify all subsequent instances of the activity.

In both these examples, conventional meanings of Jacob's expressions clashed with his idiosyncratic ones. Adults therefore often failed to understand him or if they succeeded in doing so, they 'corrected' him, providing conventional signifiers for the signifieds he named. These clashes between idiosyncratic and conventional meanings were not uncommon among the toddlers. They are the focus of the next section.

When conflicts arise

It was several months into my observations that Jacob began to call me by name. When he first did, he used the wrong one. Jacob was trying to tell me to come follow him. He repeated the same sentence over and over numerous times, before he gave up trying to verbalize. (He eventually came over to me and tugged my shirt.) One set of sounds I was able to distinguish was 'wee-wa.' I asked Ann if she knew what Jacob meant. She said, "That's Lisa, my sister." When Jacob again called me "Wee-wa," this time in Ann's presence, she told him, "No, that's not Lisa, that's Helene." Jacob echoed, "Haween."

On my next visit, Jacob again called me 'Wee-wa.' This time, Carl was there and corrected him. The following visit, Jacob began to use my correct name, 'Haween' when talking directly to me. However, during a game of hide and seek, when he couldn't find me, Jacob said aloud, "Where Haween? Where Wee-wa go?" A few visits later, the video camera caught him using both names together, saying "Here, Wee-wa, Haween" while handing me a toy, even though by that time, he was otherwise exclusively calling me 'Haween,' except when deliberately in a game.

When Jacob initially attached the signifier 'Wee-wa' to me, it was a reasonable choice. 'Wee-wa' indicated a female, adult person, rather than a truck for instance. Jacob may have internally thought of me as 'Wee-wa' for some time before articulating this. Upon meeting with resistance, he began to use a conventionally accepted sign for me, 'Haween,' but 'Wee-wa' continued to figure in his mind. In fact, for Jacob both signifiers referred to me and at times, he used both in tandem. However, Jacob predominantly used the conventional sign in social context and by the end of my visits, 'Wee-wa' only appeared out loud in a game. Eventually, 'Haween' replaced 'Wee-wa' altogether as a singular signifier for me.

Jacob initiated the game during the second visit in which he called me 'Wee-wa.' When I told Carl about Jacob's 'misnaming,' he told him, "That's not Lisa, that's Helene." Jacob contradicted him, saying, "No, Wee-wa." Both Carl and I protested, saying, "No, Helene." Jacob again responded, "No, Wee-wa." Jacob and I continued this exchange a few rounds with Jacob laughing at each of his turns. At one point, I had given up the struggle and Jacob had turned his attention to his lunch, but when he finished eating, Jacob tried to resume the game by again saying, "No, Wee-wa," several times.

The 'No, Wee-wa' game made reappearances on the next four visits. Then on the fifth, the following occurred:

> Jacob pushes the play button on the answering machine next to the stove where he was standing. A message plays. It's Lisa. She says her name on the message. Ann asks Jacob, "Who's that?" Jacob says, "Haween." Ann says, "No, who was talking on the answering machine? That was Lisa." Jacob says, "No, Haween." Ann says, "No, that's Lisa." "No, Haween." "No, Lisa." When Ann tells Jacob, "No, Lisa" she uses the same voice as when telling him, "You're being silly." Then Jacob begins to tease me, calling me "Wee-wa." I answer, "No, Helene." We do this exchange several times.

Both games -- Jacob calling me 'Wee-wa' and Lisa 'Haween' -- reflect that Jacob knew the 'correct' signifier for each of us. However, he saw that he could tease by deliberately using the 'wrong' or unaccepted sign. That Jacob knew he was teasing was evident in his frequent laughter, at times forced, during the game. This interpretation is confirmed by other evidence. Once after several acts of Jacob teasing me by first handing me my pen, then pulling it away giggling, he launched 'No, Wee-wa,' the first interaction reminding him of the second. The same thing occurred following Jacob teasing his mom by clicking a pen open a few times after she clicked it closed.

While 'Wee-wa' was Jacob's initial signifier for me, and it likely remained so in his mind for quite some time, Jacob recognized that in order to be understood, he needed to adopt the convention in social discourse. His idiosyncratic signifier would only be accepted by others in play. Jacob capitulated to social pressure.

George faced a similar situation in which he gave an idiosyncratic meaning to a conventional signifier. Unlike Jacob, George outwardly resisted social pressure, at least initially. The video camera captured George looking at a book containing abstract illustrations titled, ***Brown Bear, Brown Bear, What Do You See?*** It had illustrations of bears on both the front and back covers and in the text.

> Looking at the picture in the text, George says, "lion" in his awed voice, and turning to me says, "It's a big lion." I echo him, asking, "Is that a big lion?" George says, "no, no mine" and then seeming to wait for a reaction from me says again, "lion, lion."
>
> Lynn enters the room and says, "That's not a lion. That's a bear. Do you want to see a picture of a lion?" As she steps out, George closes the book and says of the back cover, "here lion, here lion." I ask, "What's that? Is that the bear saying goodbye?"
>
> Lynn returns with a photo book of animals, open to a page of numerous specimens. She places it in front of George and says, "Show Helene where the lion is." George points to a photo of a lion and says in a voice filled with awe, "lion." He makes a scared face and growls, then says, "big, big lion."
>
> George then points to a polar bear on the same page and says "big, big lion." Lynn responds, "That's a bear." Lynn then tries to interest George in some other animals pictured on the page. George offers names and sounds that they make. He points to the polar bear again, this time saying, "da" looking at me. I name, "It's a bear."

> Lynn turns the page and asks about animals pictured there. George points to a photo of a brown bear and says, "big, big lion." Lynn says about that animal too, "No, that's a bear."
>
> George closes the photo book and returns to the first one. He flips through the pages, naming the animals and stops at the bear again. He says, "whoa, big lion." Then he closes the book and says of the front cover, "here lion, here lion." Lynn responds, "That's a bear." George says, "no, no," and picking up the book comes over to me. He shows me the picture and says, "here lion." I respond, "Mommy said it was a bear. I agree with her. I think it's a bear." Walking away with the book he says, "no, mama."

During the episode, Lynn interpreted, "Interesting how he confuses the bear and the lion, 'cause he always made the same noise for both of them too." However, to George, it all made perfect sense. A growl was George's idiosyncratic name for both lion and bear. This was accepted and even encouraged by his parents and other adults. Upon learning the conventional name for one of the animals ('lion'), he substituted it for his idiosyncratic one (a growl) and used it in all cases in which his idiosyncratic name had applied.

George decided that *bears* should be called 'lion' and was very serious about it. He reiterated his point by naming various images of *bear* 'lion' (five distinct ones) and repeatedly bringing up the issue. In the end, he directly expressed disagreement by saying, "no." He also tried to win me over to his view of things.

George was particularly insistent as to his naming of the abstract images of bears. It was them he continually named and renamed and for which he stated his final opinion before letting the argument rest. By contrast with the photos, he once said, "da," and let me name the image 'bear.' His body language also showed a more willing acceptance of 'bear' for the photos. The photos showed clearly different animals than the one for which the name 'lion' met with acceptance by Lynn and me, although they were also different from each other (polar and brown bears). George likely saw them as different animals from *lion* so he could more easily accept the use of a different name. However, the abstract images were another matter.

Abstract art is intended to be interpreted by the viewer. Do the intentions of the artist really matter? The artist for the book intended to depict a bear for the text says so. But George could not read. To him, the image was a 'lion,' and that was what he wanted to call it. Although George was perhaps willing to accept *bear* for realistic photos, he wished to continue to interpret the illustrations as he chose.[11]

Another instance of direct substitution of a conventional signifier for a previously accepted idiosyncratic one occurred with Jacob. As mentioned, Jacob used the idiosyncratic word, 'momo' to signify both the signifieds, *lawnmower* and *motorcycle*. However, towards the end of my observations, Jacob was playing with his toy lawnmower and called it 'motorcycle.' Carl corrected him, telling him its conventional name.

Jacob had learned that one object he called 'momo' was conventionally called 'motorcycle' so as with George, he substituted 'motorcycle' for all cases in which 'momo' had applied. Jacob's naming his *lawnmower*, 'motorcycle' managed to entice me. It appeared that way in my notes. A week later my notes again describe Jacob naming his lawnmower, but this time he returned to his idiosyncratic, 'momo.'

Modality of symbolic expression

The previous sections included a seeming uneven treatment of the three toddlers. All are mentioned. All engaged in both symbolic adoption and initiative. However, the forms they

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[11] Two weeks later, George told me the polar bear photo was called 'bear.' The issue of the name of the illustration never resurfaced.

used and the modes in which they expressed themselves symbolically varied from toddler to toddler. Gesture primarily comprised a mode of symbolic adoption for George, but for Jeremy, it was a mode in which vast initiative and inventive expression were undertaken. Jacob engaged in some symbolic initiative in the gestural mode, yet he undertook far more initiative with verbal signs. In particular, he derived idiosyncratic verbal signs from conventional ones and dared to interpret conventional signs idiosyncratically (Although probably not intentionally. Jacob courageously dared to express himself with conventional signs before he was absolutely sure of their socially proper usage). George uniquely invented arbitrary verbal signs. Both George and Jeremy made use of onomatopoeia and auditory gestures (which could be considered quite similar modes of expression). Jacob curiously did not. Even when pretending with objects for which conventional forms of onomatopoeia existed in the language around him, Jacob rarely used them (see Chapter V).

These differences among the toddlers' symbolic expression could simply reflect differences among them as individuals -- their personalities, interests, attentions, and ways of thinking. For example, Jeremy paid particular attention to sounds in his environment. It is therefore only 'natural' that he would try to imitate those sounds and learn the conventional ways sounds are imitated. Jeremy's interests may have then led him to extend the symbolizing of sounds to the symbolizing of activities and objects through sound, even if the signifieds did not intrinsically embody sound themselves (e.g., being bounced up and down signified by 'hoo, hoo').[12] Jeremy's attraction to onomatopoeia was so great that even though he had used the conventional signifier 'clock' to signify *clocks* and *watches* for months, he began to name watches 'beep, beep.'[13]

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[12] 'Hoo, hoo' likely came from Birgit saying, "whoo hoo" at times while bouncing Jeremy. Adults also said, "whee," as they spun him around. Jeremy latched on to sounds that he could associate with soundless activities.

[13] Birgit told me she did not even know how Jeremy began calling watches 'beep, beep.' She knew no one with a watch that beeped.

Jacob seemed particularly interested in adopting conventional social behavior as well as language, hence his use of 'thank you.' This also could explain his eager efforts at adopting conventional language, even at the risk of making 'mistakes.' He allowed his 'mistakes' to be corrected and made every attempt to adopt convention or otherwise socially acceptable language (e.g., 'momo' was an acceptable signifier for lawnmower, although 'motorcycle' was not) even if his own meanings may have made more sense to him. Jacob's turning his 'Wee-wa' and 'Haween' errors into games may even reflect an attempt to 'cover them up.'

Despite plausible explanations linking individual differences to modes of symbolic expression, one might still wonder why given Jeremy's great symbolic initiative and creativity, he refrained from taking symbolic initiative in the strictly verbal realm. All his verbal signs reflected onomatopoeia. Halliday (1977) notes:

> Children vary enormously at this age (6-18 months) in the extent to which they attempt to construct some kind of imitative phonological system... [C]hildren differ very much as regards...how far they are perfectionists -- there are some children who appear simply not to attempt things, at least in public, until they are satisfied with their own performance (p.10).

So it may be that Jeremy was a perfectionist. He did not wish to attempt conventional words until he could say them very well. He neither tried approximations, nor invented his own. Indeed, although Jeremy had fewer conventional words in his vocabulary than Jacob or George (whether in English or Chinese) he pronounced them all nearly perfectly.[14]

It is also notable that Jeremy was the only toddler of the three being raised bilingual. Perhaps this played a role as well. If Jeremy were to invent a word, to which language

[14] For example, Jeremy did not begin to say my name until several months after I had finished observing him, when he was nearly two years old. When he did, he said, "Aleen." While his 'H' may have been silent, his pronounciation was closer to convention than Jacob's 'Haween' or George's 'A-een.'

would it belong? Would it be derived or somehow resemble a Chinese word or an English word? Would it incorporate consonants, vowels and tones from Chinese or the different consonants and vowels from English? Could it contain elements from both? Jeremy showed no hesitation in taking symbolic initiative. Perhaps he just limited himself to modes of invention intelligible to both speakers of Chinese and English, ones that traversed language boundaries.[15]

Modality for symbolic initiative might also be related to characteristics of signifieds. Jeremy's gestures signified actions almost exclusively, in other words, verbs. His use of onomatopoeia included names for both actions and objects, the former named by the 'sounds' of the actions, the latter by the 'sounds' the objects made. George's use of onomatopoeia also signified objects, animals in fact -- animate, sound-generating beings. Although George could have used onomatopoeia for vehicles, he gave them conventional names. Both of George's and Jacob's idiosyncratic verbal signs named objects, nouns. George's idiosyncratic auditory gestures were the only invented signs that signified properties or adjectives.

I have drawn distinctions between types and modes of symbolizing. However, the toddlers used signs from their entire repertoire as needed, whether conventional or idiosyncratic, verbal or gestural, onomatopoeic or articulatory gesture. They used signs from different modes in tandem and moved from one mode to the other. For example in 'Dancing Dinosaurs,' Jeremy signified dancing with a conventional word, as well as with multiple idiosyncratic gestures and onomatopoeia, sometimes together in the same 'utterance.'

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[15] Both of Jeremy's parents spoke both languages to him. However, his daily babysitter spoke only Chinese and there were others in his life, like me, who spoke only English.

Mathematical connections

Ernest (1998) quotes Wittgenstein as describing mathematics as "a game with signs according to rules," or in other words a 'language game,' indeed a 'motley' of language games, whose rules are not arbitrary but rather "their form and acceptance have evolved with context-related linguistic and social practices" (pp. 76-78). This chapter's description and analysis of Jacob's, Jeremy's and George's symbolizing relates their complex induction into the language game(s) of spoken English (and for Jeremy, Chinese as well) and indeed the genesis of their own idiosyncratic language games. Their experience can make us aware of factors that may apply to other children generally and to children's complex induction into the language game(s) of mathematics as they encounter them in the classroom.

Learning words, engaging in *symbolic adoption* is clearly not so straightforward. Determining which aspects of experience, which signifieds are named by what signs is not just a matter of imitation. It requires active effort to make sense as to how things are named. There seem to be no overarching rules or short-cuts. The specifics and generals to which signs refer must be experienced in order for a child to adopt conventional usage.

Examples were given on how George and Jacob used the same verbal signifier to signify different items. This difference is not necessarily obvious to adults, since we see ***through*** the words directly to their referents.

> Words...are frequently so familiar to us as adults that we fail to notice them as symbols. We are so 'at home' with them that as we speak and write, 'the words don't get in the way' (Pimm, 1995, p. 5).

For example, in the 'door' episode, to me those objects were both obviously called 'door.' At the time, I could not even really *hear* George. I was completely unaware that he was trying to point out something novel and interesting. It took a couple times of viewing the video for me to understand what George had been trying to say. He must have realized that these two very different objects had the same name. Perhaps George even generalized as to the function of *door*, such as 'something that swings open and shut, permitting entry.' Generalization as to function surely seemed a part of naming with 'cup.' All items so labeled were capable of holding liquids, including a toy egg shell.

Naming different things with the same name also occurred with idiosyncratic words. Jacob used his idiosyncratic word 'mimi' to refer to a whole class of objects, for which he, like George, likely arrived at some generalization, such as 'objects that emit sounds' or perhaps 'sounds that come from objects.' The conventional usage of 'music' had a probable influence.

Making generalizations and ignoring distinctions are important in naming. My eyes, George's eyes, the 'eyes' on the rubber duck, and the 'eyes' on the picture strip were all very different, yet had the same name. In fact, two pairs of eyes were real while the others were rather abstract, pictorial representations of eyes.

It is notable that both real objects and signs for them get named the same. Nothing in our language provides for distinction between sign and signified (e.g., '3' and three real objects are both named 'three'). Learning to ignore distinctions such as these is not at all straightforward. This situation is often apparent in the mathematics classroom, especially when children desire to attend to particulars of a problem context rather than to only the 'salient' mathematical features and symbols.[16]

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[16] Another example of failing to ignore differences comes from Mimica's (1988) study of the mathematics of the Iqwaye of Melanesia. She reports how they refuse to accept a damaged bank note in exchange for a crisp one of the same value. "What matters...is not the abstract quantity as such but the exact and concrete material properties of the exchanged quantities....[The Iqwaye] insist and make sure [the quantities] are indeed the same--down to infinitesimal detail" (pp. 20-21).

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Mathematics involves a great blurring of distinctions and collapsing of boundaries. In some sense that is the essence of abstracting -- letting particularities melt away until only structure remains. We no longer have three apples, four tomatoes and six oranges, but only three, four and six or 3, 4 and 6, or 13, or one ten and three ones, or a prime number or.... Each example represents a greater abstraction and a distancing of relation between symbol and referent -- from signs that describe an actual context to number words (that signify quantities) to numerals (that signify numbers) to a sum (that erases individual components) to the structure of the sum in base ten, to a property of the sum afforded it within the language game(s) of mathematics.

Presmeg (1997) describes the phenomenon reflected in the example above of signifiers turned into signifieds through being named by new signifiers as 'semiotic chaining.' She offers an example of five people first signified by five names then in turn signified by five fingers, one for each name, the fingers then counted, signified by number words. Upon arriving at number words, they can in turn be signified by numerals. Once here the particulars of the situation can be forgotten. '5' might signify five individuals, but it could also signify the five drinks they are drinking, the five coats they wore, or a totally unrelated five steps up to the front door.

Semiotic chaining gives a mathematical symbol such as '5' an economy and elegance highly valued in mathematics as well as density. A simple symbol like '5' carries so much potential, so many possibilities of referents, so many meanings or possible interpretations. Mathematics includes symbols of even greater density. For example, π has very extensive and variegated meanings. It signifies relationships relevant to length, area or volume, to size and distance in multiple dimensions. It can be renamed by an approximate fraction or by a decimal notation without end. Naming and its accompanying abstraction begin the semiotic

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semiotic chaining, begin the blurring of boundaries necessary for dense symbols like π to come into being.

Poincaré gives the abstracting power of names due credit for mathematics' development; as he notes:

> Perhaps I have already said somewhere that mathematics is the art of giving the same name to different things. It is proper that these things differing in matter, be alike in form, that they may, so to speak, run in the same mold. When the language has been well chosen, we are astonished to see that all the proofs made for a certain object apply immediately to many new objects; there is nothing to change, not even the words, since the names have become the same (1982, p. 375).

While names help to blur boundaries in mathematics, at other times differences are very important and also make their way into names. 5 is not 4. A triangle is not a square. Similarly a truck is not a car, nor a bear a lion.

Mathematics clearly involves giving the same name to different things, even vastly different -- ***three*** snails, ***three*** tons, ***three*** light-years, ***three*** mega-bites, ***three*** donut holes -- for in arithmetic only the ***three*** will matter (if no change of units is required). However, in certain situations, mathematics involves giving different names to the same thing. For example, I might need to rename ***three tons***, 'three thousand pounds' or ***three fourths***, 'six eighths.' Depending on context and mathematical requirements, I could choose from a myriad of possible names for the same thing. Pimm (1995) takes note of the importance of this aspect of mathematical naming. "Multiplicity of names, often deemed equivalent for our purposes (such as 2(x+1) and 2x+2), is a core phenomenon in mathematics" (p. 5).

The toddlers showed ready ability to engage in multiple namings. George called his mom both 'Mama' and 'Lynn.' Jacob called me both 'Haween' and 'Wee-wa.' Jeremy offered num

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numerous equivalent names for 'dance.' The toddlers also used multiple signs together for added emphasis or perhaps to increase the chances of being understood. Thus George used his deep intonation and also said "big, big." Jacob gave his idiosyncratic gestural sign and also said, "come here." Jeremy raised his arms and also said, "done."

As with conventional signs, the toddler's idiosyncratic signs afford the opportunity to see how the boys named different things the same and the same things multiply, but they additionally provide examples of symbolic genesis -- the forms and processes of invention. All three boys made some use of metonymy, yet they used it in different ways. Lakoff and Johnson (1980) describe metonymy as follows:

> Metonymy...has primarily a referential function, that is, it allows us to use one entity to *stand for* another. But metonymy is not merely a referential device. It also serves the function of understanding. For example, in the case of metonymy THE PART FOR THE WHOLE [traditionally called *synecdoche*] there are many parts that can stand for the whole. Which part we pick out determines which aspect of the whole we are focusing on (p. 36).

Jacob made use of metonymy when he let the sound *mo* repeated, that is, 'momo,' stand for *mo*torcycle and lawn*mo*wer. He took a part of the conventional symbol and allowed it to replace the whole. The same occurred with 'mimi' and 'hee-hee.' Jacob made use of metonymy along the lines of its conventional usage, namely a changing of names, working as he did at the symbolic level.

George and Jeremy made use of a more organic metonymy, grounded in experience. Rather than attending to the conventional names given to objects, both of them attended to sounds the objects made, imitating them by use of idiosyncratic or conventional onomatopoeia. Onomatopoeia thus served to signify; they *stood for* names. This was true for George particularly in his naming of animals. His use of onomatopoeia waned

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somewhat as conventional names for animals entered his vocabulary. (For example, upon learning the convention, George would name an owl, 'owl' and only say 'hoo-hoo' when asked for its sound.) However, for Jeremy use of onomatopoeia was more widespread, not limited to a particular domain. Onomatopoeia remained alongside convention (e.g., 'dink' and 'dance,' 'beep, beep' and 'clock') and Jeremy even imparted sound where none intrinsically existed.

Other examples that could be considered metonymy are Jacob's 'want to see Daddy' and 'want to build house.' Again operating at the symbolic level, Jacob used expressions that referred to the first instances of activities (looking through the video camera, playing with the foam squares) to refer to all subsequent instances of the activities. This led to a conflict between Jacob's idiosyncratic meanings and the conventional meanings signified by the expressions.

Children may wish to use metonomy in the mathematics classroom as well, naming by association, first instance or characteristic. Metonomy may even be a way by which children make sense of conventional mathematics language. Orr (1987) describes students who constantly incorporate referents and solution processes from the first problems where certain signifiers appear in all subsequent problems where those signifiers reappear, even though the original referents are totally irrelevant to the new problems (e.g., x now stands for something else).

When encountering children in the classroom who make 'mistakes' with conventional language as Jacob did, teachers are apt to consider them confused. Rather than try to understand how 'misnaming' arises, they are prone to just correct it. However, misnaming does not occur randomly, but happens because children are forming generalizations about their experiences. They generate rules and attempt to use language systematically. The rules they

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they generate may not be the same ones operating in conventional language. Broken eggshells are not called 'cup' even though they could function as such. However, the rules children generate are mathematical in nature and rather than just bulldoze children's 'mistakes' in the classroom, teachers can try to uncover and understand their rules, maybe even utilize them. 'Lion-bear' is a good case in point.

George originally used metonymy to idiosyncratically name both *lion* and *bear* . He growled, giving the same sound for both. Upon learning a conventional name, namely 'lion,' why could he not directly substitute it for his idiosyncratic one? Use of mathematical notation might make this process clearer. It also helps to emphasize the rule-based nature of George's 'confusion.' Let a be a lion, b be a bear, c be a growl, d be the word 'lion,' '$<$' indicate 'named by,' and '$=$' indicate identity. Let $a < c$ and $b < c$ (the original naming). If $c = d$ (the two names mean the same) then $a < d$ and $b < d$ (direct substitution, giving the new name in place of the old). It made sense 'mathematically' for *bear* to be called 'lion.' Jacob also made this sort of direct substitution of conventional name for an idiosyncratic one when he replaced 'mo-mo' with 'motorcycle.' However, unlike George, Jacob did not resist correction.

In the misnaming instances, the toddlers likely did not think the signifieds were *the same* on the experiential level, simply because they named them by the same signifiers. All of George's cups were quite different as were the doors. However, they composed the same category. *Cups* are all called 'cup' because they share common characteristics. Multiple distinct specifics of cup compose a general category of cup. The toddlers' misnamings may similarly reflect a categorization.

Jacob likely did not think that motorcycles and lawnmowers were the same. You can ride one to town while the other you push and it cuts grass. When Jacob called a *lawnmower*, 'moto

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'motorcycle' he simply made a substitution at the symbolic level. On an experiential level, both objects have particularly loud, relatively apparent motors, which could be a basis for a category.

George likely saw *bears* and *lions* as different too. For example, on numerous occasions, George pretended to be afraid of the lion photo (see Chapter V), but never of any of the bear images. George differentiated but nevertheless classified together *bears* and *lions*. Both are large, hulking animals capable of inspiring fear. And both growl.

'Wee-wa, Haween' is another case in point. Jacob likely did not think I really was Lisa even though he chose to call me by the same name. We were two distinct people who nonetheless belonged to the same category. Jacob did not call me by just any name he knew. For example, he did not call me, 'Daddy' or 'Mommy,' 'Barney' or 'Gran'ka' (Grandpa), which were names he used for people (or big purple dinosaurs as the case may be). He gave me the name used for another female person about his mother's age.

Our names also bear phonetic resemblance, especially in Jacob's articulation of them, and as with the common element of 'mo' in lawnmower and motorcycle, this too could have played a role. 'Haween' and 'Wee-wa' both share the sound 'wee.' The sounds 'ha' and 'wa' are quite close. The two words can be seen as inversions of each other, like mirror images, 'Wee-wa,' 'Haween.' For Jacob, the syllables themselves could have been the salient parts of the name with order irrelevant. The names 'Wee-wa' and 'Haween' thus are the *same* in all ways that matter in accord with Jacob's rule. For a time, he articulated them in a particular order, and then under pressure, reversed it.

I have used the term 'misnaming' to describe occurrences when my subjects used conventional language in idiosyncratic ways that directly countered convention and thus faced challenges from others. Misnaming has to do with using conventional signifiers to refer to 'incorrect' signifieds. I have argued that misnamings are not mere mistakes, not mere slips as it were. They have everything to do with active thinking, with making generalizations, generating categories, constructing and applying rules. When children misname in the mathematics classroom, teachers are apt to call them 'mistakes,' or even likelier, 'misconceptions.'

The term 'misconception' captures the incongruity between a conventional idea or concept and a child's idiosyncratic concept or 'misconcept.' I have discussed how concept formation and signification are dialectical and interdependent processes. However, the term 'misconception,' as used by educators, implies faulty thinking, confusion, definitely something requiring immediate and active correction. Using the term 'misnaming' could perhaps shift the issue from one that is at its core conceptual to one that is equally about language. Rather than wondering, 'why do students think that way?' or more likely, 'why don't they get it?' teachers could wonder and even ask students directly, "why do you call it that?" The latter question could carry the assumption that children"s namings and misnamings follow very logical, *mathematical* rules that are worth uncovering and exploring for their own sake.

Not misnaming only involves the use of rules and principles. The invention of names does as well. Jeremy formed numerous gestural signs following the principle 'do as I do.' Although he could use various different signifiers to name the same signified -- as he did with dancing -- they all obeyed the principle. For playing music on the stereo, Jeremy codified a gesture that also fit the principle. Jeremy additionally showed that although the 'play music' sign clearly grew out of mimicking a particular action, he could extend and generalize its use. His sign came to signify 'make music,' regardless of how.

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While children use generalization (or categorization), rules (direct substitution, part for whole, first instance for all instances, sound for sound-maker, etc.) and features of signifieds to determine how to use conventional signs and to invent their own, invention can also involve the arbitrary assignment of symbols. Arbitrary assignment occurred with George's idiosyncratic words, 'gaga,' and 'nana.' To children much if not all mathematical words and notation must appear arbitrary, even words such as 'quadrilateral' that are named metonymically (by the particular attribute of having four sides). We could expect this arbitrariness to be difficult for children, yet it need not be. If left to their own devises, some may invent equally arbitrary names. However, it is notable that out of the three boys, only George's idiosyncratic language included arbitrary assignment of symbol to referent.

Another characteristic of these signs and of many of the toddlers' use of onomatopoeia is the presence of exact repetition. Repetition was widespread in all the boys' idiosyncratic namings, which may reflect a need for multiple syllables or rhythm or rhyme. Alternatively it could be a way to ensure comprehension: 'I'll say it twice or three times to make sure you got it.' Cassirer (1953) describes this phenomenon as 'reduplication' and views it as reflecting repetition on a sensory level. He notes that in some languages, "...a noun is changed to a verb...by the mere reduplication of a sound or syllable" (p. 197).

Although reduplication may not feature in mathematical language, perhaps it could. A teacher could exploit reduplication to emphasize and draw attention to sensory level repetition. For example, a triangle could also be called 'side, side, side,' the number twelve, 'six, six,' multiplication 'add, add, add, add....'

Another seemingly arbitrary idiosyncratic sign was George's 'da.' Its singularity of utterance may be due to its introduction, by George, when he was assimilating conventional words into his vocabulary, ones that did not carry reduplication. 'Da' is a particu

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particularly interesting case, for rather than directly standing in for a conventional word (from the adult viewpoint) it is a word with idiosyncratic meaning, making it more difficult to understand and 'translate.' Its idiosyncratic meaning is particularly interesting as well.

'Da' functioned as a way for George to name the unnamed. Anything and everything could be identified as 'da' until it was named. 'Da' can be seen as similar to 'x' when it functions as an unknown, waiting to be known (as in the equation $3 + x = 7$). Beyond naming a particular unknown (as when 'da' was used to name a particular object of unknown name similar to 'x' naming the unknown but specific value '4' in the given equation) 'da' also hints at the infinite. George possessed a finite vocabulary. 'Da' carried the potential to name the infinite number of things for which he had no name. In mathematical terms, let 'da' be an element of set A with A being all things with potential names. Set A is surely an infinite set. Thus 'da' has infinite potential, naming towards infinity one item at a time.

There are arbitrary names as well as linguistic symbols that fail to name. Grandin (1995) speaks of the difficulty autistic children face in learning words that do not clearly name pictures. Verbs, adverbs and especially intangibles (such as 'getting along with people') are quite difficult for autistics to understand. One could expect the use of words that name nothing to be even more difficult to grasp, even for non-autistic children. However, Jacob's ready use of 'thank you' showed that he had no difficulty in making use of an utterance whose purpose was mere social convention.

Despite their failure to name, symbols that are social conventions can be very important for communication. Mathematics has its own social conventions. However, in mathematical language, more crucial than the possibility of appearing impolite, ignoring conventions can cause grave mis-communication. For example, a fraction (meaning how many parts out of the whole; e.g., 3 out of 4 students are girls) is written with a horizontal bar separating the

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two numerals (3/4) whereas a ratio (how many parts of one kind in relation to how many parts of another, e.g., 3 girl students to 1 boy) uses a colon to separate the numerals (3:1). Using the notation for one to mean the other (as children often try to do -- they will write ratios with fraction notation) can confuse at the very least. Reading and writing equations from left to right is another convention in mathematics (one that failing to follow does not really matter for communication's sake) as is the order of operations convention (one must multiply, then add or subtract, e.g. given 5 + 4 x 7, one must first multiply 4 and 7 then add five to the product).

'Thank you' reflects attention to social rules, rules 'about language.' This adherence to rules is in contrast to those the toddlers generated regarding use and invention of particular signs, rules 'within language.' Use of metonomy was discussed as such a rule, or rule-based process for generating idiosyncratic signs. Metaphor can also be viewed as a process or means by which the toddlers named and formed categories. This object is like a cup, so it shall be called 'cup.' A bear is like a lion, so it shall be called 'lion.' Haween is like Wee-wa, so she shall be called 'Wee-wa.'

As with any language, metaphor is present in mathematical language. Lakoff and Nunez (1997) discuss how metaphors are evoked when thinking about mathematical ideas. For example, arithmetic is alternatively viewed metaphorically as 'Object Collection,' 'Object Construction' or 'Motion.' Walkerdine (1988) discusses how mathematical notation may be metaphor-free, yet as soon as that notation is read, metaphor appears. She offers the example:

> [W]e could articulate [2+3=5] as: 'two plus three equals five' or 'two add three makes five' or some other combination of these or other terms. The metaphorical implications of makes and equals, for example, are quite different and certainly allow the speaker/hearer (implicitly) to link mathematical with other discourses...(p. 184).

The link of mathematical discourse to other discourses can sometimes be seen more clearly through comparison of metaphors used in different languages. For example, in English we call certain numbers 'even' and others 'odd' based on the characteristic of whether or not they are divisible by 2. These names are metaphorical although not necessarily obviously so to the average English speaker. 'Even' refers to the notion of balance, there being the same amount on each side, making the total 'even.' 'Odd' refers to the presence of a remainder, an 'odd-man out' after dividing the number by 2. The more common meanings of 'even' and 'odd' may interfere with the meanings relevant to mathematics. For example 'odd' more commonly means *strange*. Children may wonder, why is the number 3 strange?

In Hebrew and Arabic, even numbers are essentially called 'paired' or 'coupled' and odd numbers 'non-paired.' These terms evoke a more obvious and accessible metaphor, e.g., 'a pair of shoes,' 'a young couple.' One can imagine people pairing up to dance or walk down the hall. Either everyone is paired or not. And pairing is an extremely practical activity. It is clear why one would 'want to know that.' In contrast, the metaphors, 'odd' and 'even,' are essentially dead at best for students and probably their teachers as well.

These are examples of using metaphor to link mathematical concepts to ideas outside of mathematics. Metaphorical namings take place within mathematics as well. Pimm (1988) discusses how naming new mathematical concepts through metaphorical reference to old stresses certain aspects of the new concept (e.g., bears growl as lions do). He offers the example of 'spherical triangle,' which is not really a 'triangle' according our standard definition of the term in Euclidean geometry. However, using the word 'triangle' offers a metaphor that emphasizes commonalties that spherical triangles share with triangles, help

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helping people see them as "three segments of great circles meeting pairwise in three points" (p. 33).

However, as with the example of 'odd' and 'even,' a metaphor may fail to evoke the relevant commonalty or any notion at all. Although the advanced student of mathematics is very familiar with the idea of *triangle*, to children first encountering mathematics in the classroom a word such as 'triangle' may fail to be *transparent* (Pimm, 1995). To a child, 'triangle' appears arbitrary, although it is in fact metonymical. The name 'triangle' contains 'tri' and 'angle,' calling attention to a certain property of the shape, namely its possession of three angles. Interestingly, the definition frequently offered to children of triangle calls attention to a different property, that a triangle has three sides. If indeed it is the sides of a triangle to which young children attend, why then is it not named a 'three-sider' in early elementary school? Such a name embodies a more relevant metonym and would permit a more obvious connection of sign to referent.

Young children may wish to name triangles metaphorically as well and choose names such as 'birthday hat' or 'pumpkin nose.' Such names can be limiting in that they could enforce a restricted view of such shapes, allowing for consideration of only isosceles or even equilateral triangles. Yet in fact, these are the types of triangles most encountered in elementary school mathematics and children come away with such a view of triangles anyway. Allowing children to name mathematical objects metaphorically could permit a greater initial connection of sign to referent, for to them, 'birthday hat' is transparent whereas some fancy definition is not. Finding a triangle that is not reminiscent of a birthday hat (e.g., a scalene triangle) could help reveal in what ways that name fails to serve. Naming and concept formation go hand in hand. Challenging a name challenges the concept it names.

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If children are given the permission and power to name in the mathematics classroom, their thinking would be more available to inspection since they would 'call it as they see it.' A teacher's job of dealing with 'misconceptions' would then become easier. If triangles are 'birthday hats,' then a teacher knows her students need to encounter non-birthday hat triangles in order to develop a full appreciation of them as mathematical objects.

New triangles may give rise to new names. Allowing multiple names to exist simultaneously would permit emphasis of various aspects of a concept and hence an expanded view. Choosing names could be an open and collective process. In what ways is this shape like or not like a 'birthday hat?' Through namings and renamings a complex idea would become sufficiently understood so that in the end, a conventional name could suffice (and perhaps be readily translated into multiple idiosyncratic names in children's minds). The toddlers too had multiple names for the same referent. Perhaps this multiplicity plays an important role in developing strong concepts.

Naming by way of metaphor involves signs in the strictly verbal realm (using conventional names to signify novel referents). However, the toddlers invented names in other modalities as well. Jeremy was particularly attracted to gesture. Both he and George used onomatopoeia. Their expression reflects attention to certain aspects of experience -- motions of the body, the sounds things make. In contrast, Jacob's inventions were primarily made up of words (i.e., even his idiosyncratic verbal signs were derived from conventional ones). Beyond revealing certain attentions, the toddlers' inventions show that for each child some media of expression gave rise to 'symbolic initiative,' whereas for others only 'symbolic adoption' reigned. This can be compared to one artist who prefers to paint whereas another creates with clay.

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Children may carry these media of attention and expression with them into school. Could signs in modalities other than the strictly verbal be accepted in the mathematics classroom? How about gestures for addition and subtraction? A possible sign for addition could be motioning the arms as if pushing things together and for subtraction, scattering away. What about signs that use onomatopoeia? For example, addition could be the 'Plink, plink, plink' of marbles filling up a jar, subtraction the 'eeee...pkhuuu' of a tree crashing to the ground. George's auditory gestures of big and small voice are quite applicable to classroom mathematics. When comparing the relative size of various numbers, the bigger number could be said in a loud voice and the smaller in a soft voice. Numbers could even be ordered by voices growing relatively louder or softer.

It was noted with the toddlers that gesture seems particularly well suited to signifying action, i.e., verbs. Hughes (1986) reports on findings that further support the prospect of children using gestures and other alternative media of expression to signify mathematical actions. He found that children (under 9) had considerable difficulty expressing certain notions with written notation. He describes tasks where children were asked to represent addition and subtraction in some written way. Most subjects represented final quantities alone (with tally marks, etc.). While some children successfully represented changes in quantity in ingenious ways, none used the conventional signs (i.e., + and -) even though all but the youngest ones encountered them every day in school. Hughes concludes:

> [These findings] point to a serious mismatch between the system of symbols children are required to learn and their own spontaneous conceptualizations. It seems that the whole notion of representing these transformations on paper is something which children find very hard to grasp...(p. 78).

Thus not only could children use gestures to signify arithmetic transformations, but perhaps they should; gestures may prove more meaningful signifiers to children.

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The toddlers' creativity and/or ease with particular symbolic media may additionally reflect issues of relative abstraction. For example, Jeremy may find that use of onomatopoeia better 'captures' the referents they signify. Similarly with gestural signs. On the other hand, he may find verbal signs to be more abstract, positioned 'at a greater distance' from referents. However, Jacob may find use of onomatopoeia to be utter nonsense, having no trouble making use of arbitrary verbal signs.

When I think about teaching mathematics, the toddlers' experiences demand that I attend to the relative ability of symbols to *be* signs for children, to truly refer to the objects of intent. Which signs are transparent for some children may not be for others. Thus, generalizations made about all children could be faulty. Reality can run counter to intuition. For example, for some children Dienes blocks may fail to represent number in base ten whereas numerals may succeed.

I am also reminded of the value of opening classrooms to idiosyncratic and varied expression. Students would benefit by signifying in ways that make sense to them, rather than only through signs imposed arbitrarily from outside. As a teacher, I would benefit from the window thereby opened to students' conceptions. Skemp (1987) agrees:

> We should use transitional, informal notations as bridges to the formal, highly condensed notations of established mathematics. By allowing children to express thoughts in their own ways to begin with, we are using symbols already well-attached to their conceptual structure (p. 188).

Even if teachers choose not to take up children's invented language, rules, and meanings, children's inventions may linger in their minds long after they publicly acknowledge convention. Denied and undeveloped, idiosyncratic ideas could turn sour, causing confusion and self-doubt. While they differ from convention, idiosyncratic ideas have a log

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logical basis, but children may never learn this. Rather than recognizing that they do think logically, they may come away from mathematics classrooms considering themselves stupid and incompetent. On the other hand, validating children's namings may give them confidence and a real sense of 'mathematical power.'

Educators may resist a call to develop children's idiosyncratic language out of concern that if these systems are articulated and developed, children's learning of convention could be thwarted. In other words, children could become even more confused or worse, remain in their own idiosyncratic world views. This outcome need not be the case. Multiple sign systems can exist side by side and in fact enrich experience and understanding (e.g., English and Chinese, base 10 and base 2, etc.). In addition, if children shared their namings with others in the classroom, they could become communal rather than only individual. Idiosyncratic namings would thus become part of a 'language game.' In this way, children may not only develop stronger understandings of particular concepts, but also of signification itself, of 'language games' and the position and need for linguistic conventions.

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Chapter V: Symbolizing the Imagined

The previous chapter included descriptions of the toddlers at play. However, in this chapter, I will discuss *play* as such, because it is young children's imaginative play, pretending, make believe, role play or whatever other name one chooses that has captured the attention of theorizers of play. Such play perhaps represents something quintessential about childhood, where creativity is expressed with a seeming disregard for the confines imposed by ordinary reality. Often the word 'play' is used alone to mean pretending as if no other sort of play exists. Regardless, no other sort of play practiced by toddlers has seemed to inspire the deep inspection and speculation awarded to pretending. In his philosophy of play, Huizinga (1950) expresses a sense of wonder and deep respect for this form of childhood play:

> The child is *making an image* of something different, something more beautiful or more sublime, or more dangerous than what he usually is...[He] is quite literally 'beside himself' with delight, transported beyond himself...(p. 14).

Psychoanalysts have been interested in uncovering the purposes and personal meanings behind imaginary play and see it as a fruitful avenue for therapy:

> Like dreams, play provides a 'royal road to the unconscious' and, therefore, a means of unraveling the child's unconscious fantasies and impulses. It also provides a mode of coping with and transforming these experiences (Slade & Wolf, 1994, p. 1).

Imaginary play has naturally drawn the attention of developmental psychologists, who see it as quite sophisticated, particularly when it involves multiple roles and social negotiation:

> Because sociodramatic play requires abilities such as representational thought, role enactment, high levels of organization, planning, and verbal abilities, some investigators have argued that the inception of sociodramatic

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> play indicates a broad developmental advance in young children's social cognition (Weinberger & Starkey, 1994, p. 328).

For Vygotsky,

> [A] child's greatest achievements are possible in play -- achievements which tomorrow will become his average level of real action and morality...Play...creates the zone of proximal development in the child...in play it is as though he were a head taller than himself (1976; pp. 548, 552).

To Bruner (1976) play provides important opportunities for learning, in other species as well as humans. Play is "able to reduce or neutralize the pressure of goal-directed action, the 'push' to successful[ly] complet[e] an act" (p. 15), thereby making learning of an act less stressful and more possible.

Piaget is certainly not outside the fray. He developed a good deal of theory about pretend play and subsequent researchers have to a great extent followed his lead. Despite my difficulty with aspects of his theory, particularly its orientation to developmental stages, I find other elements quite useful. For one, Piaget termed pretending 'symbolic play.'

Piaget viewed children as constructing 'motivated symbols' that bore resemblance to the signifieds they were attempting to represent. Symbolic play provides an important cognitive arena for a child that is the opposite of imitation of reality or "accommodation to external models." Instead, pretending involves "assimilation of reality to the self":

> The child...needs a means of self-expression, that is a system of signifiers constructed by him and capable of being bent to his wishes. Such is the system of symbols characteristic of symbolic play (Piaget, 1995, p. 493).

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For Piaget, the symbol arises when a child substitutes an object for another in evoking a familiar behavior but without its real objective. There is play. There is pretense. He offers examples from his own children:

> [I]n observation 64 J. mimes sleep while she is holding a cloth, a coat collar, or even a [toy] donkey's tail, instead of a pillow, and in observation 65 L. does the same thing, pretending to be holding a pillow when her hands are empty (1976, p. 559).

Piaget views symbolic play as fulfilling an important role in the learning of conventional and 'arbitrary' signs that make up collective language. The activity of 'assimilation of reality to the self' is that which gives meanings to imitated models. "Then it is that language, a system of collective signs, becomes possible..."(1962, p. 3).

In constructing a taxonomy of symbolic play, Piaget and Piagetians have made useful distinctions among different forms of pretending. Generally, the forms are positioned within a hierarchy or progression, in that a certain type of pretense is the first to emerge and others follow developmentally. For example, in Bretherton (1984) the author only mentions one distinction that may have to do with 'individual style' rather than development, that of children who are 'object-dependent' and 'object-independent' -- i.e., children capable of pretending with 'invisible' objects such Piaget's L., mentioned above. I prefer to see all of these 'developmental' distinctions as having to do with individual expression for all can be found among my subjects collectively although not individually, and I do not view any one of my subjects to be 'more developed' than any other, only different. In addition, forms of pretending from various 'stages' can be found to occur simultaneously. Since my data are discussed in light of these as well as other distinctions, I will review them here, as offered in Bretherton (1984). I will not necessarily use the same terms in my analysis.

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Piaget begins with 'self-representation' -- pretending to do something the child does in reality but outside of its purposeful context. The examples of J. and L. above are included in this category. 'Other-representation' occurs when children pretend to do that which is clearly outside their usual behavior such as to cook dinner. Then there is 'others as passive recipients of the child's action.' This refers to situations when a child pretends to feed mommy or a doll. Bretherton sees using animates (e.g., mommy) as recipients as occurring first. 'Parallel roles' occur when children include themselves and partners in the same action such as everyone taking a drink. 'Use of replica as an agent' describes children animating inanimate objects. 'Assuming another person's role' seems nearly identical to 'other-representation,' but here the child becomes another rather than only acts like another (e.g., not just 'I am doing what daddy does' but 'I am daddy,' although without verbalization the distinction cannot be clear; and maybe for children there is none). 'Use of doll as active partner' involves a child both taking on a role in relation to a doll and animating the doll, playing two roles simultaneously. In 'simple collaborative role play,' a child plays a role and gives others instructions to play opposite roles. Bretherton describes additional forms of role play, but they did not relate to my data so I do not discuss them here.

I have specific interests in examining pretend play. Rather than its general function in learning or development or its relation to language learning, I am interested in how pretending relates to doing mathematics. I, therefore, maintain a particular view on pretending and pay attention to certain aspects of the enterprise.

I see pretending as activity in which children abstract from experience, create and hold in their minds imagined situations and spontaneously invent and make use of 'motivated symbols,' using them to symbolize the imagined, call it 'to life.' My analysis of the toddlers is directed at uncovering those aspects of experience to which the toddlers attend and symbolize, what situations they choose to imagine and create, and what forms their symbols take. I also take note of others' abilities to 'read' the toddlers' symbols and take part along side them in their play.

The title of this chapter is 'Symbolizing the Imagined,' not the *imaginary*. However, it is the imaginary with which pure mathematicians play. Likewise, some may wish to reserve the term 'pretending' to denote make-believe, the pretending of the imaginary -- being King Arthur rather than a mundane daddy or self. However, like mathematics (which includes applied mathematics and everyday calculating), pretend play in children can be seen as following a continuum from acting out (symbolizing, modeling) the real to the fantastic. What is constant among all instances of pretending is that there is imagination, the holding of images in the mind. Those images are operated upon, with the aid of symbols as proxy, *as if* they were actual.

Unlike when naming a visible object such as a car, pretending involves symbolizing something not immediately visible or tangible, something done in the past or by someone else or by no one. With fantastic pretending taking its place at one end of the continuum, I wish to place at the other end what may be termed *narrative* or *storytelling* -- symbolizing an actual event that perhaps just occurred and was witnessed by others.

I think it pointless to draw a clear distinction between storytelling and pretending. All pretending can be seen as telling stories. I am telling the story of how I sleep or how mommy cooks. Stories can be fiction, fantasy -- the story of how I visit Santa Claus. So all the instances I relate here are stories and all are pretend play (since in telling a story a child can be pretending that an event that just occurred is occurring again), but I will call 'pretend play' that which is commonly recognized as *pretend play* and 'stories' those which are clearly *stories* -- attempts to symbolize actual events that occurred. And all can be

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be subsumed under 'symbolizations of the imagined,' since all involve symbolizing that which is remembered or made up in the mind.

Dramatic symbols

The signs with which children 'speak' when pretending are mostly dramatic in nature. They are Piaget's 'motivated symbols' -- signifiers that bear strong resemblance to their signifieds, and intentionally so. Any actor must do his best to portray an action, event, emotion as convincingly as possible to his audience. When pretending, children often have audiences and very much 'put on a show.' Even when the audience is just themselves, there is the need to make the imagined 'real': thus the portrayal must be realistic.

In an episode caught on tape called 'Big, Big Lion,' George had an audience -- his mother and me. George would frequently growl, 'becoming' a lion and 'scaring' others. However, in this episode, after a brief stint as a 'lion,' he decided to let a photo of a lion symbolize a lion and George was the one to be afraid. He showed us how fear should look.

Beyond being more dramatic than his mom or I (or others I witnessed him 'scaring') managed to be, George demonstrated great variations in his symbolizing of fear. In the few minute episode of 'Big, Big Lion' he a) stiffened his body, b) made facial expressions that showed fear, c) clasped his hands to his chest, d) 'spoke' in a breathless, scared voice, e) ran away, f) hid, g) exclaimed "oh no!," h) shrieked and i) breathed heavily. He also used verbal signs, declaring, "lion" and "big, big lion."

In 'Big, Big Lion,' George revealed what he had noticed about fear through his ability to turn his observations into signs. His observations may have come from his own exp

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experiences of being afraid. However, it is likely that at least some came from his observing others (being afraid or pretending to be afraid) or from 'professional' dramatizations of fear in films he saw (which are also forms of pretense and role-playing). George was certainly entertaining (we enjoyed his performance and laughed) and even surprising. At his panting, Lynn asked, "Where did you learn to breathe like that?"

Lynn's question reflects the fact that we were not always privy to what George happened to see -- to all of his experiences of fear (including others' pretending of fear, such as in films). As George never really was afraid of a lion in the exact way he portrayed it, George can be understood as having symbolized something imaginary -- something that he created in his mind, although he surely drew upon numerous and varied experiences. Although 'Big, Big Lion' demonstrates George's creativity and acting ability, it does not permit comparison between signs and the events signified, and therefore does not offer a complete window into the symbolizing process -- from attending to abstracting to representing. Only the end product is visible -- the symbols themselves. That they succeeded at capturing that which was signified permitted a successful reading (i.e., we knew what George was 'saying').

An episode involving Jeremy offers a different opportunity. In 'Spilt Milk,' Jeremy 'tells a story' of something that had just occurred. In fact, almost all of the episode, from event to story told just afterwards was caught on video tape. I relate it here in full:

> Jeremy, Birgit and I are all heading outside to play in the playground. Birgit is carrying a cup of milk and Jeremy asks for it. She hands it to him thinking he wants a drink, but he proceeds to dump it out on the balcony in front of the apartment. Birgit goes inside to get a pitcher of water to wash away the milk. This is when the camera goes on.
>
> Jeremy looks at the puddle of milk, gasps and 'talks' (i.e., makes sounds like talking, but ones that are not recognizable as words) and makes verbalizations that at times sound like 'oh, no.' There is a worried tone to his voice. He stands next to the spill and looks up towards me. He 'talks'

and gestures -- points to the milk and then raises his arms up above his shoulders and waves them around.

Jeremy walks over to a deep part of the spill and starts to stamp in it gently, watching his feet and the milk. Birgit notices and she and I tell him to stop. Jeremy backs out of the spill and bends over and wipes off the tops of his shoes. Then he notices the milk streaming slowly towards him (from the place where he was stamping).

Jeremy looks at the approaching milk and says, "Hey." Then he says, "No!" and again, "No!" with an upset tone to his voice, backing away. He lifts his arms behind him, palms facing backwards and looks around him as if searching for a place to hide. He runs over towards the fence at the edge of the balcony and grabs on. He watches the milk as it moves for a second or two more and then runs away, whimpering. He grabs on to the fence again some distance away.

I call Jeremy back and he returns, just in time to watch Birgit pour water on the milk to wash it away, over the edge of the balcony. Jeremy watches intently as the water mixes with the milk and the liquid moves towards him and under his shoes. He says, "ooooh" with distinct enjoyment and laughs. Meanwhile, Birgit has urged him to get out of the way. He runs away like before, but this time shrieks and grabs onto the fence at a much closer distance.

Jeremy continues to watch the action. He runs back towards the liquid and makes gentle steps in it, watching his feet. He laughs and then bends over and pats the water with his hands. Jeremy stops and stands up and watches through the open door as Birgit approaches with another pitcher of water. As she gets closer, Jeremy shrieks and backs up. He watches her pour the water and as it thickens under his feet, he bends over and pats it. Birgit tells him to stop and moves him gently out of the way as she finishes pouring the water. Jeremy goes over to the edge of the balcony and watches the water drip off the edge, bending down to get a better look.

Jeremy turns away from the fence, watching through the open door again. Then he turns his head to me and begins to 'talk.' He makes pointers with both his arms, his index fingers extended. As he 'talks,' Jeremy points towards the ground and then moves his arms all around in sweeping, sometimes circular motions, with knees bent. He bends down and touches the ground then gets up and faces me, continuing to 'talk.' He bends down once more and touches the ground with his forefingers, five times, punctuated by simultaneous, "eh, eh, eh, eh, eh."

Birgit returns with another pitcher of water. Jeremy stops talking to watch her pour. He examines the water on the ground for a moment and then falls down on his knees, getting his pants all wet. Jeremy is now concerned about his wet pants and goes to get them changed.

There were actually two stories being told here, one each time Jeremy turned to me and 'talked' (as well as gestured). There was clear pretending as well -- Jeremy pretended to be afraid of the milk, perhaps imagining it chasing him (see the fourth paragraph of the narrative above). Regarding the stories, events themselves were visible as was Jeremy's symbolizing of the events.

With both stories, Jeremy's 'talking' symbolized the act of storytelling, however his gestures made clear the stories' particulars. In the first story, Jeremy 'told' how the milk moved and spread all around on the ground, by moving his arms as such. In the second one, Jeremy again symbolized the movement of the liquid but also seemed to emphasize the place where the movement occurred (by touching the ground) and perhaps the numerous times liquid was poured (by touching the ground many times in succession).

In 'Spilt Milk,' the actual events can be compared with the ways in which Jeremy symbolized them. In doing so, what Jeremy saw, chose to replay, chose to represent, and how is revealed. Jeremy's abstraction of the raw events may have been different from that of another observer.

One way to discern Jeremy's viewpoint is to juxtapose what he 'told' with what he did not. For one, Jeremy did not 'tell' about how the milk and later the water was *poured* onto the ground. He did not mime pouring in any way (only the impact of the liquid on the ground and what it did once there). Perhaps Jeremy did not much *see* the pouring *per se*, or perhaps it was uninteresting -- he had seen liquid being poured into vessels many times before. What was novel and worthy of attention and telling had to do with the behavior of liquid falling on a flat surface.

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Things falling was the focus of other stories Jeremy 'told.' He once 'told' of how a caddie full of pens had fallen and spilled out onto the floor. Upon seeing the caddie some time after the event, Jeremy 'talked' in a concerned voice, said, "uh oh," and made accompanying facial expressions.

Jeremy appeared to tell three different stories about falling on a single day. When I first arrived that day, Jeremy sat on his mom's lap. He lifted his arms with palms facing upward, elbows bent. He 'talked' for about the duration of a sentence and then suddenly fell back very forcefully, landing on the sofa behind him. I laughed and he repeated this sequence of actions more or less several times. Birgit explained to me that Jeremy had done the same thing a few days prior, after falling off a chair, except that he hung onto an arm of the chair and leaned over to one side to portray falling. Perhaps he was now telling me the same story.

Later on, Jeremy fell in reality. He was bouncing up and down on a toy horse on wheels when he suddenly fell backwards. He hit his head hard on an old telephone on the floor (a large, heavy one that his mom had gotten for him to play with) and cried. After being comforted, Jeremy immediately got back on the horse, 'talked' and leaned backwards, telling us what had happened right away. Birgit and I told him, "We know, you don't need to do it again." Later, sitting at the table and having a snack, Jeremy stopped several times to tell us the story again. He would 'talk' and lean back in his chair. At times he made faces and also hitting motions as if getting back at the culprit. However, Birgit pointed out that Jeremy refused to tell about having cried. When Birgit asked him about crying, Jeremy just initiated a frequently played word game (see Chapter VI). Jeremy had also refused to report crying after falling off the chair a few days earlier.

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Jeremy had been having cereal as part of his snack and he knocked the bowl off the table spilling it onto the floor. He showed great concern over it. Once down off his chair, Jeremy bent over and pointed to where the cereal was on the floor. He then got down on his hands and knees and crawled under the table. He pointed to the spill and 'talked,' appearing to tell about how the cereal fell and spilled.

This last story did not contain much by way of gesture. I understood Jeremy because I had witnessed the events and the result was still present -- the spilt cereal and bowl. The other stories made use of the props that were part of the actual events the stories related (the chair, toy horse, milk) and gestures that mimicked actions. The gestural signs were invented and employed spontaneously. Jeremy's attention and spontaneous symbolic invention can be 'witnessed' in these episodes.

Jeremy made symbolic use of props that played actual roles in the signified events. As such they were signifieds as well as signifiers. Yet, as signifiers, they did not merely signify themselves in a direct one-to-one fashion; rather, they signified events, actions in which they had played a role. Thus, the signifieds can be seen as being greater than and containing the signifiers that refer to them. The signifiers condense and economically signify something larger than themselves.

There were other times when Jeremy was not as successful in telling stories as during the episodes related above. There were numerous occurrences when I felt that Jeremy was telling me a story, explaining something and/or pretending, but I was unsure as to what he was 'saying.' The referents for his symbols were unknown or unclear and his symbols did not successfully function as signs for me -- they did not point to anything I could determine. However, that Jeremy was expressing using *symbols* ('talking,' bodily gestu

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gestures and physical objects) was clear to me, or at least became so over time (i.e., I realized he was not just engaging in random behaviors).

Making use of givens

By this section title, I am not referring to the point made above of using as signifiers things that are also thereby signified. In such situations, a child is transforming an object into a sign, making it signify in particular ways, appropriating it into a system of signs. The object is not intrinsically symbolic. Milk is milk. Jeremy let milk signify itself and the actions in which it took part. In this section, I wish to focus on objects in the toddler's environment that are intrinsically symbolic, fashioned, constructed, introduced as signs. The toddlers 'read' them and made use of them in pretending. Hence, they are given. They are *a priori* members of the toddlers' environment -- signs which the toddlers drew upon for inspiration and use in symbolic constructions. Once this door is opened, it is surprising how wide it swings, for the vast majority of toys the toddlers played with are symbolic -- they were designed to signify non-toys.

Jacob loved cars and trains and would play with his toy versions often. George too loved vehicles, especially cars and trucks. At day care, he would push or ride a toy truck up and down a hallway many times daily, ritualistically in fact. At home he played with matchbox cars -- sending them down a ramp and wheeling them into their toy garage. He also had a kid-sized toy car and truck that he sat in and 'drove.'

Another researcher may choose to type play with toy vehicles as 'sensori-motor play' rather than 'imaginative play,' for what is observed is largely physical movement on either

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a gross or fine motor level.[1] Yet, this form of play is most surely symbolic. If a child were pushing around a box and, when asked, said it was a car, there would be no doubt that he was pretending -- imagining the box were a car. A toy car is also not a real car. It has been constructed by toy manufacturers to symbolize a car, sometimes quite realistically and at other times as caricature. When children play with it, it becomes a sign for a real car -- they imagine it to be a car. In fact, Grandin (1995) notes that failure to play appropriately with symbolic toys constitutes a symptom of autism: "The child may spend long periods of time spinning the wheel of a toy car instead of driving it around on the floor" (p. 45). In other words to autistics, the toy is merely an object and not a sign; they do not give it a role in an imagined situation.

The toddlers revealed pretense in their play with givens in a variety of ways. The video camera recorded Jacob playing with toy train cars on a track.

> The track is configured with two loops connected by a bridge. From where Jacob sits, he can only reach part of the track. Jacob plays with several train cars, held together by magnets. One is an engine, the front car of a train. Jacob moves the train on the track in its frontwards direction as far as he can reach. Then, rather than moving it backwards along the same portion of track, Jacob turns the train around. This is cumbersome. He asks for my help and he makes sure that all the wheels fit in the track grooves before proceeding. Jacob again moves the train forward to the limit of his reach and once again turns it around, asking for help. He does this several times.

Jacob seemed to know from observation that real trains did not move backwards (or off the tracks for that matter).[2] In playing with a toy train, Jacob preserved the integrity of how a

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[1] Indeed, in the literature I have encountered on pretend play, I have not seen what I have termed 'play with givens' discussed. Bretherton's categories do not include it. One could similarly see building with blocks not only as 'constructive play' but also as involving pretense -- pretending to build a building that a child may observe in reality. There is little constructive play in my data. When it did occur, it was with the aid of adults and often initiated by them.

[2] I observed Jacob play with his train on other occasions, although a symbolic engine car was not always present (i.e., all the cars were 'middle' ones). Jacob particularly made sure to keep the cars on the track. Great dexterity was frequently required of him to do so. Jacob moved the cars slowly and replaced their wheels in the grooves whenever necessary. He expressed frustration when the cars came off the track

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real train would operate. In other words, it was not merely a toy that could be played with any old way. For Jacob, it symbolized a real train and had to behave as such. Through use of this given symbol, Jacob imagined real trains.

George too revealed pretense in his play with toy cars and trucks. Cars and trucks are means of transport for people. As a person on or in a vehicle, George thereby became a driver or passenger. Every time George played with the truck at day care, upon entering the playroom from the hall he would call out, "hi" and upon leaving would say, "bye," sometimes also waving or blowing kisses. At home, with his kid-sized car and truck, George would clearly 'drive.' Here is an example:

> George goes to his car out on the screened porch with a set of plastic keys that must have been interesting in themselves at an earlier age. He gets in the car and closes the door. He holds one key in his hand, with the rest of the set dangling, and puts the tip in the center of the wheel at the horn. He says, "bye-bye," to John and me, and waves, and then blows a kiss with a hand. John explains to me that these days, George always needs to have his keys when he goes to play with his car and truck.

George did not play this way with his matchbox cars, but that does not mean he was not pretending. Perhaps he imagined tiny people inside the cars even though he never *became* one. Indeed, on occasion he would wave and say 'bye' as an outsider to a car as if talking to such imaginary people.

George not becoming a passenger of a matchbox car speaks of boundaries. Curiously, both George and Jacob seemed to have additional boundaries when playing with toy vehicles. The stereotype of pretend play with trains for example is to make sounds for trains -- the engine, 'chugga, chugga, chugga...' and the whistle, 'whoo, whoo!' However, neither boy ever did this. Jacob sometimes narrated, e.g., 'choo-choo down the

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frequently. Adults would often respond by changing the layout of tracks so that Jacob's task of moving the train within the track was made easier.

track,' ('choo-choo' was his *name* for train) but there were no 'beep, beep's' no 'vrmmm, vrmmm's.' I will further discuss this issue of setting boundaries for pretend play in a later section.

George's and Jacob's lack of sound effects may indicate that for them the toy vehicles sufficiently signified the referents of intent. Further dramatization was unnecessary to demonstrate the symbols 'were' what they appeared to be. This interpretation is supported by the few times I heard Jacob say 'beep, beep.' When Jacob played with a tower of Legos on wheels, he said repeatedly, "big cuck, beep beep." While playing with a movable drawing of a car wheel and keys in a pop-up book, he also said, "beep, beep." In these instances, the meaning of the symbols was perhaps not obvious. Naming the Lego construction 'cuck' and saying, 'beep, beep,' helped to *make* it a truck. Repeating 'beep, beep' helped *keep* it a truck, lest it revert to merely being blocks. On the other hand, toy vehicles constructed to *be* symbolic sufficiently met their objective. There was no need for elaboration.

There were numerous other symbolic givens with which the toddlers pretended. I will briefly review some of them here. Some have made prior appearances. Others will make appearances later in the text.

In 'Big, big lion,' a photograph of a lion inspired the play. It *became* a lion and in his pretending, George ran from it and returned to it again and again. George seemed to need to see it in order to maintain the pretense that a lion was really there in the middle of the living room. The photograph provided George fresh inspiration at each viewing.

For Jeremy, photos of food inspired him to pretend to eat them, e.g., cheese, eggs and a sandwich. Both George and Jeremy pretended with toy telephones. Jacob had a toy lawnmower which he pushed on the grass, often along side his dad using a real one. Because of George's interest in the dust mop John used, John fashioned a toy one for him by sawing off the handle of a real one. I witnessed George pretend to mop with it as John had just done, including shaking it off on the porch. Jeremy pretended with a baby doll. He 'fed' it cereal with a spoon, then made a crying sound and picked up the doll. He hugged and rocked it saying, "no, baby."

One day at George's day care was 'food day.' There were numerous realistic, plastic symbols for various foods as well as cooking and eating implements. George pretended to eat cake and ice cream and drink milk and juice. Another time, George pretended to feed a small lion figure (from the *Lion King* film) by putting juice, cheese and ice cream by the figure's mouth, as well as saying, "lion gonna eat." He also pointed to the figure's rear, sniffed and said, "poop, lion, poop."

Objects designed for children -- toy and pictorial symbols -- inspired pretending *as if* they were real. Real, functional objects did as well. When Jacob's mom would cook, Jacob often liked to 'cook' along side her, stirring some water in a pan with a wooden spoon. Jacob often pretended to 'go to the office,' saying so along with 'bye, bye,' and grabbing a dob kit of his father's, which likely symbolized bags parents carried upon leaving the house. Jeremy pretended with a real telephone alongside a toy one:

> Jeremy plays with his toy telephone, pretending to talk in it. Lao picks up the real phone and pretends to talk to Jeremy. Jeremy goes up to him and trades phones so that he has the real one. He points to it and says, "this, this." They both continue pretending to talk. Jeremy hands the phone to Birgit. She talks and gives it back to him. Lao gives Birgit his phone and she pretends to talk to Jeremy.

Like Jacob, Jeremy too pretended to cook and feed his parents using real pots and pans.

Although the toddlers clearly pretended with symbolic objects as well as real ones *as themselves*, for developmental psychologists, a definitive mark of symbolic play is the use of an object to symbolize something *else*.

Substitution

Piaget (1976) views substitution of one object for another (e.g., a coat collar for a pillow) as requisite for consciousness of make-believe -- to claim that symbolism occurs. For Vygotsky (1976), substitution of objects is also central to play. In pretending, Vygotsky sees the child as letting meaning dominate over objects. It is meaning that the child imparts onto objects, rather than actual features of objects, that determines behavior. However, without objects, meanings cannot 'float' to them.

> Separating words [or meanings] from things requires a pivot in the form of other things....([A child] cannot sever meaning from an object, or a word from an object except by finding a pivot in something else, i.e., by the power of one object to steal another's name.) (p. 547).

The 'pivot' is a provocative concept in Vygotsky's theory. My data include numerous instances of the toddlers enabling objects to 'steal' the names or meanings of others. However, Vygotsky underestimates children. In addition to his claiming they require pivots to pretend (which my data counter), he states that "play with an imaginary situation is...impossible for a child under three" (p. 544). All my subjects were well under three and engaged in such play. Even Piaget's (1976) data contradict Vygotsky's position on pivots. L. pretended with invisible objects at one year of age. Vygotsky further states that for children, pivots must bear some resemblance to the objects they symbolize. "[A]ny stick can be a horse, but, for example, a postcard can never be a horse for a child" (*ibid.*, p. 547). I think Piaget would have concurred, since even his definition of a symbol requires that it be 'motivated.' Collective signs and only collective signs are arbitrary. However, Jeremy's activity challenges this premise. He let objects substitute for others and signify in creative and unlikely ways, examples of which I discuss below.

Interestingly, Jacob did not engage in much object substitution. Some instances have already been mentioned -- the Lego construction as truck, the dob kit as briefcase. At times, he animated stuffed animals (through actions he performed on them and narration) which can be viewed as substitution of a sort (e.g., perhaps Pooh became a boy) or instead as 'bringing them to life' (which can also be viewed as something living substituting for something not living).

Another example was reported by Ann, Jacob's mother. After an impressionable encounter with a grandfather clock, Jacob would recall it by making other objects 'turn into' it. For example, Jacob swung a basket hanging from the hand of a toy snowman as if it were a clock pendulum and said, "ding dong."

George used object substitution more often, particularly around food. He pretended to eat a rock and a block. However, he was much more likely to pretend to eat invisible food (and feed it to others) which he did frequently. Even on the 'food day' at day care with numerous readily available symbols for food, George also pretended to eat and feed me invisible food.

Jeremy had a genuine penchant for object substitution. He pretended to eat an eraser and a music box, 'turned' a blanket into a hat, a ball into a dance partner and a cylindrical vacuum attachment into a horn. And there was more. After pulling an object out of my camera bag and hearing me call it a 'microphone,' Jeremy stuck one end in his ear and said, "Hello," 'making' it a telephone. A couple weeks later, he pulled a large, rectangular battery out of my bag and as with the microphone before, put it to his ear and said, "Hello." After pretending a toy cow was drinking from a toy trough, Jeremy dipped a toy length of fence into the trough and 'made it' slurp and swallow!

One time, Jeremy made his sip-cup signify in multiple ways. He drank juice from it, but then 'turned' it into a gun or another weapon of some sort. He moved it forward and back (in my general direction) and said, "piew, piew...." Then Jeremy turned to his mom and played with her long hair. Picking the cup up again, he inverted it and shook it on her head, as if it contained hair oil, and continued to 'arrange' her hair. Birgit complained about the juice spilling on her. As if to appease her, Jeremy 'slurped' the juice off her sleeve!

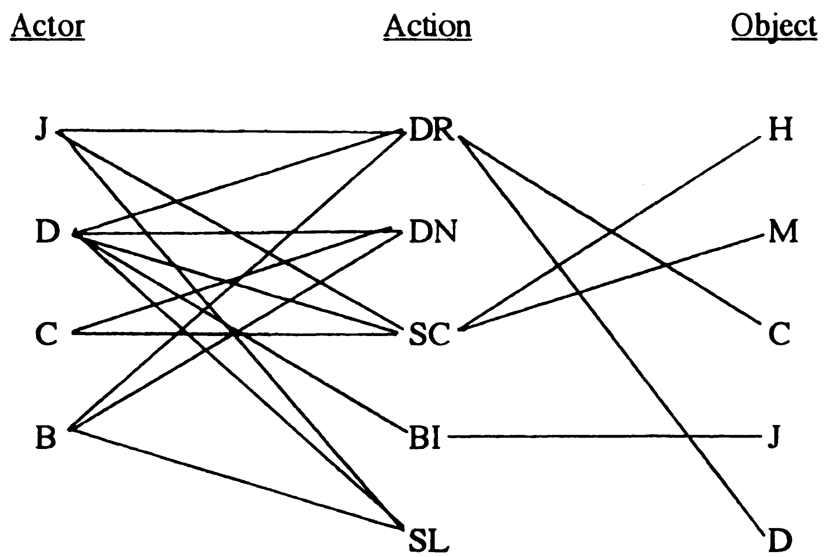
Jeremy can here be seen as "moving in the field of meaning" (Vygotsky, 1976, p. 550). His juice cup first *meant* what it really was, but then another meaning was substituted -- that of a weapon, and then another yet again. Birgit's reaction to Jeremy's play led him to re-acknowledge its material meaning (it *was* juice), however, he let his mom's arm now 'become' a cup (or some other device for holding juice).

This episode differs from some others in that while the symbol was held constant, what it signified varied from one meaning to another and another (rather than a particular signifier signifying a particular signified, e.g., a battery for a telephone). Another especially rich episode of Jeremy pretending shows a reverse sort of movement. In 'Dancing Dinosaurs' (caught on tape and previously mentioned in Chapter IV, pp. 86-87), there was a particular set of meanings (or activities or roles) and various characters moved into and out of them. The roles were fixed but different objects took them on.

> 'Dancing Dinosaurs' begins with Jeremy making a scary face and growling alternately at me and Birgit. We both pretend to be scared. At one point I leave the scene, as Birgit brings in a dinosaur puppet (D) and a stuffed Barney doll (B). Jeremy also drinks juice from a cup (C). Birgit animates

Barney frequently in the ways which Jeremy requests, while Jeremy animates the other dinosaur. Jeremy (J) also participates sometimes as himself. The various players engage in the following actions (or play the following roles): drink (DR), dance (DN), scare (SC), bite (BI) and sleep (SL). There were also objects of some actions, which can be thought of as fulfilling the role of *acted upon* -- Jeremy (J), the juice cup (C), the dinosaur (D), Mom (M) and me (H).

Activity that Jeremy either does or directs include the following: Jeremy (J) scares (SC) me (H) and his mom (M). So does the dinosaur (D). The dinosaur maybe tries to scare Barney but that is unclear since Barney does not 'act' scared. The juice cup (C) also once growls (SC). Barney (B), the dinosaur (D) and the juice cup (C) all dance (DN). Barney (B), the dinosaur (D) and Jeremy (J) all 'drink' (DR) juice (C) although Jeremy does it for real. Jeremy (J) also 'drinks' (DR) the dinosaur (D) a couple times, putting his mouth on its nose, making slurping sounds and afterwards says, "ahh." The dinosaur (D) once bites (BI) Jeremy (J). Jeremy (J), the dinosaur (D) and Barney (B) all sleep (SL). See the following diagram showing all the above.



In this scenario, a structure or grammar can be discerned, with different objects (or actions) capable of filling different positions within the structure. A role can be substituted one for another (e.g., Barney plays the role of dancer and then the juice cup does), but also an action can be substituted one for another (e.g., Barney dances and then drinks). Thus, there can be multiple kinds of substitution -- *role* and *action* as well as *meaning* (i.e., a cup 'becomes' a gun, also known as 'object substitution').

Role play, the taking on of a role, can also be seen as using oneself as a pivot, substituting another character for oneself. Jeremy *became* a scary monster. In pretending to do what his parents do, Jacob *became* Mommy or Daddy. Role play sometimes ends there. However, often role play involves playing a role in opposition to other roles. Not only does one make oneself a pivot but another person does as well and/or another participates in imparting meaning to the pivot. In either case, the toddlers communicated that they wanted others to 'play opposite' them, to participate in constructing a fantastic world. Pretending thus became social play -- social role play involving social negotiation.

Social negotiation

Weinberger & Starkey (1994) characterize 'sociodramatic play' as the most advanced type of pretense, in part due to the social negotiation it requires. Bretherton's (1984) hierarchy culminates with 'collaborative role play,' the term 'collaborative' also highlighting the involvement of social negotiation. Negotiations involved in the toddlers' social role play ranged from the subtle to the direct.

George would often pretend to be an animal. A lion was his favorite and most frequent role, which he would enact by growling. However, he also 'became' a sweet puppy dog, crawling on hands and knees and emitting a soft 'mhm, mhm, mhm...' sound. George's interaction with others when taking on these roles could be seen as that of putting on a performance for an audience, much like with 'Big, Big Lion.' However, with 'Big, Big Lion,' a photo played a role opposite George in a space very much like a stage (i.e., at a distance and set apart from the 'audience') with others external to the action, filling roles only of passive viewers. On the other hand, when George was a lion or a dog, he came in close proximity to others and looked at them expectantly. Others complied by acting scared or sometimes petting the puppy, offering words of praise. When Patty, George's day care provider asked several times to see a monkey, he refused. Perhaps a monkey failed to inspire opposite roles.

Parents sometimes initiated pretend play and the toddlers played along, accepting the scenario and taking on a role. This dynamic occurred with Jeremy in the telephone episode, described in the first section of this chapter. (Lao initiated Jeremy's role in the play.) Once, Birgit pretended to cry and Jeremy then stroked her cheek. In 'Dancing Dinosaurs,' Birgit introduced Barney into the scene; however, at first she made him growl. Jeremy 'told him' to dance instead. That time Birgit accepted Jeremy's instructions, but another time she did not. Jeremy reached out his juice cup to Birgit wanting her to pretend to drink. She told him, "don't want." Jeremy held his cup near her mouth anyway and made smacking noises followed by "ahhh." Since Birgit failed to play her role as Jeremy had asked, he animated her just as he had his dinosaur. Another time several others did play the roles Jeremy requested. Jeremy pretended to drink from an empty beer bottle, then gave Birgit, Lao and me each a 'drink' in turn.

George engaged in similar pretend scenarios. He, too, would follow a parent's lead in pretending, for example, answering, "Santa," in response to Lynn's question of where he was going in his car. He, too, enticed others into roles in a distributive fashion, as the following example shows:

> George gets out of the car and goes to the truck. He puts his hands down in the truck and takes them up, in positions as if he's holding something small and invisible. He goes over to the porch entrance where Lynn is sitting on the step. John is inside the house nearby. George acts to 'take a bite,' then puts his hands by Lynn's mouth, she 'takes a bite,' then by my mouth and I 'take a bite.' Then he 'puts the food down' on the step and climbs up inside. He turns around, bends down and picks up the pretend invisible food and goes to John to have him 'taste it.'

Since I was able to observe George at day care, I was privy to the ways he followed the lead of other children. Although I did not witness him initiating social role play with children, he accepted the roles they established for him and participated in negotiating the particulars.

On the 'food day' at day care, an older playmate pretended to pour milk for George from a toy milk carton into a toy sip cup. She told George to wait and he did so patiently as she 'poured' into the cup and replaced the cap. She handed the cup to George. He put it to his lips and pretended to drink. Then George held up the cup and declared, "milk." Later that same day, the playmate initiated a different pretend scenario in which George also 'successfully' played his role:

> She brings a play cake to the table, pretending it's George's birthday. She sings, "happy birthday," and places the cake in front of George. He says, "wow." She hands George a piece of cake (the plastic cake is divided into four pieces with icing that also separates). George pretends to eat it. Soon after, she tells George he needs to give it back to her, also saying, "all done." George says, "all done," and hands it back to her. She takes the cake back to the oven to "cook it some more." Sometime later, she brings it back to the table, placing it in front of George. George again says, "wow." He also says "thank you" when she hands him a piece. He 'eats' as if a few bites, then hands it back to her and says, "all done."

An interaction between George and a different playmate played out differently:

> She is walking around with a basket containing several differently shaped plastic blocks. She goes by George and hands one to him. He throws it. Patty's [George's day care provider's] daughter tells him, "no." The playmate picks it up and hands it back to George. George begins 'munching' on it. She is passing out blocks to other kids too. She returns to George and holds out her basket to him. George reaches a free hand in as if to take another. She quickly pulls away the basket and says, "no." George continues to 'munch' on his block for quite some time.

In this last example, George was given an object but not told the meaning his playmate wished to attribute to it. George relied on his creativity to make one up. He attended to social cues to determine if his meaning was acceptable and his behavior within the bounds set by others in the pretend scenario.

Creativity is reflected in all of the toddlers' pretending. Jeremy's seems unbounded.[3] However, both Jacob and George set limits to their imagined worlds, or at least limits as to how they chose to symbolize them.

Undrama: placing limits

The most creative and dramatic forms of pretending George and Jeremy displayed were absent from Jacob's repertoire. Likewise, there were ways that Jeremy pretended that George did not. I do not believe these differences are due to a lack of imagination; rather they primarily reflect choices and boundaries of outward representation.

As I mentioned previously, the times I heard Jacob say, "beep, beep," were when he played with objects where the meaning of car or truck was not necessarily obvious. When he played with symbolic vehicles, he moved them around but otherwise did not animate them -- no motors, horns or whistles. To Jacob, these dramatizations may have been unnecessary in order to convey signification. The case of the grandfather clock further supports this interpretation (e.g., animation was employed since a toy snowman did not obviously signify a clock). It could be that Jacob did imagine horns and motors or other features or activities, but he was somehow able to hold them in his 'mind's eye' without symbolizing them. He had no need for this form of symbolic expression.

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[3] However, structure can be detected underneath apparent randomness. This is further discussed in Chapter VI.

Jacob's stance towards animation extended to stuffed animals. At times, he would pretend they were sleeping, but only by playing a role opposite them -- putting them in bed, turning off the light, saying, "night, night." He never snored for them. Jacob also narrated what he imagined his toys doing, saying, "puppy is crying," and "bunny is eating," but he did not make sounds or movements to demonstrate these actions. Perhaps conventional verbal symbols provided sufficient cognitive support for Jacob to imagine the toys did what he said. He *told* and therefore did not need to *show*.

However, Jacob's paucity in animating the inanimate seems to go beyond the issue of cognitive necessity. I believe this absence of animation reflects a stance Jacob held towards pretend and reality. On my final observation with Jacob, I tried to lure him into certain forms of pretending. Jacob had drunk water from 'daddy's cup' and after finishing it, pretended to drink. I joined in, pretending to drink from an invisible cup and eat an invisible sandwich. Jacob looked at me curiously but warily. I offered Jacob some of my 'sandwich.' He played along and opened his mouth, but did not know quite where to 'bite' and did not join me in loud, dramatic chomping, chewing and slurping.

I think that Jacob saw me as being somewhat ridiculous. Why pretend to eat something invisible? He pretended to drink from his father's glass in order to *become* him. What is the point in pretending to drink for its own sake? The psychoanalysts would be pleased; Jacob seemed to pretend for a reason. Grandfather clocks *impressed* him. But what is impressive about eating or sleeping? Furthermore, Jacob had words to talk about these things. He could speak of them in economical ways. He had no need to act them out dramatically, no need for 'motivated symbols.' Of course, if Jacob performed this way on a stage, he would be booed off.

In addition, I see a setting of boundaries. Jacob could imagine inanimate objects ‘coming to life’ but not him ‘becoming’ them, which perhaps he would do if he spoke for them. He stayed on the outside, did not venture in. He played *opposite* them, not them. In contrast, Jeremy played opposite the baby doll, feeding it and hugging it, but simultaneously ‘was’ the doll, ‘crying’ for it.

Interestingly, George had a similar boundary. Unlike Jacob, George ate invisible food and clearly had a taste for the dramatic. But like Jacob, George did not *become* inanimates. George himself could become a lion and while his imagination extended to a lion picture and doll, he would act opposite or narrate their imagined activities, but not animate them. George would ride in or on a truck, but not *become* the truck.

For both Jacob and George, there was a boundary between pretense and reality, one they did not cross. Perhaps it is an issue of space, bodily boundaries, but it may also express a desire for limitations. While some things change, others must stay constant. While aspects of reality are transformed by imagination, other aspects must be preserved. Parameters are set within which play is permitted.

Mathematical connections

Mathematics is essentially play with a pretend world. It may not seem that way when using arithmetic to balance a checkbook -- a very real and utilitarian task. Even so, although more prized than monopoly money, real money is also part of a game, an economic game akin to Wittgenstein’s ‘language games.’ Society collectively agrees to give it value, to exchange goods and services for it. Things could and have been otherwise. No object has an intrinsic monetary value. Without the game, there would be no quantification of value, no money. And no checkbook.

The worlds of pure mathematics and surely that of school mathematics are even more obviously pretend. Mathematicians create pretend worlds by making assumptions expressed symbolically. They then play around within that world, with pretend objects and ideas. Through their use of symbols, they bring imagined objects into being. They arrive at theorems and construct proofs. The language they use reflects that they are pretending. A theorem can be written: If...(such a thing existed or were true), then...(such another thing would exist or be true). In a proof, an idealized example is imagined: Let ABCD be a rectangle....

Although school math is taken very seriously by society, it too very much takes its place within pretend games. Gerofsky (1996) discusses how the word problems found in school math belong to "fictive" utterances that "pretend *that*" a situation exists, even the most realistic, "real-life" problems. She explains that these stories have no "truth value" and raises the question of why include them at all in school mathematics, since they are "at heart an arithmetic or algebraic formulation which has been 'dressed up' in words" (p. 39).

While I recognize the problematic nature of word problems and their use in school math, I do believe they can serve important educational functions if used in particular ways. I will not explore those functions and ways of use here, except to say that perhaps it would be wise to publicly recognize the fictional character of word problems and introduce them with, 'pretend,' 'imagine,' 'make-believe.' If children would realize they were *only* playing games, perhaps doing mathematics would not be so rife with anxiety. 'Imagine that Bob has five lollipops and Walter has three lollipops....' 'Let's pretend that each of these cubes is a lollipop....' 'Make believe a train leaving from San Francisco traveled at a rate of 75 mph....'[4]

A central aspect of the pretending of mathematics is the use of symbols. Imagined situations are encoded and expressed in symbols and re-expressed through semiotic chaining, then operated upon. Blocks signify imagined lollipops, numbers signify the blocks, numerals signify numbers, addition is performed with numerals and so forth. Mathematics involves symbolizing something imagined as with any other sort of pretend play.

There were numerous, diverse aspects to the toddlers' pretending that shed light on the enterprise of symbolizing the imagined generally and offer insight for mathematics as well. Just as there is a continuum between symbolizing a true story and a fantastic situation, there is a continuum between using mathematical symbols to solve actual problems and to create completely fantastic mathematical worlds. Perhaps it is important to recognize where on the continuum a particular problem lies, especially for teachers and students. However, once in the symbolic realm, reality is to some degree dispensed with; indeed, it needs to be. Symbols can signify reality, but not be one and the same with reality. In mathematics, symbols often signify other symbols in which case reality is further removed. In pure mathematics, symbols are all that exist, relating only to each other (although on some level derived from human interactions and perceptions of reality) much as George's character related only to the imagined lion signified by the lion photo in his fantasy play. How real boys might behave around real lions was irrelevant.

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4 This last example, although highly common, is particularly problematic. For one, it assumes the speed of the train is constant, which is not just pretend, but impossible. A real train's speed varies throughout a journey, which includes stops along the way. For a real average speed to be determined for the journey of a real train, it must be done by dividing total distance by total time, one of which a problem like this would leave to be found. In the pretend problem, the givens and the 'to finds' are reversed.

Another continuum worth considering is the one ranging from 'motivated symbols' (signifiers that bear strong resemblance to signifieds) to completely arbitrary signs. Personal dispositions likely led Jeremy to act out imaginary situations with great drama and Jacob to sometimes be satisfied with narration, using conventional symbol strings to signify. Children in a classroom may display similar dispositions. One child may wish to draw a detailed picture of Bob and Walter holding lollipops, while another would write '5 + 3 = 8' and be done. Generally, mathematics values economy of expression -- the tighter the better, the more elegant, the more beautiful. Both historically and in school curriculum, mathematics moves towards ever more compact and abstract symbolism. However, it must be remembered that in such movement, specificity and detail are lost along the way and if moving too quickly, understanding too can be lost.

When children learn new ideas, it may be helpful for most to begin at the 'motivated symbol' end of things and move towards the 'arbitrary sign.' Skemp (1987) expressed this as a movement from the "informal" to the "formal." Findings from Hughes (1986) support this notion. In a game where children needed to identify in writing the number of objects contained inside a box, pre-school children constructed their own systems of representation that were pictographic (drawings depicting actual objects) or iconic (e.g., tally marks), which are 'motivated,' rather than make use of numerals, which are arbitrary.

To take an example from school mathematics, learning fractions may be very difficult by only using symbols such as '4/5' or even 'a/b' at a greater level of abstraction. Illustrative situations of sharing and partitioning, graphic representations, even physical acts of cutting and manipulating pieces for comparison may be necessary for most children to understand the ideas involved and subsequently be able to move towards more abstract symbolism. However, it must be remembered that all representations and illustrations of fractions *are* symbols and that fractions -- part-whole relationships -- are ways of seeing that are symbolically mediated even if initially only by every day spoken language.[5] These ways of seeing are by no means intrinsic or automatic and do not exist in every culture.[6]

"Names stress and ignore" (Pimm, 1995; p. xiii); so do toddlers' spontaneous signs. In telling his story of 'Spilt Milk,' Jeremy chose to stress some aspects of the events and ignore others. What he attended to, what he abstracted and how he symbolized was apparent. School children engaging in mathematical tasks will surely stress some aspects of a situation and ignore others, not necessarily the ones the teacher or curriculum has in mind. Children could get carried away by context if, for example, they become concerned with the flavors and colors of lollipops or with the details of Bob's and Walter's clothes. Likewise children could ignore the mathematical features of a problem. However, children's attentions should not be rejected outright. They reflect how they see, how they think and how they symbolize (including which symbols mean or fail to mean for them).

For example, if a child only likes red lollipops and not green or yellow ones, the total number of lollipops Bob and Walter have may be irrelevant since only the number of red ones matters. Rather than dismissing this perspective as 'unmathematical,' as a teacher I could see capitalizing on it and complexifying the problem. Perhaps Bob and Walter each have different numbers of different colors of lollipops. Who has more red? More green? How many red do they have together? What if Walter only likes red? What if Bob likes all colors equally? Could they trade lollipops and arrive at an arrangement preferable to both? What would happen if Sally showed up with 6 purple lollipops? The example may be

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[5] Pimm (1995) draws a distinction between 'illustrations' and 'representations,' the former being situations used to illustrate a mathematical idea, such as sharing brownies in the case of fractions. He warns, "[T]here is the possibility of becoming absorbed in the detail of the illustration at the expense of attention to what it is supposedly illustrating" (p. 26).

[6] Denny (1986) discusses how Inuit languages have no means of expressing numerical division. While they can conceive of dividing produce and meat into shares, and do so, they follow anatomical structure. Part to whole as an abstract numerical relationship (which includes the quantiative equality of each share) has no place.

farfetched. I do not think children's preference for a certain color of lollipop should prevent them from doing the math of the original problem. However, I hope I have illustrated that what children stress and ignore can be relevant to how they engage with particular problems or mathematical ideas, and that variation and creativity in children's attention can be approached as an asset and not an inherent liability.

'Spilt Milk' offers further food (or drink) for thought. The self-referential function of the milk meaning itself and also more than itself has its parallels in mathematics. The artist M. C. Escher explored this notion in his artwork. For example, in *Print Gallery* (1956), a print in a gallery depicting a town turns out to *be* one and the same as the town. In *Drawing Hands* (1948), a drawn hand draws another, which is drawing it (see Hofstadter, 1980). In mathematics, symbols frequently refer to symbols and self-reference is common. To solve an equation, one frequently rewrites it several times. All are equivalent statements. Each refers back to each other (which are mathematically the *same* as itself) as do individual symbols. For example, the a here refers to the a in the previous statement and in the previous statement.

Although signs signify themselves, like the milk they can signify more than themselves. In an equation, a refers to a, a particular number. While at times a can signify a specific, though unknown number, at other times a signifies a member of a whole class of numbers. Let a be an integer, let a be prime, let a be even, etc. Even in elementary school, this aspect of symbolizing is present. For example, in order to make sense of even numbers, children explore (or are shown) examples, e.g., 4, 6, 18. They then make (or are provided) generalizations and draw conclusions about all even numbers. Thus, in such a situation, 4 and 6 signify not only themselves, but all even numbers.

There were other ways in which signifier and signified were identical (or nearly so) in the toddlers' pretending. Symbolic cars signified real ones and telephones signified, well, telephones. Such givens were thoroughly present in the toddlers' environment. Givens are also thoroughly present in school (and other) mathematics -- given symbols, given concepts, given problem specifics. That the toddlers made use of givens in pretend play -- not merely as a 'reflex,' but in conjunction with imagination and creativity -- intimates that children generally can make meaningful use of givens in the pretend game of mathematics.

However, as Jacob helps to point out, what may seem given to an adult may not be so for a child, and for one child, not so for another. In other words, the meaning of a given symbol may not always be clear. A realistically designed toy car as well as a cartoonish car picture were both obviously 'car' to Jacob, yet a Lego construction only 'became' a truck when supported by added symbolism. A '5' may not obviously *be* 'five' (the idea of five) to all children, nor a circle with a vertical line drawn down the center *be* 'one half.' Symbols may appear obviously 'motivated' or transparent to some students (and teachers) and not others. When a teacher notices a student does not share the same meaning of a symbol that she does -- or no meaning -- she will likely try to explain or demonstrate further. Of course, she must use additional symbols (words, pictures, etc.) to do so. Those too may or may not be transparent to students. The toddlers make me aware of the need to continually attend to the degree givens really function as givens (i.e., carry intended meanings) for children in classrooms.

Then there are mathematical symbols that function very much as Vygotsky's 'pivots.' These are known in mathematical language as 'variables' (e.g., in '$a + b = b + a$,' any number can be substituted for a and for b) and 'unknowns' (e.g., in '$a + 5 = 7$,' only one value for *a* can make the equation true) (Usiskin, 1988).[7] In a sense, we can see variables and unknowns as place holders, holding the place of specific or multiple possible values. Vygotsky too saw pivots as holding the place of other objects or of meanings that belonged to other objects. To work effectively with pivots, both in pretend play and in mathematics, children must substitute one object, meaning or value for another.

The toddlers used pivots both as unknowns, when for example, a toy snowman was given the particular meaning/value of a grandfather clock, and as variables. Jeremy demonstrated this latter usage when his cup took on numerous different meanings. George used a variable pivot even more generally when he let invisible food be any food (sometimes specifically named, e.g., grapes, and other times unnamed and likely unspecified to George). His motions alone denoted the 'presence' of food -- cupping hands, taking bites, chewing, etc. This can be seen as akin to the empty shapes frequently used as placeholders for unknowns in elementary school introductions to algebra. The shape outline is George's cupped hands and the empty space is like the invisible food that could be anything (any number, until decided).

In Jeremy's 'Dancing Dinosaurs,' there were no concrete, visible pivots that took on various meanings, no obvious placeholders as such. There were numerous actors performing numerous actions, who also at times filled the position of being ***acted upon.*** If Vygotsky is right, that children require some sort of pivot, perhaps Jeremy held an imaginary, abstract one (or indeed several) in his mind. Perhaps Jeremy utilized something akin to an *a* in algebra. Something completely abstract held positions that various objects or actions could fill. The episode can be represented thus: let *a* be an actor, ✳ an action and *b* an acted upon. Jeremy's pretending took the forms of 'a ✳' and 'a ✳ b.' In the absence of

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[7] That is, in conventional number usage. If we performed arithmetic on a clock, an infinite number of values could be substituted for *a* and the equation would be true, *a* being hours: 2, 14, 26, 38, etc.

a visible pivot, substitution of one thing for another implies the position of a pivot, one that can be denoted, 'held' with an abstract symbol.

'Dancing Dinosaurs' involved the social negotiation of roles as did other episodes of the toddlers' pretending. A central part of the negotiation had to do with the meanings of symbols. One of the examples with George showed that he accepted the meanings his playmate attributed to the symbolic objects and actions she used -- she 'poured him milk' and 'brought him birthday cake.' The playmate communicated very clearly the meanings she held and George happily complied. However, in the example that followed, George's playmate failed to communicate specific meanings. Through trial and error, George sought to arrive at one himself, one that was socially acceptable.

With Jeremy and 'Dancing Dinosaurs,' the negotiation had less to do with meanings of particular objects and more to do with the roles they had to fill. (Although this can also be interpreted as negotiating the meanings of symbols, e.g., Barney as dancer, dinosaur puppet as scarer, etc.) Jeremy rejected Barney as filling the role of a scarer and successfully asked that he dance instead. His own dinosaur did not dance until I asked if he could. Jeremy accepted my suggestion, letting his dinosaur dance too.

The meanings of mathematical symbols, be they signs for quantities, shapes or operations (akin to actions) are also subject to social negotiation. Indeed, mathematical symbols function as signs in a classroom only because they carry agreed-upon meanings. Their use and meanings arise not only through the social negotiations in particular classrooms, but have arisen historically over centuries of social negotiation across great distances of time and space. This negotiation has involved give and take and even conflict and resistance.[8]

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[8] See e.g., Tahta (1991) for a discussion of the resistance of European abacists to Arabic numerals and algorithms used to calculate with them.

The toddlers show that elementary children can participate in negotiations over meanings. They can introduce their own meanings and accept those advanced by others. As with any pretend game, cohesion is of prime importance. Whatever meanings are decided must be maintained; otherwise, the game falls apart. If children wish to continue to play the game, they must either maintain prior agreements or renegotiate.

An example comes from Ball's (1990) teaching of third grade mathematics. A student (pseudonym, Sean) claimed that certain numbers could be *both* odd and even. This contradicted the meanings the class had up to that point attributed to the terms 'odd' and 'even,' and to the recognition that these signs signified mutually exclusive categories. The class resisted Sean's new meanings but he persisted in forwarding his idea. Eventually, the class agreed to maintain the previously agreed upon meanings for 'odd' and 'even,' but decided to accept Sean's concept and give it a new name. The numbers he described were thereafter known as 'Sean numbers.'[9]

Negotiation of symbol meanings among students is not common in math classrooms. Nevertheless, while primarily top-down (from teacher to student), I am certain that negotiation does occur. As with George and his block, students may attempt to arrive at meanings by bumping up against the parameters they encounter. However, the meanings they thereby construct may not necessarily be the meanings teachers hold in their minds. As with the playmate who gave George a block, teachers' and curricula's exact meanings may remain a mystery to students.

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[9] See also Ball (1992) for a description of Sean's idea.

George had no difficulty operating within parameters. He set his own, as did Jacob. Encountering parameters in school mathematics might similarly 'make sense' to them. They could readily accept for example, that given a particular system, '10' means *ten* and that given another, '10' means *four* (e.g., in base four). Jeremy might present a different situation. Given his creativity and predisposition to substitute meanings, often without capitulation to boundaries, he might need to be 'reined in.'

If allowed, a child like Jeremy may wish to make '10' mean *four*, then *three*, then *twenty-eight,* willy nilly. Such behavior need not operate as an impediment to the child's or class's learning. As a teacher, I imagine taking advantage of such creativity while simultaneously establishing the need for clearly stated, agreed upon parameters within which play can occur. For example, how could 'ten' mean *twenty-eight* ? Suppose we counted on our bodies and used twenty-eight body parts. The '1' could then stand for *one* whole person and the '0' for no parts left over.[10]

While Jeremy showed great flexibility in the ways he would symbolize, George and Jacob set particular boundaries. This could show an unwillingness and even difficulty with certain forms of symbolizing (although I do not think any of these would interfere with their engagement with school mathematics, perhaps, the opposite in fact). This difference may also reflect various needs for symbols as cognitive support.

Without symbols, very little mathematics could be done. Pimm (1995) explains, "[S]ymbols allow us to manipulate, by proxy, things that are not easily handled, or which are even impossible to handle, by our physical selves" (p. 109). In doing mathematics, we must manipulate symbols, if not on paper, at least in our minds. For example, without

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[10] This example draws upon indigenous body tally counting systems found in Papua New Guinea and elsewhere. See e.g., Saxe (1981), Lancy (1983), Ifrah (1985).

some symbolic representation, be it Arabic numerals, Roman numerals, or discrete pebbles, humans would not be able to grasp the magnitude of one hundred (or even twenty), let alone arithmetic patterns and properties of addition.

Jeremy, George and Jacob exhibited differing needs for symbols as cognitive support in regards to number of symbols used simultaneously, domains, and diversity of forms. Again, children in mathematics classrooms could exhibit similar variability. Rather than restrict children to a limited usage of the current corpus of given mathematical symbols (as is common practice), it may be wise to allow them to select, create, embellish and extend symbols as they see fit for engaging in mathematical pretending. If children were to use mathematical symbols in genuine conjunction with their imaginations, perhaps they would thereby succeed in calling mathematical ideas 'to life.'

Chapter VI: Systematizing and Playing with Symbols

Naming explored how the toddlers came to name their world, link signifiers with signifieds, gain fluency with signs. *Symbolizing the Imagined* focused on a different use of signs. Rather than signifying in ordinary ways, signifiers took on roles in pretend contexts and acquired meanings within those contexts through relationships with other signifiers. Signs went beyond serving as a medium for interaction with the world; they enabled the toddlers to construct imagined worlds, to enter symbolic realms separate from material reality. Signs were on their way to becoming *symbols*.

This chapter focuses on the end of the toddlers' symbolic continuum: on their play with symbols. Rather than signify for purposes of earnest communication or to birth imagined situations, in the episodes discussed here, the toddlers' symbols existed more or less for their own sake. They formed relationships, took positions within patterns, were governed by rules the toddlers set and were manipulated systematically. Most of all, symbols were objects of play. At times, they were derived from signs. At others, they resembled signifiers in form but never pointed to external meanings. At others still, they were material objects. Their relevance to symbolizing lies in the roles they took in relationships, in that they participated in forming patterns.

Mathematics is commonly understood as being ordered and systematic. It involves investigation of systems and structures and is frequently described as 'the science of patterns.' Resnick (1997) maintains that mathematical objects do not primarily derive their characteristics through resemblance to real objects but from the patterns they form. In other words, mathematical objects, denoted symbolically, derive meaning from patterns. Conversely, patterns require objects, symbols in order to exist and be 'seen.'

> [W]e do not literally see or intuit patterns. After all, there is nothing to see but their positions. Seeing a pattern is more a matter of *seeing that* certain of its instances fit it or satisfy its defining conditions (p. 225).

Definitions and rules have roles to play too. They create parameters or boundaries and govern how patterns are to behave or structures constructed. Hence, in mathematics, there is a dialectical and reciprocal relationship between symbols and forces of systematicity, be they rules that define and describe patterns or the patterns themselves.[1]

This relationship can be illustrated by the ubiquitous mathematical symbols of numerals and number words. Throughout the world today, numerals (commonly called *Arabic*) follow a base ten *system*. They express numbers as multiples of powers of ten. For example, '341' denotes ***three times ten squared, plus four times ten to the power one, plus one times ten to the zero power***. English number words embody the same base ten structure for the most part. Exceptions are the unique words 'eleven' and 'twelve' (rather than 'one-teen' and 'two-teen') and the teens to some degree, which place the signifier for tens after the signifier for ones in contrast to the decades which follow (e.g., 'thir***teen***' verses '***twenty***-three'). French has unique (related but no mention of ten in the second decade) number words through sixteen and uses 'twenty' in higher combinations, exhibiting an even greater mismatch between the systems embodied in number words verses numerals. The number word systems in Asian and Semitic languages have a greater degree of regularity and congruity with base ten numerals. Other languages use different systems altogether. For example, numerous tribes in Papua New Guinea use the names for body parts as number names, which are derived from mapping particular body parts to quantities in a fixed way (Lancy, 1983). Regardless of the system, most languages

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[1] Rules and patterns can be seen as inextricably linked, one implying the other. For example, given a pattern, there are particular rules that it follows, which a mathematician may attempt to express. Given rules for a pattern, the pattern can be constructed. Although surely linked and related, rules and patterns represent different points of departure and differing emphasis and attention on aspects of a structure. These differing emphases also led to different forms in the toddlers' play.

throughout the world use systems to generate number names, systems the names in turn reveal (Ascher, 1991).

Symbol and systematicity interact in mathematics, likewise in the toddlers' play. The toddlers attended to systematicity in the world around them and engaged in systematizing. They adopted rituals and created new ones. They ordered their environments, establishing rules for themselves and others to follow. Their play was patterned and patterns were objects with which to play. In all these activities, symbols had a role.

Systematizing

> Essentially cognition is always oriented toward this essential aim, the articulation of the particular into a universal law and order (Cassirer, 1953, p. 77).

From birth on, humans attempt to create order out of the chaos around them, to generate rules about how things work, identify patterns, use patterns to predict future events, and group unique occurrences into general categories. Language owes itself to this disposition, as does mathematics. In this section, I will discuss episodes of the toddlers that elucidate their varied expressions and dispositions toward systematizing.

The toddlers displayed evidence of trying to find structure in the world around them. Jacob's grandparents reported that when they took him to a familiar restaurant, he mentioned 'choo-choo train.' He knew the tracks ran past the establishment and waited with anticipation for a train to pass by. Jeremy showed a similar awareness as to where train tracks crossed through town. When approaching railroad crossings in a car (whether or not a train was coming), Jeremy generally announced, "choo-choo!"

When playing hide and seek, Jacob would always look again in the last place the hider had hid. He even once looked for me in my frequent hiding place when he did not immediately see me, although we had not been playing hide and seek! There was also an incident when Jacob's dad had left the room and gone outside. The next time Carl 'disappeared,' Jacob again looked outside and seemed to insist that he must be out there. It took some effort to convince him to look downstairs where Carl actually was.

The world's organization also involved forbidden places. Jacob told me about a closet, "there no." George moved me out of a chair he identified as 'dada's' and into one that must not have 'belonged' to a particular person. On several occasions, he asked me to sit down in specific spots.

Things belonging to people was also part of the world's structure. George identified shoes (not on feet) and cups placed on a table as belonging to, or being drunk by his mom, dad, him or me. Jacob named types of music according to their listeners and cars according to their drivers. To the toddlers, the world contained places linked to events and things linked to people. It also contained routines and rituals.

Rituals in human life vary from complex, highly revered religious rituals involving multiple participants, objects and actions to simple, individual rituals such as brushing one's teeth before going to bed. What all rituals hold in common is structured, routinized performance, generally enacted under certain conditions. Mass is held on Sundays. The rain dance is performed if rain is needed. I lock the front door whenever I am last to leave the house. I mentioned previously that George was particularly attracted to the ritual of clinking glasses when drinking. He insisted on doing this at nearly every meal.

George created and enacted other rituals with his parents. When he and his dad would exit the house from the side porch, he would point or pick up three differently-sized rocks in turn and John would name them, 'little,' 'large,' and 'medium-sized rock,' respectively. When looking at books of images (and the bath picture strip), his parents would ask a series of: "Where is the _____ (object name)?" "What's this?" and with animals, "What does the _____ (animal name) say?" and George would answer. Jacob had a similar ritual around a book, Richard Scarry's *Busytown*. His mom (and once I) would ask him repeatedly, "Where is a _____ ?" filling the blanks with names of vehicles. Jacob would flip through the book until he found an example and then point to it. Some vehicles had only one example in the book (e.g., train, bus), and Jacob had to find the particular page. Other names had numerous choices of referents. Jacob seemed to generally select the first example he found. During the height of his interest, Jacob would interact with this book in this way several times daily.

Other rituals in the toddlers' lives were all their own. Jacob made sure to turn off the light whenever he came up from downstairs. At day care, George always played with a particular truck in a particular way, riding it up and down the hallway, usually saying, "bye," upon leaving the main room and, "hi," upon entering. Patty noted that George began every morning with this activity.

At home, George had a set of matchbox vehicles with which he played in ritualized ways. For example, he sent the vehicles down a ramp one at a time, always waiting for each car to stop before sending down the next one. He also always placed them in a particular spot when putting them away -- on the far right end of the shelf from which he launched them. Around the time I began observing George, he initiated the ritual of lining up his vehicles on that spot on the shelf, always side by side in a straight row against the wall, facing out, as if parked in a parking lot.

Roughly a month after George began to 'line up his cars,' he began to explore other ways to arrange and order them. Lynn reported that he placed his vehicles bumper to bumper and moved them forward a small distance at a time, in order, making them 'snake around.' While departing from his usual ritual (although not abandoning it -- he continued to enact it), and in a sense, freeing himself from its strictures, George nonetheless played with order and established new rules (and/or carried over old ones) to govern this play.

A week after George's explorations began, I caught him on video 'lining up his cars' for the first time. He first appeared to line them up any old way, but quickly rearranged them so that the only non-cars, a van and a truck, were adjacent and in the front. However, he then found a stray car and placed it in front of them. Two weeks later, the camera captured George exploring order with his vehicles rather extensively. I call this episode, 'Arranging Cars,' as described from the video:

> George begins by arranging six of his vehicles in a straight line, bumper to bumper. He then undoes this arrangement one car at a time, beginning from the back. He wheels them into new places as if they are real vehicles, first 'driving' them in reverse, curving them back and then 'driving' them forward and into position. The new arrangement has three vehicles side by side under the ramp of his toy garage as if parked there. The three remaining vehicles are in a straight line immediately in front of the others.
>
> George proceeds with making other arrangements. In all, he ends up making seven completely ordered arrangements (i.e., all the vehicles are in a single line or row) and three 'unordered' arrangements. In the ordered arrangements, the van and truck are always adjacent and at one end. In unordered arrangements, they are always next to each other.

George's behavior implies a rule that the van and truck must always be together and in a place of prominence, a rule that George seemed to have followed earlier and which may have governed more generally his play with the toy vehicles.

In making new arrangements, for the most part, George 'undid' prior ones, peeling off the last vehicles placed, one at a time, and positioning them anew. George also continued to move the vehicles as they were if real, 'driving' them along the shelf, until making the final two arrangements. For these, George picked up the vehicles and lifted them in the air to their new places.

In making the final two arrangements, George can be seen as abandoning the need to be 'real.' Initially, his vehicles were signs, signifying real vehicles. George moved them accordingly while exploring structures. However, at one point, the vehicles' verisimilitude stopped being important. Structure became paramount. The vehicles became objects whose individual meanings no longer mattered. What mattered was that George could place them in relationship to one another, could position them in patterns. They had become *symbols.*

I witnessed another time in which George seemed to establish parameters for exploring structures. He played with a pop-up toy at day care, one that had four characters in a row. Each could be 'popped' by manipulating a trigger in front of it. George seemed to explore different orders of popping up the characters -- once in order, another time the end two first, then two simultaneously, etc. However, he always closed them one by one in order of lineup. After some time of only alternating orders of opening, George began to alter orders of closing too. However, even though he was exploring in both directions, George maintained the constant of all closed before opening and all open before closing.

Rules set parameters for exploratory play, yet they permeate games. 'Games' can be identified in large part because of the rules that players must follow. There are no games without rules. All of the toddlers played the well known game of hide and seek, yet each played it a bit differently. Different rules governed. Jacob was generally the seeker. An adult would hide, he would find the adult, the adult would come out of hiding chasing him, he would run away, the adult would hide again, etc. If Jacob were to hide, he would be *the hider*. Roles did not switch. Once when I tried to play in such a way that we took turns at the roles of hider and seeker, Jacob quickly abandoned the game.

With Jeremy, it was more or less the opposite. He hid in closets and the adult would find him. Sometimes he remained 'hidden' for quite some time, denying that he had been found. Upon emerging, he would go to another closet. The game never lasted more than a few rounds. There also were only three closets in the apartment and Jeremy always switched among them, never returning to the same place twice.

When George played 'hide and seek,' the players took turns and the rules were generally more extensive. George would also sometimes play with more than one adult simultaneously. Generally, George would call out whose turn it was to hide, e.g., "Helene hide." The adult would usually follow unless George had asked them to hide several times in row in which case a debate might ensue. George would also sometimes call out his own turn, "baby hide," although he most often just took a turn, but never two turns in a row. This is also how George initiated the game by suddenly hiding in one of the usual hiding places. George frequently ensured that there was turn-taking, that hiding was evenly distributed among the two or three players. When George was the seeker, upon finding the hider he would laughingly call, "ahaaa!"and then announce who should take the next turn.

Once, while playing 'hide and seek' at day care, George initiated a new and local rule (one that could not be applied at home). Among the places I hid was behind the door to the playroom. Whenever I hid there, George called out to Patty, "Tata! Where Helene go?" She would answer with a similar question, then George would pull the door, revealing me behind it. He did not enact this sequence of actions when I hid in other places. During the game, Patty left the room. Subsequently, whenever I hid behind the door, George asked the same question omitting Patty's name. He also, of course, did not wait for an answer.

The toddlers invented other rules to govern play. Jeremy would sometimes dance with a bear. Others could only dance with it with permission. He decided when time was up, removing the bear from their arms and returning it to the sofa. Once, while his mom bounced him on her leg, Jeremy insisted on holding his juice cup. It was not easy to hold on to it and maintain balance on her leg. Naturally, he fell off quite a bit, but even so, he managed to hang on to the cup. The few times he dropped it, Jeremy made sure to pick it up before climbing back on Birgit's leg. When playing on the basketball court near his home one day, every time he threw his ball off the court, Jeremy squeezed through the poles of the hoop on his way to retrieve the ball, and sometimes upon his return as well.

George set rules that others also needed to follow. He sometimes enforced them with stern scoldings of "no, no." Early on in my observations, Lynn informed me that George liked to climb up and down stairs 'all the time.' She saw him as working on mastering this physical feat. He also set rules to this activity. No one was allowed to step on the stairs while George was in the middle of a trip, either up or down, neither parents nor playmates. If it so happened that this rule was broken, George returned to his point of origin, either the top or bottom, and began the trip all over again. I once had a chance to witness this rule in action. George climbed the stairs up from the basement. He said, "no, no," several times as he approached and on route as if warning Lynn and me. Once George reached the top, Lynn and I were able to climb without objection.

One day in the bathtub, Lynn tried to hand George his toothbrush. He told her "no, mama," so she put it on the edge of the tub. George picked it right up. Lynn seemed to already know there was a rule that George was to pick up the toothbrush himself. Also in the bath, George was used to playing with two 'fish cups,' one red and one yellow, one in each hand. Apparently there was also a blue one that had been missing for a while. I witnessed two occasions in which Lynn tried to give George all three cups. Both times George returned the blue one, also admonishing, "no, no." At day care, George told me "no, no" as he reconnected a trailer to a truck that I had taken apart. It seems I was not supposed to do that.

Jacob developed rules in conjunction with a certain way of playing with his Lego blocks. One day, Jacob's grandmother built a ring out of Legos and guided Jacob to stand inside it. Jacob sat down and pushed the disconnected ends of the ring together so that he sat completely enclosed by it. A week later, I built a similar ring. Jacob stepped inside and sat down. I tried to hand him a toy but he would not permit its entry. When Jacob wanted to get out of the ring, he seemed a bit perplexed. I held his hand for him to step over the ring without breaking it.

Two months passed without any play with Lego rings that I knew of. Then, the video camera caught an episode which I call 'Lego Ring,' in which Ann made a Lego ring for Jacob. A description from the video follows.

> Ann is constructing the ring and she tells me what happened when she made a similar ring the night before. "At first he seemed upset because he couldn't get out. Then I picked one [block] up and said, 'open the door.' 'Open the door' became the way to get out. He wouldn't leave unless he took one piece out and walked out. But he would step in but would not necessarily step out of the fence."
>
> Jacob sits in the incomplete ring as Ann continues to build. He asks me to hand him a 'car' (Lego block on wheels). Ann tells him to wait until she makes the tunnel first. Jacob says, "okay" and sits patiently. Ann tells me she'll try to make a tunnel that is big enough for Jacob to crawl through as well. When Ann completes the structure, there is a ring of Legos containing an arch high enough for Jacob to crawl under. Jacob pushes the 'car' out, under the arch.

> Ann tells Jacob, "There. Look how big that door is. Can you crawl through the door?" Jacob reaches to break the ring as he had done the night before saying, "See door." Ann stops him. She and I point to the arch saying, "Here's the door," encouraging him to exit that way. Jacob stays put. Then Jacob goes ahead and breaks the ring against Ann's objections. He crawls out.
>
> Pointing to the break, Jacob says "See door?" many times. Ann asks about the arch, "This door is just for cars?" She reconnects the ring where Jacob broke it. He steps over the ring and sits back down.
>
> Jacob again makes a 'door' and exits, talking about it. Ann asks Jacob a couple times if he can now crawl through 'this door,' pointing to the arch. Jacob stands as if in contemplation about it. Ann points again through the arch with her foot asking, "Can you go through here?" After a few more seconds contemplation, Jacob crawls back into the ring through the arch. I say, "Jacob went in through the tunnel." He answers, "Yea."
>
> I then ask if Jacob can go out through the tunnel. He responds by breaking the fence and crawling out that way. Ann reconnects the 'door' and Jacob crawls back in through the tunnel. Ann again asks Jacob, "Can you go back out through the tunnel?" Jacob says, "No," and makes a 'door' as before. Ann reconnects it and Jacob steps back in this time.
>
> Now Ann asks Jacob, "Can you step out without using a door?" Jacob again makes a 'door' and crawls out. He crawls in through the tunnel and again exits by making a 'door.' Ann asks, "Can you go back in through the door?" Jacob answers, "no, door." Ann asks, "The door's just for out?" As Ann failed to reconnect the 'door' as usual, Jacob does it himself. He then steps back in, makes a 'door' and crawls back out again. This is the seventh time Jacob has exited the ring by way of a 'door.'
>
> Jacob stands outside the still open 'door' for a few moments. Ann asks about playing with the trucks and the tunnel. Contrary to his prior restrictions, Jacob returns 'inside' the still disconnected ring by walking through the 'door.' He then begins to play with vehicles, pushing small ones under small elevations in the now broken ring and big ones through the arch.

Jacob established particular and strict rules for his play with the 'Lego Ring.' These were apparent to Ann; she challenged them in nearly every possible way. The rules were: a) Exiting the ring is possible only by making a 'door' defined as taking apart a link in the ring; b) once there is a 'door,' exiting is done by crawling through it; c) a 'door' may not be used to enter; d) entry is permitted only after the 'door' has been 'closed,' so that the ring is unbroken; e) entry may be done by different means as long as it is not done by way of a 'door.' In particular, entering the ring can be done by stepping over it or by crawling through the arch.

Jacob's rules were part of a system that he created for play with the 'Lego ring.' Jacob can be seen as pretending that his behavior was bound by certain restrictions; he entered into an imagined system that governed his actions. This play was different from the pretend play discussed in the previous chapter, as there was no link to anything clearly 'real,' no attempt to signify something outside the system. Rather the system was invented to exist in its own space, separate from any external meaning. And yet there was meaning within the system itself, meaning created out of relationships generated within the system.

Symbols played a key role in the system's construction. 'Door' was a symbol derived from a sign in spoken language, but here it had a specific and in some ways very different definition from the standard one. Jacob's initial reluctance to crawl through the arch may have been due to Ann's and my calling it a 'door.' Within Jacob's system, 'door' meant something else. Only after we stopped referring to the arch as 'door' did Jacob include it in his system. Calling the arch by another name, 'tunnel,' may have fostered its initial and continued inclusion.

There were a good many other symbols in Jacob's system, although perhaps not always so obvious. Resnick's (1997) discussion of patterns and positions maintains that positions are all that we see. Thus, the visibles in Jacob's system can be understood as being symbols. So for example, Jacob's actions were symbols: crawling out, crawling in, stepping over, sitting down. Legos configured in particular ways were also symbols: the broken link as 'door,' the arch as 'tunnel,' the unbroken ring. These actions and objects were all symbols because they held roles within the system; they were key to its construction and took meaning from within it.

That Jacob constructed a system of symbols and rules for their interaction becomes more obvious in comparison with what 'reality' permits. For example, nothing except his will stopped Jacob from crawling into an unbroken ring, distorting its shape and knocking it apart. Nothing except his will stopped him from stepping out as well as in the ring or stepping 'through' the arch, knocking it over. And nothing except his will stopped Jacob from walking into the ring while still broken. Actually, Jacob did take this action in the end.

That Jacob 'broke' a rule he previously established could be interpreted as showing that this rule never existed. Jacob behaved inconsistently. If rules change willy nilly, they cannot be considered to be rules. However, I believe another interpretation better fits the data. At the point that Jacob walked in through the broken ring, he abandoned the system. Jacob's behavior had been restricted by a system *he* imagined. There were no material constraints on his actions. He decided the game was over and it was time to move onto something else -- to playing with trucks and tunnels. The prior rules vanished; they no longer applied. The symbols vanished as well; some no longer existed at all while others lost the meanings the system had bestowed upon them.

Pretend play that involved external signification also created structures and was bound by rules (examples were included in Chapter V).The episode with Jeremy and the 'Dancing Dinosaurs' created a structure out of actors and actions. Play with given symbolic objects was governed by rules required of the real objects they signified. Jacob made sure the toy train moved along its track. George made sure he used keys with his toy car and truck on the porch at the beginning of play, before 'going anywhere.' Of course, keys were not materially necessary to make the vehicles go.

Pattern play

The previous section discussed attempts by the toddlers to organize, systematize and give structure to the world they encountered and to their own play. Here, I explore a particular and quite common form of play that can be seen as a part of this general effort, which I call 'pattern play.'

'Pattern' is a term frequently used in describing mathematics. I have quoted its use by a few who have dared to try to pin down what mathematics *is* (Resnick, 1997; Thurston, 1994), and there are others who have used it similarly (e.g., Steen, 1990). I used 'pattern' as part of my own description of mathematics in Chapter II. However, 'pattern' is not used *within* mathematics as far as I know. Mathematics does not define 'pattern' as it does 'set,' 'sequence' and even '2.' There is no specific or general mathematical object called a 'pattern.' Indeed, if there were, it would be difficult for 'pattern' to describe all of what mathematics studies. Given the lack of an explicit definition, 'pattern' has various interpretations, some specific, others more amorphous. This situation makes my use of the term problematic, and yet I choose to use it. It is still the best existing English term for naming the episodes of the toddlers I wish to describe here, in part, because it does allow for a flexible and general interpretation. I wish to claim different forms of play as related to each other and falling within a general type, a type which I call 'pattern play.' To make my use of the term helpful, I will offer an explanation as to what I mean by it.

In Chapter II, I wrote, "Mathematics involves the attempt to understand structures and patterns...." This is consistent with my emphasis on processes that give rise to mathematics. I do not exactly define the cultural artifact 'mathematics,' but claim that it is the product of particular processes, which this project does attempt to begin to pin down. In concordance with a fallibilist epistemology as well, 'pattern' is something for which people search and attempt to understand; in a sense, it is a *way of seeing*. Yet, a way of seeing also creates objects even if only within the mind. I look for a pattern, find a pattern and thus create a pattern. Creation may also come first. I create a pattern and then see it as a pattern. I think this may often be the case with the toddlers, although it may be a desire to see patterns that leads them to create them in the first place. So what is this object 'pattern' that I see and create?

In a few words, I would say that pattern is the structure of relationships. Indeed, I use 'pattern' and 'structure' together and often interchangeably. Does this mean that any relationship is a pattern, mother and child, boy and lion? Perhaps yes, but not entirely. A single relationship involves a single mapping between two people or things. Although there is one mapping, this may constitute a basic building block for a pattern or a trivial instance of a pattern, just as the number one is a number but cannot by itself constitute a number system. Patterns are systems that involve regularity as well as predictability. As systems, they require multiple elements or multiple instances of the same element. In this sense, 'mother and child' could constitute a pattern since the term describes a relationship repeated millions and millions of times over. Calling two people I had never met before 'mother and child' leads me to predict certain features of that relationship. However, again, I must emphasize pattern as a way of seeing. 'Mother and child' is only a pattern for me if I see it ***in*** the multiple instances I encounter, hence the emphasis on structure. Rather than focusing on characteristics of particular instances, *pattern* indicates an attention to underlying structure. In using 'pattern' to describe certain instances of the toddlers' play, I am claiming the toddlers are creating relationships and placing their attention on the structures formed by those relationships. Because of the toddlers' attention to structure, the things forming relationships are *symbols*. They hold the positions within the patterns.

In a pure sense, a 'pattern' in and of itself need not include repetition. A dress pattern is a 'pattern' and so is a circle. They involve relationships among line, space and distance in a fixed way. They also highlight underlying structure. A dress itself is not a 'pattern.' However, these 'patterns' have the potential of repeatability. I can cut a hundred dresses from the same pattern. I can also see a hundred instances of circles in the world, with an ideal circle as my pattern for seeing. That they are repeated or repeatable perhaps helps us see them, establish them as patterns. Perhaps this is why it is expected that patterns repeat, that they involve repetition with infinite potential.

Repetitive play is a bold mark of the early years of childhood. Babies and toddlers are frequently observed enacting the same behavior over and over again. The behavior may be pleasurable or part of an attempt to master some skill, hence children's desire to repeat it. This may have been the case with Jacob, Jeremy and George as well. There were times when the activity was in focus for its own sake and was repeated because it warranted repetition. However, at other times, the activities themselves seemed to fade in importance as structure took to the fore. Activities formed particular relationships, structured in fixed ways. As structure became a focus, activities left the mundane world of material action and instead took positions in 'patterns.' The toddlers thereby played with patterns in which activities held positions, became symbols. Pattern play took different forms among the three toddlers. I call the first form 'sequence.'

'Sequence' has a particular definition in mathematics, that of an ordered list. A sequence need not contain repetition within itself, although it may. Regardless of internal repetition, as with other patterns, a sequence is structured and is repeatable. I use 'sequence' to describe activities of the toddlers that involved a fixed and ordered set of actions. That the actions were arranged in a fixed order is apparent because they were repeated again in the same order. Thus, 'sequence' describes instances in which the toddlers repeated a fixed and ordered set of actions.

The toddlers sometimes performed sequences individually, although one could argue that the presence of an audience (i.e., me and/or others) made them not entirely individual. They also performed sequences with a partner or sometimes two. Some of these are rather interesting to explore, but first a few words about individual sequences.

Jeremy was the least interested of the toddlers in performing individual sequences. I only have six recorded instances of his enacting them and all but one were performed briefly (four rounds or less). The one exception occurred one day when Jeremy was playing outside. On three separate occasions, he jumped down sets of stairs one stair at a time, holding onto his mom's hand. This sequence can be seen as repeating a single element again and again, not very interesting structurally. The activity itself was compelling to Jeremy.

I have ten instances of individual sequences for Jacob, half were performed many (five or more) times. George enacted fourteen individual sequences. Eleven were performed many times. One of those sequences made frequent appearances (also outside my presence according to Lynn). On a particular bed at home, George would stand on the bed, holding onto the headboard, raise his arms in the air, yell, "doh!" and fall straight back on the bed, then repeat. Once at day care, he did these actions on a sofa, without a headboard, of course.

George enacted one individual sequence repeatedly for 22 minutes. (It was caught on video.) Lynn had brought out a small box containing even smaller blocks that George had never seen before. He dumped them on the floor, then stacked them back inside the box.

He lifted the box and inverted it, calling out "ahhh" as the blocks fell back out on the floor. George stacked them again. At every enactment of the sequence, George made sure that all the blocks were back in the box before dumping it. If some blocks fell prematurely as he tried to get a grip on the box, George made sure to replace them before continuing.

George had plenty of time and opportunity to engage in exploration with the blocks, into ways to arrange them as he did with cars in 'Arranging Cars,' or into their properties as objects. Lynn tried to encourage this, once beginning to build a tower with the blocks, but George quickly changed the game back to enacting the sequence. The primary elements of repetition -- manipulating the blocks in various ways -- was less important than the structure of the sequence -- exact repetition of events in a fixed and constant order.

George enacted roughly as many sequences with partners as by himself. There were others of an ambiguous status, such as his ritual of riding up and down the hall at day care. In many ways, it was an individual activity; however, he generally expected a response to his saying, "hi" and "bye," at least when I was present. He would also frequently play in parallel with another child. With Jacob and Jeremy, there was a clear preference for sequences with partners. Each developed a partner sequence game.

Jacob and I invented a game we metonymically named 'Boom,' choosing a word that was part of the game itself. We played 'boom' a number of times, each time a little differently. The first time involved us taking turns falling face down on a bean bag chair from a kneeling position. Before we fell we each said, "ah, ah, ah..." a number of times and then "boom" while falling. We each got back up before the next person took a turn. I tried to institute a fixed number of 'ah's' on different occasions, but Jacob never adopted this regularity.

We played again a week and a half later. I asked Jacob if he would like to play 'Boom.' This time, Jacob had us fall simultaneously. He would watch me and make sure he fell exactly when I did. Once Jacob did not get up immediately, so I 'fell' on top of him. Thereafter we played so that Jacob fell first, then I fell on top of him. The vocalizations remained as before. A few days later when we played again, we played the same way with my 'falling' on Jacob.

The last time we played 'Boom' a few days later was caught on video. We returned to the original form of the game and took turns falling. I tried to test the rules. During one turn, I failed to say any 'ah's' and just fell right away saying 'boom.' On Jacob's next turn, he seemed to emphasize the 'ah's,' saying more 'ah's' than usual and looking right at me. He also paused a moment after saying them before falling with 'boom.' I played 'correctly' the next time. At a later turn, I waited before enacting the sequence. This seemed to get Jacob upset; he whined and said, "No." He left the game soon after.

Jeremy's game was called 'Five,' also named metonymically by a word contained in the game. He played it with his parents before my observations began. Jeremy would tell a partner he wanted to play by holding up an open hand and saying, "five." His partner had to be sitting on the floor. Sometimes, Jeremy had to drag her there. The partner would raise her hand, Jeremy would hit it, and the partner was to fall back. Then the partner got up and the sequence began again.

I also tried to test Jeremy while playing the game, to see what deviations he would permit and how he might enforce the sequence. One time playing 'Five,' I refused to fall immediately upon Jeremy's hitting my hand. When this happened, he hit my hand repeatedly until I did fall. On another occasion, I sometimes refused to get up right away. Jeremy pulled at my feet for me to get up. Jeremy dragged his mom into the game at some point. He played back and forth between us although not in a fixed way. Now, when I did not get up on my own, Jeremy just let me lay there and played with his mom instead.

A couple months later, Jeremy played with both his mom and me and I again failed to get right up. Jeremy made no attempt to 'get me up'; he just looked at me, so I sat up on my own. Jeremy pulled me to go sit on the sofa and continued playing with his mom. Jeremy did not necessarily work to enforce rules in the game, but if I broke the rules, he stopped playing with me.

Another episode confirms this interpretation. Jeremy and I were enacting a sequence that began with him hiding his face behind a toy. I called out "Where are you?" Jeremy revealed his face and I said, "There he is!" Jeremy laughed and reinitiated the sequence. After playing numerous rounds to Jeremy's delight, I decided to pause during my turn to see what Jeremy would do. Jeremy just put down the toy and turned his attention to something else.

Jacob and George reacted differently when a sequence was not followed. Rather than leave the game, they worked to make it continue. In consonance with George's enforcing rules generally, he tried to enforce the rules of a sequence -- to maintain the presence of all its elements and in order. This took the form of his playing his part many times until the partner responded, e.g., continuing to call, "bye," from his truck at day care until I answered, or calling the person's name to remind her of her turn. For example, George enacted a sequence with Patty and me to some extent in which he sat on my lap and lay back, then sat up again. While lying back, Patty tickled him under his chin. Patty was not a researcher and did not deliberately neglect her turn. She was often distracted by things going on with the other children. However, George was sure to call out her name if there were any delay.

Jacob's efforts of enforcement were generally more subtle, as with the previous example of 'Boom,' in which he 'told' me more or less by demonstration what I should do. Alternatively, if his partner failed to play her role, Jacob might play it for her. For example, one day Jacob and I were tossing his teddy bear back and forth. Every time I threw it to Jacob, I called out, "Here he comes!" On the occasions that I failed to say my line, Jacob said it instead. I also waited to throw the bear a number of times. Jacob would just look at me expectantly and patiently, waiting for me to take my turn. I felt he treated it as if I were teasing him.

Across the three toddlers, sequences generally contained specific, fixed elements, executed in order multiple times, without deviation. However, Jeremy additionally enacted a different sort of sequence that I think of as a 'semi-random sequence.' I noted five examples of 'semi-random sequences' with Jeremy. These took the form of a set of repeated actions interspersed with unstructured explorations. For example, one day Jeremy played on his parents' bed. He enacted a sequence of turning on and off the lamp next to the bed, which he completed four times. However, in between these particular actions he did other and varying things: lay down on the bed, seemed to pretend to sleep, 'talked' to me about the paintings on the wall, walked around, rolled around and fell back on the bed, and played with objects he found on the lamp table -- a piece of tape and a bobby pin.

Although Jeremy's actions might be interpreted as reflecting an inability to focus on a particular activity, that Jeremy also participated in the more standard form of sequences, albeit with less weddedness than George or Jacob, belies this notion. Jeremy could instead be viewed as possessing great flexibility. He was able to maintain a sequence while simultaneously deviate from it. He adhered to an overall constant structure, but interrupted it with seemingly random explorations, unrelated to the sequence or to each other. 'Semi-random' is actually the form of sequence most present in daily life. Actions do not generally get repeated over and over again in immediate succession. Although life does contain fixed, repeated activities, they are interspersed with random and unpredictable events.

Another form of pattern that Jeremy found particularly attractive is one I have termed, 'distribution.' It was mentioned in Chapter V in conjunction with Jeremy and George having everyone present pretend to do the same thing. Jeremy and George engaged in distribution rather frequently and in varying ways. I did not witness Jacob enact this behavior.

George frequently distributed things (or imaginary things) and names. For example, he gave each of his rubber ducks a kiss. Some time later he declared that they were sad, kissed them each once again, then had them 'kiss' each other. On several occasions, he distributed descriptions of people's feet, either by naming their states himself or eliciting them from others, e.g., "shoes," "no shoes." On a number of occasions, he was also sure to distribute 'bye' to everyone in the room.

George's distributions were almost exclusively of a singular nature in the sense of 'one for you, one for you, one for me.' The only exceptions were when George got two or more, because he started with himself and ended with himself. For example, at day care one day, he hopped, then asked that Patty and I each hop in turn, then hopped himself once more. Also, when he gave everyone a lick of his ice cream, he got to finish it himself.

One time, I tried to challenge George's rule of one and only one turn per person.

> George goes over to the wallpaper by the stairs. He acts as if he's grabbing something off the wall and comes over and feeds me, saying, "gapes,"

> (grapes). I pretend to eat one off his hand. Then I say, "Can I have another?" as he heads back towards the wall. George tells me, "No." He walks over to Patty with a 'grape' and feeds it to her. Then he goes back to the wall and 'picks' off one for himself and 'eats' it.[2]

In contrast, Jeremy's acts of distribution generally involved multiple turns per person and as such turned into sequences; he would repeat the distribution in order again and again. For example, after I put a pen in my mouth and took it out, Jeremy took the pen from me and put it in his mouth, then he returned it to my mouth, etc., repeating this back and forth action several times. Jeremy played 'Five' with Birgit and me, scared Birgit and me, gave drinks to Birgit and me, going back and forth between us several times.

Distribution with Jeremy also included continuous actions. For example, one day, Jeremy, Birgit and I were all drawing on the same piece of paper. When Birgit and I each stopped (at different times) and put down our crayons, Jeremy handed a crayon back to each of us, 'telling' us to resume coloring. Jeremy was especially insistent that everyone dance when his favorite dance songs were playing, as the following episode shows:

> Lao goes and turns on the country CD we usually listen to. Jeremy and Lao start dancing while I try to write in my notebook. Jeremy comes over to me and pulls at my finger and says, "Dance." I get up and dance with them. When Jeremy isn't looking, I sit down to write again. He notices and comes over to me and asks me to dance again. I get up and dance. Then Lao sits down. Jeremy goes over to him and gets him to dance. We all dance and then Jeremy sits down. Both Lao and I go over to him and say, "Dance, dance." Jeremy gets up. The song ends and a slow one comes on. Jeremy moves his feet back and forth, and then arches his back and says, "Aahh," as if annoyed and stops moving. The song is apparently too slow. Dad goes over and puts on another fast song. We all dance. I try to sit down one more time, but Jeremy won't let me.

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[2] The wallpaper had no pictures of grapes or any fruit for that matter. It was a floral pattern. Patty had explained that George used to do this with a playmate -- 'pick grapes' off the wall.

Implied in these examples of distribution of continuous action is the rule that everyone must participate. Jeremy was the main enforcer of this rule. He had Birgit and me resume coloring and returned Lao and me to the dance floor. Interestingly in the dancing example, Jeremy demonstrated that he knew we knew the rule as well. He had taught it to us. We were thereby also capable of enforcing it. Jeremy's sitting down was perhaps a 'test' that we had indeed learned the rule and would not permit him to violate it either.

This interpretation corresponds to Jeremy's actions during other times I broke the rules, as in the previous examples given for sequence games. He knew I knew the rules. He did not need to 'teach' me each time. If I broke them, perhaps that meant I was not interested in playing and that was fine with him.

Although Jeremy did not work to enforce rules with non-distributive sequences, he did with distributive ones. Distribution patterns mattered more to him somehow. He did not permit me to stop coloring or dancing. In another example, he was not successful at enforcing the distribution exactly, so he compensated in another way.

Jeremy had him and Birgit each pretend to drink from a toy trough, several times in succession. He started, so it went: Jeremy, Birgit, Jeremy, Birgit, Jeremy. It was then Birgit's turn but she refused. Jeremy tried to make her 'drink,' pulling her down, but she insisted she was no longer thirsty. She picked up a toy pig and made that 'drink.' This seemed to satisfy Jeremy; the 'pig drank' in substitution. Jeremy took another turn, then handed Birgit a toy cow for it to take a turn in her stead.

George too worked to enforce distribution. While eating ice cream, he gave his mom and me each a lick. He then offered some to John, but he refused. George raised up his ice cream and called, "Dada," four times, but then finally gave up. That George tried to enforce rules with distribution is not surprising as he was generally insistent on following rules. However, rule enforcement of distribution seemed less important to George than in other situations (e.g., the stairs), while with Jeremy it was more important.

A final form of pattern play was again of varying interest among the toddlers -- mirror play. Mirror play involves the mimicking of another's actions and/or utterances. For some of this play, a mirror is an apt metaphor as when Jeremy made the same pose as a baby in a photo. He reflected back what he saw as if in a mirror. A good deal of mirror play is the simultaneous mimicking of actions as if making movements in front of a mirror. Jacob played in this way a good deal. He generally lead and I followed although once his mom joined the game too. We would sometimes play standing, but sitting at a table seemed to offer potential for the most interesting games. He would choose among a number of different motions, not in any particular order, and always watch that I followed. Banging on the table, yelling "ahhh," was a favorite element that was frequently included.

Jeremy's mirror play was generally more restricted. He would enact only one or two motions during any one game, although if there were two adults present (say at the table), he would be sure to get both of us to follow in consonance with his predilection for distribution.

I only have one recorded instance of true mirror play with George. Rather than simultaneously reflected images, it involved simultaneously reflected sounds. While swinging in his swing in the back yard, as he came forward George called out a nonsense sound, Eee-ya!" I imitated and a mirror game began. George and I simultaneously said, "Ee-ya" again and again. George seemed to wait until I started with the 'ee' before saying 'ya' so that we could stay in sync. He later introduced other nonsense sounds, "Yyyo!" and "Nnno!" that I also mimicked.

Mirror play can quickly take on the same characteristics of distribution and sequence play when reflections do not occur simultaneously. For example, I once jumped, opening my legs into a 'V' and Jacob imitated me. I closed my legs and he imitated again. We continued these same actions several times. While mimicked, since the actions do not occur together, they can be seen as being distributed across players, as if tossed back and forth. Since they are repeated in order, they form a sequence.

This form of mirroring was even more prevalent with sounds. Jeremy and I said, "Um" back and forth. Jacob and his mom said, "I'm sorry" back and forth. George and I said each other's names back and forth.[3] These examples give the sense of a word being tossed among players like a ball, like an echo.

I believe that I generally initiated mirror games with the toddlers. Adults are given to mimic infants and toddlers, mirroring back to them their earliest attempts at speech and gesture. This provides a way for babies to communicate -- to express and be heard -- as well as a way to be playful. However, it is interesting the different degrees to which the three toddlers made this play possible, picked it up, extended it, and initiated it on their own.

Forms of attention may have played a role. Mirror play requires watching someone watch you and seeing that the person is doing what you are doing. Of course, it also requires the production of symbols, utterances and gestures that can be copied, an ability which rests on there not being external meaning for those particular expressions (otherwise they are responded to, not copied). Mirroring is a game of marks removed from communication. The lack of mirror play in George's repertoire may simply reflect a lack of production of

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[3] This may not be considered mirroring as the element being 'mirrored' was not exactly identical, although it did have an identical relationship in the structure of the game. We each said the other's name.

meaningless marks and/or a lack of attention to my actions if I happened to mirror one back to him. Whatever the reasons may be, mirror play was most extensive and varied with Jacob, less so with Jeremy, and nearly nonexistent with George.

There were instances, however, of George tossing words back and forth, so to speak. The words were not identical, however, and thus the instances do not qualify as 'mirror play.' The elements George employed had more complex relationships than 'a = a,' which stemmed from their meanings as signs. Play with symbols derived from signs as well as with structures formed among symbols is the focus of the next section.

Symbol play

One day, Jeremy was rubbing his mom's back and said, "Nice." Birgit asked, "Is that nice? Is Mommy's skin smooth?" Hearing "smooth," Jeremy said, "moooo." He suddenly left his mother and headed for his toy barn. He took it out and began to play with it. Birgit commented, laughing, "*Smooth*, sounds like *moo*, sounds like *cow*, sounds like *barn.*"

Birgit noticed the chain of associations Jeremy was making. 'Smooth' does indeed sound like 'moo.' 'Moo' is the sound a cow makes. A cow can be found in a barn. Some of Jeremy's associations may have been based on experience, as in connecting 'moo' to cow to barn, but symbolizing was involved to some extent in all of them. The objects of toy barn and animals with which Jeremy began to play were themselves symbolic. The initial association was even more clearly made at the symbolic level, out of the aural resemblance between the words 'smooth' and 'moo' as objects in their own right.

Although the above example involved a spontaneous association among symbols, association of words at the symbolic level made its way into games as well. Jeremy had a game in which he said the Chinese words for 'cry' and 'laugh' and followed each with a demonstration of (pretend) crying and laughing respectively, often repeating these actions as a sequence once or twice. Jeremy's parents sometimes instigated the game and may have initiated its invention.[4]

That 'cry' and 'laugh' were opposites was no accident. Signification helped to position them together and as opposites, on two levels in fact: by way of words from the symbolic world of a 'language game' and through use of 'motivated symbols' from pretending's repertoire. One rarely experiences crying and laughing together and would not necessarily see them as opposites or inversions of each other without language's mediation. Indeed, actual experiences with laughing and crying did not trigger the game for Jeremy, yet hearing one of the verbal signs for them did. The game thus involved a pairing of related *symbols.*

George had a similar game with symbolic pairings. One time while George was taking a bath, Lynn suddenly said "Up." George immediately responded, "Down." Lynn and George continued to say these words back and forth several times. Lynn once tried to switch roles saying George's word, "Down," but George would not play along. He only resumed the game once Lynn returned to her prior role.

The words, 'up' and 'down' were said without context. They had no meanings here; they were sounds, objects with which to play. And yet, they had a relationship, opposites again and again, no accident. While *up* and *down* may be opposites on an experiential level, their signifiers are opposites on a symbolic level. In this incident, only the symbolic level

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[4] Jeremy had not yet begun behavior where he would merely repeat what adults said. From the total of his behavior, it can be assumed that every sign he expressed came from active efforts to signify and symbolize. His words were his own. This game likely arose from interplay between Jeremy and his parents. Even if his parents had completely and independently invented the game, which is unlikely, Jeremy would only have adopted it if it had made sense and been interesting to him.

mattered. Although I never witnessed it, by their automatic assumption of actions and roles, Lynn and George appeared to have played the game before. It was a game played with symbols.

While 'language games' contain words that can be paired as opposites, George additionally learned that 'no' and 'not' could be joined with a word to mean its opposite. For example, he named people as having 'shoes' or 'no shoes' pointing to their feet. When John described a bowl of porridge in a storybook as 'hot,' George brought the book over to me telling me "not hot, not hot," encouraging me to 'take a bite.' One time when Lynn was not home, George said, "Lynn," and then "no Lynn," which was in fact the case.

Using 'no' or 'not' in front of a word is a move at the level of symbol. The above examples make this dynamic especially clear, since both the word and its negative inversion were said together. Rather than ascribe two distinct signs to signify the occurrence of two diametric states on a material level, one merely has to place 'no' in front of one sign to produce its opposite.

George had two other symbol games. They both followed the same structure and contained similar relationships. In one, he would say, "mama, dada, mama, dada..." and then he or someone else would soon interject, "baby!" and end the game. In the other, he would say "car, van, car, van..." with the interjection of "truck!" These were clearly games with words played at the level of symbol. The words had related meanings in the material world, but as they were placed in a meaningless structure in regard to grammar, context or purposes of communication (i.e., these were not 'meaningful sentences'), their relationship as symbols was placed to the fore. That they followed the same underlying as well as relational structure further supports this view. They were both patterns of the form 'a, b, a, b...c!' with 'a,' 'b,' 'c' all belonging to the same category (family members in one case and vehicles in the other) and 'c' being the most important member.

Jacob's game (discussed in Chapter IV) of calling me 'Wee-wa' and the real Lisa, 'Haween' can also be considered a symbol game. Jacob switched names, mapping each onto the other signified and he used this move to tease. George had a similar game; however, unlike Jacob's game which arose out of an unintentional 'misnaming,' George's game was intentional. Like several previous examples, it made use of opposites. It was initiated as follows:

> John tells me how the other evening riding around, he and Lynn were pointing out the moon to George whenever it would appear. He likes to look for it and see it very much. They would say, "Look, George, the moon." At one point, George called it, "sun." John would reply, "That's not the sun, that's the moon." George would insist, "sun." Then earlier today, when they were outside, the sun appeared for a little while from behind the clouds. George called it 'moon.' They say he knows what they are called and is clearly doing this deliberately to tease. As they tell me the story, George points towards the outside and says, "moon," with a subtle grin on his face, waiting for a response. John says, "No, that's the sun." George repeats, "moon."

George seemed to play this game often and I witnessed it every couple weeks until the end of my observations. He generally played it with his dad, but also played once with me when his parents were absent. He called the sun 'moon' and also said, "no sun, moon," reflecting a view of them as opposites. He even once managed to trip me up and get me to agree with him!

Piaget did not believe it possible for young children to engage in such play with symbols, play that involves detaching words from things and certainly not play that involves mapping signifiers onto 'incorrect' signifieds. In *The Child's Conception of the World*, Piaget (1929) explains that children possess a:

> systematic difficulty of distinguishing the sign from the thing signified....[W]ords are regarded as a part of things and are situated within things....There is thus nominal realism...namely that the thing includes its name in its intrinsic character although it is invisible (pp. 44, 55, 73).

Piaget offers numerous examples of children's 'nominal realism' and symbolic 'confusion' up through age 11. Here is one that George directly counters: "Fert (7):...And could the moon have been called 'sun' and the sun 'moon'? -- *No* -- Why not? -- ***Because the sun makes it warm and the moon gives light***" (p. 81, Piaget's italics designate the child's voice and plain text the researcher's voice).

I cannot explain Piaget's data; however, as has been found with particular tasks of his, the subjects may not have completely understood the questions and their contexts; they made up answers because the researcher expected answers (see e.g., Donaldson, 1978). It is correct to say that the moon and sun cannot be called by their opposite names when playing by the rules of the 'language game.' Calling signifiers by 'incorrect' signifieds would thwart communication. However, if one is trying to tease, to be funny, it is perfectly legitimate to break the rules. In fact, one must. That is how one makes a joke. The popularizing mathematician John Allen Paulos points out:

> In humor...logic is often inverted, patterns are distorted, rules are misunderstood, and structures are confused. Yet these transformations are not random and must still make sense on some level. Understanding the 'correct' logic, pattern, rule or structure is essential to understanding what is incongruous in a given story -- to 'getting the joke' (1980, p. 11).

And to make a joke, one must understand the structures in order to transform them. In his 'sun-moon game,' George distorted the rules that link signifiers to signifieds and along particular lines. Calling the sun 'ball' or the moon 'cloud' would not have been as funny.

On the day that George teased me, his babysitter Milady, who was not aware of the game, explained to him patiently and clearly (from the video record):

> At nighttime there's no sun. When you go to bed there's a moon. But during the day when George is up and playing, there's the sun. At nighttime, there's the moon.

George did not attempt the game after that. Well, if you're not going to get the joke.... George did not mean to be literal.

Examples of Jacob engaged in teasing were mentioned in connection with the 'Haween, Wee-wa game' (see Chapter IV). The incidents involved playing with expectations. Jacob was behaving in certain ways that he knew would lead to certain expectations in my mind -- I expected the completion of a familiar pattern -- and then he misled me by distorting the pattern. Jeremy also distorted expected patterns to tease or make a joke. While playing ball with Birgit and me, at times he would act as if he were throwing the ball to me, but then quickly switch directions, throwing it over his head behind him instead. Some of Jeremy's object substitution when pretending (discussed in Chapter V) also had the ring of a joke, as when he had a toy fence -- which signified an inanimate object -- 'drink' from a toy trough. He transposed the structure of animates being animate and inanimates not animate, a structure that is expected even among pretend symbolic objects.

Other structure transpositions included Jeremy calling a picture of his mom, 'Baba,' (Chinese for 'Daddy') and holding his dinosaur puppet by its head, 'making it growl' from its rear. On a couple of occasions, Jeremy inverted the structure of his game 'Five.' He hit the adult 'five' as usual and then suddenly and unexpectedly fell himself. Another inversion occurred when Jeremy spent a long time slowly drinking juice from a glass with a spoon. Then he suddenly turned to a bowl of dry cereal and lifted it to his mouth as if drinking it. All of these examples played with expectations, thwarted predictions and caused observers to laugh.

Mathematical connections

> Humor, let me reiterate, though it may use formal devices, depends ultimately on one's sensitivity to the interplay among the various 'levels' of meaning. It is a very complex skill, this ability to distinguish levels of meaning, perceive their relationship, evaluate their relative importance given the context, then almost simultaneously form a global impression. Appreciating humor -- even recognizing it -- requires human skills of the highest order: no computer comes close to having them (Paulos, 1980; p. 50).

Indeed, in all of their systematizing, the toddlers betrayed impressive 'human skills.' Whether constructing rules for a game or breaking rules to make a joke, the toddlers needed to operate at a meta-level -- to build systems of symbols, to maintain the systems' integrity and to transform them.

In 'Arranging Cars' and 'Lego Ring,' while on the surface George and Jacob were manipulating concrete objects, their play was about much more -- about relationships formed among the materials and the actions made on them. George and Jacob played *with* abstract structures; the materials served as place-holders, as visible positions, as symbols. Their play was not about the materials themselves; it was about structure.

There is a pervasive Piaget-based belief that through the early elementary years, children 'think concretely.' Teachers are called to stock their classrooms with interesting physical objects, with 'manipulatives.' They are asked to represent abstract ideas materially, to 'make the symbolic concrete.' However, as George and Jacob intimate, when manipulating objects, even objects not designed to be 'manipulatives,' children may just as readily concentrate on their manipulations of the objects, on building abstract structures, as on the objects themselves. Rather than the symbolic becoming concrete, the concrete becomes symbolic. Children think abstractly like everyone else, using signs and symbols as mediators and place-holders, not as repositories of thought.

The toddlers seemed to show no difficulty operating at an abstract level. They made ready associations and generalizations about the world around them, places and things linked to people, events linked to places, activities linked to prohibitions. Their errors occurred not because of a lack of ability to generalize but perhaps in a propensity to over-generalize. That Jacob expected to find people again in the same place they were most recently found was certainly an incorrect assumption. In some instances, it may have served him well statistically (in that the missing person was just as, if not more likely, to be in that spot); in other instances, it failed him completely (as when a hider switched locations sequentially so Jacob was always wrong), and yet he did not revise his assumption. Perhaps experience is necessary -- and nothing better than the quintessential counter-example -- to temper and qualify generalizations. And perhaps it is dispositions to over-generalize, shared by children and adults alike -- rather than a supposed inability to abstract -- about which teachers need to be concerned.

Recognizing that certain generalizations hold only under particular conditions is a part of developing more nuanced understandings. 'Under particular conditions' is a key feature of ritual enactment as well. Ritual is routinized performance -- an invariable sequence of events, executed in order, performed repeatedly over time and in specific circumstances. Rituals take on an 'if, then' structure. ***If*** certain conditions hold, ***then*** the ritual is performed. This structure was present in the toddlers' rituals. ***If*** we are all sitting at the table drinking, ***then*** we clink glasses. ***If*** we look at this particular book, ***then*** we interact with it in a specific way. Rituals made their way into games as well, although it may be more accurate to say that in games they had a 'when, then' structure. ***When*** I find the hider, ***then*** I call out "ahaa!" ***When*** Helene hides behind the door, ***then*** I call to Patty and ask, "Where Helene go?"

'If, then' or 'when, then' structures are pervasive in mathematics. They help to set up parameters for mathematical worlds and frequently frame the theorems that govern and explain those worlds. They are also part of the rituals or routines of mathematics, known as *algorithms* and *problem-solving routines.*

Algorithms are rituals in that they are highly routinized means of carrying out computations. They are also used under certain conditions, e.g., there is a particular algorithm for multi-digit addition, a different one for multi-digit multiplication, etc. Algorithms are meant to become automatic; one must gain fluency and execute them without thought. Other sorts of rituals are similarly automatic in their enactment. They are performed often enough that they require no conscious awareness or mental control. However, this does not mean that rituals are without meaning, be they mathematical or otherwise, or that they can be learned in the same automatic, effortless fashion in which frequent practitioners perform them. *Rite* must not be confused with *rote.*

Schoenfeld (1988) offers an example that highlights the dangers of this 'confusion,' not of a mathematical ritual *per se*, but of a ritualistic and unreflective adherence to conventional form. He explains how in high school geometry classes the two-column-proof format has taken on 'sacred status.' Students adhere to it completely, unthinkingly, independent of meaning, even giving it priority over meaning. Students spend more time and effort making sure a proof is in proper form than on the substance of the proof itself. Much of instruction is devoted to issues of form as well.

For a ritual to be effective and used appropriately, it must be meaningful. For example, when learning algorithms, it is important to understand relationships among the symbols, how carrying relates to adding, for example -- in order to ensure correct initial performance. Algorithms learned without understanding can become 'buggy.' Yet, even more crucial is knowing when to apply a particular algorithm or problem-solving procedure, recognizing conditions of 'if.' Once algorithms become automatic, for the most part one must still consciously weigh conditions of 'if.' Tricks such as, "*if* the story problem contains the words 'all together,' ***then*** add," lead to serious pitfalls. Add what? Which numbers exactly? Besides, the protagonists of the story may have just been "all together happy."

Other than being used incorrectly, rituals can be applied too generally and unnecessarily, as in the two-column proof example. To multiply '17 x 10,' one need not use the algorithm for multi-digit multiplication and yet many people would!

Although algorithms' automaticity can lead to a false sense of security (on the part of teachers as well) that one can use them completely automatically, there is no need to dismiss automaticity altogether. It is cumbersome and taxing of mental resources to invent ways to solve recurring problems each and every time. And the toddlers show by example that children should have no difficulty adopting routines simply because they are routines. On the contrary, routines can be a source of enjoyment and confidence, offering a sense of mastery and certainty -- an ability to predict, to know what comes next. And yet, there is a danger of over-automaticity just as there is with over-generalization. Indeed, over use and attraction to ritual calls to mind obsessive compulsive behavior. If a child resists doing worksheet upon worksheet of the same type of problem over and over again (e.g., two digit by two digit multiplication), rather than regarding this behavior negatively, perhaps it should be taken as a sign of good mental health.

Although some of the toddlers' rituals appeared more or less automatic, others involved elements of surprise. Although the naming rituals around images of animals, vehicles, etc., were enacted in a repetitive, iterative fashion, there was no predicting what exactly would be named each time. Algorithms can be treated similarly in classrooms, as predictable routines infused with elements of surprise: which algorithms to use, in which situations, with what numbers? To generate interest in algorithms, they could be compared wirh rituals in daily life in the classroom, for example. Mathematical comparison could contribute to the successful performance of other rituals. Mathematics could infuse daily life. Children might just perform rituals around going to lunch or switching lessons with the same explicit exactness and regard for order that mathematics requires.

Rituals are governed by particular rules, those of exactness, order and conditions for performance. However, rules can govern other sorts of activity, including activities that are less predictable than rituals. George engaged in rule-bounded exploration. Rules provided outward parameters and governed how exploration was to take place. For example, in 'Arranging Cars,' George moved the vehicles within a bounded physical space and held to constants such as the van and truck are always together. He also 'undid' arrangements beginning with the last placed vehicles first and generally moved them one by one. With mathematical explorations, one must similarly establish parameters and conduct them systematically so that discernible patterns may emerge.

Rules are certainly key in the construction and playing of games as well. Rules were important to all of the toddlers' versions of hide and seek, although each version had different rules. There is not one way to play hide and seek, not just one set of rules that must apply. However, once rules are established, they must remain fixed throughout the game; they cannot change mid-stream. The same holds for mathematical 'games.' Rules must be set at the outset and followed throughout. However, there is no problem with changing a rule to create a new version or new 'game' in which to play mathematically. Non-Euclidean geometry provides a quintessential example. Simply change Euclid's parallel postulate and create a whole new geometry!

Rules govern 'game worlds' and must be observed when one is in the game. However, one can easily leave a game and then the rules no longer apply (although it could be argued that some 'games' are quite difficult if not impossible to leave, such as one's native 'language game' or the 'economic game'). Jacob showed that the imagined system he created in 'Lego Ring' did not have a permanent hold on him. He could abandon it whenever he desired. Perhaps it would do well if people could similarly abandon mathematical 'games,' if they could step outside of them and see them for what they are -- models of reality (though not always), good and useful, even extraordinary ones, but with limits. MacNeal (1994) bemoans that so many people have trouble stepping outside even very narrow math-related rules. They will claim with all stubborn earnestness that they cannot add apples and oranges. Why not? What on earth is stopping them?

An important component of rules in mathematical games is the assignment of signifieds to signifiers, in other words, the definition of words, the creation of signs.[5] Mathematics takes words that play roles in wider 'language games' and gives them particular, more specific and sometimes different definitions. Walkerdine (1988) discusses 'findings' that young children do not 'correctly' understand terms such as 'big' and 'little' in mathematics lessons or experimental tasks. She explains: "The terms [used at home] may therefore be the same signifiers as those used in school mathematics, but they clearly do not form the same signs" (p. 70). At home, the signifiers point to different signifieds.

This difference of signification in the 'language game' of mathematics versus other discourses can pose serious obstacles to students' comprehension of mathematics. Orr (1987) describes cases of African-American students for whom words like 'by' and 'of'

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[5] Recall that a sign is formed by 'fusing' a signifier with a signified.

made no sense whatsoever when used in mathematical discourse, since they either had completely different meanings or else did not exist in the students' native 'language game.'

On the other hand, Jacob had no trouble ascribing, let alone understanding, a very particular meaning for 'door' in 'Lego Ring.' This situation may have been because Jacob was active in defining the term as well as the context for its use, or because he clearly knew he had created an imagined world and was playing a game. Jacob's experience may offer clues for dealing with students' semantic difficulties in the mathematics classroom. If students knew they were entering a game world and actively participated in mathematical namings and definings, they may have less semantic confusion. Repeated use of terms in context could help as well, just as when learning any language for the first time. It certainly helped Ann and me learn Jacob's definition for 'door.'

Rules play roles in patterns as well. A rule for a sequence could be: do a, b, c in order, then repeat. A rule for distribution could be: give everyone one and only one, which George seemed to use. A rule for 'mirroring' or symmetry could be: Do what I do, when I do it. Patterns are further bounded. One does not choose just any action to enact and mirror, but chooses from a bounded domain. One domain for Jacob in a given mirroring game could have been 'movements of the upper body that can be done sitting down.'

The toddlers' patterns made use of actions and utterances. Some patterns primarily involved utterances and incorporated relationships among symbols. The toddlers enacted sequences on their own, but seemed to favor patterns they could do jointly with others, whether sequences, distributions or mirrorings. All of these features could inform exploration of pattern in early elementary mathematics.

The kindergarten classroom commonly features simple, repetitive sequences with only two elements, e.g., 'clap, jump, clap, jump...' Every child enacts the sequence simultaneously. This is a relatively banal and uninteresting pattern, even for the toddlers in this study. Why not enact patterns like the game of 'Five' with different individuals taking different roles in the overall construction of the sequence? There still is a unified system, yet made up of parts. This is closer to material systems in that while they may function as wholes, parts play different roles. One student could clap while the other jumps. Cooperation and perhaps greater attention to the whole is required to produce the pattern successfully. The pattern has become more complex as well.

Comparisons can also be made among patterns and in relation to domains. Jacob played the mirroring game with different elements whether standing up or sitting at a table, and depending on which table (i.e., a short, child-sized table and a dining table offered different possibilities), yet it was essentially the same game. What if the two-element sequence described above drew only on the domain of the upper body and then only on the lower? What if each was then mapped onto each other, perhaps with attention to material relations among the symbols? How could one jump with one's hands or clap with one's feet? Such activities could draw children's attention to structure, the interchangability of symbols and the rich relations that can be expressed within and between patterns. George showed attention to these issues with his patterns, 'Mama, dada, mama, dada....baby!' and 'Car, van, car, van...truck!'

Patterns in classrooms could also incorporate degrees of variability as in Jeremy's 'semi-random sequences' and even Jacob's 'Boom' game. Attempts to symbolize variability could lead to the need for variables -- symbols that can stand for a range of possible meanings, rather than a specific one.

'Boom' varied in its structure somewhat from enactment to enactment, but the 'ah, ah...' portion varied within each game as to the number of 'ah's.' There could be two or five or any number in between. What was important was that the 'ah's' were said. However, it did seem that at least two 'ah's' were required. There was plurality. A singular 'ah' did not suffice. While differentiation of number or quantity was not obviously evident with the toddlers, this example seems to be an exception. Jacob's differentiation in 'boom' between one and more than one could be related to his learning of English grammar, which uses the same distinction to determine the singularity and plurality of nouns and verbs. Two is plural. Other languages do not necessarily make the same distinctions: Arabic, for instance, has three different noun forms that distinguish among the numerosities of one, two and more than two.

Attention to number and plurality can be seen in other patterns of the toddlers. Mirroring can be viewed as paying particular attention to two-ness. Distribution patterns involved a one-to-one relationship of object (word, action, thing) to person (or symbolic object, e.g., rubber duck). Sometimes they stopped there as happened with George most frequently. At other times, they cycled through recipients again and again as was Jeremy's predilection. The cycling form of distribution recalls sharing or iterative division -- one for you, one for me, one for you, one for me -- until the total has been shared equally. The 'one and only one' form of distribution is reminiscent of counting, mapping a single number word onto each single object being counted.

Counting also comes to mind in George's play with stairs. George stepped on each step exactly once until he reached the top (or bottom). That George prohibited anyone else from stepping on a step while he was in the middle of trip, in which case the step would have been stepped on twice, can be seen as similar to the prohibition against counting an object more than once during a given count. There is also a sense of accumulation, accumulated stairs passed, accumulated distance. Repetition is present here, too, in that George employed the identical action over and over again, albeit with a different stair each time. Repetition is similarly present in every act of counting, in that the same action of touching or pointing or just visually pointing is repeated again and again until the count is complete, although the objects get tagged with different words each time.

Researchers into young children's counting have identified a number of how-to-count principles that children employ from their first efforts at counting (e.g., Gelman & Gallistel, 1978). The toddlers made use of similar principles in their pattern play, such as one-to-one mapping and repeating sequences in a fixed and stable order. The toddlers did not use number words, however, and it is unclear whether their activities included a sense of numerosity. Did repeated behaviors, such as climbing stairs or distributing sips involve attention to accumulation, to more and more with each movement or cycle? When George reached the top of a set of stairs, did he have an impression of a final total? Other research has found that translating the act of counting into a total is quite difficult for the youngest counters:

> When asked to count a set of 4 objects, young children may correctly count, "1-2-3-4," but then fail to report that the set has a cardinal value of 4. When the experimenter then asks, "How many things are there?" young children often count the set again instead of stating the final number name used in the count (Ginsburg, Klein & Starkey, 1998, p. 415).

This finding could be attributed to issues of language. Young children's experience may have led them to think that 'how many?' means *count*. As with Walkerdine's (1988) analysis of size words, the children and the researchers may have different signs (i.e., the same signifier points to different signifieds). However, this finding could also reflect a different perspective on number.

The data on Jacob, Jeremy and George taken together show a great propensity for expression through action as well as for activity involving repetition and order. The toddlers' motions that bore resemblance to counting may have been more relevant than any objects being 'counted,' attention to actions rather than things, just as Jacob's actions around the Legos may have held his primary attention rather than the Legos themselves. The counting-like motions were characterized by repetition and order; hence, an ordinal awareness of numerosity may have been more likely than a cardinal one.

The English language takes an opposite perspective, attributing cardinality to things that are essentially ordinal in nature -- things that are quantifiable, but that cannot be experienced and counted all at once, such as sunsets or paces. This need not be the case. Whorf (1956) informs:

> [A] likeness of cyclicity to aggregates is not unmistakably given by experience prior to language, or it would be found in all languages, and it is not...[In Hopi], 'They stayed ten days' becomes 'they stayed until the eleventh day' or 'they left after the tenth day' (pp. 139, 140).

Some young toddlers may regard ordinal experiences ordinally, like the Hopi. They have not yet begun to think in all the ways their language dictates. Additionally in young children, an ordinal perspective may supersede a cardinal one even with regard to discrete, countable sets. Children as well as adults need to count sets with numerosities greater than five in order to determine their cardinality (Dehaene, 1997; Ginsburg, Klein & Starkey, 1998; unless perhaps the items are arranged in a familiar configuration, such as a carton of a dozen eggs). The repetitive and ordered act of counting may form a metaphor for understanding discrete number, as with other 'metaphor-actions' (Donnelly, 2000). Hence, young children may more readily view discrete sets in an ordinal rather than a cardinal way, particularly once they count them.

Another hint at number arose in games involving opposites. 'Up' and 'down,' 'sun' and 'moon' call to mind points placed equidistant from some axis. One person says, 'Up,' the other is positioned opposite and says, 'Down.' The axis is akin to an origin with 'up' and 'down' negatives or inversions of each other. 'Shoes, no shoes,' 'hot, not hot,' have an even greater ring of pairing something across from its opposite. 'No' and 'not' function like the symbol, "-," that when placed in front of a numeral signifies its negative.

Numerals form symbol systems, whether they signify the counting number words or positive and negative integers. Perhaps, then, it is no surprise to find parallels between the toddlers' play with symbol systems and numbers. However, there is one last parallel to mathematics I would like to mention, which involves the toddlers' humor. In humor, structure itself becomes an object of play, to be manipulated, transposed, turned on its head.

Telling jokes by manipulating structure is rather crucial to mathematical production. Much mathematics proceeds from taking a joke seriously. "What *if* 'something ridiculous' were true? ***Then***, what would happen?" Proofs by contradiction take this form. What is believed not to be true is assumed and taken seriously throughout the course of the proof. In the end, something which directly contradicts the given parameters is proven. Thus, what was assumed must be false and its opposite true.

Whole branches of mathematics may have begun with a joke. Negative integers: What if there were a number I could add to five to get zero? Non-Euclidean geometry: What if I made a contrary assumption to the parallel postulate?[6] In addition to its utility to creators of mathematics, humor might be a welcome and positive force in the classroom.

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[6] Humor and jokes take different forms, from knock-knock jokes to provocative, wholesale farce. Mathematical 'jokes' more likely resemble the latter. Devising an axiom contrary to one of Euclid's basic axioms and building a new geometry from there was surely a daring and provocative act.

Featherstone (2000) offers an incident in which a third grade student wrote in his notebook, "-Pat + Pat = 0." The student reflected a playful spirit; he was "playing with -- and in -- the math assignment." The example also shows that the student understood the interchangability of symbols, that anything can go where the 'Pat' is and an attention to the pattern or rule of anything plus its negative equals zero -- understandings that number sentences alone would not necessarily have revealed. The example can also be seen as involving a sense of humor. The student told a joke. He played with expectations that only numerals were to be used and instead put a person's name into the equation. In doing so, he retained part of the prior structure (that of the equation) as he transformed another (domain for elements). In telling his joke, he betrayed (and perhaps arrived at) an even deeper mathematical understanding than through generation of mundane number sentences alone.

Humor could be an invaluable resource for arriving at deep and flexible understandings of mathematical structure, as well as provide a levity much needed in the anxiety-ridden mathematics classroom. It can also offer alternative directions for handling errors. I recall from my own teaching of elementary level mathematics an incident in which a student accidentally said, "1 + 1 = 1." The rest of the class laughed in ridicule. Naturally, it bothered me to see the student treated this way and I did not want him to take it to heart. After some thought, I entered class the next day and asked, "What if one plus one did equal one? Could we think of some examples where this could be true?" We came up with 'one mound of mashed potatoes plus another mound of mashed potatoes equals a mound of mashed potatoes' and 'one drop of water drop plus another drop of water equals one drop of water.' In turning a mistake into a joke *and taking the joke seriously*, we explored some important notions about mathematics: parameters can be changed, relationships redefined, and alternative mathematical systems created. Who knows where else humor could lead?

I would like to make one final point related to the toddlers' systematizing, which relates to their different attractions to types of structure and by degrees. Jacob had the most varied and prevalent mirror play. George had almost none. Jeremy had a predilection for cyclical distribution as well as a tenacity to enforce it that he showed for no other systematic activity. George had an overall interest in rules and structure. Some of this difference among the toddlers may be due to differences in cognitive dispositions, especially when a certain form of play was present in one or two of the toddlers and not the others. However, it may also relate to differences in what might be considered personality characteristics, such as singlemindedness and determination.

George showed the highest level of determination. He could focus and pursue an individual activity for a great length of time. He also set and enforced rules that applied to adults' behavior. Jeremy showed the greatest lack of attachment to routine, the greatest flexibility and willingness to give up something he enjoyed and move on to something else. Jacob held a middle ground, attracted to systematizing and yet willing to change things based on the input of others, which may reflect strong sociability and an interest in social interaction or pleasing others.

These 'personality traits' could play out in different ways in a mathematics classroom. As a teacher, I think it important to take note of the potential advantages or pitfalls of these traits, do what I could to cultivate all children's strengths, and support their learning math that fit, as much as possible, who they are as people.

Chapter VII: Conclusions

In Chapter III, I presented my research questions as follows:

> What characterizes the symbolizing of young toddlers and what relevance does it hold for understanding mathematical symbolizing and thought?
>
> What conceptual lenses are effective in analyzing toddlers' symbolizing that simultaneously offer insight into mathematical symbolizing?

The last three chapters have addressed these questions. Here, I will offer a brief review of my responses, giving primary attention to the second question and priority to general and cross-cutting lenses rather than particular ones.

The role of action

One general perspective this study presents is *action* as a basis for formation and/or expression of symbols and symbol systems. The toddlers used, created and expressed symbols and symbol systems in action. Signs and their accompanying concepts were action-based as well. Actions comprised the data that were analyzed. Actions also form the basis of *doing* and *knowing* mathematics. For example, an understanding of number words develops from the action of counting.

This view relates to Donnelly's (2000) discussion of 'metaphor-actions.' Infants and toddlers enact and through action develop conceptual approaches to the world. Through action, they also symbolize their conceptions. I will take this opportunity to look across the three toddlers' various forms of symbolic expression and speculate as to their individual, dominant, conceptual approaches to the world.

Jacob seemed particularly predisposed to creating sequences. He repeated sets of actions again and again. The actions themselves did not necessarily need to be so enjoyable or interesting. Jacob's attention was drawn to form. His interest in mirror play (in which players simultaneously enact the same actions), while not involving a fixed and ordered pattern of actions, shows a similar attraction to form. While the particular elements in the structure could vary, although generally chosen from a fixed domain, the form of moving from one action to another, choosing an action from a set of possibles and placing it in a blank spot, the action additionally being mirrored, this form remained constant. Perhaps Jacob's attention to form also underlaid his use of full sentences, linguistic conventions and grammatical forms such as the past tense and pronouns at relatively early ages, as well as his attraction and adeptness with puzzles and other toys that involved fitting shapes into spaces, positioning elements into forms.

Jeremy showed a marvelous predilection for associating things that did not obviously 'go together.' At face value, Jeremy seemed to engage in random, chaotic play, moving from one thing to another without apparent rhyme or reason. And yet, this may have been his very purpose, to associate one thing or action with another 'drawn from domains far apart.' His play was about creating metaphors, maybe testing them, acting out the question, what if I associated this thing with this other thing, where might that lead? Jeremy not only worked to learn language that involved bringing things together in categories, but he also brought unlikely, even implausible things together, and explored ways to connect them. Therein lay his creativity and humor.

George presents a bit more difficulty than the other two boys to detect dominant modes of acting and thinking. George did show a strong interest in mastering tasks and setting and enforcing rules. However, these do not appear to resemble or offer potential for becoming *conceptual* approaches to the world, as does Jacob's search for form and Jeremy's search for association.[1] Which tasks did George choose to master and where? How and with whom did he set and enforce rules?

The answers to these questions indicate that George showed an interest in certain contradictions. Perhaps he acted to explore contradictions and overcome them. Surely George chose to master things that were challenging, but perhaps they were additionally things he saw as belonging to adult competence. George was a 'baby'; he even referred to himself as 'baby' and not by his given name. Yet George desired to act as did the adults around him. Similarly, adults and not babies set rules and prohibitions, and yet George explored his ability to set rules that adults would follow. George turned the tables on 'the way things are' and created contradictions. George was a baby but not always, not when he sought adult power and abilities.

Another of George's compelling interests fits this interpretation. George adored vehicles, those on wheels that moved freely across the terrain -- cars, vans, trucks. Trains, boats and planes were not much part of his world; neither were bikes or motorcycles for that matter. When I asked George's parents and day care provider for their opinion as to why these vehicles were so compelling to George, they said they thought it was the motion. But trains and motorcycles move as well.

George's interest in particular vehicles may have been based on experience with them. George's parents reported they took long car trips when he was an infant and 'car' was one of his first words. George may never have traveled in a train or ridden a bike. However,

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[1] For example, Donnelly (1998) discusses her two sons, one of whom was compelled to dig as a young child and showed a similar 'digging into' creative and intellectual activities, whereas the other son showed a great interest in connectors and connecting things and similarly connected ideas one to the other across a vast terrain. I might dare to speculate that my daughter may arrange and rearrange thoughts, placing them in different configurations and contexts, checking to make sure she has included each one, just as she does now with blocks, barrettes and rubber ducks.

trains crisscrossed all through the town where the toddlers lived and Jacob and Jeremy showed great interest in them. George had surely experienced trains if only as a witness.

Perhaps George was not attracted to trains because they differ from cars in their freedom. Trains move along a track; they are confined to a fixed space or trajectory. On the other hand, cars, trucks and vans have a much greater range of movement over a vast network of roads and even wider range if one dares to drive across a field.

Coupled with the great freedom and ability to move through space that cars, trucks and vans enjoy is a contrasting experience of staying in the same space. In order to travel through space with a car, one must sit inside the car. A driver or passenger of a car stays still within a very confined space while simultaneously traveling across a wide terrain. This contradiction is similar to possessing the seemingly vast power and ability of an adult while simultaneously remaining a limited baby. Thus, George can be understood as searching out points of contradiction. His interest in juxtaposing and playing with opposites (opposite signifiers for heat, 'shoeness,' positions in space, celestial bodies) may relate to this disposition.

As with the dispositions mentioned in Chapter VI of single-mindedness, determination and weddedness to rules, these predominant cognitive dispositions could likewise play roles in the toddlers' approaches to mathematics and other school subjects once they enter school. The toddlers also showed varying dispositions in their symbolic expression, from the means they used to generate signs to media of expression to ways of creating systems of symbols. These varying dispositions may also relate to cognition (e.g., symbols as cognitive support and the relative abstraction of different types of symbols) as words (and other signs) reflect and constitute thought.

Means of sign generation included metonymy, metaphor, 'do as I do,' forming generalizations and categories, drawing distinctions, 'motivation' (i.e., signifier bears some resemblance to signified) and arbitrary assignment of phonemes. Some of these processes led to 'misnamings.' Media of expression included words, gestures, auditory gestures, onomatopoeia and reduplication of syllables. Forms of systematizing included rule governed sets of actions, rule bounded exploration, rituals, 'defining of terms,' symbols as unknowns and variables (such as 'pivots' in pretend play), using signs to generate imagined situations, sequences, semi-random sequences, distributions, mirrorings, creating structures informed by sign relationships and manipulating structure.

In identifying these different means, forms and expressions, I identified differences among the individual toddlers, as not all the categories applied to all of them. I set out to find individual differences in cognition and symbolizing and I found them. Yet, I hope it is clear that these differences vary and cut across individuals in no clear cut, predictable ways. Differences cannot be easily categorized and used to label the toddlers the way some teachers like to peg students as 'visual,' 'auditory' or 'tactile learners.' The multiplicity that only three subjects revealed hints that there may be a great deal more. Although there may be some limitation to, for example, 'media of symbolic expression' (even though certain areas were clearly not explored such as writing), dominating conceptual metaphors may number in the thousands or more, or be as unique as a fingerprint.

The symbolic continuum

These issues cut across the 'symbolic continuum,' whereby there is a distinction as well as relationship between sign and symbol. In this section, I will discuss this notion in some depth, but I first would like to say a few words about metonymy and metaphor as they relate to the symbolic continuum.

Metonymy took many different forms in the toddlers' symbolizing. Metonymy was a means for generating verbal signs in several different ways, but also a means for generating fixed and spontaneous gestures, including those used in pretend play. George *became* a dog by signifying certain characteristics of a dog, its sound, walk and panting. He did not signify every aspect of a dog, did not *become* a dog in its entirety. This is a form of metonymy, naming by way of characteristic. Metonymy also relates to the issue of 'symbol containment,' when a symbol becomes a sign for something in which it is a part, signifies something greater than itself. When 'milk' signifies the events in which it participated or '6' signifies the even numbers, a part 'names' the whole.

These examples help to illustrate that metonymy can occur nearly anywhere within the symbolic continuum. Metonymy can use 'motivated symbols' that greatly resemble actual characteristics of signifieds, as with onomatopoeia, the signs of pretend play and even Jeremy's gesture for *make music* (in which he associated a particular action of adjusting the stereo with playing music and singing). Metonymy can also draw upon arbitrary signs that signify parts or characteristics (as with 'triangle'). Metonymy can additionally involve taking parts of signs, as with Jacob's 'momo,' or even parts of symbol systems (to name the system), as with '6.' Thus, metonymy can make use of varying relations among experience, signs and sign or symbol systems. Metonymic namings can arise out of direct namings of experience, choosing existing signs that name experience, or by drawing on symbols without immediately attending to experience.

Metaphors involve a different sort of relationship among experience and signs. Association takes place at the experiential level, but naming takes place at the level of sign only. A sign for one experience signifies an associated experience. Experience is named circumlocutorily, not directly.

To focus in on the symbolic continuum, signs arise out of desires to signify experience and then take relationships in sign systems or 'language games.' Within sign systems, signs take on relationships with other signs. They form familiar expressions, take positions in a structure or grammar and make use of 'symbols' that fulfill roles of social or linguistic convention (e.g., 'thank you,' 'not') rather than signify a material experience. When signs are created for the purposes of pretend play, they can also be understood as forming systems. It is the system, the relationships among signs, that help to birth imagined situations, not the individual signs themselves. A stick cannot *be* a 'horse' without other signs that help to convey this meaning (e.g., words, actions, sounds, other objects, etc.), which together form a system of signs.

It is at the point when signification is somewhat dispensed with that signs become *symbols*. In such situations, emphasis is placed on relationships between signs and underlying structure, rather than on meaning external to the system. Meaning comes primarily from within. Signs in a sign system can become symbols with a change of attention, but symbols also arise through 'semiotic chaining,' whereby a sign becomes the signified of a signifier that is signified again, etc., until arriving at a system where signification 'fades away' and signs become symbols. Symbols may also arise independent of signification merely by taking roles in a system, such as a ring of Legos (in 'Lego Ring'), banging on a table (as in a mirror game) or a parenthesis (which has a particular role in written English and written mathematics). These are all symbols in a system but have no meaning independent of that system.

Although I find it helpful to think of a sign-symbol continuum, in some ways this characterization is too simplistic. 'Continuum' conveys an image of linearity, as if there is one smooth, continuous line connecting 'sign' to 'symbol' and all else in-between. The sign-symbol relationship can also be understood as involving levels with connections among them. I see Paulos (1980) hinting at these levels when he speaks of negotiating 'levels of meaning' in appreciating humor.

There is the experiential level, the 'concrete' in which, for example, a boy directly experiences fear from a lion. Then there is the connection of experience to a level of signification, whereby a boy is signified by the word 'boy' (or the boy, George) and a lion is signified by the word 'lion' (or a photo of a lion, a lion doll, a juice cup, etc.). These signs then form a system, yet another level, for only taken together and in the company of other signs can they signify intended meaning, e.g., 'The boy is afraid of the lion.' The signs can be regarded as symbols if I attend to aspects of structure conveyed by the larger sign system. For example, 'boy' is a noun and bears phonetic resemblance, i.e., rhymes with 'toy,' 'joy' and 'coy.' This then is another level, that of a symbols and symbol systems.

The signs can also connect to the level of symbol by way of semiotic chaining, e.g., the word 'one' can signify 'boy' and another 'one,' 'lion,' which in turn can be signified by numerals, '1' and '1.' While semiotic chaining creates additional levels, they are levels within the 'connection to signification.' '1' and '1' are still individual signs, not yet part of a system. However, once I say '1 + 1 = 2,' they do enter a system and become symbols, because they are now part of a symbol system.

Perhaps I should instead say, '1 + 1 = 1.' The boy ran away or else the lion ate the boy. This construction may actually be more accurate, because if you put a lion and a boy together, it will not be long before only one is left, most likely the lion. Yet, in saying, '1 + 1 = 1,' I am also making a 'joke.' '1' in this context is to take its meaning from the symbol system of the natural numbers according to standard rules of addition. In this system, '1 +

1 = 2.' I made a 'joke' by transforming structure. I let '1' signify material objects from whence it came through semiotic chaining. As such '1' signifies a boy as well as a lion. '1 + 1 = 1' thus signifies a material experience of only a lion remaining once a boy and a lion are placed together. '1' is thus a sign and not a symbol. There is no symbol *system* that I know of where this is true. Herein lies the transformation and the 'joke.' '1' is not meant to be a sign. This illustrates an essential point I would like to make.

School mathematics places particular emphasis on understanding and working within the symbol systems of mathematics. This goal is surely important. However, it may be just as important to be able to move from symbol to sign and back again, to negotiate and even understand the structures and processes of signification. Mathematical creativity, like jokes, may rest on this ability, to recognize, for example, that one plus one could equal one and then to attempt to create a system where this equality would hold. Mathematical applications surely rests on this ability, for applications requires continual 'translation' back and forth between symbol systems and material experience. Do the results apply? Do the models fit? There is the famous, nationally administered bus and soldier problem in which students properly performed long division but then 'left' 12 soldiers without transportation (Schoenfeld, 1988). Once one leaves a symbol system and turns symbols into signs, there is the problem: 'What do the signs mean?' Any practical use of mathematics is going to involve this problem. Solving it seems one of the greatest difficulties students have in mathematics classrooms today.

Although I have drawn distinctions between levels of symbol systems, sign systems and experience, these distinctions are also simplistic, for all are experienced. Systems of signification are experienced and enacted, are rooted in action. Furthermore, from birth onwards there is no experience that does not involve signification to some degree. Even a boy having a sudden fearful reaction to a lion experiences fear in particular ways because of knowledge that signification has participated in constructing. Thus, experience and the products of symbolizing continually interrelate, even though it may be conceptually helpful to see them as separate at times.

Mathematics as a game

The remaining perspective I would like to discuss is that of mathematics as a game, a 'language game,' a pretend game, a rule-based game.[2] All games require rules and rules create systems, spaces, 'playing fields' in which games are played, 'language games' included. One 'steps into' a linguistic system with its accompanying modes of thought. There can be some creativity, some new construction but only that which is permitted by the rules, the structures and meanings already inherent in the system. Mathematics functions very much like a 'language game,' especially as it is taught in school, although a frequently dysfunctional one. Children do not learn the mathematics 'language game' the way they learn their native tongue.

This concern may underlie the call for children to be 'enculturated' into the discipline of mathematics (e.g., Brown, Collins & Duguid, 1989). Surely, if children mastered mathematics as they do everyday spoken language, there would be a great deal more success as well as creativity. Yet, even this prospect requires attention to the organic ways children interact with mathematical language. For even as they learn to participate in their dominant, environmental 'language game,' all children simultaneously invent unique ones. No two people really speak the *same* language.

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[2] This talk of games should not be confused with 'math games' as they are currently known, games played within mathematics classrooms, generally to practice or review skills. These are games played with pieces of mathematics. They do not treat mathematics itself as a game.

Genuine 'enculturation' into the mathematics discipline involves attention to systems, to games. One can enter a game, leave a game or change a game by changing its rules. Different games can be played simultaneously and compared one with the other. Perhaps most effective at conveying this perspective is to view mathematics as a 'pretend game.' A situation is set up, 'ifs' are established, and then one plays around in the 'thens,' seeing where they might lead. Since it is a 'pretend game,' it can be set up and played out in multiple ways, as well as exited and viewed from the outside. And pretending is *playing*. One can be playful and make 'jokes.'

Featherstone (2000) describes students who were playing in classroom mathematics, that is, until they were reminded it is 'serious business.' Then the usual anxiety and need to 'get it' entered in. Up until then they were able to playfully explore a student's idea. They joined her as 'playmates' in a mathematical 'playground.' Games are often taken seriously, but at their heart they are playful. If mathematics were treated playfully and as a pretend game, students might approach it with enthusiasm rather than the usual dread. In addition, as Featherstone puts it, play may place students "in a position to invent mathematics their teachers have not thought to lead them towards."

In conclusion

Studies conclude with suggestions for future research and here is a prime question I would like to see explored. What if the mathematics classroom became a place to 'play' mathematics, if mathematics were treated as a 'pretend game?' And what if mathematics pedagogy and curriculum took into account the specific issues of symbolizing that this study explored? What would result in terms of classroom activity and students' understandings, success, creativity and self-esteem? What if the central role of symbolizing in mathematics were recognized? What if symbolizing and the activities and structures involved became a topic of mathematical study? What if learning mathematics truly became about attempts to understand, represent and symbolize structures and 'live within' symbolic creations? How would these emphases affect classroom practice and student experience?

I could see these questions explored with teachers in classrooms, say if I were a teacher or worked with teachers in a master's course or a collaborative research project, or if someone else did the same. The input of classrooms would not only help to answer these questions but surely lead to changes and expansions of theoretical concepts.

Whether or not a classroom were set up to explore these issues from a curricular and pedagogical perspective (i.e., how would classroom practice change?), they could still be studied. A 'regular classroom' would still offer opportunities to explore issues of students' symbolizing as they encounter mathematics, issues that could relate but also likely differ from those arising with toddlers.

I also advocate additional research with toddlers. Simply other children, girls, children in different family or socio-economic circumstances, all would yield additional and perhaps very different insights into symbolizing and mathematical thinking more generally. Different researchers with mathematical backgrounds different from mine may find different things as well. I see this study very much as 'the tip of the iceberg.' There is so much more to learn.

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