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By

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ABSTRACT

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The existence of time periodic solutions of nonlinear wave equations $u_{tt} - \Delta_n u + (\frac{n-1}{2})^2 u = g(u) - f(t,x)$ on n-dimensional spheres is considered. The corresponding functional of the equation is studied by the convexity in suitable subspaces, minimax arguments for almost symmetric functional, some comparison principles and Morse theory. The existence of infinitely many time periodic solutions is obtained with suitable assumptions on the growth of the nonlinear term $g(u)$ when the non-symmetric perturbation f is not small.

To my parents.

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Introduction

During the past three decades, the initial value or Cauchy problem has played the central role in the theory of evolutionary differential equations, which describe many fundamental physical processes of interaction. The Cauchy problem has been studied extensively with considerable success. In spite of a great deal of recent activity, many physically and mathematically important difficult problems still remain, even when global existence and unicity have been well established. Among the most interesting problems of this type are those of the existence, regularity and stability of time-periodic solutions.

This dissertation is focused on the nonlinear wave equation

$$Au = g(u) - f(t,x),\ (t,x) \in S^1 \times S^n,\ n > 1, \tag{1}$$

where $Au = u_{tt} - \triangle_n u + (\frac{n-1}{2})^2 u$, and $f(t,x)$ is 2π-periodic function in t. We are concerned with the existence of multiple 2π-periodic solutions for the case where g is superlinear, i.e., $g(\xi)/\xi \to \infty$ as $|\xi| \to \infty$.

The existence of time-periodic solutions has always played an important role in the theory of differential equations and mathematical physics. Even the existence of the periodic solutions for nonlinear ordinary differential equations is nontrivial, requiring Poincare-Bendixson theory to study the periodic orbits of general 2-dimensional autonomous systems. In this case, periodic orbits together with the steady state significantly influence the behavior of all other orbits. There is no question that the existence of periodic solutions for partial differential equations is a much harder problem.

The reason we choose the compact space S^n instead of the usual $\mathbb{R}^n$ is motivated by several considerations. First, S^n is a naturally curved physical space going back to Einstein space. There is no reason to believe that the complete flat $\mathbb{R}^n$ is a better choice for consideration. Secondly, the usual Minkowski space $\mathbb{R} \times \mathbb{R}^n$ can be conformally embedded into the Einstein Universe $\mathbb{R} \times S^n$, with the usual wave operator $\Box_0 = \partial_t^2 - \Delta_{\mathbb{R}^n}$ is transforming into the operator A in this dissertation. Third, some recent developments in constructive quantum field theory [13, 14, 21, 22, 26] are based on the analysis of the Einstein Universe $\mathbb{R} \times S^n$, which requires us to understand the classical differential equations on it. Also, we want to point out that many simple interactions like $g(u) = u^3$ on $\mathbb{R} \times \mathbb{R}^n$ have no periodic solutions because of the necessary decay properties of their solutions.

The main difficulty of problem (1) is the lack of compactness. When n is odd, the null space of A is infinite dimensional, and the component of u in this eigenspace is very difficult to control. This fact makes the problem much harder than an elliptic equation $\Delta u = g(x, u)$, or than a Hamiltonian system in which every eigenspace is finite dimensional. The associated functional of (1) is indefinite in a very strong sense. In particular, it is not bounded from above or from below, and it does not satisfy the Palais-Smale compactness condition in any reasonable space.

In the case of $n = 1$, Bahri, Brezis, Coron, Nirenberg and Rabinowitz [5, 8, 9, 10, 16] have proved the existence of nontrivial periodic solutions of (1) under reasonable assumptions on g(u) at u=0 and u at infinity, and the monotonicity of g. For $n > 1$, Benci and Fortunato [7] proved by using the dual variational method that (1) possesses infinitely many 2π-periodic solutions in L^p in the case $g(u) = |u|^{p-2}u$, $2 < p < 2 + \frac{2}{n}$ and $f = 0$. The existence of a nontrivial periodic solution in the case of $g(0) = 0$

and $f = 0$, and the existence of multiple, in some cases infinitely many, time periodic solutions for several classes of nonlinear terms which satisfy symmetry and growth conditions were established in Zhou [29, 30]. These conditions include time translation invariance or oddness; $f = 0$ and $g(u) \sim |u|^{p-2}u$ as $u \to \infty$, $(2 < p < \frac{2(n+1)}{n-1})$. Their proofs involved variational methods; a suitable and complicated approximation scheme; index and pseudo-index theory; Sobolev type embedding theorem for the operator A and the best estimate on the spherical harmonics obtained by Sogge [23]. The monotonicity of g played an essential role in their proof to compensate for the infinite dimensional null space of A.

In this dissertation, we are going to study the effect of perturbations which are not small, destroy the symmetry with $f \neq 0$, and show how multiple solutions persist despite these nonsymmetric perturbations, provided the growth of the nonlinear term at infinity is suitably controlled. Our method is based on the following ingredients. (1) The elimination of the null space from the underlying Hilbert space to establish the Palais-Smale condition for a new functional; (2) A variational technique developed for nonlinear nonsymmetric elliptic equations by Rabinowitz [17]; (3) The construction of a comparison functional which can be used to estimate the size of the critical values; and (4) The estimate of Morse index at the critical points.

Our main result is the following

Theorem 0.1 *Suppose that* $2 < p < \frac{7n+1+\sqrt{25n^2-2n+9}}{2(3n-1)}$ *and* $g(\xi) \in C(\mathbb{R}, \mathbb{R})$ *satisfies*

(g1) $[g(\xi_1) - g(\xi_2)](\xi_1 - \xi_2) \geq a_1|\xi_1 - \xi_2|^p$;

(g2) there exists $r > 0$ *such that*

$$0 < pG(\xi) \equiv p\int_0^\xi g(\tau)d\tau \leq \xi g(\xi) \;\; for \;\; |\xi| \geq r;$$

(g3) there exists $a_2 > 0$ *such that*

$$|g(\xi)| \leq a_2(|\xi|^{p-1} + 1) \;\; for \;\; \xi \in \mathbb{R};$$

(g4) $g(\xi) = o(|\xi|)$ *at* $\xi = 0$.

Then for any $f(t,x) \in L^{p/(p-1)}(S^1 \times S^n)$, 2π*-periodic in* t, *the above non-linear wave equation (1) has infinitely many periodic weak solutions in* $L^p(S^1 \times S^n) \cap H(S^1 \times S^n)$.

Remark 0.1 *By a weak solution of (1), we mean a function* $u(t,x)$ *satisfying*

$$\int_{S^1 \times S^n} [u(\phi_{tt} - \Delta_n \phi + (\frac{n-1}{2})^2 \phi) + g(u)\phi - f\phi] \, dx \, dt = 0$$

for all $\phi \in C^\infty(S^1 \times S^n)$.

Remark 0.2 *For the* p *as in the theorem 0.1 and* $1 < q < p$, *the following types of functions*

$$g(x,z) = h(x)|z|^{p-2}z + g(x)|z|^{p-1} + k(x)|z|^{q-2}z,$$

where $h(x)$, $g(x)$, $k(x) \in C^0(S^n, (0,\infty))$, *satisfy conditions* $(g1) - (g4)$. *In Chapter 2, we will deal with the simplest case* $g(u) = |u|^{p-2}u$, $p > 2$ *which shows the ideas involved, but the estimates are much easier to obtain than the general case.*

Remark 0.3 *In general we cannot expect the equation (1) to have nontrivial solution if* g *is not super-linear [29].*

Remark 0.4 *The regularity results in the case of* $n = 1$ *are obtained by Brezis and Nirenberg [10] for asymptotically linear* g *and by Rabinowitz [16] for superlinear* g. *For* n *even regularity results are obtained by Jerison, Sogge and Zhou [11] and for* $n = 3$ *by Zhao and Zhou [28] for the spherically symmetric solutions. However for* $n > 3$ *and* n *odd, the regularity of weak solutions of the equation (1) is still open.*

In [29], the existence result is proved for the case g is an odd function and for $2 < p < \frac{1}{2}(1 + (\frac{9n-1}{n-1})^{1/2})$, where finite-dimensional approximation is used to overcome the lack of compactness mentioned above. Using, however, Tanaka's idea [24], we get around these difficulties by maximizing the original functional $F(u)$ associated with the equation (1) with respect to N. That is, we consider the functional $I(u) = \max_{v \in N} F(u+v)$ for u in the orthogonal complement of N. Due to a compact embedding theorem 1.1 for this new space, we can prove that $I(u)$ has the desired compactness properties. And it is easy to see that each critical point of $I(u)$ corresponds to a unique critical point of $F(u)$.

We are able to improve on p without the restriction of oddness on g. In the case of $f(t,x) \equiv 0$, the equation (1) has a natural symmetry and the functional $F(u)$ is S^1-invariant. We will address the case where $f(t,x)$ is not identically 0 as a perturbation from symmetry by using the ideas from [17]. The situation for the wave equation is more complicated since the operator A has infinitely many positive and infinitely many negative eigenvalues. The idea is based on some topological linking theorems. The key in this argument is to estimate the size of some explicitly constructed critical values. To do this, we will introduce a symmetric comparison functional $K(u)$ defined only on the positive eigenspace. Using the symmetry we will construct critical values of $K(u)$, and will establish the relations between critical values of $I(u)$ and $K(u)$. An argument by Morse index theory on $K(u)$ will finally prove the needed estimates.

This dissertation will be organized as follows. We will give some preliminaries in Chapter 1, to serve as background in understanding later presentation. In Chapter 2, we will consider the case where $g(u) = |u|^{p-2}u$, $p > 2$, which is relatively easier than the more general case of Theorem 0.1 because of the explicit form of the nonlinear term and, more importantly, that we just need to consider the Z_2-action instead of the S^1-action. Finally in Chapter 3, using the ideas in Chapter 2, we carefully will show how to modify the functional, check the Palais-Smale conditions of the modified functional, establish the S^1 covariant version of Borsuk-Ulam theorem, and find the connection between the Morse index of some critical points and their critical values.

CHAPTER 1

Preliminaries

Let A the linear wave operator such that

$$Au = u_{tt} - \Delta_n u + (\frac{n-1}{2})^2 u,$$

where $(t, x) \in S^1 \times S^n, n > 1$. It is well known that the eigenvalues of A are

$$\lambda(l, j) = (l + \frac{n-1}{2} - j)(l + \frac{n-1}{2} + j), \qquad l, j = 0, 1, 2, ...,$$

and the corresponding eigenfunctions in $L^2(S^1 \times S^n)$ are

$$\phi_{l,m}(x)\, sin\, jt, \quad \phi_{l,m}(x)\, cos\, jt, \qquad m = 1, 2, ..., \; M(l, n),$$

where $\phi_{l,m}(x),\ m = 1, 2, ..., M(l, n)$, are spherical harmonics of degree 1 on S^n and

$$M(l, n) = \frac{(2l + n - 1)\Gamma(l + n - 1)}{\Gamma(l + 1)\,\Gamma(n)} = O(l^{n-1}).$$

Then $u \in L^2(S^1 \times S^n)$ can be written as

$$u = \sum_{l,j,m} u_{l,j,m} e^{ijt} \phi_{l,m}(x),$$

where $u_{l,j,m}$ are the Fourier coefficients with $u_{l,j,m} = \bar{u}_{l,-j,m}$. Hence

$$(Au, u)_{L^2} = \sum_{l,j,m} \lambda(l, j) |u_{l,j,m}|^2.$$

And the Sobolev space we will work on is defined as

$$H = \{u \in L^2(S^1 \times S^n) : ||u||_H^2 = \sum_{l,j,m} |\lambda(l,j)||u_{l,j,m}|^2 + \sum_{\lambda(l,j)=0} |u_{l,j,m}|^2 < \infty\}.$$

Clearly H is a Hilbert space with the inner product

$$\langle u, v\rangle_H = \sum_{l,j,m} |\lambda(l,j)| u_{l,j,m}\bar{v}_{l,j,m} + \sum_{\lambda(l,j)=0} u_{l,j,m}\bar{v}_{l,j,m}.$$

We decompose H into invariant subspaces:

$$N = \{u \in H | u_{l,j,m} = 0 \text{ for } \lambda(l,j) \neq 0\},$$
$$E^+ = \{u \in H | u_{l,j,m} = 0 \text{ for } \lambda(l,j) \leq 0\},$$
$$E^- = \{u \in H | u_{l,j,m} = 0 \text{ for } \lambda(l,j) \geq 0\}.$$

As can be seen from the expression of the eigenvalues, if the space S^n is odd dimensional, i.e., n odd, the kernel N of the operator A is infinite dimensional and $||u||_H = ||u||_{L^2}$ for $u \in N$. Consequently, a compact embedding theorem of the type $E \hookrightarrow L^p$, $(p > 2)$ for $E = E^+ \oplus E^-$ the orthogonal complement of N:

Theorem 1.1 *(Zhou [30]) For any* $2 \leq p < \dfrac{2n+2}{n-1}$, $E \hookrightarrow L^p$ *is compact.*

Remark 1.1 *The surprising fact is the exponent* $\frac{2n+2}{n-1}$, *which is almost optimal. Note that* $\|u\|_H$ *is much smaller than* $\|u\|_{L^2} + \|\nabla u\|_{L^2} = \|u\|_{W^{1,2}(S^1 \times S^n)}$. *And we have the continuous the embedding* $W^{1,2} \hookrightarrow L^p$ *for* $2 \leq p \leq \frac{2n+2}{n-1}$, *and the compact embedding* $W^{1,2} \hookrightarrow L^p$ *only for* $2 \leq p < \frac{2n+2}{n-1}$.

Remark 1.2 *Unlike the* 1*-dimensional case where the existence result is obtained for all of* $2 < p < \infty$ *(Tanaka [24], Zhou [29]), the above embedding theorem 1.1 presents a crucial restriction on* p *for any existence results of wave equations on* S^n, $n > 1$. *Note that in* 1*-dimension the compact embedding* $E \hookrightarrow L^p$ *works for all of* $2 < p < \infty$ *([10, 27, 29]).*

Remark 1.3 *If n is even, then $N = \emptyset$ and $H = E$, and hence problems are much easier to handle [29].*

Next we introduce some definitions on group actions that will be used throughout the paper. Let G be a compact Lie-group and X a topological space. An *action* of G on X is a map $\phi : G \times X \to X, \quad \phi(g,x) = gx$ with the following properties:

(i) $\mathbf{1}x = x$ for each $x \in X$, $\mathbf{1}$ is the unit element in G,

(ii) $g_1(g_2x) = (g_1g_2)x$, g_1, $g_2 \in G$, $x \in X$.

We denote by $O_x = \{gx \mid g \in G\}$ the *orbit* of x. A subspace X_1 of X is called *invariant* under the action of G if $O_x \subset X_1$ for all $x \in X_1$. The closed subgroup $G_x = \{g \mid gx = x\}$ is called the *isotropy group* of x. If $G_x = G$, we say that x is a *fixed point* under the action of G, we will denote by $Fix(G)$ all the fixed points of X under the action of G.

A functional $F : X \to \mathbb{R}$ is said to be *G-invariant* if $F(gx) = F(x)$ for each $x \in X$ and $g \in G$. If X and Y are two G-spaces, we say that a function $F : X \to Y$ is *G-equivariant* if $F(gx) = gF(x)$ for each $x \in X$ and $g \in G$. In this paper we will use two groups, $\mathbb{Z}_2 = \{id, -id\}$ and $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$. For example, any linear topological space is $\mathbb{Z}_2$ space and any Hilbert space H is an S^1-space if we define a group action T_θ on H as

$$(T_\theta u)(x,t) = u(x+\theta,t) \ \text{ for any } \ \theta \in [0,2\pi) \ \text{ and } \ u \in H.$$

Finally we set up a variational formulation for the equation (1). The functional corresponding to the equation (1) for $u \in H$ is given by

$$F(u) = \frac{1}{2}\langle Lu, u\rangle_H - \int_\Omega (G(u) - f \cdot u)dt\, dx,$$

where $G(\xi) = \int_0^\xi g(\tau)\, d\tau$, $\Omega = S^1 \times S^n$, and L is the continuous self-adjoint operator in H associated with the operator A, i.e.,

$$\langle Lu, v\rangle_H = (Au, v) = \sum_{l,j,m} \lambda(l,j) u_{l,j,m} \bar{v}_{l,j,m}.$$

Using the Hilbert Space norm defined above, for $u = u^+ + u^- \in E$, $u^+ \in E^+$, $u^- \in E^-$ and $v \in N$, $F(u)$ can be written as

$$F(u+v) = \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \int_\Omega (G(u+v) - f \cdot (u+v))dt\,dx, \tag{1.1}$$

which is in $C^2(E \oplus N, \mathbb{R})$.

We close this section by introducing the notion of Palais-Smale compactness condition $(P.S.)$ which plays an essential role in applying minimax methods.

Definition 1.1 *A differentiable functional $F(u)$ on a Hilbert space H is said to satisfy Palais-Smale compactness condition (P.S.) if the following holds: whenever $\{u_j\}$ is a sequence in H such that $F(u_j)$ is uniformly bounded and $F'(u_j) \to 0$ in H^* as $j \to \infty$, then $\{u_j\}$ is precompact in H.*

Remark 1.4 *Note that often (P.S.) is not satisfied even by simple smooth functions, e.g., consider $F : \mathbb{R} \to \mathbb{R}$ with $F(u) = \cos u$ or $F(u) = c$, c some constant and take $u_j = j\pi$ for each $j \in \mathbb{N}$.*

CHAPTER 2

The simple case $g(u) = |u|^{p-2}u$

In this chapter we consider the simpler case where $g(u) = |u|^{p-2}u$ and we prove the following theorem.

Theorem 2.1 *For the same* $2 < p < \frac{7n+1+\sqrt{25n^2-2n+9}}{2(3n-1)}$ *as in Theorem 0.1 and* $f(t,x) \in L^{\frac{p}{p-1}}$ 2π*-periodic in t, the following non-linear wave equation*

$$Au = |u|^{p-2}u - f(t,x), \quad (t,x) \in S^1 \times S^n, \; n > 1, \tag{2.1}$$

has infinitely many periodic weak solutions in L^p.

The procedure for the proof of the above theorem is motivated by Tanaka [24], where the existence of infinitely many solutions of the 1-dimensional wave equation (for $2 < p < \infty$) is obtained using Morse theory and eigenvalue estimates. Although most of the proofs in Tanaka works for S^n with some n-dimensional modifications, we found that the eigenvalue estimates in his paper using interpolation theory did not work for the n-dimensional case due to the big multiplicity of the eigenvalues of A in n-dimensions. We take a different approach in Section 2.5 to prove the result. We will treat this Chapter as preparation for Chapter 3.

We will assume $2 < p < \frac{2n+2}{n-1}$ throughout in this thesis in consideration of the compact embedding Theorem 1.1. First we formulate the variational scheme for the proof of the theorem 2.1.

2.1 Variational Scheme

2.1.1 Introduction of a new variational formulation

As mentioned in the preliminary Chapter 1 the corresponding functional to the equation (2.1) is given by, for $w = u + v \in H, u \in E$ and $v \in V$,

$$F(w) = \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \frac{1}{p}\,||u+v||_p^p + (f, u+v). \tag{2.2}$$

We instead study the functional $I(u)$ on E,

$$I(u) = \max_{v \in N} F(u+v) = \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q(u), \tag{2.3}$$

where

$$Q(u) = \min_{v \in N}[\frac{1}{p}\,||u+v||_p^p - (f, u+v)], \tag{2.4}$$

which is easier to handle due to the compact embedding Theorem 1.1 on E. In Section 2.5, we will show that critical points of $I(u)$ are also those of $F(u)$. First, in the following lemmas we study the functional $Q(u)$ in detail to prepare for the proof of a compactness result for the functional $I(u)$.

Lemma 2.1 *(i) For all $u \in L^{p+1}$, there exists a unique $v(u) \in N$ such that*

$$Q(u) = \frac{1}{p}\,||u + v(u)||_p^p - (f, u + v(u)). \tag{2.5}$$

(ii) The map $v : L^p \to N$ is continuous.

(iii) $Q : E \to \mathbb{R}$ is in C^1 and for all $u, h \in E$,

$$\langle Q'(u), h\rangle = (|u + v(u)|^{p-2}(u + v(u)) - f,\, h). \tag{2.6}$$

Moreover, $Q' : E \to E^*$ is compact and there are constants $C_1, C_2 > 0$ depending on $||f||_{p/(p-1)}$ such that for all $u \in E$,

$$||Q'||_{E^*} \leq C_1(|Q(u)|^{\frac{p-1}{p}} + 1), \tag{2.7}$$

$$|\langle Q'(u), u\rangle - pQ(u)| \leq C_2(|Q(u)|^{\frac{1}{p}} + 1). \tag{2.8}$$

Proof: (i) Because the map $v \mapsto \frac{1}{p}||u+v||_p^p - (f, u+v)$ is strictly convex and coercive on N, there is a minimum at, say, $v(u)$ by the generalized Weierstrass Theorem.

(ii) Suppose that $u_j \to u$ in L^p. We will show that $v(u_j) \to v(u)$ strongly in $N \cap L^p$. Since $v(u_j)$ is the minimizer for u_j, we have

$$\frac{1}{p}||u_j + v(u)||_p^p - (f, u_j + v(u)) \geq \frac{1}{p}||u_j + v(u_j)||_p^p - (f, u_j + v(u_j)).$$

Then $\{v(u_j)\}$ is bounded in L^p and hence there is a subsequence $\{u_{j'}\}$ of $\{u_j\}$ such that $v(u_{j'}) \rightharpoonup \bar{v}$ in N. We will denote $\{u_{j'}\}$ by $\{u_j\}$ for simplicity. Letting $j \to \infty$ in the above inequality, we get

$$\begin{aligned} \frac{1}{p}||u + v(u)||_p^p - (f, u + v(u)) &\geq \overline{\lim_{j\to\infty}}\left(\frac{1}{p}||u_j + v(u_j)||_p^p - (f, u_j + v(u_j))\right) \\ &\geq \frac{1}{p}||u + \bar{v}||_p^p - (f, u + \bar{v}). \end{aligned}$$

This implies $\overline{\lim}\, ||u + v(u_j)||_p = ||u + v(u)||_p$ and $\bar{v} = v(u)$ by the uniqueness of $v(u)$. Hence $v(u_j) \to v(u)$ strongly in $N \cap L^p$.

(iii) By the convexity of the function $v \mapsto \frac{1}{p}||u+v||_p^p - (f, u+v)$, we have for all $u, h \in E$ and $\tau > 0$,

$$\begin{aligned} Q(u + \tau h) - Q(u) &= \frac{1}{p}(||u + \tau h + v(u + \tau h)||_p^p - ||u + v(u)||_p^p) \\ &\quad -(f, \tau h + v(u + \tau h) - v(u)) \\ &\geq (|u + v(u)|^{p-2}(u + v(u)) - f, \tau h + v(u + \tau h) - v(u)). \end{aligned}$$

Noting that $v(u + \tau h) - v(u) \in N$, we get

$$Q(u + \tau h) - Q(u) \geq \tau(|u + v(u)|^{p-2}(u + v(u)) - f, h).$$

By interchanging the role of $Q(u + \tau h)$ and $Q(u)$, we have

$$\begin{aligned} Q(u + \tau h) - Q(u) \leq\ & \tau(|u + \tau h + v(u + \tau h)|^{p-2} \\ & (u + \tau h + v(u) + \tau h)) - f, h). \end{aligned}$$

Taking limit $\tau \to 0$ in the above two inequalities, we obtain the derivative formula (2.6). Therefore $Q \in C^1(E, \mathbb{R})$. Moreover from the compact embedding Theorem 1.1 and the continuity of $v(u) : L^{p+1} \to N$, we conclude that $Q'(u) : E \to E^*$ is compact. On the other hand by (2.6) and Theorem 1.1,

$$\begin{aligned} ||Q'(u)||_E^* &= \sup_{||h||_E=1} (|u+v(u)|^{p-2}(u+v(u)) - f, h) \\ &\leq c_p|| \, |u+v(u)|^{p-2}(u+v(u)) - f||_{p/(p-1)}. \end{aligned}$$

Applying Hölder's inequality and (2.5), we get

$$||Q'(u)||_E^* \leq C(\frac{1}{p}||u+v(u)||_p^{p-1} + 1) \leq C(|Q(u)|^{(p-1)/p} + 1).$$

Inequality (2.8) can be easily obtained from (2.6), (2.7) and Hölder's inequality. In all we have obtained the desired results. □

For later use we introduce $Q_0 \in C^1(E, \mathbb{R})$ defined by

$$Q_0(u) = \min_{v \in N} \frac{1}{p}||u+v||_p^p = \frac{1}{p}||u+v_0(u)||_p^p, \tag{2.9}$$

where $v_0(u)$ can be given uniquely as in Lemma 2.1. In the following we list some properties of Q_0 that will be needed in constructing a modified functional in Section 2.1.2. First by setting $f = 0$ in Lemma 2.1, we obtain, for $u, h \in E$,

$$\langle Q_0'(u), h\rangle = (|u+v_0(u)|^{p-2}(u+v_0(u)), h), \tag{2.10}$$

$$||Q_0'(u)||_{E^*} \leq C(Q_0(u)^{(p-1)/p} + 1), \tag{2.11}$$

$$\langle Q_0'(u), u\rangle \leq p\,Q_0(u) + C(Q_0(u)^{1/p} + 1). \tag{2.12}$$

Similarly, as in the proof of the previous lemma, we can easily show the following relations between $Q(u)$ and $Q_0(u)$.

Lemma 2.2 *There is a constant $C > 0$ depending on $||f||_{p/(p-1}$ such that for $u \in E$,*

$$|Q(u)| \le C(Q_0(u) + 1), \tag{2.13}$$

$$|Q(u) - Q_0(u)| \le C(Q_0(u)^{1/p} + 1). \tag{2.14}$$

Now we verify the Palais-Smale compactness condition ($P.S.$) for $I(u)$ which plays a crucial role in applying minimax methods to $I(u)$.

Proposition 2.1 *$I(u) \in C^1(E, \mathbb{R})$ satisfies ($P.S.$).*

Proof: Let $M > 0$. Suppose $I(u_j) \le M$ for all j and $I'(u_j) \to 0$ in E^*. We have for $u_j = u_j^+ + u_j^- \in E^+ \oplus E^- = E$,

$$\langle I'(u_j), h\rangle = \langle u_j^+ - u_j^-, h\rangle - \langle Q'(u_j), h\rangle \text{ for } h \in E.$$

First we will show that $\{u_j\}$ is bounded in E. Then the compactness of Q' will immediately prompt the existence of a convergent subsequence of $\{u_j\}$. Setting $h = u_j$ or $h = u_j^+ - u_j^-$, we get

$$|||u_j^+||_E^2 - ||u_j^-||_E^2 - \langle Q'(u_j), u_j\rangle| \le m||u_j||_E, \tag{2.15}$$

$$|||u_j||_E^2 - \langle Q'(u_j), u_j^+ - u_j^-\rangle| \le m||u_j||_E, \tag{2.16}$$

where $m = \sup ||I'(u_j)||_{E^*}$. From $I(u_j) \le M$, we have

$$\frac{1}{2}||u_j^+||_E^2 - \frac{1}{2}||u_j^-||_E^2 - Q(u_j) \le M,$$

which combined with (2.15) leads to

$$\frac{1}{2}\langle Q'(u_j), u_j\rangle - Q(u_j) \le M + m||u_j||_E.$$

Thus by (2.8), we get

$$(\frac{p}{2} - 1)Q(u_j) - C_2(|Q(u_j)|^{1/p} + 1) \le M + m||u_j||_E,$$

which implies

$$Q(u_j) \leq C(||u_j||_E + 1), \quad \text{for all } j, \tag{2.17}$$

where $C > 0$ is independent of j. Now it follows from (2.7) and (2.17) that

$$\begin{aligned} |\langle Q'(u_j), u_j^+ - u_j^- \rangle| &\leq ||Q'(u_j)||_{E^*} ||u_j||_E \\ &\leq C(|Q(u_j)|^{(p-1)/p} + 1)||u_j||_E \\ &\leq C(||u_j||_E^{(p-1)/p} + 1)||u_j||_E. \end{aligned}$$

This, substituted into (2.16), yields

$$\begin{aligned} ||u_j||_E^2 &\leq m||u_j||_E + \langle Q'(u_j), u_j^+ - u_j^- \rangle \\ &\leq m||u_j||_E + C(||u_j||_E^{(p-1)/p} + 1)||u_j||_E. \end{aligned}$$

Thus $\{u_j\}$ is bounded in E.

Finally note that $I'(u_j) = u_j^+ - u_j^- - Q'(u_j)$ where $Q' : E \to E^*$ is compact and $I'(u_j) \to 0$ as $j \to \infty$. We can easily see that $\{u_j\}$ is precompact in E. □

2.1.2 Modified functional

Next we replace $I(u)$ by a modified functional $J(u)$ for which it is easier to construct the critical values. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\chi(\tau) = 1$ for $\tau \leq 1$, $\chi(\tau) = 0$ for $\tau \geq 2$ and $-2 \leq \chi'(\tau) \leq 0$, $0 \leq \chi(\tau) \leq 1$, for $\tau \in \mathbb{R}$. For $u = u^+ + u^- \in E^+ \oplus E^- = E$ and $a = max\{1, \frac{12}{p-1}\}$, let

$$\Phi(u) = a(I(u)^2 + 1)^{1/2}, \quad \psi(u) = \chi(\Phi(u)^{-1} Q_0(u)).$$

Define

$$J(u) = \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)),$$

where $Q_0(u)$ is as defined in Section 2.1.1.

The reason for introducing $J(u)$ is that the first assertion of the following proposition which says $J(u)$ is almost an even functional, holds for $J(u)$ but not for $I(u)$. Using the following proposition, we will then show that large critical values of $J(u)$ are also critical values of $I(u)$.

Proposition 2.2 *(Proposition 1.2 of Tanaka [24]) The functional $J(u) \in C^1(E, \mathbb{R})$ satisfies the following:*

(i) there is $\alpha = \alpha(\|f\|_{p/(p-1)}) > 0$ such that for $u \in E$,

$$|J(u) - J(-u)| \leq \alpha(|J(u)|^{\frac{1}{p}} + 1), \textit{ and}$$

(ii) there is $M_0 > 0$ such that $J(u) \geq M_0$ and $\|J'(u)\|_{E^*} \leq 1$ implies $J(u) = I(u)$.

Proof: (i) From the definition of $J(u)$, we have

$$|J(-u) - J(u)| \leq \psi(u)|Q(u) - Q_0(u)| + \psi(-u)|Q(-u) - Q_0(-u)|. \tag{2.18}$$

Suppose that $-u \in \mathit{supp}\ \psi$, i.e., $Q_0(u) \leq 2\Phi(-u) = 2a(I(-u)^2 + 1)^{1/2}$. From the definition of $J(u)$,

$$I(-u) = J(u) + (Q_0(u) - Q(-u)) - \psi(u)(Q(u) - Q_0(u)).$$

By Lemma 2.2, we get

$$\begin{aligned} |I(-u)| &\leq |J(u)| + C(Q_0(u)^{1/p} + 1) \\ &\leq |J(u)| + C\Phi(-u)^{1/p}. \end{aligned}$$

Using Young's inequality, we deduce that

$$|I(-u)| \leq 2|J(u)| + C.$$

Hence we get for $-u \in \mathit{supp}\ \psi$,

$$\begin{aligned} Q_0(u) &\leq 2\Phi(-u) = 2a(I(-u)^2 + 1)^{1/2} \\ &\leq C|J(u)| + C. \end{aligned} \tag{2.19}$$

Similarly we have for $u \in supp\, \psi$,

$$Q_0(u) \leq C|J(u)| + C. \tag{2.20}$$

It follows (2.13), (2.18), (2.19) and (2.20) that for $u \in E$

$$\begin{aligned} |J(-u) - J(u)| &\leq C(\psi(u) + \psi(-u))(Q_0(u)^{1/p} + 1) \\ &\leq \alpha(|J(u)|^{1/p} + 1). \end{aligned}$$

This proves the first assertion of the proposition.

(ii) To prove the second assertion of the proposition, we need the following two lemmas.

Lemma 2.3 *There is a constant* $M_1 = M_1(||f||_{p/(p-1)}) > 0$ *such that* $J(u) \geq M_1$ *and* $u \in supp\, \psi$ *imply* $I(u) \geq \frac{1}{3}J(u)$.

Proof: From the definition of $J(u)$,

$$\begin{aligned} J(u) &= I(u) - (1 - \psi(u))(Q(u) - Q_0(u)) \\ &\leq I(u) + C(Q_0(u)^{1/p} + 1). \end{aligned}$$

By definition of ψ, we get for $u \in supp\, \psi$,

$$\begin{aligned} J(u) &\leq I(u) + C(|I(u)|^{1/p} + 1) \\ &\leq I(u) + \frac{1}{2}|I(u)| + C_1. \end{aligned}$$

Choosing $M_1 = 2C_1$, we get the desired result. □

Lemma 2.4 *For all* $u = u^+ + u^- \in E = E^+ \oplus E^-$ *and* $h \in E$,

$$\begin{aligned} \langle J'(u), h\rangle &= (1 + T_1(u))\langle u^+ - u^-, h\rangle - (1 + T_2(u))\langle Q_0'(u), h\rangle \\ &\quad -(\psi(u) + T_1(u))\langle Q'(u) - Q_0'(u), h\rangle, \end{aligned}$$

where $T_1(u), T_2(u) \in C(E, \mathbb{R})$ *are functionals satisfying*

$$\sup\{|T_i(u)|; u \in E, J(u) \geq M_2,\ i = 1,2\} \to 0 \ \ as \ \ M_2 \to \infty. \tag{2.21}$$

Proof: For all $u = u^+ + u^- \in E$ and $h \in E$, we have

$$\begin{aligned}
\langle J'(u), h\rangle = \langle u^+ - u^-, h\rangle \ &- \ \langle Q_0'(u), h\rangle - \langle \psi'(u), h\rangle (Q(u) - Q_0(u)) \\
&- \ \psi(u)\langle Q'(u) - Q_0'(u), h\rangle, \ \ where
\end{aligned}$$

$$\begin{aligned}
\langle \psi'(u), h\rangle &= \chi'(\Phi(u)^{-1}Q_0(u))\Phi(u)^{-3} \\
&\quad \times[-a^2 I(u)\langle I'(u), h\rangle Q_0(u) + \Phi(u)^2\langle Q_0'(u), h\rangle], \\
\langle I'(u), h\rangle &= \langle u^+ - u^-, h\rangle - \langle Q_0'(u), h\rangle - \langle Q'(u) - Q_0'(u), h\rangle.
\end{aligned}$$

By regrouping terms, we get the desired expression for $\langle J'(u), h\rangle$ for

$$\begin{aligned}
T_1(u) &= a^2\chi'(\cdot)\Phi(u)^{-3}I(u)Q_0(u)(Q(u) - Q_0(u)), \\
T_2(u) &= T_1(u) + \chi'(\cdot)\Phi(u)^{-1}(Q(u) - Q_0(u)).
\end{aligned}$$

Now suppose that $u \in E$ satisfies $J(u) \geq M_2$. From (2.14), we get

$$|T_1(u)| \leq C\,|\chi'(\cdot)|\,\Phi(u)^{-2}Q_0(u)\,(Q_0(u)^{1/p} + 1).$$

If $u \notin$ *supp* ψ, then $T_1 = 0$. Otherwise, by the definition of $\psi(u)$, we have $Q_0(u) \leq 2\Phi(u)$. On the other hand we get from Lemma 2.3, $\Phi(u) \geq I(u) \geq \frac{1}{3}J(u) \geq \frac{1}{3}M_2$. Hence we obtain

$$|T_1(u)| \leq C\Phi(u)^{-(p-1)/p} \leq CM_2^{-(p-1)/p} \to 0 \ \ as \ \ M_2 \to \infty.$$

Similarly we have $T_2(u) \to 0$ as $M_2 \to \infty$. $\square$

Let us now prove the second assertion of the proposition. Recall that by definition of $J(u)$ it suffices to show that $\psi(u) = 1$. Thus we need to show

$$Q_0(u) \leq \Phi(u), \tag{2.22}$$

for $u \in E$ such that $J(u) \geq M_0$ and $||J'(u)||_{E^*} \leq 1$. For sufficiently large $M_0 > 0$, we can assume by Lemma 2.4 that $J(u) \geq M_0$ implies $|T_1(u)| \leq \frac{1}{2}$, $|T_2(u)| \leq 1$ and

$$\frac{p(1+T_2(u))}{2(1+T_1(u))} - 1 > \frac{p-2}{4} \equiv b.$$

From (2.21), we obtain

$$\begin{aligned}
I(u) \quad &- \quad \frac{1}{2(1+T_1(u))}\langle J'(u), u\rangle \\
&= \quad -Q(u) + \frac{1+T_2(u)}{2(1+T_1(u))}\langle Q_0'(u), u\rangle \\
&\quad\ + \frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\langle Q'(u) - Q_0'(u), u\rangle \\
&= \quad \left(\frac{p(1+T_2(u))}{2(1+T_1(u))} - 1\right)Q_0(u) - (Q(u) - Q_0(u)) \\
&\quad\ + \frac{\psi(u)+T_1(u)}{2(1+T_1(u))}\langle Q'(u) - Q_0'(u), u\rangle \\
&\equiv \quad (I) + (II) + (III).
\end{aligned}$$

But by (2.14) we easily see that

$$|(II)| \leq C(Q_0(u)^{1/p} + 1).$$

On the other hand it follows from (2.8), (2.13), (2.14), and (2.12) that

$$\begin{aligned}
|\langle Q'(u) - Q_0'(u), u\rangle| \quad &\leq \quad |pQ(u) - \langle Q'(u), u\rangle| + p|Q(u) - Q_0(u)| \\
&\qquad + C(Q_0(u)^{1/p} + 1) \\
&\leq \quad C(Q_0(u)^{1/p} + 1),
\end{aligned}$$

which implies

$$|(III)| \leq C(Q_0(u)^{1/p} + 1).$$

Thus we have

$$\begin{aligned}
&I(u) - \frac{1}{2(1+T_1(u))}\langle J'(u), u\rangle \\
\geq\ &\left(\frac{p(1+T_2(u))}{2(1+T_1(u))} - 1\right)Q_0(u) - C(Q_0(u)^{1/p} + 1) \\
\geq\ &bQ_0(u) - C. \qquad (2.23)
\end{aligned}$$

Now letting $h = u^+ - u^-$ in Lemma 2.4, we get

$$\begin{aligned}\langle J'(u), u^+ - u^-\rangle &= (1+T_1(u))||u||_E^2 - (1+T_2(u))\langle Q_0'(u), u^+ - u^-\rangle \\ &\quad -(\psi(u)+T_1(u))\langle Q'(u) - Q_0'(u),\ u^+ - u^-\rangle.\end{aligned}$$

We estimate the second and third term on the right hand side of the previous equation. By (2.11), we have

$$|\langle Q_0'(u), u^+ - u^-\rangle| \le ||Q_0'(u)_{E^*}||u||_E \le C(Q_0(u)^{(p-1)/p} + 1)||u||_E.$$

Similarly by (2.7) and (2.13),

$$|\langle Q'(u), u^+ - u^-\rangle| \le C(Q_0(u)^{(p-1)/p} + 1)||u||_E.$$

Recalling that $|T_1(u)| \le \frac{1}{2}$ and the assumption that $||J'(u)||_{E^*} \le 1$, we then have

$$\begin{aligned}\frac{1}{2}||u||_E^2 &\le ||J'(u)||_{E^*}||u||_E + C(Q_0(u)^{(p-1)/p} + 1)||u||_E \\ &\le C(Q_0(u)^{(p-1)/p} + 1)||u||_E,\end{aligned}$$

which leads us to

$$||u||_E \le C(Q_0(u)^{(p-1)/p} + 1). \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\begin{aligned}I(u) &\ge \frac{1}{2(1+T_1(u))}\langle J'(u), u\rangle + bQ_0(u) - C \\ &\ge -C||J'(u)||_{E^*}||u||_E + bQ_0(u) - C \\ &\ge bQ_0(u) - C(Q_0(u)^{(p-1)/p} + 1) \\ &\ge bQ_0(u)/2 - C_0.\end{aligned}$$

Finally we remark that

$$\inf\{Q_0(u); ||J'(u)||_{E^*} \le 1 \ \mathit{and}\ \ J(u) \ge M\} \to \infty \ \ \mathit{as}\ \ M \to \infty.$$

This follows from (2.24) since $J(u) \to \infty$ implies $b\,\frac{Q_0(u)}{6} - C_0 \geq 0$; hence $I(u) \geq b\,Q_0(u)/3$. Combining these estimates yields

$$Q_0(u) \leq aI(u) \leq \Phi(u).$$

Thus the proof of the lemma is completed. $\square$

Immediate consequences of the above proposition are the following two corollaries which ensures that large critical values of $J(u)$ are also critical values of $I(u)$, and that the $(P.S.)$ condition holds for large values of $J(u)$.

Corollary 2.1 *If $J'(u) = 0$ and $J(u) \geq M_0$ for $u \in E$, then $I(u) = J(u)$ and $I'(u) = 0$.*

Corollary 2.2 *$J(u)$ satisfies $(P.S.)$ on the set $\{u \mid J(u) \geq M_0\}$.*

2.2 Minimax methods

2.2.1 Construction of critical values

We rearrange the positive eigenvalues of the wave operator A as $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$, and let $e_1, e_2, e_3, \cdots$ be the corresponding orthonormal eigenfunctions. Then the positive eigenspace E^+ can be written as

$$E^+ = \overline{span}\,\{e_j : j \in N\}.$$

Define

$$E_k^+ = \overline{span}\,\{e_j : 1 \leq j \leq k\}.$$

Note that $||u||_E \le \mu_k^{1/2} ||u||_{L^2}$ for $u \in E_k^+$. For $u = u^+ + u^- \in E_k^+ \oplus E^-$, by Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
J(u) &= \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)) \\
&\le \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q_0(u) + C(Q_0(u)^{1/p} + 1) \\
&\le \frac{1}{2}||u^+||_E^2 - \frac{1}{2}Q_0(u) - \frac{1}{2}||u^-||_E^2 + C \\
&= \frac{1}{2}||u^+||_E^2 - \frac{1}{2p}||u + v_0(u)||_p^p - \frac{1}{2}||u^-||_E^2 + C \\
&\le \frac{1}{2}||u^+||_E^2 - c||u^+u^- + v_0(u)||_2^p - \frac{1}{2}||u^-||_E^2 + C \\
&\le \frac{1}{2}||u^+||_E^2 - c||u^+||_2^p - \frac{1}{2}||u^-||_E^2 + C \\
&\le \frac{1}{2}||u^+||_E^2 - c\mu_k^{-p/2}||u^+||_E^p - \frac{1}{2}||u^-||_E^2 + C.
\end{aligned}
$$

Hence there is an $R_k > 0$ such that $J(u) \le 0$ for all $u \in E_k^+ \oplus E^-$ with $||u||_E \ge R_k$. We may assume that $R_k < R_{k+1}$ for each $k \in \mathbb{N}$.

Now we construct minimax values following Rabinowitz's procedure [17]. Let B_R denote the closed unit ball of radius R in E, $D_k = B_{R_k} \cap (E_k^+ \oplus E^-)$ and

$\Gamma_k = \{\gamma \in C(D_k, E); \gamma$ satisfies $(\gamma_1) - (\gamma_3)\}$, where

(γ_1) γ is odd in D_k,

(γ_2) $\gamma(u) = u$ for all $u \in \partial D_k$,

(γ_3) $\gamma(u) = \alpha^+(u)u^+ + \alpha^-(u)u^- + \kappa(u)$, where $\alpha^+ \in C(D_k, [0,1])$ and $\alpha^- \in C(D_k, [1, \bar{\alpha}])$ are even functionals ($\bar{\alpha} \ge 1$ depends on γ) and κ is a compact operator such that on ∂D_k, $\alpha(u) = \alpha^+(u) + \alpha^-(u) = 1$ and $\kappa(u) = 0$.

Define

$$b_k = \inf_{\gamma \in \Gamma} \sup_{u \in D_k} J(\gamma(u)), \quad k \in \mathbb{N}.$$

If $f \equiv 0$ and J is even, it can be shown as in [1] that the numbers b_k are critical values of J. If f is not identically 0, that need not be the case. However we will use these numbers as the basis for a comparison argument. To construct a sequence of critical values of J, we must define another set of minimax values. Let

$U_k = D_{k+1} \cap \{u \in E; \langle u, e_{k+1}\rangle \geq 0\}$;

$\Lambda_k = \{\lambda \in C(U_k, E);\ \lambda$ satisfies $(\lambda_1) - (\lambda_3)\}$, where

(λ_1) $\lambda|_{D_k} \in \Gamma_k$,

(λ_2) $\lambda(u) = u$ on $\partial U_k \setminus D_k$,

(λ_3) $\lambda(u) = \tilde{\alpha}^+(u)u^+ + \tilde{\alpha}^-(u)u^- + \tilde{\kappa}(u)$, where $\tilde{\alpha}^+ \in C(U_k, [0,1])$ and $\tilde{\alpha}^- \in C(U_k, [1, \tilde{\alpha}])$ are even functionals ($\tilde{\alpha} \geq 1$ depends on λ) and $\tilde{\kappa}$ is a compact operator such that $\tilde{\alpha}(u) = 1$ and $\tilde{\kappa}(u) = 0$ on $\partial U_k \setminus D_k$.

Now define

$$c_k = \inf_{\lambda \in \Lambda} \sup_{u \in U_k} J(\lambda(u)) \quad k \in \mathbb{N}.$$

By definition of b_k and c_k we easily see that $c_k \geq b_k$. The key to this construction is that we have the following existence result.

First recall that J satisfies the *(P.S.)* condition (Corollary 2.2) on $\{u \in E;\ J(u) \geq M_0\}$ and $J'(u)$ is an operator of the form:

$$J'(u) = (1 + T_1(u))(u^+ - u^-) + \textit{compact},$$

where $|T_1(u)| \leq 1/2$ on $\{u \in E;\ J(u) \geq M_0\}$ (see proof of Proposition 2.2). Thus we have the following deformation lemma.

Lemma 2.5 *(cf.[18, 19]) Suppose that $c > M_0$ is a regular value of $J(u)$, that is, $J'(u) \neq 0$ when $J(u) = c$. Then for any $\tilde{\varepsilon} > 0$, there exist an $\varepsilon \in (0, \tilde{\varepsilon}]$ and $\eta \in C([0,1] \times E,\ E)$ such that*

(i) $\eta(t, \cdot)$ is odd for all $t \in [0,1]$ if $f(t,x) \equiv 0$;

(ii) $\eta(t, \cdot)$ is a homeomorphism of E onto E for all t;

(iii) $\eta(0, u) = u$ for all $u \in E$;

(iv) $\eta(t, u) = u$ if $J(u) \notin [c - \tilde{\varepsilon}, c + \tilde{\varepsilon}]$;

(v) $J(\eta(1, u)) \leq c - \varepsilon$ if $J(u) \leq c + \varepsilon$;

(vi) $\eta(1, u)$ satisfies $(\lambda 3)$.

Proposition 2.3 *Suppose $c_k > b_k \geq M_0$. Let $\delta \in (0, c_k - b_k)$ and*

$$\Lambda_k(\delta) = \{\lambda \in \Lambda_k; J(\lambda) \leq b_k + \delta \ \ on \ \ D_k\}.$$

Then

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \sup_{u \in U_k} J(\lambda(u)) \ (\geq c_k)$$

is a critical value of $I(u)$.

Proof: By Corollary 2.1, it is enough to show that $c_k(\delta)$ is a critical value of $J(u)$. First note that by definition of b_k and Λ_k, $\Lambda_k(\delta) \neq \emptyset$. Choose $\bar{\varepsilon} = \frac{1}{2}(c_k - b_k - \delta) > 0$. Now suppose that $c_k(\delta)$ is not a critical value of J. Then by Lemma 2.5 there exist $\varepsilon \in (0, \bar{\varepsilon}]$ and η as in the lemma. Choose $H \in \Lambda_k(\delta)$ such that

$$\max_{U_k} J(H(u)) \leq c_k(\delta) + \varepsilon.$$

Let $\bar{H} = \eta(1, H)$. We need to show $\bar{H} \in \Lambda_k$. Clearly $\bar{H} \in C(U_k, E)$. (λ_1) and (λ_2) easily follow from the choice of H and (iv) of Lemma 2.5. Since H satisfies (λ_3), so does $\bar{H}$ by Lemma 2.5. Moreover on D_k, $J(H(u)) \leq c_k(\delta) - \bar{\varepsilon}$ and hence $J(\bar{H}(u)) = J(H(u)) \leq b_k + \delta$ on D_k, again by (iv) of Lemma 2.5. Therefore $\bar{H}(u) \in \Lambda_k(\delta)$ and by (v) of Lemma 2.5,

$$\max_{U_k} J(H(u)) \leq c_k(\delta) - \varepsilon,$$

which contradicts to the definition of $c_k(\delta)$. Hence $c_k(\delta)$ is a critical value of J □

Therefore, to establish the existence of critical values, it suffices to show that there exists a subsequence $\{k_j\}$ such that

$$c_{k_j} > b_{k_j} \geq M_0 \ \text{ for } \ j \in \mathbb{N} \ \text{ and } \ b_{k_j} \to \infty \ \text{ as } \ j \to \infty. \tag{2.25}$$

Arguing indirectly we have the following proposition.

Proposition 2.4 *If $c_k = b_k$ for all $k \geq k_0$, then there exists a constant $\bar{C} > 0$ such that*

$$b_k \leq \bar{C}\, k^{p/(p-1)} \quad \textit{for all } k \in \mathbb{N}. \tag{2.26}$$

Proof: We refer [17] for the proof using the property of almost symmetry of $J(u)$ ((i) of Proposition 2.2). □

Our goal in the rest of Chapter 2 is showing the existence of subsequence $\{k_j\}$ with the property (2.25). In fact, by Proposition 2.4, we will prove that there exists $\{k_j\}$, $\varepsilon > 0$ and C_ε satisfying

$$b_{k_j} > C_\varepsilon k_j^{p/(p-1-\varepsilon)} \text{ for all } j \in \mathbb{N}. \tag{2.27}$$

2.2.2 Comparison functional $K(u)$

To show (2.27), we introduce a comparison functional. By the definition of $Q_0(u)$ and (2.14) for $u = u^+ + u^- \in E = E^+ \oplus E^-$,

$$\begin{aligned}
J(u) &= \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q_0(u) - \psi(u)(Q(u) - Q_0(u)) \\
&\geq \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - 2Q_0(u) - a_1 \\
&= \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \frac{2}{p}|u^+ + u^- + v_0(u)||_p^p - a_1 \\
&\geq \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \frac{2}{p}||u^+ + u^-||_p^p - a_1 \\
&\geq \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \frac{a_0}{p}||u^+||_p^p - \frac{a_0}{p}||u^-||_p^p - a_1,
\end{aligned}$$

where $a_0 > 0,\ a_1 > 0$ are constants independent of u. For $u \in E^+$, set

$$K(u) = \frac{1}{2}||u^+||_E^2 - \frac{a_0}{p}||u^+||_p^p \in C^2(E^+, \mathbb{R}).$$

Then we can easily see the following.

Lemma 2.6 *i)* $J(u) \geq K(u) - a_1$ *for all* $u \in E^+$.

ii) $K(u)$ *satisfies the (P.S.) on* E^+.

In the next section, we will construct critical values β_k of $K(u)$ such that $\beta_k \leq b_k + a_1$ and we will deal with β_k instead of b_k to prove (2.27).

2.3 Critical values β_k of $K(u)$

2.3.1 Bahri-Berestycki's max-min value β_k; [3, 4]

For $m > k$, k, $m \in \mathbb{N}$, set

$$A_k^m = \{\sigma \in C(S^{m-k}, E_m^+); \sigma(-x) = -\sigma(x) \text{ for all } x \in S^{m-k}\}$$

and

$$\beta_k^m = \sup_{\sigma \in A_k^m} \min_{x \in S^{m-k}} K(\sigma(x)).$$

We list some properties of β_k^m in the following proposition.

Proposition 2.5 *(i)* $0 \leq \beta_k^m \leq \beta_{k+1}^m < \infty$ *for all* m, $k \in \mathbb{N}$;

(ii) for all $k \in N$, *there exists* $\nu(k)$ *and* $\tilde{\nu}(k)$ *such that*

$$0 \leq \nu(k) \leq \beta_k^m \leq \tilde{\nu}(k) < \infty \text{ for all } m \geq k+1;$$

(iii) moreover, $\nu(k) \to \infty$ *as* $j \to \infty$.

Proof: (i) For any $\sigma \in A_k^m$, it is clear that there is a $\tilde{\sigma} \in A_{k+1}^m$ with $\tilde{\sigma}(S^{m-k-1}) \subset \sigma(S^{m-k})$. Hence we have $\beta_k^m \leq \beta_{k+1}^m$. To prove (ii) and (iii) of the proposition, we need the following lemmas.

Lemma 2.7 *For all* $\sigma \in A_k^m$,

$$\sigma(S^{m-k}) \cap E_k^+ \neq \emptyset.$$

Proof: Applying Lemma 2.9 in Section 2.3.2 to $h = \sigma : S^{m-k} \to E_m^+$ *and* $g = id : E_k^+ \to E_m^+$, we easily get the result. $\square$

Lemma 2.8 *For all $\theta \in (0, 1/p)$, there is a $C_\theta > 0$ independent of $k \in \mathbb{N}$ such that*

$$\|u\|_p \leq C_\theta \mu_k^{-\theta} \|u\|_E \ \ for \ \ u \in (E_k^+)^\perp,$$

where $(E_k^+)^\perp = \{v \in E^+; \langle v, e_i \rangle = 0 \; for \; i = 1, 2, ..., k\}$.

Proof: We have by the definition of $\| \cdot \|_E$ *and* μ_k

$$||u||_2 \leq \mu_k^{-1/2} ||u||_E \ \text{ for } u \in (E_k^+)^\perp.$$

On the other hand, by Theorem 1.2 (Compact Embedding)

$$||u||_q \leq C_q ||u||_E \quad \text{for all} \quad u \in E \quad \text{and} \quad q \in [2, (2n+2)/(n-1)).$$

Using Hölder's inequality, we get for $q \in (p, (2n+2)/(n-1))$

$$||u||_p \leq ||u||_2^\tau ||u||_q^{1-\tau} \; for \; u \in E^+,$$

where $\tau = \frac{2(q-p)}{p(q-2)} \in (0, \frac{2}{p})$. Thus

$$\|u\|_p \leq C_q^{1-\tau} \mu_k^{-\tau/2} \|u\|_E \quad \text{for} \quad u \in (E_k^+)^\perp,$$

which is the desired result. $\square$

(ii) We now prove the existence of $\tilde{\nu}(k)$. By the linking Lemma 2.7 we have for all $\sigma \in A_k^m$,

$$\min_{x \in S^{m-k}} K(\sigma(x)) \leq \sup_{u \in E_k^+} K(u). \tag{2.28}$$

Recalling that $||u||_E \leq \mu_k^{1/2} ||u||_2$ on E_k^+, we have on E_k^+

$$K(u) \leq \frac{1}{2}||u||_E^2 - C||u||_2^p \leq \frac{1}{2}||u||_E^2 - C\mu_k^{-p/2}||u||_E^p.$$

Thus the right-hand side of (2.28) is finite and independent of σ and m. Set

$$\tilde{\nu}(k) = \sup_{u \in E_k^+} K(u) < \infty,$$

which implies

$$\beta_k^m = \sup_{\sigma \in A_k^m} \min_{x \in S^{m-k}} K(\sigma(x)) \leq \tilde{\nu}(k).$$

Then we prove the existence of $\nu(k)$. First we define a map $\sigma : S^{m-k} \to E_m^+ \backslash \{0\}$ by

$$\sigma(x) = a_0^{-1/(p-2)} ||w(x)||_p^{-p/(p-2)} \omega(x),$$

where $w(x) = \sum_{i=k}^{m} x_i e_i$ and S^{m-k} is understood as

$$S^{m-k} = \{x = (x_k, ..., x_m) \in R^{m-k+1}; \sum_{i=k}^{m} x_i^2 = 1\}.$$

Then obviously $\sigma \in A_k^m$. Since $||w(x)||_E = 1$ on S^{m-k}, we have

$$K(\sigma(x)) = (\frac{1}{2} - \frac{1}{p}) a_0^{-2/(p-2)} ||w(x)||_p^{-2p/(p-2)}.$$

Since $w(x) \in (E_{k-1}^+)^\perp$, $||w(x)||_E = 1$ for all $x \in S^{m-k}$, it follows from Lemma 2.8 that

$$||w(x)||_p \leq C_\theta \, \mu_{k-1}^{-\theta} \quad \text{for } x \in S^{m-k},$$

where $\theta \in (0, 1/p)$ and C_θ is a constant independent of k and x. Thus

$$K(\sigma(x)) \geq C'_\theta \, \mu_{k-1}^{2\theta p/(p-2)} \quad \text{for all } x \in S^{m-k}.$$

The right-hand side is independent of m. Set $\nu(k) = C'_\theta \, \mu_{k-1}^{2\theta p/(p-2)}$. Then we have

$$\beta_k^m \geq \min_{x \in S^{m-k}} K(\sigma(x)) \geq \nu(k) \quad \text{for } m > k,$$

which completes the proof of (ii) of the proposition.

(iii) From the definition of $\nu(k)$, it is easy to see that $\nu(k) \to \infty$ as $n \to \infty$ since $\mu(k) \to \infty$ as $k \to \infty$. □

As in Proposition 2.1 we can verify the following compactness conditions $(P.S.)_*$, $(P.S.)_m$ for $K(u)$.

$(P.S.)_*$: If $\{u_m\}_{m=1}^{\infty} \subset E^+$ satisfies $u_m \in E_m^+, K(u_m) \leq C$ and $\|(K|_{E_m^+})'(u_m)\|_{E_m^{+*}} \to 0$ as $m \to \infty$, then $\{u_m\}$ is relatively compact in E^+;

$(P.S.)_m$: If $\{u_j\}_{j=1}^{\infty} \subset E_m^+$ satisfies $K(u_j) \leq C$ and $(K|_{E_m^+})'(u_j) \to 0$ as $j \to \infty$, then $\{u_j\}$ is relatively compact in E_m^+.

Since K is an even functional satisfying above $(P.S.)_*$ and $(P.S.)$, we have the following result via standard argument.

Proposition 2.6 *Suppose $\nu(k) > 0$. Then β_k^m is a critical value of $K|_{E_m^+}$, and the limit of any convergent subsequence of β_k^m as $m \to \infty$ is a critical value of K.*

By (ii) of Proposition 2.5 choose a sequence $\{m_j\}$ such that $m_j \to \infty$ as $j \to \infty$ and

$$\beta_k = \lim_{j \to \infty} \beta_k^{m_j} \text{ exists for all } k \in \mathbb{N}.$$

Then we have the following facts about the β_k's due to Proposition 2.5 and 2.6:

Corollary 2.3 *i) β_k's are critical values of $K \in C^2(E^+, \mathbb{R})$ for each $k \in \mathbb{N}$;*

ii) $\beta_k \leq \beta_{k+1}$ for all $k \in \mathbb{N}$;

iii) $\beta_k \to \infty$ as $k \to \infty$.

2.3.2 The relation between β_k and other minimax values

To estimate b_k we establish the following relation between b_k and β_k.

Proposition 2.7 *For all $k \in \mathbb{N}$,*

$$b_k \geq \beta_k - a_1, \tag{2.29}$$

where a_1 is the number in Lemma 2.6.

To prove this proposition, we need several topological linking lemmas. We first state a version of the Borsuk-Ulam theorem.

Lemma 2.9 *Let $a, b \in \mathbb{N}$. Suppose $h \in C(S^a, \mathbb{R}^{a+b})$ and $g \in C(\mathbb{R}^b, \mathbb{R}^{a+b})$ are odd functions and there exists $r_0 > 0$ such that $g(y) = y$ for $|y| \geq r_0$. Then $h(S^a) \cap g(\mathbb{R}^b) \neq \emptyset$.*

Proof: We choose $R \geq \gamma_0$ such that $R > \max_{x \in S^a} |h(x)|$. Write

$$D^{a+1} = \{tx \in \mathbb{R}^{a+1}; t \in [0,1], x \in S^a\},\ D^b = \{y \in \mathbb{R}^b; |y| \leq R\}.$$

Define $F \in C(\partial(D^{a+1} \times D^b), \mathbb{R}^{a+b})$ by

$$F(tx, y) = th(x) - g(y).$$

This is well defined and odd on $\partial(D_{a+1} \times D^b)$. Note that $\partial(D^{a+1} \times D^b) \simeq S^{a+b}$ (odd homeomorphic). Thus by the Borsuk-Ulam theorem, there is a $(t_0 x_0, y_0) \in \partial(D^{a+1} \times D^b)$ such that

$$F(t_0 x_0, y_0) = 0, \ \textit{i.e.}, \ t_0 h(x_0) = g(y_0).$$

Since $\partial(D^{a+1} \times D^b) = S^a \times D^b \cup D^{a+1} \times \partial D^b$, the following two cases should be considered:

i) $t_0 = 1, x_0 \in S^a$ and $y_0 \in D^b$;

ii) $t_0 \in [0, 1), x_0 \in S^a$ and $y_0 \in \partial D^b$.

Case 1. We have $h(x_0) = g(y_0)$. So we have $h(S^a) \cap g(\mathbb{R}^b) \neq \emptyset$. This is the desired result.

Case 2. Since $g(y) = y$ on ∂D^b, we have $|g(y_0)| = |y_0| = R$. On the other hand, by the choice of R, we get $|t_0 h(x_0)| < R$. These are incompatible with $t_0 h(x_0) = g(y_0)$. So this case cannot take place. $\square$

From the above lemma, we can deduce the following.

Lemma 2.10 *For all $\gamma \in \Gamma_k$ and for all $\sigma \in A_k^m$,*

$$((p_m\gamma)(D_k) \cup \{u \in E_k^+ \oplus E^- : ||u||_E \geq R_k\}) \cap \sigma(S^{m-k}) \neq \emptyset,$$

where $P_m : E \to E_m^+ \oplus E^-$ is the usual orthogonal projection.

Proof: Let $\gamma \in \Gamma_k = \{\gamma \in C(D_k, E); \gamma \text{ satisfies } (\gamma_1)-(\gamma_3)\}, D_k = B_{R_k} \cap (E_k^+ \oplus E^-)$. We extend γ to $\tilde{\gamma} \in C(E_k^+ \oplus E^-, E)$ by $\tilde{\gamma}(u) = \gamma(u)$ if $||u||_E \leq R_k$, and $\tilde{\gamma}(u) = u$ if $||u||_E \geq R_k$. Obviously, $\tilde{\gamma}(u)$ is well defined and odd in $E_k^+ \oplus E^-$ and since $m > k$,

$$P_m\tilde{\gamma}(E_k^+ \oplus E^-) = P_m\gamma(D_k) \cup \{u \in E_k^+ \oplus E^-; ||u||_E \geq R_k\}.$$

Therefore, it suffices to prove $P_m\tilde{\gamma}(E_k^+ \oplus E^-) \cap \sigma(S^{m-k}) \neq \emptyset$. We rearrange $\{\phi_{l,m}(x)cos\, jt, \phi_{l,m}(x)sin\, jt : \lambda(l,j) < 0, m = 1, \ldots, M(l,n)\}$ as follows, denoted by $f_1, f_2, f_3, \cdots$. We set for $l \in N$,

$$E_l^- = span\,\{f_j; 1 \leq j \leq l\}$$

and let $P_{m,l} : E = E^- \oplus E^+ \to E_m^+ \oplus E_l^-$ be the orthogonal projection. Consider the operators

$$\sigma : S^{m-k} \to E_m^+ \subset E_m^+ \oplus E_l^-, \;\; P_{m,l}\tilde{\gamma} : E_k^+ \oplus E_l^- \to E_m^+ \oplus E_l^-.$$

Applying Lemma 2.9 for $h = \sigma$ and $g = P_{m,l}\tilde{\gamma}$, there exists $x_l \in S^{m-k}$ and $u_l \in E_k^+ \oplus E_l^-$ such that

$$\sigma(x_l) = P_{m,l}\tilde{\gamma}(u_l). \tag{2.30}$$

Since S^{m-k} is compact, there is a subsequence x_{l_j} such that

$$x_{l_j} \to x \text{ in } S^{m-k}, \;\; \sigma(x_{l_j}) \to \sigma(x) \text{ in } E_m^+.$$

On the other hand, by (γ_3),

$$P_{m,l}\tilde{\gamma}(u_l) = P_{m,l}[\alpha^+(u)u_l^+ + \alpha^-(u)u_l^- + \kappa(u_l)] = \alpha^+(u)u_l^+ + \alpha^-(u)u_l^- + P_{m,l}\kappa(u_l),$$

where $\alpha^-(u) \geq 1$ on $E_- \oplus E_k^+$ and $\overline{\kappa(E^- \oplus E_k^+)} = \overline{\kappa(D_k)}$ is compact. Hence we have

$$u_l^- = \frac{1}{\alpha^-(u_l)} P_{E_l^-}[\sigma(x_l) - \kappa(u_l)]$$

and $\{u_l^-\}$ has a convergent subsequence $\{u_{l_j}^-\}$. From the boundedness of u_l and $Dim(E_m^+) < \infty$, u_l has a convergent subsequence. Passing to the limit in (2.30), we obtain

$$P_m \tilde{\gamma}(u) = \sigma(x), \quad \text{i.e.,} \quad P_m \tilde{\gamma}(E_k^+ \oplus E^-) \cap \sigma(S^{m-k}) \neq \emptyset.$$

This completes the proof. □

Let us define

$$b_k^m = \inf_{\gamma \in \Gamma_k} \sup_{u \in D_k} J(P_m \gamma(u))$$

and recall that $b_k = \inf_{\gamma \in \Gamma_k} \sup_{u \in D_k} J(\gamma(u))$. Then we have

Lemma 2.11 *For $k \in \mathbb{N}$, $b_k = \lim_{m \to \infty} b_k^m$.*

Proof: Since $P_m \Gamma_k = \{P_m \gamma;\ \gamma \in \Gamma_k\} \subset \Gamma_k$, it is clear that $b_k \leq b_k^m$ for all $m > k$. Let's prove the other direction i.e., $b_k \geq \limsup_{m \to \infty} b_k^m$ for $k \in \mathbb{N}$. From the definition of b_k, for any $\varepsilon > 0$ there is a $\gamma \in \Gamma_k$ such that

$$\sup_{u \in D_k} J(\gamma(u)) \leq b_k + \epsilon.$$

By (γ_3), $\gamma(u) = \alpha^+(u)u^+ + \alpha^-(u)u^- + \kappa(u)$, where $\alpha^\pm$ satisfies the condition in (γ_3) and $\overline{\kappa(D_k)}$ is compact. Since

$$P_m \kappa(u) \to \kappa(u) \text{ as } m \to \infty \text{ uniformly in } D_k,$$

we have

$$P_m \gamma(u) = \alpha^+(u)u^+ + \alpha^-(u)u^- + P_m \kappa(u) \to \alpha^+(u)u^+ + \alpha^-(u)u^- + \kappa(u) = \gamma(u) \text{ uniformly in } D_k.$$

Hence

$$\sup_{u\in D_k} J(P_m\gamma(u)) \to \sup_{u\in D_k} J(\gamma(u)) \text{ as } m\to\infty.$$

Thus we obtain

$$\limsup_{m\to\infty} b_k^m \le \limsup_{m\to\infty} \sup_{u\in D_k} J(P_m\gamma(u)) = \sup_{u\in D_k} J(\gamma(u)) \le b_k + \varepsilon.$$

Since the above inequality holds for any $\varepsilon > 0$, we get the desired result. □

Using above lemmas, we now prove Proposition 2.7.

Proof: Since $J(u) \le 0$ on $\{u \in E_m^+ \oplus E^- : \|u\|_E \ge R_k\}$, Lemma 2.10 concludes that

$$\min_{x\in S^{m-k}} J(\sigma(x)) \le \sup_{u\in D_k} J(P_m\gamma(u)),$$

for all $\gamma\in\Gamma_k$ and all $\sigma\in A_k^m$. Thus

$$\min_{x\in S^{m-k}} K(\sigma(x)) - a_1 \le \sup_{u\in D_k} J(P_m\gamma(u)),$$

which implies

$$\sup_{\sigma\in A_k^m} \min_{S^{m-k}} K(\sigma(x)) - a_1 \le \inf_{\gamma\in\Gamma_k} \sup_{D_k} J(P_m\gamma(u)).$$

Thus $\beta_k^m - a_1 \le b_k^m$ and by letting $m = m_j \to \infty$, we get

$$\beta_k - a_1 \le \limsup_{m\to\infty} b_k^m = b_k.$$

This establishes the proof. □

2.4 Estimate of β_k using Morse Index

In this section some index properties of β_k are discussed. The lower bound for the index of K'' obtained here and the upper bound estimate in the next section give the growth estimate (2.27) that we are looking for.

Definition 2.1 *For $u \in E^+$, we define an index of $K''(u)$ by*

$$\begin{aligned} \textit{index}\, K''(u) &= \textit{the number of nonpositive eigenvalues of}\, K''(u) \\ &= \max\{\dim S; S \leq E^+ \textit{ such that } \langle K''(u)h, h\rangle \leq 0 \textit{ for all } h \in S\}. \end{aligned}$$

Here "$A \leq B$" in the bracket means A is a subspace of B.

Proposition 2.8 *Suppose $\beta_k < \beta_{k+1}$. Then there exists $u_k \in E^+$ such that*

$$K(u_k) \leq \beta_k,$$
$$K'(u_k) = 0,$$
$$\textit{index}\, K''(u_k) \geq k.$$

By definition of β_k is a critical value of $K(u)$, the result without the last assertion is obvious. To prove the last assertion, we first consider finite dimensional case.

Proposition 2.9 *Suppose $\beta_k^m < \beta_{k+1}^m$, $m > k+1$. Then there exists a $u_k^m \in E_m^+$ such that*

$$K(u_k^m) \leq \beta_k^m,$$
$$(K|_{E_m^+})'(u_k^m) = 0,$$
$$\textit{index}\,(K|_{E_m^+})''(u_k^m) \geq k.$$

To prove the above proposition, we will use a theorem from Morse theory, i.e., a result concerning the relationship between certain homotopy groups of level sets of a functional and its critical points. First we need a theorem to treat the case where critical points may be degenerate.

Proposition 2.10 *(Marino-Prodi [15]) Let U be a C^2 open subset in some Hilbert space H and $\phi \in C^2(U, \mathbb{R})$. Assume ϕ'' is a Fredholm operator (of null index) on the critical set $Z(\phi) = \{x \in U; \phi'(x) = 0\}$, ϕ satisfies (P.S.) and $Z(\phi)$ is compact. Then,* *for any $\varepsilon > 0$, there exists $\psi \in C^2(U, \mathbb{R})$ satisfying (P.S.) and with the following properties:*

(i) $\psi(x) = \phi(x)$ if distance $\{x, Z(\phi)\} \geq \varepsilon$;

(ii) $|\psi(x) - \phi(x)|, \|\psi'(x) - \phi'(x)\|, \|\psi''(x) - \phi''(x)\| \leq \varepsilon$ for all $x \in U$;

(iii) the critical points of ψ are finite in number and nondegenerate.

It is easy to see that $K|_{E_m^+}$ satisfies all the assumptions of the above Proposition. That is,

1) $K|_{E_m^+} \in C^2(E_m^+, \mathbb{R})$ satisfies $(P.S.)$ and Fredholm.

2) All critical value of $K|_{E_m^+}$ are non-negative because

$$K(u) = K(u) - \frac{1}{2}\langle (K|_{E_m^+})'(u), u\rangle = (\frac{1}{2} - \frac{1}{p})\, a_0\, ||u||_p^p \geq 0.$$

3) $Z(K|_{E_m^+})$ is compact. In fact, note that there exists $\tilde{R}_m > 0$ such that $K(u) < 0$ for $u \in E_m^+$ with $||u||_E \geq \tilde{R}_m$; hence $Z(K|_{E_m^+})$ is bounded.

Applying Proposition 2.10 to $K|_{E_m^+}$, for all $\varepsilon > 0$ there exists $\phi_\varepsilon \in C^2(E_m^+, \mathbb{R})$ satisfying $(P.S.)$ and for all $\in E_m^+$,

$$\begin{aligned} |\phi_\varepsilon(u) - K(u)| \quad &< \varepsilon, \\ ||\phi_\varepsilon'(u) - (K|_{E_m^+})'(u)|| \quad &< \varepsilon, \\ ||\phi_\varepsilon''(u) - (K|_{E_m^+})''(u)|| \quad &< \varepsilon; \end{aligned} \tag{2.31}$$

$$\text{the critical points of } \phi_\varepsilon \text{ are finite and non-degenerate.} \tag{2.32}$$

For $m > k$ and $\varepsilon > 0$, let

$$\beta_k^m(\varepsilon) = \sup_{\sigma \in A_k^m} \min_{x \in S^{m-k}} \phi_\varepsilon(\sigma(x)).$$

Then by (2.31),

$$\beta_k^m - \varepsilon \leq \beta_k^m(\varepsilon) \leq \beta_k^m + \varepsilon.$$

Moreover we have

Lemma 2.12 *Suppose that* $a_\varepsilon \in \mathbb{R}$ *satisfies* $\beta_k^m(\varepsilon) < a_\varepsilon - 2\varepsilon < a_\varepsilon < \beta_{k+1}^m(\varepsilon)$. *Then*

$$\pi_{m-k-1}([\phi_\varepsilon \geq a_\varepsilon]_m, p) \neq 0 \textit{ for some } p \in [\phi_\varepsilon \geq a_\varepsilon]_m,$$

where $[\phi_\varepsilon \geq a_\varepsilon]_m = \{u \in E_m^+;\ \phi_\varepsilon(u) \geq a_\varepsilon\}$ *and* π_m *is the* $m-th$ *homotopy group.*

Proof: We argue by contradiction. Suppose that

$$\pi_{m-k-1}([\phi_\varepsilon \geq a_\varepsilon]_m, p) = 0 \text{ for all } p \in [\phi_\varepsilon \geq a_\varepsilon]_m.$$

By the definition of $\beta_{k+1}^m(\varepsilon)$, there is a $\sigma \in A_{k+1}^m$ such that $\sigma(S^{m-k-1}) \subset [\phi_\varepsilon \geq a_\varepsilon]_m$. Since $\pi_{m-k-1}([\phi_\varepsilon \geq a_\varepsilon]_m, p) = 0$, there is a homotopy

$$H : [0,1] \times S^{m-k-1} \to [\phi_\varepsilon \geq a_\varepsilon]_m$$

such that

$$H(0,x) = \sigma(x),\ H(1,x) = p \text{ for all } x \in S^{m-k-1}.$$

Write

$$S^{m-k} = \{(t,x); x \in \mathbb{R}^{m-k},\ t \in \mathbb{R},\ |x|^2 + t^2 = 1\}.$$

Define $\tilde{\sigma} : S^{m-k} \to E_m^+$ by

$$\tilde{\sigma}(t,x) = \begin{cases} p & \text{if t=1, x=0,} \\ H(t, x/|x|) & \text{if } 0 < t \leq 1, \\ -H(-t, -x/|x|) & \text{if } -1 < t \leq 0, \\ -p & \text{if } t = -1,\ x = 0. \end{cases}$$

Obviously $\tilde{\sigma}(S_+^{m-k}) \subset [\phi_\varepsilon \geq a_\varepsilon]_m$, where we denote $S_\pm^{m-k} = \{(t,x) \in S^{m-k}; t > 0(< 0)\}$. On the other hand, we obtain from (2.31) and evenness of $K(u)$ that

$$|\phi_\varepsilon(-u) - \phi_\varepsilon(u)| \leq 2\varepsilon \textit{ for } u \in E_m^+.$$

So we have $\tilde{\sigma}(S_-^{m-k}) \subset [\phi_\varepsilon \geq a_\varepsilon - 2\varepsilon]_m$. Consequently, we have $\tilde{\sigma}(S^{m-k}) \subset [\phi_\varepsilon \geq a_\varepsilon - 2\varepsilon]_m$. From the definition of $\beta_k^m(\varepsilon)$,

$$\beta_k^m(\varepsilon) \geq \min_{x \in S^{m-k}} \phi_\varepsilon(\tilde{\sigma}(x)) \geq a_\varepsilon - 2\varepsilon,$$

which contradicts with the assumption. Thus the proof is completed. □

Using property (2.32), we can apply a classical theorem from Morse theory to ϕ_ε and we obtain

Lemma 2.13 *For a regular value $a \in \mathbb{R}$ of ϕ_ε, set*

$$L(\varepsilon; a) = \max\{\, \mathit{index}\, \phi_\varepsilon''(x); \phi_\varepsilon(x) \leq a,\ \phi_\varepsilon'(x) = 0\}.$$

Then

$$\pi_l([\phi_\varepsilon \geq a]_m, p) = 0 \quad \textit{for all } \ p \in [\phi_\varepsilon \geq a]_m,\ \ l \leq m - L(\varepsilon; a) - 2.$$

Proof: Let $b \in \mathbb{R}$, $b < a$ be such that ϕ_ε has no critical values in $(-\infty, b]$. By the "noncritical neck principle" (cf. Theorem 4.67 of Schwartz [20]), $[\phi_\varepsilon \geq b]_m$ is a deformation retract of E_m^+. Hence

$$\pi_l([\phi_\varepsilon \geq b]_m, p) = 0 \text{ for all } l \in \mathbb{N} \text{ and for all } p.$$

Using theorem 7.3 in Schwartz [20] ,

$$\pi_l([\phi_\varepsilon \geq b]_m, [\phi_\varepsilon \geq a]_m) = 0 \text{ for } l \leq m - L(\varepsilon; a) - 1.$$

Considering the homotopy exact sequence:

$$\begin{aligned} \to \pi_{l+1}([\phi_\varepsilon \geq b]_m, [\phi_\varepsilon \geq a]_m) &\to \pi_l([\phi_\varepsilon \geq a]_m, p) \to \pi_l([\phi_\varepsilon \geq b]_m, p) \\ &\to \pi_l([\phi_\varepsilon \geq b]_m, [\phi_\varepsilon \geq a]_m) \to \dots, \end{aligned}$$

we get the result. □

Now we can prove Proposition 2.9.

Proof: Since $\beta_k^m < \beta_{k+1}^m$ and the critical points of ϕ_ε are finite and nondegenerate, by Sard's theorem there exists a sequence $a_\varepsilon \in \mathbb{R}$ $(0 < \varepsilon \le \varepsilon_0)$ such that a_ε is a regular value of ϕ_ε and

$$\beta_k^m(\varepsilon) < a_\varepsilon - 2\varepsilon < a_\varepsilon < \beta_{k+1}^m(\varepsilon), \quad a_\varepsilon \to \beta_k^m \text{ as } \varepsilon \to 0.$$

By Lemma 2.12 and 2.13, we have $L(\varepsilon; a_\varepsilon) \ge k$ for $0 < \varepsilon < \varepsilon_0$ and hence there exists $u_\varepsilon \in E_m^+$ such that

$$\phi_\varepsilon(u_\varepsilon) \le a_\varepsilon,\ \phi_\varepsilon'(u_\varepsilon) = 0,\ \operatorname{index} \phi_\varepsilon''(u_\varepsilon) \ge k.$$

It follows from (2.31) that (u_ε) satisfies

$$K(u_\varepsilon) \text{ is bounded as } \quad \varepsilon \to 0,$$
$$(K|_{E_m^+})'(u_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Since $K|_{E_m^+}$ satisfies $(P.S.)$ on E_m^+, we can choose a convergent subsequence $u_{\varepsilon_j} \to u_k^m$ for some $u_k^m \in E_m^+$. Then we have

$$K(u_k^m) \le \beta_k^m,\ K|_{E_m^+}'(u_k^m) = 0 \text{ and } \operatorname{index}(K|_{E_m^+})''(u_k^m) \ge k.$$

This completes the proof. □

Finally we prove Proposition 2.8, the main result in this section:
Since $\beta_k < \beta_{k+1}$, we have $\beta_k^{m_j} < \beta_{k+1}^{m_j}$ for sufficiently large j. By Proposition 2.9, there exists $u_k^{m_j} \in E_m^+$ satisfying, $K(u_k^m) \le \beta_k^m$ and $(K|_{E_m^+})'(u_k^m) = 0$. Since $K \in C^2(E^+, \mathbb{R})$ satisfies $(P.S.)_*$, $(u_k^{m_j'}) \to u_k$ for some subsequence $m_{j'}$ of m_j, we have $K(u_k) \le \beta_k$ and $K'(u_k) = 0$

Let us prove the last assertion: index $K''(u_k) \ge k$.
First of all, we have

$$\operatorname{index} K''(u_k^m) \ge \operatorname{index}(K|_{E_m^+})''(u_k^m) \text{ for all } m \in \mathbb{N}.$$

On the other hand, we observe that $K''(u_k)$ is an operator of type: $K''(u_k) = id + \kappa$ where κ is a compact operator. Note that

$$\langle K''(u_k)h, h\rangle \leq 0 \text{ if and only if } \langle \kappa(u_k)h, h\rangle \leq -\langle h, h\rangle$$

and $\overline{\lambda_j} \to 0$ where $\overline{\lambda_j}$ are eigenvalues of κ. Hence there exists an $\varepsilon > 0$ such that for $h \in E^+$,

$$\operatorname{index} K''(u_k) = \operatorname{index}(K''(u_k) - \varepsilon).$$

Since $K \in C^2(E^+, \mathbb{R})$, we have for some j_0',

$$||K''(u_k^{m_j'}) - K''(u_k)|| < \varepsilon \text{ for } j' \geq j_0'.$$

Thus for $j' \geq j_0'$ and $h \in E^+$,

$$\langle K''(u_k)h, h\rangle - \varepsilon||h||_E^2 \leq \langle K''(u_k^{m_j'})h, h\rangle,$$

i.e.,

$$\operatorname{index}\,(K''(u_k) - \varepsilon) \geq \operatorname{index} K''(u_k^{m_j'}).$$

Now by Proposition 2.9, we have

$$\operatorname{index} K''(u_k) \geq k,$$

which completes the proof of Proposition 2.8

2.5 Proof of the existence of the solutions

By Proposition 2.3 and Proposition 2.4, we know that (2.27), the growth estimate on β_k's, ensures the existence of an unbounded sequence of critical values. We now prove (2.27). First note by Proposition 2.8 that there exists $\{u_{k_j}\}$ such that

$$\beta_{k_j} \geq K(u_{k_j}) = \frac{1}{2}||u_{k_j}||_E^2 - \frac{a_0}{p}||u_{k_j}||_p^p = (\frac{1}{2} - \frac{1}{p})a_0||u_j||_p^p. \tag{2.33}$$

Thus, by Proposition 2.8 again, we need to get an upper bound of index $K''(u_{k_j})$ in terms of $||u_{k_j}||_p^p$ in proving (2.27).

For $u, h, w \in E^+$, $K''(u)$ is given by

$$\langle K''(u)w, h\rangle = \langle w, h\rangle - (p-1)a_0(|u|^{p-2}h, h).$$

Thus by the definition of index,

$$\text{index}\, K''(u) = \max\{\,\text{dim}S;\ S \le E^+,\ (p-1)a_0(|u|^{p-2}h, h) \ge ||h||_E^2, h \in S\}.$$

Define an operator $D : L^2 \to E^+$ such that for $v(x,t) = \sum v_{l,j,m}\phi_{l,m}e^{ijt}$,

$$(Dv)(x,t) = \sum_m \sum_{\lambda(l,j)>0} |\lambda(l,j)|^{-1/2} v_{l,j,m}\phi_{l,m}e^{ijt}.$$

Remark 2.1 *D is an isometry from $L_+^2 = \overline{span}_{L^2}\{\phi_{l,m}e^{ijt}; \lambda(l,j) > 0\}$ to E^+ and $D = 0$ on $\overline{span}_{L^2}\{\phi_{l,m}e^{ijt}; \lambda(l,j) \le 0\}$.*

Remark 2.2 *Setting $h = Dv$ in the above expression of index, we get*

$$\begin{aligned} \text{index}\, K''(u) &= \max\{\,\text{dim}S;\ S \le L^2 \text{ s.t. } (p-1)a_0(|u|^{p-2}Dv, Dv) \ge ||v||_2^2, v \in S\} \\ &= \#\{\,\mu_j : \mu_j \ge 1,\ \text{eigenvalues of } D^*((p-1)a_0|u|^{p-2})D\}. \end{aligned}$$

Proposition 2.11 *There exist $C > 0$ such that for $u \in E^+$,*

$$\text{index}\, K''(u_j) \le C\|u\|_s^r,$$

where $r = \frac{2(p-2)nq}{n+1-(n-1)q}$ and $s = \frac{(p-2)q}{q-1}$.

Proof: We try to find a big enough l such that

$$(p-1)a_0(|u|^{p-2}Dv, Dv) \le ||v||_2^2, \text{ on } E^+ \backslash E_{l-1}^+,$$

which implies index $K''(u) \leq l$. First we have the following estimate on $E^+ \backslash E^+_{l-1}$

$$\begin{aligned}
\int_\Omega |Dv|^2 |u|^{p-2} &\leq C\Big(\int \Omega |Dv|^{2q}\Big)^{\frac{1}{q}}\Big(\int \Omega |u|^{(p-2)\frac{q}{q-1}}\Big)^{\frac{q-1}{q}}, \\
&= C||Dv||^2_{2q} ||u||^{p-1}_{(p-2)\frac{q}{q-1}}, \\
&\leq C||Dv||^{2s}_2 ||Dv||^{2(1-s)}_{\bar{q}} ||u||^{p-2}_{(p-2)\frac{q}{q-1}}, \\
&\leq C\frac{1}{\lambda_l^s} ||v||^{2s}_2 ||v||^{2(1-s)}_2 ||u||^{p-2}_{(p-2)\frac{q}{q-1}}, \\
&= C\frac{1}{\lambda_l^s} ||v||^2_2 ||u||^{p-2}_{(p-2)\frac{q}{q-1}},
\end{aligned}$$

where $\bar{q} = \dfrac{2n+2}{n-1}$, $\dfrac{1}{2q} = \dfrac{s}{2} + \dfrac{1-s}{\bar{q}}$ and to get the second last inequality, we used the facts $||Dv||^2_E \leq |\lambda_l|^{-1} ||v||^2_{L^2}$ on $E^+ \backslash E^+_{l-1}$ and $||Dv||^2_E = ||v||^2_{L^2}$ and the compact embedding theorem 1.1. Thus to have $\int |Dv|^2 |u|^{p-2} \leq ||v||^2_2$, we need

$$||u||^{p-2}_{(p-2)\frac{q}{q-1}} \leq |\lambda_l|^s \sim C|l|^{s/n}, \quad s = (n+1-(n-1)q)/2q, \ i.e.,$$

$$\alpha := C||u||^{(p-2)n\frac{2q}{(n+1)-(n-1)q}}_{(p-2)\frac{q}{q-1}} \sim l.$$

Let $l = [\alpha + 1]$. Then

$$\int |Dv|^2 |u|^{p-1} \leq ||v||^2_{L^2} \ \text{ for all } \ v \in E^+ \backslash E^+_{l-1}.$$

and therefore index $K''(u) \leq l = [\alpha+1] \leq C\alpha = C||u||^{(p-2)\frac{2nq}{n+1-(n-1)q}}_{(p-2)\frac{q}{q-1}}$ □

We now prove $b_{k_j} > C\, k_j^{\frac{p}{p-1-\epsilon}}$ (2.27) : From Proposition 2.11 and Proposition 2.8 we have

$$j \leq \text{index}\, K''(u_{k_j}) \leq C||u_{k_j}||^{(p-2)\frac{2nq}{(n+1)-(n-1)q}}_{(p-2)\frac{q}{q-1}}, \qquad 2 < p < \frac{2(n+2)}{n-1}.$$

Note that

$$||u_{k_j}||^p_p \geq C||u_{k_j}||^p_{(p-2)\frac{q}{q-1}} \ \text{ if } \ q \geq \frac{p}{2},$$

so that

$$||u_{k_j}||^p_p \geq j^{p/((p-2)\frac{2nq}{(n+1)-(n-1)q}} \ \text{ if } \ q \geq \frac{p}{2}.$$

In order to have (2.27) it needs

$$(p-2)\frac{2nq}{(n+1)-(n-1)q} < (p-1).$$

Since $\frac{2nq}{(n+1)-(n-1)q}$ is an increasing function of q, choose $q = \frac{p}{2}$. Then we finally obtain

$$2 < p < \frac{7n+1+\sqrt{25n^2-2n+9}}{2(3n-1)},$$

for which (2.27) is satisfied.

Remark 2.3 *This upper bound of p may not be optimal and we are still trying to improve it.*

Now there exists a sequence $u_k \subset E$ of critical points of $I(u)$ such that as $k \to \infty$

$$I(u_k) = \frac{1}{2}||u_k^+||_E^2 - \frac{1}{2}||u_k^-||_E^2 - \frac{1}{p}||u_k + v(u_k)||_p^p - (f, u_k + v(u_k)) \to \infty.$$

Since $I'(u_k) = 0$, we have

$$\langle I'(u_k), u_k\rangle = ||u_k^+||_E^2 - ||u_k^-||_E^2 - (|u_k + v(u_k)|^{p-2}(u_k + v(u_k)) + f, u_k + v(u_k)) = 0.$$

Above two equations combined gives

$$(\frac{1}{2} - \frac{1}{p})||u_k + v(u_k)||_p^p + \frac{1}{2}(f, u_k + v(u_k)) \to \infty \quad \text{as } n \to \infty. \tag{2.34}$$

By direct calculation we can easily see that the $\{u_k + v(u_k)\}$ are critical points of $F(u)$, so it follows from (2.34) that

$$||u_k + v(u_k)||_p \to \infty \; as \; k \to \infty.$$

This ensures the existence of a unbounded sequence of critical points for $F(u)$, which is a unbounded sequence of the weak solutions of the nonlinear wave equation (1) on S^n.

We have proved the result for the simple case where $g(u) = |u|^{p-2}u$. Now we turn to the more general case where $g(u)$ satisfies the conditions $(g1) - (g4)$ of Theorem 0.1.

CHAPTER 3

The existence for general nonlinearity

Here we apply similar ideas as in Chapter 2 to prove Theorem 0.1, but we have to use S^1 index theory to replace Z_2-action and hence estimates are much more complicated. We first state Theorem 0.1 again.

Theorem 3.1 *Suppose that* $2 < p < \frac{7n+1+\sqrt{25n^2-2n+9}}{2(3n-1)}$ *and* $g(\xi) \in C(\mathbb{R}, \mathbb{R})$ *satisfies*

(g1) $[g(\xi_1) - g(\xi_2)](\xi_1 - \xi_2) \geq a_1|\xi_1 - \xi_2|^p$;

(g2) there exists $r > 0$ *such that*

$$0 < pG(\xi) \equiv p\int_0^\xi g(\tau)d\tau \leq \xi g(\xi) \;\; for \;\; |\xi| \geq r;$$

(g3) there exists $a_2 > 0$ *such that*

$$|g(\xi)| \leq a_2(|\xi|^{p-1} + 1) \;\; for \;\; \xi \in \mathbb{R};$$

(g4) $g(\xi) = o(|\xi|)$ *at* $\xi = 0$.

Then for any $f(t, x) \in L^{p/(p-1)}(S^1 \times S^n)$, 2π*-periodic in t, the above non-linear wave equation (1) has infinitely many periodic weak solutions in* $L^p(S^1 \times S^n) \cap H(S^1 \times S^n)$.

3.1 Variational Scheme

3.1.1 A new variational formulation

As we did in Chapter 2.1, we introduce a new functional on E.

$$I(u) = \max_{v \in N} F(u+v) = \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q(u), \tag{3.1}$$

where

$$Q(u) = \min_{v \in N} \int_\Omega (G(u+v) - f \cdot (u+v))\, dt\, dx, \quad \Omega = S^1 \times S^n. \tag{3.2}$$

It turns out that the functional $I(u)$ is in $C^1(E, \mathbb{R})$ and much easier to handle in proving the Palais-Smale $(P.S.)$ condition due to the compact embedding Theorem 1.1. Moreover, it is shown in Section 3.5 that the critical points of $I(u)$ are also the critical points of $F(u)$.

By Properties $(g2)$ and $(g3)$, we have the following facts:

Remark 3.1 $(g2')$ $\quad c_1|\xi|^p \le G(\xi) + c_2 \le \frac{1}{p}(\xi g(\xi) + c_3)$.

Remark 3.2 $(g3')$ $\quad |g(\xi)|^{p/(p-1)} \le c_4(\xi g(\xi) + 1)$.

We will use $(g2')$ and $(g3')$ to verify the $(P.S.)$ condition for $I(u)$.

Lemma 3.1 *(i) For all $u \in L^p$, there exists a unique $v(u) \in N_p$ such that*

$$Q(u) = \int_\Omega (G(u+v(u)) - f \cdot (u+v(u)))\, dt\, dx, \tag{3.3}$$

(ii) Suppose $u_j \to u$ in E. Then $v(u_j) \rightharpoonup v(u)$ in L^p and $g(u_j + v(u_j)) \rightharpoonup g(u+v(u))$ in $L^{\frac{p}{p-1}}$,

(iii) $Q(u)$ is of class C^1 on E and for all $u, h \in E$,

$$\langle Q'(u), h\rangle = \int_\Omega (g(u+v(u)) - f) \cdot h\, dt\, dx\,. \tag{3.4}$$

In particular, $Q'(u) : E \to E^*$ is a compact operator.

From now on we denote by C various constants which depend on $||f||_{p/(p-1)}$ and are independent of $u \in E$.

Proof: (i) Fix $u \in E$. Then the functional $v \mapsto \int_\Omega [G(u+v) - f \cdot (u+v)]\, dt\, dx$ is strictly convex and coercive and hence there exists unique $v(u)$ that minimizes this functional.

(ii) Suppose $u_j \to u$ in E. Then $u_j \to u$ in L^p by the compact embedding Theorem 1.1. Since $v(u_j)$ is the minimizer for $\{u_j\}$, we have

$$\begin{aligned}\int_\Omega & [G(u_j + v(u)) - f \cdot (u_j + v(u))]\, dt\, dx \\ & \geq \int_\Omega [G(u_j + v(u_j)) - f \cdot (u_j + v(u_j))]\, dt\, dx. \end{aligned} \tag{3.5}$$

Since (g3) and (g4) imply

$$G(\xi) \leq \frac{1}{\mu}(\xi\, g(\xi) + C) \leq C(|\xi|^p + |\xi|) + C_1,$$

the left hand side of 3.5 is bounded. Further (g1) concludes that

$$\begin{aligned} M & \geq \int_\Omega [G(u_j + v(u_j)) - f \cdot (u_j + v(u_j))]\, dt\, dx, \\ & \geq \int_\Omega [C|u_j + v(u_j)|^{p+1} - f \cdot (u_j + v(u_j)) + C]\, dt\, dx. \end{aligned}$$

Thus $v(u_j)$ is bounded in L^p and hence $v(u_j) \rightharpoonup \overline{v}$ in L^p. Also, the left hand side of (3.5) converges to $\int_\Omega [G(u + v(u)) - f \cdot (u + v(u))]\, dt\, dx$ as $j \to \infty$ and so we have

$$\begin{aligned}\int_\Omega & [G(u + v(u)) - f \cdot (u + v(u))]\, dt\, dx \\ & \geq \liminf_{j \to \infty} \int_\Omega [G(u_j + v(u_j)) - f \cdot (u_j + v(u_j))]\, dt\, dx \\ & \geq \int_\Omega [G(u + \overline{v}) - f \cdot (u + \overline{v})]\, dt\, dx, \end{aligned}$$

where last inequality is due to the weak lower semi-continuity of $w \mapsto \int_\Omega [G(w) - f \cdot w]\, dt\, dx$. Thus $v(u) = \overline{v}$ by the uniqueness of $v(u)$, which implies $v(u_j) \rightharpoonup v(u)$ in L^p.

Now we will show $g(u_j+v(u_j)) \rightharpoonup g(u+v(u))$ in $L^{p/(p-1)}$. Note that $g(u_j+v(u_j))$ is bounded in $L^{p/(p-1)}$ by (g3). Thus $g(u_j+v(u_j)) \rightharpoonup \eta$ in $L^{p/(p-1)}$. We have to show $g(u+v(u)) = \eta$, which will be done by Minty's trick:

For any $w \in L^p$, we have from the monotonicity of $g(\xi)$

$$(g(u_j+v(u_j)) - g(u_j+w), v(u_j) - w) \geq 0. \tag{3.6}$$

Since $g(u_j+v(u_j)) - f \in N^{\perp}_{p/(p-1)}$, $(g(u_j+v(u_j)) - f, v(u_j)) = 0$ and hence $\eta - f \in N^{\perp}_{p/(p-1)}$. Thus

$$(g(u_j+v(u_j)), v(u_j)) = (f, v(u_j)) \rightarrow (f, v(u)) = (\eta, v(u)).$$

Taking limit in (3.6), we have

$$(\eta - g(u+w), v(u) - w) \geq 0 \;\text{ for all }\; w \in L^p.$$

Set $w = v(u) - \tau\tilde{w}(\tau > 0, \tilde{w} \in L^p)$, divide by τ and let $\tau \to 0$,

$$(\eta - g(u+v(u)), \tilde{w}) \geq 0 \quad \text{for all } \tilde{w} \in L^p.$$

Therefore $\eta = g(u+v(u))$.

(iii) By the convexity of $G(\xi)$, we have for all $u, h \in E$ and $z \in \mathbb{R}$

$$\begin{aligned} Q(u+\tau h) - Q(u) &= \int_\Omega [G(u+\tau h + v(u+\tau h)) \\ &\quad -G(u+v(u)) - f\cdot(\tau h + v(u+\tau h) - v(u)]\,dt\,dx \\ &\geq \int_\Omega (g(u+v(u) - f)(\tau h + v(u+\tau h) - v(u)\,dt\,dx \\ &= \tau\int_\Omega (g(u+v(u) - f)h\,dt\,dx, \end{aligned}$$

since $g(u+v(u)) - f \in N^{\perp}_{p/(p-1)}$ and $v(u+\tau h) - v(u) \in N_p$.

Similarly we have by interchanging the role of $Q(u+\tau h)$ and $Q(u)$

$$Q(u+\tau h) - Q(u) \leq \tau\int_\Omega (g(u+\tau h + v(u+\tau h)) - f)h\,dt\,dx.$$

Letting $\tau \to 0$, we get

$$\langle Q'(u), h\rangle = \int_\Omega (g(u + v(u) - f)h\, dt\, dx \quad \text{for all } u, h \in E.$$

Hence by (ii) and the compact embeddings of $E \hookrightarrow L^p$ and its dual $(L^p)^* = L^{p/(p-1)} \hookrightarrow E^*$, $Q' : E \to E^*$ is continuous and compact. □

Proposition 3.1 *Under the conditions (g1)-(g3) and* $f \in L^{p/(p-1)}$, $I(u) \in C^1(E, \mathbb{R})$ *satisfies the Palais-Smale compactness condition (P.S.).*

Proof: From the assumptions of (*P.S.*), we have

$$I(u_j) = \frac{1}{2}(||u_j^+||^2 - ||u_j^-||^2) - Q(u_j) \le M, \tag{3.7}$$

$$|\langle I'(u_j), h\rangle| = |\langle u_j^+ - u_j^-, h\rangle - \langle Q'(u_j), h\rangle| \le m||h||_E, \tag{3.8}$$

where $m = \sup \|I'(u_j)\|_{E^*}$.

Setting $h = u_j$ in (3.8), we have

$$|\ ||u_j^+||^2 - ||u_j^-||^2 - \langle Q'(u_j), u_j\rangle| \le m||u_j||.$$

This then combined (3.7) with gives

$$|\frac{1}{2}\langle Q'(u_j), u_j\rangle - Q(u)| \le M + m||u_j||. \tag{3.9}$$

Since $\int_\Omega (g(u_j + v(u_j)) - f)v(u_j)\, dt\, dx = 0$, it follows from (3.9) that

$$\begin{aligned}\int_\Omega [\frac{1}{2}(u_j + v(u_j))g(u_j + v(u_j)) - G(u_j + v(u_j)) \ &+\ \frac{1}{2}f \cdot (u_j + v(u_j))]\, dt\, dx \\ &\le\ M + m||u_j||. \qquad (3.10)\end{aligned}$$

We want to get an estimation of the left-hand side of (3.10).

From $(g2)$ the first term of left-hand side is estimated as

$$\int_\Omega (u_j + v(u_j))g(u_j + v(u_j))\,dt\,dx$$
$$\leq C\int_\Omega [\frac{1}{2}(u_j + v(u_j))g(u_j + v(u_j)) - G(u_j + v(u_j))]\,dt\,dx + C.$$

and from $(g5)$ the second term as

$$\begin{aligned} |\int_\Omega f\cdot(u_j + v(u_j))| &\leq ||f||_{p/(p-1)}||u_j + v(u_j)||_p \\ &\leq C(\int_\Omega (u_j + v(u_j))g(u_j + v(u_j))\,dt\,dx)^{\frac{1}{p}}. \end{aligned}$$

Two terms combined yield

$$\begin{aligned} C\int_\Omega (u_j + v(u_j))g(u_j + v(u_j))\,dt\,dx &- C(\int_\Omega (u_j + v(u_j))g(u_j + v(u_j))\,dt\,dx)^{\frac{1}{p}} \\ &\leq M + m||u_j||. \end{aligned}$$

Thus

$$\int_\Omega (u_j + v(u_j))g(u_j + v(u_j)) \leq C||u_j||_E + C.$$

Using $(g3')$, we have

$$||g(u_j + v(u_j))||_{\frac{p}{p-1}}^{\frac{p}{p-1}} \leq C\int_\Omega (u_j + v(u_j))g(u_j + v(u_j))\,dt\,dx + C \leq C||u_j||_E + C.$$

Consequently,

$$||g(u_j + v(u_j))||_{p/(p-1)} \leq C(||u_j||_E^{(p-1)/p} + 1). \tag{3.11}$$

Let $h = u_j^+ - u_j^-$ in (3.8) to get

$$\begin{aligned} ||u_j||_E^2 - \langle Q'(u_j), u_j^+ - u_j^-\rangle &= ||u_j||_E^2 - \int_\Omega (g(u_j + v(u_j)) - f)(u_j^+ - u_j^-)\,dt\,dx \\ &\leq m||u_j||_E. \end{aligned}$$

Thus

$$||u_j||_E^2 - ||g(u_j + v(u_j)) - f||_{p/(p-1)}||u_j^+ - u_j^-||_p \leq m||u_j||_E.$$

By (3.11) and the embedding theorem 1.1,

$$||u_j||_E^2 - c'(||u_j||_E^{(p-1)/p} + 1)||u_j||_E \leq m||u_j||_E.$$

Thus $\{u_j\}$ is bounded in E. Now from

$$u_j^+ - u_j^- = I'(u_j) + Q'(u_j),$$

where $I'(u_j) \to 0$ and Q' is compact, we can conclude that $\{u_j\}$ has a convergent subsequence. □

For later use we define

$$Q_0(u) = \min_{v \in N_p} \int_\Omega G(u+v)\,dt\,dx.$$

Note that $Q_0(u)$ is an S^1-invariant and satisfies the following properties as $Q(u)$ does:

(i) for all $u \in E$, there exists a unique $v_0(u) \in N_{p+1}$ such that

$$Q_0(u) = \int_\Omega G(u + v_0(u))\,dt\,dx,$$

(ii) if $u_n \to u$ in E, then $v_0(u_n) \to v_0(u)$ in N_p and

$$g(u_n + v_0(u_n)) \rightharpoonup g(u + v_0(u)) \text{ in } L^{p/(p-1)},$$

(iii) $Q_0(u)$ is of class C^1 on E, $Q_0' : E \to E^*$ is compact and for all $u, h \in E$,

$$\langle Q_0'(u), h\rangle = \int_\Omega g(u + v_0(u))h\,dt\,dx$$

We show the following relations between $Q(u)$ and $Q_0(u)$:

Lemma 3.2 *There are constants $C_1, C_2, \cdots, C_7 > 0$ such that for all $u \in E$,*

$$\|u + v(u)\|_p^p \leq C_1 Q(u) + C_2, \tag{3.12}$$

$$\|u + v_0(u)\|_p^p \leq C_1 Q_0(u) + C_2, \tag{3.13}$$

$$\langle Q'(u), u\rangle \geq p\,Q(u) - C_3(|Q(u)|^{1/p} + 1), \tag{3.14}$$

$$\langle Q_0'(u), u\rangle \geq p\,Q_0(u) - C_3, \tag{3.15}$$

$$|Q(u)| \leq C_4(Q_0(u) + 1), \tag{3.16}$$

$$Q_0(u) \leq C_4(|Q(u)| + 1), \tag{3.17}$$

$$|Q(u) - Q_0(u)| \leq C_5(Q_0(u)^{1/p} + 1), \tag{3.18}$$

$$|Q(u) - Q_0(u)| \leq C_5(|Q(u)|^{1/p} + 1), \tag{3.19}$$

$$||g(u + v(u))||_{p/(p-1)}^{p/(p-1)} \leq C_6\langle Q'(u), u\rangle + C_7(Q_0(u)^{1p} + 1), \tag{3.20}$$

$$||g(u + v_0(u))||_{p/(p-1)}^{p/(p-1)} \leq C_6\langle Q_0'(u), u\rangle + C_7. \tag{3.21}$$

Proof: We will prove (3.12), (3.14), (3.16), (3.18) and (3.20). The rest immediately follows from these with $f \equiv 0$. To prove (3.12) it is enough to get

$$|\xi|^p \leq C(G(\xi) - f \cdot \xi) + C|f|^{p/(p-1)} + C \; for \; \xi \in \mathbb{R},$$

which follows from (g_2') and Young's inequality on $f \cdot \xi$. We again use (g_2') to get

$$\begin{aligned}
pQ(u) &= p\int_\Omega [G(u + v(u)) - f \cdot (u + v(u)]\, dt\, dx \\
&\leq \int_\Omega (u + v(u)) \cdot g(u + v(u)) - pf \cdot (u + v(u))\, dt\, dx + C \\
&\leq \langle Q'(u), u\rangle + C||f||_{p/(p-1)}||u + v(u)||_p + C \\
&\leq \langle Q'(u), u\rangle + C|Q(u)|^{1/p} + C \quad \text{by} \;\; (3.12),
\end{aligned}$$

which implie (3.14). Next we prove (3.16). First by the definition of $Q(u)$, we get

$$\begin{aligned}
Q(u) \;\; &- \;\; Q_0(u) \\
&\leq \int_\Omega G(u + v_0(u)) - f \cdot (u + v_0(u))\, dt\, dx - \int_\Omega G(u + v_0(u))\, dt\, dx \\
&= -\int f \cdot (u + v_0(u))\, dt\, dx \\
&\leq ||f||_{p/(p-1)}||u + v_0(u)||_p \\
&\leq C(Q_0(u)^{1/p} + 1) \quad \text{by} \;\; (3.13).
\end{aligned}$$

Thus we have (3.16) by Young's inequality. And (3.18) can be similarly proved . It follows from $(g3')$, (3.12) and (3.16) that

$$\begin{aligned} ||g(u+v(u))||_{p/(p-1)}^{p/(p-1)} &= \int_\Omega |g(u+v(u)|^{\frac{p}{p-1}}\, dt\, dx \\ &\le C\int_\Omega (u+v(u)\cdot g(u+v(u))\, dt\, dx + C \\ &\le C\langle Q'(u), u\rangle + C\,||f||_{p/(p-1)}\cdot ||u+v(u)||_p + C \\ &\le C\langle Q'(u), u\rangle + C\,|Q(u)|^{1/p} + C \\ &\le C\langle Q'(u), u\rangle + C\,Q_0(u)^{1/p} + C, \end{aligned}$$

which yields (3.20). □

3.1.2 Modified functional

As in Rabinowitz [17], we replace $I(u)$ by a modified functional $J(u) \in C^1(E, \mathbb{R})$. For $u = u^+ + u^- \in E$, we set

$$A(u) = \frac{1}{2}\,\|u^+\|_E^2 - \frac{1}{2}\,\|u^-\|_E^2$$

and $a_1 \equiv 4/(p+2) \in (0,1)$. Let $\delta > 0$ be a constant such that $\delta_0 \equiv a_1(1+\delta)^3 \in (0,1)$ and set $a_0 \equiv a_1(1+\delta)^2$. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a function such that $\chi(\tau) = 1$ for $\tau \le 1$, $\chi(\tau) = 0$ for $\tau \ge 1+\delta$ and $0 \le \chi(\tau) \le 1$ for all $\tau \in \mathbb{R}$. Further we set

$$\psi_1(u) = \chi(\frac{Q(u)+b}{a_1(A(u)^2+1)^{1/2}}),$$

$$\psi_0(u) = 1 - \chi(\frac{Q_0(u)+b_0}{a_0(A(u)^2+1)^{1/2}}),$$

where b, $b_0 > 0$ are constants such that

$$Q_0(u) + b_0 \ge 1, \quad Q(u) + b \ge 1,$$

$$Q_0(u) + b_0 \le (1+\delta)(Q(u)+b) \text{ for all } u \in E.$$

Note that the existence of b, $b_0 > 0$ is ensured by (3.18). By the choice of b, b_0 and the definitions of $\psi_1(u)$ and $\psi_0(u)$, we observe that

$$supp\, \psi_0(u) \cap supp\, \psi_1(u) = \emptyset \tag{3.22}$$

and for $u \in supp\, \psi_1(u) \cup supp\, (1 - \psi_0(u))$,

$$|Q_0(u)|, |Q(u)| \leq \delta_0 |A(u)| + C. \tag{3.23}$$

We now define for $u \in E$,

$$\begin{aligned} J(u) &= \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \frac{1}{2}(1 + \psi_0(u) - \psi_1(u))Q_0(u) \\ & \quad -\frac{1}{2}(1 - \psi_0(u) + \psi_1(u))Q(u) \ \in \ C^1(E, \mathbb{R}). \end{aligned}$$

First we state an inequality that will be often used. For all $u \in E$, it follows from (3.18) that

$$|J(u) - (\frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q_0(u))| \leq C(Q_0(u)^{1/p} + 1), \tag{3.24}$$

where $C > 0$ is a constant independent of $u \in E$. The reason for introducing $J(u)$ is that the first assertion of the following proposition, which says J is almost invariant, holds for $J(u)$ but not for $I(u)$. Using the following proposition, we will show that large critical values of $J(u)$ are also critical values of $I(u)$.

Proposition 3.2 *The functional $J(u) \in C^1(E, \mathbb{R})$ satisfies:*

(i) there is a constant $\alpha > 0$ such that for $u \in E$ and $\theta \in [0, 2\pi)$,

$$|J(T_\theta u) - J(u)| \leq \alpha(|J(u)|^{1/p} + 1),$$

where $(T_\theta u)(t, x) = u(t + \theta, x)$ for $\theta \in [0, 2\pi) \simeq S^1$.

(ii) there is a constant $M_0 > 0$ such that $J(u) \geq M_0$ and $||J'(u)||_E^* \leq$ $\min[J(u)^{-p/(p-1)}, 1]$ imply that $J(u) = I(u)$.

Proof: Since $Q_0(T_\theta u) = Q_0(u)$ and $\psi_0(T_\theta u) = \psi_0(u)$ for all θ and u, we have from the definition of $J(u)$,

$$\begin{aligned} J(T_\theta u) - J(u) = & -\frac{1}{2}(1-\psi_0(u))(Q(T_\theta u) - Q(u)) + \frac{1}{2}\psi_1(u)(Q(u) - Q_0(u)) \\ & - \frac{1}{2}\psi_1(T_\theta u)(Q(T_\theta u) - Q_0(u)). \end{aligned}$$

By (3.18), we have

$$|Q(T_\theta u) - Q(u)|, |Q(u) - Q_0(u)|, |Q(T_\theta u) - Q_0(u)| \le C(Q_0(u)^{1/p} + 1).$$

Hence we get

$$|J(T_\theta u) - J(u)| \le C((1-\psi_0(u)) + \psi_1(u) + \psi_1(T_\theta u)(Q_0(u)^{1/p} + 1).$$

We may suppose that $u \in$ *supp* $(1-\psi_0(\cdot)) \cup$ *supp* $\psi_1(\cdot) \cup$ *supp* $\psi_1(T_\theta \cdot)$. Otherwise we have $J(T_\theta u) = J(u)$. It follows from the definition of $J(u)$ and (3.24) (note that $\delta_0 \in (0,1)$) that

$$\begin{aligned} |J(u)| & \ge |A(u)| - Q_0(u) - C(Q_0(u) + b_0)^{1/p} \\ & \ge |A(u)| - \delta_0|A(u)| - C(|A(u)| + 1)^{1/p} \\ & \ge C_1|A(u)| - C_2 \ge C_1' Q_0(u) - C_2', \end{aligned}$$

which leads us to the conclusion:

$$|J(T_\theta u) - J(u)| \le C'(Q_0(u)^{1/p} + 1) \le C''(|J(u)|^{1/p} + 1).$$

Proof of (ii) of the proposition using the following lemma can be similarly done as in [25] (see also the proof of Proposition 2.2).

Lemma 3.3 *(Tanaka[25]) For $u = u^+ + u^- \in E$ and $h \in E$,*

$$\begin{aligned} \langle J'(u), h\rangle = & (1 + T_2(u))\langle u^+ - u^-, h\rangle \\ & -\frac{1}{2}(1 + \psi_0(u) - \psi_1(u) + T_0(u))\langle Q_0'(u), h\rangle \\ & -\frac{1}{2}(1 - \psi_0(u) + \psi_1(u) + T_1(u))\langle Q'(u), h\rangle, \end{aligned}$$

where $T_0(u)$, $T_1(u)$, $T_2(u) \in C(E, \mathbb{R})$ *and*

$$\mathit{supp}\, T_i \subset \mathit{supp}\, \psi_i \quad \mathit{for}\ i = 0, 1,$$

$$\sup \{|T_i(u)|;\ i = 0, 1, 2, J(u) \geq M\} \to 0 \ \ \mathit{as} \ \ M \to \infty.$$

□

As corollaries to (ii) of Proposition 3.2 and Proposition 3.1, we have

Corollary 3.1 *Whenever* $u \in E$ *satisfies* $J'(u) = 0$ *and* $J(u) \geq M_0$, *then* $I(u) = J(u)$ *and* $I'(u) = 0$.

Corollary 3.2 $J(u)$ *satisfies the* (*P.S.*) *condition on* $\{u \in E; J(u) \geq M_0\}$.

Corollary 3.1 ensures that large critical values of $J(u)$ are critical values of $I(u)$. Hence in what follows we will seek for critical points of $J(u)$ with large critical values.

3.2 Minimax methods and existence result

First we define an S^1-action on E which plays an important role in constructing critical values of the functional $I(u)$.

Definition 3.1 *We define a group action* T_θ *on* E *by* $(T_\theta u)(t,x) = u(t+\theta, x)$ *for* $\theta \in [0, 2\pi) \simeq S^1$ *and* $u \in E$. *Suppose* $\bar{E}$ *is a subspace of E, then E is said to be invariant under the action* T_θ *if* $T_\theta(\bar{E}) = \bar{E}$.

We rearrange positive eigenvalues of A in the following order counting multiplicity, denoted by

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots.$$

Recall that $\phi_{l,m}(x)\, sin\, jt$ and $\phi_{l,m}(x)\, cos\, jt$ corresponds to the same eigenvalues of A. We may arrange eigenfunctions corresponding to the positive eigenvalues in such a way $e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)}, \cdots$ that

i) $e_k^{(1)} = \phi_{l,m}(x)\, sin\, jt$; $e_k^{(2)} = \phi_{l,m}(x)\, cos\, jt$ for same l, m, j, and $k = 1, 2, 3, \cdots$, and

ii) corresponding eigenvalues are such that

$$0 < \mu_1^{(1)} = \mu_1^{(2)} \le \mu_2^{(1)} = \mu_2^{(2)} \le \cdots .$$

We define subspaces E_k^+ $(k \in \mathbb{N})$ by

$$E_k^+ = \overline{span}\{e_1^{(1)},\ e_1^{(2)},\ e_2^{(1)},\ e_2^{(2)}, \cdots, e_k^{(1)},\ e_k^{(2)}\}.$$

Then E_k^+ is S^1-invariant, $\overline{\bigcup_{k=1}^{\infty} E_k^+} = E^+$, and $||u||_E \le \mu_k ||u||_2$ for $u \in E_k^+$. For $u = u^+ + u^- \in E_k^+ \bigoplus E^-$, we have, by (3.24) and (3.13),

$$\begin{aligned}
J(u) &\le \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - Q_0(u) + C(Q_0(u)^{1/p} + 1) \quad \text{by (3.24)} \\
&\le \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - \frac{1}{2}Q_0(u) + C \\
&\le \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - C_1||u + v_0(u)||_p^p + C_2 \quad \text{by (3.13)} \\
&\le \frac{1}{2}||u^+||_E^2 - \frac{1}{2}||u^-||_E^2 - C_1'||u + v_0(u)||_2^p + C_2 \\
&\le \frac{1}{2}||u^+||_E^2 - C_1'\mu_k^{-p/2}||u^+||_E^p - \frac{1}{2}||u^-||_E^2 + C_2.
\end{aligned}$$

Hence there is a constant $R_k > 0$ such that

$$J(u) < 0 \ \text{ for all } \ u \in E_k^+ \oplus E^- \ \text{ with } \ ||u||_E \ge R_k.$$

We may assume $R_k < R_{k+1}$ for all k. To construct a family of minimax sets, we introduce another(simpler) S^1-action $\hat{T}_\theta$ on E by

$$(\hat{T}_\theta u)(t,x) = \sum_{l,j,m} u_{l,j,m} e^{i\theta sign(j)} \phi_{l,m}(x) e^{ijt}.$$

Recall that $\phi_{l,m}(x)\, sin\, jt$ and $\phi_{l,m}(x)\, cos\, jt$ corresponds to the same eigenvalues of A. We may arrange eigenfunctions corresponding to the positive eigenvalues in such a way $e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)}, \cdots$ that

i) $e_k^{(1)} = \phi_{l,m}(x)\, sin\, jt;\ \ e_k^{(2)} = \phi_{l,m}(x)\, cos\, jt$ for same l, m, j, and $k = 1, 2, 3, \cdots$,

and

ii) corresponding eigenvalues are such that

$$0 < \mu_1^{(1)} = \mu_1^{(2)} \le \mu_2^{(1)} = \mu_2^{(2)} \le \cdots .$$

We define subspaces E_k^+ $(k \in \mathbb{N})$ by

$$E_k^+ = \overline{span}\{e_1^{(1)},\, e_1^{(2)},\, e_2^{(1)},\, e_2^{(2)}, \cdots, e_k^{(1)},\, e_k^{(2)}\}.$$

Then E_k^+ is S^1-invariant, $\overline{\bigcup_{k=1}^{\infty} E_k^+} = E^+$, and $||u||_E \le \mu_k ||u||_2$ for $u \in E_k^+$. For $u = u^+ + u^- \in E_k^+ \bigoplus E^-$, we have, by (3.24) and (3.13),

$$\begin{aligned}
J(u) &\le \frac{1}{2}\,||u^+||_E^2 - \frac{1}{2}\,||u^-||_E^2 - Q_0(u) + C(Q_0(u)^{1/p} + 1) \quad \text{by } (3.24)\\
&\le \frac{1}{2}\,||u^+||_E^2 - \frac{1}{2}\,||u^-||_E^2 - \frac{1}{2}\,Q_0(u) + C\\
&\le \frac{1}{2}\,||u^+||_E^2 - \frac{1}{2}\,||u^-||_E^2 - C_1||u + v_0(u)||_p^p + C_2 \quad \text{by } (3.13)\\
&\le \frac{1}{2}\,||u^+||_E^2 - \frac{1}{2}\,||u^-||_E^2 - C_1'||u + v_0(u)||_2^p + C_2\\
&\le \frac{1}{2}\,||u^+||_E^2 - C_1'\mu_k^{-p/2}||u^+||_E^p - \frac{1}{2}\,||u^-||_E^2 + C_2.
\end{aligned}$$

Hence there is a constant $R_k > 0$ such that

$$J(u) < 0 \ \text{ for all } \ u \in E_k^+ \oplus E^- \ \text{ with } \ ||u||_E \ge R_k.$$

We may assume $R_k < R_{k+1}$ for all k. To construct a family of minimax sets, we introduce another(simpler) S^1-action $\hat{T}_\theta$ on E by

$$(\hat{T}_\theta u)(t, x) = \sum_{l,j,m} u_{l,j,m} e^{i\theta sign(j)} \phi_{l,m}(x) e^{ijt}.$$

We denote by $X = (E, \hat{T}_\theta)$ the space E with S^1-action $\hat{T}_\theta$ and $E = (E, T_\theta)$ the space E with S^1-action T_θ. We also denote by X^+, X_n^+, X^- the spaces E^+, E_n^+, E^- with S^1-action $\hat{T}_\theta$. Let

$$\operatorname{Fix} S^1 = \{u \in X : \hat{T}_\theta u = u \text{ for all } \theta \in [0, 2\pi)\}.$$

Definition 3.2 *A mapping $h : X \to E$ is said to be S^1- equivariant if and only if*

$$(h \circ \hat{T}_\theta)(u) = (T_\theta \circ h)(u) \text{ for } u \in X \text{ and } \theta \in [0, 2\pi] \approx S^1.$$

The usual identity map is not $S^1(X, E)$-equivariant. Let us define a new map $\eta : X \to E$ which is $S^1(X, E)$-equivariant and will play the role of the identity map. For $u = \sum_{l,j,m} \gamma_{l,j,m} e^{i\theta_{l,j,m}} \phi_{l,m}(x) e^{ijt}$, let

$$(\eta(u))(t, x) = \sum_{l,j,m} \gamma_{l,j,m} e^{i\theta_{l,j,m}|j|} \phi_{l,m}(x) e^{ijt},$$

where

$$\gamma_{l,j,m} = \gamma_{l,-jm} \text{ non-negative for all } l, j, m,$$

$$\theta_{l,-j,m} = -\theta_{l,j,m} \text{ for all } l, j, m \text{ and } \theta_{l,j,m} \in [0, 2\pi] \text{ for } j > 0.$$

Note that the mapping $\eta : X \to E$ is linear and isometry and it is easy to see the following properties of η:

Lemma 3.4 *(i) $\eta(u) \in C(X, E)$;*

(ii) $\eta(u)$ is S^1-equivariant;

(iii) $\eta(X_k^+ \oplus X^-) = E_k^+ \oplus E^-$ and $||\eta(u)||_E = ||u||_E$ for all $u \in X$;

(iv) If K is precompact in E, the $\eta^{-1}(K)$ is also precompact in X.

Now, similarly as in Chapter 2 we can define a family of minimax sets. Let B_R is the closed unit ball of radius R in E about 0, $D_k = \{B_{R_k} \cap (X_k^+ \oplus X^-)\}$, and

$$\Gamma_k = \{\gamma \in C(D_k, E) : \gamma \text{ satisfies } (\gamma_1) - (\gamma_3)\},$$

where

(γ_1) γ is S^1- equivariant,

(γ_2) $\gamma(u) = \eta(u)$ for all $u \in (D_k \cap \partial B_{R_k}) \cup (D_k \cap FixS_1)$,

(γ_3) $\gamma(u) = \alpha^+(u)\eta(u)^+ + \alpha^-(u)\eta(u)^- + \beta(u)$ for all $u \in D_k$, where $\alpha^+ \in C(D_k, [0,1])$ and $\alpha^- \in C(D_k, [1, \bar{\alpha}))$ is an S^1-invariant functional ($\bar{\alpha} > 1$ depends on γ) and $\beta \in C(D_k, E)$ is a compact and S^1-equivariant mappings such that $\alpha(u) = 1$ and $\beta(u) = 0$ on $(D_k \cap \partial B_{R_k}) \cup (D_k \cap FixS^1)$.

Moreover, set

$$U_k = \{u \in D_{k+1};\ u = x + \rho e_{k+1}^{(1)},\ x \in X_k^+ \oplus X^-,\ \rho \geq 0\},$$

$$\Lambda_k = \{\lambda \in C(U_k, E) : \lambda \text{ satisfies } (\lambda_1) - (\lambda_3) \text{ in the following }\},$$

(λ_1) $\lambda|_{D_k} \in \Gamma_k$,

(λ_2) $\lambda(u) = \eta(u)$ for all $u \in (U_k \cap \partial B_{R_{k+1}}) \cup (U_k \cap (X_k^+ \oplus X^-) \setminus D_k) \cup (U_k \cap FixS_1)$,

(λ_3) $\lambda(u) = \alpha^+(u)\eta(u)^+ + \alpha^-(u)\eta(u)^- + \beta(u)$ for all $u \in U_n$, where $\alpha^+ \in C(U_k, [0,1])$ and $\alpha^- \in C(U_k, [1, \bar{\alpha}))$ ($\alpha > 0$ depends on γ) and $\beta \in C(U_k, E)$ is a compact mapping such that $\alpha(u) = 1$ and $\beta(u) = 0$ on $(U_k \cap \partial B_{_{k+1}}) \cup ((U_k \cap (X_k^+ \oplus X^-) \setminus D_k) \cup (U_k \cap FixS^1)$.

Note that $\Gamma_k \neq \emptyset$ and $\Lambda_k \neq \emptyset$ since $\eta|_{D_k} \in \Gamma_k$ and $\eta|_{U_k} \in \Lambda_k$. Define

$$b_k = \inf_{\gamma \in \Gamma_k} \sup_{u \in D_k} J(\gamma(u)), \quad c_k = \inf_{\lambda \in \Lambda_k} \sup_{u \in U_k} J(\lambda(u)).$$

Then we easily see that $c_k \geq b_k$, moreover if $c_k > b_k$, we have the following existence result.

First we have the same Deformation Lemma as in Chapter 2 since J satisfies $(P.S.)$ condition (Corollary 3.2) and $J'(u)$ is an operator of the form

$$J'(u) = (1 + T_1(u))(u^+ - u^-) + compact,$$

where $|T_1(u)| \leq 1/2$ on $\{u \in E;\ J(u) \geq M_0\}$ (see proof of Lemma 3.3). We state it again here.

Lemma 3.5 *(cf. [18], [19]) (Deformation Lemma) Suppose $C > M_0$ is a regular value of $J(u)$. Then for any $\bar{\varepsilon} > 0$ there exist an $\varepsilon \in (0, \bar{\varepsilon})$ and an one parameter family of homeomorphisms $\Phi(t, \cdot)$ of E, $0 \le t \le 1$ with the properties:*

(i) $\Phi(t,u) = u$, if $t = 0$, or $|J(u) - c| \ge \bar{\varepsilon}$;

(ii) $\Phi(1, A_{c+\varepsilon}) \le A_{c-\varepsilon}$ where $A_c = \{u \in E : J(u) \le c\}$;

(iii) $\Phi(1,u) = \alpha^+(u)u^+ + \alpha^-(u)u^- + \kappa(u)$, where $\alpha^+ \in C(E, [0,1))$, $\alpha^- \in C(E, [1, \bar{\alpha}))$ ($\bar{\alpha} > 1$ constant) and κ is a compact operator.

By standard contradiction argument using this Lemma, we get the critical values $\{c_k(\delta)\}$ of $J(u)$ as in the following lemma.

Lemma 3.6 *Suppose $c_k > b_k \ge M_2$. Let $\delta \in (0, c_k - b_k)$ and*

$$\Lambda_k(\delta) = \{\lambda \in \Lambda_k : J(\lambda) \le b_k + \delta \text{ on } D_k\}.$$

Let

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \max_{u \in U_k} J(\lambda(u)) \ \ (\ge c_k).$$

Then $c_k(\delta)$ is a critical value of $I(u)$.

Therefore the existence of a subsequence of $\{c_k\}_{k=1}^{\infty}$ which satisfy $c_{k_j} > b_{k_j} \ge M_0$ ensures the existence of critical values of $I(u)$. In what follows, we will show that there is a subsequence $\{k_j\}_{j=1}^{\infty}$ such that

$$c_{k_j} > b_{k_j} \ \ \text{for } j \in \mathbb{N},$$

$$b_{k_j} \to \infty \ \ \text{as } j \to \infty.$$

Arguing alternatively, we have

Proposition 3.3 *Assume $c_k = b_k$ for all $k \ge k_0$, then there is a constant $C > 0$ such that*

$$b_k \le C k^{p/(p-1)} \ \ \text{for all } k \in \mathbb{N}.$$

Proof: Note that $D_{k+1} = \bigcup_{\theta\in[0,2\pi)} \hat{T}_\theta(U_k)$ and for any $u \in D_{k+1} \setminus (X_k^+ \oplus X^-)$ there is a unique $(x,\theta) \in (U_k \backslash D_k) \times [0,2\pi)$ such that $T_\theta x = u$. For any given $\lambda \in \Lambda_k$, we define $\gamma : D_{k+1} \to E$ by

$$\lambda(u) = T_\theta(\lambda(x)) \text{ for } u = \hat{T}_\theta x \in D_{k+1} \text{ where } (x,\theta) \in U_k \times [0,2\pi).$$

We can see λ is well-defined (by (λ_1)), continuous and belongs to Γ_{k+1}. Moreover by (i) of Proposition 3.2, we have

$$\begin{aligned} b_{k+1} &\leq \sup_{u\in D_{k+1}} J(\lambda(u)) = \sup_{x\in U_k,\theta\in[0,2\pi)} J(T_\theta\lambda(x)) \\ &\leq \sup_{x\in U_k} [J(\lambda(x)) + \alpha\,(|J(\lambda(x))|^{1/p} + 1)]. \end{aligned}$$

Since $\lambda \in \Lambda_k$ is arbitrary, we deduce

$$b_{k+1} \leq c_k + \alpha(c_k^{1/p} + 1) \ \ for\ all\ k.$$

If $c_k = b_k \ for\ k \geq k_0$, we obtain

$$b_{k+1} \leq b_k + \alpha(b_k^{1/p} + 1) \ \ for\ k \geq k_0.$$

An induction argument yields the desired result. □

Our goal in the next two sections is proving that there exists a subsequence $\{k_j\}_{j=1}^{\infty}, \varepsilon > 0$ and $C_\varepsilon > 0$ satisfying

$$b_{k_j} > C_\epsilon k_j^{p/(p-1-\varepsilon)} \text{ for all } j \in \mathbb{N}. \tag{3.25}$$

3.3 Critical values β_k of a comparison functional $K(u)$

3.3.1 Introduction of comparison functional $K(u)$

To estimate b_k, we introduce a new comparison functional $K(u)$ here. By (3.24), the definition of $Q_0(u)$ and $(g3)$, we have for $u^+ \in E^+$

$$\begin{aligned} J(u^+) &\geq \frac{1}{2}||u^+||_E^2 - CQ_0(u^+) - C \\ &\geq \frac{1}{2}||u^+||_E^2 - C\int_\Omega G(u^+)\,dt\,dx - C \\ &\geq \frac{1}{2}||u^+||_E^2 - \frac{\bar{a}_1}{p}||u^+||_p^p - \bar{a}_2, \end{aligned}$$

where $C, \bar{a}_1, \bar{a}_2 > 0$ are constants independent of $u^+ \in E^+$. We define a comparison functional K on E^+ by

$$K(u) = \frac{1}{2}||u^+||_E^2 - \frac{\bar{a}_1}{p}||u^+||_p^p.$$

Then $K(u) \in C^2(E^+, \mathbb{R})$ and it is easy to show that $K(u)$ satisfies the $(P.S.)$ condition. So we have the following lemma:

Lemma 3.7 *(i)* $J(u) \geq K(u) - \bar{a}_2$ *for all* $u \in E^+$.

(ii) K satisfies the Palais-Smale condition $(P.S.)$.

3.3.2 Bahri-Berestycki's max-min value β_k

First let us define a family of max-min sets for $K(u)$. Recall that

$$S^{2m-2k+1} = \{z \in \mathbb{C}^{m-k+1}; |z| = 1\},$$

and the group $S^1 = \{e^{i\theta}\}$ acts naturally on it by

$$e^{i\theta}z = (e^{i\theta}z_1, e^{i\theta}z_2, ..., e^{i\theta}z_{m-k+1}) \text{ for } z = (z_1, z_2, ..., z_{m-k+1}) \in S^{2m-2k+1}.$$

For $m > k,\ k,\ m \in \mathbb{N}$, set

$$A_k^m = \{\sigma \in C(S^{2m-2k+1}, E_m^+) : \sigma(e^{i\theta}x) = T_\theta\sigma(x) \text{ for all } x \in S^{2m-2k+1}\},$$

$$\beta_k^m = \sup_{\sigma\in A_k^m} \min_{x\in S^{2m-2k+1}} K(\sigma(x)).$$

We will prove that $\beta_k = \lim_{j\to\infty}\beta_k^{m_j}$ is a sequence of critical values of $K(u)$ and $b_k \geq \beta_k + C$. To get some estimates on β_k^m's, we need several lemmas. First we state a version of a Borsuk-Ulam lemma.

Lemma 3.8 *Let $a, b, N \in \mathbb{N}$. Suppose that $g \in C(\mathbb{R}^N \times \mathbb{C}^a, \mathbb{R}^N \times \mathbb{C}^{a+b})$ and $h \in C(S^{2b+1}, \mathbb{R}^N \times \mathbb{C}^{a+b})$ satisfy the following conditions:*

(i) $g = (g_1 ..., g_N, g_{N+1}, ..., g_{N+a+b})$ and $h = (h_1, ..., h_{N+a+b})$ are S^1 -equivariant in the following sense: for all $1 \leq j \leq N$ and $1 \leq l \leq a+b$,

$$g_j(x, e^{i\theta}y) = g_j(x,y),\ g_{N+l}(x, e^{i\theta}y) = e^{ik_l\theta}g_{N+l}(x,y),$$
$$h_j(e^{i\theta}z) = h_j(z),\ h_{N+1}(e^{i\theta}z) = e^{ik_l\theta}h_{N+l}(z)$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{C}^a$ and $z \in S^{2b+1}$, where $k_l \neq 0$ are integers;

(ii) $g(x,0) = (x,0)$ for all $x \in \mathbb{R}^N$;

(iii) there is a $\gamma_0 > 0$ such that

$$|g(x,y)|^2 = |x|^2 + |y|^2 \ \ for \ \ |x|^2 + |y|^2 \geq \gamma_0^2.$$

Then

$$h(S^{2b+1}) \cap g(\mathbb{R}^N \times \mathbb{C}^a) \neq \emptyset.$$

Proof: Consider the following S^1-equivariant continuous mapping.

$F : \mathbb{R}^N \times \mathbb{C}^a \times \mathbb{C}^{b+1} \to \mathbb{R}^N \times \mathbb{C}^a \times \mathbb{C}^b$; $F(x,y,tz) = g(x,y) - th(z)$, where $x \in \mathbb{R}^N$, $y \in \mathbb{C}^a$ and $tz \in \mathbb{C}^{b+1} = \{tz; t \geq 0, z \in S^{2b+1}\}$. Set $R = \max\{\gamma_0, \max\{|h(z)|; z \in S^{2b+1}\}\} + 1$ and

$$\Omega = \{(x,y,tz); |x|^2 + |z|^2 < R^2, t \in [0,1), z \in S^{2b+1}\}.$$

Applying S^1-version of Borsuk-Ulam theorem to $F : \partial\Omega \to \mathbb{R}^N \times \mathbb{C}^a \times \mathbb{C}^b$, there exists $(x_0,\ y_0,\ t_0z_0) \in \partial\Omega = \{(x, y, tz) \in \bar{\Omega}; |x|^2 + |y|^2 = R^2 \text{ or } t = 1\}$ such that

$$F(x_0,\ y_0,\ t_0z_0) = 0,\ \ i.e.,\ \ g(x_0,\ y_0) = t_0h(z_0).$$

From the choice of R, $F(x, y, tz) \neq 0$ on $\partial\Omega \cap \{(x, y, tz); |x|^2 + |y|^2 = R^2\}$. Therefore we have $t_0 = 1$ and $g(x_0, y_0) = h(z_0)$. □

We also need the following technical lemma(same as Lemma 2.8).

Lemma 3.9 *For all $\theta \in (0, 1/p)$, there is a $C_\theta > 0$ independent of $k \in \mathbb{N}$ such that*

$$||u||_p \leq C_\theta \mu_k^{-\theta} ||u||_E \ \ for\ \ u \in (E_k^+)^\perp,$$

where $(E_k^+)^\perp = \{v \in E^+; \langle v, e_i\rangle = 0 \ \ for\ \ i = 1, 2, ..., k\}$.

Now we can prove the following estimates on β_k^m's(see Proposition 2.5).

Proposition 3.4 *(i) $0 \leq \beta_k^m \leq \beta_{k+1}^m < \infty$ for all $m,\ k \in \mathbb{N}$;*

(ii) For all $k \in \mathbb{N}$, there exists $\nu(k)$ and $\tilde{\nu}(k)$ such that

$$0 \leq \nu(k) \leq \beta_k^m \leq \tilde{\nu}(k) < \infty \ \ for\ all\ \ m \geq k + 1;$$

(iii) $\nu(k) \to \infty$ as $k \to \infty$.

Proof: (i) For any $\sigma \in A_k^m$, $\sigma|_{S^{2m-2k-1}} \in A_{k+1}^m$ and $\sigma|_{S^{2m-2k-1}}(S^{2m-2k-1}) \subset \sigma(S^{2m-2k+1})$. Hence we have $\beta_k^m \leq \beta_{k+1}^m$.

(ii) First we prove the existence of $\tilde{\nu}(k)$. Applying Lemma 3.8 to $h = \sigma : S^{2m-2k+1} \to E_m^+$ *and* $g = id : E_k^+ \to E_m^+$, we can see that

$$\sigma(S^{2m-2k+1}) \cap E_k^+ \neq \emptyset \ \ \text{for all}\ \ \sigma \in A_k^m.$$

Thus we have for all $\sigma \in A_k^m$,

$$\min_{x\in S^{2m-2k+1}} K(\sigma(x)) \leq \sup_{u\in E_k^+} K(u). \tag{3.26}$$

For $u \in E_k^+$, we have

$$\begin{aligned} K(u) &= \frac{1}{2}||u||_E^2 - \frac{a_0}{p}||u||_p^p \leq \frac{1}{2}||u||_E^2 - C||u||_2^p \\ &\leq \frac{1}{2}||u||_E^2 - C\mu_k^{-p/2}||u||_E^p. \end{aligned}$$

Thus the right-hand side of (3.26) is finite and independent of σ and m. Set

$$\tilde{\nu}(k) = \sup_{u\in E_k^+} K(u) < \infty,$$

then we obtain

$$\beta_k^m = \sup_{\sigma\in A_k^m} \min_{x\in S^{2m-2k+1}} K(\sigma(x)) \leq \tilde{\nu}(k).$$

Now we show the existence of $\nu(k)$. We construct a special $\sigma \in A_k^m$ as follows: write

$$S^{2m-2k+1} = \{x = (x_{2k}, ..., x_{2m+1}) \in \mathbb{R}^{2m-2k+2};\ \sum_{i=2k}^{2m+1} x_i^2 = 1\}$$

and set $\sigma : S^{2m-2k+1} \to E_m^+\backslash 0$ by

$$\sigma(x) = a_0^{-1/(p-2)}||w(x)||_p^{-p/(p-2)}\omega(x),$$

where $w(x)$ is defined by $w(x) = \sum_{i=2k}^{2m+1} x_i e_i$. Obviously we have $\sigma \in A_k^m$. Since $||w(x)||_E = 1$ on $S^{2m-2k+1}$, we have

$$K(\sigma(x)) = (\frac{1}{2} - \frac{1}{p})\, a_0^{-2/(p-2)}||w(x)||_p^{-2p/(p-2)}.$$

On the other hand $w(x) \in (E_{k-1}^-)^\perp$, $||w(x)||_E = 1$ for all $x \in S^{2m-2k+1}$, and hence it follows from Lemma 3.9 that

$$||w(x)||_p \leq C_\theta\, \mu_{k-1}^{-\theta} \text{ for } x \in S^{2m-2k+1},$$

where $\theta \in (0, 1/p)$ and C_θ is a constant independent of k and x. Therefore

$$K(\sigma(x)) \geq C'_\theta \, \mu_{k-1}^{2\theta p/(p-2)} \text{ for all } x \in S^{2m-2k+1}.$$

The right-hand side the above inequality is independent of m. Set $\nu(k) = C'_\theta \, \mu_{k-1}^{2\theta p/(p-2)}$. Then we have

$$\beta_k^m \geq \min_{x \in S^{2m-2k+1}} K(\sigma(x)) \geq \nu(k) \text{ for } m > n.$$

(iii) Since $\mu(k) \to \infty$ as $n \to \infty$, we obtain $\nu(k) \to \infty$ as $k \to \infty$. □

As in Proposition 3.1, we can prove the following compactness conditions $(P.S.)_m$, $(P.S.)_*$ for $K(u)$:

$(P.S.)_m$: If $\{u_j\} \subset E_m^+$ satisfies $K(u_j) \leq C$ and $(K|_{E_m^+})'(u_j) \to 0$ as $j \to \infty$, then $\{u_j\}$ is relatively compact in E_m^+.

$(P.S.)_*$: If $\{u_m\} \subset E^+$ satisfies $u_m \in E_m^+$, $K(u_m) \leq C$ and $\|(K|_{E_m^+})'(u_m)\|_{(E_m^+)^*} \to 0$ as $m \to \infty$, then $\{u_m\}$ is relatively compact in E^+.

Since K is an even functional, we have the following results via standard argument. (Bahri and Berestycki [4])

Proposition 3.5 *Suppose $\nu(k) > 0$. Then β_k^m is a critical value of $K|_{E_m^+}$. And the limit of any convergent subsequence of β_k^m as $m \to \infty$ is a critical value of K.*

By (ii) of Proposition 3.4, choose a sequence $\{m_j\}$ such that $m_j \to \infty$ as $j \to \infty$ and

$$\beta_k = \lim_{j \to \infty} \beta_k^{m_j} \text{ exists for all } k \in \mathbb{N}.$$

Then by the above Proposition 3.5, we have the following properties for β_k.

Proposition 3.6 *i) β_k is a critical value of $K \in C^2(E^+, \mathbb{R})$ for each $k \in \mathbb{N}$;*

ii) $\beta_k \leq \beta_{k+1}$ for all $k \in \mathbb{N}$;

iii) $\beta_k \to \infty$ as $k \to \infty$.

Here we establish the comparison result between critical values of $J(u)$ and $K(u)$.

Proposition 3.7 *For all* $k \in \mathbb{N}$,

$$b_k \geq \beta_k - \bar{a}_2,$$

where $\bar{a}_2$ *is the number appeared in Lemma 3.7.*

First we state a linking lemma which can be proved using Borsuk-Ulam Lemma 3.8.

Lemma 3.10 *For all* $\gamma \in \Gamma_k$ *and for all* $\sigma \in A_k^m$,

$$((P_m\gamma)(D_k) \cup \{u \in E_k^+ \oplus E^- : ||u||_E \geq R_k\}) \cap \sigma(S^{2m-2k+1}) \neq \emptyset,$$

where $P_m : E \to E_m^+ \oplus E^-$ *is an orthogonal projection.*

Proof: Let $\gamma \in \Gamma_k$ and extend γ to $\tilde{\gamma} \in C(X_k^+ \oplus X^+, X)$ by $\tilde{\gamma}(u) = \gamma(u)$ if $||u||_E \leq R_k$, and $\tilde{\gamma}(u) = \eta(u)$ if $||u||_E \geq R_k$. Obviously, $\tilde{\gamma}(u)$ is well defined and S^1-equivariant. Since $m > k$, by definition of $\eta(u)$ we have

$$P_m\tilde{\gamma}(X_k^+ \oplus X^-) = P_m\gamma(D_k) \cup \{u \in E_k^+ \oplus E^-; ||u||_E \geq R_k\}.$$

Therefore it suffices to prove $P_m\tilde{\gamma}(X_k^+ \oplus X^-) \cap \sigma(S^{2m-2k+1}) \neq \emptyset$. We rearrange negtive eigenfunctions and denote by $f_1, f_2, f_3, \ldots$. We set for $l \in \mathbb{N}$,

$$E_l^- = span\{f_j; 1 \leq j \leq l\}$$

and let $P_{m,l} : E = E^+ \oplus E^- \to E_m^+ \oplus E_l^-$ be the orthogonal projection. Consider the operators

$$\sigma : S^{2m-2k+1} \to E_m^+ \subset E_m^+ \oplus E_l^-, \quad P_{m,l}\tilde{\gamma} : X_m^+ \oplus X_l^- \to E_m^+ \oplus E_l^-.$$

Applying Lemma 3.8 for $h = \sigma$ and $g = P_{m,l}\tilde{\gamma}|_{X_k^+ \oplus X_l^-}$, we get some $x_l \in S^{2m-2k-1}$ and $u_l \in E_k^+ \oplus E_l^-$,

$$\sigma(x_l) = P_{m,l}\tilde{\gamma}(u_l). \tag{3.27}$$

Since $S^{2m-2k+1}$ is compact, there is a subsequence $\{x_{l_j}\}$ such that

$$x_{l_j} \to x \text{ in } S^{2m-2k+1}, \quad \sigma(x_{l_j}) \to \sigma(x) \text{ in } E_m^+.$$

Now, using (γ_3) similarly as in the proof of Lemma 2.10 we can show that there exists $u \in X_k^+ \oplus X^-$ such that $P_m \tilde{\gamma}(u) = \sigma(x)$. This completes the proof. □

Now we prove Proposition 3.7, the main result of this section.

Proof: First we recall that $J(\gamma(u)) \leq 0$ for $u \in E_k^+ \oplus E^-$ with $||u||_E \geq R_k$ by the choice of R_k. Using Lemma 3.10 and Lemma 3.7, we can see

$$\begin{aligned} b_k^m \equiv \inf_{\gamma \in \Gamma_k} \sup_{u \in D_k} J(P_m \gamma(u)) &\geq \sup_{\sigma \in A_k^m} \min_{x \in S^{2m-2k+1}} J(\sigma(x)) \\ &\geq \sup_{\sigma \in A_k^m} \min_{x \in S^{2m-2k+1}} K(\sigma(x)) - \bar{a}_2, \end{aligned}$$

that is,

$$b_k^m \geq \beta_k^m - \bar{a}_2 \text{ for all } m > k.$$

Hence we have

$$\liminf_{m \to \infty} b_k^m \geq \beta_k - \bar{a}_2. \tag{3.28}$$

On the other hand, we have

$$\limsup_{m \to \infty} b_k^m \leq b_k. \tag{3.29}$$

In fact, it follows from (γ_3) that for $\gamma \in \Gamma_k$

$$P_m \gamma(u) = \alpha^+ \eta(u)^+ \alpha^- \eta(u)^- + P_m \beta(u) \longrightarrow \alpha^+ \eta(u)^+ \alpha^- \eta(u)^- + \beta(u) = \gamma(u),$$

uniformly in D_k as $m \to \infty$. Hence we have

$$\sup_{u \in D_k} J(P_m \gamma(u)) \to \sup_{u \in D_k} J(\gamma(u)) \text{ as } m \to \infty.$$

Choosing $\gamma \in \Gamma_n$ such that $\sup_{u \in D_k} J(\gamma(u)) \leq b_k + \varepsilon$, we obtain

$$\limsup_{m \to \infty} b_k^m \leq \limsup_{m \to \infty} \sup_{u \in D_k} J(P_m \gamma(u)) = \sup_{u \in D_k} J(\gamma(u)) \leq b_k + \varepsilon.$$

Thus (3.29) holds since the above inequality holds for any $\varepsilon > 0$. Combining (3.28) and (3.29), we get the estimate of the proposition. □

3.4 Morse index and β_k

We want to get $\beta_{k_j} \geq C_\epsilon k_j^{p/(p-1-\epsilon)}$ for all $j \in \mathbb{N}$. Estimates of Morse index at β_{k_j}'s will give the result. We proceed similarly as in Chapter 2.

Definition 3.3 *For $u \in E^+$, we define a index of $K''(u)$ by*

$$\begin{aligned} \textit{index}\, K''(u) &= \textit{the number of nonpositive eigenvalues of}\, K''(u) \\ &= \max \{\dim S; S \leq E^+ \textit{ such that } \langle K''(u)h, h\rangle \leq 0,\ h \in S\}. \end{aligned}$$

Here "$A \leq B$" in the bracket means A is a subspace of B.

Proposition 3.8 *Suppose $\beta_k < \beta_{k+1}$. Then there exists $u_k \in E^+$ such that*

$$K(u_k) \leq \beta_k,$$
$$K'(u_k) = 0,$$
$$\textit{index}\, K''(u_k) \geq 2k - 1.$$

By definnitin of β_k, the result without the last assertion is obvious. To get the last assertion, we first consider finite dimensional case.

Proposition 3.9 *Suppose $\beta_k^m < \beta_{k+1}^m$, $m > n+1$. Then there exists $u_k^m \in E_m^+$ such that*

$$K(u_k^m) \leq \beta_k^m,$$
$$(K|_{E_m^+})'(u_k^m) = 0,$$
$$\textit{index}\,(K|_{E_m^+})''(u_k^m) \geq 2k - 1.$$

To prove the above proposition, we will use a theorem from Morse theory, i.e., a result concerning the relationship between certain homotopy groups of level sets of a functional and its critical points. We proceed as in Chapter 2. First we need a theorem to treat the case where critical points may be degenerate.

Proposition 3.10 *(cf. Marino-Prodi [15]) Let U be a C^2 open subset in some Hilbert space H and $\phi \in C^2(U, \mathbb{R})$. Assume ϕ'' is a Fredholm operator (of null index) on the critical set $Z(\phi) = \{x \in U; \phi'(x) = 0\}$, ϕ satisfies (P.S.) and $Z(\phi)$ is compact. Then, for any $\varepsilon > 0$, there exists $\psi \in C^2(U, \mathbb{R})$ satisfying (P.S.) and with the following properties :*

(i) $\psi(x) = \phi(x)$ if distance $\{x, Z(\phi)\} \geq \varepsilon$;

(ii) $|\psi(x) - \phi(x)|, \|\psi'(x) - \phi'(x)\|, \|\psi''(x) - \phi''(x)\| \leq \varepsilon$ for all $x \in U$;

(iii) the critical points of ψ are finite in number and nondegenerate.

We can easily prove that $K|_{E_m^+}$ satisfies all the assumptions of the above proposition, that is,

1) $K|_{E_m^+} \in C^2(E_m^+, \mathbb{R})$ satisfies $(P.S.)$ and Fredholm.

2) All critical value of $K|_{E_m^+}$ are non-negative because

$$K(u) = K(u) - \frac{1}{2}\langle (K|_{E_m^+})'(u), u\rangle = (\frac{1}{2} - \frac{1}{p})\, a_0\, ||u||_p^p \geq 0.$$

3) $Z(K|_{E_m^+})$ is compact. In fact, note that there exists $\tilde{R}_m > 0$ such that $K(u) < 0$ for $u \in E_m^+$ with $||u||_E \geq \tilde{R}_m$; hence $Z(K|_{E_m^+})$ is bounded.

Thus by Proposition 3.10, for all $\varepsilon > 0$ there exists $\phi_\varepsilon \in C^2(E_m^+, \mathbb{R})$ satisfying $(P.S.)$ and

$$\begin{aligned} |\phi_\varepsilon(u) - K(u)| &< \varepsilon, \\ ||\phi_\varepsilon'(u) - (K|_{E_m^+})'(u)|| &< \varepsilon, \\ ||\phi_\varepsilon''(u) - (K|_{E_m^+})''(u)|| &< \varepsilon; \end{aligned} \tag{3.30}$$

the critical points of ϕ_ε are finite and non-degenerate. (3.31)

For $m > k$ *and* $\varepsilon > 0$, let

$$\beta_k^m(\varepsilon) = \sup_{\sigma \in A_k^m} \min_{x \in S^{2m-2k+1}} \phi_\varepsilon(\sigma(x)).$$

By (2.31),

$$\beta_n^m - \varepsilon \le \beta_k^m(\varepsilon) \le \beta_k^m + \varepsilon.$$

Moreover, we have

Lemma 3.11 *Suppose that* $a_\varepsilon \in \mathbb{R}$ *satisfies* $\beta_k^m(\varepsilon) < a_\varepsilon - 2\varepsilon < a_\varepsilon < \beta_{k+1}^m(\varepsilon)$. *Then*

$$\pi_{2m-2k-1}([\phi_\varepsilon \ge a_\varepsilon]_m, w) \ne 0 \ \textit{for some}\ w \in [\phi_\varepsilon \ge a_\varepsilon]_m,$$

where $[\phi_\varepsilon \ge a_\varepsilon]_m = \{u \in E_m^+;\ \phi_\varepsilon(u) \ge a_\varepsilon\}$.

Proof: We argue by contradiction. Suppose that

$$\pi_{2m-2k-1}([\phi_\varepsilon \ge a_\varepsilon]_m, w) = 0 \ \text{ for all } \ w \in [\phi_\varepsilon \ge a_\varepsilon]_m.$$

Then there is a homotopy

$$H : [0,1] \times S^{2m-2k-1} \longrightarrow [\phi_\varepsilon \ge a_\varepsilon]_m$$

such that $H(0,x) = \sigma(x)$, $H(1,x) = w_0$ for all $x \in S^{2m-2k-1}$. Write

$$S^{2m-2k+1} = \{\xi = (\zeta, \rho e^{i\theta});\ \zeta \in \mathbf{C}^{m-k},\ \rho \in \mathbb{R},\ |\zeta|^2 + \rho^2 = 1\}.$$

By the definition of $\beta_{k+1}^m(\varepsilon)$, there is a $\sigma \in A_{k+1}^m$ such that $\sigma(S^{2m-2k-1}) \subset [\phi_\varepsilon \ge a_\varepsilon]_m$. Define $\tilde{\sigma} : S^{2m-2k-1} \longrightarrow E_m^+$ by

$$\tilde{\sigma}(\zeta, \rho e^{i\theta}) = \begin{cases} \sigma(\zeta) & \text{if } \rho = 1,\ |\zeta| = 1, \\ T_\theta H(e^{-i\theta} \frac{\zeta}{\rho|\zeta|}) & \text{if } \rho \ne 0,\ \zeta \ne 0, \\ T_\theta w_0 & \text{if } \rho = -1,\ \zeta = 0. \end{cases}$$

Then we can easily check that $\tilde{\sigma} \in A_k^m$. Since K is invariant under the action T_θ, by (3.30), we have

$$|\phi_\epsilon(u) - \phi_\epsilon(T_\theta u)| \le 2\epsilon \ \text{ for } \ u \in E_m^+.$$

Thus $\phi_\epsilon(\tilde{\sigma}(\zeta, \rho e^{i\theta})) \geq a_\epsilon - 2\epsilon$, i.e., $\tilde{\sigma}(S^{2m-2k+1}) \subset [\phi_\varepsilon \geq a_\varepsilon - 2\varepsilon]_m$. From the definition of $\beta_n^m(\varepsilon)$,

$$\beta_n^m(\varepsilon) \geq \min_{x \in S^{2m-2k+1}} \phi_\varepsilon(\tilde{\sigma}(x)) \geq a_\varepsilon - 2\varepsilon.$$

But this contradicts with the assumption. Thus the proof is completed. □

Now the proofs of Proposition 3.9 and Proposition 3.8 can be similarly done as those of Propositon 2.9 and Proposition 2.9 using Lemma 3.11 and the following Lemma 3.12.

Lemma 3.12 *For a regular value $a \in \mathbb{R}$ of ϕ_ε , set*

$$L(\varepsilon; a) = \max\{\, index\ \phi_\varepsilon''(x); \phi_\varepsilon(x) \leq a,\ \phi_\varepsilon'(x) = 0\}.$$

Then

$$\pi_l([\phi_\varepsilon \geq a]_m, w) = 0 \quad for\ all\ \ p \in [\phi_\varepsilon \geq a_\varepsilon]_m,\ \ l \leq 2m - L(\varepsilon; a) - 2.$$

3.5 Proof of the Main Theorem

By Lemma 3.6 and Proposition 3.3, we know that (3.25), the growth estimate on β_k's, ensures the existence of an unbounded sequence of critical values. We now prove (3.25). First note by Proposition 3.8 that there exits u_{k_j} such that

$$\beta_{k_j} \geq K(u_{k_j}) = \frac{1}{2}||u_{k_j}||_E^2 - \frac{a_0}{p}||u_{k_j}||_p^p = (\frac{1}{2} - \frac{1}{p})\, a_0\, ||u_{k_j}||_p^p.$$

Due to Proposition 3.8, we can get an upper bound of index $K''(u_j)$ same as in Proposition 2.11.

Proposition 3.11 *There exist $C > 0$ such that for $u \in E^+$,*

$$index\, K''(u_j) \leq C||u||_s^r,$$

where $r = \frac{2(p-2)nq}{n+1-(n-1)q}$ *and* $s = \frac{(p-2)q}{q-1}$.

Then by the same proof as in the case of $g(u) = |u|^{p-2}u$, we get (3.25) for the same p's satisfying $2 < p < \frac{7n+1+\sqrt{25n^2-2n+9}}{2(3n-1)}$.

This establishes the existence of a sequence $\{u_k\} \subset E$ of critical points of $I(u)$ such that as $k \to \infty$,

$$I(u_k) \to \infty \text{ and } I'(u_k) = 0.$$

Let $\bar{u}_k = u_k + v(u_k)$. Then it can be shown that $\bar{u}_k$ is a critical point of $F(u)$ by direct calculation. On the other hand since $I'(u_k) = 0$, we have

$$I(u_k) = \int_\Omega \frac{1}{2} g(\bar{u}_k)\,\bar{u}_k - G(\bar{u}_k) + \frac{1}{2} f\,\bar{u}_k\,dx\,dt \to \infty.$$

Finally it follows from ($g3$) that $\{\bar{u}_k\}$ is a unbounded sequence in L^p. We have proved that there exists a unbounded sequence of critical points for $F(u)$, which is a unbounded sequence of the weak solutions of the nonlinear wave equation (1) on S^n.

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