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ABSTRACT

The depth of blow-up rings of ideals

By

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Let R be a local Cohen-Macaulay ring and let I be an R-ideal. The Rees algebra $\mathcal{R}$, the associated graded ring $\mathcal{G}$ and the fiber cone $\mathcal{F}$ are graded algebras that reflect various algebraic and geometric properties of the ideal I. The Cohen-Macaulay property of $\mathcal{R}$ and $\mathcal{G}$ has been extensively studied by many authors, but not much is known about the Cohen-Macaulayness of $\mathcal{F}$. We give an estimate for the depth of $\mathcal{R}$ and $\mathcal{G}$ when these rings fail to be Cohen-Macaulay. We assume that I has small reduction number, sufficiently good residual intersection properties, and satisfies local conditions on the depth of some powers. We also study the Serre properties of $\mathcal{R}$ and $\mathcal{G}$ and how they are related. In particular the S_1 property for $\mathcal{G}$ leads to criteria for when $I^n = I^{(n)}$, where $I^{(n)}$ is the n-th symbolic power of I. We prove a quite general theorem on the Cohen-Macaulayness of $\mathcal{F}$ that unifies and generalizes several known results. We also relate the Cohen-Macaulay property of $\mathcal{F}$ to the Cohen-Macaulay property of $\mathcal{R}$ and $\mathcal{G}$.

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CHAPTER 1

Introduction

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let I be an R-ideal. The *Rees algebra* $\mathcal{R} = R[It] \cong \oplus_{i\geq 0} I^i$ and the *associated graded ring* $\mathcal{G} = gr_I(R) = \mathcal{R} \otimes_R R/I \cong \oplus_{i\geq 0} I^i/I^{i+1}$ are two graded algebras that reflect various algebraic and geometric properties of the ideal I. For example, Proj$(\mathcal{R})$ is the blow-up of Spec(R) along $V(I)$ and Proj$(\mathcal{G})$ corresponds to the exceptional fiber of the blow-up.

Many authors have extensively studied the Cohen-Macaulay property of $\mathcal{R}$ and $\mathcal{G}$, since besides being interesting in its own right, it greatly facilitates the study of several other properties of these algebras, such as their normality ([6]), the depth of their graded pieces ([16]), the Castelnuovo-Mumford regularity ([41]) or the number and degrees of their defining equations ([3], [39]).

A very useful tool in the study of blow-up rings is the notion of reduction of an ideal, with the reduction number measuring how closely the two ideals are related. This approach is due to Northcott an Rees ([44]). An ideal $J \subset I$ is called a reduction of I if the morphism $R[Jt] \hookrightarrow R[It]$ is finite. When investigating $\mathcal{R}$ or $\mathcal{G}$ one tries to simplify I by passing to a minimal reduction J; one of the big advantages is that J contains a lot of information about I, but often requires fewer generators. More precisely, if k is infinite, every minimal reduction of I is generated by ℓ elements, where ℓ is the analytic spread of I. (We refer to Chapter 2 for the terminology and the definitions.) Using the finiteness of the morphism above one can recover some of the properties of $R[It]$ from those of $R[Jt]$, expecting even better results on $R[It]$ when J has a "nice" structure and when the reduction is relatively small. This approach was first used by Huckaba and Huneke ([27], [28]) to prove the Cohen-Macaulayness of $\mathcal{R}$ and $\mathcal{G}$ for ideals with reduction number one and analytic deviation at most two. Inspired by this work, there have been several other results in the literature giving sufficient conditions for $\mathcal{R}$ and $\mathcal{G}$ to be Cohen-Macaulay, when the ideal I has small reduction number and small analytic deviation (see for example [55], [51], [21], [22], [2], [1], [49], [50]). Johnson and Ulrich proved a theorem that unifies and generalizes those results.

Theorem ([39, Theorem 3.1]) *Let R be a local Cohen-Macaulay ring of* *dimension d with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, and reduction number r, let $k \geq 1$ be an integer with $r \leq k$, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$ and that* $\operatorname{depth}(R/I^j) \geq d - \ell + k - j$ *for $1 \leq j \leq k$. Then $\mathcal{G}$ is Cohen-Macaulay, and if $g \geq 2$, $\mathcal{R}$ is Cohen-Macaulay.*

The G_ℓ property is a local condition on the number of generators of I up to codimension $\ell - 1$, and we say that I satisfies the condition $AN^-_\bullet$ if certain residual intersections of I are Cohen-Macaulay. We study these so-called Artin-Nagata properties in Section 2.4. Although the definition is rather technical, these properties are satisfied by a large class of ideals that include for instance perfect ideals of height two and perfect Gorenstein ideals of height three.

The Artin-Nagata properties play an important role in the proof of the above theorem as well as in the proof of the main results of this work, since, together with the G_ℓ property, they guarantee the existence of a "nice" generating set for a minimal reduction of I. This facilitates the computations of several intersections and ideal quotients (see Lemma 3.1.1 and the other technical results presented in Section 3.1 for details). Indeed, most of the previous works also made use of these residual intersection techniques, but only implicitly.

A similar result has been shown by Goto, Nakamura and Nishida in [23, Theorem 1.1]. They weakened the $AN_{\bullet}^{-}$ assumptions and substituted the G_{ℓ} property with a local condition on reduction numbers.

Another interesting issue is to estimate the depth of blow-up rings when they fail to be Cohen-Macaulay. Recently, Cortadellas and Zarzuela have come up with such formulas in [12], but only in very special cases. They proved that if I is an equimultiple ideal with reduction number at most one, then depth $\mathcal{G}$ = depth $R/I + g$. They also showed that if I has analytic deviation one and reduction number at most two and if some additional assumptions on local reduction numbers are satisfied (see Theorem 3.2.8 for the precise statement), then $\min\{\text{depth } R/I, \text{depth } R/I^2\} + g \leq$depth $\mathcal{G} \leq \min\{\text{depth } R/I, \text{depth } R/I^2\} + g + 1$.

It is natural then to look for a general estimate of depth $\mathcal{G}$ involving the depth of the powers of I. This is done in Chapter 3. If R is a Noetherian ring and I an R-ideal with analytic spread ℓ, we easily get that depth $\mathcal{G} \leq \inf\{\text{depth } R/I^j \mid j \geq 1\} + \ell$ (see Remark 3.2.20). Our main result is the following theorem, which gives the desired lower bound for depth $\mathcal{G}$.

Theorem 3.2.10 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal of grade g, analytic* *spread* ℓ, *and reduction number* r, *let* $k \geq 0$ *be an integer with* $r \leq k$, *assume that* I *satisfies* G_ℓ, $AN^-_{\ell-k-1}$ *and that for every* $p \in V(I)$, $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j\}$ *whenever* $1 \leq j \leq k-1$. *Then* $\operatorname{depth} \mathcal{G} \geq \min(\{d\} \cup \{\operatorname{depth}\ R/I^j + \ell - k + j \mid 1 \leq j \leq k\})$.

In the setup of the theorem we do not have any restriction on the analytic deviation, but the assumption on the depth of the powers forces the reduction number to be "sufficiently small", namely $r \leq \ell - g + 1$. Section 3.2 is devoted to the proof of the theorem. The main tool is a combination of the residual intersection techniques used by Johnson and Ulrich in [39] and of the local cohomology techniques used by Goto, Nakamura and Nishida in [23]. In particular the theorem generalizes their results, as well as the formulas obtained by Cortadellas and Zarzuela. Theorem 3.2.10 also gives an estimate for depth $\mathcal{R}$, since if $\mathcal{G}$ is not Cohen-Macaulay, we have that depth $\mathcal{R} =$ depth $\mathcal{G} + 1$ ([29, 3.10]).

The theorem is particularly interesting when the reduction number is small. In this case the assumptions are much simpler and we get a precise formula for depth $\mathcal{G}$.

Corollary 3.2.23 *Let* R *be a local Cohen-Macaulay ring of dimension* d *with infinite residue field, let* I *be an* R*-ideal with grade* g, *analytic spread* ℓ, *and reduction number* $r \leq 1$. *Assume that* I *satisfies* G_ℓ *and*

$AN^-_{\ell-2}$. *Then* depth $\mathcal{G} = \min\{d, \text{depth } R/I + \ell\}$.

Corollary 3.2.25 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread $\ell \geq g+1$, and reduction number $r \leq 2$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-3}$ and that R/I is Cohen-Macaulay. Then* depth $\mathcal{G} = \min\{d, \text{depth } R/I^2 + \ell\}$.

Since in our setup $\mathcal{R}$ and $\mathcal{G}$ are not necessarily Cohen-Macaulay, it is natural to ask which Serre properties are still satisfied in this case. In Chapter 4 we obtain such results. The main idea is to use the estimates of depth $\mathcal{R}$ and of depth $\mathcal{G}$ from Theorem 3.2.10. For example, under assumptions similar to those of Theorem 3.2.10 we get the following result for $\mathcal{G}$.

Theorem 4.0.1 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, let $k \geq 0$ be an integer with $r \leq k$, assume that I satisfies G_ℓ, $AN^-_{\ell-k-1}$ and that for some integer $t \geq 1$,* $\text{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + t\}$ *whenever $p \in V(I)$ and $1 \leq j \leq k$. Then $\mathcal{G}$ is S_t.*

The S_1 property for $\mathcal{G}$ is particularly important because it leads to criteria for when the power I^n of I coincides with its symbolic power $I^{(n)}$. Indeed, if $\mathcal{G}$ is S_1, under the additional assumption that locally at every non minimal prime p in $V(I)$, the analytic spread is less than the height of p, we obtain the equality of all the regular and the symbolic powers of I. This issue has been addressed by many authors, for example by Huneke ([30]) and Hochster ([26]), especially if the ideal I is prime.

From Theorem 4.0.1 we obtain that $\mathcal{G}$ is S_1 in the following cases.

Corollary 4.0.2 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number $r \leq 1$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-2}$, and that R/I has no associated primes of height $\geq \ell + 1$. Then $\mathcal{G}$ is S_1.*

Corollary 4.0.3 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread $\ell \geq g+1$, and reduction number $r \leq 2$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-3}$, that R/I is Cohen-Macaulay, and that R/I^2 has no associated primes of height $\geq \ell + 1$. Then $\mathcal{G}$ is S_1.*

Of particular interest is also the S_2 property for $\mathcal{R}$, because together with the condition that $\mathcal{R}$ is regular at every prime of height one, it is equivalent to the normality of $\mathcal{R}$. This property is very important and therefore has been studied by many authors (see for example [6], [38] and the literature cited there). For instance, if R is a normal ring, the normality of $\mathcal{R}$ is equivalent to the normality of the ideal I, which means that every power of I is integrally closed. We have the following result.

Theorem 4.0.8 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade $g \geq 2$, analytic spread ℓ, and reduction number r, let $k \geq 0$ be an integer with $r \leq k$. Assume that I satisfies G_ℓ, $AN^-_{\ell-k-1}$ and that* $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + 2\}$ *whenever $p \in V(I)$ and $1 \leq j \leq k$. Furthermore assume that I is $\ell - 2$-residually S_2 locally up to height $\ell + 1$. Then $\mathcal{R}$ is S_2.*

Here the residually S_2 assumption is a requirement that locally some residual intersections of I satisfy S_2.

It is natural to ask how the Serre properties for $\mathcal{R}$ and $\mathcal{G}$ are related. Brumatti, Simis and Vasconcelos in [6, Theorem 1.5] related the S_2 property for $\mathcal{R}$ with the S_1 property for $\mathcal{G}$. With the methods of their proof we generalized their result in Theorem 4.0.10. In particular we obtain the following equivalence.

Corollary 4.0.11 *Let R be a Cohen-Macaulay ring and let I be an R-ideal with* $\operatorname{ht} I \geq t$, *for some integer $t \geq 1$. Then $\mathcal{R}$ satisfies S_t if and only if $\mathcal{G}$ satisfies S_{t-1}.*

In the same setup, another interesting blow-up ring is the fiber cone of I, $\mathcal{F} = \mathcal{F}(I) = \oplus_{i \geq 0} I^i / \mathfrak{m} I^i \cong \mathcal{R} \otimes_R k \cong \mathcal{G} \otimes_R k$. This graded object is important from a geometric point of view, since $\mathrm{Proj}(\mathcal{F})$ corresponds to the fiber over the closed point of the blow-up of $\mathrm{Spec}(R)$ along $V(I)$. The fiber cone encodes a lot of information about the ideal I. For example, its Hilbert function gives the minimal number of generators of the powers of I and, if the residue field is infinite, its Krull dimension coincides with the minimal number of generators of any minimal reduction of I. We assume from now on that k is infinite. It is particularly interesting to know when $\mathcal{F}$ is Cohen-Macaulay. If J is a minimal reduction of I, then it is well known that $\mathcal{F}(J)$ is a polynomial ring over k and that $\mathcal{F}(I)$ is a finite extension of $\mathcal{F}(J)$. Hence $\mathcal{F}(I)$ is a free $\mathcal{F}(J)$-graded module if and only if $\mathcal{F}(I)$ is a Cohen-Macaulay graded ring. This greatly facilitates the computation of the Hilbert function. Also, the Cohen-Macaulayness of $\mathcal{F}$ implies, under some additional assumptions, that the ideal $\mathfrak{m} I^n$ is integrally closed for every n. This property plays an important role in the study of evolutions (see [33], [35]). Furthermore, if char $k = 0$ and $\mathcal{F}$ is S_2, then the reduction number of I is less than the multiplicity of $\mathcal{F}$ ([56]).

If I is generated by a regular sequence, or more generally when I is generated by analytically independent elements; i.e., when $r(I) = 0$, then $\mathcal{F}$ is trivially Cohen-Macaulay. So the first interesting case is that of ideals with reduction number one. Under this assumption, if the ring R is Cohen-Macaulay, $\mathcal{F}$ was shown to be Cohen-Macaulay by Huneke and Sally ([34]) when I is $\mathfrak{m}$-primary, by Shah ([47], [48]) when I is equimultiple, by Cortadellas and Zarzuela ([10]) when I has analytic deviation one and is generically a complete intersection. Later, in [11] Cortadellas and Zarzuela proved a more general theorem for ideals with reduction number one, which covers all the previous results (see Chapter 5 for the precise statement). Other cases have been studied in [43], [17], [15], [14], [35]. Recently Huneke and Hübl proved a result ([33, Theorem 2.1]) on the Cohen-Macaulayness of $\mathcal{F}$ for ideals having analytic deviation one, but any reduction number r. They assume that $\mathcal{F}$ has homogeneous generating relations only in degree greater than r. This is the only assumption involving the reduction number.

We study the Cohen-Macaulayness of $\mathcal{F}$ in Chapter 5. The work of Huneke and Hübl inspired our main result, Theorem 5.0.3, which generalizes their theorem, as well as Cortadellas and Zarzuela's. The main idea is that good "intersection properties" guarantee that we can find $\dim \mathcal{F}$ elements that are a regular sequence on $\mathcal{F}$.

In particular, if I has good residual intersection properties and sufficiently small reduction number, as in the setup of the previous chapters, we have the following.

Corollary 5.0.10 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r. Assume that I satisfies G_ℓ, $AN^-_{\ell-r-1}$ and that for every $p \in V(I)$,* $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + r - j, r - j\}$ *whenever $1 \leq j \leq r-1$. Assume that $\mathcal{F}$ has at most two homogeneous generating relations in degree $\leq r$. Then $\mathcal{F}$ is Cohen-Macaulay.*

In particular the above assumptions are satisfied by strongly Cohen-Macaulay ideals with the "expected" reduction number, and so we obtain the following result.

Corollary 5.0.13 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a strongly Cohen-Macaulay ideal of grade g, analytic spread ℓ, and reduction number r with $r \leq \ell - g + 1$. Assume that I satisfies G_ℓ. Then $\mathcal{F}$ is Cohen-Macaulay.*

It is natural to ask how the Cohen-Macaulayness of $\mathcal{F}$ is related to the Cohen-Macaulayness of $\mathcal{R}$ and $\mathcal{G}$. In general these notions are independent: An example of D'Anna, Guerrieri and Heinzer (Example 5.0.11) shows that the Cohen-Macaulayness of $\mathcal{R}$ or $\mathcal{G}$ does not imply the same property for $\mathcal{F}$. On the other hand, it is easy to build examples in which $\mathcal{F}$ is Cohen-Macaulay, but $\mathcal{R}$ and $\mathcal{G}$ are not (see for instance Example 5.0.17).

However, under additional assumptions, as a special case of Theorem 5.0.3, we have that $\mathcal{G}$ Cohen-Macaulay implies $\mathcal{F}$ Cohen-Macaulay.

Corollary 5.0.9 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, and reduction number $r \geq \ell - g$. Assume that I satisfies G_ℓ and that $\mathcal{F}$ has at most two homogeneous generating relations of degree $\leq r$. If $\mathcal{G}$ is Cohen-Macaulay, then $\mathcal{F}$ is Cohen-Macaulay.*

A class of ideals which is particularly interesting in this context is that of perfect ideals of height two. From Corollary 5.0.13 we easily get the following.

Corollary 5.0.16 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a perfect R-ideal of height 2 and analytic spread ℓ. Assume that I satisfies G_ℓ. If $\mathcal{R}$ is Cohen-Macaulay, then $\mathcal{F}$ is Cohen-Macaulay.*

In light of [52, Corollary 5.4], it is easy to decide if $\mathcal{R}$ is Cohen-Macaulay, just by checking if the presentation matrix satisfies the so-called "row condition". This fact has been very useful in building examples that show that the converse of Corollary 5.0.16 is not true: even for perfect ideals of height two the Cohen-Macaulayness of $\mathcal{F}$ does not imply the Cohen-Macaulayness of $\mathcal{R}$.

CHAPTER 2

Preliminaries

In this chapter we fix the notation and give the basic definitions that we will use in this work. For undefined terminology or further details we refer to [7] and [54].

2.1 Basic Notation

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k and let I be a proper R-ideal. We denote the grade of I by g, the height of I by $\operatorname{ht} I$, and the minimal number of generators of I by $\mu(I)$. We say that I is a *complete intersection* if $\mu(I) = g$. We say that I is *generically a complete intersection* if it is a complete intersection locally at every associated prime. We say that I is *unmixed* if every associated prime has the same height.

2.2 Blow-up Rings and Reductions

The *Rees algebra* of I is

$$\mathcal{R} = R[It] \cong \oplus_{i \geq 0} I^i,$$

the *associated graded ring* of I is

$$\mathcal{G} = gr_I(R) = \mathcal{R} \otimes_R R/I \cong \oplus_{i \geq 0} I^i / I^{i+1},$$

and the *fiber cone* of I is

$$\mathcal{F} = \oplus_{i \geq 0} I^i / \mathfrak{m} I^i \cong \mathcal{R} \otimes_R k \cong \mathcal{G} \otimes_R k.$$

The *analytic spread* ℓ of I is defined to be the Krull dimension of $\mathcal{F}$,

$$\ell(I) = \dim \mathcal{F} = \dim \mathcal{R} \otimes_R k = \dim \mathcal{G} \otimes_R k.$$

We have that

$$\text{ht } I \leq \ell(I) \leq \min\{\dim R, \mu(I)\}.$$

The *analytic deviation* of I is defined as $\ell(I) - \text{ht } I$. An ideal with analytic deviation zero; i.e, with $\ell(I) = \text{ht } I$, is called *equimultiple*. The *second analytic deviation* is the difference $\mu(I) - \ell(I)$.

The *symmetric algebra* of I is

$$S(I) = \oplus_{i \geq 0} S_i(I),$$

where $S_i(I)$ denotes the jth symmetric power of I. There is a natural surjective homomorhism $\alpha : S(I) \to \mathcal{R}$. If α is an isomorphism, we say that I is of *linear type*. If grade $I > 0$, this is equivalent to $S(I)$ being R-torsionfree.

A very useful tool in the study of blow-up rings is the notion of reduction of an ideal. An ideal $J \subset I$ is called a *reduction* of I if the morphism

$$R[Jt] \hookrightarrow R[It]$$

is module finite, or equivalently if $I^{r+1} = JI^r$ for some $r \geq 0$. The least such r is denoted by $r_J(I)$ and it is called *the reduction number of I with respect to J*. We have that $\ell(I) \leq \mu(J)$ for every reduction J of I. A reduction is *minimal* if it is minimal with respect to inclusion. If the residue field k is infinite, we have that $\ell(I) = \mu(J)$ for every minimal reduction J of I. In this case we define the *reduction number* r of I by

$$r(I) = \min\{r_J(I) \mid J \text{ a minimal reduction of } I\}.$$

Notice that $r(I) = 0$ if and only if $\ell(I) = \mu(I)$.

2.3 Strongly Cohen-Macaulay Ideals

Let $I = (x_1, \ldots, x_n)$ be an ideal of a local Cohen-Macaulay ring R. By $H_i(x_1, \ldots, x_n, R)$ we denote the ith Koszul homology of the Koszul complex built on $x_1, \ldots, x_n$. We say that I is *strongly Cohen-Macaulay* (SCM), if $H_i(x_1, \ldots, x_n, R)$ are Cohen-Macaulay R-modules for every i, and that I satisfies *sliding depth* (SD) if depth $H_i(x_1, \ldots, x_n, R) \geq \dim R - n + i$ for every i. Since R is Cohen-Macaulay, the property of being SCM or SD is independent of the chosen generating set of I. Also, we have that either $H_i(x_1, \ldots, x_n, R) = 0$ or $\dim H_i(x_1, \ldots, x_n, R) = \dim R/I$, and that $H_i(x_1, \ldots, x_n, R) = 0$ for every $i > n -$ grade I. Hence, if I is strongly Cohen-Macaulay, then I satisfies sliding depth. By [31] we have that if I is an ideal in the linkage class of a complete intersection, then I is strongly Cohen-Macaulay. So, in particular, if R is Gorenstein and I is perfect of grade two or perfect of grade three with R/I Gorenstein, then I is strongly Cohen-Macaulay. We recall that I is *perfect* if R/I is Cohen-Macaulay and has finite projective dimension.

2.4 Artin-Nagata Properties

The notions presented in this section will play a very important role throughout this work.

Definition 2.4.1 Let R be a local Cohen-Macaulay ring, let I be an R-ideal of height g, and let $s \geq g$ be an integer.

1. An *s-residual intersection* of I is a proper R-ideal $K = \mathfrak{a} : I$ where $\mathfrak{a} \subset I$ with $\mu(\mathfrak{a}) \leq s \leq \operatorname{ht} K$.

2. An s-residual intersection K of I is called a *geometric* s-residual intersection if $\operatorname{ht}(I + K) \geq s + 1$.

If R is Gorenstein and I is unmixed, then K is a g-residual intersection of I if and only if K is linked to I, and K is a geometric g-residual intersection of I if and only if K is geometrically linked to I. Hence the notion of residual intersections, essentially introduced by Artin and Nagata in [4], generalizes the concept of linkage to the case where the two linked ideals need not have the same height. Unlike for linkage, it is not clear when residual intersections are Cohen-Macaulay. Ulrich in [51] introduced the term "Artin-Nagata" (AN) for the Cohen-Macaulayness of residual intersections up to a given height.

Definition 2.4.2 Let R be a local Cohen-Macaulay ring, let I be an R-ideal of height g, and let $s \geq g$ be an integer.

1. We say that I satisfies AN_s if for every $g \leq i \leq s$ and every i-residual intersection K of I, R/K is Cohen-Macaulay.

2. We say that I satisfies AN_s^- if for every $g \leq i \leq s$ and every geometric i-residual intersection K of I, R/K is Cohen-Macaulay.

To guarantee the existence of s-residual intersections and geometric $(s-1)$-residual intersections, one usually assumes the condition G_s (see [4]).

Definition 2.4.3 Let R be a local Cohen-Macaulay ring, let I be an R-ideal, and let s be an integer.

We say that I satisfies property G_s, if $\mu(I_p) \leq \dim R_p$ for any prime ideal $p \in V(I)$ with $\dim R_p \leq s-1$. We say that I satisfies G_∞, if I satisfies G_s for every s.

Now we recall two important results, the first due to Herzog,Vasconcelos and Villareal, and the second due to Ulrich, that guarantee that an ideal satisfies the Artin-Nagata properties.

Theorem 2.4.4 ([25]) *Let R be a local Cohen-Macaulay ring and let I be an R-ideal satisfying G_s and sliding depth. Then I satisfies AN_s.*

Theorem 2.4.5 ([51]) *Let R be a local Gorenstein ring of dimension d, let I be an R-ideal of grade g, and assume that I satisfies G_s and that* depth $R/I^j \geq d-g-j+1$ *for* $1 \leq j \leq s-g+1$. *Then I satisfies AN_s.*

Notice that if I is strongly Cohen-Macaulay and satisfies G_s, then by Theorem 2.4.4 I satisfies AN_s. This result provides a large class of ideals satisfying the Artin-Nagata properties.

CHAPTER 3

The Depth of Blow-Up Rings of Ideals

In this chapter we study the depth of the associated graded ring and of the Rees algebra of ideals having good residual intersection properties and sufficiently small reduction number. In Section 1 we present some technical results that will play a crucial role in the rest of the chapter. Section 2 is devoted to the proof of the main result, Theorem 3.2.10. We then give several corollaries.

Throughout this chapter we use the following notation: R is a local Cohen-Macaulay ring of dimension d with infinite residue field, I is a proper R-ideal of grade g, analytic spread ℓ and reduction number r, $\mathcal{R}$ and $\mathcal{G}$ denote the Rees algebra and the associated graded ring of I.

3.1 Preliminary Results

We begin with a lemma, due to Ulrich, which contains some basics facts about Artin-Nagata properties.

Lemma 3.1.1 ([39, Lemma 2.3]) *Let R be a local Cohen-Macaulay ring with infinite residue field, let $\mathfrak{a} \subset I$ be (not necessarily distinct) R-ideals with $\mu(\mathfrak{a}) \leq s \leq \text{ht } \mathfrak{a} : I$ and assume that I satisfies G_s.*

(a) *There exists a generating sequence $a_1, \ldots, a_s$ of $\mathfrak{a}$ such that for every $0 \leq i \leq s-1$ and for every subset $\{\nu_1, \ldots, \nu_i\}$ of $\{1, \ldots, s\}$, $\text{ht } (a_{\nu_1}, \ldots, a_{\nu_i}) : I \geq i$ and $\text{ht } I + (a_{\nu_1}, \ldots, a_{\nu_i}) : I \geq i + 1$.*

(b) *Assume that I satisfies AN_t^- for some $t \leq s - 1$ and that $\mathfrak{a} \neq I$, write $\mathfrak{a}_i = (a_1, \ldots, a_i)$, $K_i = \mathfrak{a}_i : I$ and let '-' denote images in R/K_i. Then for $0 \leq i \leq t + 1$:*

(i) *$K_i = \mathfrak{a}_i : (a_{i+1})$ and $\mathfrak{a}_i = I \cap K_i$, if $i \leq s - 1$.*

(ii) depth $R/\mathfrak{a}_i = d - i$.

(iii) *K_i is unmixed of height i.*

(iv) ht $\bar{I} = 1$, *if $i \leq s - 1$.*

Remark 3.1.2 The above result is very useful when applied to reductions. Let I be an R-ideal with analytic spread ℓ. If $s \geq \ell$, then s general elements in I generate a reduction and, if I satisfies G_s, s general elements $a_1, \ldots, a_s$ in I have the property that ht $(a_1, \ldots, a_s) : I \geq s$. So Lemma 3.1.1 says that we can choose the generators of the reduction so that they have "good properties". In particular $a_1, \ldots, a_g$ form a regular sequence in R.

A similar lemma, in the sense of being able to choose a convenient set of generators for a reduction, has been proven by Goto.

Lemma 3.1.3 ([23, Lemma 2.1]) *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal, and let J be a reduction of I, generated by s elements. Then there exists a system of generators $a_1, \ldots, a_s$ for J satisfying:*

(a) $(a_1, \ldots, a_i)_p$ *is a reduction of* I_p, *if* $p \in V(I)$ *and* $i = \text{ht } p \leq s$.

(b) $a_i \notin p$ *if* $p \in Ass(R/(a_1, \ldots, a_{i-1})) \setminus V(I)$ *for any* $1 \leq i \leq s$.

Definition 3.1.4 ([23]) In the setup of the previous lemma, let $J_i = (a_1, \ldots, a_i)$. We define $r_i = \{\max r_{J_{i_p}}(I_p) \mid p \in V(I) \text{ and ht } p = i\}$ for $g \leq i \leq s$.

The ideals J_i defined by Goto, and the ideals $\mathfrak{a}_i$ of Lemma 3.1.1 enjoy similar properties.

Lemma 3.1.5 ([23, Lemma 2.4, Lemma 2.5, Corollary 2.6, Lemma 6.2 and their proof]) *In the setup of Lemma 3.1.3, let* $k \geq 0$ *be an integer with* $r_J(I) \leq k$*, assume that* I *satisfies* AN_t^- *for some* $t \leq s-1$*, and that* $r_i \leq \max\{0, i-t-1\}$ *for all* $g \leq i < s$*. Write* $J_i = (a_1, \ldots, a_i)$ *and* $K_i = J_i : I$*. Then for* $0 \leq i \leq t+1$*:*

(a) $\mathrm{ht}\ J_i : I \geq i$.

(b) $\mathrm{ht}\ (J_i : I) + I \geq i+1$, *if* $i \leq s-1$.

(c) $K_i = J_i : (a_{i+1})$ *and* $J_i = I \cap K_i$, *if* $i \leq s-1$.

(d) $\mathrm{depth}\ R/J_i \geq d - i$.

The following lemma is a generalization of [39, Lemma 2.5].

Lemma 3.1.6 *Let* R *be a local Cohen-Macaulay ring of dimension* d *with infinite residue field, let* $\mathfrak{a} \subset I$ *be* R*-ideals with* $\mu(\mathfrak{a}) \leq s \leq \mathrm{ht}\ \mathfrak{a} : I$*, let* t *be an integer with* $t \leq s-1$*. Assume that* I *satisfies* G_s *and* AN_t^- *and that* $[\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i I^{j-1}$ *for* $0 \leq i \leq s-1$*,* $j \geq \max\{1, i-t\}$*, where* $a_1, \ldots, a_s$ *and* $\mathfrak{a}_i$ *are as defined in Lemma 3.1.1* (a)*. Then*

$$\mathrm{depth}\ R/\mathfrak{a}_i I^j \geq \min(\{d-i\} \cup \{\mathrm{depth}\ R/I^{j-n} - n \mid 0 \leq n \leq i-1\}),$$

for $0 \leq i \leq s$ *and* $j \geq \max\{0, i-t-1\}$.

Proof. We use induction on i. The assertion is trivial for $i = 0$; so we may assume that $0 \leq i \leq s-1$. We need to show that the inequality holds for $i+1$. For $j = 0$ (which can only occur if $i+1 \leq t+1$), we have that depth $R/\mathfrak{a}_{i+1} = d - i - 1$ by Lemma 3.1.1 (b)(ii), and so our assertion follows. Thus we may suppose that $j \geq 1$. But then by assumption,

$$\begin{aligned} \mathfrak{a}_i I^j \cap a_{i+1} I^j &= a_{i+1}[(\mathfrak{a}_i I^j : (a_{i+1})) \cap I^j] \\ &\subset a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^j] \\ &= a_{i+1}\mathfrak{a}_i I^{j-1} \\ &\subset \mathfrak{a}_i I^j \cap a_{i+1} I^j. \end{aligned}$$

Hence we obtain an exact sequence

$$0 \to a_{i+1}\mathfrak{a}_i I^{j-1} \to \mathfrak{a}_i I^j \oplus a_{i+1} I^j \to \mathfrak{a}_{i+1} I^j \to 0. \quad (1)$$

On the other hand, by the assumption for $i = 0$,

$$[0 : (a_{i+1})] \cap \mathfrak{a}_i I^{j-1} \subset [0 : (a_{i+1})] \cap I^j = 0,$$

and therefore $a_{i+1}\mathfrak{a}_i I^{j-1} \cong \mathfrak{a}_i I^{j-1}$, $a_{i+1} I^j \cong I^j$. The required depth estimate for $R/\mathfrak{a}_{i+1} I^j$ follows from (1) and the induction hypothesis.□

The next lemma gives conditions that imply the intersections in the assumptions of Lemma 3.1.6. It is a generalization of [39, Lemma 2.8].

Lemma 3.1.7 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, let J be a reduction of I generated by s elements with* $\mathrm{ht}\ J : I \geq s$, $r = r_J(I)$, *let $k \geq 0$ be an integer with $r \leq k$, assume that I satisfies G_s, AN^-_{s-k-1} and that for every $p \in V(I)$,* $\mathrm{depth}(R/I^j)_p \geq \min\{\dim R_p - s + k - j, k - j\}$ *whenever $1 \leq j \leq k-1$. Write $\mathcal{G} = gr_I(R)$, for $a \in I$ let a' denote the image of a in $[\mathcal{G}]_1$, and for $\mathfrak{a} = J$, let $a_1, \dots, a_s$ and $\mathfrak{a}_i$ be defined as in Lemma 3.1.1* (a). *Then:*

(a) *For every $p \in Spec(R)$,* $\mathrm{depth}(R/\mathfrak{a}_i I^j)_p \geq \min\{\dim R_p - s + k - j, k-j\}$, *whenever* $0 \leq i \leq s-1$ *and* $\max\{0, i-s+k\} \leq j \leq k-1$.

(b) $[\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever $0 \leq i \leq s-1$ and $j \geq$* $\max\{1, i-s+k+1\}$.

(c) $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever $0 \leq i \leq s$ and $j \geq i - s + k + 1$.*

(d) *$a'_1, \dots, a'_g$ form a $\mathcal{G}$-regular sequence and* $[(a'_1, \dots, a'_i) :_{\mathcal{G}} (a'_{i+1})]_j = [(a'_1, \dots, a'_i)]_j$ *whenever $g \leq i \leq s-1$ and* $j \geq \max\{1, i-s+k+1\}$.

Proof. First we show (a), (b) and (c) by induction on j. Notice that for $j = 0$, (which can only occur if $i \leq s-k$), (a) holds by Lemma 3.1.1 (b)(ii) with $t = s - k - 1$. Also, (b) holds for $j = 1$ by Lemma 3.1.1 (b)(i) and (c) trivially holds for $j = 1$.

Claim 1. If (b) holds for j, then so does (a).

Proof. This follows from Lemma 3.1.6, with $t = s - k - 1$, using the assumption on the depth of the powers of I.

Now we can assume that (a), (b) and (c) hold for $j \geq 1$ and we show that they hold for $j + 1$.

CLAIM 2. If (a) holds for j, then (b) and (c) hold for $j + 1$ and the maximal value of i; namely $i = s - k + j$.

Proof. If $j \geq k$, then $i = s - k + j \geq s$; so there is nothing to show in (b) and $I^{j+1} = JI^j = \mathfrak{a}_s I^j$, which implies (c). Let $j \leq k-1$. In order to show (b), let $p \in \operatorname{Ass}(R/\mathfrak{a}_i I^j)$. By (a), $\min\{\dim R_p - s + k - j, k - j\} \leq 0$; hence $\dim R_p \leq s - k + j = i$. If $I_p = R_p$, then $\mathfrak{a}_i \subset p$, since $p \in \operatorname{Ass}(R/\mathfrak{a}_i I^j)$. Since ht $\mathfrak{a}_{i+1} : I \geq i + 1$ by Lemma 3.1.1 (a), we have that $(a_{i+1})_p = R_p$ and (b) follows. If $I_p \neq R_p$, since ht $I + \mathfrak{a}_i : I \geq i + 1$ by Lemma 3.1.1 (a), we have that $I_p = (\mathfrak{a}_i)_p$ and again we get the desired equality of (b). Since

$$\mathfrak{a}_i \cap I^{j+1} \subset [\mathfrak{a}_i : (a_{i+1})] \cap I^{j+1} = \mathfrak{a}_i I^j,$$

(c) holds.

CLAIM 3. If (b) and (c) hold for $j + 1$ and maximal i, then (b) and (c) hold for $j + 1$ and any i.

Proof. Take $i < s - k + j + 1$. By decreasing induction on i, $\mathfrak{a}_{i+1} \cap I^{j+1} = \mathfrak{a}_{i+1} I^j$. Hence

$$\begin{aligned}
\mathfrak{a}_i \cap I^{j+1} &= \mathfrak{a}_i \cap \mathfrak{a}_{i+1} \cap I^{j+1} \\
&= \mathfrak{a}_i \cap \mathfrak{a}_{i+1} I^j \\
&= \mathfrak{a}_i \cap (\mathfrak{a}_i I^j + a_{i+1} I^j) \\
&= \mathfrak{a}_i I^j + \mathfrak{a}_i \cap a_{i+1} I^j \\
&= \mathfrak{a}_i I^j + a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^j] \\
&= \mathfrak{a}_i I^j + a_{i+1} \mathfrak{a}_i I^{j-1} \\
&= \mathfrak{a}_i I^j.
\end{aligned}$$

This shows (c).

Since

$$[\mathfrak{a}_i : (a_{i+1})] \cap I^{j+1} \subset [\mathfrak{a}_i : (a_{i+1})] \cap I^j \subset \mathfrak{a}_i,$$

we have that

$$[\mathfrak{a}_i : (a_{i+1})] \cap I^{j+1} \subset \mathfrak{a}_i \cap I^{j+1} = \mathfrak{a}_i I^j;$$

hence (b) holds.

The first claim of (d) follows from [53, Corollary 2.7] and part (c) with $i = g$. Now let $u \in [(a'_1, \ldots, a'_i) : a'_{i+1}]_j$. Picking an element $x \in I^j$ with $x + I^{j+1} = u$, we have $a_{i+1} x \in \mathfrak{a}_i + I^{j+2}$, and so by part (c),

$$
\begin{aligned}
a_{i+1}x &\in \mathfrak{a}_{i+1} \cap (\mathfrak{a}_i + I^{j+2}) \\
&= \mathfrak{a}_i + \mathfrak{a}_{i+1} \cap I^{j+2} \\
&= \mathfrak{a}_i + \mathfrak{a}_{i+1} I^{j+1} \\
&= \mathfrak{a}_i + a_{i+1} I^{j+1}.
\end{aligned}
$$

Thus $a_{i+1}(x-y) \in \mathfrak{a}_i$ for some $y \in I^{j+1}$. Since $x-y+I^{j+1} = x+I^{j+1} = u$, we may replace x by $x-y$ to assume that $a_{i+1}x \in \mathfrak{a}_i$. But then $x \in [\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i I^{j-1}$ by (b), which implies $u \in (a_1', \ldots, a_i')$. □

Now we are ready for the last technical result.

Lemma 3.1.8 *Let s and t be integers. Assume $[(a_1', \ldots, a_i') : a_{i+1}']_j = [(a_1', \ldots, a_i')]_j$ whenever $0 \leq i \leq s-1$ and $j \geq i-t$, where $a_1', \ldots, a_s'$ are defined as in Lemma 3.1.7. Then*

$$
\text{depth } [\mathcal{G}/(a_1', \ldots, a_i')]_j \geq
$$

$$
\min(\{\text{depth } R/I^n + n - j - 1 \mid j-i+1 \leq n \leq j+1\} \cup \{\text{depth } R/I^{j-i} - i + 1\}),
$$

whenever $0 \leq i \leq s$ and $j \geq i-t$.

Proof. We show by induction on i that

$$
\text{depth } [\mathcal{G}/(a_1', \ldots, a_i')]_j \geq \min\{\text{depth } I^n/I^{n+1} + n - j \mid j-i \leq n \leq j\},
$$

whenever $0 \leq i \leq s$ and $j \geq i - t$. The assertion being trivial for $i = 0$, we may assume that $0 \leq i \leq s - 1$. We need to show that the inequality holds for $i + 1$.

By assumption we have an exact sequence

$$0 \to [\mathcal{G}/(a'_1, \dots, a'_i)]_j \to [\mathcal{G}/(a'_1, \dots, a'_i)]_{j+1} \to$$

$$[\mathcal{G}/(a'_1, \dots, a'_{i+1})]_{j+1} \to 0 \qquad (2)$$

whenever $0 \leq i \leq s - 1$ and $j \geq i - t$.

Applying a depth chase and the induction hypothesis to (2) the conclusion follows. □

Remark 3.1.9 ([23, Lemma 2.7, Lemma 3.1, Corollary 3.2, Corollary 3.3 and their proof]) Lemma 3.1.6 and Lemma 3.1.7 are still satisfied if the assumption "I satisfies G_s" is replaced by "$r_i \leq \max\{0, i-t-1\}$ for all $g \leq i < s$" (with $t = s-k-1$ in Lemma 3.1.7), where r_i's are as in Definition 3.1.4. In this context $a_1, \dots, a_s$ are defined as in Lemma 3.1.3. In fact, in the proofs of Lemma 3.1.6 and Lemma 3.1.7, the condition "I satisfies G_s" is used only to apply Lemma 3.1.1. With the new assumption we use Lemma 3.1.5 instead. Also, Lemma 3.1.8 still holds if $a'_1, \dots, a'_s$ are the images of $a_1, \dots, a_s$, with $a_1, \dots, a_s$ as in Lemma 3.1.3.

3.2 An Estimate of the Depth of the Associated Graded Ring

Many authors have extensively studied the Cohen-Macaulay property of $\mathcal{R}$ and $\mathcal{G}$. One of the most general results for ideals with good residual intersection properties and sufficiently small reduction number is the following theorem of Johnson and Ulrich.

Theorem 3.2.1 ([39, Theorem 3.1]) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, and reduction number r, let $k \geq 1$ be an integer with $r \leq k$, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell-1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$ and that* $\operatorname{depth}(R/I^j) \geq d-\ell+k-j$ *for $1 \leq j \leq k$. Then $\mathcal{G}$ is Cohen-Macaulay, and if $g \geq 2$, $\mathcal{R}$ is Cohen-Macaulay.*

Goto, Nakamura and Nishida proved a similar result, weakening the Artin-Nagata assumptions. The condition G_ℓ is replaced by a local condition on the reduction numbers.

Theorem 3.2.2 ([23, Theorem 1.1]) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, and reduction number r, let $k \geq 0$ be an* *integer with* $r \leq k$. *Assume that* $r_i \leq \max\{0, i-\ell+k\}$ *for all* $g \leq i < \ell$, *where* r_i*'s are defined as in Definition 3.1.4. Assume that* I *satisfies* $AN^-_{\ell-k-1}$ *and that* $\operatorname{depth}(R/I^j) \geq d-\ell+k-j$ *for* $1 \leq j \leq k$. *Then* $\mathcal{G}$ *is Cohen-Macaulay. Morover,* $\mathcal{R}$ *is Cohen-Macaulay if* $g > 0$ *and* $r < \ell$.

Remark 3.2.3 There are many ideals satisfying the assumptions on the depth of the powers in Theorem 3.2.1 and in Theorem 3.2.2. For example, if I is strongly Cohen-Macaulay and satisfies G_s, it can be seen from the Approximation Complex ([24, the proof of 5.1]) that $\operatorname{depth}(R/I^j) \geq d-g-j+1$ whenever $1 \leq j \leq s-g+1$.

Theorem 3.2.1 and Theorem 3.2.2 generalize several previous results about the Cohen-Macaulyness of blow-up rings.

The relationship between the Cohen-Macaulay property of $\mathcal{R}$ and $\mathcal{G}$ is well understood, for example by the following results of Huneke, Lipman, and Simis, Ulrich and Vasconcelos.

Theorem 3.2.4 ([30, Proposition 1.1]) *Let* R *be a Cohen-Macaulay ring and let* I *be an* R*-ideal with positive height. If* $\mathcal{R}$ *is Cohen-Macaulay, then* $\mathcal{G}$ *is Cohen-Macaulay.*

Remark 3.2.5 Lipman showed in [42] that if R is a regular ring, the converse of the above theorem holds. In this case, if I has positive height, $\mathcal{R}$ is Cohen-Macaulay if and only if $\mathcal{G}$ is Cohen-Macaulay.

Theorem 3.2.6 ([49, Corollary 3.6]) *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade $g > 0$, analytic spread ℓ, and reduction number r. Assume that I satisfies G_ℓ. Then the following are equivalent:*

(a) $\mathcal{R}$ is Cohen-Macaulay.

(b) $\mathcal{G}$ is Cohen-Macaulay and $r < \ell$.

Another interesting issue is to estimate the depth of blow-up rings when they fail to be Cohen-Macaulay, weakening the assumptions of Theorem 3.2.1 and Theorem 3.2.2. We can focus our attention on the study of depth $\mathcal{G}$, because an estimate of depth $\mathcal{R}$ follows by this result of Huckaba and Marley.

Theorem 3.2.7 ([29, Theorem 3.10]) *Let R be a local ring and let I be an R-ideal. Suppose* $\operatorname{depth} \mathcal{G} < \operatorname{depth} R$. *Then* $\operatorname{depth} \mathcal{R} = \operatorname{depth} \mathcal{G} + 1$.

Cortadellas and Zarzuela have come up with formulas for the depth of $\mathcal{G}$ in [12], in the case of ideals with reduction number at most two and analytic deviation at most one. More precisely:

Theorem 3.2.8 ([12, Theorem 4.1]) *Let R be a local Cohen-Macaulay ring and let I be an equimultiple ideal with reduction number ≤ 1. Then*

$$\text{depth } \mathcal{G} = \text{depth } R/I + g.$$

Theorem 3.2.9 ([12, Theorem 5.8]) *Let R be a local Cohen-Macaulay ring and let I be an ideal with analytic deviation one and reduction number ≤ 2. Assume that I is unmixed and that $r_g(I) \leq 1$, where* $r_g(I) = \max\{r(I_p) \mid p \in V(I) \text{ with ht } p = g\}$. *Then*

$$\min\{\text{depth } R/I, \text{depth } R/I^2\} + g \leq \text{depth } \mathcal{G} \leq$$
$$\min\{\text{depth } R/I, \text{depth } R/I^2\} + g + 1.$$

These results motivate the search for a lower bound and an upper bound for depth $\mathcal{G}$ in terms of the depth of the powers of I up to the reduction number, in more general cases. The upper bound is easy to find because if R is a Noetherian ring and I is an R-ideal with analytic spread ℓ, we have that depth $\mathcal{G} \leq \inf\{\text{depth } R/I^j \mid j \geq 1\} + \ell$ (see Remark 3.2.20).

Theorem 3.2.10, which is the main result of this chapter, gives an effective lower bound for depth $\mathcal{G}$. In particular the theorem unifies and generalizes the above mentioned results by Johnson-Ulrich, Goto-Nakamura-Nishida and Cortadellas-Zarzuela.

The goal of this section is to prove the following assertion.

Theorem 3.2.10 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, let J be a reduction of I generated by s elements with* $\operatorname{ht} J : I \geq s$, $r = r_J(I)$, *and let $k \geq 0$ be an integer with $r \leq k$. Assume that I satisfies G_s, AN^-_{s-k-1}, and that for every $p \in V(I)$,* $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - s + k - j, k - j\}$ *whenever $1 \leq j \leq k-1$. Then*

$$\operatorname{depth} \mathcal{G} \geq \min(\{d\} \cup \{\operatorname{depth} R/I^j + s - k + j \mid 1 \leq j \leq k\}).$$

Remark 3.2.11 Notice that the assumption "for every $p \in V(I)$, depth $(R/I^j)_p \geq \min\{\dim R_p - s + k - j, k - j\}$ whenever $1 \leq j \leq k-1$" in Theorem 3.2.10 implies that $k \leq s - g + 1$. This is trivial if $k \leq 1$. If $k \geq 2$, let p be a minimal prime of I with $\operatorname{ht} p = g$. One has $0 = \operatorname{depth}(R/I)_p \geq \min\{g - s + k - 1, k - 1\}$ and so $k \leq s - g + 1$. Hence the reduction number is forced to be "sufficiently small".

We will make strong use of the technical results proven in the previous section. The proof of the theorem consists of three main steps:

STEP 1: We show that we can reduce the problem to the case where $g = 0$.

STEP 2: We prove the theorem in the case where $g = 0$ and $s \leq k$. This is the crucial part of the proof. We will need some more preliminary results.

STEP 3: We prove the general case. It follows rather easily from the previous step.

Proof of the Theorem.

Let $J = (a_1, \ldots, a_s)$, where $a_1, \ldots, a_s$ are defined as in Lemma 3.1.1 (a), and let $a'_1, \ldots, a'_s$ be their images in $[\mathcal{G}]_1$.

STEP 1. Reduction to the case $\mathbf{g = 0}$.

The following lemma shows that we may assume $g = 0$ in order to prove the theorem.

Lemma 3.2.12 *Using the notation of Theorem 3.2.10, suppose that $g > 0$ and let $R^* = R/\mathfrak{a}_g$, $I^* = I/\mathfrak{a}_g$, $J^* = J/\mathfrak{a}_g$, $\mathcal{G}^* = \mathcal{G}(I^*)$. Then we have the following:*

(a) *R^* is a local Cohen-Macaulay ring of dimension $d - g$ and I^* is an R^*-ideal of grade 0.*

(b) *J^* is a reduction of I^* generated by $s - g$ elements, ht $J^* : I^* \geq s - g$, and $r_{J^*}(I^*) \leq r_J(I)$, so that k may be taken unchanged.*

(c) *I^* satisfies G_{s-g}.*

(d) *I^* satisfies $AN^-_{s-g-k-1}$.*

(e) *For every* $p^* \in V(I^*)$, *depth*$(R^*/I^{*j})_{p^*} \geq \min\{\dim R_{p^*} - (s-g) + k - j, k - j\}$ *whenever* $1 \leq j \leq k-1$.

(f) *depth* $\mathcal{G} =$ *depth* $\mathcal{G}^* + g$, *and if depth* $\mathcal{G}^* \geq \min(\{d-g\} \cup \{\text{depth } R^*/I^{*j} + s - g - k + j \mid 1 \leq j \leq k\})$, *then depth* $\mathcal{G} \geq \min(\{d\} \cup \{\text{depth } R/I^j + s - k + j \mid 1 \leq j \leq k\})$.

Proof. Parts (a) and (b) are clear, and (d) holds by [36, Lemma 1.1.6].

In order to prove (c), let $p^* \in V(I^*)$ with ht $p^* \leq s-g-1$. Then $p^* = p/\mathfrak{a}_g$, where $p \in V(I)$ and ht $p \leq s-1$. By Lemma 3.1.1 (a), $I_p = (a_1, \ldots, a_i)_p$ for all $p \in V(I)$ with ht $p \leq i \leq s-1$; so that $\mu(I^*)_{p^*} \leq$ ht p^*.

In order to show (e), notice that since $R^*/I^{*j} \cong R/\mathfrak{a}_g + I^j$, by Lemma 3.1.7 (c) for every $j \geq 1$ we have an exact sequence

$$0 \to R/\mathfrak{a}_g I^{j-1} \to R/\mathfrak{a}_g \oplus R/I^j \to R^*/I^{*j} \to 0. \qquad (3)$$

By (3), Lemma 3.1.7 (b), and Lemma 3.1.6 with $t = s-k-1$, we have that $\text{depth}(R^*/I^{*j})_{p^*} \geq \min\{\text{depth}(R/I^{j-n})_p - n \mid 0 \leq n \leq g\}$ whenever $j > 1$. Since the inequality holds also when $j = 1$, we get the desired condition on the depth of the powers.

Finally, we have that $\mathcal{G}^* = \mathcal{G}/(a'_1, \ldots, a'_g)$ and that depth $\mathcal{G} =$ depth $\mathcal{G}^* + g$ by Lemma 3.1.7 (d). If

depth $\mathcal{G}^* \geq \min(\{d-g\} \cup \{\text{depth } R^*/I^{*j} + s - g - k + j \mid 1 \leq j \leq k\})$, again using the fact that

$$\operatorname{depth}(R^*/I^{*j}) \geq \min\{\operatorname{depth} R/I^{j-n} - n \mid 0 \leq n \leq g\}$$

for $j \geq 1$, we conclude that

$$\operatorname{depth} \mathcal{G} \geq \min(\{d\} \cup \{\operatorname{depth} R/I^j + s - k + j \mid 1 \leq j \leq k\})$$

and this finishes the proof of the lemma and the reduction to the case $g = 0$. □

STEP 2. The proof in the case s ≤ k.

In the setup of Theorem 3.2.10, we now assume $g = 0$ and $s \leq k$. We are going to prove the theorem in this special case. Since $k \leq s+1$ by Remark 3.2.11, either $k = s$ or $k = s+1$. If $s = 0$, then I is nilpotent and $k \leq 1$; so we have either $\mathcal{G} = R$ or $\mathcal{G} = R/I \oplus I$ and depth $\mathcal{G}=$ depth R/I. In any case the theorem holds; hence we can assume $s > 0$.

The main idea of the proof is to use suitable truncations of the graded $\mathcal{G}$-modules $\mathcal{G}/(a'_1, \ldots, a'_i)$ in order to have "convenient" short exact sequences. More precisely, for $0 \leq i \leq s$ consider the graded $\mathcal{G}$-modules

$$M_{(i)} = [\mathcal{G}/(a'_1, \ldots, a'_i)]_{\geq i-s+k+1} = \mathcal{G}_+^{i-s+k+1}/(a'_1, \ldots, a'_i)\mathcal{G}_+^{i-s+k}$$

and

$$N_{(i)} = \mathcal{G}_+^{i-s+k}/(a'_1, \ldots, a'_{i-1})\mathcal{G}_+^{i-s+k-1} + a'_i\mathcal{G}_+^{i-s+k}.$$

Notice that $M_{(i)}$ can be obtained as a truncation of $N_{(i)}$, namely

$$M_{(i)} = [N_{(i)}]_{\geq i-s+k+1}.$$

In addition $M_{(i-1)}$ coincides with $N_{(i)}$ in degree $i-s+k$; i.e.,

$$[N_{(i)}]_{i-s+k} = [G/(a'_1, \ldots, a'_{i-1})]_{i-s+k}.$$

Hence for $0 \leq i \leq s$ we have exact sequences

$$0 \to M_{(i)} \to N_{(i)} \to [G/(a'_1, \ldots, a'_{i-1})]_{i-s+k} \to 0. \qquad (4)$$

On the other hand, if $0 \leq i \leq s-1$, then

$$N_{(i+1)} = M_{(i)}/a'_{i+1}M_{(i)}$$

and by Lemma 3.1.7 (d) we have that $0 :_{M_{(i)}} (a'_{i+1}) = 0$. Thus, in the range $0 \leq i \leq s-1$ we have exact sequences

$$0 \to M_{(i)}(-1) \to M_{(i)} \to N_{(i+1)} \to 0, \qquad (5)$$

where the first map is given by multiplication by a'_{i+1}. Furthermore $M_{(s)} = 0$, since $I^{k+1} = JI^k$.

The exact sequences (4) and (5) are an essential tool for the proof of the theorem. We are going to apply a depth chase and the local cohomology functor to these sequences, to get an estimate for depth $M_{(i)}$, starting from $i = s$ and using decreasing induction on i (Lemma 3.2.15). Eventually we will obtain an inequality for depth

$M_{(0)}$. Since $M_{(0)} = [\mathcal{G}]_{\geq 1}$ if $k = s$, or $M_{(0)} = [\mathcal{G}]_{\geq 2}$ if $k = s+1$, we will get the required estimate for depth $\mathcal{G}$.

First we need a lemma, that we are going to prove in a more general context.

Let S be a homogeneous Noetherian ring with S_0 local and homogeneous maximal ideal $\mathfrak{M}$, let $H^\bullet(-)$ denote local cohomology with support in $\mathfrak{M}$.

For a graded S-module N and an integer j we put $a_j(N) = \max\{n \mid [H^j(N)]_n \neq 0\}$ and we call it the j-th a-invariant of N.

Lemma 3.2.13 *Let $0 \to A \to B \to C \to 0$ be an exact sequence of graded S-modules, let n and j be integers.*

(a) *If $a_j(A) \leq n$ and $a_j(C) \leq n$, then $a_j(B) \leq n$.*

(b) (i) *If $H^j(A) = 0$, then $a_j(C) \geq a_j(B)$.*

(ii) *If $H^j(B) = 0$, then $a_{j+1}(A) \geq a_j(C)$.*

(iii) *If $H^j(C) = 0$, then $a_{j+1}(B) \geq a_{j+1}(A)$.*

Proof. The assertions follow from the long exact sequence of local cohomology

$$\dots \to H^j(A) \to H^j(B) \to H^j(C) \to H^{j+1}(A) \to H^{j+1}(B) \to \dots$$

and the definition of $a_j(-)$. □

Since we will use the following fact several times, we recall it here for convenience.

Lemma 3.2.14 ([20, Lemma 2.2]) *Let $S = \oplus_{n \geq 0} S_n$ be a homogeneous Noetherian ring with $(S_0, \mathfrak{m}_0)$ a local ring. Let N be a finitely generated graded S-module with $N_n = 0$ for all $n >> 0$. Then for any integers i, n we have an isomorphism $[H^i_M(N)]_n \cong H^i_{\mathfrak{m}_0}(N_n)$ of S_0-modules, where M is the maximal homogeneous ideal of S.*

Let

$$\lambda = \min(\{d\} \cup \{\text{depth } R/I^j + s - k + j \mid 1 \leq j \leq k\}).$$

Recall that we want to show that depth $\mathcal{G} \geq \lambda$. The next lemma gives an estimate of depth $M_{(i)}$ in terms of λ.

Lemma 3.2.15 *In addition to the assumptions of Theorem 3.2.10, assume that $g = 0$ and $s \leq k$. Let $M_{(i)}$ be defined as above. Then:*

(a) $a_j(M_{(i)}) \leq i - s + k$ *for any integer j and $0 \leq i \leq s$.*

(b) depth $M_{(i)} \geq \lambda - i - 1$, *and if* depth $M_{(i)} = \lambda - i - 1$ *then* $a_{\lambda - i - 1}(M_{(i)}) = i - s + k$.

Proof. First we prove (a) by decreasing induction on i. If $i = s$, the assertion is obvious since $M_{(s)} = 0$. Suppose $0 \leq i \leq s - 1$ and assume $a_j(M_{(i+1)}) \leq i+1-s+k$ for any integer j. By Lemma 3.2.14 we have that, for any integer j, $[H^j([\mathcal{G}/(a'_1,\dots,a'_i)]_{i+1-s+k})]_n = 0$, if $n \neq i+1-s+k$. Hence, by Lemma 3.2.13 (a) applied to the sequence

$$0 \to M_{(i+1)} \to N_{(i+1)} \to [\mathcal{G}/(a'_1,\dots,a'_i)]_{i+1-s+k} \to 0 \qquad (6)$$

we have that $a_j(N_{(i+1)}) \leq i+1-s+k$ for any j. Applying the local cohomology functor to the sequence

$$0 \to M_{(i)}(-1) \to M_{(i)} \to N_{(i+1)} \to 0 \qquad (7)$$

we get the exact sequence

$$\cdots \to H^{j-1}(N_{(i+1)}) \to H^j(M_{(i)}(-1)) \to H^j(M_{(i)}) \to H^j(N_{(i+1)}) \to \dots$$

Now, if $n > i-s+k$, we have

$$0 = [H^{j-1}(N_{(i+1)})]_{n+1} \to [H^j(M_{(i)})]_n \to$$

$$[H^j(M_{(i)})]_{n+1} \to [H^j(N_{(i+1)})]_{n+1} = 0.$$

It follows that for any j, $[H^j(M_{(i)})]_n = 0$ whenever $n > i-s+k$. Hence $a_j(M_{(i)}) \leq i-s+k$ and the proof of (a) is complete.

Now we prove (b), again using decreasing induction on i. If $i = s$ the assertion is trivial since $M_{(s)} = 0$. Assume now that $0 \leq i \leq s-1$, that depth $M_{(i+1)} \geq \lambda - i - 2$, and that if depth $M_{(i+1)} = \lambda - i - 2$, then $a_{\lambda-i-2}(M_{(i+1)}) = i+1-s+k$. By Lemma 3.1.7 (d) and Lemma 3.1.8 with $t = s-k-1$, we have that depth $[\mathcal{G}/(a'_1, \ldots, a'_i)]_{i+1-s+k} \geq \min(\{\text{depth } R/I^j + j - i + s - k - 2 \mid k - s + 2 \leq j \leq i - s + k + 2\} \cup \{\text{depth } R/I^{k-s+1} - i + 1\})$. If $0 \leq i \leq s-2$, then $j \leq k$ and so

$$\text{depth } [\mathcal{G}/(a'_1, \ldots, a'_i)]_{i+1-s+k} \geq$$

$$\min\{\text{depth } R/I^j + j - i + s - k - 2 \mid 1 \leq j \leq k\}$$

$$\geq \lambda - i - 2.$$

If $i = s - 1$, then $j \leq k + 1$. By Lemma 3.1.7 (b) and Lemma 3.1.6 with $t = s - k - 1$, we have that

$$\begin{aligned} \text{depth } R/I^{k+1} &= \text{depth } R/JI^k \\ &\geq \min(\{d - s\} \\ &\cup \{\text{depth } R/I^{k-n} - n \mid 0 \leq n \leq s - 1\}) \\ &\geq \lambda - s. \end{aligned}$$

So we have that depth $[\mathcal{G}/(a'_1, \ldots, a'_{s-1})]_k \geq \lambda - s - 1$. Hence in any case it follows that

$$\text{depth } [\mathcal{G}/(a'_1, \ldots, a'_i)]_{i+1-s+k} \geq \lambda - i - 2,$$

and so by (6) we have that depth $N_{(i+1)} \geq \lambda - i - 2$. Since $N_{(i+1)} = M_{(i)}/a'_{i+1}M_{(i)}$ and a'_{i+1} is $M_{(i)}$-regular, we conclude that depth $M_{(i)} \geq \lambda - i - 1$ and this proves the first assertion of (b).

If depth $M_{(i)} = \lambda - i - 1$, then depth $N_{(i+1)} = \lambda - i - 2$ and so $H^{\lambda-i-2}(N_{(i+1)}) \neq 0$. Applying the local cohomology functor to the sequence (6) we get the exact sequence

$$\cdots \to H^{\lambda-i-2}(M_{(i+1)}) \to H^{\lambda-i-2}(N_{(i+1)}) \to$$

$$H^{\lambda-i-2}[\mathcal{G}/(a'_1, \dots, a'_i)]_{i+1-s+k} \to \dots$$

If depth $M_{(i+1)} > \lambda - i - 2$, then $H^{\lambda-i-2}(M_{(i+1)}) = 0$; hence $H^{\lambda-i-2}(N_{(i+1)}) \cong [H^{\lambda-i-2}(N_{(i+1)})]_{i+1-s+k}$ and so $a_{\lambda-i-2}(N_{(i+1)}) = i + 1 - s + k$. If depth $M_{(i+1)} = \lambda - i - 2$, then by assumption $a_{\lambda-i-2}(M_{(i+1)}) = i + 1 - s + k$. Since depth $[\mathcal{G}/(a'_1, \dots, a'_i)]_{i+1-s+k} \geq \lambda - i - 2$, applying Lemma 3.2.13 (b)(iii) with $j = \lambda - i - 3$ to (6), we get that $a_{\lambda-i-2}(N_{(i+1)}) \geq i + 1 - s + k$. In any case $a_{\lambda-i-2}(N_{(i+1)}) \geq i + 1 - s + k$. Finally, since depth $M_{(i)} = \lambda - i - 1$, applying Lemma 3.2.13 (b)(ii) with $j = \lambda - i - 2$ to the sequence (7), we conclude that $a_{\lambda-i-1}(M_{(i)}) \geq i - s + k$. □

In the next corollary and lemma we obtain some useful results about the a-invariants of $\mathcal{G}$, that we will need to conclude the proof of the theorem.

Corollary 3.2.16 *With the assumptions of Lemma 3.2.15, we have that $a_j(\mathcal{G}) \leq k - s$ for any integer j.*

Proof. We have an exact sequence $0 \to M_{(0)} \to \mathcal{G} \to C \to 0$, where

$$C = \begin{cases} R/I & \text{if } k = s \\ R/I \oplus I/I^2 & \text{if } k = s+1. \end{cases}$$

Since for any integer j, $a_j(M_{(0)}) \le k-s$ by Lemma 3.2.15, and $a_j(C) \le k-s$ by Lemma 3.2.14, using Lemma 3.2.13 (a) we conclude that $a_j(\mathcal{G}) \le k-s$ for any integer j. □

Lemma 3.2.17 *If* depth $\mathcal{G} = t < d$, *then* $a_t(\mathcal{G}) < \max\{0, a_{t+1}(\mathcal{G})\}$. *In particular, with the assumptions of Corollary 3.2.16, one has* $a_t(\mathcal{G}) < k - s$.

Proof. Suppose $t < d$. Then we have depth $\mathcal{R} = t+1$ by Theorem 3.2.7, and so $H^t(\mathcal{R}) = 0$. Hence, applying the local cohomology functor to

$$0 \to \mathcal{R}_+(1) \to \mathcal{R} \to \mathcal{G} \to 0$$

we get the exact sequence

$$0 \to H^t(\mathcal{G}) \to H^{t+1}(\mathcal{R}_+(1)) \to H^{t+1}(\mathcal{R}) \to H^{t+1}(\mathcal{G}). \qquad (8)$$

Let $m = \max\{0, a_{t+1}(\mathcal{G})\}$. If $n > m$, then $[H^{t+1}(\mathcal{G})]_n = 0$ and so $[H^{t+1}(\mathcal{R}_+)]_{n+1}$ maps onto $[H^{t+1}(\mathcal{R})]_n$. Applying the local cohomology functor to

$$0 \to \mathcal{R}_+ \to \mathcal{R} \to R \to 0$$

we get $[H^{t+1}(\mathcal{R}_+)]_n \cong [H^{t+1}(\mathcal{R})]_n$, since, for every integer j, $H^j(R) = [H^j(R)]_0 \cong H^j_{\mathfrak{m}}(R)$ by Lemma 3.2.14. Hence $[H^{t+1}(\mathcal{R}_+)]_n = 0$ for any $n > m$. From (8) we conclude that $a_t(\mathcal{G}) < m$. □

Now we are ready to finish the proof of the theorem.

Conclusion of the case $\mathbf{s \leq k}$.

Consider the exact sequence

$$0 \to M_{(0)} \to \mathcal{G} \to C \to 0.$$

Recall that

$$C = \begin{cases} R/I & \text{if } k = s \\ R/I \oplus I/I^2 & \text{if } k = s+1 \end{cases}$$

and that

$$\lambda = \min(\{d\} \cup \{\operatorname{depth} R/I^j + s - k + j \mid 1 \leq j \leq k\}).$$

Notice that $\operatorname{depth}_{\mathcal{R}} C = \operatorname{depth}_R C \geq \lambda - 1$. This is obvious if $k = s$, and it easily follows from the exact sequence $0 \to I/I^2 \to R/I^2 \to R/I \to 0$ if $k = s+1$. Notice that, if $k = s+1$, we have that depth $R/I > \lambda - 1$. Since depth $M_{(0)} \geq \lambda - 1$ by Lemma 3.2.15, it follows that depth $\mathcal{G} \geq \lambda - 1$. If depth $\mathcal{G} = \lambda - 1$, then $a_{\lambda-1}(\mathcal{G}) < k - s$ by Lemma 3.2.17. The sequence above yields

$$H^{\lambda-1}(M_{(0)}) \to H^{\lambda-1}(\mathcal{G}) \to H^{\lambda-1}(C). \qquad (9)$$

If depth $M_{(0)} > \lambda - 1$, then $H^{\lambda-1}(M_{(0)}) = 0$. If $k = s$, then $a_{\lambda-1}(\mathcal{G}) < 0$ and so $[H^{\lambda-1}(C)]_{a_{\lambda-1}(\mathcal{G})} = 0$, a contradiction. If $k = s+1$, then $a_{\lambda-1}(\mathcal{G}) < 1$ and so $[H^{\lambda-1}(C)]_{a_{\lambda-1}(\mathcal{G})} = [H^{\lambda-1}(R/I)]_{a_{\lambda-1}(\mathcal{G})} = 0$, since depth $R/I > \lambda - 1$, and again we get a contradiction. If depth $M_{(0)} = \lambda - 1$, then $a_{\lambda-1}(M_{(0)}) = k - s$ by Lemma 3.2.15. Applying Lemma 3.2.13 (b)(iii) with $j = \lambda - 2$ to the sequence $0 \to M_{(0)} \to \mathcal{G} \to C \to 0$, we get that $a_{\lambda-1}(\mathcal{G}) \geq k - s$, a contradiction. Hence depth $\mathcal{G} \geq \lambda$ and this proves our result in the case $g = 0$ and $s \leq k$.

STEP 3. Proof of the general case.

Let $\delta = \delta(I) = s - g + 1 - k$ and recall that $\delta \geq 0$ by Remark 3.2.11. We are going to induct on δ. By step 1, we can assume that $g = 0$; hence $\delta = s + 1 - k$. Since by step 2 the theorem holds if $\delta = 0$ or $\delta = 1$ (i.e., $k = s$ or $k = s + 1$), we can assume that $\delta \geq 2$ and so $s - k - 1 \geq 0$. Hence I satisfies AN_0^-. Write $K = 0 : I$ and let '-' denote images in $\bar{R} = R/K$.

We will show that our assumptions are preserved, and that δ decreases when passing from R to $\bar{R}$. Since I satisfies G_s and $s \geq k+1 \geq 1$, I satisfies G_1. Hence $I_p = 0$ for every $p \in V(I)$ with $\dim R_p = 0$. It follows that ht $I + K > 0$, and so K is a geometric 0-residual intersection of I. Hence $\bar{R}$ is Cohen-Macaulay, since I satisfies AN_0^-. By

Lemma 3.1.1 (b) $I \cap K = 0$ and grade $\bar{I} = 1$. Furthermore $\dim \bar{R} = \dim R = d$, $\bar{J} = (\bar{a}_1, \ldots, \bar{a}_s)$ is a reduction of $\bar{I}$ with ht $\bar{J} : \bar{I} \geq s$, $r_{\bar{J}}(\bar{I}) \leq r_J(I)$, and thus k may be taken to remain unchanged. Clearly $\bar{I}$ satisfies G_s since $\bar{R}$ is equidimensional of the same dimension as R. Also, by [39, Lemma 2.4 (b)], $\bar{I}$ satisfies AN^-_{s-k-1}. Since $I \cap K = 0$ we have an exact sequence

$$0 \to K \to \mathcal{G} \to gr_{\bar{I}}(\bar{R}) \to 0 \qquad (10)$$

where depth $K = d$ since depth $\bar{R} = d$.

Now using the degree 0 piece of the sequence (10), we have that depth $\bar{R}/\bar{I} \geq \min\{d-1, \text{depth } R/I\}$. Also, for $j \geq 1$ we have the isomorphisms

$$\bar{I}^j/\bar{I}^{j+1} \cong I^j/(I^{j+1} + I^j \cap K) \cong I^j/I^{j+1}.$$

Using the exact sequences

$$0 \to I^j/I^{j+1} \to R/I^{j+1} \to R/I^j \to 0$$

and

$$0 \to \bar{I}^j/\bar{I}^{j+1} \to R/\bar{I}^{j+1} \to R/\bar{I}^j \to 0$$

it follows, by induction on j, that whenever $j \geq 1$,

$$\text{depth } \bar{R}/\bar{I}^j \geq \min(\{d-1\} \cup \{\text{depth } R/I^t \mid 1 \leq t \leq j\}).$$

Applying this in the ring $R = R_p$, we have that whenever $1 \le j \le k-1$, for every $\bar{p} \in V(\bar{I})$

$$\begin{aligned} \text{depth } (\bar{R}/\bar{I}^j)_{\bar{p}} &\ge \min(\{\dim R_p - 1\} \\ &\cup \{\text{depth} R_p/I_p^t \mid 1 \le t \le j\}) \\ &\ge \min\{\dim R_p - s + k - j, k - j\} \\ &= \{\dim \bar{R}_{\bar{p}} - s + k - j, k - j\}. \end{aligned}$$

This shows that the condition on the depth of the powers is preserved when passing from R to $\bar{R}$.

Now

$$\delta(\bar{I}) = s - \text{grade } \bar{I} + 1 - k < s + 1 - k = \delta(I),$$

thus we may use our induction hypothesis to conclude that

$$\text{depth } gr_{\bar{I}}(\bar{R}) \ge \min(\{d\} \cup \{\text{depth } \bar{R}/\bar{I}^j + s - k + j \mid 1 \le j \le k\}).$$

Then by (10) and using again the inequality

$$\text{depth } \bar{R}/\bar{I}^j \ge \min(\{d-1\} \cup \{\text{depth } R/I^t \mid 1 \le t \le j\})$$

for $j \ge 1$, we have that

$$\text{depth } \mathcal{G} \ge \min(\{d\} \cup \{\text{depth } R/I^j + s - k + j \mid 1 \le j \le k\})$$

and the proof of the theorem is complete. □

Remark 3.2.18 If in the setting of Theorem 3.2.10 we have stronger assumptions on the depth of the powers of I; namely, if $\operatorname{depth}(R/I^j) \geq d - \ell + k - j$ whenever $1 \leq j \leq k$, then depth $\mathcal{G} = d$; i.e.' $\mathcal{G}$ is Cohen-Macaulay. Hence Theorem 3.2.10 generalizes Theorem 3.2.1.

Remark 3.2.19 The theorem still holds if the condition "I satisfies G_s" is replaced by "$r_i \leq \max\{0, i - s + k\}$ for all $g \leq i < s$", where r_i's are as in Definition 3.1.4. In this context we choose the reduction $J = (a_1, \ldots, a_s)$ with $a_1, \ldots, a_s$ as in Lemma 3.1.3. Notice that all the technical results of the previous section hold (see Remark 3.1.9). Furthermore the condition $r_i \leq \max\{0, i - s + k\}$ for all $g \leq i < s$ is still satisfied when we factor out $(a_1, \ldots, a_g)$ to assume $g = 0$ (see [23, Lemma 3.4]), and when we factor out $K = 0 : I$ (see [23, Lemma 5.1]). Since in the rest of the proof the G_s property is needed only to be able to use Lemma 3.1.6, Lemma 3.1.7, and Lemma 3.1.8, it follows that the result still holds. Hence Theorem 3.2.10 recovers Theorem 3.2.2.

The following remark gives an upper bound for depth $\mathcal{G}$.

Remark 3.2.20 *Let R be a Noetherian local ring, and let I be an R-ideal with analytic spread ℓ. Then* depth $\mathcal{G} \leq \inf\{\operatorname{depth} R/I^j \mid j \geq 1\} + \ell$.

Proof. Since $\mathcal{G}$ is a Noetherian ring, we have that $\text{depth}_{\mathfrak{m}\mathcal{G}}\mathcal{G} = \inf\{\text{depth } I^j/I^{j+1} \mid j \geq 1\} = \inf\{\text{depth } R/I^j \mid j \geq 1\}$. But $\text{depth}_{\mathfrak{m}\mathcal{G}}\mathcal{G} \geq \text{depth } \mathcal{G} - \dim \mathcal{G}/\mathfrak{m}\mathcal{G} = \text{depth } \mathcal{G} - \ell$. □

Remark 3.2.21 Let R be a Noetherian local ring, and let I be an R-ideal with analytic spread ℓ. Burch's inequality ([8]) states that

$$\inf\{\text{depth } R/I^j \mid j \geq 1\} + \ell \leq \dim R.$$

Hence, if $\mathcal{G}$ is Cohen-Macaulay, we have that

$$\text{depth } \mathcal{G} = \inf\{\text{depth } R/I^j \mid j \geq 1\} + \ell.$$

We call $\inf\{\text{depth } R/I^j \mid j \geq 1\}$ the *Burch number* of I and we denote it by $B(I)$.

Corollary 3.2.22 *With the assumptions of Theorem 3.2.10 we have that*

$\min(\{d - \ell\} \cup \{\text{depth } R/I^j - k + j \mid 1 \leq j \leq k\}) \leq B(I) \leq d - \ell$.

Proof. By Theorem 3.2.10 and Remark 3.2.20 we have that $\min(\{d\} \cup \{\text{depth } R/I^j + \ell - k + j \mid 1 \leq j \leq k\}) \leq \text{depth } \mathcal{G} \leq B(I) + \ell$. The conclusion follows since $B(I) = \text{depth}_{\mathfrak{m}\mathcal{G}}\mathcal{G} \leq \dim \mathcal{G} - \dim \mathcal{G}/\mathfrak{m}\mathcal{G}$. □

We will mainly use Theorem 3.2.10 when $s = \ell$ and J is a minimal reduction of I such that $\text{ht } J : I \geq \ell$ and $r_J(I) = r$. The next corollaries are special cases of Theorem 3.2.10, for small reduction number.

These special cases are particularly interesting because we can get a precise formula for depth $\mathcal{G}$.

Corollary 3.2.23 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number $r \leq 1$. Assume that I satisfies G_ℓ and $AN^-_{\ell-2}$. Then* depth $\mathcal{G} = \min\{d, \text{depth } R/I + \ell\}$.

Proof. The assertion follows from Theorem 3.2.10 with $s = \ell$, J a minimal reduction of I such that ht $J : I \geq \ell$ and $r_J(I) = r$, $k = 1$, and from Remark 3.2.20. $\square$

Remark 3.2.24 If we apply the previous corollary with $\ell = g$, the conditions G_ℓ and $AN^-_{\ell-2}$ are automatically satisfied. Hence Corollary 3.2.23 covers Theorem 3.2.8.

Corollary 3.2.25 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread $\ell \geq g+1$, and reduction number $r \leq 2$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-3}$ and that R/I is Cohen-Macaulay. Then* depth $\mathcal{G} = \min\{d, \text{depth } R/I^2 + \ell\}$.

Proof. The assertion follows from Theorem 3.2.10 with $s = \ell$, J a minimal reduction of I such that ht $J : I \geq \ell$ and $r_J(I) = r$, $k = 2$, and from Remark 3.2.20. $\square$

Remark 3.2.26 If we apply Theorem 3.2.10 with $s = g+1$ and $k = 2$, we get that $\min\{\text{depth } R/I + g, \text{depth } R/I^2 + g + 1\} \leq \text{depth } \mathcal{G} \leq \min\{\text{depth } R/I + g + 1, \text{depth } R/I^2 + g + 1\}$. So, by Remark 3.2.19, we have that Theorem 3.2.10 covers Theorem 3.2.9.

For reduction number 3 we get the following estimate of depth $\mathcal{G}$.

Corollary 3.2.27 *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread $\ell \geq g + 2$, and reduction number $r \leq 3$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-4}$, that R/I is Cohen-Macaulay, and that R/I^2 has no associated primes of height $\geq \ell$. Then* $\min\{d, \text{depth } R/I^2 + \ell - 1, \text{depth } R/I^3 + \ell\} \leq \text{depth } \mathcal{G} \leq \min\{\text{depth } R/I^2 + \ell, \text{depth } R/I^3 + \ell\}$.

Proof. The assertion follows from Theorem 3.2.10 with $s = \ell$, J a minimal reduction of I such that ht $J : I \geq \ell$ and $r_J(I) = r$, $k = 3$, and from Remark 3.2.20. □

3.3 Examples

In this section we give examples of classes of ideals to which Theorem 3.2.10 can be applied in order to compute the depth of the associated graded ring.

Example 3.3.1 Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with analytic spread ℓ, satisfying G_ℓ and $AN^-_{\ell-2}$, and let J be a minimal reduction of I. Since I satisfies G_ℓ and $AN^-_{\ell-2}$, by [51, Proposition 1.11] we have that ht $J : I \geq \ell$, and therefore by [51, Remark 1.12] J satisfies G_ℓ and $AN^-_{\ell-2}$. Clearly $r(J) = 0$. Let $\mathcal{G}(J)$ be the associated graded ring of J. Then by Corollary 3.2.23, we have that

$$\text{depth } \mathcal{G}(J) = \min\{d, \text{depth } R/J + \ell\}.$$

Now we present a class of ideals whose associated graded ring is not Cohen-Macaulay and we can use our results to compute its depth.

Example 3.3.2 Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with grade g, analytic spread $\ell \leq g+2$ and reduction number $r \neq 0$. Assume that I satisfies $G_{\ell+1}$ and that R/I is Cohen-Macaulay. Let J be a minimal reduction of I. By [51, Proposition 1.11] we have that ht $J : I \geq \ell+1$. As $J : I \neq R$ it follows that some associated prime of R/J has height at least $\ell+1$. Therefore depth $R/J \leq d - \ell - 1$. Let $\mathcal{G}(J)$ be the associated graded ring of J. By Example 3.3.1, we have that

$$\text{depth } \mathcal{G}(J) = \min\{d, \text{depth } R/J + \ell\} \leq d - 1.$$

In particular $\mathcal{G}(J)$ is not Cohen-Macaulay.

As a special case of the previous example, we have the following:

Example 3.3.3 Let $R = k[[x_1, \ldots, x_8]]$, where k is an infinite field. Let

$$\phi = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{pmatrix}$$

and let I be the ideal generated by the 2 by 2 minors of ϕ,

$$I = (x_1x_6 - x_2x_5, x_1x_7 - x_3x_5, x_1x_8 - x_4x_5, x_2x_7 - x_3x_6,$$

$$x_2x_8 - x_4x_6, x_3x_8 - x_4x_7).$$

The ideal I has grade 3 and analytic spread 5. Since I is a complete intersection on the punctured spectrum of R, I satisfies G_8. Furthermore, I satisfies AN_3^-, since R/I is Cohen-Macaulay. The ideal

$$J = (x_1x_6 - x_2x_5, x_1x_7 - x_3x_5, x_1x_8 - x_4x_5 + x_2x_7 - x_3x_6,$$

$$x_2x_8 - x_4x_6, x_3x_8 - x_4x_7)$$

is a minimal reduction of I. Since I and J coincide on the punctured spectrum of R, we have that $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathrm{Ass}(R/J)$ and so depth $R/J = 0$. By Example 3.3.2, we have that depth $\mathcal{G}(J) = 5$.

Now we present a class of equimultiple ideals of reduction number one whose associated graded ring is not Cohen-Macaulay.

Example 3.3.4 Let R be a local Gorenstein ring with infinite residue field, let p be a prime ideal of height $g \geq 2$ such that R_p is regular, and let $t \geq 1$ be an integer. Let $\alpha_1, \ldots, \alpha_g$ be a regular sequence contained in $p^{(t)}$, where $p^{(t)}$ denotes the t-th symbolic power of p; i.e., $p^{(t)} = p^t R_p \cap R$. Write $J = (\alpha_1, \ldots, \alpha_g)$ and set $I = J : p^{(t)} = J : p^t$. If either $g = 2$ or $t = 1$, assume that at least 2 of the α_i's are contained in $p^{(t+1)}$. By [46, Corollary 4.2] we have that $I^2 = JI$. Hence I is equimultiple of reduction number one. Assume that $R/p^{(t)}$ is not Cohen-Macaulay. Since I is linked to $p^{(t)}$, it follows that R/I is not Cohen-Macaulay. Let $\mathcal{G}$ be the associated graded ring of I. By Corollary 3.2.23, we have that

$$\text{depth } \mathcal{G} = \min\{d, \text{depth } R/I + g\} = \text{depth } R/I + g < d.$$

In particular $\mathcal{G}$ is not Cohen-Macaulay.

As a special case of the previous example, we have the following.

Example 3.3.5 Let $R = k[[x_1, x_2, x_3, x_4]]$, where k is an infinite field, and let p be the defining ideal of $k[[t^4, t^3 s, t s^3, s^4]]$,

$$p = (x_1 x_4 - x_2 x_3, x_1^2 x_3 - x_2^3, x_1 x_3^2 - x_4 x_2^2, x_3^3 - x_4^2 x_2).$$

The ideal p is prime of grade 2 and R/p is not Cohen-Macaulay. Let

$$J = ((x_1 x_4 - x_2 x_3)^2, (x_3^3 - x_4^2 x_2)^2) \subset p^2$$

and let

$$I = J : p =$$

$$((x_1x_4 - x_2x_3)^2, (x_3^3 - x_4^2x_2)^2, x_1x_4^5x_2 - x_1x_4^3x_3^3 - x_4^2x_2x_3^4 + x_3^7,$$

$$x_1x_4^4x_2x_3 - x_4^3x_2^2x_3^2 - x_1x_4^2x_3^4 + x_4x_2x_3^5, x_1x_4^3x_2x_3^2 - x_4^2x_2^2x_3^3 - x_1x_4x_3^5 + x_2x_3^6).$$

The ideal I is equimultiple of reduction number one. By Example 3.3.4, we have that depth $\mathcal{G}(I) = 3$.

CHAPTER 4

The Serre Properties of Blow-Up Rings of Ideals

In this chapter we study the Serre properties of the associated graded ring and of the Rees algebra of ideals having good residual intersection properties and sufficiently small reduction number.

Recall that a ring R satisfies the Serre condition S_t, where $t \geq 0$ is an integer, if for every $p \in \text{Spec } R$, depth $R_p \geq \min\{t, \dim R_p\}$.

Clearly if R is S_t, then R is S_k for every $k \leq t$, and if R is Cohen-Macaulay, then R is S_t for every t. Since in the setup of the previous chapter the associated graded ring $\mathcal{G}$ and the Rees algebra $\mathcal{R}$ are not necessarily Cohen-Macaulay, we ask which Serre properties are still satisfied in this case. We use the estimates for depth $\mathcal{G}$ and depth $\mathcal{R}$ from Theorem 3.2.10.

We first see how assumptions similar to those of Theorem 3.2.10

imply the Serre properties for $\mathcal{G}$.

Theorem 4.0.1 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, let J be a reduction of I generated by s elements with* $\text{ht } J : I \geq s$, $r = r_J(I)$, *and let $k \geq 0$ be an integer with $r \leq k$. Furthermore assume that I satisfies G_s, AN^-_{s-k-1} and that for some integer $t \geq 1$,* $\text{depth}(R/I^j)_p \geq \min\{\dim R_p - s + k - j, k - j + t\}$ *whenever $p \in V(I)$ and $1 \leq j \leq k$. Then $\mathcal{G}$ is S_t.*

Proof. We need to show that for every $\mathcal{P} \in \text{Spec } \mathcal{G}$, depth $\mathcal{G}_{\mathcal{P}} \geq \min\{t, \dim \mathcal{G}_{\mathcal{P}}\}$. Let q denote the contraction of $\mathcal{P}$ to R. By Theorem 3.2.10 we have that

$$\begin{aligned} \text{depth } \mathcal{G}_q &\geq \min(\{\text{ht } q\} \\ &\quad \cup \{\text{depth } R_q/I_q^j + s - k + j \mid 1 \leq j \leq k\}) \\ &\geq \min\{\text{ht } q, t + s\}. \end{aligned}$$

If ht $q \leq t + s$, then $\mathcal{G}_q$ is Cohen-Macaulay and so $\mathcal{G}_{\mathcal{P}}$ is Cohen-Macaulay. Hence we may assume that ht $q > t + s$. Since $\mathcal{G}_{\mathcal{P}}$ is a localization of $\mathcal{G}_q$, we have that

$$\dim \mathcal{G}_{\mathcal{P}} - \text{depth } \mathcal{G}_{\mathcal{P}} \leq \dim \mathcal{G}_q - \text{depth } \mathcal{G}_q.$$

As $\mathcal{G}_q$ is equidimensional and catenary, it follows that

$$\dim \mathcal{G}_q = \dim \mathcal{G}_q/\mathcal{P}\mathcal{G}_q + \dim \mathcal{G}_{\mathcal{P}}.$$

Hence

$$\text{depth } \mathcal{G}_{\mathcal{P}} \geq \text{depth } \mathcal{G}_q - \dim \mathcal{G}_q/\mathcal{P}\mathcal{G}_q.$$

But since $\mathcal{P}$ contracts to q, we have that

$$\dim \mathcal{G}_q/\mathcal{P}\mathcal{G}_q \leq \dim \mathcal{G}_q/q\mathcal{G}_q = \ell(I_q) \leq \ell \leq s.$$

So we conclude that depth $\mathcal{G}_{\mathcal{P}} \geq t + s - s = t$. □

In particular, if the reduction number is small, we have simpler assumptions that imply the S_1 property for $\mathcal{G}$.

Corollary 4.0.2 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number $r \leq 1$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-2}$, and that R/I has no associated primes of height $\geq \ell + 1$. Then $\mathcal{G}$ is S_1.*

Proof. The assertion follows from Theorem 4.0.1 with $s = \ell$, J a minimal reduction of I such that ht $J : I \geq \ell$ and $r_J(I) = r$, $k = 1$ and $t = 1$. □

Corollary 4.0.3 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread $\ell \geq g+1$, and reduction number $r \leq 2$. Further assume that I satisfies G_ℓ, $AN^-_{\ell-3}$, that R/I is Cohen-Macaulay, and that R/I^2 has no associated primes of height $\geq \ell + 1$. Then $\mathcal{G}$ is S_1.*

Proof. The assertion follows from Theorem 4.0.1 with $s = \ell$, J a minimal reduction of I such that ht $J : I \geq \ell$ and $r_J(I) = r$, $k = 2$ and $t = 1$. □

The S_1 property is particularly interesting because it leads to criteria for when $I^n = I^{(n)}$, where $I^{(n)}$ is the n-th symbolic power of I. We recall that $I^{(n)}$ is the intersection over all isolated primary components of the ordinary power I^n.

Let R be a Cohen-Macaulay ring and let I be an R-ideal. If $p \in V(I)$ is the contraction of a minimal prime $\mathcal{P}$ of $\mathcal{G}$, then $\ell(I_p) \geq \dim \mathcal{G}_p/\mathcal{P}_p\mathcal{G}_p = \dim \mathcal{G}_p$. The last equality holds since $\mathcal{G}_p$ is equidimensional. Hence $\ell(I_p) = \text{ht } p$.

Remark 4.0.4 *Let R be a Cohen-Macaulay ring and let I be an R-ideal. If $\mathcal{G}$ is S_1 and $\ell(I_p) < \text{ht } p$ for every non minimal prime p in $V(I)$, then $I^n = I^{(n)}$ for all $n \geq 1$.*

Proof. Since the inclusion $I^n \subset I^{(n)}$ always holds, it suffices to show that $I_p^{(n)} \subset I_p^n$ for every $p \in \text{Ass}(I^{i-1}/I^i)$ and $1 \leq i \leq n$. Let $p \in \text{Ass}(I^{i-1}/I^i)$. Then $p \subset \mathcal{P}$ for some $\mathcal{P} \in \text{Ass } \mathcal{G}$. Since $\mathcal{G}$ is S_1, $\mathcal{P}$ is a minimal prime of $\mathcal{G}$ and so its contraction p to R satisfies $\ell(I_p) = \text{ht } p$. Hence by assumption p is a minimal prime of I and so $I_p^{(n)} = I_p^n$, as desired. □

Remark 4.0.5 Let R be a Noetherian ring and let I be an R-ideal. Recall that I is *normally torsionfree* if $\mathcal{G}$ is R/I-torsionfree. If $I^n = I^{(n)}$ for every $n \geq 1$, then $\mathrm{Ass}(R/I^n) \subset \mathrm{Ass}(R/I)$ for every $n \geq 1$, and so if $x \in R/I$ is a zero divisor on $\mathcal{G}$, then x is a zero divisor on R/I; i.e., I is normally torsionfree. If I is a prime ideal the converse also holds and, if in addition I is generically a complete intersection, the two conditions are equivalent to $\mathcal{G}$ being a domain.

Combining Theorem 4.0.1, Corollary 4.0.2 and Corollary 4.0.3 with Remark 4.0.4 we get the following criterion for the equality of regular and symbolic powers of I.

Corollary 4.0.6 *If in addition to the assumptions of Theorem 4.0.1 with $t = 1$, or of Corollary 4.0.2, or of Corollary 4.0.3, we have that $\ell(I_p) <$ ht p for every non-minimal prime p in $V(I)$, then $I^n = I^{(n)}$ for all $n \geq 1$ and I is normally torsion free.*

Now we study the Serre properties of the Rees algebra. First we need the following definition.

Definition 4.0.7 Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g, and let $s \geq g$ be an integer.We say that I is *$s-$residually* S_2 if for every $g \leq i \leq s$ and every i-residual intersection K of I, R/K satisfies Serre's condition S_2.

To obtain the S_t property for $\mathcal{R}$, in addition to the assumptions that imply the S_t property for $\mathcal{G}$, we need a local condition on the residual intersections. Namely:

Theorem 4.0.8 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade $g \geq 2$, analytic spread ℓ, and reduction number r, let $k \geq 0$ be an integer with $r \leq k$. Assume that I satisfies G_ℓ, $AN^-_{\ell-k-1}$ and that for some integer $t \geq 1$,* $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + t\}$ *whenever $p \in V(I)$ and $1 \leq j \leq k$. Furthermore assume that I is $\ell - 2$-residually S_2 locally up to height $\ell + t - 1$. Then $\mathcal{R}$ is S_t.*

Proof. We need to show that for every $\mathcal{P} \in \operatorname{Spec} \mathcal{R}$, depth $\mathcal{R}_\mathcal{P} \geq \min\{t, \dim \mathcal{R}_\mathcal{P}\}$. Denote by q the contraction of $\mathcal{P}$ to R. By Theorem 3.2.10 we have that depth $\mathcal{G}_q \geq \min\{\text{ht } q, t + \ell\}$. If ht $q \geq t + \ell$, then depth $\mathcal{R}_q \geq$ depth $\mathcal{G}_q \geq t + \ell$. Since $\mathcal{R}_\mathcal{P}$ is a localization of $\mathcal{R}_q$, we have that

$$\dim \mathcal{R}_\mathcal{P} - \text{depth } \mathcal{R}_\mathcal{P} \leq \dim \mathcal{R}_q - \text{depth } \mathcal{R}_q.$$

As $\mathcal{R}_q$ is equidimensional and catenary, it follows that

$$\dim \mathcal{R}_q = \dim \mathcal{R}_q/\mathcal{P}\mathcal{R}_q + \dim \mathcal{R}_\mathcal{P}.$$

But since $\mathcal{P}$ contracts to q, we have that

$$\dim \mathcal{R}_q/\mathcal{P}\mathcal{R}_q \leq \dim \mathcal{R}_q/q\mathcal{R}_q = \ell(I_q) \leq \ell.$$

It follows that depth $\mathcal{R}_{\mathcal{P}} \geq t$.

If ht $q \leq t + \ell - 1$, then $\mathcal{G}_q$ is Cohen-Macaulay. We claim that also $\mathcal{R}_q$ is Cohen-Macaulay, which implies the Cohen-Macaulayness of $\mathcal{R}_{\mathcal{P}}$. By Theorem 3.2.6 to prove our claim we only need to show that $r(I_q) < \ell(I_q)$. If $\ell(I_q) = \ell$, then by assumption $r(I_q) \leq r \leq \ell - g + 1 \leq \ell - 1$, and so we are done. If $\ell(I_q) < \ell$, then $r(I_q) = 0 < \ell(I_q)$ by [9, 2.1(g)].

□

Now we analyze the relationship between the Serre properties for $\mathcal{R}$ and for $\mathcal{G}$. Brumatti, Simis and Vasconcelos related the property S_2 for $\mathcal{R}$ to the property S_1 for $\mathcal{G}$ in the following theorem.

Theorem 4.0.9 ([6, Theorem 1.5]) *Let R be a Noetherian ring satisfying S_2, and let I be an R-ideal of positive grade. The following two conditions are equivalent:*

(1) *$\mathcal{R}$ satisfies S_2.*

(2) (i) *$\mathcal{G}$ satisfies S_1, and*

(ii) *For every $p \in \operatorname{Spec}(R)$ with $\operatorname{ht} p = 1$, I_p is principal.*

The following theorem generalizes their result.

Theorem 4.0.10 *Let R be an equidimensional and universally catenary Noetherian ring satisfying S_t, and let I be an R-ideal of positive height. The following two conditions are equivalent:*

(1) $\mathcal{R}$ *satisfies* S_t.

(2) (i) $\mathcal{G}$ *satisfies* S_{t-1}, *and*

(ii) *If* $q \in V(I)$ *and* $\ell(I_q) = \text{ht } q \leq t-1$, *then* $r(I_q) < \ell(I_q)$.

Furthermore, if I *is* G_ℓ, *then* (2)(ii) *can be replaced by*

(2) (ii′) *If* $q \in V(I)$ *and* $\ell = \ell(I_q) = \text{ht } q \leq t-1$, *then* $r(I_q) < \ell(I_q)$.

Proof. (1) $\Rightarrow$ (2). First we verify that (1) implies (2)(i). Let P be a prime ideal of $\mathcal{G}$ and denote by $\mathcal{P}$ its inverse image in $\mathcal{R}$. Localize R at $p = R \cap \mathcal{P}$ and denote the resulting local ring by R. We want to show that depth $\mathcal{G}_P \geq \min\{t-1, \dim \mathcal{G}_P\}$. Since R is equidimensional and universally catenary and ht $I > 0$, we have that ht $\mathcal{P} = \text{ht } P + 1$. Furthermore ht $\mathcal{P} \leq \text{ht } p + 1$. Hence, if $\dim \mathcal{G}_P \geq t-1$, then $\dim \mathcal{R}_{\mathcal{P}} \geq t$ and so depth $\mathcal{R}_{\mathcal{P}} \geq t$ by (1). Also, depth $R_{\mathcal{P}} \geq t-1$ since R satisfies S_t. From the exact sequences

$$0 \to (It)\mathcal{R} \to \mathcal{R} \to R \to 0$$

and

$$0 \to I\mathcal{R} \to \mathcal{R} \to \mathcal{G} \to 0$$

it follows that depth $\mathcal{G}_P \geq t-1$. If dim $\mathcal{G}_P < t-1$, then dim $\mathcal{R}_{\mathcal{P}} < t$, and so $\mathcal{R}_{\mathcal{P}}$ is Cohen-Macaulay by (1). It follows that $\mathcal{G}_P$ is Cohen-Macaulay and the proof of (2)(i) is complete.

In the setting of (2)(ii) we have that $\dim R_q \leq t-1$, and so $\dim \mathcal{R}_q \leq t$. Hence $\mathcal{R}_q$ is Cohen-Macaulay by (1). The conclusion follows from [41, Theorem 2.3].

(2) $\Rightarrow$ (1). We may assume $t \geq 0$. Let $\mathcal{P}$ be a prime ideal of $\mathcal{R}$ and let $p = R \cap \mathcal{P}$. We want to show that depth $\mathcal{R}_{\mathcal{P}} \geq \min\{t, \dim \mathcal{R}_{\mathcal{P}}\}$. If $I \not\subset p$ we have $\mathcal{R}_p = R_p[t]$ which satisfies S_t since R_p does, and so $\mathcal{R}_{\mathcal{P}}$ satisfies S_t. Hence we may assume that $I \subset p$. If $It \not\subset \mathcal{P}$, by the usual prime avoidance argument there exists a nonzero divisor $x \in I$ such that $xt \notin \mathcal{P}$. We have that x is a regular element of $\mathcal{R}$. Since $t^{-1} \in \mathcal{R}_{xt}$, it follows that

$$(\mathcal{R}/x\mathcal{R})_{xt} = (R[It, t^{-1}]/xR[It, t^{-1}])_{xt} =$$

$$(R[It, t^{-1}]/t^{-1}R[It, t^{-1}])_{xt} = (\mathcal{G})_{xt}.$$

So the assumption (2)(i) implies that depth $\mathcal{R}_{\mathcal{P}} \geq \min\{t, \dim \mathcal{R}_{\mathcal{P}}\}$.

If $It \subset \mathcal{P}$, then $\mathcal{P}_p$ is the irrelevant maximal ideal of $\mathcal{R}_p$. If $\dim \mathcal{R}_p > t$, then $\dim \mathcal{G}_p > t-1$ and so depth $\mathcal{G}_p \geq t-1$ by (2)(i). If $\mathcal{G}_p$ is Cohen-Macaulay, then depth $\mathcal{R}_p \geq$ depth $\mathcal{G}_p > t-1$, if $\mathcal{G}_p$ is not Cohen-Macaulay, then depth $\mathcal{R}_p =$ depth $\mathcal{G}_p + 1$ by Theorem 3.2.7. In any case depth $\mathcal{R}_p \geq t$, and so depth $\mathcal{R}_{\mathcal{P}} \geq t$. If $\dim \mathcal{R}_p \leq t$, then $\dim R_p \leq t-1$. Hence R_p and $\mathcal{G}_p$ are Cohen-Macaulay. In this case $\mathcal{R}_p$ is Cohen-Macaulay by (2)(ii) and [41, Theorem 2.3]. If I is G_ℓ (2)(ii′),[41, Theorem 2.3] and [49, Theorem 2.4] imply that $\mathcal{R}_p$ is

Cohen-Macaulay and the proof is complete. □

Notice that condition (ii) is empty if $t \leq \text{ht } I$, and, if I is G_ℓ, condition (ii′) is empty if $t \leq \ell$. In these cases we have a simpler version of the theorem.

Corollary 4.0.11 *Let R be an equidimensional and universally catenary Noetherian ring satisfying S_t for some integer $t \geq 1$, and let I be an R-ideal with* ht $I \geq t$. *Then $\mathcal{R}$ satisfies S_t if and only if $\mathcal{G}$ satisfies S_{t-1}.*

Corollary 4.0.12 *Let R be an equidimensional and universally catenary Noetherian ring satisfying S_t, and let I be an R-ideal with positive height and analytic spread $\ell \geq t$. Assume that I satisfies G_ℓ. Then $\mathcal{R}$ satisfies S_t if and only if $\mathcal{G}$ satisfies S_{t-1}.*

Combining Theorem 4.0.1 with Theorem 4.0.10 we obtain another result on the Serre properties of $\mathcal{R}$ (compare with Theorem 4.0.8).

Theorem 4.0.13 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade $g > 0$, analytic spread ℓ, reduction number r, and let $k \geq 0$ be an integer with $r \leq k$. Furthermore assume that I satisfies G_ℓ, $AN^-_{\ell-k-1}$ and that for some integer*

$t \geq 1$, depth $(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + t - 1\}$ *whenever* $p \in V(I)$ *and* $1 \leq j \leq k$. *Finally suppose that if* $q \in V(I)$ *and* $\ell = \ell(I_q) =$ *ht* $q \leq t - 1$, *then* $r(I_q) < \ell(I_q)$. *Then* $\mathcal{R}$ *satisfies* S_t.

The S_2 property is very important in the study of the normality of $\mathcal{R}$. By Serre's normality criterion, a Noetherian ring R is normal if and only if R satisfies R_1 and S_2. We recall that a Noetherian ring R satisfies R_1 if R_p is regular for every $p \in \operatorname{Spec}(R)$ with ht $p \leq 1$.

Remark 4.0.14 ([6, Remark 2.3], Theorem 4.0.8). Let R be a polynomial ring in n variables (localized at the maximal irrelevant ideal) over an infinite perfect field, let I be an R-ideal with grade $g \geq 2$, analytic spread ℓ and reduction number r, and let $k \geq 0$ be an integer with $r \leq k$. Let J be the ideal of the presentation of $\mathcal{R}$; i.e., $\mathcal{R} \cong R[T_1, \ldots, T_m]/J$. Let $h_1, \ldots, h_s$ be a set of generators of J and consider the Jacobian matrix $M = \partial(h_1, \ldots, h_s)/\partial(x_1, \ldots, x_n, T_1, \ldots, T_m)$. Let N be the ideal generated by all $(m-1) \times (m-1)$ minors of M. If $\operatorname{ht}(J, N) \geq m + 1$, then $\mathcal{R}$ satisfies R_1. If further we assume that I satisfies G_ℓ and $AN^-_{\ell-k-1}$, that I is $\ell - 2$ residually S_2 locally up to height $\ell+1$, and that $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + 2\}$ for every $p \in V(I)$ and whenever $1 \leq j \leq k$, then $\mathcal{R}$ satisfies S_2 and hence is normal.

Recall that $B(I)$, the Burch number of I, is the value inf $\{\text{depth } R/I^j \mid j \geq 1\}$. We know that if $\mathcal{G}$ is Cohen-Macaulay, then $B(I) = \text{depth } \mathcal{G}$, (see Remark 3.2.21). Now we want to see how $B(I)$ is related to the S_t property for $\mathcal{G}$.

Lemma 4.0.15 *Let R be a local Cohen-Macaulay ring of dimension d and let I be an R-ideal with analytic spread ℓ. If $\mathcal{G}$ satisfies S_t for some positive integer $t \leq d - \ell$, then $B(I) \geq t$.*

Proof. Since $\mathcal{G}$ is equidimensional and catenary, we have that ht $\mathfrak{m}\mathcal{G} = d - \ell$; hence $B(I) = \text{depth}_{\mathfrak{m}\mathcal{G}}\mathcal{G} = \min\{\text{depth } \mathcal{G}_P \mid P \in V(\mathfrak{m}\mathcal{G})\} \geq t$. □

From Theorem 4.0.1 and Lemma 4.0.15 we get the following result.

Corollary 4.0.16 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, reduction number r, and let $k \geq 0$ be an integer with $r \leq k$. Furthermore assume that I satisfies G_ℓ, $AN^-_{\ell-k-1}$ and that for some integer t with $1 \leq t \leq d - \ell$,* depth $(R/I^j)_p \geq \min\{\dim R_p - s + k - j, k - j + t\}$ *whenever $p \in V(I)$ and $1 \leq j \leq k$. Then $B(I) \geq t$.*

Now we want to use the results obtained in this chapter to present another theorem on the Cohen-Macaulayness of $\mathcal{G}$ (compare with Theorem 3.2.1).

First we recall that a Noetherian ring R is *quasi Gorenstein* if $R \cong \omega_R$, where ω_R denotes the canonical module of R. If in addition R is Cohen-Macaulay, then R is *Gorenstein*.

We want to use the following criterion for the Cohen-Macaulayness of a ring R.

Lemma 4.0.17 ([32, Lemma 5.8]) *Let R be a quasi Gorenstein local ring, assume that for every $p \in \operatorname{Spec}(R)$ with* ht $p \geq 2$, depth $R_p \geq 1 + 1/2 \dim R_p$. *Then R is Cohen-Macaulay.*

Theorem 4.0.18 *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, reduction number r, and let $k \geq 0$ be an integer with $r \leq k$. Further assume that I is unmixed, generically a complete intersection, that I satisfies G_ℓ and $AN^-_{\ell-k-1}$, that depth $(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j,\ 1/2(\dim R_p - \ell + 1) + k - j\}$ whenever $p \in V(I)$ and $1 \leq j \leq k$, and that $\ell(I_p) <$ ht p for every non-minimal prime p in $V(I)$. Then $\mathcal{G}$ is Cohen-Macaulay.*

Proof. First notice that our assumption on the depth of the powers implies that for every $p \in V(I)$, $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j + 1\}$ whenever $1 \leq j \leq k$. Hence I is normally torsion free by

Corollary 4.0.6, and so $R[It, t^{-1}]$ is quasi Gorenstein by [40, the proof of Theorem 3.2].

Next we show that for every $\mathcal{P} \in \operatorname{Spec} R[It, t^{-1}]$ with ht $\mathcal{P} \geq 2$,

$$\operatorname{depth} R[It, t^{-1}]_{\mathcal{P}} \geq 1 + 1/2 \dim R[It, t^{-1}]_{\mathcal{P}}.$$

If $t^{-1} \notin \mathcal{P}$, then $R[It, t^{-1}]_{\mathcal{P}} = R[t, t^{-1}]_{\mathcal{P}}$ is Cohen-Macaulay and so the inequality is trivially satisfied. Hence we may assume that $t^{-1} \in \mathcal{P}$ and so $R[It, t^{-1}]_{\mathcal{P}}/(t^{-1})_{\mathcal{P}} \cong \mathcal{G}_{\mathcal{P}} \neq 0$. Let q denote the contraction of $\mathcal{P}$ to R, and notice that $I \subset q$.

By Theorem 3.2.10 we have that depth $\mathcal{G}_q \geq \min(\{\text{ht } q\} \cup \{1/2(\text{ht } q + \ell + 1) \mid 1 \leq j \leq k\})$. If ht $q \leq \ell + 1$, then $R[It, t^{-1}]_q$ is Cohen-Macaulay, and so $R[It, t^{-1}]_{\mathcal{P}}$ is Cohen-Macaulay. If ht $q > \ell + 1$, then the inequality

$$\dim R[It, t^{-1}]_{\mathcal{P}} - \operatorname{depth} R[It, t^{-1}]_{\mathcal{P}} \leq \dim R[It, t^{-1}]_q - \operatorname{depth} R[It, t^{-1}]_q$$

shows that

$$\begin{aligned}
\operatorname{depth} R[It, t^{-1}]_{\mathcal{P}} &\geq \dim \mathcal{G}_{\mathcal{P}} + 1 - \text{ht } q + \operatorname{depth} \mathcal{G}_q \\
&\geq \dim \mathcal{G}_{\mathcal{P}} + 3/2 - 1/2 \text{ ht } q + 1/2\, \ell \\
&= 1/2(\dim \mathcal{G}_{\mathcal{P}} + 1) + 1 \\
&+ 1/2(\dim \mathcal{G}_{\mathcal{P}} - \text{ht } q + \ell) \\
&\geq 1/2 \dim R[It, t^{-1}]_{\mathcal{P}} + 1,
\end{aligned}$$

where the last inequality holds since

$$\dim \mathcal{G}_{\mathcal{P}} - \dim \mathcal{G}_q = -\dim \mathcal{G}_q/\mathcal{P}\mathcal{G}_q \geq -\ell.$$

It follows from Lemma 4.0.17 that $R[It, t^{-1}]$ is Cohen-Macaulay; hence $\mathcal{G}$ is Cohen-Macaulay. □

Notice that the depth assumptions of Theorem 4.0.18 are weaker than those of Theorem 3.2.1, since in the above theorem we assume $\operatorname{depth}(R/I^j)_p \geq 1/2(\dim R_p - \ell + 1) + k - j$ for $1 \leq j \leq k$, if ht $p \geq \ell + 1$.

Monomial Varieties of Codimension 2

Let $k[u_1, \ldots, u_n]$ be a polynomial ring over an infinite field k. Consider the semigroup ring $k[u_1^{a_1}, u_2^{a_2}, \ldots, u_n^{a_n}, u_1^{c_1} \ldots u_n^{c_n}, u_1^{b_1} \ldots u_n^{b_n}] \subset k[u_1, \ldots, u_n]$, where $a_j, b_j, c_j \in \mathbf{N}_0$, $a_j > 0, (b_j, c_j) \neq (0,0)$ for $1 \leq j \leq n$, and, further $(b_1, \ldots, b_n) \neq (0, \ldots, 0)$ and $(c_1, \ldots, c_n) \neq (0, \ldots, 0)$. Let $I \subset R = k[x_1, \ldots, x_n, y, z]$ denote the defining ideal of this semigroup ring. Following Giménez, Morales and Simis ([19]) we say that I defines a *monomial variety of codimension* 2. The ideals defining monomial varieties of codimension 2 are a subset of the toric ideals; i.e., presentation ideals of semigroup algebras.

We want to apply our results to these ideals. We can prove that $\mathcal{G}$ is Cohen-Macaulay. Furthermore I is normally torsionfree if and only if I is a complete intersection locally in codimension 3.

The ideal I is prime of height two, and the variety in question is affine of codimension two. Furthermore, the analytic spread of I is equal to two if I is a complete intersection and equal to three in all the remaining cases ([18, Theorem 4.2]). Also, I has reduction number one ([5, Corollary 3.4]) and depth $R/I \geq n-1$ ([45, Theorem 2.3]). If I is a complete intersection, then $\mathcal{G}$ is trivially Cohen-Macaulay and normally torsionfree, since it is a polynomial ring over R/I. If I is not a complete intersection, notice that I satisfies G_3, since R is a regular ring. It follows from Corollary 3.2.23 that depth $\mathcal{G}$ = depth $R/I + 3 \geq n+2 = \dim R$. Hence $\mathcal{G}$ is Cohen-Macaulay.

If I is normally torsionfree, then by Remark 4.0.5 and by [30, Theorem 2.2] we have that $\ell(I_p) <$ ht p for every non minimal prime p in $V(I)$. It follows that if $p \in V(I)$ and ht $p = 3$, then $\ell(I_p) = 2$. Hence I_p is equimultiple and generically a complete intersection, and so by [13] it is a complete intersection.

If I is a complete intersection locally in codimension 3, then $\ell(I_p) <$ ht p for every non-minimal prime p of I. Since $\mathcal{G}$ satisfies S_1, it follows from Remark 4.0.4 and Remark 4.0.5 that I is normally torsionfree, that $I^n = I^{(n)}$ for every $n \geq 1$, and that $\mathcal{G}$ is a domain. Also, by [26, Proposition], $\mathcal{G}$ is Gorenstein.

For example, if $I \subset k[x,y,z,w]$ is the homogeneous ideal of a monomial curve in $\mathbf{P}^3$ lying on the quadric surface $xy - wz$, then I is a complete intersection on the punctured spectrum of R and so I is normally torsionfree, which recovers [43, Proposition 2.3]. In general a monomial curve in $\mathbf{P}^3$ need not be normally torsionfree. For example the ideal $I \subset k[x, y, z, w]$ defining $k[t^5, t^4u, t^3u^2, u^5]$ is not normally torsionfree, indeed $I^2 \neq I^{(2)}$.

Now let $R := k[x, y, z, w]$ and let $I \subset R$ be the homogeneous ideal of a projective monomial curve defined by

$$x = {u_1}^{a_1}, y = {u_2}^{a_1}, z = {u_1}^{b_1}{u_2}^{a_1 - b_1}, w = {u_1}^{c_1}{u_2}^{a_1 - c_1},$$

$(a_1 > b_1 > c_1)$. We want to find necessary and sufficient conditions for I to be normally torsionfree, in terms of the exponents a_1, b_1 and c_1.

Since I is normally torsionfree if and only if I is a complete intersection locally in codimension 3, take $p \in \mathrm{V}(I)$ with ht $p = 3$ and denote by $\bar{p}$ its image in R/I. Since ht $\bar{p} = 1$, either $u_1^{a_1} \notin \bar{p}$ or $u_2^{a_1} \notin \bar{p}$. If $u_1^{a_1} \notin \bar{p}$, then

$$k[u_1^{a_1}, u_2^{a_1}, {u_1}^{b_1}{u_2}^{a_1 - b_1}, {u_1}^{c_1}{u_2}^{a_1 - c_1}]_{\bar{p}} = k[x^{a_1}, x^{a_1 - b_1}, x^{a_1 - c_1}]_{\bar{p}^{\mathrm{a}}},$$

if $u_2^{a_1} \notin \bar{p}$, then

$$k[u_1^{a_1}, u_2^{a_1}, {u_1}^{b_1}{u_2}^{a_1 - b_1}, {u_1}^{c_1}{u_2}^{a_1 - c_1}]_{\bar{p}} = k[x^{a_1}, x^{b_1}, x^{c_1}]_{\bar{p}^{\mathrm{a}}},$$

where we denote by $\bar{p}^{\mathrm{a}}$ the dehomogenization of the ideal $\bar{p}$. The ideal

$(I^{\mathrm{a}})_{\bar{p}^{\mathrm{a}}}$ is a complete intersection if and only if $k[x^{a_1}, x^{a_1-b_1}, x^{a_1-c_1}]$ and $k[x^{a_1}, x^{b_1}, x^{c_1}]$ are Gorenstein.

We denote by $<v_1, \ldots, v_n>$ the semigroup S generated by the integers $v_1, \ldots, v_n$, following Bruns and Herzog ([7, page 178]). The conductor $c = c(S)$ of S is defined by

$$c = \max\{a \in \mathbf{N} \mid a - 1 \notin S\}.$$

We say that the semigroup S is *symmetric* if, for all i with $0 \leq i \leq c-1$, one has $i \in S$ if and only if $c - i - 1 \notin S$. By [7, Theorem 4.4.8], S is symmetric if and only if $k[t^{v_1}, \ldots, t^{v_n}]$ is Gorenstein.

Hence we have that I is normally torsion free if and only if $<a_1, b_1, c_1>$ and $<a_1, a_1 - b_1, a_1 - c_1>$ are symmetric.

Notice that if I is the above mentioned ideal defining $k[t^5, t^4u, t^3u^2, u^5]$, then $<a_1, b_1, c_1> = <5,\ 4,\ 3>$ is not symmetric. Hence I is not normally torsionfree.

CHAPTER 5

Cohen-Macaulayness of the Fiber Cone

Let $\mathcal{F}$, $\mathcal{R}$ and $\mathcal{G}$ denote the fiber cone, the Rees algebra and the associated graded ring of an ideal I in a local Cohen-Macaulay ring R. In this chapter we study the Cohen-Macaulay property of $\mathcal{F}$ and we relate it to the Cohen-Macaulayness of $\mathcal{R}$ and $\mathcal{G}$. In particular we will give examples for perfect ideals of height two. For most of the computations we used MACAULAY.

Recall that if $\mu(I) = n$, then

$$\mathcal{R} \cong R[T_1, \ldots, T_n]/Q$$

where Q is an ideal of $R[T_1, \ldots, T_n]$, and

$$\mathcal{F} \cong R[T_1, \ldots, T_n]/(\mathfrak{m}, Q) \cong k[T_1, \ldots, T_n]/\bar{Q},$$

where "$^-$" denotes images in $k = R/\mathfrak{m}$, the residue field of R. In particular, since $[\mathcal{R}]_1 = [S(I)]_1$, all the linear polynomials in Q have coefficients in $\mathfrak{m}$. Hence all the homogeneous relations of $\mathcal{F}$ have degree at least two.

When $r(I) = 0$, then $\mathcal{F}$ is a polynomial ring over k, $\mathcal{F} = k[T_1, \ldots, T_\ell]$, where ℓ is the analytic spread of I. In this case $\mathcal{F}$ is trivially Cohen-Macaulay. Some results have been obtained in the literature for ideals with reduction number at most one. One of the most general ones is the following theorem of Cortadellas and Zarzuela.

Theorem 5.0.1 ([11, Theorem 3.2]) *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number at most one. Assume that I satisfies $AN_{\ell-2}^{-}$, and that $r_i \leq \max\{0, i-\ell+1\}$ for all $g \leq i < \ell$, where r_i's are defined as in Definition 3.1.4. Furthermore assume that* depth $R/I \geq d - \ell$. *Then $\mathcal{F}$ is Cohen-Macaulay.*

Recently Huneke and Hübl proved the following theorem for ideals with analytic deviation one, which does not have any restriction on the reduction number.

Theorem 5.0.2 ([33, Theorem 2.1]) *Let R be a local Cohen-Macaulay ring of dimension d, let I be an unmixed R-ideal of height $d-1$ and analytic spread d. Assume that I is generically a complete intersection and that for all minimal reductions J of I, $\mathcal{F}$ has no homogeneous* *generating relations in degree* $\leq r_J(I)$. *Furthermore assume that the grade of* $\mathcal{G}_+$ *is* $d-1$, *where* $\mathcal{G}_+$ *is the ideal generated by homogeneous elements in* $\mathcal{G}$ *of positive degree. Then* $\mathcal{F}$ *is Cohen-Macaulay.*

The main result of this chapter unifies and generalizes the above theorems.

Theorem 5.0.3 *Let* R *be a Noetherian local ring with infinite residue field, let* I *be an* R*-ideal with analytic spread* ℓ, *minimal number of generators* n, *reduction number* r, *and let* $k \geq 0$ *be an integer with* $r \leq k$. *Let* $a_1, \ldots, a_\ell$ *be general elements in* I *and let* $\mathfrak{a}_i = (a_1, \ldots, a_i)$. *Assume that* $[\mathfrak{a}_i I^j : a_{i+1}] \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever* $0 \leq i \leq \ell - 1$ *and* $j \geq k$. *Furthermore assume that* $\mathcal{F}$ *has at most two homogeneous generating relations in degree* $\leq k$ *if* $n - \ell \geq 2$, *and that* $\mathcal{F}$ *has at most one homogeneous generating relations in degree* $\leq k$ *if* $n = \ell + 1$. *Then* $\mathcal{F}$ *is Cohen-Macaulay.*

Proof. Write $\mathcal{F} = k[T_1, \ldots, T_n]/J$ and let $(J_{\leq k})$ denote the ideal generated by the forms in J that have degree less than or equal to k. If $k = 0$, then $\mathcal{F}$ is a polynomial ring over a field, and so it is Cohen-Macaulay. Hence we may assume that $k > 0$. If $n - \ell \geq 2$, then by assumption $\mu(J_{\leq k}) \leq 2$. Hence we have that the projective dimension of $k[T_1, \ldots, T_n]/(J_{\leq k})$ is less than or equal to 2 and so depth

$k[T_1, \dots, T_n]/(J_{\leq k}) \geq n-2 \geq \ell$. Similarly if $n = \ell + 1$, we have that $\mu(J_{\leq k}) \leq 1$, and depth $k[T_1, \dots, T_n]/(J_{\leq k}) \geq n-1 = \ell$. In any case we may assume that the images $a'_1, \dots, a'_\ell$ of $a_1, \dots, a_\ell$ in $[\mathcal{F}]_1 = I/\mathfrak{m}I$ form a regular sequence in the ring $k[T_1, \dots, T_n]/(J_{\leq k})$.

CLAIM: $\mathfrak{a}_i I^j \cap \mathfrak{m} I^{j+1} = \mathfrak{a}_i \mathfrak{m} I^j$ whenever $0 \leq i \leq \ell$ and $j \geq 0$.

Proof of the claim by induction on j.

Case 1: $j \leq k-1$.

Let $\lambda_1 a_1 + \cdots + \lambda_i a_i \in \mathfrak{m} I^{j+1}$, with $\lambda_1, \dots, \lambda_i \in I^j$. We denote by $\lambda'_1, \dots, \lambda'_i$ the images of $\lambda_1, \dots, \lambda_i$ in $I^j/\mathfrak{m}I^j$. Then $\lambda'_1 a'_1 + \cdots + \lambda'_i a'_i = 0$ in $k[T_1, \dots, T_n]/(J_{\leq k})$, since $j+1 \leq k$. As $a'_1, \dots, a'_i$ form a regular sequence in this ring, there is an alternating $i \times i$ matrix A with entries in I^{j-1} so that $[\lambda'_1, \dots, \lambda'_i] = [a'_1, \dots, a'_i]A'$. Here A' denotes the image of A in $I^{j-1}/\mathfrak{m}I^{j-1}$. Hence $[\lambda_1, \dots, \lambda_i] = [a_1, \dots, a_i]A$ modulo $\mathfrak{m}I^j$. As $[a_1, \dots, a_i]A[a_1, \dots, a_i]^t = 0$, it follows that $\lambda_1 a_1 + \cdots + \lambda_i a_i \in \mathfrak{a}_i \mathfrak{m} I^j$.

Case 2: $j \geq k$.

We use decreasing induction on i. If $i = \ell$, we are done since $j \geq r$.

If $i < \ell$, then

$$\begin{aligned}
\mathfrak{a}_i I^j \cap \mathfrak{m} I^{j+1} &= \mathfrak{a}_i I^j \cap \mathfrak{m} I^{j+1} \cap \mathfrak{a}_{i+1} I^j \\
&= \mathfrak{a}_i I^j \cap \mathfrak{a}_{i+1} \mathfrak{m} I^j \\
&= \mathfrak{a}_i \mathfrak{m} I^j + a_{i+1}[(\mathfrak{a}_i I^j : a_{i+1}) \cap \mathfrak{m} I^j] \\
&\subset \mathfrak{a}_i \mathfrak{m} I^j + a_{i+1}(\mathfrak{a}_i I^{j-1} \cap \mathfrak{m} I^j) \\
&= \mathfrak{a}_i \mathfrak{m} I^j + a_{i+1} \mathfrak{m} I^{j-1} \\
&= \mathfrak{a}_i \mathfrak{m} I^j,
\end{aligned}$$

and this finishes the proof of the claim.

In order to prove that $\mathcal{F}$ is Cohen-Macaulay, since $a'_1, \ldots, a'_\ell$ are a regular sequence on the ring $k[T_1, \ldots, T_n]/(J_{\leq k})$, it suffices to show that for $1 \leq i \leq \ell$ and $j \geq k$,

$$[(a'_1, \ldots, a'_{i-1}) :_{\mathcal{F}} a'_i]_j = [(a'_1, \ldots, a'_{i-1})]_j$$

in $\mathcal{F}$, or equivalently that

$$[(\mathfrak{a}_{i-1} I^j + \mathfrak{m} I^{j+1}) : a_i] \cap I^j \subset \mathfrak{a}_{i-1} I^{j-1} + \mathfrak{m} I^j.$$

But

$$\begin{aligned}
[(\mathfrak{a}_{i-1} I^j + \mathfrak{m} I^{j+1}) : a_i] \cap I^j &\subset [(\mathfrak{a}_{i-1} I^j + \mathfrak{a}_i \mathfrak{m} I^j) : a_i] \cap I^j \\
&= [(\mathfrak{a}_{i-1} I^j + a_i \mathfrak{m} I^j) : a_i] \cap I^j \\
&\subset (\mathfrak{a}_{i-1} I^j : a_i) \cap I^j + \mathfrak{m} I^j \\
&\subset \mathfrak{a}_{i-1} I^{j-1} + \mathfrak{m} I^j.
\end{aligned}$$

□

Notice that if I has second analytic deviation one and k is the smallest degree of a generating relation in $\mathcal{F}$, then $r(I) = k - 1$. Hence $\mathcal{F}$ does not have any generating relation in degree $\leq r$, and we have a simpler version of the previous theorem.

Theorem 5.0.4 *Let R be a Noetherian local ring with infinite residue field, let I be an R-ideal with analytic spread ℓ, minimal number of generators n with $n = \ell + 1$, reduction number r, let $a_1, \ldots, a_\ell$ be general elements in I and let $\mathfrak{a}_i = (a_1, \ldots, a_i)$. Assume that $[\mathfrak{a}_i I^j : a_{i+1}] \cap I^j = \mathfrak{a}_i I^{j-1}$ whenever $0 \leq i \leq \ell - 1$ and $j \geq r$. Then $\mathcal{F}$ is Cohen-Macaulay.*

Remark 5.0.5 Theorem 5.0.3 covers Theorem 5.0.1 and Theorem 5.0.2. Indeed the assumptions of Theorem 5.0.1 and of Theorem 5.0.2 imply the desired intersections $[\mathfrak{a}_i I^j : a_{i+1}] \cap I^j = \mathfrak{a}_i I^{j-1}$ (see [11, Lemma 2.5] and [33, the proof of Theorem 2.1]).

We now recall two lemmas that imply the assumptions of Theorem 5.0.3.

Lemma 5.0.6 ([37, Lemma 2.2]) *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal of grade g, analytic* *spread ℓ. Assume that I satisfies G_ℓ, let $\mathfrak{a}_i$ be the ideals defined in Theorem 5.0.3, and assume that $\mathcal{G}$ is Cohen-Macaulay. Then $[\mathfrak{a}_i : a_{i+1}] \cap I^j = \mathfrak{a}_i I^{j-1}$ whenever $0 \leq i \leq \ell - 1$ and $j \geq i - g + 1$.*

Lemma 5.0.7 (Lemma 3.1.7) *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, reduction number r, let $k \geq 0$ be an integer with $r \leq k$, assume that I satisfies G_ℓ, $AN^-_{\ell-k-1}$ and that for every $p \in V(I)$,* $\operatorname{depth}(R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j\}$ *whenever $1 \leq j \leq k-1$. Let $\mathfrak{a}_i$ be the ideals defined in Theorem 5.0.3. Then $[\mathfrak{a}_i : a_{i+1}] \cap I^j = \mathfrak{a}_i I^{j-1}$ whenever $0 \leq i \leq \ell - 1$ and $j \geq \max\{1, i-\ell+k+1\}$.*

Combining Theorem 5.0.3 with Lemma 5.0.6 we get the following corollary, that relates the Cohen-Macaulayness of $\mathcal{G}$ with the Cohen-Macaulayness of $\mathcal{F}$.

Corollary 5.0.8 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, minimal number of generators n and reduction number r. Assume that I satisfies G_ℓ. Furthermore assume that $\mathcal{F}$ has at most two homogeneous generating relations in degree $\leq \max\{r, \ell - g\}$ if $n - \ell \geq 2$, and that $\mathcal{F}$ has at most one homogeneous generating relation in degree*

$\leq \max\{r, \ell - g\}$ *if* $n = \ell + 1$. *If* $\mathcal{G}$ *is Cohen-Macaulay, then* $\mathcal{F}$ *is Cohen-Macaulay.*

Proof. The statement follows from Lemma 5.0.6 and Theorem 5.0.3 with $k = \max\{r, \ell - g\}$. □

Suppose that $r \geq \ell - g$. This is the case for instance if I is equimultiple or has analytic deviation one. Again we point out that if I has second analytic deviation one, then the assumption "$\mathcal{F}$ has at most one homogeneous generating relation in degree $\leq r$" is automatically satisfied. So we get a simpler version of the previous corollary.

Corollary 5.0.9 *Let* R *be a local Cohen-Macaulay ring with infinite residue field, let* I *be an* R*-ideal with grade* g, *analytic spread* ℓ, *and reduction number* $r \geq \ell - g$. *Assume that* I *satisfies* G_ℓ *and that* $\mathcal{F}$ *has at most two homogeneous generating relations in degree* $\leq r$. *If* $\mathcal{G}$ *is Cohen-Macaulay, then* $\mathcal{F}$ *is Cohen-Macaulay.*

Combining Theorem 5.0.3 with Lemma 5.0.7 we get the following corollary.

Corollary 5.0.10 *Let* R *be a local Cohen-Macaulay ring with infinite residue field, let* I *be an* R*-ideal with grade* g, *analytic spread* ℓ, *minimal number of generators* n, *reduction number* r, *and let* $k \geq 0$ *be* *an integer with* $r \leq k$. *Assume that* I *satisfies* G_ℓ, $AN^-_{\ell-k-1}$ *and that for every* $p \in V(I)$, $\operatorname{depth}\ (R/I^j)_p \geq \min\{\dim R_p - \ell + k - j, k - j\}$ *whenever* $1 \leq j \leq k-1$. *Assume that* $\mathcal{F}$ *has at most two homogeneous generating relations in degree* $\leq k$ *if* $n - \ell \geq 2$, *and that* $\mathcal{F}$ *has at most one homogeneous generating relation in degree* $\leq k$ *if* $n = \ell + 1$. *Then* $\mathcal{F}$ *is Cohen-Macaulay.*

Next, we use an example of D'Anna, Guerrieri and Heinzer to point out that the assumption "$\mathcal{F}$ has at most two homogeneous generating relations in degree $\leq \max\{r, \ell - g\}$" in Corollary 5.0.8 can not be removed or weakened.

Example 5.0.11 ([14, Example 2.3]) Let $R = k[t^6, t^{11}, t^{15}, t^{31}]$, where k is an infinite field, and let $I = (t^6, t^{11}, t^{31})$. R is a Cohen-Macaulay ring, I is an ideal of grade 1, analytic spread 1, reduction number 2, and second analytic deviation 2. $\mathcal{G}$ is Cohen-Macaulay but $\mathcal{F}$ is not Cohen-Macaulay. One has

$$\mathcal{F} = k[T_1, T_2, T_3]/(T_2^3, T_1T_3, T_2T_3, T_3^2);$$

so $\mathcal{F}$ has 3 generating relations in degree 2.

In particular the above example shows that in general $\mathcal{G}$ Cohen-Macaulay does not imply $\mathcal{F}$ Cohen-Macaulay. In Example 5.0.11 $\mathcal{R}$

is not Cohen-Macaulay; so it is natural to ask if in general $\mathcal{R}$ Cohen-Macaulay implies $\mathcal{F}$ Cohen-Macaulay. We get a negative answer to this question.

Example 5.0.12 Let R and I be as in Example 5.0.11. By adding two variables x and y we obtain the ideal $I' = (I, x, y) \subset R[x, y]$. Now I' has grade 3, analytic spread 3, reduction number 2, and $\mathcal{G}(I')$ is a polynomial ring over $\mathcal{G}(I)$; thus Cohen-Macaulay. Hence $\mathcal{R}(I')$ is Cohen-Macaulay by Theorem 3.2.6, but $\mathcal{F}(I')$ is a polynomial ring over $\mathcal{F}(I)$, and so it is not Cohen-Macaulay.

Let R be a local Cohen-Macaulay ring and let I be a strongly Cohen-Macaulay ideal with grade g and analytic spread ℓ, satisfying G_ℓ. By [24, the proof of Theorem 4.6] the first $\ell - g + 1$ symmetric powers of I have no torsion; i.e., $\mathcal{F}$ does not have any relation in degree less than or equal to $\ell - g + 1$. Hence we obtain better results for strongly Cohen-Macaulay ideals, and in particular for perfect ideals of height two. We have the following corollary.

Corollary 5.0.13 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a strongly Cohen-Macaulay ideal of grade g, analytic spread ℓ, and reduction number r with $r \leq \ell - g + 1$. Assume that I satisfies G_ℓ. Then $\mathcal{F}$ is Cohen-Macaulay.*

Proof. The statement follows from Theorem 2.4.4, Remark 3.2.3 and Corollary 5.0.10 with $k = \ell - g + 1$. □

The next example shows that the above result is not true even for perfect ideals of height two with second analytic deviation one, if the reduction number is not the "expected" one.

Example 5.0.14 Let

$$\phi = \begin{pmatrix} x^3 & 0 & 0 \\ y^2 & 0 & yz \\ 0 & y^2 & z^2 \\ 0 & z^2 & x^2 \end{pmatrix}$$

be a matrix with entries in $k[[x, y, z]]$, where k is an infinite field. Let I be the ideal generated by the 3 by 3 minors of ϕ. I is perfect of height two, it has analytic spread three, and reduction number five. Also, I satisfies G_3. The fiber cone

$$\mathcal{F} = k[T_1, T_2, T_3, T_4]/(T_2^5 T_4, T_2^4 T_4^2)$$

is not Cohen-Macaulay.

Next, we recall an important result for perfect ideals of height two, that has been very useful in building examples.

Theorem 5.0.15 ([52, Corollary 5.4]) *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect R-ideal of height 2 with analytic spread ℓ, reduction number r, let ϕ be an n by $n-1$ matrix presenting I, and let ϕ' be the $n-\ell$ by $n-1$ matrix consisting of the last $n-\ell$ rows of ϕ. Assume that I satisfies G_ℓ. The following are equivalent:*

(a) *$\mathcal{R}$ is Cohen-Macaulay.*

(b) *$r < \ell$ (in which case $r = 0$ or $r = \ell - 1$).*

(c) *After elementary row operations on ϕ, $I_{n-\ell}(\phi') = I_{n-\ell}(\phi)$.*

We refer to the condition (c) of Theorem 5.0.15 as the "row condition".

In particular, by Corollary 5.0.13 and Theorem 3.2.6 we have.

Corollary 5.0.16 *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a perfect R-ideal of grade 2 and analytic spread ℓ. Assume that I satisfies G_ℓ. If $\mathcal{R}$ is Cohen-Macaulay, then $\mathcal{F}$ is Cohen-Macaulay.*

The converse of the above result is not true; i.e., for perfect ideals of height two satisfying G_ℓ, $\mathcal{F}$ Cohen-Macaulay does not imply that $\mathcal{R}$ is Cohen-Macaulay. It is easy to build counterexamples for ideals with second analytic deviation one, because in this case, if I is generated by homogeneous polynomials of the same degree in a power series ring over a field, $\mathcal{F}$ is an hypersurface ring and so it is always Cohen-Macaulay. However, $\mathcal{R}$ is not Cohen-Macaulay if the row condition is not satisfied (see Example 5.0.17).

We recall that since in the following examples we work in power series rings over a field, which are regular, the Cohen-Macaulayness of $\mathcal{R}$ and $\mathcal{G}$ are equivalent (see Remark 3.2.5).

Example 5.0.17 Let $I \subset k[[x,y]]$, where k is an infinite field, be the ideal generated by the 2 by 2 minors of

$$\phi = \begin{pmatrix} 0 & y^2 \\ y^2 & x^2 \\ x^2 & xy \end{pmatrix}.$$

Then

$$\mathcal{F} = k[T_1, T_2, T_3]/(T_1^3T_2 - T_2^4 + 2T_1T_2^2T_3 - T_1^2T_3^2)$$

is Cohen-Macaulay, but $\mathcal{R}$ and $\mathcal{G}$ are not Cohen-Macaulay, since the row condition of Theorem 5.0.15 is not satisfied.

However, for perfect ideals of grade two satisfying G_ℓ, $\mathcal{F}$ Cohen-Macaulay does not imply that $\mathcal{R}$ is Cohen-Macaulay even if the second analytic deviation is greater than one.

Example 5.0.18 Let $I \subset k[[x,y]]$, where k is an infinite field, be the ideal generated by the 3 by 3 minors of

$$\phi = \begin{pmatrix} x^2 & 0 & xy \\ y^2 & x^2 & 0 \\ 0 & y^2 & x^2 \\ 0 & 0 & y^2 \end{pmatrix}.$$

Then

$$\mathcal{F} = k[T_1, T_2, T_3, T_4]/(T_3^2 - T_2T_4, T_2^3 - 2T_1T_2T_3 + T_1^2T_4 - T_3T_4^2)$$

is Cohen-Macaulay, but $\mathcal{R}$ and $\mathcal{G}$ are not Cohen-Macaulay, because the row condition of Theorem 5.0.15 is not satisfied, since $\mu(I_2(\phi)) > 3$.

Notice that in the previous example $\mathcal{F}$ is a complete intersection. So it is natural to ask what happens if $\mathcal{F}$ is not a complete intersection. Still, $\mathcal{F}$ Cohen-Macaulay does not imply that $\mathcal{R}$ is Cohen-Macaulay.

Example 5.0.19 Let $I \subset k[[x,y]]$, where k is an infinite field, be the ideal generated by the 3 by 3 minors of

$$\phi = \begin{pmatrix} x^3 & 0 & x^2y \\ y^3 & x^3 & 0 \\ 0 & y^3 & x^3 \\ 0 & 0 & y^3 \end{pmatrix}.$$

Then $\mathcal{F} = k[T_1, T_2, T_3, T_4]/(T_3^2 - T_2T_4, T_2^4T_3 - 3T_1T_2^2T_4 + 3T_1^2T_2T_3T_4 - T_1^3T_4^2 + T_2T_4^4, T_2^5 - 3T_1T_2^3T_3 + 3T_1^2T_2^2T_4 - T_1^3T_3T_4 + T_2T_3T_4^3)$ is Cohen-Macaulay, not a complete intersection, but $\mathcal{R}$ and $\mathcal{G}$ are not Cohen-Macaulay, because the row condition of Theorem 5.0.15 is not satisfied, since $\mu(I_2(\phi)) > 3$.

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