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SEIBERG-WITTEN INVARIANTS OF 4-MANIFOLDS WITH CIRCLE ACTIONS

By

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ABSTRACT

SEIBERG-WITTEN INVARIANTS OF 4-MANIFOLDS WITH CIRCLE ACTIONS

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The main result of this paper is a formula for calculating the Seiberg-Witten invariants of 4-manifolds with fixed-point free circle actions. This is done by showing under suitable conditions a diffeomorphism between the moduli space of the 4-manifold and the moduli space of the quotient 3-orbifold. Two corollaries include $b_+ > 1$ 4-manifolds with fixed-point free circle actions are simple type and a new proof that $\mathcal{SW}_{Y^3 \times S^1} = \mathcal{SW}_{Y^3}$. Using the formula, we show how to construct a nonsymplectic 4-manifold with a free circle action whose orbit space fibers over circle. We also describe a nontrivial 3-manifold which is not the orbit space of any symplectic 4-manifold with a free circle action. An infinite number of $b_+ = 1$ 4-manifold where the Seiberg-Witten invariants are still diffeomorphism invariants are constructed and studied. As an application of the main results, we derive a formula for the 3-dimensional Seiberg-Witten invariants of the total space of a circle bundle over a surface.

To my loving wife Lisa

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CHAPTER 1

Introduction

The main idea of this work is to systematically study 4-manifolds that admit an S^1-action and classify them using Seiberg-Witten gauge theory. When the action on X^4 is free, the quotient by the S^1-action is a smooth 3-manifold Y and the manifold with given circle action is classified by the Euler class $\chi \in H^2(Y;\mathbb{Z})$. When the circle action is not free there will be non-trivial isotropy groups, which forces the orbit space to be an orbifold rather than a manifold. The main result of this paper is a formula for calculating the Seiberg-Witten invariants of any 4-manifold with a fixed point free circle action. We derive the formula by proving the existence of a diffeomorphism between the moduli space of the 4-manifold with the moduli space of the quotient 3-orbifold.

A given manifold may admit more than one circle action. So while the 3-manifold and Euler class are fully sufficient to classify a free circle action, the Seiberg-Witten invariants are stronger in that they are invariant of the underlying space up to diffeomorphism regardless of the circle action.

The first theorem we prove puts a restriction on the set of Spin^c structures with nontrivial Seiberg-Witten invariants for manifolds which admit a fixed point free circle action. (See Chapters 2 and 3 for descriptions of Spin^c structures and Seiberg-Witten invariants.)

Theorem A. *Let ξ be a* Spin^c *structure on $b_+ \neq 1$ 4-manifold X with a fixed point free circle action such that $SW_X(\xi) \neq 0$. Then the* Spin^c *structure ξ is pulled back from a* Spin^c *structure on Y.*

See section 4.1 for the statement when $b_+(X) = 1$. This theorem is already enough to imply that X is SW simple type – that the expected dimension of the moduli space for all Spin^c structures with nontrivial invariants is zero.

Let $\pi : X \to Y$ be the projection map from a smooth 4-manifold with a fixed point free circle action to its quotient orbifold. The manifold X can be thought of as a orbifold circle bundle over Y. If $\mathbf{i}\eta$ is the connection 1-form of the circle bundle and g_Y is any orbifold metric, we can form the metric $g_X = \eta \otimes \eta + \pi^*(g_Y)$ on X. After perturbing the Seiberg-Witten equations on Y by a closed orbifold 2-form δ and on X by its self-dual pullback $\pi^*(\delta)^+ = \frac{1}{2}(1+\star)\pi^*(\delta)$, there is a moduli space of irreducible solutions to the Seiberg-Witten equations $\mathcal{M}^*(X, g_X, \pi^*(\delta)^+)$ associated with X and $\mathcal{M}^*(Y, g_Y, \delta)$ associated with Y (see sections 2.4 and 3.3 for definitions). Let $\mathcal{N}^*(X, g_X, \pi^*(\delta)^+)$ be the subcomponent of $\mathcal{M}^*(X, g_X, \pi^*(\delta)^+)$ which are the Spin^c structures that are pulled back from Y. Theorem A tells us that these are the only Spin^c structures that are useful to study. We can now state the main theorem of this paper:

Theorem B. *The pullback map π^* induces a homeomorphism*

$$\pi^* : \mathcal{M}^*(Y, g_Y, \delta) \to \mathcal{N}^*(X, g_X, \pi^*(\delta)^+).$$

Furthermore, if either of the two moduli spaces is a smooth manifold, then both of them are smooth, and π^* is a diffeomorphism.

The approach to the proof of Theorem B was inspired by similar work done in [MOY].

As in the free case, a manifold with a fixed point free S^1-action can still be considered as a unit circle bundle, but now it is a unit circle bundle of an orbifold line bundle over a 3-orbifold. In this setup, $H^2(Y;\mathbb{Z})$ is replaced by a group called $\mathrm{Pic}^t(Y)$ which records local data around the singular set (see section 2.2). Our main results express the Seiberg-Witten invariants of X in terms of the Seiberg-Witten invariants of the orbifold Y and the orbifold Euler class χ:

Theorem C. *Let X be a closed smooth 4-manifold with $b_+>1$ and a fixed point free circle action. Let Y^3 be the orbifold quotient space and suppose that $\chi \in Pic^t(Y)$ is the orbifold Euler class of the circle action. If ξ is a* Spin^c *structure over X with $SW_X^4(\xi) \neq 0$, then $\xi = \pi^*(\xi_0)$ for some* Spin^c *structure on Y and*

$$SW_X^4(\xi) = \sum_{\xi' \equiv \xi_0 \mod \chi} SW_Y^3(\xi'),$$

where $\xi' - \xi_0$ is a well-defined element of $Pic^t(Y)$. When $b_+=1$, the formula holds for all Spin^c *structures pulled back from Y.*

This results produces two immediate corollaries. One is a corresponding formula for manifolds with free circle actions. This corollary is useful for calculating examples. The second corollary is a proof of the well known fact that the Seiberg-Witten invariants of $Y^3 \times S^1$ are the same as the Seiberg-Witten invariants of Y^3.

Theorem C, together with the conjectured formula when X contains fixed points (see chapter 9), would completely calculate the Seiberg-Witten invariants for all $b_+ > 1$ 4-manifolds with circle actions. These calculations are useful beyond just distinguishing manifolds. When Seiberg-Witten invariants are combined with C. Taubes's results on symplectic manifolds (c.f. [T]), the formulas become an easy and powerful way of calculating an obstruction for an S^1-manifold to admit a symplectic structure.

As an example of the main result, we produce a nonsymplectic 4-manifold with a free circle action whose orbit space is a 3-manifold which fibers over S^1. This example runs counter to intuition since there is a well-known conjecture/question of Taubes that $M^3 \times S^1$ admits a symplectic structure if and only if M^3 fibers over S^1. Furthermore, there is evidence [FGM] which suggests that many such 4-manifolds are, in fact, symplectic. As another example of our formula, we construct a 3-manifold which is not the orbit space of any symplectic 4-manifold with a free circle action.

Theorem B can also be used to study moduli spaces in the case when $b_+(X) = 1$. Normally when $b_+(X) = 1$, the Seiberg-Witten invariant depends on the "chamber" of the metric used to calculate it. A theorem of T. J. Li and A. Liu [LL] shows how the numerical invariant changes when the metric moves from one chamber into another. Under certain conditions, their theorem says that the invariant does not change (making it a diffeomorphism invariant again). We show how to construct an infinite number of $b_+ = 1$ manifolds with this property and study their moduli spaces using Theorem B. This theorem provides a way to see explicitly why the invariants do not change when a chamber wall is crossed.

Another application of the Theorem C is a formula for the Seiberg-Witten invariant of the total space of a circle bundle over a surface. This formula can be thought of as the 3 dimensional analog of the 4 dimensional formula.

This dissertation is organized as follows.

- In **Chapter 2** we show how to define the Seiberg-Witten equations and invariant on a 3-orbifold.
- **Chapter 3** shows the relationship between 4-manifolds with fixed point free circle actions and orbifold line bundles over a 3-orbifold.
- We prove Theorem A in **Chapter 4** by showing a relationship between solutions and topology.

- In **Chapter 5** we prove Theorem B by showing that solutions to the Seiberg-Witten equations on the 4-manifold are circle invariant.
- **Chapter 6** contains a proof of Theorem C and its corollaries.
- **Chapter 7** is an alternate proof of Theorem C in the case where the 4-manifold admits a free circle action.
- **Chapter 8** describes some examples and applications of both Theorem B and Theorem C.
- In the last chapter, **Chapter 9**, we conjecture what the Seiberg-Witten invariants are for a 4-manifold which admits a circle action that has fixed points.

CHAPTER 2

Seiberg-Witten on 3-orbifolds

We show that all of the usual notions of gauge theory hold for 3-dimensional real orbifolds. Throughout, we assume that all orbifolds are oriented, connected, and closed unless otherwise specified. We start with the definition of orbifolds (c.f. [S]).

2.1 Definitions

An *n-dimensional orbifold* Y is a Hausdorff space $|Y|$ together with a system $\Xi = (\{U_i\}, \{\varphi_i\}, \{\tilde{U}_i\}, \{G_i\}, \{\tilde{\varphi}_{ij}\})$ which satisfies

1. $\{U_i\}$ is locally finite.

2. $\{U_i\}$ is closed under finite intersections.

3. For each U_i, there exist a finite group G_i acting smoothly and effectively on a connected open subset $\tilde{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\varphi_i : \tilde{U}_i/G_i \to U_i$.

4. If $U_i \subset U_j$, there exist a monomorphism $f_{ij} : G_i \to G_j$ and a smooth embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ such that for all $g \in G_i$, $x \in \tilde{U}_i$, $\tilde{\varphi}_{ij}(g \cdot x) = f_{ij}(g) \cdot \tilde{\varphi}_{ij}(x)$ making

the following diagram commute:

$$\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\varphi}_{ij}"] \arrow[d, "r_i"'] & \tilde{U}_j \arrow[d, "r_j"] \\
\tilde{U}_i/G_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\varphi_i"'] & \tilde{U}_j/G_j \arrow[d, "\varphi_j"] \\
U_i \arrow[r] & U_j
\end{tikzcd}$$

where φ_{ij} are induced by the monomorphisms and the r_i's are the natural projections.

The system Ξ is called an *atlas* and each $\varphi_i \circ r_i : \tilde{U}_i \to U_i$ is called a *local chart.* An orbifold Y is *connected* and *closed* if the underlying space $|Y|$ is. Two atlases give the same orbifold structure if there is a common refinement.

Let $x \in |Y|$ and $\tilde{U}_x \to U$ be a local chart containing x. The local group at x, denoted G_x, is the isotropy group of G of any point in U corresponding to x (well-defined up to isomorphism). Set $\Sigma Y = \{x \in |Y| \mid G_x \neq 1\}$. This set is closed and nowhere dense, and in fact it is easily shown that $\dim \Sigma Y \leq n-2$. After removing the singular set, $Y \setminus \Sigma Y$ becomes a manifold.

All theorems henceforth will be stated and proved for 3-dimensional orbifolds Y where ΣY is a finite disjoint set of smooth circles $l_1, \ldots, l_n$ that are assigned integral multiplicities $\alpha_1, \ldots, \alpha_n$ given by their local isotropy group $\mathbb{Z}_{\alpha_i} = \mathbb{Z}/\alpha_i\mathbb{Z}$. Let D be the standard complex disk and consider a $\mathbb{Z}_{\alpha_i}$ action on it by rotation. We will take a convenient atlas in all of the atlases which give the same orbifold structure. Equip $|Y|$ with an atlas of coordinate charts

$$\phi_i : (S^1 \times D, S^1 \times 0) \to (U_i, l_i) \qquad i = 1, \ldots, n$$

$$\phi_x : D_x^3 \to U_x \quad x \in Y \setminus \{l_1, \ldots, l_n\},$$

where the ϕ_i induce homeomorphisms from $(S^1 \times D/\mathbb{Z}_{\alpha_i}, S^1 \times 0)$ to (U_i, l_i), the ϕ_x are homeomorphisms, the U_i are all pairwise disjoint, $U_x \cap \Sigma Y = \emptyset$, and the transition functions are all diffeomorphisms.

Example 2.1 *The triple $Y = (S^3, K, n)$ where K is a knot in S^3, K is the singular locus $\Sigma Y = K$, and the isotropy group around K is $\mathbb{Z}_n$, is an example of a 3-orbifold.*

Define an n-dimensional *orbifold bundle* over Y in the following manner. Set $U_x \times V^n$ over each U_x for an n-dimensional vector space V^n. Over U_i the vector bundle is given by the quotient $(S^1 \times D \times V^n)/\mathbb{Z}_{\alpha_i}$ where $(S^1 \times D \times V^n)$ is a $\mathbb{Z}_{\alpha_i}$-equivariant vector bundle specified up to isometry by giving a representation $\sigma_i : \mathbb{Z}_{\alpha_i} \to GL_n(V)$. The vector bundle over Y is then specified by a 1-cocycle of transition functions over the overlaps.

2.2 Orbifold line bundles

Under tensor product the topological isomorphism classes of orbifold line bundles form a group $\mathrm{Pic}^t(Y)$ called the *topological Picard group*. We describe this group in this section.

We can record the information in $\mathrm{Pic}^t(Y)$ by using a generalization of equivariant cohomology. Think of Y as the union of $Y \setminus \{l_1, \ldots, l_n\}$ and $\amalg\ (l_i \times D/\mathbb{Z}_{\alpha_i})$. Define Y_V to be the union of $Y \setminus \{l_1, \ldots, l_n\}$ and $\amalg\ \big(l_i \times (D \times_{\mathbb{Z}_{\alpha_i}} E\mathbb{Z}_{\alpha_i})\big)$ glued using sections of

$l_i \times (D \setminus \{0\} \times_{\mathbb{Z}_{\alpha_i}} E\mathbb{Z}_{\alpha_i}) \to U_i \setminus l_i$. These sections are unique up to homotopy because the fibers of the bundle are contractible.

The following theorem is contained in [FuS].

Theorem 2.2 *The following groups are isomorphic:*

1. $H^1(Y_V;\mathbb{Z}) \cong H^1(|Y|;\mathbb{Z})$,

2. $H^2(Y_V;\mathbb{Z}) \cong Pic^t(Y)$.

Remark 2.3 *In the literature, the group* $H_V^*(Y) := H^*(Y_V)$ *is often called the* V-cohomology ring of Y.

Here is another way to describe $\mathrm{Pic}^t(Y)$. Define an orbifold line bundle over Y to be a trivial line bundle $A = (Y \setminus l_i) \times \mathbb{C}$ and over U_i it is given by $B = (S^1 \times D \times \mathbb{C}_\xi)/\mathbb{Z}_{\alpha_i}$ where $a \in \mathbb{Z}_{\alpha_i}$ acts using the standard representation

$$a \cdot (\gamma, w, z) \mapsto (\gamma, \mathbf{e}^{\frac{2\pi i a}{\alpha_i}} w,\ \mathbf{e}^{\frac{2\pi i a}{\alpha_i}} z).$$

The bundle is glued together using a transition function $\varphi_{BA}(\gamma, w) = w$ on the overlap $S^1 \times (D \setminus \{0\})$. For each l_i, create such a line bundle called E_i.

Let L be an orbifold line bundle over Y. There is a collection of integers $\beta_1, \ldots \beta_n$ satisfying

$$0 \leq \beta_i < \alpha_i$$

such that the bundle $L \otimes E_1^{-\beta_1} \otimes \cdots \otimes E_n^{-\beta_n}$ is a trivial orbifold line bundle over each neighborhood of the l_i's. By forgetting the orbifold structure, it can be naturally identified with a smooth line bundle (denoted by $|L|$) over the smooth manifold $|Y|$.

Theorem 2.4 *The isomorphism classes of orbifold line bundles on Y with specified isotropy representations $\xi_1^{\beta_1}, \ldots, \xi_n^{\beta_n}$ along $l_1, \ldots, l_n$ respectively are in bijective correspondence with $\chi \in H^2(|Y|;\mathbb{Z})$.*

The proof below generalizes [F2] to the case of an arbitrary orbifold line bundle.

Proof: Given $L \in \mathrm{Pic}^t(Y)$, we construct $L \otimes E_1^{-\beta_1} \otimes \cdots \otimes E_n^{-\beta_n}$ and its desingularization $|L|$ explicitly. Let $\pi : X \to Y$ be the unit circle bundle of L. Set $Q = \cup U_i$ in Y and $P = \pi^{-1}(Q)$ with $P_i = \pi^{-1}(U_i)$. Then $X' = X \setminus P$ is a principal S^1-bundle over $Y' = Y \setminus Q$.

In general, the unit circle bundle X is an orbifold rather than a manifold. P_i is a quotient of $l_i \times D^2 \times S^1$ by the action of $\mathbb{Z}_\alpha$ defined by $\xi : (\gamma, w, t) \mapsto (\gamma, \xi w, \xi^\beta t)$. It follows that the isotropy group of a point in the quotient of $l_i \times \{0\} \times S^1$ is $\{\xi \in \mathbb{Z}_\alpha : \xi^\beta = 1\}$ for all points $p \in l$. When the isotropy group is trivial ($gcd(\alpha, \beta) = 1$) the quotient is smooth. In the case that $\beta \equiv 0 \bmod_\alpha$, L is a usual line bundle around that loop, but the 4-manifold still has a nontrivial orbifold structure. Set $d = gcd(\alpha_i, \beta_i)$.

Let $m_i = \partial(\{0\} \times D \times \{1\})$ be the meridian loop of l_i before the quotient is taken. Denote the class it represents in the quotient by $\tilde{m}_i$. The homeomorphism $\varphi_i : \partial P_i \to \partial X'$ determines a section $s : \partial Y' \to \partial X_0$ which is specified up to homology by the relation:

$$\varphi_*[\tilde{m}_i] = \left(\frac{\alpha_i}{d}\right) s_*[m_i'] + \left(\frac{\beta_i}{d}\right) [f'],$$

where m_i' is the meridian of l_i in Y', f' is a fiber of $\partial X'$, and $0 \leq \beta_i \leq \alpha_i$. The local invariants (α_i, β_i) specify P_i up to orientation-preserving equivariant homeomorphism.

The bundle X' can be extended to the unit circle bundle of $|L|$ by equivariantly attaching $l_i \times D \times S^1$ with a bundle isomorphism ϕ_i. Bundle isomorphisms covering the identity are classified up to vertical equivariant isotopy by homotopy classes of maps in $[\partial(S^1 \times D), S^1] = \mathbb{Z} \oplus \mathbb{Z}$. However we can change ϕ_i by a bundle automorphism classified by $[S^1 \times D, S^1] = H^1(S^1 \times D; \mathbb{Z}) = \mathbb{Z}$; these maps change $(\phi_i)_*([l_i])$ by a multiple of the fiber. Therefore the resulting bundle $X' \cup_{\phi_i} (l_i \times D \times S^1)$ can be

completely specified by the map

$$(\phi_i)_*[m_i] = s_*[m'] + r[f']$$

for some $r \in \mathbb{Z}$. Thus we determine the principal S^1 bundle of $|L|$ by specifying that $r = 0$.

In summary:

1. The unit circle bundle of L is obtained by gluing the quotient P using maps

$$(\varphi_i)[\tilde{m}_i] = \left(\frac{\alpha_i}{d}\right) s_*[m_i'] + \left(\frac{\beta_i}{d}\right) [f'].$$

   Note that this bundle depends only on the section $s_*[m_i']$ as well because bundle automorphisms of P_i correspond to

$$[U_i, S^1] = H^1(l_i \times (D \times_{\mathbb{Z}_{\alpha_i}} E\mathbb{Z}_{\alpha_i}); \mathbb{Z}) = H^1(l_i \times B\mathbb{Z}_{\alpha_i}; \mathbb{Z}) = \mathbb{Z}.$$

2. The unit circle bundle of $L \otimes E_1^{-\beta_1} \otimes \cdots \otimes E_n^{-\beta_n}$ is obtained by gluing in the quotient $\amalg_i\, l_i \times (D/\mathbb{Z}_{\alpha_i}) \times S^1$ into X' using maps $(\phi_i)_*[\tilde{m}_i] = s_*[m_i']$.

3. The unit circle bundle of the desingularization $|L|$ is obtained by gluing in $\amalg_i\, l_i \times D \times S^1$ using maps $(\phi_i)_*[m_i] = s_*[m_i']$.

Next we show that two orbifold line bundles L_1 and L_2 with the same isotropy representations and equivalent desingularizations $|L_1| = |L_2|$ are equivalent as orbifold line bundles.

Construct two principal S^1-bundles X_1 and X_2 from X' to form unit circle bundles $|L_1|$ and $|L_2|$. The construction depends on choices of the class $\sum_{i=1}^n s_j[m_i'] \in H_1(\partial X'; \mathbb{Z})$ coming from sections $s_j : \partial Y' \to \partial X_j'$ for $j = 1, 2$. Let $\theta_j \in H^2(Y', \partial Y')$

be the obstruction to extending these sections over X_j'. Let $\tau \in H^1(\partial Y')$ be the primary difference of s_1 and s_2. A diagram chase

$$\begin{tikzcd}
H^1(\amalg_i l_i \times D; \mathbb{Z}) \arrow[r, "\delta_1"] \arrow[d, "i^*"] & H^2(Y, \amalg_i l_i \times D; \mathbb{Z}) \arrow[r, "j_1^*"] \arrow[d, "\lambda", "\cong"'] & H^2(Y; \mathbb{Z}) \\
H^1(\partial Y'; \mathbb{Z}) \arrow[r, "\delta_2"] \arrow[d, "\delta_3"] & H^2(Y', \partial Y'; \mathbb{Z}) & \\
H^2(\amalg_i l_i \times D, \partial; \mathbb{Z}) & &
\end{tikzcd}$$

shows that $j_1^*\lambda^{-1}\delta_2(\tau) = j_1^*\lambda^{-1}(\theta_1 - \theta_2) = c_1(|L_1|) - c_1(|L_2|) = 0$. Thus there is an element $\tau' \in H^1(\amalg_i l_i \times D)$ such that $\delta_1\tau' = \lambda^{-1}\delta_2\tau$, and $\delta_3(\tau - i^*\tau') = 0$. Therefore $\tau \in i^*(H^1(\amalg_i l_i \times D; \mathbb{Z}))$ implying that $(s_1)_*[m_i']$ is homotopic to $(s_2)_*[m_i']$ through a homotopy in $l_i \times D$. Since the construction of the unit circle bundle of the orbifold line bundle in (1) depended only on these sections, L_1 and L_2 are equivalent. □

The above theorem means that a given orbifold line bundle L over Y is specified by the data

$$(c_1(|L|), \beta_1, \ldots, \beta_n)$$

called the *Seifert invariant of L over Y*. (This data, of course, is not unique).

2.3 Spin^c Structures on 3-orbifolds

The Spin^c structures on a 3-orbifold Y are defined by a pair $\xi = (W, \rho)$ consisting of a rank 2 complex orbifold bundle W with a hermitian metric (the spinor bundle) and an action ρ of orbifold 1-forms on spinors,

$$\rho : T^*Y \to \mathrm{End}(W),$$

which satisfies the property that, if e^1, e^2, e^3 are an orthonormal coframe at a point in Y, then the endomorphisms $\rho(e^i)$ are skew-adjoint and satisfy the Clifford relations

$$\rho(e^i)\rho(e^j) + \rho(e^j)\rho(e^i) = -2\delta_{ij}.$$

We also require that the volume form $e^1 \wedge e^2 \wedge e^3$ acts by

$$\rho(e^1 \wedge e^2 \wedge e^3) = -\mathbf{Id}_W.$$

We will write $c_1(\xi)$ for the first Chern class of $\det W$.

Theorem 2.5 *The tangent bundle T^*Y of an orbifold always lifts to an orbifold* $\mathrm{Spin}^c(3)$*-bundle.*

Proof: If we can split TY into a 1-dimensional real line bundle and a complex orbifold line bundle L, then $w_2(TY) = w_2(\mathbb{R}\oplus L) = w_2(L)$ is just the mod 2 reduction of $c_1(L)$ for some orbifold line bundle L. Hence TY lifts.

We need to find a nowhere zero section of TY. Note that each $l_i \times D/\mathbb{Z}_{\alpha_i}$ comes with an $\mathbb{Z}_{\alpha_i}$-invariant oriented nonzero vector field that is tangent to l_i at each point in $D/\mathbb{Z}$. This vector field induces a nonzero section $s : \partial Y' \to TY|_\partial$. Remove an extra $S^1 \times D$ from the interior of Y' and put a similar nonzero section on the boundary. The obstruction to extending the section into the interior of

$$Y'' = Y \setminus \left((S^1 \times D) \;\; \cup \;\; \amalg_i \, l_i \times D/\mathbb{Z}_{\alpha_i}\right)$$

is an element of $H^3(Y'', \partial Y''; \pi_2(S^2)) = \mathbb{Z}$. Using the homology relation

$$[\partial(S^1 \times D)] = -\sum_i [\partial(l_i \times D/\mathbb{Z}_{\alpha_i})],$$

the obstruction can be removed by changing the framing on the boundary of $S^1 \times D$. Thus TY admits a nowhere zero vector field. $\square$

Remark 2.6 *In [S], I. Satake treated the V-Euler class as the index of a unit vector field on TY with singularities and showed that $\chi_V(Y) = 0$ for odd dimensional orbifolds. Thus it is not surprising that nonzero vector fields exists on 3-orbifolds.*

Theorem 2.7 *The set of* Spin^c *structures lifting the frame bundle of a* 3*-orbifold Y is a principal homogeneous space over $Pic^t(Y)$: The difference of two* Spin^c *structures ξ_1, ξ_2 is an orbifold line bundle.*

Proof: Let ξ_1 and ξ_2 be two Spin^c structures which are lifts of the frame bundle. Away from the l_i's, the difference of two Spin^c structures is a complex line bundle as in the smooth case. Because

$$(c_1(\xi_1) - c_1(\xi_2))\,[\partial(l_i \times D/\mathbb{Z}_{\alpha_i})] = 0$$

for all l_i, we can extend the complex line bundle over the desingularization $|Y|$ using techniques in Theorem 2.4. Thus we can investigate locally to show that any two lifts of isotropy representations into $\mathrm{Spin}^c(3)$ differ by a representation into S^1. Note that this is not immediately obvious because there are many different representations of $\mathbb{Z}_\alpha$ into $\mathrm{Spin}^c(3) = U(2)$.

Let Θ_i be the unit vector field on $l_i \times D/\mathbb{Z}_\alpha$ which is tangent to the circle l_i at each point. We use the fact that $\rho : \mathbb{Z}_\alpha \to SO(3)$ is a rotation which leaves the nonzero vector field Θ_i invariant. Identify $SU(2)$ with the unit quaternions. The map $Ad : SU(2) \to SO(3)$ is given by

$$g \mapsto gh\bar{g}$$

for all $h \in \mathrm{Im}\mathbb{H}$ and is the double cover of $SO(3)$. Thus $SO(3)$ can be thought of as the unit quaternions modulo the equivalence $h \sim -h$. Without loss of generality, we may assume that the invariant vector field Θ_i is generated by $\mathbf{i} \in \mathrm{Im}\mathbb{H}$ at each point in l_i.

It is easy to see that elements of $SO(3)$ which rotate the second two components while leaving $\mathbf{i}$ invariant are of the form $e^{\mathbf{i}\theta} \in \mathbb{H}$. Hence $\rho(1) = \lambda^\tau$ where λ is a 2α-root of unity in $\mathbb{C}$ and $0 \leq \tau < \alpha$.

The Spin^c representation $\sigma : \mathrm{Spin}^c \to \mathrm{End}(\mathbb{H})$ is given by $\sigma(g, e^{i\theta})h = ghe^{i\theta}$ for all $h \in \mathbb{H}$. Here we have used the fact $\mathrm{Spin}^c(3) = SU(2) \times S^1 \,/\, (-1,-1)$. Using this identification, Spin^c projects to $SO(3)$ by the adjoint map as well,

$$(g, e^{i\theta}) \mapsto gh\bar{g}$$

for all $h \in \mathrm{Im}\mathbb{H}$. Thus the representation ρ lifts to $\tilde{\rho}$:

$$\begin{tikzcd}
 & S^1 \arrow[d] \\
 & \mathrm{Spin}^c \arrow[d] \\
\mathbb{Z}_\alpha \arrow[r, "\rho"'] \arrow[ur, "\tilde{\rho}"] \arrow[uur, "\hat{\rho}"] & SO(3)
\end{tikzcd}$$

given by $\tilde{\rho}(1) = (\lambda^\tau, \hat{\rho}(1))$ (or equivalently $(-\lambda^\tau, -\hat{\rho}(1))$) for some representation $\hat{\rho} : \mathbb{Z}_\alpha \to S^1$. The representation $\hat{\rho}$ is given by $\hat{\rho}(1) = \lambda^\kappa$ for some $0 \leq \kappa < \alpha$. Hence the difference of two Spin^c structures $\xi_1 - \xi_2$ locally is a representation $(\hat{\rho}_1 - \hat{\rho}_2) : \mathbb{Z}_{\alpha_i} \to S^1$.

Globally, $\xi_1 - \xi_2$ differs by a complex line bundle over $|Y|$ and local isotropy representations into S^1, i.e., an element in $\mathrm{Pic}^t(Y)$ as described in Theorem 2.4. $\square$

2.4 Seiberg-Witten Equations on 3-orbifolds

Fix an orbifold $SO(3)$-connection $^\circ\nabla$ on the cotangent bundle T^*Y and a Spin^c structure $\xi = (W, \rho)$.

Definition 2.8 *A Hermitian connection* ∇ *on* W *is called* spinorial *with respect to* $^\circ\nabla$ *if it is compatible with Clifford multiplication, i.e.,*

$$\nabla(\rho(v)\psi) = \rho(^\circ\nabla v)\psi + \rho(v)(\nabla\psi). \tag{2.1}$$

The set of all spinorial connections will be denoted by $\mathcal{A}(W)$.

Given a trivialization for W, the connection matrix of any $^\circ\nabla$-spinorial connection ∇ can be written with respect to this trivialization as

$$\frac{1}{4}\sum \omega^i_j \otimes \rho(e^i \wedge e^j) + \mathbf{i}b\mathbf{Id}_W$$

where ω^i_j are the connection matrices for $^\circ\nabla$, and $b \in \Omega^1(Y, \mathbb{R})$ is an orbifold 1-form. We will often think of spinorial connections on a Spin^c structure as $U(1)$ connections on det W coupled with the Levi-Civita connection ∇_Y on T^*Y. A spinorial connection ∇ defines a Dirac operator $D_A : \Gamma(Y, W) \to \Gamma(Y, W)$ on the space of orbifold sections of W which is self-adjoint. The perturbed Seiberg-Witten equations are the following pair of equations for (A, Ψ) where A is a $U(1)$ orbifold connection on det W and Ψ is an orbifold section of W:

$$\begin{aligned} F_A + \mathbf{i}\delta - \star\tau(\Psi) &= 0 \\ D_A(\Psi) &= 0. \end{aligned} \tag{2.2}$$

Here $\tau : \Gamma(Y, W) \to \Omega^1(Y; \mathbf{i}\mathbb{R})$ is the adjoint to Clifford multiplication, defined by

$$\langle \rho(\mathbf{i}b)\Psi, \Psi\rangle_W = 2\langle \mathbf{i}b, \tau(\Psi)\rangle_{\Lambda^1},$$

for all orbifold 1-forms b and all $\Psi \in \Gamma(Y, W)$. The δ is a closed orbifold 2-form used to perturb the equations.

For a fixed metric g_Y and perturbation term δ, the moduli space $\mathcal{M}(Y, \xi, g_Y, \delta)$ is the space of solutions to (2.2) modulo the action of the gauge group $\mathcal{G} = Map(Y, S^1)$. Let $\mathcal{M}^*(Y, \xi, g_Y, \delta)$ denote the set of irreducible solutions (i.e., where $\Psi \not\equiv 0$). For a generic perturbation, the moduli space is a compact, smooth manifold containing no reducible solutions. In that case, the fundamental class $[\mathcal{M}(Y, \xi, g_Y, \delta)]$ is essentially the Seiberg-Witten invariant. Evaluating it against some universal classes defines a map $SW^3(\xi) \in \mathbb{Z}$ which is independent of the Riemannian metric and perturbation when $b_1(Y) > 1$ (c.f., [M]). Denote the union over all distinct Spin^c structures by $\mathcal{M}(Y, g_Y, \delta)$.

CHAPTER 3

4-Manifolds with fixed point free circle actions

In this chapter we study manifolds with fixed point free circle actions. We describe the cohomology of these manifolds and show that under some circumstances, line bundles with connections can be pushed forward to orbifold line bundles with connection on the quotient. Finally, we describe how to pullback Spin^c structures from Spin^c structures on the quotient.

3.1 Homology

A 4-manifold with fixed point free S^1-action can be viewed as the boundary of a disk bundle or the unit circle bundle of an orbifold line bundle L over a 3-orbifold Y. Henceforth, we will assume that X is a unit circle orbifold line bundle L over Y where each local invariant β_i is relatively prime to α_i. Denote $\pi : X \to Y$ for the projection map.

When X is smooth, then $X_V \to Y_V$ is an honest S^1-bundle and we have the Gysin

sequence:

$$
\begin{tikzcd}
0 \arrow[r] & H^1_V(Y) \arrow[r] \arrow[d, equal] & H^1_V(X) \arrow[r] \arrow[d, equal] & H^0_V(Y) \arrow[r] \arrow[d, equal] & H^2_V(Y) \arrow[d, equal] \\
0 \arrow[r] & H^1(|Y|) \arrow[r] & H^1(X) \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Pic}^t(Y) \\
 & & & 1 \arrow[r] & {[L]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\arrow[r] & H^2_V(X) \arrow[r] \arrow[d, equal] & H^1_V(Y) \arrow[r] \arrow[d, equal] & H^3_V(Y) \arrow[d, equal] \\
\arrow[r] & H^2(X) \arrow[r] & H^1(|Y|) \arrow[r] & H^3_V(Y)
\end{tikzcd}
$$

Theorem 3.1 *If X is a 4-manifold with a fixed-point free circle action over Y given by the sphere bundle of a line bundle L over Y, then*

$$
\begin{aligned}
H^1(X,\mathbb{Z}) &\cong \begin{cases} H^1(|Y|;\mathbb{Z}), & [L] \text{ is not torsion} \\ H^1(|Y|,\mathbb{Z}) \oplus \mathbb{Z}, & [L] \text{ is torsion} \end{cases} \\
H^2(X;\mathbb{Z}) &\cong (Pic^t(Y)/<[L]>) \oplus \ker(\cdot \cup [L]) : H^1(|Y|;\mathbb{Z}) \to H^3_V(Y;\mathbb{Z}).
\end{aligned}
$$

In particular, since the kernel of $(\cdot \cup [L]) : H^1 \to H^3$ is torsion free, all torsion classes must come from pullbacks in $\pi^*(Pic^t(Y))$.

When $[L]$ is not torsion, the rank of $\mathrm{Pic}^t(Y)/<[L]>$ and $\ker(\cdot \cup [L])$ are both equal to $b_1(|Y|) - 1$. A basis for the former space can be represented by the Poincaré duals of tori of the form $\pi^{-1}(loop)$ for smooth loops in $Y \setminus \Sigma Y$. A basis for the later space can be represented by surfaces in X, which, after integrating over the fiber, are the Poincaré duals of surfaces in $|Y|$. The simple intersection relationship between loops and surfaces in $|Y|$ implies that the intersection form Q_X should be simple as well.

In fact, since the signature is zero (c.f. [HP]), the classification of intersection forms says that Q_X is equivalent to the direct sum of matrices of the form (where d

an integer)

$$\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$$

with respect to a basis $\{A, B\}$ where $A \in \pi^*(\mathrm{Pic}^t(Y))$ is a class pulled back from Y. Pulled back classes always have square zero by the naturality of the cup product and the fact that the product of 2-forms on Y is always zero.

3.2 Line bundles over X

Orbifold line bundles E over Y pullback to usual line bundles $\pi^*(E)$ over X. Except for the case $X = |Y| \times S^1$, this is a many to one correspondence. Nonetheless, it can be made faithful in the following way. Given a line bundle E with connection A over X with the following two properties:

1. The curvature two form of A pulls up from Y, i.e.,

$$\iota_{\mathrm{T}} F_A = 0,$$

   where T is the everywhere non-zero vector field generated by the circle action on X.

2. There exists a point $x \in Y \setminus \Sigma Y$ such that holonomy of A around $\pi^{-1}(x)$ is trivial.

Then (E, A) can be pushed forward to an orbifold line bundle with connection on Y (up to gauge equivalence). If one such point $x \in Y \setminus \Sigma Y$ satisfies the second condition, then all points outside the critical set do. Such connections are said to have *trivial fiberwise holonomy.*

We state Proposition 5.1.3 from [MOY].

Proposition 3.2 *There is a natural one-to-one correspondence between orbifold line bundles with connection over Y and usual line bundles with connection over X, whose curvature forms pull up from Y and whose fiberwise holonomy is trivial. Furthermore, this correspondence induces an identification between orbifold sections of the orbifold bundle over Y with fiberwise constant sections of its pullback over X.*

Pull back connections π^*A are characterized by $\nabla_{\mathrm{T}}^{\pi^*A}\Psi = 0$ for all pulled back sections Ψ.

3.3 Seiberg-Witten Equations of Smooth 4-manifolds

A Spin^c structure $\xi = (W, \sigma)$ on an oriented 4-manifold X is a hermitian vector bundle W of rank 4, together with a Clifford multiplication $\sigma : T^*X \to End(W)$. The bundle W decomposes into two bundles of rank 2, $W^+ \oplus W^-$, with $\det W^+ = \det W^-$. The bundle W^- is the subspace annihilated by the action of self-dual 2-forms. We set $c_1(\xi)$ to be the first Chern class of $\det W^+$.

There is a natural way to pullback a Spin^c structure from Y to X. Let η denote the connection 1-form of the circle bundle $\pi : X \to Y$, and let g_Y be a metric on Y, then endow X with the metric $g_X = \eta \otimes \eta + \pi^*(g_Y)$. Using this metric, there is an orthogonal splitting

$$T^*X \cong \mathbb{R}\eta \oplus \pi^*(T^*Y).$$

If $\xi = (W, \rho)$ is a Spin^c structure over Y, define the pullback of ξ to be $\pi^*(\xi) = (\pi^*(W) \oplus \pi^*(W), \sigma)$ where the action

$$\sigma : T^*X \to \mathrm{End}(\pi^*(W) \oplus \pi^*(W))$$

is given by

$$\sigma(b\eta + \pi^*(a)) = \begin{pmatrix} 0 & \pi^*(\rho(a)) + b\mathbf{Id}_{\pi^*(W)} \\ \pi^*(\rho(a)) - b\mathbf{Id}_{\pi^*(W)} & 0 \end{pmatrix}.$$

This defines a Spin^c structure on X.

Choosing a Spin^c structure $\xi_0 = (W_0, \rho)$ on Y gives rise to a one-to-one correspondence between Hermitian orbifold line bundles and Spin^c structures on Y via $E \mapsto W_0 \otimes E$. Likewise, the pullback Spin^c structure $\xi = \pi^*(\xi_0)$ induces a one-to-one correspondence between Hermitian line bundles and Spin^c structures on X.

Remark 3.3 *In this way we can think of a* Spin^c *structure with respect to* ξ_0 *or* ξ *as a choice of line bundle on* Y *or* X *respectively. This allows us to push-forward a* Spin^c *structure with a trivial fiberwise connection on* $\det W^+$ *from* X *to* Y *via Proposition 3.2.*

There is a natural connection on X which is compatible with the reduction $T^*X = \mathbb{R}\eta \oplus \pi^*(T^*Y)$. Let ∇^Y denote the Levi-Civita connection on Y and set ${}^\circ\nabla = d \oplus \pi^*(\nabla^Y)$. This is a compatible connection which satisfies

$$ {}^\circ\nabla\eta = 0, \quad \text{and} \quad {}^\circ\nabla(\pi^*(\beta)) = \pi^*(\nabla^Y \beta). \tag{3.1}$$

It is more convenient to use this reducible $SO(4)$-connection instead of the Levi-Civita connection. By coupling it with a $U(1)$-connection A on $\det W^+$ we can define a spinorial connection on W^+. Define a Dirac operator $\not{D}_A^+ : \Gamma_X(W^+) \to \Gamma_X(W^-)$ from the space of smooth sections of W^+ to W^-. The 4-dimensional perturbed Seiberg-Witten equations for a section $\Psi \in \Gamma_X(W^+)$ and a $U(1)$-connection A on $\det W^+$

are:

$$\begin{aligned} F_A^+ + \mathbf{i}\delta - q(\Psi) &= 0, \\ \not{D}_A^+(\Psi) &= 0. \end{aligned} \tag{3.2}$$

Here F_A^+ is the projection of the curvature onto the self-dual two forms, δ is self-dual 2-form used to perturb the equations, and $q : \Gamma_X(W^+) \to \Omega^+(X, \mathbf{i}\mathbb{R})$ defined by $q(\Psi) = \Psi \otimes \Psi^* - \frac{1}{2}|\Psi|^2$ is the adjoint of Clifford multiplication by self-dual 2-forms,i.e,

$$\langle \sigma(\mathbf{i}\beta)\Psi, \Psi \rangle_{W^+} = 4\langle \mathbf{i}\beta, q(\Psi) \rangle_{\mathbf{i}\Lambda^+} \tag{3.3}$$

for all self-dual 2-forms β and all sections Ψ.

Similar to the 3-dimensional case, the moduli space $\mathcal{M}(X, \xi, g_X, \delta)$ is the space of solutions (A, Ψ) modulo the action of the gauge group. We are using a reducible connection $^\circ\nabla$ instead of the Levi-Civita connection on T^*X, but this alternative compatible connection is an allowable perturbation of the usual Seiberg-Witten equations and can be used to calculate the Seiberg-Witten invariants (see section 4 of [OS2]). Under suitable generic conditions the moduli space is a compact, oriented, smooth manifold of dimension

$$d(\xi) = \frac{1}{4}\left(c_1(\xi)^2 - 2\chi(X) - 3\sigma(X)\right) \tag{3.4}$$

which is independent of metric and perturbation when $b_+(X) > 1$.

The Seiberg-Witten invariant $SW_X(\xi)$ is a suitable count of solutions. Fix a base point in M and let $\mathcal{G}^0 \subset Map(X, S^1)$ denote the group of maps which map that point to 1. The base moduli space, denoted by $\mathcal{M}^0$, is the quotient of the space of solutions by $\mathcal{G}^0$. When the moduli space $\mathcal{M}(X, \xi, g_X, \delta)$ is smooth, $\mathcal{M}^0$ is a principle S^1-bundle over $\mathcal{M}(X, \xi, g_X, \delta)$. For a given Spin^c structure ξ, the 4-dimensional Seiberg-Witten

invariant $SW_X(\xi)$ is defined to be 0 when $d(\xi) < 0$, the sum of signed points when $d(\xi) = 0$, or if $d(\xi) > 0$, it is the pairing of the fundamental class of $\mathcal{M}(X, \xi, g_X, \delta)$ with the maximal cup product of the Euler class of the S^1-bundle $\mathcal{M}^0$.

The dimension formula (3.4) simplifies when the manifold has a fixed point free circle action. Because X has a nonzero vector field T, the Euler class is zero. As mentioned previously, the signature of X is also zero.

Proposition 3.4 *Suppose that X is a 4-manifold with a fixed point free circle action. The expected dimension of the moduli space for a* Spin^c *structure ξ is*

$$d(\xi) = \frac{1}{4} c_1(\xi)^2.$$

CHAPTER 4

Spin^c structures and SW solutions

We continue to work with a circle bundle $\pi : X \to Y$ with an S^1-invariant metric $g_X = \eta^2 + \pi^*(g_Y)$. The perturbation $\delta \in \Omega^2(Y, \mathbf{i}\mathbb{R})$ is a closed orbifold 2-form used to perturb the 3-dimensional equations which is then pulled back and projected on to the self-dual 2-forms of X to perturb the 4-dimensional equations.

4.1 Restrictions on Spin^c structures

First, we make some basic observations. If $SW_X(\xi) \neq 0$ for some Spin^c structure ξ, then the expected dimension of the moduli space is nonnegative, implying

$$c_1(\xi)^2 \geq 0. \tag{4.1}$$

If $b_+(X) = 1$, then the metric g_X induces a splitting $\mathcal{H}^2(X; \mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+$ is one dimensional. Let $c_1(\xi)^+$ be the L^2 projection onto the self-dual subspace $\mathcal{H}^+$. When $c_1(\xi)^+$ is nonzero, it provides an orientation for $\mathcal{H}^+$. In this situation the Seiberg-Witten invariant depends on the chamber of $2\pi c_1(\xi) + \pi^*(\delta)$. We will say that $\alpha \in H^2(X; \mathbb{R})$ lies in the positive chamber if $\alpha^+ \cdot c_1(\xi)^+ > 0$. Denote the Seiberg-Witten invariant calculated for $\alpha = (2\pi c_1(\xi) + \pi^*(\delta))$ in this chamber by

$SW^+_X(\xi)$ and denote the invariant of the other chamber by $SW^-_X(\xi)$.

When $c_1(\xi)^+ = 0$ there no distinguished chamber. However, if $SW_X(\xi) \neq 0$ in either chamber, the dimension of the moduli is nonzero and

$$0 \leq c_1(\xi)^2 = (c_1(\xi)^-)^2 \leq 0.$$

Since the intersection form on $\mathcal{H}^-$ is definite, $c_1(\xi)$ is a torsion class and pulled back from Y by Theorem 3.1.

With this as background, we can state:

Theorem A. *Let ξ be a* Spin^c *structure on 4-manifold X with a fixed point free circle action such that $SW_X(\xi) \neq 0$ (in either chamber when $b_+ = 1$).*

1. *If $b_+(X)>1$ or $b_+(X)=0$, then $c_1(\xi)$ is pulled back from Y.*
2. *If $b_+(X)=1$, then either $c_1(\xi)$ is pulled back from Y, or $SW^+_X(\xi) = 0$.*

Remark 4.1 *In case 2b, the Seiberg-Witten invariant of the other chamber can be calculated using the wall crossing formula of [LL].*

Corollary 4.2 *If $b_+(X) > 1$ then $c_1(\xi)^2 = 0$ and X is SW simple type.*

Recall that a 4-manifold is SW simple type if the dimension of the moduli space is 0 for all Spin^c structures with nonzero Seiberg-Witten invariants.

Theorem A follows easily from the following formula about Seiberg-Witten solutions.

Theorem 4.3 *Let (A, Ψ) be any solution in $\mathcal{M}(X, \xi, g_X, \pi^*(\delta)^+)$. Then*

$$2\pi c_1(\xi) \cdot \pi^*(\delta) \;=\; \int_X |\nabla_{\mathrm{T}}\Psi|^2 + |D^+\Psi|^2 + |\iota_{\mathrm{T}} F_A|^2 + 2\pi^2 c_1(\xi)^2.$$

The vector field T *is the everywhere nonzero vector field generated by the circle action.*

Remark 4.4 *The equation in Theorem 4.3 only holds for perturbations which are pulled back from Y. It does not hold for a general self-dual 2-form on X.*

The rest of this section contains a proof of Theorem A assuming Theorem 4.3 above. We will then come back and prove Theorem 4.3 in the next section. We prove each case separately.

Proof of case 1: When $b_+(X) > 1$, the moduli space is nonempty for all generic metric and perturbation pairs. Since generic pairs are dense in the space of metrics and self-dual 2-forms, we can take a sequence of generic pairs which converge to the pair $(g_X, 0)$. By compactness, solutions of the generic pairs converge to a solution $(A, \Psi) \in \mathcal{M}(X, \xi, g_X, 0)$ and it satisfies

$$0 = \int_X |\nabla_{\mathrm{T}}\Psi|^2 + |D^+\Psi|^2 + |\iota_{\mathrm{T}} F_A|^2 + 2\pi^2 c_1(\xi)^2 \tag{4.2}$$

by Theorem 4.3. Using equation (4.1) we conclude that all terms in equation (4.2) vanish; in particular, $c_1(\xi)^2 = 0$ and

$$\iota_{\mathrm{T}} F_A = 0.$$

Since $dF_A = 0$, this equation implies $\mathcal{L}_{\mathrm{T}} F_A = 0$ by Cartan's formula. Together the equations $\iota_{\mathrm{T}} F_A = \mathcal{L}_{\mathrm{T}} F_A = 0$ imply that F_A is pulled back from Y. Since $c_1(\xi) = \frac{i}{2\pi} F_A$, case 1 follows.

When $b_+(X) = 0$ we have that $b_2(X) = 0$ is also zero because the signature is zero. Thus $c_1(\xi)$ is always a torsion class and this is pulled back by Theorem 3.1. □

Proof of case 2: Assume that $c_1(\xi)$ is not pulled back. By the argument proceeding the statement of Theorem A, $c_1(\xi)^+ \neq 0$.

We proceed by contradiction. Suppose that $SW_X^+(\xi) \neq 0$. In this chamber, the

moduli space will be nonempty for all generic pairs of metrics and perturbations. Note that the unperturbed Seiberg-Witten equations ($\delta = 0$) are in this chamber because $(c_1(\xi) - 0)^+ \cdot c_1(\xi)^+ > 0$. Hence we can use the same argument as in case 1 to show that $c_1(\xi)$ is pulled back from Y – contradicting our assumption. Thus $SW_X^+(\xi) = 0$. $\square$

4.2 Solutions to the SW equations

In this section we prove Theorem 4.3. The idea is to prove a Weitzenböck-type decomposition for the Dirac operator we constructed in section 3.3. Before we prove this decomposition, however, we need to show that the full Dirac operator $\not{D}_A : \Gamma_X(W^+ \oplus W^-) \to \Gamma_X(W^+ \oplus W^-)$ is self-adjoint. The following technical lemma accomplishes this.

Lemma 4.5 *Let $\xi = (W, \sigma)$ be a* Spin^c *structure over X. Let ∇ be a spinorial connection created by coupling a connection $A \in \mathcal{A}(\det W^+)$ with the $SO(3)$-connection $^\circ\nabla$ defined in section 3.3. Similarly, let $\nabla^{L.C.}$ be the spinorial connection created by coupling the same connection A with the Levi-Civita connection $^\circ\nabla^{L.C.}$. Then*

$$\not{D}_A^{L.C.} = \not{D}_A - \frac{1}{2}\sigma(\eta \wedge d\eta). \tag{4.3}$$

Since $\not{D}^{L.C.}$ and Clifford multiplication by 3-forms are both self-adjoint operators, $\not{D}_A$ is self-adjoint.

Proof: Extend η to an orthonormal coframe $\{\eta = e^0, e^1, e^2, e^3\}$ on a patch of X so that $e^0 = \eta$, and $\{e^1, e^2, e^3\}$ are horizontal lifts of an orthonormal coframe $\{\bar{e}^1, \bar{e}^2, \bar{e}^3\}$ on Y. Let $\{e_0 = \mathrm{T}, e_1, e_2, e_3\}$ be the dual vector fields with respect to the metric g_X.

The difference 1-form $\omega = \nabla^{L.C.} - {}^{\circ}\nabla \in \Omega^1(\mathfrak{so}(T^*X))$ can be thought of as an element in $\Omega^1(\Lambda^2 T^*X)$ via the vector space isomorphism

$$i : \mathfrak{so}(T^*X) \to \Lambda^2(T^*X)$$

defined by

$$i(a_j^k) = \frac{1}{2}\sum_{j<k} a_k^j e^j \wedge e^k.$$

The action of $\mathfrak{so}(T^*X)$ on the bundle $\dot{W}$ is modeled on $\sigma_{\Lambda^2} \circ i$. Thus we can Clifford multiply the Λ^2 component of $\omega \in \Omega^1(\Lambda^2 T^*X)$ to get

$$\not{D}_A^{L.C.} = \not{D}_A + \sigma_{\Lambda^1 \otimes \Lambda^2}(\omega)$$

where $\sigma_{\Lambda^1 \otimes \Lambda^2} : \Lambda^1 \otimes \Lambda^2 \to \mathrm{End}(W)$ is a linear map defined by

$$\sigma_{\Lambda^1 \otimes \Lambda^2}(\alpha \otimes \beta) = \sigma(\alpha)\sigma(\beta)$$

for a basis element $\alpha \otimes \beta \in \Lambda^1 \otimes \Lambda^2$. This map can be conveniently reformulated as

$$\sigma_{\Lambda^1 \otimes \Lambda^2}(\alpha \otimes \beta) = -\sigma(\iota_{\alpha^\flat}\beta) + \sigma(\alpha \wedge \beta),$$

where $\iota_{\alpha^\flat}$ is contraction with the vector field which is g_X-dual to α.

Let $\{\zeta_{12}, \zeta_{13}, \zeta_{23}\}$ be the functions defined by

$$d\eta = 2\zeta_{12}e^1 \wedge e^2 + 2\zeta_{13}e^1 \wedge e^3 + 2\zeta_{23}e^2 \wedge e^3. \tag{4.4}$$

We can use equation (4.4) and the first Cartan Structure equation

$$de^i = \sum_j e^j \wedge w_j^i$$

to calculate the connection matrix for ${}^{\circ}\nabla^{L.C.}$. For example, we can write $d\eta$ as

$$d\eta = e^1 \wedge (\zeta_{12}e^2 + \zeta_{13}e^3) + e^2 \wedge (-\zeta_{12}e^1 + \zeta_{13}e^3) + e^3 \wedge (-\zeta_{13}e^1 - \zeta_{23}e^2)$$

to get the top row of the connection matrix

$$\begin{pmatrix} 0 & \zeta_{12}e^2 + \zeta_{13}e^3 & -\zeta_{12}e^1 + \zeta_{23}e^3 & -\zeta_{13}e^1 - \zeta_{23}e^2 \\ -\zeta_{12}e^2 - \zeta_{13}e^3 & 0 & -\zeta_{12}e^0 + \omega_2^1 & -\zeta_{13}e^0 + \omega_3^1 \\ \zeta_{12}e^1 - \zeta_{23}e^3 & \zeta_{12}e^0 - \omega_2^1 & 0 & -\zeta_{23}e^0 + \omega_3^2 \\ \zeta_{13}e^1 + \zeta_{23}e^2 & \zeta_{13}e^0 - \omega_3^1 & \zeta_{23}e^0 - \omega_3^2 & 0 \end{pmatrix} \tag{4.5}$$

The ω_j^i's in the second, third, and forth row are pulled-back from the connection 1-form for the Levi-Civita connection on Y. The connection matrix for ${}^{\circ}\nabla$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_2^1 & \omega_3^1 \\ 0 & -\omega_2^1 & 0 & \omega_3^2 \\ 0 & -\omega_3^1 & -\omega_3^2 & 0 \end{pmatrix}. \tag{4.6}$$

Using the isomorphism i, the difference $\nabla^{L.C.} - \nabla$ can be written as

$$\omega = \frac{1}{2}\sum_{i=1}^{3} e^i \otimes \eta \wedge \iota_{e_i}(d\eta) + \frac{1}{2}\eta \otimes d\eta.$$

A straight forward calculation gives $\sigma_{\Lambda^1 \otimes \Lambda^2}(\omega) = -\frac{1}{2}\sigma(\eta \wedge d\eta)$. $\square$

Remark 4.6 *The operators $\not{D}_A^{L.C.}$ and $\not{D}_A$ have the same index.*

Lemma 4.7 *The square of the Dirac operator decomposes into*

$$(\not{D}_A^+)^* \not{D}_A^+ \;=\; -(\nabla_{\mathrm{T}})^2 + (D^+)^* D^+ + \frac{1}{2}\sigma((\eta \wedge \iota_{\mathrm{T}} F_A)^+). \tag{4.7}$$

where $(\ )^+$ *is the projection onto self-dual 2-forms.*

Proof: We work with the full Dirac operator first. By using the definition of $^\circ\nabla$ from equation (3.1), we see from

$$\begin{aligned}
\langle \sigma(\eta)\nabla_{\mathrm{T}}\Psi, \Phi\rangle &= \langle \Psi, \nabla_{\mathrm{T}}(\sigma(\eta)\Phi)\rangle \\
&= \langle \Psi, {}^\circ\nabla_{\mathrm{T}}(\eta)\Phi\rangle + \langle \Psi, \sigma(\eta)\nabla_{\mathrm{T}}\Phi\rangle \\
&= \langle \Psi, \sigma(\eta)\nabla_{\mathrm{T}}\Phi\rangle
\end{aligned}$$

that $\sigma(\eta)\nabla_{\mathrm{T}}$ is L^2 self-adjoint.

The Dirac operator decomposes into a sum of two self-adjoint operators:

$$\not{D}_A = \sigma(\eta)\nabla_{\mathrm{T}} + D,$$

where $D = \sum_{i=1}^3 \sigma(e^i)\nabla_{e_i}$.

Squaring and noting that $\nabla_T \eta = 0$ and $\sigma(\eta)\sigma(\eta) = -Id$ yields

$$\not{D}_A^2 = -(\nabla_{\mathrm{T}})^2 + D^2 + \{\sigma(\eta)\nabla_{\mathrm{T}}, D\}.$$

The last term simplifies using Clifford relations and the equations (2.1) and (3.1):

$$\{\sigma(\eta)\nabla_{\mathrm{T}}, D\} \;=\; \sum_{i=1}^{3} \sigma(\eta \wedge e^i)[\nabla_{\mathrm{T}}, \nabla_{e_i}].$$

One can use the connection matrix (4.5) to calculate that

$$[\mathrm{T}, e_i] = {}^\circ\nabla^{L.C.}_{\mathrm{T}} e_i - {}^\circ\nabla^{L.C.}_{e_i} T = 0 \tag{4.8}$$

for $i = 1, 2, 3$. In this situation the curvature reduces to

$$F_\nabla(\mathrm{T}, e_i) = [\nabla_{\mathrm{T}}, \nabla_{e_i}], \tag{4.9}$$

and we can see that

$$\{\sigma(\eta)\nabla_{\mathrm{T}}, D\} \;=\; \sum_{i=1}^{3} \sigma(\eta \wedge e^i) F_\nabla(\mathrm{T}, e_i) = \sigma(\eta)\sigma(\iota_{\mathrm{T}} F_\nabla). \tag{4.10}$$

By the definition of ${}^\circ\nabla$, the action of $[\nabla_{\mathrm{T}}, \nabla_{e_i}]$ for $i = 1, 2, 3$ commutes with Clifford multiplication. Therefore $F_\nabla(\mathrm{T}, e_i)$ is a scalar endomorphism, so

$$F_\nabla(\mathrm{T}, e_i) = \frac{1}{2} F_A(\mathrm{T}, e_i). \tag{4.11}$$

Restricting attention to W^+ gives the formula. □

Proof of Theorem 4.3: Take a solution $(A, \Psi) \in \mathcal{M}(X, g_X, \pi^*(\delta)^+)$, apply $(\not{D}_A^+)^* \not{D}_A^+$, and take the inner product with Ψ. After applying Lemma 4.7 and integrating over X we get

$$\begin{aligned} 0 &= \int_X \langle (\not{D}_A^+)^* \not{D}_A^+ \Psi, \Psi \rangle \\ &= \int_X \langle (\sigma(\eta)\nabla_{\mathrm{T}})^2 \Psi, \Psi \rangle + \langle (D^+)^* D^+ \Psi, \Psi \rangle + \frac{1}{2} \langle \sigma((\eta \wedge \iota_{\mathrm{T}} F_A)^+) \Psi, \Psi \rangle \\ &= \int_X |\nabla_{\mathrm{T}} \Psi|^2 + |D^+ \Psi|^2 + 2 \langle (\eta \wedge \iota_{\mathrm{T}} F_A), q(\Psi) \rangle. \end{aligned} \tag{4.12}$$

In the last step we used the adjoint property of $q(\Psi)$ from equation (3.3) and the fact that $q(\Psi)$ is self-dual.

Substituting the $q(\Psi) = F_A^+ + \pi^*(\mathbf{i}\delta)^+$ from (3.2), we get

$$\begin{aligned}
2\langle(\eta \wedge \iota_{\mathrm{T}} F_A), q(\Psi)\rangle &= 2\langle(\eta \wedge \iota_{\mathrm{T}} F_A), F_A^+ + \pi^*(\mathbf{i}\delta)^+)\rangle \\
&= \langle \iota_{\mathrm{T}} F_A, \ \iota_{\mathrm{T}} \left(F_A + \star F_A + \pi^*(\mathbf{i}\delta) + \star\pi^*(\mathbf{i}\delta)\right)\rangle \\
&= |\iota_{\mathrm{T}} F_A|^2 + \langle \iota_{\mathrm{T}} F_A, \iota_{\mathrm{T}} \star \left(F_A + \star\pi^*(\mathbf{i}\delta)\right)\rangle \\
&= |\iota_{\mathrm{T}} F_A|^2 + \frac{1}{2}\mathbf{i}F_A \wedge \mathbf{i}F_A - \mathbf{i}F_A \wedge \pi^*(\delta)
\end{aligned} \tag{4.13}$$

The last equality is true by the following calculation. Let F_{ij} be the functions defined by

$$F_A = \sum_{0 \le i < j} \mathbf{i}F_{ij} e^i \wedge e^j.$$

Then

$$\iota_{\mathrm{T}} F_A = \mathbf{i}F_{01}e^1 + \mathbf{i}F_{02}e^2 + \mathbf{i}F_{03}e^3$$

and

$$\iota_{\mathrm{T}} \star F_A = \mathbf{i}F_{12}e^3 - \mathbf{i}F_{13}e^2 + \mathbf{i}F_{23}e^1.$$

Taking their inner product gives

$$\langle \iota_{\mathrm{T}} F_A, \iota_{\mathrm{T}} \star F_A \rangle = F_{01}F_{23} - F_{02}F_{13} + F_{03}F_{12} = \frac{1}{2}\mathbf{i}F_A \wedge \mathbf{i}F_A.$$

A similar calculation shows that $\langle \iota_{\mathrm{T}} F_A, \ \iota_{\mathrm{T}} \star \pi^*(\delta)\rangle = -\mathbf{i}F_A \wedge \pi^*(\delta)$.

Integrating equation (4.13) over X gives the lemma. $\square$

CHAPTER 5

Diffeomorphic moduli spaces

In this chapter we prove Theorem B. We continue to work with the circle bundle $\pi : X \to Y$ with the S^1-invariant metric $g_X = \eta \otimes \eta + \pi^*(g_Y)$ as in 3.3. Fix a closed orbifold 2-form δ to perturb the 3-dimensional equations and pull it back to get an S^1-invariant 2-form on X. Perturb the 4-dimensional equations by projecting $\pi^*(\delta)$ onto the self-dual 2-forms to get $\pi^*(\delta)^+$.

The total moduli space $\mathcal{M}(X, g_X, \pi^*(\delta)^+)$ is a disjoint collection of moduli spaces, one component for each Spin^c structure ξ on X. Define

$$\mathcal{N}(X, g_X, \pi^*(\delta)^+)$$

to be the components of the total moduli space whose cohomology class $c_1(\xi)$ is pulled back from Y.

Theorem A implies that we need only look at these components to calculate the Seiberg-Witten invariants when $b_+ > 1$. This restriction on the total moduli space is done to rule out Spin^c structures covered in Theorem A case 2b.

Note that for any Spin^c structure ξ whose $c_1(\xi)$ class is pulled back and for any

2-form $\pi^*(\delta)$,

$$\begin{aligned} c_1(\xi)^2 &= 0 \\ c_1(\xi) \cdot \pi^*(\delta) &= 0. \end{aligned} \tag{5.1}$$

In particular, the expected dimension of the moduli space is 0 by Proposition 3.4.

A pullback of a solution (A_0, Ψ_0) to (2.2) on Y is the solution $(A, \Psi) = \pi^*(A_0, \Psi_0)$ to (3.2) on X. Pick an orthonormal coframe on a patch of X

$$\{e^0, e^1, e^2, e^3\}$$

so that $e^0 = \eta$, and $\{e^1, e^2, e^3\}$ are horizontal lifts of an orthonormal coframe $\{\bar{e}^1, \bar{e}^2, \bar{e}^3\}$ on Y. Let $\{e_0 = \mathrm{T}, e_1, e_2, e_3\}$ be the dual vector fields with respect to the metric g_X. In this case the Dirac operator can be written as

$$\not{D}_A^+ = \sigma(\eta)\nabla_{\mathrm{T}} + D_A \tag{5.2}$$

where ∇ is a connection on W^+ created by coupling A with the reducible connection ${}^\circ\nabla$ (see section 3.3) and $D = \sum \sigma(e^i)\nabla_{e_i}$ for $i = 1, 2, 3$. From the construction of the pulled back Spin^c structure, it is immediately clear that $\pi^*(\Psi)$ is harmonic since it is constant along the fiber and comes from a harmonic spinor on Y. The first equation of (3.2) is satisfied by pulling up the first equation and projecting each term onto the self-dual 2-forms. Since a gauge transformation on ξ pulls back to a gauge transformation of $\pi^*(\xi)$, we get a well defined map on the level of moduli spaces.

Theorem B. *The pullback map π^* induces a homeomorphism*

$$\pi^* : \mathcal{M}^*(Y, g_Y, \delta) \to \mathcal{N}^*(X, g_X, \pi^*(\delta)^+).$$

Furthermore, if either of the two moduli spaces is a smooth manifold, then both of them are smooth, and π^* is a diffeomorphism.

One remark: There is no restriction on $b_+(X)$ in the above theorem.

The next three sections contains the proof of this theorem. We show that π^* is a homeomorphism in the first two sections. In the final section we show that $d\pi^*$ is an isomorphism on the kernel of the linearizations. This is sufficient to prove that the moduli spaces are diffeomorphic because the expected dimension of each is zero. Bochner vanishing arguments are used to prove that π^* and $d\pi^*$ are surjective.

5.1 π^* is injective

Suppose we have two irreducible solutions to the 3-dimensional equations whose pullbacks (A, Ψ) and (A', Ψ') differ by a gauge transformation $g \in Map(X, S^1)$,

$$g(A, \Psi) = (A', \Psi').$$

We wish to show that g is in fact pulled back from $Map(Y, S^1)$. We think of g as a section of $\mathrm{End}(\det W^+) = \mathrm{End}(\pi^*(\det W)) = \pi^*(\mathrm{End}(\det W))$. Use A to create a connection ∇^{End} on $\mathrm{End}(\pi^*(W^+))$ which has trivial fiberwise holonomy. Then

$$\begin{aligned} (\nabla^{\mathrm{End}}_{\mathrm{T}} g)\Psi &= \nabla^A_{\mathrm{T}}(g\Psi) - g\nabla^A_{\mathrm{T}}\Psi \\ &= 0 \end{aligned}$$

because $\Psi' = g\Psi$ are pulled back sections. By the unique continuation theorem for elliptic operators, $\Psi \neq 0$ on a dense open set, hence

$$\nabla_{\mathrm{T}}^{\mathrm{End}} g \equiv 0$$

on X. Thus g is a fiberwise constant section of the line bundle $\pi^*(\mathrm{End}(\det W))$ and by Proposition 3.2, it can be pushed forward to a section of $\mathrm{End}(\det W)$ on Y, i.e., a gauge transformation on Y.

5.2 π^* is surjective

Take a solution $(A, \Psi) \in \mathcal{N}^*(X, g_X, \pi^*(\delta)^+)$ to the Seiberg-Witten equations (3.2). We will show that the solution is pulled up from a solution (A_0, Ψ_0) on Y.

Combining the formula in Theorem 4.3 with the fact that Spin^c structures from $\mathcal{N}^*(X, g_X, \pi^*(\delta)^+)$ satisfy equations (5.1), we get

$$\underbrace{2\pi c_1(\xi) \cdot \pi^*(\delta)}_{0} = \int_X |\nabla_{\mathrm{T}}\Psi|^2 + |D^+\Psi|^2 + |\iota_{\mathrm{T}} F_A|^2 + \underbrace{2\pi^2 c_1(\xi)^2}_{0}. \tag{5.3}$$

and that the following terms must be identically zero:

$$0 = \nabla_{\mathrm{T}}\Psi, \tag{5.4}$$

$$0 = D^+\Psi, \tag{5.5}$$

$$0 = \iota_{\mathrm{T}} F_A, \tag{5.6}$$

Equation (5.6) implies $\mathcal{L}_{\mathrm{T}} F_A = 0$ and together the equations imply that F_A is circle invariant and pulled up from Y. Equation (5.4) and the fact that $\Psi \not\equiv 0$ means that A has trivial fiberwise holonomy. Therefore we can apply Proposition 3.2 and

Remark 3.3 to ξ with connection A to conclude that Ψ corresponds to an orbifold section Ψ_0 on a Spin^c structure ξ_0 with connection A_0 on Y.

In this situation, D^+ is the Dirac operator on the orbifold Y, so by equations (5.5) and (5.4), the second Seiberg-Witten equation of (2.2) is satisfied for (A_0, Ψ_0). It is easy to check that (A_0, Ψ_0) also satisfies the first Seiberg-Witten equation.

Therefore the map π^* is a homeomorphism of moduli spaces. □

5.3 The kernels are isomorphic

Consider an irreducible solution $\mathsf{S} = (A, \Psi)$ to the 4-dimensional Seiberg-Witten equations for a fixed metric and perturbation $(g_X, \pi^*(\delta)^+)$ in the Spin^c structure ξ. In the previous section we saw that (A, Ψ) was pulled back from a solution $\mathsf{S}_0 = (A_0, \Psi_0)$ to the 3-dimensional equations on Y in the Spin^c structure ξ_0.

We now describe the tangent space at the solution S. The following sequence of operators (for a fixed $k \geq 5$)

$$0 \longrightarrow T_1 L^2_{k+2}(Y, S^1) \xrightarrow{\mathcal{L}_\mathsf{S}} T_\mathsf{S} L^2_{k+1}(\mathrm{i}T^*Y \oplus W^+) \xrightarrow{LSW^4} L^2_k(\mathrm{i}\Lambda_+ T^*Y \oplus W^-) \longrightarrow 0$$

is called the *deformation complex at* S. The map $\mathcal{L}_\mathsf{S}$ is the infinitesimal action of the gauge group at S described by its differential at the identity:

$$\mathcal{L}_\mathsf{S} : \mathbf{i}f \mapsto (-2\mathbf{i}df, \mathbf{i}f\Psi),$$

and LSW^4 is the linearization of the 4-dimensional Seiberg-Witten equations with fixed perturbation $\pi^*(\delta)^+$. We can wrap LSW^4 and $\mathcal{L}_\mathsf{S}$ into one operator

$$\mathcal{T}_\mathsf{S} : L^2_{k+1}(\mathbf{i}T^*X \oplus W^+) \to L^2_k(\mathbf{i}\Lambda_+ T^*X \oplus W^- \oplus \mathbf{i}\Lambda^0 T^*Y)$$

by setting $\mathcal{T}_{\mathsf{S}} = LSW^4 + \mathcal{L}_{\mathsf{S}}^*$. Then the $\ker \mathcal{T}_{\mathsf{S}}$ is the set of (a, ψ) which satisfy

$$\begin{aligned} d^+ a - q(\psi, \Psi) - q(\Psi, \psi) &= 0, \\ \not{D}_A \psi + \tfrac{1}{2}\sigma(a)\Psi &= 0, \\ -2d^* a + \mathbf{i}\mathrm{Im}\langle \psi, \Psi \rangle &= 0. \end{aligned} \tag{5.7}$$

The last equation is a slice condition for the gauge group action.

Let $\mathcal{H}_{\mathsf{S}}^*$ denote the cohomology of the deformation complex at S. We can now state Lemma 2.2.11 from [N, page 129]:

Lemma 5.1 *The deformation complex at* S *is Fredholm, that is, the coboundary maps have closed ranges and the cohomology spaces are finite dimensional. Moreover,*

$$\mathcal{H}_{\mathsf{S}}^0 \cong \ker \mathcal{L}_{\mathsf{S}}, \quad \mathcal{H}_{\mathsf{S}}^1 \cong \ker \mathcal{T}_{\mathsf{S}}$$

and

$$\operatorname{coker} \mathcal{T}_{\mathsf{S}} \cong \mathcal{H}_{\mathsf{S}}^0 \oplus \mathcal{H}_{\mathsf{S}}^2.$$

In particular, the expected dimension of the moduli space for the Spin^c *structure* ξ *is*

$$d(\xi) = -\chi_{\mathbb{R}}(\mathcal{H}_{\mathsf{S}}^*) = -\dim \mathcal{H}_{\mathsf{S}}^0 + \dim \mathcal{H}_{\mathsf{S}}^1 - \dim \mathcal{H}_{\mathsf{S}}^2.$$

A metric and perturbation (g, δ) is called a *good pair* if $\mathcal{H}_{\mathsf{S}}^0 = \mathcal{H}_{\mathsf{S}}^2 = 0$ for every solution to the Seiberg-Witten equations in the Spin^c structure ξ. If $(g_X, \pi^*(\delta)^+)$ is good, then the moduli space $\mathcal{M}(X, \xi, g_X, \pi^*(\delta)^+)$ is a smooth manifold of dimension $d(\xi)$, its formal tangent space can be identified with $\mathcal{H}_{\mathsf{S}}^1$ at the point S, and we can use it to calculate the Seiberg-Witten invariants of ξ. For a careful treatment of these ideas, see pages 127-135 of [N].

There is a similar complex for the solution S_0 on Y. It too can be described by an operator

$$\mathcal{T}_{\mathsf{S}_0} : L^2_{k+1}(\mathbf{i}T^*Y \oplus W) \to L^2_k(\mathbf{i}\Lambda^2 T^*Y \oplus W \oplus \mathbf{i}\Lambda^0 T^*Y)$$

given by the map

$$\begin{bmatrix} a_0 \\ \psi_0 \end{bmatrix} \overset{\mathcal{T}_{\mathsf{S}_0}}{\to} \begin{bmatrix} d^+ a_0 - \tau(\psi, \Psi) - \tau(\Psi, \psi) \\ \not{D}_{A_0}\psi_0 + \frac{1}{2}\rho(a_0)\Psi_0 \\ -2d^* a_0 + \mathbf{i}\mathrm{Im}\langle \psi_0, \Psi_0 \rangle \end{bmatrix}.$$

It also has a complex at S_0, and an associated cohomology denoted by $\mathcal{H}^*_{\mathsf{S}_0}$ which can be described using $\mathcal{T}_{\mathsf{S}_0}$ and a similar statement as Lemma 5.1 above.

By definition S is irreducible if and only if $\mathcal{H}^0_{\mathsf{S}} = 0$ (and likewise for S_0). Hence solutions in $\mathcal{N}^*(X, \xi, g_X, \pi^*(\delta)^+)$ satisfy

$$0 = d(\xi) = \dim \mathcal{H}^1_{\mathsf{S}} - \dim \mathcal{H}^2_{\mathsf{S}} \tag{5.8}$$

by Proposition 3.4, equation (5.1), and the previous lemma. Therefore $\mathcal{H}^2_{\mathsf{S}}$ vanishes for these solutions precisely when $\dim \mathcal{H}^1_{\mathsf{S}} = 0$. We will use this fact and the following theorem to show when $\dim \mathcal{H}^2_{\mathsf{S}} = 0$.

Theorem 5.2 *Let $\mathsf{S} = (A, \Psi) \in \mathcal{N}^*(X, \xi, g_X, \pi^*(\delta)^+)$ be a irreducible solution to the Seiberg-Witten equations and let $\mathsf{S}_0 = (A_0, \Psi_0) \in \mathcal{M}^*(Y, \xi_0, g_Y, \delta)$ be the solution such that $\mathsf{S} = \pi^*(\mathsf{S}_0)$. Then*

$$\pi^*(\mathcal{H}^1_{\mathsf{S}_0}) = \mathcal{H}^1_{\mathsf{S}},$$

i.e., the kernels of $\mathcal{T}_{\mathsf{S}_0}$ and $\mathcal{T}_{\mathsf{S}}$ are naturally isomorphic via π^*.

Because the expected dimension of the moduli space on the 3-manifold is always zero, when there is a good pair (g_Y, δ) such that $\dim \mathcal{H}^0_{\mathsf{S}_0} = \dim \mathcal{H}^2_{\mathsf{S}_0} = \dim \mathcal{H}^1_{\mathsf{S}_0} = 0$ for all solutions in $\mathcal{M}(Y, g_Y, \delta)$ we get by Theorem 5.2 that the dimension of $\mathcal{H}^1_{\mathsf{S}}$ will be zero for the pulled back solutions as well. Hence $\mathcal{H}^2_{\mathsf{S}} = 0$ by equation (5.8) for all irreducible solutions S implying that $\mathcal{N}^*(X, g_X, \pi^*(\delta)^+)$ is a smooth manifold. Thus Theorem 5.2 finishes the proof of Theorem B. If, in addition, $\mathcal{N}(X, g_X, \pi^*(\delta)^+)$ does not contain any reducible solutions, then $(g_X, \pi^*(\delta)^+)$ will be a good pair for any Spin^c structure pulled back from Y.

The rest of this section contains the proof of Theorem 5.2. We use a Bochner vanishing argument similar to equation (5.3).

Certainly a solution to $\mathcal{T}_{\mathsf{S}_0}(a_0, \psi_0){=}0$ pulls back to a solution of $\mathcal{T}_{\mathsf{S}}(\pi^*(a_0), \pi^*(\psi_0)) = 0$. We need to show that π^* is surjective, i.e., for each solution (a, ψ) of the equations (5.7), we will prove that

$$\nabla_{\mathrm{T}}\psi = 0 \qquad \text{and} \qquad a \in \pi^*(\Omega^1(Y; i\mathbb{R})).$$

Use η to decompose a into $a = f\eta + c$ where $f \in \Omega^0(X; \mathbf{i}\mathbb{R})$ and $c \in \Omega^1(X; \mathbf{i}\mathbb{R})$. Since (a, ψ) satisfies $\not{D}^+_A\psi + \frac{1}{2}\sigma(a)\Psi = 0$, we have

$$\begin{aligned} 0 &= \int_X |\not{D}^+_A\psi + \frac{1}{2}\sigma(a)\Psi|^2 \\ &= \int_X |(\sigma(\eta)\left(\nabla_{\mathrm{T}}\psi + \frac{1}{2}f\Psi)\right) + (D^+\psi + \frac{1}{2}\sigma(c)\Psi)|^2 \\ &= \int_X |\nabla_{\mathrm{T}}\psi + \frac{1}{2}f\Psi|^2 + |D^+\psi + \frac{1}{2}\sigma(c)\Psi|^2 + \\ &\qquad 2\mathrm{Re}\langle\sigma(\eta)\nabla_{\mathrm{T}}\psi, D^+\psi\rangle + \mathrm{Re}\langle\sigma(\eta)\nabla_{\mathrm{T}}\psi, \sigma(c)\Psi\rangle + \\ &\qquad \mathrm{Re}\langle f\sigma(\eta)\Psi, D^+\psi\rangle + \mathrm{Re}\langle f\sigma(\eta)\Psi, \frac{1}{2}\sigma(c)\Psi\rangle. \end{aligned} \tag{5.9}$$

Two of the cross terms in equation (5.9) are zero as follows. First, since $\nabla_{\mathrm{T}}\eta = 0$ we have

$$
\begin{aligned}
2\int_X \mathrm{Re}\langle\sigma(\eta)\nabla_{\mathrm{T}}\psi, D^+\psi\rangle &= \int_X \langle\sigma(\eta)\nabla_{\mathrm{T}}\psi, D^+\psi\rangle + \langle D^+\psi, \sigma(\eta)\nabla_{\mathrm{T}}\psi\rangle \\
&= \int_X \langle\psi, \sigma(\eta)\nabla_{\mathrm{T}}(D^+\psi)\rangle + \langle\psi, D^-\left(\sigma(\eta)\nabla_{\mathrm{T}}\psi\right)\rangle \\
&= \int_X \langle\psi, \{\sigma(\eta)\nabla_{\mathrm{T}}, D\}\psi\rangle
\end{aligned}
$$

But by equations (4.10), (4.11), and (5.6),

$$
\{\sigma(\eta)\nabla_{\mathrm{T}}, D\} \;=\; \frac{1}{2}\eta\wedge\iota_{\mathrm{T}}F_A \;=\; 0.
$$

Similarly, we can use the fact that $f\eta$ and c are both self-adjoint to show

$$
\begin{aligned}
2\int_X \mathrm{Re}\langle\sigma(f\eta)\Psi, \sigma(c)\Psi\rangle &= \int_X \langle(\sigma(c)\sigma(f\eta) + \sigma(f\eta)\sigma(c)\Psi, \Psi\rangle \\
&= -2\int_X \langle c, f\eta\rangle|\Psi|^2 \;=\; 0.
\end{aligned}
$$

The remaining two cross terms in equation (5.9) are analyzed in the following lemma.

Lemma 5.3 *In the situation above,*

$$
\int_X Re\langle\sigma(\eta)\nabla_{\mathrm{T}}\psi, \sigma(c)\Psi\rangle + Re\langle f\sigma(\eta)\Psi, D^+\psi\rangle = \int_X |\iota_{\mathrm{T}}da|^2. \tag{5.10}
$$

Proof: Let $\{\eta = e^0, e^1, e^2, e^3\}$ be a local coframe where e^1, e^2, e^3 are pulled back from the base. First we take the adjoints of both terms on the left hand side of equation (5.10).

Applying the adjoint of $\sigma(\eta)\nabla_{\mathrm{T}}$ in the first term of equation (5.10) gives

$$
\begin{aligned}
\sigma(\eta)\nabla_{\mathrm{T}}(\sigma(c)\Psi) &= \sum_{i=1}^{3}\sigma(\eta)\sigma({}^\circ\nabla_{\mathrm{T}}(c_i e^i))\Psi + \sigma(\eta)\sigma(c)(\nabla_{\mathrm{T}}\Psi) \\
&= \sum_{i=1}^{3}\sigma(\eta)\sigma({}^\circ\nabla_{\mathrm{T}}(c_i e^i))\Psi \\
&= \sum_{i=1}^{3}\sigma\left((\mathrm{T}c_i)\eta\wedge e^i\right)\Psi + \sigma(\eta)\sigma(c_i{}^\circ\nabla_{\mathrm{T}}(e^i))\Psi \\
&= \sum_{i=1}^{3}\sigma\left((\mathrm{T}c_i)\eta\wedge e^i\right)\Psi \qquad (5.11)
\end{aligned}
$$

We used equation (2.1) in the first line, and (5.4) in the second. We also used the definition of ${}^\circ\nabla$ from equation (3.1).

Similarly, we take the adjoint of D in the second term of equation (5.10) to find

$$D(f\sigma(\eta)\Psi) = \sigma(df\wedge\eta)\Psi + \sigma(f\eta)D\Psi = \sigma(df\wedge\eta)\Psi. \qquad (5.12)$$

Next we show that the sum of the right hand sides of equations (5.11) and (5.12) is equal to

$$\sigma(\eta\wedge\iota_{\mathrm{T}}(da))\Psi.$$

First, note that for $i = 0, 1, 2, 3$,

$$\iota_T d\eta = 0 \qquad \text{and} \qquad \iota_{\mathrm{T}} de^i = 0. \qquad (5.13)$$

This holds for $e^0 = \eta$ since $d\eta$ is the curvature of a principal orbifold circle bundle so is pulled back from Y; it holds for the remaining i since e^1, e^2, e^3 are pulled back from Y. Hence,

$$\begin{aligned}
\eta \wedge \iota_{\mathrm{T}}(da) &= \eta \wedge \iota_{\mathrm{T}}(d(f\eta + c)) \\
&= \eta \wedge \iota_{\mathrm{T}} \left((df \wedge \eta + f d\eta) + \sum_{i=1}^{3} (dc_i \wedge e^i + c_i \wedge de^i) \right) \\
&= df \wedge \eta + \sum_{i=1}^{3} (\mathrm{T}c_i) \eta \wedge e^i
\end{aligned} \tag{5.14}$$

Combining equations (5.11), (5.12), and (5.14) and projecting onto the self-dual 2-forms we get:

$$\int_X \mathrm{Re}\langle \sigma(\eta)\nabla_{\mathrm{T}}\psi, \sigma(c)\Psi\rangle + \mathrm{Re}\langle f\sigma(\eta)\Psi, D^+\psi\rangle = \int_X \mathrm{Re}\langle \psi, (\eta \wedge \iota_{\mathrm{T}} da)^+\Psi\rangle$$

Using equation (5.7), we can reduce further

$$\begin{aligned}
\int_X \mathrm{Re}\langle \psi, (\eta \wedge \iota_{\mathrm{T}} da)^+\Psi\rangle &= \int_X \frac{1}{2}\langle \psi, (\eta \wedge \iota_{\mathrm{T}} da)^+\Psi\rangle + \frac{1}{2}\langle (\eta \wedge \iota_{\mathrm{T}} da)^+\Psi, \psi\rangle \\
&= \int_X 2\langle (\eta \wedge \iota_{\mathrm{T}} da)^+, q(\Psi, \psi) + q(\psi, \Psi)\rangle \\
&= \int_X 2\langle (\eta \wedge \iota_{\mathrm{T}} da), d^+a\rangle \\
&= \int_X |\iota_{\mathrm{T}} da|^2 + \frac{1}{2}\int_X \mathbf{i}da \wedge \mathbf{i}da.
\end{aligned}$$

The last equality is the same calculation as in equation (4.13). □

Combining equations (5.9-5.10), gives the sum of non-negative terms. Hence we conclude that the following terms are identically zero:

$$\nabla_{\mathrm{T}}\psi + \frac{1}{2}f\Psi = 0, \tag{5.15}$$

$$D^+\psi + \frac{1}{2}\sigma(c)\Psi = 0, \tag{5.16}$$

$$\iota_{\mathrm{T}} da = 0. \tag{5.17}$$

Notice that the equation (5.17) is equivalent to

$$^{\circ}\nabla_T a = df. \tag{5.18}$$

We investigate equation (5.15) more carefully in the next lemma.

Lemma 5.4

$$\int_X |\nabla_{\mathrm{T}}\psi + \frac{1}{2} f\Psi|^2 \;=\; \int_X |\nabla_{\mathrm{T}}\psi|^2 + \frac{1}{4} f^2|\Psi|^2 + 2|df|^2.$$

Since $f = \iota_{\mathrm{T}} a$ and $\Psi \neq 0$ almost everywhere, we conclude that

$$\nabla_{\mathrm{T}}\psi \;=\; 0, \tag{5.19}$$

$$\iota_{\mathrm{T}} a \;=\; 0. \tag{5.20}$$

Equation (5.19) implies that the spinor is circle invariant while equations (5.17) and (5.20) imply that a is pulled back from Y. These two facts together imply that (a, ψ) is pulled back from some (a_0, ψ_0) on Y. Equation (5.16) shows that (a_0, ψ_0) satisfies the last equation of $\mathcal{T}_{\mathsf{S}_0}$. It is easy to verify that (a_0, ψ_0) satisfies the other two equations of $\mathcal{T}_{\mathsf{S}_0}$. Hence (a, ψ) is in $\pi^*(\ker \mathcal{T}_{\mathsf{S}_0})$ and this completes the proof of Theorem B.

Proof of Lemma 5.4: We must show that the cross term satisfies

$$\int_X \mathrm{Re}\langle \nabla_{\mathrm{T}}\psi, f\Psi\rangle = \int_X 2|df|^2.$$

Integrating by parts and noting that $\nabla_{\mathrm{T}}\Psi = 0$,

$$\int_X \mathrm{Re}\langle \nabla_{\mathrm{T}}\psi, f\Psi\rangle \quad = \quad \int_X \mathrm{Re}\langle \psi, (-{}^{\circ}\nabla_{\mathrm{T}} f)\Psi\rangle.$$

Pulling out the imaginary valued function ${}^{\circ}\nabla_{\mathrm{T}} f$, using equation (5.7), and integrating by parts again,

$$\begin{aligned} \int_X \mathrm{Re}\langle \psi, (-{}^{\circ}\nabla_{\mathrm{T}} f)\Psi\rangle &= -2\int_X \langle {}^{\circ}\nabla_{\mathrm{T}} f, d^* a\rangle \\ &= 2\int_X \langle f, {}^{\circ}\nabla_{\mathrm{T}} d^* a\rangle. \end{aligned} \tag{5.21}$$

The results follows once we show ${}^{\circ}\nabla_{\mathrm{T}} d^* a = \Delta f$. We first calculate d^*a at a point $p \in X$ over $p_0 = \pi(p) \in Y$. Choose a coframe $\{e^0 = \eta, e^1, e^2, e^3\}$ at p such that the $\{e^1, e^2, e^3\}$ are pulled back from a coframe on Y chosen such that the pull back of the connection 1-forms satisfy $\omega^i_j(p) = 0$ in the matrix (4.5). Then

$$d^* a \quad = \quad -\sum_{i=0}^{3} \iota_{e_i} {}^{\circ}\nabla^{L.C.} a \quad = \quad -\sum_{i=0}^{3} \langle {}^{\circ}\nabla^{L.C.}_{e_i} a, e^i\rangle.$$

Differentiating this with respect to ${}^{\circ}\nabla_{\mathrm{T}}$,

$$\begin{aligned} {}^{\circ}\nabla_{\mathrm{T}} d^* a &= -\sum_{i=0}^{3} {}^{\circ}\nabla_{\mathrm{T}} \langle {}^{\circ}\nabla^{L.C.}_{e_i} a, e^i\rangle \\ &= -\sum_{i=0}^{3} \langle {}^{\circ}\nabla_{\mathrm{T}} {}^{\circ}\nabla^{L.C.}_{e_i} a, e^i\rangle. \end{aligned} \tag{5.22}$$

Next we will show using the connection matrices (4.5) and (4.6), and equation (4.8) that

$$\sum_{i=0}^{3} \langle [{}^{\circ}\nabla_{\mathrm{T}}, {}^{\circ}\nabla^{L.C.}_{e_i}] a, e^i\rangle \quad = \quad 0. \tag{5.23}$$

By setting $a = \sum a_k e^k$ and using the fact that ${}^\circ\nabla_{\mathrm{T}} e^i = 0$,

$$\begin{aligned}
\sum_{i=0}^{3} \langle [{}^\circ\nabla_{\mathrm{T}}, {}^\circ\nabla^{L.C.}_{e_i}] a, e^i \rangle &= \sum_{i=0}^{3} \langle {}^\circ\nabla_{\mathrm{T}} \left((e_i \cdot a_k) e^k + a_k {}^\circ\nabla^{L.C.}_{e_i} e^k \right) - {}^\circ\nabla^{L.C.}_{e_i} \left((\mathrm{T} \cdot a_k) e^k \right), e^i \rangle \\
&= \sum_{i=0}^{3} \mathrm{T} \cdot e_i \cdot a_i + (\mathrm{T} \cdot a_k) \langle {}^\circ\nabla^{L.C.}_{e_i} e^k, e^i \rangle + a_k \langle {}^\circ\nabla_{\mathrm{T}} {}^\circ\nabla^{L.C.}_{e_i} e^k, e^i \rangle \\
&\qquad - e_i \cdot \mathrm{T} \cdot a_i - (\mathrm{T} \cdot a_k) \langle {}^\circ\nabla^{L.C.}_{e_i} e^k, e^i \rangle.
\end{aligned}$$

The first and fourth term cancel because $[\mathrm{T}, e_i] = 0$ by equation (4.8). The second and last term also cancel. The third term is equal to

$$a_k \mathrm{T} \cdot \langle {}^\circ\nabla^{L.C.}_{e_i} e^k, e^i \rangle \tag{5.24}$$

because ${}^\circ\nabla_T$ is compatible with the metric and ${}^\circ\nabla_{\mathrm{T}} e^i = 0$. But

$$\sum_{i=0}^{3} \langle {}^\circ\nabla^{L.C.}_{e_i} e^k, e^i \rangle = -\sum_{i=0}^{3} \langle e^k, {}^\circ\nabla^{L.C.}_{e_i} e^i \rangle.$$

Using the fact that $\omega^i_j(p) = 0$ for $i, j = 1, 2, 3$, we can see that ${}^\circ\nabla^{L.C.}_{e_i} e^i = 0$ by inspecting the connection matrix (4.5). Since this term vanishes, equation (5.24) vanishes giving equation (5.23).

Therefore we can commute ${}^\circ\nabla_{\mathrm{T}}$ with ${}^\circ\nabla^{L.C.}_{e_i}$ in equation (5.22), and apply equation (5.18) to get:

$${}^\circ\nabla_{\mathrm{T}} d^* a = -\sum_{i=0}^{3} \langle {}^\circ\nabla^{L.C.}_{e_i} df, e^i \rangle = \Delta f.$$

This statement is independent of frame, so we can substitute it into equation (5.21). The lemma now follows by integration by parts.

CHAPTER 6

Results

We are now ready to prove the formula for calculating the Seiberg-Witten invariants of a 4-manifold with a fixed point free circle action and state some immediate corollaries.

Theorem C. *Let X be a closed smooth 4-manifold with $b_+ > 1$ and a fixed point free circle action. Let Y^3 be the orbifold quotient space and suppose that $\chi \in Pic^t(Y)$ is the orbifold Euler class of the circle action. If ξ is a* Spin^c *structure over X with $SW^4_X(\xi) \neq 0$, then $\xi = \pi^*(\xi_0)$ for some* Spin^c *structure on Y and*

$$SW^4_X(\xi) = \sum_{\xi' \equiv \xi_0 \mod \chi} SW^3_Y(\xi'),$$

where $\xi' - \xi_0$ is a well-defined element of $Pic^t(Y)$. When $b_+ = 1$, the formula holds for all Spin^c *structures which are pulled back from Y.*

Remark 6.1 *In the $b_+(X) = 1$ case, the numerical invariant may still depend on the chamber structure of Y if $b_1(Y) = 1$.*

Proof: Recall that for a generic choice of metric and perturbation (g_Y, δ) the moduli space satisfies $\mathcal{H}^0_{\mathsf{S}_0} = \mathcal{H}^1_{\mathsf{S}_0} = \mathcal{H}^2_{\mathsf{S}_0} = 0$ for all solutions $\mathsf{S}_0 = (A_0, \Psi_0)$ to the 3-dimensional Seiberg-Witten equations (see section 5.3 for more details). For this good pair the moduli space $\mathcal{M}(Y, g_Y, \delta)$ is a smooth manifold with no reducible solutions. Since we can choose a perturbation generically such that the projection of $F_{A_0} + \delta$ onto the harmonic 2-forms is not a multiple of the harmonic representative of χ for all solutions in $\mathcal{M}(Y, g_Y, \delta)$, we have that

$$(\pi^*(F_{A_0}) + \pi^*(\delta))^+ \neq 0$$

on X as well, hence $\mathcal{N}(X, g_X, \pi^*(\delta)^+)$ does not contain reducible solutions either. By Theorem B, $\mathcal{N}(X, g_X, \pi^*(\delta)^+)$ is diffeomorphic to a smooth manifold without reducible solutions. We have in effect shown that $(g_X, \pi^*(\delta)^+)$ is a good pair and that this moduli space can be used to calculate the SW invariant.

Choose a specific Spin^c structure ξ on X such that $c_1(\xi)$ is pulled back and $SW_X(\xi) \neq 0$. There exists a Spin^c structure ξ_0 on Y such that $\xi = \pi^*(\xi_0)$ by Theorem B, and

$$\mathcal{N}(X, \xi, g_X, \pi^*(\delta)^+) = \coprod_{\xi' \cong \xi_0 \mod \chi} \mathcal{M}(Y, \xi', g_Y, \delta).$$

From this the formula follows. □

When the action is free, the theorem above reduces to the formula:

Corollary 6.2 *Let X be a closed smooth 4-manifold with $b_+ > 1$ and a free circle action. Then the orbit space Y^3 is a smooth 3-manifold and suppose that $\chi \in H^2(Y; \mathbb{Z})$ is the first Chern class of the circle action on X. If ξ is a* Spin^c *structure over X with $SW_X^4(\xi) \neq 0$, then $\xi = \pi^*(\xi_0)$ for some* Spin^c *structure on Y and*

$$SW_X^4(\xi) = \sum_{\xi' \equiv \xi_0 \mod \chi} SW_Y^3(\xi'),$$

where $\xi' - \xi_0$ is a well-defined element of $H^2(Y;\mathbb{Z})$.

Because of this formula, it is particularly easy to calculate the Seiberg-Witten invariants for manifolds with free circle actions. We will use this version of Theorem C in Chapter 7. In the next chapter we give an alternative proof of this corollary using a gluing theorem and assuming the next corollary, which is also a consequence of Theorem C:

Corollary 6.3 (c.f. Donaldson [D]) *Let $X \cong Y^3 \times S^1$ with $b_+(X) > 1$. If a* Spin^c *structure ξ has $SW_X(\xi) \neq 0$, then there is one* Spin^c *structure ξ_0 on Y such that $\xi = \pi^*(\xi_0)$ and*

$$SW_X^4(\xi) = SW_Y^3(\xi_0).$$

The usual route used to explain the corollary above is to consider the cyclic covering of X by $Y^3 \times \mathbb{R}$. There is a natural way to pullback solutions of (3.2) to solutions on $Y^3 \times \mathbb{R}$ for Spin^c structures pulled up from Y^3. After putting the solution in temporal gauge it satisfies the 3-dimensional Seiberg-Witten equations because it is a constant gradient-flow of the Chern-Simons-Dirac functional [CM]. Thus for each ξ on X such that $SW_X(\xi) \neq 0$ there is a Spin^c structure on Y whose moduli space is nonempty for all generic metrics and perturbations. This corollary shows that this moduli space can actually be identified with the moduli space of X and can be used to calculate the Seiberg-Witten invariant.

CHAPTER 7

An Alternate Proof

In this chapter we prove Corollary 6.2 assuming Corollary 6.3. It shows that the Seiberg-Witten invariants can be computed using the gluing formula in [MMS]. This proof cannot be generalized to the fixed point free case because the argument breaks down: the sum on the left hand side of equation (7.2) cannot be reduced to a single term as in the free case.

7.1 Classifying free circle actions

Let X be an oriented connected 4-manifold carrying a smooth free S^1-action. Its orbit space Y is a 3-manifold whose orientation is determined so that, followed by the natural orientation on the orbits, the orientation of X is obtained. Choose a smooth connected loop l representing the Poincaré dual $PD(\chi) \in H_1(Y;\mathbb{Z})$. Remove a tubular neighborhood $N \cong D^2 \times l$ of l from Y, and set $X' = (Y \setminus N) \times S^1$. View X' as an S^1-manifold whose action is given by rotation in the last factor. Let m' be the meridian of l in X', and let f' be an orbit in X'. We then have:

Lemma 7.1 *The manifold X is diffeomorphic (by a bundle isomorphism) to the man-*

ifold

$$X(l) = X' \cup_\varphi D^2 \times T^2 \tag{7.1}$$

where $\varphi : T^3 \to \partial X'$ *is an equivariant diffeomorphism which evaluates* $\varphi_*([\partial(D^2 \times pt)] = [m' + f']$ *in homology.*

When gluing $D^2 \times T^2$ into the boundary of a manifold, the resulting closed manifold is determined up to diffeomorphism by the image in homology of $[\partial(D^2 \times pt)]$. (For example, see [MMS].)

The proof follows immediately from construction (3) in Theorem 2.4 where the section on the boundary $s : \partial(Y \backslash N) \to \partial X'$ is given by $s_*[\partial D^2] = m' + f'$. Henceforth, we shall work with $X(l)$ and refer to it as X.

7.2 Gluing along T^3

Since we have $X = X' \cup_\varphi (D^2 \times T^2)$ we may apply the gluing theorem of Morgan, Mrowka, and Szabó [MMS].

Theorem 7.2 (Morgan, Mrowka, and Szabó) *If the* Spin^c *structure* ξ *over* X *restricts nontrivially to* $D^2 \times T^2$, *then* $SW_X(\xi) = 0$. *For each* Spin^c *structure* $\xi_0 \to X'$ *that restricts trivially to* $\partial X'$, *let* $V_X(\xi_0)$ *denote the set of isomorphism classes of* Spin^c *structures over* X *whose restriction to* X' *is equal to* ξ_0. *Then we have*

$$\sum_{\xi \in V_X(\xi_0)} SW_X(\xi) = \sum_{\xi \in V_{Y \times S^1}(\xi_0)} SW_{Y \times S^1}(\xi) + \sum_{\xi \in V_{X_{0/1}}(\xi_0)} SW_{X_{0/1}}(\xi), \tag{7.2}$$

where the manifold $X_{0/1} = X' \cup_{\varphi_{0,1}} D^2 \times T^2$ *is defined by the map* $\varphi_{0,1}$ *which maps* $[\partial(D^2 \times 1 \times 1)] \mapsto [f']$ *in homology.*

In our situation, this formula simplifies significantly. Let i denote the inclusion of $\partial X'$ into X_0. A study of the long exact sequences in homology shows that the

left hand side consists of a single term when $i_*[m' + f']$ is indivisible. Since $i_*[f']$ is independent of $i_*[m']$ and $i_*[f']$ is a primitive class in $H_1(X_0;\mathbb{Z})$, $i_*[m'+f']$ is such a class. Therefore, the formula enables the calculation of the SW invariants of X in terms of the SW invariants of $Y \times S^1$ and a manifold $X_{0/1}$.

The manifold $X_{0/1}$ admits a semi-free S^1-action whose fixed point set is a torus. Its orbit space is $Y \setminus N$, and $\partial(Y \setminus N) = \partial N$ is the image of the fixed point set. The condition $b_+(X) \geq 2$ of the Corollary 6.2 implies that $b_+(X_{0/1}) > 1$ and that

$$\text{rank } H_1(Y \setminus N, \partial(Y \setminus N);\mathbb{Z}) > 1.$$

The two statements are proved as follows. The Gysin sequence

$$H^2(Y;\mathbb{Z}) \xrightarrow{\pi^*} H^2(X;\mathbb{Z}) \longrightarrow H^1(Y;\mathbb{Z}) \xrightarrow{\cdot\cup\chi} H^3(Y;\mathbb{Z}) \tag{7.3}$$

implies

$$H^2(X;\mathbb{Z}) \quad \cong \quad \left(H^2(Y;\mathbb{Z})/ <\chi>\right) \;\oplus\; \ker\left(\cup\chi : H^1(Y;\mathbb{Z}) \to H^3(Y;\mathbb{Z})\right). \tag{7.4}$$

Each component of the direct sum above has rank $b_1(Y) - 1$. The bilinear form of X is the direct sum of hyperbolic pairs which implies that $b_+(X) = b_1(Y) - 1$. Since $[l]$ is not a torsion element, removing N from Y implies the rank of $H_1(Y \setminus N, \partial(Y \setminus N);\mathbb{Z})$ is also $b_1(Y) - 1$. The second statement now follows because $b_1(Y) - 1 = b_+(X) > 1$. The first statement requires the following Mayer-Vietoris sequence

$$H_3(T^3;\mathbb{Z}) \to H_2(X';\mathbb{Z}) \oplus H_2(D^2 \times T^2;\mathbb{Z}) \to H_2(X_{0/1};\mathbb{Z}) \xrightarrow{0} H_1(T^3;\mathbb{Z}).$$

The rank of $H_2(X';\mathbb{Z})$ is $2b_1(Y) - 1$ and the rank of the image of the first map is 2. Therefore $b_2(X_{0/1}) = 2b_1(Y) - 2$. Since the bilinear form of $X_{0/1}$ is also a direct sum

of hyperbolic pairs, $b_+(X_{0/1}) > 1$.

We can now apply the following general theorem about manifolds with semi-free circle actions (whose local isotropy group at a point is either trivial or S^1) to the manifold $X_{0/1}$.

Proposition 7.3 *Let X be a smooth closed oriented 4-manifold with a smooth semi-free circle action and $b_+(X) > 1$. Let $X^* = X/S^1$ be its orbit space. Suppose that X^* has a nonempty boundary and* rank $H_1(X^*, \partial X^*; \mathbb{Z}) > 1$. *Then $SW_X \equiv 0$.*

Proof: Let F denote the fixed point set of X and F^* its image in X^*. Then $\partial X^* \subset F^*$. The restriction of the circle action to $X \setminus F$ defines a principal S^1-bundle whose Euler class lies in $H^2(X^* \setminus F^*; \mathbb{Z})$. Let $\chi' \in H_1(X^*, F^*; \mathbb{Z})$ denote its Poincaré dual. Consider the exact sequence

$$0 \to H_1(X^*, \partial X^*; \mathbb{Z}) \xrightarrow{i_*} H_1(X^*, F^*; \mathbb{Z}) \to H_0(F^*, \partial X^*; \mathbb{Z}) \to H_0(X^*, \partial X^*; \mathbb{Z}).$$

Since the rank of $H_1(X^*, \partial X^*; \mathbb{Z})$ is greater than 1, there is a class in $i_*(H_1(X^*, \partial X^*; \mathbb{Z}))$ which is primitive and not a multiple of χ'. This class may be represented by a path α in X^* which starts and ends on ∂X but is otherwise disjoint from F^*.

The preimage $S = \pi^{-1}(\alpha)$ is a 2-sphere of self-intersection 0 in X. It has self-intersection 0 because the path α can be perturbed slightly to another path α' which is disjoint from α; hence $S' = \pi^{-1}(\alpha')$ is homologous but disjoint from S. The Gysin sequence gives:

$$H_3(X^*, F^*, \mathbb{Z}) \to H_1(X^*, F^*, \mathbb{Z}) \xrightarrow{\rho} H_2(X, F, \mathbb{Z}) \to H_2(X^*, F^*, \mathbb{Z})$$

where $\rho_*(i_*[\alpha]) = [S]$. The image of $H_3(X^*, F^*, \mathbb{Z}) \cong \mathbb{Z}$ in $H_1(X^*, F^*, \mathbb{Z})$ is generated by χ'. Since $i_*[\alpha]$ is primitive and not a multiple of χ', the class $[S] \in \mathrm{Im}\rho \subset$

$H_2(X, F, \mathbb{Z})$ is not torsion; hence $[S]$ is nontorsion as an element of $H_2(X;\mathbb{Z})$.

It now follows from [FS1] that $\mathrm{SW}_X \equiv 0$. $\square$

Proposition 7.3 implies that the formula (7.2) simplifies to

$$SW_X(\xi) = \sum_{\xi' \in V_{Y\times S^1}(\xi|_{X'})} SW_{Y\times S^1}(\xi'). \tag{7.5}$$

7.3 Spin^c structures which are not pullbacks

There are Spin^c structures on X which do not arise from Spin^c structures that are pulled up from Y. In this section we show that the Seiberg-Witten invariants vanish for these Spin^c structures.

Fix a Spin^c structure $\xi_0 = (W_0, \rho)$ on Y and consider its pullback $\xi = \pi^*(\xi_0)$ over X (see section 3.3). The other pulled back Spin^c structures are now obtained by the addition of classes $\pi^*(e)$ for $e \in H^2(Y;\mathbb{Z})$.

Looking at the Gysin sequence (7.3), if a class $e \in H^2(X;\mathbb{Z})$ is not in the image of π^*, then $\xi + e$ is not a Spin^c structure which is pulled back from Y.

Lemma 7.4 *If ξ is a* Spin^c *structure on X which is not pulled back from Y, then $SW_X(\xi) = 0$.*

Proof: We claim that there exists an embedded torus with self-intersection 0 which pairs nontrivially with $c_1(\xi)$. Then by the adjunction inequality [KM] the Spin^c structure ξ has Seiberg-Witten invariant equal to zero. Let

$$\mathbf{H} = \ker(\cdot \cup \chi : H^1(Y;\mathbb{Z}) \to H^3(Y;\mathbb{Z}))$$

in equation (7.4), and consider for a moment the projection of $c_1(\xi)$ onto the first factor of $\mathbf{H} \oplus \pi^*(H^2(Y;\mathbb{Z}))$ by changing the Spin^c structure by an element of

$\pi^*(H^2(Y;\mathbb{Z}))$. Since ξ is not pulled back from Y, $c_1(\xi)|_{\mathbf{H}} \neq 0$, and since $H^1(Y;\mathbb{Z})$ is a free abelian group, $c_1(\xi)|_{\mathbf{H}}$ is not a torsion class.

Examining the Gysin sequence, $c_1(\xi)|_{\mathbf{H}} \in H^2(X;\mathbb{Z})$ maps to a class $\beta \in H^1(Y;\mathbb{Z})$, $\beta \cup \chi = 0$. Thus the Poincaré dual of β can be represented by a surface b, and there is a 1-cycle λ in $Y \setminus N$ rel ∂ such that $[\lambda] \cdot [b] \neq 0$. Since ∂N is connected, $[\lambda]$ is actually represented by a loop λ in $Y \setminus N$. The preimage $\pi^{-1}(\lambda) = \lambda \times S^1$ in X is a torus with self-intersection 0, and $c_1(\xi)|_{\mathbf{H}} \cdot [\pi^{-1}(\lambda)] = [b] \cdot [\lambda] \neq 0$.

On the other hand, if $A \in \pi^* H^2(Y;\mathbb{Z})$ then its Poincaré dual is represented by a loop α in Y which may be chosen disjoint from λ. Thus $A \cdot [\pi^{-1}(\lambda)] = 0$. This means that $c_1(\xi) \cdot [\pi^{-1}(\lambda)] \neq 0$, as required. $\square$

7.4 Identifying the set $V_{Y \times S^1}(\xi|_{X'})$

We identify the Spin^c structures in the set $V_{Y\times S^1}(\xi|_{X'})$.

According to the previous lemma, the only nontrivial Seiberg-Witten Spin^c structures are those pulled up from Y. Thus far we have seen that for such a Spin^c structure $\xi = \pi^*(\xi^*)$ with $\xi_0 = \xi|_{X'}$, we have

$$SW_X(\xi) = \sum_{\xi' \in V_{Y\times S^1}(\xi_0)} SW_{Y\times S^1}(\xi').$$

Let $\tilde{\pi} : Y \times S^1 \to Y$ be the projection. We identify the set $V_{Y\times S^1}(\xi_0)$ of isomorphism classes of Spin^c structures over $Y \times S^1$ which restrict on X' to ξ_0.

Lemma 7.5 $V_{Y\times S^1}(\xi_0) = \{\, \tilde{\pi}^*\,(\xi^* + n \cdot \chi) \;\mid n \in \mathbb{Z}\,\}$.

Proof: The diagram

$$\begin{tikzcd}
X \arrow[ddr, "\pi"'] & X' \arrow[l, "inc"'] \arrow[r, "inc"] \arrow[d, "\tilde{\pi}|_{Y\setminus N}"] & Y \times S^1 \arrow[ddl, "\tilde{\pi}"] \\
& Y \setminus N \arrow[d, "inc"] & \\
& Y &
\end{tikzcd}$$

induces Spin^c structures on X, X', and $Y \times S^1$ which satisfy

$$inc^*(\pi^*(\xi^*)) = \xi_0 = inc^*(\tilde{\pi}^*(\xi^*).$$

Recall that ξ is the only Spin^c structure induced on X by ξ_0 since $i_*[m' + f']$ is indivisible. Since $\tilde{\pi}^*(\xi^*) \in V_{Y\times S^1}(\xi_0)$, the set of Spin^c structures on $Y \times S^1$ is $\{\tilde{\pi}^*(\xi^*) + e | e \in H^2(Y \times S^1; \mathbb{Z})\}$. Now $\tilde{\pi}^*(\xi^*) + e$ lies in $V_{Y\times S^1}(\xi_0)$ if and only if $inc^*(\pi^*(\xi^*) + e) = \xi_0$, i.e. if and only if $inc^*(e) = 0$. Therefore,

$$V_{Y\times S^1}(\xi_0) = \{\tilde{\pi}^*(\xi^*) + e \mid inc^*(e) = 0\}\,. \tag{7.6}$$

The kernel of inc^* is equal to the image of j^* in the diagram below.

$$\begin{tikzcd}
H^2(Y \times S^1, (Y \setminus N) \times S^1; \mathbb{Z}) \arrow[r, "j^*"] \arrow[d, "PD"] & H^2(Y \times S^1; \mathbb{Z}) \arrow[r, "inc^*"] \arrow[d, "PD"] & H^2(X'; \mathbb{Z}) \arrow[d, "PD"] \\
H_2(D^2 \times T^2; \mathbb{Z}) \arrow[r] & H_2(Y \times S^1; \mathbb{Z}) \arrow[r] & H_2(X', \partial X'; \mathbb{Z}) \\
n[T^2] \arrow[r, "j_*"'] & n[l \times f'] \arrow[r] & 0
\end{tikzcd}$$

However $j_*[\mathrm{pt} \times T^2] = [l \times f']$, and since $\tilde{\pi}^*(\chi) = PD^{-1}[l \times f']$, the lemma follows. □

Applying Corollary 6.3 completes the proof.

CHAPTER 8

Examples

In this chapter we discuss many interesting examples related to a question/conjecture of C. Taubes:

> *If Y is a 3-manifold such that $Y \times S^1$ admits a symplectic structure, must Y fiber over the circle?*

A well-known technique of Thurston shows that if Y is a 3-manifold which fibers over the circle (with homologically essential fiber) then $Y \times S^1$ admits a symplectic structure. One might ask whether one could use similar techniques to construct a symplectic structure on X, a 4-manifold with a free S^1-action, whose orbit space is a 3-manifold Y which fibers over the circle with essential fibers. Partial positive results to this question have been posted by Fernández, Gray, and Morgan [FGM]. However, using the formula of the above theorem, some knot theory, and work of Taubes [T], we construct in Example 1 a 4-manifold with a free circle action whose quotient fibers over the circle (with essential fiber) but which admits no symplectic structure.

Another example related to Taubes' question is a 3-manifold which cannot be the orbit space of any symplectic 4-manifold with a free circle action. In Example 2, we demonstrate a 3-manifold which has the property that each 4-manifold which is a principal S^1-fiber bundle (i.e. admits a free S^1-action) over it has a SW polynomial

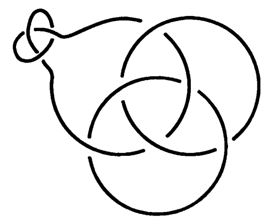

Figure 8.1. Y_K before surgery.

whose coefficients are all greater than 1 in absolute value. Since Taubes has shown that the canonical class of a symplectic 4-manifold has Seiberg-Witten invariant equal to 1, these 4-manifolds cannot admit a symplectic structure.

First we describe the main construction for all of our examples.

8.1 A construction and a calculation

The following construction is similar to but simpler than the main construction in [FS2]. Let Z_K denote the manifold resulting from 0-surgery on a knot K in S^3. Let m' be a meridian of the knot in Z_K. Let m_1, m_2, m_3 be loops that correspond to the S^1 factors of T^3. Construct a new manifold

$$Y_K = T^3 \#_{m_1=m} Z_K = [T^3 \setminus (m_1 \times D^2)] \cup [Z_K \setminus (m \times D^2)]$$

by removing tubular neighborhoods of m and m_1 and fiber summing the two manifolds along the boundary such that $m = m_1$ and ∂D^2 is sent to ∂D^2.

This is a familiar construction. If one forms a link L from the Borromean link by taking the composite of the first component with the knot K (see Figure 1), then Y_K is the result of surgery on L with each surgery coefficient equal to 0. If K is a fibered knot, then the resulting manifold $T^3 \#_{m_1=m} Y_K$ is a fibered 3-manifold.

Consider the formal variables $t_\beta = \exp(PD(\beta))$ for each $\beta \in H_1(Y;\mathbb{Z})$ which satisfies the relation $t_{\alpha+\beta} = t_\alpha t_\beta$. The Seiberg-Witten polynomial $\mathcal{SW}$ of X is a Laurent polynomial with variables t_β and coefficients equal to the Seiberg-Witten invariant of the Spin^c structure defined by t_β.

Theorem 8.1 (Meng and Taubes [MT]) *For a closed oriented 3-manifold Y with $b_1 > 0$, the Seiberg-Witten polynomial is given by the Milnor torsion of Y. In the situation above we can simplify this to:*

$$\mathcal{SW}^3_{Y_K} = \Delta_K(t^2_{m_1}) \tag{8.1}$$

where Δ_K is the symmetrized Alexander polynomial of K.

For example, the manifold Y_K in Figure 8.1 where K is the trefoil knot has Seiberg-Witten polynomial

$$\mathcal{SW}^3_{Y_K}(t_{m_1}) = -t^{-2}_{m_1} + 1 - t^2_{m_1}.$$

8.2 Example 1: Non symplectic X^4 whose quotient fibers over S^1

We first produce an example of a nonsymplectic 4-manifold which admits a free circle action whose orbit space is a 3-manifold which is fibered over the circle. Our construction generalizes easily to produce a large class of such manifolds with this property. Let K_1 and K_2 be any fibered knots. Form the fiber sum of the complements of K_1 and K_2 with neighborhoods of the first and second meridians of T^3, i.e.,

$$Y_{K_1K_2} = (S^3 \setminus K_1)\#_{m=m_1}T^3\#_{m_2=m}(S^3 \setminus K_2)$$

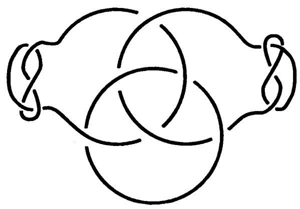

Figure 8.2. $Y_{K_1K_2}$ before surgery

where m is the meridian of the corresponding knot. Since both K_1 and K_2 are fibered, the manifold $Y_{K_1K_2}$ is a fibered 3-manifold. By the Meng-Taubes theorem, the Seiberg-Witten polynomial of this manifold is

$$\mathcal{SW}^3_{Y_{K_1K_2}}(t_{m_1}, t_{m_2}) = \Delta_{K_1}(t^2_{m_1})\Delta_{K_2}(t^2_{m_2}).$$

Let $X_{K_1K_2}(l)$ be the 4-manifold with free circle action that has $Y_{K_1K_2}$ for its orbit space and $PD[l]$ for the Euler class of the circle action. Taking both K_1 and K_2 to be the figure eight knot (see Figure 8.2), we get a manifold with the Seiberg-Witten polynomial:

$$\mathcal{SW}^3_{Y_{K_1K_2}} = t^{-2}_{m_1}t^{-2}_{m_2} - 3t^{-2}_{m_2} + t^2_{m_1}t^{-2}_{m_2} - 3t^{-2}_{m_1} + 9 - 3t^2_{m_1} + t^{-2}_{m_1}t^2_{m_2} - 3t^2_{m_2} + t^2_{m_1}t^2_{m_2}.$$

The Seiberg-Witten polynomial of the manifold $X_{K_1K_2}(4m_1)$ can be calculated from Corollary 6.2,

$$\mathcal{SW}^4_{X_{K_1K_2}(4m_1)} = 2t^{-2}_{m_1+m_2} - 3t^{-2}_{m_2} + 9 - 6t^2_{m_1} + 2t^2_{m_1+m_2} - 3t^2_{m_2},$$

where $t_\beta = \exp(\pi^*(PD(\beta)))$ is the pullback of the Spin^c structure on $Y_{K_1K_2}$.

A theorem of Taubes [T] implies that the first Chern class c_1 of a symplectic 4-manifold must have a Seiberg-Witten invariant ± 1. We thus see that the manifold $X_{K_1K_2}(4m_1)$ admits no symplectic structure with either orientation. This is not the only free S^1-manifold over $Y_{K_1K_2}$ with this property. The manifolds $X_{K_1K_2}(-4m_1)$, $X_{K_1K_2}(4m_2)$, and $X_{K_1K_2}(-4m_2)$ also admit no symplectic structures.

8.3 Example 2: Y^3 which is not a quotient of a symplectic X^4

Next we produce an example of a 3-manifold which is not the orbit space of any symplectic 4-manifold with a free circle action. Let $K_1 = K_2$ be the nonfibered knot 5_2 (see [R]). Note that $H^2(Y_{K_1K_2};\mathbb{Z})$ has no torsion. The Seiberg-Witten polynomial of $Y_{K_1K_2}$ is

$$\begin{aligned}\mathcal{SW}^3_{Y_{K_1K_2}} = 4t_{m_1}^{-2}t_{m_2}^{-2} - 6t_{m_2}^{-2} + 4t_{m_1}^{2}t_{m_2}^{-2} - 6t_{m_1}^{-2} + 9 \\ - 6t_{m_1}^{2} + 4t_{m_1}^{-2}t_{m_2}^{2} - 6t_{m_2}^{2} + 4t_{m_1}^{2}t_{m_2}^{2}.\end{aligned}$$

By Corollary 6.3 we see that $X_{K_1K_2}(0) = Y_{K_1K_2} \times S^1$ does not admit a symplectic structure. For all other principal S^1-bundles above $Y_{K_1K_2}$, we need to calculate as in Example 1. There are only finitely many free S^1 manifolds $X_{K_1K_2}(l)$ which need to be checked because for all $l = am_1 + bm_2$ with $|a|, |b| > 4$ the Seiberg-Witten polynomial $\mathcal{SW}^4$ is equal to the 3-dimensional polynomial (only the meaning of the variables will change). An example calculation of the Seiberg-Witten invariant for the $t_{m_1}^{-2}t_{m_2}^{-2}$ Spin^c

structure of $X_{K_1K_2}(6m_1)$ is

$$\begin{aligned} SW^4_{X_{K_1K_2}(6m_1)}(t^{-2}_{m_1}t^{-2}_{m_2}) &= \sum_{n=-\infty}^{\infty} SW^3_{Y_{K_1K_2}}(t^{-2+6n}_{m_1}t^{-2}_{m_2}) \\ &= \cdots + SW^3_{Y_{K_1K_2}}(t^{-8}_{m_1}t^{-2}_{m_2}) + SW^3_{Y_{K_1K_2}}(t^{-2}_{m_1}t^{-2}_{m_2}) + \\ &\qquad SW^3_{Y_{K_1K_2}}(t^{4}_{m_1}t^{-2}_{m_2}) + \cdots \\ &= \cdots + 0 + 4 + 0 + \cdots \\ &= 4. \end{aligned}$$

The $6m_1$ pairs the Spin^c structure $t^{-2}_{m_1}t^{-2}_{m_2}$ with Spin^c structures that are outside the set of Spin^c structures with nontrivial invariants. For principal S^1-bundles with $|a|, |b| \leq 4$, calculations show that the Seiberg-Witten invariant for each Spin^c structure is greater than one in absolute value. For instance, the Seiberg-Witten polynomial for $X_{K_1K_2}(2m_1)$ is

$$\mathcal{SW}^4_{X_{K_1K_2}(2m_1)} = 2t^{-2}_{m_2} - 3 + 2t^{2}_{m_2}.$$

Hence for all $l \in H_1(Y_{K_1K_2};\mathbb{Z})$ the principal S^1-bundle $X_{K_1K_2}(l)$ does not admit a symplectic structure. Therefore, $Y_{K_1K_2}$ is not the orbit space of any symplectic 4-manifold with a free circle action.

Remark 8.2 *The above two examples show:*

1. *There exist nonsymplectic free S^1-manifolds with fibered orbit space.*

2. *There exists a nontrivial 3-manifold which is not the orbit space of any symplectic 4-manifold with a free S^1-action.*

8.4 Example 3: $b_+ = 1$ diffeomorphism invariants

In this section we construct a $b_+(X)$=1 4-manifold with free circle actions whose Seiberg-Witten invariants are still diffeomorphism invariants. In this situation we can

use Theorem C to calculate its Seiberg-Witten polynomial. We then use Theorem B to study the moduli spaces of X and its quotient Y and explain why the invariants do not change when crossing a "wall."

Recall the construction from section 8.1. Instead of the Borromean link, use the Whitehead link in S^3 and compose each component with the knots K_1 and K_2 (see Figure 1). Then the 3-manifold $Y_{K_1 K_2}$ is the result of surgery on this new link with each surgery coefficient equal to 0. Because the Whitehead link is fibered, when and K_1 and K_2 are fibered knots, the resulting 3-manifold fibers over the circle.

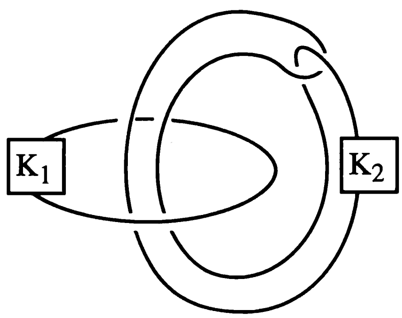

Figure 8.3. $Y_{K_1 K_2}$ before surgery.

Define the $X_{K_1 K_2}(L)$ to be the unit circle bundle of a line bundle L over $Y_{K_1 K_2}$. When $c_1(L)$ is nontorsion, we get the following facts:

1. $b_1(Y_{K_1 K_2}) = 2$, $b_1(X_{K_1 K_2}(L)) = 2$, and $b_+(X_{K_1 K_2}(L)) = 1$.

2. The cup product pairing

$$\cup : H^1(X_{K_1 K_2}(L); \mathbb{Z}) \otimes H^1(X_{K_1 K_2}(L); \mathbb{Z}) \to H^2(X_{K_1 K_2}(L); \mathbb{Z})$$

is trivial. This can be computed from the cup product on $Y_{K_1 K_2}$ using the

isomorphism $\pi^* : H^1(Y_{K_1K_2};\mathbb{Z}) \to H^1(X_{K_1K_2}(L);\mathbb{Z})$.

The two facts above are exactly the conditions needed to show that the wall crossing number is zero for all Spin^c structures [LL]. Hence Seiberg-Witten invariants are still diffeomorphism invariants for these manifolds. In fact, any unit circle bundle over a three manifold which satisfies the conditions above will be such an example. The manifolds constructed above are also particularly easy for calculating the Seiberg-Witten polynomial using Theorem C. We give one example.

Let $Y = Y_{K_1K_2}$ be the manifold where K_1 and K_2 are the fibered 6_3 knot in [R] (see Figure 2). Then the Seiberg-Witten polynomial

$$\mathcal{SW}_Y^3(x,y) = (x^{-4} - 3x^{-2} + 5 - 3x^2 + x^4)(y^{-4} - 3y^{-2} + 5 - 3y^2 + y^4)$$

is calculated using Milnor torsion (Theorem 8.1). In this setup $x = \exp(PD(m_1))$ and $y = \exp(PD(m_2))$ are formal variables where $m_1, m_2 \in H_1(Y;\mathbb{Z})$ represent the meridian loops of each component of the Whitehead link. Thus the term $9x^2y^2$ in the polynomial above means that the Seiberg-Witten invariant for the Spin^c structure identified with $PD(2m_1 + 2m_2)$ is 9.

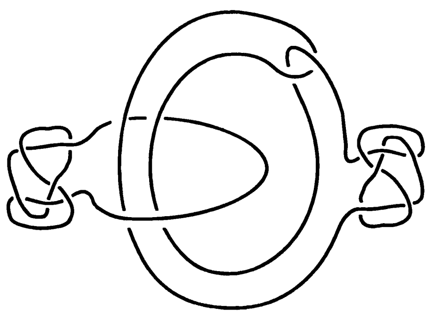

Figure 8.4. Y constructed out of 6_3 knots.

Let $X = X_{K_1K_2}(L)$ be the unit circle bundle of a line bundle L which satisfies $c_1(L) = 4PD(m_1)$. Since the Seiberg-Witten invariants for X are independent of the wall crossing, we can use a similar argument as above to show that $c_1(\xi)$ is pulled back from Y. Thus $d(\xi) = 0$ and condition (5.1) holds for Spin^c structures with nontrivial SW invariants. We can apply Theorem C to get

$$\mathcal{SW}_X^4(x,y) = 7y^{-4}-6x^2y^{-4}-21y^{-2}+18x^2y^{-2}+35-30x^2-21y^2+18x^2y^2+7y^4-6x^2y^4$$

where the formal variables are defined by $x = \exp(\pi^*(PD(m_1)))$ and $y = \exp(\pi^*(PD(m_2)))$ and represent the pullback of Spin^c structures on Y. X is also another example of a nonsymplectic 4-manifold with a circle action whose quotient fibers over the circle.

The power of Theorem C is that one can actually visualize why the Seiberg-Witten invariant does not change when crossing a wall. Let G_X be the product space of metrics and $\Gamma(\Lambda_+)$, then $(g_X, \delta) \in G_X$ is called a *good pair* if the moduli space $\mathcal{M}(X, \xi, g_X, \delta)$ is a smooth manifold without reducible solutions. When $b_+ > 1$ the wall of bad pairs is at least codimension 2 and a cobordism can be constructed between the two moduli spaces of good pairs. However, when $b_+(X) = 1$ it is possible that two good pairs cannot be connected through a generic smooth path in G_X without crossing a wall of bad pairs where reducible solutions occur. Passing through a bad pair could cause a singularity to occur in the cobordism. For a general $b_+ = 1$ manifold, this will often break the invariance of the Seiberg-Witten invariant.

Suppose that we had two good pairs that can not be connected without going through a bad pair. Connect the two good pairs with smooth generic paths to good pairs of the form $(g_X = \eta^2 + g_Y, \pi^*(F \pm \eta)^+)$. Here g_X is fixed, $||\eta||$ is sufficiently small, and F is the harmonic curvature form which represents $2\pi \mathbf{i} c_1(\xi_0)$ for some Spin^c structure on Y. Suppose for the sake of argument that $\{\gamma(t) = (g_X, \pi^*(F + t\eta)^+) \mid -$

$1 \leq t \leq 1\}$ is a smooth generic path in G_X connecting the good pairs. Then a bad pair occurs in both G_X and G_Y precisely when $t = 0$. While the wall has codimension $b_+(X) = 1$ in G_X and hence unavoidable, the wall in G_Y has codimension $b_1(Y) = 2$. Thus it is possible to perturb the path in G_Y to a smooth generic path which avoids the bad pairs. The moduli spaces $\mathcal{M}(Y, \xi_0, g_Y, F \pm \eta)$ are then cobordant, and

$$SW_Y(\xi_0, g_Y, F - \eta) = SW_Y(\xi_0, g_Y, F + \eta).$$

This can be done for each Spin^c structure ξ' on Y such that $\xi = \pi^*(\xi')$, so by Theorem C

$$SW_X(\xi, g_X, \pi^*(F - \eta)^+) = SW_X(\xi, g_X, \pi^*(F + \eta)^+),$$

i.e., the Seiberg-Witten invariant is independent of metric and perturbation.

Note that the perturbed path in G_Y will correspond to a perturbed path in G_X which will still go through a bad pair. The moduli space for X will have reducible solutions at the bad pair, but they do not change the value of the Seiberg-Witten invariant.

The same analysis holds for any b_+=1 4-manifold with a fixed point free circle action and $b_1(Y) = 2$. Therefore we get the following corollary to Theorem C.

Corollary 8.3 *Let X be a b_+=1 4-manifold with a fixed point free circle action whose quotient Y satisfies $b_1(Y) = 2$. If $\xi = \pi^*(\xi_0)$ is a* Spin^c *structure which is pulled back from a* Spin^c *structure ξ_0 on Y, then*

$$SW_X^4(\xi) = \sum_{\xi' \equiv \xi_0 \mod \chi} SW_Y^3(\xi')$$

and the numerical invariant does not depend on the chamber in which it was calculated.

8.5 Application: A Formula for circle bundles over surfaces

A corollary to Theorem C is the calculation of the 3-dimensional Seiberg-Witten invariants for the total space of a circle bundle over a surface. The following corollary can also be derived from [MOY] using different techniques.

Corollary 8.4 *Let $\pi : Y \to \Sigma_g$ be a smooth 3-manifold which is the total space of a circle bundle over a surface of genus $g > 0$. Let $c_1(Y) = n\lambda \in H^2(\Sigma_g; \mathbb{Z})$ where λ is the generator and $n \neq 0$. The only invariants which are not zero on Y come from* Spin^c *structures which are pulled back $\pi : Y \to \Sigma_g$. Hence,*

$$SW_Y(\pi^*(s\lambda)) = \sum_{t \equiv s \mod n} SW_{\Sigma_g \times S^1}(\tilde{\pi}^*(t\lambda))$$

where $\tilde{\pi} : \Sigma_g \times S^1 \to \Sigma_g$.

Proof: Let $\pi : Y \to \Sigma_g$ be the total space of a circle bundle over Σ with Euler class $n\lambda$. Then the manifold $Y \times S^1$ can be thought of as a smooth 4-manifold with a free circle action for which the orbit space is $\Sigma_g \times S^1$. The Euler class of the action is $\tilde{\pi}^*(n\lambda))$. Applying the Corollary 6.2 gives

$$SW^4_{Y \times S^1}((\pi, id)^*(\tilde{\pi}^*(s\lambda))) = \sum_{\tilde{\pi}^*(t\lambda) \equiv \tilde{\pi}^*(s\lambda) \mod \tilde{\pi}^*(n\lambda)} SW^3_{\Sigma \times S^1}(\xi')$$

the right hand side of the equation. Applying Corollary 6.3 shows that $SW^4 = SW^3$ in this case. □

The Seiberg-Witten polynomial for the product of a surface with a circle,

$$\mathcal{SW}_{\Sigma_g \times S^1}(t) = (t - t^{-1})^{2g-2},$$

follows from the Seiberg-Witten invariants of $\Sigma_g \times T^2$ in [FM2]. Combining this with the previous results gives a formula for the Seiberg-Witten polynomial in terms of the Euler class and the genus of the surface.

Corollary 8.5 *Let $\pi : Y \to \Sigma_g$ be the total space of a circle bundle over surface with $g > 0$. Assume $c_1(Y) = n\lambda$ where $\lambda \in H^2(\Sigma_g; \mathbb{Z})$ is the generator and n is an even number $n = 2l \neq 0$, then the Seiberg-Witten polynomial of Y is*

$$\mathcal{SW}_Y(t) = sign(n) \sum_{i=0}^{|l|-1} \sum_{k=-(2g-2)}^{k=2g-2} (-1)^{(g-1)+i+k|l|} \binom{2g-2}{(g-1)+i+k|l|} t^{2i}$$

where $t = exp(\pi^*(\lambda))$ and defining the binomial coefficient $\binom{p}{q} = 0$ for $q < 0$ and $q > p$. For the formula where n is odd, replace l by n and t^{2i} by t^i.

This formula highlights the fact that principal S^1-bundles over surfaces are simple examples that illustrate the difference between Milnor torsion and Turaev torsion. If one uses [MT] to calculate the Milnor torsion for a circle bundle Y over a surface, one finds that the invariant is identically 0. This is because all Spin^c structures on Y with nontrivial invariants have torsion first Chern class. Turaev introduced another type of torsion in [Tu1, Tu2] and a combinatorially defined function on the set of Spin^c structures $T : \mathcal{S}(Y) \to \mathbb{Z}$ derived from this torsion, and showed that this function was the Seiberg-Witten polynomial up to sign. Therefore Turaev torsion is not 0 for principal S^1-bundles over surfaces. Note that this can also be seen by calculating both Milnor torsion and Turaev torsion directly.

CHAPTER 9

Final remarks

Theorem C together with an affirmative answer to the following conjecture would establish a way to calculate Seiberg-Witten invariants for *any* $b_+>1$ 4-manifold with a circle action.

Conjecture 9.1 *If X is a $b_+ > 1$ smooth closed 4-manifold with a circle action that has fixed points, then $\mathcal{SW}_X \equiv 0$.*

There is already considerable evidence which suggest that this is true. For simply connected 4-manifolds carrying a circle action, we can apply the classification result of R. Fintushel [F1, F2].

Theorem 9.2 (Fintushel) *Modulo the 3-dimensional Poincaré conjecture, a simply connected 4-manifold carrying a smooth S^1-action must be a connected sum of copies of S^4, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, and $S^2 \times S^2$.*

This classification result is enough to show that in the $b_+ > 1$ case, X is the connected sum of two $b_+ > 0$ pieces, and hence $\mathcal{SW}_X \equiv 0$.

The conjecture also follows from Proposition 7.3 for 4-manifolds with smooth semi-free actions whose orbit space Y has a nonempty boundary and the rank $H_1(Y, \partial Y; \mathbb{Z}) > 1$.

A counter example to the conjecture above would be just as interesting as the proof.

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