This is to certify that the

dissertation entitled

Mathematical Analysis of Maxwell's Equations

presented by

Aurelia Minut

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Major professor

Date May 29, 2001

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.
MAY BE RECALLED with earlier due date if requested.

DATE DUE	DATE DUE	DATE DUE

Mathematical Analysis of Maxwell's Equations

By

Aurelia Minut

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2001

Abstract

Mathematical Analysis of Maxwell's Equations

By

Aurelia Minut

The goal of this dissertation is to establish interior L^p-type estimates for the solutions of Maxwell's equations with source term in a domain with two different materials separated by a C^2 interface. Such an estimate is employed to solve a nonlinear optics problem. The usual elliptic estimates cannot be applied directly, due to the singularity of the dielectric permittivity. A special curl-div decomposition is introduced for the electric field to reduce the problem to an elliptic equation in divergence form with jump coefficients. The potential analysis and the jump condition lead to the L^p estimates which are superior to the straightforward Nash-Moser estimates. The reduction procedure is expected to be useful for future numerical simulation. The problem arises in the modeling of surface enhanced second-harmonic generation of nonlinear diffractive optics in periodic structures (gratings).

To My Family

ACKNOWLEDGMENTS

First and foremost I would like to express my deep gratitude to my advisor *Prof. Zhengfang Zhou* for his patient guidance through this research, for sharing his knowledge and insight. Numerous discussions and suggestions lead to ideas that shaped this research. I am also grateful to *Prof. Gang Bao* for his invaluable expertise and insight in Applied Mathematics, for his help and suggestions. I would also like to thank the other members of my committee, *Prof. Baisheng Yan*, *Prof. Clifford Weil*, *Prof. Dennis Dunninger* and *Prof. Michael Frazier.*

Table of Contents

Chapter 1

Introduction

The invention of the laser, a quarter of a century ago, created many fascinating new fields. Among them nonliner optics has the broadest scope. The field started with the experimental work of P. A. Franken on optical second-harmonic generation (SHG) in 1961 and the theoretical work of N. Bloembergen on optical wave mixing in 1962. Since that time the field has grown at such an explosive rate that today it has already found exciting applications in many areas of science.

Nonlinear optical phenomena have many important applications, for instance finding a method, through SHG, to obtain coherent radiation at a wavelength shorter than that of available lasers. Ordinary light consists of the resultant of uncorrelated contributions to the wave field from an enormous number of independentely radiating atomic systems. Because of this, ordinary light is not coherent; i.e., well-defined phase relationships do not persist among the wave fields observed at different points of time or space. Laser radiation, on the other hand, is highly coherent, because the simulated emisions of different radiating atoms are syncronized. This high degree of coherence of the laser beam is essential to nonlinear optics. Coherence enables the concentration of the radiation by focusing a laser beam onto a small area, the minimum size of which is limited only by diffraction and by the optical quality of the laser and the focusing system. In this way, it is possible to obtain extremely intense local radiation fields, but in small volumes. Coherence also enables the combination of the weak contributions of nonlinear interactions from widely separated parts of an extended medium so as to obtain an appreciable resultant. With ordinary, incoherent light, both approaches are limited, and it is possible to observe many interesting nonlinear optical phenomena only with the aid of laser sources. However, nonlinear optical effects are in general so weak that the observation of nonlinear phenomena in the optical region can only be made using high intensity laser beams.

Enhancement of nonlinear optical effects presents a great challenge to the optical science community. Recently, a partial differential equation (PDE) was introduced in [20], [21] and [17] to model nonlinear SHG in periodic structures. Moreover, Reinisch and Neviere [20] and [21] showed that using diffraction gratings or periodic structures can greatly enhance SHG and that the PDE model can accurately predict the field propagation.

The existence and regularity of solutions of a system of Maxwell's equations in periodic structures is currently the subject of experimental and theoretical research with a goal to model the surface-enhanced second harmonic generation of nonlinear optics. The research on Maxwell's equations is of great interest because of its applications in plasma-physics, semiconductor-superconductor modeling and other industrial problems. Results on existence and uniqueness for Maxwell's equations in linear media with periodic structures were obtained by Chen and Friedman [7] by assuming a piecewise constant dielectric coefficient. Little is known concerning the questions of existence and uniqueness for nonlinear Maxwell equations. Bao and Dobson [3] and [4] have recently obtained results on existence and uniqueness in two simpler cases where the Maxwell equations can be reduced to a system of nonlinear Helmholtz equations. By an integral-equation approach Dobson and Friedman [8] showed the existence and uniqueness of the solutions of linear Maxwell's equations in a periodic structure that separates two homogeneous materials and is piecewise $\mathcal{C}^2$. Computational results have also been obtained using a combination of finite element method and the fixed-point iteration algorithm.

In this dissertation we investigate the regularity of the solutions of the PDE that governs SHG. We present interior and global L^p type estimates for the solutions of Maxwell equations with source term in a domain filled with two different materials that are separated by a $\mathcal{C}^2$ interface. Such an estimate is crucial in the regularity study of nonlinear Maxwell's equations that arise for instance in the modeling of nonlinear optics. This work is motivated by recent research on gratings (periodic structures) enhanced nonlinear optical effects of nonlinear optics.

We focus on the problem

$$\begin{aligned} \nabla \times E &= -i\,\omega\,\mu\,H, \\ \nabla \times H &= i\,\omega\,\epsilon E + g, \end{aligned}$$

where E is the electric field, H is the magnetic field, g is the source term, ϵ is the dielectric permittivity, μ is the magnetic permeability and ω is the frequency.

Consider a surface S defined by $z = \phi(x,y)$, $S \in \mathcal{C}^2$. Above S the dielectric coefficient is a constant ϵ^+ and below it, the dielectric coefficient is another constant ϵ^-, $\epsilon^+ \neq \epsilon^-$. Hence ϵ is defined by

$$\epsilon = \begin{cases} \epsilon^+ & \text{in} \quad B^+ \\ \epsilon^- & \text{in} \quad B^-. \end{cases}$$

The magnetic permeability μ is assumed to be constant throughout $\mathbb{R}^3$. The presence of the source term g is motivated by the SHG effects.

The main result of this dissertation may be stated as follows.

Theorem 1.1 *Let $1 < p < \infty$, let B be a ball in $\mathbb{R}^3$ and let $g \in L^p(B)$. Suppose $E \in L^p(B)$ and $H \in W^{1,p}(B)$ are the solution of Maxwell's equations with source term g; specifically*

$$\begin{aligned} \nabla \times E &= -i\,\omega\,\mu\,H, \\ \nabla \times H &= i\,\omega\,\epsilon\,E + g. \end{aligned} \tag{1.1}$$

Then we have the following interior apriori estimate.

$$\| E \|_{L^p(B')} + \| H \|_{W^{1,p}(B')} \leq C\,(\| H \|_{L^p(B)} + \| g \|_{L^p(B)} + \| E \|_{W^{-1,p}(B)}),$$

where C is a constant depending only on p and $\bar{B}' \subset B$.

The proof is based on a combination of potential theory, compact operator theory, elliptic regularity results and a special curl-div decomposition of the electric field.

To prove the boundary estimates, we need to impose a transparent boundary conditions for the problem. The usual Sommerfeld or Silver-Müller radiation condition is no longer valid due to the infinite property of the periodic structure. We use the boundary condition introduced in [5], which is not a Dirichlet or Neumann condition. The proof of these estimates is based on a nonstandard extention of the solution to the boundary, maximum principle for elliptic equations and the Riesz Convexity Theorem.

Our work is organized as follows. In Chapter 2 we review the physiscs of Maxwell's equations and nonlinear optics. We also introduce SHG and its mathematical model. In Chapter 3 we state without proof some elliptic theory results that we will use in our study of Maxwell's equations. Finally, in Chapter 4, we start with the study of the regularity of the fields E and H on the interface S. This study is bassed on potential theory and integral equation approaches. Once the regularity on S is established, the regularity above and below the surface S is obtained. We then derive interior and global L^p estimates for the two fields E and H. Our main results are the interior and global estimates of Theorem 4.1 and Theorem 4.2. Using these theorems, we establish existence theory for our model problem (1.1). Our theorems indicate that the magnetic field H is continuous throughout the entire domain, whereas the electric field E has discontinuities across the interface S.

Chapter 2

Physics Background

2.1 Electromagnetic Theory of Light

Over the last century, there have been several attempts to explain the nature of light. Some scientists tried to explain it using Newtonian mechanics and making the assumption that the space was filled with an obscure material called ether. To Newton, the lack of any evidence of diffraction was proof that light can not be a wave. The opponents of his corpuscular hypothesis, lead by Huyghens, were impressed with the properties of light, such as the transparency of many dense media and the lack of evidence for interaction between two light beams traversing the same portion of a medium simultaneously; properties that are characteristic of waves.

In the end, when the dust settled, it turned out that indeed, light had a dual wave-corpuscular nature. Its wave properties are fully described by a set of four coupled partial differential equations (PDE) called Maxwell's equations. It was Maxwell that conceived the possibility of electromagnetic waves and showed that their velocity in vacuum should be equal to the measured velocity of light. For a better understanding of Maxwell's equations, some basic laws of physics will be presented briefly in the next sections.

2.1.1 Electric and Magnetic Fields

The understanding of the electric and magnetic fields is very important because, as we will see in the next sections, they enter the theory of light. Given two separated objects in space, electromagnetism deals with the interaction between them in the sense that it pays attention to what is happening in the space between the two objects, rather than to the two objects and the effect of one body on the other. We can think that this space is filled with a 'field' whose properties determine how the objects interact. Consider an electric charge q_1. This charge is obviously affecting its surroundings. If another electric charge q_2 is added, the repulsion or attraction of q_2 reveals this influence, but it would still be there even if q_2 were taken away. We say that q_1 produces an electric field E in the space around it, and we define it as the force acting on electric charges.

The magnetic induction B is defined as the force exerted on a unit test charge, in the absence of an electric field, as it traverses the field point at unit velocity. We define both electric (E) and magnetic (H) quantities in terms of the force F on a unit test charge, using the single equation

$$F = \mathbf{e}(E + v \times B),$$

where v is the velocity of the test charge, B is the magnetic induction, and $\mathbf{e} = -1.602e^{-19}C$. The electric field E is measured by a stationary test charge; the magnetic induction B contributes an additional, velocity-dependent force on moving charges.

These fields have two properties: direction and magnitude (usually called intensity). The direction of E at any point corresponds to the direction of the line of force passing through that point. The intensity D of the field is defined as the number of lines of force passing through a unit area at right angles to the direction of the lines. In empty space, the electric field intensity E is directly proportional to the electric induction. This fact is usually written as

$$D = \epsilon_0 E.$$

To complete the description of an electromagnetic field, it is necessary to relate B to its sources. The magnetic quantity analogous to D is the field intensity H and the proportionality corresponding to (2.1.1) for magnetic fields in vacuum is

$$B = \mu_0 H.$$

2.1.2 Maxwell's Equations

James Clerk Maxwell, a Scottish scientist of the last century, brought together and formulated mathematically four laws dealing with electricity and magnetism – Maxwell theory. These laws are the basis of electromagnetic theory and they give a complete description of electromagnetism. They deal with the interaction of electric or magnetic or both charged bodies. The bodies and their charges may be stationary or in motion and they may be situated in a vacuum or in any other media. Unfortunately, there exists a fundamental difference between electricity and magnetism, that doesn't make electromagnetism a unified whole. Isolated electrical charges can be easily produced, and experimented on, in the laboratory, whereas isolated magnetic poles have so far never been identified. However, in theory, for symmetry reasons, it has been assumed that both electricity and magnetism can be represented by poles.

Maxwell's equations originated from the following experimental laws of electricity and magnetism discovered in the nineteenth century. The first two equations

$$\begin{aligned} \nabla \cdot E &= \frac{1}{\epsilon_0}\rho, \\ \nabla \cdot B &= 0 \end{aligned}$$

are differential forms of Gauss' law and connect the stationary electric and magnetic fields. The first equation expresses the inverse-square law for the field of a point charge; the second equation contains the assertion that free magnetic poles do not exist.

Ampere's law, gives the dependence of the magnetic field on the density J of the moving current (the source):

$$\nabla \times B = \frac{1}{\epsilon_0 c^2} J. \tag{2.1}$$

The equations introduced so far were concerned with stationary charges. The first law that took into account time dependent effects, called Faraday's law, addresses the case of time dependent magnetic fields, namely

$$\nabla \times E = -\frac{\partial B}{\partial t}.$$

Maxwell observed that these laws are interrelated and that time dependent electric fields generate magnetic fields. The result consists of the above mentioned four PDE's which, taken together, are called Maxwell's equations:

$$\begin{aligned} \nabla \cdot D &= \frac{1}{\epsilon_0}\rho, \\ \nabla \cdot B &= 0, \\ \nabla \times E &= -\frac{\partial B}{\partial t}, \\ \nabla \times H &= \frac{1}{c^2}\frac{\partial D}{\partial t} + \frac{1}{\epsilon_0 c^2}J. \end{aligned} \tag{2.3}$$

These equations form the basis of all electromagnetic phenomena. They are written for fields in a vacuum, where there exist an electrical charge of density ρ and an electrical current; that is, a moving charge of density J. Note that these equations are not symmetric in H and E which is due to the presence of the electric charge and of the moving current. The constant ϵ_0 was introduced into Maxwell's equations to be consistent with the SI ('International System') units used in electromagnetic calculations. In numerical terms,

$$\epsilon_0 = \frac{1}{4\pi}\frac{10^{-9}}{9}Fm^{-1},$$

and

$$\mu_0 = 4\pi \times 10^{-7}Hm^{-1},$$

It is customary to write

$$\frac{1}{\epsilon_0\mu_0} = c^2.$$

The last Maxwell equation contains Ampere's law (2.1), but with the important additional statement that magnetic fields arise, not only from circulating electric charges, of density J, but also from time variations in the electric displacement D. It is this phenomenon of *displacement currents*, as Maxwell called it, that give rise to electromagnetic waves, and so, to light.

Every experiment performed so far confirmed the exactness of Maxwell's theory and today, Maxwell's equations are accepted as the correct theory describing light. They imply that light is nothing else but electromagnetic wave. For more details, see [12].

2.2 Nonlinear Optics

Optics is the branch of physics that describes the generation, composition, transmission and interaction of light and relates these to other physical phenomena. More precisely, it deals with the propagation and interaction of the electric and magnetic fields composing light, with matter. Much of the subject matter of optics concerns the changes which occur in electromagnetic waves at interfaces between distinct media.

Optical nonlinearity is obtained by changes in the optical properties of a medium as the intensity of the light is increased, or by introducing one or more other light waves. The electric and magnetic properties of the media are characterized by two functions: ϵ called dielectric permittivity (coefficient) and μ called magnetic permeability. In linear, isotropic media, these quantities are simple scalar constants, independent of the direction and magnitude of the electric field, but not of its frequency. If ϵ or μ, or both are nonlinear functions of the electric field, then the corresponding subfield of optics is called nonlinear optics.

The constitutive relations

$$\begin{aligned} B &= \mu_0 H + 4\pi M, \\ D &= \epsilon_0 E + 4\pi P, \\ J &= \sigma E, \end{aligned} \tag{2.4}$$

connect the various fields which enter Maxwell's equations and the media.

Here P and M denote the electric and magnetic dipole per unit volume. An applied electric field E displaces the bound charges of a dielectric medium against restoring forces, thereby creating internal dipole moments which give rise to additional lines of electric induction. The sum of the internal dipole moments induced per unit volume is called the polarization P. The laboratory fields of interest are small compared to the electric fields experienced by the electrons in the atoms and molecules from which dense matter is constructed. In this circumstance, we can expand the dipole moment per unit volume P in a Taylor series in powers of the macroscopic field E. We can write

$$P_i = \sum_j \chi_{ij}^{(1)} E_j + \sum_{jk} \chi_{ijk}^{(2)} E_j E_k + \ldots , \tag{2.5}$$

where $\chi_{ij}^{(1)}$ is the susceptibility tensor of ordinary dielectric theory, $\chi_{ijk}^{(2)}$ is referred to as the second order susceptibility and there is a summation over the repeating indices. Alternatively, the dielectric constant ϵ is defined as

$$\epsilon = \epsilon_0 + 4\pi\chi^{(\omega)}. \tag{2.6}$$

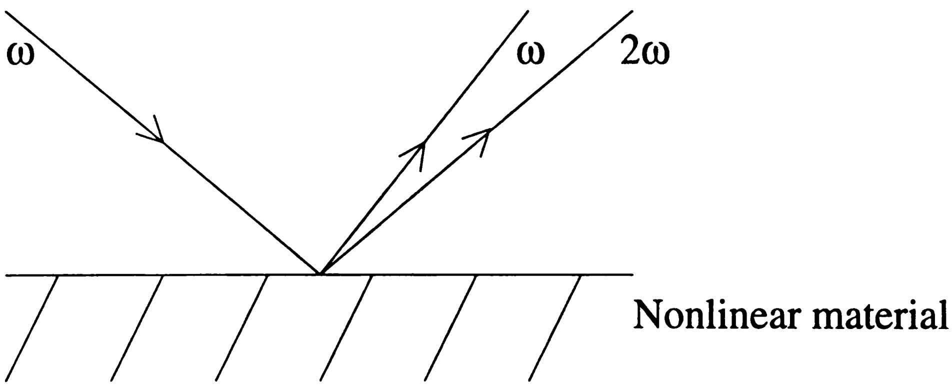
In the optical frequency range, it is almost always true that $\mu = \mu_0$. However, ϵ or σ, and occasionally both, have values distinctively different from the vacuum case. In vacuum, ϵ is denoted by ϵ_0. Throughout this dissertation, we shall confine our attention entirely to non-magnetic media; i.e., $M = 0$ and to spaces containing no free charges; i.e., $J = 0$.

Recent research in optics has shown that the linearity of the wave equations in optical media is only an approximation, but a very good one. However, the coefficients of the nonlinear terms, though extremely small, lead to striking effects under proper circumstances. The existence, character and magnitude of these nonlinearities provide detailed information about the structure of matter.

2.2.1 Second Harmonic Generation

The theory of harmonic generation provides the simplest example of the light wave propagation problem in nonlinear media.

The physics of SHG may be described as follows. Given an incident plane wave of frequency ω on a nonlinear medium, the interaction of the incident wave with the nonlinear medium generates diffracted waves of frequencies ω and 2ω. The fact that new frequency components are present is the most striking difference between nonlinear and linear optics. An intuitive picture follows.



The essential feature of SHG is the existence of a perceptible dependence of the electronic polarization on the square of the electric field intensity at optical frequencies, in addition to the usual direct dependence. When SHG is considered, the i-th component of the total charge polarization P in a medium contains two contributions

$$P_i = P_i^{(\omega)} + P_i^{(2\omega)}, i = 1, 2, 3,$$

where

$$P_i^{(\omega)} = \chi^{(\omega)} E_j^{(\omega)}, \; i,j = 1,2,3, \tag{2.7}$$

and

$$P_i^{(2\omega)} = \chi^{(2\omega)} E_j^{(\omega)} E_k^{(\omega)}, \; i,j,k = 1,2,3. \tag{2.8}$$

The first contribution $P_i^{(\omega)}$, accounts for the linear part of the medium's response. It is assumed in (2.7) that the intensity of the second harmonic radiation is very weak and the depletion of the primary wave due to a conversion of a portion of its energy into second harmonic is ignored, which in literature [20], [21], is known as the undepleted-pump approximation. The second part, $P_i^{(2\omega)}$, is quadratic in the electric field, and introduces the third-rank tensor $\chi_{ijk}^{(2\omega)}$. The superscripts (ω) and (2ω) have now become necessary for distinguishing the frequencies at which the respective quantities must be evaluated. Equation (2.8) is often written in the following more condensed form.

$$P_i^{(2\omega)} = \chi^{(2\omega)} : E^{(\omega)} E^{(\omega)}.$$

Note that the first-order linear susceptibility $\chi_{ij}^{(\omega)}$ is a dimensionless ratio, independent of the system of units in which E and P are measured. It contains the essential optical properties of the dielectrics. The values of the coefficients in the second-order susceptibility tensor $\chi_{ijk}^{(2\omega)}$, which describe the nonlinearity, are not independent of the system of units; they have dimensions inverse to those of the electric field intensity.

2.2.2 Modeling of the Problem

Consider a plane, polarized, monochromatic wave, denoted by the subscript i, incident on an medium and that SHG effects are present. Assume that the medium is non-magnetic, $M = 0$, with constant magnetic permeability μ and that there are no free charges; i.e., $J = 0$. The dielectric coefficient is denoted by ϵ and it is piecewise constant. Then the time harmonic (time dependence $e^{-i\omega t}$) Maxwell's equations (2.3), (2.4) that govern SHG take the form.

$$\begin{aligned} -\partial_t B &= \nabla \times E, \\ \partial_t D &= \nabla \times H, \end{aligned} \tag{2.9}$$

along with the constitutive equations

$$\begin{aligned} B &= \mu\ H, \\ D &= \epsilon_0\ E + 4\pi P, \end{aligned} \tag{2.10}$$

where E is the electric field, H is the magnetic field, B is the magnetic induction, D is the electric displacement, ϵ is the dielectric coefficient, μ is the magnetic permeability, ω is the frequency and P is the nonlinear polarization vector. The nonlinear polarization vector P can be expressed through the electric-field vector components, as in (2.5) by

$$P_i = \sum_j \chi_{ij}^{(1)} E_j + \sum_{jk} \chi_{ijk}^{(2)} E_j E_k + \cdots . \tag{2.11}$$

We write the total electric field as the sum of the two fields, one term for each of the two frequencies by

$$E = E^{(\omega)} + E^{(2\omega)}, \tag{2.12}$$

where

$$E^{(\omega)} = E_0^{(\omega)} e^{-i\omega t} + \bar{E}_0^{(\omega)} e^{i\omega t} \tag{2.13}$$

is the incident wave and $\bar{E}$ denotes the complex conjugate of E. Similarly, we can write

$$H = H^{(\omega)} + H^{(2\omega)} \tag{2.14}$$

with

$$H^{(\omega)} = H_0^{(\omega)} e^{-i\omega t} + \bar{H}_0^{(\omega)} e^{i\omega t}. \tag{2.15}$$

Then

$$E_j E_k = E_{0j}^{(\omega)} E_{0k}^{(\omega)} e^{-2i\omega t} + \bar{E}_{0j}^{(\omega)} \bar{E}_{0k}^{(\omega)} e^{2i\omega t} + E_{0j}^{(2\omega)} \bar{E}_{0k}^{(\omega)} e^{-i\omega t} + \bar{E}_{0j}^{(2\omega)} E_{0k}^{(\omega)} e^{i\omega t},$$

and the polarization P defined in (2.11) becomes

$$\begin{aligned} P_i = &\sum_j \chi_{ij}^{(1)} (E_{0j}^{(\omega)} e^{-i\omega t} + \bar{E}_{0k}^{(\omega)} e^{i\omega t} + E_{0j}^{(2\omega)} e^{-i2\omega t} + \bar{E}_{0k}^{(2\omega)} e^{i2\omega t}) \\ &+ \sum_{jk} \chi_{ijk}^{(2)} (E_{0j}^{(\omega)} E_{0k}^{(\omega)} e^{-2i\omega t} + \bar{E}_{0j}^{(\omega)} \bar{E}_{0k}^{(\omega)} e^{2i\omega t} \\ &+ E_{0j}^{(2\omega)} \bar{E}_{0k}^{(\omega)} e^{-i\omega t} + \bar{E}_{0j}^{(2\omega)} E_{0k}^{(\omega)} e^{i\omega t}) \end{aligned} \tag{2.17}$$

where we used that fact that the combination of two fields of frequencies ω produces a field with their sum frequency 2ω and the difference frequency of harmonic and fundamental is equal to the fundamental frequency. Note that the difference arises from the product of one of the wave factors with the conjugate of the other. We are not interested in the sum, which is a third harmonic.

Substituting the polarization (2.17), the decompositions of E (2.12), (2.13) and H (2.14), (2.15) into Maxwell's equations (2.9), (2.10) and (2.6) and separating the components with frequency ω and the components with frequency 2ω, we get the following two different forms of Maxwell's equations written at each of the two frequencies.

For frequency ω,

$$\begin{aligned} \nabla \times E^{(\omega)} &= -i\,\omega\,\mu\,H^{(\omega)}, \\ \nabla \times H^{(\omega)} &= i\,\omega\,\epsilon\,E^{(\omega)} + \chi_{jk}^{(\omega)}\,i\,\omega\,\bar{E}_{x_j}^{(\omega)}\,E_{x_k}^{(2\omega)} \end{aligned} \tag{2.18}$$

and for frequency 2ω,

$$\begin{aligned} \nabla \times E^{(2\omega)} &= -i\,2\omega\,\mu\,H^{(2\omega)}, \\ \nabla \times H^{(2\omega)} &= i\,2\omega\,\epsilon\,E^{(2\omega)} + \chi_{jk}^{(2\omega)}\,i\,2\omega\,E_{x_j}^{(\omega)}\,E_{x_k}^{(\omega)}. \end{aligned} \tag{2.19}$$

Note that the undepleted-pump approximation is not used to get the equation (2.18). If the undepleted-pump approximation is assumed, then (2.18) becomes

$$\begin{aligned} \nabla \times E^{(\omega)} &= -i\,\omega\,\mu\,H^{(\omega)}, \\ \nabla \times H^{(\omega)} &= i\,\omega\,\epsilon\,E^{(\omega)} \end{aligned} \tag{2.20}$$

but the equation (2.19) remains unchanged.

The nonlinear polarization present at the frequency ω can be viewed as a source of electromagnetic radiation at that frequency in (2.19) .

We shall take equation (2.19) as the subject of our investigation in the next chapters and (2.20) and (2.19) when we establish the existence theory in the last section of Chapter 4.

Chapter 3

Mathematical Background

3.1 Sobolev Spaces

Throughout this section Ω will denote a bounded domain in $\mathbb{R}^n$. For $p \geq 1$, we let $L^p(\Omega)$ denote the classical Banach space consisting of measurable functions on Ω that are p-integrable. The norm in $L^p(\Omega)$ is defined by

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p}.$$

For $p = \infty, L^\infty(\Omega)$ denotes the Banach space of bounded functions on Ω with the norm

$$||u||_{L^\infty(\Omega)} = \sup_\Omega |u|.$$

In the following we shall use $||u||_p$ for $||u||_{L^p(\Omega)}$ when there is no ambiguity.

The $W^{k,p}(\Omega)$ spaces are Banach spaces where continuous differentiability is replaced by weak differentiability. For $p \geq 1$ and k a non-negative integer, we let

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega); D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}.$$

The space $W^{k,p}(\Omega)$ is clearly linear. A norm is introduced by defining

$$||u||_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u|^p \, dx\right)^{1/p}.$$

An equivalent norm would be

$$||u||_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} ||D^\alpha||_p.$$

We denote by $W^{-1,q}(\Omega)$ the dual space of $W^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and we use $<,>$ to denote the pairing between the two spaces.

If $u \in W^{-1,p}(\Omega)$, we define the norm

$$||u||_{W^{-1,p}(\Omega)} = \sup\{| < f, u > || \, u \in W^{1,p}(\Omega), ||u||_{W^{1,p}(\Omega)} \leq 1\}.$$

We shall need the following inequalities in dealing with integral estimates. *Young's Inequality.*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

This holds for positive real numbers a, b, p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Hölder's inequality.

$$\int_\Omega uv\, dx \le ||u||_p ||v||_q.$$

This holds for functions $u \in L^p(\Omega), v \in L^q(\Omega), 1/p + 1/q = 1.$

Interpolation inequality in L^p spaces.

$$||u||_q \le \varepsilon ||u||_r + \varepsilon^{-\mu} ||u||_p,$$

where $p < q < r$ and

$$\mu = \left(\frac{1}{p} - \frac{1}{q}\right) \Big/ \left(\frac{1}{q} - \frac{1}{r}\right).$$

Theorem 3.1 *Interpolation inequality in $W^{k,p}$ spaces. Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and $u \in W^{k,p}(\Omega)$. Then for any $\varepsilon > 0, 0 < |\beta| < k$,*

$$||D^\beta u||_{L^p(\Omega)} \le \varepsilon ||u||_{W^{k,p}(\Omega)} + C\varepsilon^{\frac{|\beta|}{|\beta|-k}} ||u||_{L^p(\Omega)} \tag{3.1}$$

where $C = C(k, \Omega)$.

Theorem 3.2 *If $\Omega \subset \mathbb{R}^n$ has the uniform cone property or if it is bounded and has the cone property, and if $1 \le p \le \infty$, then there exists a constant $K = K(m,p,\Omega)$ such that for $1 \le j \le m$ and any $u \in W^{m,p}(\Omega)$,*

$$||u||_{W^{j,p}} \le K ||u||_{W^{m,p}}^{\frac{j}{m}} ||u||_{W^{0,p}}^{\frac{m-j}{m}}.$$

Theorem 3.3 *General embedding theorem for $W^{k,p}(\Omega)$.*

Let Ω be a $C^{0,1}$ domain in $\mathbb{R}^n$.

1. *If $kp < n$, then $W^{k,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega), p^* = \frac{np}{n-kp}$, and compactly embedded in $L^q(\Omega)$ for any $q < p^*$.*

2. *If $0 \leq m < k - \frac{n}{p} < m+1$, then $W^{k,p}(\Omega)$ is continuously embedded in $C^{m,\alpha}(\bar{\Omega}), \alpha = k - \frac{n}{p} - m$, and compactly embedded in $C^{m,\beta}(\bar{\Omega})$ for any $\beta < \alpha$.*

Theorem 3.4 *Trace theorem. If $u \in W^{m,p}(\Omega)$, then the trace $v = u|_{\partial\Omega}$ belongs to $W^{m-\frac{1}{p},p}(\partial\Omega)$ and*

$$||v||_{m-\frac{1}{p},p,\partial\Omega} \leq K||u||_{m,p,\Omega}.$$

3.2 Regularity Theory for Second Order Elliptic Equations

Definition 3.1 *An operator L of the form*

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$$

with coefficients a^{ij}, b^i, c, where $i,j = 1, \cdots, n$, defined on a domain $\Omega \subset \mathbb{R}^n$, is elliptic in the domain Ω if the coefficient matrix $\mathcal{A} = [a^{ij}]$ is positive everywhere in Ω.

Theorem 3.5 *Maximum principle. Let L be elliptic in the bounded domain Ω with*

$c \leq 0$. *Suppose that in* Ω, $Lu \geq 0$, *for* $u \in C^0(\bar{\Omega})$. *Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} |u|.$$

If $Lu = 0$ *in* Ω, *then*

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|.$$

Theorem 3.6 *Let* $1 < p < \infty$, *let* Ω *be an open set in* $\mathbb{R}^n$ *and let* $f \in L^p(\Omega)$. *Suppose* $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$ *is a strong solution of the equation* $Lu = f$ *in* Ω *where the coefficients of* L *satisfy, for positive constants* λ, Λ,

$$a^{ij} \in C^0(\Omega), b^i, c \in L^\infty(\Omega);$$

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n;$$

$$|a^{ij}|, |b^i|, |c| \leq \Lambda,$$

where $i, j = 1, \ldots, n$. *Therefore for any domain* $\Omega' \subset\subset \Omega$,

$$||u||_{W^{2,p}(\Omega')} \leq C(||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)}),$$

where C *depends on* $n, p, \lambda, \Lambda, \Omega', \Omega$ *and the moduli of continuity of the coefficients* a^{ij} *on* Ω'.

3.3 Potential Theory

Let $D_- \subset \mathbb{R}^m$ be a bounded domain of class C^2. By $\Gamma := \partial D_-$ we denote its boundary and by $D_+ := \mathbb{R}^m \backslash \bar{D}_-$ its open complement. We assume the unit normal n to the boundary to be directed into the exterior D_+.

Let

$$\Phi(x,y) := \frac{1}{4\pi} \frac{1}{|x-y|}$$

denote the fundamental solution of Laplace's equation.

Definition 3.2 *Given a function $\varphi \in C(\Gamma)$ the function*

$$u(x) := \int_\Gamma \varphi(y)\, \Phi(x,y)\, ds(y), \qquad x \in \mathbb{R}^m \backslash \Gamma, \tag{3.2}$$

is called single-layer potential with density φ.

Definition 3.3 *Given a function $\varphi \in C(\Gamma)$ the function*

$$v(x) := \int_\Gamma \varphi(y)\, \frac{\partial \Phi(x,y)}{\partial n}\, ds(y), \qquad x \in \mathbb{R}^m \backslash \Gamma, \tag{3.3}$$

is called double-layer potential with density φ.

For a proof of the next theorem, see [14].

Theorem 3.7 *Let Γ be of class C^2 and $\varphi \in C(\Gamma)$. Then the single-layer potential u with density φ is continuous throughout $\mathbb{R}^m$. For $x \in \Gamma$*

$$u(x) = \int_\Gamma \varphi(\xi)\, \Phi(x,\xi)\, ds(\xi).$$

The double-layer potential v with density φ can be continuously extended from D_+ to $\bar{D}_+$ and from D_- to $\bar{D}_-$ with limiting values

$$v_\pm(x) = \int_\Gamma \varphi(\xi)\, \frac{\partial \Phi(x,\xi)}{\partial n}\, ds(\xi) \;\pm\; \frac{1}{2}\,\varphi(x), \qquad x \in \Gamma,$$

where $v_\pm(x) := \lim_{\gamma \to 0^+} v\,(x \pm \gamma\, n(x))$ and where the integrals exist as improper integrals.

A proof of the following theorem may be found in [18].

Theorem 3.8 *The operators S, D defined for $x \in \Gamma$ by*

$$(Su)(x) = \int_\Gamma \varphi(\xi)\; \Phi(x,\xi)\; ds(\xi)$$

and

$$(Du)(x) = \int_\Gamma \varphi(\xi)\, \frac{\partial \Phi(x,\xi)}{\partial n}\, ds(\xi)$$

are continuous from $W^{m,p}(\Gamma)$ into $W^{m+1,p}(\Gamma)$ for all p such that $1 < p < \infty$, m positive integer.

Chapter 4

L^p Estimates for Maxwell's Equations with Source Term

4.1 Surface Estimates

To obtain the desired estimates, we need some basic facts on potential theory and jump conditions of solutions. In this section, we develop these necessary facts that we shall use later to prove the estimates. Throughout this section, B denotes a ball in $\mathbb{R}^3$ and S denotes a C^2 surface embedded in $\mathbb{R}^3$ given by $z = \phi(x, y)$ such that S divides B into two connected components, B^+ and B^-. Also ϵ is defined by

$$\epsilon = \begin{cases} \epsilon^+ & \text{in } B^+, \\ \\ \epsilon^- & \text{in } B^-, \end{cases} \tag{4.1}$$

where ϵ^+ and ϵ^- are constants, $\epsilon^+ \neq \epsilon^-$.

Lemma 4.1 *Let $1 < p < \infty$, let $h \in W^{1,p}(B)$ and let $f \in L^p(B)$. Suppose*

$$\nabla \cdot (\epsilon \, \nabla h - f) = 0$$

with ϵ defined as in (4.1). Then

$$\left[\!\left[\epsilon \, \frac{\partial h}{\partial n} - f \cdot n \right]\!\right] = 0 \text{ on } S,$$

where $[\![\cdot]\!]$ is defined by $[\![u]\!] = u^+ - u^-$ for any u defined on B. Note that u^+ and u^- denote $u_\pm(x) = \lim_{y \to x \in B^\pm} u(y), x \in S$.

Proof. Let $\varphi \in C_0^\infty(B)$. Let $\langle \cdot, \varphi \rangle$ denote the linear functional on the distribution space determined by φ. Then

$$\langle \nabla \cdot (\epsilon \, \nabla h - f), \; \varphi \rangle = 0 \text{ in } B,$$

which implies that

$$\begin{aligned} 0 = \langle \epsilon \, \nabla h - f, \; \nabla \cdot \varphi \rangle &= \int_B (\epsilon \, \nabla h(\xi) - f(\xi)) \;\; \nabla \cdot \varphi(\xi) \, d\xi \\ = \;\; &\int_{B^+} \left(\epsilon^+ \, \nabla h(\xi) - f(\xi)\right) \;\; \nabla \cdot \varphi(\xi) \, d\xi + \int_{B^-} \left(\epsilon^- \, \nabla h(\xi) - f(\xi)\right) \;\; \nabla \cdot \varphi(\xi) \, d\xi. \end{aligned}$$

Applying Green's first identity and using the fact that $\varphi|_{\partial B} = 0$, we get

$$\begin{aligned} 0 = &\left\langle \nabla \cdot (\epsilon^+ \, \nabla h(\xi) - f(\xi)), \; \varphi(\xi) \right\rangle_{B^+} + \left\langle (\epsilon^+ \, \nabla h(\xi) - f(\xi)) \cdot n^+, \; \varphi(\xi) \right\rangle_S \\ &+ \left\langle \nabla \cdot (\epsilon^- \, \nabla h(\xi) - f(\xi)), \; \varphi(\xi) \right\rangle_{B^-} + \left\langle (\epsilon^- \, \nabla h(\xi) - f(\xi)) \cdot n^-, \; \varphi(\xi) \right\rangle_S \end{aligned} \tag{4.2}$$

Here n^+ and n^- denote the normals on S directed into the exterior of B^+ and B^-, respectively.

We know that the equation

$$\nabla \cdot (\epsilon^+ \, \nabla h - f) = 0,$$

is satisfied in B^+ and in B^-. Hence (4.2) becomes

$$0 = \left\langle \epsilon^+ \frac{\partial h}{\partial n^+}(\xi) - f(\xi) \cdot n^+, \; \varphi(\xi) \right\rangle_S - \left\langle \epsilon^- \frac{\partial h}{\partial n^-}(\xi) - f(\xi) \cdot n^-, \; \varphi(\xi) \right\rangle_S,$$

which proves the jump condition for h across the interface S and hence Lemma 4.1.

□

Next, we study the regularity of h introduced in Lemma 4.1 with the additional assumption that $h \mid_{\partial B} = 0$.

Lemma 4.2 *Let $h \in W^{1,p}(B)$, $h \mid_{\partial B} = 0$ and let $f \in W^{-1,p}(B)$ satisfy the equation*

$$\nabla \cdot (\epsilon \, \nabla h) = f, \tag{4.4}$$

in B. Then, on the interface S, h is given by

$$\langle G, \; f \rangle = \frac{\epsilon^+ + \epsilon^-}{2} \, h(x_0) - (\epsilon^+ - \epsilon^-) \int_S h(\xi) \, \frac{\partial G(x_0, \xi)}{\partial n} \, dS(\xi), \tag{4.5}$$

where G is the Green's function for Δ on B ($\Delta G = \delta$ and $G(x, \cdot) \mid_{\partial B} = 0$).

Before proving this lemma, we make the following remarks.

Remark 4.1 *If ϵ is a constant or a C^1 function, then* Lemma 4.2 *is a standard elliptic regularity result* [10]. *For the model problem in this dissertation, ϵ is only piecewise constant. For general $\epsilon \in L^\infty$, the Nash-Moser technique implies only $h \in C^\alpha$ for some small α and for $f = 0$. The $W^{1,p}$ regularity established in* Lemma 4.2 *implies that $h \in C^\alpha$ for any $\alpha \in (0,1)$ and for any right hand side f.*

Remark 4.2 *In the case $h \in L^p(B^+)$ or $h \in L^p(B^-)$, the standard elliptic regularity results* [10] *indicate the $W^{1,p}$ regularity of h away from a tubular neighborhood of the interface S due to the definition of ϵ. Thus it suffices to prove the regularity result near the interface S. Note that* Lemma 4.2 *gives an explicit formula for h on S. Using this formula, the behavior of h near the interface S can be easily obtained.*

Proof. We now present our proof of Lemma 4.2. Using the definition of the weak derivative, we get

$$\begin{aligned}
\langle G,\ f\rangle &= \langle G,\ \nabla\cdot(\epsilon\,\nabla h)\rangle = -\langle \nabla G,\ \epsilon\,\nabla h\rangle \\
&= -\int_{B^+} \nabla G(x,\xi)\epsilon^+\,\nabla h(\xi)\,d\xi - \int_{B^-} \nabla G(x,\xi)\epsilon^-\,\nabla h(\xi)\,d\xi.
\end{aligned}$$

Using Green's first identity, we get

$$\begin{aligned}
\langle G,\ f\rangle = \epsilon^+ \int_{B^+} h(\xi)\triangle G(x,\xi)\,d\xi - \epsilon^+ \int_{\partial B^+} h(\xi)\,\frac{\partial G(x,\xi)}{\partial n^+}\,dS \\
+ \epsilon^- \int_{B^-} h(\xi)\triangle G(x,\xi)\,d\xi - \epsilon^- \int_{\partial B^-} h(\xi)\,\frac{\partial G(x,\xi)}{\partial n^-}\,dS.
\end{aligned} \tag{4.6}$$

where n^+ and n^- denote the normals to S directed into the exterior of B^+ and B^-, respectively. Using $G\,|_{\partial B}= 0$ and denoting $n = n^+$, (4.6) becomes

$$\begin{aligned}\langle G,\ f\rangle &= \epsilon^+ \int_{B^+} h(\xi)\triangle G(x,\xi)\ d\xi - \epsilon^+ \int_S h(\xi)\ \frac{\partial G(x,\xi)}{\partial n}\ dS \\ &\quad + \epsilon^- \int_{B^-} h(\xi)\triangle G(x,\xi)\ d\xi + \epsilon^- \int_S h(\xi)\ \frac{\partial G(x,\xi)}{\partial n}\ dS.\end{aligned}$$

Assume $x \in B^+$. Then

$$\int_{B^+} h(\xi)\triangle G(x,\xi)\ d\xi = h(x)$$

and

$$\int_{B^-} h(\xi)\triangle G(x,\xi)\ d\xi = 0.$$

Hence

$$\langle G,\ f\rangle = \epsilon^+\ h(x) - \epsilon^+ \int_S h(\xi)\ \frac{\partial G(x,\xi)}{\partial n}\ dS + \epsilon^- \int_S h(\xi)\ \frac{\partial G(x,\xi)}{\partial n}\ dS.$$

Now let $x \longrightarrow x_0 \in S$, $x \in B^+$. Theorem 3.7 implies,

$$\begin{aligned}\langle G,\ f\rangle &= \epsilon^+\ h(x_0) \\ &\quad - \epsilon^+ \left(\int_S h(\xi)\ \frac{\partial G(x_0,\xi)}{\partial n}\ dS + \frac{1}{2}\ h(x_0)\right) \\ &\quad + \epsilon^- \left(\int_S h(\xi)\ \frac{\partial G(x_0,\xi)}{\partial n}\ dS + \frac{1}{2}\ h(x_0)\right),\end{aligned}$$

or

$$\langle G,\ f\rangle = \frac{\epsilon^+ + \epsilon^-}{2}\, h(x_0)$$
$$- \epsilon^+ \int_S h(\xi)\, \frac{\partial G(x_0,\xi)}{\partial n}\, dS + \epsilon^- \int_S h(\xi)\, \frac{\partial G(x_0,\xi)}{\partial n}\, dS.$$

Hence

$$\langle G,\ f\rangle = \frac{\epsilon^+ + \epsilon^-}{2}\, h(x_0) - (\epsilon^+ - \epsilon^-) \int_S h(\xi)\, \frac{\partial G(x_0,\xi)}{\partial n}\, dS,$$

and the lemma is proved.

Note that a similar argument yields the same formula in the case $x \in B^-$. $\square$

Remark 4.3 *Set*

$$I = (\epsilon^+ - \epsilon^-) \int_S h(\xi)\, \frac{\partial G(x_0,\xi)}{\partial n} dS.$$

From Theorem 3.8, *I is a continuous operator from $W^{1,p}(S)$ into $W^{2,p}(S)$. Note that* Theorem 3.3 *gives that $W^{2,p}(S)$ is compactly embedded in $W^{1,p}(S)$. Hence I is a compact operator from $W^{1,p}(S)$ into $W^{1,p}(S)$.*

Corollary 4.1 *If h satisfies the conditions in* Lemma 4.2, *then*

$$\| h \|_{W^{1,p}(B')} \leq C\ (||h||_{L^p(B)} + \| f \|_{W^{-1,p}(B)}),$$

for any $B' \subset\subset B$.

Corollary 4.2 *For any $f \in L^p(B)$, there exists $h \in W_0^{1,p}(B)$ that solves*

$$\nabla \cdot (\epsilon \nabla h) = \nabla \cdot f \ \textit{in } B, \qquad h\,|_{\partial B} = 0. \tag{4.8}$$

Proof. For any $f \in L^p(B)$, we can find $\tilde{h} \in W^{1,p}\,|_S$ solution of (4.5). By the Fredholm alternative, we can solve

$$\epsilon\,\triangle h = \nabla \cdot f \text{ in } B^{\pm},$$

$$h\,|_{\partial B^{\pm}\setminus S} = 0,$$

$$h\,|_S = \tilde{h}\,|_S,$$

to get $W^{1,p}(B^{\pm})$ solutions $\tilde{h}^+$ and $\tilde{h}^-$. Then,

$$h = \begin{cases} h^+ & \text{in } B^+, \\ h^- & \text{in } B^-, \\ \tilde{h} & \text{on } S, \end{cases}$$

is the $W^{1,p}(B)$ solution of (4.8).

4.1.1 Cutoff Argument

In the previous section, we proved Lemma 4.2 with the assumption that $h\,|_{\partial B} = 0$. However, to prove the main results of this dissertation, we need its conclusion to hold for arbitary $h \in W^{1,p}(B)$. This can be achieved by introducing the following cutoff argument.

Let $\tilde{h} = \chi h$, where

$$\chi = \begin{cases} 1 & \text{in } B_\delta, \\ 0 & \text{outside } B_{2\delta}, \end{cases}$$

where $\delta > 0$ and B_δ denotes a ball of radius δ. Then

$$\begin{aligned} \nabla \cdot (\epsilon \nabla \tilde{h}) &= \nabla \cdot (\epsilon \nabla(\chi h)) \\ &= \nabla \cdot (\epsilon \chi \nabla h) + \nabla \cdot (\epsilon h \nabla \chi) \\ &= \chi \nabla \cdot (\epsilon \nabla h) + \epsilon \nabla h \cdot \nabla \chi + \nabla \cdot (\epsilon h \nabla \chi) \\ &= \chi f + \epsilon \nabla h \cdot \nabla \chi + \nabla \cdot (\epsilon h \nabla \chi) \\ &\stackrel{def}{=} \tilde{f}. \end{aligned}$$

It is easy to see that the first and the third terms of $\tilde{f}$ are in $W^{-1,p}$. For the second term,

$$\begin{aligned} \epsilon \nabla h \nabla \chi &= \nabla(\epsilon h) \nabla \chi - h \nabla \epsilon \cdot \nabla \chi \\ &= \nabla(\epsilon h) \nabla \chi - h \partial_t \epsilon \partial_t \chi - h \partial_n \epsilon \partial_n \chi. \end{aligned}$$

Note that ϵ is piecewise constant separated by the interface S. Hence $\partial_t \epsilon = 0$ for any tangential direction t. If we have $\partial_n \chi \mid_S = 0$, then

$$\epsilon \nabla h \nabla \chi = \nabla(\epsilon h) \nabla \chi.$$

Now $\epsilon\, h \in L^p(B)$ so $\nabla(\epsilon\, h)\nabla\chi \in W^{-1,p}(B)$, which proves that the second term in $\tilde{f}$ (and thus $\tilde{f}$ itself) is in $W^{-1,p}(B)$. Therefore,

$$\begin{aligned}\| h \|_{W^{1,p}(B')} = \| \tilde{h} \|_{W^{1,p}(B')} &\leq C \| \tilde{f} \|_{W^{-1,p}(B)} \\ &= C\, (\| \nabla \cdot f \|_{W^{-1,p}(B)} + \| h \|_{L^p(B)}),\end{aligned}$$

since $\tilde{h} \mid_{\partial B} = 0$.

Next, we only need to construct $\chi \in C_0^\infty(B)$ such that $\partial_n \chi \mid_S = 0$.

Without loss of generality, we may assume that $(0,0,0) \in S \cap B$.

Note that when $S = \{z = 0\}$, the construction of χ is very easy. For example,

$$\chi(x, y, z) = \eta(x^2 + y^2 + z^2)$$

with any $\eta \in \mathbf{C}_0^\infty(\mathbb{R}_+)$ and

$$\eta(t) = \begin{cases} 1 & \text{for } t < \delta, \\ 0 & \text{for } t > 2\delta. \end{cases}$$

For general S, we introduce a new variable $(\tilde{x}, \tilde{y}, \tilde{z})$ such that

$$\begin{aligned} x &= k(\tilde{x}, \tilde{y}, \tilde{z}), \\ y &= l(\tilde{x}, \tilde{y}, \tilde{z}), \\ z &= \tilde{z} + \phi(x, y). \end{aligned} \tag{4.9}$$

for some functions k and l. This will make $S = \{\tilde{z} = 0\}$. We introduce

$$\tilde{\chi}(\tilde{x}, \tilde{y}, \tilde{z}) = \chi(x(\tilde{x}, \tilde{y}, \tilde{z}), y(\tilde{x}, \tilde{y}, \tilde{z}), z(\tilde{x}, \tilde{y}, \tilde{z})).$$

Then

$$\partial_{\tilde{z}} \tilde{\chi} = \chi_x \, x_{\tilde{z}} + \chi_y \, y_{\tilde{z}} + \chi_z \, z_{\tilde{z}}.$$

In order to find χ such that $\partial_n \chi | S = 0$, with the new set of variables (4.9), we find $\tilde{\chi}(\tilde{x}, \tilde{y}, \tilde{z})$ such that $\partial_{\tilde{z}} \tilde{\chi} = 0$ and $\partial_n \chi = -C \, \partial_{\tilde{z}} \tilde{\chi}$ where $C = C(\tilde{x}, \tilde{y}, \tilde{z})$. We find $\partial_n \chi | S$:

$$\partial_n \chi \mid_S = \nabla \chi \cdot n = \nabla \chi \, (x, y, z) \cdot (\phi_x, \phi_y, -1) = \chi_x \, \phi_x + \chi_y \, \phi_y - \chi_z.$$

Now $\partial_n \chi = -C \, \partial_{\tilde{z}} \tilde{\chi}$ implies

$$\begin{aligned} x_{\tilde{z}} &= -C \, \phi_x, \\ y_{\tilde{z}} &= -C \, \phi_y, \\ z_{\tilde{z}} &= C, \end{aligned} \tag{4.10}$$

Also, from (4.9),

$$\begin{aligned} x_{\tilde{z}} &= k_{\tilde{z}}, \\ y_{\tilde{z}} &= l_{\tilde{z}}, \\ z_{\tilde{z}} &= 1 + \phi_x \, k_{\tilde{z}} + \phi_y \, l_{\tilde{z}}, \end{aligned}$$

and using (4.10), we get

$$
\begin{aligned}
C &= 1 + \phi_x\, k_{\tilde{z}} + \phi_y\, l_{\tilde{z}}, \\
k_{\tilde{z}} &= -\phi_x(1 + \phi_x\, k_{\tilde{z}} + \phi_y\, l_{\tilde{z}}), \\
l_{\tilde{z}} &= -\phi_y(1 + \phi_x\, k_{\tilde{z}} + \phi_y\, l_{\tilde{z}}),
\end{aligned}
$$

or

$$
\begin{aligned}
(1 + \phi_x^2)\, k_{\tilde{z}} + \phi_x\, \phi_y\, l_{\tilde{z}} &= -\phi_x, \\
\phi_x\, \phi_y\, k_{\tilde{z}} + (1 + \phi_y^2)\, l_{\tilde{z}} &= -\phi_y.
\end{aligned}
$$

Set

$$
A = \begin{pmatrix} 1 + \phi_x^2 & \phi_x\, \phi_y \\ \phi_x\, \phi_y & 1 + \phi_y^2 \end{pmatrix}.
$$

Then, $\det(A) = 1 + \phi_x^2 + \phi_y^2 \neq 0$ gives the existence of $k_{\tilde{z}}$ and $l_{\tilde{z}}$, say

$$
\begin{aligned}
k_{\tilde{z}} &= F(k(\tilde{x}, \tilde{y}, \tilde{z}), l(\tilde{x}, \tilde{y}, \tilde{z})), \\
l_{\tilde{z}} &= G(k(\tilde{x}, \tilde{y}, \tilde{z}), l(\tilde{x}, \tilde{y}, \tilde{z}))
\end{aligned}
\tag{4.11}
$$

for some F, G.

We need to find a solution k and l of (4.11) having prescribed data on the surface $z = \phi(x, y)$ or $\tilde{z} = 0$. To show the local existence of k and l satisfying (4.11), we need to show that the surface $\tilde{\phi} : \{\tilde{z} = 0\}$ is a noncharacteristic surface. Let B denote the matrices of the coefficients of the system (4.11). Then

$$\det(\sum_{|\alpha|=1} B_\alpha (D\tilde{\phi})^\alpha) = \det(I) \neq 0,$$

where $D\tilde{\phi} = (0,0,1)$ and I is the the 2×2 identity matrix. Therefore, we get local existence for k and l for any initial values on the surface $\tilde{z} = 0$ and hence for χ and the cutoff argument is proved.

4.2 Interior Estimates

In this section, we start our regularity study by proving the interior estimates.

Theorem 4.1 *Let $1 < p < \infty$, let B be an open ball in $\mathbb{R}^3$ and let $g \in L^p(B)$. Let $E \in L^p(B)$ and $H \in W^{1,p}(B)$ be a solution of Maxwell's equations with source term g*

$$\begin{aligned} \nabla \times E &= -i\,\omega\,\mu\,H, \\ \nabla \times H &= i\,\omega\,\epsilon\,E + g. \end{aligned} \tag{4.12}$$

Let S be a C^2 surface embedded in $\mathbb{R}^3$ such that S divides B into two connected components B^+ and B^-. Assume that the magnetic permeability μ is constant throughout B and the electric permittivity ε is defined by

$$\epsilon = \begin{cases} \epsilon^+ & \text{in } B^+, \\ \epsilon^- & \text{in } B^-. \end{cases}$$

Then for any B' with $\bar{B}' \subset B$,

$$\| E \|_{L^p(B')} + \| H \|_{W^{1,p}(B')} \leq C \, (\| H \|_{L^p(B)} + \| g \|_{L^p(B)} + \| E \|_{W^{-1,p}(B)}),$$

where C is a constant depending only on p.

Proof. We prove this theorem in two steps. In the first step we prove that

$$\| H \|_{W^{1,p}(B')} \leq C \, (\| H \|_{L^p(B)} + \| E \|_{L^p(B)} + \| g \|_{L^p(B)}) \tag{4.14}$$

and in the second step we prove that

$$\| E \|_{L^p(B')} \leq C \, (\| H \|_{L^p(B)} + \| g \|_{L^p(B)} + \| E \|_{W^{-1,p}(B)}). \tag{4.15}$$

Proof of step 1.

Taking $\nabla \times$ in the second equation of (4.12), we get

$$\nabla \times \nabla \times H = \nabla \times (i \, \omega \, \epsilon \, E + g). \tag{4.16}$$

Let $\tilde{g} = i \, \omega \, \epsilon \, E + g$. Since $E, g \in L^p(B)$ and $\varepsilon \in L^\infty$, it follows that $\tilde{g} \in L^p(B)$. Using the vector identity

$$\nabla \times \nabla \times u = -\triangle u + \nabla(\nabla \cdot u), \tag{4.17}$$

(4.16) becomes

$$-\triangle H + \nabla(\nabla \cdot H) = \nabla \times \tilde{g}.$$

Now taking $\nabla\cdot$ in the first equation of (4.12) and using the vector identity

$$\nabla \cdot (\nabla \times u) = 0, \tag{4.18}$$

we get that

$$\nabla \cdot H = 0. \tag{4.19}$$

With the above equation, (4.16) becomes

$$\triangle H = -\nabla \times \tilde{g}. \tag{4.20}$$

From the standard interior elliptic estimates Theorem 3.6,

$$\begin{aligned}
||H||_{W^{1,p}(B')} &\leq C\,(||H||_{L^p(B)} + ||\nabla \times \tilde{g}||_{W^{-1,p}(B)}) \\
&\leq C\,(||H||_{L^p(B)} + ||\tilde{g}||_{L^p(B)}) \\
&\leq C\,(||H||_{L^p(B)} + ||E||_{L^p(B)} + ||g||_{L^p(B)}).
\end{aligned}$$

which completes the proof of the first step.

Proof of step 2.

To prove the second step, we introduce the following curl-div decomposition for the field E.

$$E = i\,\omega\,\mu\,\nabla \times (G(x,\xi) * H) + F \tag{4.21}$$

where $G(x,\xi)$ is the Green's function for $\triangle$ on B (i.e., $\triangle G = \delta$, and $G|_{\partial B=0}$) and F is determined from the Maxwell's equations (4.12) as follows. Substituting the above decomposition of E in the first equation of (4.12), we get

$$\begin{aligned} \nabla \times E &= \nabla \times (i\,\omega\,\mu\,\nabla \times (G * H) + F) \\ &= i\,\omega\,\mu\,\nabla \times \nabla \times (G * H) + \nabla \times F. \end{aligned} \tag{4.22}$$

Using (4.17),

$$\begin{aligned} \nabla \times E &= i\,\omega\,\mu(-\triangle(G * H) + \nabla(\nabla \cdot (G * H))) + \nabla \times F \\ &= -i\,\omega\,\mu\,\triangle G * H + \nabla(\nabla \cdot (G * H)) + \nabla \times F \\ &= -i\,\omega\,\mu\,H + \nabla(\nabla \cdot (G * H)) + \nabla \times F \end{aligned}$$

since $\triangle G * H = \delta * H = H$. Using Green's identity, the symmetry of $G(x,\xi)$, $G|_{\partial B} = 0$ and (4.19), we get

$$\begin{aligned} \nabla \cdot (G * H) &= \nabla \cdot \int_B G(x,\xi) H(\xi)\,d\xi = \int_B \nabla_x \cdot (G(x,\xi)H(\xi))\,d\xi \\ &= \int_B \nabla_\xi \cdot (G(x,\xi)H(\xi))\,d\xi \\ &= \int_{\partial B} G(x,\xi)H(\xi) \cdot n\,dS - \int_B G(x,\xi)\,\nabla \cdot H(\xi)\,d\xi = 0 \end{aligned}$$

Hence (4.22) becomes

$$\nabla \times E = -i\,\omega\,\mu\,H + \nabla \times F. \tag{4.23}$$

Therefore equations (4.12) and (4.23) give

$$\nabla \times F = 0.$$

Hence $F = \nabla h$ for some h.

Substituting E in the second equation of (4.12), we get

$$\nabla \times H = i\,\omega\,\epsilon\,(i\,\omega\,\mu\,\nabla \times (G * H) + \nabla h) + g.$$

Taking $\nabla\cdot$ in the above equation and using (4.18), yields

$$0 = -\omega^2\,\mu\,\nabla \cdot (\epsilon\,\nabla \times (G * H)) + i\,\omega\,\nabla \cdot (\epsilon\,\nabla h) + \nabla \cdot g,$$

or

$$i\,\omega\,\nabla \cdot (\epsilon\,\nabla h) = \omega^2\,\mu\,\nabla \cdot (\epsilon\,\nabla \times (G * H)) - \nabla \cdot g,$$

or

$$\nabla \cdot (\epsilon\,\nabla h) = \frac{\omega\,\mu}{i}\,\nabla \cdot (\epsilon\,\nabla \times (G * H)) - \frac{1}{i\,\omega}\,\nabla \cdot g = \nabla \cdot (\frac{\omega\,\mu}{i}\,\epsilon\,\nabla \times (G * H) - \frac{1}{i\,\omega}\,g).$$

If we denote $\frac{\omega\mu}{i}\,\epsilon\,\nabla\times(G*H)-\frac{1}{i\,\omega}\,g=f$, the above equation becomes

$$\nabla\cdot(\epsilon\,\nabla h)=\nabla\cdot f.$$

From the above definition of f, we get that $f\in L^p(B)$ since $g\in L^p(B),\epsilon\in L^\infty(B)$ and $H\in W^{1,p}(B)$.

From Lemma 4.2, $h\in W^{1,p}(B)$ and hence $F=\nabla h\in L^p(B)$. Then $E\in L^p(B)$ since $\nabla\times(G*H)\in W^{2,p}(B)$ and $F\in L^p(B)$. Furthermore, from the decomposition (4.21) of E, we get

$$\| E \|_{L^p(B')}\leq C\,(\|\nabla\times(G*H)\|_{L^p(B')}+\|\nabla h\|_{L^p(B')}). \tag{4.24}$$

We start by estimating the first term of (4.24).

$$\|\nabla\times(G*H)\|_{L^p(B')}\leq C\;\|G*H\|_{W^{1,p}(B')}\leq C\;\|H\|_{L^p(B)}\,. \tag{4.25}$$

To estimate the second term of (4.24), we use Corollary (4.1), the definition of f, (4.25) and we get

$$\begin{aligned}\|\nabla h\|_{L^p(B')} &\leq C\,(\|\nabla\cdot f\|_{W^{-1,p}(B)}+\|h\|_{L^p(B)})\\ &\leq C\,(\|f\|_{L^p(B)}+\|h\|_{L^p(B)})\\ &\leq C\,(\|\nabla\times(G*H)\|_{L^p(B)}+\|g\|_{L^p(B)}+\|h\|_{L^p(B)}),\\ &\leq C\,(\|H\|_{L^p(B)}+\|g\|_{L^p(B)}+\|h\|_{L^p(B)}).\end{aligned} \tag{4.26}$$

We used $||G * H||_{W^{1,p}} \leq ||H||_{L^p}$ which follows from standard elliptic regularity results [10] given that

$$\triangle(G * H) = \triangle G * H = \delta * H = H.$$

From the decomposition (4.21) of E, from

$$\nabla h = E - i\omega\mu\nabla(G * H),$$

from Corollary 4.1 and from

$$||h||_{L^p(B)} \leq ||Dh||_{L^p(B)} \leq ||h||_{W^{1,p}(B)},$$

we get

$$||h||_{L^p(B)} \leq C(||E||_{W^{-1,p}(B)} + ||\nabla(G * H)||_{W^{-1,p}(B)}).$$

Hence

$$\begin{aligned} \| h \|_{L^p(B)} &\leq C\,(\| E \|_{W^{-1,p}(B)} + \| \nabla \times (G * H) \|_{W^{-1,p}(B)}) \\ &\leq C\,(\| E \|_{W^{-1,p}(B)} + \| G * H \|_{L^p(B)}) \qquad (4.27) \\ &\leq C\,(\| E \|_{W^{-1,p}(B)} + \| H \|_{L^p(B)}) \end{aligned}$$

using the estimate

$$||f * g||_{L^p} \leq ||f||_{L^1}\, ||g||_{L^p}.$$

With the above estimate (4.27), (4.26) becomes

$$\| \nabla h \|_{L^p(B')} \leq C\,(\| H \|_{L^p(B)} + \| g \|_{L^p(B)} + \| E \|_{W^{-1,p}(B)}). \tag{4.29}$$

Therefore, from (4.25) and (4.29),

$$\| E \|_{L^p(B')} \leq C\,(\| H \|_{L^p(B)} + \| g \|_{L^p(B)} + \| E \|_{W^{-1,p}(B)}),$$

which completes the proof of the second step.

Finally, (4.14) and (4.15) give

$$\| E \|_{L^p(B')} + \| H \|_{W^{1,p}(B')} \leq C\,(\| H \|_{L^p(B)} + \| g \|_{L^p(B)} + \| E \|_{W^{-1,p}(B)})$$

and the proof of the theorem is complete. □

4.3 Boundary Estimates

In this section, we extend Theorem (4.1) to the boundary $\partial\Omega$ in order to establish the existence theory for scattering problems later. We start by outlying the basic setup for the scattering problem. Consider the scattering of electromagnetic waves by a nonmagnetic periodic structure. The structure, a C^2 interface S, separates the medium into two connected components. Above the interface, the medium has a constant dielectric coefficient ϵ^+, and below another constant dielectric coefficient ϵ^-, with $\epsilon^+ \neq \epsilon^-$. The medium is assumed nonmagnetic with a constant magnetic permeability throughtout and no free charges are present.

There are two constants Λ_1 and Λ_2, such that for any $n_1, n_2 \in \mathbb{Z}$ and for almost all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the dielectric coefficients $\epsilon^{\pm}$ satisfy

$$\epsilon^{\pm}(x_1 + n_1\Lambda_1, x_2 + n_2\Lambda_2, x_3) = \epsilon^{\pm}(x_1, x_2, x_3). \tag{4.30}$$

In order to solve the system of differential equations, we need boundary conditions in the x_3 direction. These conditions may be derived from the radiation condition, the periodicity of the structure and Green's functions. If we denote the top boundary of the structure by Γ_t and the bottom by Γ_b, we write the boundary conditions as in [5].

Lemma 4.3 *There exist boundary pseudo-differential operators B of order one, such that*

$$\nu \times (\nabla \times H) = B(P(H)) \text{ on } \Gamma_b,$$

where ν is the unit vector $\nu = (0, 0, \pm 1)$ and the operator B is defined by

$$Bf = -i \sum_{n \in \Lambda} \frac{1}{\beta^{(n)}} \{ (\beta^{(n)})^2 (f_1^{(n)}, f_2^{(n)}, 0) + (\alpha_n \cdot f^{(n)})\, \alpha_n \}\, e^{i\alpha_n \cdot x}, \tag{4.31}$$

where P is the projection onto the plane orthogonal to ν; i. e.,

$$Pf = -\nu \times (\nu \times f),$$

$$f^{(n)} = \Lambda_1^{-1} \Lambda_2^{-1} \int_0^{\Lambda_1} \int_0^{\Lambda_2} f(x) e^{-i\alpha_n \cdot x}\, dx_1\, dx_2,$$

$\alpha_n = (\frac{2\pi n_1}{\Lambda_1}, \frac{2\pi n_2}{\Lambda_2}, 0)$, *and* $x = (x_1, x_2, x_3)$.

A similar boundary condition may be written for Γ_t, by replacing the field H with $H - H_I$, where H_I denotes the incident field. Without loss of generality, we shall consider here the case of Γ_b, for simplicity denoted by Γ with $\Gamma = \{x_3 = 0\}$. Note that we don't need boundary conditions in the x_1, x_2 directions since the structure is periodic in these two directions.

We are trying to extend the apriori estimates of Theorem 4.1 to the boundary Γ.

Recall (4.20),

$$\triangle H = -\nabla \times \tilde{g},$$

where $\tilde{g} = i\,\omega\,\epsilon\,E + g$. Note that above (or below) the interface S, the dielectric coefficient ϵ is constant and the above equation becomes

$$\begin{aligned}\triangle H &= -i\,\omega\,\epsilon\,\nabla \times E - \nabla \times g \\ &= -i\,\omega\,\epsilon\,(-i\,\omega\,\mu\,H) - \nabla \times g \\ &= -\omega^2\,\epsilon\,\mu\,H - \nabla \times g \\ &= -k^2\,H - \nabla \times g\end{aligned}$$

where we used Maxwell's equations (4.12) and $k^2 = \omega^2\,\epsilon\,\mu$.

Hence, in the regions above and below the interface S, H satisfies

$$\triangle H + k^2 H = -\nabla \times g.$$

We may formulate the global boundary estimate problem as follows.

Theorem 4.2 *Let $T > 0$ and let*

$$\Omega = \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq \Lambda_1, 0 \leq x_2 \leq \Lambda_2, -T \leq x_3 < T\},$$

with a boundary portion $\Gamma \subset \Gamma_t \cup \Gamma_b$ and let $1 < p < \infty$. Assume $g \in L^p(\Omega)$. Let $H \in W^{1,p}(\Omega)$ be a solution of the equation

$$\triangle H + k^2 H = -\nabla \times g \textit{ in } \Omega$$

$$\nu \times (\nabla \times H) = B(P(H)) \textit{ on } \Gamma,$$

where B is defined in (4.31). Then for any domain Ω' with $\Omega' \subset\subset \Omega \cup \Gamma$,

$$\| H \|_{W^{1,p}(\Omega')} \leq C\, (\| H \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)}),$$

where C is a constant depending on $p, \Gamma, \Omega, \Omega'$.

Proof.

Note that in the region Ω^-, between the interface S and $\Gamma = \{x_3 = 0\}$, we have

$$\triangle H + k^2 H = -\nabla \times g \text{ in } \Omega$$

$$\nu \times (\nabla \times H) = B(P(H)) \text{ on } \Gamma,$$

where B is the pseudo-differential operator of order one defined above in (4.31).

Since H is Λ periodic, we can expand H in a Fourier series and write

$$H(x) = \sum_{n=-\infty}^{\infty} U_n(x_3) e^{i\alpha_n \cdot x}$$

where $x = (x_1, x_2, x_3)$, $\alpha_n = (\frac{2\pi n_1}{\Lambda_1}, \frac{2\pi n_2}{\Lambda_2}, 0)$ and

$$U_n(x_3) = \frac{1}{\Lambda_1 \Lambda_2} \int_0^{\Lambda_1} \int_0^{\Lambda_2} H(x) e^{-i\alpha_n \cdot x} \, dx_1 dx_2$$

We extend H to $\Omega^+ = \{x \mid 0 \leq x_3 \leq T\}$, for some $T > 0$, by

$$H_e = \sum_{n=-\infty}^{\infty} U_n e^{i\beta_n x_3} \, e^{i\alpha_n \cdot x}$$

where

$$\beta_n = \sqrt{k^2 - |\alpha_n|^2},$$

and $U_n = U_n(0)$ for notational simplicity.

Define

$$\tilde{H} = \begin{cases} H & \text{in } \Omega^-, \\ H_e & \text{in } \Omega^+. \end{cases}$$

Then $\tilde{H}$ satisfies the boundary condition on Γ from the derivation of the operator B, and

$$\triangle \tilde{H} + k^2 \tilde{H} = \tilde{G} \quad \text{in} \quad \Omega,$$

where

$$\tilde{G} = \begin{cases} G & \text{in} \quad \Omega^-, \\ 0 & \text{in} \quad \Omega^+, \end{cases}$$

and $G = -\nabla \times g$.

From the elliptic regularity result, Theorem 3.6,

$$||\tilde{H}||_{W^{1,p}(\Omega_1)} \leq C(||\tilde{H}||_{L^p(\Omega_2)} + ||\tilde{G}||_{W^{-1,p}(\Omega_2)}),$$

where

$$\Omega_1 = \{x \mid 0 \leq x_1 \leq \Lambda_1, 0 \leq x_2 \leq \Lambda_2, -T_1 \leq x_3 < T_1\},$$

$$\Omega_2 = \{x \mid 0 \leq x_1 \leq \Lambda_1, 0 \leq x_2 \leq \Lambda_2, -T_2 \leq x_3 < T_2\},$$

such that $0 < T_1 < T_2 < T$ and $\Omega_1, \Omega_2 \subset \Omega^+ \cup \Omega^-$ (see Figure 4.1).

Using the definition of $\tilde{G}$, we get

$$||\tilde{H}||_{W^{1,p}(\Omega_1)} \leq C(||\tilde{H}||_{L^p(\Omega_2)} + ||G||_{W^{-1,p}(\Omega_2^-)})$$

Since $\Omega_1^- = \Omega_1 \cap \Omega^- \subset \Omega_1$ and from the definition of $\tilde{H}$,

$$||H||_{W^{1,p}(\Omega_1^-)} \leq C||\tilde{H}||_{W^{1,p}(\Omega_1)}.$$

Hence

$$||H||_{W^{1,p}(\Omega_1^-)} \leq C(||\tilde{H}||_{L^p(\Omega_2)} + ||G||_{W^{-1,p}(\Omega_2^-)})$$

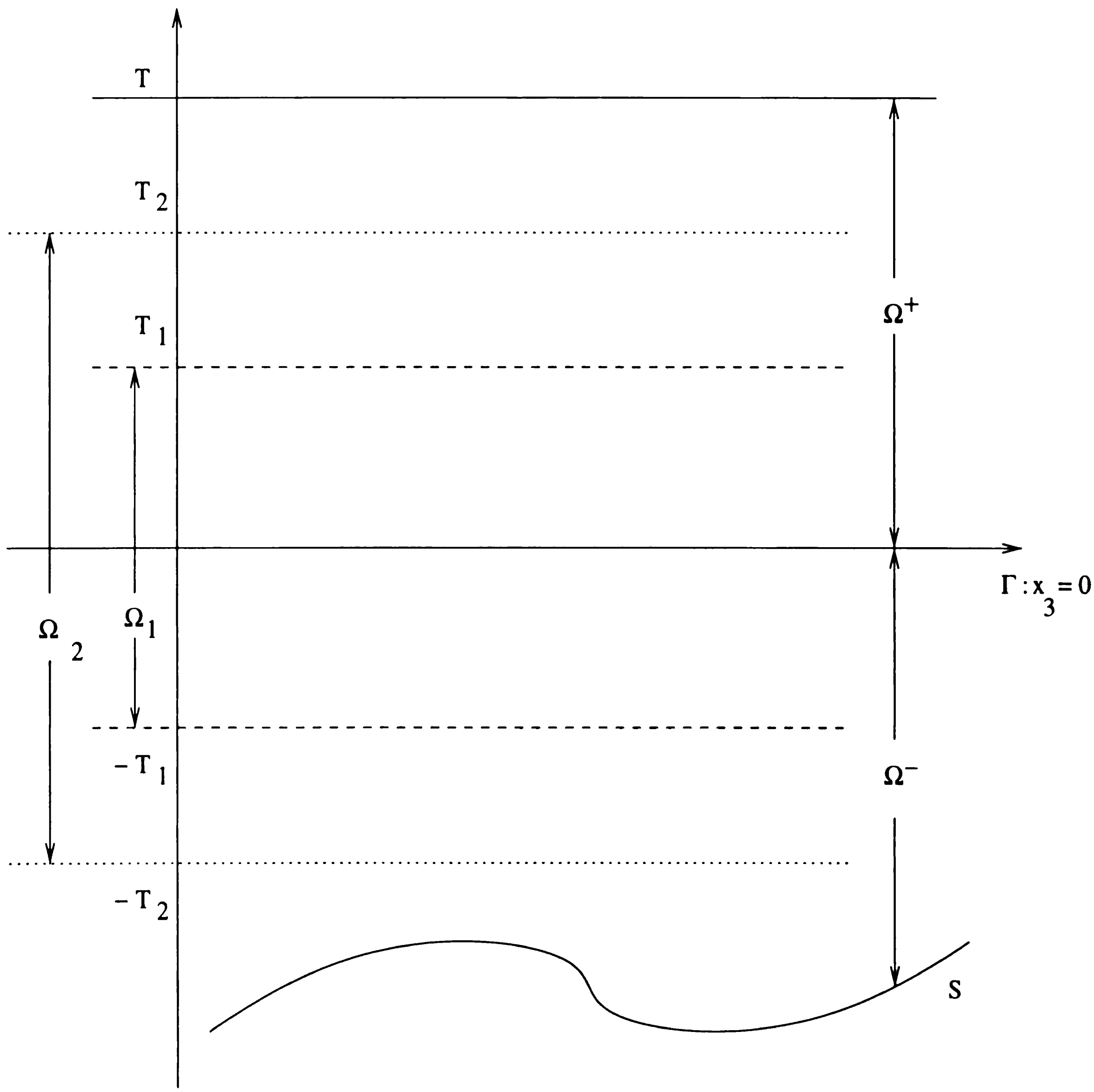


Figure 4.1:

and

$$||H||_{W^{1,p}(\Omega_1^-)} \leq C(||H||_{L^p(\Omega_2^-)} + ||H_e||_{L^p(\Omega_2^+)} + ||g||_{L^p(\Omega_2^-)}).$$

CLAIM 1: For $2 \leq p \leq \infty$, we have

$$||H_e||_{L^p(\Omega_2^+)} \leq C||H||_{L^p(\Gamma_1)},$$

where $\Gamma_1 = \{x \,||\, 0 \leq x_1 \leq \Lambda_1, 0 \leq x_2 \leq \Lambda_2, x_3 = 0\}$

Assuming this claim for the moment, we obtain

$$||H||_{W^{1,p}(\Omega_1^-)} \leq C(||H||_{L^p(\Omega_2^-)} + ||H||_{L^p(\Gamma_1)} + ||g||_{L^p(\Omega_2^-)}).$$

By the trace theorem, Theorem 3.4, Sobolev inequality, Theorem 3.3 and the interpolation inequality, Theorem 3.2, for $p = \frac{2q}{3-q}$ we get

$$\begin{aligned} ||H||_{L^p(\Gamma_1)} &\leq C||H||_{W^{1-\frac{1}{q},q}(\Omega_1^-)} \leq C||H||_{W^{1,q}(\Omega_1^-)} \\ &\leq C||H||^a_{W^{1,p}(\Omega_1^-)}||H||^{1-a}_{L^p(\Omega_1^-)} \leq \epsilon\, ||H||_{W^{1,p}(\Omega_1^-)} + C_\epsilon\, ||H||_{L^p(\Omega_1^-)}, \end{aligned}$$

where a is a positve constant less than 1, ϵ is any positive constant, C_ϵ is a constant depending on the choice of ϵ. Young's inequality (3.1) was used in the last estimate. Combining these estimates and choosing a suitably small ϵ, we have

$$||H||_{W^{1,p}(\Omega_1^-)} \leq C(||H||_{L^p(\Omega_2^-)} + ||g||_{L^p(\Omega_2^-)}),$$

which is the boundary estimate we required.

Next let us turn our attention to the proof of Claim 1.

Case 1: For $p = 2$, it is easy to see, by using the orthogonality of $e^{i\alpha_n \cdot x}$, that

$$||H_e||^2_{L^2(\Omega_2^+)} = \sum_{n=-\infty}^{\infty} ||U_n||^2 \int_0^{T_2} |e^{i\beta_n x_3}|^2 dx_3 \leq C \sum_{n=-\infty}^{\infty} ||U_n||^2 = ||H||^2_{L^2(\Gamma_1)}.$$

Case 2: $p = \infty$. Let $H_e = H^{(1)} + H^{(2)}$, where

$$H^{(1)} = \sum_{n \in S_1} U_n e^{i\beta_n x_3} e^{i\alpha_n \cdot x}, H^{(2)} = H - H^{(1)} = \sum_{n \notin S_1} U_n e^{i\beta_n x_3} e^{i\alpha_n \cdot x},$$

where $S_1 = \{n \mid k^2 - |\alpha_n|^2 \geq -k^2\}$. Note that S_1 is a finite set, and

$$|U_n| = \frac{1}{\Lambda_1 \Lambda_2} \left| \int \int e^{-i\alpha_n \cdot x} H(x_1, x_2, 0) dx_1\, dx_2 \right| \leq C \|H\|_{L^\infty(\Gamma_1)}.$$

For any n, we obviously have

$$||H^{(1)}||_{L^\infty(\Omega_2^+)} \leq \sum_{n \in S^1} |U_n| \leq C|S^1| \;\; \|H\|_{L^\infty(\Gamma_1)}.$$

Next we want to show that $\|H^{(2)}\|_{L^\infty(\Omega_2^+)} \leq C\|H\|_{L^\infty(\Gamma_1)}$. To do this, we use the maximum principle for elliptic equations. But we can not use it directly because of the wrong sign before the term H_e in the equation $\triangle H_e + k^2 H_e = 0$. To overcome this technical difficulty, let us consider a new function $V = H^{(2)} f(x_3)$.

CLAIM 2: There exists a function f defined on $[0, T]$ for a suitable $T > 0$ such that $1 \leq f(x_3) \leq 2$ for all $x_3 \in [0, T]$ and the equation for V becomes

$$\Delta V + A(x_3)V_{x_3} = 0, \;\; x_3 \in [0, T].$$

Again, let us use the claim first. From the maximum principle for elliptic equations, Theorem 3.5, for $x \in \Omega_2^+$, V will attain its maximum on the boundary of Ω_2^+. Since H is Λ periodic in the $x = (x_1, x_2)$ direction, V can only have a maximum on the bottom or top parts of the boundary unless V is constant, in which case the estimate is trivial.

If V has a maximum on the bottom part,

$$||H^{(2)}||_{L^\infty(\Omega_2^+)} \leq ||V||_{L^\infty(\Omega_2^+)} = f(0)||H^{(2)}||_{L^\infty(\Gamma_1)} \leq C||H||_{L^\infty(\Gamma_1)}.$$

If V has a maximum on the top part, we see that

$$\begin{aligned} ||H^{(2)}||_{L^\infty(\Omega_2^+)} &\leq ||V||_{L^\infty(\Omega_2^+)} \leq 2\,||H^{(2)}(\cdot, T_2)||_{L^\infty} \\ &= 2\,|| \sum_{i\beta_n \leq -k^2} U_n e^{i\alpha_n \cdot x} e^{i\beta_n T_2} ||_{L^\infty} \leq 2 \sum_{i\beta_n \leq -k^2} |U_n| e^{i\beta_n T_2} \\ &\leq 2 \sup\{|U_n|\} \sum_{i\beta_n \leq -k^2} e^{i\beta_n T_2} \leq C\,||H||_{L^\infty(\Gamma_1)}, \end{aligned}$$

where we have used the facts $\sum_{n \notin S^1} e^{i\beta_n T_2}$ is bounded, and $|U_n| \leq C\,||H||_{L^\infty(\Gamma_1)}$ for each n.

To finish the argument of this case, we need a proof of Claim 2. Since $V =$

$H_e f(x_3)$, a direct computation shows that

$$\begin{aligned}
\triangle V &= \triangle H_e f(x_3) + f''(x_3)H_e + 2(H_e)_{x_3} f'(x_3) \\
&= -k^2 H_e f(x_3) + f''(x_3)H_e + 2(H_e)_{x_3} f'(x_3) \\
&= -k^2 V + f''(x_3)H_e + 2[H_e f'(x_3)]_{x_3} - 2H_e f''(x_3) \\
&= -k^2 V - \frac{f''(x_3)}{f(x_3)}V + 2[\frac{f'(x_3)}{f(x_3)}]'V + 2\frac{f'(x_3)}{f(x_3)}V_{x_3} \\
&= [-k^2 - \frac{f''(x_3)}{f(x_3)} + 2(\frac{f'(x_3)}{f(x_3)})']V + 2\frac{f'(x_3)}{f(x_3)}V_{x_3}.
\end{aligned}$$

To prove Claim 2, we need to solve the ordinary differential equation

$$k^2 + \frac{f''(x_3)}{f(x_3)} - 2\left(\frac{f'(x_3)}{f(x_3)}\right)' = 0,$$

which is the same as

$$f''f - 2(f')^2 = k^2 f^2.$$

Let $u = f'/f$. We should have

$$u' = \frac{f''f - f'^2}{f^2} = k^2 + u^2.$$

We know that $u(x_3) = k\tan(kx_3)$ is a solution. Consequently,

$$f(x_3) = \exp\left(k\int_0^{x_3} \tan ks\, ds\right) = \frac{1}{\cos kx_3}$$

is a solution, which is also our candidate in Claim 2.

From the Riesz Convexity Theorem [19], we get

$$||H_e||_{L^p(\Omega_2^+)} \le C||H||_{L^p(\Gamma_1)}$$

for $2 \le p \le \infty$, which concludes the proof of Claim 1.

We only need to prove that

$$||H_e||_{L^p(\Omega_2^+)} \le C||H||_{L^p(\Gamma_1)}$$

for $1 < p < 2$, since the case $2 \le p < \infty$ is covered by Claim 1.

For any $g \in L^q(\Omega_2^+)$, with $\frac{1}{q} + \frac{1}{p} = 1$, let $L = \sqrt{-\partial_{x_1}^2 - \partial_{x_2}^2 - k^2}$. Then

$$\begin{aligned}
\int_{\Omega_2^+} H_e\, g\, dx_1\, dx_2\, dx_3 &= \int_{\Gamma_1}\int_0^{T_2} e^{-x_3 L} H|_{x_3=0}\, g(x_1, x_2, x_3)\, dx_3\, dx_1\, dx_2 \\
&= \int_{\Gamma_1}\int_0^{T_2} H|_{x_3=0}\, e^{-x_3 L}\, g(\cdot, x_3)\, dx_3\, dx_1\, dx_2 \\
\le \left(\int_{\Gamma_1} |H(x_1, x_2, 0)|^p\right)^{\frac{1}{p}} &\left(\int_{\Gamma_1} |\int_0^{T_2} e^{-x_3 L}\, g(\cdot, x_3) dx_3|^q\, dx_1\, dx_2\right)^{\frac{1}{q}}.
\end{aligned}$$

Note that for $q > 2$, the previous estimate implies that

$$\begin{aligned}
\left\| \int_0^{T_2} e^{-x_3 L}\, g(\cdot, x_3)\, dx_3 \right\|_q &\le \int_0^{T_2} ||e^{-x_3 L}\, g(\cdot, x_3)||_q\, dx_3 \\
&\le C \int_0^{T_2} ||g(\cdot, x_3)||_q\, dx_3 \le C\, ||g||_{L^q(\Omega_2^+)}.
\end{aligned}$$

Therefore,

$$\int_{\Omega_2^+} (H_e\, g)\, dx_1\, dx_2\, dx_3 \le C\, ||H(x_1, x_2, 0)||_{L^p(\Gamma_1)} ||g||_{L^q(\Omega_2^+)}.$$

Hence for $1 < p < \infty$

$$||H_e||_{L^p(\Omega_2^+)} \leq C\, ||H||_{L^p(\Gamma_1)},$$

and the theorem is proved.

4.4 Existence Theory

In this section we establish the existence of solutions for Maxwell's equations with source term (2.20) and (2.19).

We will show that the linearized SHG problem has a $W^{1,p} \cup L^p$ solution (H, E).

Theorem 4.3 *For $T > 0$, let*

$$\Omega = \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq \Lambda_1, 0 \leq x_2 \leq \Lambda_2, -T \leq x_3 < T\}.$$

Let $\Gamma_t = \{x_3 = T\}$ and $\Gamma_b = \{x_3 = -T\}$ denote the top and bottom boundaries of Ω. Assume that H and E are periodic, with period Λ_1 in the x_1 direction, and period Λ_2 in the x_2 direction.

Let S denote a C^2 surface embedded in $\mathbb{R}^3$ given by $x_3 = \phi(x_1, x_2)$ such that S divides Ω into two connected components, Ω^+ and Ω^-, and let ϵ be piecewise constant; i.e.,

$$\epsilon = \begin{cases} \epsilon^+ & \text{in } \Omega^+, \\ \epsilon^- & \text{in } \Omega^-. \end{cases}$$

Assume μ is constant throughtout Ω and let B be the first order pseudo-differential *operator defined in (4.31). Then the SHG model problem*

$$\nabla \times E^{(\omega)} = -i\,\omega\,\mu\,H^{(\omega)},$$

$$\nabla \times H^{(\omega)} = i\,\omega\,\epsilon\,E^{(\omega)},$$

$$\nu \times (\nabla \times (H^{(\omega)} - H_I^{(\omega)})) = B(P(H^{(\omega)} - H_I^{(\omega)})) \text{ on } \Gamma_t,$$

$$\nu \times (\nabla \times H^{(\omega)}) = B(P(H^{(\omega)})) \text{ on } \Gamma_b,$$

and

$$\nabla \times E^{(2\omega)} = -i\,2\omega\,\mu\,H^{(2\omega)},$$

$$\nabla \times H^{(2\omega)} = i\,2\omega\,\epsilon\,E^{(2\omega)} + \chi_{jk}^{(2\omega)}\,i\,2\omega\,E_{x_j}^{(\omega)}\,E_{x_k}^{(\omega)},$$

$$\nu \times (\nabla \times H^{(2\omega)}) = B(P(H^{(2\omega)})) \text{ on } \Gamma_t,$$

$$\nu \times (\nabla \times H^{(2\omega)}) = B(P(H^{(2\omega)})) \text{ on } \Gamma_b,$$

has a solution $(H^{(\omega)}, H^{(2\omega)}) \in W^{1,p}$ *and* $(E^{(\omega)}, E^{(2\omega)}) \in L^p$*, for any* $1 < p < \infty$ *and all frequencies* ω *except a countable set.*

Proof.

In order to study our problem (4.34), we need to know the regularity of the field $E^{(\omega)}$ which appears in the right hand side of the second equation.

From Theorem 4.3 in [5] we know that for all but possibly a discrete set of ω, there is a unique weak solution $H^{(\omega)} \in W^{1,2}(\Omega)$ for

$$\nabla \times E^{(\omega)} = -i\, \omega\, \mu\, H^{(\omega)},$$

$$\nabla \times H^{(\omega)} = i\, \omega\, \epsilon\, E^{(\omega)},$$

$$\nu \times (\nabla \times (H^{(\omega)} - H_I^{(\omega)})) = B(P(H^{(\omega)} - H_I^{(\omega)})) \text{ on } \Gamma_t,$$

$$\nu \times (\nabla \times H^{(\omega)}) = B(P(H^{(\omega)})) \text{ on } \Gamma_b,$$

where $H_I^{(\omega)}$ denotes the incident field.

From the regularity result, we see that $H^{(\omega)} \in W^{1,p}(\Omega)$ for any $1 < p < \infty$. Hence $E^{(\omega)} \in L^p(\Omega)$, for $1 < p < \infty$. Therefore, $g = \chi_{jk}^{(2\omega)}\, i\, 2\omega\, E_{x_j}^{(\omega)}\, E_{x_k}^{(\omega)} \in L^p(\Omega)$, for any $1 < p < \infty$.

An argument similar to the one in [5] yields a $W^{1,2}(\Omega)$ solution $H^{(2\omega)}$ since $g \in L^2(\Omega)$. The regularity theorem we established implies that $H^{(2\omega)} \in W^{1,p}(\Omega)$ and $E^{(2\omega)} \in L^p(\Omega)$ and the theorem is proved.

Bibliography

[1] R. A. Adams, *Sobolev Spaces*, New York Academic Press, 1975.

[2] G. C. Baldwin, *An Introduction to Nonlinear Optics*, Plenum Press, New York, 1969.

[3] G. Bao and D. Dobson, *Second harmonic generation in nonlinear optical films*, J. Math. Phys., 35 (1994), pp. 1622–1633.

[4] G. Bao and D. Dobson, *Diffractive optics in nonlinear media with periodic structures*, European J. Appl. Math., 6 (1995), pp. 573–590.

[5] G. Bao and D. Dobson, *On the scattering by a biperiodic structure*, Proc. Amer. Math. Soc., 128 (2000), pp. 2715–2723.

[6] G. Bao and Z. Zhou, *An inverse problem for scattering by a doubly periodic structure*, Trans. Amer. Math. Soc., 350 (1998), pp. 4089–4103.

[7] X. Chen and A. Friedman, *Maxwell's equations in a periodic structure*, Trans. Amer. Math. Soc., 323 (1991), pp. 465–507.

[8] D. Dobson and A. Friedman, *The time-harmonic Maxwell equations in a doubly periodic structure*, J. Math. Anal. Appl. 166 (1992), pp. 507–528.

[9] L. Evans, *Partial Differential Equations*, Grad. Stud. in Math., vol. 19, AMS, 1998.

[10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.

[11] L. Hormander, *The Analysis of Linear Partial Differential Operators*, Springer-Verlag, New York, 1990.

[12] W. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, Inc., New York, 1965.

[13] F. John, *Partial Differential Equations*, Springer-Verlag, New York, 1982.

[14] R. Kress, *Linear Integral Equations*, Springer-Verlag, New York, 1989.

[15] D. L. Mills, *Nonlinear Optics*, Springer-Verlag, Germany, 1998.

[16] R. Palais, *Foundations of Global Non-Linear Analysis*, W. A. Benjamin, Inc., New York, 1968.

[17] E. Popov and M. Neviere, *Surface-enhanced second-harmonic generation in non-linear corrugated dielectrics: New theoretical approaches*, J. Opt. Soc. Amer. B, 11 (1994), pp. 1555–1564.

[18] J. C. Nédélec, *Ondes acoustiques et electromagnetiques. Equations integrales*, Cours de dea, Ecole Polytechnique, Universite Paris VI, Paris XI, 1996.

[19] L. Nirenberg, *Functional Analysis. Notes by L. Sibner*, Courant Institute of Mathematical Sciences, New York University, 1960–1961.

[20] R. Reinish and M. Neviere, *Electromagnetic theory of diffraction in nonlinear optics and surface-enhanced nonlinear optical effects*, Phys. Rev. B, 28 (1983), pp. 1870–1885.

[21] R. Reinish, M. Neviere, H. Akhouayri, J. Coutaz, D. Mayster and E. Pic, *Grating enhanced second harmonic generation through electromagnetic resonances* , Opt. Engrg., 27 (1998), pp. 961–971.

[22] Y. R. Shen, *The Principles of Nonlinear Optics*, John Wiley, New York, 1984.

[23] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston, 1991.