This is to certify that the

dissertation entitled

THE LOWEST EIGENVALUE OF THE NEGATIVE LAPLACIAN IN TWO DIMENSIONS: A MODIFIED PERTURBATION METHOD

presented by

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has been accepted towards fulfillment of the requirements for

Ph.D. degree in Mathematics

Major professor

Date 12/22/00

THE LOWEST EIGENVALUE OF THE NEGATIVE LAPLACIAN IN TWO DIMENSIONS: A MODIFIED PERTURBATION METHOD

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2000

ABSTRACT

THE LOWEST EIGENVALUE OF THE NEGATIVE LAPLACIAN IN TWO DIMENSIONS: A MODIFIED PERTURBATION METHOD

By

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The eigenvalue problem for the negative Laplace operator in two dimensions is classical in mathematics and physics. Nevertheless, analytical methods for estimating the eigenvalues are still of much current interest. In this work, a modified perturbation method is formulated by applying perturbation method, reflection method, and the Fredholm alternative theorem. The method provides the asymptotic expansion formulas of the lowest eigenvalue to bounded doubly connected regions having the inner boundary which encloses a region with the maximum dimension of $2c$, $c \ll 1$. The first three order terms of the asymptotic expansion formulas are found explicitly by correcting the inner and outer boundary conditions alternatively and by applying the generalized Green's functions. The relations between the first three order terms of the asymptotic expansion formulas and geometric properties of the regions are also investigated.

To my Mother

ACKNOWLEDGMENTS

I would like to express my deep gratitude to my advisor *Professor C. Y. Wang* for introducing me to the world of research applied mathematics in the best possible way, for sharing his knowledge and insight, and for his belief in me. I also would like to express my thanks to each of the committee members, *Prof. Gang Bao*, *Dr. Chichia Chiu*, *Prof. Charles R. MacCluer*, and *Prof. Jerry D. Schuur* for their assistance and time.

I thank my family and friends for their encouragement and support.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

The eigenvalue problem for the negative Laplace operator in two dimensions is

$$- \triangle \psi = \mu\psi \ in \ \bar{R} \ , \tag{1}$$

$$\psi = 0 \ on \ \bar{C} \ , \tag{2}$$

where $\bar{R}$ is a bounded region with boundary $\bar{C}$ in two dimensional space. It arises from separating the time variable out of the wave equation, so it occurs in many applications; particularly in applications to vibrations of membranes and to acoustic and electromagnetic waveguides. For instance [12], we can consider the case of a fixed, uniform, flexible membrane $\bar{R}$, of mass ρ per unit area, stretched under uniform tension T per unit length. The equation of motion is the wave equation

$$\triangle\bar{\Psi} - \frac{1}{\sigma^2}\frac{\partial^2\bar{\Psi}}{\partial t^2} = 0 \ ; \ \ \sigma^2 = \frac{T}{\rho} \ , \tag{3}$$

where $\bar{\Psi}$ is the vertical displacement of the membrane from its equilibrium position. The boundary condition is

$$\bar{\Psi} = 0 \ on \ \bar{C} \times [0, \infty) \ . \tag{4}$$

By requiring simple harmonic dependence on time, we can separate out the time factor:

$$\bar{\Psi} = \psi e^{-i\omega t} \ , \ \omega \ is \ the \ vibrational \ frequency \ , \tag{5}$$

where

$$- \triangle \psi = \mu\psi \ in \ \bar{R} \ ; \ \mu = \frac{\omega^2}{\sigma^2} \ , \tag{6}$$

$$\psi = 0 \ on \ \bar{C} \ . \tag{7}$$

Eqs(1),(2) has a spectrum of infinitely many positive eigenvalues

$$0 \ < \ \mu_1 \ < \ \mu_2 \ \leq \ \mu_3 \ \leq \ \cdots \ , \tag{8}$$

with no finite accumulation point [8]. The closed form solutions for the eigenvalues μ of $-\triangle$ exist only in few geometric regions. In two-dimensional space, they exist only in regions [12] which can be described by rectangular, parabolic, polar, or elliptic coordinates, regions such as rectangles, circles, ellipses, annular circles, and confocal ellipses.

Numerical techniques such as the finite difference method, finite element method, point matching method, and eigenfunction matching method are often used to solve this problem. However, for doubly connected regions with the region bounded by inner boundary which has maximum dimension of $2c$, $c \ll 1$, the disadvantages of numerical techniques, such as repetition of the evaluation for each different c and serious scaling problems due to the small size, encourage us to develop the asymptotic expansion formula for μ.

The dissertation consists of three parts:

In Chapter 2, the formulation of the modified perturbation method and the resulting asymptotic expansion formulas are presented; it is performed by applying the perturbation method [1], the reflection method [7], and the Fredholm alternative theorem [4].

In Chapter 3 and 4, the applications of the modified perturbation method to special cases are executed, such as regions with an inner circular boundary, regions with an inner linear boundary, annular circular regions, and circular regions with a centered strip ; it is achieved by correcting the inner and outer boundary conditions alternatively and by applying the generalized Green's functions [4, 12]. The first three order terms of the asymptotic expansion formulas are found explicitly.

In Chapter 5, the accuracy of the first three order terms of the asymptotic expansion formulas resulting from the modified perturbation method is compared. The first three order terms of the asymptotic expansion formulas to general regions are exhibited explicitly. Relations between the first three order terms of the asymptotic expansion formulas and geometric properties of the regions are also investigated.

CHAPTER 1

Perturbation Formulation

Consider a membrane having region R_0 enclosed by a boundary S_0, with an internal core which has a boundary S_B. The governing Helmholtz equation is

$$\triangle W(x,y) + K^2 W(x,y) = 0 \ , \tag{1.1}$$

where W is the normalized vertical displacement and K is the normalized vibrational frequency, $K = \omega \cdot L \cdot \sqrt{\frac{\rho}{T}}$. The symbol L is a characteristic length defined by $\sqrt{(\text{area of } R_0)/\pi}$. Let region bounded by S_0 and S_B be the region R. Consider

$$\triangle U(x,y) + K^2 U(x,y) = 0 \ in \ R \ , \tag{1.2}$$

$$U(x,y) = 0 \ on \ S_0 \cup S_B \ , \tag{1.3}$$

and

$$\triangle \breve{U}(x,y) + \breve{K}^2 \breve{U}(x,y) = 0 \ in \ R_0 \ , \tag{1.4}$$

$$\breve{U}(x,y) = 0 \ on \ S_0 \ . \tag{1.5}$$

Let K_0 be the fundamental frequency to eqs(1.4),(1.5) and U_0 be the corresponding eigenfunction. K_0 is *simple* [8]. According to eq(A.1), we assume that [1, 7] the fundamental frequency K to eqs(1.2),(1.3) and its corresponding eigenfunction are

$$U(x,y) = U_0(x,y) + \sum_{l=1}^{\infty} \lambda^l U_l(x,y) \ , \tag{1.6}$$

$$K = K_0 + \sum_{l=1}^{\infty} \lambda^l F_l \ , \tag{1.7}$$

where the parameter λ is introduced as a formal way of separating out approximate solutions of various orders in eq(1.7) and as a sequencing tool in eq(1.6).

Substituting eqs(1.6),(1.7) into eq(1.2) yields

$$\begin{aligned}
&\left(\triangle U_0(x,y) + K_0^2 U_0(x,y)\right) + \lambda \left(\triangle U_1(x,y) + K_0^2 U_1(x,y) + 2K_0 F_1 U_0(x,y)\right) \\
&+ \sum_{m=2}^{\infty} \lambda^m \Bigg[\triangle U_m(x,y) + K_0^2 U_m(x,y) + 2K_0 \sum_{j=1}^{m} F_j U_{m-j}(x,y) \\
&\qquad + \sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t U_{m-(s+t)}(x,y) \Bigg] = 0 \ .
\end{aligned} \tag{1.8}$$

For similar orders of λ, this leads to

$$\triangle U_0(x,y) + K_0^2 U_0(x,y) = 0 \ , \tag{1.9}$$

$$\triangle U_1(x,y) + K_0^2 U_1(x,y) = -2K_0 F_1 U_0(x,y) \ , \tag{1.10}$$

and

$$\begin{aligned}
&\triangle U_m(x,y) + K_0^2 U_m(x,y) = \\
&-2K_0 \sum_{j=1}^{m} F_j U_{m-j}(x,y) - \sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t U_{m-(s+t)}(x,y) \ , \\
&\qquad\qquad m = 2,3,4,\ldots \ .
\end{aligned} \tag{1.11}$$

Let

$$U_n(x,y) = V_n(x,y) + W_n(x,y) \ , \ n = 0,1,2,\dots \ , \tag{1.12}$$

where $V_0(x,y) = 0$, V_n's are defined in R, W_n's are defined in R_0, and [7]

$$W_0(x,y) = 0 \ on \ S_0 \ , \tag{1.13}$$

$$W_0(x,y) + \lambda V_1(x,y) = 0 \ on \ S_B \ , \tag{1.14}$$

$$W_0(x,y) + \lambda\left(V_1(x,y) + W_1(x,y)\right) = 0 \ on \ S_0 \ , \tag{1.15}$$

$$\begin{aligned} &W_0(x,y) + \lambda\left(V_1(x,y) + W_1(x,y)\right) + \cdots \\ &+\lambda^{m-1}\left(V_{m-1}(x,y) + W_{m-1}(x,y)\right) + \lambda^m V_m(x,y) = 0 \ on \ S_B \ , \\ &\qquad m = 2,3,4,\dots \ , \end{aligned} \tag{1.16}$$

$$\begin{aligned} &W_0(x,y) + \lambda\left(V_1(x,y) + W_1(x,y)\right) + \cdots \\ &+\lambda^m\left(V_m(x,y) + W_m(x,y)\right) = 0 \ on \ S_0 \ , \\ &\qquad m = 2,3,4,\dots \ . \end{aligned} \tag{1.17}$$

Set $\lambda = 1$. Then eqs(1.6),(1.7) become

$$\begin{aligned} U(x,y) = {} & W_0(x,y) + \left(V_1(x,y) + W_1(x,y)\right) + \cdots \\ & + \left((V_m(x,y) + W_m(x,y)\right) + \cdots \ , \end{aligned} \tag{1.18}$$

$$K = K_0 + F_1 + \cdots + F_m + \cdots \ , \tag{1.19}$$

where

$$\triangle W_0(x,y) + K_0^2 W_0(x,y) = 0 \ in \ R_0 \ , \tag{1.20}$$

$$W_0(x,y) = 0 \ on \ S_0 \ , \tag{1.21}$$

$$\triangle V_1(x,y) + K_0^2 V_1(x,y) = 0 \ in \ R \ , \tag{1.22}$$

$$V_1(x,y) = -W_0(x,y) \ on \ S_B \ , \tag{1.23}$$

$$\triangle W_1(x,y) + K_0^2 W_1(x,y) = -2K_0 F_1 W_0(x,y) \ in \ R_0 \ , \tag{1.24}$$

$$W_1(x,y) = -V_1(x,y) \ on \ S_0 \ , \tag{1.25}$$

$$\triangle V_2(x,y) + K_0^2 V_2(x,y) = -2K_0 F_1 V_1(x,y) \ in \ R \ , \tag{1.26}$$

$$V_2(x,y) = -W_1(x,y) \ on \ S_B \ , \tag{1.27}$$

$$\begin{gathered}\triangle W_2(x,y) + K_0^2 W_2(x,y) = \\ -2K_0 F_1 W_1(x,y) - 2K_0 F_2 W_0(x,y) - F_1^2 W_0(x,y) \ in \ R_0 \ ,\end{gathered} \tag{1.28}$$

$$W_2(x,y) = -V_2(x,y) \ on \ S_0 \ , \tag{1.29}$$

$$\begin{gathered}\triangle V_m(x,y) + K_0^2 V_m(x,y) = \\ -2K_0 \sum_{j=1}^{m} F_j V_{m-j}(x,y) - \sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t V_{m-(s+t)}(x,y) \ in \ R \ , \\ m = 3, 4, 5, \dots \ ,\end{gathered} \tag{1.30}$$

$$V_m(x,y) = -W_{m-1}(x,y) \ on \ S_B \ , \ m = 3, 4, 5, \dots \ , \tag{1.31}$$

$$\begin{gathered}\triangle W_m(x,y) + K_0^2 W_m(x,y) = \\ -2K_0 \sum_{j=1}^{m} F_j W_{m-j}(x,y) - \sum_{s=1}^{m-1} \sum_{t=1}^{m-s} F_s F_t W_{m-(s+t)}(x,y) \ in \ R_0 \ , \\ m = 3, 4, 5, \dots \ ,\end{gathered} \tag{1.32}$$

$$W_m(x,y) = -V_m(x,y) \ on \ S_0 \ , \ m = 3, 4, 5, \dots \ . \tag{1.33}$$

By the Fredholm alternative theorem, the existence conditions [4] of W_n, $n = 1, 2, 3, \ldots$, give

$$\int_{R_0} 2K_0 F_1 W_0(x,y) U_0(x,y)\, dA = -\oint_{S_0} \frac{\partial U_0(x,y)}{\partial n} V_1(x,y)\, ds \;, \tag{1.34}$$

$$\int_{R_0} \left(2K_0F_1W_1(x,y) + 2K_0F_2W_0(x,y) + F_1^2W_0(x,y)\right) U_0(x,y)\, dA$$

$$= -\oint_{S_0} \frac{\partial U_0(x,y)}{\partial n} V_2(x,y)\, ds \;, \tag{1.35}$$

$$\int_{R_0} \left(2K_0 \sum_{j=1}^{m} F_j W_{m-j}(x,y) + \sum_{s=1}^{m-1}\sum_{t=1}^{m-s} F_s F_t W_{m-(s+t)}(x,y)\right) U_0(x,y)\, dA$$

$$= -\oint_{S_0} \frac{\partial U_0(x,y)}{\partial n} V_m(x,y)\, ds \;,\; m = 3, 4, 5, \ldots \tag{1.36}$$

Thus, the corrections to the fundamental frequency are found

$$F_1 = \frac{\oint_{S_0} \frac{\partial U_0(x,y)}{\partial n} V_1(x,y)\, ds}{-2K_0 \int_{R_0} U_0^2(x,y)\, dA} \;, \tag{1.37}$$

$$F_2 = \frac{\int_{R_0} \left(2K_0F_1W_1 + F_1^2W_0(x,y)\right) U_0(x,y)\, dA + \oint_{S_0} \frac{\partial U_0(x,y)}{\partial n} V_2(x,y)\, ds}{-2K_0 \int_{R_0} U_0^2(x,y)\, dA} \;, \tag{1.38}$$

$$F_m = \frac{\int_{R_0} \left(2K_0 \sum_{j=1}^{m-1} F_j W_{m-j}(x,y)\right) U_0(x,y)\, dA}{-2K_0 \int_{R_0} U_0^2(x,y)\, dA} - \frac{\oint_{S_0} \frac{\partial U_0(x,y)}{\partial n} V_m(x,y)\, ds}{2K_0 \int_{R_0} U_0^2(x,y)\, dA}$$

$$- \frac{\int_{R_0} \left(\sum_{s=1}^{m-1}\sum_{t=1}^{m-s} F_s F_t W_{m-(s+t)}(x,y)\right) U_0(x,y)\, dA}{2K_0 \int_{R_0} U_0^2(x,y)\, dA} \;, m = 3, 4, 5, \ldots \;. \tag{1.39}$$

Remark :

$$F_m \ , \ m = 1, 2, 3, \ldots$$

are unique up to a constant multiplier to U_0 .

CHAPTER 2

S_B is a Circle of Radius c Centered at (x_0, y_0)

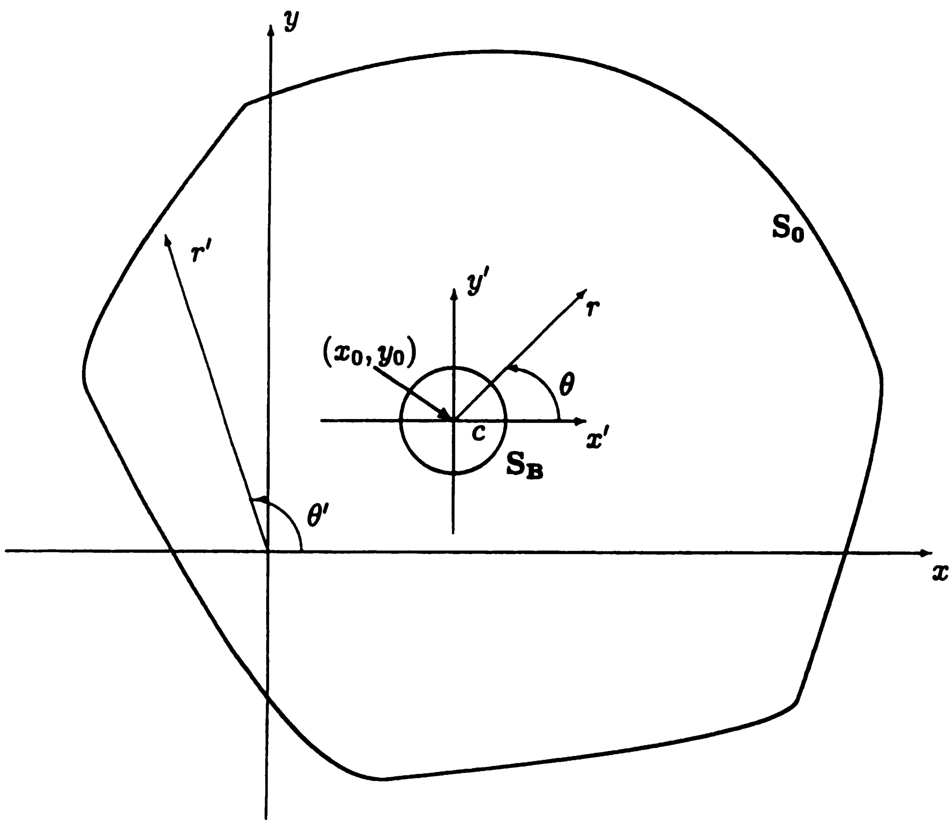


Figure 2.1: S_B is a circle of radius c centered at (x_0, y_0)

The coordinates (x', y'), (x, y), (r, θ), and (r', θ'), as in Figure 2.1, are related by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}, \tag{2.1}$$

$$x = r' \cos\theta' \quad , \quad y = r' \sin\theta' \ , \tag{2.2}$$

and

$$x' = r\cos\theta \quad , \quad y' = r\sin\theta \ . \tag{2.3}$$

Due to the invariability of the governing Helmholtz equation under the translation of coordinates, the governing Helmholtz equation can be written as

$$\frac{\partial^2 U(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U(r,\theta)}{\partial \theta^2} + K^2 U(r,\theta) = 0 \ . \tag{2.4}$$

Eq(2.4) can be separated by $U(r,\theta) = \Psi(r)\Phi(\theta)$ resulting in *Bessel* equations of order b [12]

$$z^2\frac{d^2\Psi(z)}{dz^2} + z\frac{d\Psi(z)}{dz} + (z^2 - b^2)\Psi(z) = 0 \ , \ where \ z = Kr \tag{2.5}$$

and

$$\frac{d^2\Phi(\theta)}{d\theta^2} + b^2\Phi(\theta) = 0 \ , \tag{2.6}$$

where b is a separation constant. The periodic solutions to eq(2.6) are

$$\sin(n\theta) \ , \ \cos(n\theta) \ , \quad n = 0, 1, 2, \ldots \ . \tag{2.7}$$

The corresponding solutions [12] to eq(2.5) are J_n, the n^{th} order *Bessel* function, and Y_n, the n^{th} order *Neumann* function, where

$$J_n(z) = \sum_{l=0}^{\infty}(-1)^l \frac{\left(\frac{z}{2}\right)^{(n+2l)}}{l!(n+l)!} \ , \ n = 0, 1, 2, \ldots \ , \tag{2.8}$$

$$Y_0(z) = \frac{2}{\pi}\left(\ln\frac{z}{2} + \gamma\right) J_0(z) - \frac{2}{\pi}\sum_{l=1}^{\infty}(-1)^l \frac{\left(\frac{z}{2}\right)^{2l}}{(l!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{l}\right) \ , \tag{2.9}$$

$$Y_m(z) = \frac{2}{\pi}\left(\ln\frac{z}{2}\right)J_m(z) - \frac{1}{\pi}\sum_{j=0}^{m-1}\frac{(m-j-1)!}{j!}\left(\frac{2}{z}\right)^{(m-2j)}$$

$$-\frac{1}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l\left(\frac{z}{2}\right)^{(m+2l)}(\psi(m+l+1)+\psi(l+1))}{l!(m+l)!}\ , \tag{2.10}$$

$$m = 1, 2, 3, \dots\ ,$$

where $\gamma \approx 0.5772$, $\psi(m+l+1) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{m+l}\right) - \gamma$, and $\psi(1) = -\gamma$.

2.1 Membrane With a Circular Core of Radius c Centered at (x_0, y_0)

$U_0(r,\theta)$ is finite in R_0, we assume that

$$U_0(r,\theta) = B_0 J_0(K_0 r) + \sum_{m=1}^{\infty} J_m(K_0 r)\left(A_m \sin(m\theta) + B_m \cos(m\theta)\right) \tag{2.11}$$

with appropriate constant coefficients B_0, B_m, and A_m determined by the boundary condition, eq(1.5).

$\{1,\ \cos(m\theta),\ \sin(m\theta)\}_{m=1}^{\infty}$ is a complete orthogonal set of functions and

$$\int_0^{2\pi} \sin^2(m\theta)\ d\theta = \pi\ , \tag{2.12}$$

$$\int_0^{2\pi} \cos^2(m\theta)\ d\theta = \pi\ . \tag{2.13}$$

Thus, eqs(1.22),(1.23) give

$$V_1(r,\theta) = \tilde{B}_0(c)\left(J_0(K_0 r) - \frac{J_0(K_0 c)}{Y_0(K_0 c)}Y_0(K_0 r)\right) - B_0\frac{J_0(K_0 c)}{Y_0(K_0 c)}Y_0(K_0 r)$$

$$+\sum_{m=1}^{\infty}\left[\tilde{A}_m(c)\left(J_m(K_0 r) - \frac{J_m(K_0 c)}{Y_m(K_0 c)}Y_m(K_0 r)\right) - A_m\frac{J_m(K_0 c)}{Y_m(K_0 c)}Y_m(K_0 r)\right]\sin(m\theta)$$

$$+\sum_{m=1}^{\infty}\left[\tilde{B}_m(c)\left(J_m(K_0r)-\frac{J_m(K_0c)}{Y_m(K_0c)}Y_m(K_0r)\right)-B_m\frac{J_m(K_0c)}{Y_m(K_0c)}Y_m(K_0r)\right]\cos(m\theta), \tag{2.14}$$

To correct the boundary condition on S_0 to $O(\frac{1}{\ln c})$ for (U_0+V_1), $\tilde{B}_0(c)$, $\tilde{A}_m(c)$, and $\tilde{B}_m(c)$ must be at most $O(\frac{1}{\ln c})$. Green's 2^{nd} identity [4, 11] and eqs(1.20),(1.21),(1.22),(1.23) give

$$\oint_{S_0} V_1(x,y)\frac{\partial U_0(x,y)}{\partial n}\,ds=\oint_{S_B} U_0(x,y)\left(\frac{\partial V_1(x,y)}{\partial n}+\frac{\partial U_0(x,y)}{\partial n}\right)ds, \tag{2.15}$$

then eq(1.37) becomes

$$F_1=\frac{\oint_{S_B} U_0(x,y)\left(\frac{\partial V_1(x,y)}{\partial n}+\frac{\partial U_0(x,y)}{\partial n}\right)ds}{-2K_0\int_{R_0} U_0^2(x,y)\,dA}. \tag{2.16}$$

[10] gives

$$\mathcal{W}\left(J_m(z),Y_m(z)\right)=\frac{2}{\pi z}, \tag{2.17}$$

where $\mathcal{W}\left(J_m(z),Y_m(z)\right)$ is the Wronskian of $J_m(z)$ and $Y_m(z)$.

$$\frac{J_0(K_0c)}{Y_0(K_0c)}=\frac{-\pi}{2}\frac{1}{(-\ln c)}+\frac{\pi}{2}\left(\ln 2-\gamma-\ln K_0\right)\frac{1}{(\ln c)^2}+\cdots. \tag{2.18}$$

Eqs(2.11),(2.12),(2.13),(2.14),(2.16),(2.17),(2.18) yield

$$F_1=\frac{\int_0^{2\pi} U_0(c,\theta)\left(\frac{\partial V_1(r,\theta)}{\partial r}\,|_{r=c}+\frac{\partial U_0(r,\theta)}{\partial r}\,|_{r=c}\right)c\,d\theta}{2K_0\int_{R_0} U_0^2(r,\theta)\,dA}$$

$$=\frac{2\tilde{B}_0(c)B_0\frac{J_0(K_0c)}{Y_0(K_0c)}+\sum_{m=1}^{\infty}\left(\tilde{A}_m(c)A_m+\tilde{B}_m(c)B_m\right)\frac{J_m(K_0c)}{Y_m(K_0c)}}{-K_0\int_{R_0} U_0^2(r,\theta)\,dA}$$

$$+\frac{2B_0^2\frac{J_0(K_0c)}{Y_0(K_0c)}+\sum_{m=1}^{\infty}\left(A_m^2+B_m^2\right)\frac{J_m(K_0c)}{Y_m(K_0c)}}{-K_0\int_{R_0} U_0^2(r,\theta)\,dA}$$

$$= \frac{2\tilde{B}_0(c)B_0\frac{J_0(K_0c)}{Y_0(K_0c)} + \sum_{m=1}^{\infty}\left(\tilde{A}_m(c)A_m + \tilde{B}_m(c)B_m\right)\frac{J_m(K_0c)}{Y_m(K_0c)}}{-K_0\int_{R_0} U_0^2(r,\theta)\,dA}$$

$$+\left(\frac{\pi B_0^2}{K_0\int_{R_0} U_0^2(r,\theta)dA}\right)\frac{1}{|\ln c|} + \left(\frac{\pi(\ln K_0 + \gamma - \ln 2)B_0^2}{K_0\int_{R_0} U_0^2(r,\theta)dA}\right)\frac{1}{|\ln c|^2} + \cdots . \tag{2.19}$$

$\tilde{B}_0(c)$, $\tilde{A}_m(c)$, and $\tilde{B}_m(c)$ are at most $O(\frac{1}{\ln c})$, the simplest choice is to set $\tilde{B}_0(c)$, $\tilde{A}_m(c)$, and $\tilde{B}_m(c)$ equal to *zero*. Other choices would only lead to a higher order correction to the first order result F_1, eq(2.19). Thus,

$$V_1(r,\theta) = \frac{-B_0J_0(K_0c)}{Y_0(K_0c)}Y_0(K_0r) - \sum_{m=1}^{\infty}\frac{J_m(K_0c)}{Y_m(K_0c)}Y_m(K_0r)\left(A_m\sin(m\theta) + B_m\cos(m\theta)\right) \tag{2.20}$$

and

$$F_1 = \left(\frac{\pi B_0^2}{K_0\int_{R_0} U_0^2(r,\theta)dA}\right)\frac{1}{|\ln c|} + \left(\frac{\pi(\ln K_0 + \gamma - \ln 2)B_0^2}{K_0\int_{R_0} U_0^2(r,\theta)dA}\right)\frac{1}{|\ln c|^2} + \cdots, \tag{2.21}$$

where $\gamma \approx 0.5772$.

The Green's 2^{nd} identity [4, 11] and the generalized Green's function $G(r,\theta;\tilde{r},\tilde{\theta})$,eq(B.5), yield

$$W_1(r,\theta) = EU_0(r,\theta) + \oint_{S_0}\frac{\partial G(r,\theta;\tilde{r},\tilde{\theta})}{\partial n}V_1(\tilde{r},\tilde{\theta})\,ds \tag{2.22}$$

$$= EU_0(r,\theta) + \sum_{N=1}^{\infty}\sum_{j=1}^{l(N)}\frac{U_{N,j}(r,\theta)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2}\oint_{S_0}\frac{\partial U_{N,j}(r,\theta)}{\partial n}V_1(r,\theta)\,ds\ , \tag{2.23}$$

where E is a constant and

$$U_{N,j}(r,\theta) = B_0(N,j)J_0(K_Nr) + \sum_{m=1}^{\infty}J_m(K_Nr)\left(A_m(N,j)\sin(m\theta) + B_m(N,j)\cos(m\theta)\right)$$

$$(2.24)$$

with appropriate constant coefficients $B_0(N,j)$, $B_m(N,j)$, and $A_m(N,j)$ determined by the boundary condition, eq(1.5).

Eqs(2.18),(2.20) yield

$$\oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} V_1(r,\theta)\, ds = \left(\frac{\pi B_0}{2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds \right) \frac{1}{|\ln c|} + \dots . \tag{2.25}$$

To correct the boundary condition on S_B to $O(\frac{1}{\ln c})$ for $(U_0 + V_1 + W_1)$, the constant E must be *zero*. Thus,

$$W_1(r,\theta) = \oint_{S_0} \frac{\partial G(r,\theta;\tilde{r},\tilde{\theta})}{\partial n} V_1(\tilde{r},\tilde{\theta})\, ds \tag{2.26}$$

$$= \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{U_{N,j}(r,\theta)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} V_1(r,\theta)\, ds . \tag{2.27}$$

Let

$$V_2(x,y) = V_2^i(x,y) + V_2^h(x,y) , \tag{2.28}$$

where

$$\triangle V_2^i(x,y) + K_0^2 V_2^i(x,y) = -2K_0 F_1 V_1(x,y) \textit{ in } R , \tag{2.29}$$

$$\triangle V_2^h(x,y) + K_0^2 V_2^h(x,y) = 0 \textit{ in } R , \tag{2.30}$$

$$V_2^h(x,y) = -W_1(x,y) - V_2^i(x,y) \textit{ on } S_B . \tag{2.31}$$

For the non-homogeneous equation

$$R'' + \frac{1}{r} R' + \left(K_0^2 - \frac{m^2}{r^2} \right) R = Y_m(K_0 r) , \quad m = 0,1,2,3,\dots \tag{2.32}$$

the particular solution is

$$R = \frac{rY'_m(K_0 r)}{-2K_0} \ , \quad m = 0, 1, 2, 3, \ldots \tag{2.33}$$

Thus,

$$V_2^i(r,\theta) = F_1 \left[\frac{-B_0 J_0(K_0 c)}{Y_0(K_0 c)} rY'_0(K_0 r) \right.$$
$$\left. - \sum_{m=1}^{\infty} \frac{J_m(K_0 c)}{Y_m(K_0 c)} rY'_m(K_0 r) \left(A_m \sin(m\theta) + B_m \cos(m\theta) \right) \right] . \tag{2.34}$$

Eqs(2.12),(2.13),(2.30),(2.31),(2.34) yield

$$V_2^h(r,\theta) = \tilde{D}_0(c) \left(J_0(K_0 r) - \frac{J_0(K_0 c)}{Y_0(K_0 c)} Y_0(K_0 r) \right)$$

$$+ \sum_{m=1}^{\infty} \left(\tilde{C}_m(c) \sin(m\theta) + \tilde{D}_m(c) \cos(m\theta) \right) \left(J_m(K_0 r) - \frac{J_m(K_0 c)}{Y_m(K_0 c)} Y_m(K_0 r) \right)$$

$$+ D_0 Y_0(K_0 r) + \sum_{m=1}^{\infty} Y_m(K_0 r) \left(C_m \sin(m\theta) + D_m \cos(m\theta) \right) \ , \tag{2.35}$$

where

$$D_0 = \frac{-1}{2\pi Y_0(K_0 c)} \int_0^{2\pi} W_1(c,\theta) \ d\theta + \frac{B_0 F_1 c J_0(K_0 c) Y'_0(K_0 c)}{Y_0^2(K_0 c)} \ , \tag{2.36}$$

$$C_m = \frac{-1}{\pi Y_m(K_0 c)} \int_0^{2\pi} \sin(m\theta) W_1(c,\theta) \ d\theta + \frac{A_m F_1 c J_m(k_0 c) Y'_m(K_0 c)}{Y_m^2(K_0 c)} \ , \tag{2.37}$$

$$D_m = \frac{-1}{\pi Y_m(K_0 c)} \int_0^{2\pi} \cos(m\theta) W_1(c,\theta) \ d\theta + \frac{B_m F_1 c J_m(k_0 c) Y'_m(K_0 c)}{Y_m^2(K_0 c)} \ . \tag{2.38}$$

Eqs(2.24),(2.25),(2.27) yield

$$\int_0^{2\pi} W_1(c,\theta) \ d\theta = \left[\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0 B_0(N,j) J_0(K_N c)}{(K_N^2 - K_0^2) \|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r) \ ds \right] \frac{1}{|\ln c|}$$
$$+ \cdots \ ,$$

(2.39)

$$\int_0^{2\pi} \sin(m\theta) W_1(c,\theta)\, d\theta = \left[\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0 A_m(N,j) J_m(K_N c)}{2(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds \right] \frac{1}{|\ln c|} + \cdots ,$$

(2.40)

$$\int_0^{2\pi} \cos(m\theta) W_1(c,\theta)\, d\theta = \left[\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0 B_m(N,j) J_m(K_N c)}{2(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds \right] \frac{1}{|\ln c|} + \cdots .$$

(2.41)

To correct the boundary condition on S_0 to $O(\frac{1}{|\ln c|^2})$ for $(U_0 + V_1 + W_1 + V_2)$, $\tilde{D}_0(c)$, $\tilde{D}_m(c)$, and $\tilde{C}_m(c)$ must be at most $O(\frac{1}{|\ln c|^2})$. Green's 2^{nd} identity [4, 11] gives

$$\oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} V_2(r,\theta)\, ds = \oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} V_2^i(r,\theta)\, ds$$

$$- \oint_{S_B} \left(\frac{\partial U_0(r,\theta)}{\partial n} V_2^h(r,\theta) - \frac{\partial V_2^h(r,\theta)}{\partial n} U_0(r,\theta) \right) ds. \qquad (2.42)$$

Thus, eq(1.38) becomes

$$F_2 = \frac{F_1^2}{-2K_0} - \frac{\oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} V_2^i(r,\theta)\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} + \frac{\oint_{S_B} \left(\frac{\partial U_0(r,\theta)}{\partial n} V_2^h(r,\theta) - \frac{\partial V_2^h(r,\theta)}{\partial n} U_0(r,\theta) \right) ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} .$$

(2.43)

Eqs(2.18),(2.21),(2.34) yield

$$\oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} V_2^i(r,\theta)\, ds = -F_1 \left\{ \frac{B_0 J_0(K_0 c)}{Y_0(K_0 c)} \oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} r Y_0'(K_0 r)\, ds \right.$$

$$+\sum_{m=1}^{\infty}\frac{J_m(K_0c)}{Y_m(K_0c)}\oint_{S_0}\frac{\partial U_0(r,\theta)}{\partial n}\left(A_m\sin(m\theta)+B_m\cos(m\theta)\right)rY_m'(K_0r)\,ds\Bigg\}$$

$$=\left(\frac{\pi^2B_0^3}{2K_0\int_{R_0}U_0^2(r,\theta)\,dA}\oint_{S_0}\frac{\partial U_0(r,\theta)}{\partial n}rY_0'(K_0r)\,ds\right)\frac{1}{|\ln c|^2}+\cdots\,. \tag{2.44}$$

Eqs(2.11),(2.17),(2.35),(2.36),(2.39) yield

$$\oint_{S_B}\left(\frac{\partial U_0(r,\theta)}{\partial n}V_2^h(r,\theta)-\frac{\partial V_2^h(r,\theta)}{\partial n}U_0(r,\theta)\right)ds$$

$$=\int_0^{2\pi}\left(\frac{-\partial U_0(r,\theta)}{\partial r}\Big|_{r=c}V_2^h(c,\theta)+\frac{\partial V_2^h(r,\theta)}{\partial r}\Big|_{r=c}U_0(c,\theta)\right)c\,d\theta$$

$$=-4B_0\tilde{D}_0(c)\frac{J_0(K_0c)}{Y_0(K_0c)}-2\sum_{m=1}^{\infty}\left(A_m\tilde{C}_m(c)+B_m\tilde{D}_m(c)\right)\frac{J_m(K_0c)}{Y_m(K_0c)}$$

$$+4B_0D_0+2\sum_{m=1}^{\infty}\left(A_mC_m+B_mD_m\right)$$

$$=-4B_0\tilde{D}_0(c)\frac{J_0(K_0c)}{Y_0(K_0c)}-2\sum_{m=1}^{\infty}\left(A_m\tilde{C}_m(c)+B_m\tilde{D}_m(c)\right)\frac{J_m(K_0c)}{Y_m(K_0c)}$$

$$+\left[\sum_{N=1}^{\infty}\sum_{j=1}^{l(N)}\frac{\pi^2B_0^2B_0(N,j)}{(K_N^2-K_0^2)\|U_{N,j}\|^2}\oint_{S_0}\frac{\partial U_{N,j}(r,\theta)}{\partial n}Y_0(K_0r)\,ds\right]\frac{1}{|\ln c|^2}+\cdots\,. \tag{2.45}$$

$\tilde{D}_0(c)$, $\tilde{D}_m(c)$, and $\tilde{C}_m(c)$ are at most $O(\frac{1}{|\ln c|^2})$, the simplest choice is to set $\tilde{D}_0(c)$, $\tilde{D}_m(c)$, and $\tilde{C}_m(c)$ equal to *zero* . Other choices would only lead to a higher order correction to the 2^{nd} order result F_2, eq(2.43). Thus,

$$V_2^h(r,\theta)=D_0Y_0(K_0r)+\sum_{m=1}^{\infty}Y_m(K_0r)\left(C_m\sin(m\theta)+D_m\cos(m\theta)\right)\;, \tag{2.46}$$

where

$$D_0=\frac{-1}{2\pi Y_0(K_0c)}\int_0^{2\pi}W_1(c,\theta)d\theta+\frac{B_0F_1cJ_0(K_0c)Y_0'(K_0c)}{Y_0^2(K_0c)}\;, \tag{2.47}$$

$$C_m = \frac{-1}{\pi Y_m(K_0 c)} \int_0^{2\pi} \sin(m\theta) W_1(c,\theta) d\theta + \frac{A_m F_1 c J_m(k_0 c) Y'_m(K_0 c)}{Y_m^2(K_0 c)} , \tag{2.48}$$

$$D_m = \frac{-1}{\pi Y_m(K_0 c)} \int_0^{2\pi} \cos(m\theta) W_1(c,\theta) d\theta + \frac{B_m F_1 c J_m(k_0 c) Y'_m(K_0 c)}{Y_m^2(K_0 c)} . \tag{2.49}$$

and, by eqs(2.21),(2.43),(2.44),(2.45),

$$F_2 = \left\{ \frac{\pi^2 B_0^4}{-2K_0^3 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} + \frac{\pi^2 B_0^3 \oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} r Y'_0(K_0 r)\, ds}{-4K_0^2 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} \right.$$

$$\left. + \frac{\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0^2 B_0(N,j)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right\} \frac{1}{|\ln c|^2}$$

$$+ \cdots . \tag{2.50}$$

Eqs(1.19),(2.21),(2.50) give

$$K = K_0 + \left(\frac{\pi B_0^2}{K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right) \frac{1}{|\ln c|} + \left[\frac{\pi(\ln K_0 + \gamma - \ln 2) B_0^2}{K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right.$$

$$- \frac{\pi^2 B_0^4}{2K_0^3 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} - \frac{\pi^2 B_0^3 \oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} r Y'_0(K_0 r)\, ds}{4K_0^2 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2}$$

$$\left. + \frac{\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0^2 B_0(N,j)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right] \frac{1}{|\ln c|^2}$$

$$+ \cdots , \tag{2.51}$$

where $\gamma \approx 0.5772$.

2.2 Circular Membrane With a Circular Core of Radius c Centered at (x_0, y_0)

The geometry of the concerned region is with the outer boundary where S_0 is $r' = 1$ and the inner boundary where S_B is $r = c$.

For a circular membrane with the boundary where S_0 is $r' = 1$, the frequencies $\check{K}$ and the corresponding eigenfunctions [8, 12] $\check{U}$ to eqs(1.4),(1.5) are

$$K_{0,m} \ , \ m = 1, 2, 3, \dots \ , \tag{2.52}$$

$$K_{p,m} \ , \ p, m = 1, 2, 3, \dots \ , \tag{2.53}$$

and

$$J_0(K_{0,m}r') \ , \ m = 1, 2, 3, \dots \ , \tag{2.54}$$

$$J_p(K_{p,m}r') \sin(p\theta') \ , \ J_p(K_{p,m}r') \cos(p\theta') \ , \ p, m = 1, 2, 3, \dots \ , \tag{2.55}$$

respectively, where $K_{n,m}$ is the m^{th} zero of J_n , $n = 0, 1, 2, 3, \dots$, $m = 1, 2, 3, \dots$. $K_0 = K_{0,1} \approx 2.4048$ and $U_0 = J_0(K_0 r')$.

Translational addition theorems for circular cylindrical wave functions [5, 6, 12] give

$$J_0(Kr') = \sum_{l=-\infty}^{\infty} J_l(Kr_0) J_l(Kr) \cos\left(l\theta - l(\theta_0 + \pi)\right) \ , \tag{2.56}$$

$$J_p(Kr') \sin(p\theta') = \sum_{l=-\infty}^{\infty} J_{l-p}(Kr_0) J_l(Kr) \sin\left(l\theta - (l-p)(\theta_0 + \pi)\right) \ , \quad p = 1, 2, 3, \dots \ , \tag{2.57}$$

$$J_p(Kr') \cos(p\theta') = \sum_{l=-\infty}^{\infty} J_{l-p}(Kr_0) J_l(Kr) \cos\left(l\theta - (l-p)(\theta_0 + \pi)\right) \ , \quad p = 1, 2, 3, \dots \ , \tag{2.58}$$

where r_0 is the distance between $O = (0,0)$ and $O_1 = (x_0, y_0)$ and θ_0 is the angle from the x axis to $\overline{OO_1}$. [10] gives

$$J_{-i}(z) = (-1)^i J_i(z) \quad , \quad i = 1, 2, 3, \dots \ . \tag{2.59}$$

Thus,

$$\begin{aligned} J_0(K_{0,m}r') &= J_0(K_{0,m}r_0)J_0(K_{0,m}r) \\ &+ \sum_{i=1}^{\infty} \left[(-1)^i 2J_i(K_{0,m}r_0)\sin(i\theta_0)\right] J_i(K_{0,m}r)\sin(i\theta) \\ &+ \sum_{i=1}^{\infty} \left[(-1)^i 2J_i(K_{0,m}r_0)\cos(i\theta_0)\right] J_i(K_{0,m}r)\cos(i\theta) \ , \\ &\qquad m = 1, 2, 3, \dots \ , \end{aligned} \tag{2.60}$$

$$\begin{aligned} J_p(K_{p,m}r')\sin(p\theta') &= J_p(K_{p,m}r_0)\sin(p\theta_0)J_0(K_{p,m}r) \\ &+ \sum_{i=1}^{\infty} \left[(-1)^{(i-p)} J_{i-p}(K_{p,m}r_0)\cos\left((i-p)\theta_0\right)\right. \\ &\left. +(-1)^{(i+1)} J_{i+p}(K_{p,m}r_0)\cos\left((i+p)\theta_0\right)\right] J_i(K_{p,m}r)\sin(i\theta) \\ &+ \sum_{i=1}^{\infty} \left[(-1)^{(i-p+1)} J_{i-p}(K_{p,m}r_0)\sin\left((i-p)\theta_0\right)\right. \\ &\left. +(-1)^i J_{i+p}(K_{p,m}r_0)\sin\left((i+p)\theta_0\right)\right] J_i(K_{p,m}r)\cos(i\theta) \ , \\ &\qquad p, m = 1, 2, 3, \dots \ , \end{aligned} \tag{2.61}$$

$$J_p(K_{p,m}r')\cos(p\theta') = J_p(K_{p,m}r_0)\cos(p\theta_0)J_0(K_{p,m}r)$$

$$+\sum_{i=1}^{\infty}\Big[(-1)^{(i-p)}J_{i-p}(K_{p,m}r_0)\sin\left((i-p)\theta_0\right)$$

$$+(-1)^{i}J_{i+p}(K_{p,m}r_0)\sin\left((i+p)\theta_0\right)\Big]\,J_i(K_{p,m}r)\sin(i\theta)$$

$$+\sum_{i=1}^{\infty}\Big[(-1)^{(i-p)}J_{i-p}(K_{p,m}r_0)\cos\left((i-p)\theta_0\right)$$

$$+(-1)^{i}J_{i+p}(K_{p,m}r_0)\cos\left((i+p)\theta_0\right)\Big]\,J_i(K_{p,m}r)\cos(i\theta)\ , \tag{2.62}$$

$$p,m=1,2,3,\dots\ .$$

Integrals of products of *Bessel* functions [10] give

$$\int_0^z tJ_0^2(t)\ dt=\frac{z^2}{2}\left[J_0^2(z)+J_1^2(z)\right]\ , \tag{2.63}$$

$$\int_0^z tJ_{n-1}^2(t)\ dt=2\sum_{l=0}^{\infty}(n+2l)J_{n+2l}^2(z)\ , \tag{2.64}$$

$$n=2,3,4,\dots\ .$$

Thus,

$$\int_{R_0} J_0^2(K_{0,m}r')\ dA=\pi\left[J_0^2(K_{0,m})+J_1^2(K_{0,m})\right]$$

$$=\pi J_1^2(K_{0,m})\ , \tag{2.65}$$

$$m=1,2,3,\dots\ ,$$

$$\|J_p(K_{p,m}r')\sin(p\theta')\|^2=\int_{R_0}\left(J_p(K_{p,m}r')\sin(p\theta')\right)^2\ dA$$

$$=\frac{2\pi}{K_{p,m}^2}\sum_{l=0}^{\infty}(p+1+2l)J_{p+1+2l}^2(K_{p,m})\ , \tag{2.66}$$

$$p,m=1,2,3,\dots\ ,$$

$$\|J_p(K_{p,m}r')\cos(p\theta')\|^2 = \int_{R_0} \left(J_p(K_{p,m}r')\cos(p\theta')\right)^2 \, dA$$

$$= \frac{2\pi}{K_{p,m}^2} \sum_{l=0}^{\infty} (p+1+2l) J_{p+1+2l}^2(K_{p,m}) \ , \qquad (2.67)$$

$$p, m = 1, 2, 3, \dots \ .$$

Law of Cosine :

$$r^2 = r'^2 + r_0^2 - 2r_0 r' \cos(\theta' - \theta_0) \ . \qquad (2.68)$$

$$J_0'(z) = -J_1(z) \quad , \quad Y_0'(z) = -Y_1(z) \ . \qquad (2.69)$$

Then,

$$\oint_{S_0} \frac{\partial J_0(K_0 r')}{\partial n} r Y_0'(K_0 r) \, ds$$

$$= J_1(K_0)K_0 \int_0^{2\pi} \sqrt{1 + r_0^2 - 2r_0\cos\theta'} Y_1(K_0\sqrt{1 + r_0^2 - 2r_0\cos\theta'}) \, d\theta' \ . \qquad (2.70)$$

Translational addition theorems for circular cylindrical wave functions [5, 6, 12] give

$$Y_0(Kr) = \sum_{l=-\infty}^{\infty} J_l(Kr_0) Y_l(Kr') \cos\left(l\theta' - l\theta_0\right) \ , \qquad (2.71)$$

where r_0 is the distance between $O = (0,0)$ and $O_1 = (x_0, y_0)$, θ_0 is the angle from the x axis to $\overline{OO_1}$, and the formula holds for points lying ouside the circle with the diameter $\overline{OO_1} = r_0$. Then,

$$\oint_{S_0} \frac{\partial J_0(K_{0,n}r')}{\partial n} Y_0(K_0 r) \, ds = -2\pi K_{0,n} J_0(K_0 r_0) Y_0(K_0) J_1(K_{0,n}) \ , \qquad (2.72)$$

$$n = 2, 3, 4, \dots \ ,$$

$$\oint_{S_0} \frac{\partial J_p(K_{p,m}r')\sin(p\theta')}{\partial n} Y_0(K_0 r) \, ds = 2\pi K_{p,m} J_p(K_0 r_0) Y_p(K_0) J_p'(K_{p,m}) \sin(p\theta_0) \ ,$$

$$p, m = 1, 2, 3, \dots \ , \qquad (2.73)$$

$$\oint_{S_0} \frac{\partial J_p(K_{p,m}r')\cos(p\theta')}{\partial n} Y_0(K_0 r)\, ds = 2\pi K_{p,m} J_p(K_0 r_0) Y_p(K_0) J'_p(K_{p,m}) \cos(p\theta_0) \ ,$$
$$p, m = 1, 2, 3, \dots \ . \tag{2.74}$$

Eqs(2.51),(2.60),(2.61),(2.62),(2.65),(2.66),(2.67), (2.70),(2.72),(2.73),(2.74) give

$$K = K_0 + \left(\frac{J_0^2(K_0 r_0)}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|} + \left[\frac{(\ln K_0 + \gamma - \ln 2)J_0^2(K_0 r_0)}{K_0 J_1^2(K_0)} - \frac{J_0^4(K_0 r_0)}{2K_0^3 J_1^4(K_0)}\right.$$
$$- \frac{J_0^3(K_0 r_0)\int_0^{2\pi} \sqrt{1 + r_0^2 - 2r_0\cos\theta'}\, Y_1(K_0\sqrt{1 + r_0^2 - 2r_0\cos\theta'})\, d\theta'}{4K_0 J_1^3(K_0)}$$
$$+ \frac{\pi J_0^2(K_0 r_0)}{2K_0 J_1^2(K_0)} \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{K_{p,m}^3 J_p(K_{p,m} r_0) J'_p(K_{p,m}) J_p(K_0 r_0) Y_p(K_0)}{\left(K_{p,m}^2 - K_0^2\right)\sum_{l=0}^{\infty}(p + 1 + 2l) J_{p+1+2l}^2(K_{p,m})}$$
$$\left. - \frac{\pi J_0^3(K_0 r_0) Y_0(K_0)}{K_0 J_1^2(K_0)} \sum_{n=2}^{\infty} \frac{K_{0,n} J_0(K_{0,n} r_0)}{\left(K_{0,n}^2 - K_0^2\right) J_1(K_{0,n})}\right] \frac{1}{|\ln c|^2}$$
$$+ \cdots \ . \tag{2.75}$$

where r_0 is the distance between $O = (0,0)$ and $O_1 = (x_0, y_0)$, $K_0 = K_{0,1} \approx 2.4048$, $\gamma \approx 0.5772$, and $K_{0,m}$ is the m^{th} zero of J_0, $J_{p,m}$ is the m^{th} zero of J_p, $m, p = 1, 2, 3, \dots$.

2.3 Annular Circular Membrane With Outer Radius 1 and Inner Radius c

The geometry of the concerned region is with the outer boundary where S_0 is $r' = r = 1$ and the inner boundary where S_B is $r' = r = c$. $r_0 = 0$.

$$J_0(0) = 1 \quad , \quad J_p(0) = 0 \ , \ p = 1, 2, 3, \dots \ . \tag{2.76}$$

Eq(2.75) gives

$$K = K_0 + \left(\frac{1}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|} + \left[\frac{(\ln K_0 + \gamma - \ln 2)}{K_0 J_1^2(K_0)} - \frac{1}{2K_0^3 J_1^4(K_0)}\right.$$

$$\left. - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)} - \frac{\pi Y_0(K_0)}{K_0 J_1^2(K_0)} \sum_{n=2}^{\infty} \frac{K_{0,n}}{\left(K_{0,n}^2 - K_0^2\right) J_1(K_{0,n})}\right] \frac{1}{|\ln c|^2}$$

$$+ \cdots \ , \tag{2.77}$$

where $K_0 = K_{0,1} \approx 2.4048$, $\gamma \approx 0.5772$, and $K_{0,p}$ is the p^{th} zero of J_0, $p = 1, 2, 3, \ldots$.

Alternatively, eqs(B.7),(B.8) [4] become

$$\frac{\partial^2 \tilde{G}(r,\theta;\tilde{r},\tilde{\theta})}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{G}(r,\theta;\tilde{r},\tilde{\theta})}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{G}(r,\theta;\tilde{r},\tilde{\theta})}{\partial \theta^2} + \tilde{K}^2 \tilde{G}(r,\theta;\tilde{r},\tilde{\theta})$$

$$= -\frac{\delta(r,\tilde{r})\delta(\theta,\tilde{\theta})}{r} \ in \ R_0 \ , \tag{2.78}$$

$$\tilde{G}(1,\theta;\tilde{r},\tilde{\theta}) = 0 \ , \ 0 \le \theta, \tilde{\theta} \le 2\pi, 0 \le \tilde{r} \le 1 \ . \tag{2.79}$$

$\{1, \ \sin(m\theta), \ \cos(m\theta)\}_{m=1}^{\infty}$ is an orthogonal complete set of functions, we assume that

$$\tilde{G}(r,\theta;\tilde{r},\tilde{\theta}) = H_0(\tilde{\theta})P_0(r,\tilde{r}) + \sum_{m=1}^{\infty} H_m(\tilde{\theta})P_m(r,\tilde{r})\cos(m\theta)$$

$$+ \sum_{m=1}^{\infty} L_m(\tilde{\theta})Q_m(r,\tilde{r})\sin(m\theta) \ . \tag{2.80}$$

Eq(2.78) yields

$$H_0(\tilde{\theta})\left(r\frac{\partial^2 P_0(r,\tilde{r})}{\partial r^2} + \frac{\partial P_0(r,\tilde{r})}{\partial r} + r\tilde{K}^2 P_0(r,\tilde{r})\right)$$

$$+ \sum_{m=1}^{\infty} H_m(\tilde{\theta})\left(r\frac{\partial^2 P_m(r,\tilde{r})}{\partial r^2} + \frac{\partial P_m(r,\tilde{r})}{\partial r} + \left(r\tilde{K}^2 - \frac{m^2}{r}\right)P_m(r,\tilde{r})\right)\cos(m\theta)$$

$$+ \sum_{m=1}^{\infty} L_m(\tilde{\theta})\left(r\frac{\partial^2 Q_m(r,\tilde{r})}{\partial r^2} + \frac{\partial Q_m(r,\tilde{r})}{\partial r} + \left(r\tilde{K}^2 - \frac{m^2}{r}\right)Q_m(r,\tilde{r})\right)\sin(m\theta)$$

$$= -\delta(r,\tilde{r})\delta(\theta,\tilde{\theta}) \ . \tag{2.81}$$

Multiplying by 1, $\cos(m\theta)$, $\sin(m\theta)$ and integrating from 0 to 2π on eq(2.81), $m = 1, 2, 3, \ldots$ yields

$$H_0(\tilde{\theta}) = \frac{1}{2\pi} \ , \quad H_m(\tilde{\theta}) = \frac{1}{\pi}\cos(m\tilde{\theta}) \ , \quad L_m(\tilde{\theta}) = \frac{1}{\pi}\sin(m\tilde{\theta}) \ , \qquad m = 1,2,3,\ldots \ , \tag{2.82}$$

$$\mathcal{L}_n(P_n(r,\tilde{r})) = -\delta(r,\tilde{r}) \ , \quad n = 0,1,2,\ldots \ , \tag{2.83}$$

$$\mathcal{L}_m(Q_m(r,\tilde{r})) = -\delta(r,\tilde{r}) \ , \quad m = 1,2,3,\ldots \ , \tag{2.84}$$

where $\mathcal{L}_n$ is the *Sturm-Liouville* operator of order n [12] ;

$$\mathcal{L}_n(W) = rW'' + W' + r\left(\tilde{K}^2 - \frac{n^2}{r^2}\right)W \ , \quad n = 0,1,2,\ldots \ . \tag{2.85}$$

$\{1, \ \sin(m\theta), \ \cos(m\theta)\}_{m=1}^{\infty}$ is an orthogonal set of functions, eqs(2.79),(2.82) yield

$$P_n(1,\tilde{r}) = 0 \ , \quad P_n(0,\tilde{r}) \ is \ finite \ , \quad n = 0,1,2,\ldots \ , \tag{2.86}$$

$$Q_m(1,\tilde{r}) = 0 \ , \quad Q_m(0,\tilde{r}) \ is \ finite \ , \quad m = 1,2,3,\ldots \tag{2.87}$$

Thus,

$$P_n(r,\tilde{r}) = \begin{cases} \dfrac{\pi}{2}J_n(\tilde{K}r)\left(\dfrac{Y_n(\tilde{K})}{J_n(\tilde{K})}J_n(\tilde{K}\tilde{r}) - Y_n(\tilde{K}\tilde{r})\right) \ , & r \le \tilde{r} \\ \\ \dfrac{\pi}{2}J_n(\tilde{K}\tilde{r})\left(\dfrac{Y_n(\tilde{K})}{J_n(\tilde{K})}J_n(\tilde{K}r) - Y_n(\tilde{K}r)\right) \ , & r \ge \tilde{r} \end{cases} \ , \qquad n = 0,1,2,\ldots \ , \tag{2.88}$$

$$Q_m(r,\tilde{r}) = P_m(r,\tilde{r}) \ , \ m = 1,2,3,\ldots \ . \tag{2.89}$$

Eqs(2.80),(2.82),(2.88),(2.89) give

$$\tilde{G}(r,\theta;\tilde{r},\tilde{\theta}) = \frac{1}{2\pi}P_0(r,\tilde{r}) + \frac{1}{\pi}\sum_{m=1}^{\infty}\cos\left(m(\theta-\tilde{\theta})\right)P_m(r,\tilde{r}) \ . \tag{2.90}$$

Thus, by eq(B.6),

$$G(r,\theta;\tilde{r},\tilde{\theta}) = \begin{cases} -\frac{1}{4}\left[J_0(K_0r)Y_0(K_0\tilde{r}) + \frac{Y_0(K_0)J_0(K_0r)J_0(K_0\tilde{r})}{K_0J_1(K_0)}\right. \\ \qquad -\frac{rY_0(K_0)J_0(K_0\tilde{r})J_1(K_0r)}{J_1(K_0)} \\ \qquad \left. -\frac{(J_0(K_0r)J_0(K_0\tilde{r})Y_1(K_0) + \tilde{r}J_0(K_0r)Y_0(K_0)J_1(K_0\tilde{r}))}{J_1(K_0)}\right] \\ -\frac{1}{2}\sum_{m=1}^{\infty}\cos\left(m(\theta-\tilde{\theta})\right)\left[J_m(K_0r)Y_m(K_0\tilde{r}) - \frac{J_m(K_0r)Y_m(K_0)J_m(K_0\tilde{r})}{J_m(K_0)}\right] \\ \qquad ,r \le \tilde{r} \\ \textit{interchange } r \textit{ and } \tilde{r} \textit{ in the above result of } r \le \tilde{r} \\ \qquad ,r \ge \tilde{r} \ . \end{cases} \tag{2.91}$$

Eq(2.54) yields

$$U_0(r,\theta) = J_0(K_0r) \ . \tag{2.92}$$

Then ,by eq(2.20),

$$V_1(r,\theta) = -\frac{J_0(K_0c)}{Y_0(K_0c)}Y_0(K_0r) \ . \tag{2.93}$$

Recurrence relations of *bessel* functions [10] give

$$J_1'(K_0) = \frac{1}{2}(J_0(K_0) - J_2(K_0)) = -\frac{1}{2}J_2(K_0) \ . \tag{2.94}$$

Thus,

$$W_1(r,\theta) = \oint_{S_0} \frac{\partial G(r,\theta;\tilde{r},\tilde{\theta})}{\partial n} V_1(\tilde{r},\tilde{\theta})\ ds$$

$$= \int_0^{2\pi} \frac{\partial G(r,\theta;\tilde{r},\tilde{\theta})}{\partial \tilde{r}}\,|_{\tilde{r}=1} \left(-\frac{J_0(K_0c)}{Y_0(K_0c)} Y_0(K_0)\right)\ d\tilde{\theta}$$

$$= -\pi Y_0(K_0)\frac{J_0(K_0c)}{Y_0(K_0c)}\left[\frac{K_0J_0(K_0r)Y_1(K_0)}{2} + J_0(K_0r)Y_0(K_0) - \frac{K_0Y_0(K_0)rJ_1(K_0r)}{2}\right.$$

$$\left. - \frac{\left(K_0Y_1(K_0)J_1(K_0)J_0(K_0r) + \frac{K_0}{2}J_2(K_0)Y_0(K_0)J_0(K_0r)\right)}{2J_1(K_0)}\right] . \tag{2.95}$$

Eqs(2.21),(2.65),(2.92) give

$$F_1 = \left(\frac{1}{K_0J_1^2(K_0)}\right)\frac{1}{|\ln c|} + \left(\frac{\ln K_0 + \gamma - \ln 2}{K_0J_1^2(K_0)}\right)\frac{1}{|\ln c|^2} + \cdots . \tag{2.96}$$

Eqs(2.46),(2.47),(2.48),(2.49),(2.92),(2.95),(2.96) give

$$V_2^h(r,\theta) = D_0Y_0(K_0r) , \tag{2.97}$$

where

$$D_0 = \frac{-1}{2\pi Y_0(K_0c)}\int_0^{2\pi} W_1(c,\theta)d\theta + \frac{F_1cJ_0(K_0c)Y_0'(K_0c)}{Y_0^2(K_0c)}$$

$$= \frac{\pi^3}{4}Y_0(K_0)\left[Y_0(K_0) + \frac{K_0}{2}Y_1(K_0) - \frac{\left(K_0J_1(K_0)Y_1(K_0) + \frac{K_0}{2}Y_0(K_0)J_2(K_0)\right)}{2J_1(K_0)}\right]\frac{1}{|\ln c|^2} + \cdots . \tag{2.98}$$

Thus,

$$\oint_{S_B} \left(\frac{\partial U_0(r,\theta)}{\partial n} V_2^h(r,\theta) - \frac{\partial V_2^h(r,\theta)}{\partial n} U_0(r,\theta) \right) ds$$

$$= \int_0^{2\pi} \left(\frac{-\partial J_0(K_0 r)}{\partial r} \mid_{r=c} D_0 Y_0(K_0 c) + \frac{\partial D_0 Y_0(K_0 r)}{\partial r} \mid_{r=c} J_0(K_0 c) \right) c \, d\theta$$

$$= 2\pi c K_0 D_0 \left(-J_0'(K_0 c) Y_0(K_0 c) + Y_0'(K_0 c) J_0(K_0 c) \right)$$

$$= 4D_0$$

$$= \pi^3 Y_0(K_0) \left[Y_0(K_0) + \frac{K_0}{2} Y_1(K_0) - \frac{\left(K_0 J_1(K_0) Y_1(K_0) + \frac{K_0}{2} Y_0(K_0) J_2(K_0) \right)}{2 J_1(K_0)} \right] \frac{1}{|\ln c|^2} + \cdots . \tag{2.99}$$

Eqs(2.34),(2.92) yield

$$V_2^i(r,\theta) = F_1 \left(\frac{-J_0(K_0 c)}{Y_0(K_0 c)} r Y_0'(K_0 r) \right) . \tag{2.100}$$

Thus,

$$\oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} V_2^i(r,\theta) \, ds = \int_0^{2\pi} \frac{\partial J_0(K_0 r)}{\partial r} \mid_{r=1} V_2^i(1,\theta) \, d\theta$$

$$= -2\pi K_0 J_1(K_0) Y_1(K_0) F_1 \frac{J_0(K_0 c)}{Y_0(K_0 c)}$$

$$= \pi^2 \frac{Y_1(K_0)}{J_1(K_0)} \frac{1}{|\ln c|^2} + \cdots . \tag{2.101}$$

Eqs(2.43),(2.65),(2.92),(2.96),(2.99),(2.101) give

$$F_2 = \left\{ \frac{-1}{2K_0^3 J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)} + \right.$$

$$\left.\frac{\pi^2 Y_0(K_0)\left[2Y_0(K_0)+K_0Y_1(K_0)-\dfrac{\left(K_0J_1(K_0)Y_1(K_0)+\dfrac{K_0}{2}Y_0(K_0)J_2(K_0)\right)}{J_1(K_0)}\right]}{4K_0J_1^2(K_0)}\right\}\frac{1}{|\ln c|^2}$$

$$+\cdots .$$

(2.102)

Eqs(1.19),(2.96),(2.102) give

$$K = K_0 + F_1 + F_2 + \cdots$$

$$= K_0 + \left(\frac{1}{K_0J_1^2(K_0)}\right)\frac{1}{|\ln c|} + \left\{\frac{(\ln K_0+\gamma-\ln 2)}{K_0J_1^2(K_0)} - \frac{1}{2K_0^3J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0J_1^3(K_0)} + \right.$$

$$\left.\frac{\pi^2 Y_0(K_0)\left[2Y_0(K_0)+K_0Y_1(K_0)-\dfrac{\left(K_0J_1(K_0)Y_1(K_0)+\dfrac{K_0}{2}Y_0(K_0)J_2(K_0)\right)}{J_1(K_0)}\right]}{4K_0J_1^2(K_0)}\right\}\frac{1}{|\ln c|^2}$$

$$+\cdots ,$$

(2.103)

Recurrence relations for cross-products of *bessel* functions [10] give

$$M_0T_0 - N_0G_0 = \frac{4}{\pi^2K_0^2} , \tag{2.104}$$

where

$$M_0 = J_0(K_0)Y_0(K_0) - J_0(K_0)Y_0(K_0) = 0 , \tag{2.105}$$

$$N_0 = J_0(K_0)Y_0'(K_0) - J_0'(K_0)Y_0(K_0) = J_1(K_0)Y_0(K_0) \ , \tag{2.106}$$

$$G_0 = J_0'(K_0)Y_0(K_0) - J_0(K_0)Y_0'(K_0) = -J_1(K_0)Y_0(K_0) \ , \tag{2.107}$$

$$T_0 = J_0'(K_0)Y_0'(K_0) - J_0'(K_0)Y_0'(K_0) = 0 \ . \tag{2.108}$$

Then,

$$\frac{\pi}{2}Y_0(K_0) = \frac{1}{K_0 J_1(K_0)} \ . \tag{2.109}$$

Recurrence relations of *bessel* functions [10] give

$$\frac{2}{K_0}J_1(K_0) = J_0(K_0) + J_2(K_0) = J_2(K_0) \ . \tag{2.110}$$

Thus,

$$\begin{aligned} K = K_0 &+ \left(\frac{1}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|} \\ &+ \left[\frac{(\ln K_0 + \gamma - \ln 2)}{K_0 J_1^2(K_0)} + \frac{1}{2K_0^3 J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)}\right]\frac{1}{|\ln c|^2} + \cdots \ , \end{aligned} \tag{2.111}$$

where $K_0 = K_{0,1} \approx 2.4048$ and $\gamma \approx 0.5772$.

CHAPTER 3

S_B is a Strip of Length $2c$ Centered at (x_0, y_0)

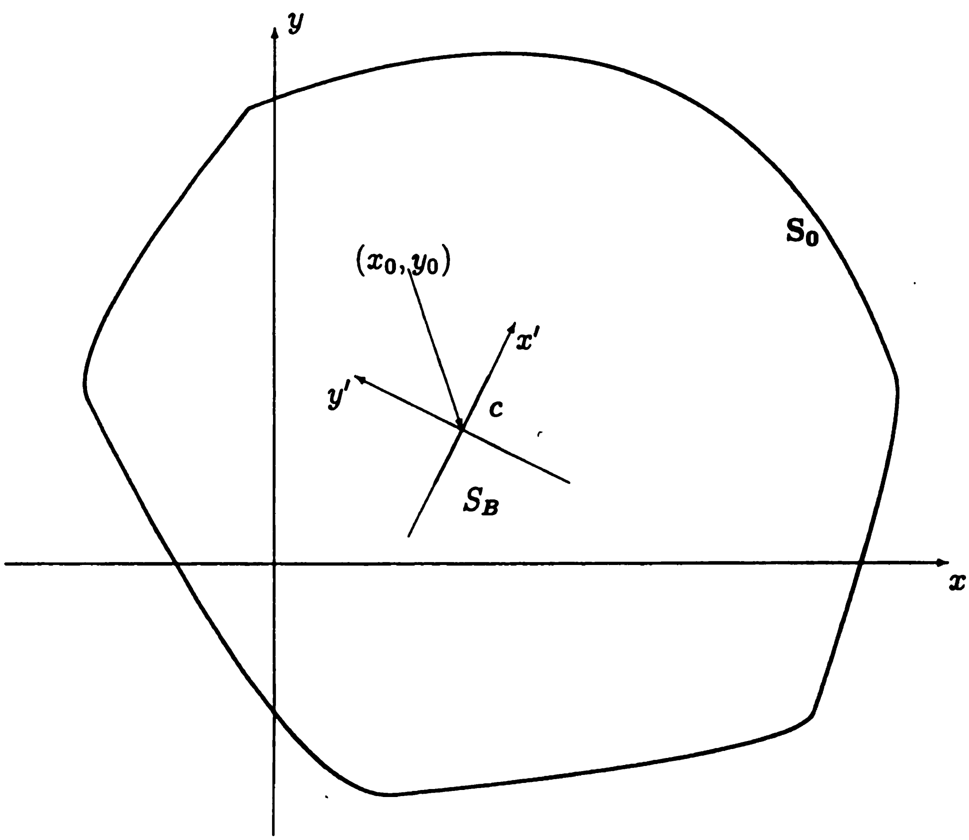


Figure 3.1: S_B is a strip of length $2c$ centered at (x_0, y_0)

The cartesian coordinates (x', y') and (x, y), as in Figure 3.1, are related by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi_0 & \sin\varphi_0 \\ -\sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}, \tag{3.1}$$

where $\varphi_0 = \theta_0 + \theta_1$, θ_0 is the angle from x axis to $\overleftrightarrow{OO_1}$, θ_1 is the angle from $\overleftrightarrow{OO_1}$ to x' axis, and $O = (0,0)$, $O_1 = (x_0, y_0)$. Let (ξ',η') be elliptic coordinates related to the cartesian coordinates (x,y) by

$$x = c\cosh\xi'\cos\eta' \quad , \quad y = c\sinh\xi'\sin\eta' \ , \tag{3.2}$$

and (ξ,η) be elliptic coordinates related to the cartesian coordinates (x',y') by

$$x' = c\cosh\xi\cos\eta \quad , \quad y' = c\sinh\xi\sin\eta \ , \tag{3.3}$$

where $2c$ is the distance between the foci. Due to the invariability of the governing Helmholtz equation under the translation and rotation of coordinates, the governing Helmholtz equation can be written as

$$\frac{\partial^2 U(\xi,\eta)}{\partial\xi^2} + \frac{\partial^2 U(\xi,\eta)}{\partial\eta^2} + \frac{K^2c^2}{2}\left(\cosh(2\xi) - \cos(2\eta)\right)U(\xi,\eta) = 0 \ . \tag{3.4}$$

Eq(3.4) can be separated by $U(\xi,\eta) = \Psi(\xi)\Phi(\eta)$ resulting in *Mathieu* equations [12]

$$\frac{d^2\Psi}{d\xi^2} + \left[h^2\cosh^2\xi - b\right]\Psi = 0 \ , \tag{3.5}$$

$$\frac{d^2\Phi}{d\eta^2} + \left[b - h^2\cos^2\eta\right]\Phi = 0 \ , \tag{3.6}$$

where $h = Kc$ and b is a separation constant. The periodic solutions [12] to eq(3.6) are S_{o_p}, the p^{th} order *odd angular Mathieu* function, and S_{e_q}, the q^{th} order *even angular Mathieu* function, where

$$S_{o_p}(h,\cos\eta) = \sum_{l=1}^{\infty} B_{2l}^{o}(h,p)\sin(2l\eta); \qquad \sum_{l=1}^{\infty}(2l)B_{2l}^{o}(h,p) = 1$$

$$if\ p\ is\ even\ , \tag{3.7}$$

$$S_{o_p}(h,\cos\eta) = \sum_{l=0}^{\infty} B^o_{2l+1}(h,p)\sin\left((2l+1)\eta\right); \quad \sum_{l=0}^{\infty}(2l+1)B^o_{2l+1}(h,p) = 1$$
$$if\ p\ is\ odd\ , \tag{3.8}$$

$$S_{e_q}(h,\cos\eta) = \sum_{l=0}^{\infty} B^e_{2l}(h,q)\cos(2l\eta); \qquad \sum_{l=0}^{\infty} B^e_{2l}(h,q) = 1$$
$$if\ q\ is\ even\ , \tag{3.9}$$

$$S_{e_q}(h,\cos\eta) = \sum_{l=0}^{\infty} B^e_{2l+1}(h,q)\cos\left((2l+1)\eta\right); \qquad \sum_{l=0}^{\infty} B^e_{2l+1}(h,q) = 1$$
$$if\ q\ is\ odd\ , \tag{3.10}$$

$$p = 1,2,3,\dots\ ,\ q = 0,1,2,\dots\ .$$

The corresponding solutions [13] to eq(3.5) are J_{o_p}, the p^{th} order *odd radial Mathieu* function of the *first* kind, N_{o_p}, the p^{th} order *odd radial Mathieu* function of the *second* kind, J_{e_q},the q^{th} order *even radial Mathieu* function of the *first* kind, and N_{e_q}, the q^{th} order *even radial Mathieu* function of the *second* kind, where

$$J_{o_p}(h,\cosh\xi) = \sqrt{\frac{\pi}{2}}\tanh\xi\sum_{l=1}^{\infty}\left\{(-1)^{l-\frac{p}{2}}(2l)B^o_{2l}(h,p)J_{2l}(h\cosh\xi)\right\}$$
$$= \frac{\sqrt{\frac{\pi}{2}}}{B^o_2(h,p)}\sum_{l=1}^{\infty}\Big\{(-1)^{l-\frac{p}{2}}B^o_{2l}(h,p)$$
$$\left[J_{l-1}(\frac{1}{2}he^{-\xi})J_{l+1}(\frac{1}{2}he^{\xi}) - J_{l+1}(\frac{1}{2}he^{-\xi})J_{l-1}(\frac{1}{2}he^{\xi})\right]\Big\}\ if\ p\ is\ even\ , \tag{3.11}$$

$$J_{o_p}(h,\cosh\xi) = \sqrt{\frac{\pi}{2}}\tanh\xi\sum_{l=0}^{\infty}\left\{(-1)^{l-\frac{p-1}{2}}(2l+1)B^o_{2l+1}(h,p)J_{2l+1}(h\cosh\xi)\right\}$$

$$= \frac{\sqrt{\frac{\pi}{2}}}{B_1^o(h,p)} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{p-1}{2}} B_{2l+1}^o(h,p)$$

$$\Big[J_l(\frac{1}{2}he^{-\xi}) J_{l+1}(\frac{1}{2}he^{\xi}) - J_{l+1}(\frac{1}{2}he^{-\xi}) J_l(\frac{1}{2}he^{\xi}) \Big] \Big\} \ \textit{if p is odd} \ , \tag{3.12}$$

$$J_{e_q}(h, \cosh\xi) = \sqrt{\frac{\pi}{2}} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q}{2}} B_{2l}^e(h,q) J_{2l}(h\cosh\xi) \Big\}$$

$$= \frac{\sqrt{\frac{\pi}{2}}}{B_0^e(h,q)} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q}{2}} B_{2l}^e(h,q) J_l(\frac{1}{2}he^{\xi}) J_l(\frac{1}{2}he^{-\xi}) \Big\} \ \textit{if q is even} \ , \tag{3.13}$$

$$J_{e_q}(h, \cosh\xi) = \sqrt{\frac{\pi}{2}} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q-1}{2}} B_{2l+1}^e(h,q) J_{2l+1}(h\cosh\xi) \Big\}$$

$$= \frac{\sqrt{\frac{\pi}{2}}}{B_1^e(h,q)} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q-1}{2}} B_{2l+1}^e(h,q)$$

$$\Big[J_l(\frac{1}{2}he^{-\xi}) J_{l+1}(\frac{1}{2}he^{\xi}) + J_{l+1}(\frac{1}{2}he^{-\xi}) J_l(\frac{1}{2}he^{\xi}) \Big] \Big\} \ \textit{if q is odd} \ , \tag{3.14}$$

$$N_{o_p}(h, \cosh\xi) = \sqrt{\frac{\pi}{2}} \tanh\xi \sum_{l=1}^{\infty} \Big\{ (-1)^{l-\frac{p}{2}} (2l) B_{2l}^o(h,p) N_{2l}(h\cosh\xi) \Big\}$$

$$= \frac{\sqrt{\frac{\pi}{2}}}{B_2^o(h,p)} \sum_{l=1}^{\infty} \Big\{ (-1)^{l-\frac{p}{2}} B_{2l}^o(h,p)$$

$$\Big[J_{l-1}(\frac{1}{2}he^{-\xi}) N_{l+1}(\frac{1}{2}he^{\xi}) - J_{l+1}(\frac{1}{2}he^{-\xi}) N_{l-1}(\frac{1}{2}he^{\xi}) \Big] \Big\} \ \textit{if p is even} \ , \tag{3.15}$$

$$N_{o_p}(h, \cosh\xi) = \sqrt{\frac{\pi}{2}} \tanh\xi \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{p-1}{2}} (2l+1) B_{2l+1}^o(h,p) N_{2l+1}(h\cosh\xi) \Big\}$$

$$= \frac{\sqrt{\frac{\pi}{2}}}{B_1^o(h,p)} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{p-1}{2}} B_{2l+1}^o(h,p)$$
$$\Big[J_l(\frac{1}{2}he^{-\xi}) N_{l+1}(\frac{1}{2}he^{\xi}) - J_{l+1}(\frac{1}{2}he^{-\xi}) N_l(\frac{1}{2}he^{\xi}) \Big] \Big\} \quad if\ p\ is\ odd\ , \tag{3.16}$$

$$N_{e_q}(h, \cosh\xi) = \sqrt{\frac{\pi}{2}} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q}{2}} B_{2l}^e(h,q) N_{2l}(h\cosh\xi) \Big\}$$
$$= \frac{\sqrt{\frac{\pi}{2}}}{B_0^e(h,q)} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q}{2}} B_{2l}^e(h,q) N_l(\frac{1}{2}he^{\xi}) J_l(\frac{1}{2}he^{-\xi}) \Big\} \quad if\ q\ is\ even\ , \tag{3.17}$$

$$N_{e_q}(h, \cosh\xi) = \sqrt{\frac{\pi}{2}} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q-1}{2}} B_{2l+1}^e(h,q) N_{2l+1}(h\cosh\xi) \Big\}$$
$$= \frac{\sqrt{\frac{\pi}{2}}}{B_1^e(h,q)} \sum_{l=0}^{\infty} \Big\{ (-1)^{l-\frac{q-1}{2}} B_{2l+1}^e(h,q)$$
$$\Big[J_l(\frac{1}{2}he^{-\xi}) N_{l+1}(\frac{1}{2}he^{\xi}) + J_{l+1}(\frac{1}{2}he^{-\xi}) N_l(\frac{1}{2}he^{\xi}) \Big] \Big\} \quad if\ q\ is\ odd\ , \tag{3.18}$$

$$p = 1, 2, 3, \dots\ ,\ q = 0, 1, 2, \dots\ .$$

3.1 Membrane With a Strip of Length $2c$ Centered at (x_0, y_0)

$U_0(\xi, \eta)$ is finite in R_0, we assume that

$$U_0(\xi,\eta) = \sum_{m=1}^{\infty} A_{2m}^o(c, \varphi_0) S_{o_{2m}}(h_0, \cos\eta) J_{o_{2m}}(h_0, \cosh\xi)$$

$$+\sum_{n=0}^{\infty}\Big[A^{o}_{2n+1}(c,\varphi_0)S_{o_{2n+1}}(h_0,\cos\eta)J_{o_{2n+1}}(h_0,\cosh\xi)$$

$$+A^{e}_{2n}(c,\varphi_0)S_{e_{2n}}(h_0,\cos\eta)J_{e_{2n}}(h_0,\cosh\xi)$$

$$+A^{e}_{2n+1}(c,\varphi_0)S_{e_{2n+1}}(h_0,\cos\eta)J_{e_{2n+1}}(h_0,\cosh\xi)\Big] \tag{3.19}$$

with coefficients $A^{o}_{2m}(c,\varphi_0)$, $A^{o}_{2n+1}(c,\varphi_0)$, $A^{e}_{2n}(c,\varphi_0)$, and $A^{e}_{2n+1}(c,\varphi_0)$ determined by the boundary condition, eq(1.5), where $h_0 = K_0c$.

$\{S_{o_{2(n+1)}}(h,\cos\eta),\ S_{o_{2n+1}}(h,\cos\eta),\ S_{e_{2n}}(h,\cos\eta),\ S_{e_{2n+1}}(h,\cos\eta)\}_{n=0}^{\infty}$ is a complete orthogonal set of functions and [13]

$$\int_0^{2\pi} S^2_{o_{2(n+1)}}(h,\cos\eta)\ d\eta = \pi\sum_{l=1}^{\infty}(B^{o}_{2l}(h,2(n+1)))^2 = M^{o}_{2(n+1)}(h)\ , \tag{3.20}$$

$$\int_0^{2\pi} S^2_{o_{2n+1}}(h,\cos\eta)\ d\eta = \pi\sum_{l=0}^{\infty}(B^{o}_{2l+1}(h,2n+1))^2 = M^{o}_{2n+1}(h)\ , \tag{3.21}$$

$$\int_0^{2\pi} S^2_{e_{2n}}(h,\cos\eta)\ d\eta = 2\pi\sum_{l=0}^{\infty}\left(\frac{1}{\epsilon_l}\right)(B^{e}_{2l}(h,2n))^2 = M^{e}_{2n}(h)\ , \tag{3.22}$$

$$\int_0^{2\pi} S^2_{e_{2n+1}}(h,\cos\eta)\ d\eta = \pi\sum_{l=0}^{\infty}(B^{e}_{2l+1}(h,2n+1))^2 = M^{e}_{2n+1}(h)\ , \tag{3.23}$$

where

$$\epsilon_l = \begin{cases} 1 & if\ l = 0 \\ 2 & if\ l \neq 0\ . \end{cases} \tag{3.24}$$

[13] gives

$$J_{o_p}(h_0,1) = 0\ ,\quad p = 1,2,3,\ldots\ . \tag{3.25}$$

Thus, eqs(1.22),(1.23) give

$$V_1(\xi,\eta) = \sum_{m=1}^{\infty}\tilde{A}^{o}_{2m}(c,\varphi_0)S_{o_{2m}}(h_0,\cos\eta)\left(J_{o_{2m}}(h_0,\cosh\xi) - \frac{J_{o_{2m}}(h_0,1)}{N_{o_{2m}}(h_0,1)}N_{o_{2m}}(h_0,\cosh\xi)\right)$$

$$+\sum_{n=0}^{\infty}\Bigg[\tilde{A}^{o}_{2n+1}(c,\varphi_0)S_{o_{2n+1}}(h_0,\cos\eta)\left(J_{o_{2n+1}}(h_0,\cosh\xi)-\frac{J_{o_{2n+1}}(h_0,1)}{N_{o_{2n+1}}(h_0,1)}N_{o_{2n+1}}(h_0,\cosh\xi)\right)$$

$$+\tilde{A}^{e}_{2n}(c,\varphi_0)S_{e_{2n}}(h_0,\cos\eta)\left(J_{e_{2n}}(h_0,\cosh\xi)-\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}N_{e_{2n}}(h_0,\cosh\xi)\right)$$

$$+\tilde{A}^{e}_{2n+1}(c,\varphi_0)S_{e_{2n+1}}(h_0,\cos\eta)\left(J_{e_{2n+1}}(h_0,\cosh\xi)-\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}N_{e_{2n+1}}(h_0,\cosh\xi)\right)\Bigg]$$

$$-\sum_{m=1}^{\infty}A^{o}_{2m}(c,\varphi_0)\frac{J_{o_{2m}}(h_0,1)}{N_{o_{2m}}(h_0,1)}N_{o_{2m}}(h_0,\cosh\xi)S_{o_{2m}}(h_0,\cos\eta)$$

$$-\sum_{n=0}^{\infty}\Bigg[A^{o}_{2n+1}(c,\varphi_0)\frac{J_{o_{2n+1}}(h_0,1)}{N_{o_{2n+1}}(h_0,1)}N_{o_{2n+1}}(h_0,\cosh\xi)S_{o_{2n+1}}(h_0,\cos\eta)$$

$$+A^{e}_{2n}(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}N_{e_{2n}}(h_0,\cosh\xi)S_{e_{2n}}(h_0,\cos\eta)$$

$$+A^{e}_{2n+1}(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}N_{e_{2n+1}}(h_0,\cosh\xi)S_{e_{2n+1}}(h_0,\cos\eta)\Bigg]$$

$$=\sum_{m=1}^{\infty}\tilde{A}^{o}_{2m}(c,\varphi_0)S_{o_{2m}}(h_0,\cos\eta)J_{o_{2m}}(h_0,\cosh\xi)$$

$$+\sum_{n=0}^{\infty}\Big[\tilde{A}^{o}_{2n+1}(c,\varphi_0)S_{o_{2n+1}}(h_0,\cos\eta)J_{o_{2n+1}}(h_0,\cosh\xi)$$

$$+\tilde{A}^{e}_{2n}(c,\varphi_0)S_{e_{2n}}(h_0,\cos\eta)\left(J_{e_{2n}}(h_0,\cosh\xi)-\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}N_{e_{2n}}(h_0,\cosh\xi)\right)$$

$$+\tilde{A}^{e}_{2n+1}(c,\varphi_0)S_{e_{2n+1}}(h_0,\cos\eta)\left(J_{e_{2n+1}}(h_0,\cosh\xi)-\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}N_{e_{2n+1}}(h_0,\cosh\xi)\right)\Bigg]$$

$$-\sum_{n=0}^{\infty}\Bigg[A^{e}_{2n}(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}N_{e_{2n}}(h_0,\cosh\xi)S_{e_{2n}}(h_0,\cos\eta)$$

$$+A^e_{2n+1}(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}N_{e_{2n+1}}(h_0,\cosh\xi)S_{e_{2n+1}}(h_0,\cos\eta)\Bigg] \ . \tag{3.26}$$

[13] gives

$$\mathcal{W}\left(J_{e_q}(h,\cosh\xi),N_{e_q}(h,\cosh\xi)\right)=1\ , \quad q=0,1,2,\dots\ , \tag{3.27}$$

where $\mathcal{W}\left(J_{e_q}(h,\cosh\xi),N_{e_q}(h,\cosh\xi)\right)$ is the Wronskian of $J_{e_q}(h,\cosh\xi)$ and $N_{e_q}(h,\cosh\xi)$. Thus, by eqs(2.16),(3.19),(3.22),(3.23),(3.25),(3.26),(3.27),

$$\begin{aligned}
F_1 &= \frac{\oint_{S_B} U_0(x,y)\left(\frac{\partial V_1(x,y)}{\partial n}+\frac{\partial U_0(x,y)}{\partial n}\right)ds}{-2K_0\int_{R_0}U_0^2(x,y)\,dA} \\
&= -\frac{\int_0^{2\pi} U_0(0,\eta)\left(\frac{\partial V_1(\xi,\eta)}{\partial \xi}\big|_{\xi=0}+\frac{\partial U_0(\xi,\eta)}{\partial \xi}\big|_{\xi=0}\right)d\eta}{-2K_0\int_{R_0}U_0^2(\xi,\eta)\,dA} \\
&= \frac{1}{-2K_0\int_{R_0}U_0^2(\xi,\eta)\,dA}\sum_{n=0}^{\infty}\Bigg[\tilde{A}^e_{2n}(c,\varphi_0)A^e_{2n}(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}M^e_{2n}(h_0) \\
&\qquad +\tilde{A}^e_{2n+1}(c,\varphi_0)A^e_{2n+1}(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}M^e_{2n+1}(h_0)\Bigg] \\
&\quad +\frac{1}{-2K_0\int_{R_0}U_0^2(\xi,\eta)\,dA}\sum_{n=0}^{\infty}\Bigg[\left(A^e_{2n}\right)^2(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}M^e_{2n}(h_0) \\
&\qquad +\left(A^e_{2n+1}\right)^2(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}M^e_{2n+1}(h_0)\Bigg] \ .
\end{aligned} \tag{3.28}$$

$\tilde{A}^o_{2m}(c,\varphi_0)$ and $\tilde{A}^o_{2n+1}(c,\varphi_0)$ do not affect F_1, the simplest choice is to set them equal to *zero*. [13] gives

$$B^o_{2l}(h,p)=O(h^{|2l-p|})\ , \tag{3.29}$$

$$B^{o}_{2l+1}(h,p) = O(h^{|2l+1-p|}) \; , \tag{3.30}$$

$$B^{e}_{2l}(h,q) = O(h^{|2l-q|}) \; , \tag{3.31}$$

$$B^{e}_{2l+1}(h,q) = O(h^{|2l+1-q|}) \; . \tag{3.32}$$

$$\frac{J_{e_{2n+1}}(h,1)}{N_{e_{2n+1}}(h,1)} = O(h^{4n+2}) \; , \tag{3.33}$$

$$\frac{J_{e_{2n}}(h,1)}{N_{e_{2n}}(h,1)} = \begin{cases} O(\frac{1}{\ln h}) & n=0 \\ \\ O(h^{4n}) & n \neq 0 \; . \end{cases} \tag{3.34}$$

$$A^{e}_{2n}(c,\varphi_0) = O(1) \quad , \quad A^{e}_{2n+1}(c,\varphi_0) = O(1) \; . \tag{3.35}$$

$h_0 \cosh\xi$ is $O(1)$ on S_0. Thus, to correct the boundary condition on S_0 to $O(\frac{1}{\ln c})$ for (U_0+V_1), $\tilde{A}^{e}_{2n}(c,\varphi_0)$ and $\tilde{A}^{e}_{2n+1}(c,\varphi_0)$ must be at most $O(\frac{1}{\ln c})$, the simplest choice is to set them equal to *zero*. The other choices would only lead to a higher order correction to the first order result F_1, eq(3.28). Thus,

$$V_1(\xi,\eta) = -\sum_{n=0}^{\infty}\Bigg[A^{e}_{2n}(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}N_{e_{2n}}(h_0,\cosh\xi)S_{e_{2n}}(h_0,\cos\eta)$$

$$+A^{e}_{2n+1}(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}N_{e_{2n+1}}(h_0,\cosh\xi)S_{e_{2n+1}}(h_0,\cos\eta)\Bigg] \; , \tag{3.36}$$

and

$$F_1 = \frac{\sum_{n=0}^{\infty}\left[\left(A^{e}_{2n}\right)^2(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}M^{e}_{2n}(h_0) + \left(A^{e}_{2n+1}\right)^2(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}M^{e}_{2n+1}(h_0)\right]}{-2K_0\int_{R_0} U_0^2(\xi,\eta)\, dA} \; .$$

$$(3.37)$$

The expansion formulas connecting the circular cylindrical wave functions with the concentric elliptical ones [5] give

$$J_p(Kr)\sin\left(p(\theta-\varphi_0)\right) = \frac{\sqrt{8\pi}}{2}\sum_{m=1}^{\infty}\frac{B_p^o(h,m)}{M_m^o(h)}S_{o_m}(h,\cos\eta)J_{o_m}(h,\cosh\xi)\ , \tag{3.38}$$

$$p = 1,2,3,\dots\ ,$$

where p and m are both *even* or *odd.*

and

$$J_q(Kr)\cos\left(q(\theta-\varphi_0)\right) = \frac{\sqrt{8\pi}}{\epsilon_q}\sum_{n=0}^{\infty}\frac{B_q^e(h,n)}{M_n^e(h)}S_{e_n}(h,\cos\eta)J_{e_n}(h,\cosh\xi)\ , \tag{3.39}$$

$$q = 0,1,2,\dots\ ,$$

where q and n are both *even* or *odd.* Then,

$$A_0^e(c,\varphi_0) = \sqrt{8\pi}B_0\frac{B_0^e(h_0,0)}{M_0^e(h_0)} + \sqrt{2\pi}\sum_{p=1}^{\infty}\left[\frac{B_{2p}^e(h_0,0)}{M_0^e(h_0)}\left(A_{2p}\sin(2p\varphi_0)+B_{2p}\cos(2p\varphi_0)\right)\right]\ . \tag{3.40}$$

[13] gives

$$B_0^e(h,0) = 1+\frac{h^2}{8}+\frac{7h^4}{512}+\cdots\ , \tag{3.41}$$

$$M_0^e(h) = \pi\left(2+\frac{1}{2}h^2+\frac{13}{128}h^4+\cdots\right). \tag{3.42}$$

$$\frac{\left(B_0^e(h,0)\right)^2}{M_0^e(h)} = \frac{1}{2\pi}+0h^2+\cdots\ . \tag{3.43}$$

$$\frac{J_{e_0}(h,1)}{N_{e_0}(h,1)} = \frac{\pi}{2}\frac{1}{\ln c}+\frac{\pi}{2}\left(\ln 4-\gamma-\ln K_0\right)\frac{1}{(\ln c)^2}+\cdots\ . \tag{3.44}$$

Thus,

$$F_1 = \frac{\sum_{n=0}^{\infty}\left[\left(A_{2n}^{e}\right)^2(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}M_{2n}^{e}(h_0) + \left(A_{2n+1}^{e}\right)^2(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}M_{2n+1}^{e}(h_0)\right]}{-2K_0\int_{R_0} U_0^2(\xi,\eta)\,dA}$$

$$= \frac{\left[\left(A_0^{e}\right)^2(c,\varphi_0)\frac{J_{e_0}(h_0,1)}{N_{e_0}(h_0,1)}M_0^{e}(h_0)\right]}{-2K_0\int_{R_0} U_0^2(\xi,\eta)\,dA} + \cdots$$

$$= \frac{\left\{\sqrt{8\pi}B_0\frac{B_0^{e}(h_0,0)}{M_0^{e}(h_0)} + \sqrt{2\pi}\sum_{p=1}^{\infty}\left[\frac{B_{2p}^{e}(h_0,0)}{M_0^{e}(h_0)}\left(A_{2p}\sin(2p\varphi_0)+B_{2p}\cos(2p\varphi_0)\right)\right]\right\}^2}{-2K_0\int_{R_0} U_0^2(\xi,\eta)\,dA}$$

$$\frac{J_{e_0}(h_0,1)}{N_{e_0}(h_0,1)}M_0^{e}(h_0) + \cdots$$

$$= \frac{\pi B_0^2}{K_0\int_{R_0} U_0^2(\xi,\eta)\,dA}\frac{1}{|\ln c|} + \frac{\pi B_0^2\left(\ln K_0+\gamma-\ln 4\right)}{K_0\int_{R_0} U_0^2(\xi,\eta)\,dA}\frac{1}{|\ln c|^2} + \cdots\ ,$$

$$\textit{where } \gamma \approx 0.5772. \tag{3.45}$$

Eq(2.23) yields

$$W_1(\xi,\eta) = EU_0(\xi,\eta) + \sum_{N=1}^{\infty}\sum_{j=1}^{l(N)}\frac{U_{N,j}(\xi,\eta)}{(K_N^2-K_0^2)\|U_{N,j}\|^2}\oint_{S_0}\frac{\partial U_{N,j}(\xi,\eta)}{\partial n}V_1(\xi,\eta)\,ds\ , \tag{3.46}$$

where E is a constant and

$$U_{N,j}(\xi,\eta) = \sum_{m=1}^{\infty} A_{2m}^{o}(c,\varphi_0,N,j)S_{o_{2m}}(h_N,\cos\eta)J_{o_{2m}}(h_N,\cosh\xi)$$

$$+\sum_{n=0}^{\infty}\left[A_{2n+1}^{o}(c,\varphi_0,N,j)S_{o_{2n+1}}(h_N,\cos\eta)J_{o_{2n+1}}(h_N,\cosh\xi)\right.$$

$$+A_{2n}^{e}(c,\varphi_0,N,j)S_{e_{2n}}(h_N,\cos\eta)J_{e_{2n}}(h_N,\cosh\xi)$$

$$+A^{e}_{2n+1}(c,\varphi_0,N,j)S_{e_{2n+1}}(h_N,\cos\eta)J_{e_{2n+1}}(h_N,\cosh\xi)\Big]$$

(3.47)

with coefficients $A^{o}_{2m}(c,\varphi_0,N,j)$, $A^{o}_{2n+1}(c,\varphi_0,N,j)$, $A^{e}_{2n}(c,\varphi_0,N,j)$, and $A^{e}_{2n+1}(c,\varphi_0,N,j)$ determined by the boundary condition, eq(1.5), where $h_N = K_N c$. The expansion formulas connecting the circular cylindrical wave functions with the concentric elliptical ones [5] give

$$S_{e_q}(h,\cos\eta)N_{e_q}(h,\cosh\xi) = \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty} B^{e}_{n}(h,q)Y_n(Kr)\cos(n(\theta-\varphi_0)) \ , \qquad (3.48)$$

$$q = 0,1,2,\dots \ ,$$

where q and n are both *even* or *odd*.

Thus, by eqs(3.36),(3.40),(3.41),(3.42),(3.44),(3.48),

$$\oint_{S_0} \frac{\partial U_{N,j}(\xi,\eta)}{\partial n} V_1(\xi,\eta)\, ds$$

$$= -\sqrt{\frac{\pi}{2}}\sum_{q=0}^{\infty}\Bigg\{A^{e}_{2q}(c,\varphi_0)\frac{J_{e_{2q}}(h_0,1)}{N_{e_{2q}}(h_0,1)}\sum_{n=0}^{\infty}B^{e}_{2n}(h_0,2q)$$

$$\oint_{S_0} Y_{2n}(K_0 r)\cos(2n(\theta-\varphi_0))\,\frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds$$

$$+A^{e}_{2q+1}(c,\varphi_0)\frac{J_{e_{2q+1}}(h_0,1)}{N_{e_{2q+1}}(h_0,1)}\sum_{n=0}^{\infty}B^{e}_{2n+1}(h_0,2q+1)$$

$$\oint_{S_0} Y_{2n+1}(K_0 r)\cos((2n+1)(\theta-\varphi_0))\,\frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds\Bigg\}$$

$$= -\sqrt{\frac{\pi}{2}}A^{e}_{0}(c,\varphi_0)\frac{J_{e_0}(h_0,1)}{N_{e_0}(h_0,1)}\sum_{n=0}^{\infty}B^{e}_{2n}(h_0,0)$$

$$\oint_{S_0} Y_{2n}(K_0 r)\cos(2n(\theta-\varphi_0))\,\frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds + \cdots$$

$$= \left[\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\left(\lim_{c\to 0} A_0^e(c,\varphi_0)\right)\oint_{S_0} Y_0(K_0 r)\frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds\right]\frac{1}{|\ln c|}+\cdots$$

$$= \left(\frac{\pi B_0}{2}\oint_{S_0} Y_0(K_0 r)\frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds\right)\frac{1}{|\ln c|}+\cdots\ . \tag{3.49}$$

To correct the boundary condition on S_B to $O(\frac{1}{\ln c|})$ for $(U_0+V_1+W_1)$, the constant E must be *zero*. Thus,

$$W_1(\xi,\eta) = \sum_{N=1}^{\infty}\sum_{j=1}^{l(N)}\frac{U_{N,j}(\xi,\eta)}{(K_N^2-K_0^2)\|U_{N,j}\|^2}\oint_{S_0}\frac{\partial U_{N,j}(\xi,\eta)}{\partial n}V_1(\xi,\eta)\, ds\ . \tag{3.50}$$

Let

$$V_2(x,y) = V_2^i(x,y)+V_2^h(x,y)\ , \tag{3.51}$$

where

$$\triangle V_2^i(x,y)+K_0^2V_2^i(x,y) = -2K_0F_1V_1(x,y)\ in\ R\ , \tag{3.52}$$

$$\triangle V_2^h(x,y)+K_0^2V_2^h(x,y) = 0\ in\ R\ , \tag{3.53}$$

$$V_2^h(x,y) = -W_1(x,y)-V_2^i(x,y)\ on\ S_B\ . \tag{3.54}$$

The method of variation of parameters [3] gives

$$V_2^i(\xi,\eta) = 2K_0F_1\sum_{n=0}^{\infty}\left\{A_{2n}^e(c,\varphi_0)\frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}S_{e_{2n}}(h_0,\cos\eta)\right.$$

$$[F_{2n}^e(\xi)N_{e_{2n}}(h_0,\cosh\xi)-G_{2n}^e(\xi)J_{e_{2n}}(h_0,\cosh\xi)]$$

$$+A_{2n+1}^e(c,\varphi_0)\frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}S_{e_{2n+1}}(h_0,\cos\eta)$$

$$\left.\left[F_{2n+1}^e(\xi)N_{e_{2n+1}}(h_0,\cosh\xi)-G_{2n+1}^e(\xi)J_{e_{2n+1}}(h_0,\cosh\xi)\right]\right\}\ , \tag{3.55}$$

where

$$F_q^e(\xi) = \int^{\xi} J_{e_q}(h_0, \cosh\mu) N_{e_q}(h_0, \cosh\mu)\ d\mu\ , \tag{3.56}$$

$$G_q^e(\xi) = \int^{\xi} N_{e_q}^2(h_0, \cosh\mu)\ d\mu\ , \tag{3.57}$$

$$q = 0, 1, 2, \dots\ .$$

Alternatively, eqs(2.33),(3.36),(3.48) give

$$V_2^i(\xi,\eta) = -\sqrt{\frac{\pi}{2}} F_1 r \sum_{q=0}^{\infty} \left\{ A_{2q}^e(c,\varphi_0) \frac{J_{e_{2q}}(h_0,1)}{N_{e_{2q}}(h_0,1)} \sum_{n=0}^{\infty} B_{2n}^e(h_0,2q) Y_{2n}'(K_0 r) \cos(2n(\theta-\varphi_0)) \right.$$

$$+ A_{2q+1}^e(c,\varphi_0) \frac{J_{e_{2q+1}}(h_0,1)}{N_{e_{2q+1}}(h_0,1)} \sum_{n=0}^{\infty} B_{2n+1}^e(h_0,2q+1) Y_{2n+1}'(K_0 r)$$

$$\left. \cos((2n+1)(\theta-\varphi_0)) \right\}\ . \tag{3.58}$$

Eqs(3.20),(3.21),(3.22),(3.23),(3.53),(3.54),(3.55) yield

$$V_2^h(\xi,\eta) = \sum_{m=1}^{\infty} \tilde{C}_{2m}^o(c,\varphi_0) S_{o_{2m}}(h_0,\cos\eta) J_{o_{2m}}(h_0,\cosh\xi)$$

$$+ \sum_{n=0}^{\infty} \left[\tilde{C}_{2n+1}^o(c,\varphi_0) S_{o_{2n+1}}(h_0,\cos\eta) J_{o_{2n+1}}(h_0,\cosh\xi) \right.$$

$$+ \tilde{C}_{2n}^e(c,\varphi_0) S_{e_{2n}}(h_0,\cos\eta) \left(J_{e_{2n}}(h_0,\cosh\xi) - \frac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)} N_{e_{2n}}(h_0,\cosh\xi) \right)$$

$$\left. + \tilde{C}_{2n+1}^e(c,\varphi_0) S_{e_{2n+1}}(h_0,\cos\eta) \left(J_{e_{2n+1}}(h_0,\cosh\xi) - \frac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)} N_{e_{2n+1}}(h_0,\cosh\xi) \right) \right]$$

$$+ \sum_{m=1}^{\infty} C_{2m}^o(c,\varphi_0)(h_0,\cosh\xi) S_{o_{2m}}(h_0,\cos\eta)$$

$$+\sum_{n=0}^{\infty}\Big[C^o_{2n+1}(c,\varphi_0)N_{o_{2n+1}}(h_0,\cosh\xi)S_{o_{2n+1}}(h_0,\cos\eta)$$

$$+C^e_{2n}(c,\varphi_0)N_{e_{2n}}(h_0,\cosh\xi)S_{e_{2n}}(h_0,\cos\eta)$$

$$+C^e_{2n+1}(c,\varphi_0)N_{e_{2n+1}}(h_0,\cosh\xi)S_{e_{2n+1}}(h_0,\cos\eta)\Big] \ , \tag{3.59}$$

where

$$C^o_{2m}(c,\varphi_0) = \frac{-\int_0^{2\pi} W_1(0,\eta)S_{o_{2m}}(h_0,\cos\eta)\ d\eta}{M^o_{2m}(h_0)N_{o_{2m}}(h_0,1)} \ , \tag{3.60}$$

$$C^o_{2n+1}(c,\varphi_0) = \frac{-\int_0^{2\pi} W_1(0,\eta)S_{o_{2n+1}}(h_0,\cos\eta)\ d\eta}{M^o_{2n+1}(h_0)N_{o_{2n+1}}(h_0,1)} \ , \tag{3.61}$$

$$C^e_{2n}(c,\varphi_0) = \frac{-\int_0^{2\pi} W_1(0,\eta)S_{e_{2n}}(h_0,\cos\eta)\ d\eta}{M^e_{2n}(h_0)N_{e_{2n}}(h_0,1)}$$

$$-\frac{2K_0F_1A^e_{2n}(c,\varphi_0)\dfrac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}\left[F^e_{2n}(0)N_{e_{2n}}(h_0,1)-G^e_{2n}(0)J_{e_{2n}}(h_0,1)\right]}{N_{e_{2n}}(h_0,1)} \ , \tag{3.62}$$

$$C^e_{2n+1}(c,\varphi_0) = \frac{-\int_0^{2\pi} W_1(0,\eta)S_{e_{2n+1}}(h_0,\cos\eta)\ d\eta}{M^e_{2n+1}(h_0)N_{e_{2n+1}}(h_0,1)}$$

$$-\frac{2K_0F_1A^e_{2n}(c,\varphi_0)\dfrac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}\left[F^e_{2n+1}(0)N_{e_{2n+1}}(h_0,1)-G^e_{2n+1}(0)J_{e_{2n+1}}(h_0,1)\right]}{N_{e_{2n+1}}(h_0,1)} \ . \tag{3.63}$$

Eqs(3.7),(3.8),(3.9),(3.10),(3.47),(3.49),(3.50) give

$$\int_0^{2\pi} W_1(0,\eta) S_{e_{2n}}(h_0, \cos\eta)\, d\eta$$

$$= \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\left(\oint_{S_0} \frac{\partial U_{N,j}(\xi,\eta)}{\partial n} V_1(\xi,\eta)\, ds \right)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \left[\sum_{q=0}^{\infty} A_{2q}^e(c, \varphi_0, N, j) J_{e_{2q}}(h_N, 1) \right.$$

$$\left. \left(2\pi B_0^e(h_N, 2q) B_0^e(h_0, 2n) + \pi \sum_{l=1}^{\infty} B_{2l}^e(h_N, 2q) B_{2l}^e(h_0, 2n) \right) \right]$$

$$= \left\{ \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\left(\frac{\pi B_0}{2} \oint_{S_0} Y_0(K_0 r) \frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds \right)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \left[\sum_{q=0}^{\infty} A_{2q}^e(c, \varphi_0, N, j) J_{e_{2q}}(h_N, 1) \right. \right.$$

$$\left. \left. \left(2\pi B_0^e(h_N, 2q) B_0^e(h_0, 2n) + \pi \sum_{l=1}^{\infty} B_{2l}^e(h_N, 2q) B_{2l}^e(h_0, 2n) \right) \right] \right\}$$

$$\frac{1}{|\ln c|} + \cdots , \tag{3.65}$$

$$\int_0^{2\pi} W_1(0,\eta) S_{e_{2n+1}}(h_0, \cos\eta)\, d\eta$$

$$= \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\left(\oint_{S_0} \frac{\partial U_{N,j}(\xi,\eta)}{\partial n} V_1(\xi,\eta)\, ds \right)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \left[\sum_{q=0}^{\infty} A_{2q+1}^e(c, \varphi_0, N, j) J_{e_{2q+1}}(h_N, 1) \right.$$

$$\left. \left(\pi \sum_{l=0}^{\infty} B_{2l+1}^e(h_N, 2q+1) B_{2l+1}^e(h_0, 2n+1) \right) \right]$$

$$= \left\{ \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\left(\frac{\pi B_0}{2} \oint_{S_0} Y_0(K_0 r) \frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds \right)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \left[\sum_{q=0}^{\infty} A_{2q+1}^e(c, \varphi_0, N, j) J_{e_{2q+1}}(h_N, 1) \right. \right.$$

$$\left(\pi \sum_{l=0}^{\infty} B^e_{2l+1}(h_N, 2q+1) B^e_{2l+1}(h_0, 2n+1)\right)\Bigg]\Bigg\}$$

$$\frac{1}{|\ln c|} + \cdots ,$$

(3.66)

$$\int_0^{2\pi} W_1(0,\eta) S_{o_{2m}}(h_0, \cos\eta)\, d\eta = 0 , \tag{3.67}$$

$$\int_0^{2\pi} W_1(0,\eta) S_{o_{2n+1}}(h_0, \cos\eta)\, d\eta = 0 . \tag{3.68}$$

Thus, by eqs(3.25),(3.27),

$$\oint_{S_B} \left(\frac{\partial U_0(\xi,\eta)}{\partial n} V_2^h(\xi,\eta) - \frac{\partial V_2^h(\xi,\eta)}{\partial n} U_0(\xi,\eta) \right) ds$$

$$= \int_0^{2\pi} \left(-\frac{\partial U_0(\xi,\eta)}{\partial \xi} \Big|_{\xi=0} V_2^h(0,\eta) + \frac{\partial V_2^h(\xi,\eta)}{\partial \xi} \Big|_{\xi=0} U_0(0,\eta) \right) d\eta$$

$$= \sum_{n=0}^{\infty} \left[\frac{-1}{N_{e_{2n}}(h_0,1)} A^e_{2n}(c,\varphi_0) \tilde{C}^e_{2n}(c,\varphi_0) M^e_{2n}(h_0) \right.$$

$$\left. + \frac{-1}{N_{e_{2n+1}}(h_0,1)} A^e_{2n+1}(c,\varphi_0) \tilde{C}^e_{2n+1}(c,\varphi_0) M^e_{2n+1}(h_0) \right]$$

$$+ \sum_{n=0}^{\infty} \left[A^e_{2n}(c,\varphi_0) C^e_{2n}(c,\varphi_0) M^e_{2n}(h_0) + A^e_{2n+1}(c,\varphi_0) C^e_{2n+1}(c,\varphi_0) M^e_{2n+1}(h_0) \right]$$

(3.69)

$\tilde{C}^o_{2n}(c,\varphi_0)$ and $\tilde{C}^o_{2n+1}(c,\varphi_0)$ do not affect the second order result F_2, eq(2.43), the simplest choice is to set them equal to *zero*.

$$N_{o_p}(h,1) = O(h^{-p}) , \ p = 1,2,3,\ldots ,$$

$$N_{e_{2q+1}}(h,1) = O(h^{-2q-1}) \ , \ q = 0,1,2,\dots \ ,$$

$$N_{e_{2q}}(h,1) = \begin{cases} O(\ln h) & if\ q = 0 \\ O(h^{-2q}) & if\ q \neq 0 \end{cases} . \tag{3.70}$$

$$\frac{J_{e_{2n+1}}(h,1)}{N_{e_{2n+1}}(h,1)} = O(h^{4n+2}) \ , \tag{3.71}$$

$$\frac{J_{e_{2n}}(h,1)}{N_{e_{2n}}(h,1)} = \begin{cases} O(\frac{1}{\ln h}) & n = 0 \\ O(h^{4n}) & n \neq 0 \ . \end{cases} \tag{3.72}$$

$$F^e_q(0) = 0 + \cdots \ , \tag{3.73}$$

$$G^e_q(0) = 0 + \cdots \ , \tag{3.74}$$

$$q = 0,1,2,\dots \ .$$

$h_0 \cosh\xi$ is $O(1)$ on S_0. Thus, to correct the boundary condition on S_0 to $O(\frac{1}{|\ln c|^2})$ for $(U_0 + V_1 + W_1 + V_2)$, $\tilde{C}^e_{2n}(c,\varphi_0)$ and $\tilde{C}^e_{2n+1}(c,\varphi_0)$ must be at most $O(\frac{1}{|\ln c|^2})$, the simplest choice is to set them equal to *zero*. The other choices would only lead to a higher order correction to the second order result F_2, eq(2.43). Thus,

$$V_2^h(\xi,\eta) = \sum_{n=0}^{\infty} [C^e_{2n}(c,\varphi_0) N_{e_{2n}}(h_0,\cosh\xi) S_{e_{2n}}(h_0,\cos\eta)$$

$$+ C^e_{2n+1}(c,\varphi_0) N_{e_{2n+1}}(h_0,\cosh\xi) S_{e_{2n+1}}(h_0,\cos\eta)\Big] \ , \tag{3.75}$$

where

$$C^e_{2n}(c,\varphi_0) = \frac{-\int_0^{2\pi} W_1(0,\eta) S_{e_{2n}}(h_0,\cos\eta)\, d\eta}{M^e_{2n}(h_0) N_{e_{2n}}(h_0,1)}$$

$$
-\frac{2K_0F_1A_{2n}^e(c,\varphi_0)\dfrac{J_{e_{2n}}(h_0,1)}{N_{e_{2n}}(h_0,1)}\left[F_{2n}^e(0)N_{e_{2n}}(h_0,1)-G_{2n}^e(0)J_{e_{2n}}(h_0,1)\right]}{N_{e_{2n}}(h_0,1)}\,, \tag{3.76}
$$

$$
C_{2n+1}^e(c,\varphi_0)=\frac{-\displaystyle\int_0^{2\pi}W_1(0,\eta)S_{e_{2n+1}}(h_0,\cos\eta)\,d\eta}{M_{2n+1}^e(h_0)N_{e_{2n+1}}(h_0,1)}
$$

$$
-\frac{2K_0F_1A_{2n}^e(c,\varphi_0)\dfrac{J_{e_{2n+1}}(h_0,1)}{N_{e_{2n+1}}(h_0,1)}\left[F_{2n+1}^e(0)N_{e_{2n+1}}(h_0,1)-G_{2n+1}^e(0)J_{e_{2n+1}}(h_0,1)\right]}{N_{e_{2n+1}}(h_0,1)}\,. \tag{3.77}
$$

And

$$
\oint_{S_B}\left(\frac{\partial U_0(\xi,\eta)}{\partial n}V_2^h(\xi,\eta)-\frac{\partial V_2^h(\xi,\eta)}{\partial n}U_0(\xi,\eta)\right)\,ds
$$

$$
=\sum_{n=0}^{\infty}\left[A_{2n}^e(c,\varphi_0)C_{2n}^e(c,\varphi_0)M_{2n}^e(h_0)+A_{2n+1}^e(c,\varphi_0)C_{2n+1}^e(c,\varphi_0)M_{2n+1}^e(h_0)\right]\,. \tag{3.78}
$$

[13] gives

$$
B_0^e(h,0)=1+\frac{h^2}{8}+\frac{7h^4}{512}+\cdots\,, \tag{3.79}
$$

$$
M_0^e(h)=\pi\left(2+\frac{1}{2}h^2+\frac{13}{128}h^4+\cdots\right)\,, \tag{3.80}
$$

$$
J_{e_0}(h,1)=\sqrt{\frac{\pi}{2}}+\cdots\,, \tag{3.81}
$$

and

$$
N_{e_0}(h,1)=\sqrt{\frac{2}{\pi}}\ln h+\cdots\,. \tag{3.82}
$$

$$A_0^e(c,\varphi_0) = \sqrt{8\pi} B_0 \frac{B_0^e(h_0,0)}{M_0^e(h_0)} + \sqrt{2\pi} \sum_{p=1}^{\infty} \left[\frac{B_{2p}^e(h_0,0)}{M_0^e(h_0)} \left(A_{2p} \sin(2p\varphi_0) + B_{2p} \cos(2p\varphi_0) \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} B_0 + \cdots .$$

$$(3.83)$$

Thus, by eqs(3.65),(3.76),(3.78),(3.79),(3.81),(3.82),(3.83),

$$\oint_{S_B} \left(\frac{\partial U_0(\xi,\eta)}{\partial n} V_2^h(\xi,\eta) - \frac{\partial V_2^h(\xi,\eta)}{\partial n} U_0(\xi,\eta) \right) ds$$

$$= A_0^e(c,\varphi_0) C_0^e(c,\varphi_0) M_0^e(h_0) + \cdots$$

$$= \left\{ B_0^2 \pi^2 \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{B_0(N,j)}{(K_N^2 - K_0^2) \|U_{N,j}\|^2} \oint_{S_0} Y_0(K_0 r) \frac{\partial U_{N,j}(r,\theta)}{\partial n} \, ds \right\} \frac{1}{|\ln c|^2} + \cdots .$$

$$(3.84)$$

Eqs(3.19),(3.45),(3.58),(3.79),(3.81),(3.82),(3.83) give

$$\oint_{S_0} \frac{\partial U_0(\xi,\eta)}{\partial n} V_2^i(\xi,\eta) \, ds$$

$$= -\sqrt{\frac{\pi}{2}} F_1 \sum_{q=0}^{\infty} \left\{ A_{2q}^e(c,\varphi_0) \frac{J_{e_{2q}}(h_0,1)}{N_{e_{2q}}(h_0,1)} \right.$$

$$\left[\sum_{n=0}^{\infty} B_{2n}^e(h_0,2q) \oint_{S_0} r Y_{2n}'(K_0 r) \cos\left(2n(\theta - \varphi_0)\right) \frac{\partial U_0(r,\theta)}{\partial n} \, ds \right]$$

$$+ A_{2q+1}^e(c,\varphi_0) \frac{J_{e_{2q+1}}(h_0,1)}{N_{e_{2q+1}}(h_0,1)}$$

$$\left. \left[\sum_{n=0}^{\infty} B_{2n+1}^e(h_0,2q+1) \oint_{S_0} r Y_{2n+1}'(K_0 r) \cos\left((2n+1)(\theta - \varphi_0)\right) \frac{\partial U_0(\xi,\eta)}{\partial n} \, ds \right] \right\}$$

$$= -\sqrt{\frac{\pi}{2}} \frac{\pi B_0^2}{K_0 \int_{R_0} U_0^2(\xi,\eta)\, dA} \frac{1}{|\ln c|} \sqrt{\frac{2}{\pi}} B_0 \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{\frac{2}{\pi}} \ln c} \oint_{S_0} r Y_0'(K_0 r) \frac{\partial U_0(r,\theta)}{\partial n}\, ds + \cdots$$

$$= \left\{ \frac{\pi^2 B_0^3 \oint_{S_0} r Y_0'(K_0 r) \frac{\partial U_0(r,\theta)}{\partial n}\, ds}{2K_0 \int_{R_0} U_0^2(\xi,\eta)\, dA} \right\} \frac{1}{|\ln c|^2} + \cdots . \tag{3.85}$$

Thus, by eqs(2.43),(3.45),(3.84),(3.85),

$$F_2 = \frac{F_1^2}{-2K_0}$$

$$- \frac{\oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} V_2^i(r,\theta)\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} + \frac{\oint_{S_B} \left(\frac{\partial U_0(r,\theta)}{\partial n} V_2^h(r,\theta) - \frac{\partial V_2^h(r,\theta)}{\partial n} U_0(r,\theta) \right) ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA}$$

$$= \left\{ \frac{\pi^2 B_0^4}{-2K_0^3 \left(\int_{R_0} U_0^2(\xi,\eta)\, dA \right)^2} + \frac{\pi^2 B_0^3 \oint_{S_0} r Y_0'(K_0 r) \frac{\partial U_0(r,\theta)}{\partial n}\, ds}{-4K_0^2 \left(\int_{R_0} U_0^2(\xi,\eta)\, dA \right)^2} \right.$$

$$\left. + \frac{B_0^2 \pi^2 \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{B_0(N,j)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} Y_0(K_0 r) \frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right\} \frac{1}{|\ln c|^2} + \cdots .$$

$$\tag{3.86}$$

Thus, eqs(1.19),(3.45),(3.86) give

$$K = K_0 + F_1 + F_2 + \cdots$$

$$= K_0 + \left(\frac{\pi B_0^2}{K_0 \int_{R_0} U_0^2(\xi,\eta)\, dA} \right) \frac{1}{|\ln c|} + \left[\frac{\pi (\ln K_0 + \gamma - \ln 4) B_0^2}{K_0 \int_{R_0} U_0^2(\xi,\eta)\, dA} \right.$$

$$- \frac{\pi^2 B_0^4}{2K_0^3 \left(\int_{R_0} U_0^2(\xi,\eta)\, dA \right)^2} - \frac{\pi^2 B_0^3 \oint_{S_0} r Y_0'(K_0 r) \frac{\partial U_0(r,\theta)}{\partial n}\, ds}{4K_0^2 \left(\int_{R_0} U_0^2(\xi,\eta)\, dA \right)^2}$$

$$
+\frac{B_0^2\pi^2 \sum_{N=1}^{\infty}\sum_{j=1}^{l(N)} \frac{B_0(N,j)}{(K_N^2-K_0^2)\|U_{N,j}\|^2} \oint_{S_0} Y_0(K_0 r)\frac{\partial U_{N,j}(r,\theta)}{\partial n}\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA}\Bigg] \frac{1}{|\ln c|^2}
$$

$$
+\cdots , \tag{3.87}
$$

where $\gamma \approx 0.5772$.

3.2 Circular Membrane With a Centered Strip of Length $2c$

The geometry of the concerned region is with the outer boundary where S_0 is $r = r' = 1$ and the inner boundary where S_B is $\xi = \xi' = 0$.

Eqs(2.51),(2.111),(3.87) give

$$
K = K_0 + \left(\frac{1}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|}
$$

$$
+\left[\frac{(\ln K_0 + \gamma - \ln 4)}{K_0 J_1^2(K_0)} + \frac{1}{2K_0^3 J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)}\right]\frac{1}{|\ln c|^2} + \cdots , \tag{3.88}
$$

where $K_0 = K_{0,1} \approx 2.4048$ and $\gamma \approx 0.5772.$

CHAPTER 4

Conclusions

The main contribution of the dissertation is two-fold. First, from the computational point of view, a general formula to the asymptotic approximations of the fundamental frequencies K of membranes with an internal core of maximum dimension $2c$, $c \ll 1$, is derived. It is convergent in asymptotic sense. Moreover, the first three order terms of the asymptotic approximations are carried out explicitly. Second, from the point of view of the inverse problem, relations between the first three order terms of the asymptotic approximations and geometric properties of the regions are investigated from the explicit formula. These are summarized in Theorem 4.1 and Corollary 4.2 respectively.

Theorem 4.1 A general formula to the asymptotic approximations of the fundamental frequencies K of membranes with an internal core of maximum dimension $2c$, $c \ll 1$, is formed as in eqs(1.19),(1.37),(1.38),(1.39). Moreover, the asymptotic expansion of K for membrane with an internal core of maximum dimension $2c$, derived from eq(2.51), eq(3.87) and the minimax principle [8], is

$$K = K_0 + \frac{K_1}{|\ln c|} + \frac{K_2}{|\ln c|^2} + \cdots , \tag{4.1}$$

where K_0 is the fundamental frequency of the membrane without an internal core,

$$K_1 = \frac{\pi B_0^2}{K_0 \int_{R_0} U_0^2(r,\theta)\, dA}, \tag{4.2}$$

and

$$\begin{aligned}
&\left[\frac{\pi(\ln K_0 + \gamma - \ln 4) B_0^2}{K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right. \\
&- \frac{\pi^2 B_0^4}{2K_0^3 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} - \frac{\pi^2 B_0^3 \oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} r Y_0'(K_0 r)\, ds}{4K_0^2 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} \\
&\left. + \frac{\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0^2 B_0(N,j)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right] \\
&\leq \qquad K_2 \qquad \leq \\
&\left[\frac{\pi(\ln K_0 + \gamma - \ln 2) B_0^2}{K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right. \\
&- \frac{\pi^2 B_0^4}{2K_0^3 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} - \frac{\pi^2 B_0^3 \oint_{S_0} \frac{\partial U_0(r,\theta)}{\partial n} r Y_0'(K_0 r)\, ds}{4K_0^2 \left(\int_{R_0} U_0^2(r,\theta)\, dA \right)^2} \\
&\left. + \frac{\sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{\pi^2 B_0^2 B_0(N,j)}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \oint_{S_0} \frac{\partial U_{N,j}(r,\theta)}{\partial n} Y_0(K_0 r)\, ds}{2K_0 \int_{R_0} U_0^2(r,\theta)\, dA} \right].
\end{aligned} \tag{4.3}$$

Corollary 4.2 Eqs(4.1),(4.2),(4.3) show that the geometry of internal core starts to affect K at K_2 while the position of internal core starting to affect K at K_1 and the geometry of membrane without an internal core starting to affect K at K_0.

The asymptotic expansion formula of K obtained by the modified perturbation method is highly accurate and is valid for small c. This is summarized in Lemma 4.3.

Lemma 4.3 (1a) Eqs(2.111),(C.13) show that the first three order terms in the asymptotic expansion of K, the fundamental frequency of the annular circular membrane, obtained by the modified perturbation method agree with those in the *exact* series.

(1b) Eqs(3.88),(C.18) show that the first three order terms in the asymptotic expansion of K, the fundamental frequency of the circular membrane with a centered strip, obtained by the modified perturbation method agree with those in the *exact* series of K_e, the fundamental frequency of the elliptic membrane with an internal confocal strip, i.e. $K_0 \approx 2.4048$, $K_1 \approx 1.5429$, $K_2 \approx 0.1208$.

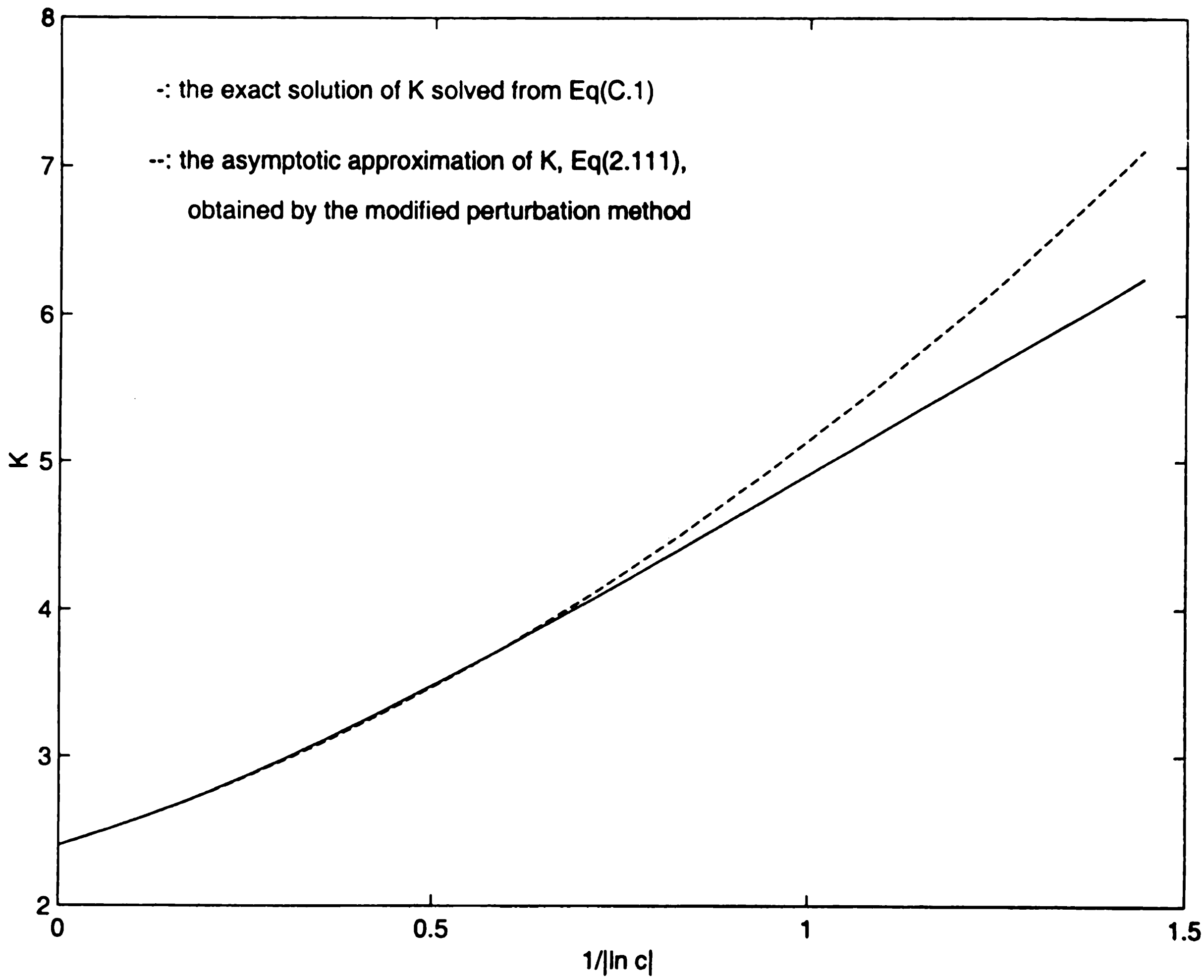
(1c) The asymptotic expansion [9] of K_{nu}, the fundamental frequency of the circular membrane with a centered strip, computed by the eigenfunction matching method is of the form

$$K_{nu} = K_0 + \frac{K_1}{|\ln c|} + \frac{K_2}{|\ln c|^2} + \cdots , \tag{4.4}$$

where $K_0 = 2.4048$, $K_1 = 1.55$, and $K_2 = -0.012$ are computed by a least squares fit on our numerical results for the range $c = 10^{-2}$ to $c = 10^{-6}$. (Numerical instability occurs for $c < 10^{-6}$).

(2) The comparison between the first three order terms in the asymptotic expansion of K (eq(2.111)), the fundamental frequency of the annular circular membrane, obtained by the modified perturbation method and the *exact* solution is shown in Figure 4.1. The error is less than 1 % as c is less than 0.25 and less than 5 % as c is less than 0.4.

Figure 4.1: The comparison of the asymptotic approximation and the exact solution



Propositon 4.4 Eqs(2.77),(2.111) show that

$$\sum_{n=2}^{\infty} \frac{K_{0,n}}{\left(K_{0,n}^2 - K_0^2\right) J_1(K_{0,n})} = -\frac{\pi}{4} Y_0(K_0) \ , \tag{4.5}$$

where $K_0 = K_{0,1} \approx 2.4048$ and $K_{0,p}$ is the p^{th} zero of J_0, $p = 1, 2, 3, \ldots$.

remark: The series in eq(4.5) converge slowly.

The future research related to the dissertation would be studies in the convergence of the modified perturbation method, extensions of the method to higher frequencies, and extensions of the method to membranes with arbitrary many internal cores.

APPENDICES

APPENDIX A

The Pin-point Phenomenon of Simply Connected Membranes

Proposition A.1

$$\lim_{c \to 0} K = K_0 \ . \tag{A.1}$$

Proof:

First, consider the case where S_B is the circle of radius c centered at (x_0,y_0) . Green's 2^{nd} identity [4, 11] gives

$$\int_R \left(U(x,y) \triangle U_0(x,y) - U_0(x,y) \triangle U(x,y)\right) dA$$

$$= \oint_{S_0 \cup S_B} \left(U(x,y) \frac{\partial U_0(x,y)}{\partial n} - U_0(x,y) \frac{\partial U(x,y)}{\partial n} \right) ds \ , \tag{A.2}$$

where the normal derivative on S_0 is with respect to the outer normal direction and the normal derivative on S_B is with respect to the inner normal direction. Eqs(1.2),(1.3),(1.4),(1.5),(A.2) give

$$(K^2 - K_0^2) \int_R U(x,y) U_0(x,y) \, dA = - \oint_{S_B} U_0(x,y) \frac{\partial U(x,y)}{\partial n} \, ds \ . \tag{A.3}$$

Let $zero^{th}$ order approximation of $U(x,y)$ be $\tilde{U}(x,y)$. Consider $\tilde{U}(x,y)$ on the annular circle with the inner boundary, S_B, and the outer boundary, $r = O(c)$, then $\tilde{U}(x,y)$ is independent on θ, so

$$\frac{\partial^2 \tilde{U}(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{U}(r)}{\partial r} = 0 \ . \tag{A.4}$$

Thus,

$$\tilde{U}(r) = C_1 + C_2 \ln r \ , \tag{A.5}$$

where C_1 and C_2 are constants.

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The boundary condition , $\tilde{U}(c) = 0$, yields

$$C_2 = \frac{-C_1}{\ln c} \quad , \quad C_1 = O(1) \ . \tag{A.6}$$

Thus,

$$\oint_{S_B} U_0(x,y)\frac{\partial \tilde{U}(x,y)}{\partial n}\ ds = -\int_0^{2\pi} U_0(c,\theta)\frac{\partial \tilde{U}(r,\theta)}{\partial r}\ |_{r=c}\ c\ d\theta = \frac{C_1}{\ln c}\int_0^{2\pi} U_0(c,\theta)\ d\theta \ , \tag{A.7}$$

so eq(A.3) becomes

$$(K^2 - K_0^2)\int_{R_0} U_0^2(x,y)\ dA = \frac{-C_1}{\ln c}\int_0^{2\pi} U_0(c,\theta)\ d\theta + l.o.t. \ . \tag{A.8}$$

The proposition is proved by the minimax principle [8] and eq(A.8). □

APPENDIX B

The Generalized Green's Functions

A generalized Green's function [4, 12] is needed for problems of the type

$$\triangle W(x,y) + K_0^2 W(x,y) = -f(x,y) \ in \ R_0 \ , \tag{B.1}$$

$$W(x,y) = h(x,y) \ on \ S_0 \ , \tag{B.2}$$

where K_0^2 is an eigenvalue of Eqs(1.4),(1.5).

For the case where the eigenvalue K_0^2 is *simple*, the generalized Green's function $G(x,y;\tilde{x},\tilde{y})$ is a solution [4, 12] of

$$\triangle_{(x,y)} G(x,y;\tilde{x},\tilde{y}) + K_0^2 G(x,y;\tilde{x},\tilde{y}) = -\delta(x,y;\tilde{x},\tilde{y}) + \frac{U_0(x,y)U_0(\tilde{x},\tilde{y})}{\|U_0\|^2} \ in \ R_0 \ , \tag{B.3}$$

$$G(x,y;\tilde{x},\tilde{y}) = 0 \ on \ S_0 \ , \tag{B.4}$$

where U_0 is an eigenfunction corresponding to the eigenvalue K_0^2 .

The generalized Green's function $G(x,y;\tilde{x},\tilde{y})$ can be expressed as an expansion in eigenfunctions with terms associated with the eigenvalue K_0^2 omitted; [4, 12]

$$G(x,y;\tilde{x},\tilde{y}) = \sum_{N=1}^{\infty} \sum_{j=1}^{l(N)} \frac{U_{N,j}(x,y)U_{N,j}(\tilde{x},\tilde{y})}{(K_N^2 - K_0^2)\|U_{N,j}\|^2} \ , \tag{B.5}$$

where K_N^2 , $l(N)$, $U_{N,j}$ are the eigenvalues to eqs(1.4),(1.5), the multiplicities of K_N^2, and the corresponding orthogonal eigenfunctions to K_N^2 respectively, $N = 1, 2, 3, 4, \dots$.

An alternative [4, 12] is to derive more compact series expressions directly from the differential equation. The solution $G(x, y; \tilde{x}, \tilde{y})$ to eqs(B.3),(B.4) can be obtained from the usual Green's function $\tilde{G}(x, y; \tilde{x}, \tilde{y})$ by using

$$G(x, y; \tilde{x}, \tilde{y}) = \lim_{\tilde{K} \to K_0} \frac{\partial}{\partial(\tilde{K}^2)} \left[(\tilde{K}^2 - K_0^2) \tilde{G}(x, y; \tilde{x}, \tilde{y}) \right] , \tag{B.6}$$

$$\triangle_{(x,y)} \tilde{G}(x, y; \tilde{x}, \tilde{y}) + \tilde{K}^2 \tilde{G}(x, y; \tilde{x}, \tilde{y}) = -\delta(x, y; \tilde{x}, \tilde{y}) \ in \ R_0 , \tag{B.7}$$

$$\tilde{G}(x, y; \tilde{x}, \tilde{y}) = 0 \ on \ S_0 , \tag{B.8}$$

where $\tilde{K}^2$ is not an eigenvalue to eqs(1.4),(1.5).

APPENDIX C

The ***Exact*** Series Expansion of K

C.1 Annular Circular Membrane With Outer Radius 1 and Inner Radius c

The characteristic equation [14] is

$$Y_0(K)J_0(Kc) - J_0(K)Y_0(Kc) = 0 \ . \tag{C.1}$$

We assume that [2]

$$K = K_0 + K_1 \frac{1}{|\ln c|} + K_2 \delta(c) + o(\delta(c)) \ , \tag{C.2}$$

where

$$K_0 = K_{0,1} \approx 2.4048 \ : \quad \textit{the first zero of } J_0 \ ,$$

$$K_1 = \frac{\pi}{2} \frac{Y_0(K_0)}{J_1(K_0)} \ , \quad \textit{and} \quad \lim_{c \to 0} \frac{\delta(c)}{\left(\frac{1}{|\ln c|}\right)} = 0 \ .$$

Addition theorems for *bessel* functions [10] give

$$Y_0(K) = Y_0(K_0) - K_1 Y_1(K_0) \frac{1}{|\ln c|} - Y_1(K_0) K_2 \delta(c) + \cdots \qquad , \tag{C.3}$$

$$J_0(Kc) = 1 - \frac{K_0^2 c^2}{4} + \cdots \qquad , \tag{C.4}$$

$$J_0(K) = -K_1 J_1(K_0)\frac{1}{|\ln c|} + \frac{K_1^2 J_2(K_0)}{4}\frac{1}{|\ln c|^2} - K_2 J_1(K_0)\delta(c) + \cdots \ , \tag{C.5}$$

$$Y_0(Kc) = \frac{2}{\pi}\ln c + \frac{2}{\pi}(\ln K_0 + \gamma - \ln 2) + \cdots \qquad , \tag{C.6}$$

where $\gamma \approx 0.5772$. Eq(C.1) becomes

$$\left[Y_0(K_0) - K_1 Y_1(K_0)\frac{1}{|\ln c|} - Y_1(K_0) K_2 \delta(c) + \cdots\right]$$

$$-\left[-K_1 J_1(K_0)\frac{1}{|\ln c|} + \frac{K_1^2 J_2(K_0)}{4}\frac{1}{|\ln c|^2} - K_2 J_1(K_0)\delta(c) + \cdots\right]$$

$$\left[\frac{2}{\pi}\ln c + \frac{2}{\pi}(\ln K_0 + \gamma - \ln 2) + \cdots\right] = 0 \ . \tag{C.7}$$

Balancing the leading orders, one finds

$$\delta(c) = \frac{1}{|\ln c|^2} \tag{C.8}$$

and

$$K_2 = \frac{-\frac{\pi}{2}K_1 Y_1(K_0) + \frac{K_1^2 J_2(K_0)}{4} + (\ln K_0 + \gamma - \ln 2)\, K_1 J_1(K_0)}{J_1(K_0)} \ . \tag{C.9}$$

Eqs(C.2),(C.9) give

$$K = K_0 + \frac{\pi}{2}\frac{Y_0(K_0)}{J_1(K_0)}\frac{1}{|\ln c|}$$

$$+\frac{-\frac{\pi}{2}K_1 Y_1(K_0) + \frac{K_1^2 J_2(K_0)}{4} + (\ln K_0 + \gamma - \ln 2)\, K_1 J_1(K_0)}{J_1(K_0)}\frac{1}{|\ln c|^2} + \cdots \ . \tag{C.10}$$

$$\frac{\pi}{2}Y_0(K_0) = \frac{1}{K_0 J_1(K_0)} \ . \tag{C.11}$$

$$\frac{2}{K_0}J_1(K_0) = J_2(K_0) \ . \tag{C.12}$$

Thus,

$$K = K_0 + \left(\frac{1}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|}$$
$$+ \left[\frac{(\ln K_0 + \gamma - \ln 2)}{K_0 J_1^2(K_0)} + \frac{1}{2K_0^3 J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)}\right]\frac{1}{|\ln c|^2} + \cdots \ , \tag{C.13}$$

where $K_0 = K_{0,1} \approx 2.4048$ and $\gamma \approx 0.5772$.

C.2 Elliptic Membrane of Area π With an Internal Confocal Strip of Length $2c$

The characteristic equation [13] is

$$N_{e_0}(h,1)J_{e_0}(h,\cosh c) - J_{e_0}(h,1)N_{e_0}(h,\cosh c) = 0 \ , \tag{C.14}$$

where $h = Kc$. The asymptotic expansion [9] of K is

$$K = K_0 + K_1\frac{1}{|\ln c|} + K_2\frac{1}{|\ln c|^2} + \cdots \ , \tag{C.15}$$

where

$$K_0 = K_{0,1} \approx 2.4048 \ : \ \textit{the first zero of } J_0 \ ,$$

$$K_1 = \frac{\pi}{2}\frac{Y_0(K_0)}{J_1(K_0)} \approx 1.5429 \ ,$$

$$K_2 = \frac{-\frac{\pi}{2}K_1 Y_1(K_0) + \frac{K_1^2 J_2(K_0)}{4} + J_1(K_0)K_1(\ln K_0 - \ln 4 + \gamma)}{J_1(K_0)} \approx 0.1208 \ .$$

Eqs(C.11),(C.12) give

$$K_1 = \frac{\pi}{2}\frac{Y_0(K_0)}{J_1(K_0)} = \frac{1}{K_0 J_1^2(K_0)} \approx 1.5429 \ , \tag{C.16}$$

$$K_2 = \frac{-\frac{\pi}{2}K_1 Y_1(K_0) + \frac{K_1^2 J_2(K_0)}{4} + J_1(K_0)K_1(\ln K_0 - \ln 4 + \gamma)}{J_1(K_0)}$$

$$= \frac{(\ln K_0 + \gamma - \ln 4)}{K_0 J_1^2(K_0)} + \frac{1}{2K_0^3 J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)}$$

$$\approx 0.1208 \qquad . \tag{C.17}$$

Thus,

$$K = K_0 + \left(\frac{1}{K_0 J_1^2(K_0)}\right)\frac{1}{|\ln c|}$$

$$+ \left[\frac{(\ln K_0 + \gamma - \ln 4)}{K_0 J_1^2(K_0)} + \frac{1}{2K_0^3 J_1^4(K_0)} - \frac{\pi Y_1(K_0)}{2K_0 J_1^3(K_0)}\right]\frac{1}{|\ln c|^2} + \cdots \ , \tag{C.18}$$

where $K_0 = K_{0,1} \approx 2.4048$ and $\gamma \approx 0.5772$.

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