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Time Discretization of Transition Layer Dynamics in Viscoelastic Systems

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ABSTRACT

Time Discretization of Transition Layer Dynamics in Viscoelastic Systems

By

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We investigate how evolution occurs as the strain Du of a viscoelastic system $u_{tt} = \mathrm{Div}(\sigma(Du) + Du_t) - u$ goes towards a state of equilibrium. The physical description of the system is an elastic material with a nonconvex double-well energy density and a viscous stress placed on a rigid elastic foundation subject to a zero displacement boundary condition. The time limit of Du eventually exhibits a finite number of discontinuous interfaces if the strain starts from the continuous initial data whose transition layers are steep enough and the initial energy is sufficiently small. The system conserves the number of phases and the transition layers stay within the initial interfaces. We first consider the one-dimensional case of the problem by using the implicit time discretization method and the Andrews-Pego transformed equations. Numerical computations are conducted and the results are extended to the two-dimensional system.

To my parents.

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I would like to dedicate this work to my family in Korea, especially my parents, Youngsik Lim and Jaebun Kim to express my gratefulness for their constant love, prayer and support.

Finally, I would like to thank God for blessing me to accomplish all these things.

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Introduction

There are various results on the phase transitions of microstructured elastic crystals [1, 3, 6, 12, 13, 16, 23, 25, 26, 28, 29]. Nonconvex double-well free energy induces hysteretic behavior of the fine microstructures of the material. The usual approach involves the minimization of the elastic energy. Due to the lack of convexity in the free energy functional, every minimizing sequence fails to attain the minimizer. In this situation, a minimizing sequence will undergo finer and finer oscillations [6, 7, 26]. However, the energy dissipation prevents such behavior and the sequence converges to the minimizer of the energy [4, 15, 25].

This dissertation focuses on the viscoelastic system

$$u_{tt} = \mathrm{Div}(\sigma(Du) + Du_t) - u, \tag{0.1a}$$

where u is a mapping from $\Omega \times (0, \infty) \subset \mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^N$ for some open bounded domain Ω satisfying the following boundary and initial conditions

$$u = 0 \quad \text{on} \quad \partial\Omega \times [0,\ \infty), \tag{0.1b}$$

$$u = u_0, \quad u_t = v_0 \quad \text{in} \quad \bar{\Omega} \times \{0\} \tag{0.1c}$$

and $\sigma(X) = \frac{\partial W(X)}{\partial X}$ for some stored energy function $W : M^{N\times n} \to \mathbb{R}$.

The system describes a time dependent elastic material with a nonconvex energy W and a viscous stress Δu_t with zero displacement boundary conditions. The material interacts with an elastic foundation u. In other words, the material is placed on a system of linearly elastic springs [28].

Many global existence results for the solutions of similar systems are available [2, 4, 5, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 20, 21, 22, 24, 25, 27]. The existence of the weak solution for the viscoelastic type materials was developed for the cases without assuming the ellipticity of the free energy W [25], the convexity of W or the Lipschitz continuity of σ [15]. In all three cases, the viscous dissipation term plays a significant role in the strong convergence of the minimizing sequences. In the higher dimensional case, G. Friesecke and G. Dolzmann [15] approached the result by an approximation, called the time discretization method, on each sufficiently small time interval.

The dynamics of the transition layers on the viscoelastic system (0.1) is the main topic in this dissertation. Transition layers are defined by the part of the graph of the strain Du where the norm of Du is sufficiently small and the graph changes the sign, that is, the small neighborhoods of the solution u where it has local extrema. For the dynamics of layers in our system, the continuous initial strain must have transition layers which are steep enough, that is the norm of $\text{Div}(Du_0)$ should be sufficiently large. The time limit of the strain Du usually experiences a discontinuity at a finite number of points. More precisely, the finitely many layers of the strain Du get steeper as time increases and eventually become discontinuous at the time limit. Away from these finitely many points, the solution remains continuous. The number of transition layers and the number of zeros of Du remain the same. The layers of the solution are always within the intervals of initial layers, which is a comparable result to the stick-slip motion of layers in a system with nonzero time-dependent displacement boundary conditions [29]. In [29], it was proven that the layers do not stay in the initial intervals and will move both forward and backward. G. Friesecke and J. B. McLeod [16] proved this jump discontinuity of the transition layers at the time limit using the weak solution of the system. In this dissertation, we use the time discretized solutions discussed in [15] and the Andrews-Pego transformed equations which were introduced in [2, 23] to show the phenomenon described above. This approach has some advantages over the method in [16]. It was proven that the time discretized solutions aid in the proof of existence of the limit of the minimizing sequences to the energy functional [15] and several estimates which are essential for the proof of the results are more easily verified.

The interaction of the material with an elastic foundation u induces a finely layered microstructure [5]. It has also been shown using the bifurcation analysis that the elastic foundation induces oscillations in the one-dimensional case of the static problem [28]. Nevertheless, under the assumption of low initial energy, the results still hold without the elastic foundation u and only minor change is needed in the proof. In fact, it can be easily proven that without the u term, the absolute value of the solution approaches 1 as time goes infinity except for the finitely many isolated points where the discontinuity occurs, while with the u term, there is a neighborhood that the time limit of the strain is not 1.

The finite difference methods (FDM) and the finite element methods (FEM) are used for the numerical observation of the dynamics of transition layers in one and two dimensional cases. The methods will be described in Chapters 2 and 3 along with the discussion of efficiency.

In Chapter 1, we use the method of time discretization [15] to prove that the solution approaches the equilibrium state as time goes to infinity and to describe the transition layer dynamics in the one-dimensional case of the system

$$u_{tt} - (\sigma(u_x) + u_{xt})_x + u = 0, \tag{0.2}$$

where u maps from $\Omega \times (0, \infty) \subset \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and $\Omega = (0, 1)$ with the same boundary and initial conditions as $(0.1b)$ and $(0.1c)$.

Let $m > 0$ be fixed and sufficiently small. Let $j \in \mathbb{N}$. For each time interval $((j-1)m,\ jm]$, we consider the minimizer $u^{m,j}$ of the functional defined inductively by

$$J^{m,j}[u] := \int_0^1 \left[\frac{1}{2m^2}|u - 2u^{m,j-1} + u^{m,j-2}|^2 + W(u_x) + \frac{1}{2m}|u_x - u_x^{m,j-1}|^2 + \frac{1}{2}|u|^2\right] dx, \tag{0.3}$$

assuming $u^{m,0} := u_0,\ \ v^{m,0} := v_0,\ \ u^{m,-1} := u_0 - mv_0$. The minimizer $u^{m,j}$ is known as the time discretized solution of system (0.2) since it can be proven to be the weak solution of the time approximated equation of the system

$$\frac{1}{m^2}(u - 2u^{m,j-1} + u^{m,j-2}) - (\sigma(u_x))_x - \frac{1}{m}(u_x - u_x^{m,j-1})_x + u = 0$$

in each time interval $((j-1)m,\ jm]$, $j \in \mathbb{N}$. In [15], it was shown that if $m \to 0$, a subset of $u^{m,j}$ converges to a weak solution of (0.1). Note that $u^{m,j}$ is only a function of x and is in the Sobolev space $W_0^{1,p}(\Omega, \mathbb{R})$, where p is the coercivity exponent of W which is greater than or equal to 2. If W is a convex function or σ is Lipschitz continuous, a standard argument from partial differential equations easily proves the existence of a minimizer of functional (0.3). However, without such hypotheses, the third term of the right hand side of (0.3), which is physically interpreted as a viscous stress, allows the proof of the existence of the functional minimizer. Therefore, in this problem, we do not have to assume the hypotheses given above.

Like in previous works [16, 25], the decay of the energy functional

$$E(u^{m,j}, v^{m,j}) = \int_0^1 \left[\frac{1}{2}(u^{m,j}(x))^2 + W(u_x^{m,j}(x)) + \frac{1}{2}(v^{m,j}(x))^2\right] dx$$

is the crucial point of the proof. An important assumption here is that the initial energy $E(u_0, v_0)$ should be sufficiently small. We prove that the transition layers approach a jump discontinuity as time goes to infinity ($j \to \infty$) by showing that a finite number of intervals where the time discretized strain $u_x^{m,j}$ is steep enough are decreasing to a finite number of isolated points as j goes to infinity. Unfortunately, the intervals in Ω where the norm of $u_x^{m,j}$ is sufficiently small, denote the intervals as $I(u_x^{m,j})$, do not decrease monotonically as $j \to \infty$ in general, that is, $I(u_x^{m,j+1}) \not\subset I(u_x^{m,j})$. Instead, we introduce the time discretized version of Andrews-Pego transformed equations

$$p^{m,j}(x) := \frac{1}{m}\int_0^x [u^{m,j}(y) - u^{m,j-1}(y)]dy - \frac{1}{m}\int_0^1\int_0^z [u^{m,j}(y) - u^{m,j-1}(y)]dydz,$$

$$q^{m,j}(x) := u_x^{m,j}(x) - p^{m,j}(x)$$

and consider the finite number of intervals in Ω where the norm of $q^{m,j}$ is sufficiently small, denote them as $I(q^{m,j})$. We show that the $I(q^{m,j})$ decrease monotonically and exponentially in a nested fashion ($I(q^{m,j+1}) \subset I(q^{m,j})$) to the isolated points as $j \to \infty$ and the intervals $I(u_x^{m,j})$ are contained in the $I(q^{m,j})$ for each $j \in \mathbb{N}$. The solution approaches the jump discontinuities as $j \to \infty$ because of the decrease of the $I(q^{m,j})$ and the fact that the $I(u_x^{m,j})$ are contained in the $I(q^{m,j})$. We also prove the existence and the equilibrium state of the time limit of the discretized solution in order to show the continuity of the time limit of the strain except for the finitely many points which prove to be the zeros of the time limit of $u_x^{m,j}$.

In Chapter 2, we show the convergence analysis for the numerical results of the one-dimensional system after obtaining the matrix equation using the FDM. Next we derive the matrix equation from the FEM and compare the convergence rates and the accuracy to the FDM. We introduce several finite elements such as linear, quadratic and Hermite cubic elements and also discuss the convergence rates and efficiency of these elements. Two difficulties arise in deriving a numerical algorithm for the viscoelastic system. First, in the FDM, the central difference approximation of the nonlinear term $\sigma(u_x)_x$ produces a significant truncation error as time increases due to the lack of smoothness of u near the position of transition layers. The matrix derived from the central difference approximation becomes nonsymmetric because of the term $\sigma(u_x)_x$. However, in the FEM case, numerical integration gives the averaging effect of the solution and the error is reduced. After modifying the FDM algorithm by averaging the nonlinear term, the matrix becomes symmetric and the accuracy is improved. Second, the direct iteration method for the nonlinear term for both methods is computationally expensive. The alternating direction implicit (ADI) Method [19], one of the locally one-dimensional (LOD) methods, for the system is introduced. The computation cost is reduced since the ADI method is a non-iterative method.

The a-priori estimate on the strain $u_x^{m,j}$ given in Chapter 1 confirms the boundedness of the strain. The advantage of the method of discretizing the time interval not only eases the analytical proof but it is also effective in the numerical simulation. The numerical results show a strong agreement with the theoretical predictions. Here, we discuss the importance of small initial energy in the formation of the jump discontinuity at the time limit. Examples of large initial energy do not exhibit this type of behavior. Usually, $u_x^{m,j}$ does not obey the results of the layer dynamics given in Chapter 1 under the initial energy whose norm exceeds some critical point. However, $u_x^{m,j}$ decreases quickly and then follows the theory of the layer dynamics even though it does not preserve the number of transition layers of the initial data. Furthermore, we discuss more examples such as using the Neumann boundary conditions instead of the Dirichlet boundary conditions or without assuming the steepness of the initial transition layers. Surprisingly, the results obtained from the numerical simulations show that the main results of the transition layer dynamics hold while not theoretically proven.

In Chapter 3, We extend the numerical results of the discontinuity of layers at the time limit to the two-dimensional case which is a more physically appropriate way to model experiments. The matrix equations of the system using the FDM and FEM are derived and the convergence analysis of the numerical solution is discussed for the matrix equation from the FDM. We also examine the convergence rates by comparing the results from the numerical computations using both methods. As in the case of the one-dimensional system, the accuracy is better in the FEM than before averaging the nonlinear term in the FDM. After the averaging, the FDM error is reduced and nearly the same as the FEM. The computation cost is also discussed by comparing the performance of the codes using the direct iteration method and the ADI method.

CHAPTER 1

Time Discretization of Transition Layer Dynamics in one-dimensional Viscoelastic Systems

Transition layers, the portion of the strain u_x where the norm is sufficiently small and the graph changes the sign, become steeper and eventually discontinuous as time goes to infinity. The number of transition layers of the strain is preserved. Moreover, transition layers stay in the intervals where the initial layers occur. Important assumptions are that the initial energy is low, and the transition layers are sufficiently steep. Time discretized solutions of the viscoelastic system are introduced in section 1.2 and are used for the proof of the dynamical behavior of the transition layers.

1.1 The initial-boundary value problem and hypotheses

In this section, we introduce the initial-boundary value problem in the one dimensional case and list the hypotheses which will be assumed throughout this Chapter.

Consider the initial-boundary value problem

$$\begin{cases} u_{tt} - (\sigma(u_x) + u_{xt})_x + u = 0, \\ u|_{x=0} = u|_{x=1} = 0 \qquad (t \in [0,\infty)), \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = v_0 \quad (x \in [0,1]), \end{cases} \tag{1.1}$$

where u is a function from $(0,1) \times (0,\infty) \subset \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, $\sigma = W'$ and W is a stored energy function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the following conditions

$(H1)$ $W \in C^2(\mathbb{R})$,

$(H2)$ There exist $c > 0$, $C > 0$, and $p \geq 2$ such that

$$c|z|^p - C \leq W(z) \leq C(|z|^p + 1), \quad |\sigma(z)| \leq C(|z|^{p-1} + 1),$$

$(H3)$ $W(z_\pm) = 0$ for some $z = z_\pm$ and $W(z) > 0$ elsewhere. $\sigma(z) = z \cdot \widetilde{\sigma}(z)$ for some $\widetilde{\sigma} \in C^1(\mathbb{R})$. There exist $z_{1,2}$, where $z_- < z_1 < 0 < z_2 < z_+$ such that $W''|_{(z_1,z_2)} < 0$ and $W''|_{\mathbb{R}\setminus[z_1,z_2]} > 0$.

Assumption $(H3)$ indicates that W is a double-well nonconvex function. It is usually a fourth order polynomial and the most common example is $W(z) = \frac{1}{4}(z^2-1)^2$, where $z_\pm = \pm 1$, $z_1 = -\frac{1}{\sqrt{3}}$ and $z_2 = \frac{1}{\sqrt{3}}$. Moreover, assume

$(A1)$ $u_0 \in C^2$, $v_0 \in W_0^{1,2}$, $||(u_0)_x||_{L^\infty} + ||v_0||_{W^{1,2}} \leq M$,

$(A2)$ $E(u_0, v_0) < \epsilon$, where

$$E(u,v) := \int_0^1 \left(\frac{1}{2}u^2 + W(u_x) + \frac{1}{2}v^2 \right) dx,$$

$(A3)$ $\mathcal{L}_\rho(0) := \{x \in [0,1] : |(u_0)_x(x)| \leq \rho\} \subset (0,1)$,

$(A4)$ $|(u_0)_{xx}(x)| \geq K$ in $\mathcal{L}_\rho(0)$

for some $M, \quad \epsilon, \quad \rho, \quad K > 0$. Here, $\epsilon, \ \rho$ are sufficiently small numbers and K is a sufficiently large number.

Let the connected components of $\mathcal{L}_\rho(0)$ be denoted by $[(a_0)_i, (b_0)_i]$, $i = 1, \cdots, N$ $(0 < (a_0)_1 < (b_0)_1 < \cdots < (a_0)_N < (b_0)_N < 1)$. Note that by the assumption $(A4)$, there exists only one zero of $(u_0)_x$ in each interval $[(a_0)_i, (b_0)_i]$. Let the zeros of $(u_0)_x$ be $(x_0)_i, \ \ (x_0)_i \in [(a_0)_i, (b_0)_i], \ \ i = 1, \cdots, N.$

Before proceeding to the main results, we introduce the time discretized version of the solution of (1.1) in the next section.

1.2 The time discrete scheme for the solution

Let $m > 0$ be fixed. $m \ll 1$. The m will be the time step size of our problem. Let $u^{m,0} := u_0, \ \ v^{m,0} := v_0, \ \ u^{m,-1} := u_0 - mv_0$. For $j \in \mathbb{N}$, define the following functional inductively

$$J^{m,j}[u] := \int_0^1 \left[\frac{1}{2m^2} |u - 2u^{m,j-1} + u^{m,j-2}|^2 + W(u_x) + \frac{1}{2m} |u_x - u_x^{m,j-1}|^2 + \frac{1}{2} |u|^2 \right] dx$$

on the Sobolev space $W_0^{1,p}(\Omega, \mathbb{R})$, where $\Omega = (0,1)$ and p is the coercivity exponent of W in $(H2)$. It was shown that $J^{m,j}$ attains a minimum $u^{m,j}$ if W satisfies the hypotheses $(H1)$, $(H2)$ and $(H3)$ since the first and forth integrands are convex and the nonconvex term $W(u_x)$ combined with the energy dissipation term $\frac{1}{2m}|u_x - u_x^{m,j-1}|^2$ provides the sequentially weakly lower semi-continuity [15]. It can be easily verified that for each $j \in \mathbb{N}$, $u^{m,j}(x)$, which is only the function of x, satisfies the following equation

$$\frac{1}{m^2}(u - 2u^{m,j-1} + u^{m,j-2}) - (\sigma(u_x))_x - \frac{1}{m}(u_x - u_x^{m,j-1})_x + u = 0, \tag{1.2}$$

which is the time approximated equation of (1.1). The $u^{m,j}$, $j \in \mathbb{N}$ are thus called the time discretized solutions of problem (1.1). We next define the linear interpolation function $u^j(x,t)$ of $u^{m,j}(x)$ as follows

$$u^j(x,t) := \left(\frac{mj-t}{m}\right) u^{m,j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) u^{m,j}(x), \quad t \in ((j-1)m,\ jm]$$

for all $j \in \mathbb{N}$. Since $u^j(x,t)$ is the piecewise linear function of the time t, it enables us to differentiate the time discretized solutions with respect to t on each subinterval. It is now important to define the functions which are called the time discretized version of Andrews-Pego transformed equations. The equations will play a crucial role for the proof of main results. Define

$$\begin{aligned}
p_0(x) &:= \int_0^x v_0(y)dy - \int_0^1 \int_0^z v_0(y)dydz, \\
q_0(x) &:= (u_0)_x(x) - p_0(x), \\
p^{m,j}(x) &:= \frac{1}{m}\int_0^x [u^{m,j}(y) - u^{m,j-1}(y)]dy - \frac{1}{m}\int_0^1 \int_0^z [u^{m,j}(y) - u^{m,j-1}(y)]dydz, \\
q^{m,j}(x) &:= u_x^{m,j}(x) - p^{m,j}(x)
\end{aligned}$$

for all $j \in \mathbb{N}$. Note that $p_x^{m,j}(x) = \frac{u^{m,j}(x)-u^{m,j-1}(x)}{m}$. Denote it by $v^{m,j}(x)$. For all $j \in \mathbb{N}$ and $(j-1)m < t \leq jm$, define the interpolation functions of $p^j(x,t)$, $q^j(x,t)$ and $v^j(x,t)$ of $p^{m,j}(x)$, $q^{m,j}(x)$ and $v^{m,j}(x)$ in the same way

$$\begin{aligned}
p^j(x,t) &:= \left(\frac{mj-t}{m}\right) p^{m,j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) p^{m,j}(x), \\
q^j(x,t) &:= \left(\frac{mj-t}{m}\right) q^{m,j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) q^{m,j}(x), \\
v^j(x,t) &:= \left(\frac{mj-t}{m}\right) v^{m,j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) v^{m,j}(x).
\end{aligned}$$

Next, we state the main results on this chapter.

1.3 Main results

The following theorem is the main results on Chapter 1 and this describes the dynamical behavior of the transition layers.

Theorem 1.1 *Suppose the stored energy function W and the initial data $(u_0, v_0) \in W_0^{1,\infty} \times L^2$ are assumed to satisfy (H1)-(H3), (A1)-(A4). Then the following holds*

(P1) (Conservation of number of zeros). The number of zeros of $u_x^{m,j}$ which is denoted by $N(j)$ is finite, positive for all $j \in \{0\} \cup \mathbb{N}$, and it is independent of j. Let the zeros be denoted by $0 < x_1^m(j) < \cdots < x_N^m(j) < 1 \ \forall j \in \mathbb{N}$.

(P2) (Conservation of number of intervals of transition layers). The number of connected components of $\mathcal{L}_{\frac{\rho}{2}}(j) := \{x \in (0,1) : |u_x^j(x,t)| \le \frac{\rho}{2}\}$ is finite, positive for all $j \in \mathbb{N}$, and it is independent of j, and in each connected component, $u_x^j(x,t)$ is strictly monotone and has exactly one zero. Let the connected components of $\mathcal{L}_{\frac{\rho}{2}}(j)$ be denoted by $[a_i^m(j), b_i^m(j)]$, $i = 1, \cdots, N \ \forall j \in \mathbb{N}$ $(0 < a_1^m(j) < b_1^m(j) < \cdots < a_N^m(j) < b_N^m(j) < 1)$.

(P3) (Lock-in and exponential steepening of transition layers). For all $j \in \mathbb{N}$, $i = 1, 2, \cdots, N$ and for some $C_0 > 0$, $C_0 \approx 1$,

$$[a_i^m(j), b_i^m(j)] \subset [(a_0)_i, (b_0)_i], \ \textit{in particular} \ x_i^m(j) \in [(a_0)_i, (b_0)_i],$$

$$|u_{xx}^j(x,t)| \ge \frac{KC_0}{2} e^{\sigma_0 jm} \quad \forall x \in \mathcal{L}_{\frac{\rho}{2}}(j) = \bigcup_{i=1}^{N} [a_i^m(j), b_i^m(j)],$$

$$|b_i^m(j) - a_i^m(j)| \le \frac{2\rho}{KC_0} e^{-\sigma_0 jm},$$

where $\sigma_0 := \min_{[-\rho,\rho]} |\sigma'| > 0$.

(P4) (Convergence of phases). $\lim_{j\to\infty} x_i^m(j) =: (x_\star)_i^m$ exists for all $i = 1, 2, \cdots, N$ and $(x_\star)_i^m \in [(a_0)_i, (b_0)_i]$ (in particular, $0 < (x_\star)_1^m < \cdots < (x_\star)_N^m < 1$).

(P5) (Jump discontinuity of the limit state). $\lim_{j\to\infty} u_x^{m,j} =: (u_\star^m)_x$ *(which exists as an L^p limit) is continuous on* $(0,1)\backslash\{(x_\star)_1^m, \cdots, (x_\star)_N^m\}$ *but discontinuous at every* $(x_\star)_i^m$ $\forall i = 1, 2, \cdots, N$.

1.4 Energy decay and a-priori estimates

We first prove the decay of the energy functional

$$E(t) = E(u^j, v^j) = \int_0^1 \left[\frac{1}{2}(u^j(x,t))^2 + W(u_x^j(x,t)) + \frac{1}{2}(v^j(x,t))^2\right] dx$$

for $t \in ((j-1)m,\ jm],\ j \in \mathbb{N}$. It is difficult to show the proof of decay of $E(t)$ since the time derivatives of u^j, u_x^j and v^j are constant with respect to time. The functional becomes the combination of the time discretized solutions and their interpolation functions. Thus, one must be careful in the calculation of the following lemma.

Lemma 1.1 *$E(t)$ is non-increasing, bounded by the initial data on $((j-1)m,\ jm]$ for all $j \in \mathbb{N}$.*

PROOF. Recall that $u^{m,j}$ satisfies (1.2). That is, the following equation

$$v_t^j - \sigma(u_x^{m,j})_x - v_{xx}^{m,j} + u^{m,j} = 0 \tag{1.3}$$

is satisfied for all $j \in \mathbb{N}$. Then the following estimate holds for $(j-1)m < t \leq jm$

$$\frac{d}{dt}E(t) = \int_0^1 (v^j \cdot v_t^j + \sigma(u_x^j) \cdot u_{xt}^j + u^j \cdot u_t^j)dx$$

$$= \int_0^1 [v^{m,j} \cdot v_t^j + \sigma(u_x^{m,j}) \cdot u_{xt}^j + u^{m,j} \cdot u_t^j + (v^j - v^{m,j})v_t^j \tag{1.4}$$
$$+ (\sigma(u_x^j) - \sigma(u_x^{m,j})) \cdot u_{xt}^j + (u^j - u^{m,j}) \cdot u_t^j]dx$$

$$= \int_0^1 [v^{m,j}\{v_t^j - \sigma(u_x^{m,j})_x + u^{m,j}\} + (v^j - v^{m,j})v_t^j \tag{1.5}$$
$$+ (\sigma(u_x^j) - \sigma(u_x^{m,j})) \cdot u_{xt}^j + (u^j - u^{m,j}) \cdot u_t^j]dx$$

$$= \int_0^1 \left[v^{m,j} \cdot v_{xx}^{m,j} + \frac{(t-jm)}{m^2} \cdot |v^{m,j} - v^{m,j-1}|^2 \right.$$
$$\left. + (\sigma(u_x^j) - \sigma(u_x^{m,j})) \cdot u_{xt}^j + \frac{(t-jm)}{m^2} \cdot |u^{m,j} - u^{m,j-1}|^2 \right] dx$$

$$= -\int_0^1 |v_x^{m,j}|^2 dx + \frac{(t-jm)}{m^2} \int_0^1 |v^{m,j} - v^{m,j-1}|^2 dx \tag{1.6}$$
$$+ \int_0^1 (\sigma(u_x^j) - \sigma(u_x^{m,j})) \cdot u_{xt}^j dx + (t-jm) \int_0^1 |v^{m,j}|^2 dx.$$

The first three terms of the integrand of (1.4) are the same as the first term of the integrand of (1.5) since $u^{m,j}$ satisfy the same boundary conditions as (1.1) and $u_t^j = v^{m,j}$. By using the Mean Value Theorem and by the fact that the function σ' is bounded below by a negative number, that is for all $y \in \mathbb{R}$, $\sigma'(y) \geq -L$ for some $L > 0$, the integrand of the third term of (1.6) is estimated in the following way

$$
\begin{aligned}
(\sigma(u_x^j) - \sigma(u_x^{m,j})) \cdot u_{xt}^j &= \sigma'(c_1^{m,j})(u_x^j - u_x^{m,j}) \cdot u_{xt}^j \\
&= \sigma'(c_1^{m,j})(t - jm)\frac{(u_x^{m,j} - u_x^{m,j-1})^2}{m^2} \\
&= (jm - t)\{-\sigma'(c_1^{m,j})\}\frac{(u_x^{m,j} - u_x^{m,j-1})^2}{m^2} \\
&\leq m \cdot \max_{y\in\mathbb{R}}\{-\sigma'(y)\}\frac{(u_x^{m,j} - u_x^{m,j-1})^2}{m^2} \\
&\leq mL\frac{(u_x^{m,j} - u_x^{m,j-1})^2}{m^2} \\
&= mL(v_x^{m,j})^2 \qquad (1.7)
\end{aligned}
$$

for some $c_1^{m,j}$ between u_x^j and $u_x^{m,j}$. Moreover, both the second and the forth term of (1.6) are negative since $t - jm \leq 0$. Since m is sufficiently small, the following inequalities on the energy $E(t)$ are derived

$$
\begin{aligned}
\frac{d}{dt}E(t) &\leq (-1 + mL)\int_0^1 |v_x^{m,j}|^2 dx \\
&\leq -\frac{1}{2}\|v_x^{m,j}\|_{L^2}^2
\end{aligned}
$$

for all $t \in ((j-1)m, jm]$ and this proves Lemma 1.1.

Note that by taking an integral from $(j-1)m$ to jm on both sides of the above inequality, we get

$$
E(jm) - E((j-1)m) = \int_{(j-1)m}^{jm} \frac{d}{dt}E(t)dt \leq -\frac{1}{2}m\|v_x^{m,j}\|_{L^2}^2.
$$

After taking a summation from $j = 1$ to $j = S$, the following estimate is established

$$
E(Sm) - E(0) = \sum_{j=1}^{S}[E(jm) - E((j-1)m)] \leq -\frac{m}{2}\sum_{j=1}^{S}\|v_x^{m,j}\|_{L^2}^2.
$$

Therefore,

$$\frac{m}{2}\sum_{j=1}^{S}\|v_x^{m,j}\|_{L^2}^2 \le E(0) - E(Sm) < E(0) < \epsilon \tag{1.8}$$

for all $S \in \mathbb{N}$. Next, we show the several estimates on the functions which will play a significant role for the proof of Theorem 1.1.

Lemma 1.2 *The following a-priori estimates hold*

(a) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \|p^j(\cdot,t)\|_{L^\infty} \le \eta,$

(b) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \left\|\pi_a\left(\int_0^x u^j(y,t)dy\right)\right\|_{L^\infty} \le \sigma_0\eta,$ *where* $\pi_a(f) := f - \int_0^1 f,$

(c) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \|q^j(\cdot,t)\|_{L^\infty} \le K_0,$

(d) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \|u^j(\cdot,t)\|_{L^\infty} \le K_0,$

(e) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \left|\int_0^1 \sigma(u_x^j(x,t))dx\right| \le \sigma_0\eta,$

(f) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \|v^j(\cdot,t)\|_{L^\infty} = \sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\le jm} \|p_x^j(\cdot,t)\|_{L^\infty} \le K_0,$

for some constants K_0, $\eta > 0$, *where* η *is a sufficiently small constant.*

Proof. Since $\int_0^1 p^j(x,t)dx = 0$, $p^j(x',t) = 0$ for some x' in $(0,1)$. Hence, the following holds

$$|p^j(x,t)| = \left|\int_{x'}^x p_x^j(y,t)dy\right| \le \int_0^1 |p_x^j(y,t)|dy \le \left(\int_0^1 |p_x^j(y,t)|^2dy\right)^{\frac{1}{2}}.$$

Since $\|p_x^j(\cdot,t)\|_{L^2} = \|v^j(\cdot,t)\|_{L^2} \le \sqrt{E(u^j,v^j)} \le \sqrt{E(u_0,v_0)} \le \sqrt{\epsilon}$, (a) is accomplished by choosing $\eta > 0$ such that $\eta > \max\{\sqrt{\epsilon}, 2\sqrt{\epsilon}/\sigma_0, C_5\sqrt{\epsilon}/\sigma_0\}$, where C_5 will be chosen later.

Similarly,

$$\begin{aligned}
\left\| \int_0^x u^j(y,t)dy \right\|_{L^\infty} &= \left\| \int_0^x \int_0^y (p^j(z,t) + q^j(z,t))dzdy \right\|_{L^\infty} \\
&\leq \left\| \int_0^x (p^j(y,t) + q^j(y,t))dy \right\|_{L^2} \\
&= \|u^j(\cdot,t)\|_{L^2} \\
&\leq \sqrt{\epsilon} \leq \frac{\sigma_0 \eta}{2},
\end{aligned}$$

which proves (b).

By using equation (1.3),

$$\begin{aligned}
q_t^j &= \frac{q^{m,j}(x) - q^{m,j-1}(x)}{m} \\
&= \frac{u_x^{m,j}(x) - u_x^{m,j-1}(x)}{m} - \int_0^x v_t^j + \int_0^1 \int_0^x v_t^j \\
&= v_x^{m,j}(x) - \sigma(u_x^{m,j}(x)) + \sigma(u_x^{m,j}(0)) - v_x^{m,j}(x) + v_x^{m,j}(0) \\
&\quad + \pi_a \left(\int_0^x u^{m,j} \right) + \int_0^1 \sigma(u_x^{m,j}) - \sigma(u_x^{m,j}(0)) + \int_0^1 v_x^{m,j} - v_x^{m,j}(0) \\
&= -\pi_a(\sigma(u_x^{m,j})) + \pi_a \left(\int_0^x u^{m,j} \right) && (1.9) \\
&= -\sigma(p^{m,j} + q^{m,j}) + e_1^{m,j}, && (1.10)
\end{aligned}$$

where $e_1^{m,j} = \int_0^1 \sigma(u_x^{m,j}(x))dx + \pi_a(\int_0^x u^{m,j}(y)dy)$. From the hypotheses $(H2)$ and $(H3)$, $\sigma(z) \leq W(z) + C_1$ for some $C_1 > 0, \ \ z \in \mathbb{R}$.

This and the estimate (b) imply

$$\begin{aligned}
|e_1^{m,j}| &\leq \left| \int_0^1 \sigma(u_x^{m,j}) \right| + \left| \pi_a \left(\int_0^x u^{m,j} \right) \right| \\
&\leq \int_0^1 (|W(u_x^{m,j})| + C_1) + \sigma_0 \eta \\
&\leq \epsilon + C_1 + \sigma_0 \eta < C_2
\end{aligned}$$

for some $C_2 > 0$. Here, the energy estimate $\int_0^1 |W(u_x^{m,j})|dx \leq E(t) \leq E(0) < \epsilon$ was used. Therefore,

$$\|e_1^{m,j}\|_{L^\infty} \leq C_2 \tag{1.11}$$

for all $j \in \mathbb{N}$. Since $\|p^j\|_{L^\infty} < \eta$, from (1.10) and (1.11), $q_t^j < 0$ when $q^j \geq K_1$ and $q_t^j > 0$ when $q^j \leq -K_1$ for sufficiently large $K_1 > 0$. Let $K_0 > \max\{\eta + K_1, K_2\}$, where K_2 will be chosen later. Hence, q^j is bounded by K_0 which completes the proof of (c). Note

$$\|u_x^j\|_{L^\infty} \leq \|p^j\|_{L^\infty} + \|q^j\|_{L^\infty} \leq \eta + K_1 < K_0. \tag{1.12}$$

Now, (d) clearly follows from (a) and (c) since

$$|u^j| \leq \int_0^1 |u_x^j| \leq \int_0^1 |p^j| + \int_0^1 |q^j| \leq \eta + K_1 < K_0.$$

Note that for all $j \in \mathbb{N}$,

$$|\sigma(u_x^j)| \leq C_3 \tag{1.13}$$

for some $C_3 > 0$ since $\|u_x^j\|_{L^\infty}$ is uniformly bounded for all $j \in \mathbb{N}$ by (1.12). Since q_t^j satisfies (1.10), (1.13) combined with (1.11) implies that q_t^j is uniformly bounded by $C_2 + C_3$ for all $j \in \mathbb{N}$ in L^∞ norm. Also, (1.12) implies that

$$|\sigma'(u_x^j)| \leq C_4 \tag{1.14}$$

for all $j \in \mathbb{N}$ and for some $C_4 > 0$.

From the conditions $(H2)$, $(H3)$ on W, σ and by (1.12) and (1.13),

$$C_5 := \sup_{z\in[-K_0,K_0]\setminus\{z_-,z_+\}} \frac{|\sigma(z)|}{\sqrt{W(z)}}$$

is well defined and

$$\begin{aligned} \left|\int_0^1 \sigma(u_x^j(x,t))dx\right| &\leq \|\sigma(u_x^j)\|_{L^2} \\ \leq \; C_5\left(\int_0^1 |W(u_x^j)|\right)^{\frac{1}{2}} &< C_5\sqrt{\epsilon} \leq \sigma_0\eta, \end{aligned}$$

which proves (e).

It will be shown next that $\|p_{xx}^j\|_{L^2}$ is uniformly bounded for all $j \in \mathbb{N}$ in order to prove (f).

Since

$$p_{xx}^j(x,t) = r^j(x,t) + s^j(x,t),$$

where

$$\begin{aligned} r^j(x,t) &:= \left(\frac{mj-t}{m}\right)p_t^{j-1}(x) + \left(\frac{t-m(j-1)}{m}\right)p_t^j(x), \\ s^j(x,t) &:= \left(\frac{mj-t}{m}\right)q_t^{j-1}(x) + \left(\frac{t-m(j-1)}{m}\right)q_t^j(x), \end{aligned}$$

and $\|q_t^j\|_{L^\infty}$ is uniformly bounded for all $j \in \mathbb{N}$, one would only need to show that $\|p_t^j\|_{L^2}$ is uniformly bounded for all $j \in \mathbb{N}$. p_t^j satisfies the following equations

$$\begin{aligned} p_t^j &= u_{xt}^j - q_t^j \\ &= p_{xx}^{m,j} + \pi_a\left[\sigma(p^{m,j}+q^{m,j}) - \int_0^x\int_0^{x'}(p^{m,j}+q^{m,j})\right]. \end{aligned} \tag{1.15}$$

The last equality follows from (1.9) and the identity $u^j_{xt} = p^{m,j}_{xx}$. Let

$$f(p^{m,j}) := -q^j_t = \pi_a \left[\sigma(p^{m,j} + q^{m,j}) - \int_0^x \int_0^{x'} (p^{m,j} + q^{m,j}) \right].$$

Note that for all $j \in \mathbb{N}$,

$$\|f(p^{m,j})\|_{L^\infty} < M_1, \tag{1.16}$$

where $M_1 = C_2 + C_3$ since q^j_t is uniformly bounded. From (1.15),

$$p^{m,j} - p^{m,j-1} = m\Delta p^{m,j} + mf(p^{m,j}),$$

which implies

$$(1 - m\Delta)p^{m,j} = p^{m,j-1} + mf(p^{m,j}).$$

Therefore,

$$\begin{aligned} p^{m,j} &= \frac{p^{m,j-1}}{(1-m\Delta)} + \frac{m}{(1-m\Delta)} f(p^{m,j}) \\ &= \frac{1}{(1-m\Delta)} \left[\frac{p^{m,j-2}}{(1-m\Delta)} + \frac{m}{(1-m\Delta)} f(p^{m,j-1}) \right] + \frac{m}{(1-m\Delta)} f(p^{m,j}) \\ &= \frac{p^{m,j-2}}{(1-m\Delta)^2} + m \left[\frac{f(p^{m,j-1})}{(1-m\Delta)^2} + \frac{f(p^{m,j})}{(1-m\Delta)} \right] \\ &\dots \\ &= \frac{p_0}{(1-m\Delta)^j} + m \left[\frac{f(p^{m,1})}{(1-m\Delta)^j} + \cdots + \frac{f(p^{m,j})}{(1-m\Delta)} \right]. \end{aligned}$$

Thus,

$$p^j_t = \frac{p^{m,j} - p^{m,j-1}}{m} = \frac{\Delta p_0}{(1-m\Delta)^j} + m \sum_{k=1}^{j-1} \frac{\Delta f(p^{m,k})}{(1-m\Delta)^{j+1-k}} + \frac{f(p^{m,j})}{(1-m\Delta)}.$$

Note that the second term on the right hand side is the same as

$$m \sum_{k=1}^{j-1} \frac{1}{(1-m\Delta)^{j-k}} \cdot \frac{\Delta}{(1-m\Delta)} \cdot f(p^{m,k}).$$

By incorporating the inequality $\|\frac{\Delta}{(1-m\Delta)}\|_{L^2} \leq 1$ and (1.16), the following inequalities occur

$$\begin{aligned}
\|p_t^j\|_{L^2} &\leq \|\Delta p_0\|_{L^2} + m \sum_{k=1}^{j-1} \left\| \frac{1}{(1-m\Delta)^{j-k}} \cdot \frac{\Delta}{(1-m\Delta)} \cdot f(p^{m,k}) \right\|_{L^2} + \|f(p^{m,j})\|_{L^2} \\
&\leq \|\Delta p_0\|_{L^2} + mM_1 \cdot \sum_{k=1}^{j-1} \left\| \frac{1}{(1-m\Delta)^{j-k}} \right\|_{L^2} + \|f(p^{m,j})\|_{L^2} \\
&\leq \|\Delta p_0\|_{L^2} + mM_1 \cdot \sum_{k=1}^{j-1} \frac{1}{(1-m\lambda_1)^{j-k}} + M_1 \\
&\leq \|\Delta p_0\|_{L^2} + \frac{M_1}{\lambda_1} \cdot \left[\frac{1}{(1-m\lambda_1)^{j-1}} - 1 \right] + M_1 \\
&\leq \|\Delta p_0\|_{L^2} + \left(-\frac{1}{\lambda_1} + 1 \right) \cdot M_1 \\
&\leq M_2
\end{aligned}$$

for some $M_2 > 0$. Here, $\lambda_1 < 0$ is the largest eigenvalue of Δ. Let $K_2 = M_1 + M_2$. Then $\|p_x^j\|_{L^\infty} \leq \|p_{xx}^j\|_{L^2} \leq K_2 \leq K_0$. Therefore, $\|p_t^j\|_{L^2}$ is uniformly bounded for all $j \in \mathbb{N}$ and this proves the estimate (f). Proof of Lemma 1.2 is now completed.

The equilibrium state of the time limit of the discretized solution is proven in the next section. This result is sufficient to prove the last part of the main results.

1.5 Equilibrium state of the time limit of the solution

We now introduce the following function φ, which is called the phase function. This function will play an important role for proving the equilibrium state of the solution at the time limit. Fix $r > 0$ such that for $\lambda \in [-r,\ r]$, the equation $\sigma(z) = \lambda$ has three different solutions $z_1(\lambda) < z_2(\lambda) < z_3(\lambda)$. Define

$$\varphi(z) := \begin{cases} i, & z \in \bigcup_{\lambda \in [-r,\ r]} z_i(\lambda), \quad i = 1, 2, 3, \\ \infty, & \text{elsewhere.} \end{cases}$$

The next proposition states that the discretized solution $u^{m,j}$ converges in $W_0^{1,p}$ to an equilibrium state as time goes to infinity.

Proposition 1.1 *Suppose (H1)-(H3), (A1)-(A4) hold. Then the solution* $(u^{m,j}, v^{m,j})$ *of (1.3) converges strongly in* $W_0^{1,p} \times L^2$ $(1 \leq p < \infty)$ *to some equilibrium state* $(u_*^m, 0) \in W_0^{1,\infty} \times L^2$ *as* $j \to \infty$.

PROOF. The proof consists of several Lemmas. The following lemma states that under some appropriate conditions on the elastic stress $\sigma(u_x^{m,j}(x)) - \int_0^x u^{m,j}$ and the phase function φ, the strain $u_x^{m,j}$ converges to an equilibrium state. We must be careful when choosing the pointwise representatives of $u_x^{m,j}$ since in the measure zero sets of $(0,1)$, we never know the behavior of the strain $u_x^{m,j}$. It is important to choose the good representatives so that the limit state is continuous except for the finitely many points which are the zeros of the limit state.

Lemma 1.3 *Assume there exists a full measure subset* $\tilde{\Omega} \in (0,1)$*(Measure of* $\tilde{\Omega}$ *is* 1.*) and pointwise representatives* $\bar{w}^{m,j}$ *of* $u_x^{m,j}$ *such that*

(B1) $\sigma(\bar{w}^{m,j}(x)) - \int_0^x u^{m,j} =: \lambda_j^m(x) \to \lambda^m$ *as* $j \to \infty$ *for some* $\lambda^m \in (-r, r)$ *and all* $x \in \tilde{\Omega}$,

(B2) $\lim_{j \to \infty} \varphi(\bar{w}^{m,j}(x))$ *exists and is finite for all* $x \in \tilde{\Omega}$.

Then $\lim_{j \to \infty} \bar{w}^{m,j}(x) =: \bar{w}^m(x)$ *exists for all* $x \in \tilde{\Omega}$. *Moreover, the equivalence class* $\hat{w}^m$

of $\bar{w}^m$ *satisfies*

$$\|\hat{w}^m\|_{L_\infty} \le K_2, \quad and \quad u_\star^m(x) := \int_0^x \hat{w}^m \quad is\ in \quad W_0^{1,\infty}$$

and satisfies

$$\sigma((u_\star^m)_x(x)) - \int_0^x u_\star^m \equiv \lambda^m \quad a.e., \quad \varphi((u_\star^m)_x(x)) = \lim_{j\to\infty} \varphi(u_x^{m,j}(x)) \quad a.e.,$$

and

$$u^{m,j} \longrightarrow u_\star^m \quad in \quad W_0^{1,p} \quad (1 \le p < \infty) \quad as \ j \to \infty.$$

Proof. Recall that $\sup_{j\in\mathbf{N}} \|u_x^j\|_{L^\infty} < K_2$ by (1.12). We first show that $\int_0^x u^{m,j}$ is convergent in $C([0,\ 1])$. Define

$$\chi_i^{m,j}(x) := \begin{cases} 1, & x \in \tilde{\Omega} \text{ and } \varphi(\bar{w}^{m,j}(x)) = i \in \{1,2,3\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi_\infty^{m,j}(x) := \begin{cases} 1, & x \in \tilde{\Omega} \text{ and } \varphi(\bar{w}^{m,j}(x)) = \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\bar{w}^{m,j}(x) = z_i(\int_0^x u^{m,j} + \lambda_j^m(x))$ and $\chi_\infty^{m,j}(x) = 0$ if $\varphi(\bar{w}^{m,j}(x)) = i, \ \ i = 1,2,3$ for $x \in \tilde{\Omega}$, the following equation holds in $\tilde{\Omega}$

$$\bar{w}^{m,j}(x) - \bar{w}^{m,k}(x) = \sum_{i=1}^{3}\left[\chi_i^{m,j}(x)\cdot z_i\left(\int_0^x u^{m,j} + \lambda_j^m(x)\right) - \chi_i^{m,k}(x)\cdot z_i\left(\int_0^x u^{m,k} + \lambda_k^m(x)\right)\right]$$
$$+\chi_\infty^{m,j}(x)\cdot \bar{w}^{m,j}(x) - \chi_\infty^{m,k}(x)\cdot \bar{w}^{m,k}(x). \tag{1.17}$$

Note that since $1 = \frac{d}{d\lambda^m}(\lambda^m) = \frac{d}{d\lambda^m}(\sigma(z_i(\lambda^m))) = \sigma'(z_i(\lambda^m))\cdot z_i'(\lambda^m)$,

$$|z_i(a) - z_i(b)| \le \sup_{x\in[-r,\, r]} |z_i'(x)|\cdot|a-b| \le \sup_{x\in[-r,\, r]} \frac{1}{|\sigma'(z_i(x))|}\cdot|a-b| \le \frac{1}{\bar{M}}|a-b|, \tag{1.18}$$

where $\bar{M} := \min_{z\in\sigma^{-1}([-r,r])} |\sigma'(z)|$. Let $\xi_{j,k}^m(x) = \int_0^x |u^{m,j} - u^{m,k}|$. Then the following holds

$$\begin{aligned} 0 &\le \frac{d}{dx}\xi_{j,k}^m(x) = |u^{m,j}(x) - u^{m,k}(x)| \\ &= \left|\int_0^x (u_x^{m,j} - u_x^{m,k})\right| \\ &= \Big|\int_0^x [\sum_{i=1}^{3}\chi_i^{m,k}(x')\{\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')\} + \sum_{i=1}^{3}\{\chi_i^{m,j}(x') - \chi_i^{m,k}(x')\}\bar{w}^{m,j}(x') \\ &\quad +\chi_\infty^{m,j}(x')\cdot\bar{w}^{m,j}(x') - \chi_\infty^{m,k}(x')\cdot\bar{w}^{m,k}(x')]dx'\Big|. \end{aligned} \tag{1.19}$$

Note that by (1.12),

$$\int_0^x \{\sum_{i=1}^{3}(\chi_i^{m,j} - \chi_i^{m,k})\bar{w}^{m,j}\} \le 2K_2|\{x\in(0,1) : \varphi(\bar{w}^{m,j}(x)) \ne \varphi(\bar{w}^{m,k}(x))\}|$$

and

$$\begin{aligned} \int_0^x \left(\chi_\infty^{m,j}\cdot\bar{w}^{m,j} - \chi_\infty^{m,k}\cdot\bar{w}^{m,k}\right) &\le K_2|\{x\in(0,1) : \varphi(\bar{w}^{m,j}(x)) \ne \varphi(\bar{w}^{m,k}(x))\}| \\ &+ 2K_2|\{x\in(0,1) : \varphi(\bar{w}^{m,j}(x)) = \varphi(\bar{w}^{m,k}(x)) = \infty\}|. \end{aligned}$$

Therefore, the last three terms in (1.19) are dominated by

$$3K_2|\{x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) \neq \varphi(\bar{w}^{m,k}(x))\}|$$
$$+2K_2|\{x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) = \varphi(\bar{w}^{m,k}(x)) = \infty\}| =: \delta^m_{j,k}.$$

By the assumption $(B2)$, for fixed $x \in (0,1)$, if j,k are sufficiently large, $\varphi(\bar{w}^{m,j}(x)) = \varphi(\bar{w}^{m,k}(x)) = i(x)$ for some $i(x) = 1,2,3$. Therefore, $\delta^m_{j,k} \to 0$ as $\min\{j,k\} \to \infty$. For each $i \in \{1,2,3\}$,

$$\begin{aligned}\int_0^x \chi_i^{m,k}(x')(\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x'))dx' &\leq \int_{J_1} |\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')|dx' \\ &\quad + \int_{J_2} |\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')|dx',\end{aligned}$$

where

$$J_1 := \{x' \in (0,x) : \chi_i^{m,j}(x') = \chi_i^{m,k}(x') = 1\},$$

$$J_2 := \{x' \in (0,x) : \chi_i^{m,j}(x') = 1,\ \chi_i^{m,k}(x') = 0 \ \text{ or } \ \chi_i^{m,j}(x') = 0,\ \chi_i^{m,k}(x') = 1\}.$$

In the set J_1,

$$\begin{aligned}|\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')| &= \left| z_i\left(\int_0^{x'} u^{m,j}(s)ds + \lambda_j^m(x')\right) - z_i\left(\int_0^{x'} u^{m,k}(s)ds + \lambda_k^m(x')\right)\right| \\ &\leq \frac{1}{\bar{M}}\left[\int_0^{x'} |u^{m,j}(s) - u^{m,k}(s)|ds + |\lambda_j^m(x') - \lambda_k^m(x')|\right] \\ &= \frac{1}{\bar{M}}[\xi^m_{j,k}(x') + |\lambda_j^m(x') - \lambda_k^m(x')|].\end{aligned}$$

Note that $J_2 \subset \{x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) \neq \varphi(\bar{w}^{m,k}(x))\}$. Hence,

$$\begin{aligned}
\frac{d}{dx}\xi^m_{j,k}(x) &\leq 2 \cdot \delta^m_{j,k} + \frac{1}{\bar{M}} \int_0^1 |\lambda^m_j(x') - \lambda^m_k(x')| dx' + \int_{J_1} \frac{1}{\bar{M}} \xi^m_{j,k}(x') dx' \\
&\leq 2 \cdot \delta^m_{j,k} + \frac{1}{\bar{M}} \|\lambda^m_j - \lambda^m_k\|_{L^1} + \frac{1}{\bar{M}} \xi^m_{j,k}(x) \\
&\leq \epsilon^m_{j,k} + \frac{1}{\bar{M}} \xi^m_{j,k}(x),
\end{aligned}$$

where $\epsilon^m_{j,k} := 2 \cdot \delta^m_{j,k} + \frac{1}{M}\|\lambda^m_j - \lambda^m_k\|_{L^1}$. Since $\|\lambda^m_j - \lambda^m_k\|_{L^1} \to 0$ as $\min\{j,k\} \to \infty$ by the assumption $(B1)$, $\epsilon^m_{j,k} \to 0$ as $\min\{j,k\} \to \infty$. By Gronwall's Inequality,

$$\xi^m_{j,k}(x) \leq \epsilon^m_{j,k} \cdot \bar{M} \cdot (exp\left(\frac{1}{\bar{M}}x\right) - 1) \to 0 \;\text{ as }\; \min\{j,k\} \to \infty.$$

Therefore,

$$\left| \int_0^x u^{m,j} - \int_0^x u^{m,k} \right| \leq \xi^m_{j,k}(x) \to 0$$

as $\min\{j,k\} \to \infty$. By combining this with the assumption $(B1)$, we get

$$\left| \left(\int_0^x u^{m,j} + \lambda^m_j(x) \right) - \left(\int_0^x u^{m,k} + \lambda^m_k(x) \right) \right| \to 0 \text{ a.e.}$$

as $\min\{j,k\} \to \infty$. By the assumption $(B2)$, for fixed $x \in (0,1)$ and for some $i(x) = 1,2,3,\;\; \chi^{m,j}_i(x) = \chi^{m,k}_i(x) = 1$ for sufficiently large j,k and this implies

$$z_i\left(\int_0^x u^{m,j} + \lambda^m_j(x) \right) = z_i\left(\int_0^x u^{m,k} + \lambda^m_k(x) \right) \;\text{ and }\; \chi^{m,j}_\infty(x) = \chi^{m,k}_\infty(x) = 0.$$

Therefore, the right hand side of (1.17) converges to zero and thus $\bar{w}^{m,j}(x) - \bar{w}^{m,k}(x) \to 0$ for all $x \in \tilde{\Omega}$ as $\min\{j,k\} \to \infty$. Hence,

$$\lim_{j\to\infty} \int_0^x u^{m,j} =: U^m \;\text{ exists for all }\; x \in [0,1],$$

$$\lim_{j\to\infty} \bar{w}^{m,j} =: \bar{w}^m \;\text{ exists for all }\; x \in \tilde{\Omega}.$$

This implies that $u_x^{m,j}$ converges to the equivalence class $\hat{w}^m$ of $\bar{w}^m$ in L^1 as $j \to \infty$. Let $u_\star^m(x) := \int_0^x \hat{w}^m$. Then since

$$u_\star^m(1) = \int_0^1 \hat{w}^m = \lim_{j\to\infty} \int_0^1 u_x^{m,j} = \lim_{j\to\infty} (u^{m,j}(1) - u^{m,j}(0)) = 0,$$

$u_\star^m$ satisfies the Dirichlet boundary conditions. Moreover, since

$$u^{m,j}(x) = \int_0^x u_x^{m,j} \to \int_0^x \hat{w}^m = u_\star^m(x) \ \text{ in } \ C([0,1]) \text{ as } j \to \infty,$$

$U^m = \int_0^x u_\star^m$. Thus

$$u^{m,j} \to u_\star^m \ \text{ in } \ W^{1,p} \ \ textas \ \ j \to \infty, \ \ 1 \le p < \infty.$$

Since $u^{m,j}$, $u_x^{m,j}$ are uniformly bounded, $u_\star^m \in W_0^{1,\infty}$. Thus

$$u_x^{m,j} \to (u_\star^m)_x \quad \text{boundedly a.e.,}$$

Since $\sigma(u_x^{m,j}) - \int_0^x u^{m,j} \to \sigma((u_\star^m)_x) - \int_0^x u_\star^m$ boundedly a.e., by the assumption $(B1)$,

$$\lambda^m = \sigma((u_\star^m)_x) - \int_0^x u_\star^m \quad \text{a.e.} \tag{1.20}$$

Since $\sigma((u_\star^m)_x)$ lies in one of the three intervals $\bigcup_{\lambda\in[-r,\,r]} z_i(\lambda^m) \ \ i \in \{1,2,3\}$ a.e., we can choose the nice pointwise representatives $\bar{w}^{m,j}$ of $u_x^{m,j}$ such that (1.20) holds for the set $(0,1)$ except for the finitely many points which are the limits $(x_\star)_i^m$ of finitely many zeros $x_i^m(j)$, $i = 1, 2, \cdots, N$ of $u^{m,j}$ in $(P4)$. Hence, we can conclude that $(u^{m,j},\ v^{m,j})$ converges to an equilibrium state $(u_\star^m, 0)$ strongly in $W_0^{1,p} \times L^2$. This proves Lemma 1.3.

In the next lemmas, we will show that under the low initial energy, the assumptions

$(B1)$ and $(B2)$ are satisfied. The following Lemma shows that the convergence of mean elastic stress $\int_0^1 (\sigma(u_x^{m,j}) - \int_0^x u^{m,j})dx$ implies the convergence of elastic stress $\sigma(u_x^{m,j}) - \int_0^x u^{m,j}$.

Lemma 1.4 *Let $u^{m,j}$, $j \in \mathbb{N}$ be a solution of (1.2). Assume that*

$$\lim_{j\to\infty} \int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j} \right) dx =: \lambda^m \quad \textit{exists.}$$

Then

$$\sigma(u_x^{m,j}) - \int_0^x u^{m,j} \longrightarrow \lambda^m \quad a.e. \ \ as \ \ j \to \infty.$$

PROOF. Recall that

$$q_t^j = -\pi_a(\sigma(u_x^{m,j})) + \pi_a(\int_0^x (u^{m,j}))$$

from (1.9). Thus, the sufficient condition to our conclusion is when q_t^j goes to zero a.e., as $j \to \infty$. Define the following modification of the energy functional $E(t)$

$$\widetilde{E}(t) := \int_0^1 \left[W(u_x^j(x,t)) + \frac{1}{2}(u^j(x,t))^2 + p^j(x,t)\omega^j(x,t) \right] dx,$$

where $\omega^j(x,t) = \left(\frac{mj-t}{m}\right) q_t^{j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) q_t^j(x)$. Note that $\widetilde{E}(t)$ is uniformly bounded and moreover sufficiently small since the first two terms are the part of energy functional $E(t)$ and the third term is small since $p^j(x,t)$ is small enough by the estimate (a) of Lemma 1.2 and $\omega^j(x,t)$ are the interpolation functions of uniformly

bounded functions $q_t^j \ \ \forall j \in \mathbb{N}$. By equation (1.3),

$$\begin{aligned}
\frac{d}{dt}\widetilde{E}(t) &= \int_0^1 [\sigma(u_x^j)\cdot u_{xt}^j + u^j\cdot u_t^j + p_t^j\cdot\omega^j + p^j\cdot\omega_t^j]dx \\
&= \int_0^1 [\sigma(u_x^j)\cdot u_{xt}^j + u^j\cdot u_t^j + p_t^j\cdot q_t^j + p_t^j(\omega^j - q_t^j) + p^j\cdot\omega_t^j]dx \\
&= \int_0^1 [(\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j + (u^j - u^{m,j})\cdot u_t^j \\
&\quad -\sigma(u_x^{m,j})_x\cdot u_t^j + u^{m,j}\cdot u_t^j + u_{xt}^j\cdot q_t^j - (q_t^j)^2 \\
&\quad +p_t^j\left(\left(\frac{mj-t}{m}\right)q_t^{j-1} + \left(\frac{t-m(j-1)-m}{m}\right)q_t^j\right) + p^j\omega_t^j]dx \\
&\leq mL\|v_x^{m,j}\|_{L^2}^2 + (t-mj)\|v^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx \qquad (1.21)\\
&\quad + \int_0^1 [(v_{xx}^{m,j} - v_t^j)\cdot u_t^j + u_{xt}^j u_{xt}^j - u_{xt}^j p_t^j \\
&\qquad +p_t^j\left(\frac{t-mj}{m}\right)\cdot(q_t^j - q_t^{j-1}) + p^j\omega_t^j]dx \\
&\leq mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx \\
&\quad + \int_0^1 [-u_{xt}^j v_x^{m,j} - u_t^j v_t^j + u_{xt}^j u_{xt}^j + u_t^j p_{xt}^j \qquad (1.22)\\
&\qquad +p_t^j\cdot(t-mj)\cdot\omega_t^j + p^j\omega_t^j]dx \\
&= mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx + \int_0^1 \omega_t^j[(t-mj)\cdot p_t^j + p^j]dx \\
&= mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx + \int_0^1 \omega_t^j\left[(t-mj)\cdot\left(\frac{p^{m,j}-p^{m,j-1}}{m}\right)\right. \\
&\quad \left. + \left(\frac{mj-t}{m}\right)p^{m,j-1} + \left(\frac{t-m(j-1)}{m}\right)p^{m,j}\right]dx \\
&= mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx + 2\cdot\int_0^1 \omega_t^j\cdot p^j dx - \int_0^1 \omega_t^j\cdot p^{m,j}dx.
\end{aligned}$$

The first term of (1.21) follows from (1.7). (1.22) vanishes because of the identities

$u^j_{xt} = v^{m,j}_x$, $p^j_{xt} = v^j_t$ and the boundary conditions of system (1.1). Since

$$
\begin{aligned}
\left|\int_0^1 \omega^j_t \cdot p^{m,j} dx\right| &\le \|p^{m,j}\|_{L^2} \cdot \|\omega^j_t\|_{L^2} \\
&\le \|p^{m,j}_{xx}\|_{L^2} \cdot \|\omega^j_t\|_{L^2} \\
&= \|v^{m,j}_x\|_{L^2} \cdot \left\| \frac{q^j_t - q^{j-1}_t}{m} \right\|_{L^2} \\
&= \|v^{m,j}_x\|_{L^2} \cdot \left\| \pi_a \left(\frac{\sigma(u^{m,j}_x) - \sigma(u^{m,j-1}_x)}{m} \right) - \left(\int_0^x \frac{u^{m,j} - u^{m,j-1}}{m} \right) \right\|_{L^2} \\
&\le \|v^{m,j}_x\|_{L^2} \cdot \left(\left\| \pi_a \left(\sigma'(c) \cdot \frac{u^{m,j}_x - u^{m,j-1}_x}{m} \right) \right\|_{L^2} + \left\| \frac{u^{m,j}_x - u^{m,j-1}_x}{m} \right\|_{L^2} \right) \\
&\le M_3 \cdot \|v^{m,j}_x\|^2_{L^2},
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_0^1 \omega^j_t \cdot p^j dx\right| &\le \|p^j\|_{L^2} \cdot \|\omega^j_t\|_{L^2} \\
&\le \|p^j_{xx}\|_{L^2} \cdot \|\omega^j_t\|_{L^2} \\
&= \|v^j_x\|_{L^2} \cdot \|\omega^j_t\|_{L^2} \\
&\le M_4 \cdot (\|v^{m,j-1}_x\|_{L^2} + \|v^{m,j}_x\|_{L^2}) \cdot \|v^{m,j}_x\|_{L^2} \\
&\le M_4 \cdot (\|v^{m,j-1}_x\|_{L^2} \cdot \|v^{m,j}_x\|_{L^2} + \cdot \|v^{m,j}_x\|^2_{L^2})
\end{aligned}
$$

and

$$
\|v^{m,j-1}_x\|_{L^2} \cdot \|v^{m,j}_x\|_{L^2} \le M_5 \cdot (\|v^{m,j-1}_x\|^2_{L^2} + \|v^{m,j}_x\|^2_{L^2})
$$

for some M_3, M_4 and $M_5 > 0$, the following estimate on $\frac{d}{dt}\widetilde{E}(t)$ holds

$$
\frac{d}{dt}\widetilde{E}(t) \le mL\|v^{m,j}_x\|^2_{L^2} - \int_0^1 (q^j_t)^2 dx + M_6 \cdot (\|v^{m,j-1}_x\|^2_{L^2} + \|v^{m,j}_x\|^2_{L^2}) \qquad (1.23)
$$

for some $M_6 > 0$. By taking an integral from $(j-1)m$ to jm on both sides of inequality

(1.23), we obtain the following estimate

$$\begin{aligned}
\widetilde{E}(jm) - \widetilde{E}((j-1)m) &= \int_{(j-1)m}^{jm} \frac{d}{dt}\widetilde{E}(t) \\
&\leq -m\int_0^1 (q_t^j)^2 + (mL + M_6)m\|v_x^{m,j}\|_{L^2}^2 + mM_6\|v_x^{m,j-1}\|_{L^2}^2.
\end{aligned}$$

By taking the summation $j = 1, \cdots, S$, we get

$$\widetilde{E}(Sm) - \widetilde{E}(0) \leq -m\sum_{j=1}^{S}\int_0^1 (q_t^j)^2 + (mL + M_7)\sum_{j=1}^{S} m\|v_x^{m,j}\|_{L^2}^2 + mM_6\|(v_0)_x\|_{L^2}^2$$

for some $M_7 > 0$. By inequality (1.8),

$$\begin{aligned}
m\sum_{j=1}^{S}\int_0^1 (q_t^j)^2 &\leq \widetilde{E}(0) - \widetilde{E}(Sm) + 2(mL + M_7)\epsilon + M_6 m\|(v_0)_x\|_{L^2}^2 \\
&\leq |\widetilde{E}(0)| + |\widetilde{E}(Sm)| + 2(mL + M_7)\epsilon + \epsilon_1 \\
&\leq \delta
\end{aligned}$$

for some $\epsilon_1, \delta \ll 1$. Therefore, $m\sum_{j=1}^{\infty}\int_0^1 (q_t^j)^2 \leq \delta$ and this implies $q_t^j \to 0$ a.e. as $j \to \infty$ and this completes the proof of Lemma 1.4.

The next Lemma shows the convergence of the phase function under the assumptions of the convergence of mean elastic stress.

Lemma 1.5 *Let $u^{m,j}$ be the solution of (1.2). Assume*

$$\lim_{j\to\infty}\left|\int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right) dx\right| =: \lambda^m$$

exists. Then the assumption (B2) in Lemma 1.3 holds.

PROOF. By the estimates (b) and (e), mean elastic stress $\int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right) dx$ is sufficiently small. Then by Lemma 1.4, $\limsup_{j\to\infty}\left|\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right|$ is sufficiently small a.e. Note that these inequalities also hold for the interpolation functions $u^j(x,t)$

and their derivatives with respect to x for all $j \in \mathbb{N}$. Combining this and the estimate (b) of Lemma 1.2, $\limsup_{j\to\infty} |\sigma(u_x^j)| \leq \frac{2r}{3}$ a.e. This implies that for almost every x, there exists $J(x) \in \mathbb{N}$ such that

$$\{u_x^j(x,t) : j \geq J(x)\} \subseteq \sigma^{-1}([-r,\, r]) = \bigcup_{i=1}^{3} \bigcup_{\lambda\in[-r,\, r]} z_i(\lambda).$$

Since $\{u_x^j(x,t) : j \geq J(x)\}$ is connected, for all $j \geq J(x)$, $u_x^j(x,t) \in \bigcup_{\lambda\in[-r,\, r]} z_i(\lambda)$ for some $i(x) = 1, 2$, or 3 and this implies that $\lim_{j\to\infty} \varphi(u_x^j(x,t))$ exists and finite a.e. Consequently, $\lim_{j\to\infty} \varphi(u_x^{m,j}(x))$ exists and finite a.e.

The next lemma shows the convergence of mean elastic stress and this completes the proof of Proposition 1.1.

Lemma 1.6 *Let $u^{m,j}$ be the solution of (1.2). Then,*

$$\lim_{j\to\infty} \underbrace{\left[\int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right) dx\right]}_{=:\, c(j)} \quad \textit{exists.}$$

PROOF. Suppose this fails. Then there exists a subsequence $j_k \to \infty$ such that $c(j_k) \to \lambda^m$ and another subsequence $j_s \to \infty$ such that $c(j_s) \to \bar{\lambda}^m$ for some λ^m, $\bar{\lambda}^m \in [-\frac{2r}{3}, \frac{2r}{3}]$ and $\lambda^m < \bar{\lambda}^m$. Then by Lemma 1.4, $\sigma(u_x^{m,j_k}) - \int_0^x u^{m,j_k} \to \lambda^m$ a.e. as $j_k \to \infty$ and $\sigma(u_x^{m,j_s}) - \int_0^x u^{m,j_s} \to \bar{\lambda}^m$ a.e. as $j_s \to \infty$. Also by Lemma 1.5, $\lim_{j_k\to\infty} \varphi(u_x^{m,j_k})$, $\lim_{j_s\to\infty} \varphi(u_x^{m,j_s})$ exist and finite, respectively. Thus, these satisfy the assumptions $(B1)$, $(B2)$ of Lemma 1.3 and therefore, there exist u^m, $\bar{u}^m \in W^{1,\infty}$ such that

$$\sigma(u_x^m)_x - u^m \equiv \lambda^m \text{ a.e.,} \quad \sigma(\bar{u}_x^m)_x - \bar{u}^m \equiv \bar{\lambda}^m \text{ a.e.,}$$

$$\varphi(u_x^m)(x) = \lim_{j_k\to\infty} \varphi(u_x^{m,j_k}(x)) \text{ a.e.,} \quad \varphi(\bar{u}_x^m)(x) = \lim_{j_s\to\infty} \varphi(u_x^{m,j_s}(x)) \text{ a.e.}$$

Note that $\varphi(u_x^m)(x) = \varphi(\bar{u}_x^m)(x) =: \varphi_\infty(x)$ since the limit of the phase function is independent of λ^m and $\bar{\lambda}^m$.

Consider the case where $\varphi_\infty(x) \in \{1, 3\}$ a.e. That is, the measure of the set $\Omega_u := \{x \in (0,1) : \varphi_\infty(x) = 2\}$ is zero. Now we introduce the following principle whose proof was done in [16].

Comparison Principle for weak solutions of the ordinary differential equation $\sigma(u_x)_x = u$.

Assume that $u,\ \bar{u} \in W^{1,\infty}$ satisfy

$$\sigma(u_x)_x - u \equiv \lambda \text{ a.e.}, \quad \sigma(\bar{u}_x)_x - \bar{u} \equiv \bar{\lambda} \text{ a.e.},$$

$\lambda < \bar{\lambda}$, $u(0) = \bar{u}(0) = 0$, $\sigma(u_x)$ and $\sigma(\bar{u}_x) \in [-r, r]$ a.e., $\varphi(u_x) = \varphi(\bar{u}_x)$ a.e., $\varphi(u_x) \in \{1, 3\}$ a.e. Then $u(x) < \bar{u}(x)$ for all $x \in (0, 1]$.

Since $\lambda^m < \bar{\lambda}^m$, u^m, $\bar{u}^m$ satisfies the above comparison principle, therefore, $u^m(1) < \bar{u}^m(1)$. This contradicts to the boundary conditions of (1.1). In the case when the measure of Ω_u is not zero, contradiction arises from the following modified principle which was also proven in [16].

Refined Comparison Principle for weak solutions of the ordinary differential equation $\sigma(u_x)_x = u$.

Under the same assumptions as the Comparison Principle but the condition $\varphi(u_x) \in \{1, 3\}$ a.e. is replaced by $\int_0^1 W(u_x) < \epsilon_1$ and $|\Omega_u| \neq 0$, the following inequality

$$u(1) < \bar{u}(1)$$

holds.

Now the Proposition 1.1 is finally completed.

In the next step, we will show that the set of x where the transition layers are steep, is decreasing to the finitely many isolated points.

1.6 Dynamical behavior of the transition layers

If the set

$$\mathcal{L}_{\frac{\rho}{2}}(j) = \{x \in (0,1) : |u_x^j(x,t)| \leq \frac{\rho}{2}\}$$

is monotonically decreasing to the finitely many isolated points as j goes to infinity, we obtain the desired conclusion since this is equivalent to the fact that the layers are getting steeper and steeper as $j \to \infty$ and eventually become discontinuous. But unfortunately, the set $\mathcal{L}_{\frac{\rho}{2}}(j)$ is not always decreasing as $j \to \infty$. We define the following set $\widetilde{\mathcal{L}}(j)$ instead and show our set $\mathcal{L}_{\frac{\rho}{2}}(j)$ is contained in the newly defined set $\widetilde{\mathcal{L}}(j)$. We will show then the set $\widetilde{\mathcal{L}}(j)$ is decreasing to the finitely many isolated points. Let $\eta \in (0, \frac{\rho}{4})$. Set $\rho_0 := \rho - \eta$. Define

$$\widetilde{\mathcal{L}}(j) := \{x \in (0,1) : |q^j(x,t)| \leq \rho_0\}.$$

The following Lemma states that the set of transition layers are always in the set $\widetilde{\mathcal{L}}(j)$ and furthermore in the set of initial transition layers $\mathcal{L}_\rho(0)$. This Lemma plays an important role for showing the preservation of the number of transition layers.

Lemma 1.7

$$\mathcal{L}_{\frac{\rho}{2}}(j) \subseteq \widetilde{\mathcal{L}}(j) \subseteq \mathcal{L}_\rho(0) \quad \forall j \in \mathbb{N}.$$

PROOF. If $x \in \mathcal{L}_{\frac{\rho}{2}}(j)$, then $|u_x^j(x,t)| \leq \frac{\rho}{2}$. Therefore, by the estimate (a) of Lemma 1.2,

$$|q^j(x,t)| = |u_x^j - p^j| \leq \frac{\rho}{2} + \eta < \frac{\rho}{2} + \frac{\rho}{4} = \rho - \frac{\rho}{4} < \rho - \eta = \rho_0$$

which implies

$$|(u_0)_x| \leq |p^j(x,0)| + |q^j(x,0)| \leq \eta + \rho_0 = \rho.$$

This proves Lemma 1.7.

We show next the set $\widetilde{\mathcal{L}}(j)$ is exponentially decreasing to the finitely many isolated points.

Lemma 1.8 *For all $j \in \mathbb{N}$ and for some $C_0 > 0$, $C_0 \approx 1$,*

(i) $|q^j_x(x,t)| \geq C_0 e^{jm\sigma_0} |(q_0)_x|$ if $x \in \widetilde{\mathcal{L}}(j)$ (exponential growth),

(ii) $\widetilde{\mathcal{L}}(j+1) \subseteq \widetilde{\mathcal{L}}(j)$ (monotonicity).

PROOF. We will show (i) by induction. Fix $j \in \mathbb{N}$ and fix $x \in \widetilde{\mathcal{L}}(j)$. Then $x \in \mathcal{L}_\rho(0)$ by Lemma 1.7. By the hypothesis $(A4)$, $|(u_0)_{xx}(x)| \geq K$. Suppose $(u_0)_{xx}(x) \geq K$. Since $(p_0)_x(x)$ is bounded by the estimate (f) of Lemma 1.2, $(q_0)_x(x) = (u_0)_{xx}(x) - (p_0)_x(x) > 0$. By differentiating both sides of equation (1.9) with respect to x for $j = 1$, and by using the estimates (d), (f) of Lemma 1.2 and equation (1.14), we get the following estimate

$$\begin{aligned}
q^{m,1}_x(x) - q^{m,0}_x(x) &= \{-[\sigma(u^{m,1}_x(x))]_x + u^{m,1}(x)\}m \\
&= \{-\sigma'(u^{m,1}_x(x))(p^{m,1}_x(x) + q^{m,1}_x(x)) + u^{m,1}(x)\}m \\
&\geq -\sigma'(u^{m,1}_x(x))q^{m,1}_x(x)m - C_4K_3m - K_2m \\
&\geq -\sigma'(u^{m,1}_x(x))q^{m,1}_x(x)m - C_6m
\end{aligned}$$

for some $C_6 > 0$. Thus,

$$(1 + \sigma'(u^{m,1}_x(x))m)q^{m,1}_x(x) \geq q^{m,0}_x(x) - C_6m.$$

Since m is sufficiently small and $q_x^{m,0} = (q_0)_x > 0$, $q_x^{m,1}(x)$ is also positive. Therefore, the following inequality holds

$$(1-\sigma_0 m)q_x^{m,1}(x) \geq (1+\sigma'(u_x^{m,1}(x))m)q_x^{m,1}(x) \geq q_x^{m,0}(x) - C_6 m.$$

Recall that $\sigma_0 = \min\limits_{[-\rho,\rho]} |\sigma'|$. By induction, suppose $q_x^{m,j-1} > 0$. Then,

$$(1+\sigma'(u_x^{m,1}(x))m)q_x^{m,j}(x) \geq q_x^{m,j-1}(x) - C_6 m$$

which implies that $q_x^{m,j} > 0$ and the following inequality holds

$$(1-\sigma_0 m)q_x^{m,j}(x) \geq q_x^{m,j-1}(x) - C_6 m.$$

By iterating this, we get the following inequality

$$\begin{aligned}
q_x^{m,j} &\geq \frac{1}{1-\sigma_0 m}\cdot q_x^{m,j-1} - C_6 m \cdot \frac{1}{1-\sigma_0 m} \\
&\geq \frac{1}{1-\sigma_0 m}\cdot\left[\frac{1}{1-\sigma_0 m}\cdot q_x^{m,j-2} - C_6 m\cdot\frac{1}{1-\sigma_0 m}\right] - C_6 m\cdot\frac{1}{1-\sigma_0 m} \\
&= \frac{1}{(1-\sigma_0 m)^2}\cdot q_x^{m,j-2} - C_6 m\left[\frac{1}{1-\sigma_0 m} + \frac{1}{(1-\sigma_0 m)^2}\right] \\
&\ \ \cdots \\
&= \frac{1}{(1-\sigma_0 m)^j}\cdot (q_0)_x - C_6 m\left[\frac{1}{1-\sigma_0 m} + \cdots + \frac{1}{(1-\sigma_0 m)^j}\right] \\
&= \frac{1}{(1-\sigma_0 m)^j}\cdot (q_0)_x - C_6 m\left[\frac{\frac{1}{1-\sigma_0 m} - \frac{1}{(1-\sigma_0 m)^{j+1}}}{\frac{-\sigma_0 m}{1-\sigma_0 m}}\right] \\
&= \frac{1}{(1-\sigma_0 m)^j}\cdot (q_0)_x + \frac{1-\frac{1}{(1-\sigma_0 m)^j}}{\sigma_0}\cdot C_6 \\
&= \frac{1}{(1-\sigma_0 m)^j}\cdot\left((q_0)_x - \frac{C_6}{\sigma_0}\right) + \frac{C_6}{\sigma_0}.
\end{aligned}$$

This implies

$$q_x^{m,j} \geq e^{jm\sigma_0}\cdot (q_0)_x.$$

Similarly,

$$q_x^{m,j-1} \geq e^{(j-1)m\sigma_0} \cdot (q_0)_x.$$

Since m is sufficiently small, $e^{-m\sigma_0} \approx 1$. Therefore, we can establish the exponential growth of q_x^j, that is

$$|q_x^j(x,t)| \geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x|$$

for some $C_0 \approx 1$. Similarly, for the case $(u_0)_{xx}(x) \leq -K$, $q_x^{m,j} < 0$ for all $j \in \{0\} \bigcup \mathbb{N}$ and the following inequalities hold

$$(1-\sigma_0 m)q_x^{m,j}(x) \leq q_x^{m,j-1}(x) + C_6 m,$$

$$q_x^{m,j} \leq e^{jm\sigma_0} \cdot (q_0)_x, \quad q_x^{m,j-1} \leq e^{(j-1)m\sigma_0} \cdot (q_0)_x.$$

Hence, we get the same conclusion

$$|q_x^j(x,t)| \geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x|$$

and this proves (i) of Lemma 1.8.

Note that for $K > 4K_0$,

$$\begin{aligned} |q_x^j(x,t)| &\geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x| \\ &\geq C_0 e^{jm\sigma_0}(|(u_0)_{xx}| - K_0) \\ &\geq \frac{3}{4} K C_0 e^{jm\sigma_0}. \end{aligned} \tag{1.24}$$

Also, Notice that

$$|u_x^j| = |p^j + q^j| \leq \eta + \rho_0 = \rho \tag{1.25}$$

when $|q^j| \le \rho_0$. If $q^j = \rho_0$, then $u_x^j = p^j + q^j = p^j + \rho_0 \ge -\eta + \rho_0 > 0$ and if $q^j = -\rho_0$, then $u_x^j = p^j - \rho_0 \le \eta - \rho_0 < 0$, which implies $sign(u_x^j) = sign(q^j)$ at $|q^j| = \rho_0$. By using this and equation (1.9) and also by using the estimates (b), (e) of Lemma 1.2, we have at $|q^j| = \rho_0$ and for some $c_2^{m,j}$ between 0 and $u_x^{m,j}$,

$$\begin{aligned}
\frac{d}{dt}|q^j(x,t)| &= sign(q^j(x,t)) \cdot \left[\frac{q^{m,j}(x) - q^{m,j-1}(x)}{m}\right] \\
&= sign(u_x^j(x,t)) \cdot \left[\frac{q^{m,j}(x) - q^{m,j-1}(x)}{m}\right] \\
&= sign(u_x^j(x,t)) \cdot [\sigma(0) - \sigma(u_x^{m,j}(x)) \\
&\quad + \int_0^1 \sigma(u_x^{m,j})] + \pi_a(\int_0^x u^{m,j})] \\
&\ge -\sigma'(c_2^{m,j}) \cdot u_x^{m,j} \cdot sign(u_x^j) - \left|\int_0^1 \sigma(u_x^{m,j})\right| - \left\|\pi_a\left(\int_0^x u^{m,j}\right)\right\|_{L^\infty} \\
&\ge -\sigma'(c_2^{m,j}) \cdot u_x^j \cdot sign(u_x^j) - \sigma'(c_2^{m,j}) \cdot (u_x^{m,j} - u_x^j) \cdot sign(u_x^j) \\
&\quad - \left|\int_0^1 \sigma(u_x^{m,j})\right| - \left\|\pi_a(\int_0^x u^{m,j})\right\|_{L^\infty} \\
&\ge \sigma_0 \cdot |u_x^j| - \sigma'(c_2^{m,j}) \cdot sign(u_x^j) \cdot \frac{jm-t}{m}(u_x^{m,j} - u_x^{m,j-1}) - 2\sigma_0\eta \\
&\ge \sigma_0 \cdot (|q^j| - |p^j|) - \sigma'(c_2^{m,j}) \cdot sign(u_x^j) \cdot \frac{jm-t}{m}(u_x^{m,j} - u_x^{m,j-1}) - 2\sigma_0\eta \\
&\ge \sigma_0 \cdot (\rho_0 - \eta) - 2\sigma_0\eta - \sigma'(c_2^{m,j}) \cdot sign(u_x^j) \cdot \frac{jm-t}{m}(u_x^{m,j} - u_x^{m,j-1}) \\
&\ge \sigma_0 \cdot (\rho - 4\eta) - \sigma'(c_2^{m,j}) \cdot sign(u_x^j) \cdot \frac{jm-t}{m}(u_x^{m,j} - u_x^{m,j-1}). \qquad (1.26)
\end{aligned}$$

Note that $|\frac{jm-t}{m}| < 1$. By the estimate (a) of Lemma 1.2, $|p^{m,j} - p^{m,j-1}| \le 2\eta \ll 1$ and $|q^{m,j} - q^{m,j-1}| = m|q_t^j| \le mM_1 \ll 1$ by (1.16) and $m \ll 1$. Hence, $|u_x^{m,j} - u_x^{m,j-1}| \le |p^{m,j} - p^{m,j-1}| + |q^{m,j} - q^{m,j-1}| \ll 1$ and this enables the second term of equation (1.26) to be small. Now we can say

$$\frac{d}{dt}|q^j(x,t)| \ge 0$$

which implies $|q^j(x,t)| \le |q^{j+1}(x,t)|$ for all $j \in \mathbb{N}$ when $|q^j(x,t)| = \rho_0$. By (i), q^j is strictly increasing or decreasing on $\widetilde{\mathcal{L}}(j)$ which implies

$$\widetilde{\mathcal{L}}(j+1) \subseteq \widetilde{\mathcal{L}}(j).$$

Now (ii) is proved and this completes the proof of Lemma 1.8.

From the part (i) of Lemma 1.8, the estimate (f) of Lemma 1.2 and the hypothesis (A4), if $x \in \mathcal{L}_{\frac{\varrho}{2}}(j)$ and $K > \max\{4K_0, \frac{4K_0}{C_0}\}$,

$$\begin{aligned}
|u^j_{xx}(x,t)| &\geq |q^j_x(x,t)| - |p^j_x(x,t)| \\
&\geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x| - K_0 \\
&\geq C_0 e^{jm\sigma_0} \cdot (|(u_0)_{xx}| - |(p_0)_x|) - K_0 \\
&\geq C_0 e^{jm\sigma_0} \cdot (|(u_0)_{xx}| - K_0 - \frac{K_0}{C_0}) \\
&\geq C_0 e^{jm\sigma_0} \cdot (K - \frac{K}{4} - \frac{K}{4}) \\
&\geq \frac{1}{2} K C_0 e^{jm\sigma_0}.
\end{aligned} \tag{1.27}$$

From Lemma 1.7, inequality (1.27) and the fact that $\|u^{m,j}\|_{C^2} < \infty \ \forall j \in \mathbb{N}$, $\mathcal{L}_{\frac{\varrho}{2}}(j)$ has a finite number of components $[a^m_i(j), b^m_i(j)]$, $0 < a^m_1(j) < b^m_1(j) < \cdots < a^m_{N(j)}(j) < b^m_{N(j)}(j) < 1$, and in each of which, $u^j_x(x,t)$ is strictly monotone and has exactly one zero $x^m_i(j)$. Also, $N(j) \geq 1$ since $u^j(0,t) = u^j(1,t) = 0 \ \forall j \in \mathbb{N}$.

Lemma 1.9 $N(j) \equiv$ *const.* $\forall j \in \mathbb{N}$.

PROOF. For all $j \in \mathbb{N}$, define

$$g^j(x,t) := u^j_x(x,t), \quad (j-1)m < t \leq jm.$$

Since $g^j, g^j_x \in C((0,1)\times((j-1)m, jm])$ and at each zero $(x_0(j), t_0(j))$ of g^j, $|g^j_x(x,t)| \geq \frac{KC_0}{2} > 0$ by inequality (1.27), for each $t_0(j)$, $\{g^j(x_0(j), t_0(j)) | (x_0(j), t_0(j))$ is a zero of $g^j\}$ does not contain a critical value of $g^j(\cdot, t(j))$. By Implicit Function Theorem, the number of zeros of $g^j(\cdot, t)$ is independent of t for $(j-1)m < t \le jm$. Since this holds for all $j \in \mathbb{N}$, number of zeros of $g^{m,j}(x) := u_x^{m,j}(x)$ is independent of j which implies $N(j) \equiv const$. This proves Lemma 1.9. Similarly, by defining

$$g^j(x,t) := u_x^j(x,t) - \frac{\rho}{2}, \;\; g^j(x,t) := u_x^j(x,t) + \frac{\rho}{2}, \;\; (j-1)m < t \le jm,$$

the number of connected components of $\mathcal{L}_{\frac{\rho}{2}}(j)$ is independent of j.

Now, the proof of $(P1)$, $(P2)$ is completed.

From Lemma 1.7, $[a_i^m(j), b_i^m(j)] \subseteq [(a_0)_i, (b_0)_i]$, $i = 1, 2, \cdots, N$. Moreover,

$$\begin{aligned} \rho &= |u_x^j(b_i^m(j), t) - u_x^j(a_i^m(j), t)| \\ &= \int_{a_i^m(j)}^{b_i^m(j)} |u_{xx}^j| dx \\ &\ge \frac{1}{2} K C_0 e^{jm\sigma_0} \cdot |b_i^m(j) - a_i^m(j)|, \end{aligned}$$

which implies

$$|b_i^m(j) - a_i^m(j)| \le \frac{2\rho}{KC_0} \cdot e^{-jm\sigma_0} \quad \forall i = 1, 2, \cdots, N$$

for fixed j. This proves the last part of $(P3)$. The rest of $(P3)$ was already proved.

From (1.24) and from similar analysis as in the case $\mathcal{L}_{\frac{\rho}{2}}(j)$, $\widetilde{\mathcal{L}}(j)$ has a finite number of components $[\alpha_i^m(j), \beta_i^m(j)]$, $0 < \alpha_1^m(j) < \beta_1^m(j) < \cdots < \alpha_N^m(j) < \beta_N^m(j) < 1$. By Lemma 1.7, $x_i^m(j) \in [a_i^m(j), b_i^m(j)] \subseteq [\alpha_i^m(j), \beta_i^m(j)] \subseteq [a_i^0, b_i^0]$. By (ii) of Lemma 1.8, $[\alpha_i^m(j), \beta_i^m(j)] \subseteq [\alpha_i^m(j), \beta_i^m(j)]$. Therefore, the set of $[\alpha_i^m(j), \beta_i^m(j)]$ forms a nested family of intervals. Thus

$$\begin{aligned} 2\rho > 2\rho_0 &= |q^j(\beta_i^m(j),t) - q^j(\alpha_i^m(j),t)| \\ &= \int_{\alpha_i^m(j)}^{\beta_i^m(j)} |q_x^j| dx \\ &\geq \frac{3}{4} K C_0 |\beta_i^m(j) - \alpha_i^m(j)| \cdot e^{jm\sigma_0} \end{aligned}$$

which finishes the proof of $(P4)$.

$(P3)$ and $(P4)$ automatically imply that $(u_\star^m)_x$ is discontinuous at every $(x_\star)_i^m$. It remains to show that $(u_\star^m)_x$ is continuous on $(0,1)\backslash\{(x_\star)_1^m, \cdots, (x_\star)_N^m\}$. Since $u_\star^m$ is an equilibrium state, it satisfies the following equation

$$\sigma((u_\star^m)_x(x)) = \int_0^x (u_\star^m) + \lambda^m$$

for some constant $\lambda^m > 0$. We know that the first term on the right hand side of the above equation is small by the estimate (b) of Lemma 1.2. Furthermore, λ^m is sufficiently small on $(0,1)\backslash\{(x_\star)_1^m, \cdots, (x_\star)_N^m\}$ from the proof of Lemma 1.5. Therefore, $(u_\star^m)_x$, the inverse image of σ is continuous on those intervals which proves $(P5)$ and Theorem 1.1 is completed.

Remark. The result of transition layer dynamics works for the viscoelastic system without the elastic foundation term u, that is, for the system

$$u_{tt} - (\sigma(u_x) + u_{xt})_x = 0.$$

The proof is similar to the proof of the original system. Only the minor change of the proof of energy decay (Lemma 1.1), the estimate of (c) of Lemma 1.2, Section 1.5 and the estimate of $\frac{d}{dt}|q^j(x,t)|$ in Section 1.6 is needed.

CHAPTER 2

Convergence Analysis of Numerical Solutions in One-dimensional Systems

We obtained the theoretical results on the dynamics of the strain u_x in Chapter 1. Next we will derive the numerical results on the behavior of u_x by using the finite difference methods (FDM) and the finite element methods (FEM) using the linear, quadratic and Hermite cubic elements.

The nonlinear term $\sigma(u_x)_x$ is treated by the direct iteration method for both schemes. However, two types of problems arise. First, the steepness of transition layers affected by the term $\sigma(u_x)_x$ leads to the truncation error as time approaches infinity. Second, iteration in each time step is computationally expensive. The first case is treated in Section 2.3 by averaging the term $\sigma(u_x)_x$ and the error is greatly reduced. In the second case, the alternating direction implicit (ADI) method combined with the explicit method for the system is derived in Section 2.4 and it is shown that the computational cost is reduced.

Even though the computational cost is high, using a higher order element in the FEM can reduce the order of error in the space dimension. In our problem, due to the time derivative of a energy dissipation term u_{xxt}, the error for the quadratic element is of order 1 and is of order 3 for the Hermite cubic elements in the space direction. However, the fact that the time limit of the solution u which is only in $C^0(\Omega)$ and its derivative has a singularity prohibits the improvement of the error for the higher elements. The numerical results presented in Section 2.5 shows that there is not much difference between the three types of elements.

In the first two sections of this chapter, we derive the matrix equation for the one-dimensional systems which is obtained from the FDM using the standard second order central difference approximation for $\sigma(u_x)_x$ and prove the convergence of the solution.

2.1 Derivation of the matrix equation using the Finite Difference Methods

Let m, j be fixed. Recall that the governing equation for the discretized solution is

$$\frac{1}{m^2}(u^{m,j} - 2u^{m,j-1} + u^{m,j-2}) = (\sigma(u_x^{m,j}))_x + \frac{1}{m}(u_x^{m,j} - u_x^{m,j-1})_x - u^{m,j}. \tag{2.1}$$

Note that $W, \sigma = W', \sigma'$ are all bounded for small $u_x^{m,j}$. Divide the interval $(0,1)$ into n subintervals (x_{k-1}, x_k) with length $\frac{1}{n}$, $k = 1, \cdots, n$, $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. By using the central difference scheme, the following approximation will be used

$$
\begin{aligned}
u_x^{m,j}(x_k) &\approx \frac{u^{m,j}(x_{k+1}) - u^{m,j}(x_{k-1})}{2h}, \\
u_{xx}^{m,j}(x_k) &\approx \frac{u^{m,j}(x_{k+1}) - 2u^{m,j}(x_k) + u^{m,j}(x_{k-1})}{h^2}, \\
\sigma(u_x^{m,j}(x_k))_x &= \sigma'(u_x^{m,j}(x_k)) \cdot u_{xx}^{m,j}(x_k) \\
&\approx \sigma'\left(\frac{u^{m,j}(x_{k+1}) - u^{m,j}(x_{k-1})}{2h}\right) \frac{u^{m,j}(x_{k+1}) - 2u^{m,j}(x_k) + u^{m,j}(x_{k-1})}{h^2}
\end{aligned} \tag{2.2}
$$

for $k = 1, \cdots, n-1$, where $h = \dfrac{1}{n}$. By substituting these equations for the terms in equation (2.1) and multiplying by m^2 on both sides, we get the following equation

$$
\begin{aligned}
& (1+m^2) \cdot u^{m,j}(x_k) - \frac{m}{h^2} \cdot (u^{m,j}(x_{k+1}) - 2u^{m,j}(x_k) + u^{m,j}(x_{k-1})) \\
& -\frac{m^2}{h^2} \cdot \sigma'\left(\frac{u^{m,j}(x_{k+1}) - u^{m,j}(x_{k-1})}{2h}\right) \cdot (u^{m,j}(x_{k+1}) - 2u^{m,j}(x_k) + u^{m,j}(x_{k-1})) \\
= & -\frac{m}{h^2} \cdot (u^{m,j-1}(x_{k+1}) - 2u^{m,j-1}(x_k) + u^{m,j-1}(x_{k-1})) + 2u^{m,j-1}(x_k) - u^{m,j-2}(x_k).
\end{aligned}
$$

After arranging the above equation in terms of $u^{m,j}(x_{k-1})$, $u^{m,j}(x_k)$ and $u^{m,j}(x_{k+1})$, we get

$$
\begin{aligned}
& A(u^{m,j}(x_k)) \cdot u^{m,j}(x_{k-1}) + B(u^{m,j}(x_k)) \cdot u^{m,j}(x_k) + A(u^{m,j}(x_k)) \cdot u^{m,j}(x_{k+1}) \\
= \;& F(u^{m,j-1}(x_k)),
\end{aligned} \tag{2.3}
$$

for $k = 1, \cdots, n-1$ where

$$A(u^{m,j}(x_k)) := -\frac{m}{h^2} - \frac{m^2}{h^2} \cdot \sigma' \left(\frac{u^{m,j}(x_{k+1}) - u^{m,j}(x_{k-1})}{2h} \right), \tag{2.4}$$

$$B(u^{m,j}(x_k)) := 1 + m^2 + \frac{2m}{h^2} + \frac{2m^2}{h^2} \cdot \sigma' \left(\frac{u^{m,j}(x_{k+1}) - u^{m,j}(x_{k-1})}{2h} \right), \tag{2.5}$$

$$\begin{aligned} F(u^{m,j-1}(x_k)) := &-\frac{m}{h^2} \cdot (u^{m,j-1}(x_{k-1}) + u^{m,j-1}(x_{k+1})) \\ &+ \left(2 + \frac{2m}{h^2}\right) \cdot u^{m,j-1}(x_k) - u^{m,j-2}(x_k) \end{aligned} \tag{2.6}$$

for $k = 2, \cdots, n-2$. Note that for $k = 1, n-1$,

$$\begin{aligned} A(u^{m,j}(x_1)) &= -\frac{m}{h^2} - \frac{m^2}{h^2} \cdot \sigma' \left(\frac{u^{m,j}(x_2)}{2h} \right), \\ A(u^{m,j}(x_{n-1})) &= -\frac{m}{h^2} - \frac{m^2}{h^2} \cdot \sigma' \left(\frac{-u^{m,j}(x_{n-2})}{2h} \right) \end{aligned}$$

since $u^{m,j}(x_0) = u^{m,j}(x_n) = 0$ for all $j \in \mathbb{N}$. The similarities hold for $B(u^{m,j}(x_1)), B(u^{m,j}(x_{n-1})), F(u^{m,j-1}(x_1))$ and $F(u^{m,j-1}(x_{n-1}))$. Note also that the definitions of $A(u^{m,j}(x_k)), B(u^{m,j}(x_k))$ indicate they depend on $u^{m,j}(x_{k-1})$, $u^{m,j}(x_{k+1})$ and $F(u^{m,j-1}(x_k))$ depends on $u^{m,j-1}(x_{k-1})$, $u^{m,j-1}(x_k)$, $u^{m,j-1}(x_{k+1})$ and $u^{m,j-2}(x_k)$. Equation (2.3) now becomes the following matrix equation

$$\mathbb{C}(u^{m,j})\{u^{m,j}\} = \{\mathbb{F}(u^{m,j-1})\},$$

where $\mathbb{C}(u^{m,j}) = \{C_{l,s}(u^{m,j}(x_k))\}_{(n-1)\times(n-1)}$ is the tri-diagonal matrix

$$\begin{pmatrix} B(u^{m,j}(x_1)) & A(u^{m,j}(x_1)) & 0 & & \cdots & 0 \\ A(u^{m,j}(x_2)) & B(u^{m,j}(x_2)) & A(u^{m,j}(x_2)) & 0 & \cdots & 0 \\ 0 & A(u^{m,j}(x_3)) & B(u^{m,j}(x_3)) & A(u^{m,j}(x_3)) & \cdots & 0 \\ & & \cdots & & & \\ & & & \cdots & & \\ 0 & \cdots & A(u^{m,j}(x_{n-3})) & B(u^{m,j}(x_{n-3})) & A(u^{m,j}(x_{n-3})) & 0 \\ 0 & \cdots & 0 & A(u^{m,j}(x_{n-2})) & B(u^{m,j}(x_{n-2})) & A(u^{m,j}(x_{n-2})) \\ 0 & \cdots & & 0 & A(u^{m,j}(x_{n-1})) & B(u^{m,j}(x_{n-1})) \end{pmatrix}$$

and

$$\begin{aligned} \{u^{m,j}\} &= \{u^{m,j}(x_1), \cdots, u^{m,j}(x_{n-1})\}^T, \\ \{\mathbb{F}(u^{m,j-1})\} &= \{F(u^{m,j-1}(x_1), \cdots, F(u^{m,j-1}(x_{n-1})\}^T. \end{aligned}$$

Note that the matrix $\mathbb{C}(u^{m,j})$ is nonsymmetric.

The matrix $\mathbb{C}(u^{m,j})$ involves the nonlinear term σ' which is the function of the unknown present time solution $u^{m,j}$. Thus in each time step $j \in \mathbb{N}$, we assume that $\mathbb{C}(u_i^{m,j}) = \mathbb{C}(u_{i-1}^{m,j})$, $u_0^{m,j} = u^{m,j-1}$ for $i \in \mathbb{N}$ which allows the use of an iterative solution method to compute $u^{m,j}$. We stop when the norm of the difference of $u_i^{m,j}$ and $u_{i-1}^{m,j}$ is sufficiently small. This method is called the direct iteration method.

2.2 Existence of the Finite Difference solution

We now prove the existence of the solution of the matrix equation which we obtained from the previous section.

Theorem 2.1 *The solution of the matrix equation*

$$\mathbb{C}(u^{m,j})\{u^{m,j}\} = \{\mathbb{F}(u^{m,j-1})\},$$

exists.

PROOF. Let $j \in \mathbb{N}$ be fixed. Consider the following iteration

$$\mathbb{C}(u_{i-1}^{m,j})\{u_i^{m,j}\} = \{\mathbb{F}(u^{m,j-1})\} \quad \forall i \in \mathbb{N},$$

where $u_0^{m,j} = u^{m,j-1}$. By subtracting the equation $\mathbb{C}(u_{i-1}^{m,j})\{u_i^{m,j}\} = \{\mathbb{F}(u^{m,j-1})\}$ from $\mathbb{C}(u_i^{m,j})\{u_{i+1}^{m,j}\} = \{\mathbb{F}(u^{m,j-1})\}$, we get

$$\mathbb{C}(u_i^{m,j})\{u_{i+1}^{m,j} - u_i^{m,j}\} = (\mathbb{C}(u_{i-1}^{m,j}) - \mathbb{C}(u_i^{m,j}))\{u_i^{m,j}\}. \tag{2.7}$$

The k^{th} element of the right hand side of the above equation is

$$\sum_{s=1}^{n-1}\{C_{k,s}(u_{i-1}^{m,j}(x_k)) - C_{k,s}(u_i^{m,j}(x_k))\} \cdot u_i^{m,j}(x_s).$$

By the Mean Value Theorem, it becomes

$$\sum_{s=1}^{n-1}\sum_{l=1}^{n-1}[C_{k,s}(u_{i_*}^{m,j}(x_k))]_{,X_l} \cdot (u_{i-1}^{m,j}(x_l) - u_i^{m,j}(x_l)) \cdot u_i^{m,j}(x_s),$$

where the value $u_{i_*}^{m,j}(x_k)$ is in between $u_{i-1}^{m,j}(x_k)$ and $u_i^{m,j}(x_k)$ and $[C_{k,s}(u_{i_*}^{m,j}(x_k))]_{,X_l}$ represents the partial derivative of $C_{k,s}(u_{i_*}^{m,j}(x_k))$ with respect to $u_{i_*}^{m,j}(x_l)$. Now, the right hand side of equation (2.7) becomes

$$\begin{aligned} & \sum_{l=1}^{n-1}\sum_{s=1}^{n-1} u_i^{m,j}(x_s) \cdot [C_{k,s}(u_{i_*}^{m,j}(x_k))]_{,X_l} \cdot (u_{i-1}^{m,j}(x_l) - u_i^{m,j}(x_l)) \\ = \;& \mathbb{D}(u_{i_*}^{m,j})\{u_{i-1}^{m,j} - u_i^{m,j}\}, \end{aligned}$$

where $\mathbb{D}(u_{i_*}^{m,j})$ is the following matrix

$$\begin{pmatrix} \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{1,s}(u_{i_*}^{m,j}(x_1))]_{,X_1} & \cdot\cdot & \sum_{s=1}^{n-1} u_i^{m,j}(x_s))[C_{1,s}(u_{i_*}^{m,j}(x_1))]_{,X_{n-1}} \\ \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{2,s}(u_{i_*}^{m,j}(x_2))]_{,X_1} & \cdot\cdot & \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{2,s}(u_{i_*}^{m,j}(x_2))]_{,X_{n-1}} \\ \cdots & \cdot\cdot & \cdots \\ \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{n-2,s}(u_{i_*}^{m,j}(x_{n-2}))]_{,X_1} & \cdot\cdot & \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{n-2,s}(u_{i_*}^{m,j}(x_{n-2}))]_{,X_{n-1}} \\ \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{n-1,s}(u_{i_*}^{m,j}(x_{n-1}))]_{,X_1} & \cdot\cdot & \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{n-1,s}(u_{i_*}^{m,j}(x_{n-1}))]_{,X_{n-1}} \end{pmatrix}.$$

Since $C_{1,s}(u_{i_*}^{m,j}(x_1)),\ s = 1, \cdots, n-1$ only contains $u_{i_*}^{m,j}(x_2)$,

$$D_{1,l}(u_{i_*}^{m,j}) = \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{1,s}(u_{i_*}^{m,j}(x_1))]_{,X_l} = 0$$

if $l \neq 2$. When $l = 2$,

$$\begin{aligned} D_{1,2}(u_{i_*}^{m,j}) &= \sum_{s=1}^{n-1} u_i^{m,j}(x_s)[C_{1,s}(u_{i_*}^{m,j}(x_1))]_{,X_2} \\ &= u_i^{m,j}(x_1)[C_{1,1}(u_{i_*}^{m,j}(x_1))]_{,X_2} + u_i^{m,j}(x_2)[C_{1,2}(u_{i_*}^{m,j}(x_1))]_{,X_2} \\ &= \frac{m^2}{h^3} \cdot \sigma'' \left(\frac{u_{i_*}^{m,j}(x_1)}{2h} \right) \cdot u_i^{m,j}(x_1) - \frac{m^2}{2h^3} \cdot \sigma'' \left(\frac{u_{i_*}^{m,j}(x_1)}{2h} \right) \cdot u_i^{m,j}(x_2) \\ &\leq h \left(\frac{m}{h^2} \right)^2 |\sigma''| \cdot \left(|u_i^{m,j}(x_1)| + \frac{1}{2} |u_i^{m,j}(x_2)| \right). \end{aligned} \tag{2.8}$$

Let $r = \frac{m}{h^2}$. Since m and h^2 are both small, we can make r bounded by letting m be sufficiently small. The first equality comes from the equations (2.4), (2.5) which are the components of the matrix $\mathbb{C}(u_{i_*}^{m,j})$. Here, h is sufficiently small and $u^{m,j}$ is bounded from the a-priori estimate (d) of Lemma 1.2. Moreover, σ'' is bounded and these enable (2.8) to be bounded by a small number. Therefore, all the elements in the first row of the matrix $\mathbb{D}(u_{i_*}^{m,j})$ are zero except for the second element and the second element is sufficiently small. Similarly, $C_{n-1,s}(u_{i_*}^{m,j}(x_{n-1}))$ only contains $u_{i_*}^{m,j}(x_{n-2})$, $s=1,\cdots,n-1$, which implies

$$D_{n-1,l}(u_{i_*}^{m,j})=0$$

if $l\neq n-2$. When $l=n-2$,

$$\begin{aligned}
D_{n-1,n-2}(u_{i_*}^{m,j}) &= u_i^{m,j}(x_{n-2})[C_{n-1,n-2}(u_{i_*}^{m,j}(x_{n-1}))]_{,X_{n-2}} \\
&\quad +u_i^{m,j}(x_{n-1})[C_{n-1,n-1}(u_{i_*}^{m,j}(x_{n-1}))]_{,X_{n-2}} \\
&= \frac{m^2}{2h^3}\cdot\sigma''\left(\frac{-u_{i_*}^{m,j}(x_{n-2})}{2h}\right)\cdot u_i^{m,j}(x_{n-2}) \\
&\quad -\frac{m^2}{h^3}\cdot\sigma''\left(\frac{-u_{i_*}^{m,j}(x_{n-2})}{2h}\right)\cdot u_i^{m,j}(x_{n-1}) \\
&\leq hr^2|\sigma''|\left(|u_i^{m,j}(x_{n-1})|+\frac{1}{2}|u_i^{m,j}(x_{n-2})|\right) \\
&\leq \epsilon_1
\end{aligned}$$

for some $\epsilon_1\ll 1$. Thus all the elements in the $(n-1)^{th}$ row of the matrix $\mathbb{D}(u_{i_*}^{m,j})$ are zero except for the $(n-2)^{th}$ element and the $(n-2)^{th}$ element is bounded by a small number.

Since only $u_{i_*}^{m,j}(x_{k-1})$ and $u_{i_*}^{m,j}(x_{k+1})$ are in the $C_{k,s}(u_{i_*}^{m,j}(x_k))$, $k\neq 1,n-1$, $s=1,\cdots,n-1$,

$$D_{k,l}(u_{i_*}^{m,j})=0$$

if $l \neq k-1,\ k+1$. When $l = k-1$,

$$
\begin{aligned}
D_{k,k-1}(u_{i_*}^{m,j}) &= u_i^{m,j}(x_{k-1})[C_{k,k-1}(u_{i_*}^{m,j}(x_k))]_{,X_{k-1}} + u_i^{m,j}(x_k)[C_{k,k}(u_{i_*}^{m,j}(x_k))]_{,X_{k-1}} \\
&\quad + u_i^{m,j}(x_{k+1})[C_{k,k+1}(u_{i_*}^{m,j}(x_k))]_{,X_{k-1}} \\
&= \frac{m^2}{2h^3} \cdot \sigma'' \left(\frac{u_{i_*}^{m,j}(x_{k+1}) - u_{i_*}^{m,j}(x_{k-1})}{2h} \right) \cdot (u_i^{m,j}(x_{k-1}) + u_i^{m,j}(x_{k+1})) \\
&\quad - \frac{m^2}{h^3} \cdot \sigma'' \left(\frac{u_{i_*}^{m,j}(x_{k+1}) - u_{i_*}^{m,j}(x_{k-1})}{2h} \right) \cdot u_i^{m,j}(x_k) \\
&\leq hr^2 |\sigma''| \left(\frac{1}{2} |u_i^{m,j}(x_{k-1})| + |u_i^{m,j}(x_k)| + \frac{1}{2} |u_i^{m,j}(x_{k+1})| \right) \\
&\leq \epsilon_2
\end{aligned}
$$

for some $\epsilon_2 \ll 1$. Similarly,

$$
\begin{aligned}
D_{k,k+1}(u_{i_*}^{m,j}) &= -\frac{m^2}{2h^3} \cdot \sigma'' \left(\frac{u_{i_*}^{m,j}(x_{k+1}) - u_{i_*}^{m,j}(x_{k-1})}{2h} \right) \cdot (u_i^{m,j}(x_{k-1}) + u_i^{m,j}(x_{k+1})) \\
&\quad + \frac{m^2}{h^3} \cdot \sigma'' \left(\frac{u_{i_*}^{m,j}(x_{k+1}) - u_{i_*}^{m,j}(x_{k-1})}{2h} \right) \cdot u_i^{m,j}(x_k) \\
&\leq \epsilon_2.
\end{aligned}
$$

Therefore, the matrix $\mathbb{D}(u_{i_*}^{m,j})$ is of the following form

$$
\begin{pmatrix}
0 & D_{1,2}(u_{i_*}^{m,j}) & 0 & & \cdots & 0 \\
D_{2,1}(u_{i_*}^{m,j}) & 0 & D_{2,3}(u_{i_*}^{m,j}) & 0 & \cdots & 0 \\
0 & D_{3,2}(u_{i_*}^{m,j}) & 0 & & \cdots & 0 \\
 & & \cdots & & & \\
 & & & \cdots & & \\
0 & \cdots & & 0 & D_{n-3,n-2}(u_{i_*}^{m,j}) & 0 \\
0 & \cdots & 0 & D_{n-2,n-3}(u_{i_*}^{m,j}) & 0 & D_{n-2,n-1}(u_{i_*}^{m,j}) \\
0 & \cdots & & 0 & D_{n-1,n-2}(u_{i_*}^{m,j}) & 0
\end{pmatrix},
$$

where the nonzero elements are sufficiently small. Thus, the norm of the matrix $\mathbb{D}(u_{i_*}^{m,j})$, which is defined by $\sup_{x\in\mathbb{R}^{n-1}} \|\mathbb{D}(u_{i_*}^{m,j})x\|_{L^2}$, is small and bounded.

Now, it remains to show that $\mathbb{C}(u_i^{m,j})$ is invertible and bounded away from zero. Consider the matrix $\mathbb{C}(u_i^{m,j})$ as the sum of three matrices $\mathbb{J}, \mathbb{K}$ and $\mathbb{L}(u_i^{m,j})$, where $\mathbb{J}, \mathbb{K}$ are the following matrices, respectively

$$\begin{pmatrix} 1+m^2 & 0 & & \cdots & 0 \\ 0 & 1+m^2 & 0 & \cdots & 0 \\ & \cdots & & & \\ & & \cdots & & \\ & & & \cdots & \\ 0 & \cdots & 0 & 1+m^2 & 0 \\ 0 & \cdots & & 0 & 1+m^2 \end{pmatrix}, \begin{pmatrix} 2r & -r & 0 & & \cdots & 0 \\ -r & 2r & -r & 0 & \cdots & 0 \\ & \cdots & & & & \\ & & \cdots & & & \\ & & & & \cdots & \\ 0 & \cdots & 0 & -r & 2r & -r \\ 0 & \cdots & & 0 & -r & 2r \end{pmatrix}$$

and $\mathbb{L}(u_i^{m,j})$ is the following tri-diagonal matrix

$$m \cdot r \cdot \begin{pmatrix} 2L(x_1) & -L(x_1) & 0 & & \cdots & 0 \\ -L(x_2) & 2L(x_2) & -L(x_2) & 0 & \cdots & 0 \\ & \cdots & & & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & & & & \cdots & \\ 0 & \cdots & 0 & -L(x_{n-2}) & 2L(x_{n-2}) & -L(x_{n-2}) \\ 0 & \cdots & & 0 & -L(x_{n-1}) & 2L(x_{n-1}) \end{pmatrix},$$

where $L(x_k) := \sigma'\left(\frac{u^{m,j}(x_{k+1}) - u^{m,j}(x_{k-1})}{2h}\right)$.

The matrix $\mathbb{K}$ is positive definite since eigenvalues of the matrix are positive by the following inequality

$$|\lambda_i - k_{ii}| \leq \sum_{j \neq i} |k_{ij}|,$$

where $k_{ii} = 2r$ and $\sum_{j \neq i} |k_{ij}| \geq 2r$.

The matrix $\mathbb{C}(u_i^{m,j})$ is positive definite and bounded away from zero since

$$\begin{aligned}
\xi^T \cdot \mathbb{C}(u_i^{m,j}) \cdot \xi &= (1+m^2) \cdot |\xi|^2 + \xi^T \cdot \mathbb{K} \cdot \xi + \xi^T \cdot \mathbb{L}(u_i^{m,j}) \cdot \xi \\
&\geq (1+m^2) \cdot |\xi|^2 - 3m \cdot r \cdot max|\sigma'| \cdot |\xi|^2 \\
&= (1+m^2 - 3m \cdot r \cdot max|\sigma'|)|\xi|^2 \\
&> \frac{1}{2}|\xi|^2 \\
&> 0.
\end{aligned}$$

The proof of Theorem 2.1 is now complete.

The following pictures are some examples of the dynamical behavior of the solution u and the strain u_x for the polynomial function $u_1(x,0) = 10x^4 - 21x^3 + 13.4x^2 - 2.4x$ and the sine function $u_2(x,0) = \frac{1}{60}\sin(10\pi x(x+1))$.

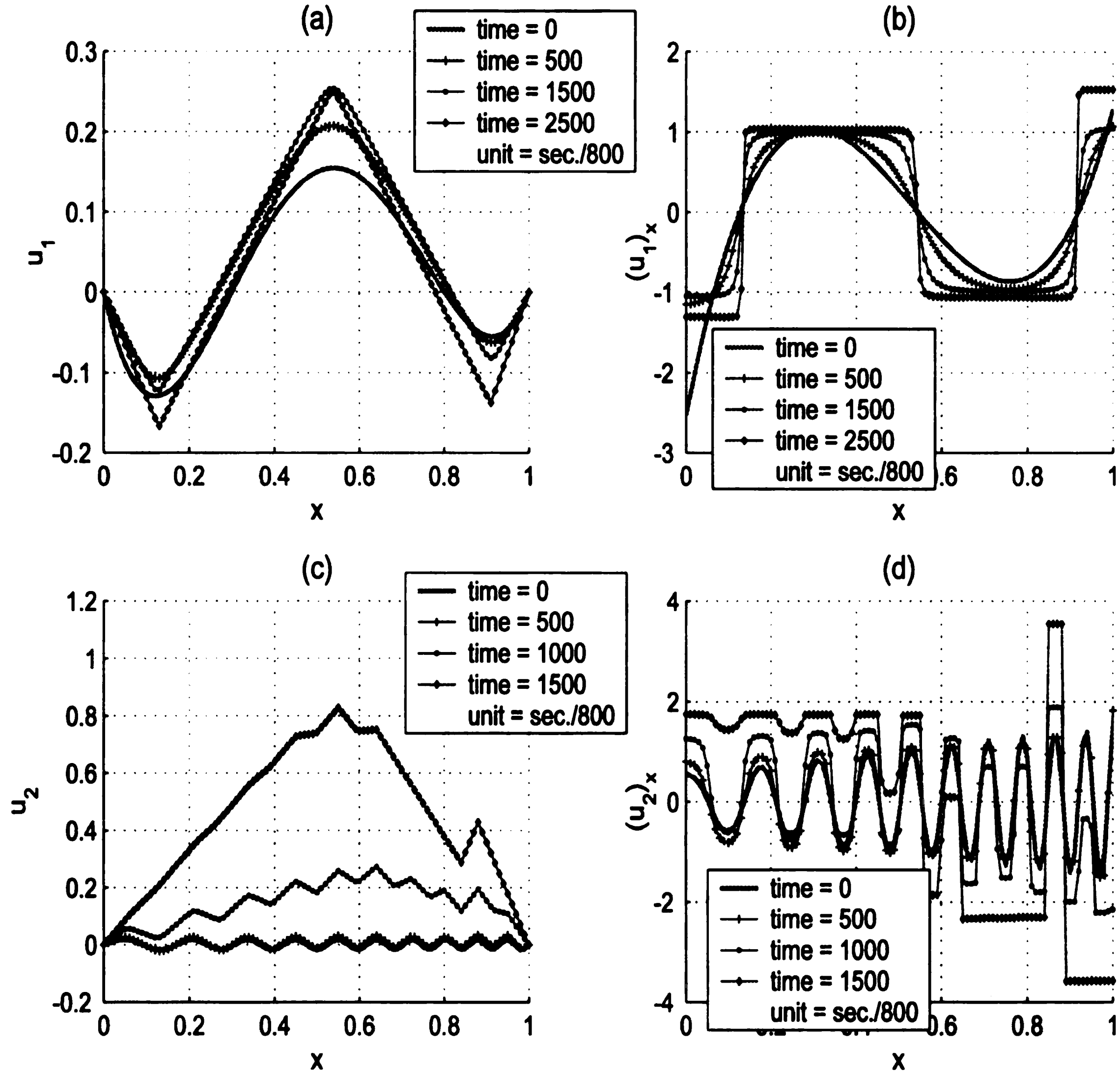

Figure 2.1. Transition Layer Dynamics for (a) $u_1(x,0) = 10x^4 - 21x^3 + 13.4x^2 - 2.4x$, (b) $(u_1)_x(x,0)$ (c) $u_2(x,0) = \frac{1}{60}\sin(10\pi x(x+1))$ and (d) $(u_2)_x(x,0)$ using the FDM.

We can see the steepness behavior of the transition layers from Figure 2.1. From (b) and (d) of Figure 2.1., we can also see that on the compliment of transition layers, the graphs are decreasing or increasing near 1 or -1 until 1500 time steps and this is because of the energy decay which is proven in Lemma 1.1 in Chapter 1. Note that the energy functional $E(t)$ has minima near 1 and -1 since the stored energy function

$W(u_x^{m,j})$ has minima at 1 and -1. The other integrands of $E(t)$ are negligible since they are sufficiently small.

The graph of $u_x^{m,j}$ except for the finitely many zeros is not exactly approaching 1 or -1 and the reason is the following. Recall that by Proposition 1.1, the time limit $u_\star^m$ of the discretized solution $u^{m,j}$ is in the equilibrium state and satisfies the following equation

$$\sigma((u_\star^m)_x)_x - u_\star^m = 0 \text{ a.e.}$$

Thus, this indicates that there is a neighborhood such that $(u_\star^m)_x \neq 1$. For the viscoelastic system without the elastic foundation term $u^{m,j}$, the absolute value of graph is approaching 1 and it is also clear since $(u_\star^m)_x$ satisfies

$$\sigma((u_\star^m)_x)_x = 0 \text{ a.e.}$$

At 2500 time steps in (b), the graph moves above and below 1 and -1. This type of behavior is even worse in the sine function (d). It blows up even after 1000 time steps. This is because of the truncation error of the numerical solution due to the effect of the nonlinear term $\sigma(u_x^{m,j})_x$ on the transition layers. We improve the error in the next section by averaging of the term $\sigma(u_x^{m,j})_x$ in the FDM algorithm.

2.3 Average approximation of $\sigma(u_x^{m,j})_x$

When the time is sufficiently large, the approximation of $\sigma(u_x^{m,j})_x$ (2.2) produces a significant error since the difference between $u^{m,j}(x_{k+1})$ and $u^{m,j}(x_{k-1})$ can be very large near the position of transition layers. Since $\sigma(u_x^{m,j})_x = (\widetilde{\sigma}(u_x^{m,j}) \cdot u_x^{m,j})_x$, the nonlinear coefficient $\widetilde{\sigma}(u_x^{m,j})$ of the Laplacian $u_{xx}^{m,j}$ leads to the nonsymmetry of the matrix $\mathbb{C}(u^{m,j})$ in Section 2.1. Therefore, we modify (2.2) by using the average approximation instead of the central difference approximation to recover the symmetry of $\mathbb{C}(u^{m,j})$.

Let $x_{k+\frac{1}{2}}$ be the point in the middle of x_k and x_{k+1}. Similarly, let $x_{k-\frac{1}{2}}$ be the point in the middle of x_{k-1} and x_k. Then $u_x^{m,j}(x_{k+\frac{1}{2}}) \approx \frac{u^{m,j}(k+1)-u^{m,j}(k)}{h}$, $u_x^{m,j}(x_{k-\frac{1}{2}}) \approx \frac{u^{m,j}(k)-u^{m,j}(k-1)}{h}$ and the following average approximation holds

$$
\begin{aligned}
\sigma(u_x^{m,j}(x_k))_x &= (\widetilde{\sigma}(u_x^{m,j}(x_k)) \cdot u_x^{m,j}(x_k))_x \\
&\approx \frac{\widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}})) \cdot u_x^{m,j}(x_{k+\frac{1}{2}}) - \widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}})) \cdot u_x^{m,j}(x_{k-\frac{1}{2}})}{h} \\
&\approx \frac{\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}}))}{h^2} \cdot u^{m,j}(x_{k-1}) \\
&\quad - \frac{(\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}})) + \widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}})))}{h^2} \cdot u^{m,j}(x_k) \\
&\quad + \frac{\widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}}))}{h^2} \cdot u^{m,j}(x_{k+1}),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}})) &\approx \frac{\widetilde{\sigma}(u_x^{m,j}(x_{k-1})) + \widetilde{\sigma}(u_x^{m,j}(x_k))}{2} \\
\widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}})) &\approx \frac{\widetilde{\sigma}(u_x^{m,j}(x_k)) + \widetilde{\sigma}(u_x^{m,j}(x_{k+1}))}{2}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\widetilde{A}(u^{m,j}(x_k)) \cdot u^{m,j}(x_{k-1}) + \widetilde{B}(u^{m,j}(x_k)) \cdot u^{m,j}(x_k) + \widetilde{A}(u^{m,j}(x_{k+1})) \cdot u^{m,j}(x_{k+1}) \\
= \;& F(u^{m,j-1}(x_k)),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{A}(u^{m,j}(x_k)) &:= -\frac{m}{h^2} - \frac{m^2}{h^2} \cdot \widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}})), \\
\widetilde{B}(u^{m,j}(x_k)) &:= 1 + m^2 + \frac{2m}{h^2} + \frac{m^2}{h^2} \cdot [\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}})) + \widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}}))]
\end{aligned}
$$

and $F(u^{m,j-1}(x_k))$ is the same as (2.6) for $k = 1, \cdots, n-1$. We now derive the following modified matrix equation for (2.1)

$$\widetilde{\mathbb{C}}(u^{m,j})\{u^{m,j}\} = \{\mathbb{F}(u^{m,j-1})\},$$

where $\widetilde{\mathbb{C}}(u^{m,j}) = \{\widetilde{C}_{l,s}(u^{m,j}(x_k))\}_{(n-1)\times(n-1)}$ is the following symmetric tri-diagonal matrix

$$\begin{pmatrix}
\widetilde{B}(u^{m,j}(x_1)) & \widetilde{A}(u^{m,j}(x_2)) & 0 & & \cdots & 0 \\
\widetilde{A}(u^{m,j}(x_2)) & \widetilde{B}(u^{m,j}(x_2)) & \widetilde{A}(u^{m,j}(x_3)) & 0 & \cdots & 0 \\
0 & \widetilde{A}(u^{m,j}(x_3)) & \widetilde{B}(u^{m,j}(x_3)) & \widetilde{A}(u^{m,j}(x_4)) & \cdots & 0 \\
 & & \cdots & & & \\
 & & & \cdots & & \\
0 & \cdots & \widetilde{A}(u^{m,j}(x_{n-3})) & \widetilde{B}(u^{m,j}(x_{n-3})) & \widetilde{A}(u^{m,j}(x_{n-2})) & 0 \\
0 & \cdots & 0 & \widetilde{A}(u^{m,j}(x_{n-2})) & \widetilde{B}(u^{m,j}(x_{n-2})) & \widetilde{A}(u^{m,j}(x_{n-1})) \\
0 & \cdots & & 0 & \widetilde{A}(u^{m,j}(x_{n-1})) & \widetilde{B}(u^{m,j}(x_{n-1}))
\end{pmatrix}.$$

We compare the standard central difference scheme to the average approximation in the next pictures.

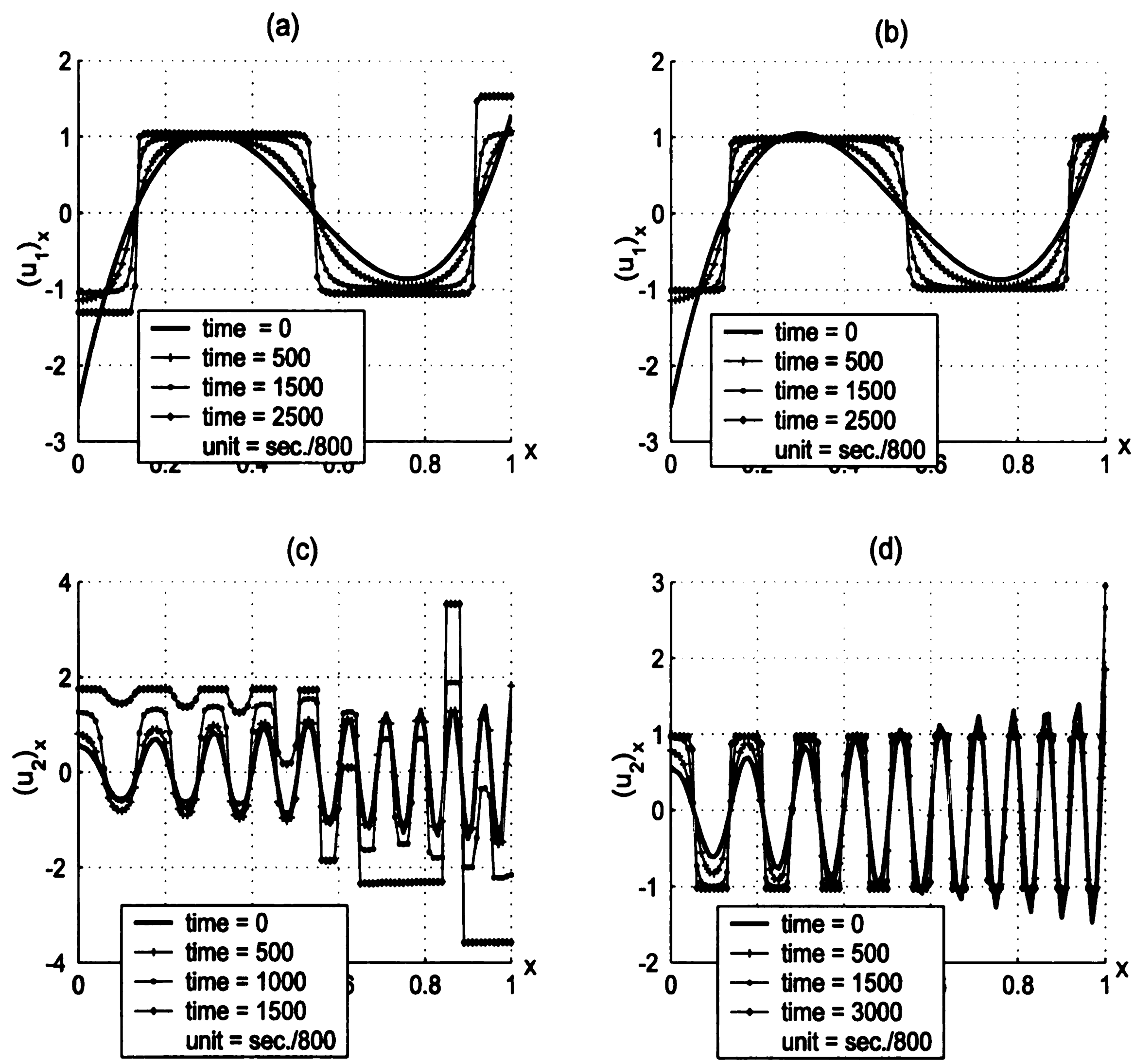

Figure 2.2. Dynamics using the central difference scheme for (a) $u_1(x,0) = 10x^4 - 21x^3 + 13.4x^2 - 2.4x$, (c) $u_2(x,0) = \frac{1}{60}\sin(10\pi x(x+1))$, the average approximation for (b) $u_1(x,0)$ and (d) $u_2(x,0)$.

At 2500 time steps for the polynomial function (a), the central difference approximation produces an error and it starts blowing up. However, the graph (b) for the same function using the average approximation still converges near 1 and -1. We can see from (c) and (d) that the average approximation is more efficient in the sine function. Unlike (c), the absolute value of the graph converges near 1 in (d) even after 3000 time steps.

Note that the matrix $\widetilde{\mathbb{C}}(u^{m,j})$ still involves the current time step which increases the computation time. In the next section, we introduce a new noniterative method which saves the computation cost.

2.4 The Alternating Direction Implicit (ADI) Method

While it is common to use the explicit methods to the hyperbolic equations, it is useful to apply the ADI method since it can reduce the order of error and improve the stability. In [19, Chapter 8], the improved ADI method which is combined with the explicit procedure was introduced for the following second-order wave equation

$$\frac{\partial^2 u}{\partial t^2} + \bar{A}u = f, \quad x \in \Omega,\ t > 0,$$

where $\bar{A} = \sum_{i=1}^{l} \bar{A}_i$, l is the dimension of x. Given $z^0, \cdots, z^{j-1}$, the approximated solution z^j at the time t^j is given by the following explicit formula

$$\frac{z^{j,0} - 2z^{j-1} + z^{j-2}}{m^2} + \bar{A}z^{j-1} = f^{j-1} \tag{2.9}$$

and the implicit stepping

$$\begin{aligned} \frac{z^{j,\kappa} - z^{j,\kappa-1}}{m^2} + \alpha\bar{A}_\kappa(z^{j,\kappa} - 2z^{j-1} + z^{j-2}) &= 0, \ \ \kappa = 1, \cdots, l, \\ z^j &= z^{j,l}, \end{aligned} \tag{2.10}$$

where $\alpha \in [.25, .5]$. By applying this ADI procedure to our system, we get

$$\begin{aligned} u^{m,\star}(x_k) &= \widetilde{u}^{m,j-1}(x_k) - \widehat{A}(u^{m,j-1}(x_k)) \cdot u^{m,j-1}(x_{k-1}) \\ &\quad -(\widehat{B}(u^{m,j-1}(x_k) + m^2) \cdot u^{m,j-1}(x_k) \\ &\quad -\widehat{A}(u^{m,j-1}(x_{k+1})) \cdot u^{m,j-1}(x_{k+1}) + \widehat{F}(u^{m,j-1}(x_k)), \end{aligned}$$

$$\begin{aligned} &\left(\alpha\widehat{A}(u^{m,j-1}(x_k)) - \frac{m}{h^2}\right) \cdot u^{m,j}(x_{k-1}) + \left(1 + \alpha\widehat{B}(u^{m,j-1}(x_k)) + \frac{2m}{h^2}\right) \cdot u^{m,j}(x_k) \\ &+ \left(\alpha\widehat{A}(u^{m,j-1}(x_{k+1})) - \frac{m}{h^2}\right) \cdot u^{m,j}(x_{k+1}) \\ = \;& u^{m,\star}(x_k) + \alpha\widehat{A}(u^{m,j-1}(x_k)) \cdot \widetilde{u}^{m,j-1}(x_{k-1}) + \alpha\widehat{B}(u^{m,j-1}(x_k)) \cdot \widetilde{u}^{m,j-1}(x_k) \\ &+\alpha\widehat{A}(u^{m,j-1}(x_{k+1})) \cdot \widetilde{u}^{m,j-1}(x_{k+1}), \end{aligned}$$

where

$$\begin{aligned} \widetilde{u}^{m,j-1}(x_k) &:= 2u^{m,j-1}(x_k) - u^{m,j-2}(x_k), \\ \widehat{F}(u^{m,j-1}(x_k)) &:= F(u^{m,j-1}(x_k)) - \widetilde{u}^{m,j-1}(x_k), \\ \widehat{A}(u^{m,j-1}(x_k)) &:= -\frac{m^2}{h^2} \cdot \widetilde{\sigma}(u_x^{m,j-1}(x_{k-\frac{1}{2}})), \\ \widehat{B}(u^{m,j-1}(x_k)) &:= \frac{m^2}{h^2} \cdot [\widetilde{\sigma}(u_x^{m,j-1}(x_{k-\frac{1}{2}})) + \widetilde{\sigma}(u_x^{m,j-1}(x_{k+\frac{1}{2}}))] \end{aligned}$$

for $k = 1, \cdots, n-1$. We can compare the direct iteration method to the ADI method from the following table that shows the computation time.

Methods	$t = 500$	$t = 1500$	$t = 2500$	$t = 3500$
Direct Iteration Method	24.11	71.52	119.03	168.29
ADI Method	18.07	52.21	90.46	127.27

Table 2.1. CPU time (sec.), unit of t = sec./800, $h = \frac{1}{100}$, $m = \frac{1}{800}$.

Table 2.1 shows that the computation time in the ADI method is faster than the Direct iteration Method.

2.5 Derivation of the matrix equation using the Finite Element Methods

Recall that the discretized solution $u^{m,j}$ satisfies equation (1.2) in Chapter 1. Since we consider the Dirichlet boundary conditions $u^{m,j}(0) = u^{m,j}(1) = 0$, these become the essential boundary conditions for constructing the weak formulation for the system.

Multiply equation (1.2) by a test function $w \in W_0^{1,\infty}(\Omega)$, where $\Omega = (0,1)$ and integrate over the finite element $(x_e,\ x_{e+1})$ with the length h. Then we get the following expression

$$\int_{x_e}^{x_{e+1}} (wv_t^j - w\sigma(u_x^{m,j})_x + wu^{m,j} - wv_{xx}^{h,j})dx = 0$$

and this implies

$$\int_{x_e}^{x_{e+1}} (wv_t^j + w_x\sigma(u_x^{m,j}) + wu^{m,j} + w_x v_x^{m,j})dx = 0 \tag{2.11}$$

for all $j \in \mathbb{N}$. The boundary terms after integration by parts are zero since the test function w satisfies essential boundary conditions.

Let $u^{m,j}(x) := \sum_{s=1}^{n_e} u_s^{m,j}\psi_s(x)$, $\quad w(x) := \psi_k(x)$, $k = 1, \cdots, n_e$. Here, $u_s^{m,j}$, $s = 1, \cdots, n_e$ are undetermined constants and $\psi_s(x)$ are the interpolation functions. Then equation (2.11) becomes

$$\sum_{s=1}^{n_e}\left[\int_0^h \psi_k(\bar{x})\psi_s(\bar{x})d\bar{x}\right](\ddot{u}_s^{m,j}+u_s^{m,j})+\sum_{s=1}^{n_e}\left[\int_0^h \frac{d\psi_k(\bar{x})}{d\bar{x}}\frac{d\psi_s(\bar{x})}{d\bar{x}}d\bar{x}\right]\dot{u}_s^{m,j}$$
$$+\int_0^h \frac{d\psi_k(\bar{x})}{d\bar{x}}\cdot\sigma\left(\sum_{j=1}^{n_e}u_s^{m,j}\frac{d\psi_s(\bar{x})}{d\bar{x}}\right)d\bar{x}=0, \tag{2.12}$$

where $\dot{u}_s^{m,j} \approx \dfrac{u_s^{m,j}-u_s^{m,j-1}}{m}$, $\ddot{u}_s^{m,j} \approx \dfrac{u_s^{m,j}-2u_s^{m,j-1}+u_s^{m,j-2}}{m^2}$. Define the (k,s) components of $n_e \times n_e$ matrices $\mathbb{A}^e$, $\mathbb{B}^e$ as follows

$$\{\mathbb{A}^e\}_{k,\,s} := \int_0^h \psi_k(\bar{x})\psi_s(\bar{x})d\bar{x},\quad \{\mathbb{B}^e\}_{k,\,s} := \int_0^h \frac{d\psi_k(\bar{x})}{d\bar{x}}\frac{d\psi_s(\bar{x})}{d\bar{x}}d\bar{x}$$

and let the k^{th} component of $n_e \times 1$ vector $\mathbb{F}^e(u^{m,j})$ be the following

$$\{\mathbb{F}^e(u^{m,j})\}_k := \int_0^h \frac{d\psi_k(\bar{x})}{d\bar{x}}\cdot\sigma\left(\sum_{j=1}^{n_e}u_s^{m,j}\frac{d\psi_s(\bar{x})}{d\bar{x}}\right)d\bar{x}.$$

After multiplying the both sides of (2.12) by m^2 and moving the previous time solutions to the right hand side, we get the following matrix equation

$$\begin{aligned}[(1+m^2)\mathbb{A}^e+m\mathbb{B}^e]\{u^{m,j}\} &= \mathbb{A}^e(2\{u^{m,j-1}\}-\{u^{m,j-2}\})+m\mathbb{B}^e\{u^{m,j-1}\}\\ &\quad -m^2\mathbb{F}^e(u^{m,j})\end{aligned} \tag{2.13}$$

on each finite element $(x_e,\ x_{e+1})$. It is known that for the linear elements, $\mathbb{A}^e$ and $\mathbb{B}^e$ are the following matrices

$$\mathbb{A}^e=\frac{h}{6}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix},\quad \mathbb{B}^e=\frac{1}{h}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

For the quadratic elements, the following are the matrices A^e and B^e

$$\mathbb{A}^e = \frac{h}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}, \quad \mathbb{B}^e = \frac{1}{3h}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}.$$

For the hermit cubic elements,

$$\mathbb{A}^e = \begin{pmatrix} \frac{13h}{35} & -\frac{11h^2}{210} & \frac{9h}{70} & \frac{13h^2}{420} \\ -\frac{11h^2}{210} & \frac{h^3}{105} & -\frac{13h^2}{420} & -\frac{h^3}{140} \\ \frac{9h}{70} & -\frac{13h^2}{420} & \frac{13h}{35} & \frac{11h^2}{210} \\ \frac{13h^2}{420} & -\frac{h^3}{140} & \frac{11h^2}{210} & \frac{h^3}{105} \end{pmatrix}, \quad \mathbb{B}^e = \begin{pmatrix} \frac{6}{5h} & -\frac{1}{10} & -\frac{6}{5h} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2h}{15} & \frac{1}{10} & -\frac{h}{30} \\ -\frac{6}{5h} & \frac{1}{10} & \frac{6}{5h} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{h}{30} & \frac{1}{10} & \frac{2h}{15} \end{pmatrix}.$$

For n subintervals (x_{k-1}, x_k) of $(0,\ 1)$ with length $h = \frac{1}{n}$, $k = 1, \cdots, n$, $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ we get the following global matrix equation of $n-1$ by $n-1$ for our system after assembling the local element equations (2.13).

$$[(1+m^2)\mathbb{A} + m\mathbb{B}]\{u^{m,j}\} = \mathbb{A}\{2u^{m,j-1} - u^{m,j-2}\} + m\mathbb{B}\{u^{m,j-1}\} - m^2\mathbb{F}(u^{m,j}), \tag{2.14}$$

where $\{u^{m,j}\} := \{u_1^{m,j}, \cdots, u_{n-1}^{m,j}\}^T$ and the similarity holds for $\{u^{m,j-1}\}$, $\{u^{m,j-2}\}$.

$\mathbb{A}$, $\mathbb{B}$ are the following global stiffness matrices of $(n-1) \times (n-1)$

$$\mathbb{A} = \frac{h}{6} \begin{pmatrix} 4 & 1 & 0 & & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ & & & \cdots & & \\ 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & \cdots & & 0 & 1 & 4 \end{pmatrix}, \quad \mathbb{B} = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ & & & \cdots & & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{pmatrix}$$

for the linear elements. Define $u_0^{m,j} = u_n^{m,j} = 0$. Then $\mathbb{F}(u^{m,j})$ is the following vector,

$$\{F_1(u^{m,j}), \cdots, F_{n-1}(u^{m,j})\}^T,$$

where

$$\begin{aligned} F_s(u^{m,j}) \quad :=& \quad \int_0^h \frac{d\psi_2(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{s-1}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_s^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} \right) d\bar{x} \\ & + \int_0^h \frac{d\psi_1(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_s^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{s+1}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} \right) d\bar{x} \\ =& \quad \sigma \left(\frac{u_s^{m,j} - u_{s-1}^{m,j}}{h} \right) - \sigma \left(\frac{u_{s+1}^{m,j} - u_s^{m,j}}{h} \right) \end{aligned}$$

for $s = 1, \cdots, n-1$ since $\frac{d\psi_1(\bar{x})}{d\bar{x}} = -\frac{1}{h}$, $\frac{d\psi_2(\bar{x})}{d\bar{x}} = \frac{1}{h}$. For the quadratic elements, $\mathbb{A}$, $\mathbb{B}$

are the following, respectively.

$$\mathbb{A} = \frac{h}{30}\begin{pmatrix} 16 & 2 & 0 & & & & & \cdots & & & 0 \\ 2 & 8 & 2 & -1 & 0 & & & \cdots & & & 0 \\ 0 & 2 & 16 & 2 & 0 & & & \cdots & & & 0 \\ 0 & -1 & 2 & 8 & 2 & -1 & 0 & & \cdots & & 0 \\ 0 & 0 & 0 & 2 & 16 & 2 & 0 & & \cdots & & 0 \\ & & & & & \cdots & & & & & \\ 0 & & \cdots & & 0 & 2 & 16 & 2 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & -1 & 2 & 8 & 2 & -1 & 0 \\ 0 & & & \cdots & & & 0 & 2 & 16 & 2 & 0 \\ 0 & & & \cdots & & & 0 & -1 & 2 & 8 & 2 \\ 0 & & & \cdots & & & & & 0 & 2 & 16 \end{pmatrix},$$

$$\mathbb{B} = \frac{1}{3h}\begin{pmatrix} 16 & -8 & 0 & & & & & \cdots & & & 0 \\ -8 & 14 & -8 & 1 & 0 & & & \cdots & & & 0 \\ 0 & -8 & 16 & -8 & 0 & & & \cdots & & & 0 \\ 0 & 1 & -8 & 14 & -8 & 1 & 0 & & \cdots & & 0 \\ 0 & 0 & 0 & -8 & 16 & -8 & 0 & & \cdots & & 0 \\ & & & & & \cdots & & & & & \\ 0 & & \cdots & & 0 & -8 & 16 & -8 & 0 & 0 & 0 \\ 0 & & \cdots & & 0 & 1 & -8 & 14 & -8 & 1 & 0 \\ 0 & & & \cdots & & & 0 & -8 & 16 & -8 & 0 \\ 0 & & & \cdots & & & 0 & 1 & -8 & 14 & -8 \\ 0 & & & \cdots & & & & & 0 & -8 & 16 \end{pmatrix}.$$

The vector $\mathbb{F}(u^{m,j})$ has the following components

$$
\begin{aligned}
F_s(u^{m,j}) \quad := \quad & \int_0^h \frac{d\psi_2(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{s-1}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_s^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} + u_{s+1}^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& \text{if } \; s = 2k-1, \;\; k = 1, \cdots, \frac{n}{2},
\end{aligned}
$$

$$
\begin{aligned}
F_s(u^{m,j}) \quad := \quad & \int_0^h \frac{d\psi_3(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{s-2}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{s-1}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} + u_s^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& + \int_0^h \frac{d\psi_1(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_s^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{s+1}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} + u_{s+2}^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& \text{if } \; s = 2k, \;\; k = 1, \cdots, \frac{n-2}{2}.
\end{aligned}
$$

For the Hermite cubic elements, let $u_{2k-1}^{m,j} = u_x^{m,j}(x_{k-1}), \; k = 1, \cdots, n, \quad u_{2k} = u^{m,j}(x_k), \; k = 1, \cdots, n-1, \quad u_{2n}^{m,j} = u_x^{m,j}(x_n)$. Then we get the $2n \times 2n$ matrix equation (2.14) where

$$
\mathbb{A} = \left(\begin{array}{ccccccccccc}
\frac{h^3}{105} & -\frac{13h^2}{420} & -\frac{h^3}{140} & 0 & & & \cdots & & & & 0 \\
-\frac{13h^2}{420} & \frac{26h}{35} & 0 & \frac{9h}{70} & \frac{13h^2}{420} & 0 & & & \cdots & & 0 \\
-\frac{h^3}{140} & 0 & \frac{2h^3}{105} & -\frac{13h^2}{420} & -\frac{h^3}{140} & 0 & & & \cdots & & 0 \\
0 & \frac{9h}{70} & -\frac{13h^2}{420} & \frac{26h}{35} & 0 & \frac{9h}{70} & \frac{13h^2}{420} & 0 & \cdots & & 0 \\
0 & \frac{13h^2}{420} & -\frac{h^3}{140} & 0 & \frac{2h^3}{105} & -\frac{13h^2}{420} & -\frac{h^3}{140} & 0 & \cdots & & 0 \\
 & & & & \cdots & & & & & & \\
0 & \cdots & 0 & \frac{9h}{70} & -\frac{13h^2}{420} & \frac{26h}{35} & 0 & \frac{9h}{70} & \frac{13h^2}{420} & 0 & \\
0 & \cdots & 0 & \frac{13h^2}{420} & -\frac{h^3}{140} & 0 & \frac{2h^3}{105} & -\frac{13h^2}{420} & -\frac{h^3}{140} & 0 & \\
0 & & \cdots & & 0 & \frac{9h}{70} & -\frac{13h^2}{420} & \frac{26h}{35} & 0 & \frac{13h^2}{420} & \\
0 & & \cdots & & 0 & \frac{13h^2}{420} & -\frac{h^3}{140} & 0 & \frac{2h^3}{105} & -\frac{h^3}{140} & \\
0 & & & \cdots & & & 0 & \frac{13h^2}{420} & -\frac{h^3}{140} & \frac{h^3}{105} &
\end{array} \right),
$$

$$
\mathbb{B} = \begin{pmatrix}
\frac{2h}{15} & \frac{1}{10} & -\frac{h}{30} & 0 & & & \cdots & & & 0 \\
\frac{1}{10} & \frac{12}{5h} & 0 & -\frac{6}{5h} & -\frac{1}{10} & 0 & & \cdots & & 0 \\
-\frac{h}{30} & 0 & \frac{4h}{15} & \frac{1}{10} & -\frac{h}{30} & 0 & & \cdots & & 0 \\
0 & -\frac{6}{5h} & \frac{1}{10} & \frac{12}{5h} & 0 & -\frac{6}{5h} & -\frac{1}{10} & 0 & \cdots & 0 \\
0 & -\frac{1}{10} & -\frac{h}{30} & 0 & \frac{4h}{15} & \frac{1}{10} & -\frac{h}{30} & 0 & \cdots & 0 \\
 & & & & \cdots & & & & & \\
0 & \cdots & 0 & -\frac{6}{5h} & \frac{1}{10} & \frac{12}{5h} & 0 & -\frac{6}{5h} & -\frac{1}{10} & 0 \\
0 & \cdots & 0 & -\frac{1}{10} & -\frac{h}{30} & 0 & \frac{4h}{15} & \frac{1}{10} & -\frac{h}{30} & 0 \\
0 & & \cdots & & 0 & -\frac{6}{5h} & \frac{1}{10} & \frac{12}{5h} & 0 & -\frac{1}{10} \\
0 & & \cdots & & 0 & -\frac{1}{10} & -\frac{h}{30} & 0 & \frac{4h}{15} & -\frac{h}{30} \\
0 & & & \cdots & & & 0 & -\frac{1}{10} & -\frac{h}{30} & \frac{2h}{15}
\end{pmatrix}
$$

and

$$
F_1(u^{m,j}) := \int_0^h \frac{d\psi_2(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_1^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} + u_2^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} + u_3^{m,j} \frac{d\psi_4(\bar{x})}{d\bar{x}} \right) d\bar{x},
$$

$$
F_{2n}(u^{m,j}) := \int_0^h \frac{d\psi_4(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{2n-2}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{2n-1}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} + u_{2n}^{m,j} \frac{d\psi_4(\bar{x})}{d\bar{x}} \right) d\bar{x},
$$

$$
\begin{aligned}
F_s(u^{m,j}) := & \int_0^h \frac{d\psi_4(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{s-3}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{s-2}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} \right. \\
& \left. + u_{s-1}^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} + u_s^{m,j} \frac{d\psi_4(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& + \int_0^h \frac{d\psi_2(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{s-1}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_s^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} \right. \\
& \left. + u_{s+1}^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} + u_{s+2}^{m,j} \frac{d\psi_4(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& \text{if } s = 2k+1, \quad k = 1, \cdots, n-1,
\end{aligned}
$$

$$
\begin{aligned}
F_s(u^{m,j}) \quad := \quad & \int_0^h \frac{d\psi_3(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_{s-2}^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{s-1}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} \right. \\
& \left. + u_s^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} + u_{s+1}^{m,j} \frac{d\psi_4(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& + \int_0^h \frac{d\psi_1(\bar{x})}{d\bar{x}} \cdot \sigma \left(u_s^{m,j} \frac{d\psi_1(\bar{x})}{d\bar{x}} + u_{s+1}^{m,j} \frac{d\psi_2(\bar{x})}{d\bar{x}} \right. \\
& \left. + u_{s+2}^{m,j} \frac{d\psi_3(\bar{x})}{d\bar{x}} + u_{s+3}^{m,j} \frac{d\psi_4(\bar{x})}{d\bar{x}} \right) d\bar{x} \\
& \text{if } s = 2k, \quad k = 1, \cdots, n-1.
\end{aligned}
$$

As in the case of FDM, the difficulty arises for solving the matrix equation (2.14) since it involves the vector $\mathbb{F}(u^{m,j})$ containing the nonlinear terms, but again the numerical solution is obtained using the direct iteration method by treating the term $u^{m,j}$ which is inside the nonlinear term σ as the known solution in the previous time step.

The truncation error is reduced in the FEM case since the numerical integration of $F_s(u^{m,j})$ has an effect of averaging the nonlinear term $\sigma(u_x^{m,j})_x$. The following pictures are the examples using the FEM with the linear, quadratic and Hermite cubic elements and using the FDM without averaging the nonlinear term, respectively.

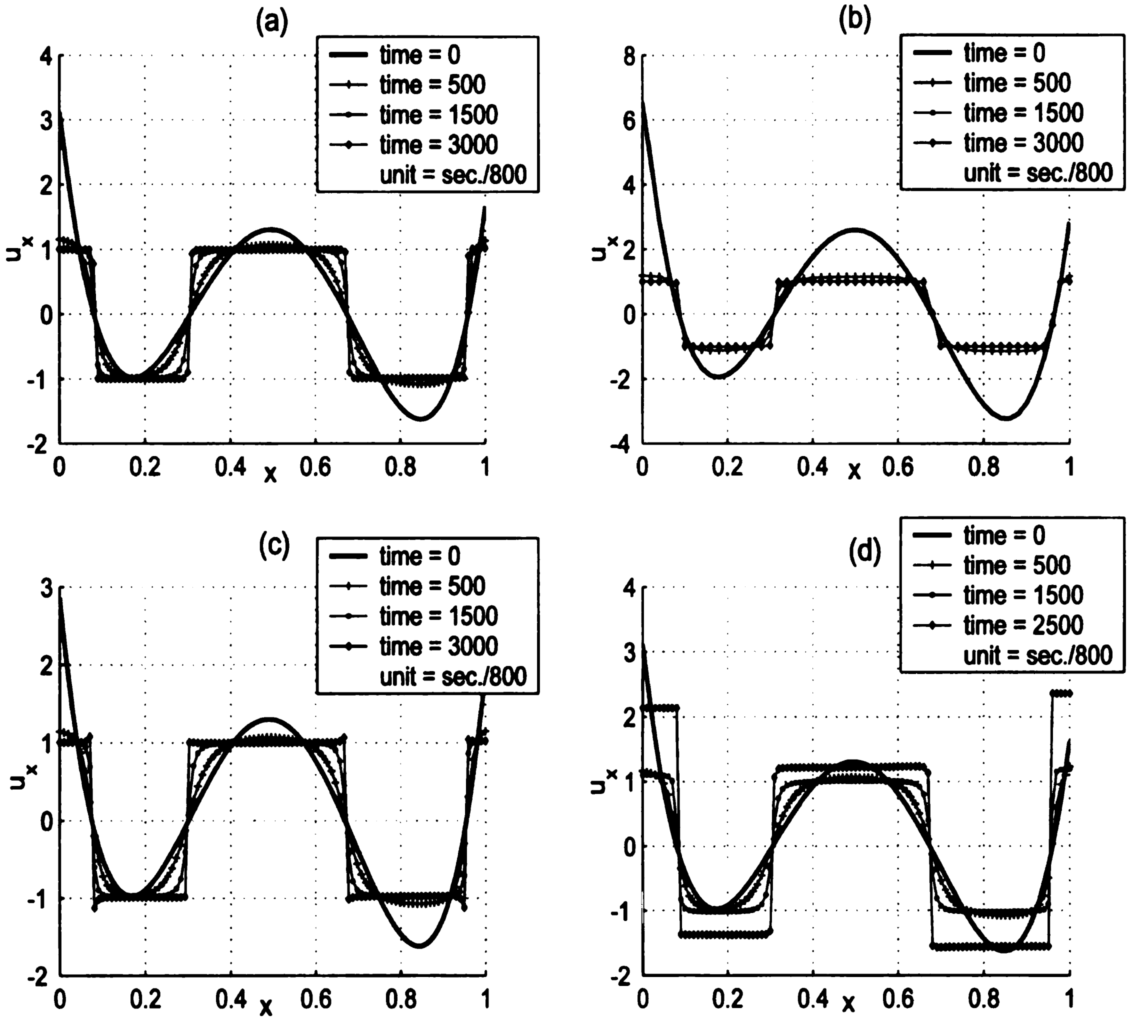

Figure 2.3. Transition Layer Dynamics using the (a) linear elements , (b) quadratic elements (c) Hermite cubic elements for the FEM and (d) FDM without averaging the nonlinear term $\sigma(u_x)_x$.

From Figure 2.3, one can see that the three finite elements have almost the same behavior. However, the graphs are still convergent at 3000 time steps while in (d), using the central difference scheme induces a blow up at 2500 time steps. This indicates that the FEM is still much more accurate than the FDM without averaging the term $\sigma(u_x)_x$.

2.6 Examples

Note that if initial energy is large, the dynamics no longer follow such a behavior and the following pictures show the importance of this assumption.

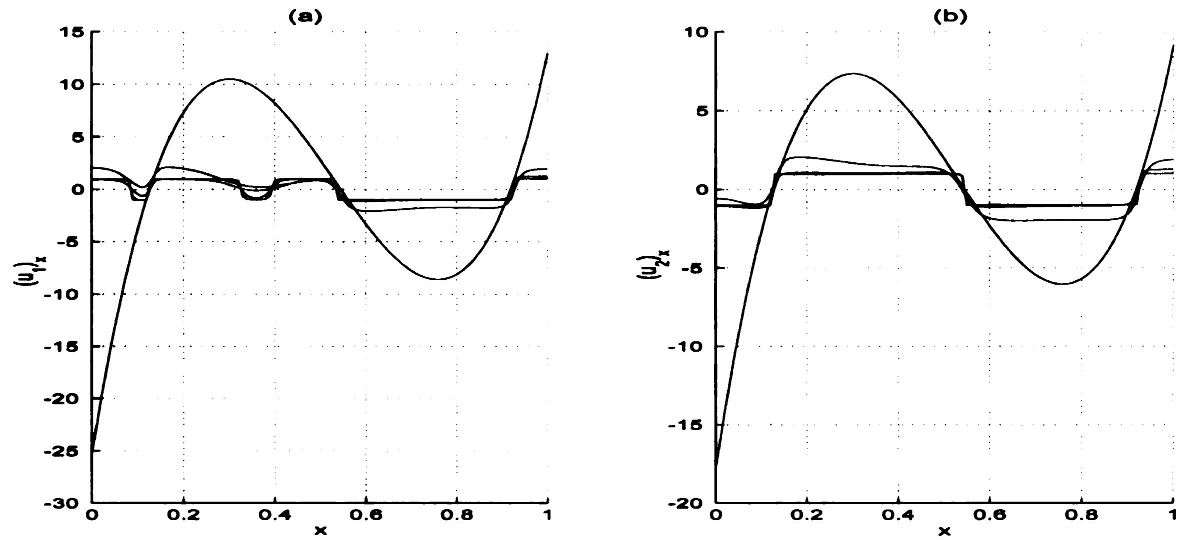

Figure 2.4. Transition Layer Dynamics for the (a) strain when $|(u_1)_x(x,0)| > 20$ for some $x \in (0,1)$, (b) strain when $|(u_2)_x(x,0)| < 20$.

The graph in part (b) of Figure 2.4 decrease fast and still has the same number of zeros and the transition layers are getting steeper. However, in the graph in part (a), the shape of the initial value $(u_1)_x(x,0)$ is the same but its absolute value is greater than 20 for some $x \in (0,1)$, the graph does not preserve the number of zeros and develops new zeros and transition layers. Thus, in this example, the critical point is when $u_x(x,0) = 20$. The equation for the initial data is $(u_1)_x(x,0) = 400x^3 - 630x^2 + 268x - 24$ for (a) and $(u_2)_x(x,0) = 280x^3 - 441x^2 + 187.6x - 16.8$ for (b).

We next consider the Neumann boundary conditions instead of the Dirichlet boundary conditions. Although we did not prove the dynamics for this type of problem theoretically, we can see that the dynamics also hold for the Neumann problems in Figure 2.5. We used the initial data $u_x(x, 0) = 500x^5 - 1250x^4 + 1080x^3 - 370x^2 + 40x$.

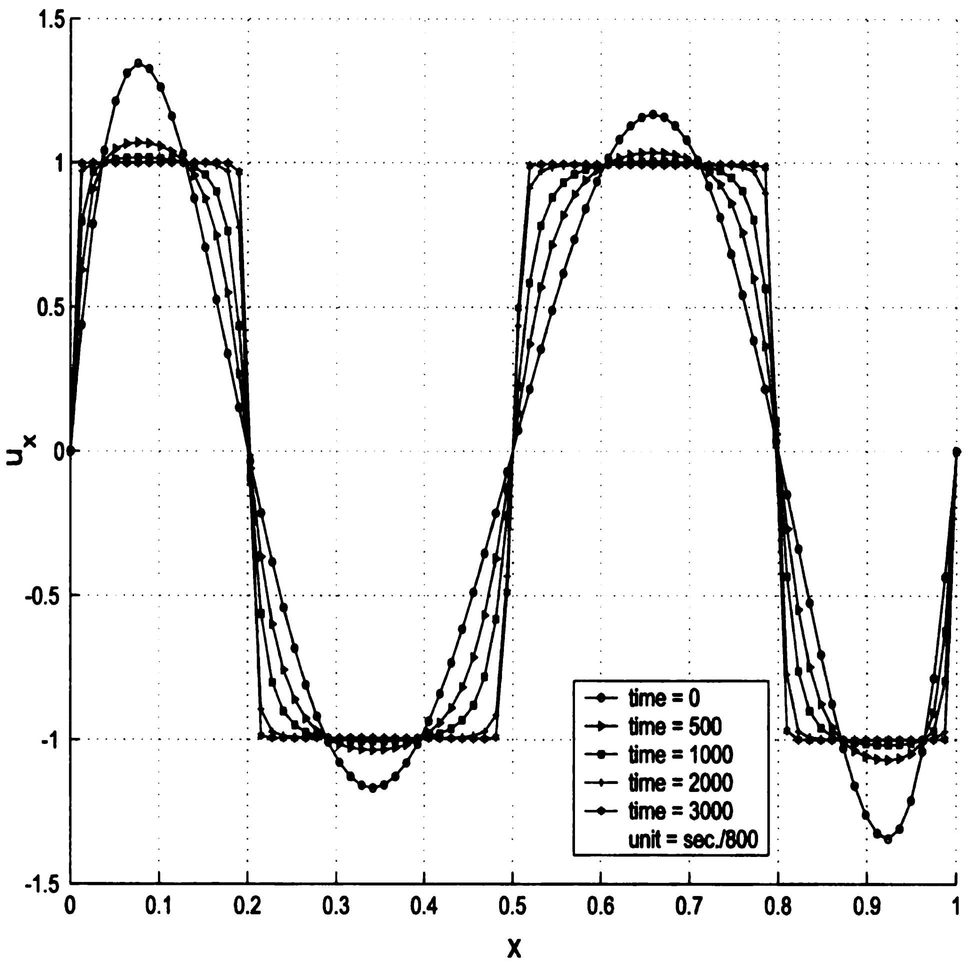

Figure 2.5. Transition Layer Dynamics using the Neumann boundary conditions.

One more interesting example is when the assumption (A4) is omitted. We used the polynomial function $u_x(x,0) = 60x^5 - 120x^4 + 80x^3 - 21.12x^2 + 2.2080x - 0.0640$ as an initial data. Note that $u_x(x,0)$ has a local minimum at $x = 0.2$ which is also a zero of $u_x(x,0)$. At $x = 0.2$, the graph in Figure 2.6 pushes down to the negative values and develops the transition layers.

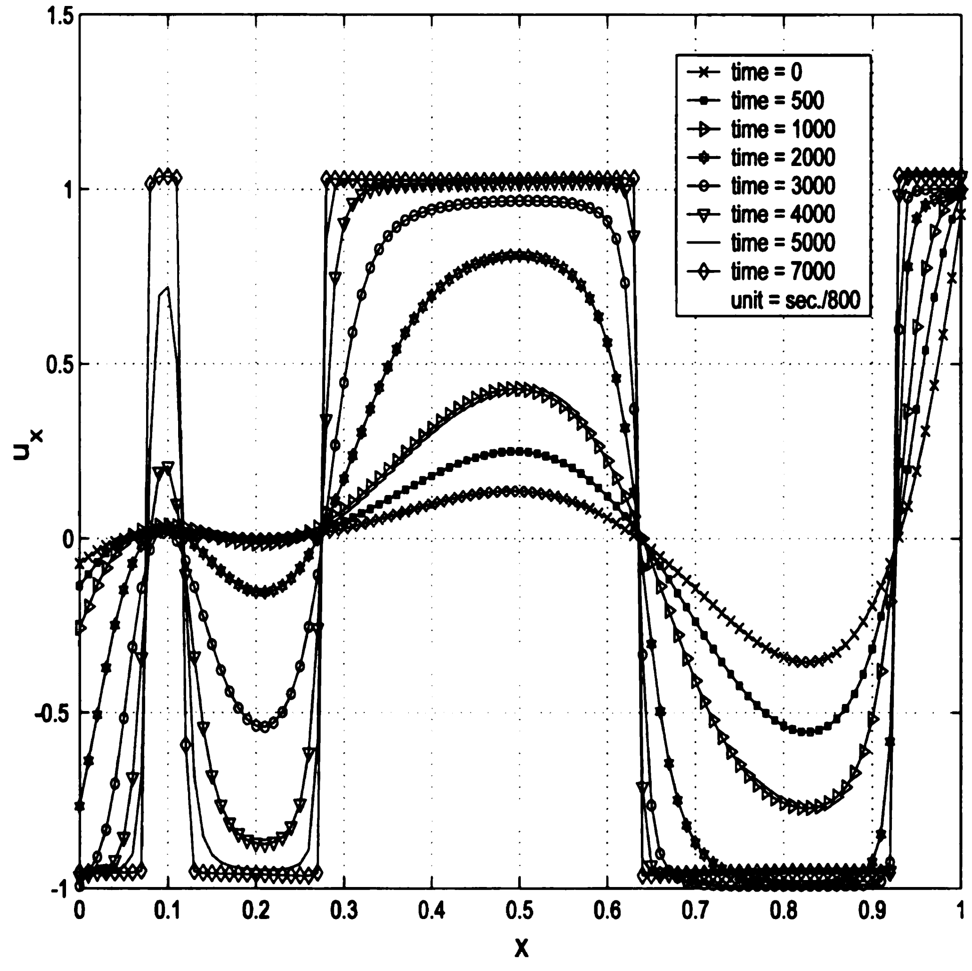

Figure 2.6. Transition Layer Dynamics for the problem for the strain whose initial data does not obey $(A4)$.

CHAPTER 3

Convergence Analysis of Numerical Solutions in Two-dimensional Systems

Even though we have not achieved the analytic proof of the transition layer dynamics for the multi-dimensional systems, the extended numerical methods to the two-dimensional systems from the one-dimensional systems also show the transition layer dynamics. The finite number of portions of the surfaces become steeper and discontinuous at the time limit. In this chapter, we derive the matrix equations from both FDM and FEM. We also prove the convergence of the numerical solution for the two-dimensional systems from the FDM with the standard central difference approximation. Average approximation for the nonlinear term $\mathrm{Div}(\sigma(Du^{m,j}))$ and the ADI method are discussed in Section 3.3 and 3.4.

3.1 Derivation of the matrix equation using the Finite Difference Methods

Consider the two-dimensional viscoelastic system,

$$\begin{cases} u_{tt} = \mathrm{Div}(\sigma(Du) + Du_t) - u, \\ u|_{\partial\Omega} = 0 \qquad (t \in [0,\infty)), \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = v_0 \quad (x \in \bar{\Omega}), \end{cases} \tag{3.1}$$

where u is a mapping from $\Omega \times (0,\infty)$ to $\mathbb{R}$, $\Omega = (0,1) \times (0,1)$. It was proven [15] that for higher dimensions, the discretized solution $u^{m,j}$, $j \in \mathbb{N}$, of system (3.1) is the minimizer of the following functional

$$J^{m,j}[u] := \int_\Omega \left(\frac{1}{2m^2} |u - 2u^{m,j-1} + u^{m,j-2}|^2 + W(Du) + \frac{1}{2m} |Du - Du^{m,j-1}|^2 + \frac{1}{2}|u|^2 \right) dx.$$

Therefore, $u^{m,j}$, $j \in \mathbb{N}$ satisfies the following equation

$$\frac{1}{m^2}(u - 2u^{m,j-1} + u^{m,j-2}) - \mathrm{Div}(\sigma(Du)) - \frac{1}{m} \cdot \Delta(u - u^{m,j-1}) + u = 0. \tag{3.2}$$

The second term of the above equation can be rewritten by

$$\mathrm{Div}(\sigma(Du)) = \frac{\partial^2 W(Du)}{(\partial u_x)^2} \cdot u_{xx} + \frac{2\partial^2 W(Du)}{(\partial u_x)(\partial u_y)} \cdot u_{xy} + \frac{\partial^2 W(Du)}{(\partial u_y)^2} \cdot u_{yy}. \tag{3.3}$$

For simplicity, we use the following stored energy function

$$W(Du) = \frac{1}{4}(u_x^2 - 1)^2 + \frac{1}{2} u_y^2. \tag{3.4}$$

Hence, the second term of equation (3.3) is zero and

$$\frac{\partial^2 W(Du)}{(\partial u_x)^2} = 3u_x^2 - 1, \quad \frac{\partial^2 W(Du)}{(\partial u_y)^2} = 1.$$

As in Chapter 2, divide the interval $(0,1)$ in the x direction into n subintervals (x_{k-1}, x_k) of uniform length $\frac{1}{n}$, $k = 1, \cdots, n$, $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ and divide the interval $(0,1)$ in the y direction into the same number of subintervals (y_{k-1}, y_k) of uniform length $\frac{1}{n}$, $k = 1, \cdots, n$, $0 = y_0 < y_1 < \cdots < y_{n-1} < y_n = 1$. By using the same method as in Chapter 2 (Central difference scheme for the FDM), the following approximations hold.

$$\begin{aligned}
u_x^{m,j}(x_k, y_s) &\approx \frac{u^{m,j}(x_{k+1}, y_s) - u^{m,j}(x_{k-1}, y_s)}{2h}, \\
u_y^{m,j}(x_k, y_s) &\approx \frac{u^{m,j}(x_k, y_{s+1}) - u^{m,j}(x_k, y_{s-1})}{2h}, \\
u_{xx}^{m,j}(x_k, y_s) &\approx \frac{u^{m,j}(x_{k+1}, y_s) - 2u^{m,j}(x_k, y_s) + u^{m,j}(x_{k-1}, y_s)}{h^2}, \\
u_{yy}^{m,j}(x_k, y_s) &\approx \frac{u^{m,j}(x_k, y_{s+1}) - 2u^{m,j}(x_k, y_s) + u^{m,j}(x_k, y_{s-1})}{h^2}.
\end{aligned}$$

Thus, equation (3.2) becomes

$$\begin{aligned}
&(1+m^2)u^{m,j}(x_k, y_s) - \frac{m}{h^2}(u^{m,j}(x_{k+1}, y_s) - 2u^{m,j}(x_k, y_s) + u^{m,j}(x_{k-1}, y_s)) \\
&-\frac{m}{h^2}(u^{m,j}(x_k, y_{s+1}) - 2u^{m,j}(x_k, y_s) + u^{m,j}(x_k, y_{s-1})) \\
&-\frac{m^2}{h^2}\left[3\left(\frac{u^{m,j}(x_{k+1}, y_s) - u^{m,j}(x_{k-1}, y_s)}{2h}\right)^2 - 1\right](u^{m,j}(x_{k+1}, y_s) - 2u^{m,j}(x_k, y_s) \\
&+u^{m,j}(x_{k-1}, y_s)) - \frac{m^2}{h^2}(u^{m,j}(x_k, y_{s+1}) - 2u^{m,j}(x_k, y_s) + u^{m,j}(x_k, y_{s-1})) \\
= \;& 2u^{m,j-1}(x_k, y_s) - u^{m,j-2}(x_k, y_s) - \frac{m}{h^2}[u^{m,j-1}(x_{k+1}, y_s) + u^{m,j-1}(x_{k-1}, y_s) \\
&+u^{m,j-1}(x_k, y_{s+1}) + u^{m,j-1}(x_k, y_{s-1}) - 4u^{m,j-1}(x_k, y_s)],
\end{aligned}$$

for $k,\ s = 1, \cdots, n-1$. After arranging the above equation in terms of

$$u^{m,j}(x_{k-1}, y_s),\ u^{m,j}(x_k, y_s),\ u^{m,j}(x_{k+1}, y_s),\ u^{m,j}(x_k, y_{s-1}) \text{ and } u^{m,j}(x_k, y_{s+1}),$$

we get the following equation

$$\begin{aligned} & G_2(u^{m,j}(x_k, y_s))u^{m,j}(x_{k-1}, y_s) + G_2(u^{m,j}(x_k, y_s))u^{m,j}(x_{k+1}, y_s) \\ & + G_1(u^{m,j}(x_k, y_s))u^{m,j}(x_k, y_s) + G_3^m u^{m,j}(x_k, y_{s-1}) + G_3^m u^{m,j}(x_k, y_{s+1}) \\ = \ & H(u^{m,j-1}(x_k, y_s)), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} G_1(u^{m,j}(x_k, y_s)) &:= 1 + m^2 + \frac{4m}{h^2} + \frac{2m^2}{h^2} \\ & + \frac{2m^2}{h^2} \cdot \left[3 \left(\frac{u^{m,j}(x_{k+1}, y_s) - u^{m,j}(x_{k-1}, y_s)}{2h} \right)^2 - 1 \right], \\ G_2(u^{m,j}(x_k, y_s)) &:= -\frac{m}{h^2} - \frac{m^2}{h^2} \cdot \left[3 \left(\frac{u^{m,j}(x_{k+1}, y_s) - u^{m,j}(x_{k-1}, y_s)}{2h} \right)^2 - 1 \right] \\ G_3^m &:= -\frac{m}{h^2} - \frac{m^2}{h^2}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} H(u^{m,j-1}(x_k, y_s)) &:= 2u^{m,j-1}(x_k, y_s) - u^{m,j-2}(x_k, y_s) - \frac{m}{h^2} \cdot [u^{m,j-1}(x_{k+1}, y_s) \\ & + u^{m,j-1}(x_{k-1}, y_s) + u^{m,j-1}(x_k, y_{s+1}) + u^{m,j-1}(x_k, y_{s-1}) \\ & - 4u^{m,j-1}(x_k, y_s)]. \end{aligned} \tag{3.7}$$

Equation (3.5) becomes the following matrix equation

$$\mathbb{G}(u^{m,j})\{u^{m,j}\} = \{\mathbb{H}(u^{m,j-1})\},$$

where $\mathbb{G}(u^{m,j})$ is a $(n-1)^2 \times (n-1)^2$ matrix and can be represented by the matrix of the $(n-1) \times (n-1)$ submatrices $\mathbf{G}_{l,s}(u^{m,j}),\ \ l,s = 1,\cdots,n-1$. That is,

$$\begin{pmatrix} \mathbf{G}_{1,1}(u^{m,j}) & \cdots & \mathbf{G}_{1,n-1}(u^{m,j}) \\ \mathbf{G}_{2,1}(u^{m,j}) & \cdots & \mathbf{G}_{2,n-1}(u^{m,j}) \\ \cdots & \cdots & \cdots \\ \mathbf{G}_{n-2,1}(u^{m,j}) & \cdots & \mathbf{G}_{n-2,n-1}(u^{m,j}) \\ \mathbf{G}_{n-1,1}(u^{m,j}) & \cdots & \mathbf{G}_{n-1,n-1}(u^{m,j}) \end{pmatrix}.$$

The main diagonal entries $\mathbf{G}_{l,l}(u^{m,j}),\ \ l = 1,\cdots,n-1$ and the other diagonal entries $\mathbf{G}_{l,l-1}(u^{m,j}),\ \ \mathbf{G}_{l-1,l}(u^{m,j}),\ \ l = 2,\cdots,n-1$ are the only nonzero elements. We denote $G_i(u^{m,j}(x_{k(l)}, y_{s(l)}))$, the nonzero elements of $\mathbf{G}_{l,l}(u^{m,j}), \mathbf{G}_{l,l-1}(u^{m,j})$ or $\mathbf{G}_{l-1,l}(u^{m,j})$, by $(G_i^{m,j})_{k(l),s(l)},\ i = 1,2$. Then, the $\mathbf{G}_{l,l}(u^{m,j}),\ \ l = 1,\cdots,n-1$ is the following $(n-1) \times (n-1)$ tri-diagonal matrix.

$$\begin{pmatrix} (G_1^{m,j})_{(l-1)n-(l-2),1} & G_3^m & 0 & & \cdots & 0 \\ G_3^m & (G_1^{m,j})_{(l-1)n-(l-3),2} & G_3^m & 0 & \cdots & 0 \\ & & \ddots & & & \\ & & & \ddots & & \\ 0 & \cdots & 0 & G_3^m & (G_1^{m,j})_{l(n-1)-1,n-2} & G_3^m \\ 0 & \cdots & & 0 & G_3^m & (G_1^{m,j})_{l(n-1),n-1} \end{pmatrix}.$$

The two other diagonal elements $\mathbf{G}_{l,l-1}(u^{m,j}),\quad \mathbf{G}_{l-1,l}(u^{m,j}),\quad l = 2,\cdots,n-1$ are the following diagonal matrices, respectively

$$\begin{pmatrix} (G_2^{m,j})_{(l-1)n-(l-2),1} & 0 & & \cdots & 0 \\ 0 & (G_2^{m,j})_{(l-1)n-(l-3),2} & 0 & \cdots & 0 \\ & \cdots & & & \\ & & \ddots & & \\ & & & \cdots & \\ 0 & \cdots & 0 & (G_2^{m,j})_{l(n-1)-1,n-2} & 0 \\ 0 & \cdots & & 0 & (G_2^{m,j})_{l(n-1),n-1} \end{pmatrix},$$

$$\begin{pmatrix} (G_2^{m,j})_{(l-2)n-(l-3),1} & 0 & & \cdots & 0 \\ 0 & (G_2^{m,j})_{(l-2)n-(l-4),2} & 0 & \cdots & 0 \\ & \cdots & & & \\ & & \ddots & & \\ & & & \cdots & \\ 0 & \cdots & 0 & (G_2^{m,j})_{(l-1)(n-1)-1,n-2} & 0 \\ 0 & \cdots & & 0 & (G_2^{m,j})_{(l-1)(n-1),n-1} \end{pmatrix}.$$

The $\{u^{m,j}\}$ is a $(n-1)^2 \times 1$ vector and it is also considered as the $(n-1) \times 1$ vector of $n-1$ subvectors $\{\mathbf{u}_i^{m,j}\},\quad i = 1,\cdots,n-1$. That is,

$$\{u^{m,j}\} = \{\{\mathbf{u}_1^{m,j}\},\cdots,\{\mathbf{u}_{n-1}^{m,j}\}\}^T,$$

where

$$\{\mathbf{u}_i^{m,j}\} := \{u^{m,j}(x_i,y_1),\cdots,u^{m,j}(x_i,y_{n-1})\}^T.$$

Similarly,

$$\{\mathbb{H}(u^{m,j-1})\} = \{\{\mathbf{H}_1^{m,j}\},\cdots,\{\mathbf{H}_{n-1}^{m,j}\}\}^T,$$

where

$$\{\mathbf{H}_i^{m,j}\} := \{H^{m,j}(x_i, y_1), \cdots, H^{m,j}(x_i, y_{n-1})\}^T.$$

3.2 Existence of the Finite Difference solution

Theorem 3.1 *The solution of the matrix equation*

$$\mathbb{G}(u^{m,j})\{u^{m,j}\} = \{\mathbb{H}(u^{m,j-1})\},$$

exists.

PROOF. As in Theorem 2.1, consider the following equation

$$\mathbb{G}(u_i^{m,j})\{u_{i+1}^{m,j} - u_i^{m,j}\} = (\mathbb{G}(u_{i-1}^{m,j}) - \mathbb{G}(u_i^{m,j}))\{u_i^{m,j}\}. \tag{3.8}$$

By using the same argument as in the one-dimensional case, the right hand side of the above equation becomes the following matrix equation

$$\mathbb{P}(u_{i_*}^{m,j})\{u_{i-1}^{m,j} - u_i^{m,j}\},$$

where $\mathbb{P}(u_{i_*}^{m,j})$ is the following matrix

$$\begin{pmatrix} 0 & P_{1,2}^{m,j} & 0 & & \cdots & 0 \\ P_{2,1}^{m,j} & 0 & P_{2,3}^{m,j} & 0 & \cdots & 0 \\ & & & \cdots & & \\ 0 & \cdots & 0 & P_{(n-1)^2-1,(n-1)^2-2}^{m,j} & 0 & P_{(n-1)^2-1,(n-1)^2}^{m,j} \\ 0 & \cdots & & 0 & P_{(n-1)^2,(n-1)^2-1}^{m,j} & 0 \end{pmatrix}.$$

Here, denote $P_{l,s}(u_{i_*}^{m,j})$ by $P_{l,s}^{m,j}$.

$$\begin{aligned} P_{1,2} &\leq hr^2 \cdot const. \cdot \left|\frac{\partial^3 W}{\partial u_x^3}\right| \cdot |u_i^{m,j}(x_1, y_1) + u_i^{m,j}(x_1, y_2) + u_i^{m,j}(x_2, y_1)| \\ &\leq \epsilon_3. \end{aligned}$$

Recall that $r = \frac{m}{h^2}$. Similarly,

$$\begin{aligned} P_{(n-1)^2,(n-1)^2-1} &\leq hr^2 \cdot const. \cdot \left|\frac{\partial^3 W}{\partial u_x^3}\right| \cdot |u_i^{m,j}(x_{n-2}, y_{n-1}) + u_i^{m,j}(x_{n-1}, y_{n-2}) \\ &+ u_i^{m,j}(x_{n-1}, y_{n-1})| \\ &\leq \epsilon_4, \end{aligned}$$

for some $\epsilon_3,\ \epsilon_4 \ll 1$.

For general $P_{l,l+1}(u_{i_*}^{m,j})$,

$$\begin{aligned} P_{l,l+1} &\leq hr^2 \cdot const. \cdot \left|\frac{\partial^3 W}{\partial u_x^3}\right| \cdot |u_i^{m,j}(x_{l-1}, y_s) + u_i^{m,j}(x_l, y_s) + u_i^{m,j}(x_{l+1}, y_s)| \\ &\leq \epsilon_5. \end{aligned}$$

Similarly,

$$P_{l+1,l} \leq \epsilon_6,$$

for some $\epsilon_5,\ \epsilon_6 \ll 1$. This proves that the matrix $\mathbb{P}(u_{i_*}^{m,j})$ is bounded and small.

Now, it remains to show that $\mathbb{G}(u_i^{m,j})$ is invertible and bounded away from zero. Consider the matrix $\mathbb{G}(u_i^{m,j})$ as the sum of three matrices $\widetilde{\mathbb{J}}, \widetilde{\mathbb{K}}$ and $\widetilde{\mathbb{L}}(u_i^{m,j})$, where $\widetilde{\mathbb{J}}, \widetilde{\mathbb{K}}$ are the following matrices, respectively.

Here, denote $P_{l,s}(u_{i_*}^{m,j})$ by $P_{l,s}^{m,j}$.

$$\begin{aligned} P_{1,2} &\leq hr^2 \cdot const. \cdot \left|\frac{\partial^3 W}{\partial u_x^3}\right| \cdot |u_i^{m,j}(x_1,y_1) + u_i^{m,j}(x_1,y_2) + u_i^{m,j}(x_2,y_1)| \\ &\leq \epsilon_3. \end{aligned}$$

Recall that $r = \frac{m}{h^2}$. Similarly,

$$\begin{aligned} P_{(n-1)^2,(n-1)^2-1} &\leq hr^2 \cdot const. \cdot \left|\frac{\partial^3 W}{\partial u_x^3}\right| \cdot |u_i^{m,j}(x_{n-2},y_{n-1}) + u_i^{m,j}(x_{n-1},y_{n-2}) \\ &+ u_i^{m,j}(x_{n-1},y_{n-1})| \\ &\leq \epsilon_4, \end{aligned}$$

for some $\epsilon_3,\ \epsilon_4 \ll 1$.

For general $P_{l,l+1}(u_{i_*}^{m,j})$,

$$\begin{aligned} P_{l,l+1} &\leq hr^2 \cdot const. \cdot \left|\frac{\partial^3 W}{\partial u_x^3}\right| \cdot |u_i^{m,j}(x_{l-1},y_s) + u_i^{m,j}(x_l,y_s) + u_i^{m,j}(x_{l+1},y_s)| \\ &\leq \epsilon_5. \end{aligned}$$

Similarly,

$$P_{l+1,l} \leq \epsilon_6,$$

for some $\epsilon_5,\ \epsilon_6 \ll 1$. This proves that the matrix $\mathbb{P}(u_{i_*}^{m,j})$ is bounded and small.

Now, it remains to show that $\mathbb{G}(u_i^{m,j})$ is invertible and bounded away from zero. Consider the matrix $\mathbb{G}(u_i^{m,j})$ as the sum of three matrices $\widetilde{\mathbb{J}}, \widetilde{\mathbb{K}}$ and $\widetilde{\mathbb{L}}(u_i^{m,j})$, where $\widetilde{\mathbb{J}}, \widetilde{\mathbb{K}}$ are the following matrices, respectively.

$$
\begin{pmatrix}
1+m^2 & 0 & & \cdots & 0 \\
0 & 1+m^2 & 0 & \cdots & 0 \\
 & \cdots & & & \\
 & & \cdots & & \\
 & & & \cdots & \\
0 & \cdots & 0 & 1+m^2 & 0 \\
0 & \cdots & & 0 & 1+m^2
\end{pmatrix},
$$

$$
\begin{pmatrix}
4r & -r & 0 & \cdots & \cdots & 0 & -r & 0 & \cdots & \cdots & 0 \\
-r & 4r & -r & 0 & \cdots & 0 & 0 & -r & 0 & \cdots & 0 \\
 & & \cdots & & & & & & \cdots & & \cdots \\
 & & & \cdots & & & & & & \cdots & \\
 & & & & \cdots & & & & 0 & -r & 0 \\
-r & 0 & & & & \cdots & & & & 0 & -r \\
0 & -r & 0 & & & & \cdots & & & & \\
 & & \cdots & & & & & \cdots & & & \\
\cdots & & & \cdots & & & & & \cdots & & \\
0 & \cdots & 0 & -r & 0 & 0 & \cdots & 0 & -r & 4r & -r \\
0 & \cdots & \cdots & 0 & -r & 0 & \cdots & \cdots & 0 & -r & 4r
\end{pmatrix},
$$

where in the first row, the second $-r$ appears at n^{th} column. Similarly, in the first column, the second $-r$ appears at n^{th} row.

The matrix $\widetilde{\mathbb{L}}(u_i^{m,j})$ has the similar form. It has non-zero terms in the tri-diagonal positions and nonzero terms appears at n^{th} column of first row and n^{th} row of first column. That is,

$$
hr\begin{pmatrix}
2\widetilde{L}(\cdot) & -\widetilde{L}_2(\cdot) & 0 & \cdots & -\widetilde{L}_1(\cdot) & 0 & & \cdots & \cdots & 0 \\
-\widetilde{L}_2(\cdot) & 2\widetilde{L}(\cdot) & -\widetilde{L}_2(\cdot) & 0 & \cdots & -\widetilde{L}_1(\cdot) & 0 & & \cdots & 0 \\
 & & \cdots & & & & \cdots & & & \\
-\widetilde{L}_1(\cdot) & & & \cdots & & & & & & \\
 & \cdots & & & \cdots & & & & & \\
 & & & & & \cdots & & & \cdots & \\
 & & & & & & \cdots & & & -\widetilde{L}_1(\cdot) \\
 & & & \cdots & & & & \cdots & & \\
0 & \cdots & & 0 & -\widetilde{L}_1(\cdot) & \cdots & 0 & -\widetilde{L}_2(\cdot) & 2\widetilde{L}(\cdot) & -\widetilde{L}_2(\cdot) \\
0 & \cdots & \cdots & & 0 & -\widetilde{L}_1(\cdot) & \cdots & 0 & -\widetilde{L}_2(\cdot) & 2\widetilde{L}(\cdot)
\end{pmatrix},
$$

where

$$
\begin{aligned}
\widetilde{L}_1(x_k, y_s) &= \frac{\partial^3 W}{\partial u_x^3}\left(\frac{u(x_{k+1}, y_s) - u(x_{k-1}, y_s)}{2h}\right), \\
\widetilde{L}_2(x_k, y_s) &= \frac{\partial^3 W}{\partial u_y^3}\left(\frac{u(x_k, y_{s+1}) - u(x_k, y_{s-1})}{2h}\right), \\
\widetilde{L} &= \widetilde{L}_1 + \widetilde{L}_2.
\end{aligned}
$$

Note that this matrix is not symmetric since each row depends on the components x_k and y_s. The matrix $\widetilde{\mathbb{K}}$ is positive definite since eigenvalues of the matrix are positive by the following inequality.

$$
|\lambda_i - \widetilde{k}_{ii}| \leq \sum_{j \neq i} |\widetilde{k}_{ij}|,
$$

where $\widetilde{k}_{ii} = 4 \cdot r$, $\sum_{j \neq i} |\widetilde{k}_{ij}| \geq 4 \cdot r$.

Now, the matrix $\mathbb{G}(u_i^{m,j})$ is positive definite and bounded away from zero since

$$\begin{aligned}
\xi^T \cdot \mathbb{G}(u_i^{m,j}) \cdot \xi &= (1+m^2) \cdot |\xi|^2 + \xi^T \cdot \widetilde{\mathbb{K}} \cdot \xi + \xi^T \cdot \widetilde{\mathbb{L}}(u_i^{m,j}) \cdot \xi \\
&\geq (1+m^2) \cdot |\xi|^2 - 5m \cdot r \cdot max|\sigma'| \cdot |\xi|^2 \\
&= (1+m^2 - 5m \cdot r \cdot max|\sigma'|)|\xi|^2 \\
&> \frac{1}{2}|\xi|^2 \\
&> 0.
\end{aligned}$$

This proves Theorem 3.1.

3.3 Average approximation of Div$(\sigma(Du^{m,j}))$

As in the one-dimensional case, we modify (3.3) by applying the average approximation to the term Div$(\sigma(Du^{m,j}))$. Since $\sigma(Du^{m,j}) = (\widetilde{\sigma}(u_x^{m,j}) \cdot u_x^{m,j}, u_y^{m,j})$, where $\widetilde{\sigma}(u_x^{m,j}) = (u_x^{m,j})^2 - 1$,

$$\text{Div}(\sigma(Du^{m,j})) = (\widetilde{\sigma}(u_x^{m,j}) \cdot u_x^{m,j})_x + (u_y^{m,j})_y.$$

As in Chapter 2,

$$\begin{aligned}
\sigma(u_x^{m,j}(x_k, y_s))_x \approx\ & \frac{\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}}, y_s))}{h^2} \cdot u^{m,j}(x_{k-1}, y_s) \\
& - \frac{(\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}}, y_s)) + \widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}}, y_s)))}{h^2} \cdot u^{m,j}(x_k, y_s) \\
& + \frac{\widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}}, y_s))}{h^2} \cdot u_x^{m,j}(x_{k+1}, y_s)
\end{aligned}$$

Therefore, we get the modified equation of (3.5)

$$\begin{aligned}
& \widetilde{G}_2(u^{m,j}(x_k, y_s)) \cdot u^{m,j}(x_{k-1}, y_s) + \widetilde{G}_2(u^{m,j}(x_{k+1}, y_s)) \cdot u^{m,j}(x_{k+1}, y_s) \\
& + \widetilde{G}_1(u^{m,j}(x_k, y_s)) \cdot u^{m,j}(x_k, y_s) + G_3^m \cdot u^{m,j}(x_k, y_{s-1}) + G_3^m \cdot u^{m,j}(x_k, y_{s+1}) \\
= \; & H(u^{m,j-1}(x_k, y_s)),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{G}_1(u^{m,j}(x_k, y_s)) &:= 1 + m^2 + \frac{4m}{h^2} + \frac{2m^2}{h^2} \\
&\quad + \frac{m^2}{h^2} \cdot [\widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}}, y_s)) + \widetilde{\sigma}(u_x^{m,j}(x_{k+\frac{1}{2}}, y_s))], \\
\widetilde{G}_2(u^{m,j}(x_k, y_s)) &:= -\frac{m}{h^2} - \frac{m^2}{h^2} \cdot \widetilde{\sigma}(u_x^{m,j}(x_{k-\frac{1}{2}}, y_s))
\end{aligned}$$

and G_3^m and $H(u^{m,j-1}(x_k, y_s))$ are the same as (3.6) and (3.7), respectively for $k, s = 1, \cdots, n-1$. The ADI method is more useful in the two dimensional case since this locally one-dimensional method enables us to solve the two one-dimensional $(n-1) \times (n-1)$ tri-diagonal matrix equations instead of dealing with the $(n-1)^2 \times (n-1)^2$ banded matrix with 5 nonzero diagonals.

3.4 The ADI method in two-dimensions

We apply the explicit formula and the ADI approximation (2.9), (2.10) to the two-dimensional system (3.1). Since $\kappa = 2$, we get the following 3 equations

$$
\begin{aligned}
u^{m,\star}(x_k,y_s) &= \widetilde{u}^{m,j-1}(x_k,y_s) - \widehat{G}_2(u^{m,j-1}(x_k,y_s))\cdot u^{m,j-1}(x_{k-1},y_s)\\
&\quad -\widehat{G}_2(u^{m,j-1}(x_{k+1},y_s))\cdot u^{m,j-1}(x_{k+1},y_s)\\
&\quad -(\widehat{G}_1(u^{m,j-1}(x_k,y_s)) + m^2)\cdot u^{m,j-1}(x_k,y_s) + \widehat{G}_3^m\cdot u^{m,j-1}(x_k,y_{s-1})\\
&\quad -2\widehat{G}_3^m\cdot u^{m,j-1}(x_k,y_s) + \widehat{G}_3^m\cdot u^{m,j-1}(x_k,y_{s+1}) + \widehat{H}(u^{m,j-1}(x_k,y_s)),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\alpha\widehat{G}_2(u^{m,j-1}(x_k,y_s)) - \frac{m}{h^2}\right)\cdot u^{m,\star\star}(x_{k-1},y_s)\\
&+\left(1+\alpha\widehat{G}_1(u^{m,j-1}(x_k,y_s)) + \frac{2m}{h^2}\right)\cdot u^{m,\star\star}(x_k,y_s)\\
&+(\alpha\widehat{G}_2(u^{m,j-1}(x_{k+1},y_s)) - \frac{m}{h^2})\cdot u^{m,\star\star}(x_{k+1},y_s)\\
=\ & u^{m,\star}(x_k,y_s) + \alpha\widehat{G}_2(u^{m,j-1}(x_k,y_s))\cdot\widetilde{u}^{m,j-1}(x_{k-1},y_s)\\
&+\alpha\widehat{G}_1(u^{m,j-1}(x_k,y_s))\cdot\widetilde{u}^{m,j-1}(x_k,y_s) + \alpha\widehat{G}_2(u^{m,j-1}(x_{k+1},y_s))\cdot\widetilde{u}^{m,j-1}(x_{k+1},y_s),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\alpha\widehat{G}_3^m - \frac{m}{h^2}\right)\cdot u^{m,j}(x_k,y_{s-1}) + \left(1 - 2\alpha\widehat{G}_3^m + \frac{2m}{h^2}\right)\cdot u^{m,j}(x_k,y_s)\\
&+\left(\alpha\widehat{G}_3^m - \frac{m}{h^2}\right)\cdot u^{m,j}(x_k,y_{s+1})\\
=\ & u^{m,\star\star}(x_k,y_s) + \alpha\widehat{G}_3^m\cdot\widetilde{u}^{m,j-1}(x_k,y_{s-1})\\
&-2\alpha\widehat{G}_3^m\cdot\widetilde{u}^{m,j-1}(x_k,y_s) + \alpha\widehat{G}_3^m\cdot\widetilde{u}^{m,j-1}(x_k,y_{s+1}),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{u}^{m,j-1}(x_k, y_s) &:= 2u^{m,j-1}(x_k, y_s) - u^{m,j-2}(x_k, y_s), \\
\widehat{H}(u^{m,j-1}(x_k, y_s)) &:= H(u^{m,j-1}(x_k, y_s)) - \widetilde{u}^{m,j-1}(x_k, y_s), \\
\widehat{G}_1(u^{m,j-1}(x_k, y_s)) &:= \frac{m^2}{h^2} \cdot [\widetilde{\sigma}(u_x^{m,j-1}(x_{k-\frac{1}{2}}, y_s) + \widetilde{\sigma}(u_x^{m,j-1}(x_{k+\frac{1}{2}}, y_s))], \\
\widehat{G}_2(u^{m,j-1}(x_k, y_s)) &:= -\frac{m^2}{h^2} \cdot \widetilde{\sigma}(u_x^{m,j-1}(x_{k-\frac{1}{2}}, y_s)), \\
\widehat{G}_3^m &:= -\frac{m^2}{h^2}.
\end{aligned}
$$

3.5 Derivation of the matrix equation using the Finite Element Methods

Multiply equation (3.2) by the test function $w \in W_0^{1,\infty}(\Omega)$, where $\Omega = (0,1) \times (0,1)$ and integrate over the finite element Ω_e. Then the final weak formulation holds

$$
\int_{\Omega_e} w \cdot \left[\frac{u^{m,j} - 2u^{m,j-1} + u^{m,j-2}}{m^2} - \mathrm{Div}(\sigma(Du^{m,j})) - \frac{\mathrm{Div}(Du^{m,j} - Du^{m,j-1})}{m} + u^{m,j} \right] = 0
$$

for all $j \in \mathbb{N}$. Since w satisfies essential boundary conditions, the boundary terms vanish after the integration by parts of the above equation and we get the following equation.

$$
\begin{aligned}
&\int_{\Omega_e} \left(w \cdot \frac{u^{m,j} - 2u^{m,j-1} + u^{m,j-2}}{m^2} + w_x \cdot \sigma_1(Du^{m,j}) + w_y \cdot \sigma_2(Du^{m,j}) + wu^{m,j} \right. \\
&\left. + w_x \cdot \frac{u_x^{m,j} - u_x^{m,j-1}}{m} + w_y \cdot \frac{u_y^{m,j} - u_y^{m,j-1}}{m} \right) dxdy = 0, \qquad (3.9)
\end{aligned}
$$

where σ_1, σ_2 are x and y components of σ, respectively. By (3.4), these are given by

$$
\sigma_1 = u_x^{m,j}((u_x^{m,j})^2 - 1), \quad \sigma_2 = u_y^{m,j}.
$$

Similarly, as in the one dimensional case, define

$$u^{m,j}(x,y) := \sum_{s=1}^{n_e} u_s^{m,j}\psi_s(x,y), \quad w(x,y) := \psi_k(x,y).$$

Here, $u_s^{m,j}$ are undetermined constants and $\psi_s(x,y)$ are interpolation functions. Then equation (3.9) becomes

$$\begin{aligned}
&\sum_{s=1}^{n_e}\left[\int_{\Omega_e} \psi_k(\bar{x},\bar{y})\psi_s(\bar{x},\bar{y})d\bar{x}d\bar{y}\right](\ddot{u}_s^{m,j} + u_s^{m,j}) \\
+ &\sum_{s=1}^{n_e}\left[\int_{\Omega_e}\left(\frac{\partial\psi_k(\bar{x})}{\partial\bar{x}}\frac{\partial\psi_s(\bar{x})}{\partial\bar{x}} + \frac{\partial\psi_k(\bar{y})}{\partial\bar{y}}\frac{\partial\psi_s(\bar{y})}{\partial\bar{y}}\right)d\bar{x}d\bar{y}\right]\dot{u}_s^{m,j} \\
+ &\int_{\Omega_e}\left(\frac{\partial\psi_k(\bar{x})}{\partial\bar{x}}\sigma_1\left(\sum_{j=1}^{n_e} u_s^{m,j}\frac{\partial\psi_s(\bar{x})}{\partial\bar{x}}\right) + \frac{\partial\psi_k(\bar{y})}{\partial\bar{y}}\sigma_2\left(\sum_{j=1}^{n_e} u_s^{m,j}\frac{\partial\psi_s(\bar{y})}{\partial\bar{y}}\right)\right)d\bar{x}d\bar{y} = 0,
\end{aligned} \tag{3.10}$$

where $\dot{u}_s^{m,j}$, $\ddot{u}_s^{m,j}$ have the same meanings as in Chapter 2. For $k,\ s = 1,\cdots n_e$, let

$$\begin{aligned}
\{\mathbb{A}^e\}_{k,s} &:= \int_{\Omega_e} \psi_k(\bar{x},\bar{y})\psi_s(\bar{x},\bar{y})d\bar{x}d\bar{y}, \\
\{\mathbb{B}^e\}_{k,s} &:= \int_{\Omega_e}\left(\frac{\partial\psi_k(\bar{x})}{\partial\bar{x}}\frac{\partial\psi_s(\bar{x})}{\partial\bar{x}} + \frac{\partial\psi_k(\bar{y})}{\partial\bar{y}}\frac{\partial\psi_s(\bar{y})}{\partial\bar{y}}\right)d\bar{x}d\bar{y}, \\
\{\mathbb{F}^e(u^{m,j})\}_k &:= \int_{\Omega_e}\left(\frac{\partial\psi_k(\bar{x})}{\partial\bar{x}}\cdot\sigma_1\left(\sum_{j=1}^{n_e} u_s^{m,j}\frac{\partial\psi_s(\bar{x})}{\partial\bar{x}}\right)\right. \\
&\qquad \left. + \frac{\partial\psi_k(\bar{y})}{\partial\bar{y}}\cdot\sigma_2\left(\sum_{j=1}^{n_e} u_s^{m,j}\frac{\partial\psi_s(\bar{y})}{\partial\bar{y}}\right)\right)d\bar{x}d\bar{y}.
\end{aligned}$$

After multiplying the both sides of (3.10) by m^2 and moving the previous time solutions to the right hand side, we get the same type of matrix equation on each finite element Ω_e, as in the one dimensional case as follows

$$\begin{aligned}
\{(1+m^2)\mathbb{A}^e + m\mathbb{B}^e\}\{u^{m,j}\} &= \mathbb{A}^e(2\{u^{m,j-1}\} - \{u^{m,j-2}\}) + m\mathbb{B}^e\{u^{m,j-1}\} \\
&\quad - m^2\{\mathbb{F}^e(u^{m,j})\}.
\end{aligned} \tag{3.11}$$

The assembly of the global stiffness matrix from this finite element equation depends on the elements. Let $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3)$ be the three components of the triangle. Then the following functions are interpolation functions for the triangular elements

$$\psi_s^e(x,y) = \frac{1}{2Ar_e}(\alpha_s^e + \beta_s^e x + \gamma_s^e y), \quad (s = 1,2,3), \quad \text{where}$$

$$\alpha_1^e = x_2x_3 - x_3x_2, \quad \alpha_2^e = x_3x_1 - x_1x_3, \quad \alpha_3^e = x_1x_2 - x_2x_1,$$

$$\beta_1^e = y_2 - y_3, \quad \beta_2^e = y_3 - y_1, \quad \beta_3^e = y_1 - y_2,$$

$$\gamma_1^e = x_3 - x_2, \quad \gamma_2^e = x_1 - x_3, \quad \gamma_3^e = x_2 - x_1,$$

$$Ar_e = \text{Area of the triangle.}$$

Choose the right triangle on each equilengthed cubic of length h. Then we can obtain the local matrices $\mathbf{A}^e$, $\mathbb{B}^e$ as follows

$$\mathbb{A}^e = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \mathbb{B}^e = \frac{h^2}{24}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The following pictures are the layer dynamics of the two-dimensional surface using FDM and FEM linear triangular elements, respectively. We used the initial data $u_0(x,y) = -4(5x^3 - 7x^2 + 2x)(x^2 - x)$.

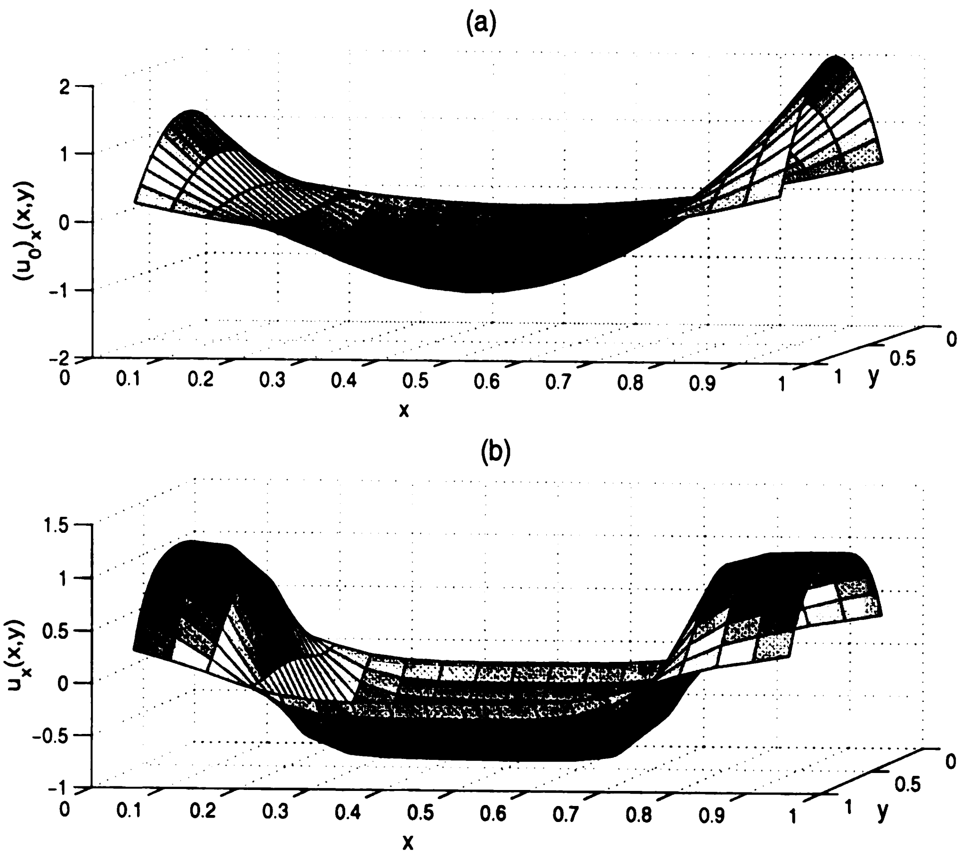

Figure 3.1. Transition Layer Dynamics in two-dimensions for (a) initial data $(u_0)_x$ and (b) u_x after 1000 (sec./800) time steps using the FDM.

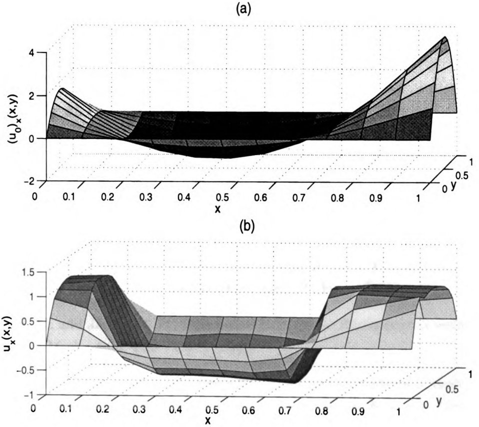

Figure 3.2. Transition Layer Dynamics in two-dimensions for (a) initial data $(u_0)_x$ and (b) u_x after 200 (sec./800) time steps using the FEM linear triangular elements

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