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# DIFFUSION APPROXIMATION FOR SOLUTIONS OF PERTURBED DIFFERENTIAL EQUATIONS

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# ABSTRACT

## DIFFUSION APPROXIMATION FOR SOLUTIONS OF PERTURBED DIFFERENTIAL EQUATIONS

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We consider the operator differential equation perturbed by a fast Markov process:

$$\frac{d}{dt}u_\epsilon(t) = A\big(y\big(\tfrac{t}{\epsilon}\big)\big)u_\epsilon(t), \quad t > 0$$

$$u_\epsilon(0) = u_0$$

in a separable Hilbert space $H$. Here $y$ is an ergodic jump Markov process in phase space $Y$ satisfying some mixing conditions and $\{A(y),\, y \in Y\}$ is a family of closed linear operators. We study the asymptotic behavior of the distributions of $u_\epsilon(t/\epsilon)$. For the case when the operators $A(y)$ commute, Salehi and Skorokhod (1996) proved that the distributions of $u_\epsilon(t/\epsilon)$ asymptotically coincide with the distributions of some Gaussian random field with independent increments.

We do not assume that the operators $A(y)$ commute, but we impose some conditions on the structure of these operators. We study the asymptotic behavior of the stochastic process $z_\epsilon(t) = e^{-t\bar{A}}u_\epsilon(t)$, where $\bar{A} = \int A(y)\rho(dy)$, and $\rho(\cdot)$ is the ergodic distribution of the Markov process $y(t)$, $t \geq 0$. We prove that the stochastic process $z_\epsilon(t/\epsilon)$ converges weakly as $\epsilon \to 0$ to a diffusion process $\tilde{z}(t)$, $t \geq 0$, which is described using its generator. The proof is based on the theorem on weak convergence of $H$-valued stochastic processes to a diffusion process.

*To my family*

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# Contents

# Chapter 1

# Introduction

## 1.1 Introduction

Randomly perturbed dynamical systems and differential equations were studied by many authors: Krylov and Bogolubov, Gikhman (1950, 1951, 1964), Khasminskii (1966), Papanicolaou and Varadhan (1973), Papanicolaou, Strook and Varadhan (1977), Papanicolaou (1978), Krylov and Rozovskii (1979), Rozovskii (1990), Hoppensteadt, Salehi and Skorokhod (1995), Salehi and Skorokhod (1994, 1996) as well as other authors. There are several types of problems that are considered for such equations.

Averaging theorems state the convergence of the solution of the perturbed equation to the solution of the "averaged" equation. The first averaging theorems were proved by N.M. Krylov and N.N. Bogolubov in 1920s and 1930s. They considered the equation of the form

$$\frac{dx_\epsilon(t)}{dt} = \frac{1}{\epsilon}a(x_\epsilon(t), t, \frac{t}{\epsilon}),$$

where function $a$ depends on two times: fast (third argument) and slow (second argument). If there exists the average of $a$ in fast time

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T a(x,t,\tau)d\tau = \bar{a}(x,t),$$

then the solution of the perturbed equation $x_\epsilon(t)$ converges to $\bar{x}(t)$, which is the solution of the averaged equation

$$\frac{d\bar{x}(t)}{dt} = \bar{a}(x,t).$$

Gikhman (1950, 1951) established a general averaging theorem for randomly perturbed equations. An averaging theorem for the randomly perturbed Volterra integral equation

$$x_\epsilon(t) = \varphi(t) + \int_0^t K(s,t,y(\frac{s}{\epsilon}),x_\epsilon(s))ds$$

was proved by Hoppensteadt, Salehi and Skorokhod (1995). The kernel $K$ depends on a fast Markov process $y$ with ergodic distribution $\rho$, which is assumed to satisfy some mixing conditions. One of the results of the above mentioned article is the convergence of $x_\epsilon(t)$ to $\bar{x}(t)$, the solution of the averaged equation

$$\bar{x}(t) = \varphi(t) + \int_0^t \bar{K}(s,t,\bar{x}(s))ds,$$

where

$$\bar{K}(s,t,x) = \int_Y K(s,t,y,x)\rho(dy).$$

More precisely, it was proved that

$$P\left\{\lim_{\epsilon\to 0}\sup_{t\le T}\|x_\epsilon(t)-\bar{x}(t)\|=0\right\}=1.$$

An averaging theorem for dynamical systems in Hilbert space was proved by Hoppensteadt, Salehi and Skorokhod (1996).

Hoppensteadt, Salehi and Skorokhod (1995) also present a result of another type: a theorem on normal deviations. From the averaging theorem the difference between the solution of the perturbed equation and the solution of the corresponding averaged equations tends to zero. But appropriately normed, this difference (the deviation) is asymptotically Gaussian. Namely, it is proved that the finite-dimensional distributions of the process $\epsilon^{-1/2}(x_\epsilon(t)-\bar{x}(t))$ converge weakly as $\epsilon\to 0$ to a Gaussian process, which can be described as the solution of some stochastic linear integral equation. A theorem on normal deviation for the difference equations was proved by Hoppensteadt, Salehi and Skorokhod (1997).

The theorems on the approximation by diffusions deal with the asymptotic behavior of $x_\epsilon(t/\epsilon)$. From the averaging theorems and theorems on normal deviations we know that the solution of the perturbed equation is close to the solution of the averaged equation and the deviation is approximately Gaussian with mean zero and variance of order $\epsilon$. When we consider $x_\epsilon(t)$ for large time, that is time of order $O(1/\epsilon)$, these deviations accumulate and $x_\epsilon(t/\epsilon)$ converges weakly as $\epsilon\to 0$ to a diffusion process.

Theorems of such type were proved by Hoppensteadt, Salehi and Skorokhod (1995) for perturbed Volterra equations, Salehi and Skorokhod (1994) for perturbed wave equations, and Hoppensteadt, Salehi and Skorokhod (1997) for difference equations. Salehi and Skorokhod (1996) consider a very general setup: operator differential equations in a separable Hilbert space $H$:

$$\frac{du_\epsilon(t)}{dt} = A(y(\frac{t}{\epsilon}))u_\epsilon(t), \quad u_\epsilon(0) = u_0 \tag{1.1}$$

and

$$\frac{d^2u_\epsilon(t)}{dt^2} = A(y(\frac{t}{\epsilon}))u_\epsilon(t), \quad u_\epsilon(0) = u_0, \quad \left.\frac{du_\epsilon(t)}{dt}\right|_{t=0} = v_0, \tag{1.2}$$

where $y(t)$ is an ergodic homogeneous Markov process in phase space $Y$ with ergodic distribution $\rho$ satisfying some mixing conditions, and $\{A(y), y \in Y\}$ is a family of closed linear operators with a common dense domain. The theory of differential equations with operator-valued coefficients is contained in Krein (1982) and Kato (1984).

Under the assumption that the operators $A(y)$ commute and $A^*(y) = -A(y)$, where $A^*$ denotes the adjoint to $A$, the finite-dimensional distributions of $u_\epsilon(t/\epsilon)$ coincide asymptotically with those of a Gaussian process $\hat{u}_\epsilon(t)$. A similar result is obtained for equation (1.2) under the assumption that all $A(y)$ are symmetric non-positive self-adjoint operators. Also, theorems on normal deviations are established for both equations.

The method of proof is based on spectral decompositions for the operators $A(y)$. Since these operators commute, they can be represented as spectral integrals with respect to the same resolution of identity. This method does not work if the operators $A(y)$ do not commute, which is the case considered in the present work.

We assume that $A(y) = A_0(y) + A_1(y)$, where $A_0(y)$ commute with $\bar{A} = \int_Y A(y)\rho(dy)$, and $A_1(y)$ are finite-dimensional with the same finite-dimensional range for all $y \in Y$. Since operators $A(y)$, $y \in Y$ do not commute, we can not use the method of Salehi and Skorokhod (1996).

A different approach using a sufficient condition for the convergence to a diffusion process found in Skorokhod (1989). Such method is implemented in Hoppensteadt, Salehi and Skorokhod (1997) for the difference equations in $R^d$. It is based on martingale characterization of multidimensional diffusion processes (see Strook and Varadhan (1979)). Another important tool for proving the weak convergence is a sufficient condition for the weak compactness of a sequence of the stochastic processes, which can be found in Parthasarathy (1967) for the case of $H$-valued processes.

In the present work we generalize the method used in the paper of Hoppensteadt, Salehi and Skorokhod (1997) to the case of infinite-dimensional Hilbert space using the results of Daletskii (1967, 1983). The limiting diffusion process is an infinite-dimensional diffusion process. Such processes arise in the theory of stochastic partial differential equations. Stochastic differential equations have received a considerable attention in the recent years. Here we mention just a few books and papers that are devoted to them: Varadhan (1980), Rozovskii (1990), Protter (1990), Krylov and Rozovskii (1979), Papanicolaou and Varadhan ((1973), Papanicolaou (1978).

## 1.2 Summary

Chapter 2 contains the results that are necessary to prove the weak convergence to a diffusion process in a separable Hilbert space $H$. Section 2.1 contains the theorem on weak compactness of a sequence of $H$-valued stochastic processes, a condition that ensures that the diffusion coefficients specify the transition probability of a Markov process and a proposition on the martingale characterization of diffusion processes. These results are used in section 2.2 to formulate and prove the theorem on weak convergence to a diffusion process, which is the main result of Chapter 2.

In Chapter 3 we consider the differential equation (1.1) with operator-valued coefficients in a separable Hilbert space $H$. The coefficients are perturbed by an ergodic jump Markov process $y$ in a phase space $Y$ with ergodic distribution $\rho$. We do not assume that the coefficients commute, but assume that $A(y) = A_0(y) + A_1(y)$, where $A_0(y)$ commute with $\bar{A} = \int_Y A(y)\rho(dy)$, and $A_1(y)$ are finite-dimensional with the same finite-dimensional range for all $y \in Y$. This assumption is crucial for the proof of Lemma 3.7 (section 3.4). It ensures that all the integrals involving the resolution of identity are well defined. We also suppose that $A_0(y) = a_0(y)D$, where $a_0$ is a real-valued function and $D$ is a non-random operator. This is a more restrictive assumption than the one made in the paper by Salehi and Skorokhod (1996), where a general family of closed commuting operators is considered. If $A_0(y) = 0$ for all $y \in Y$, then we have a system of linear differential equations, which were studied by Khasminskii (1966).

We think that it is possible to extend the results obtained in this dissertation to the case when $A_0(y) = \sum_{k=1}^{n} a_{k0}(y)D_k$, $n > 1$, $a_{k0}$ $k = 1, \ldots, n$ are real-valued functions and $D_k$, $k = 1, \ldots, n$ are non-random operators, and also to the case of general closed operators $A_0(y)$. These are the problems for future research.

Under the above assumptions on operators $A_0$ and $A_1$ the commutativity condition means that $A_0(y)$ commute with $\bar{A}_1$ for all $y \in Y$. This condition is satisfied in many applications when $\bar{A}_1 = 0$. An example of such kind is given in section 3.5, where we consider a perturbed partial differential equation. When $\bar{A}_1 \neq 0$, the commutativity condition is not satisfied automatically. There are some applications of the results of the thesis in this case as well (for example, to the systems of differential and partial differential equations).

The main result is formulated in section 3.3. The proof is in section 3.4 and it is based on the results from Chapter 2 and some special representations obtained using the properties of Markov process $y$ and spectral decomposition for the operator $\bar{A}$.

# Chapter 2

# Theorem on weak convergence to a diffusion process

## 2.1 A sufficient condition for weak compactness of a sequence of stochastic processes

Let $H$ be a separable Hilbert space and $\xi_n(t)$, $t \in R_+$ be a sequence of $H$-valued stochastic processes. We say that $\xi_n$ converges weakly to a stochastic process $\xi$ if the finite dimensional distributions of $\xi_n$ converge weakly to those of the process $\xi$, i.e.

$$\lim_{n\to\infty} Ef(\xi_n(t_1), \xi_n(t_2), \dots \xi_n(t_k)) = f(\xi(t_1), \xi(t_2), \dots \xi(t_k))$$

for all $k \geq 1$, $t_1, t_2, \dots t_k \in R_+$ and $f : H^k \to R$ that is bounded and continuous.

We say that the sequence $\{\xi_n(t),\ n = 1, 2, \dots\}$ is weakly compact if any subsequence $\{n_k, k \geq 1\}$ admits a further subsequence $\{n_{l_k}, k \geq 1\}$ such that $\xi_{n_{l_k}}$ is weakly convergent in the sense of the above definition.

To prove the theorem on weak convergence to the diffusion process we need a theorem on weak compactness of a sequence of $H$-valued stochastic processes.

**Theorem 2.1.** *Let the sequence of stochastic processes $\xi_n$ satisfy the conditions:*

*a) there exists a positive compact linear operator $Q : H \to H$ such that its range contains $\xi_n(t)$ for all $n \geq 1$ and $t \in R_+$ and*

$$\lim_{r\to\infty} \limsup_{n\to\infty} \sup_{t\leq T} P\{\|Q^{-1}\xi_n(t)\| > r\} = 0, \text{ for all } T > 0,$$

*b)*

$$\lim_{h\to 0} \limsup_{n\to\infty} \sup_{t\leq T} \sup_{|t-t'|\leq h} P\{\|\xi_n(t) - \xi_n(t')\| > \epsilon\} = 0$$

*for all $\epsilon > 0$ and $T > 0$.*

*Then the sequence $\xi_n$ is weakly compact.*

Proof of this theorem follows from the condition of compactness of measures in Hilbert space (see Parthasarathy (1967), ch. VI, p. 151).

## 2.2 Theorem on weak convergence to a diffusion process

We consider a Markov process $\xi(t)$, $t \in R_+$ in $H$ with transition probability $P(s, x, t, B)$, $x \in H$, $0 \leq s < t < \infty$, $B \in \mathcal{B}(H)$. It is called a diffusion process if there exist continuous functions $a$: $R_+ \times H \to H$ and $B$: $R_+ \times H \to L_+(H)$, where $L_+(H)$ is the space of all continuous non-negative linear operators from $H$ to $H$, such that

$$\int g(x')P(s,x,t,dx') - g(x) = \int_s^t \int L_u g(x')P(s,x,u,dx')du \tag{2.1}$$

where $g$ is a function from H to R that has bounded first and second derivatives, $0 \leq s < t$ and

$$L_u g(x) = \big(g'(x), a(u,x)\big) + \frac{1}{2}Tr\big(g''(x)B(u,x)\big) \tag{2.2}$$

and $TrB$ denotes the trace of the operator $B$. The operator $L_u$ is called the generator of the Markov process.

**Proposition 2.1.** *Let functions a and B satisfy the condition: for any $r > 0$ there exists a constant $l_r$ for which*

$$\|a(t,x) - a(t,x')\| + \left[Tr(B(t,x) - B(t,x'))^2\right]^{1/2} \leq l_r\|x - x'\| \tag{2.3}$$

*if $\|x\| \leq r$, $\|x'\| \leq r$, $t \leq r$. Then the transition probability is determined by functions a and B through the formula:*

$$P(s,x,t,A) = P\{\tilde{\xi}_{s,x}(t) \in A\}, \quad A \in \mathcal{B}(H),\ x \in H,\ t \geq s \geq 0,$$

*where the process $\tilde{\xi}_{s,x}$ is the solution of the stochastic differential equation*

$$d\tilde{\xi}_{s,x}(t) = a(t,\tilde{\xi}_{s,x}(t))dt + B^{1/2}(t,\tilde{\xi}_{s,x}(t))dW(t) \tag{2.4}$$

*on the interval* $[s, \infty)$ *satisfying the initial condition*

$$\tilde{\xi}_{s,x}(s) = x,$$

*where* $W(t)$ *is the generalized Wiener process in* $H$ *for which* $E(W(t), z) = 0$, $E(W(t), z)^2 = t\|z\|^2$, $z \in H$ *and* $B^{1/2}$ *is a linear operator such that* $(B^{1/2})^* B^{1/2} = B$, *where* $B^*$ *denotes a conjugate to* $B$. *Under condition (2.3) the stochastic differential equation (2.4) has a unique solution for any initial condition.*

This proposition was proved by Daletskii (1967), Theorem 2.1, p. 33. Also see Daletskii (1983).

**Proposition 2.2.** *Let* $\tilde{\xi}(t)$, $t \in R_+$ *be a measurable* $H$*-valued stochastic process and let* $(\mathcal{F}_t,\ t \in R_+)$ *be the filtration generated by* $\tilde{\xi}$. *If for any function* $g : H \to R$ *with bounded first and second derivatives*

$$g(\tilde{\xi}(t)) - g(\tilde{\xi}(0)) - \int_0^t L_u g(\tilde{\xi}(u))du$$

*is a local martingale, where* $L_u$ *is defined by (2.2) and functions* $a$ *and* $B$ *satisfy the condition of Proposition 2.1, then* $\tilde{\xi}$ *admits a continuous modification* $\xi$, *which is a Markov random function with transition probability* $P(s, x, t, B)$.

This proposition is proved in Strook and Varadhan (1979) for $R^d$-valued stochastic processes. The proof for the case of $H$-valued processes is the same.

Denote by $C^{(2)}(H)$ the set of all functions from $H$ to $R$ with bounded first and second derivatives.

**Theorem 2.2.** *Let $\xi_n(t)$, $t \in R_+$, $n = 1, 2, \dots$ be a sequence of measurable $H$-valued processes. Suppose that*

*1) the distributions of $\xi_n(0)$ converge weakly to some distribution $m_0(\cdot)$ on $\mathcal{B}(H)$;*

*2) there exists a compact positive operator $Q$ for which*

$$\lim_{r\to\infty} \limsup_{n\to\infty} \sup_{t\le T} P\{\|Q^{-1}\xi_n(t)\| > r\} = 0$$

*for all $T > 0$;*

*3) there exists a subset $D \subset C^{(2)}(H)$ that is dense in $C^{(2)}(H)$ and the generator of a diffusion process $L_t$ with the coefficients satisfying the conditions of Proposition 2.1 for which*

$$\lim_{n\to\infty} E\left(G(\xi_n(t_1), \dots \xi_n(t_k))\left[g(\xi_n(t+h)) - g(\xi_n(t)) - \int_t^{t+h} L_u g(\xi_n(u))du\right]\right) = 0$$

*for all $k \ge 1$, $0 \le t_1 \le \dots t_k < t < t+h$, $G \in C(H^k)$, $g \in D$.*

*Then $\xi_n$ converges weakly to a Markov random function $\tilde{\xi}$ with the transition probability P(s, x, t, B) that is determined by relation (2.1) and the distribution of $\tilde{\xi}(0)$ equals $m_0(\cdot)$.*

*Proof.* It is easy to check that the sequence $\{\xi_n, n = 1, 2, \dots\}$ is weakly compact, since the assumptions of Theorem 2.1 are satisfied.

Let $\{n_l', l \ge 1\}$ be a subsequence for which the sequence $\{\xi_{n_l}, l \ge 1\}$ converges weakly to some stochastic process $\tilde{\xi}$. The distribution of $\tilde{\xi}(0)$ is $m_0$ and $\tilde{\xi}$ is a stochastically continuous process, i.e. $P\{\|\tilde{\xi}(t) - \tilde{\xi}(s)\| > \epsilon\} \to 0$ as $s \to t$. Therefore $\tilde{\xi}$ has a measurable modification, so we can assume that $\tilde{\xi}$ is measurable.

It follows from assumption 3) of the theorem and stochastic continuity of $\tilde{\xi}$ that

$$E\Big(G(\tilde{\xi}(t_1),\dots\tilde{\xi}(t_k))\Big[g(\tilde{\xi}(t+h)) - g(\tilde{\xi}(t)) - \int_t^{t+h} L_u g(\tilde{\xi}(u))du\Big]\Big) =$$

$$\lim_{n\to\infty} E\Big(G(\xi_n(t_1),\dots\xi_n(t_k))\Big[g(\xi_n(t+h)) - g(\xi_n(t)) - \int_t^{t+h} L_u g(\xi_n(u))du\Big]\Big) = 0.$$

Therefore

$$E\Big(g(\tilde{\xi}(t+h)) - g(\tilde{\xi}(t)) - \int_t^{t+h} L_u g(\tilde{\xi}(u))du \Big/ \mathcal{F}_t\Big) = 0,$$

and so the limit process satisfies the conditions of Proposition 2.2.

Since the sequence $\{\xi_n,\ n \geq 1\}$ is weakly compact and all convergent subsequences have the same limit, we conclude that $\xi_n \to \tilde{\xi}$ weakly as $n \to \infty$.

$\square$

# Chapter 3

# Theorem on diffusion approximation for solution of perturbed differential equation

## 3.1 Introduction

We consider operator differential equation in a separable Hilbert space $H$:

$$\begin{aligned} &\tfrac{d}{dt}u_\epsilon(t) = A\big(y\big(\tfrac{t}{\epsilon}\big)\big)u_\epsilon(t), \quad t > 0 \\ &u_\epsilon(0) = u_0, \end{aligned} \tag{3.1}$$

where $\{y(t),\, t \geq 0\}$ is an ergodic homogeneous Markov process in a measurable space $(Y, \mathcal{C})$ satisfying some mixing conditions and $\{A(y),\, y \in Y\}$ is a family of closed linear operators with a common dense domain $D$, $u_0$ is a fixed element of $D$. We denote by $(\cdot,\cdot)$ the scalar product in $H$. Differential equations with operator-valued coefficients are studied in Kato (1984) and Krein (1982).

Let $\rho$ be the ergodic distribution of the process $y$. We assume that for all $x \in D$ the integral $\int A(y)x\rho(dy) = \bar{A}x$ is defined, and we consider the averaged equation for (3.1):

$$\begin{aligned} &\tfrac{d}{dt}\bar{u}(t) = \bar{A}\bar{u}(t), \quad t > 0 \\ &\bar{u}(0) = u_0. \end{aligned} \tag{3.2}$$

We will investigate the asymptotic behavior of $u_\epsilon(t/\epsilon)$ as $\epsilon \to 0$.

## 3.2 Assumptions

**I.** Let $\{\bar{U}(t), t \geq 0\}$ be a family of linear operators from $H$ to $H$ satisfying the differential equation

$$\begin{aligned} &\tfrac{d}{dt}\bar{U}(t) = \bar{A}\bar{U}(t), \quad t > 0 \\ &\bar{U}(0) = I, \end{aligned} \tag{3.3}$$

where $I$ is the identity operator.

The solution of equation (3.3) defines a semigroup of operators in $H$:

$$\bar{U}(t+s) = \bar{U}(t)\bar{U}(s), \quad t, s \geq 0.$$

We extend this semigroup to a group by setting

$$\bar{U}(-t) = \left(\bar{U}(t)\right)^{-1}, \quad t \geq 0.$$

Assume that this group is unitary, i.e. for any $f \in D$

$$(\bar{U}(t)f, \bar{U}(t)f) = (f, f).$$

Also suppose that $\{\bar{U}(t),\ t \in R\}$ is weakly continuous.

**II.** Suppose that $y(t)$, $t \geq 0$ is a jump Markov process with transition probability $P(t, y, C)$, $t \geq 0$, $y \in Y$, $C \in \mathcal{C}$ satisfying the relation

$$\lim_{t \to 0} \frac{1}{t}(P(t, y, C) - 1_C(y)) = \Pi(y, C)$$

and $\sup_y Var\Pi(y, \cdot) < \infty$.

**III.** SMC (Strong mixing condition). Set

$$R(t, y, B) = P(t, y, B) - \rho(B), \quad t \geq 0,\ y \in Y,\ B \in \mathcal{C}.$$

Assume that

$$\int_0^\infty |R(t, y, B)| dt < \infty \quad \text{for all } y \in Y, B \in \mathcal{C}.$$

Set

$$R(y, B) = \int_0^\infty R(t, y, B) dt.$$

Under assumption I the group $\{\bar{U}(t),\ t \in R\}$ admits the following representation (see Dunford and Schwartz (1963), v.2, sec. XII.6.1, p. 1243, Stone's Theorem):

$$\bar{U}(t) = e^{itS} = \int_R e^{it\lambda} dE_\lambda.$$

The resolution of identity $E_\lambda$ and the symmetric operator $S$ are determined uniquely by the group $\{\bar{U}(t),\, t \in R\}$.

Set $\tilde{A}(y) = A(y) - \bar{A}$, $y \in Y$.

**IV.** Suppose that $\tilde{A}(y) = \tilde{A}_0(y) + \tilde{A}_1(y)$, $\tilde{A}_0(y) = a_0(y)D$, where $a_0$ is a function from $H$ to $R$ for which

$$\int_Y |a_0(y)|\rho(dy) < \infty, \quad \int_Y\int_Y a_0(y)a_0(y')R(y, dy')\rho(dy) < \infty,$$

$D$ is a linear operator from $H$ to $H$ such that $D^* = -D$ and $\tilde{A}_0(y)E_\lambda = E_\lambda \tilde{A}_0(y)$ for all $y \in Y$.

Denote $\bar{a}_0 = \int_Y a_0(y)\rho(dy)$.

**V.** Assume that $\tilde{A}_1(y) = i\tilde{S}_1(y)$, and all $\tilde{S}_1(y)$, $y \in Y$ are symmetric and finite-dimensional with the same range $\tilde{R}$ of dimension $n$. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis in $\tilde{R}$ and $S_{kj}(y) = (\tilde{S}_1(y)e_k, e_j)$, $k, j = 1, \ldots, n$.

## 3.3 Results

Set $z_\epsilon(t) = \bar{U}(-t)u_\epsilon(t)$.

**Theorem 3.1.** *Suppose that conditions I–V are fulfilled. Then there exists a positive* *compact operator $Q$ such that the stochastic process $Q\tilde{z}_\epsilon(t) = Qz_\epsilon(t/\epsilon)$ converges weakly as $\epsilon \to 0$ to the diffusion process $\tilde{z}(t)$ with the generator $L$ determined on functions $\Phi : H \to R$ with bounded first and second derivatives by the relation*

$$L\Phi(Qz) = (\Phi'(Qz), Q\tilde{a}(z)) + \frac{1}{2}Tr\Phi''(Qz)Q\tilde{B}(z)Q \tag{3.4}$$

*where*

$$\tilde{a}(z) = \int_Y \int_Y \iint_{\{\lambda=\mu\}} dE_\lambda \tilde{A}(y)\tilde{A}(y')dE_\mu z R(y,dy')\rho(dy), \tag{3.5}$$

$$\tilde{B}(z) = B_{00}(z) + B_{01}(z) + B_{10}(z) + B_{11}(z), \tag{3.6}$$

$$B_{00}(z) = 2\int_Y \int_Y < \tilde{A}_0(y')z \circ \tilde{A}_0(y)z > R(y,dy')\rho(dy), \tag{3.7}$$

$$B_{01}(z) = 2\int_Y \int_Y \iint_{\{\lambda=\mu\}} < \tilde{A}_0(y')z \circ dE_\lambda \tilde{A}_1(y)dE_\mu z > R(y,dy')\rho(dy), \tag{3.8}$$

$$B_{10}(z) = 2\int_Y \int_Y \iint_{\{\lambda=\mu\}} < dE_\lambda \tilde{A}_1(y')dE_\mu z \circ \tilde{A}_0(y)z > R(y,dy')\rho(dy), \tag{3.9}$$

$$B_{11}(z) = 2\int_Y \int_Y \iint_{\{\lambda'+\mu-\lambda-\mu'=0\}} < dE_{\lambda'} \tilde{A}_1(y')dE_{\mu'} z \circ dE_\lambda \tilde{A}_0(y)dE_\mu z > \times R(y,dy')\rho(dy). \tag{3.10}$$

*Here $< a \circ b >$ denotes the tensor product of vectors $a, b \in H$, namely for any $x \in H$ the following relation holds $< a \circ b > x = (a,x)b$.*

**Remark 3.1.** *For a function $\Phi : H \to R$, its derivative $\Phi'$ at point $z$ is defined in the following way: we consider $\Phi(z+tu)$, $t \in R$, $u \in H$ as a function of $t$ acting from*

*R to R. If the weak differential*

$$\mathcal{D}\Phi(z,u) = \frac{\partial}{\partial t}\Phi(z+tu)\bigg|_{t=0}$$

*depends on u linearly, then* $\mathcal{D}\Phi(z,u) = (\Phi'(z), u)$. *Vector* $\Phi'(z)$ *is called the weak derivative of* $\Phi$ *at point* $z$.

*To define the second derivative consider*

$$\frac{\partial^2}{\partial t_1 \partial t_2}\Phi(z + t_1u_1 + t_2u_2)\bigg|_{t_1=0, t_2=0}, \quad u_1, u_2 \in H,\ t_1, t_2 \in R.$$

*This expression (the second differential) is a bilinear function of* $u_1$, $u_2 \in H$, *and it defines an operator* $\Phi''(z)$ *acting from* $H$ *to* $H$, *this operator is called the second derivative of function* $\Phi$ *at point* $z$. *Its boundedness means that the bilinear function and the operator are bounded.*

*The third derivative is defined through*

$$\frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3}\Phi(z + t_1u_1 + t_2u_2 + t_3u_3)\bigg|_{t_1=0, t_2=0, t_3=0} = V(u_1, u_2, u_3),$$

$u_1, u_2, u_3 \in H$, $t_1, t_2, t_3 \in R$.

*The boundedness of the third derivative means that the trilinear form* $V$ *defined on* $H \times H \times H$ *is bounded, i.e.*

$$\sup_{||u_k|| \leq 1,\, k=1,2,3} |V(u_1, u_2, u_3)| = ||V|| < \infty.$$

**Remark 3.2.** *Under conditions IV and V, formulas (3.5) and (3.7)-(3.10) can be rewritten as follows:*

$$\tilde{a}(z) = D_0 z + \iint_{\{\lambda=\mu\}} \sum_{k,j=1}^{n} \Big[ (dE_\mu z, e_j)\hat{A}_{kj} dE_\lambda e_k +$$

$$dE_\lambda e_k(\hat{C}_{kj} z, dE_\mu e_j) + \hat{S}_{jk} dE_\lambda e_k (dE_\mu z, e_j) \Big],$$

$$B_{00}(z) = 2 \int_Y \int_Y \tilde{a}_0(y)\tilde{a}_0(y') R(y, dy')\rho(dy) < Dz \circ Dz >,$$

$$B_{01}(z) = 2 \sum_{k,j=1}^{n} \iint_{\{\lambda=\mu\}} (dE_\mu z, e_j) < \hat{A}_{kj} z \circ dE_\lambda e_k >,$$

$$B_{10}(z) = 2 \sum_{k,j=1}^{n} \iint_{\{\lambda=\mu\}} (dE_\mu z, e_j) < dE_\lambda e_k \circ \hat{C}_{kj} z >,$$

$$B_{11}(z) = 2 \sum_{k,j,l,m=1}^{n} \iiiint_{\{\lambda'+\mu-\lambda-\mu'=0\}} S_{kjlm}(dE_\mu z, e_j)(dE_{\mu'} z, e_m) \times$$

$$< dE_{\lambda'} e_l \circ dE_\lambda e_k >,$$

*where*

$$D_0 = \int_Y \int_Y \tilde{A}_0(y')\tilde{A}_0(y) R(y, dy')\rho(dy) =$$

$$- \int_Y \int_Y \tilde{a}_0(y')\tilde{a}_0(y) R(y, dy')\rho(dy) DD,$$

$$\hat{A}_{kj} = i \int_Y \int_Y \tilde{A}_0(y') S_{kj}(y) R(y, dy')\rho(dy) =$$

$$i \int_Y \int_Y \tilde{a}_0(y') S_{kj}(y) R(y, dy')\rho(dy) D,$$

$$\hat{C}_{kj} = i \int_Y \int_Y \tilde{A}_0(y) S_{kj}(y') R(y, dy')\rho(dy) =$$

$$i\int_Y\int_Y \tilde{a}_0(y)S_{kj}(y')R(y,dy')\rho(dy)D,$$

*where* $\tilde{a}_0(y) = a_0(y) - \bar{a}_0$,

$$S_{kjlm} = -\int_Y\int_Y S_{kj}(y)S_{lm}(y')R(y,dy')\rho(dy),$$

$$\hat{S}_{jk} = \sum_{l=1}^{n} S_{ljkl}.$$

**Remark 3.3.** *Operator $Q$ is chosen so that $Q\tilde{a}(z)$ and and $Q\tilde{B}(z)Q$ satisfy the assumptions of Proposition 2.1. Therefore $Q\tilde{a}(z)$ and $Q\tilde{B}(z)Q$ determine the transition probability of the process $\tilde{z}(t)$ as described in Proposition 2.1.*

## 3.4 Proofs

The proof of Theorem 3.1 follows from Theorem 3.2 formulated below and the Theorem 2.2 on weak convergence to a diffusion process.

**Theorem 3.2.** *Let $\Phi : H \to R$ have bounded $\Phi'$, $\Phi''$, $\Phi'''$, $D\Phi'$, $DD\Phi'$, $D\Phi''D$ and $\frac{\partial}{\partial z}(D\Phi''(z)Dx, x)$, and let assumptions I–V be fulfilled. Then for any $0 \le t_1 < t_2$*

$$E\big(\Phi(\tilde{z}_\epsilon(t_2)) - \Phi(\tilde{z}_\epsilon(t_1))/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\big) = E\left(\int_{t_1}^{t_2} L\Phi(\tilde{z}_\epsilon(\tau))d\tau \Big/ \mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\right) + o(1),$$

*where $\mathcal{F}^\epsilon_t$ is the $\sigma$-algebra generated by $\{y(s/\epsilon),\, s \le t\}$, the operator $L$ is given by (3.4) with the coefficients defined by formulas (3.5)–(3.10).*

The proof of Theorem 3.2 is based on the following lemmas:

**Lemma 3.1.** *The process $z_\epsilon$ is bounded, namely $\|z_\epsilon(t)\| = \|u_0\|$ for all $t > 0$.*

*Proof.* It is easy to see that $z_\epsilon(t)$ satisfies the following differential equation:

$$\begin{aligned} &\tfrac{d}{dt} z_\epsilon(t) = B\big(t, y(\tfrac{t}{\epsilon})\big) z_\epsilon(t), \quad t > 0 \\ &z_\epsilon(0) = u_0, \end{aligned} \tag{3.11}$$

where $B(t,y) = \bar{U}(-t)\tilde{A}(y)\bar{U}(t)$. Indeed,

$$\frac{d}{dt} z_\epsilon(t) = \frac{d}{dt}\bar{U}(-t)u_\epsilon(t) = -\bar{A}\bar{U}(-t)u_\epsilon(t) + \bar{U}(-t)A(y(t/\epsilon))u_\epsilon(t) =$$

$$\bar{U}(-t)(A(y(t/\epsilon)) - \bar{A})\bar{U}(t)z_\epsilon(t).$$

Note that $B^*(t,y) = -B(t,y)$ for all $t \geq 0$, $y \in Y$, where $B^*$ denotes the conjugate to $B$, and so $(B(t,y)z, z) = 0$ for all $t \geq 0$, $y \in Y$, $z \in H$. Therefore

$$(z_\epsilon(t), z_\epsilon(t)) - (z_\epsilon(0), z_\epsilon(0)) = 2\int_0^t (z_\epsilon(s), B(s, y(\frac{s}{\epsilon}))z_\epsilon(s))ds = 0.$$

□

Let $g$ be a measurable bounded function from $H$ to $R$. Consider the linear operators $\Pi$ and $R$ that act on a function $g$ as follows:

$$\Pi g(y) = \int_Y [g(y') - g(y)]\Pi(y, dy')$$

$$Rg(y) = \int_Y g(y')R(y, dy'),$$

where $\Pi(y, C)$ and $R(y, C)$, $y \in Y$, $C \in \mathcal{C}$ were defined in assumptions II and III respectively.

**Lemma 3.2.** *Let $g : Y \to R$ be measurable, bounded and satisfy $\int_Y g(y)\rho(dy) = 0$. Then $\Pi R g = -g$.*

*Proof.* Consider the semigroup of operators $\{T_t,\ t \geq 0\}$ generated by the transition probability of the Markov process $y(t)$:

$$T_t g(y) = \int_Y g(y') P(t, y, dy'),$$

then according to assumption II

$$\Pi g = \lim_{t \to 0} \frac{1}{t}(T_t g - g).$$

We calculate $\Pi R g$ under the condition that $\int_Y g(y)\rho(dy) = 0$. Note that $\int_0^\infty T_t g(y) dt = Rg(y)$ and therefore

$$\Pi R g = \lim_{h \to 0} \frac{1}{h}(T_h R g - R g) = \lim_{h \to 0} \frac{1}{h}\left(\int_0^\infty T_h T_t g(y) dt - \int_0^\infty T_t g(y) dt\right) =$$

$$\lim_{h \to 0} \frac{1}{h}\left(\int_h^\infty T_t g(y) dt - \int_0^\infty T_t g(y) dt\right) = \lim_{h \to 0} -\frac{1}{h}\int_0^h T_t g(y) dt = -g(y)$$

because $T_t \to I$ as $t \to 0$, where $I$ is the identity operator.

□

**Lemma 3.3.** *The process $(z_\epsilon(t), y(\frac{t}{\epsilon}))$ is a homogeneous Markov process in the phase space $H \times Y$ with generator $G_\epsilon$ determined on functions $f : H \times Y \to R$, which are* *measurable and bounded with bounded partial derivative in the first argument, by the relation:*

$$G_\epsilon(t)f(z,y) = \lim_{h\to 0}\frac{1}{h}\left[E\left(f(z_\epsilon(t+h), y(\frac{t+h}{\epsilon}))/z_\epsilon(t)=z, y(\frac{t}{\epsilon})=y\right) - f(z,y)\right] =$$

$$(f'_z(z,y), B(t,y)z) + \frac{1}{\epsilon}\Pi f(z,y) = (f'_z(z,y), B(t,y)z) + \frac{1}{\epsilon}\int_Y [f(z,y') - f(z,y)]\Pi(y,dy').$$

The proof of this lemma is the same as in the finite-dimensional case. The next lemma is a generalization of Dynkin's formula.

**Lemma 3.4.** *Let $f : R_+ \times H \times Y \to R$ be measurable and bounded with bounded partial derivatives in first and second arguments. Then for any $0 \le t_1 < t_2 < \infty$*

$$E\big(f(t_2, z_\epsilon(t_2), y(\frac{t_2}{\epsilon})) - f(t_1, z_\epsilon(t_1), y(\frac{t_1}{\epsilon}))/\mathcal{F}^\epsilon_{t_1}\big) =$$

$$E\left(\int_{t_1}^{t_2} [f'_s(s, z_\epsilon(s), y(\frac{s}{\epsilon})) + G_\epsilon(s) f(s, z_\epsilon(s), y(\frac{s}{\epsilon}))ds \Big/ \mathcal{F}^\epsilon_{t_1}\right).$$

**Lemma 3.5.** *Let conditions I–V be fulfilled. Let $\Phi$ be a bounded measurable function such that $\Phi'$ and $D^*\Phi'$ are bounded. Then for any $0 \le t_1 < t_2 < \infty$*

$$E\big(\Phi(z_\epsilon(t_2)) - \Phi(z_\epsilon(t_1))/\mathcal{F}^\epsilon_{t_1}\big) =$$

$$\epsilon E\Bigg(\int_{t_1}^{t_2}\int_Y \Big[\big(\Phi''(z_\epsilon(u))B\big(u, y(\frac{u}{\epsilon})\big)z_\epsilon(u), B(u,y')z_\epsilon(u)\big) +$$

$$+\big(\Phi'(z_\epsilon(u)), B(u,y')B\big(u, y(\frac{u}{\epsilon})\big)z_\epsilon(u)\big) + \big(\Phi'(z_\epsilon(u)), B_1(u,y')\bar{A}z - \bar{A}B_1(u,y')z_\epsilon(u)\big)\Big] \times$$

$$R\left(y(\frac{u}{\epsilon}), dy'\right)du \Big/ \mathcal{F}^{\epsilon}_{t_1}\Big) + O(\epsilon),$$

*where* $B_1(s,y) = \bar{U}(-s)\tilde{A}_1(y)\bar{U}(s)$ *and* $O(\epsilon)$ *does not depend on* $t_1$ *and* $t_2$.

*Proof.* Using the fact that $z_\epsilon$ satisfies the differential equation (3.11) we can write

$$\Phi(z_\epsilon(t_2)) - \Phi(z_\epsilon(t_1)) = \int_{t_1}^{t_2} g(s, z_\epsilon(s), y(\frac{s}{\epsilon}))ds$$

where $g(s,z,y) = (\Phi'(z), B(s,y)z)$. Set $f(s,z,y) = Rg(s,z,y)$, then since $\int_Y g(s,z,y)\rho(dy) = 0$ from Lemma 3.2 we have that $\Pi f(s,z,y) = -g(s,z,y)$ (here operator $\Pi$ acts on $g$ as function of $y$). From Lemma 3.4 we have

$$E\Big(\int_{t_1}^{t_2} g(s, z_\epsilon(s), y(\frac{s}{\epsilon}))ds \Big/ \mathcal{F}^{\epsilon}_{t_1}\Big) = -\epsilon\big(f(t_2, z_\epsilon(t_2), y(\frac{t_2}{\epsilon})) - f(t_1, z_\epsilon(t_1), y(\frac{t_1}{\epsilon}))\big) +$$

$$\epsilon E\Big(\int_{t_1}^{t_2}\Big[(f'_z(s, z_\epsilon(s), y(\frac{s}{\epsilon})), B(s, y(\frac{s}{\epsilon}))z_\epsilon(s)) + f'_s(s, z_\epsilon(s), y(\frac{s}{\epsilon}))\Big]ds \Big/ \mathcal{F}^{\epsilon}_{t_1}\Big) =$$

$$\epsilon E\Big(\int_{t_1}^{t_2}\int_Y \Big[\big(\Phi''(z_\epsilon(u))B\big(u, y(\frac{u}{\epsilon})\big)z_\epsilon(u), B(u,y')z_\epsilon(u)\big) +$$

$$+ \big(\Phi'(z_\epsilon(u)), B(u,y')B\big(u, y(\frac{u}{\epsilon})\big)z_\epsilon(u)\big) + \big(\Phi'(z_\epsilon(u)), B_1(u,y')\bar{A}z - \bar{A}B_1(u,y')z_\epsilon(u)\big)\Big] \times$$

$$R\big(y(\frac{u}{\epsilon}), dy'\big)du \Big/ \mathcal{F}^{\epsilon}_{t_1}\Big) + O(\epsilon)$$

since under the assumptions of the lemma function $f$ is bounded.

□

**Lemma 3.6.** *Suppose that conditions I–V are fulfilled. Then for a function* $\Phi : H \to R$ *such that* $\Phi'$, $\Phi''$, $\Phi'''$ $D^*\Phi'$, $D^*D^*\Phi'$, $D^*\Phi''D$ *and* $\frac{\partial}{\partial z}(D^*\Phi''(z)Dx, x)$ *are* *bounded, the following representation is valid:*

$$E\Big(\Phi\big(z_\epsilon(\frac{t_2}{\epsilon})\big)-\Phi\big(z_\epsilon(\frac{t_1}{\epsilon})\big)\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\Big)=E\big(\Phi(\tilde z_\epsilon(t_2))-\Phi(\tilde z_\epsilon(t_1))/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\big)=$$

$$E\Big(\int_{t_1}^{t_2}\bar K\big(\frac{\tau}{\epsilon},\tilde z_\epsilon(\tau)\big)d\tau\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\Big)+O(\epsilon),$$

*where*

$$\bar K(\tau,z)=\bar K_1(\tau,a)+\bar K_2(\tau,z),$$

$$\bar K_1(\tau,z)=\int_Y\int_Y\big(\Phi''(z)B(\tau,y)z,B(\tau,y')z\big)R(y,dy')\rho(dy),$$

$$\bar K_2(\tau,z)=\int_Y\int_Y\big(\Phi'(z),B(\tau,y')B(\tau,y)z\big)R(y,dy')\rho(dy).$$

*Proof.* Set

$$K(u,z,y)=\int_Y\Big[\big(\Phi''(z)B(u,y)z,B(u,y')z\big)+$$

$$\big(\Phi'(z),B(u,y')B(u,y)z\big)+\big(\Phi'(z),B_1(u,y')\bar Az-\bar AB_1(u,y')z\big)\Big]R(y,dy'),$$

$$\bar K(u,z)=\int_Y K(u,y,z)\rho(dy).$$

Then we can rewrite the statement of the previous lemma as follows:

$$E\big(\Phi(\tilde z_\epsilon(t_2))-\Phi(\tilde z_\epsilon(t_1))/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\big)=\epsilon E\Big(\int_{\frac{t_1}{\epsilon}}^{\frac{t_2}{\epsilon}}K\big(s,z_\epsilon(s),y(\frac{s}{\epsilon})\big)ds\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\Big)+O(\epsilon).$$

Denote $\tilde K(s,z,y)=K(s,z,y)-\bar K(s,z)$. We need to prove that

$$\epsilon E\Big(\int_{\frac{t_1}{\epsilon}}^{\frac{t_2}{\epsilon}}\tilde K\big(s,z_\epsilon(s),y(\frac{s}{\epsilon})\big)ds\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\Big)\to 0 \text{ as } \epsilon\to 0.$$

Since $\int_Y \tilde{K}(s,z,y)\rho(dy) = 0$ for all $s \geq 0$, $z \in H$, $\Pi R\tilde{K} = -\tilde{K}$ by Lemma 3.2 (the operators $\Pi$ and $R$ act on $\tilde{K}$ as a function of its last argument, $y$).

Set $f(s,z,y) = R\tilde{K}(s,z,y)$. Recall that $B(s,y) = a_0(y)D + B_1(s,y)$, and $B_1$ is a bounded operator: $B_1(s,y) = \bar{U}(-s)\tilde{A}_1(y)\bar{U}(s)$. Under assumptions of the lemma, $\tilde{K}$ has bounded partial derivatives in first and second arguments since $z_\epsilon$ is bounded (Lemma 3.1):

$$K'_s(s,z,\hat{y}) = \int_Y \Big[(\Phi''(z)(B_1(s,\hat{y})\bar{A} - \bar{A}B_1(s,\hat{y})z, B(s,y')z)+$$

$$(\Phi''(z)B(s,\hat{y})z, (B_1(s,\hat{y})\bar{A} - \bar{A}B_1(s,\hat{y})z)+$$

$$(\Phi'(z), B(s,y')((B_1(s,\hat{y})\bar{A} - \bar{A}B_1(s,\hat{y})z) + (\Phi'(z), (B_1(s,y')\bar{A} - \bar{A}B_1(s,y')z)B(s,\hat{y})z)+$$

$$(\Phi'(z), B_1(s,y'\bar{A}\bar{A}z - 2\bar{A}B_1(s,y')\bar{A}z + \bar{A}\bar{A}B((s,y')z)\Big] R(\hat{y},dy'),$$

$$K'_z(s,z,y) = \int_Y B^*(s,y')B(s,y)z\Phi'(z) + \Phi''(z)B(s,y')B(s,y)+$$

$$[\bar{A}z - \bar{A}B_1(s,y')]^*\Phi'(z) + \Phi''(z)[\bar{A}z - \bar{A}B_1(s,y')]z+$$

$$B(s,y')^*\Phi''(z)B(s,y)z + B(s,y)^*\Phi''(z)B(s,y') + \{\frac{\partial}{\partial z}(\Phi''(z)B(s,y)x, B(s,y')x)\}|_{x=z}.$$

From Lemma 3.4 we have

$$E\Big(\int_{t_1}^{t_2} \tilde{K}(s, z_\epsilon(s), y(\frac{s}{\epsilon}))ds \Big/ \mathcal{F}^\epsilon_{t_1}\Big) = O(\epsilon)+$$

$$\epsilon E\Big(\int_{t_1}^{t_2} \Big[(f'_z(s, z_\epsilon(s), y(\frac{s}{\epsilon})), B(s, y(\frac{s}{\epsilon}))z_\epsilon(s)) + f'_s(s, z_\epsilon(s), y(\frac{s}{\epsilon}))\Big] ds \Big/ \mathcal{F}^\epsilon_{t_1}\Big).$$

So

$$E\Big(\int_{\frac{t_1}{\epsilon}}^{\frac{t_2}{\epsilon}} \tilde{K}(s, z_\epsilon(s), y(\frac{s}{\epsilon}))ds \Big/ \mathcal{F}^\epsilon_{t_1}\Big) = O(\epsilon) + O(t_2 - t_1).$$

□

**Lemma 3.7.** *Under the assumptions of the previous lemma there exists the limit*

$$lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{K}(\tau, z)d\tau = \tilde{K}(z),$$

$\tilde{K}(z) = \tilde{K}_1(z) + \tilde{K}_2(z)$, *and*

$$\tilde{K}_1(z) = (\mathcal{A}_0(\Phi''(z))z, z) + \iint_{\{\lambda=\mu\}} (\mathcal{A}_1(\Phi''(z)dE_\lambda)dE_\mu z, z) +$$

$$\iint_{\{\lambda=\mu\}} (\mathcal{A}_2(dE_\lambda \Phi''(z))z, dE_\mu z) +$$

$$\iiiint_{\{\lambda'+\mu-\lambda-\mu'=0\}} (\mathcal{A}_3(dE_{\lambda'}\Phi''(z)dE_\lambda)dE_\mu z, dE_{\mu'} z), \tag{3.12}$$

$$\tilde{K}_2(z) = \iint_{\{\lambda=\mu\}} \int_Y \int_Y (\Phi'(z), dE_\lambda \tilde{A}(y)\tilde{A}(y')dE_\mu z)R(y, dy')\rho(dy) =$$

$$(\Phi'(z), D_0 z) + \iint_{\{\lambda=\mu\}} \sum_{k,j=1}^{n} \Big[(dE_\mu z, e_j)(\Phi'(z), \hat{A}_{kj} dE_\lambda e_k) +$$

$$(\Phi'(z), dE_\lambda e_k)(\hat{C}_{kj} z, dE_\mu e_j) + \hat{S}_{jk}(\Phi'(z), dE_\lambda e_k)(dE_\mu z, e_j)\Big], \tag{3.13}$$

*where for a linear operator $C$ from $H$ to $H$*

$$\mathcal{A}_0(C) = \int_Y \int_Y \tilde{A}_0(y')^* C \tilde{A}_0(y) R(y, dy')\rho(dy), \tag{3.14}$$

$$\mathcal{A}_1(C) = \int_Y \int_Y \tilde{A}_0(y')^* C \tilde{A}_1(y) R(y, dy') \rho(dy), \tag{3.15}$$

$$\mathcal{A}_2(C) = \int_Y \int_Y \tilde{A}_1(y')^* C \tilde{A}_0(y) R(y, dy') \rho(dy), \tag{3.16}$$

$$\mathcal{A}_3(C) = \int_Y \int_Y \tilde{A}_1(y')^* C \tilde{A}_1(y) R(y, dy') \rho(dy). \tag{3.17}$$

*and*

$$D_0 = \int_Y \int_Y \tilde{A}_0(y') \tilde{A}_0(y) R(y, dy') \rho(dy).$$

*The operators $\hat{A}_{kj}$, $\hat{C}_{kj}$, $k, j = 1, \ldots, n$ and coefficients $S_{kjlm}$, $\hat{S}_{jk}$, $k, j, l, m = 1, \ldots n$ are defined in Remark 3.2.*

**Remark 3.4.** *Assumption V ensures that the integrals in formulas (3.12) and (3.13) are well-defined:*

$$(\mathcal{A}_1(\Phi''(z) dE_\lambda) dE_\mu z, z) = \sum_{k,j=1}^{n} (\Phi''(z) dE_\lambda e_k, \hat{A}_{kj} z)(dE_\mu z, e_j), \tag{3.18}$$

$$\mathcal{A}_2(dE_{\lambda'} \Phi''(z) z, dE_{\mu'} z) = \sum_{k,j=1}^{n} (dE_{\mu'} z, e_j)(\Phi''(z) \hat{C}_{kj} z, dE_{\lambda'} e_k), \tag{3.19}$$

$$\mathcal{A}_3(dE_{\lambda'} \Phi''(z) dE_\lambda) dE_\mu z, dE_{\mu'} z) = \sum_{k,j,l,m=1}^{n} S_{kjlm} (dE_\mu z, e_j)(dE_{\mu'} z, e_m) \times$$

$$(\Phi''(z) dE_\lambda e_k, dE_{\lambda'} e_l), \tag{3.20}$$

*since for any vectors $x_1, x_2 \in H$ the expression $(E_\lambda x_1, x_2)$ treated as a function of $\lambda$, has bounded variation on $R$.*

*Proof.* Recall that

$$\bar{K}_1(\tau, z) = \int_Y \int_Y \big(\Phi''(z)B(\tau, y)z, B(\tau, y')z\big)R(y, dy')\rho(dy),$$

and $B(\tau, y) = \bar{U}(-\tau)\tilde{A}(y)\bar{U}(\tau)$. Using the spectral decomposition for $\bar{U}(\tau)$ we can write

$$\bar{K}_1(\tau, z) = \int_Y \int_Y (\Phi''(z) \int e^{-i\tau\lambda} dE_\lambda \tilde{A}(y) \int e^{i\tau\mu} dE_\mu z,$$
$$\int e^{-i\tau\lambda'} dE_{\lambda'} \tilde{A}(y') \int e^{i\tau\mu'} dE_{\mu'} z\big) R(y, dy')\rho(dy) =$$
$$\int_Y \int_Y \iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')} \big(\tilde{A}(y')^* dE_{\lambda'} \Phi''(z) dE_\lambda \tilde{A}(y) dE_\mu z, dE_{\mu'} z\big).$$

For an operator $C$ set

$$\mathcal{A}(C) = \iint \tilde{A}(y')^* C \tilde{A}(y) R(y, dy')\rho(dy),$$

then

$$\bar{K}_1(\tau, z) = \iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')} \big(\mathcal{A}(dE_{\lambda'} \Phi''(z) dE_\lambda) dE_\mu z, dE_{\mu'} z\big).$$

Under assumption IV $\tilde{A}(y) = \tilde{A}_0(y) + \tilde{A}_1(y)$, and therefore $\mathcal{A}(C)$ can be written in the form

$$\mathcal{A}(C) = \mathcal{A}_0(C) + \mathcal{A}_1(C) + \mathcal{A}_2(C) + \mathcal{A}_3(C),$$

where $\mathcal{A}_i(C)$, $i = 0, 1, 2, 3$ are given by formulas (3.14)–(3.17) and satisfy the following properties:

1) $E_\lambda \mathcal{A}_0(C) E_\mu = \mathcal{A}_0(E_\lambda C E_\mu)$,

2) $E_\lambda \mathcal{A}_1(C) = \mathcal{A}_1(E_\lambda C)$,

3) $\mathcal{A}_2(C)E_\mu = \mathcal{A}_2(CE_\mu)$.

Using these properties we rewrite $\bar{K}_1(\tau, z)$ as follows:

$$\bar{K}_1(\tau, z) = \iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')}\big(\mathcal{A}_0(\Phi''(z))dE_\lambda dE_\mu z, dE_{\lambda'}dE_{\mu'}z\big)+$$

$$\iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')}\big(\mathcal{A}_1(\Phi''(z)dE_\lambda)dE_\mu z, dE_{\lambda'}dE_{\mu'}z\big)+$$

$$\iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')}\big(\mathcal{A}_2(dE_{\lambda'}\Phi''(z))dE_\lambda dE_\mu z, dE_{\mu'}z\big)+$$

$$\iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')}\big(\mathcal{A}_3(dE_{\lambda'}\Phi''(z)dE_\lambda)dE_\mu z, dE_{\mu'}z\big) =$$

$$\iint \big(\mathcal{A}_0(\Phi''(z))dE_\lambda z, dE_{\lambda'}z\big)+$$

$$\iiint e^{i\tau(\mu-\lambda)}\big(\mathcal{A}_1(\Phi''(z)dE_\lambda)dE_\mu z, dE_{\lambda'}z\big)+$$

$$\iiint e^{i\tau(\lambda'-\mu')}\big(\mathcal{A}_2(dE_{\lambda'}\Phi''(z))dE_\lambda z, dE_{\mu'}z\big)+$$

$$\iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')}\big(\mathcal{A}_3(dE_{\lambda'}\Phi''(z)dE_\lambda)dE_\mu z, dE_{\mu'}z\big) =$$

$$(\mathcal{A}_0(\Phi''(z))z, z) + \iint e^{i\tau(\mu-\lambda)}\big(\mathcal{A}_1(\Phi''(z)dE_\lambda)dE_\mu z, z\big)+$$

$$\iint e^{i\tau(\lambda'-\mu')}\big(\mathcal{A}_2(dE_{\lambda'}\Phi''(z))z, dE_{\mu'}z\big)+$$

$$\iiiint e^{i\tau(\lambda'+\mu-\lambda-\mu')}\big(\mathcal{A}_3(dE_{\lambda'}\Phi''(z)dE_\lambda)dE_\mu z, dE_{\mu'}z\big).$$

Now we average with respect to $\tau$, that is, compute

$$\frac{1}{T}\int_0^T \bar{K}_1(\tau,z)d\tau.$$

Note that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\tau(\mu-\lambda)}d\tau = 0 \quad \text{if } \mu\neq\lambda \text{ and it equals } 1 \text{ otherwise.}$$

Therefore

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{K}(\tau,z)d\tau = (\mathcal{A}_0(\Phi''(z))z,z)+$$

$$\iint_{\{\lambda=\mu\}}(\mathcal{A}_1(\Phi''(z)dE_\lambda)dE_\mu z,z) + \iint_{\{\lambda=\mu\}}(\mathcal{A}_2(dE_\lambda\Phi''(z))z,dE_\mu z)+$$

$$\iiiint_{\{\lambda'+\mu-\lambda-\mu'=0\}}(\mathcal{A}_3(dE_{\lambda'}\Phi''(z)dE_\lambda)dE_\mu z,dE_{\mu'}z).$$

Formula (3.12) is proven.

Formula (3.13) is obtained similarly:

$$\bar{K}_2(\tau,z) = \int_Y\int_Y(\Phi'(z),B(\tau,y')B(\tau,y)z)R(y,dy')\rho(dy) =$$

$$\int_Y\int_Y\iint\left(\Phi'(z),e^{-i\tau\lambda}dE_\lambda\tilde{A}(y')\tilde{A}(y)e^{i\tau\mu}dE_\mu z\right)R(y,dy')\rho(dy) =$$

$$\int_Y\int_Y(\Phi'(z),\tilde{A}_0(y')\tilde{A}_0(y)z)R(y,dy')\rho(dy)+$$

$$\int_Y\int_Y\iint e^{i\tau(\lambda-\mu)}(\Phi'(z),\tilde{A}_0(y')dE_\lambda\tilde{A}_1(y)dE_\mu z)R(y,dy')\rho(dy)+$$

$$\int_Y\int_Y\iint e^{i\tau(\lambda-\mu)}(\Phi'(z),dE_\lambda\tilde{A}_1(y')dE_\mu\tilde{A}_0(y)z)R(y,dy')\rho(dy)+$$

$$\int_Y \int_Y \iint e^{i\tau(\lambda-\mu)} (\Phi'(z), dE_\lambda \tilde{A}_1(y')\tilde{A}_1(y) dE_\mu z) R(y, dy')\rho(dy) =$$

$$(\Phi'(z), D_0 z) + \int_Y \int_Y \iint e^{i\tau(\lambda-\mu)} \Big[ i \sum_{k,j=1}^{n} (\tilde{A}_0^*(y')\Phi'(z), S_{kj}(y)(dE_\mu z, e_j) dE_\lambda e_k) +$$

$$i \sum_{k,j=1}^{n} (\Phi'(z), S_{kj}(y)(dE_\mu \tilde{A}_0(y)z, e_j) dE_\lambda e_k) +$$

$$i \sum_{k,j=1}^{n} (\Phi'(z), S_{kj}(y)(dE_\mu z, e_j) dE_\lambda \tilde{A}_1(y') e_k) \Big] R(y, dy')\rho(dy) =$$

$$(\Phi'(z), D_0 z) + \int_Y \int_Y \iint e^{i\tau(\lambda-\mu)} \Big[ i \sum_{k,j=1}^{n} (dE_\mu z, e_j)(\Phi'(z), \tilde{A}_0(y') S_{kj}(y) dE_\lambda e_k) +$$

$$i \sum_{k,j=1}^{n} (\tilde{A}_0(y)z, e_j)(\Phi'(z)(z), S_{kj}(y') dE_\lambda e_k) -$$

$$\sum_{k,j,l,m=1}^{n} S_{kj}(y)(dE_\mu z, e_j) S_{lm}(y')(e_k, e_m)(\Phi'(z), dE_\lambda e_l) \Big] R(y, dy')\rho(dy).$$

Taking into account the definitions of $\hat{A}_{kj}$, $\hat{C}_{kj}$ and $S_{kjlm}$ given in Remark 3.2 we have

$$\bar{K}_2(\tau, z) = (\Phi'(z), D_0 z) + \int_Y \int_Y \iint e^{i\tau(\lambda-\mu)} \sum_{kj=1}^{n} \Big[ (dE_\mu z, e_j)(\Phi'(z), \hat{A}_{kj} dE_\lambda e_k) +$$

$$(\Phi'(z), dE_\lambda e_k)(\hat{C}_{kj} z, dE_\mu e_j) + \hat{S}_{jk}(dE_\mu z, e_j)(\Phi'(z), dE_\lambda e_k) \Big] R(y, dy')\rho(dy).$$

Averaging with respect to $\tau$, we get formula (3.13).

□

**Lemma 3.8.** *Suppose that the assumptions of Lemma 3.7 are fulfilled. Then*

$$E\big(\Phi(\tilde{z}_\epsilon(t_2)) - \Phi(\tilde{z}_\epsilon(t_1))/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\big) = E\Big(\int_{t_1}^{t_2} \tilde{K}(\tilde{z}_\epsilon(\tau)) d\tau / \mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\Big) + o(1),$$

*where* $\tilde{K}(\cdot)$ *was introduced in the previous Lemma.*

*Proof.* We need to show that

$$\limsup_{\epsilon\to 0}\left|E\left(\int_{t_1}^{t_2}[\bar{K}(\frac{s}{\epsilon},\tilde{z}_\epsilon(s))-\tilde{K}(\tilde{z}_\epsilon(s)]ds\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\right)\right|=0.$$

For an $h>0$ consider

$$E_{\frac{t}{\epsilon},z,y}\int_t^{t+h}[\bar{K}(\frac{s}{\epsilon},\tilde{z}_\epsilon(s))-\bar{K}(\tilde{z}_\epsilon(s))]ds=E_{\frac{t}{\epsilon},z,y}\int_t^{t+h}[\bar{K}(\frac{s}{\epsilon},z)-\tilde{K}(z)]ds+$$

$$E_{\frac{t}{\epsilon},z,y}\int_t^{t+h}[(\bar{K}(\frac{s}{\epsilon},\tilde{z}_\epsilon(s))-\bar{K}(\tilde{z}_\epsilon(s)))-(\bar{K}(\frac{s}{\epsilon},z)-\tilde{K}(z))]ds. \tag{3.21}$$

From Lemma 3.7

$$\frac{1}{h}\int_t^{t+h}[\bar{K}(\frac{s}{\epsilon},z)-\tilde{K}(z)]ds=$$

$$\frac{\epsilon}{h}\int_{\frac{t}{\epsilon}}^{\frac{t+h}{\epsilon}}[\bar{K}(s,z)-\tilde{K}(z)]ds\to 0\quad\text{as } \epsilon\to 0 \text{ for any } h>0. \tag{3.22}$$

Fix $s>0$ and consider $\psi(z)=\bar{K}(\frac{s}{\epsilon},z)-\tilde{K}(z)$ as a function of $z$. The function $\psi$ satisfies the assumptions of Lemma 3.5, and thus we have that

$$E\big(\psi(\tilde{z}_\epsilon(s))-\psi(\tilde{z}_\epsilon(t))/\mathcal{F}^\epsilon_{\frac{t}{\epsilon}}\big)=E\big((\psi'_z(\theta),\tilde{z}_\epsilon(s))-(\psi'_z(\theta),\tilde{z}_\epsilon(t))/\mathcal{F}^\epsilon_{\frac{t}{\epsilon}}\big)=O(\epsilon+h).$$

Therefore for some constant $c_1>0$

$$P_{\frac{t}{\epsilon},z,y}\{|(\psi'_z(\theta),\tilde{z}_\epsilon(s)-\tilde{z}_\epsilon(t))|>\delta\}\le\frac{c_1}{\delta}(\epsilon+h).$$

Thus

$$\left|E_{\frac{t}{\epsilon},z,y}\int_t^{t+h}\left[(\bar{K}(\frac{s}{\epsilon},\tilde{z}_\epsilon(s))-\bar{K}(\tilde{z}_\epsilon(s)))-(\bar{K}(\frac{s}{\epsilon},z)-\tilde{K}(z))\right]ds\right|\leq$$

$$h(\delta+\frac{c_1}{\delta}(\epsilon+h)).$$

The last inequality together with formulas (3.21) and (3.22) imply that

$$\limsup_{\epsilon\to 0} E_{\frac{t}{\epsilon},z,y}\frac{1}{h}\left|\int_t^{t+h}\left[\bar{K}(\frac{s}{\epsilon},\tilde{z}_\epsilon(s))-\bar{K}(\tilde{z}_\epsilon(s))\right]ds\right|\leq\delta+\frac{c_1}{\delta}h$$

for any $h>0$, $\delta>0$.

Set $t_k=t_1+kh$, $k=0,1,2,\ldots,n$, then

$$\limsup_{\epsilon\to 0}\left|E\left(\int_{t_1}^{t_2}\left[\bar{K}(\frac{s}{\epsilon}\tilde{z}_\epsilon(s))-\tilde{K}(\tilde{z}_\epsilon(s)\right]ds\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\right)\right|=$$

$$\limsup_{\epsilon\to 0}\left|E\left(h\sum_{k=0}^{n-1}\frac{1}{h}\int_{t_k}^{t_{k+1}}\left[\bar{K}(\frac{s}{\epsilon}\tilde{z}_\epsilon(s))-\tilde{K}(\tilde{z}_\epsilon(s)\right]ds\Big/\mathcal{F}^\epsilon_{\frac{t_1}{\epsilon}}\right)\right|\leq(t_2-t_1)(\delta+\frac{c_1}{\delta}h)$$

for any $\delta>0$, $h>0$. The proof is completed by letting $h\to 0$, then $\delta\to 0$.

□

Lemma 3.8 implies the statement of Theorem 3.2 if we note that for any vectors $a,b\in H$ and linear operator $C$ from $H$ to $H$

$$(Ca,b)=TrC<b\circ a>,$$

and use formulas (3.12)–(3.17).

To complete the proof of Theorem 3.1 first we observe that the set of functions $\Phi$

that satisfy the assumptions of Theorem 3.2 is dense in $C^{(2)}(H)$. Therefore the first part of condition 3) is satisfied.

Unfortunately, the drift $\tilde{a}$ and the diffusion operator $\tilde{B}$ do not necessarily satisfy the conditions of Proposition 2.1, and therefore do not necessarily define a diffusion process. Therefore we consider the process $Q\tilde{z}_\epsilon(t)$, where $Q$ is a compact positive operator, and we apply Theorem 3.2 to this process. Its generator

$$L\Phi(Qz) = (\Phi'(Qz), Q\tilde{a}(z)) + \frac{1}{2}TrQ\Phi''(Qz)Q\tilde{B}(z) =$$

$$(\Phi'(Qz), Q\tilde{a}(z)) + \frac{1}{2}Tr\Phi''(Qz)Q\tilde{B}(z)Q.$$

Note that $\tilde{a}(z)$ is linear in $z$, since it has the structure $F_1 z$, where $F_1$ is a linear operator. The diffusion operator $\tilde{B}(z)$ is bilinear in $z$ and has the structure $< F_2 z \circ z >$, where $F_2$ is a linear operator. Thus

$$Q\tilde{a}(z) = QF_1 z, \quad TrQ\tilde{B}(z)Q = TrQ < F_2 z \circ z > Q = (z, QF_2 Qz).$$

We choose $Q$ so that $QF_1$ and $QF_2Q$ are bounded. Then $Q\tilde{a}(z)$ and $Q\tilde{B}(z)Q$ satisfy the condition of Proposition 2.1 and therefore define a diffusion process.

## 3.5 Example

Let $H$ be $L_2(R^3)$, that is the space of complex-valued functions f such that

$$\int_{R^3} |f(x)|^2 dx < \infty, \quad x = (x_1, x_2, x_3).$$

Let $\Delta$ be the Laplace operator. We consider the equation

$$\frac{\partial u_\epsilon(t,x)}{\partial t} = -ia_0(\theta(\frac{t}{\epsilon}))\Delta + A_1(\theta(\frac{t}{\epsilon}))u_\epsilon(t,x), \tag{3.23}$$

where $\theta(t)$ is a Markov process in phase space $\Theta$ that has ergodic distribution $\rho$ and satisfies conditions II and III of section 3.2, the function $a_0$ satisfies condition IV, and $\{A_1(\theta),\ \theta \in \Theta\}$ is a family of finite-dimensional operators. For example, let $A_1(\theta)$ be an integral operator with the kernel

$$K_1(x,y,\theta) = i\sum_{k=1}^{n} a_k(x)b_k(y,\theta),$$

where $a_k$, $b_k$, $k = 1, 2, \ldots, n$ are real-valued functions and

$$A_1(\theta)u(t,x) = i\sum_{k=1}^{n} a_k(x)\int_{R^3} b_k(y,\theta)u(t,y)dy.$$

Since the range of operator $A_1(\theta)$ is contained in the span of functions $a_1, \ldots, a_n$, this operator is finite-dimensional. Hence the condition V of section 3.2 is fulfilled.

For convenience assume that

$$\int_{\Theta} a_0(\theta)\rho(d\theta) = 1.$$

Also assume that $\int_{\Theta} b_k(y,\theta)\rho(d\theta) = 0$ for all $y \in R^3$ and $k = 1, 2, \ldots, n$. The last assumption ensures that condition IV of section 3.2 is satisfied: the operator $\bar{A} = -i\Delta$ commutes with $A_0(\theta) = -ia_0(\theta)\Delta$.

The operator $S$ introduced in section 3.2 equals $-\Delta$, it has the spectrum $[0, \infty)$. Its Green function (resolvent kernel) is

$$\Gamma_z(x, y) = \frac{1}{4\pi} \frac{e^{i\sqrt{z}|x-y|}}{|x-y|},$$

$x, y \in R^3$, $x \neq y$, $z \notin [0, \infty)$. The resolvent $R_z$ satisfies

$$(R_z u, v) = \int_{R^3} \int_{R^3} \Gamma_z(x, y) u(y) \overline{v(x)} dx dy.$$

Denote by $E$ the resolution of identity of operator $S$. Using the expression for the Green function and the fact that $E(\{a\}) = 0$ for all $a \in R$, it can be shown that the resolution of identity for the operator $S$ satisfies

$$(E(C)u, v) = \int_{R^3} \int_{R^3} \left( \int_C \frac{1}{4\pi^2} \frac{\sin(\sqrt{\lambda}|x-y|)}{|x-y|} d\lambda \right) u(y) \overline{v(x)} dx dy. \qquad (3.24)$$

Here $C \subset [0, \infty)$, $u, v \in C^{(}2)(R^3)$ and have bounded support.

Consider $z_\epsilon(t, x) = e^{-t\bar{A}} u_\epsilon(t, x)$. Let $e_j, j = 1, 2, \ldots$ be an orthonormal base in $L_2$.

The Fourier coefficients of $z_\epsilon(t,x)$ can be computed as follows:

$$(z_\epsilon(t), e_j) = z_{\epsilon,j}(t) = (e^{it\Delta}u(t), e_j) =$$

$$\int_0^\infty e^{-it\lambda}(dE_\lambda u_\epsilon(t), e_j) = \int_0^\infty e^{-it\lambda}\int_{R^3}\int_{R_3}\frac{1}{4\pi^2}\frac{\sin(\sqrt{\lambda}|x-y|)}{|x-y|}u(t,y)\overline{e_j(x)}d\lambda dx dy.$$

According to the Theorem 3.1, there exists a positive compact operator $Q$ such that the stochastic process $Qz_\epsilon(t/\epsilon)$ converges weakly to the diffusion process that is determined by its generator. Let us compute the diffusion coefficients using formulas given in Remark 3.2:

$$\tilde{a}(z) = c\Delta\Delta z, \quad c = -\int_\Theta\int_\Theta (a_0(\theta') - 1)(a_0(\theta) - 1)R(\theta, d\theta')\rho(d\theta),$$

$$\tilde{B}(z) = B_{00}(z) = 2c < \Delta z \circ \Delta z >,$$

here

$$< \Delta z \circ \Delta z > f(x) = (\Delta z, f)\Delta z(x) = \int_{R^3}\Delta z(y)\overline{f(y)}dy\Delta z(x).$$

**Remark 3.5.** *According to Remark 3.2 in the case of equation (3.23)* $B_{01}(z) = B_{10}(z) = B_{11}(z) = 0$ *and* $\tilde{a}(z)$, $\tilde{B}(z) = B_{00}(z)$ *do not depend on the operators* $A_1(\theta)$, *since* $E(\{a\}) = 0$ *for all* $a \in R$. *So the limit process will be the same for all equations of the form (3.23) as long as operators* $A(\theta)$ *are finite-dimensional and the operator* $\bar{A}$ *commutes with* $\Delta$.

Now let us choose an operator $Q$. It has to be chosen so that $Q\Delta\Delta$ is a bounded operator. We can do it in the following way: we try to find a kernel $K$ such that for

$u \in L_2$

$$Qu(x) = \int_{R^3} K(x,y)u(y)dy.$$

Since $Q$ has to be a compact operator, the kernel has to be in $L_2(R^3 \times R^3)$, also it has to be symmetric and nonnegative, and the following relation must hold:

$$\|Q\Delta\Delta u\| \leq c_1 \|u\| \tag{3.25}$$

for some constant $c_1$. If the equation (3.23) is considered in a bounded region $\alpha \subset R^3$, with the boundary $\partial\alpha$ ( and this is usually the case for the equations of such form), then we set $K$ and $\Delta K$ together with their normal derivatives equal to 0 outside of $\alpha$ and on its boundary. Then according to the Green formula for the Laplace operator we obtain

$$Q\Delta\Delta u(x) = \int_\alpha K(x,y)\Delta\Delta u(y)dy =$$

$$\int_\alpha \Delta_y K(x,y)\Delta u(y)dy + \int_{\partial\alpha}\left(\frac{\partial\Delta u}{\partial n}K - \Delta u\frac{\partial K}{\partial n}\right)ds =$$

$$\int_\alpha \Delta_y K(x,y)\Delta u(y)dy = \int_\alpha \Delta_y\Delta_y K(x,y)u(y)dy,$$

where subscript $y$ indicates that operator $\Delta$ acts on $K$ as a function of $y$, $ds$ is an element of the boundary, and $\frac{\partial}{\partial n}$ denotes derivative in the normal direction to the boundary. If $\Delta\Delta K$ is bounded, then formula (3.25) is valid.

If equation (3.23) is considered in an unbounded region, then $K$ and its derivatives up to the third order have to decrease rapidly at infinity.

**Remark 3.6.** *The example considered above can be generalized to the case of a system* *of partial differential equations. Let $H$ be $(L_2(R^3))^r$, that is the space of functions $u: R^3 \to C^r$, $u = (u_1, u_2, \dots u_r)'$ such that*

$$\int_{R^3} |u(x)|^2 dx < \infty,$$

*here $x = (x_1, x_2, x_3)$, $|u| = \sqrt{|u_1|^2 + |u_2|^2 + \dots |u_r|^2}$ is the Euclidian norm. We consider a system of partial differential equations*

$$\frac{\partial u_{\epsilon,k}(t,x)}{\partial t} = -ia_k^{(0)}(\theta(\frac{t}{\epsilon}))\Delta u_{\epsilon,k}(t,x) + i\sum_{j=1}^{r} a_{jk}^{(1)}(x)\int_{R^3} b_{jk}(y,\theta(\frac{t}{\epsilon}))u_{\epsilon,j}(t,y)dy.$$

*or in vector notation*

$$\frac{\partial u_\epsilon(t,x)}{\partial t} = -ia^{(0)}(\theta(\frac{t}{\epsilon}))\Delta u_\epsilon(t,x) + A^{(1)}(\theta(\frac{t}{\epsilon}))u_\epsilon(t,x),$$

*here $a^{(0)} = (a_1^{(0)}, \dots, a_r^{(0)})'$ is a vector-column. We assume that $\int_\Theta b_{jk}(y,\theta)\rho(d\theta) = 0$ for all $j,k = 1,2,\dots r$ and $y \in R^3$. Then condition IV of section 3.2 is satisfied.*

*Again, $A^{(1)}(\theta)$ can be any finite-dimensional operators (not necessarily integral) as long as operator $\bar{A}$ commutes with $\Delta$. Note that since the action of $\Delta$ on $(u_1, \dots u_r)$ is coordinate-wise, the diffusion coefficients of the limit process can be computed in the same way as above, and they do not depend on operators $A^{(1)}(\theta)$.*

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