This is to certify that the

dissertation entitled

Family Gromov-Witten Invariants
for Kahler Surfaces

presented by

Junho Lee

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Thomas H Parker
Major professor

Date August 14, 2001

Family Gromov-Witten Invariants for Kähler Surfaces

By

Junho Lee

AN ABSTRACT OF A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2001

Professor Thomas H. Parker

ABSTRACT

Family Gromov-Witten Invariants for Kähler Surfaces

By

Junho Lee

The usual Gromov-Witten invariants are zero for Kähler surfaces with $p_g \geq 1$. In this paper we use analytic methods to define Family Gromov-Witten Invariants for Kähler surfaces. We prove that these are well-defined invariants of the deformation class of the Kähler structure and develop methods for computing them, including a version of the TRR formula and the symplectic sum formula. Finally, we explicitly compute some of these family GW invariants for elliptic surfaces.

To my parents.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude and thanks to my advisor, Professor Thomas H. Parker for his constant and patient help, encouragement, and excellent advice.

I would also like to thank Professors Ronald Fintushell, John D. McCarthy, Ivanov Nikolai, and Akbulut Selman for their time and valuable suggestions.

TABLE OF CONTENTS

Introduction

Gromov-Witten invariants are counts of holomorphic curves in a symplectic manifold X. To define them using the analytic approach one chooses an almost complex structure J compatible with the symplectic structure and considers the set of maps $f : \Sigma \to X$ from Riemann surfaces Σ which satisfy the (nonlinear elliptic) J-holomorphic map equation

$$\overline{\partial}_J f = 0. \tag{0.1}$$

After compactifying the moduli space of such maps, one imposes constraints, requiring, for example, that the image of the map passes through specified points. With the right number of constraints and a generic J, the number of such maps is finite. That number is a GW invariant; it depends only on the symplectic structure of X.

There are some beautiful conjectures about what the counts of holomorphic curves on Kähler surfaces *ought to be* ([V],[KP],[YZ],[G]). However, as currently defined, the corresponding GW invariants of Kähler surfaces with $p_g \geq 1$ are all zero! This discrepancy occurs because GW invariants count curves for *generic* almost complex structures J, whereas Kähler structures are very special — Donaldson details this in [D]. They can have whole families of curves which disappear when the Kähler J is perturbed to a generic J. For example, a generic $K3$ surface ($p_g = 1$) has no holomorphic curves at all, whereas *algebraic* K3 surfaces do admit holomorphic curves.

Clearly a new version of the invariants is needed — one which counts the relevant holomorphic curves. Work in that direction is just beginning. Bryan and Leung ([BL1],[BL2]) defined such invariants for K3 and abelian surfaces by using the Twistor family; they were also able to calculate their invariants in important cases. In a preprint to appear shortly, Behrend-Fantechi [BF] have define invariants for a more general class of algebraic surfaces using algebraic geometry, but have not yet made calculations. We approach the same issues using the geometric analysis approach to GW invariants.

Given a Kähler manifold (X, ω, J, g) we constructs a $2p_g$-dimensional family of elements $K_J(f, \alpha)$ in $\Omega^{0,1}(f^*TX)$, where α is a real part of a holomorphic 2 form. We then modifies the J-holomorphic map equation (1) by considering the pairs (f, α) satisfying

$$\overline{\partial}_J f = K_J(f, \alpha). \tag{0.2}$$

The solutions of this equation form a moduli space whose dimension is $2p_g$ larger than the dimension of the usual GW moduli space.

Because α range over a vector space compactness is an issue. Here things get interesting because there are instances when the moduli space for (0.2) is *not* compact. In fact, when the map represents a component of a canonical divisor the moduli space is *never* compact. Nevertheless, there is a simple analytic criterion — the uniform boundedness of the energy of the map and the L^2 norm of α — that ensures that the moduli space is compact.

Theorem 0.0.1 *Let (X, J) be a Kähler surface and fix a genus g and a class $A \in H_2(X, \mathbb{Z})$. Denote by $C(J)$ the supremum of $E(f) + ||\alpha||_{L^2}$ over all (J, α)-holomorphic maps from genus g curves into X which represent A. If $C(J)$ is finite, then the family*

GW invariants

$$GW_{g,k}^{J,\mathcal{H}}(X, A)$$

are well-defined. They are invariant under deformations $\{J_t\}$ *of the Kähler structure with* $C(J_t)$ *bounded. Furthermore, if* A *is a* $(1,1)$ *class then all the maps which contribute to these invariants are in fact* J*-holomorphic.*

The last sentence of Theorem 0.0.1 means that the invariants for $(1,1)$ classes are counts of holomorphic curves in (X, J). That is not the same as saying the invariants are enumerative, since there is no claim that each curve is counted with multiplicity one. But it does mean that the family GW invariants, which *a priori* are counts of maps which are holomorphic with respect to families of almost complex structures on X, are in fact calculable from the complex geometry of (X, J) alone.

Theorem 0.0.1 yields well-defined family GW invariants provided there is a finite energy bound $C(J)$. Following the Kodaira classification of surfaces, we verify the energy bound case-by-case using geometric arguments. That yields the following cases where the family GW invariants are well-defined.

Proposition 0.0.2 *The moduli space for a class* A *is compact, and hence the family GW invariants are well-defined, when* (X, J) *is*

(a) a K3 or abelian surface with $A \neq 0$,

(b) a minimal elliptic surface $\pi : E \to C$ *with Kodaira dimension 1 with* $-A \cdot$ *(fiber class)* $\neq$ *deg* $\pi_*(A)$ *, and*

(c) a minimal surface of general type and A *is in a certain subspace of the (1,1) classes (see Proposition 4.0.26).*

The second half of this paper develops computational methods. We extend several existing techniques for calculating GW invariants to the family GW invariants.

In particular, the 'TRR formula' applies to the family invariants, and at least some special cases of the symplectic sum formula [IP3] apply, with appropriate minor modifications to the formula. Those formulas enable us to enumerate the curves in the elliptic surfaces $E(n)$ for the class $A=$ section plus multiples of the fiber.

Theorem 0.0.3 *Let $E(n)$ be a standard elliptic surface with a section s of self-intersection $-n$. Denote by S and F the homology class of the section and the fiber. Then the $g=0$ family GW invariants for the classes $A=S+dF$ are well-defined and are given by the generating function*

$$\sum_{d\geq 0} GW_{0,0}^{\mathcal{H}_n}\left(E(n), S+dF\right) t^d = \prod_{d\geq 0}\left(\frac{1}{1-t^d}\right)^{12n}. \tag{0.3}$$

Bryan and Leung used algebraic methods to show (0.3) for $K3$ surfaces (i.e. $n=2$) [BL1]. This provided a verification of the well-known Yau-Zaslow Conjecture [YZ] for those cases when the homology class A is primitive. On the other hand, the above formula for $n=1$ gives the ordinary GW-invariants of rational elliptic surface $E(1)$, which was shown by Ionel and Parker [IP3]. They related TRR formula and their sum formula for the relative invariants to obtain a quasi-modular form as in (0.3). We follow the same argument — relating TRR formula and sum formula — to show Theorem 0.0.3.

Chapter 1 gives the definition of a (J,α)-holomorphic map and some of the analytic consequences of that defintion, most notably an expression for the energy in terms of pullbacks of the symplectic form and the form α. Chapter 2 begins by describing the relation between a complete linear system $|C|$ — or more generally a Severi variety — and the moduli space of (J,α)-holomorphic maps. That leads us to consider the family of (J,α)-holomorphic maps in which α is the real part of holomorphic 2-form;

the corresponding family moduli space should be an analytic version of the Severi variety. As partial justification of that view, we prove the last statement of Theorem 0.0.1: any (J, α)-holomorphic map which represents a (1,1) class is in fact holomorphic (theorem 2.0.12).

Chapter 3 summarizes the analytic results which lead to the definition of the family GW-invariants. That involves constructing the virtual moduli cycle by adapting the method of Li and Tian [LT]. Thus defined, the family invariants satisfy a Divisor Axiom and a Composition Law analogous to those of ordinary GW-invariants. To keep the exposition flowing the main results are stated in Chapter 3 and their technical proofs are deferred until Chapter 5.

Chapter 4 contains examples of Kähler sufaces with $p_g \geq 1$ with well-defined family invariants. We focus on minimal surfaces and establish the results summarized in Proposition 0.0.2 above. For the case of K3 and Abelian surfaces we prove that our family GW-invariants agree with the invariants defined by Bryan and Leung. That is done in the course of the proof of Theorem 4.0.23 by relating the holomorphic 2-forms to the twistor family.

Chapter 5 contains the analysis which proves that the family GW invariants are well-defined. Slightly modifying the arguments of Li and Tian, we consider the product of the space of C^ℓ stable maps and the parameter space $\mathcal{H}$ for α. The (J, α)-holomorphic map equation defines a section Φ of a generalized bundle $E \to B$ whose zero set is the moduli space of (J, α)-holomorphic maps. In general that space is neither smooth nor compact. For the ordinary GW-invariants Li and Tian showed that after perturbing Φ its zero set becomes smooth and compact and defines a virtual moduli cycle. Our case requires more care because the parameter space $\mathcal{H}$ is not compact.

The construction consists of two main steps. First, using the Fredholm property of the section Φ of $E \to B$, one can construct a collection of finite dimensional subbundles $E_i \to U_i = \Phi^{-1}(E_i)$ whose union contains the moduli space and has each restriction $\Phi_i = \Phi|_{U_i}$ being smooth. Compactness of moduli space, which follows from the energy bound $C(J)$ of Theorem 0.0.1, ensures that there is such a finite collection. Second, one can perturb the moduli space locally in each U_i in such a way that the local perturbations fit together to produce a well-defined cycle, the virtual moduli cycle. This second part is very general procedure and is proved in Theorem 1.2 of [LT].

Thus the bulk of Chapter 5 is devoted to working through the first step. The arguments parallel the proofs for J-holomorphic maps in [LT]. The key step is establishing uniform estimates for the linearization of holomorphic map equation and its adjoint operator. Those estimates are still true for (J, α)-holomorphic maps and are locally uniform in α. The exposition ends up being rather lengthy because of the need to recall the extensive notation of [LT] and because we have taken the trouble of filling in missing details and some fixing minor errors in [LT]. At the end of Chapter 5 we prove the two properties of family GW-invariants: the Divisor Axiom and the Composition Law.

Turning to the computations, we give an overview of the proof of Theorem 0.0.3 in Chapter 6. This argument is an extension of the elegant argument used by Ionel and Parker to compute the GW-invariants of $E(1)$ [IP3]. It involves computing the generating function for the invariants in two ways, first using the so-called TRR formula, and second using a syplectic sum formula as in [IP3]. Roughly, the only modification needed is a shift in the dimension counts. But to justify the computation we need to extend both the TRR formula and the symplectic sum formula to apply to the family GW invariants. The extended TRR formula is proved in Chapter 7 and sum formula is established in the last four Chapters.

Chapter 8 gives an alternative definition of the family invariants for $E(n)$ based on the idea of perturbing the (J, α)-holomorphic map equations as in [RT1] and [RT2]. This alternative definition is better suited to adapt the analytic arguments in [IP2] and [IP3] to a family version of sum formula. The proof of the sum formula begins by studying holomorphic maps into a degeneration of $E(n)$. Because $E(n)$ is a Kähler surface we are able to degenerate within a holomorphic family, rather than the symplectic family used in [IP3].

The degeration family Z is constructed in Chapter 9. It is a family $\lambda : Z \to D^2$ whose fiber Z_λ at $\lambda \neq 0$ is a copy of $E(n)$ and whose center fiber is a union of $E(n)$ with $T^2 \times S^2$ along a fixed elliptic fiber V. As $\lambda \to 0$ maps into Z_λ converge to maps into Z_0, and by bumping α to zero along the fiber V we can ensure that the limits satisfy a simple matching condition along V (there is a single matching condition for the classes A that we consider). Conversely, if a map into Z_0 satisfies the matching condition then it can be smoothed to produce a map into Z_λ for small λ. That smoothing is described in Chapter 11 and then used prove the required sum formula for the family invariants of $E(n)$.

The appendix contains a brief discussion of how the family GW invariants defined here relate to those defined by Behrend and Fantachi in [BF].

CHAPTER 1

(J, α)-holomorphic maps

A J-holomorphic map into an almost complex manifold (X, J) is a map $f : \Sigma \to X$ from a complex curve Σ (a closed Riemann surface with complex structure j) whose differential is complex linear. Equivalently, f is a solution of the J-holomorphic map equation

$$\bar{\partial}_J f = 0 \qquad \text{where} \qquad \bar{\partial}_J f = \frac{1}{2}(df + Jdfj).$$

In this Chapter we will show that when X is four-dimensional there is natural infinite-dimensional family of almost complex structures parameterized the J-anti-invariant 2-forms on X.

Let (X, J) be a 4-dimensional almost Kähler manifold with the hermitian triple (ω, J, g). Using J, we can decompose $\alpha \in \Omega^2(X)$ as $\alpha = \alpha_+ + \alpha_-$ where

$$\alpha_+(u, v) = \frac{\alpha(u, v) + \alpha(Ju, Jv)}{2} \qquad \alpha_-(u, v) = \frac{\alpha(u, v) - \alpha(Ju, Jv)}{2} \tag{0.1}$$

Definition 1.0.4 *A 2-form α is called J-anti-invariant if $\alpha = \alpha_-$. Denote the set of all J-anti-invariant 2-forms by $\Omega^-_J(X)$. Each $\alpha \in \Omega^-_J(X)$ defines an endomorphism K_α of TX by the equation*

$$\langle u, K_\alpha v\rangle = \alpha(u, v). \tag{0.2}$$

It follows that

$$\langle K_\alpha u, v\rangle = -\langle u, K_\alpha v\rangle, \quad JK_\alpha = -K_\alpha J, \quad \text{and} \quad \langle Ju, K_\alpha u\rangle = 0. \tag{0.3}$$

Definition 1.0.5 *For $\alpha \in \Omega_J^-(X)$, a map $f : \Sigma \to X$ is called (J,α)-holomorphic if*

$$\overline{\partial}_J f = K_J(f,\alpha) \tag{0.4}$$

where $K_J(f,\alpha) = K_\alpha(\partial f \circ j) = \frac{1}{2}K_\alpha(df - Jdfj)j$.

The next proposition and its corollary list some pointwise relations involving the quantities that appear in the (J,α)-holomorphic equation. We state these first for general C^1 maps, then specialize to (J,α)-holomorphic maps.

Proposition 1.0.6 *Fix a metric within the conformal class j and let dv be the associated volume form. Then for any C^1 map f we have the pointwise equalities*

(a) $|\overline{\partial}_J f|^2\, dv = \frac{1}{2}|df|^2\, dv - f^*\omega$, (b) $\langle \overline{\partial}_J f\,,\, K_J(f,\alpha)\rangle\, dv = f^*\alpha$,

(c) $K_\alpha^2 = -|\alpha|^2 I$, (d) $|K_J(f,\alpha)|^2\, dv = |\alpha|^2\left(\frac{1}{2}|df|^2\, dv + f^*\omega\right)$.

Proof. Fix a point $p \in \Sigma$ and an orthonormal basis $\{e_1, e_2 = je_1\}$ of $T_p\Sigma$. Setting $v_1 = df(e_1)$ and $v_2 = df(e_2)$, we have $2\overline{\partial}_J f(e_1) = v_1 + Jv_2$ and $2K_J(f,\alpha)(e_1) = K_\alpha v_2 - JK_\alpha v_1$, and similarly $2\overline{\partial}_J f(e_2) = v_2 - Jv_1$ and $2K_J(f,\alpha)(e_2) = -K_\alpha v_1 - JK_\alpha v_2$. Therefore,

$$\begin{aligned} 4|\overline{\partial}_J f|^2 = |v_1 + Jv_2|^2 + |v_2 - Jv_1|^2 &= 2(|v_1|^2 + |v_2|^2) + 4\langle v_1, Jv_2\rangle \\ &= 2|df|^2 - 4f^*\omega(e_1, e_2). \end{aligned}$$

That gives (a), and (b) follows from the similar computation

$$\begin{aligned}
4\langle\overline{\partial}_J f, K(f,\alpha)\rangle &= \langle v_1 + Jv_2, K_\alpha v_2 - JK_\alpha v_1\rangle + \langle v_2 - Jv_1, -K_\alpha v_1 - JK_\alpha v_2\rangle \\
&= \langle v_1, K_\alpha v_2\rangle - \langle v_1, JK_\alpha v_1\rangle + \langle Jv_2, K_\alpha v_2\rangle - \langle Jv_2, JK_\alpha v_1\rangle \\
&\qquad -\langle v_2, K_\alpha v_1\rangle - \langle v_2, JK_\alpha v_2\rangle + \langle Jv_1, K_\alpha v_1\rangle + \langle Jv_1, JK_\alpha v_2\rangle \\
&= 4\langle v_1, K_\alpha v_2\rangle \\
&= 4f^*\alpha(e_1, e_2).
\end{aligned}$$

Next fix $x \in X$ and an orthonormal basis $\{w^1, w^2, w^3, w^4\}$ of T_x^*X with $w^2 = -Jw_1$ and $w^4 = -Jw^3$. Then the six forms

$$w^1 \wedge w^2 \pm w^3 \wedge w^4, \quad w^1 \wedge w^3 \pm w^2 \wedge w^4, \quad w^1 \wedge w^4 \pm w^2 \wedge w^3$$

give an orthonormal basis of $\Lambda^2(T_x^*X)$, and two of these span the subspace of J anti-invariant forms. Hence

$$\alpha = a(w^1 \wedge w^3 - w^2 \wedge w^4) + b(w^1 \wedge w^4 + w^2 \wedge w^3)$$

for some a and b, and in this basis K_α is the matrix

$$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & b & -a \\ -a & -b & 0 & 0 \\ -b & a & 0 & 0 \end{pmatrix}$$

Consequently, $K_\alpha^2 = -(a^2+b^2)I = -|\alpha|^2 I$. Lastly, since K_α is skew-adjoint, (c) shows that

$$|K_J(f,\alpha)|^2 = -\langle \partial f \circ j,\ K_\alpha^2(\partial f \circ j)\rangle = |\alpha|^2\,|\partial f|^2.$$

Equation (d) then follows from (a) because $|df|^2 = |\partial f|^2 + |\overline{\partial}_J f|^2$. □

Corollary 1.0.7 *Suppose the map $f : \Sigma \to X$ is (J,α)-holomorphic. Then* *(a)* $|\overline{\partial}_J f|^2 \, dv = f^*\alpha$,

(b) $(1-|\alpha|^2)\, |df|^2 \, dv = 2\,(1+|\alpha|^2)\, f^*\omega$, *and*

(c) $|\alpha|^2\, |df|^2 = (1+|\alpha|^2)\, |\overline{\partial}_J f|^2$.

Proof. Since f is (J,α)-holomorphic, $|\overline{\partial}_J f|^2 = \langle \overline{\partial}_J f, K_J(f,\alpha)\rangle = |K_J(f,\alpha)|^2$, so (a) follows from Proposition 1.0.6b while (b) and (c) follow from Proposition 1.0.6 (a) and (d). □

There is a second way of writing the (J,α)-holomorphic equation (0.4). For each $\alpha \in \Omega^-_J(X)$, $I + JK_\alpha$ is invertible since JK_α is skew-adjoint. Hence

$$J_\alpha = (I + JK_\alpha)^{-1} J (I + JK_\alpha) \tag{0.5}$$

is an almost complex structure. A map $f : \Sigma \to X$ is (J,α)-holomorphic if and only if f is J_α-holomorphic, i.e. satisfies

$$\overline{\partial}_{J_\alpha} f \;=\; \frac{1}{2}\,(df + J_\alpha df j) \;=\; 0. \tag{0.6}$$

From this perspective, a solution of the (J,α)-holomorphic equation is a J_α holomorphic map with J_α lying in the family (0.5) parameterized by $\alpha \in \Omega^-_J(X)$. In particular, we see from (0.6) that the (J,α)-holomorphic equation is elliptic.

Proposition 1.0.8 *For any $\alpha \in \Omega^J_-(X)$, the almost complex structure J_α on X satisfies*

$$\langle J_\alpha u, J_\alpha v\rangle = \langle u, v\rangle \qquad \text{and} \qquad J_\alpha = \frac{1-|\alpha|^2}{1+|\alpha|^2}\, J - \frac{2}{1+|\alpha|^2}\, K_\alpha \tag{0.7}$$

Proof. From (0.3), the endomorphisms $A_+ = I + JK_\alpha$ and $A_- = I - JK_\alpha$ are transposes, and $A_+ J = JA_-$ and $A_+ K_\alpha = K_\alpha A_-$. Consequently, A_+^{-1} and A_-^{-1} are transposes, with $A_-^{-1}J = JA_+^{-1}$ and $A_-^{-1}K_\alpha = K_\alpha A_+^{-1}$ and therefore $A_-^{-1}A_+ = A_+A_-^{-1}$. Consequently,

$$\begin{aligned}\langle J_\alpha u, J_\alpha v\rangle &= \langle A_+^{-1}JA_+u, A_+^{-1}JA_+v\rangle = \langle JA_-^{-1}A_+u, JA_-^{-1}A_+v\rangle \\ &= \langle A_-^{-1}A_+u, A_-^{-1}A_+v\rangle = \langle u, A_-A_+^{-1}A_-^{-1}A_+v\rangle \\ &= \langle u, v\rangle.\end{aligned}$$

On the other hand, noting that $K_\alpha^2 = -|\alpha|^2 I$, it is easy to verify that

$$(I + JK_\alpha)^{-1} = \frac{1}{1+|\alpha|^2} I - \frac{1}{1+|\alpha|^2} JK_\alpha. \tag{0.8}$$

With that, the second part of (0.7) follows from the definition of J_α. □

In summary, (J,α)-holomorphic maps can be regarded as solutions of the J_α-holomorphic map equation $\overline{\partial}_{J_\alpha} f = 0$ for a family of almost complex structures parameterized by α as in (0.6). We will frequently move between these two viewpoints.

CHAPTER 2

Curves and Canonical Families of (J, α) Maps

Given a Kähler surface X, we would like to use (J, α)-holomorphic curves to solve the following problem in enumerative geometry:

Enumerative Problem Give a $(1, 1)$ homology class A, count the curves in X that represent A, have a specified genus g, and pass through the appropriate number of generic points.

We begin this Chapter with some dimension counts which show that in order to interpret this problem in terms of holomorphic maps we need to consider families of maps of dimension p_g. We then show that there is a very natural family of (J, α)-holomorphic maps with exactly that many parameters. We conclude the Chapter with a theorem showing that such maps do indeed represent holomorphic curves in X.

One can phrase the above enumerative problem in terms of the *Severi variety* $V_g(C) \subset |C|$, which is defined to be the closure of the set of all curves with geometric genus g. Assuming that $C - K$ is ample, it follows from the Riemann-Roch theorem that the dimension of the complete linear system $|C|$ is given in terms of

$p_g = \dim_{\mathbb{C}} H^{0,2}(X)$ and $q = \dim_{\mathbb{C}} H^{0,1}(X)$ by

$$\dim_{\mathbb{C}}|C| \;=\; \frac{C^2 - C\cdot K}{2} \;+\; p_g \;-\; q$$

and the formal dimension of the Severi variety is

$$\dim_{\mathbb{C}} V_g(C) \;=\; -K\cdot C \;+\; g-1 \;+\; p_g \;-\; q. \tag{0.1}$$

The right-hand side of (0.1) is the 'appropriate number' of point constraints to impose; the set of curves in $V_g(C)$ through that many generic points should be finite, making the enumerative problem well-defined.

Now let $\mathcal{M}_g(X,A)$ be the moduli space of holomorphic maps from Riemann surfaces of genus g, which represent homology class A. Then its virtual dimension is given by

$$\dim{}_{\mathbb{C}}\mathcal{M}_g(X,A) \;=\; -K\cdot A \;+\; g-1. \tag{0.2}$$

The image of a map in $\overline{\mathcal{M}}_g(X,[C])$ might be not a divisor in $|C|$, instead it is a divisor in some other complete linear system $|C'|$ with $[C'] = [C]$. As in [BL1], we define the parameterized Severi variety

$$V_g([C]) = \coprod_{[C']=[C]} V_g(C')$$

Its expected dimension is now given by

$$\dim_{\mathbb{C}} V_g([C]) \;=\; -K\cdot C \;+\; g-1 \;+\; p_g. \tag{0.3}$$

We still have p_g dimensional difference between (0.3) and (0.2). Therefore, the cut-down moduli space by (0.3) many point constraints is empty when $p_g \geq 1$. This implies that the corresponding Gromov-Witten invariants is zero, whenever $p_g \geq 1$.

We show that there is a natural — in fact obvious — p_g-dimensional family of (J,α)-holomorphic maps associated with every Kähler surface.

Definition 2.0.9 *Given a Kähler surface on X, define the parameter space $\mathcal{H}$ by*

$$\mathcal{H} = \{\alpha + \overline{\alpha} \mid \alpha \in H^{2,0}(X)\} \tag{0.4}$$

Here $H^{2,0}(X)$ means the set of holomorphic $(2,0)$ forms on X. Note that all forms $\alpha \in H^{2,0}(X)$ are closed since $d\alpha = \partial\alpha + \overline{\partial}\alpha = \partial\alpha$ is a $(3,0)$ form and hence vanishes because X is a complex surface. Thus $\mathcal{H} \subset \Omega^-_J(X)$ is a $2p_g$-dimensional real vector space of closed forms. We give it the (real) inner product defined by the L^2 inner product of forms:

$$\langle\alpha,\ \beta\rangle = \int_X \alpha \wedge \beta. \tag{0.5}$$

We can use the forms $\alpha \in \mathcal{H}$ to parameterize the right-hand side of the (J, α)-holomorphic map equation (1.0.5).

Definition 2.0.10 *Henceforth the term '(J,α)-holomorphic map' means a map satisfying (1.0.5) for α in the above family $\mathcal{H}$.*

Lemma 2.0.11 *The zero divisor $Z(\alpha)$ of each nonzero $\alpha \in \mathcal{H}$ represents the canonical class.*

Proof. Write $\alpha = \beta + \overline{\beta}$ with $\beta \in H^{2,0}(X)$. Since β is a section of the canonical bundle, this means that $Z(\alpha) = Z(\beta)$ represents the canonical divisor with appropriate multiplicities. □

Next, using this $2p_g$ dimensional parameter space $\mathcal{H}$, we define the family moduli space

$$\overline{\mathcal{M}}_g^{\mathcal{H}}(X, [C]) = \{\ (f,\alpha) \mid \overline{\partial}_{J_\alpha} f = 0,\ \ [\mathrm{Im}\, f] = [C],\ \text{and}\ \alpha \in \mathcal{H}\ \}$$

Since we just parameterize the $\bar{\partial}$-operator by $2p_g$ dimensional parameter space, the formal dimension of the family moduli space is given by

$$(\text{Formal})\ \dim_{\mathbf{C}}\overline{\mathcal{M}}_g^{\mathcal{H}}(X,[C]) \ = \ -K\cdot C \ + \ g-1 \ + \ p_g$$

On the other hand, we define *a component of the canonical class* to be a homology class of a component of some canonical divisor.

Theorem 2.0.12 *If f is a (J,α)-holomorphic map which represents a class $A \in H^{1,1}(X)$. Then f is, in fact, holomorphic. Furthermore, if A is not a linear combination of components of the canonical class, then $\alpha = 0$.*

Proof. Since $\alpha \in H^{2,0}(X) \oplus H^{2,0}(X)$ is closed and $A \in H^{1,1}(X)$, it follows from Corollary 1.0.7a that

$$\int_{\Sigma} |\bar{\partial}_J f|^2 = \alpha(A) = 0.$$

Thus f is holomorphic, that is, $\bar{\partial}_J f \equiv 0$. Consequently, $|\alpha|^2|df|^2 \equiv 0$ by Corollary 1.0.7c. Since df has at most finitely many zeros, we can conclude that $\alpha = 0$ along the image of f. Hence $\alpha = 0$, otherwise it contradicts to the assumption on A by Lemma 2.0.11. □

CHAPTER 3

Family GW-Invariants

Let X be a complex surface with a Kähler structure (ω, J, g). In this Chapter we will define the Family Gromov-Witten Invariants associated to (X, J) and the parameter space $\mathcal{H}$ of (0.4). We also state some properties of these invariants. To keep the discussion clear we defer the proofs and some technical definitions until later Chapters.

Our approach is to extend the analytic arguments of Li and Tian [LT] to show that the moduli space of (J, α)-holomorphic maps carries a virtual fundamental class whenever it is compact. While compactness is automatic for the usual Gromov-Witten invariants, it must be verified case-by-case for the family GW invariants (see Example 3.5 below). Thus compactness appears as a hypothesis in the results of this Chapter.

First, we recall the notion of C^ℓ stable maps as defined in [LT]. Fix an integer $l \geq 0$ and consider pairs $(f; \Sigma, x_1, \cdots, x_k)$ consisting of

1. a connected nodal curve $\Sigma = \bigcup_{i=1}^{m} \Sigma_i$ of arithmetic genus g with distinct smooth marked points $x_1, \cdots, x_k$, and

2. a continuous map $f : \Sigma \to X$ so that each restriction $f_i = f_{|\Sigma_i}$ lifts to a C^l-map from the normalization $\tilde{\Sigma}_i$ of Σ into X.

Definition 3.0.13 *A* stable C^l map *of genus g with k marked points is a pair $(f;\Sigma,x_1,\cdots,x_k)$ as above which satisfies the stability condition:*

- *If the homology class $[f_i]\in H_2(X,\mathbb{Q})$ is trivial, then the number of marked points in Σ_i plus the arithmetic genus of Σ_i is at least three.*

Two stable maps $(f,\Sigma;x_1,\cdots,x_k)$ and $(f',\Sigma';x'_1,\cdots,x'_k)$ are equivalent if there is a biholomorphic map $\sigma:\Sigma\mapsto\Sigma'$ such that $\sigma(x_i)=x'_i$ for $1\le i\le k$ and $f'=f\circ\sigma$. We denote by

$$\overline{\mathcal{F}}^l_{g,k}(X,A)$$

the space of all equivalence classes $[f;\Sigma,x_1,\cdots,x_k]$ of C^l-stable maps of genus g with k marked points and with total homology class A. The topology of $\overline{\mathcal{F}}^l_{g,k}(X,A)$ is defined by sequential convergence as in Chapter 2 of [LT]. There are two continuous maps from $\overline{\mathcal{F}}^l$. First, there is an evaluation map

$$ev:\overline{\mathcal{F}}^l_{g,k}(X,A)\to X^k \tag{0.1}$$

which records the images of the marked points. Second, for $2g+k\ge3$, collapsing the unstable components of the domain gives a stabilization map

$$st:\overline{\mathcal{F}}^l_{g,k}(X,A)\to\overline{\mathcal{M}}_{g,k} \tag{0.2}$$

to the compactified Deligne-Mumford space of genus g curves with k marked points. For $2g+k<3$ we define $\overline{\mathcal{M}}_{g,k}$ to be the topological space of consisting of a single point and define (0.2) to be the map to that point.

We next construct a 'generalized bundle' E over $\overline{\mathcal{F}}^l_{g,k}(X,A)\times\mathcal{H}$, again following [LT]. Recall that each $\alpha\in\mathcal{H}$ defines an almost complex structure J_α on X by (0.5). Denote by $\mathrm{Reg}(\Sigma)$ the set of all smooth points of Σ. For each $([f;\Sigma,x_1,\cdots,x_k],\alpha)$, define

$$\Lambda_{j_\Sigma J_\alpha}(f^*TX)$$

to be the set of all continuous sections ν of $\mathrm{Hom}(T\mathrm{Reg}(\Sigma), f^*TX)$ with $\nu \circ j_\Sigma = -J_\alpha \circ \nu$ such that ν extends continuously across the nodes of Σ. We take E to be the bundle whose fiber over $([f, \Sigma; x_1, \cdots, x_k], \alpha)$ is $\Lambda_{j_\Sigma J_\alpha}(f^*TX)$ and give E the continuous topology as in Chapter 2 of [LT]. We then define a section $\Phi : \overline{\mathcal{F}}^{1}_{g,k}(X, A) \times \mathcal{H} \to E$ by

$$\Phi([f, \Sigma; x_1, \cdots, x_k], \alpha) = df + J_\alpha df j_\Sigma. \tag{0.3}$$

The right-hand side of (0.3) vanishes for J_α-holomorphic maps. Thus $\Phi^{-1}(0)$ is the moduli space of (J, α)-holomorphic maps. The following is a family version of Proposition 2.2 in [LT].

Proposition 3.0.14 *Suppose that the set $\Phi^{-1}(0)$ is compact. Then the section Φ gives rise to a generalized Fredholm orbifold bundle with a natural orientation and with index*

$$r = 2c_1(X)[A] + 2(g-1) + 2k + \dim \mathcal{H}. \tag{0.4}$$

We will postpone both the proof of Proposition 3.0.14, and the definitions of the terms in its statement until Chapter 5. Until then we will accept it, and continue following the construction of Li-Tian.

By Theorem 1.2 of [LT], the bundle E has a rational homology "Euler class" in $\overline{\mathcal{F}}^{1}_{g,k}(X, A) \times \mathcal{H}$; in fact, since $\mathcal{H}$ is contractible this Euler class lies in $H_r(\overline{\mathcal{F}}^{1}_{g,k}(X, A); \mathbb{Q})$ where r is the index (0.4). We call this class the *virtual fundamental cycle* of the moduli space of family holomorphic maps parameterized by $\mathcal{H}$ and denote it by

$$[\mathcal{M}^{J,\mathcal{H}}_{g,k}(X, A)]^{\mathrm{vir}}. \tag{0.5}$$

In particular,

$$\dim\, [\mathcal{M}^{J,\mathcal{H}}_{g,k}(X, A)]^{\mathrm{vir}} = 2c_1(X)[A] + 2(g-1) + 2k + 2p_g. \tag{0.6}$$

The next issue is whether the virtual fundamental cycle is independent of the Kähler structure on X. The next proposition is analogous to the Proposition 2.3 in [LT]. It shows that the virtual fundamental cycle depends only on certain deformation class of the Kähler structure.

Proposition 3.0.15 *Let (ω_t, J_t, g_t), $0 \leq t \leq 1$, be a continuous family of Kähler structures on X. Let $\mathcal{H}_t$ be the corresponding continuous family of finite subspaces defined by (0.4) and let Φ_t be the corresponding family of sections of E_t over $\overline{\mathcal{F}}^J_{g,k}(X, A) \times \mathcal{H}_t$. If $\Phi_t^{-1}(0)$ is compact for all $0 \leq t \leq 1$, then*

$$[\mathcal{M}^{J_0,\mathcal{H}_0}_{g,k}(X, A)]^{\mathrm{vir}} = [\mathcal{M}^{J_1,\mathcal{H}_1}_{g,k}(X, A)]^{\mathrm{vir}}.$$

We also postpone the proof of Proposition 3.0.15 to Chapter 5.

The family GW invariants can now be defined by pulling back cohomology classes by the evaluation and stabilization maps and integrating over the virtual fundamental cycle. That of course requires that the virtual fundamental cycle exists, so we must assume that we are in a situation where $\Phi_t^{-1}(0)$ is compact.

Definition 3.0.16 *Whenever the virtual fundamental cycle $[\mathcal{M}^{J,\mathcal{H}}_{g,k}(X, A)]^{\mathrm{vir}}$ exists, we define the family GW invariants of (X, J) to be the map*

$$GW^{J,\mathcal{H}}_{g,k}(X, A) : [H^*(X; \mathbb{Q})]^k \times H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}) \mapsto \mathbb{Q}$$

defined on $\alpha_1, \cdots, \alpha_k \in H^*(X; \mathbb{Q})$ and $\beta \in H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$ by

$$GW^{J,\mathcal{H}}_{g,k}(X, A)(\beta; \alpha_1, \cdots, \alpha_k) = [\mathcal{M}^{J,\mathcal{H}}_{g,k}(X, A)]^{\mathrm{vir}} \cap (st^*(\beta) \cup ev^*(\pi_1^*\alpha_1 \wedge \cdots \wedge \pi_k^*\alpha_k)).$$

We will use the shorter notation

$$GW^{J,\mathcal{H}}_{g,k}(X, A)(\alpha_1, \cdots, \alpha_k)$$

for the special case when $\beta = 1 \in H^0(\overline{\mathcal{M}}_{g,k})$; this corresponds to imposing no constraints on the complex structure of the domain.

The condition that $\Phi^{-1}(0)$ is compact must be checked "by hand". In general, $\Phi^{-1}(0)$ is compact for some choices of A, but not for others.

Example 0.17 Let (X, J) be a Kähler surface with $p_g > 1$. Then there is a non-zero element $\beta \in H^{2,0}$ whose zero set $Z(\beta)$ is non-empty, represents the canonical class K, and whose irreducible components can be parameterized by holomorphic maps. Fix a parameterization $f : \Sigma \to X$ of one such component; this represents a non-zero class $A \in H_2(X, \mathbb{Z})$. Then $\alpha = \beta + \overline{\beta}$ lies in the space $\mathcal{H}$ of (2.0.9) and $\Phi(f, \lambda\alpha) = 0$ for all real λ. Thus on any Kähler surface with $p_g > 1$, the set $\Phi^{-1}(0)$ is not compact for an component of the canonical class A.

On the other hand, in the next Chapter we will give examples of classes A in Kähler surfaces with $p_g > 1$ for which $\Phi^{-1}(0)$ *is* compact.

Theorem 3.0.18 *If there is a constant C, depending only on (X, ω, J, g) such that $E(f) + ||\alpha|| < C$ for all (J, α)-holomorphic maps into (X, J), then $\Phi^{-1}(0)$ is compact and hence the family GW invariants are well-defined.*

Proof. Consider a sequence (f_n, α_n) of J_α-holomorphic maps. The uniform bound on $||\alpha_n||$ implies that the J_α lie in a compact family. Since $E(f_n) < C$ the proof of Gromov's Compactness Theorem (see [PW] and [IS]) shows that $\{(f_n, \alpha_n)\}$ has a convergent subsequence. Consequently, $\Phi^{-1}(0)$ is compact as in the hypothesis of Proposition 3.0.14. That means that the virtual fundamental cycle (0.5) is well-defined. The family GW invariants are then given by Definition 3.0.16. □

We conclude this Chapter by listing two important properties of the family GW invariants. These are analogous to divisor axiom and composition laws of ordinary GW invariants. Again, the proofs appear in Chapter 5.

Proposition 3.0.19 (Divisor Axiom) *If $\alpha_k \in H^2(X, Z)$ then*

$$GW^{J,\mathcal{H}}_{g,k}(X, A)(\alpha_1, \cdots, \alpha_k) = \alpha_k(A)\ GW^{J,\mathcal{H}}_{g,k-1}(X, A)(\alpha_1, \cdots, \alpha_{k-1}). \qquad (0.7)$$

The second property generalizes the composition law of ordinary Gromov-Witten invariants. For that we consider maps from a domain Σ with node p and relate them to maps whose domain is the normalization of Σ at p. When the node is separating the genus and the number of marked points decompose as $g = g_1 + g_2$ and $k = k_1 + k_2$ and is a natural map

$$\sigma : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \mapsto \overline{\mathcal{M}}_{g,k} \tag{0.8}$$

defined by gluing (k_1+1)-th marked point of the first component to the first marked point of the second component. We denote by $PD(\sigma)$ the Poincaré dual of the image of this map σ.

Given any decomposition $A = A_1 + A_2$, $g = g_1 + g_2$, and $k = k_1 + k_2$ let $E_1 \oplus E_2^t$ be the generalized bundle over

$$\overline{\mathcal{F}}_{g_1,k_1+1}(X, A_1) \times \overline{\mathcal{F}}_{g_2,k_2+1}(X, A_2) \times \mathcal{H}$$

whose fiber over $([f_1, \Sigma_1; \{x_i\}], [f_2, \Sigma_2; \{y_j\}], \alpha)$ is $\Lambda^{0,1}_{j_{\Sigma_1} J_\alpha} \oplus \Lambda^{0,1}_{j_{\Sigma_2} J_{t\alpha}}$. The formula

$$\Psi_t([f_1, \Sigma_1; \{x_i\}], [f_2, \Sigma_2; \{y_j\}], \alpha) = (df_1 + J_\alpha df_1 j_{\Sigma_1}, df_2 + J_{t\alpha} df_2 j_{\Sigma_2}) \tag{0.9}$$

defines a section of $E_1 \oplus E_2^t$.

On the other hand, for non-separating nodes there is another natural map

$$\theta : \overline{\mathcal{M}}_{g-1,k+2} \mapsto \overline{\mathcal{M}}_{g,k} \tag{0.10}$$

defined by gluing the last two marked points. We also write $PD(\theta)$ for the Poincaré dual of the image of θ. The composition law is then the following two formulas.

Proposition 3.0.20 (Composition Law) *Let $\{H_\gamma\}$ be any basis of $H^*(X; Z)$ and $\{H^\gamma\}$ be its dual basis and suppose that $GW^{J,\mathcal{H}}_{g,k}(X, A)$ is defined.*
(a) Given any decomposition of (A, g, k), if the set $\Psi_t^{-1}(0)$ is compact for all $0 \le t \le 1$, then

$GW_{g,k}^{J,\mathcal{H}}(X,A)(PD(\sigma);\alpha_1,\cdots,\alpha_k)$
$= \sum_{\gamma, A=A_1+A_2} GW_{g_1,k_1+1}^{J,\mathcal{H}}(X,A_1)(\alpha_1,\cdots,\alpha_{k_1},H_\gamma)GW_{g_2,k_2+1}(X,A_2)(H^\gamma,\alpha_{k_1+1},\cdots,\alpha_k)$

(b) $GW_{g,k}^{J,\mathcal{H}}(X,A)(PD(\theta);\alpha_1,\cdots,\alpha_k) = \sum_\gamma GW_{g-1,k+2}^{J,\mathcal{H}}(X,A)(\alpha_1,\cdots,\alpha_k,H_\gamma,H^\gamma)$

That completes our overview of the family GW invariants. We next look at some examples, namely the various types of minimal Kähler surfaces. There we can use the specific geometry of the space to verify that the moduli space is compact and hence the family GW invariants are well-defined.

CHAPTER 4

Kähler surfaces with $p_g \geq 1$

In this Chapter we will focus on the family GW-invariants for minimal Kähler surfaces X with $p_g \geq 1$. The Enriques-Kodaira Classification [BPV] separates such surfaces into the following three types.

1. X is K3 or Abelian surface with canonical class $K = 0$. In this case, $p_g = 1$.

2. X is an elliptic surface $\pi : X \to C$ with Kodaira dimension 1. If the multiple fibers B_i have multiplicity m_i, then a canonical divisor is

$$K = \pi^* D + \sum (m_i - 1) B_i \qquad \text{where} \quad \deg D = 2g(C) - 2 + \chi(\mathcal{O}_X) \tag{0.1}$$

3. X is a surface of general type with $K^2 > 0$.

We will examine these cases one at a time. For each we will show that the family invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined. By Theorem 3.0.18 the key issue is bounding the energy $E(f)$ and the pointwise norm $|\alpha|$ uniformly for all (J, α)-holomorphic maps into X.

K3 and Abelian Surfaces

Let (X, J) be a K3 or Abelian surface. Since the canonical class is trivial, Yau's proof of the Calabi conjecture implies that (X, J) has a Kähler structure (ω, J, g) whose metric g is Ricci flat. For such a structure all holomorphic $(0, 2)$ forms are parallel, and hence have pointwise constant norm (see [B]). Thus $\mathcal{H} \cong \mathbb{C}$ consists of closed forms α with $|\alpha|$ constant. Furthermore, the structure is also hyperkähler, meaning that there is a three-dimensional space of Kähler structures which is isomorphic as an algebra to the imaginary quaternions. The unit two-sphere in that space is the so-called *Twistor Family* of complex structures.

Consider the set $\mathcal{T}_0 = \{ J_\alpha \mid \alpha \in \mathcal{H}\}$. Since α has no zeros, equation (0.7) shows that $J_\alpha \to -J$ uniformly as $|\alpha| \to \infty$. We can therefore compactify $\mathcal{T}_0$ to $\mathcal{T} \cong \mathbb{P}^1$ by adding $-J$ at infinity.

Proposition 4.0.21 *$\mathcal{T}$ is the Twistor Family induced from the hyperkähler metric g.*

Proof. Let $\alpha \in \mathcal{H}$ with $|\alpha| = 1$. It then follows from Proposition 1.0.8 that $J_\alpha = -K_\alpha$ and (α, J_α, g) is a Kähler structure on X. On the other hand, we define α' by $\alpha'(u, v) = \alpha(u, Jv)$. Then $|\alpha'| = 1$ and $\alpha' \in \mathcal{H}$ since β' is holomorphic for each holomorphic 2-form β. Moreover, by definition we have

$$J_{\alpha'} = -K_{\alpha'} = -JK_\alpha = JJ_\alpha.$$

Since $(\alpha', J_{\alpha'}, g)$ is also Kähler and $JJ_\alpha J_{\alpha'} = -Id$, the Kähler structures $\{J, J_\alpha, J_{\alpha'}\}$ multiply as unit imaginary quaternions. It follows that $\mathcal{T}$ is the Twistor Family induced from the hyperkhler metric g. □

Lemma 4.0.22 *Let A be a nontrivial homology class with $\omega(A) \geq 0$. Then there exits a constant C_A such that every (J, α)-holomorphic map $f : C \to X$ representing A with $\alpha \in \mathcal{H}$ satisfies*

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2 < \omega(A) + C_A \qquad \textit{and} \qquad |\alpha| \leq 1.$$

Proof. Since $|\alpha|$ is a constant, we can integrate Corollary 1.0.7b to conclude that $|\alpha| \leq 1$. Let C_A be an upper bound for the function $\alpha \mapsto |\alpha(A)|$ on the set of $\alpha \in \mathcal{H}$ with $|\alpha| \leq 1$. Because α is closed, Proposition 1.0.6a and Corollary 1.0.7a imply that

$$E(f) = \frac{1}{2}\int_C |df|^2 = \int_\Sigma f^*(\omega + \alpha) = \omega(A) + \alpha(A) \leq \omega(A) + C_A. \quad \square$$

Theorem 4.0.23 *Let (X, J) be a K3 or Abelian surface. For each non-trivial $A \in H_2(X, \mathbb{Z})$, the invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined and independent of J. Furthermore, if $A = mB$ and $A' = mB'$ where B and B' are primitive with the same square, then*

$$GW^{J,\mathcal{H}}_{g,k}(X, A) = GW^{J,\mathcal{H}}_{g,k}(X, A').$$

Proof. For any nontrivial homology class A, we can choose a Ricci flat Kähler structure (ω, J, g) such that $\omega(A) \geq 0$ (if $\omega(A) < 0$, then we choose $(-\omega, -J, g)$). It then follows from Lemma 4.0.22 and Theorem 3.0.18 that $GW^{J,\mathcal{H}}_{g,k}(X, A)$ is well-defined.

Bryan and Leung have applied the machinery of Li and Tian to define family GW invariants associated to the Twistor Family $\mathcal{T}$ [BL1, BL2]. Their invariants, which we denote by

$$\Phi^{\mathcal{T}}_{g,k}(X, A),$$

are actually independent of the Twistor Family since the moduli space of complex structures on X is connected. On the other hand, if $A = mB$ and $A' = mB'$ where B and B' are primitive with the same square, then there is an orientation preserving diffeomorphism of X which sends the class B to the class B'. That implies that $\Phi^{\mathcal{T}}_{g,k}(X, A) = \Phi^{\mathcal{T}}_{g,k}(X, A')$.

To complete the proof it suffices to show that

$$GW^{J,\mathcal{H}}_{g,k}(X, A) = \Phi^{\mathcal{T}}_{g,k}(X, A). \tag{0.2}$$

For that, recall from Theorem 1.2 of [LT] that the moduli cycle is defined from a section s of a generalized Fredholm orbifold bundle $E \to B$ and is represented by a cycle that lies in an arbitrarily small neighborhood of $s^{-1}(0)$. Both sides of (0.2) are defined in that way using the same Fredholm bundle E over the space of Kähler structures. In the first case B is $\{J_\alpha \,|\, \alpha \in \mathcal{H}\}$ and $s^{-1}(0)$ is the set of of all (f, α) where f is a J_α-holomorphic map, and in the second case $B = \mathcal{T}$ is the Twistor Family and $s^{-1}(0)$ is the set of J_α-holomorphic maps for $J_\alpha \in \mathcal{T}$. By Proposition 4.0.21 $\{J_\alpha \,|\, \alpha \in \mathcal{H}\}$ parameterizes the Twistor Family after adding a point at infinity to $\mathcal{H}$. But since $\omega(A) \geq 0$, Lemma 4.0.22 shows that $|\alpha| \leq 1$ for all J_α holomorphic maps representing the homology class A with $\alpha \in \mathcal{H}$. Thus the moduli cycle is bounded away from the point at infinity, so the two definitions of the moduli cycle are exactly equal. That gives (0.2) $\square$

Elliptic Surfaces

First, we recall the well-known facts about minimal elliptic surfaces X with Kodaira dimension 1 [FM].

1. X is elliptic in a unique way.

2. Every deformation equivalence is through elliptic surfaces.

Therefore, there is a unique elliptic structure $\pi : (X, J) \to C$. Moreover, for the fiber class F and any homology class $A \in H_2(X; Z)$, the integer

$$F \cdot A + \deg(\pi_* A) \tag{0.3}$$

is well-defined for each complex structure J and it is invariant under the deformation of complex structure J.

Let (ω, J, g) be a Kähler structure on X and $\mathcal{H}$ be as in (0.4). For $\alpha \in \mathcal{H}$, let $\|\alpha\|$ denote the L^2 norm as in (0.5).

Lemma 4.0.24 *Let $A \in H_2(X;Z)$ such that the integer (0.3) is positive. Then, there exit uniform constants E_0 and N such that for any J_α-holomorphic map $f:\Sigma \to X$, representing homology class A, with $\alpha \in \mathcal{H}$, we have*

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2 \le E_A, \qquad ||\alpha|| \le N.$$

Proof. It follows from (0.1) and Lemma 2.0.11 that for any nonzero $\alpha \in \mathcal{H}$, the zero set of α lies in the union of fibers F_i. Let $N(\alpha)$ be a (non-empty) union of ε-tubular neighborhoods of the F_i. Denote by $\mathcal{S}$ the unit sphere in $\mathcal{H}$ and set

$$m(J) = \min_{\alpha\in\mathcal{S}} \min_{x\in X\setminus N(\alpha)} |\alpha| \quad \text{and} \quad N = \frac{2}{m(J)}.$$

We can always choose a smooth fiber $F \subset X \setminus N(\alpha)$ such that f is transversal to F. Let $f^{-1}(F) = \{p_1, \cdots, p_n\}$ and for each i fix a small holomorphic disk D_i normal to F at $f(p_i)$. We can further assume that f is transversal to each D_i at $f(p_i)$.

Define $\text{sgn}(r)$ to be the sign of a real number r if $r \neq 0$, and 0 if $r = 0$. Denote by $I(S,f)_p$ the local intersection number of the map f and a submanifold $S \hookrightarrow X$ at $f(p)$. In terms of bases $\{e_1, e_2 = j\, e_1\}$ of $T_{p_i}\Sigma$, $\{v_1, v_2 = j\, v_1\}$ of $T_{f(p_i)}F$, and $\{v_3, v_4 = j\, v_3\}$ of $T_{f(p_i)}D_i$ we have

$$I(F,f)_{p_i} = \text{sgn}\left((v^1\wedge v^2\wedge v^3\wedge v^4)(v_1,v_2,f_*e_1,f_*e_2)\right) = \text{sgn}\left((v^3\wedge v^4)(f_*e_1,f_*e_2)\right),$$

$$I(D_i,f)_{p_i} = \text{sgn}\left((v^1\wedge v^2\wedge v^3\wedge v^4)(f_*e_1,f_*e_2,v_3,v_4)\right) = \text{sgn}\left((v^1\wedge v^2)(f_*e_1,f_*e_2)\right).$$

Comparing with $\text{sgn}\, f^*\omega(e_1,e_2) = \text{sgn}\left((v^1\wedge v^2)(f_*e_1,f_*e_2) + (v^3\wedge v^4)(f_*e_1,f_*e_2)\right)$ shows that

$$I(F,f)_{p_i} + I(D_i,f)_{p_i} \;=\; \text{sgn}\;(f^*\omega)(e_1,e_2). \tag{0.4}$$

Now suppose $m(J)||\alpha|| \ge 2$. Then $|\alpha| \ge 2$ along each F_i, so by (0.4) and Corollary 1.0.7b

$$\sum_i \left(I(f,F)_{p_i} + I(f,D_i)_{p_i}\right) < 0.$$

This contradicts to our assumption $A \cdot f + \deg(\pi_* A) > 0$ since by definition $\sum_i I(f, F)_{p_i} = A \cdot f$ and $\sum_i I(f, D_i)_{p_i} = \deg(\pi_* A)$. Therefore $||\alpha|| < N$ with N as above. The energy bound follows exactly same arguments as in the proof of Lemma 4.0.22. □

Proposition 4.0.25 *For any homology class A with (0.3) positive, the invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined and depend only on the deformation class of (X, J).*

Proof. It follows from Lemma 4.0.24 and Theorem 3.0.18 that the invariants $GW^{J,\mathcal{H}}_{g,k}(X, A)$ are well-defined. On the other hand, (0.3) is invariant under the deformation of J. Therefore, applying Proposition 3.0.15, we can conclude that the invariants only depends on the deformation equivalence class of J. □

Surfaces of General Type

Let (X, J) be a surface of general type.

Proposition 4.0.26 *If A is of type (1,1) and is not a linear combination of components of the canonical class, then we can define the invariant $GW^{J,\mathcal{H}}_{g,k}(X, A)$. They are invariant under the deformations of complex structures which preserve (1,1)-type of A.*

Proof. Lemma 2.0.12 and Theorem 3.0.18 imply that the invariants $GW^{\mathcal{H}}_{g,k}(X, A)$ are well-defined under the assumption that A is type $(1, 1)$. On the other hand, Proposition 3.0.15 also implies that the invariants $GW^{\mathcal{H}}_{g,k}(X, A)$ are invariant under deformations of the complex structure which preserve the $(1, 1)$ type of A. □

CHAPTER 5

Virtual Moduli cycles

This Chapter is devoted to the technical proofs of the analysis results stated in Chapter 3. Specially, we will prove Proposition 3.0.14, 3.0.15, 3.0.19, and 3.0.20.

We first recall the notion of orbifold bundle.

Definition 5.0.27 *A topological fibration $\pi : E \to B$ is an* **orbifold bundle** *if there is an open covering $\{U_i\}$ of B such that*

(a) each U_i is of the form $\tilde{U}_i/\Gamma_i$, where Γ_i is a finite group acting on $\tilde{U}_i$

(b) for each i, there is a topological bundle $\tilde{E}_i \to \tilde{U}_i$, such that $E_{|U_i} = \tilde{E}_i/\Gamma_i$

(c) For any i, j, there is a bundle map

$$\Phi_{ij} : \tilde{E}_j|_{\pi_j^{-1}(U_i \cap U_j)} \to \tilde{E}_i|_{\pi_i^{-1}(U_i \cap U_j)}$$

which is compatible with actions Γ_i and Γ_j and descends to the identity map of $E_{|E_i \cap E_j}$, where $\pi_k : \tilde{U}_k \to U_k$ is the natural projection

(d) for each $x \in \pi^{-1}(U_i \cap U_j)$, there is a small neighborhood U_x, such that $\Phi_{ij}|_{\pi_j^{-1}(U_x)}$ is an isomorphism from each connected component of $\pi_j^{-1}(U_x)$ onto its image

Any such a $\pi_i : \tilde{U}_i \to U_i$ is called a **local uniformization** *of B. We denote ϕ_{ij} by the induced map from $\pi_j^{-1}(U_i \cap U_j)$ to $\pi_i^{-1}(U_i \cap U_j)$. An* **orbifold section** *$s : B \to E$ is a continuous map such that for each i, $s_{|U_i}$ lifts to a section s_i of $\tilde{E}_i$ over $\tilde{U}_i$.*

We now define a generalized Fredholm orbifold bundle following [LT].

Definition 5.0.28 *An orbifold bundle $E \to B$ is called a* **generalized Fredholm orbifold bundle of index** r *if there is an orbifold section $s : B \to E$ satisfying*

(a) $s^{-1}(0)$ is compact with a finite covering $\{U_i\}$

(b) for each $s_i : \tilde{U}_i \to \tilde{E}_i$, there is a topological subbundle $\tilde{E}_{0i}$ of finite rank over $\tilde{U}_i$ such that

(i) $s_i^{-1}(\tilde{E}_{0i}) \subset \tilde{U}_i$ is smooth of dimension $r + rk(\tilde{E}_{0i})$

(ii) $E_{0i}|_{s_i^{-1}(E_{0i})}$ is a smooth bundle over $s_i^{-1}(E_{0i})$ with $s_i|_{s_i^{-1}(E_{0i})}$ smooth

(c) for each i, there is a finite dimensional vector space F_i, on which Γ_i acts, and a Γ_i-equivariant bundle homomorphism $\psi_i : \tilde{U}_i \times F_i \to \tilde{E}_i$, satisfying

(i) $\psi_i|_{s_i^{-1}(\tilde{E}_{0i}) \times F_i} : s_i^{-1}(\tilde{E}_{0i}) \times F_i \to \tilde{E}_{0i}$ is smooth and transverse to s_i along $s_i^{-1}(0) \cap \tilde{U}_i$

(ii) if $dimF_j \leq dimF_i$, then there is an injective bundle homomorphism

$$\theta_{ij} : \pi_j^{-1}(U_i \cap U_j) \times F_j \to \pi_i^{-1}(U_i \cap U_j) \times F_i$$

such that $\tilde{p}_i \circ \theta_{ij} = \phi_{ij} \circ \tilde{p}_j$, where $\tilde{p}_i : \tilde{U}_i \times F_i \to \tilde{U}_i$ is a natural projection, and $\psi_i \circ \theta_{ij} = \Phi_{ij} \circ \psi_j$ on $\pi_j^{-1}(U_i \cap U_j) \times F_j$

(iii) $dimF_k \leq dimF_j \leq dimF_i$, then $\theta_{ik} = \theta_{ij} \circ \theta_{jk}$ over $\pi_k^{-1}(U_i \cap U_j)$

(iv) for any $x \in U_i \cap U_j$, θ_{ij} is Γ_x-equivariant near $\pi^{-1}(x)$, where Γ_x is the uniformization group of B at x

For each i, (F_i, E_{0i}) is called a **resolution** *of $s_i : \tilde{U}_i \to \tilde{E}_i$.*

Proof of Proposition 3.0.14 Following Chapter 3 in [LT], we will show that

$$\Phi : \overline{\mathcal{F}}^{J}_{g,k}(X, A) \times \mathcal{H} \to E$$

satisfies Definition 5.0.28b. Namely, we will show a family version of Proposition 3.4 in [LT]. Then Proposition 3.0.14 follows from the proof of Proposition 2.2 in [LT]. The proof of above assertion consists of 4 steps. First, we recall local uniformizations of $\overline{\mathcal{F}}^l_{g,k}(X,A)$ in [LT]. Then these give the local uniformizations of $\overline{\mathcal{F}}^l_{g,k}(X,A)\times\mathcal{H}$ in an obvious way. Second, we recall the definition of approximated maps and weighted norms as in [LT]. Next, we show the main estimates for the linearization of the Cauchy-Riemann operator $\overline{\partial}_{J_\alpha}$ and its adjoint operator. These are family versions of Lemma 3.9 and 3.10 in [LT]. Finally, we use the main estimates of the previous step and the Inverse Function Theorem to conclude our assertion.

Step 1 In this step, we recall the local uniformizations of $\overline{\mathcal{F}}^l_{g,k}(X,A)$ in [LT]. In the following, we will denote by $\mathcal{C}$ a stable map $(f,\Sigma;x_1,\cdots,x_k)$. We fix $l\geq 2$. Let $[\mathcal{C}]=[f,\Sigma;x_1,\cdots,x_k]\in\overline{\mathcal{F}}^l_{g,k}(X,A)$. A component of Σ is called a *bubble* component if it collapses to a point under the stabilization of Σ. We add one or two marked points to each bubble component, to obtain a stable curve $(\Sigma;x_1,\cdots,x_k,z_1,\cdots,z_l)\in\overline{\mathcal{M}}_{g,k+l}$. Let W be a small neighborhood of $(\Sigma;x_1,\cdots,x_k,z_1,\cdots,z_l)$ in $\overline{\mathcal{M}}_{g,k+l}$ and $\tilde{W}$ be the uniformization of W, i.e. $W=\tilde{W}/\Gamma$, where $\Gamma=\mathrm{Aut}(\Sigma;\{x_i\};\{z_j\})$.

Let $\tilde{\mathcal{U}}$ be the universal family of curves over $\tilde{W}$. We fix a metric h on $\tilde{\mathcal{U}}$. Define the distance of two maps f_1 and f_2 from fibers of $\tilde{\mathcal{U}}$ over $\tilde{W}$ as follows:

$$\begin{aligned}d_{\tilde{W}}(f_1,f_2)&=\sup_{x\in\mathrm{Dom}(f_1)}\sup_{d_h(y,x)=d_h(x,\mathrm{Dom}(f_2))}d_X(f_1(x),f_2(y))\\&\quad+\sup_{y\in\mathrm{Dom}(f_2)}\sup_{d_h(x,y)=d_h(y,\mathrm{Dom}(f_1))}d_X(f_1(x),f_2(y))\end{aligned}$$

Since the homology class of any non-stable component under the map f is non-trivial, there is at least one regular value of f on each unstable component. Therefore, we can assume that $f^{-1}(f(z_j))$ consists of finitely many immersed points. We choose local hypersurfaces $H_1,\cdots,H_l$ such that H_j intersects $\mathrm{Im}(f)$ transversally at $f(z_j)$.

Fix a small $\delta > 0$, and define

$$\mathrm{Map}_\delta(W) = \{\ (\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \mid (\tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \in \tilde{W},\ d_{\tilde{W}}(\tilde{f}, f) < \delta,$$
$$\tilde{f} \text{ is } C^0 \text{ on } \tilde{\Sigma} \text{ and } C^l \text{ on } \mathrm{Reg}(\tilde{\Sigma}),\ \text{and } \tilde{f}(\tilde{z}_j) \in H_j\ \}.$$

Let K be any compact subset in $\tilde{\mathcal{U}} \backslash \mathrm{Sing}(\tilde{\mathcal{U}})$ of the form : there exits a diffeomorphism $\psi_K : (K \cap \Sigma) \times \tilde{W} \to K$ such that $\psi_K((K \cap \Sigma) \times \{t\})$ lies in the fiber of $\tilde{\mathcal{U}}$ over $t = (\tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \in \tilde{W}$. Then we define

$$\mathrm{Map}_\delta(W, K) = \{\ (\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \in \mathrm{Map}(W) \mid ||\tilde{f}\psi_{K|_{(K\cap\Sigma)\times\{t\}}} - f_{|K\cap\Sigma}||_{C^l} < \delta,$$
$$\text{where } t = (\tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \in \tilde{W}\ \}$$

By forgetting added marked points, each point in $\mathrm{Map}_\delta(W, K)$ give rise to a stable map $\mathcal{C}$ and consequently, an equivalence class $[\mathcal{C}] \in \overline{\mathcal{F}}^l_{g,k}(X, A)$. Let $p_{W,K}$: $\mathrm{Map}_\delta(W, K) \to \overline{\mathcal{F}}^l_{g,k}(X, A)$ be such a projection map and let

$$\mathrm{Map}_\delta(W_0, K) = p_W(\,\mathrm{Map}_\delta(W, K)\,).$$

Let $\mathrm{Aut}(\mathcal{C})$ be the automorphism group of the stable map $\mathcal{C}$. It is a subgroup of $\Gamma = \mathrm{Aut}(\Sigma; \{x_i\}; \{z_j\})$, so it is finite and acts on $\tilde{\mathcal{U}}$. Denote by $m(\mathcal{C})$ its order. ¿From now on, K always denotes a compact set in $\tilde{\mathcal{U}} \setminus \mathrm{Sing}(\tilde{\mathcal{U}})$ containing an open neighborhood of $\cup_j f^{-1}(f(z_j))$. Moreover, we may assume that K is invariant under the action of $\mathrm{Aut}(\mathcal{C})$.

There is an action of $\mathrm{Aut}(\mathcal{C})$ on $\mathrm{Map}_\delta(W, K)$ with $\mathrm{Map}_\delta(W_0, K)$ as its quotient: For $\tau \in \mathrm{Aut}(\mathcal{C})$ and $\mathcal{C}' = (f', \tilde{\Sigma}'; \{x'_i\}, \{z'_j\}) \in \mathrm{Map}_\delta(W, K)$ we define

$$\tau_*(\mathcal{C}') = (f'\tau^{-1}, \tau(\Sigma'); \{\tau(x'_i)\}, \{\tau(z'_j)\}).$$

Then $\mathrm{Map}_\delta(W_0, K) = \mathrm{Map}_\delta(W, K)/\mathrm{Aut}(\mathcal{C})$, see Lemma 3.1 [LT].

Let $\mathcal{V} \subset \mathcal{H}$ be a small neighborhood of α. The topological bundle $E_{|\mathrm{Map}_\delta(W_0,K)\times\mathcal{V}}$ over $\mathrm{Map}_\delta(W_0, K) \times \mathcal{V}$ lifts to the bundle

$$E_{|\mathrm{Map}_\delta(W,K)\times\mathcal{V}} \to \mathrm{Map}_\delta(W, K) \times \mathcal{V}. \tag{0.1}$$

In fact, $E \to \overline{\mathcal{F}}^{\,l}_{g,k}(X,A) \times \mathcal{H}$ is a topological orbifold bundle with the local uniformization

$$p_{W,K} \times I : \mathrm{Map}_\delta(W,K) \times \mathcal{V} \to \mathrm{Map}_\delta(W_0,K) \times \mathcal{V}.$$

Without further confusion, we simply denote the lifted bundle $E_{|\mathrm{Map}_\delta(W,K)\times\mathcal{V}}$ by E. On the other hand, the section Φ defined as in (0.3) lifts to a section, still denoted by Φ, of E over $\mathrm{Map}_\delta(W,K) \times \mathcal{V}$.

Step 2 In this step, we recall the definition of approximated maps and weighted norms in [LT]. In the following, we assume $\Phi(\mathcal{C},\alpha) = 0$, i.e. f is J_α-holomorphic.

Denote by $q_1, \cdots, q_s$ the nodes in Σ. For any q_i ($1 \le i \le s$), by shrinking $\tilde{W}$ if necessary, we may choose coordinates w_{i1}, w_{i2}, as well as t in $\tilde{W}$, near $\mathcal{C}$, such that the fiber

$$(\Sigma_t; \{x_i(t)\}, \{z_j(t)\})$$

of $\tilde{\mathcal{U}}$ over t is locally given by the equation

$$w_{i1}w_{i2} = \epsilon_i(t), \quad |w_{i1}| < 1, \ |w_{i2}| < 1,$$

where ϵ_i is a C^∞-function of t.

For any y in Σ_t, if $|w_{i1}| > L\sqrt{|\epsilon_i(t)|}$ or $|w_{i2}| > L\sqrt{|\epsilon_i(t)|}$ for all i, where L is a large number, then there is a unique $\pi_t(y)$ in $\Sigma = \Sigma_0$ such that $d_h(y, \pi_t(y)) = d_h(y, \Sigma)$. Note that if y is not in the coordinate chart given by w_{i1}, w_{i2}, then simply set $w_{i1}(y) = w_{i2}(y) = \infty$.

Introduce a complex structure $\tilde{J}_\alpha = j_{\mathcal{U}} + J_\alpha$ on $\tilde{\mathcal{U}} \times X$ and let $F : \Sigma \to \tilde{\mathcal{U}} \times X$ be the graph of f. Put $p_i = F(q_i)$. We may assume

1. $F(\{w_{i1}w_{i2} = 0 \mid |w_{i1}| < 1, \ |w_{i2}| < 1\})$ is contained in a coordinate chart $(u_1, \cdots, u_{2N})$ of $\tilde{\mathcal{U}} \times X$ near p_i.

2. $\tilde{J}_\alpha(\frac{\partial}{\partial u_j}) = \frac{\partial}{\partial u_{N+j}} + O(|u|)$ $\quad \tilde{J}_\alpha(\frac{\partial}{\partial u_{N+i}}) = -\frac{\partial}{\partial u_j} + O(|u|)$ where $|u| = \sqrt{\sum_j |u_j|^2}$.

3. In complex coordinates $u_j + \sqrt{-1}u_{N+j}$, $F(w_{i1}, w_{i2}) = (w_{i1}, w_{i2}, 0, \cdots, 0) + \mathcal{O}(|w_{i1}|^2 + |w_{i2}|^2)$.

Then we can extend F to a neighborhood of q_i using the formula in 3.

Let β be a cut-off function such that

$$\beta(x) = 0 \text{ for } |x| \leq 1, \quad \beta(x) = 1 \text{ for } |x| > 2, \text{ and } |d\beta(x)| \leq 2.$$

Definition 5.0.29 *We define $f_t(y)$, where $y \in \Sigma_t$, as follows:*

$$f_t = \begin{cases} f(\pi_t(y)) & \text{if either } |w_{i1}(y)| > 1 \text{ or } |w_{i2}(y)| > 1 \text{ for all } i \\ \pi_2(F(y)) & \text{if } |w_{i1}(y)| < \frac{1}{2} \text{ and } |w_{i2}(y)| < \frac{1}{2} \text{ for some } i \\ g_i(y) & \text{if } \frac{1}{2} \leq |w_{i1}(y)| \leq 1 \text{ or } \frac{1}{2} \leq |w_{i1}(y)| \leq 1 \end{cases}$$

where $g_i(y)$

$$= \exp_{f(q_i)} \Big(\beta(2d_h(y, q_i)) \exp^{-1}_{f(q_i)} f(\pi_t(y)) + \Big(1 - \beta(2d_h(y, q_i))\Big) \exp^{-1}_{f(q_i)} \pi_2(F(y)) \Big)$$

Next, we will define weighted norms as in [LT]. Let r be the distance function to the singular set $\mathrm{Sing}(\tilde{\mathcal{U}})$ with respect to the metric h on $\tilde{\mathcal{U}}$. In the below, all norms and covariant derivatives over Σ_t are taken with respect to the induced metric $h_{|\Sigma_t}$.

Definition 5.0.30 *For any smooth section $\xi \in \Gamma^0(\Sigma_t, f_t^*TX)$, we define*

$$||\xi||_{1,p} = \left(\int_{\Sigma_t} (|\xi|^p + |\nabla \xi|^p) d\mu_t \right)^{\frac{1}{p}} + \left(\int_{\Sigma_t} r^{-\frac{2(p-2)}{p}} |\nabla \xi|^2 d\mu_t \right)^{\frac{1}{2}}$$

$$L^{1,p}(\Sigma_t, f_t^*TX) = \{ \ \xi \in \Gamma^0(\Sigma_t, f_t^*TX) \mid ||\xi||_{1,p} < \infty \ \}$$

$$L^{1,p} = \Big\{ \ (t, \beta, \xi) \mid \xi \in L^{1,p}(\Sigma_t, f_t^*TX), \ t \in \tilde{W}, \ \beta \in \mathcal{V} \ \Big\}$$

where $p \geq 2$ and $\Gamma^0(\Sigma_t, f_t^*TX)$ is the space of continuous sections of f_t^*TX over Σ_t. If Σ_t has more than one components, then ξ consists of continuous sections of components which have the same value at each node.

Definition 5.0.31 *For $\eta \in \mathrm{Hom}(\Sigma_t, f_t^*TX)$, we define*

$$||\eta||_p = \left(\int_{\Sigma_t} |\eta|^p d\mu_t\right)^{\frac{1}{p}} + \left(\int_{\Sigma_t} r^{-\frac{2(p-2)}{p}} |\eta|^2 d\mu_t\right)^{\frac{1}{2}}$$

$$L^p\left(\wedge^{0,1}_{J_\beta}(f_t^*TX)\right) = \{\ \eta \in \mathrm{Hom}(T\Sigma_t, f_t^*TX) \mid \eta j_{\tilde{U}} = -J_\beta \eta,\ ||\eta||_p < \infty\ \}$$

$$L^p(\Lambda^{0,1}(TX)) = \left\{\ (t, \beta, \eta) \mid \eta \in L^p\left(\wedge^{0,1}_{J_\beta}(f_t^*TX)\right),\ t \in \tilde{W},\ \beta \in \mathcal{V}\ \right\}$$

Lemma 5.0.32 *For any $p > 2$, there exits a uniform constant c such that for any $t \in \tilde{W}$ and $\beta \in \mathcal{V}$*

$$||\Phi(f_t, \beta)||_p \leq c\,(\,|t|^{\frac{1}{2}} + ||\alpha - \beta||\,)$$

Proof. It follows from Lemma 3.8 in [LT] that $||\Phi(f_t, \alpha)||_p \leq c\,|t|^{\frac{1}{2}}$. It also follows from Lemma 3.7 in [LT] that $|df_t|$ is uniformly bounded. On the other hand, we have

$$\int_{\Sigma_t} r^{-\frac{2(p-2)}{p}} d\mu_t \leq c(p).$$

Therefore, we can conclude that

$$||\Phi(f_t, \beta)||_p \leq ||\Phi(f_t, \alpha)||_p + ||(J_\beta - J_\alpha) df_t\, j||_p \leq c\,(|t|^{\frac{1}{2}} + ||\alpha - \beta||\,). \quad \Box$$

Step 3 In this step, we will show main estimates. The linearization $L_{t,\beta}$ of Φ at (f_t, β) with respect to f_t is an elliptic operator

$$L_{f_t,\beta} : L^{1,p}(\Sigma_t, f_t^*TX) \to L^p\left(\Lambda^{0,1}_{J_\beta}(f_t^*TX)\right) \quad \text{given by}$$

$$L_{t,\beta}(\xi) = \nabla\xi + J_\beta \nabla\xi\, j_t + \frac{1}{2}\left(\nabla_\xi J_\beta\right)\left(df_t j_t + J_\beta df_t\right). \tag{0.2}$$

where j_t is the complex structure on Σ_t. Its adjoint operator $L^*_{t,\beta}$ with respect to the L^2-inner product is given as follows: for any $\eta \in \Omega^{0,1}_{J_\beta}(f_t^*TX)$

$$L^*_{t,\beta}(\eta) = -2\nabla_{e_1}(\eta_1) - 2\nabla_{e_2}(\eta_2) + B_{t,\beta}(\eta) \tag{0.3}$$

where $\{e_1, e_2 = j_t e_1\}$ be an orthonormal basis on Σ_t, $\eta_1 = \eta(e_1), \eta_2 = \eta(e_2)$, and $B_{t,\beta}(\eta)$ is defined by

$$\langle \xi \, , \, B_{t,\beta}(\eta) \rangle = \langle \, (\nabla_\xi J_\beta) \, f_{t*} e_2 \, , \, \eta_1 \, \rangle - \langle \, (\nabla_\xi J_\beta) \, f_{t*} e_1 \, , \, \eta_2 \, \rangle.$$

For the proof of (0.2) and (0.3), see lemma 6.3 and 6.4 in [RT1].

The next lemma follows from Lemma 3.9 in [LT]. This shows the uniform elliptic estimates for $L_{t,\beta}$.

Lemma 5.0.33 *For any fixed $p \geq 2$, there is a uniform constant c such that for any t, β, and $\xi \in L^{1,p}(\Sigma_t, f_t^* TX)$, we have*

$$||\xi||_{1,p} \leq c \, (||L_{(t,\beta)} \xi||_p + ||\xi||_{1,2}). \tag{0.4}$$

Proof. It follows from Lemma 3.9 in [LT] that (0.4) holds with some constant $c(\beta)$ which might depend on $|\nabla J_\beta|$. However, by shrinking $\mathcal{V}$, if necessary, we can choose c with $c > c(\beta)$ for any $\beta \in \mathcal{V}$. □

Fix a node q_i of Σ and choose $t \in \tilde{W}$ with $\epsilon_i(t) \neq 0$. Let $w_{i1} = \rho e^{\sqrt{-1}\,\theta}$. Then $w_{i2} = \frac{|\epsilon_i(t)|}{\rho} e^{\sqrt{-1}\,(\theta_0 - \theta)}$ and $r^2 = \rho^2 + \frac{|\epsilon_i(t)|^2}{\rho^2}$, where $\epsilon_i(t) = |\epsilon_i(t)| e^{\sqrt{-1}\,\theta_0}$. We define the neck region by

$$N_{k,t} = \Sigma_t \cap \{ \, (w_{i1}, w_{i2}, t) \in \tilde{\mathcal{U}} \mid r \leq 1/k \, \} \quad \text{and} \quad \varphi = r^{-\frac{2(p-2)}{p}}. \tag{0.5}$$

Denote by h_t the induced metric on $N_{k,t}$ and let $h_c = r^{-2} \rho^2 h_t$. Then we have

$$h_c = d\rho^2 + \rho^2 \, d\theta^2 \quad \text{and} \quad d\mu_c = r^{-2} \rho^2 d\mu_t = \frac{1}{\rho} \, d\rho \, d\theta$$

Lemma 5.0.34 *For any fixed $p > 2$, there exist k_0 and a uniform constant c, which is independent of t and β, such that if $k > k_0$, then each $\eta \in \Omega^{0,1}_{J_\beta}(f_t^* TX)$ satisfies*

$$\int_{N_{k,t}} \varphi \, |\eta|^2 \, d\mu_t \leq c \int_{N_{k,t}} \varphi \, |L^*_{t,\beta}(\eta)|^2 \, d\mu_c + c \int_{\partial N_{k,t}} \varphi \, \left(|\eta|^2 + |\nabla^c \eta|_c^2\right) \, d\theta \tag{0.6}$$

Proof. Using the metric h_c on Σ_t, for any $\eta \in \Omega^{0,1}_{J_\beta}(f_t^*TX)$, we can write it as $\eta = \eta_1\, d\rho - J_\beta\, \eta_1\, \rho\, d\theta$. In terms of ρ and θ, we also have

$$2|\,\eta_1|^2 = |\,\eta\,|_c^2 = \left(\frac{r}{\rho}\right)^2 |\,\eta\,|^2 \quad \text{and} \quad |\nabla^c \eta|_c^2 = 2\left(|\frac{\partial \eta_1}{\partial \rho}|^2 + |\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}|^2\right).$$

For fixed $p > 2$, we can choose k_0 which satisfies

$$\left(\frac{1}{k_0}\right)^{\frac{2}{p}} < \frac{2}{p}$$

For $k > k_0$, we first show the following :

$$\int_{N_{k,t}} \varphi\, |\eta|^2\, d\mu_t \;\leq\; 4\int_{\partial N_{k,t}} \varphi\, |\eta|^2\, d\theta + \int_{N_{k,t}} \varphi\, |\nabla^c \eta|_c^2\, d\mu_c. \tag{0.7}$$

Let $\psi(\rho) = \int_{\rho_0}^{\rho} \varphi\, \rho\, d\rho$, where $\rho_0 \leq \rho$ on $N_{k,t}$. Then $\psi \leq \varphi \rho^{1+q}$, where $q = \frac{p-2}{p}$. Consequently, from the integration by parts, we have

$$\begin{aligned}
\int_{N_{k,t}} \varphi\, |\eta|^2\, d\mu_t &= 2\int_{N_{k,t}} \varphi\, \rho\, |\eta_1|^2\, d\rho\, d\theta \\
&= 2\int_{\partial N_{k,t}} \psi\, |\eta_1|^2\, d\theta - 4\int_{N_{k,t}} \psi\, \langle \eta_1, \frac{\partial \eta_1}{\partial \rho}\rangle\, d\rho d\theta \\
&\leq 2\int_{\partial N_{k,t}} \varphi\, |\eta|^2\, d\theta + \int_{N_{k,t}} \varphi\, \rho |\,\eta_1|^2\, d\rho\, d\theta \;+\; 2\int_{N_{k,t}} \varphi\, \rho\, \rho^{2q} |\,\nabla^c \eta\,|_c^2\, d\rho\, d\theta.
\end{aligned}$$

Using $\rho \leq r \leq 1/k_0$, we can rearrange the above to conclude (0.7).

Next, we will show the following :

$$\int_{N_{k,t}} \varphi\, |\nabla^c \eta|_c^2\, d\mu_c \leq c\int_{N_{k,t}} \varphi\, \left(|L^*_{t,\beta}(\eta)|^2 + \frac{r^2}{\rho^2}|\,\eta_1|^2\right) d\mu_c + c\int_{\partial N_{k,t}} \varphi\, \left(|\nabla^c \eta|_c^2\right)\, d\theta \tag{0.8}$$

Since $|\nabla J_\beta|$ and $|df_t|$ are uniformly bounded,

$$|B_{f_t,\beta}(\eta)| \leq c\,|\, df_t\,|_c\,|\,\eta_1\,| \leq c\,\frac{r}{\rho}|\, df_t\,|\,|\,\eta_1\,| \leq c\,\frac{r}{\rho}|\,\eta_1\,|.$$

Consequently, we have

$$\frac{1}{2} L^*_{t,\beta}(\eta) = -\frac{\partial \eta_1}{\partial \rho} + J_\beta \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta} + \mathcal{O}(\frac{\rho}{r}|\,\eta_1\,|). \tag{0.9}$$

This implies that

$$\frac{1}{2}\int_{N_{k,t}} \varphi\,|\nabla^c \eta|_c^2\, d\mu_c = \int_{N_{k,t}} \varphi \left(|\frac{\partial \eta_1}{\partial \rho} - J_\beta \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}|^2 + 2\langle \frac{\partial \eta_1}{\partial \theta}, J_\beta \frac{1}{\rho}\frac{\partial \eta_1}{\partial \rho}\rangle \right) d\mu_c$$
$$\leq \int_{N_{k,t}} \varphi \left(c\,|\,L^*_{t,\beta}(\eta)\,|^2 + c\frac{r^2}{\rho^2}|\,\eta_1|^2 + 2\langle \frac{\partial \eta_1}{\partial \rho}, J_\beta \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\rangle \right) d\mu_c. \tag{0.10}$$

Let $J_{\beta_0} = J_\beta(f(q_i))$. If the neck region $N_{k,t}$ is sufficiently small, then $|J_\beta(f_t) - J_{\beta_0}| < 1/(4p)$ on $N_{k,t}$ for any $\beta \in \mathcal{V}$ and $t \in \tilde{W}$. Using the fact that J_β is compatible with the metric g on X, we have

$$2\left\langle \frac{\partial \eta_1}{\partial \rho}, J_\beta \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle = \left\langle (J_{\beta_0} - J_\beta)\frac{\partial \eta_1}{\partial \rho}, \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle - \left\langle \frac{\partial \eta_1}{\partial \rho}, (J_{\beta_0} - J_\beta)\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle$$
$$+\left\langle \frac{\partial \eta_1}{\partial \rho}, J_{\beta_0}\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle - \left\langle J_{\beta_0}\frac{\partial \eta_1}{\partial \rho}, \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle. \tag{0.11}$$

We have

$$\int_{N_{k,t}} \varphi \left|\left\langle (J_{\beta_0} - J_\beta)\frac{\partial \eta_1}{\partial \rho}, \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle - \left\langle \frac{\partial \eta_1}{\partial \rho}, (J_{\beta_0} - J_\beta)\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle\right| d\mu_c$$
$$\leq \frac{1}{4p}\int_{N_{k,t}} \varphi\,|\nabla^c \eta\,|_c^2\, d\mu_c. \tag{0.12}$$

On the other hand, using integration by parts, we also have

$$\left|\int_{N_{k,t}} \varphi \left\langle \frac{\partial \eta_1}{\partial \rho}, J_{\beta_0}\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle - \left\langle J_{\beta_0}\frac{\partial \eta_1}{\partial \rho}, \frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\right\rangle d\mu_c\right|$$
$$= \left|\int_{\partial N_{k,t}} \varphi \left\langle \eta_1 - a, J_{\beta_0}\frac{\partial \eta_1}{\partial \theta}\right\rangle d\theta - \int_{N_{k,t}} \varphi' \left\langle \eta_1 - a, J_{\beta_0}\frac{\partial \eta_1}{\partial \theta}\right\rangle d\rho\, d\theta\right|$$
$$\leq c\int_{\partial N_{k,t}} \varphi\,\left(|\eta_1 - a|^2 + |\nabla^c \eta|_c^2\right) d\theta$$
$$+ \frac{p-2}{p}\int_{N_{k,t}} \varphi \left(\frac{1}{\rho^2}|\,\eta_1 - a\,|^2 + |\,J_{\beta_0}\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}\,|^2\right) d\mu_c. \tag{0.13}$$

where $a(\rho) = \frac{1}{2\pi}\int \eta_1(\rho,\theta)\, d\theta$ and $\varphi' = -\frac{p-2}{p}\varphi\,\frac{2}{\rho}\,\frac{\rho^4 - |\epsilon|^2}{\rho^4 + |\epsilon|^2}$. Now, apply Wirtinger's inequality [GT] and then use (0.9) to derive the following :

$$\int_{N_{k,t}} \varphi \frac{1}{\rho^2} |\eta_1 - a|^2 d\mu_c \le \int_{N_{k,t}} \varphi |\frac{1}{\rho}\frac{\partial \eta_1}{\partial \theta}|^2 d\mu_c$$
$$\le \int_{N_{k,t}} \varphi \left(\left(1 + \frac{1}{2(p-2)}\right) |\rho \frac{\partial \eta_1}{\partial \rho}|^2 + c|L^*_{t,\beta}|^2 + c|\eta_1|^2 \right) d\mu_c. \tag{0.14}$$

Combining (0.10), (0.11), (0.12) (0.13), and (0.14), we can deduce (0.8).

On the other hand, $\varphi(r^2/\rho^2)d\mu_c = r^{\frac{2}{p}}(1/\rho)d\rho d\theta$ and $\int r^{\frac{2}{p}}(1/\rho)d\rho \le c(p)r^{\frac{2}{p}}$, where $c(p) = 2^{\frac{2}{p}}(p/4)$. Therefore, using integration by parts, we have

$$\int_{N_{k,t}} \varphi \frac{r^2}{\rho^2} |\eta_1|^2 d\mu_c \le c\int_{\partial N_{k,t}} \varphi |\eta|^2 d\theta + c\int_{N_{k,t}} r^{\frac{2}{p}} |\eta_1|^2 d\mu_c + c\int_{N_{k,t}} r^{\frac{2}{p}} |\nabla^c \eta|^2_c d\mu_c \tag{0.15}$$

Finally, (0.6) follows from (0.7), (0.8) and (0.15) since every constant in the above estimate depends only on p and we can assume r is arbitrary small. □

Step 4 In this step, we will show a family version of Proposition 3.4 in [LT]. This proves that $\Phi : \overline{\mathcal{F}}^{\,l}_{g,k}(X, A) \times \mathcal{H} \to E$ satisfies Definition 5.0.28b. Consider the vector bundle $E_{\mathcal{V}} \to \tilde{\mathcal{U}} \times X \times \mathcal{V}$ whose fiber over (q, p, β) consists of all $\eta \in \mathrm{Hom}(T_q\tilde{\mathcal{U}}, T_pX)$ with $\eta\, j_{\tilde{\mathcal{U}}} = -J_\alpha \eta$. We denote by

$$\Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_{\mathcal{V}}$$

the set of all sections of $E_{\mathcal{V}} \to \tilde{\mathcal{U}} \times X \times \mathcal{V}$, which are C^l smooth and vanish near $\mathrm{Sing}(\tilde{\mathcal{U}})$. For $\tilde{C} = (\tilde{f}, \tilde{\Sigma}; \{x_i\}, \{z_j\})$, $\beta \in \mathcal{V}$ and $\eta \in \Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_{\mathcal{V}}$, we define the restriction $\eta|_{(\tilde{C},\beta)}$ as follows: for any $x \in \tilde{\Sigma}$

$$\eta|_{(\tilde{C},\beta)}(x) = \eta(x, \tilde{f}(x), \beta).$$

Lemma 5.0.35 *There is a finite subspace $S \subset \Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_{\mathcal{V}}$ such that $S|_{(C,\alpha)}$ is transverse to $L_{f,\alpha}$, i.e., if $\eta_1, \cdots, \eta_n$ span S, then $\eta_1|_{(C,\alpha)}, \cdots, \eta_n|_{(C,\alpha)}$ and $Im(L_{f,\alpha})$ generate $L^p\left(\wedge^{0,1}_{J_\alpha}(f^*TX)\right)$, and $dim(S) = dim(S|_{(C,\alpha)})$, where $S|_{(C,\alpha)} = \{\ \eta|_{(\tilde{C},\alpha)} \mid \eta \in S\ \}$.*

Proof. Denote by $\mathrm{Coker}(L_{f,\alpha})$ the space of all $\eta \in L^2\left(\wedge^{0,1}_{J_\alpha}(f^*TX)\right)$ such that $L^*_{f,\alpha}(\eta) = 0$. Then it is a finite dimensional subspace of $L^p\left(\wedge^{0,1}_{J_\alpha}(f^*TX)\right)$ and for any $\eta \in L^p\left(\wedge^{0,1}_{J_\alpha}(f^*TX)\right)$, there are $\xi \in L^{1,2}(\Sigma, f^*TX)$ and $\eta_0 \in \mathrm{Coker}(L_{f,\alpha})$ such that $L_{f,\alpha}(\xi) = \eta - \eta_0$. Moreover, $\xi \in L^{1,p}(\Sigma, f^*TX)$.

On the other hand, the set $\Gamma_{|(C,\alpha)} = \{\eta_{|(C,\alpha)} | \eta \in \Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_\mathcal{V}\}$ is dense in $L^p\left(\wedge^{0,1}_{J_\alpha}(f^*TX)\right)$. Therefore, we can always find $\eta_1 \in \Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_\mathcal{V}$ such that the restriction $\eta_{1|C,\alpha}$ is not in $\mathrm{Im}(L_{f,\alpha}) \cup \mathrm{Coker}(L_{f,\alpha})$, if $\mathrm{Coker}(L_{f,\alpha}) \neq \{0\}$. Then, $\eta_{1\,|C,\alpha} = L_{f,\alpha}(\xi) + \eta^1$ for some ξ in $L^{1,p}(\Sigma, f^*TX)$ and η^1 in $\mathrm{Coker}(L_{f,\alpha})$. Let $\mathrm{Coker}^1(L_{f,\alpha})$ be the orthogonal complement on $\langle \eta^1 \rangle$ in $\mathrm{Coker}(L_{f,\alpha})$. Its dimension is one less than those of $\mathrm{Coker}(L_{f,\alpha})$. If $\mathrm{Coker}^1(L_{f,\alpha}) \neq \{0\}$, then we can also find $\eta_2 \in \Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_\mathcal{V}$ such that $\eta_{2\,|C,\alpha} = L_{f,\alpha}(\xi') + a\,\eta_{1\,|C,\alpha} + \eta^2$ for some ξ' in $L^{1,p}(\Sigma, f^*TX)$, some nonzero η^2 in $\mathrm{Coker}^1(L_{f,\alpha})$, and some constant a. In this way, we can use the induction on the dimension of $\mathrm{Coker}(L_{f,\alpha})$ to find $\eta_1, \cdots, \eta_n \in \Gamma^{0,1}_l(\tilde{\mathcal{U}}, TX)_\mathcal{V}$ such that $\eta_{1|(C,\alpha)}, \cdots, \eta_{n|(C,\alpha)}$ and $\mathrm{Im}(L_{f,\alpha})$ generate $L^p\left(\wedge^{0,1}_{J_\alpha}(f^*TX)\right)$. □

Denote by $C^{l+1}_0(\tilde{\mathcal{U}}, TX)$ the set of all C^{l+1}-smooth sections of $TX \to \mathcal{U} \times X$, which vanish near $\mathrm{Sing}(\mathcal{U})$. Let S be as in Lemma 5.0.35 and π_S be the orthogonal projection onto the orthogonal complement of $S_{|(t,\beta)}$ in $L^2\left(\Lambda^{0,1}_{J_\beta}(f_t^*TX)\right)$. Let P be a finite dimensional subspace in $C^{l+1}_0(\tilde{\mathcal{U}}, TX)$ such that $\dim(P)=\dim(P_f)$ and $q_S(\,\mathrm{Ker}(\pi_{S_{(f,\alpha)}} L_{(f,\alpha)})\,) = P_f$, where $q_S : L^{1,2}(\Sigma, f_t^*TX) \to P_{f_t}$ is the projection with respect to the L^2-inner product.

Lemma 5.0.36 *Let P and S be as above, and t and $||\beta - \alpha||$ be sufficiently small. Then for any $p > 2$, $\xi_0 \in P_{f_t}$ and $\eta \in L^p\left(\wedge^{0,1}_{J_\beta}(f_t^*TX)\right)$, there are unique $\xi \in L^{1,p}(f_t^*TX)$ and $\eta_0 \in S_{|(t,\beta)}$, satisfying:*

$$q_S(\xi) = \xi_0, \quad L_{t,\beta}(\xi) = \eta - \eta_0, \tag{0.16}$$

$$max\{\ ||\xi||_{1,p}, ||\eta_0||_p\ \} \leq c\, max\{\ ||\xi_0||_{1,p}, ||\eta||_p\ \}$$

where c is a uniform constant.

Proof. Its proof is similar to the proof of Lemma 3.10 in [LT]. We first show that there is ξ and η_0 such that $L_{t,\beta}(\xi) = \eta - \eta_0$ for sufficiently small $|t|$ and $||\alpha - \beta||$. Suppose not. Then we can find a sequence $\{(t_n, \beta_n)\}$ with $(t_n, \beta_n) \to (0, \alpha)$ and η_n in $\mathrm{Coker}(L_{t_n,\beta_n})$ with $||\eta_n||_p = 1$ such that η_n is perpendicular to S with respect to the L^2-metric on $L^p\left(\Lambda^{0,1}_{J_{\beta_n}}(f^*_{t_n}TX)\right)$.

It follows from the standard elliptic estimates that η_n converges to some η in $L^p\left(\Lambda^{0,1}_{J_\alpha}(f^*TX)\right)$ outside of nodes of Σ. Since η is perpendicular to S and $L^*_{f,\alpha}(\eta) = 0$, we have $\eta = 0$. This implies that $\eta_n \to 0$ on the compliment of Neck region as in (0.5) and thus $||\eta_n||_p \to 0$ by Lemma 5.0.34. This contradicts to $||\eta_n||_p = 1$.

Next, we show that there is a unique ξ and η_0 satisfying (0.16). First, choose ξ and η with $L_{t,\beta}(\xi) + \eta_0 = \eta$ and set $\xi' = \xi + q_S(\xi_0 - \xi)$ and $\eta'_0 = \eta_0 + L_{t,\beta}\left(q_S(\xi - \xi_0)\right)$. Then ξ' and η' satisfy (0.16). One can prove the uniqueness by the similar argument as above.

Finally, we show the estimate by contradiction. Suppose not. Then there is a sequence $\{\,(t_n, \beta_n)\,\}$ with $(t_n, \beta_n) \to (t, \beta)$ and ξ_n in $L^{1,p}(f^*_{t_n}TX)$ and η_{0n} in S such that

$$(i)\ \max\{\,||\xi_n||_{1,p}, ||\eta_{0n}||_p\,\} = 1, \quad \text{and} \quad (ii)\ \max\{\,||\xi_{0n}||_{1,p}, ||\eta_n||_p\,\} \to 0$$

where $q_S(\xi_n) = \xi_{0n}$ and $L_{t_n,\beta_n}(\xi_n) + \eta_{0n} = \eta_n$. By Sobolev Embedding Theorem, we may assume that ξ_n converges to some ξ and ξ_{0n} to 0 both in $L^{1,2}$-norm. We may further assume that η_{0n} converges to some η_0. Note that $L_{t,\beta}(\xi) + \eta_0 = 0$ and $q_S(\xi) = 0$. Therefore, by uniqueness we have $\xi = 0$ and $\eta_0 = 0$. On the other hand, by (i) $||\xi||_{1,p} \to 1$. It then follows from Lemma 5.0.33 that $||\xi_n||_{1,2}$ are uniformly bounded away from zero. This contradicts to $\xi_n \to \xi = 0$ in $L^{1,2}$. □

Let S be as in Lemma 5.0.35. We define E_S over $\mathrm{Map}_\delta(W, K) \times \mathcal{V}$ as follows : for any $(\tilde{\mathcal{C}}, \beta)$ in $\mathrm{Map}_\delta(W, K) \times \mathcal{V}$,

$$E_{S|_{(\tilde{\mathcal{C}},\beta)}} = S_{|_{(\tilde{\mathcal{C}},\beta)}}.$$

The following shows that $\Phi : \overline{\mathcal{F}}^{J}_{g,k}(X, A) \times \mathcal{H} \to E$ satisfies Definition 5.0.28b.

Proposition 5.0.37 *By shrinking $\mathcal{V}$ and W if necessary, if δ is sufficiently small and K is sufficiently large, $\Phi^{-1}(E_S)$ is a smooth submanifold, which contains $(\mathcal{C}, \alpha)$, in $\mathrm{Map}_\delta(W, K) \times \mathcal{V}$ and of dimension*

$$2c_1(A) + 2(g-1) + 2k + 2\,\mathrm{dim}\mathcal{H} + \mathrm{dim}S. \tag{0.17}$$

Moreover, $E_S \to \Phi^{-1}(E_S)$ is a smooth bundle with $\Phi|_{\Phi^{-1}(E_S)}$

Proof. This proof is similar to the proof of Proposition 3.4 in [LT]. By shrinking $\mathcal{V}$ and W if necessary, we can assume that for any $(t, \beta) \in \tilde{W} \times \mathcal{V}$, Lemma 5.0.36 holds. We first show that there exists an $\epsilon_0 > 0$ such that the subset

$$\{\ (t, \beta, \xi) \in L^{1,p} \mid \pi_S\, \Psi(t, \beta, \xi) = 0,\ ||\xi||_{1,p} < \epsilon_0\ \} \tag{0.18}$$

is smooth of dimension $\dim(S) + 2c_1(A) + 2(g-1) + 2k + 2l + \mathrm{dim}\mathcal{H}$.

Let $\nabla^\beta = 1/2\,(J_\beta - J_\beta \nabla J_\beta)$ and τ is the parallel translation with respect to ∇^β. Define a map $\Psi : L^{1,p} \to L^p(\wedge^{0,1}(TX))$ by $\Psi(t, \beta, \xi) = \tau\,\Phi(\exp_{f_t}(\xi), \beta)$. Then the linearization of Ψ at $(t, \beta, 0)$ is $L_{(t,\beta)}$ as in (0.2). Now, consider the following expansion :

$$\Psi(t, \beta, \xi) = \Psi(t, \beta, 0) + L_{(t,\beta)}(\xi) + H_{(t,\beta)}(\xi)$$

where $H_{(t,\beta)}$ is the higher order term satisfying $||H_{(t,\beta)}(\xi)||_p \le c||\xi||_{C^0}|||\xi||_{1,p}$ for some uniform constant c; this constant may depend on J_β, but we can still assume it is uniform on β by shrinking $\mathcal{V}$, if necessary. It also follows from the Sobolev Embedding Theorem that

$$||H_{(t,\beta)}(\xi)||_p \le c||\xi||^2_{1,p} \tag{0.19}$$

Let E_P be the bundle induced by P over $\tilde{W} \times \mathcal{V}$ with fibers $P_{(t,\beta)} = P_{f_t}$ and define a map

$$\Xi : L^{1,p} \times E_S \to L^p(\wedge^{0,1}TX) \times E_P \text{ by}$$

$$(t, \beta, \xi, \eta_0) \to (\, t, \beta, \Psi(t, \beta, \xi) + \eta_0, q_S(\xi)\,)$$

Then the linearization of Ξ at $(t, \beta, 0, 0)$ is the map

$$D\Xi : L^{1,p}(\Sigma_t, f_t TX) \times S_{(t,\beta)} \to L^p(\wedge^{0,1}_{J_\beta} f_t^* TX) \times P_{(t,\beta)} \quad \text{given by}$$

$$(\xi, \eta_0) \to (\, L_{(t,\beta)}(\xi) + \eta_0, q_S(\xi)\,)$$

By Lemma 5.0.36, it is an isomorphism with uniformly bounded inverse. Therefore, by the Inverse Function Theorem there exists an $\epsilon_0 > 0$ such that Ξ is a diffeomorphism from the region

$$\{\ (t, \beta, \xi, \eta_0) \in L^{1,p} \times E_S \mid \max\{\, ||\xi||_{1,p}, ||\eta_0||_p \,\} < \epsilon_0\ \}$$

onto its image. Furthermore, by Lemma 5.0.36, if $|t|$ and $||\alpha - \beta||$ are small, then for any $\xi_0 \in P_{(t,\beta)}$ with $||\xi_0||_{1,p} < \epsilon_0$, there is a unique (t, β, ξ, η_0) satisfying

$$\Xi(t, \beta, \xi, \eta_0) = (t, \beta, 0, \xi_0).$$

On the other hand, it also follows from (0.19) that $||\eta||_0 \le c\,||\xi||_{1,p}$ when $\Psi(t, \beta, \xi) + \eta_0 = 0$. Therefore, we can conclude that the subset (0.18) is parameterized by $\tilde{W}$, $\mathcal{V}$, and some open set of P. Note that by our choice of P and S, $\dim P - \dim S = \mathrm{ind}(L_{t,\beta})$. The subset (0.18) is thus smooth manifold of dimension $\mathrm{ind}(L_{t,\beta}) + \dim\tilde{W} + \dim S + \dim\mathcal{V}$.

Next, we will show that if δ is sufficiently small and K is sufficiently large, then $\Phi^{-1}(E_S)$ is an open subset of the following set

$$\{\ (t, \beta, \xi) \in L^{1,p} \mid \pi_S\, \Psi(t, \beta, \xi) = 0,\ ||\xi||_{1,p} < \epsilon_0,\ \exp_{f_t}\xi(z_j) \in H_j\ \} \qquad (0.20)$$

where z_j for $1 \leq j \leq l$ is the added marked points and H_j for $1 \leq j \leq l$ are local hypersurfaces given in Step 1. Note that it is a smooth manifold of dimension (0.17).

Let $(\tilde{f}, \tilde{\Sigma}; \{x_i\}, \{z_j\}, \beta)$ in $\Phi^{-1}(E_S)$. Denote by t the corresponding point $(\tilde{\Sigma}; \{x_i\}, \{z_j\}$ in $\tilde{W}$. Since $d(f, \tilde{f}) \leq \delta$, there is some ξ in $\Gamma^0(\Sigma_t, f_t^* TX)$ with $\tilde{f} = \exp_{f_t} \xi$. It follows from Lemma 3.12 in [LT] that $||\xi||_{1,p} < \epsilon_0$. This implies that $\Phi^{-1}(E_S)$ is an open set of (0.20) and thus $\Phi^{-1}(E_S)$ is a smooth manifold of dimension (0.17).

Finally, it follows from the smooth dependence of solutions of $\pi_S \Phi(\tilde{f}, \beta) = 0$ that $E_S \to \Phi^{-1}(E_S)$ is a smooth bundle with $\Phi|_{\Phi^{-1}(E_S)}$ smooth. □

Proof of Proposition 3.0.15 This proof is similar to the proof of Proposition 2.3 in [LT]. Let $\Phi_t : \overline{\mathcal{F}}^{l}_{g,k}(X, A) \times \mathcal{H}_t \to E_t$ be the generalized Fredholm orbifold induced by the Kähler structure (ω_t, J_t, g_t). We define

$$\mathcal{H} = \{ \ (t, \alpha) \mid \alpha \in \mathcal{H}_t \ \}.$$

Similarly as in Chapter 3, we then define a generalized bundle E over $\overline{\mathcal{F}}^{l}_{g,k}(X, A) \times \mathcal{H}$ and consider

$$\Psi : \overline{\mathcal{F}}^{l}_{g,k}(X, A) \times \mathcal{H} \to E \quad \text{defined by}$$

$$(\,(f, \Sigma, \{x_i\}), (t, \alpha)\,) \to df + J_\alpha(t)\, df\, j$$

where $J_\alpha(t)$ is the almost complex structure on X defines by J_t and $\alpha \in \mathcal{H}_t$ as in (0.5). By definitions, we have $\Psi|_{\overline{\mathcal{F}}^{l}_{g,k}(X,A) \times \mathcal{H}_t} = \Phi_t$. Since all $\Phi_t^{-1}(0)$ are compact, it follows from the same argument as above that $\Psi : \overline{\mathcal{F}}^{l}_{g,k}(X, A) \times \mathcal{H} \to E$ is a generalized Fredholm orbifold bundle. Moreover,Ψ gives homotopy between Φ_0 and Φ_1 as generalized Fredholm orbifold bundles. Now, this proposition follows from

Theorem 1.2 in [LT]. □

The following lemma gives two facts about the setup used by Li and Tian which are used in the course of several proofs in [LT].

Lemma 5.0.38 *Let $s : B \to E$ be a generalized Fredholm orbifold bundle.*

(a) If $p : B \to V$ be a continuous map and K be a cycle in V with $PD([K]) = \gamma$. Then $s' : p^{-1}(K) \to E'$ is also a generalized Fredholm orbifold bundle, where $s' = s_{|_{p^{-1}(K)}}$ and $E' = E_{|_{p^{-1}(K)}}$. Moreover,

$$i_* e\,(s' : p^{-1}(K) \to E') = e\,(s : B \to E) \cap p^*(\gamma)$$

where $i : p^{-1}(K) \hookrightarrow B$.

(b) If $s' : B' \to E'$ is a generalized Fredholm orbifold bundle with a continuous onto map $\pi : B \to B'$ and an injective bundle map $\tau : \pi^* E' \to E$ such that $s^{-1}(0) = (\pi^* s')^{-1}(0)$, then

$$\pi_* e\,(s : B \to E) = e\,(s' : B' \to E')$$

Proof of Proposition 3.0.19 Let $\Phi : \overline{\mathcal{F}}^{J}_{g,k}(X, A) \to E$ be a generalized Fredholm orbifold bundle as in (0.3) and $\alpha_k \in H^2(X; Z)$. Choose a cycle K which represents a Poincaré dual of α. Then by Lemma 5.0.38a, $[\mathcal{M}^{\mathcal{H}}_{g,k}(X, A)]^{\text{vir}} \cap ev_k^*(\alpha_k)$ can be regarded as a class in $H_*(ev_k^{-1}(K); \mathbb{Q})$, where ev_k is the evaluation map of the k-th marked points.

On the other hand, there is a continuous surjective map

$$\pi : ev^{-1}(K) \to \overline{\mathcal{F}}_{g,k-1}(X, A) \times \mathcal{H}$$

which forgets the k-th marked points. π satiefies the condition of Lemma 5.0.38b and hence we have

$$\pi_*\left([\mathcal{M}^{\mathcal{H}}_{g,k}(X,A)]^{\text{vir}} \cap ev_k^*(\alpha_k)\right) = [\mathcal{M}^{\mathcal{H}}_{g,k-1}(X,A)]^{\text{vir}}.$$

That implies Proposition 3.0.19 since π is a finite branched covering of order $\alpha_k(A)$.

□

Proof of Proposition 3.0.20

(a) For Ψ_t as in (0.9), the set $\Psi_t^{-1}(0)$ is compact for all $0 \le t \le 1$ by assumption. Hence, the arguments used in the proof of Proposition 3.0.14 show that for each $0 \le t \le 1$ the bundle

$$E_1 \oplus E_2^t \to \overline{\mathcal{F}}_{g_1,k_1+1}(X,A_1) \times \overline{\mathcal{F}}_{g_2,k_2+1}(X,A_1) \times \mathcal{H}$$

with a section Ψ_t is a generalized Fredholm orbifold bundle. Denote by

$$[\mathcal{M}_t]^{\text{vir}} = \left[\mathcal{M}^{\mathcal{H}}_{(g_1,k_1+1),(g_2,k_2+1)}(X,A_1,A_2,t)\right]^{\text{vir}}$$

the corresponding virtual moduli cycle. As in the proof of Proposition 3.0.15, it also follows that

$$[\mathcal{M}_1]^{\text{vir}} = [\mathcal{M}_0]^{\text{vir}} = \left[\mathcal{M}^{\mathcal{H}}_{g_1,k_1+1}(X,A_1)\right]^{\text{vir}} \otimes [\mathcal{M}_{g_2,k_2+1}(X,A_2)]^{\text{vir}} \qquad (0.21)$$

as homology classes in $H_*\left(\overline{\mathcal{F}}_{g_1,k_1+1}(X,A_1) \times \overline{\mathcal{F}}_{g_2,k_2+1}(X,A_1);\mathbb{Q}\right)$. Note that $[\mathcal{M}_{g_2,k_2+1}(X,A_2)]^{\text{vir}}$ is the cycle which defines ordinary GW-invariants.

On the other hand, there is a natural map

$$p : \overline{\mathcal{F}}_{g_1,k_1+1}(X,A_1) \times \overline{\mathcal{F}}_{g_2,k_2+1}(X,A_1) \times \mathcal{H} \to X \times X$$

defined by $\left([f_1,\Sigma_1;\{x_i\}],[f_2,\Sigma_2;\{y_j\}],\alpha\right) \to \left(f_1(x_{k_1+1}), f_2(y_1)\right)$ There is also a surjective map $\pi : \bigcup p^{-1}(\Delta) \to st^{-1}(\operatorname{Im}\sigma)$ obtained by identifying x_{k_1+1} and y_1, where the union is over all decompositions of (A,g,k), $\triangle$ is the diagonal in $X\times X$, σ is the gluing map in (0.8), and st is the stabilization map on $\overline{\mathcal{F}}^{\prime}_{g,k}(X,A)\times\mathcal{H}$. It follows from Lemma 5.0.38a that the classes

$$\sum\left[\mathcal{M}^{\mathcal{H}}_{(g_1,k_1+1),(g_2,k_2+1)}(X,A_1,A_2,t)\right]^{\text{vir}} \quad\text{and}\quad \left[\mathcal{M}^{\mathcal{H}}_{g,k}(X,A)\right]^{\text{vir}}\cap \mathrm{PD}(\sigma)$$

can be regarded as a class in $H_*(\bigcup p^{-1}(\triangle);\mathbb{Q})$ and $H_*(st^{-1}(\mathrm{Im}\,\sigma);\mathbb{Q})$, respectively. Moreover, by Lemma 5.0.38b we have

$$\begin{aligned}&\left[\mathcal{M}^{\mathcal{H}}_{g,k}(X,A)\right]^{\text{vir}}\cap \mathrm{PD}(\sigma)\\ &=\pi_*\left(\sum\left[\mathcal{M}^{\mathcal{H}}_{(g_1,k_1+1),(g_2,k_2+1)}(X,A_1,A_2,1)\right]^{\text{vir}}\cap\left(ev^*_{k_1+1}H_\gamma\wedge ev^*_1H^\gamma\right)\right)\end{aligned}\tag{0.22}$$

where ev_{k_1+1} and ev_1 are evaluation maps of x_{k_1+1} and y_1, respectively. Combining (0.21) and (0.22), we have

$$\begin{aligned}&\left[\mathcal{M}^{\mathcal{H}}_{g,k}(X,A)\right]^{\text{vir}}\cap \mathrm{PD}(\sigma)\\ &=\pi_*\sum\left(\left[\mathcal{M}^{\mathcal{H}}_{g_1,k_1+1}(X,A_1)\right]^{\text{vir}}\otimes\left[\mathcal{M}_{g_2,k_2+1}(X,A_2)\right]^{\text{vir}}\right)\cap\left(ev^*_{k_1+1}H_\gamma\wedge ev^*_1H^\gamma\right)\\ &=\pi_*\sum\left(\left[\mathcal{M}^{\mathcal{H}}_{g_1,k_1+1}(X,A_1)\right]^{\text{vir}}\cap ev^*_{k_1+1}H_\gamma\right)\otimes\left(\left[\mathcal{M}_{g_2,k_2+1}(X,A_2)\right]^{\text{vir}}\cap ev^*_1H^\gamma\right).\end{aligned}$$

That implies the first Composition Law.

(b) Similarly, we have an evaluation map of last two marked points

$$\begin{aligned}p:\overline{\mathcal{F}}_{g-1,k+2}(X,A)\times\mathcal{H}&\to X\times X\\ \left(\left[f,\Sigma;\{x_i\}\right],\alpha\right)&\to\left(f(x_{k+1}),f(x_{k+2})\right).\end{aligned}$$

There is also a surjective map $\pi : p^{-1}(\triangle)\to st^{-1}(\mathrm{Im}\,\theta)$. It also follows from Lemma 5.0.38 that

$$\left[\mathcal{M}^{\mathcal{H}}_{g,k}(X,A)\right]^{\text{vir}}\cap PD(\theta)=\pi_*\sum\left(\left[\mathcal{M}^{\mathcal{H}}_{g-1,k+2}(X,A)\right]^{\text{vir}}\cap\left(ev^*_{k+1}H_\gamma\wedge ev^*_{k+2}H^\gamma\right)\right).$$

That implies the second Composition Law. $\square$

CHAPTER 6

The Invariants of $E(n)$ — Outline

Let $\pi : E(n) \to \mathbb{P}^1$ be a standard elliptic surface with a section s of self-intersection number $-n$. Denote by S and F the homology class of the section and the fiber. We will compute family GW-invariants for the class $S + dF$ with $2p_g = 2(n-1)$ dimensional parameter space $\mathcal{H}_n$ defined as in (0.4). These invariants $GW^{\mathcal{H}_n}_{g,k}(S + dF)$ are unchanged under deformations of Kähler structure. For convenience we assemble these into the generating function

$$F(t) = \sum_{d \geq 0} GW^{\mathcal{H}_n}_{0,0}(S + dF)\, t^d. \tag{0.1}$$

In the this and the following four Chapters we will calculate the invariants $GW^{\mathcal{H}_n}_{g,k}(S + dF)$ by deriving the formula for $F(t)$ stated in Theorem 0.3. Thus our aim it to prove:

Proposition 6.0.39 *For $n \geq 1$,*

$$F(t) = \prod_{d \geq 0} \left(\frac{1}{1 - t^d} \right)^{12n} \tag{0.2}$$

As mentioned in the introduction, the cases $n = 1, 2$ have been proven by Bryan-Leung and Ionel-Parker.

This Chapter shows how Proposition 6.0.39 follows from two formulas, equations (0.4) and (0.5) below, that are proved in later Chapters. Our proof parallels the proof of Ionel and Parker [IP3] with two changes. First, we replace the use of the τ class by ψ class; that makes the argument conceptually a bit easier. Second, we must extend the TRR formula and the Symplectic gluing formula of [IP3] to family invariants.

Here is the outline the proof of (0.2). Let $G(t)$ be the generating function for the function for the sum of divisors function $\sigma(n) = \sum_{d|n}$:

$$G(t) \;=\; \sum_{d\geq 0} \sigma(d)\, t^d \;=\; \sum_{d\geq 0} \frac{dt^d}{1-t^d}$$

Following [IP3] we also consider the generating function for a genus 1 invariant, namely

$$H(t) \;=\; \sum_{d\geq 0} GW^{\mathcal{H}_n}_{1,4}\,(S+dF)\,(\psi_{(1,4);4}; F^4)\, t^d \tag{0.3}$$

where $\psi_{(g,k);i}$ denotes the first Chern class of the line bundle $L_{(g,k);i} \to \overline{\mathcal{M}}_{g,k}$ whose geometric fiber over $(C; x_1, \cdots, x_n)$ is $T^*_{x_i}C$.

We can compute $H(t)$ in two different ways. In Chapter 7, we show how to combine the composition law together with the relation between ψ class and the divisors in $\overline{\mathcal{M}}_{1,4}$ to obtain the formula

$$H(t) = \frac{1}{12}tF'(t) - \frac{1}{12}F(t) + (2-n)F(t)G(t) \tag{0.4}$$

Then, in Chapters 8–11 we develop a family version of the Gluing Theorem in [IP3] to obtain the sum formula

$$H(t) = -\frac{1}{12}F(t) + 2F(t)G(t) \tag{0.5}$$

(see Proposition 11.0.62). Equations (0.4) and (0.5) give rise to the ODE with

$$tF'(t) = 12nG(t)F(t) \tag{0.6}$$

and we show in Proposition 8.0.47 that the initial condition is $F(0) = 1$. It is well-known that the solution of this ODE is given by

$$F(t) = \prod_{d \geq 0} \left(\frac{1}{1 - t^d} \right)^{12n}.$$

That completes the proof of Proposition 6.0.39 and hence of the main Theorem 0.3 of the introduction.

CHAPTER 7

The Topological Recursion Relation (TRR)

A pinched torus can be regarded as a two-sphere with two points identified. Consequently, maps from a pinched torus are a special class of maps from the two-sphere. That observation allows one to express certain $g = 1$ GW invariants in terms of $g = 0$ invariants, and more generally express certain genus g invariants in terms of genus g invariants. Such formulas are called topological recursion relations or TRR formulas. In this Chapter we will prove formula (0.4), which is a TRR formula for the family GW invariants.

We begin by recalling the notion of the dual graph associated with a stable curve. Given a stable genus g curve with n marked points $(C; x_i, \cdots, x_k)$, its dual graph is defined as follows. Let $\pi : \tilde{C} \to C$ be the normalization of C. The dual graph G has one vertex for each component of $\tilde{C}$, and the edges of G correspond to nodal points of C; if two points on $\tilde{C}$ maps to a node, then the edge, corresponding to that node, are attached to the vertices associated to the components of $\tilde{C}$ on which the two points lie. The legs (half-edge) of G correspond to marked points of C, and these are indexed in an obvious way.

We denote by $\mathcal{M}(G)$ the moduli space of all genus g curves with n marked points whose dual graph is G. We also denote by δ_G the orbifold fundamental class of $\overline{\mathcal{M}(G)}$, that is, the fundamental class divided by the order of the automorphisms of a general element of $\mathcal{M}(G)$. Graphs with one edge correspond to degree two classes. There are two types of such graphs. One is the graph G_{irr} with one vertex of genus $g-1$. The other types are the graphs $G_{a,I}$, which have two vertices, one of genus a, with attached the legs indexed by I, and one of genus $g-a$, with attached the legs indexed by $\{1,\cdots,k\}\setminus I$.

For any $i \in \{1,\cdots,k\}$, we have

$$\psi_{(1,k);i} = \frac{1}{12}\delta_{G_{irr}} + \sum_{\substack{i\in I\\|I|\geq 2}} \delta_{G_{0,I}} \quad \text{in} \quad H^2(\overline{\mathcal{M}}_{1,k};\mathbb{Q}). \tag{0.1}$$

For the proof of (0.1), see [AC] and [G].

Proposition 7.0.40 *The generating function (0.3) satisfies*

$$H(t) = \frac{1}{12}tF'(t) - \frac{1}{12}F(t) + (2-n)F(t)G(t)$$

Proof. It follows from (0.1) that the coefficients $GW^{\mathcal{H}_n}_{1,4}(S+dF)(\psi_{(1,4);4};F^4)$ of $H(t)$ is

$$\frac{1}{12}GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{irr}};F^4) + \sum_{\substack{i\in I\\|I|\geq 2}} GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}};F^4). \tag{0.2}$$

We will apply the first Composition Law to $GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}};F^4)$ and the Second composition Law to $GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{irr}};F^4)$.

Recalling Proposition 3.0.20, the only possible decompositions of the class $S+dF$, which can appear when we apply the first Composition Law, are $S+d_1F$ and d_2F with $d_1+d_2=d$. It then follows from a dimension count and the first composition law that

$$GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}};F^4)$$
$$= \sum_{d_1+d_2=d}\sum_{\gamma} GW^{\mathcal{H}_n}_{0,|I|+1}(S+d_1F)(F^{|I|},H_\gamma)\,GW_{1,5-|I|}(d_2F)(F^{4-|I|},H^\gamma). \tag{0.3}$$

where $\{ H_\gamma \}$ and $\{ H^\gamma \}$ are bases of $H^*(E(n))$ dual by the intersection form. It also follows from Proposition 3.0.19 that, if $I = \{1, \cdots, 4\}$, then (0.3) becomes

$$\sum_{\substack{d_1+d_2=d\\ d_2>0}}\sum_{\gamma}(H_\gamma\cdot(S+d_1F))\,(H^\gamma\cdot d_2F)GW^{\mathcal{H}_n}_{0,0}(S+d_1F)\,GW_{1,0}(d_2F)$$
$$+\sum_{\gamma}(H_\gamma\cdot(S+dF))\,GW^{\mathcal{H}_n}_{0,0}(S+dF)\,GW_{1,1}(0)(H^\gamma) \tag{0.4}$$

Otherwise, (0.3) vanishes. Since $\sum_\gamma(H_\gamma A)(H^\gamma B) = AB$ and $kGW_{1,0}(kF) = (2-n)\sigma(k)$ (see [IP1]), the first sum in (0.4) becomes $(2-n)\sum_{k\geq 1} GW^{\mathcal{H}_n}_{0,0}(S+(d-k)F)\,\sigma(k)$. On the other hand, $GW_{1,1}(0)(H^\gamma) = \frac{1}{24}(KH^\gamma)$ (see [IP3]), where $K = (n-2)F$ is the canonical class. This implies that the second sum in (0.4) becomes $\frac{n-2}{24}GW^{\mathcal{H}_n}_{0,0}(S+dF)$. In summary, we have

$$\sum_{\substack{i\in I\\ |I|\geq 2}} GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{0,I}};F^4)$$
$$= (2-n)\sum_{k\geq 1} GW^{\mathcal{H}_n}_{0,0}(S+(d-k)F)\,\sigma(k) + \frac{n-2}{24}GW^{\mathcal{H}_n}_{0,0}(S+dF) \tag{0.5}$$

Note that $PD(\mathrm{Im}(\theta)) = 2\delta_{G_{irr}}$ where $\theta : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{1,4}$ as in (0.10). It then follows from the second Composition Law and Proposition 3.0.19 that

$$\begin{aligned} GW^{\mathcal{H}_n}_{1,4}(S+dF)(\delta_{G_{irr}};F^4) &= \frac{1}{2}\sum_{\gamma} GW^{\mathcal{H}_n}_{0,6}(S+dF)(F^4,H_\gamma,H^\gamma) \\ &= \frac{1}{2}\sum_{\gamma}(H_\gamma(S+dF))\,(H^\gamma(S+dF))\,GW^{\mathcal{H}_n}_{0,4}(S+dF)(F^4) \\ &= \frac{2d-n}{2}GW^{\mathcal{H}_n}_{0,0}(S+dF). \end{aligned} \tag{0.6}$$

The proof follows from (0.2), (0.5), (0.6) and the definition of $F(t)$ and $H(t)$.

□

CHAPTER 8

Ruan-Tian Invariants of $E(n)$

Instead of constructing virtual moduli cycle directly from the moduli space of stable J-holomorphic maps, Ruan and Tian [RT1, RT2] perturbed the equation (0.6) to $\overline{\partial}_J f = \nu$ where the inhomogeneous term ν can be chosen generically. For generic (J, ν), the moduli space of stable (J, ν)-holomorphic maps is then a compact smooth orbifold with all lower strata having codimension at least two. Ruan and Tian defined GW-invariants from this (perturbed) moduli space.

We can follow as similar procedure for the family invariants by introducing an inhomogeneous term into the (J, α)-holomorphic equation and vary (J, ν) and corresponding parameter space $\mathcal{H}$. In taking that approach, we immediately face two main problems: compactness and the dimension of lower strata. In general, it is difficult to show the compactness of a perturbed moduli space, even if $|\nu|$ is small and the moduli space without perturbation is compact. It is also difficult to determine the dimension of lower strata which contain bubble components. However, for the moduli space of perturbed (J, α)-holomorphic maps representing a homology class $S + dF$ in $E(n)$ with fixed complex structure J, the moduli space of (J, α)-holomorphic maps with generic perturbation is still compact and the image of lower strata under stabilization and evaluation map is contained in a set of codimension at least two. Therefore, we can define invariants from the moduli space with fixed Kähler structure and generic perturbation in the same way as for ordinary GW-invariants. This alternative definition of invariants is more geometric. In particular, using this definition of invariants we can follow the analytic arguments of Ionel and Parker in [IP2, IP3] to show sum formula (0.5) for the case at hand: the class $S + dF$ in $E(n)$.

To simplify notation in this Chapter we will set $X = E(n)$ and $A = S + dF$.

The construction of invariants starts from the perturbed equation $\overline{\partial}_J f = \nu$. Using Prym structures defined as in [L], we can lift Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$ to a finite cover

$$p_\mu : \overline{\mathcal{M}}^\mu_{g,k} \to \overline{\mathcal{M}}_{g,k}. \tag{0.1}$$

This finite cover is now a smooth manifold and has a universal family

$$\pi_\mu : \overline{\mathcal{U}}^\mu_{g,k} \to \overline{\mathcal{M}}^\mu_{g,k}$$

which is projective. Moreover, for each $b \in \overline{\mathcal{M}}^\mu_{g,k}$, $\pi_\mu^{-1}(b)$ is a stable curve isomorphic to $p_\mu(b)$.

We fix, once and for all, an embedding of $\overline{\mathcal{U}}^\mu_{g,k}$ into some $\mathbb{P}^N$. An inhomogeneous term ν is then defined as a section of the bundle $\mathrm{Hom}(\pi_1^*(T\mathbb{P}^N), \pi_2^* TX)$ which is anti-J-linear :

$$\nu(j_{\mathbb{P}}(v)) = -J(\nu(v)) \qquad \text{for any } v \in T\mathbb{P}^N \tag{0.2}$$

where $j_{\mathbb{P}}$ is the complex structure on $\mathbb{P}^N$.

For each stable map $f : \Sigma \to X$, we can specify one element $j \in p_\mu^{-1}(st(\Sigma))$. Then $\pi_\mu^{-1}(j)$ is isomorphic to the stable curve $st(\Sigma)$. In this way, we can define a map

$$\phi : \Sigma \to st(\Sigma) \cong \pi_\mu^{-1}(b) \subset \overline{\mathcal{U}}^\mu_{g,k} \hookrightarrow \mathbb{P}^N. \tag{0.3}$$

Definition 8.0.41 *A stable (J, ν, α)-holomorphic map is a stable map $f : (\Sigma, \phi) \to X$ satisfying*

$$(\, df + J_\alpha df j_\Sigma \,)(p) = \nu_\alpha(\phi(p), f(p))$$

where ϕ is defined as in (0.3), and $\nu_\alpha = (I + JK_\alpha)^{-1}\nu$. $\square$

Two stable (J, ν, α)-holomorphic maps $(f, (\phi, \Sigma); x_1, \cdots, x_k)$ and $(f', (\phi', \Sigma'); x'_1, \cdots, x'_k)$ are equivalent if

$$d_H(\phi(\Sigma), \phi'(\Sigma'))) + d_H(f(\Sigma), f'(\Sigma')) + \sum d(f(x_i), f'(x'_i)) = 0$$

where d_H is the Hausdorff distance. We then define the moduli space

$$\overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$$

as the set of all pairs $([f, (\phi, \Sigma); x_1, \cdots, x_k], \alpha)$, where $\alpha \in \mathcal{H}$ and $[f, (\phi, \Sigma); x_1, \cdots, x_k]$ is the equivalence class of (J, ν, α)-holomorphic maps with $[f(\Sigma)] = A \in H_2(X; Z)$. We denote by $\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$ the set of $([f, (\phi, \Sigma); x_1, \cdots, x_k], \alpha)$ with a smooth domain Σ. We will often abuse notation by writing (f, j, α) or simply (f, α), instead of $(f, (\phi, \Sigma), \alpha)$.

There is a stratification of $\mathcal{M}_{g,k}$ parameterized by the automorphism group of Riemann surfaces $\mathcal{M}_{g,k} = \sum_\kappa \mathbf{T}^\kappa_{g,k}$ where each strata $\mathbf{T}^\kappa_{g,k}$ is smooth and consists of the Riemann surfaces with a fixed automorphism group κ. We can also assume that

$$\mathcal{M}^\mu_{g,k} = \sum_\kappa \mathbf{T}^{\mu,\kappa}_{g,k}$$

where $\mathbf{T}^{\mu,\kappa}_{g,k} = p_\mu^{-1}(\mathbf{T}^\kappa_{g,k})$ is smooth. Let $\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_I$ consist of all (f, j, α) with j in $\mathbf{T}^{\mu,I}_{g,k}$, where I denote the trivial automorphism group.

Consider the following stabilization and evaluation maps

$$st^\mu \times ev^\mu : \mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_I \to \overline{\mathcal{M}}^\mu_{g,k} \times X^k. \tag{0.4}$$

Its Frontier is defined to be the set

$$\{ r \in \overline{\mathcal{M}}^\mu_{g,k} \times X^k \,|\, r = \lim (st^\mu \times ev^\mu)(f_n, \alpha_n)$$

$$\text{and } (f_n, \alpha_n) \text{ has no convergent subsequences} \}.$$

We denote by $\mathcal{Y}_0$ the space of all ν with $|\nu|_\infty$ is sufficiently small. Now, we are ready to state "Structure Theorem" for the moduli space.

Theorem 8.0.42 (Structure Theorem) *For generic* $\nu \in \mathcal{Y}_0$,

(a) $\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_I$ *is an oriented smooth manifold of dimension*

$$-2KA + 2(g-1) + 2k + dim(\mathcal{H}) = 2(g+k) \tag{0.5}$$

(b) the Frontier of the smooth map

$$st^\mu \times ev^\mu : \mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_I \to \overline{\mathcal{M}}^\mu_{g,k} \times X^k$$

lies in dimension 2 less that $2(g+k)$.

Proof. This proof is similar to the proof of Proposition 2.3 in [RT]. We will sketch proof, without specifying Sobolev norms.

(a) For each κ, define

$$\mathcal{X}^{\mu,\kappa} = \bigcup_{j \in \mathbf{T}^{\mu,\kappa}_{g,k}} Map_A(\Sigma_g, X) \times \{j\}$$

where $Map_A(\Sigma_g, X) = \{f : \Sigma_g \to X \mid f_*[\Sigma_g] = A\}$. Consider the vector bundle

$$\mathcal{E}^{\mu,\kappa} \to \mathcal{X}^{\mu,\kappa} \times \mathcal{H} \times \mathcal{Y}_0$$

whose fiber over (f, j, α, ν) is $\Omega^{0,1}_{j,J_\alpha}(f^*TX)$. Obviously, the (J, ν, α)-holomorphic equation defines a section Φ of $\mathcal{E}^{\mu,\kappa}$ by

$$\Phi(f, j, \alpha, \nu) = df + J_\alpha df j - \nu_\alpha.$$

The differential $D\Phi$ of Φ at (f, j, α, ν) is then an elliptic operator

$$D\Phi : \Omega^0(f^*TX) \oplus T_j\mathbf{T}^{\mu,\kappa}_{g,k} \oplus T_\alpha\mathcal{H} \oplus \mathrm{Hom}_J(T\mathbb{P}^N, TX) \to \Omega^{0,1}_{jJ_\alpha}(f^*TX)$$

given by

$$D\Phi(\xi,k,\beta,v) = L_f(\xi) + J_\alpha df\, k + L_\alpha(\beta) - A^{-1}v \quad \text{where} \tag{0.6}$$
$$L_f(\xi) = \nabla\xi + J_\alpha\nabla\xi j + \frac{1}{2}(\nabla_\xi J_\alpha)(dfj + J_\alpha df) - \tilde{\nabla}_\xi \nu_\alpha,$$
$$L_\alpha(\beta) = \big(-A^{-1}JK_\beta J_\alpha + J_\alpha A^{-1}JK_\beta\big)\,dfj,$$
$$\tilde{\nabla} = \frac{1}{2}(\nabla - J_\alpha\nabla J_\alpha), \quad \text{and} \quad A = (I + JK_\alpha)$$

Consider the universal moduli space

$$\mathcal{U}^{\mu,\kappa}_{g,k}(X,A) = \{\,(f,j,\alpha,\nu) \in \mathcal{X}^{\mu,\kappa}_{g,k} \times \mathcal{H} \times \mathcal{Y}_0 \mid \Phi(f,j,\alpha,\nu) = 0\,\}.$$

At any $(f,j,\alpha,\nu) \in \mathcal{U}^{\mu,\kappa}_{g,k}(X,A)$, the differential $D\Phi$ is surjective because of the term $A^{-1}v$. This implies that the universal moduli space is smooth.

Let $\pi : \mathcal{U}^{\mu,\kappa}_{g,k}(X,A) \to \mathcal{Y}_0$ be the projection. Its differential at (f,j,α,ν)

$$d\pi : T\mathcal{U}^{\mu,\kappa}_{g,k}(X,A) \to T\mathcal{Y}_\omega$$

is just the projection $(\xi,k,\beta,v) \to v$. It then follows that the kernel of $d\pi$ is isomorphic to the kernel of $L_f \oplus J_\alpha df \oplus L_\alpha$. Moreover, its image consists of all v with

$$A^{-1}v \in \mathrm{Im}(L_f \oplus J_\alpha df \oplus L_\alpha).$$

Note that $L_f \oplus J_\alpha df \oplus L_\alpha$ is Fredholm, and hence $\mathrm{Im}(d\pi)$ is a closed subspace of $T\mathcal{Y}_0$.

On the other hand, the map

$$p : T\mathcal{Y}_0 \to \Omega^{0,1}_{jJ_\alpha}(f^*X) \,/\, \mathrm{Im}(L_f \oplus J_\alpha df \oplus L_\alpha) \;\; \text{defined by} \;\; v \to A^{-1}v$$

is onto since $D\Phi$ is onto. Therefore, $\dim(\mathrm{Coker}(L_f \oplus J_\alpha df \oplus L_\alpha)) = \dim(T\mathcal{Y}_0 \,/\, \mathrm{Ker}(p)) = \dim(\mathrm{Coker}(d\pi))$ and hence $d\pi$ is Fredholm of the same index as the index of $(L_f \oplus J_\alpha df \oplus L_\alpha)$.

Applying Sard-Smale Theorem, we can conclude that for generic $\nu \in \mathcal{Y}_0$, the moduli space

$$\pi^{-1}(\nu) = \mathcal{M}_{g,k}(X,A,\nu,\mathcal{H},\mu)_\kappa$$

is a smooth manifold. The dimension formula follows from the Index Theorem.

For generic (J, ν), the tangent space $T_{f,j,\alpha}\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_\kappa = \mathrm{Ker}(L_f \oplus J_\alpha df \oplus L_\alpha)$, so we have

$$det\,(T_{f,j,\alpha}\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_\kappa) = det(L_f \oplus J_\alpha df \oplus L_\alpha)$$

On the other hand, there is a decomposition $L_f = L_f^{J_\alpha} + Z$, where $L_f^{J_\alpha}$ is J_α-linear and Z is the zero order term. It follows that $det(L_f \oplus J_\alpha df \oplus L_\alpha)$ is isomorphic to $det(L_f^{J_\alpha} \oplus J_\alpha df \oplus L_\alpha)$. Since both kernel and cokernel of $L_f^{J_\alpha} \oplus J_\alpha df \oplus L_\alpha$ are complex vector spaces, there is a canonical non-vanishing section of $det(L_f^{J_\alpha} \oplus J_\alpha df \oplus L_\alpha)$. Therefore, there is a nonvanishing section of $det(L_f \oplus J_\alpha df \oplus L_\alpha)$ which orients $\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_\kappa$. □

(b) This proof consists of 5 steps. In step 1, we show that the stabilization and evaluation map as in (0.4) extends continuously to the moduli space of stable maps. That follows from the compactness. In step 2, we show all possible homology classes, which can be represented by the components of stable maps. In step 3, we reduce the moduli space. The resulting reduced moduli space will have the same image as that of the moduli space under the stabilization and evaluation maps. In step 4, we stratify the reduced moduli space. The Frontier is then contained in the image of all lower strata. In the final step, we show that each image of the lower strata is contained dimension 2 less than $2(g + k)$.

Step 1 There are well-defined stabilization and evaluation map

$$st^\mu \times ev^\mu : \overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu) \to \overline{\mathcal{M}}^\mu_{g,k} \times X^k \tag{0.7}$$

where we still use the same notation for the map as in (0.4), without further confusion. It follows from Gromove Compactness Theorem, Theorem 3.0.18, and the following lemma that (0.7) extends (0.4) continuously.

Lemma 8.0.43 *There exit uniform constants* E_0 *and* N *such that for any* $(f, \Sigma, \alpha) \in \overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H})$

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2 \leq E_0 \quad \textit{and} \quad ||\alpha|| \leq N.$$

Proof. Similarly to Corollary 1.0.7, we have

$$\int_\Sigma |\overline{\partial}_J f|^2 = \int_\Sigma f^*\alpha + 2\int_\Sigma \langle \overline{\partial}_J f, \nu\rangle \tag{0.8}$$

$$(1 + |\alpha|^2) f^*\omega\, dv = \frac{1}{2}(1 - |\alpha|^2)|df|^2\, dv - 4\langle \overline{\partial}_J f, \nu\rangle\, dv + 4|\nu|^2\, dv \tag{0.9}$$

Note that f represent homology class $A = S + dF$ which is of type (1,1) with respect to the complex structure J. Therefore, it follows from (0.8) and Proposition 1.0.6a that

$$\frac{1}{2}\int_\Sigma |df|^2 \leq \omega(A) + 2\left(\int_\Sigma |df|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |\nu|^2\right)^{\frac{1}{2}}$$

We then have a uniform energy bound by using the inequality $2ab \leq \varepsilon\, a^2 + \varepsilon^{-1}\, b^2$ on the last term and absorbing the $|df|$ term on the left-hand side.

Next, we will show uniform bound of $||\alpha||$. This proof is similar to those of Lemma 4.0.24 except for using (0.9) instead of Corollary 1.0.7b. Let $\pi : X \to \mathbb{C}P^1$ be the elliptic structure for J on X and $N(\alpha)$, $m(J)$, and N be as in the proof of Lemma 4.0.24. If there is a holomorphic fiber $F \subset X \setminus N(\alpha)$ such that

(i) f is transversal to F,

(ii) at each $p \in f^{-1}(F)$, f is transversal to a holomorphic disk $D_{f(p)}$ normal to F at $f(p)$, and

(iii) $4\,|df|\,|\nu| + 4|\nu|^2 \leq \frac{1}{2}\,|df|^2$ on $f^{-1}(F)$

then the proof follows exactly as in the proof of Lemma 4.0.24. We can clearly find fibers satisfying (i) and (ii), so we need only verify that we can also obtain (iii). For that we consider the set Σ_0 of all points in Σ where $4\,|df|\,|\nu| + 4|\nu|^2 \;>\; \frac{1}{2}\,|df|^2$. Then

$|df|^2 \leq 16|\nu|$ on Σ_0, since both $|df|$ and $|\nu|_\infty$ are less than 1. Therefore

$$\int_{\Sigma_0} |\, d\pi \circ df \,|^2 \ \leq \ 16 \,\mathrm{Area}(\, st(\Sigma)\,)\, |\, d\pi\,|^2_\infty |\, \nu\,|_\infty \tag{0.10}$$

We can thus assume that (0.10) $\leq \frac{1}{3}\mathrm{Area}(\mathbb{C}P^1)$ for sufficiently small $|\nu|_\infty$. On the other hand, from the definition of $N(\alpha)$, we can also assume that $\mathrm{Area}(\,\pi(N(\alpha))\,) \leq \frac{1}{3}\mathrm{Area}(\mathbb{C}P^1)$. Therefore, we can always choose a holomorphic fiber $F = \pi^{-1}(q)$ as in the above claim with $q \in \mathbb{C}P^1 \setminus (\ \pi(N(\alpha)) \cup \pi \circ f(\Sigma_0)\,)$. $\square$

Step 2 Let (f, Σ) be a stable map. A stable bubble component which maps to a point by f is called *ghost bubble.* Now, we reduce the moduli space as follows : for a stable (J, ν, α) holomorphic map,

(i) we collapse all ghost bubbles,

(ii) we replace each multiple map from a bubble by its reduced map

(iii) we identify those bubble components which have the same image.

Denote by $\overline{\mathcal{M}}^r_{g,k}(X, A, \nu, \mathcal{H}, \mu)$ the quotient of $\overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$ by this reduction. We define the topology on it as the quotient topology.

The reduced moduli space is still compact. On the other hand, the map (0.7) descends to the reduced moduli space and by definition the image of reduced moduli space is same as that of stable moduli space under stabilization and evaluation maps.

Step 3 Let $(f, \Sigma, \alpha) \in \overline{\mathcal{M}}^r_{g,k}(X, A, \nu, \mathcal{H}, \mu)$ with $\Sigma = \cup_i \Sigma_i$ and $[f_i(\Sigma_i)] = A_i$. The following lemma shows all possible homology classes for A_i.

Lemma 8.0.44 *If $|\nu|_\infty$ is sufficiently small, then A_i is one of the following homology classes*

$$S, \quad S + d_1 F, \quad d_2 F \quad \textit{with} \ \ 0 < d_1, d_2 \leq d. \tag{0.11}$$

and each bubble component Σ_i represents either S or $d_2 F$.

Proof. Suppose not. Then there exits a sequence of (J, α_n, ν_n)-holomorphic maps (f_n, Σ_n) and a homology class C which is not one of classes in (0.11) such that (i) $\nu_n \to 0$ as $n \to \infty$, (ii) for each n, there is some component Σ_{n_i} with $[f_{n_i}(\Sigma_{n_i})] = C$. By Lemma 8.0.43 and Gromov Convergence Theorem, we may assume that $J_{\alpha_n} \to J_{\alpha_0}$ uniformly for some $\alpha_0 \in \mathcal{H}$ and f_n converges to f_0, where f_0 is (J, α_0)-holomorphic and f_0 represent the class $S + dF$. Since $S + dF$ is of type (1,1) and α_0 is closed J-anti invariant 2-form, $\alpha_0 = 0$. This implies that the homology classes C and $D = S + dF - C$ are both represented by holomorphic map with possibly reducible domain.

Note that $C \cdot F$ is either 0 or 1. Assume $C \cdot F = 0$. Then $C = d_3 F$ with $d_3 > d$ by the assumption on C and $D = S + (d - d_3)F$. Let (f_D, Σ_D) be the holomorphic map representing homology class D. Then for some component Σ_{D_i}, the restriction map f_{D_i} represent a homology class $S - d_4 F$ with $d_4 > 0$. This is impossible since $(S - d_4 F) \cdot S < 0$ and Σ_{D_i} is irreducible. Similarly, we also have the contradiction when $C \cdot F = 1$.

When Σ_i is a bubble component, ν vanishes on Σ_i and hence f_i is holomorphic. Therefore, A_i cannot be $S + d_2 F$ since Σ_i is irreducible. □

Step 4 In this step, we stratify the reduced moduli space. For each $(f, \alpha) \in \overline{\mathcal{M}}^{r}_{g,k}(X, A, \nu, \mathcal{H}, \mu)$, the normalization of the domain of the map f, without specified complex structure, is a disjoint union of smooth Riemann surfaces

$$\Sigma = P_1 \cup \cdots \cup P_n \cup B_1 \cup \cdots B_m \tag{0.12}$$

where B_i is the bubble component. We will call P_i a *principal component.*

Each component of Σ has points corresponding to the marked points of the domain, which we also call *marked points.* Each component also have points corresponding to the singular points of the domain. We call these points *intersection* *points.* Together with the normalization map, the intersection points carries the *intersection data* of the domain. Each component is also associated to a homology class in $H_2(X;Z)$, which we denote by $[P_i]$ and $[B_l]$.

Note that $[P_i]$ is one of the class in (0.11) and $[B_l]$ is either s or f by definition of reduction. Moreover, if some $[B_l] = f$, then the image of the restriction map $f_{|B_l}$ is some nodal fiber F_l and $\alpha \in \{ \beta \in \mathcal{H} \mid F_l \subset Z(\beta) \}$ by Lemma 2.0.12. Similarly, if some $[B_l] = s$, then the image of the restriction map $f_{|B_l}$ is a section S and $\alpha = 0$ by Lemma 2.0.12. In this way, the set of bubble components determines a subspace $\mathcal{H}_B \subset \mathcal{H}$. If there is no bubble component, we simply set $\mathcal{H}_B = \mathcal{H}$. Since there's no fixed component in the complete linear system $|K|$ of a canonical divisor, Lemma 2.0.11 implies that this subspace $\mathcal{H}_B$ has at least 2 real codimension whenever it is proper.

We denote by $D_\nu(f,\alpha)$ the set of following four data, (i) Σ with marked points, (ii) the intersection points with intersection data, (iii) the set of homology classes each of which is associated to a component of Σ, and (iv) the subspace $\mathcal{H}_B$ determined by bubble components. Let $\mathcal{D}_\nu$ be the set of all $D_\nu(f,\alpha)$'s. It then follows from the uniform energy bound and Lemma 8.0.44 that $\mathcal{D}_\nu$ is finite.

Fix $D \in \mathcal{D}_\nu$ such that Σ_D has more than one component. If Σ_D has just one component, proof follows from proof of (a) above. Let g_i, k_i, and d_i, be the genus of P_i, the number of marked points on P_i, and the number of intersection points on P_i with principal components, respectively. Similarly, let k^l be the number of marked points on B_l. Note that k^l might be 2 by reduction. Let $\bar{\kappa} = (\kappa_1, \cdots, \kappa_p)$, where κ_i is an automorphism group of some $j \in \overline{\mathcal{M}}_{g_1,k_i+d_i+d^i}$ and p is the number of principal component. We will use $(D,\bar{\kappa})$ to label each stratum of the reduced moduli space. We denote by

$$\mathcal{M}(D,\bar{\kappa}) \subset \overline{\mathcal{M}}^r_{g,k}(X,A,\nu,\mathcal{H},\mu)$$

the stratum labeled by $(D, \bar{\kappa})$.

Step 5 Finally, we will show that the image of each strata is contained dimension 2 less than $2(g+k)$. Consider the universal moduli space

$$\mathcal{U}_{D,\bar{\kappa}} = \{ \ (\ (f_1, j_1), \cdots, (f_n, j_n), \nu, \alpha \ ) \ \}$$

where $df_i + J_\alpha df_i j_i = \nu_i$ with $j_i \in \mathbf{T}^{\mu,\kappa_i}_{g_i,k_i+d_i}$, $\nu_i = \nu_{|P_i}$, $\alpha \in \mathcal{H}_{D_B}$, and $[df_i] = [P_i]$. We have an evaluation map

$$ev_{D_P} : \mathcal{U}_{D,\bar{\kappa}} \to X^d$$

which records the image of intersection points, where $d = \sum d_i$. We denote by $\triangle_{D_P}$ the multi-diagonal in X^d, which is determined by the intersection pattern of principal components.

Now, the inhomogeneous term ν ensure that $\mathcal{U}_{D,\bar{\kappa}}$ is smooth and ev_{D_P} is transversal to $\triangle_{D_P}$. Therefore, $\mathcal{U}_{D,\bar{\kappa}} \cap ev^{-1}_{D_P}(\triangle_{D_P})$ is smooth. Finally, the natural projection

$$\pi : \mathcal{U}_{D,\bar{\kappa}} \cap ev^{-1}_{D_P}(\triangle_{D_P}) \to \mathcal{Y}_0$$

is Fredholm. By Sard-Smale Theorem, we can then conclude that for generic ν, the moduli space

$$\mathcal{M}(D_P, \nu, \mathcal{H}_{D_B}, \kappa) = \pi^{-1}(\nu)$$

is smooth. Its dimension is obtained from the routine count and is less than or equal to

$$2g + 2\sum k_i + 2 - 2p - \text{codim}(\mathcal{H}_{D_B}) \quad \text{if } \mathcal{H}_{D_B} \neq \{0\} \tag{0.13}$$

$$2g + 2\sum k_i - 2 \quad \text{if } \mathcal{H}_{D_B} = \{0\} \tag{0.14}$$

where p is the number of principal component. Note that $2 - 2p - \text{codim}(\mathcal{H}_{D_B}) \leq -2$ since Σ_D has more than one component.

Recall that for each bubble component B_l, the homology class $[B_l]$ is either s or f. We identify each B_l with a section S or nodal fiber F according to $[B_l]$, and then fix, once and for all, a holomorphic map $h_l : \mathbb{C}P^1_l \to B_l \subset X$. Let b is the number of bubble components and set

$$\mathcal{S}(D_B) = (\mathbb{C}P^1_1)^{k^1} \times \cdots (\mathbb{C}P^1_b)^{k^b}.$$

Clearly, it is smooth of dimension $2\sum k^l$.

Let $k_1 = \sum k_i$ and $k_2 = \sum k^i$. Note that $k_1 + k_2 = k$. Each element $(j_1, \cdots, j_p)$ determines an element in $\overline{\mathcal{M}}_{g,k}$ by gluing intersection points by intersection data. Obviously, we have an evaluation map which records k_1 marked points on principal components. We also have an obvious map from $\mathcal{S}(D_B)$ into X^{k_2} by h_l's. Combing those three maps to obtain a continuous map

$$\theta \times ev_P \times h_B : \mathcal{M}(D_P, \nu, \mathcal{H}_{D_B}, \kappa) \times \mathcal{S}(D_B) \to \overline{\mathcal{M}}^{\mu}_{g,k} \times X^k$$

It then follows that its image $\mathrm{Im}(\theta \times ev_P \times h_B)$ lies in dimension $2(g+k) - 2$.

Now, it remains to show the following :

$$st^{\mu} \times ev^{\mu}(\, \mathcal{M}(D, \bar{\kappa})\,) \subset \mathrm{Im}(\theta \times ev_P \times h_B) \tag{0.15}$$

There is a decomposition of the evaluation map ev^{μ} on $\mathcal{M}(D, \bar{\kappa})$ as $ev^{\mu} = ev^{\mu}_P \times ev^{\mu}_B$, where $ev^{\mu}_P(ev^{\mu}_B)$ records marked points on principal(bubble) components. Note that $\mathrm{Im}(ev^{\mu}_B) \subset \mathrm{Im}(h_B)$ since for all $(f, \Sigma, \alpha) \in \mathcal{M}(D, \bar{\kappa})$, the image of bubbles are all same.

On the other hand, for each $(f, \Sigma, \alpha) \in \mathcal{M}(D, \bar{\kappa})$, if we forget all bubbles, then we obtain an element (f_P, Σ_P, α) in $\mathcal{M}(D_P, \nu, \mathcal{H}_{D_B}, \kappa)$. In this way, we can define a map

$$\pi_P : \mathcal{M}(D, \bar{\kappa}) \to \mathcal{M}(D_P, \nu, \mathcal{H}_{D_B}, \kappa)$$

Then $(\theta \times ev_P) \circ \pi_P = st^{\mu} \times ev^{\mu}_P$. Together with $\mathrm{Im}(ev^{\mu}_B) \subset \mathrm{Im}(h_B)$, this implies (0.15). $\square$

Now, we are ready to define invariant. Instead of using intersection theory as in [RT] , we will follow the approach in [IP2]. The above Structure Theorem implies that

$$\mathrm{Fr}(\, st^{\mu} \times ev^{\mu}\,) \subset st^{\mu} \times ev^{\mu}\left(\overline{\mathcal{M}}^{r}_{g,k}(X, A, \nu, \mathcal{H}, \mu) \setminus \mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_I\right)$$

In particular, the Frontier lies in dimension $2(g+k)-2$. It then follows Proposition 4.2 of [KM] that the image

$$st^{\mu} \times ev^{\mu}\,(\mathcal{M}_{g,k}(X, A, \nu, \mathcal{H}, \mu)_I)$$

give rise to a rational homology class. We denote it by

$$[\,\overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu)\,] \in H_*(\overline{\mathcal{M}}^{\mu}_{g,k}; \mathbb{Q}) \otimes H_*(X^k; \mathbb{Q}). \tag{0.16}$$

Definition 8.0.45 *For $2g + k \geq 3$, $\beta \in H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$, and $\alpha_1, \cdots, \alpha_k \in H^*(X^k; \mathbb{Q})$, we define invariants by*

$$\Phi_{g,k}(X, A, \mathcal{H})(\beta; \alpha_1, \cdots, \alpha_k) = \frac{1}{\lambda_{\mu}}\,(\,\beta \otimes (\alpha_1 \wedge \cdots \wedge \alpha_k)\,) \cap [\,\overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu)\,]$$

where λ_{μ} is the order of the finite cover in (0.1).

By repeating the arguments used in [RT] for the ordinary GW-invariants, we can prove that these invariants $\Phi_{g,k}(X, A, \mathcal{H})$ are independent of the inhomogeneous term ν, the finite cover p_{μ}, and the projective embedding $\overline{\mathcal{U}}^{\mu}_{g,k} \hookrightarrow \mathbb{P}^N$. Alternatively, we can simply observe that those three facts emerge as corollaries of the following proposition.

Proposition 8.0.46 $\Phi_{g,k}(X, A, \mathcal{H}) = GW^{\mathcal{H}}_{g,k}(X, A)$

Proof. As in Chapter 3, we define $\overline{\mathcal{F}}^{J}_{g,k}(X, A, \mu)$ to be the set of all equivalence classes of the stable maps of the form $(f, (\Sigma, \phi))$, where ϕ is defined as in (0.3); two stable maps $(f, (\Sigma, \phi))$ and $(f', (\Sigma', \phi'))$ are equivalent if there is a marked points preserving biholomorphic map $\sigma : \Sigma \to \Sigma'$ such that $f = f' \circ \sigma$ and $\phi = \phi' \circ \sigma$. Note that $\overline{\mathcal{F}}^{J}_{g,k}(X, A, \mu)$ is a finite cover of $\overline{\mathcal{F}}^{J}_{g,k}(X, A)$. Similarly, we define a generalized bundle E^{μ} over $\overline{\mathcal{F}}^{J}_{g,k}(X, A, \mu) \times \mathcal{H}$ and a section Φ^{μ} by $(f, (\Sigma, \phi), \alpha) \to df + J_{\alpha} df\, j$. It follows from Lemma 8.0.43 that the zero set of Φ^{μ} is compact. Therefore, by Proposition 3.0.14 there is a virtual moduli cycle which satisfying

$$\pi_*[\mathcal{M}^{\mathcal{H}_n}_{g,k}(X, A, \mu)] = \lambda_{\mu}[\mathcal{M}^{\mathcal{H}_n}_{g,k}(X, A,)] \tag{0.17}$$

where $\pi : \overline{\mathcal{F}}^{J}_{g,k}(X, A, \mu) \to \overline{\mathcal{F}}^{J}_{g,k}(X, A)$ and λ_{μ} is the order of p_{μ}.

Now, fix a generic ν as in Theorem 8.0.42. It follows from Proposition 3.0.15 and Lemma 8.0.43 that we still have the same moduli cycle as in (0.17) when we change the section Φ^{μ} by adding $-\nu$. We still use the same notation Φ^{μ} for this new section. Note that

$$U_I = \mathcal{M}(X, A, \nu, \mathcal{H}, \mu)_I \subset (\Phi^{\mu})^{-1}(0)$$

It follows from the proof of Proposition 3.0.14 that U_I is one of the open sets of the finite cover of $(\Phi^{\mu})^{-1}(0)$ as in Definition 5.0.28.

Let $n = \dim(\overline{\mathcal{M}}_{g,k} \times X^k)$ and $d = 2(g + k) = \dim(\mathcal{M}(X, A, \nu, \mathcal{H}, \mu)_I)$. Since the Frontier of $st^{\mu} \times ev^{\mu}$ lies in dimension $d - 2$, there is an arbitrary small neighborhood V of $Fr(st^{\mu} \times ev^{\mu})$ such that every homology class in $H_{n-d}(\overline{\mathcal{M}}^{\mu}_{g,k} \times X^k; Q)$ has a representative disjoint from $\overline{V}$.

We can assume that for any open set U_i in the finite cover of $(\Phi^{\mu})^{-1}(0)$ as above with $U_i \neq U_I$, the intersection $U_i \cap U_I$ lies in $(st^{\mu} \times ev^{\mu})^{-1}\left(\overline{\mathcal{M}}^{\mu}_{g,k} \times X^k \setminus \overline{V}\right)$. It then follows from the proof of Theorem 1.2 in [LT] that the cycle Z which represents the virtual moduli cycle satisfies

$$st^{\mu} \times ev^{\mu}(Z) \cap \big(\overline{\mathcal{M}}^{\mu}_{g,k} \times X^k \setminus \overline{V}\big) = st^{\mu} \times ev^{\mu}(U_I) \cap \big(\overline{\mathcal{M}}^{\mu}_{g,k} \times X^k \setminus \overline{V}\big)$$

This implies that

$$(st^{\mu} \times ev^{\mu})_*[\mathcal{M}^{\mathcal{H}_n}_{g,k}(X, A, \mu)] = [\overline{\mathcal{M}}_{g,k}(X, A, \nu, \mathcal{H}, \mu)]$$

Therefore, by Definition 8.0.45 and (0.17) we can conclude that two invariants are same. $\square$

In the below, we will not distinguish two invariants and use the same notation $GW^{\mathcal{H}_n}_{g,k}(X, A)$ for them. The following proposition shows $F(0) = 1$ which provides the initial condition for (0.6).

Proposition 8.0.47 $GW^{\mathcal{H}_n}_{0,3}(X, S)(F^3) = 1.$

Proof. Fix $\nu = 0$. Since the section class S is of type $(1,1)$, Theorem 2.0.12 implies that for any (J, α)-holomorphic map (f, α) with $[f] = S$, f is holomorphic and $\alpha = 0$. In fact, there is a unique such f since $S^2 = -n$. Now, consider the linearization as in (0.6). Propositions A.63 and A.64 of the appendix show, quite generally, that L_f is a $\overline{\partial}_J$ operator and L_0 defines a map

$$L_0 : \mathcal{H} \to \mathrm{Coker}(L_f)$$

which is injective if and only if the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X, A)$ is compact. But we just showed the moduli space is a single point, and hence compact.

On the other hand, $\mathrm{Ker}(L_f)$ is same as the Dolbeault cohomology group $H^0_{\overline{\partial}}(f^*TX)$. It is trivial since $c_1(f^*TX)[S^2] = -n + 1 < 0$. Therefore,

$$\dim(\,\mathrm{Coker}\, L_f\,) = -\mathrm{Index}(L_f) = -2(\,c_1(f^*TX) + 1\,) = 2(n-1)$$

Since L_0 is injective and $\dim(\mathcal{H}) = 2(n-1)$, $L_f \oplus L_0$ is onto. That implies $\nu = 0$ is generic in the sense of Theorem 8.0.42. Consequently, the invariant is ± 1. In this case, the sign is determined by L_f and L_f is $\overline{\partial}_J$-operator, the invariant is 1. $\square$

CHAPTER 9

Degeneration of $E(n)$

Throughout this Chapter, X always denotes the standard elliptic surface $E(n) \to \mathbb{C}P^1$ and Y always denotes $T^2 \times S^2$ with a product complex structure.

In this Chapter, we construct a degeneration of X into a singular surface which is a union of X and Y with $V = T^2$ intersection. The sum formula (0.5) will be then formulated from this degeneration. We also define the parameter space and inhomogeneous terms corresponding to this degeneration.

We fix a small constant $\epsilon > 0$ and let $D(\epsilon) \subset \mathbb{C}$ be a disk of radius ϵ. Choose a smooth fiber V in X. We then define $p : Z \to X \times D(\epsilon)$ to be the blow-up of $X \times D(\epsilon)$ along $V \times \{0\}$ and let

$$\lambda : Z \to D(\epsilon)$$

be the composition $Z@ > p >> X \times D(\epsilon) \to D(\epsilon)$, where the second map is the projection of the second factor. The central fiber $Z_0 = \lambda^{-1}(0)$ is a singular surface $X \cup_V Y$ and the fiber Z_λ with $\lambda \neq 0$ is isomorphic to X as a complex surface. Since Z is a blow-up of a Kähler manifold, it is also Kähler. Denote by (ω_Z, J_Z, g_Z) the Kähler structure on Z induced from the blow-up. We also denote by $(\omega_\lambda, J_\lambda, g_\lambda)$ the induced Kähler structure on each Z_λ with $\lambda \neq 0$.

We can describe the Z locally along $V \subset Z$ as follows : fix a normal neighborhood N of V in X. It is then a product $V \times D$, where $D \subset \mathbb{C}$ is some disk. Let x be the holomorphic coordinate of D. Then, Z is given locally along $V \subset Z$ as

$$\{ \, (v, x, \lambda, [\, l_0; l_1]) \mid v \in V, \; x\, l_1 = \lambda\, l_0 \, \} \subset N \times D^2 \times \mathbb{C}P^1$$

where $[l_0; l_1]$ are the homogeneous coordinates of $S^2 = \mathbb{C}P^1$. It is covered by two patches

$$U_0 = (l_0 \neq 0) \quad \text{and} \quad U_1 = (l_1 \neq 0).$$

On U_0, we set $y = l_1 / \, l_0$. Z is then locally given as

$$\{ \, (v, x, y) \mid v \in V \, \} \quad \text{with} \quad \lambda(p, x, y) = xy.$$

Clearly, the fiber Z_λ is given locally by the equation $xy = \lambda$. Note that we can also think of y as a holomorphic normal coordinate of the normal neighborhood of V in Y.

We now decompose Z as a union of three pieces, two ends and a neck. These are defined as follows : Let $N_X(\epsilon)\,(\, N_Y(\epsilon)\,)$ be the normal neighborhood of V in $X\,(\,Y\,)$ of the form $V \times D(\epsilon)$. We then set

$$\begin{aligned}
\mathrm{End}_X &= p^{-1}\left(\,(\,X \setminus N_X(\epsilon)\,) \times D^2\,\right) \xrightarrow{\varphi_X} (\,X \setminus N_X(\epsilon)\,) \times D^2 && (0.1)\\
\mathrm{End}_Y &= \{ \, (v, x, y') \in U_1 \mid y' = l_0 / l_1 \,, \; |y'| \leq 1/\epsilon \, \} \xrightarrow{\varphi_Y} (\,Y \setminus N_Y(\epsilon)\,) \times D^2 && (0.2)\\
U &= \{ \, (v, x, y) \in U_0 \mid |x| \leq 2\epsilon \,, \; |y| \leq 2\epsilon \, \} && (0.3)
\end{aligned}$$

where the map $\varphi_X \,(\, \varphi_Y\,)$ is the isomorphism which extends the holomorphic map

$$(v, x, y) \to (v, x, xy) \;\; (\,(v, x, y) \to (v, y, xy)\,)$$

for $(v, x, y) \in U$ and $\epsilon \leq |x| \leq 2\epsilon \,(\, \epsilon \leq |y| \leq 2\epsilon\,)$.

Next, we define the parameter space on Z as follows : choose a bumf function β on the neck region U which satisfies $\beta(|x|) = 1$ if $|x| \geq (3/2)\epsilon$ and $\beta(|x|) = 0$ if

$|x| \leq \epsilon$. We then extend β on the whole Z such that $\beta = 1$ on $\mathrm{End}_X \setminus \mathrm{Supp}(1-\beta)$ and $\beta = 0$ on End_Y. Let $\mathcal{H}$ be the parameter space of $X = E(n)$ defined as in (0.4). We consider each $\alpha \in \mathcal{H}$ as a 2-form on $X \times D(\epsilon)$. Then each $p^*\alpha$ is closed and J_Z-anti-invariant.

Definition 9.0.48 *We define the parameter space of the fibration $\lambda : Z \to D(\epsilon)$ by*

$$\mathcal{H}_Z \;=\; \{\, \beta p^*\alpha \mid \alpha \in \mathcal{H} \,\} \quad \textit{and} \quad \mathcal{H}_\lambda \;=\; \{\, \alpha_\lambda = \alpha_{|Z_\lambda} \mid \alpha \in \mathcal{H}_Z \,\} \quad \textit{when} \quad \lambda \neq 0$$

We can consider X as a Kähler submanifold of Z. It then follows from the above definition that

$$\mathcal{H}_X = \{\, \beta\alpha \mid \alpha \in \mathcal{H} \,\} = \{\, \alpha_{|X} \mid \alpha \in \mathcal{H}_Z \,\}. \tag{0.4}$$

Lemma 9.0.49 *There exit uniform constants E_0 and N, which does not depend on λ, such that*

$$E(f) = \frac{1}{2}\int_\Sigma |df|^2 \leq E_0 \quad \textit{and} \quad ||\alpha_\lambda||_2 \leq N$$

for any $(f, \Sigma, \alpha_\lambda) \in \overline{\mathcal{M}}_{g,k}(Z_\lambda, A, \nu, \mathcal{H}_\lambda)$, where $|\nu|_\infty$ is sufficiently small and $A = S + dF$.

Proof. The proof of the uniform bound of α_λ is similar to the proof of Lemma 8.0.43. We define $N(\alpha_\lambda)$ as an open neighborhood of zero set of α_λ and define $m(J_\lambda)$ as in the proof of Lemma 4.0.24. Since each α_λ is supported on the End_X, there is a constant $c > 0$ such that $m(J_\lambda) > c$ for any λ. Then, the argument in Lemma 8.0.43 shows that $N = 2/c > 2/m(J_\lambda)$.

It remains to show the uniform energy bound. Note that $\alpha_\lambda = \beta p^*\alpha$ for some $\alpha \in \mathcal{H}$. For each $p \in \Sigma$, let $\{\, e_1(p), e_2(p) = j\, e_1(p) \,\}$ be an orthonormal basis of $T_p\Sigma$. We set

$$\Sigma_- = \{\, p \in \Sigma \mid f^*p^*\alpha(\, e_1(p), e_2(p)\,) < 0 \,\}$$

Since $|\overline{\partial}_{J_\lambda} f|^2\, dv = f^*\beta p^*\alpha + 2\langle \overline{\partial}_{J_\lambda} f, \nu\rangle\, dv$, we have $|\overline{\partial}_{J_\lambda} f| \le 2|\nu|$ on Σ_-. This implies

$$-f^*p^*\alpha(\, e_1(p), e_2(p)\,) \le M|df||\nu|$$

where $p \in \Sigma_-$ and $M = \max\{\ |p^*\alpha| \mid ||p^*\alpha||_2 \le N\ \}$. Therefore, we can conclude that

$$\begin{aligned}\frac{1}{2}\int_\Sigma |df|^2 &\le \int_\Sigma f^*(\beta p^*\alpha) + \int_\Sigma |df||\nu| + \omega_\lambda(A)\\ &\le -\int_{\Sigma_-} f^*p^*\alpha + \int_\Sigma |df||\nu| + \omega_Z(A)\\ &\le (1+M)\left(\int_\Sigma |\nu|^2\right)^{\frac{1}{2}}\left(\int_\Sigma |df|^2\right)^{\frac{1}{2}} + \omega_Z(A)\end{aligned}$$

This implies the uniform energy bound independent of λ. $\square$

Finally, following [IP2], we define inhomogeneous terms on the fibration $\lambda : Z \to D(\epsilon)$. As in Chapter 8, we fix a finite cover $\overline{\mathcal{M}}^{\mu}_{g,k}$, universal family $\overline{\mathcal{U}}^{\mu}_{g,k}$ over it, and a projective embedding $\overline{\mathcal{U}}^{\mu}_{g,k} \hookrightarrow \mathbb{P}^N$. We denote by the orthogonal projection onto the normal bundle N_X (N_Y) of V by $\xi \to \xi^N$.

Definition 9.0.50 *We define an inhomogeneous term ν of the fibration $\lambda : Z \to D(\epsilon)$ to be a section of the bundle Hom($T\mathbb{P}^N, TZ$) over $\mathbb{P}^N \times Z$ which satisfies*

(i) ν is anti-J_Z-linear, i.e. $\nu\, j_{\mathbb{P}^N} = -J_Z\, \nu$,

(ii) the restriction of ν to $\mathbb{P}^N \times Z_\lambda$, we simply denote it by ν_λ, is a section of Hom($T\mathbb{P}^N, TZ_\lambda$) over $\mathbb{P}^N \times Z_\lambda$ when $\lambda \ne 0$,

(iii) ν_X (ν_Y) is also a section of Hom($T\mathbb{P}^N, TX$) (Hom($T\mathbb{P}^N, TY$)) such that ν_X^N $(\nu_Y^N) = 0$, and

(iv) for all $\xi \in N_X$ (N_X) and $v \in TV$

$$[\,\nabla_\xi \nu_X + J\nabla_{J\xi}\nu_X\,]^N = [\,(J\nabla_{\nu_X} J)\xi\,]^N \;\; \left([\,\nabla_\xi \nu_Y + J\nabla_{J\xi}\nu_Y\,]^N = [\,(J\nabla_{\nu_Y} J)\xi\,]^N\right)$$

We denote by $\mathcal{Y}^V$ the set of all inhomogeneous terms on Z.

Proposition 9.0.51 *For generic $\nu \in \mathcal{Y}^V$ and generic $\lambda \neq 0$,*

(a) $\mathcal{M}_{g,k}(Z_\lambda, A, \mathcal{H}_\lambda, \nu_\lambda)_I$ is an orientable smooth manifold of dimension $2(g+k)$, and

(b) the Frontier of the smooth map

$$st \times ev : \mathcal{M}_{g,k}(Z_\lambda, A, \mathcal{H}_\lambda, \nu_\lambda)_I \to \overline{\mathcal{M}}_{g,k} \times Z_\lambda^k.$$

lies in dimension 2 less than $g+k$.

Proof. We can consider $Z \setminus Z_0$ as a fixed smooth manifold $E(n)$ with a family of Kähler structures parameterized by $D(\epsilon) \setminus \{0\}$, namely, for each $\lambda \neq 0$,

$$(J_\lambda, \omega_\lambda, g_\lambda) = (J_Z, \omega_Z, g_Z)_{|Z_\lambda}$$

It then follows that the universal moduli space

$$\mathcal{U} = \{ \ (f, j, J_\lambda, \alpha_\lambda, \nu_\lambda) \mid f \text{ is } (J_\lambda, \alpha_\lambda, \nu_\lambda)\text{-holomorphic}, \quad [f] = s + df, \ \mathrm{Aut}(j) = I, \ \alpha_\lambda \in \mathcal{H}_\lambda \ \}$$

is smooth. On the other hand, we have a canonical projection $\pi : \mathcal{U} \to \mathcal{Y}^V$. By Sard-Smale Theorem, $\pi^{-1}(\nu)$ is smooth of dimension $2(g+k+1)$ for generic ν. Again, applying Sard-Smale Theorem to the projection $\pi^{-1}(\nu) \to D(\epsilon)$ defined by $(f, J_\lambda, \nu_\lambda, \alpha_\lambda) \to \lambda$, we can conclude (a).

In order to prove (b), we first consider the stable compactification $\overline{\mathcal{M}}_{g,k}(Z_\lambda, A, \mathcal{H}_\lambda, \nu_\lambda)$ as in Chapter 8. It follows from Lemma 9.0.49 that this is compact. We also reduce this moduli space and stratify the reduced moduli space by the same way as in the proof of Theorem 8.0.42. Note that Lemma 8.0.44 still holds for this reduced moduli space since each J_λ is a fixed complex structure. On the other hand, we consider the bump function β in the Definition 9.0.48 as a function on Z_λ. We can then assume that all singular fibers of $Z_\lambda = (E(n), J_\lambda) \to \mathbb{C}P^1$ lie in the support of $1-\beta$. That implies that if (f,α) has a bubble component, then the bubble component maps into either a singular fiber or a section. It follows that zero set of α should contain a singular fiber or a section. Therefore, we can conclude (b) using the same argument as in the proof of Theorem 8.0.42. □

We end this Chapter with the splitting argument as in [IP3]. This shows how maps into $X = E(n)$ split along the degeneration of $E(n)$. It is also a key observation for gluing of maps into X and Y, which leads to the sum formula (0.5).

Lemma 9.0.52 *Let $(f_n, \Sigma_n, \alpha_n)$ be any sequence of (J_Z, ν, α_n)-holomorphic maps such that (i) each f_n maps into Z_{λ_n}, (ii) each f_n represent the homology class $S+dF$, and (iii) $\lambda_n \to 0$ as $n \to \infty$. Then we have*

(i) f_n converges to a limit $f : \Sigma \to Z_0 = X \cup_V Y$ and α_n converges to α, after passing to some subsequence.

(ii) the limit map f can be decomposed as

$$f_1 : \Sigma_1 \to X,\ \ f_2 : \Sigma_2 \to Y,\ \ \textit{and}\ f_3 : \Sigma_3 \to V$$

where f_1 is a stable (J_X, ν_X, α)-holomorphic and $f_2\,(f_3)$ is a stable $(J_Y, \nu_Y)\ ((J_V, \nu_V))$-holomorphic

(iii) for $i = 1, 2$, each f_i transverse to V with $f_i^{-1}(V) = \{p_i\}$, where each p_i is a node of Σ.

Proof. (i) follows from Gromov Convergence Theorem and Lemma 9.0.49. Note that $\alpha = 0$ near $V \subset Z$ when $\alpha \in \mathcal{H}_Z$. Hence, $J_{\alpha_n} = J$ near $V \subset Z$. Therefore, we can apply Contact Lemma in [IP2] to conclude (ii). Lastly, (iii) follows from Contact lemma in [IP2] and lemma 3.3 in [IP3]. □

CHAPTER 10

Relative Invariants of $E(n)$

In this Chapter, following [IP2], we define relative invariants for

$$(X = E(n),\ V = T^2,\ g \le 1,\ A = S + dF,\ \nu_X,\ \mathcal{H}_X)$$
$$(Y = T^2 \times S^2,\ V = T^2,\ g \le 1,\ A = S + dF,\ \nu_Y)$$

where $\nu_X\,(\nu_Y)$ is the restriction of ν on Z to $X\,(Y)$, $\mathcal{H}_X$ is the parameter space in (0.4) and g is the genus of Riemann surfaces. As in Chapter 8, we fix the complex structure on $X\,(Y)$ and we only vary the inhomogeneous term $\nu_X\,(\nu_Y)$ to define perturbed relative moduli space. In the below, we will not specify complex structures on X and Y in the notation of moduli spaces. We also assume that we always work with a finite good cover p_μ as in (0.1) without specifying it.

For each $\nu \in \mathcal{Y}^V$ we define the relative moduli space as

$$\begin{aligned}&\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)_I\\ &= \{\ (f, j, \alpha) \in \overline{\mathcal{M}}_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) \mid f(x_{k+1}) \in V,\ \mathrm{Aut}(j) = I\ \}\end{aligned}$$

Proposition 10.0.53 *For generic $\nu \in \mathcal{Y}^V$,*

(a) $\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)_I$ is an oriented smooth manifold of dimension $2(g+k)$, and

(b) The Frontier of the map

$$st \times ev \times h : \mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)_I \to \overline{\mathcal{M}}_{g,k+1} \times X^k \times V \tag{0.1}$$

is contained in dimension 2 less than $2(g+k)$, where ev is the evaluation map of the first k marked points and h is the evaluation map of the last marked point.

Proof. Since for each $\alpha \in \mathcal{H}_X$, $\alpha = 0$ in some neighborhood of $V \subset X$, $J_\alpha = J$ on that neighborhood. Therefore, (a) follows from Lemma 4.2 in [IP2].

On the other hand, the Frontier of (0.1) is the image of

$$\mathcal{C}\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)_I \subset \overline{\mathcal{M}}_{g,k+1}(X, A, \mathcal{H}_X, \nu_X) \tag{0.2}$$

under stabilization and evaluation maps, where (0.2) is the closure of $\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)$ in $\overline{\mathcal{M}}_{g,k+1}(X, A, \mathcal{H}_X, \nu_X)$. In order to prove (b), we first reduce the closure (0.2) under the reduction as in Chapter 8 and stratify the reduced moduli space by the same way as in the proof of Theorem 8.0.42. Similarly as in the proof of Proposition 9.0.51, (i) Lemma 8.0.44 still holds, and (ii) if (f, α) has a bubble component, then the bubble component maps into either a singular fiber or the section s.

Each strata corresponds to one of the following types of stable maps : (i) f has some bubble components, (ii) f has two principle components $\Sigma_1 \cup \Sigma_2$ such that the image of Σ_2 maps entirely into V, and (iii) f is neither of type (i) nor (ii). First, consider (f, α) which is of type (i). In this case, the zero set of α should contains singular fibers or a section. This reduces the dimension of possible parameter space for the corresponding strata at least 2. Next, it follows from Lemma 6.6 in [IP2] that those strata corresponding to (ii) is empty. Lastly, note that if f is of type (iii), then it has at least 3 principle components. Therefore, (b) follows from the similar dimension count as in the proof of Theorem 8.0.42. □

It follows from the above proposition and Proposition 4.2 of [KM] that the image of (0.1) gives rise to a rational homology class. We denote it by

$$[\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X)] \in H^*(\overline{\mathcal{M}}_{g,k+1}; \mathbb{Q}) \otimes H^*(X^k; \mathbb{Q}) \otimes H^*(V; \mathbb{Q})$$

Definition 10.0.54 *For $2g + k \geq 2$, β in $H^*(\overline{\mathcal{M}}_{g,k+1}; \mathbb{Q})$, $\alpha_1, \cdots, \alpha_k$ in $H^*(X^k; \mathbb{Q})$, and γ in $H_*(V; \mathbb{Q})$, we define relative invariants by*

$$\begin{aligned} &GW^V_{g,k+1}(X, A, \mathcal{H}_X)(\,\beta; \alpha_1, \cdots, \alpha_k; C(\gamma)\,) \\ &= (\,\beta \otimes (\alpha_1 \wedge \cdots \wedge \alpha_k) \otimes C(\gamma)\,) \cap [\,\mathcal{M}^V_{g,k+1}(X, A, \mathcal{H}_X)\,] \end{aligned}$$

where $C(\gamma)$ is the Poincaré dual of γ.

Similarly as above, we set

$$\mathcal{M}^V_{g,k+1}(Y, A, \nu_Y)_I = \{\ (f, j) \in \overline{\mathcal{M}}_{g,k+1}(Y, A, \nu_Y) \mid f(y_1) \in V,\ \mathrm{Aut}(j) = I\ \} \quad (0.3)$$

$$st \times h \times ev : \mathcal{M}^V_{g,k+1}(Y, A, \nu_Y)_I \to \overline{\mathcal{M}}_{g,k+1} \times V \times Y^k \quad (0.4)$$

where h is the evaluation map of the first marked point and ev is the evaluation map of the last k marked points.

Remark 0.55 Since $p_g(Y) = 0$, the relative moduli space (0.3) is the one in [IP2]. Here, we fixed the product complex structure on Y and we only vary inhomogeneous terms. However, for a given stable map after contracting all ghost bubbles, there is at most one bubble component which maps to some holomorphic section. Using the same argument as in Chapter 6 of [IP2], we can thus show that for generic $\nu \in \mathcal{Y}_Z$, the Frontier of (0.4) is contained in dimension 2 less than the dimension of (0.3). On the other hand, for generic $\nu \in \mathcal{Y}_Z$, (0.3) is an orientable smooth manifold of dimension $2(g + k) + 2$. Therefore, we can define relative invariants as in the Definition 10.0.54. In fact, this invariants is less refined than the relative invariants in [IP2], see also Appendix in [IP3].

Finally, we set up some notations which will be used in the next Chapter. We set

$$\mathcal{M}^V(X)_I \underset{h}{\times} \mathcal{M}^V(Y)_I = \bigcup \mathcal{M}^V_{g_1,k_1+1}(X, s+d_1 f, \mathcal{H}_X)_I \underset{h}{\times} \mathcal{M}^V_{g_1,k_2+1}(Y, s+d_2 f)_I$$
$$= (h \times h)^{-1}(\Delta)$$

where the union is over all $g_1 + g_2 = 1$, $k_1 + k_2 = 4$, and $d_1 + d_2 = d$. This moduli space comes with the following maps :

$$st' \times ev : \mathcal{M}^V(X)_I \underset{h}{\times} \mathcal{M}^V(Y)_I \to \bigcup_{k_1+k_2=4} \overline{\mathcal{M}}_{1,4} \times X^{k_1} \times Y^{k_2} \tag{0.5}$$

where $st' = \sigma \circ st$ and σ is the gluing map of the domain. For generic ν, this moduli space is also smooth and the Frontier of (0.5) lies in codimension 2.

On the other hand, we set

$$\mathrm{Map}_{1,4}(Z, A) = \{\ (f, j, \alpha)\ |\ [f] = A,\ j \in \overline{\mathcal{M}}_{1,4},\ \alpha \in \mathcal{H}_Z\ \}$$

Note that $\mathcal{M}^V(X)_I \underset{h}{\times} \mathcal{M}^V(Y)_I$ and $\mathcal{M}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)_I$ are subsets of $\mathrm{Map}_{1,4}(Z, A)$. Moreover, there are following commutative diagrams :

$$\begin{array}{ccccc} \mathcal{M}^V(X)_I \underset{h}{\times} \mathcal{M}^V(Y)_I & \longrightarrow & \mathrm{Map}_{1,4}(Z, A) & \longleftarrow & \mathcal{M}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)_I \\ {\scriptstyle st' \times ev} \downarrow & & {\scriptstyle st \times ev} \downarrow & & {\scriptstyle st \times ev} \downarrow \\ \bigcup \overline{\mathcal{M}}_{1,4} \times X^{k_1} \times Y^{k_2} & \xrightarrow{\ i\ } & \overline{\mathcal{M}}_{1,4} \times Z^4 & \xleftarrow{\ j\ } & \overline{\mathcal{M}}_{1,4} \times Z^4_\lambda \end{array} \tag{0.6}$$

where $\lambda \neq 0$ and the union is over all $k_1 + k_2 = 4$.

Remark 0.56 Recall that $\overline{\mathcal{M}}_{1,4}$ is a smooth finite cover of the Deligne-Mumford space defined by Prym structures and hence it has a universal family $\overline{\mathcal{U}}_{1,4}$. The metric on $\overline{\mathcal{U}}_{1,4}$ provides a smooth family of metrics on the domain of maps in $\mathrm{Map}_{1,4}(Z, A)$. Therefore, we can define a weighted norm as in Definition 5.0.30 on $\mathrm{Map}_{1,4}(Z, A)$ to make it Banach space.

There is another way to define a topology on $\mathrm{Map}_{1,4}(Z, A)$ [IP2]. We can identify each $j \in \overline{\mathcal{M}}_{1,4}$ with $\phi : B \to \overline{\mathcal{U}}_{1,4}$, where B is a fixed smooth torus. The map ϕ

defines a complex structure on B by pulling back the complex structure on $\overline{\mathcal{U}}_{1,4}$. In this way, we can identify $\text{Map}_{1,4}(Z, A)$ with the following space

$$\text{Map}(B, Z \times \overline{\mathcal{U}}_{1,4}) = \{\ (f, \phi, \alpha) \mid f \times \phi : B \to Z \times \overline{\mathcal{U}}_{1,4},\ [f] = A,\ \alpha \in \mathcal{H}_Z\ \}.$$

For C^0 close maps $\mathcal{C}_1 = (f_1, \phi_1, \alpha_1)$ and $\mathcal{C}_2 = (f_2, \phi_2, \alpha_2)$, we can write $\mathcal{C}_2 = \exp_{\mathcal{C}_1}(\xi, h, \beta)$ and set

$$\text{dist}(\mathcal{C}_1, \mathcal{C}_2) = ||\xi||_{1,p} + ||h|| + ||\beta||$$

Taking the inf of the lenghts over all paths picewise of the above type, we can defines a distance and hence a topology on $\text{Map}_{1,4}(Z, A)$.

CHAPTER 11

Gluing Theorem

In this Chapter, we will establish a family version of Gluing Theorem as in [IP3]. Using this, we will show the sum formula (0.5). We fix a generic $\nu \in \mathcal{Y}_Z$ as in Chapter 9 and 10. In the below, we will not distinguish $\alpha \in \mathcal{H}_Z$ and its restriction α_λ to Z_λ and use the same notation for them.

Theorem 11.0.57 (Gluing Theorem) *Let* $\mathcal{C}_0 = (f_0, \Sigma_0, \alpha_0)$ *be in* $\mathcal{M}^V(X)_I \underset{h}{\times} \mathcal{M}^V(Y)_I$. *Then there are* λ_0, $\epsilon_0 > 0$, *and a small neighborhood* $\mathcal{W}$ *of* $\mathcal{C}_0$ *such that we have a continuous family of maps*

$$T_\lambda : \mathcal{W} \longrightarrow \mathcal{M}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)_I$$

for $|\lambda| < \lambda_0$, *which satisfies*

(i) T_λ *is an injective smooth map from* $\mathcal{W}$ *into* $\mathcal{M}_{1,4}(Z_\lambda, A, \lambda)_I$

(ii) $T_\lambda(\mathcal{C}_0)$ *converges to* $\mathcal{C}_0$ *as* $\lambda \to 0$

(iii) if (f, Σ, α) *in* $\mathcal{M}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)_I$ *and* $d(\mathcal{C}_0, (f, \Sigma, \alpha)) < \epsilon_0$, *then* (f, Σ, α) *is in* $T_\lambda(\mathcal{W})$,

where d *is the distance of* $Map_{1,4}(Z, A)$ *defined as in Remark 10.*

Proof. The proof of this theorem consists of 3 steps. In the first step, following [IP3] we construct approximated maps into Z_λ, each of which is associated with an element of $\mathcal{W}$. These are nearly $(J_\lambda, \nu_\lambda, \alpha)$-holomorphic. In the second step, we use the Inverse Function Theorem to perturb these approximated maps to truly $(J_\lambda, \nu_\lambda, \alpha')$-holomorphic maps. This process defines the map T_λ. The required analysis in this step is same as those in the proof of Proposition 5.0.37. In the last step, we show that the map T_λ has the desired properties as we stated.

Step 1 Let (f_1, Σ_1) and (f_2, Σ_2) be the two components of (f_0, Σ_0). Then Σ_0 lie in the image of

$$\sigma : \mathcal{M}_{g_1,k_1+1} \times \mathcal{M}_{g_2,k_2+1} \to \overline{\mathcal{M}}_{1,4}$$

where σ is defined as in Chapter 3 and $\Sigma_i \in \overline{\mathcal{M}}_{g_i,k_i+1}$ for $i = 1, 2$. Let U be an open neighborhood of Σ_0 in $\overline{\mathcal{M}}_{1,k}$. We may assume that the intersection $W = U \cap \mathrm{Im}(\sigma)$ is smooth. Let N be the tubular neighborhood of W in $\overline{\mathcal{M}}_{1,k}$. There is a trivialization $N \simeq W \times D$, where $D \subset \mathbb{C}$ is some disk. Let $\tilde{N} = \overline{\mathcal{U}}_{1,k_{|N}}$, where $\overline{\mathcal{U}}_{1,k} \to \overline{\mathcal{M}}_{1,k}$ is the universal family. Denote by $\mathcal{N}$ the set of nodes in the fiber of $\tilde{N}$ and let $V(\mathcal{N})$ be some fixed neighborhood of $\mathcal{N}$ in $\tilde{N}$. We can choose local coordinates z, w, as well as $(t, \mu) \in N$, on $V(\mathcal{N})$ such that the fiber of $\tilde{N}$ over (t, μ) is given by

$$(t, z, w) \quad \text{with} \quad zw = \mu.$$

Now, we set $\mathcal{W} = st^{-1}(W)$. This is an open neighborhood of $\mathcal{C}_0$. By shrinking $V(\mathcal{N})$ and $\mathcal{W}$, if necessary, for any $\mathcal{C} = (f, \Sigma, \alpha) \in \mathcal{W}$, we can assume the followings :

1. $f(\Sigma \cap V(\mathcal{N})) \subset U$, where U is the neck region (0.3) of Z.

2. Let $f_1 : \Sigma_1 \to X$ and $f_2 : \Sigma_2 \to Y$ be the components of the reducible map f.

It then follows from Contact Lemma in [IP2] that

$$f_1(t,z) = (\ v(z),\ az + \mathcal{O}(|z|^2)\ ) \ \text{ and } \ f_2(t,w) = (\ v(w),\ bw + \mathcal{O}(|w|^2)\ ) \ \text{ with}$$

$$1/2 \ < \ |a|,\ |b| \ < \ 2.$$

3. For $p \in \Sigma_1 \setminus V(\mathcal{N})$ ($q \in \Sigma_2 \setminus V(\mathcal{N})$) with $f_1(p)$ ($f_2(q)$) in the neck region of Z, we have

$$f_1(p) = (\,v(p), x(p), 0\,)\ (\ f_2(q) = (\,v(q), 0, y(q)\,)\ ) \ \text{ with } \ |x(p)|\,(\,|y(q)|\,) \geq \epsilon_0$$

where ϵ_0 is a uniform constant which doesn't defend on $\mathcal{C} \in \mathcal{W}$.

For each $(t,\mu) \in N$, denote by $\Sigma_t = \Sigma_{t1} \cup \Sigma_{t2}$ the fiber of $(t,0)$ in $\tilde{N}$. We also denote by $\Sigma_{(t,\mu)}$ the fiber of (t,μ) in $\tilde{N}$. We define X-side $\Sigma^X_{t,\mu}$ (Y-side $\Sigma^Y_{t,\mu}$) of $\Sigma_{(t,\mu)}$ be the set of all points $p \in \Sigma_{(t,\mu)}$ which satisfies

$$d(p,\Sigma_{t1}) < d(p,\Sigma_{t2}) \ (\ d(p,\Sigma_{t2}) < d(p,\Sigma_{t1})\ ).$$

When p is in X-side (Y-side), denote by $\pi_X(p)$ ($\pi_Y(p)$) the unique point in Σ_{t1} (Σ_{t2}) such that

$$d(p,\Sigma_{t1}) = d(\,p,\ \pi_X(p)\,) \ \ (\ d(p,\Sigma_{t2}) = d(\,p,\ \pi_Y(p)\,)\ ).$$

Let r be the distance function to $\mathcal{N}$ in $\tilde{N}$. We define a bump function β_ϵ with

$$\beta_\epsilon(r) = 0 \ \text{ if } r \geq 2\epsilon, \quad \beta_\epsilon(r) = 1 \ \text{ if } r \leq \epsilon, \quad \text{and} \quad |d\beta_\epsilon| \leq 2.$$

Definition 11.0.58 *For each* $\mathcal{C} = (f,\Sigma,\alpha) \in \mathcal{W}$ *and* $\lambda \neq 0$, *we define* $\mathcal{C}_\lambda = (F_\lambda, \Sigma_\lambda, \alpha)$ *as follows :*

1. $\Sigma_\lambda = (t,\mu)$, *where* $\mu = \lambda/ab$ *and* $\Sigma = (t,0)$, *and*

2. *let v be a local coordinate of V centered on the image of the node of Σ and we define $F_\lambda : \Sigma_\lambda \to Z_\lambda$ by*

$$F_\lambda = \begin{cases} (\,(1-\beta_\mu)v(z)\,,\ \hat{x}(z),\ \lambda/\hat{x}(z)\,) & on\ \{\ (t,z,w) \in V(\mathcal{N}) \mid |z| \geq |w|\ \} \\ \phi_{X,\lambda} \circ f_1 \circ \pi_X & on\ \Sigma_\lambda^X \cap Supp(\beta_\mu) \\ (\,(1-\beta_\mu)v(w)\,,\ \lambda/\hat{y}(w)\,,\ \hat{y}\,) & on\ \{\ (t,z,w) \in V(\mathcal{N}) \mid |w| \geq |z|\ \} \\ \phi_{Y,\lambda} \circ f_2 \circ \pi_Y & on\ \Sigma_\lambda^Y \cap Supp(\beta_\mu) \end{cases}$$

where $\hat{x}(z) = az(\,1+(1-\beta_\mu)\mathcal{O}(|z|)\,)$, $\hat{y}(w) = bw(\,1+(1-\beta_\mu)\mathcal{O}(|w|)\,)$, β_μ is the bump function defined as above with $\epsilon = |\mu|$, and $\phi_{X,\lambda}$ ($\phi_{Y,\lambda}$) is the holomorphic map which extends the following map

$$(v,x,0) \to (v,x,\lambda/x)\ \ (\,(v,0,y) \to (v,\lambda/y,y)\,)\,.$$

on the ncek region (0.3) of Z

As in Chapter 5, we use the metric on $\overline{\mathcal{U}}_{1,4}$ and metric g_Z on Z to define pointwise norm, weighted norms $||\cdot||_{1,p}$ and $||\cdot||_p$ as in Chapter 5. Note that the pointwise norm $|dF_\lambda|$ is uniformly bounded. Recall that $\nu_\alpha = (I + JK_\alpha)^{-1}\nu$.

Lemma 11.0.59 *For some $\lambda_0 > 0$, there is a uniform constant c such that for any $\mathcal{C} \in \mathcal{W}$ and $|\lambda| < \lambda_0$*

$$||\overline{\partial}_{J_\alpha} F_\lambda - \nu_\alpha||_p \leq c|\lambda|.$$

Proof. Let $\Phi(\mathcal{C}_\lambda) = \overline{\partial}_{J_\alpha} F_\lambda - \nu_\alpha$. Since π_X is holomorphic on the region $\Sigma_\lambda^X \cap \mathrm{Supp}(\beta_\mu)$ and $\phi_{X,\lambda}$ is also holomorphic, we have

$$\begin{aligned}\Phi(\mathcal{C}_\lambda) =& d\phi_{X,\lambda}(\,\overline{\partial}_{J_\alpha} f_1 - \nu_\alpha)d\pi_X + (J_\alpha - J)dF_\lambda \\ &+ d\phi_{X,\lambda}(J - J_\alpha)df_1\, d\pi_X + d\phi_{X,\lambda}\,\nu_\alpha\, d\pi_X - \nu_\alpha\end{aligned}$$

Therefore, we have $|\Phi(\mathcal{C}_\lambda)| \le c|\lambda|$ on the region $\Sigma_\lambda^X \cap \mathrm{Supp}(\beta_\mu)$.

On the region $\{\ (t,z,w) \in V(\mathcal{N}) \mid |z| \ge |w|\ \}$ we can assume F_λ maps into the region at which $\alpha = 0$. Using $zw = \mu = \lambda/ab$, we have

$$\overline{\partial}_J F_\lambda$$
$$= \left(\overline{\partial}_J(1-\beta_\mu)v, az\overline{\partial}_J[(1-\beta)\mathcal{O}(|z|)], bw\overline{\partial}_J\,(1+(1-\beta_\mu)\mathcal{O}(|z|))^{-2}\,\overline{\partial}_J[(1-\beta)\mathcal{O}(|z|)]\right)$$

All terms in $\overline{\partial}_J F_\lambda$ are bounded by $|\lambda|$, except for the term involving $\overline{\partial}_J\, v$. On the other hand, $\overline{\partial}_J\, v$ is the V-component of $\overline{\partial}_J f_1$ and hence $\nu_{f_1}^V$, where $\nu_{f_1}(z) = \nu(f_1(z), z)$. Therefore, we have

$$|\Phi(\mathcal{C}_\lambda)| \le c(\,|\nu_{f_1}^V - \nu_F^V| + |\nu_F^N| + |\lambda|\,) \le c|\lambda|$$

since the normal component ν^N vanishes along V. Using the same arguments on the other regions, we can conclude that $|\Phi(\mathcal{C}_\lambda)| \le c|\lambda|$ on Σ_λ. This implies the lemma.

□

Step 2 As above, we set $\Phi(\,F, \Sigma, \alpha\,) = \overline{\partial}_{J_\alpha} F - \nu_\alpha$. In this step, we perturb $\mathcal{C}_\lambda = (F_\lambda, \Sigma_\lambda, \alpha)$ to $\mathcal{C}'_\lambda = (F'_\lambda, \Sigma'_\lambda, \alpha')$ such that $\Phi(\mathcal{C}'_\lambda) = 0$. For doing that, we consider the linearization $D_{\mathcal{C}_\lambda}$ of Φ at $\mathcal{C}_\lambda$

$$D_{\mathcal{C}_\lambda} : L^{1,p}(F_\lambda^* TZ_\lambda) \oplus T_{\Sigma_\lambda}\overline{\mathcal{M}}_{1,4} \oplus T_\alpha \mathcal{H} \to L^p(\,\Lambda^{0,1}_{jJ_\alpha}(F_\lambda^* TZ_\lambda)\,) \tag{0.1}$$

where $L^{1,p}$ and L^p are defined by weighted norms as in Chapter 5. Using the Inverse Function Theorem, we will show that there exits a unique (ξ, k, β) such that (i) $\Phi(\,\exp_{\mathcal{C}_\lambda}(\xi, k, \beta)\,) = 0$ and (ii) the projection of (ξ, k, β) to the kernel of $D_{\mathcal{C}_\lambda}$ with respect to L^2-inner product is zero. We then define a gluing map by $T_\lambda(\mathcal{C}) = \exp_{\mathcal{C}_\lambda}(\xi, k, \beta)$.

Let $q : L^{1,p}(F_\lambda^* TZ_\lambda) \oplus T_{\Sigma_\lambda}\overline{\mathcal{M}}_{1,4} \oplus T_\alpha\mathcal{H} \to \mathrm{Ker}(D_{\mathcal{C}_\lambda})$ be the projection with respect to L^2-inner product. The following lemma is similar to Lemma 5.0.36.

Lemma 11.0.60 *There exits $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$, then for any $p > 2$, $\mathcal{C}_\lambda$ with $\mathcal{C}$ in $\mathcal{W}$, (ξ_0, k_0, β_0) in $\mathrm{Ker}(D_{\mathcal{C}_\lambda})$ and η in $L^p(\Lambda^{0,1}_{jJ_\alpha}(F^*_\lambda TZ_\lambda))$, there is a unique (ξ, k, β) which satisfies*

$$q(\xi, k, \beta) = (\xi_0, k_0, \beta_0), \quad D_{\mathcal{C}_\lambda}(\xi, k, \beta) = \eta \tag{0.2}$$

$$||\xi||_{1,p} + ||k|| + ||\beta|| \leq c \max\{\, ||\xi_0||_{1,p} + ||k_0|| + ||\beta_0||,\ ||\eta||_p \,\} \tag{0.3}$$

where c is a uniform constant.

Proof. This proof is similar to that of Lemma 5.0.36. We first show that for sufficiently small λ and any $\mathcal{C} \in \mathcal{W}$, $\mathrm{Coker}(D_{\mathcal{C}_\lambda}) = 0$. Suppose not. Then there exits a sequence $\{\, (\mathcal{C}_n, \lambda_n, \eta_n) \,\}$ such that $\lambda_n \to 0$ and $D^*_{(\mathcal{C}_n,\lambda_n)}(\eta_n) = 0$ with $||\eta||_p = 1$, where $(\mathcal{C}_n, \lambda_n)$ denotes the approximated map determined by $\mathcal{C}_n$ and λ_n. We can assume $\mathcal{C}_n \to \mathcal{C}$ and $(\mathcal{C}_n, \lambda_n) \to \mathcal{C}$. Let $\mathcal{C} = (f, \Sigma, \alpha)$. It follows from the standard elliptic estimates that η_n converges to some η outside of a node of Σ. Since $D^*_{\mathcal{C}}(\eta) = 0$ and $\mathrm{Coker}(D_{\mathcal{C}}) = 0$, we have $\eta = 0$. This implies that $\eta_n \to 0$ on the complement of neck region defined as in (0.5). On the other hand, note that $L^*_{F_n}(\eta_n) = 0$, where $F_n = F_{(\mathcal{C}_n,\lambda_n)}$. It follows from Lemma 5.0.34 that $||\eta_n||_p \to 0$. This contradicts to our assumption $||\eta_n||_p = 1$. Therefore, for sufficiently small λ and any $\mathcal{C} \in \mathcal{W}$, $\mathrm{Coker}(D_{\mathcal{C}_\lambda}) = 0$.

Consequently, there exits (ξ, k, β) with $D_{\mathcal{C}_\lambda}(\xi, k, \beta) = \eta$. Let $\zeta = (\xi, k, \beta)$ and $\zeta_0 = (\xi_0, k_0, \beta_0)$. Then $\zeta' = \zeta - q(\zeta) + \zeta_0$ satisfies (0.2). Uniqueness is obvious.

Next, we show the estimate by contradiction. Suppose not. Then there exits (ξ_n, k_n, β_n) such that (i) $||\xi_n||_{1,p} + ||k_n|| + ||\beta_n|| = 1$, (ii) $||D_n(\xi_n, k_n, \beta_n)||_p \to 0$, where $D_n = D_{(\mathcal{C}_n,\lambda_n)}$, and (iii) $||\xi_{n0}||_{1,p} + ||k_{n0}|| + ||\beta_{n0}|| \to 0$. We can assume that the approximate map $(\mathcal{C}_n, \lambda_n)$ converges to $(\mathcal{C}, \lambda) = (F, \alpha)$. By the Sobolev Embedding Theorem, we can also assume ξ_n converges to some ξ and $\xi_{n0} \to 0$ both in $L^{1,2}$-norm. We can further assume that (k_n, β_n) converges to some (k, β) and $(k_{n0}, \beta_{n0}) \to 0$.

Note that $D_{\mathcal{C},\lambda}(\xi,k,\beta) = q(\xi,k,\beta) = 0$, and hence we have $(\xi,k,\beta) = 0$ by uniqueness. Together with (i), $(k,\beta) = 0$ implies that $||\xi_n||_{1,p} \to 1$. It then follows from Lemma 5.0.33 that $||\xi_n||_{1,2}$ is uniformly bounded away from zero. This is impossible since $\xi_n \to \xi$ in $L^{1,2}$-norm and $\xi = 0$. Therefore, we have a contradiction. $\square$

As in Chapter 5, we set

$$L^{1,p} = \{ \ (\mathcal{C},\lambda,\xi,k,\beta) \mid \mathcal{C} \in \mathcal{W}, \ (\xi,k,\beta) \in L^{1,p}(F_\lambda^*TZ_\lambda) \oplus T_{\Sigma_\lambda}\overline{\mathcal{M}}_{1,4} \oplus T_\alpha\mathcal{H} \ \}$$
$$P = \{ \ (\mathcal{C},\lambda,\xi_0,k_0,\beta_0) \mid \mathcal{C} \in \mathcal{W}, \ (\xi_0,k_0,\beta_0) \in \mathrm{Ker}(D_{\mathcal{C}_\lambda}) \ \}.$$

On the other hand, let $\mathrm{Map}_{1,4}(Z_\lambda, A)$ be a subspace of $\mathrm{Map}_{1,4}(Z, A)$ which contains all maps into Z_λ and set

$$L^p = \{ \ (\mathcal{C}',\lambda,\eta) \mid \mathcal{C}' \in \mathrm{Map}_{1,4}(Z_\lambda, A) \text{ with } \mathcal{C}' = (F,j,\alpha), \ \eta \in L^p(\Lambda^{0,1}_{jJ_\alpha}F^*TZ_\lambda) \ \}$$

and define a map $\Xi : L^{1,p} \to L^p \times P$ by

$$\Xi(\mathcal{C},\lambda,(\xi,k,\beta)\,) = \big(\ \Phi(\exp_{\mathcal{C}_\lambda}(\xi,k,\beta)\,), q(\xi,k,\beta) \ \big)$$

The linearization of Ξ at $(\mathcal{C},\lambda,0)$ is the map

$$D\Xi : L^{1,p}(F_\lambda^*TZ_\lambda) \oplus T_{\Sigma_\lambda}\mathcal{M}_{1,4} \oplus T_\alpha\mathcal{H} \to L^p(\Lambda^{0,1}_{jJ_\alpha}F_\lambda^*TZ_\lambda) \times \mathrm{Ker}(D_{\mathcal{C}_\lambda})$$
$$(\xi,k,\beta) \to (\,D_{\mathcal{C}_\lambda}(\xi,k,\beta), q(\xi,k,\beta)\,)$$

By Lemma 11.0.60, it is an isomorphism with uniformly bounded inverse. Therefore, by the Inverse Function Theorem there exits $\epsilon > 0$ such that Ξ is a local diffeomorphism from the open set

$$\{ \ (\mathcal{C},\lambda,\xi,k,\beta) \in L^{1,p} \mid ||\xi||_{1,p} + ||k|| + ||\beta|| < \epsilon \ \} \tag{0.4}$$

onto its image. It then follows from Lemma 11.0.59 that for any $(\mathcal{C},\lambda)$ with $|\lambda|$ sufficiently small, there is a unique (ξ,k,β) such that

$$\Xi(\xi,k,\beta) = (\mathcal{C},\lambda,0,0). \tag{0.5}$$

Definition 11.0.61 *Fix λ with $|\lambda| < |\lambda_0|$. For each $\mathcal{C} \in \mathcal{W}$, let (ξ, k, β) be given as in (0.5). We then define*

$$T_\lambda : \mathcal{W} \to \mathcal{M}_{1,4}(Z_\lambda)_I \quad by \quad \mathcal{C} \to \exp_{\mathcal{C}_\lambda}(\xi, k, \beta).$$

Similarly as in Chapter 5, we have the following expansion

$$\Phi(\exp_{\mathcal{C}_\lambda}(\xi, k, \beta)) = \Phi(\mathcal{C}_\lambda) + D_{\mathcal{C}_\lambda}(\xi, k, \beta) + H_{\mathcal{C}_\lambda}(\xi, k, \beta)$$

where $||H_{\mathcal{C}_\lambda}(\xi, k, \beta)||_p \leq c(||\xi||_{1,p} + ||k|| + ||\beta||)^2$ for some uniform constant. Using this expansion and the estimate in Lemma 11.0.60, we can conclude that (ξ, k, β) in (0.5) satisfies

$$||\xi||_{1,p} + ||k|| + ||\beta|| \leq c|\lambda| \tag{0.6}$$

for some uniform constant.

Step 3 As a consequence of the Inverse Function Theorem, the map T_λ is smooth. It also follows from (0.6) that $T_\lambda(\mathcal{C}) \to \mathcal{C}$ as $\lambda \to 0$.

In the below, we will show the injectivity of T_λ and (iii) of Theorem 11.0.57. Denote by Q the orthogonal complement of P withe respect to L^2 norm. For each fixed λ, we also denote by $\mathcal{A}_\lambda$ the set of all appoximated maps into Z_λ. As in the proof of Lemma 11.0.60, each $\mathrm{Coker}(D_{\mathcal{C}_\lambda}) = 0$ when $|\lambda|$ is small. Hence we can deduce that

$$T_{\mathcal{C}_\lambda} L^{1,p} = T_{\mathcal{C}_\lambda} \mathcal{A}_\lambda \oplus Q_{|\mathcal{C}_\lambda}$$

Let $Q_\lambda = \cup_{\mathcal{C}} Q_{|\mathcal{C}_\lambda}$ and denote by $\exp : Q_\lambda \to \mathrm{Map}_{1,4}(Z_\lambda, A)$ the exponential map defined by

$$(\mathcal{C}, \lambda, \xi_0, k_0, \beta_0) \to \exp_{(\mathcal{C},\lambda)}(\xi_0, k_0, \beta_0). \tag{0.7}$$

Now, fix a path $\mathcal{C}_t$ starting $\mathcal{C}$ and let (ξ, k, β) be the tangent vector at $t = 0$ of the corresponding approximated maps $(\mathcal{C}_t, \lambda)$. Using parallel translation we can calculate

$$\frac{d}{dt}\exp_{(\mathcal{C}_t,\lambda)}(\xi_0, k_0, \beta_0)\big|_{t=0} = (\xi, k, \beta) + (\xi_0, k_0, \beta_0).$$

Therefore, there exits $\epsilon_0 > 0$ such that for all small $|\lambda|$ the exponential map (0.7) is a diffeomorphism from some neighborhood of zero section in Q_λ onto ϵ_0-neighborhood of $\mathcal{A}_\lambda$ in $\mathrm{Map}_{1,4}(Z_\lambda, A)$. Together with definition of T_λ and (0.6), we can conclude that T_λ is injective. On the other hand, if (f, Σ, α) in $\mathcal{M}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)$ is close to $\mathcal{C}_0$, then it is in the ϵ_0-neighborhood of $\mathcal{A}_\lambda$, which implies (f, Σ, α) in $T_\lambda(\mathcal{W})$. □

Now, we are ready to prove the sum formula (0.5).

Proposition 11.0.62

$$H(t) = -\frac{1}{12}F(t) + 2F(t)G(t)$$

Proof. By definition of generating functions $H(t)$, $F(t)$, and $G(t)$, it suffices to show that

$$GW^{\mathcal{H}_n}_{1,4}(S + dF)(\psi_{(1,4);4}; F^4) = \sum_{d_1+d_2=d} GW^{\mathcal{H}_n}_{0,0}(S + d_1F)\left(2\sigma(d_2) - \frac{1}{12}\right)$$

where $\sigma(d_2) = \sum_{k|d_2}$ as in Chapter 6.

We can choose a submanifold $F_i \subset Z$ for $i = 1, \cdots, 4$ which is in general position with respect to evaluation maps such that for $i = 1, 2$ each $F_i \cap X$ ($F_{i+2} \cap Y$) represents a fiber class in X (Y), and each $F_i \cap Z_\lambda$ represents a fiber class in $Z_\lambda = E(n)$. On the other hand, without loss of generality, we may assume there is a submanifold K in $\overline{\mathcal{M}}_{1,4}$ representing Poincaré dual of $\psi_{(1,4);4}$. We may also assume that K is in general position with respect to stabilization map.

Let $\lambda \neq 0$ be generic as in Proposition 9.0.51. Now, consider the cut-down moduli space $\mathcal{M}_\lambda$ which consists of all $(f, (j; \{x_i\}), \alpha)$ in $\overline{\mathcal{M}}_{1,4}(Z_\lambda, A, \mathcal{H}_\lambda)$ with $f(x_i)$ in F_i and

$st(j) \in K$. It then follows from Proposition 9.0.51 that $\mathcal{M}_\lambda$ is finite and $\mathrm{Aut}(j) = I$. In fact, by definition for generic λ we have

$$[\, st \times ev(\mathcal{M}_\lambda)\,] = GW_{1,4}^{\mathcal{H}_\lambda}(Z_\lambda, A)(\psi_{(1,4);4}; F^4) = GW_{1,4}^{\mathcal{H}_n}(S + dF)(\psi_{(1,4);4}; F^4) \quad (0.8)$$

where the second equality follows from Lemma 9.0.49 and Proposition 3.0.15.

Similarly as above, let $\mathcal{M}_0$ be the cut-down moduli space which consists of all

$$\Big(\big(f_1, (j_1, \{x_i\}), \alpha\big), \big(f_2, (j_2, \{y_i\})\big)\Big) \in \mathcal{M}^V(X)_I \underset{h}{\times} \mathcal{M}^V(Y)_I$$

such that $f_1(x_i) \in F_i$ and $f_2(y_{i+1}) \in F_{i+2}$ for $i = 1, 2$, and $\sigma(j_1, j_2) \in K$, where σ is the gluing map of the domain as in (0.8). It follows from Proposition 10.0.53 and Remark 10 that $\mathcal{M}_0$ is finite. Moreover, (j_1, j_2) is an element of either $\overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{0,3}$ or $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,3}$, since there are two marked points at each X-side and Y-side. Note that $\sigma^*(\psi_{(1,4);4}) = 0$, where $\sigma : \overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{1,4}$. Therefore, we have

$$\mathcal{M}_0 \subset \bigcup_{d_1+d_2=d} \mathcal{M}_{0,3}^V(X, S + d_1F, \mathcal{H}_X) \underset{h}{\times} \mathcal{M}_{1,3}^V(Y, S + d_2F) \quad (0.9)$$

Together with routine dimension count, (0.9) implies that

$$\begin{aligned}
&[\, st' \times ev\,(\mathcal{M}_0)\,] \\
&= \sum_{d_1+d_2=d} GW_{0,3}^{V,\mathcal{H}_X}(S + d_1F)(\,F^2; C(V)\,)\, GW_{1,3}^V(S + d_2F)(\,\psi_{(1,3);3}; C(pt) : F^2\,) \\
&= \sum_{d_1+d_2=d} GW_{0,3}^{\mathcal{H}_n}(S + d_1F)(\,F^3\,)\, GW_{1,3}^V(S + d_2F)(\,\psi_{(1,3);3}; C(pt); F^2\,) \\
&= \sum_{d_1+d_2=d} GW_{0,0}^{\mathcal{H}_n}(S + d_1F)\left(2\sigma(d_2) - \frac{1}{12}\right) \qquad (0.10)
\end{aligned}$$

where the second equality follows from Lemma 9.0.49, Proposition 3.0.15, and definition of relative invariants, while the third equality follows from TRR for $T^2 \times S^2$.

It remains to show that

$$i_*[\, st' \times ev\,(\mathcal{M}_0)\,] = j_*[\, st \times ev\,(\mathcal{M}_\lambda)\,] \ \text{ in } \ H_0(\overline{\mathcal{M}}_{1,4} \times Z^4\,) \quad (0.11)$$

where i and j are inclusions as in the diagram (0.6).

By Lemma 9.0.52, as $\lambda \to 0$ any sequence $(f_\lambda, j_\lambda, \alpha_\lambda) \in \mathcal{M}_\lambda$ converges to a limit (f, j, α). As above, since there are two marked points on each X-side and Y-side, j lies on one of the following images of the gluing maps :

$$\sigma_1 : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,3} \to \overline{\mathcal{M}}_{1,4}, \quad \sigma_2 : \overline{\mathcal{M}}_{1,3} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{1,4}, \quad \text{or}$$
$$\sigma_3 : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{1,4}$$

Since j also lies on K , and both $\sigma_2^*(\psi_{(1,4);4})$ and $\sigma_3^*(\psi_{(1,4);4})$ are trivial, $(f, j, \alpha) \in \mathcal{M}_0$. Hence, there is a bijective map between $\mathcal{M}_\lambda$ and $\mathcal{M}_0$ for small $|\lambda|$ by Gluing Theorem. Moreover, they are homotopic in $\text{Map}_{1,4}(Z, A)$. Therefore, by the commutative diagram (0.6), we can conclude (0.11). □

CHAPTER 11

Appendix – Relations with the Behrend-Fantechi Approach

Behrend and Fantechi [BF] have defined modified GW invariants for Kähler surfaces using algebraic geometry. While their techniques are completely different from ours, the definitions seem to be, at their core, equivalent. In this appendix we make several observations which relate their approach to ours. This is necessarily tentative because the paper [BF] is not yet available; we are relying on the terse description given in [BL3]

In algebraic geometry, the virtual fundamental class $[\overline{\mathcal{M}}_{g,k}(X,A)]^{\text{vir}}$ is obtained from the relative tangent-obstruction spaces together with the tangent-obstruction spaces of Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$. Behrend and Fantechi modified their machinery, intrinsic normal cone and obstruction complex, by replacing the relative obstruction space $H^1(f^*TX)$ by the kernel of the map

$$H^1(f^*TX) \to H^2(X, \mathcal{O}) \tag{A.1}$$

defined by dualizing of the composition

$$H^0(X, \Omega^2) \to H^0(f^*\Omega^2) \to H^0(f^*\Omega^1 \otimes f^*\Omega^1) \to H^0(f^*\Omega^1 \otimes \Omega^1). \tag{A.2}$$

In order for their machinery to work, the map (A.1) is of constant rank — in particular surjective — for every f in $\overline{\mathcal{M}}_{g,k}(X,A)$ [BL3]. Composing (A.2) with the Kodaira-Serre dual map, we have

$$H^0(X,\Omega) \to H^0(f^*\Omega^1 \otimes \Omega^1) \to H^1(f^*TX). \tag{A.3}$$

This map is given by $\beta \to K_\beta\, df\, j$.

Proposition A.63 *Let (X,J) be a Kähler surface and $A \in H^{1,1}(X,\mathbb{Z})$. Then the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact if and only if the map (A.1) is surjective for every f in $\overline{\mathcal{M}}_{g,k}(X,A)$.*

Proof. By Theorem 2.0.12 $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ consists of pairs (f,α) with $f \in \overline{\mathcal{M}}_{g,k}(X,A)$ and with the image of f contained in the zero set of α; the latter condition means that $K_\alpha = 0$ along the image, so $K_\alpha\, df\, j = 0$ for all (f,α). As usual, $\overline{\mathcal{M}}_{g,k}(X,A)$ is compact by the Gromov Compactness Theorem.

Now, suppose (A.1) is surjective. Then by duality (A.3) is injective. This implies $\alpha = 0$ and hence $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A) = \overline{\mathcal{M}}_{g,k}(X,A)$ is compact. Conversely, suppose for some $f \in \overline{\mathcal{M}}_{g,k}(X,A)$ there is a β in the kernel of (A.3). Then setting $\alpha = \beta + \bar{\beta}$ we have $\overline{\partial}_J f = tK_\alpha df j = 0$ — and hence $(f, t\alpha) \in \overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ — for all real t. That means that $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact only when (A.3) is injective or equivalently when (A.1) is surjective. □

The map (A.3) is directly related to the linearization operator of the (J,α)-holomorphic map equation.

Suppose that A is $(1,1)$ and that the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X,A)$ is compact as in Proposition A.63. Consider the linearization of the (J,α)-holomorphic map equation the (J,α)-holomorphic map equation as given in (0.6). Since J is Kähler, the linearization reduces to

$$L_f \oplus L_0 : \Omega^0(f^*TX) \oplus \mathcal{H} \to \Omega^{0,1}(f^*TX) \qquad \text{where} \qquad \begin{cases} L_f(\xi) &= \nabla\xi + J\nabla\xi\, j \\ \\ L_0(\beta) &= -2K_\beta df j \end{cases}$$

In fact, this L_f is exactly (twice) the Dolbeault derivative $\overline{\partial}$. Therefore, $\mathrm{Ker}(L_f)$ are $\mathrm{Coker}(L_f)$ are identified with the Dolbeault cohomology groups $H^0(f^*TX)$ and $H^{0,1}(f^*TX)$, respectively.

Proposition A.64 *Under either of the two equivalent conditions of Proposition A.63 there are natural identifications $H^1(f^*TX) \simeq H^{0,1}(f^*TX)$ and $H^0(X, \Omega^2) \simeq \mathcal{H}$ under which identification the map is identified with $(A.3)$ with*

$$L_0 : \mathcal{H} \to Coker(L_f).$$

By Proposition A.63 this map is injective if and only if the family moduli space $\overline{\mathcal{M}}^{\mathcal{H}}_{g,k}(X, A)$ is compact.

Proof. This follows directly by comparing the formulas for L_0 and $(A.3)$. Alternatively, we can compute the linearization from scratch as follows. Given $\xi \in \Omega^0(f^*TX)$, there is a family of maps f_t with $f_0 = f$ and $\frac{df_t}{dt}_{|t=0} = \xi$. It follows from Proposition 1.0.6b and $\langle\beta, A\rangle = 0$ that

$$0 = \frac{d}{dt}\bigg|_{t=0} \int_\Sigma f_t^*(\beta) = \frac{d}{dt}\bigg|_{t=0} \int_\Sigma \langle \overline{\partial}_J f_t, K_\beta f_{t*} j\rangle = \int_\Sigma \langle L_f(\xi), K_\beta f_* j\rangle.$$

This implies that L_0 maps $\mathcal{H}$ into $\mathrm{Coker}(L_f)$. $\square$

BIBLIOGRAPHY

BIBLIOGRAPHY

[AC] E. Arbarello and M. Cornalba, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math. No. 88 (1998), 97-127.

[B] L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin Heidelberg, 1987.

[BL1] J. Bryan and N. C. Leung, *The enumerative geometry of K3 surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), 371-410.

[BL2] J. Bryan and N. C. Leung, *Genenerating functions for the number of curves on abelian surfaces*, Duke Math. J. **99** (1999), no. 2, 311-328.

[BL3] J. Bryan and N. C. Leung, *Counting curves on irrational surfaces*, Surveys in differential geometry: differential geometry inspired by string theory, 313-339, Surv. Diff. Geom., 5, Int. Press, Boston, MA, 1999.

[BF] K. Behrend and B. Fantechi, In Preparation.

[BPV] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin Heidelberg, 1984.

[D] S. K. Donaldson, *Yang-Mills invariants of 4-manifolds* , In S.K. Donaldson and C.B. Thomas, editors, Geometry of low dimensional manifolds : Gauge Theory and Algebraic surfaces, number 150 in London Math. Soc. Lecture Note Series, Cambrige University Press, 1989.

[FM] R. Friedman, J.W. Morgan, *Smooth four-manifolds and complex surfaces*, Springer-Verlag, Berlin Heidelberg, 1994.

[G] L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, preprint, alg-geom/9711012

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, J. Wiley, New York, 1978.

[GT] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin Heidelberg, 1983.

[IP1] E. Ionel and T. Parker, *Gromov Invariants and Symplectic Maps*, Math. Annalen, **314**, 127-158 (1999).

[IP2] E. Ionel and T. Parker, *Relative Gromov-Witten Invariants*, preprint, math.SG/9907155.

[IP3] E. Ionel and T. Parker, *The Symplectic Sum Formula for Gromov-Witten Invariants*, preprint, math.SG/0010217.

[IS] S. Ivashkovich and V. Shevchishin, *Gromov compactness theorem for J-complex curves with boundary*, Internat. Math. Res. Notices **2000**, no. 22, 1167-1206.

[KM] P. Kronheimer and T. Mrowka, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995), 573-734.

[KP] S. Kleiman and R. Piene, *Enumerating singular curves on surfaces*, Algebraic Geometry : Hirzebruch 70 (Warsaw, 1998), 209-238, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999.

[L] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Alg. Geom. 3 (1994), no. 2, 283-29

[LT] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser., I, International Press, Cambridge, MA, 1998.

[P] T. Parker, *Compactified moduli spaces of pseudo-holomorphic curves* Mirror symmetry, III (Montreal, PQ, 1995), 77–113, AMS/IP Stud. Adv. Math., 10, Amer. Math. Soc., Providence, RI, 1999.

[PW] T. Parker and J. Wolfson, *Pseudo-holomorphic maps and bubble trees*, Jour. Geometric Analysis, **3** (1993) 63-98.

[RT1] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), 259-367.

[RT2] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. **130** (1997), 455-516.

[V] I. Vainsencher, *Enumeration of n-fold tangent hyperplanes to a surface*, J. Alg. Geom. **4** (1995), 503-526

[YZ] S.T. Yau and E. Zaslow, *BPS States, String Duality, and Nodal Curves on K3*, Nuclear Phys. B **471** (1996), 503-512