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MINIMALITY OF FLOWS AND ALMOST PERIODICITY OF POINTS UNDER VARIOUS CONSTRUCTIONS IN TOPOLOGICAL DYNAMICS

By

Alica Miller

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ABSTRACT

MINIMALITY OF FLOWS AND ALMOST PERIODICITY OF POINTS UNDER VARIOUS CONSTRUCTIONS IN TOPOLOGICAL DYNAMICS

By

Alica Miller

In Chapter 1 we deal with the minimality of the flows obtained by various constructions from compact minimal abelian flows. We first find a criterion for minimality of "syndetic" restrictions of compact minimal abelian flows in terms of eigenvalues (and a criterion for total minimality of compact minimal abelian flows). Using the criterion for minimality of restrictions we give a new proof of a classical theorem of W. Parry about minimality of group-extensions of compact minimal abelian flows. Then we prove a criterion for minimality of products of two compact minimal abelian flows, one of which is almost periodic, in terms of eigenvalues. For each of the criteria we give several applications. We also introduce the notion of SK groups, and use it to generalize some statements which relate total minimality, weak mixing and triviality of the structure group, as well as to improve various conditions which imply non-total-minimality of compact minimal abelian flows.

In Chapter 2 we deal with the question whether almost periodicity of a point in a flow transfers to the appropriate points in the flows obtained by various constructions

applied to the original flow, like restrictions, subflows, factors, extensions, products, etc. The most difficult is the case when we have a morphism $f : \mathcal{X} \to \mathcal{Y}$, an almost periodic point y in $\mathcal{Y}$, and a point $x \in f^{-1}(y)$. In general x is not necessarily almost periodic, but several conditions are known under which that happens. They fall into either "compact" or "noncompact" conditions, depending on whether $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be compact or not. In "noncompact" conditions other assumptions are restrictive. We find a criterion for lifting of almost periodicity of y, which generalizes both "compact" and "noncompact" statements at the same time.

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Contents

0.1 Introduction

The thesis consists of two chapters. We will first roughly describe the content of each of the two chapters; then we will give a more precise description of each section of each of the chapters.

In Chapter 1 we deal with the minimality of the flows obtained by various constructions from compact minimal abelian flows. We first find a criterion for minimality of "syndetic" restrictions of compact minimal abelian flows in terms of eigenvalues (and a criterion for total minimality of compact minimal abelian flows). Using the criterion for minimality of restrictions we give a new proof of a classical theorem of W. Parry about minimality of group-extensions of compact minimal abelian flows. Using Parry's theorem we prove a criterion for minimality of a product of two compact minimal abelian flows, one of which is almost periodic, in terms of eigenvalues. For each of the criteria we give several applications. Among other things, we introduce the notion of SK groups, well adjusted to work with the criteria, prove some properties of this notion, and then use it to generalize some statements which relate total minimality, weak mixing and triviality of the structure group, as well as to improve various conditions which imply non-total-minimality of compact minimal abelian flows.

In Chapter 2 we deal with the question whether almost periodicity of a point in a flow transfers to the appropriate points in the flows obtained by various constructions applied to the original flow, like restrictions, subflows, factors, extensions, products, etc. The most difficult is the case when we have a morphism $f : \mathcal{X} \to \mathcal{Y}$, an almost periodic point y in $\mathcal{Y}$, and a point $x \in f^{-1}(y)$. In general x is not necessarily almost periodic, but several conditions are known under which that happens. They fall into either "compact" or "noncompact" conditions, depending on whether $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be compact or not. In "noncompact" conditions other assumptions are restrictive. We find a criterion for almost periodicity of x, which generalizes both "compact" and "noncompact" statements at the same time.

The questions investigated in Chapter 1 and Chapter 2 are naturally related to each other since, in the case of compact flows, almost periodicity of a point can be expressed via minimality. In particular, in a compact minimal flow every point is almost periodic.

We now give a more precise description of each section of each chapter. First let us mention that in Section 0.2 we give the notation and the terminology we use throughout the thesis, as well as some relevant basic facts.

We first give a description of each section of Chapter 1.

In Section 1.1 we introduce the notions of the *$\mathcal{X}$-envelope* of a syndetic subgroup S of T and of an *$\mathcal{X}$-enveloped subgroup* of T. We prove some properties related to these notions and give some examples which illustrate them.

Using these notions, in Section 1.2 we formulate and prove two results, which we name as a *criterion for minimality of restrictions* and a *criterion for total minimality.* The first criterion gives necessary and sufficient conditions for a restriction $\mathcal{X}_S = \langle S, X \rangle$, S syndetic, of a compact minimal abelian flow $\mathcal{X} = \langle T, X \rangle$ to be minimal. One of the conditions is in terms of eigenvalues, the other one in terms of $\mathcal{X}$-envelopes. The second criterion gives necessary and sufficient conditions for a compact minimal abelian flow $\mathcal{X} = \langle T, X\rangle$ to be totally minimal. We apply these criteria to the cases $T = \mathbb{Z}$ and $T = \mathbb{R}$. The statement we deduce in the case $T = \mathbb{R}$ is a result of J. Egawa.

In Section 1.3 we investigate skew-extensions of compact minimal $\mathbb{Z}$-flows. In the context of a classical theorem of Parry we apply the criterion for minimality of restrictions to an appropriately defined flow and conclude that the condition coming from the criterion is equivalent to the well-known Parry's condition. In that way we get a new proof of this theorem.

In Section 1.4 we give a new proof of a more general theorem of Parry about minimality of group-extensions. In the proof we use the criterion for minimality of restrictions.

In Section 1.5 we formulate a *criterion for minimality of a product* of two compact minimal abelian flows $\mathcal{X}$ and $\mathcal{Y}$, one of which is almost periodic, in terms of eigenvalues, and prove it using Parry's theorem. Applying this criterion we deduce a new proof of a characterization of minimality of $\mathcal{X} \times \mathcal{Y}$ in terms of common factors.

In Section 1.6 we introduce the notion of *SK groups*, give examples and prove some properties of this notion. These properties will then be used in Sections 1.7 and 1.8.

In Section 1.7 we investigate the relation between weak mixing and total minimality in the context of SK acting groups (using the criteria developed in Section 1.2). We generalize a statement of N. Markley (about the equivalence of weak mixing and total minimality) and a statement of W. Gottschalk (about total minimality of $\mathcal{X}$ in terms of the structure group $\Gamma(\mathcal{X})$) by extending the class of acting groups for which their statements remain valid.

In Section 1.8 we use the criteria developed in Section 1.2 to investigate some conditions which, when imposed on compact minimal abelian flows, necessarily imply non-total-minimality. In that way we generalize and give a shorter proof of a result of H. Chu (in which for example we avoid the use of Pontryagin's duality theory), as well as several other results.

Now we give a description of each section of Chapter 2.

Morphisms of flows with not necessarily the same acting group were not seriously considered in the literature since it seems that they don't give anything more than "standard" morphisms of flows with the same acting group. In fact there is only one paper in which these "new" morphisms of flows were considered; namely [26]. There in total three propositions involving this notion were proved and no example was given. In Section 2.1 we call these morphisms "skew-morphisms" and give several natural situations where they appear. We use them in a systematic manner in the rest of this chapter.

In Section 2.2 we give facts about almost-periodicity of a point in various constructions. Some facts are stated for skew-morphisms instead of morphisms and, in some instances, it is illustrated that, using skew-morphisms, we sometimes get simpler and more natural proofs, as well as new statements. Some statements show that it is much easier to deal with almost periodicity of a point in the case of compact flows (since in that case there is a natural connection between almost periodicity of a point and minimality of flows).

Our goal is a theorem which unifies various known statements about lifting of almost periodicity of a point in both the compact, as well as not necessarily compact, case. The first important statement about lifting was given by R. Ellis in [12] for compact flows. (Applications of this statement to free abelian topological groups are given in [11].) Later Markley and others obtained some statements for not necessarily compact flows. In [27] Markley said that his results "differ from other results of this genre in that we do not assume that either space is compact." But his other assumptions were quite restrictive and were later relaxed a little bit by S. H. A. Kutaibi, F. Rhodes and others. Some other related results appeared later, like for instance a theorem of V. Pestov [30]. In order to extract what is essential in all these statements, in Section 2.3 we introduce the notion of a *skew-morphism good over a point with respect to orbit-closures* and give some natural examples.

In Section 2.4 we give several more complicated examples of skew-morphisms good over a point with respect to orbit closures.

In Section 2.5 we formulate and prove a *criterion for lifting of almost periodicity of a point*, which works for not necessarily compact flows.

In Section 2.6 we show that various other statements about lifting of almost periodicity of a point (both "compact" and "non-compact") are corollaries of our criterion. As corollaries we get results of Ellis, Markley, Kutaibi-Rhodes, Pestov.

0.2 Notations and preliminaries

0.2.1 General topology; topological groups

0.2.1. If X is a set, we denote its cardinality by $|X|$. All topological spaces are assumed to be Hausdorff. If X, Y are topological spaces, then $\mathrm{Homeo}(X)$ denotes the group of homeomorphisms of X, and $C(X,Y)$ denotes the set of continuous maps from X to Y. The map $(x,y) \mapsto x$ (resp. $(x,y) \mapsto y$) from $X \times Y$ to X is denoted by pr_1 (resp. pr_2). If T is a topological group, T_d denotes the group T equipped with the discrete topology.

0.2.2. Let X,Y be topological spaces, $f : X \to Y$ a continuous map. Then the map $g : X \to \mathrm{Gr}(f)$, defined by $g(x) = (x, f(x))$, is a homeomorphism. (Here $\mathrm{Gr}(f) = \{(x, f(x)) | x \in X\}$ is considered as a subspace of $X \times Y$.)

0.2.3. $\mathbb{T}$ will denote the topological group of complex numbers of module 1. If T is an abelian group, the continuous homomorphisms $\chi : T \to \mathbb{T}$, are called *continuous characters* of T. The set of all continuous characters of T will be denoted by $\widehat{T}$.

0.2.4. Let T_1 and T_2 be topological groups and let $\chi \in \widehat{T_1 \times T_2}$. Then for all $t_1 \in T_1$ and $t_2 \in T_2$, $\chi(t_1, t_2) = \chi(t_1, 1) \cdot \chi(1, t_2)$. If we denote by χ_1 the continuous character $t_1 \mapsto \chi(t_1, 1)$ of T_1 and by χ_2 the continuous character $t_2 \mapsto \chi(1, t_2)$ of T_2, we have $\chi(t_1, t_2) = \chi_1(t_1)\chi_2(t_2)$. Whenever no confusion can arise, we will simply write $\chi = \chi_1\chi_2$. Similarly for products of any finite number of factors.

0.2.5. We use the theory of *uniform spaces* as it is presented in [40]. We call an *entourage* what is called a *connector* in [40], or an *index* in [2], [13],[22],[38]. Entourages are denoted by small greek letters: α, β, etc. If X is a uniform space, α an entourage of X and $x \in X$, then $\alpha[x]$ denotes the set of all $y \in X$ such that $(x, y) \in \alpha$.

On a compact topological space X there is exactly one uniform structure compatible with the topology of X. The entourages of this uniform structure are all neighborhoods of the diagonal Δ in $X \times X$.

0.2.6. Let T be a topological group. A subset A of T is *syndetic* if there exists a compact subset K of T such that $T = KA$. If S is a syndetic subgroup of T, the quotient space T/S is compact. A subset A of T is *discretely syndetic* if it is a syndetic subset of T_d.

Lemma 0.2.7. *Let $h : T \to T'$ be a surjective group homomorphism. Then for every discretely syndetic subset S' of T', $h^{-1}(S')$ is discretely syndetic in T.*

Proof. There is a finite subset $F' = \{b'_1, , \cdots, b'_n\}$ of T' such that $T' = F'S'$. For every $b'_i \in F'$ let $b_i \in T$ be such that $h(b_i) = b'_i$. Let $F = \{b_1, \cdots, b_n\}$. We claim that $T = Fh^{-1}(S')$. Indeed, for $t \in T$, let $h(t) = b's'$. Put $s = b^{-1}t$. Then $h(s) = h(b)^{-1}h(t) = b'^{-1}b's' = s'$, so $s \in h^{-1}(S')$. We have $t = b \cdot b^{-1}t \in F \cdot h^{-1}(S')$. $\square$

Remark 0.2.8. Let $h : T \to T'$ be a surjective group homomorphism having the *compact-covering property* (i.e. for every compact K' in T' there is a compact K in T such that $h(K) = K'$). Then if S' is a syndetic subset of T', $h^{-1}(S')$ is a syndetic subset of T.

This statement is from [26]. The proof is analogous to the proof of Lemma 0.2.7.

Lemma 0.2.9 ([27]). *Let T be a topological group, S a syndetic subset of T, $S_1, \cdots, S_n$*

subsets of S such that $S = \bigcup_{i=1}^n S_i$, $t_1, \cdots, t_n$ elements of T. Then the set $\bigcup_{i=1}^n t_i S_i$ is syndetic.

Proof. Let K be a compact subset of T such that $T = KS$. We have: $(\bigcup_{i=1}^n K t_i^{-1}) \cdot (\bigcup_{i=1}^n t_i S_i) \supset \bigcup_{i=1}^n K t_i^{-1} t_i S_i = \bigcup_{i=1}^n K S_i = K(\bigcup_{i=1}^n S_i) = KS = T$, and the set $\bigcup_{i=1}^n K t_i^{-1}$ is compact. So the set $\bigcup_{i=1}^n t_i S_i$ is syndetic. □

0.2.10 ([4]). Let X and Y be topological spaces, $f : X \to Y$ a continuous map. We say that (X, f) is a *covering* of Y if for each point $y \in Y$ there is an open neighborhood V of y such that $f^{-1}(V)$ is a nonempty disjoint union of open subsets U_i, $i \in I$, of X, on which the restrictions $f_i : U_i \to V$ of f are homeomorphisms.

An open neighborhood V of a point $y \in Y$ is called *elementary* if it satisfies the above condition. An open neighborhood U of a point $x \in X$ is called *elementary* if there is an elementary neighborhood V of the point $y = f(x)$ such that U is one of the disjoint open subsets U_i, $i \in I$, of X, whose union is equal to $f^{-1}(V)$.

A homeomorphism $g : X \to X$, $x \mapsto gx$, is called a *deck-transformation* of the covering (X, f) if $f(gx) = f(x)$ for all $x \in X$. The deck-transformations form a group $\triangle$ under composition (written as $(g, g') \mapsto gg'$). We say that $\triangle$ is *transitive* on the fiber $f^{-1}(y)$ of a point $y \in Y$ if for any two elements $x, x' \in f^{-1}(y)$ there is an element $g \in \triangle$ such that $x' = gx$.

If (X, f) is a covering of Y, the fibers of f are discrete. Also f is a surjective local homeomorphism. In particular, f is open. (A continuous map $f : X \to Y$ is a *local homeomorphism* if for each $x \in X$ there is a neighborhood U of x such that $f(U)$ is a neighborhood of $f(x)$ in Y and the map $U \to f(U)$ which coincides with f on U is a homeomorphism.)

0.2.2 Flows

0.2.11. A triple $\mathcal{X} = \langle T, X, \pi \rangle$ consisting of a topological group T, a topological space X and a continuous action $\pi : T \times X \to X$ of T on X is called a *flow* on X. We write $t.x$ or tx for $\pi(t, x)$. We say that $\mathcal{X}$ is *compact* (respectively *abelian*), if X is compact (respectively if T is abelian). We say that $\mathcal{X}$ is *trivial* if $|X| = 1$. For $x \in X$ we denote by $\pi^x : T \to X$ the *orbital* map $t \mapsto t.x$. For $t \in T$ we denote by $\pi_t \in \mathrm{Homeo}(X)$ the *transition* homeomorphism $x \mapsto t.x$.

0.2.12. When we have a $\mathbb{Z}$-flow on X, $\mathcal{X} = \langle \mathbb{Z}, X, \pi \rangle$, then the transition homeomorphism $h := \pi_1$ completely defines the action: $\pi(n, x) = h^n(x)$. In that case we simply write $\mathcal{X} = \langle X, h \rangle$ when no confusion can arise.

0.2.13. Every flow $\mathcal{X}_S = \langle S, X, \pi \mid_{X \times S} \rangle$, where S is a subgroup of T, will be called a *restriction* of the flow $\mathcal{X} = \langle T, X, \pi \rangle$. Usually it is denoted simply by $\mathcal{X}_S = \langle S, X \rangle$. If a subset Y of X is invariant under the action of T, then the canonical flow $\langle T, Y \rangle$ is a *subflow* of $\mathcal{X}$. If $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T, Y \rangle$ are two flows with the same acting group T, then we define a T-flow on $X \times Y$ by $t.(x, y) = (tx, ty)$, for $t \in T$, $x \in X$, $y \in Y$. This flow is called the *product* of the flows $\mathcal{X}$ and $\mathcal{Y}$ and denoted by $\mathcal{X} \times \mathcal{Y}$.

0.2.14. Let $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T, Y \rangle$ be flows. A map $f : X \to Y$ is a *morphism* of flows if it is continuous and $f(tx) = tf(x)$ for all $t \in T$ and $x \in X$. If f is surjective, $\mathcal{Y}$ is a *factor* of $\mathcal{X}$, and $\mathcal{X}$ is an *extension* of $\mathcal{Y}$. *Endomorphisms*, *isomorphisms* and *automorphisms* of flows are defined in a standard way.

0.2.15. Let $\mathcal{X} = \langle T, X \rangle$ be a flow. A continuous function $\eta : X \to \mathbb{T}$ is an *eigenfunction* of $\mathcal{X}$ if there is a continuous character $\chi \in \widehat{T}$ such that $\eta(t.x) = \chi(t)\eta(x)$ for $(t, x) \in T \times X$. In that case χ is an *eigenvalue* of $\mathcal{X}$ (the eigenvalue which *corresponds* to η) and η is an eigenfunction which *corresponds* to χ. The following are equivalent:

(i) χ is trivial;

(ii) η is constant on some $T.x$ ($x \in X$);

(iii) η is constant on every $\overline{T.x}$ ($x \in X$).

If X contains a point with dense orbit, then χ is trivial iff η is constant.

0.2.16. A flow $\mathcal{X} = \langle T, X \rangle$ is *minimal* if the orbit $T.x$ of every point $x \in X$ is dense in X. It is *totally minimal* if the flow $\mathcal{X}_S$ is minimal for every syndetic (equivalently, closed syndetic) subgroup of T. If $f : \mathcal{X} \to \mathcal{Y}$ is a surjective morphism of flows, then if $\mathcal{X}$ is minimal (respectively totally minimal), $\mathcal{Y}$ is minimal (respectively totally minimal). Two compact minimal abelian flows $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T, Y \rangle$ are *disjoint* if the product $\mathcal{X} \times \mathcal{Y}$ is minimal. They are *weakly disjoint* if the product $\mathcal{X} \times \mathcal{Y}$ has a point with dense orbit.

0.2.17. Every compact flow contains a minimal set. (The proof uses Zorn's lemma ([2], [13], [22], [38]).)

0.2.18. For $x \in X$ and $U, V \subset X$, the *dwelling set* $D(U, V)$ (resp. $D(x, V)$) is the set of all $t \in T$ such that $t.U \bigcap V \neq \emptyset$ (resp. $t.x \in V$).

Lemma 0.2.19. *Let $\mathcal{X} = \langle T, X, \pi \rangle$ be a flow, $x \in X$. Then for every neighborhood V of x there are a neighborhood W of x and a neighborhood O of the unit element $e \in T$ such that $OD(x, W) \subset D(x, V)$.*

Proof. Fix a neighborhood V of x. Since $\pi : T \times X \to X$ is continuous at (e, x), there is a neighborhood W of x and a neighborhood O of e such that $OW \subset V$. We claim that then $OD(x, W) \subset D(x, V)$. Indeed, let $o \in O$ and let $t \in D(x, W)$. Then $tx \in W$, hence $o(tx) \in OW$, hence $(ot)x \in V$, i.e. $ot \in D(x, V)$. $\square$

0.2.20. The *Ellis semigroup* $E(\mathcal{X})$ of a flow $\mathcal{X} = \langle T, X, \pi\rangle$ is $\mathrm{Cl}\{\pi_t | t \in T\}$ in X^X (i.e. in $F_p(X, X)$) with the operation of composition. (Here $F_p(X, X)$ denotes the set of all maps from X to itself, equipped with the topology of pointwise convergence.) If $\mathcal{X}$ is a compact flow, $E(\mathcal{X})$ is a compact Hausdorff right semitopological semigroup ([38, p.301]).

0.2.3 x-envelopes; orbit-closures

0.2.21. Let $\mathcal{X} = \langle T, X\rangle$ be a flow. For $x \in X$ and $S \subset T$, the *x-envelope* of S , denoted by S^x, is the set $\{t \in T \mid t.x \in \overline{S.x}\}$. S^x is a closed subset of T, it contains S, and $\overline{S^x.x} = \overline{S.x}$. If S is a syndetic normal subgroup of T, then S^x is a closed subgroup of T. ([22, 2.08-2.10])

0.2.22. If S is a normal subgroup of T the following properties are easy to verify:

(i) $t.\overline{S.x} = \overline{S.t.x}$ for all $t \in T$, $x \in X$;

(ii) If $\overline{S.x}$ is a *minimal* subset of X under S, then:

(1) $t.\overline{S.x} = \overline{S.x}$ if and only if $t \in S^x$;

(2) $(\forall y \in X)\ t.y \in \overline{S.x}$ if and only if $t.\overline{S.y} = \overline{S.x}$.

0.2.23. Let $\mathcal{X} = \langle T, X\rangle$ be a flow. It is easy to see ([22, 2.23]) that the following are equivalent:

(i) the set of orbit closures under T is a partition of X;

(ii) $(\forall x \in X)\ (\forall y \in X)\ y \in \overline{T.x} \Leftrightarrow \overline{T.y} = \overline{T.x}$;

(iii) every orbit closure under T is minimal under T.

0.2.24. Let $\mathcal{X} = \langle T, X\rangle$ be a flow and S a syndetic normal subgroup of T. Then the set of orbit closures under S is a partition of X iff the set of orbit closures under T is a partition of X. In particular, if $\mathcal{X}$ is minimal, the set of orbit closures under S is a partition of X. ([22, 2.24])

We denote by $\mathcal{O}_S$ the set $\{\overline{Sx} | x \in X\}$ and by $R(\mathcal{O}_S)$ the relation $(x, y) \in R(\mathcal{O}_S) \Leftrightarrow \overline{Sx} = \overline{Sy}$ on X. If $\mathcal{X}$ is compact minimal, $R(\mathcal{O}_S)$ is an equivalence relation which is open ([22, 2.30]) and closed ([22, 2.32]).

0.2.25. Let $\mathcal{X} = \langle T, X\rangle$ be a minimal flow, S a syndetic normal subgroup of T, and K a compact subset of T such that $T = KS$. The following are easy to verify:

(i) $K.\overline{S.x} = X$ for every $x \in X$;

(ii) in particular, for every $x, y \in X$ there is a $k \in K$ such that $k.\overline{S.x} = \overline{S.y}$ (and consequently $k.x \in \overline{Sy}$).

0.2.4 Almost periodicity, proximality, distality, weak mixing

0.2.26. Let $\mathcal{X} = \langle T, X\rangle$ be a flow.

(i) A point $x \in X$ is *almost periodic* (in $\mathcal{X}$) if for every neighborhood U of x there is a syndetic subset A of T such that $Ax \subset U$, i.e. the dwelling set $D(x, U)$ is syndetic in T. A point $x \in X$ is *discretely almost periodic* if it is almost periodic in the flow $\mathcal{X}_d = \langle T_d, X\rangle$, where T_d is the group T equipped with the discrete topology. Every discretely almost periodic point is almost periodic.

(ii) A point $x \in X$ is *regularly almost periodic* if for every neighborhood U of x there is a syndetic subgroup S of T such that $Sx \subset U$. Every regularly almost periodic point is almost periodic.

(iii) A point $x \in X$ is *locally almost periodic* if for every neighborhood U of x there is a neighborhood V of x and a syndetic subset A of T such that $AV \subset U$. Every locally almost periodic point is almost periodic.

A flow $\mathcal{X}$ is *pointwise almost periodic* (resp. *pointwise regularly almost periodic*; *pointwise locally almost periodic*) if every point $x \in X$ is almost periodic (resp. regularly almost periodic; locally almost periodic). (The adjective "pointwise" is omitted in the case of regularly and locally almost periodic flows.)

A flow $\mathcal{X} = \langle T, X, \pi \rangle$ on a uniform space $(X, \mathcal{U}_X)$ is *uniformly almost periodic* (resp. *equicontinuous*) if for every $\alpha \in \mathcal{U}_X$ there exists a syndetic $A \subset T$ such that $A.x \subset \alpha[x]$ for every $x \in X$ (resp. if the family $\{\pi_t \mid t \in T\}$ of transition homeomorphisms is equicontinuous). In case of compact flows, uniformly almost periodic flows are the same as equicontinuous ones, and are simply called *almost periodic* flows.

0.2.27. Let $\mathcal{X} = \langle T, X \rangle$ be a flow, $x \in X$. The following are equivalent:

(i) x is almost periodic in $\mathcal{X}$;

(ii) for every neighborhood U of x there is a compact subset K of T such that for every $t \in T$, $Ktx \cap U \neq \emptyset$;

(iii) for every neighborhood U of x there is a compact subset K of T such that $Tx \subset KU$ ([2],[13],[22],[38]).

0.2.28. Let $\mathcal{X} = \langle T, X \rangle$ be a flow, $x \in X$. If x has a compact neighborhood, then x is almost periodic iff $\overline{Tx}$ is compact minimal. In particular, a point x in a compact flow $\mathcal{X}$ is almost periodic if and only if $\overline{Tx}$ is minimal. ([2, p.11], [13, p.10], [38, IV(1.2)])

0.2.29. Let $\mathcal{X} = \langle T, X \rangle$ be a compact flow. The following are equivalent:

(i) $\mathcal{X}$ is almost periodic;

(ii) $E(\mathcal{X})$ is a compact topological group and the canonical map $E(\mathcal{X}) \times X \to X$ is a continuous action of $E(\mathcal{X})$ on X;

(iii) $E(\mathcal{X})$ is a group and its elements are homeomorphisms of X.

If in addition $\mathcal{X}$ is minimal, these conditions are equivalent to

(iv) $E(\mathcal{X})$ is a topological group. ([2, p.60], [13, p.25], [38, IV(3.34)])

0.2.30. Let $\mathcal{X} = \langle T, X \rangle$ be a compact minimal abelian flow. The following are equivalent:

(i) $\mathcal{X}$ is almost periodic;

(ii) for every $x \in X$ there is an abelian group structure on X with the unit element x, such that the orbital map $\pi^x : T \to X$ is a continuous group homomorphism;

(iii) there is an element $e \in X$ such that there is an abelian group structure on X with the unit element e, such that the orbital map $\pi^e : T \to X$ is a continuous group homomorphism. ([36, Corollary 2.10], [38, IV(3.42)])

In particular, every nontrivial compact minimal abelian flow has a nontrivial eigenvalue ([38, p.409]).

0.2.31. Let $\mathcal{X}$ be a compact flow. There is a smallest closed invariant equivalence relation on X, denoted $S^e_{\mathcal{X}}$, such that the quotient flow $\mathcal{X}/S^e_{\mathcal{X}} = \langle T, X/S^e_{\mathcal{X}} \rangle$ is almost periodic ([2, p.125], [13, p.32] or [38, p.398]). The equivalence relation $S^e_{\mathcal{X}}$ is called the *equicontinuous structure relation* of $\mathcal{X}$. Then the Ellis semigroup $E(\mathcal{X}/S^e_{\mathcal{X}})$ is a compact topological group, called the *structure group* of $\mathcal{X}$ and denoted by $\Gamma(\mathcal{X})$.

0.2.32. Let $\mathcal{X} = \langle T, X, \pi \rangle$ be a flow on a uniform space $(X, \mathcal{U}_X)$.

(i) A pair (x, y) of points in X is *proximal* (and the points x, y are *proximal to each other*) if for every $\alpha \in \mathcal{U}_X$ there is a $t \in T$ such that $(tx, ty) \in \alpha$. The flow $\mathcal{X}$ is *proximal* if every pair of points in X is proximal.

(ii) A pair (x, y) of points in X is *regionally proximal* (and the points x, y are *regionally proximal to each other*) if there is a point $z \in X$ such that for every neighborhood V of z and every neighborhood U_1 of x and U_2 of y, $D(U_1, V) \cap D(U_2, V) \neq \emptyset$. The flow $\mathcal{X}$ is *regionally proximal* if every pair of points in X is regionally proximal.

(iii) A pair (x, y) of points in X is *distal* (and the points x, y are *distal to each other*) if either $x = y$ or (x, y) is not proximal. The flow $\mathcal{X}$ is *distal* if every pair of points in X is distal. A point $x \in X$ is *distal* if every pair (x, y), $y \in X$, is distal. If $\mathcal{X}$ is compact, $\mathcal{X}$ is *point-distal* if there is a distal point $x \in X$ with dense orbit.

0.2.33. Let $\mathcal{X}$ be a compact flow. Then if $\mathcal{X}$ is almost periodic, it is distal ([2, p.65], [13, p.36], [38, IV(2.21)]).

0.2.34. If $\mathcal{X} = \langle T, X \rangle$ is compact minimal abelian and $x \in X$, then every pair (x, tx), $t \in T$, is distal ([22, 10.07]).

0.2.35. Every nontrivial distal compact minimal flow has a nontrivial almost periodic factor ([2, p.104], [38, V(3.33)]). In particular, every nontrivial distal compact minimal abelian flow has a nontrivial eigenvalue ([2, p.105]).

0.2.36. Let $\mathcal{X}$, $\mathcal{Y}$ be distal compact minimal flows, $f : \mathcal{X} \to \mathcal{Y}$ a morphism of flows. Then f is an open map ([2, p.98], [38, V(2.3)]).

0.2.37. Let $\mathcal{X} = \langle T, X, \pi \rangle$ be a flow on a uniform space $(X, \mathcal{U}_X)$. A pair (x, y) of points in X is proximal iff it belongs to the subset

$$P_{\mathcal{X}} = \bigcap_{\alpha \in \mathcal{U}_X} T\alpha$$

of $X \times X$, which is called the ***proximal relation*** in $\mathcal{X}$. This relation is reflexive, symmetric, invariant, but is not transitive nor closed in general ([2, p.66]). In case of compact flows, if $P_{\mathcal{X}}$ is closed, it is an equivalence relation ([2, p.88]).

A pair (x, y) of points in X is regionally proximal iff it belongs to the subset

$$Q_{\mathcal{X}} = \bigcap_{\alpha \in \mathcal{U}_X} \overline{T\alpha}$$

of $X \times X$, which is called the ***regionally proximal relation*** in $\mathcal{X}$ ([38, p.283]). Every proximal pair in a compact flow is regionally proximal. The relation $Q_{\mathcal{X}}$ is reflexive, symmetric, invariant and closed, but not necessarily an equivalence relation ([38, p.401]). If $\mathcal{X}$ is compact, $S^{\mathfrak{e}}_{\mathcal{X}}$ is the smallest closed invariant equivalence relation on X which contains $Q_{\mathcal{X}}$ ([38, p.399]). In case of compact minimal abelian flows, $Q_{\mathcal{X}}$ is an equivalence relation, and hence in that case $Q_{\mathcal{X}} = S^{\mathfrak{e}}_{\mathcal{X}}$([2, p.130] or [38, p.404]).

0.2.38. A compact flow $\mathcal{X}$ is *proximally equicontinuous* if $P_{\mathcal{X}} = Q_{\mathcal{X}}$ ([38, V(1.7)2]). In that case $P_{\mathcal{X}}$ is closed, hence $P_{\mathcal{X}}$ (i.e. $Q_{\mathcal{X}}$) is an equivalence relation. So $P_{\mathcal{X}} = Q_{\mathcal{X}} = S^{\mathfrak{e}}_{\mathcal{X}}$. All compact equicontinuous (i.e. almost periodic) and all compact proximal flows are proximally equicontinuous. Also all compact (pointwise) locally almost periodic flows are proximally equicontinuous ([38, p.364]).

0.2.39. A flow $\mathcal{X} = \langle T, X \rangle$ is *weakly mixing* if for any open subsets U, U', V, V' of X there is a $t \in T$ such that at the same time $tU \cap V \neq \emptyset$ and $tU' \cap V' \neq \emptyset$, i.e. $D(U,V) \cap D(U',V') \neq \emptyset$ ([38, p.273]). If $\mathcal{X}$ is compact minimal abelian, then $\mathcal{X}$ is weakly mixing iff $S^e_{\mathcal{X}} = X \times X$ ([33, p.279], [2, p.133] or [38, V(1.19)]). Intuitively speaking, this says that weakly mixing compact minimal abelian flows are opposite to the almost periodic flows: the equicontinuous structure relation is the whole $X \times X$, i.e. maximal possible, which is opposite to the case of almost periodic flows where the equicontinuous structure relation is equal to the diagonal Δ_X, i.e. minimal possible. The only almost periodic factor of a weakly mixing compact minimal abelian flow is the trivial flow.

0.2.40. If $\mathcal{X}$ is a weakly mixing compact flow, every eigenfunction of $\mathcal{X}$ is constant. If $\mathcal{X}$ is a compact minimal abelian flow, then if every eigenfunction of $\mathcal{X}$ is constant, $\mathcal{X}$ is weakly mixing ([38, p.409]).

0.2.41. If $\mathcal{X}$ is a nontrivial compact minimal abelian flow on a metric space X, then if $\mathcal{X}$ is weakly mixing, it is not point-distal ([38, p.408]).

Chapter 1

Minimality of restrictions, group-extensions and products of compact minimal abelian flows

1.1 The notion of an $\mathcal{X}$-enveloped subgroup

Proposition 1.1.1. *Let $\mathcal{X} = \langle T, X\rangle$ be a minimal abelian flow, and let S be a syndetic subgroup of T. Then $S^x = S^y$ for every x, y from X.*

Proof. Let K be a compact subset of T such that $T = K + S$. Fix any x, y from X. There is a k in K such that $-k.x = y' \in \overline{S.y}$. Then $x = k.y'$, $y' \in \overline{S.y}$. Let $s \in S^x$. We have:

$s.y \in s.\overline{S.y} = s.\overline{S.y'} = s.\overline{S.-k.x} = s.-k.\overline{S.x} = -k.s.\overline{S.x} = -k.\overline{S.s.x} = -k.\overline{S.x} = \overline{S.-k.x} = \overline{S.y'} = \overline{S.y}$. Hence $s \in S^y$. Thus $S^x \subset S^y$. By symmetry $S^y \subset S^x$. Hence

$S^x = S^y$. $\square$

Definition 1.1.2. Let $\mathcal{X} = \langle T, X \rangle$ be a minimal abelian flow, and let S be a syndetic subgroup of T. The $\mathcal{X}$-*envelope* (or simply *envelope*) of S, denoted by S^*, is the subset of T which is equal to S^x, where x is any element of X.

By the previous proposition the notion of the $\mathcal{X}$-*envelope* of S is well defined.

Definition 1.1.3. Let $\mathcal{X}$ be a minimal abelian flow. A syndetic subgroup S of T is called an $\mathcal{X}$-*enveloped* (or simply *enveloped*) subgroup of T if $S = S^*$.

Note that a subgroup S of T may be enveloped with respect to some flow $\mathcal{X} = \langle T, X \rangle$, and at the same time not enveloped with respect to some other flow $\mathcal{Y} = \langle T, Y \rangle$.

Proposition 1.1.4. *Let $\mathcal{X} = \langle T, X \rangle$ be a minimal abelian flow and let S be a syndetic subgroup of T. Then S^* is an enveloped subgroup of T and it is the smallest enveloped subgroup of T containing S.*

Proof. S^* is syndetic and $(S^*)^* = S^*$. So S^* is an enveloped subgroup of T. Let E be an enveloped subgroup of T containing S. Then:

$$E = E^* = \{t \in T \mid t \in \overline{E.x}\} \supset \{t \in T \mid t \in \overline{S.x}\} = S^*.$$

Thus an enveloped subgroup of T, containing S, contains S^*. $\square$

Example 1.1.5. Let $\theta \neq 0$ be a real number. Consider the compact minimal abelian flow $\mathcal{X} = \langle \mathbb{R}, \mathbb{T}, \pi \rangle$, defined by $\pi(t,z) = e^{2\pi i t\theta} z$, $t \in \mathbb{R}$, $z \in \mathbb{T}$. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Consider the action of the subgroup $\mathbb{Z}\alpha$ of $\mathbb{R}$ on $\mathbb{T}$, induced by π. The orbit in this action of an element $z \in \mathbb{T}$ has the form

$$\mathbb{Z}\alpha.z = \{n\alpha.z \mid n \in \mathbb{Z}\} = \{e^{2\pi i n\alpha\theta} z \mid n \in \mathbb{Z}\}.$$

<u>1st case:</u> $\alpha\theta \in \mathbb{Q}$. Suppose $\alpha\theta = \frac{k}{l}$, $(k,l) = 1$. Then every $e^{2\pi i n\alpha\theta} = e^{2\pi i n\frac{k}{l}}$ is one of the elements $1, e^{2\pi i\frac{k}{l}}, ..., e^{2\pi i\frac{(l-1)k}{l}}$. Hence

$$\mathbb{Z}\alpha.z = \{z, e^{2\pi i\frac{k}{l}} z, ..., e^{2\pi i\frac{(l-1)k}{l}} z\}.$$

So $\overline{\mathbb{Z}\alpha.z} = \mathbb{Z}\alpha.z$. To calculate $(\mathbb{Z}\alpha)^*$ it is enough to calculate $(\mathbb{Z}\alpha)^z$ for any $z \in \mathbb{T}$, for example $(\mathbb{Z}\alpha)^1$. We have:

$\beta \in (\mathbb{Z}\alpha)^1 \Leftrightarrow e^{2\pi i\beta\theta}.1 = e^{2\pi i n\alpha\theta}$ for some $n \in \mathbb{Z} \Leftrightarrow e^{2\pi i\theta(\beta - n\alpha)} = 1 \Leftrightarrow \theta(\beta - n\alpha) = q$ for some $q \in \mathbb{Z} \Leftrightarrow \beta = \frac{1}{\theta}q + n\alpha \Leftrightarrow \beta \in \mathbb{Z}\alpha + \mathbb{Z}\frac{1}{\theta} = \mathbb{Z}\frac{k}{l\theta} + \mathbb{Z}\frac{1}{\theta} = \mathbb{Z}\frac{1}{k}\alpha$.

Thus we have $\mathbb{Z}\alpha^* = \mathbb{Z}\frac{1}{k}\alpha$, where $\alpha\theta = \frac{k}{l}, (k,l) = 1$.

<u>2nd case:</u> $\alpha\theta \notin \mathbb{Q}$. Then $\{e^{2\pi i n\alpha\theta} z \mid n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$ for every $z \in \mathbb{T}$, so $\overline{\mathbb{Z}\alpha.z} = \mathbb{T}$. Then $(\mathbb{Z}\alpha)^* = (\mathbb{Z}\alpha)^1 = \{\beta \in \mathbb{R} \mid e^{2\pi i\beta\theta} \in \mathbb{T}\} = \mathbb{R}$.

More concretely, let $\theta = 1$. Then for example: $(\mathbb{Z}\frac{2}{3})^* = \mathbb{Z}\frac{1}{3} = (\mathbb{Z}\frac{1}{3})^*$, $(\mathbb{Z}\sqrt{2})^* = \mathbb{R}$, etc. There are many subgroups of $\mathbb{R}$ that are enveloped, and many that are not.

Example 1.1.6. Let $\mathcal{X}$ be an almost periodic compact minimal abelian flow with X non-connected. X can be written as a disjoint union $X = Y \cup Z$ of two nonempty clopen sets Y and Z. Let $\alpha = (Y \times Y) \cup (Z \times Z)$. Since X is compact and α is an open neighborhood of the diagonal Δ_X, α is an entourage of the unique uniform structure on X.

<u>*Claim:*</u>

$$S := \bigcap_{x \in X} D(x, \alpha[x])$$

is a proper enveloped subgroup of T.

Since $\mathcal{X}$ is almost periodic, S is a syndetic subset of T. Note that for $y \in Y$, $\alpha[y] = Y$, and for $z \in Z$, $\alpha[z] = Z$. So $S = \{t \in T \mid (\forall y \in Y)\, t.y \in Y \text{and } (\forall z \in Z)\, t.z \in Z\} = \{t \in T \mid t.Y \subset Y \text{ and } t.Z \subset Z\} = \{t \in T \mid t.Y = Y \text{and } t.Z = Z\}$. It follows that $t_1, t_2 \in T$ implies $t_1 + t_2 \in S$, and $t \in S$ implies $-t \in S$. Thus S is a syndetic subgroup of T. Now consider any element of X, for example some $y \in Y$. Since $\mathcal{X}$ is a compact minimal abelian flow, $S^* = S^y = \{t \in T \mid t.y \in \overline{S.y}\} \subset \{t \in T \mid t.y \in Y\}$. (The inclusion holds because $S.y \subset Y$ and Y is closed.) Since this is true for any $y \in Y$, we have $S^* \subset \bigcap_{y \in Y} \{t \in T \mid t.y \in Y\} = \{t \in T \mid (\forall y \in Y)\, t.y \in Y\}$. If we do the same thing for every $z \in Z$ and combine the results, we get $S^* \subset S$. Thus $S^* = S$.

To prove that S is proper, consider any $y \in Y$. Since $\mathcal{X}$ is minimal and Z is open, there is a $t \in T$ such that $t.y \in Z$. This t does not belong to S. The claim is proved.

1.2 A criterion for minimality of restrictions and a criterion for total minimality

Proposition 1.2.1. *Let $\mathcal{X} = \langle T, X\rangle$ be a flow and S a normal syndetic subgroup of T. Let $\chi \in \widehat{T}$ be an eigenvalue of $\mathcal{X}$. The following are equivalent:*

(i) $\ker(\chi) \supset S$*;*

(ii) $\ker(\chi) \supset S^x$ *for every $x \in X$;*

(iii) $\ker(\chi) \supset S^x$ *for some $x \in X$.*

Proof. $(iii) \Rightarrow (i)$: clear, since $S^x \supset S$.

$(ii) \Rightarrow (iii)$: clear.

$(i) \Rightarrow (ii)$: Fix any $x \in X$. Let η be an eigenfunction of $\mathcal{X}$ which corresponds to χ. We have:

$$\eta(tx) = \chi(t)\eta(x) \tag{$*$}$$

for all $t \in T$, and, in particular, $\eta(sx) = \eta(x)$ for all $s \in S$. If we let $z = \eta(x)$, we have $\eta(S.x) = \{z\}$. By continuity $\eta(\overline{S.x}) = \{z\}$ and consequently $\eta(\overline{S^x.x}) = \{z\}$. Hence from $(*)$, $\chi(s) = 1$ for all $s \in S^x$, i.e. $\ker(\chi) \supset S^x$. □

Corollary 1.2.2. *Let $\mathcal{X} = \langle T, X\rangle$ be a minimal abelian flow and S a syndetic subgroup of T. Let $\chi \in \widehat{T}$ be an eigenvalue of $\mathcal{X}$. Then* $\ker(\chi) \supset S$ *if and only if* $\ker(\chi) \supset S^*$. □

Theorem 1.2.3 (criterion for minimality of restrictions). *Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a compact minimal abelian flow. Let S be a syndetic subgroup of T and let $\mathcal{X}_S =$*

$\langle S, X\rangle$. *The following statements are equivalent:*

(i) $\mathcal{X}_S$ *is a minimal flow;*

(ii) $\mathcal{X}$ *has no nontrivial eigenvalue whose kernel contains* S*;*

(iii) $S^* = T$.

Proof. By Proposition 1.2.1, (ii) is equivalent with

(ii') $\mathcal{X}$ *has no nontrivial eigenvalue whose kernel contains* S^*.

So we will prove the above theorem with (ii) replaced by (ii'). First we make some observations.

Note that $\mathcal{O}_S = \{\overline{S.x} \mid x \in X\}$ is the same as $\mathcal{O}_{S^*}$ since $\overline{S.x} = \overline{S^*.x}$. The equivalence relation $R = R(\mathcal{O}_{S^*})$, $((x, y) \in R \Leftrightarrow \overline{S^*.x} = \overline{S^*.y})$ is open and closed by 0.2.24. Hence X/R is compact Hausdorff. We denote by $p_X : X \to X/R = \widetilde{X}$ the quotient map and by $p_T : T \to T/S^*$ the canonical homomorphism. The elements of $\widetilde{X}$ will be denoted by $\tilde{x} = p_X(\overline{S^*.x})$. The map $\pi : T \times X \to X$ is compatible with the relations $(\bmod S^*) \times R$ on $T \times X$ and R on X. Hence it induces a continuous map $\tilde{\pi} : T \times X/(\bmod S^*) \times R \to \widetilde{X}$. Since $\bmod S^*$ and R are both open, we may identify $T \times X/(\bmod S^*) \times R$ with $T/S^* \times X/R = T/S^* \times \widetilde{X}$. With this identification we have

$$\tilde{\pi} \circ (p_T \times p_X) = p_X \circ \pi. \tag{1}$$

It follows that $\mathcal{Y} = \langle T/S^*, \widetilde{X}, \tilde{\pi}\rangle$, with $((t + S^*).\tilde{x} = \widetilde{t.x})$, is a flow.

Fix a point $a \in X$. Denote $\pi\,|_{T\times\{a\}}$ by φ, and $\tilde{\pi}\,|_{T/S^*\times\{\tilde{a}\}}$ by $\widetilde{\varphi}$. We have

$$(t + S^*).\tilde{a} = \tilde{a} \Leftrightarrow \widetilde{t.a} = \tilde{a} \Leftrightarrow \overline{S.t.a} = \overline{S.a} \Leftrightarrow t\overline{S.a} = \overline{S.a} \Leftrightarrow t \in S^*.$$

Hence the stabilizer of $\tilde{a}$ in the flow $\mathcal{Y}$ is the identity subgroup of T/S^*, hence $\widetilde{\varphi}$ is injective. Also $\widetilde{\varphi}$ is surjective, by 0.2.25(ii) and the equality (1). Now since $T/S^* \times \{a\}$ is compact, $\tilde{\varphi}$ is a homeomorphism.

Now using these observations we show the equivalence of the statements (i),(ii') and (iii).

$(i) \Leftrightarrow (iii)$: $\mathcal{X}_S$ is minimal iff $\widetilde{X}$ consists of one element iff $T = S^*$ (since $\tilde{\varphi}$ is bijective).

$(iii) \Rightarrow (ii')$: clear.

$(ii') \Rightarrow (iii)$: Suppose $S^* \neq T$. Define a continuous map $f : X \to T/S^*$ by $f = pr_1 \circ \tilde{\varphi}^{-1} \circ p_X$. For $t \in T$ let $\mathrm{transl}_{p_T(t)} : T/S^* \to T/S^*$ be defined by $\mathrm{transl}_{p_T(t)}(t' + S^*) = p_T(t) + t' + S^*$. Then for every $t \in T$

$$f \circ \pi_t = \mathrm{transl}_{p_T(t)} \circ f. \tag{2}$$

To prove (2), put $\tilde{f} = pr_1 \circ \tilde{\varphi}^{-1}$. For $t_1 \in T$ we have

$f(tx) = t_1 + S^* \Leftrightarrow \tilde{f}(\widetilde{tx}) = t_1 + S^* \Leftrightarrow \tilde{\varphi}(t_1 + S^*, \tilde{a}) = \widetilde{tx} \Leftrightarrow \widetilde{t_1 a} = \widetilde{tx} \Leftrightarrow \overline{St_1 a} = \overline{Stx} \Leftrightarrow t_1\overline{Sa} = t\overline{Sx} \Leftrightarrow (-t + t_1)\overline{Sa} = \overline{Sx} \Leftrightarrow ((-t + t_1)a)^{\sim} = \tilde{x} \Leftrightarrow \tilde{\varphi}(-t + t_1 + S^*, \tilde{a}) = \tilde{x} \Leftrightarrow \tilde{f}(\tilde{x}) = -t + t_1 + S^* \Leftrightarrow f(x) = (-t + S^*) + (t_1 + S^*) \Leftrightarrow (t + S^*) + f(x) = t_1 + S^*$.

Thus (2) holds. Now let $\chi : T/S^* \to \mathbb{T}$ be any nontrivial character of T/S^*. For $t \in T$ let $\mathrm{transl}_{\chi(t)} : \mathbb{T} \to \mathbb{T}$ be defined by $\mathrm{transl}_{\chi(t)}(z) = \chi(t)z$. It is easy to see that for every $t \in T$

$$\chi \circ \mathrm{transl}_{p_T(t)} = \mathrm{transl}_{(\chi \circ p_T)(t)} \circ \chi. \tag{3}$$

Let $\eta = \chi \circ f : X \to \mathbb{T}$. Then η is a continuous function which satisfies

$$\eta \circ \pi_t = \mathrm{transl}_{(\chi \circ p_T)(t)} \circ \eta$$

for all $t \in T$. (This follows from (2) and (3).) So η is an eigenfunction of $\mathcal{X}$ whose eigenvalue $\chi \circ p_T$ is nontrivial and whose kernel contains S^*. □

Corollary 1.2.4 (criterion for total minimality). *Let $\mathcal{X} = \langle T, X \rangle$ be a compact minimal abelian flow. The following statements are equivalent:*

(i) $\mathcal{X}$ is a totally minimal flow;

(ii) $\mathcal{X}$ has no nontrivial eigenvalue whose kernel is syndetic;

(iii) T has no proper $\mathcal{X}$-enveloped subgroup.

Proof. By Corollary 1.2.2, *(ii)* is equivalent to

(ii') $\mathcal{X}$ has no nontrivial eigenvalue whose kernel contains an enveloped subgroup of T.

So we will prove this corollary with (ii) replaced by (ii').

$(i) \Rightarrow (ii')$: clear from Theorem 1.2.3.

$(ii') \Rightarrow (iii)$: clear from Theorem 1.2.3.

$(iii) \Rightarrow (i)$: Suppose (iii) holds. Let S be a syndetic subgroup of T. By assumption (iii), $S^* = T$ (since S^* is enveloped). By Theorem 1.2.3, $\mathcal{X}_S$ is minimal. □

As the first applications of these criteria, we investigate $\mathbb{Z}$ and $\mathbb{R}$–flows.

Corollary 1.2.5. *Let $\mathcal{X} = \langle \mathbb{Z}, X \rangle$ be a compact minimal $\mathbb{Z}$-flow. Then the following statements are equivalent:*

(i) $\mathcal{X}$ is totally minimal;

(ii) $\mathcal{X}$ has no eigenvalue $\chi_\lambda = e^{2\pi i\lambda(\cdot)}$ such that $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$;

(iii) $\mathcal{X}$ has no eigenvalue $\chi(n) = z^n$ with $z \neq 1$ of finite order in $\mathbb{T}$.

Proof. (ii) and (iii) are clearly equivalent. We will show $(i) \Leftrightarrow (ii)$. Every character of $\mathbb{Z}$ has the form $\chi_\lambda = e^{2\pi i\lambda(\cdot)}$. We have $\ker(\chi_\lambda) = \{n \in \mathbb{Z} \mid e^{2\pi i\lambda n} = 1\} = \{n \in \mathbb{Z} \mid \lambda n = k \in \mathbb{Z}\}$. Now by the criterion for total minimality, $\mathcal{X}$ is totally minimal iff $\mathcal{X}$ has no nontrivial eigenvalue χ_λ such that $\{n \in \mathbb{Z} \mid \lambda n = k \in \mathbb{Z}\}$ is syndetic in $\mathbb{Z}$, iff $\mathcal{X}$ has no eigenvalue χ_λ with $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$. $\square$

Remark 1.2.6. The direction $(i) \Rightarrow (ii)$, i.e. $(i) \Rightarrow (iii)$, of the Corollary 1.2.5 is well known; see for example [5], p.108. The opposite direction is probably also known, but the author could not find a reference.

Corollary 1.2.7 ([9, Theorem 1]). *Let $\mathcal{X} = \langle \mathbb{R}, X \rangle$ be a compact minimal $\mathbb{R}$- flow. Let*

$$\Lambda(\mathcal{X}) = \{\lambda \in \mathbb{R} \mid \chi_\lambda = e^{2\pi i\lambda(\cdot)} \text{ is an eigenvalue of } \mathcal{X}\}$$

and let

$$\widetilde{\Lambda}(\mathcal{X}) = \{\frac{\lambda}{n} \mid \lambda \in \Lambda(\mathcal{X}),\ n \in \mathbb{Z} \setminus \{0\}\}.$$

Let $S = \mathbb{Z}\alpha$, where $\alpha > 0$ is a real number, and let $\mathcal{X}_S = \langle S, X \rangle$. Then $\mathcal{X}_S$ is minimal if and only if $\frac{1}{\alpha} \notin \widetilde{\Lambda}(\mathcal{X})$.

Proof. By the criterion for minimality of reduced flows, $\mathcal{X}_S$ is not minimal iff $\mathcal{X}$ has an eigenvalue χ_λ, $\lambda \neq 0$, such that $\{t \in \mathbb{R} \mid \lambda t \in \mathbb{Z}\} \supset \mathbb{Z}\alpha$, iff $\mathcal{X}$ has an eigenvalue χ_λ, $\lambda \neq 0$, such that $\mathbb{Z}\frac{1}{\lambda} \supset \mathbb{Z}\alpha$, iff $\mathcal{X}$ has an eigenvalue χ_λ such that $\frac{1}{\alpha} = \frac{\lambda}{n}$, $n \in \mathbb{Z} \setminus \{0\}$, iff $\frac{1}{\alpha} \in \widetilde{\Lambda}(\mathcal{X})$.

$\square$

1.3 A new proof of a theorem of Parry

We will now investigate skew-extensions of compact minimal $\mathbb{Z}$-flows. We will show that, in the special situation described in Theorem 1.3.1, the criterion for minimality of restrictions is equivalent to the well known Parry's condition. The proof illustrates the way in which one can end-up with Parry's condition after a sequence of natural steps, starting with the condition from the criterion.

Theorem 1.3.1 ([2, p.72], [28, p.98], [38, II(8.22)]). *Let G be a compact abelian topological group. Let $\mathcal{Y} = \langle Y, \tau \rangle$ be a compact minimal $\mathbb{Z}$-flow, $\varphi : Y \to G$ a continuous map, $X = Y \times G$ and let $\sigma \in$ Homeo(X) be defined by*

$$\sigma(y, g) = (\tau(y),\ \varphi(y)g). \tag{1}$$

Then the compact $\mathbb{Z}$-flow $\mathcal{X} = \langle X, \sigma \rangle$ is minimal iff

$$f(\tau(y)) = \gamma(\varphi(y))f(y)$$

has no solution $f \in C(Y, \mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$.

Proof. Define a (compact abelian) flow $\mathcal{Z} = \langle \mathbb{Z} \times G, X \rangle$ by

$$(n, g).x = (pr_1(\sigma^n(x)), pr_2(\sigma^n(x))g), \tag{2}$$

for $n \in \mathbb{Z}$, $g \in G$, $x \in X$. Writting $x = (y, g')$, we get from (1) and (2)

$$(n, g).(y, g') = (\tau^n(y), \prod_{i=1}^{n} \varphi(\tau^{n-i}(y)) \cdot g'g),$$

$$(-n,g).(y,g') = (\tau^{-n}(y), \prod_{i=1}^{n} \varphi(\tau^{-i}(y))^{-1} \cdot g'g),$$

for $n \geq 0$. From these formulas we can see (using minimality of $\mathcal{Y}$) that $\mathcal{Z}$ is minimal (the orbit $(\mathbb{Z} \times G).(y,g')$ of (y,g') has the form $D \times G$, where D is a dense subset of Y).

Let $S = \mathbb{Z} \times \{1\}$. Since $(n,1).(y,g') = \sigma^n(y,g')$, which is $n.(y,g')$ in $\mathcal{X}$, we may identify flows $\mathcal{Z}_S$ and $\mathcal{X}$. So we have

$$\mathcal{X} \text{ is not minimal} \Leftrightarrow \mathcal{Z}_S \text{ is not minimal.} \tag{$*$}$$

Now consider the following sequence of conditions:

(COND1) $\eta((n,g).(y,g')) = \chi(n,g)\eta(y,g')$, $n \in \mathbb{Z}$, g, $g' \in G$, $y \in Y$, has a solution $\eta \in C(X,\mathbb{T})$, $\chi \in \widehat{\mathbb{Z} \times G}$, with $\chi \neq 1$ and $\ker(\chi) \supset \mathbb{Z} \times \{1\}$;

(COND2) $\eta((n,g).(y,g')) = \gamma(g)\eta(y,g')$, $n \in \mathbb{Z}$, g, $g' \in G$, $y \in Y$, has a solution $\eta \in C(X,\mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$;

(COND3) $\eta((1,g).(y,g')) = \gamma(g)\eta(y,g')$, g, $g' \in G$, $y \in Y$, has a solution $\eta \in C(X,\mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$;

(COND4) $\eta((1,g).(y,1)) = \gamma(g)\eta(y,1)$, $g \in G$, $y \in Y$, has a solution $\eta \in C(X,\mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$;

(COND4') $\eta(\tau(y),\varphi(y)g) = \gamma(g)\eta(y,1)$, $g \in G$, $y \in Y$, has a solution $\eta \in C(X,\mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$;

(COND5) $\eta(\tau(y),1) = \gamma(\varphi(y))^{-1}\eta(y,1)$, $g \in G$, $y \in Y$, has a solution $\eta \in C(X,\mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$;

(COND6) $f(\tau(y)) = \gamma(\varphi(y))^{-1}f(y)$, $y \in Y$, has a solution $f \in C(Y,\mathbb{T})$, $\gamma \in \widehat{G}$, with $\gamma \neq 1$.

We have (COND1) $\Leftrightarrow$ (COND2) by 0.2.4. Let's see that (COND3) $\Rightarrow$ (COND2). If we put $g = 1$ in (COND3), we get

$$\eta(\sigma(y, g')) = \eta(y, g'),\ y \in Y,\ g' \in G.$$

Hence

$$\eta(\sigma^n(y, g')) = \eta(y, g'),\ n \in \mathbb{Z},\ y \in Y,\ g' \in G. \tag{3}$$

Then we replace (y, g') by $\sigma^{n-1}(y, g') = (n-1, 1).(y, g')$ in (COND3) and get (COND2) using (3). So (COND2) $\Leftrightarrow$ (COND3). Also (COND3) $\Rightarrow$ (COND4) $\Leftrightarrow$ (COND4') $\Rightarrow$ (COND5). Now if we put

$$f(y) = \eta(y, 1)$$

we get (COND5) $\Rightarrow$ (COND6). Also (COND6) $\Leftrightarrow$ the negation of the condition from the statement of the proposition.

Conversely, suppose that (COND6) holds and define

$$\eta(y, g') = \gamma(g')f(y). \tag{4}$$

We will show that these η, γ satisfy (COND3). We have

$\eta((1, g).(y, g')) = \eta(\tau(y), \varphi(y)g'g) = \gamma(\varphi(y'))\gamma(g')\gamma(g)f(\tau(y))$ = (from (COND6)) $\gamma(\varphi(y'))\gamma(g')\gamma(g)\gamma(\varphi(y))^{-1}f(y)$ = (by (4)) $\gamma(g')\gamma(g)\gamma(g')^{-1}\eta(y, g')$ $= \gamma(g)\eta(y, g')$.

Thus (COND6) $\Rightarrow$ (COND3) $\Leftrightarrow$ (COND1).

Now since S is a syndetic subgroup of $\mathbb{Z} \times G$, by the criterion for total minimality of reduced flows and $(*)$, $\mathcal{X}$ is not minimal iff the condition (COND1) holds. But, as we have just shown, (COND1) is equivalent to the negation of Parry's condition from the statement of the proposition. This completes the proof. $\square$

Remark 1.3.2. We could omit the proof of this version of Parry's theorem since we are giving in Section 1.4 a proof of a more general theorem of Parry (from which this one can be deduced), also by applying the criterion for minimality of restrictions. But we decided to keep this proof as well, since it illustrates how the ("natural") condition from the criterion for minimality of restrictions can be transformed, in a complicated concrete situation, to a condition which looks misterious and for which it is not clear where it is coming from. So we may say that the criterion for minimality of restrictions also sheds some light on Parry's theorem.

Remark 1.3.3. Some related types of skew-extensions are discussed in [19] and [17].

1.4 Minimality of group-extensions

In this section we give a new proof of a more general theorem of Parry about minimality of group-extensions, using the criterion for minimality of restrictions.

Definition 1.4.1 ([2], [28], [38]). Let $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T, Y \rangle$ be compact flows, K a compact topological group. An extension $p : \mathcal{X} \to \mathcal{Y}$ is called a *K-extension* if the following conditions are satisfied:

(i) there is a continuous action K on X which commutes with the action of T on X;

(ii) the fibers of p are precisely the K-orbits in X.

If in addition $\{K$ acts effectively$\}$ $\{K$ acts freely$\}$ $\{$every character of K is an eigenvalue of $\langle K, X\rangle\}$, we say that p is an $\{$*effective*$\}$ $\{$*free*$\}$ $\{$*simple*$\}$ K-extension.

If we don't want to specify the group, we say *group-extension* instead of K-extension.

Example 1.4.2. Let $\mathcal{Y} = \langle T, Y\rangle$ be a compact flow, K a compact topological group. Put $X = Y \times K$. Let $t.(y,k) = (ty, k)$ and $k.(y,k') = (y, kk')$. Let $p : X \to Y$ be defined by $p(x,y) = y$. Clearly $(\mathcal{X}, p)$ is a free K-extension of $\mathcal{Y}$. Let $\chi \in \widehat{K}$. Put $f_\chi(y,k) = \chi(k)$. Then we have

$$f_\chi(k.(y,k')) = f_\chi(y,kk') = \chi(kk') = \chi(k)\chi(k') = \chi(k)f_\chi(y,k').$$

So $(\mathcal{X}, p)$ is a simple free K-extension of $\mathcal{Y}$.

In what follows, if γ is an eigenvalue of some flow, we denote by f_γ an eigenfunction of γ.

Theorem 1.4.3 ([28]). *Let $\mathcal{X} = \langle T, X\rangle$ and $\mathcal{Y} = \langle T, Y\rangle$ be compact Abelian flows, $\mathcal{Y}$ minimal, K a compact Abelian topological group. Suppose that $(\mathcal{X}, p)$ is a simple free K-extension of $\mathcal{Y}$. Then $\mathcal{X}$ is minimal iff the functional equation*

$$\frac{f(t.p(x))}{f(p(x))} = \frac{f_\gamma(t.x)}{f_\gamma(x)} \tag{1}$$

has no solution f, f_γ, with $f \in C(Y, \mathbb{T})$ and $\gamma \in \widehat{K} \setminus \{1\}$.

Proof. Define a (compact Abelian) flow $\mathcal{Z} = \langle T \times K, X\rangle$ by

$$(t,k).x = t(kx) = k(tx),$$

for $t \in T$, $k \in K$, $x \in X$. This flow is minimal. (Indeed, let $x \in X$ and let U be an open subset of X. Since $p(U)$ is open in Y and $\mathcal{Y}$ is minimal, there is a $t \in T$ such that $tp(x) \in p(U)$, i.e. $p(tx) \in p(U)$. Then $tx \in p^{-1}(p(U)) = \bigcup_{x' \in U} Kx'$. Hence there is a $k \in K$ such that $k(tx) \in U$.)

Let $S = T \times \{1\}$. Since $(t,1).x = tx$, we may identify flows $\mathcal{Z}_S$ and $\mathcal{X}$. So $\mathcal{X}$ is not minimal iff $\mathcal{Z}_S$ is not minimal. By the criterion for minimality of restrictions of compact minimal Abelian flows and 0.2.4, $\mathcal{Z}_S$ is not minimal iff the following condition holds:

(*) $\eta((t,k).x) = \gamma(k)\eta(x)$, $t \in T$, $k \in K$, $x \in X$, has a solution $\eta \in C(X, \mathbb{T})$, $\gamma \in \widehat{K}$, with $\gamma \neq 1$.

It remains to show that the condition (*) is equivalent with the negation of the condition (1). Suppose that (*) holds. Define $f \in C(Y, \mathbb{T})$ by

$$f(p(x)) = f_\gamma(x)/\eta(x), \ x \in X$$

(This is well defined since by (*) $\frac{f_\gamma(kx)}{\eta(kx)} = \frac{\gamma(k) f_\gamma(x)}{\gamma(k)\eta(x)} = \frac{f_\gamma(x)}{\eta(x)}$.)

Since $\eta(t.x) = \eta(x)$, for $t \in T$, $x \in X$ (which follows from (*) for $k = 0$), we easily get (1).

Conversely, suppose that the negation of the condition (1) holds. Define $\eta \in C(X, \mathbb{T})$ by

$$\eta(x) = \frac{f_\gamma(x)}{f(p(x))}, \ x \in X.$$

Then we have:

$$\eta((t,k).x) = \eta(ktx) = \frac{f_\gamma(ktx)}{f(p(ktx)} = \frac{\gamma(k) f_\gamma(tx))}{f(tp(x))} = \gamma(k)\frac{f_\gamma(x)}{f(p(x))} = \gamma(k)\eta(x),$$

for $t \in T$, $x \in X$,$k \in K$. $\square$

Remark 1.4.4 ([28]). Fix $\gamma \in \widehat{K}\backslash\{1\}$. In the context of Theorem 1.4.3 the following are equivalent:

(i) there is an eigenfunction f_γ of γ such that the equation (1) has a solution f, f_γ, with $f \in C(Y, \mathbb{T})$;

(ii) for every eigenfunction f'_γ of γ, the equation (1) has a solution f', f'_γ, with $f' \in C(Y, \mathbb{T})$.

(Indeed, (ii) $\Rightarrow$ (i) is clear. Conversely, suppose that (i) holds. If f'_γ is any other eigenfunction of γ, then $(\frac{f_\gamma}{f'_\gamma})(gx) = (\frac{f_\gamma}{f'_\gamma})(x)$ for all $g \in G$, $x \in X$, so $\frac{f_\gamma}{f'_\gamma}$ can be written as $h \circ p$ for some $h \in C(Y, K)$. Therefore

$$\frac{f(t.p(x))}{f(p(x))} = \frac{f'_\gamma(tx)h(p(tx))}{f'_\gamma(x)h(p(x))} = \frac{f'_\gamma(tx)h(tp(x))}{f'_\gamma(x)h(p(x))},$$

so the equation (1) has a solution $f' = \frac{f}{h}$, f'_γ.)

Remark 1.4.5. Let $\mathcal{X} = \langle T, X\rangle$ be a simple free K-extension of $\mathcal{Y} = \langle T, Y\rangle$, where $\mathcal{X}, \mathcal{Y}$ are compact Abelian flows, and $\mathcal{Y}$ minimal. For every $\gamma \in \widehat{K}$ fix an eigenfunction f'_γ of γ (for the flow $\langle K, X\rangle$). Then the flow $\mathcal{X}$ is minimal iff the equation (1) has a solution f, f'_γ with $f \in C(Y, \mathbb{T})$ and $\gamma \in \widehat{K} \setminus \{1\}$.

(Indeed, the direction $\Leftarrow$ is clear. The direction $\Rightarrow$ follows from Remark 1.4.4.)

1.5 Minimality of a product of two compact minimal abelian flows, one of which is almost periodic

In this section we use Parry's theorem to prove a *criterion for minimality of a product* of two compact minimal abelian flows $\mathcal{X}$ and $\mathcal{Y}$, one of which is almost periodic, in terms of eigenvalues. We also give some applications of this criterion.

Theorem 1.5.1 (criterion for minimality of products). *Let $\mathcal{X} = \langle T, X, \pi \rangle$, $\mathcal{Y} = \langle T, Y, \rho \rangle$ be compact minimal abelian flows and suppose that $\mathcal{Y}$ is almost periodic. Then the product $\mathcal{X} \times \mathcal{Y}$ is minimal if and only if $\mathcal{X}$ and $\mathcal{Y}$ have no nontrivial common eigenvalue.*

Proof. Fix any $e \in Y$. Since $\mathcal{Y}$ is almost periodic, there is a compact abelian group structure on Y such that e is the identity element and the orbital map $\rho^e : T \to Y$, $t \mapsto te$, is a continuous group homomorphism (0.2.30). Denote the group operation on Y by $*$. We have $t(y * y') = ty * y'$, for $t \in T$, $y, y' \in Y$. Define an action of the group Y on $X \times Y$ by $y.(x, y') = (x, y * y')$ and a map $p : X \times Y \to X$ by $p(x, y) = x$. In this way $(\mathcal{X} \times \mathcal{Y}, p)$ becomes an Y-extension of $\mathcal{X}$. If for every $\gamma \in \widehat{Y}$ we define $f'_\gamma : X \times Y \to \mathbb{T}$ by $f'_\gamma(x, y) = \gamma(y)$, we can conclude (as in Example 1.4.2) that $\mathcal{X} \times \mathcal{Y}$ is a simple free Y-extension of $\mathcal{X}$. Now by Remark 1.4.5, $\mathcal{X} \times \mathcal{Y}$ is minimal iff the functional equation

$$\frac{f(tx)}{f(x)} = \frac{\gamma(ty)}{\gamma(y)}$$

has no solution $f \in C(X, \mathbb{T})$ with $\gamma \in \widehat{Y}$, $\gamma \neq 1$. Since $te * y = ty$ for $t \in T$, $y \in Y$, $\mathcal{X} \times \mathcal{Y}$ is not minimal iff the functional equation

$$f(tx) = \gamma(te) f(x) \tag{2}$$

has a solution $f \in C(X, \mathbb{T})$ with $\gamma \in \widehat{Y}$, $\gamma \neq 1$. We show that this condition is equivalent with $\mathcal{X}, \mathcal{Y}$ having a nontrivial common eigenvalue.

First note that for every $\gamma \in \widehat{Y}$, $\gamma \circ \rho^e$ is an eigenvalue of $\mathcal{Y}$ (since $\gamma(ty) = \gamma(te * y) = \gamma(te)\gamma(y)$). Now if (2) has a solution $f \in C(X, \mathbb{T})$ with $\gamma \in \widehat{Y}$, $\gamma \neq 1$, then $\gamma \circ \rho^e$ is a common eigenvalue of $\mathcal{X}$ and $\mathcal{Y}$, which is $\neq 1$ (since $(\gamma \circ \rho^e)(T)$ is dense in Y). Conversely, suppose that $\delta \in \widehat{T}$, $\delta \neq 1$, is a common eigenvalue of $\mathcal{X}$ and $\mathcal{Y}$. Then there is a $\gamma \in C(Y, \mathbb{T})$ such that $\gamma(ty) = \delta(t)\gamma(y)$ and we can choose γ so that $\gamma(e) = 1$. Then $\gamma(te) = \delta(t)$ and $\gamma(te * t'e) = \gamma((t + t')e) = \delta(t + t') = \gamma(te)\gamma(t'e)$. It follows that $\gamma \in \widehat{Y}$, $\gamma \neq 1$. Also there is a $f \in C(X, \mathbb{T})$ such that $f(tx) = \delta(t) f(x)$. Hence $f(tx) = \gamma(te) f(x)$, i.e. (2) has a solution $f \in C(X, \mathbb{T})$ with $\gamma \in \widehat{Y}$, $\gamma \neq 1$. □

Remark 1.5.2. A measure-theoretic analogue of Theorem 4.1 was proved in [29]: let $\mathcal{X}$ and $\mathcal{Y}$ be metric compact abelian flows which support closed ergodic invariant measures. Then $\mathcal{X}$ and $\mathcal{Y}$ are weakly disjoint iff they have no nontrivial common eigenvalue.

Remark 1.5.3. (a) The above theorem can also be proved using the criterion for minimality of restrictions instead of Parry's theorem. We would consider the flow $\mathcal{Z} = \langle T \times Y, X \times Y \rangle$, defined by $(t, y).(x, y') = (tx, ty' * y)$ and its restriction $\mathcal{Z}_S$, where $S = T \times \{e\}$. (Here $*$ and e would be the same as in the above proof.)

(b) Here is one more way to prove the easy direction ($\Rightarrow$) of Theorem 1.5.1.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ have a nontrivial common eigenvalue χ. Let $f : X \to \mathbb{T}$, $g : Y \to \mathbb{T}$ be the corresponding eigenfunctions. Then the function $f\bar{g} : X \times Y \to \mathbb{T}$, defined by $f\bar{g}(x,y) = f(x)\bar{g}(y)$, is nonconstant and invariant. Hence $\mathcal{X} \times \mathcal{Y}$ is not minimal.

Example 1.5.4. Consider almost periodic compact minimal flows $\mathcal{X} = \langle \mathbb{R}, \mathbb{T}, \pi \rangle$ and $\mathcal{Y} = \langle \mathbb{R}, \mathbb{T}, \rho \rangle$, defined by $\pi(t,z) = e^{2\pi i \alpha t} z$ and $\rho(t,z) = e^{2\pi i \beta t} z$, where $\alpha, \beta \in \mathbb{R}$. The eigenvalues of $\mathcal{X}$ (resp. $\mathcal{Y}$) are all $\chi_{n\alpha} : t \mapsto e^{2\pi i n \alpha t}$ (resp. $\chi_{n\beta} : t \mapsto e^{2\pi i n \beta t}$), $n \in \mathbb{Z}$. Hence, by Theorem 1.5.1, $\mathcal{X} \times \mathcal{Y}$ is minimal iff α and β are linearly independent over $\mathbb{Q}$.

Similarly the eigenvalues of the restriction $\mathcal{X}_{\mathbb{Z}}$ (resp. $\mathcal{Y}_{\mathbb{Z}}$) of $\mathcal{X}$ (resp. $\mathcal{Y}$) are all $\chi_{n\alpha} : k \mapsto e^{2\pi i n \alpha k}$ (resp. $\chi_{n\beta} : k \mapsto e^{2\pi i n \beta k}$), $n \in \mathbb{Z}$. Hence, by Theorem 1.5.1, $\mathcal{X}_{\mathbb{Z}} \times \mathcal{Y}_{\mathbb{Z}}$ is minimal iff α, β and 1 are linearly independent over $\mathbb{Z}$.

Corollary 1.5.5 ([2, p.161]). *Let $\mathcal{X}$, $\mathcal{Y}$ be compact minimal abelian flows, and suppose that $\mathcal{X}$ is almost periodic. Then the product $\mathcal{X} \times \mathcal{Y}$ is minimal if and only if $\mathcal{X}$ and $\mathcal{Y}$ have no nontrivial common factor.*

Proof. ($\Rightarrow$) Let $\mathcal{Z}$ be a nontrivial common factor. Then $\mathcal{Z} \times \mathcal{Z}$ is minimal, as a factor of a minimal flow $\mathcal{X} \times \mathcal{Y}$. Hence $\mathcal{Z}$ is trivial, a contradiction.

($\Leftarrow$) Let $\mathcal{X}$ and $\mathcal{Y}$ have no nontrivial common factor. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are not disjoint. Then by Theorem 1.5.1, they have a nontrivial common eigenvalue χ. Let $f : X \to \mathbb{T}$ and $g : Y \to \mathbb{T}$ be the corresponding eigenfunctions. We may assume that there are points $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = 1$, $g(y_0) = 1$. Then $f(Tx_0) = g(Ty_0) = \chi(T)$. If $\chi(T)$ is finite, then by continuity of f, $f(\overline{Tx_0}) = \chi(T)$

and similarly $g(\overline{Ty_0}) = \chi(T)$. If $\chi(T)$ is infinite (so dense in $\mathbb{T}$), then $f(\overline{Tx_0}) = g(\overline{Ty_0}) = \mathbb{T}$, since the sets $f(\overline{Tx_0})$ and $g(\overline{Tx_0})$ are compact and contain a dense subset of $\mathbb{T}$. In both cases, minimality of $\mathcal{X}$ and $\mathcal{Y}$ implies $f(X) = g(Y)$. Hence the subflow on $f(X) = g(Y)$ of the flow $\langle T, \mathbb{T}\rangle$, $(t, z) \mapsto \chi(t)z$, is a nontrivial common factor of $\mathcal{X}$ and $\mathcal{Y}$, a contradiction. $\square$

Remark 1.5.6. A different proof of the statement of this corollary is given in [2], p.161. Also note that, conversely, Theorem 1.5.1 can be deduced from this corollary.

Remark 1.5.7. If we don't assume that either of the flows $\mathcal{X}$, $\mathcal{Y}$ is almost periodic, it is possible to construct two nondisjoint compact minimal abelian flows with no nontrivial common factor. A complicated example was given in [20]. The analogous problem with eigenvalues is trivial: take any weakly mixing flow $\mathcal{X}$ and put $\mathcal{Y} = \mathcal{X}$. They are nondisjoint, but have no nontrivial common eigenvalue. Ergodic analogues of these questions are discussed in [39].

Remark 1.5.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be compact minimal abelian flows. It is known that if $\mathcal{X}$ is distal and $\mathcal{Y}$ is weakly mixing, the product $\mathcal{X} \times \mathcal{Y}$ is minimal; i.e. $\mathcal{X}$ and $\mathcal{Y}$ are disjoint ([2, p.163], [15, Theorem II.3], [38, IV(2.39)1], [41, VI.2.18]). The first proof of this fact was given by Furstenberg ([15]) who showed that a group extension of a flow disjoint from all weakly mixing compact minimal flows is itself disjoint from all weakly mixing compact minimal flows if it is minimal.

Recall that a compact minimal abelian flow $\mathcal{Y}$ is weakly mixing iff $\mathcal{Y}$ has no nontrivial eigenvalue ([33]). So it is natural to ask whether, more generally, $\mathcal{X}$ and $\mathcal{Y}$ are necessarily disjoint if we assume that $\mathcal{X}$ is distal and $\mathcal{X}$ and $\mathcal{Y}$ have no nontrivial common eigenvalue. If we want to construct a counterexample, $\mathcal{Y}$ must be non-weakly-mixing (the statement above) and also at least one of the spaces X, Y must be non-metric (this can be deduced from [2], p.161). We know of no such counterexample; i.e. we know of no example of two non-disjoint compact minimal abelian flows $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{X}$ distal, and $\mathcal{X}$ and $\mathcal{Y}$ having no nontrivial common eigenvalue.

1.6 SK groups

In this section we introduce the notion of SK groups, which will be used in Sections 1.7 and 1.8. For example, Proposition 1.6.6 below will play a role in the proof of Proposition 1.8.5(ii). The motivation for introducing SK groups comes from the criterion for minimality of restrictions and the criterion for total minimality.

Definition 1.6.1. A topological group T is said to be *SK*, if the kernel of every continuous character $\chi \in \widehat{T}$ is a syndetic subgroup of T.

Remark 1.6.2. The name "SK" means "syndetic kernels."

Example 1.6.3. (i) $\mathbb{R}$. (ii) Every compact group. (iii) Every abelian minimally almost periodic group; in particular, every abelian extremely amenable group, see [23, 23.32] for examples of such groups.

(Recall that an abelian topological group T is called *minimally almost periodic* if it has no nontrivial continuous characters. A topological group T is called *extremely amenable* if every T-flow on a compact space has a fixed point. It is easy to see that every abelian extremely amenable group is minimally almost periodic.)

Example 1.6.3(iii) shows that there are non-LCA SK groups. Also, not all LCA groups are SK, for example $\mathbb{Z}$, $\mathbb{R}_d$, $\mathbb{T}_d$, $\mathbb{R} \times \mathbb{R}_d$, etc.

Proposition 1.6.4. *A finite product of SK groups is an SK group.*

Proof. Let $T_1, ..., T_n$ be SK groups and let $T = T_1 \times ... \times T_n$. Let $\chi \in \widehat{T}$. For each $(x_1, ..., x_n) \in T$ we have $\chi(x_1, ..., x_n) = \chi(x_1, 0, ..., 0)... \chi(0, 0, ..., x_n)$. For $i = 1, 2, ..., n$ denote by χ_i the continuous character $x_i \mapsto \chi(0, ..., x_i, ..., 0)$ of T_i. So we have $\chi(x_1, ..., x_n) = \chi_1(x_1)...\chi_n(x_n)$, where $\chi_i \in \widehat{T_i}$ $(i = 1, 2, ..., n)$. Let $S_i = \ker(\chi_i)$, and $T_i = S_i + K_i$, where K_i is a compact subset of T_i $(i = 1, 2, ..., n)$. Since $\ker(\chi) \supset S_1 \times ... \times S_n$, and $S_1 \times ... \times S_n$ is syndetic ($K = K_1 \times ... \times K_n$ is compact and $S_1 \times ... \times S_n + K = T$), $\ker(\chi)$ is also syndetic.

$\square$

Corollary 1.6.5. *Every connected LCA group is SK.*

Proof. By [23, 9.14], connected LCA groups have the form $\mathbb{R}^n \times C$, where $n \geq 0$ and C is a compact connected abelian group. Since $\mathbb{R}$ and C are SK, the corollary follows from Proposition 1.6.4. $\square$

Proposition 1.6.6. *Let T be a topological group, S a subgroup of T. Let $\chi \in \widehat{T}$ be such that: (i) $\chi(S) = \mathbb{T}$, and (ii) $\ker(\chi|_S)$ is syndetic in S. Then $\ker(\chi)$ is syndetic in T.*

Proof. Let $S' = \ker(\chi\ |_S)$. We have $S = S' + K$ for some compact subset K of S. For each $t \in T$, let s_t be an element of S such that $\chi(s_t) = \chi(t)^{-1}$. Then $t + s_t + S' \subset \ker(\chi)$. (Indeed, $\chi(t + s_t + S') = \chi(t) \cdot \chi(s_t) \cdot \chi(S') = \chi(t) \cdot \chi(t)^{-1} \cdot \{1\} =$

$\{1\}$.) Thus $\ker(\chi) \supset \bigcup_{t\in T}(t + s_t + S')$. Now $\ker(\chi) + K \supset \bigcup_{t\in T}(t + s_t + S') + K = \bigcup_{t\in T}(t + s_t + S' + K) = \bigcup_{t\in T}(t + S) = T$. So $\ker(\chi)$ is syndetic in T. $\square$

Corollary 1.6.7. *Let T be a topological group which contains a connected SK subgroup S. Let $\chi \in \widehat{T}$ be such that $S \not\subset \ker(\chi)$. Then $\ker(\chi)$ is syndetic in T.*

Proof. Conditions (i) and (ii) of the previous proposition are satisfied. $\square$

Corollary 1.6.8. *Let T be an LCA group, T_0 its connected component of identity. Let $\chi \in \widehat{T}$ be such that $T_0 \not\subset \ker(\chi)$. Then $\ker(\chi)$ is syndetic in T.*

Proof. T_0 is connected and it is SK by Corollary 1.6.5. So the statement follows from Corollary 1.6.7. $\square$

Example 1.6.9. Let $T = \mathbb{R} \times \mathbb{T}_d$. Then $T_0 = \mathbb{R} \times \{0\}$. If c is the trivial character of $\mathbb{R}$, then $\ker(c \cdot id_{\mathbb{T}}) = \mathbb{R} \times \{0\}$ and this is not a syndetic subgroup of T. For any other character $\chi \in \widehat{\mathbb{R}}$, $\ker(\chi \cdot id_{\mathbb{T}})$ is a syndetic subgroup of T by Corollary 1.6.8. (For notation $\chi_1 \cdot \chi_2$ see 0.2.4.)

Remark 1.6.10. Abelian SK groups in a natural way generalize minimally almost periodic groups (which are never LCA unless trivial), but also contain connected LCA groups. The fact that in recent years it has become clear that extremely amenable (and minimally almost periodic) groups are not "exotic" ([32]), can give some importance to SK groups. In connection with this, let us mention that it is not known if minimally almost periodic groups are extremely amenable ([31]) even in the case of monothetic groups. It is proved in [18, Theorem 3.3], that an example of a polish minimally almost periodic group, which is not extremely amenable, would solve in the negative the old problem from combinatorial number theory and harmonic analysis, asking if the set $S - S$, where S is a syndetic subset of $\mathbb{Z}$, is big enough to be a neighborhood of 0 in the Bohr topology on $\mathbb{Z}$. (Recall that the *Bohr topology* on an abelian topological group T is the weakest topology on T in which all originally continuous characters of T remain continuous.)

Let us also mention that an abelian topological group T is extremely amenable iff every compact minimal T-flow is trivial, iff the universal compact minimal T-flow $\mathcal{M}_T = \langle T, M_T \rangle$ is trivial. (Recall that for every topological group T, the *universal compact minimal T-flow* is defined as a compact minimal T-flow $\mathcal{M}_T = \langle T, M_T \rangle$ such that for every compact minimal T-flow $\mathcal{X} = \langle T, X \rangle$ there exists a morphism of flows of $\mathcal{M}_T$ onto $\mathcal{X}$. It is well-known that $\mathcal{M}_T$ exists and is unique, see [2, p.115-117], [13, p.61-62], [38, IV(3.27)], and also [37, Appendix]. It is shown in [37] that $\mathcal{M}_T$ is not 3-transitive.) Obviously, if a topological group T admits at least one compact minimal non-totally-minimal flow, then $\mathcal{M}_T$ is not totally minimal.

1.7 Total minimality of $\mathcal{X}$ in terms of the structure group $\Gamma(\mathcal{X})$

Remark 1.7.1. Combining 0.2.40 and 0.2.15, for compact minimal abelian flows we have: $\mathcal{X}$ is weakly mixing iff every eigenvalue of $\mathcal{X}$ is trivial. (Another characterization of weakly mixing compact minimal abelian flows was recently given in [3].)

Proposition 1.7.2. *Let $\mathcal{X} = \langle T, X \rangle$ be a compact minimal abelian flow and suppose* *that T is an SK group. Then $\mathcal{X}$ is totally minimal iff $\mathcal{X}$ is weakly mixing.*

Proof. From 1.7.1 and the criterion for total minimality we conclude that if a compact minimal abelian flow is weakly mixing, it is totally minimal. Suppose now that $\mathcal{X}$ is totally minimal. By the criterion for total minimality, $\mathcal{X}$ has no nontrivial eigenvalue whose kernel is syndetic. Since T is SK, this implies that $\mathcal{X}$ has no nontrivial eigenvalue at all. By Remark 1.7.1 $\mathcal{X}$ is weakly mixing. $\square$

Remark 1.7.3. Let $\mathcal{X}$ be a compact minimal abelian flow. The direction "$\mathcal{X}$ weakly mixing implies $\mathcal{X}$ totally minimal" is true without T being SK (see [24, p.480] for another proof). According to [24, p.480], the equivalence "$\mathcal{X}$ is weakly mixing iff $\mathcal{X}$ is totally minimal" was first proved by N. Markley for $T = \mathbb{R}$. Proposition 1.7.2 extends this result since $\mathbb{R}$ is SK.

Remark 1.7.4. If T is not SK, Proposition 1.7.2 is not true in general. Consider for example a compact minimal (almost periodic) $\mathbb{Z}$-flow $\mathcal{X} = \langle \mathbb{Z}, \mathbb{T}, \pi \rangle$, defined by $\pi(n, z) = e^{2\pi i \theta n} z$, for $n \in \mathbb{Z}$ and $z \in \mathbb{T}$, where $\theta \in \mathbb{R}$ is irrational. This flow is totally minimal. (Follows from 1.8.1 below, but it is also easy to check directly.) However this flow is not weakly mixing. (Follows easily from the definition of weak mixing.)

More generally, any nontrivial compact minimal $\mathbb{Z}$-flow $\mathcal{X}$ on a connected space X, which satisfies $S^e_{\mathcal{X}} \neq X \times X$, is totally minimal but not weakly mixing. (Total minimality follows from 1.8.1 below. Weak mixing follows from 0.2.39.)

Corollary 1.7.5. *Let $\mathcal{X} = \langle T, X \rangle$ be a compact minimal abelian flow and suppose that T is SK. Then $\mathcal{X}$ is totally minimal if and only if the structure group $\Gamma(\mathcal{X})$ is trivial.*

Proof. For compact minimal abelian flows, $\mathcal{X}$ is weakly mixing if and only if $S^{\mathfrak{e}}_{\mathcal{X}} = X \times X$ (0.2.39). Hence (by Proposition 1.7.2) $\mathcal{X}$ is totally minimal iff $S^{\mathfrak{e}}_{\mathcal{X}} = X \times X$. Finally, since $\mathcal{X}$ is minimal, $S^{\mathfrak{e}}_{\mathcal{X}} = X \times X$ iff $\Gamma(\mathcal{X})$ is trivial. (Indeed, if $S^{\mathfrak{e}}_{\mathcal{X}} = X \times X$, then clearly $\Gamma(\mathcal{X})$ is trivial. Conversely, if $\Gamma(\mathcal{X})$ is trivial, $E(X/S^{\mathfrak{e}}_{\mathcal{X}}) = \{id_{X/S^{\mathfrak{e}}_{\mathcal{X}}}\}$. So all elements of T fix every element of $X/S^{\mathfrak{e}}_{\mathcal{X}}$. This means that for every equivalence class $C \subset X$ of the relation $S^{\mathfrak{e}}_{\mathcal{X}}$, $t.C \subset C$ for all $t \in T$. Thus C is a (closed) invariant subset of X under T. Since $\mathcal{X}$ is minimal, there is only one equivalence class; i.e. $\Gamma(\mathcal{X})$ is trivial.) $\square$

Remark 1.7.6. Thus the class of abelian topological groups for which the total minimality of every compact minimal abelian flow $\mathcal{X}$ is equivalent with the triviality of $\Gamma(\mathcal{X})$, includes abelian SK groups. (It would be interesting to characterize this class.) In the case that T is a connected LCA group, the previous corollary was stated by Gottschalk ([21, p.56]). Since, by Corollary 1.6.5, connected LCA groups are SK, we have a larger class of acting groups for which the statement holds.

1.8 Compact minimal abelian flows that are not totally minimal

1.8.1. Although total minimality is a strong condition, there are many examples of totally minimal flows. For example, the following statement holds ([22, 2.28]): *every minimal flow $\mathcal{X} = \langle T, X\rangle$, with T discrete and X connected, is totally minimal.*

Indeed, if S is a syndetic normal subgroup of T, then $T = FS = SF$ for some finite set $F \subset T$. Let $x \in X$. Then (using 0.2.21, 0.2.23 and 0.2.24) $X = \overline{T.x} = \overline{FS.x} = F.\overline{S.x} = \bigcup_{t \in F_1} t.\overline{S.x}$, where F_1 is some subset of F and the union is disjoint. Since X is connected, it cannot be a finite union of > 1 disjoint closed sets. So $X = \overline{S.x}$. Since this holds for any $x \in X$, $\mathcal{X}$ is totally minimal.

1.8.2. In [16, p.36] a family of examples of minimal $\mathbb{Z}$-flows on $\mathbb{T}^n$ ($n \geq 1$ any integer) is given. By 1.8.1, these flows are necessarily totally minimal. (See also [38, III(1.18)-III(1.20)] and [14].)

1.8.3. There exists a minimal continuous $\mathbb{R}$-flow on $\mathbb{T}^2$, with no nontrivial continuous eigenvalue ([25]). By 1.7.1(ii) and (iv), this flow is necessarily totally minimal.

1.8.4. Note that by 1.8.1 and Corollary 1.2.5, every nontrivial eigenvalue $\chi_\lambda = e^{2\pi i \lambda(\cdot)}$ of a compact minimal $\mathbb{Z}$-flow on a connected space X, satisfies $\lambda \notin \mathbb{Q}$. (But not every such flow has a nontrivial eigenvalue. However, it is proved in [14, Theorem 5.1] that every minimal $\mathbb{Z}$-flow on $\mathbb{T}^2$, $\mathcal{X} = \langle \mathbb{T}^2, h \rangle$, such that the homeomorphism h is not homotopic to the identity transformation, has a nontrivial eigenvalue.)

We will now give some conditions on compact minimal abelian flows which necessarily imply non-total-minimality.

Proposition 1.8.5. *Let $\mathcal{X} = \langle T, X, \pi \rangle$ be a compact minimal abelian flow. Then in each of the following situations $\mathcal{X}$ is not totally minimal:*

(i) $\mathcal{X}$ almost periodic, X non-connected;

(ii) $\mathcal{X}$ almost periodic, T contains a connected SK subgroup which acts nontrivially on X;

(iii) $\mathcal{X}$ *proximally equicontinuous,* T *contains a connected SK subgroup which acts nontrivially on* X*;*

(iv) $X/S^e_{\mathcal{X}}$ *non-connected;*

(v) $\mathcal{X}$ *distal,* X *totally disconnected,* $|X| > 1$*;*

(vi) $\mathcal{X}$ *point-distal,* T *SK,* X *metric,* $|X| > 1$*;*

(vii) $\mathcal{X}$ *regularly almost perodic at at least one point,* T *contains a connected SK subgroup which acts nontrivially on* X, X *metric.*

Proof. (i) Follows from Example 1.1.6 and the criterion for total minimality.

(ii) Let S be a connected SK subgroup of T and $a \in X$, and suppose that $|S.a| > 1$. By 0.2.30, X has a compact abelian group structure such that a is the identity element and the orbital map $\pi^a : T \to X$ is a continuous group homomorphism and $\overline{\pi^a(T)} = \overline{T.a} = X$. Since $\overline{S.a}$ is a nontrivial closed connected subgroup of X, there is a surjective continuous character of $\overline{S.a}$, $f_0 : \overline{S.a} \to \mathbb{T}$. Let f be a continuous character of X which extends f_0.

Define $\chi \in \widehat{T}$ by $\chi = f \circ \pi^a$. Since $\pi^a(S) = S.a$ is a nontrivial connected subgroup of X, which is dense in $\overline{S.a}$, $\chi(S) = f(\pi^a(S))$ is a nontrivial connected subgroup of $\mathbb{T}$. Hence $\chi(S) = \mathbb{T}$. By Corollary 1.6.7, $\ker(\chi)$ is a syndetic subgroup of $\mathbb{T}$. Since $f(\ker(\chi).a) = (f \circ \pi^a)(\ker(\chi)) = \{1\}$, we have $f(\overline{\ker(\chi).a}) = \{1\}$. If $t \in \ker(\chi)^a$, then $\chi(t) = f(t.a) \in f(\overline{\ker(\chi).a}) = \{1\}$. Hence $t \in \ker(\chi)$. Thus $\ker(\chi)^a \subset \ker(\chi)$. Hence $\ker(\chi)^* = \ker(\chi)^a = \ker(\chi)$. So $\ker(\chi)$ is a proper enveloped subgroup of T. Now by the criterion for total minimality ($(i) \Leftrightarrow (iii)$) $\mathcal{X}$ is not totally minimal.

(iii) Let S be a connected SK subgroup of T and $a \in X$, and suppose that

$|S.a| > 1$. Since $\mathcal{X}$ is proximally equicontinuous, $P_{\mathcal{X}} = S^{e}_{\mathcal{X}}$ (0.2.38). Consider the almost periodic flow $\langle T, X/S^{e}_{\mathcal{X}}\rangle$. This flow satisfies the conditions of (ii). Indeed, if $b \neq a$ is an element of $S.a$, then a and b are distal (0.2.34). Hence the images $\tilde{a}$ and $\tilde{b}$ of a and b in $X/S^{e}_{\mathcal{X}}$ are distinct points, and $\tilde{b} \in S.\tilde{a}$. Now by (ii), $\langle T, X/S^{e}_{\mathcal{X}}\rangle$ is not totally minimal. Consequently $\mathcal{X}$ is not totally minimal.

(iv) The flow $\langle T, X/S^{e}_{\mathcal{X}}\rangle$ is almost periodic with a non-connected phase space. By (i), $\langle T, X/S^{e}_{\mathcal{X}}\rangle$ is not totally minimal. Consequently $\mathcal{X}$ is not totally minimal.

(v) Since $\mathcal{X}$ is distal and $|X| > 1$, $S^{e}_{\mathcal{X}} \neq X \times X$ (0.2.35). The canonical map $\varphi : X \to X/S^{e}_{\mathcal{X}}$ is not only closed, but also open (0.2.36). Since X is totally disconnected, its image $X/S^{e}_{\mathcal{X}}$ under a continuous clopen map is totally disconnected. In particular, $X/S^{e}_{\mathcal{X}}$ (having more than one element) is not connected. By (iv), $\mathcal{X}$ is not totally minimal.

(vi) Follows from Proposition 1.7.2 and 0.2.41.

(vii) By [22, 5.24] $\mathcal{X}$ is locally almost periodic. Hence it is proximally equicontinuous. Now by (iii) $\mathcal{X}$ is not totally minimal.

□

Remark 1.8.6. (i) The statement 1.8.5(ii) was first proved in the case $T = \mathbb{R}$ by E. E. Floyd (see [22, 4.55 and 4.87]). It was generalized by H. Chu to non-totally-disconnected LCA groups with the connected component of the identity acting non-trivially on X ([6]). We extend the class of acting groups for which the statement is true. Also our proof is simpler than that in [6] (no need for Pontryagin's duality theory).

(ii) We could finish the proof of 1.8.5(ii) in a different way, by showing that χ is a nontrivial eigenvalue of $\mathcal{X}$ whose kernel is a syndetic subgroup of T. For this purpose it remains to show that $f(t.x) = \chi(t)f(x)$ for all $(t,x) \in T \times X$. Denote the operation in X by $*$. We have $(t_1 + t_2).a = \pi^a(t_1 + t_2) = \pi^a(t_1) * \pi^a(t_2) = t_1.a * t_2.a$ for any $t_1, t_2 \in T$. For any $x \in X$ there is a net $t_\lambda.a \to x$. Hence for any $t \in T$, $t.a * x = t.a * (\lim t_\lambda.a) = \lim(t.a * t_\lambda.a) = \lim (t + t_\lambda).a = \lim t.(t_\lambda.a) = t.(\lim t_\lambda.a) = t.x$. So $f(t.x) = f(t.a * x) = f(t.a)f(x) = \chi(t)f(x)$ for any $(t,x) \in T \times X$. Now we use the criterion for total minimality $((i) \Leftrightarrow (ii))$.

(iii) The statement 1.8.5(iii) for $\mathcal{X}$ locally almost periodic (hence proximally equicontinuous) and T non-totally-disconnected LCA group with the connected component of identity acting nontrivially on X, was proved in ([7, p.380]). We extend the class of acting groups and replace "locally almost periodic" by a weaker condition "proximally equicontinuous". The part of the proof in which the statement (iii) is reduced to the statement (ii) follows [7]. The proofs of (ii) are different.

(iv) The statement 1.8.5(v) was proved in [24, 3.2] as an application of a criterion for weak mixing that was formulated and proved there. Although 1.8.5(v) implies 1.8.5(iv), we stated both of them since the proof of 1.8.5(v) reduces to the proof of 1.8.5(iv).

(v) A complete characterization of flows which satisfy 1.8.5(vii), with $T = \mathbb{R}$, in terms of their eigenvalues, is given in [10, Theorem 2].

Remark 1.8.7. Note that proximal compact minimal abelian flows are trivial [38, IV(2.18)], in particular totally minimal.

Chapter 2

Almost periodicity of a point under various constructions

2.1 The notion of a skew-morphism of flows

Definition 2.1.1. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows. A pair of maps (h, f), where $h : T \to T'$ is a continuous group homomorphism and $f : X \to Y$ is a continuous map, is called a *skew-morphism* of flows if

$$f(tx) = h(t)f(x)$$

for all $t \in T$ and all $x \in X$. We write $(h, f) : \mathcal{X} \to \mathcal{Y}$.

A skew-morphism (h, f) is called a *skew-isomorphism* if h is an isomorphism of topological groups and f is a homeomorphism.

Example 2.1.2. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T, Y\rangle$ be two flows with the same acting group T and let $f : X \to Y$ be a morphism of flows. Then $(\mathrm{id}_T, f) : \mathcal{X} \to \mathcal{Y}$ is a skew-morphism. Also if $\mathcal{X}_d = \langle T_d, X\rangle$, then $(\mathrm{id}_T, \mathrm{id}_X) : \mathcal{X}_d \to \mathcal{X}$ is a skew-morphism (but not necessarily a skew-isomorphism).

Example 2.1.3. Let $\mathcal{X} = \langle T, X\rangle$ be a flow, $f : X \to \mathbb{T}$ be an *eigenfunction* of $\mathcal{X}$ and $\chi \in \widehat{T}$ the corresponding *eigenvalue*. Let $\mathcal{T} = \langle \mathbb{T}, \mathbb{T}\rangle$ be the flow defined by the action of the unit circle $\mathbb{T}$ on itself by multiplication. Then $(f, \chi) : \mathcal{X} \to \mathcal{T}$ is a skew-morphism.

Example 2.1.4. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism, $y \in Y$, $x \in f^{-1}(y)$. Since $f(Tx) \subset T'y$, we have $f(\overline{Tx}) \subset \overline{T'y}$. Let $f_1 : \overline{Tx} \to \overline{T'y}$ be the restriction of f to these sets. Let $\mathcal{X}' = \langle T, \overline{Tx}\rangle$ and $\mathcal{Y}' = \langle T', \overline{T'y}\rangle$ be the canonical flows. Then $(h, f_1) : \mathcal{X}' \to \mathcal{Y}'$ is a skew-morphism of flows.

Example 2.1.5. Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a flow, S a normal subgroup of T, $x \in X$, $t \in T$. Consider the canonical flows $\mathcal{Y} = \langle S, \overline{Sx}\rangle$ and $\mathcal{Z} = \langle S, \overline{Stx}\rangle$. Notice that $\overline{Stx} = t\overline{Sx}$. Let $h = \mathrm{Int}_t : S \to S$, $h(s) = tst^{-1}$, and let $f = \pi_t : X \to X$, $\pi_t(x) = tx$. Then $(h, f) = (\mathrm{Int}_t, \pi_t) : \mathcal{Y} \to \mathcal{Z}$ is a skew-isomorphism of flows. In T is abelian, $\mathrm{Int}_t = \mathrm{id}_S$, so we have a skew-isomorphism $(\mathrm{id}_S, \pi_t) : \overline{Sx} \to \overline{Stx}$.

Example 2.1.6. Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a compact minimal abelian flow, S a syndetic subgroup of T. The orbit-closures under S form a partition of X. Let R be the equivalence relation on X defined in that way, $\widetilde{X} = X/R$, $p_X : X \to X/R$ the canonical map. For $x \in X$ denote by $\tilde{x}$ the element $p_X(x)$ of $\widetilde{X}$. Let S^* be the $\mathcal{X}$-envelope of S, $p_T : T \to T/S^*$ the canonical homomorphism. The function $\tilde{\pi} : T/S^* \times X/R \to X/R$, given by $\tilde{\pi}(t + S^*, \tilde{x}) = \widetilde{tx}$, defines a flow $\widetilde{\mathcal{X}} = \langle T/S^*, X/R, \tilde{\pi}\rangle$ (follows from the proof of the criterion for minimality of restrictions). Then $(p_T, p_X) : \mathcal{X} \to \widetilde{\mathcal{X}}$ is a skew-morphism of flows.

Proposition 2.1.7. *Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T', Y \rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism.*

(i) If h is surjective, then $f(X)$ is an invariant subset of $\mathcal{Y}$ (and hence $\langle T', f(X) \rangle$ is a subflow of $\mathcal{Y}$).

(ii) If $\mathcal{X}$ is minimal and f is surjective, then $\mathcal{Y}$ is minimal.

(iii) If $\mathcal{X}$ is totally minimal, h, f are both surjective and h has the compact-covering property, then $\mathcal{Y}$ is totally minimal.

Proof. (i) and (ii) are easy.

(iii) Fix a syndetic subset S' of T' and an element $y \in Y$. By Remark 0.2.8, $S = h^{-1}(S')$ is a syndetic subset of T. Let $x \in f^{-1}(y)$. Then $\overline{Sx} = X$. Hence: $\overline{S'y} = \overline{h(S)y} = \overline{h(S)f(x)} = \overline{f(Sx)} \supset f(\overline{Sx}) = f(X) = Y$. So $\mathcal{Y}$ is totally minimal. $\square$

2.2 Almost periodicity of a point under various constructions

Proposition 2.2.1. *Let $\mathcal{X} = \langle T, X \rangle$ be a flow, $x \in X$. Let Y be an invariant subset of X which contains x and let $\mathcal{Y} = \langle T, Y \rangle$ be the subflow of $\mathcal{X}$ on Y. Then x is almost periodic in $\mathcal{X}$ if and only if x is almost periodic in $\mathcal{Y}$.*

Proof. Follows from the definition. $\square$

Remark 2.2.2. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-isomorphism, $x \in X$, $y = f(x)$. Then x is almost periodic in $\mathcal{X}$ if and only if y is almost periodic in $\mathcal{Y}$.

Proposition 2.2.3 ([2],[13],[22],[38](for morphisms)). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Let $x \in X$, $y = f(x)$. Then if x is almost periodic in $\mathcal{X}$, y is almost periodic in $\mathcal{Y}$.*

Proof. Let V be a neighborhood of y and let U be a neighborhood of x such that $f(U) \subset V$. Let S be a syndetic subset of T such that $Sx \subset U$. Then, from $f(Sx) \subset V$, $h(S)y \subset V$. Also $T' = h(T) = h(KS) = h(K)h(S)$. Since $h(K)$ is compact, $h(S)$ is syndetic. Thus y is almost periodic. $\square$

Remark 2.2.4. The above proof is the same as the proof in case of morphisms. The next three propositions however illustrate how sometimes, using skew-morphisms, we can easily get simpler and more natural proofs of known statements, as well as new statements.

Proposition 2.2.5 ([2, page 13]). *Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a flow, S a normal subgroup of T, $\mathcal{X}_S = \langle S, X\rangle$ a restriction of $\mathcal{X}$, $x \in X$. Then if x is almost periodic in $\mathcal{X}_S$, every tx, $t \in T$, is almost periodic in $\mathcal{X}_S$. (In particular, if x is almost periodic in $\mathcal{X}$, every point tx, $t \in T$, is almost periodic in $\mathcal{X}$.)*

Proof. Fix $t \in T$. Consider the canonical flows $\mathcal{Y} = \langle S, \overline{Sx}\rangle$ and $\mathcal{Z} = \langle S, \overline{Stx}\rangle$. By Example 2.1.5 and Proposition 2.2.1 we have: x is almost periodic in $\mathcal{X}_S \Leftrightarrow x$ is almost periodic in $\mathcal{Y} \Leftrightarrow tx$ is almost periodic in $\mathcal{Z} \Leftrightarrow tx$ is almost periodic in $\mathcal{X}_S$. $\square$

Proposition 2.2.6. *Let $\mathcal{X} = \langle T, X, \pi \rangle$, $\mathcal{Y} = \langle T, Y, \rho \rangle$ be two flows with the same acting group T, $r, s \in T$. Consider a continuous group homomorphism $h : T \to T \times T$, given by $h(t) = (rtr^{-1}, sts^{-1}) = (\mathrm{Int}_r(t), \mathrm{Int}_s(t))$. Suppose that the subgroup $h(T)$ of $T \times T$ has the topology induced from $T \times T$ and consider the flow $\mathcal{Z} = \langle h(T), X \times Y \rangle$, defined by*

$$(t_1, t_2)(x, y) = (\pi(t_1, x)\rho(t_2, y)) = (t_1 x, t_2 y),$$

where $(t_1, t_2) \in h(T)$ and $(x, y) \in X \times Y$. Then a point (x, y) is almost periodic in $\mathcal{X} \times \mathcal{Y}$ if and only if (rx, sy) is almost periodic in $\mathcal{Z}$.

Proof. $(h, \pi_t \times \rho_s) : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a skew-isomorphism (by 0.2.2 and a routine checking) and $(\pi_t \times \rho_s)(x, y) = (rx, sy)$. □

Corollary 2.2.7. *Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T, Y \rangle$ be two abelian flows with the same acting group T.*

(i) If a point (x, y) is almost periodic in $\mathcal{X} \times \mathcal{Y}$, then every point (rx, sy), $r, s \in T$, is almost periodic in $\mathcal{X} \times \mathcal{Y}$.

(ii) If a point x is almost periodic in $\mathcal{X}$, then every point (rx, sx), $r, s \in T$, is almost periodic in $\mathcal{X} \times \mathcal{X}$.

Proof. (i) The diagonal of $T \times T$ can be identified with T.

(ii) x is almost periodic in $\mathcal{X}$ if and only if (x, x) is almost periodic in $\mathcal{X} \times \mathcal{X}$, so (ii) follows from (i). □

Remark 2.2.8. The statement (i) is used in [1]. The statement (ii) is Lemma 8 from [8].

Proposition 2.2.9. *Let $\mathcal{X} = \langle T, X\rangle$ be a flow, $f : \mathcal{X} \to \mathcal{X}$ an endomorphism of $\mathcal{X}$. Then if x is almost periodic in $\mathcal{X}$, $(x, f(x))$ is almost periodic in $\mathcal{X} \times \mathcal{X}$.*

Proof. Consider the subflow $\mathcal{Y} = \langle T, \mathrm{Gr}(f)\rangle$ of $\mathcal{X}\times\mathcal{X}$. Let $g : X \to \mathrm{Gr}(f)$ be given by $g(x) = (x, f(x))$. Then $(\mathrm{id}_T, g) : \mathcal{X} \to \mathcal{Y}$ is a skew-isomorphism (using 0.2.2 and the assumption that f is a morphism). Hence, since x is almost periodic in $\mathcal{X}$, $(x, f(x))$ is almost periodic in $\mathcal{Y}$. By Proposition 2.2.1, $(x, f(x))$ is almost periodic in $\mathcal{X}\times\mathcal{X}$. $\square$

Remark 2.2.10. Note that, using Proposition 2.2.9, we can again deduce (ii) from Corollary 2.2.7, if we observe that in the case of an abelian flow $\mathcal{X}$ all transition homeomorphisms $x \mapsto tx$ are endomorphisms of $\mathcal{X}$.

Proposition 2.2.11 ([2],[13],[22],[38]). *Let $\mathcal{X} = \langle T, X\rangle$ be a compact flow. Then:*

(i) a point $x \in X$ is almost periodic if and only if it is discretely almost periodic;

(ii) $\mathcal{X}$ is pointwise almost periodic if and only if every orbit closure in $\mathcal{X}$ is minimal;

(iii) if $\mathcal{X}$ is minimal, every point $x \in X$ is almost periodic;

(iv) there is at least one almost periodic point of $\mathcal{X}$;

(v) let S be a syndetic normal subgroup of T, $\mathcal{X}_S = \langle S, X\rangle$ a restriction of $\mathcal{X}$, $x \in X$; then x is almost periodic in $\mathcal{X}$ if and only if x is almost periodic in $\mathcal{X}_S$.

Remark 2.2.12. All statements from Proposition 2.2.11 can be easily proved using 0.2.17 and the natural connection 0.2.28 between almost periodicity of a point and minimality in the case of compact flows.

Proposition 2.2.13 ([12], [38, II(7.10)] (for morphisms)). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two compact flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective.*

Let $y \in Y$ be an almost periodic point of $\mathcal{Y}$. Then the set $f^{-1}(y)$ contains an almost periodic point of $\mathcal{X}$.

Proof. Let $N = \overline{T'y}$. This is a minimal subset of Y by Proposition 2.2.11. The set $f^{-1}(N)$ is a nonempty closed invariant subset of X. Also $f^{-1}(N)$ is compact. Hence (by 0.2.17) $f^{-1}(N)$ contains a minimal subset M. Then $f(M)$ is a closed nonempty invariant subset of N. Hence $f(M) = N$. In particular, there is a point $x \in M$ such that $f(x) = y$. Since we must have $\overline{Tx} = M$, x is almost periodic by Proposition 2.2.11. □

Remark 2.2.14. The above proof is the same as the proof in the case of morphisms.

2.3 The notion of a skew-morphism good over a point with respect to orbit-closures

Definition 2.3.1. Let X and Y be topological spaces, $y \in Y$. A continuous map $f : X \to Y$, is said to be *good over y* if the fiber $f^{-1}(y) = \{x_i \mid i \in I\}$ is nonempty finite and if given neighborhoods U_i of x_i, $i \in I$, there exist neighborhoods W_i of x_i, $i \in I$, and V of y, such that:

(G1) $W_i \subset U_i$, $i \in I$;

(G2) if $i \neq j$ then $W_i \bigcap W_j = \emptyset$, $i, j \in I$;

(G3) $f(\bigcup_{i \in I} W_i) = V$;

(G4) $f^{-1}(V) = \bigcup_{i \in I} W_i$.

Example 2.3.2. Any *homeomorphism* $f : X \to Y$ is good over any $y \in Y$. More generally, if X is a topological space and F a finite (discrete) space, then $\mathrm{pr}_1 : X\times F \to X$ is good over any $x \in X$.

Remark 2.3.3. Let $f^{-1}(y) = \{x_1, \cdots, x_n\}$ and suppose that there exist neighborhoods U_i of x_i, $i = 1, \cdots, n$, and V of y, so that each $f : U_i \to V$, $i = 1, \cdots, n$, is a homeomorphism. Still f is not necessarily good over y. (Consider the subsets of $\mathbb{R}^2$: $X = \{(a,b) | -1 \leq a \leq 1,\ b \in \{0,1\}\} \setminus \{(0,1)\}$, $Y = \{(a,0) | -1 \leq a \leq 1\}$, the map $f = \mathrm{pr}_1$, and the point $y = (0,0)$.)

Proposition 2.3.4. *Let X and Y be compact spaces, $f : X \to Y$ a surjective continuous map, $y \in Y$. Then if the fiber $f^{-1}(y)$ is finite, f is good over y.*

Proof. Let $f^{-1}(y) = \{x_1, \cdots, x_n\}$ and let U_i be an open neighborhood of x_i, $i = 1, \cdots, n$. We may asssume that the U_i are pairwise disjoint. The set $X' = X \setminus \bigcup_{i=1}^n U_i$ is compact. For any point $z \in X'$ choose disjoint open neighborhoods O_z of y and O of $f(z)$. Then $A_z = f^{-1}(O)$ and $B_z = f^{-1}(O_z)$ are disjoint open neighborhoods of $f^{-1}(f(z))$ and $f^{-1}(y)$ respectively. The set X' is covered by $\bigcup_{z\in X'} A_z$, so there are finitely many points $z_1, \cdots, z_k \in X'$ such that $X' \subset \bigcup_{j=1}^k A_{z_j}$. Consider $\bigcap_{j=1}^k B_{z_j}$. That's a saturated (with respect to f) open neighborhood of $f^{-1}(y)$ (as an intersection of saturated neighborhoods). Also $(\bigcap_{j=1}^k B_{z_j}) \bigcap (\bigcup_{j=1}^k A_{z_j}) = \emptyset$. Since $\bigcup_{j=1}^k A_{z_j} \supset X \setminus \bigcup_{i=1}^n U_i$, we have $\bigcap_{j=1}^k B_{z_j} \subset \bigcup_{i=1}^n U_i$. Put $W_i = (\bigcap_{j=1}^k B_{z_j}) \bigcap U_i$, $i = 1, \cdots, n$. Now $\bigcup_{i=1}^n W_i = \bigcap_{j=1}^k B_{z_j} = \bigcap_{j=1}^k f^{-1}(O_{z_j}) = f^{-1}(\bigcap_{j=1}^k O_{z_j})$. Since f is surjective, $f(\bigcup_{i=1}^n W_i) = f(\bigcap_{j=1}^k B_{z_j}) = \bigcap_{j=1}^k O_{z_j}$. Put $V = \bigcap_{j=1}^k O_{z_j}$. The neighborhoods W_i, $i = 1, \cdots, n$, and V satisfy the conditions (G1)-(G4). $\square$

Remark 2.3.5. If the condition (G3) from the previous definition is replaced by

(G3') $f(W_i) = V$ for all $i \in I$,

(and if the fiber $f^{-1}(y)$ is not necessarily finite), f is said to be *locally surjective over y*. This notion was considered in [35]. Other than this definition, the line of investigation we pursue has no connections with this paper. The map f from the previous proposition is not necessarily locally surjective over y. (Consider the subsets of $\mathbb{R}^2$: $X = \{(a,0) | -1 \leq a \leq 1\} \cup \{(0,1)\}$, $Y = \{(a,0) | -1 \leq a \leq 1\}$, the map $f = \mathrm{pr}_1$, and the point $y = (0,0)$.)

Definition 2.3.6. Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T', Y \rangle$ be two flows. A skew-morphism $(h, f) : \mathcal{X} \to \mathcal{Y}$ is said to be *good over y with respect to orbit closures* if the following two conditions hold:

(GR) for any $x \in f^{-1}(y)$, the restriction $f_1 : \overline{Tx} \to \overline{T'y}$ of f is good over y;

(OC) for any $x, x' \in f^{-1}(y)$, $x' \in \overline{Tx}$ implies $x \in \overline{Tx'}$.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of flows $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T, Y \rangle$ is said to be *good over a point $y \in Y$ with respect to orbit closures* if the skew-morphism $(\mathrm{id}_T, f) : \mathcal{X} \to \mathcal{Y}$ is good over y with respect to orbit closures.

Example 2.3.7. If $(h, f) : \mathcal{X} \to \mathcal{Y}$ is a *skew-isomorphism* of flows $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T', Y \rangle$, then for any $y \in Y$, (h, f) is good over y with respect to orbit closures.

Example 2.3.8. Let $\mathcal{X} = \langle T, X \rangle$ be a flow and let $\mathcal{X}_d = \langle T_d, X \rangle$. Let $(\mathrm{id}_T, f) : \mathcal{X}_d \to \mathcal{X}$ be a skew-morphism with f a *homeomorphism*. Then for any $y \in Y$, (h, f) is good over y with respect to orbit closures.

Example 2.3.9. More generally than in the previous example, let $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T', Y \rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h *surjective* and

f a *homeomorphism.* Then for every $y \in Y$, (h, f) is good over y with respect to orbit-closures.

(Indeed, let $f^{-1}(y) = \{x\}$. Since f is a homeomorphism, $f(\overline{Tx})$ is a closed subset of Y, hence of $\overline{T'y}$. Since it contains a dense subset $T'y$ of $\overline{T'y}$, we have $f(\overline{Tx}) = \overline{T'y}$. So $f_1 : \overline{Tx} \to \overline{T'y}$ is a homeomorphism. Hence (GR) holds. Also (OC) holds since each fiber has exactly one element.)

2.4 Examples of skew-morphisms good over a point with respect to orbit-closures

Proposition 2.4.1. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that (X, f) is a covering of Y whose all fibers are finite. Let $y \in Y$. Suppose that each deck-transformation of (X, f) is an automorphism of the flow $\mathcal{X}$ and that the group of deck-transformations of (X, f) is transitive on $f^{-1}(y)$. Then (h, f) is good over y with respect to orbit closures.*

Proof. Fix any $x \in f^{-1}(y)$. Consider the restriction $f_1 : \overline{Tx} \to \overline{T'y}$ of f. Let's check that f_1 is surjective. Indeed, let $y' \in \overline{T'y}$. Suppose to the contrary, i.e. $f^{-1}(y') \cap \overline{Tx} = \emptyset$. Let $f^{-1}(y') = \{x'_i \mid i \in I\}$. Take an elementary neighborhood U'_i of each of the elements $x'_i (i \in I)$. We may assume that all of them are disjoint from $\overline{Tx}$. Let V' be the corresponding elementary neighborhood of y'. There is an element $t'y \in V'$ (since $y' \in \overline{T'y}$). Let $t \in T$ be such that $h(t) = t'$ and consider tx. We have $f(tx) = t'y$. Hence $tx \in U'_i$ for some i, a contradiction. Thus f_1 is surjective.

Let $f_1^{-1}(y) = \{x = x_1, x_2, \cdots, x_n\}$ and let U_i be a neighborhood of x_i in $\overline{Tx}$ ($i = 1, \cdots, n$). There are elementary neighborhoods W_i' of these points which all correspond to the same elementary neighborhood V' of y and are such that $W_i = W_i' \cap \overline{Tx} \subset U_i$. [Here we use finiteness of the fiber $f^{-1}(y)$.] Let $V = V' \cap \overline{T'y}$. We want to show that these W_i and V satisfy (G1)-(G4) (in that way the condition (GR) for (h, f) will be checked). Let $x' \in W_i$. Then $f(x') \in f(W_i) \subset f(W_i') = V'$. Also there is a net $t_\alpha x \to x'$. Hence $f(t_\alpha x) \to f(x')$, i.e. $h(t_\alpha)y \to f(x')$. Hence $f(x') \in \overline{T'y}$. Thus $f(x') \in V' \cap \overline{T'y} = V$. So $f(W_i) \subset V$ for $i = 1, \cdots, n$. Let now $b \in V$. Since f_1 is surjective, there is an $a \in \overline{Tx}$ such that $f_1(a) = b$. This a must belong to $f^{-1}(V')$, hence to the one of W_i', hence to the one of W_i.

Let's check (OC). Observe that for every $g \in \Delta$ and $x', x'' \in X$, $x'' \in \overline{Tx'}$ implies $gx" \in \overline{Tgx'}$ since g is an automorphism of $\mathcal{X}$. Consider $x_i \in f^{-1}(y) \cap \overline{Tx}$. Let $g \in \Delta$ be such that $gx = x_i$. From $gx \in \overline{Tx}$ we have (using the observation) $g^2x \in \overline{Tgx} \subset \overline{Tx}$. Then $g^3x \in \overline{Tg^2x} \subset \overline{Tgx}$, etc. Since all elements $x, gx, g^2x, \cdots$ are in the finite set $f^{-1}(y)$, there is a smallest $n \geq 1$ such that $g^nx = x$. We have $\overline{Tx} = \overline{Tg^nx} \subset \overline{Tg^{n-1}x} \subset \cdots \subset \overline{Tgx} \subset \overline{Tx}$. Hence $\overline{Tx} = \overline{Tgx} = \overline{Tx_i}$. Hence (OC) holds. $\square$

Lemma 2.4.2. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, let $x \in X$, $y \in Y$ and suppose that $X = \overline{Tx}$, $Y = \overline{T'y}$. Let $(h, f) : \mathcal{X} \to \mathcal{Y}$ be a skew-morphism with h surjective and $f(x) = y$. Suppose that y has a neighborhood V such that $K := f^{-1}(\overline{V})$ is compact. Then the restriction $f' : K \to \overline{V}$ of f is surjective. In particular, $\overline{V}$ is compact.*

Proof. Let $z \in \overline{V}$. Since $T'y \cap \mathrm{Int}(V)$ is dense in $\overline{V}$, there is a net $t'_\alpha y \to z$ in $\mathrm{Int}(V)$. For each t'_α let t_α be an element of T such that $h(t_\alpha) = t'_\alpha$. Since $f(t_\alpha x) = t'_\alpha y \in V$, $t_\alpha x \in K$. Since K is compact, there is a convergent subnet $t_\beta x$. Let $t_\beta x \to w \in K$. Then $f(t_\beta x) \to f(w)$, i.e. $t'_\beta y \to f(w)$. Hence $f(w) = z$. $\square$

Proposition 2.4.3. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that whenever $x_1, x_2 \in X$ are in the same fiber, their orbit-closures are either equal to each other or disjoint. Let y be a point of Y which has a neighborhood V such that $f^{-1}(\overline{V})$ is compact and let $x \in f^{-1}(y)$ be such that $\overline{Tx} \cap f^{-1}(y)$ is finite. Let $f' : \overline{Tx} \to \overline{T'y}$ be the restriction of f and let $\mathcal{X}' = \langle T, \overline{Tx}\rangle$ and $\mathcal{Y}' = \langle T', \overline{T'y}\rangle$ be the canonical flows. Then $(h, f') : \mathcal{X}' \to \mathcal{Y}'$ is good over y with respect to orbit closures.*

Proof. Let $f^{-1}(y) \cap \overline{Tx} = \{x = x_1, x_2, \cdots, x_n\}$. Since, by assumption, $\overline{Tx_i} = \overline{Tx}$ for $i = 1, 2, \cdots, n$, (h, f') is good over y with respect to orbit closures iff f' is good over y. The set $V' = V \cap \overline{T'y}$ is a neighborhood of y in $\overline{T'y}$. Note that for every subset of $\overline{T'y}$ its closures with respect to $\overline{T'y}$ and with respect to Y are the same. Since $\overline{V'} \subset \overline{V} \cap \overline{T'y}$, $f'^{-1}(\overline{V'}) \subset f^{-1}(\overline{V})$. Since $f'^{-1}(\overline{V'})$ is closed in $\overline{Tx}$, it is closed in X. Hence it is compact.

Let U_i be a neighborhood of x_i in $\overline{Tx}$ $(i = 1, 2, \cdots, n)$. Denote $K = f'^{-1}(\overline{V'})$. We may assume that $U_i \subset K$ for all i since K is a neighborhood of $f''^{-1}(y)$ in $\overline{Tx}$. The restriction $f'' : K \to \overline{V'}$ of f' is surjective by Lemma 2.4.2. Hence, by Proposition 2.3.4, f'' is good over y. So there are open neighborhoods $W_i \subset U_i$ of x_i in K and V'' of y in $\overline{V'}$, which satisfy (G1)-(G4). They are at the same time neighborhoods in $\overline{Tx}$ and in $\overline{T'y}$. So f' is good over y and consequently (h, f') is good over y with respect to orbit closures. $\square$

Proposition 2.4.4. *Let $\mathcal{X} = \langle T, X\rangle$ be a flow all of whose orbit closures are compact (for example a compact flow) and let $\mathcal{Y} = \langle T', Y\rangle$ be a compact flow. Let $(h, f) : \mathcal{X} \to \mathcal{Y}$ be a skew-morphism with h surjective and f locally injective. Let $y \in Y$ be a point with a nonempty fiber. Then if y is almost periodic in $\mathcal{Y}$, (h, f) is good over y with respect to orbit-closures.*

Proof. Suppose that y is almost-periodic in $\mathcal{Y}$. Let $x \in f^{-1}(y)$. Since $\mathcal{Y}$ is a compact flow, $\overline{T'y}$ is minimal. Hence the restriction $f_1 : \overline{Tx} \to \overline{T'y}$ of f is surjective. For each point $z \in \overline{Tx}$ we can choose an open neighborhood O_z of z in $\overline{Tx}$ such that f_1 is injective on O_z. Since $\overline{Tx}$ is compact there are finitely many points $z_1, z_2, \cdots, z_n$ such that $O_{z_1} \cup \cdots \cup O_{z_n}$ covers $\overline{Tx}$. Each of these sets can contain at most one element from $f_1^{-1}(y)$. Hence $f_1^{-1}(y)$ is finite. By Proposition 2.3.4, f_1 is good over y. So the condition (GR) is satisfied.

Let x' be another point from $f^{-1}(y)$ and suppose $x' \in \overline{Tx}$. Suppose that $x \notin \overline{Tx'}$. Let $f_1^{-1}(y) = \{x = x_1, x_2, \cdots, x' = x_m, x_{m+1}, \cdots, x_n\}$.Without loss of generality we may assume that $\overline{Tx'} \cap f_1^{-1}(y) = \{x_m, x_{m+1}, \cdots, x_n\}$. Using compactness of $\overline{Tx}$ and the fact that f_1 is good over y, we can find open pairwise disjoint neighborhoods W_i of x_i, $i = 1, 2, \cdots, n$, and V of y, so that at the same time the conditions (G1)-(G4) are satisfied, f_1 is injective on each of W_i, $i = 1, 2, \cdots, n$, and $\overline{Tx'}$ is disjoint from every $\overline{W_i}$, $i = 1, 2, \cdots, m-1$. Let $S' = D(y, V)$. Then by Proposition 2.2.11 $T' = F'S'$, where F' is a finite subset of T'. Hence by Lemma 0.2.7, $T = Fh^{-1}(S')$, where F is a finite and $S = h^{-1}(S')$ a syndetic subset of T. There is a net $t_\alpha s_\alpha x_1 \to x_m$ with $t_\alpha \in F$ and $s_\alpha \in S$. The net (t_α) in F has a convergent subnet $t_\beta \to t$. Since $t_\beta s_\beta x_1 \to x_m$, we have $t s_\beta x_1 \to x_m$. Hence $s_\beta x_1 \to t^{-1} x_m$. Since $f(s_\beta x_1) = h(s_\beta) y \in V$, $s_\beta x_1 \in \bigcup_{i=1}^n W_i = f^{-1}(V)$. At the same time $t^{-1} x_m \in \overline{Tx'}$. Since $\overline{Tx'}$ is disjoint from each $\overline{W_i}$ for $i = 1, 2, \cdots, m-1$, we have that for $\beta \geq \beta_0$ (for some β_0) all $s_\beta x_1$ are in $\bigcup_{i=m}^n W_i$. Fix some $s_\beta x_1 \in W_j$, $j \in \{m, m+1, \cdots, n\}$. For each $i = m, m+1, \cdots, n$, $s_\beta x_i \in \bigcup_{p=m}^n W_p$ (must be in $\overline{Tx'}$ and in $\bigcup_{i=1}^n W_i$ at the same time). So there are two of the points $s_\beta x_1, s_\beta x_m, s_\beta x_{m+1}, \cdots, s_\beta x_n$ in one of the sets $W_m, \cdots, W_n$. The image under f_1 of each of them is $h(s_\beta) y$. Since f_1 is injective on each of $W_m, \cdots, W_n$, these two points should be equal to each other, a contradiction. Hence $x \in \overline{Tx'}$, i.e. the condition (OC) is satisfied. $\square$

2.5 A criterion for lifting of almost periodicity of a point

Theorem 2.5.1 (criterion for lifting of almost periodicity of a point). *Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T', Y \rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Let $y \in Y$ be a point such that (h, f) is good over y with respect to orbit-closures and let $x \in f^{-1}(y)$. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.*

Proof. $(\Rightarrow)$: Suppose y is almost periodic in $\mathcal{Y}$. The restriction $f_1 : \overline{Tx} \to \overline{T'y}$ of f is good over y. In particular the fiber $f_1^{-1}(y)$ is finite. Let $f_1^{-1}(y) = \{x =$

$x_1, x_2, \cdots, x_n\}$. Fix any neighborhood U of x_1 in $\overline{Tx}$. Put $U_1 = U$ and $t_1 = e$. For each $i \in \{2, 3, \cdots, n\}$ we have $x_1 \in \overline{Tx_i}$ (since (h, f) is good over y with respect to orbit-closures). Hence for each $i \in \{2, 3, \cdots, n\}$ there is an open neighborhood U_i of x_i in $\overline{Tx}$ and $t_i \in T$ such that $t_i U_i \subset U$. Choose open neighborhoods $W_i \subset U_i$, $i = 1, 2, \cdots, n$, of the points x_i in $\overline{Tx}$ and an open neighborhood V of y in $\overline{T'y}$ so that the conditions (G1)-(G4) are satisfied. By Lemma 0.2.19, there is a neighborhood V' of y in $\overline{T'y}$ and a neighborhood O of the unit element $e_{T'}$ in T', such that $OD(y, V') \subset D(y, V)$. Also there is a compact $K' \subset T'$ such that $T' = K'D(y, V')$. We have $K' \subset F'O$ for some finite subset F' of T'. Thus $T' \subset F'OD(y, V') \subset F'D(y, V) \subset T'$, so $T' = F'D(y, V)$. By Lemma 0.2.7, there is a finite subset F of T such that $T = Fh^{-1}(D(y, V)) = FS$, so $S = h^{-1}(D(y, V))$ is syndetic in T. We have $Sx_1 \subset \bigcup_{i=1}^{n} W_i$ (since for every $s \in S$, $f(sx_1) = h(s)y \in V$). Let $S_i = \{s \in S | sx_1 \in W_i\}$, $i = 1, 2, \cdots, n$. If for $s \in S$, $sx_1 \in W_i$ for some $i = 1, 2, \cdots, n$, then $t_i s x_1 \in t_i W_i \subset t_i U_i \subset U$, hence for every $s \in S$, $s \in S_i$ implies $t_i s \in D(x_1, U)$. Cosequently $D(x_1, U) \supset \bigcup_{i=1}^{n} t_i S_i$. Since $S = \bigcup_{i=1}^{n} S_i$, the set $\bigcup_{i=1}^{n} t_i S_i$ is syndetic in T by Lemma 0.2.9. Hence $D(x_1, U)$ is syndetic and so $x = x_1$ is almost periodic.

($\Rightarrow$) : Follows from Proposition 2.2.3. $\square$

2.6 Applications of the criterion for lifting of almost periodicity of a point

Corollary 2.6.1. *Let $\mathcal{X} = \langle T, X\rangle$ be a flow whose all orbit-closures are compact and let $\mathcal{Y} = \langle T', Y\rangle$ be a compact flow. Let $(h, f) : \mathcal{X} \to \mathcal{Y}$ be a skew-morphism with h surjective and with f locally injective. Let $y \in Y$ be an almost periodic point in $\mathcal{Y}$ with a nonempty fiber. Then every $x \in f^{-1}(y)$ is an almost periodic point of $\mathcal{X}$.*

Proof. By Proposition 2.4.4, (h, f) is good over y with respect to orbit-closures. Hence by Theorem 2.5.1, every $x \in f^{-1}(y)$ is an almost periodic point of $\mathcal{X}$. □

Corollary 2.6.2 ([12, Proposition 3]). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T, Y\rangle$ be two compact flows and $f : \mathcal{X} \to \mathcal{Y}$ a surjective locally injective morphism. Let y be an almost periodic point of $\mathcal{Y}$. Then every $x \in f^{-1}(y)$ is an almost periodic point of $\mathcal{X}$.*

Proof. Follows from Corollary 2.6.1. □

Corollary 2.6.3. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective and f a homeomorphism. Let $y \in Y$ and let $x \in f^{-1}(y)$. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.*

Proof. By Example 2.3.9, (h, f) is good over y with respect to orbit closures. So the statement follows from Theorem 2.5.1. □

Corollary 2.6.4 ([30, Theorem]). *Let $\mathcal{X} = \langle T, X\rangle$ be a flow and x a point of X. Then x is almost periodic if and only if it is discretely almost periodic.*

Proof. Consider a skew-morphism $(\mathrm{id}_T, \mathrm{id}_X) : \mathcal{X}_d \to \mathcal{X}$, where $\mathcal{X}_d = \langle T_d, X\rangle$ and apply Corollary 2.6.3. □

Corollary 2.6.5 ([26, Proposition 4.3] (with $T = T'$ and $h = \mathrm{id}_T$)). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that whenever $x_1, x_2 \in X$ are in the same fiber, their orbit-closures are either equal to each other or disjoint. Let y be a point of Y which has a neighborhood V such that $f^{-1}(\overline{V})$ is compact and let $x \in f^{-1}(y)$ be such that $\overline{Tx} \cap f^{-1}(y)$ is finite. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.*

Proof. Let $f' : \overline{Tx} \to \overline{T'y}$ be the restriction of f, $\mathcal{X}' = \langle T, \overline{Tx}\rangle$, $\mathcal{Y}' = \langle T', \overline{T'y}\rangle$ the canonical flows. Then, by Proposition 2.4.3, $(h, f') : \mathcal{X}' \to \mathcal{Y}'$ is good over y with respect to orbit-closures. Hence, by Theorem 2.5.1, y is almost periodic in $\mathcal{Y}'$ iff x is almost periodic in $\mathcal{X}'$. Also, by Lemma 2.4.2, y is almost periodic in $\mathcal{Y}$ iff y is almost periodic in $\mathcal{Y}'$ and x is almost periodic in $\mathcal{X}$ iff x is almost periodic in $\mathcal{X}'$. Thus y is almost periodic in $\mathcal{Y}$ iff x is almost periodic in $\mathcal{X}$. □

Corollary 2.6.6 ([27, Theorem 2.1] (with $T = T'$ and $h = \mathrm{id}_T$)). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that (X, f) is a covering of Y all of whose fibers are finite. Let $y \in Y$ and let $x \in f^{-1}(y)$. Suppose that each deck-transformation of (X, f) is an automorphism of the flow $\mathcal{X}$ and that the group of deck-transformations of (X, f) is transitive on $f^{-1}(y)$. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.*

Proof. By Proposition 2.4.1, (h, f) is good over y with respect to orbit-closures. So the statement follows from Theorem 2.5.1. □

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