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ABSTRACT

Relative bounded cohomology and Relative ℓ_1 homology

By

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In his renowned paper 'Volume and Bounded Cohomology', M. Gromov developed the theory of bounded cohomology and ℓ_1-homology of topological spaces. His theory applies both to the absolute case and the relative one. While the theory of absolute bounded cohomology is fairly well understood algebraically from works of R. Brooks and N. Ivanov, this is not so for the relative case.

The goal of this paper is to provide the principal algebraic foundations to the theory of bounded cohomology and ℓ_1 homology in the relative case. Moreover, we give the proofs of Gromov's Equivalence theorem and Relative mapping theorem for both relative bounded cohomology and relative ℓ_1 homology, which Gromov states in his paper without proofs. We also define locally finite ℓ_1 homology of spaces in terms of relative ℓ_1 homology of spaces and prove the Gromov's Vanishing-Finiteness theorem for locally finite ℓ_1 homology.

To my parents EunSoon Chung and DongGeun Park

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Introduction

The absolute bounded cohomology was first defined for discrete groups. It appeared in a version due to P. Trauber of a theorem of Hirsch and Thurston to the effect that the bounded cohomology of an amenable group is zero. Afterwards, M. Gromov [7] defined the bounded cohomology of topological spaces and proved a number of profound theorems about it. Moreover, Gromov [7] applied the theory of bounded cohomology to Riemannian geometry, thus demonstrating the importance of this theory. The proofs in [7] are based on a specific technique developed by Gromov, which he called the theory of simplicial multicomplexes, rather than on standard ideas of algebraic topology.

The first step in understanding the theory of bounded cohomology from the point of view of homological algebra was made by R. Brooks [3]. He based his approach on the ideas of the relative homological algebra. However, his approach was incomplete in at least two respects: it did not let one construct the natural norm which one has on bounded cohomology groups precisely and it used Gromov's fundamental theorem about the bounded cohomology of simply connected spaces.

In [8] N. Ivanov improved Brook's approach using a suitable version of relative homological algebra, modified so that they take into account a natural seminorm in the bounded cohomology. He also proved the Gromov's vanishing theorem for the bounded cohomology of simply connected spaces by using the results of Dold and Thom, and an analogue of Leray's theorem about coverings in the theory of bounded cohomology by using the theory of sheaves.

The present paper extends the theories of both the absolute bounded cohomology and absolute ℓ_1 homology to the relative ones using the ideas of relative homological algebra. While Gromov defined both relative bounded cohomology and relative ℓ_1 homology only for a pair of spaces X and $Y \subset X$ as similar to the ordinary relative cohomology and homology, we define them for continuous map of spaces which also induce the standard cases.

For a topological space X, we denote by $S_n(X)$ the set of n-dimensional singular simplices in X. The ℓ_1 homology of a topological space X, denoted by $H_*^{\ell_1}(X)$, is defined as the homology of the complex $C_n^{\ell_1}(X)$ of ℓ_1 chains $c = \sum_{i=1}^{\infty} r_i\sigma_i$ with the ℓ_1 norm $\|c\|_1 = \sum_{i=1}^{\infty} |r_i| < \infty$, where $r_i \in \mathbf{R}$ and $\sigma_i \in S_n(X)$. Thus $C_n^{\ell_1}(X)$ is the norm completion of the ordinary chain complex $C_n(X)$ and so it is a Banach space. In [7] Gromov used this ℓ_1 norm mainly for defining the simplicial volume of open manifolds. By taking the dual Banach space of $C_n^{\ell_1}(X)$, we obtain a cochain complex $\mathrm{Hom}(C_n^{\ell_1}(X), \mathbf{R})$ and its cohomology is called bounded cohomology of X and is denoted by $\widehat{H}^n(X)$.

It is more convenient, in some respect, to deal with the bounded cohomology with the following independent description, which we will use, rather than with the dual space of ℓ_1 chain group. We define $\widehat{H}^n(X)$ as the cohomology of the complex $B^*(X) = \{B^n(X), \partial_n\}$, where $B^n(X)$ is the space of bounded real valued functions on $S_n(X)$ (see Section 1.2). We define the norm of $f \in B^n(X)$ by setting $\|f\| = \sup\{\, |f(\sigma)| \mid \sigma \in S_n(X)\,\}$ which turns it into a Banach space. Thus on $\widehat{H}^n(X)$ there is a natural seminorm $\|\alpha\| = \inf \|f\|$, where f runs over all bounded cocycles representing $\alpha \in \widehat{H}^n(X)$.

There is a group-theoretic analogue of bounded cohomology, which is discussed in detail in [8]. Here we briefly introduce the bounded cohomology of groups by using the standard resolution. For a discrete group G, let $B(G^n) = \{\, f \colon G^n \to \mathbf{R} \mid \|f\| < \infty \,\}$, where $\|f\| = \sup\{\,|f(g_1,\cdots,g_n)| \mid (g_1,\cdots,g_n) \in G^n\,\}$. Then $B(G^n)$ is a bounded G-module by which we mean $B(G^n)$ is a real Banach space together with the G-action $g\cdot f(g_1,\cdots,g_n) = f(g_1,\cdots,g_n g)$ such that $\|g\cdot f\| \le \|f\|$. Then there is a G-resolution of the trivial G-module $\mathbf{R}$

$$0 \to \mathbf{R} \xrightarrow{d_{-1}} B(G) \xrightarrow{d_0} B(G^2) \xrightarrow{d_1} B(G^3) \xrightarrow{d_2} \cdots$$

which is called the standard G-resolution, where the boundary operators are defined by the formulas $d_{-1}(c)(g) = c$ and for every $n \ge 0$

$$\begin{aligned} d_n(f)(g_1,\cdots,g_{n+2}) = (-1)^{n+1} & f(g_2,\cdots,g_{n+2}) \\ &+ \sum_{i=1}^{n+1}(-1)^{n+1-i} f(g_1,\cdots,g_i g_{i+1},\cdots,g_{n+2}). \end{aligned}$$

Also we let $B(G^n)^G = \{f \in B(G^n) \mid g\cdot f = f \text{ for all } g \in G\}$. Then we have a complex

$$0 \to B(G)^G \to B(G^2)^G \to B(G^3)^G \to \cdots.$$

The cohomology of this complex is called the bounded cohomology of G and is denoted by $\widehat{H}^*(G)$.

An important feature of the theory is that the bounded cohomology of a topological space and its fundamental group coincide. That makes it possible to study them simultaneously from two view points: group theory and topology.

Let $\varphi\colon Y \to X$ be a continuous map of spaces. We consider the mapping cone $B^n(X) \bigoplus B^{n-1}(Y)$ and its boundary operator defined by

$$d_n(u_n, v_{n-1}) = (\partial_n u_n,\ -\lambda^n u_n - \partial'_{n-1} v_{n-1}),$$

where ∂_* and ∂'_* are the boundary operators on $B^*(X)$ and $B^*(Y)$ respectively and $\lambda^*\colon B^*(X) \to B^*(Y)$ is a cochain map induced by φ. Then $\{B^n(X) \bigoplus B^{n-1}(Y),\, d_n\}$ is a complex. We call the n-th cohomology of this complex the n-th relative bounded cohomology of X modulo Y and denote it by $\widehat{H}^n(Y \xrightarrow{\varphi} X)$. We define the norm $\|\cdot\|$

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on $B^n(X) \bigoplus B^{n-1}(Y)$ by setting

$$\|(u_n, v_{n-1})\| = \max\{\|u_n\|, \|v_{n-1}\|\}.$$

This norm induces a seminorm $\|\cdot\|$ on $\widehat{H}^*(Y \xrightarrow{\varphi} X)$.

Similarly, for a homomorphism of groups $\varphi\colon A \to G$ we define $\widehat{H}^*(A \xrightarrow{\varphi} G)$ whose seminorm depends on the choice of a pair of G- and A-resolutions. As an important example, the group $\widehat{H}^*(A \xrightarrow{\varphi} G)$ is defined as the cohomology of the complex of mapping cone $\{B(G^{n+1})^G \bigoplus B(A^n)^A\}$ induced from the standard G- and A-resolutions. The seminorm on which is in fact the canonical one, i.e., the infimum of seminorms which arise from all pairs of G- and A-resolutions which define $\widehat{H}^*(A \xrightarrow{\varphi} G)$.

If we consider a pair of spaces X and $Y \subset X$ and an inclusion map $\varphi\colon Y \hookrightarrow X$, then there is an exact sequence

$$0 \to \ker(p^n) \hookrightarrow B^n(X) \xrightarrow{p^n} B^n(Y) \to 0,$$

where p^n is the cochain map induced by φ. It is obvious that $\{\ker(p^*)\}$ is a complex. The n-th cohomology of the complex $\{\ker(p^*)\}$ is the relative bounded cohomology group, by Gromov's definition. It is denoted by $\widehat{H}^n(X,Y)$. In fact, there is a canonical isomorphism of vector spaces

$$H^n(\beta)\colon \widehat{H}^n(X,Y) \to \widehat{H}^n(Y \xrightarrow{\varphi} X)$$

which carries the seminorm on $\widehat{H}^n(X,Y)$ to a norm equivalent to the seminorm on $\widehat{H}^n(Y \xrightarrow{\varphi} X)$. In fact, we have an explicit estimate as the following:

$$\frac{1}{n+2}\|[f]\| \le \|H^n(\beta)[f]\| \le \|[f]\| \qquad \text{for } [f] \in \widehat{H}^n(X,Y).$$

For a pair of groups G and $A \subset G$, as an analogue of Gromov's definition of $\widehat{H}^*(X,Y)$, we can define the relative bounded cohomology group $\widehat{H}^*(G,A)$ by using the exact sequence

$$0 \to \ker(p^n) \hookrightarrow B(G^{n+1})^G \xrightarrow{p^n} B(A^{n+1})^A \to 0.$$

Thus, for a pair of spaces X and $Y \subset X$, we can define $\widehat{H}^n(\pi_1 X, \pi_1 Y)$ only when the natural inclusion map $Y \hookrightarrow X$ induces an injective homomorphism $\pi_1 Y \to \pi_1 X$. In this case, there is an isomorphism

$$\widehat{H}^n(X, Y) \to \widehat{H}^n(\pi_1 X, \pi_1 Y)$$

which carries the seminorm on $\widehat{H}^n(X, Y)$ to a norm equivalent to the seminorm on $\widehat{H}^n(\pi_1 X, \pi_1 Y)$. The difficulty arises from the fact that the induced homomorphism $\pi_1 Y \to \pi_1 X$ is not injective in general.

From our definition, for a continuous map $\varphi\colon Y \to X$ and the induced homomorphism $\varphi_*\colon \pi_1 Y \to \pi_1 X$, we shall construct the cochain map

$$B^n(X) \bigoplus B^{n-1}(Y) \to B((\pi_1 X)^{n+1})^{\pi_1 X} \bigoplus B((\pi_1 Y)^n)^{\pi_1 Y}$$

and we shall see that the group $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ is canonically isomorphic with the group $\widehat{H}^n(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ and this isomorphism carries the seminorm on $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ to the canonical seminorm on $\widehat{H}^n(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$. So the relative bounded cohomology of spaces also coincides with the relative bounded cohomology of their fundamental groups. Thus it appears that our definition of relative bounded cohomology is more natural.

Amenable groups, whose definition is recalled in Section 1.3, play a special role in the theory of bounded cohomology. One of the important facts is that the bounded cohomology of an amenable group is zero. We shall see that $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ is isometrically isomorphic with $\widehat{H}^*(X)$ if the group $\pi_1 Y$ is amenable.

As similar to the relative bounded cohomology, we also define the relative ℓ_1 homology groups $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ and $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ for a continuous map $\varphi\colon Y \to X$ of spaces and for a homomorphism $\varphi\colon A \to G$ of groups respectively by using mapping cones. Then we see that the relative ℓ_1 homology of spaces also coincides with the relative ℓ_1 homology of their fundamental groups.

Now we describe the content of the paper. In Chapter 1, we review the basic definitions and results of the theory of absolute bounded cohomology following Ivanov's paper [8], which is the main source of the ideas of the present paper.

In Chapter 2, we construct the theory of the relative bounded cohomology of discrete groups following the ideas of relative homological algebra. For a group homomorphism $\varphi\colon A \to G$, we define the relative bounded cohomology of G modulo A and denote it by $\widehat{H}^*(A \xrightarrow{\varphi} G)$ [Definition 2.3]. We also define the norms $\|\cdot\|(\omega)$ on $\widehat{H}^*(A \xrightarrow{\varphi} G)$ for every $\omega \in [0, \infty]$ and prove that the canonical seminorm coincides with the seminorm induced by the standard G- and A-resolutions [Corollary 2.8]. For an amenable group A, we prove that the groups $\widehat{H}^*(G)$ and $\widehat{H}^*(A \xrightarrow{\varphi} G)$ are isometrically isomorphic for the norm $\|\cdot\|$ [Theorem 2.14] and that the norms $\|\cdot\|(\omega)$ on $\widehat{H}^*(A \xrightarrow{\varphi} G)$ are equal for every $\omega \in [0, \infty]$ [Theorem 2.15]. Also, for a subgroup A of G and the natural inclusion map $\varphi\colon A \hookrightarrow G$, we prove that the group $\widehat{H}^*(G, A)$ is isomorphic to $\widehat{H}^*(A \xrightarrow{\varphi} G)$ and the isomorphism carries the seminorm on $\widehat{H}^*(G, A)$ to the norm equivalent to the canonical seminorm on $\widehat{H}^*(A \xrightarrow{\varphi} G)$ [Theorem 2.19]. It is not known if they are actually equal.

In Chapter 3, we define the relative bounded cohomology of a space X modulo Y of a continuous map $Y \xrightarrow{\varphi} X$ and denote it by $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ [Definition 3.2]. Also we define the norms $\|\cdot\|(\omega)$ on $\widehat{H}^*(Y \xrightarrow{\varphi} X)$. The main result of this section is that $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^*\big(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X\big)$ are isometrically isomorphic [Theorem 3.3]. We also prove in Corollary 3.4 and Theorem 3.5 respectively Equivalence theorem and Relative mapping theorem which are stated in the Gromov's paper [7].

In Chapter 4, we construct a theory of ℓ_1 homology of discrete groups. We denote the ℓ_1 homology of a group G by $H_*^{\ell_1}(G)$. The main results of this section are that the canonical seminorm in $H_*^{\ell_1}(G)$ coincides with the seminorm induced by the bar resolution [Theorem 4.2] and that $H_*^{\ell_1}(G)$ is zero if G is amenable [Theorem 4.10].

In Chapter 5, we define the relative ℓ_1 homology of a group G modulo A for a group homomorphism $A \xrightarrow{\varphi} G$ and denote it by $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$[Definition 5.3]. We prove that the groups $H_*^{\ell_1}(G)$ and $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ are isometrically isomorphic for the norm $\|\cdot\|_1(\omega)$ for an amenable group A [Corollary 5.10] and that the norms $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ are equal for every $\omega \in [0,\infty]$ for an amenable group A [Theorem 5.8]. For a subgroup A of G, we also define $H_*^{\ell_1}(G,A)$ following the usual approach to the relative homology of groups. Then we prove that the groups $H_*^{\ell_1}(G,A)$ and $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ for the natural inclusion map $\varphi\colon A \hookrightarrow G$ are isomorphic and this isomorphism carries the seminorm on $H_*^{\ell_1}(G,A)$ to a norm equivalent to the canonical seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ [Theorem 5.12].

In Chapter 6, we define the relative ℓ_1 homology of a space X modulo Y of a continuous map $\varphi\colon Y \to X$ and denote it by $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ [Definition 6.3]. The main result of this section is that the groups $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ and $H_*^{\ell_1}\big(\boldsymbol{\pi}_1 Y \xrightarrow{\varphi_*} \boldsymbol{\pi}_1 X\big)$ are isometrically isomorphic [Theorem 6.4].

In Chapter 7, we define the locally finite ℓ_1 homology group of a space X as the inverse limit of an inverse system of relative ℓ_1 homology groups $H_*^{\ell_1}((X-K_j) \hookrightarrow X)$ for every compact subspaces $K_j \subset X$ and the inclusion maps $X - K_j \hookrightarrow X$. We denote it by $H_*^{\infty}(X)$ [Definition 7.1]. We also define the norms $\|\cdot\|_s$ and $\|\cdot\|_l(\omega)$ on $H_*^{\infty}(X)$ [Definition 7.2]. The group $H_*^{\infty}(X)$ is an analogue of the group $\overline{H}_*(X)$ in [7] which Gromov defined with the locally finite singular ℓ_1-chains $c = \sum_{i=1}^{\infty} r_i \sigma_i$ such that each compact subset K of X intersects only finitely many (images of) simplices σ_i. On $\overline{H}_*(X)$ there are the norms $\|\cdot\|_1$ and $\|\cdot\|_1(\theta)$ for every $\theta \in [0,\infty]$ induced by the norms $\|c\|_1 = \sum_{i=1}^{\infty} |r_i|$ and $\|c\|_1(\theta) = \|c\|_1 + \theta\|\partial c\|_1$ respectively. In fact, for every $\theta \in [0,\infty]$, Gromov defined the norm $\|h\|_1(\theta)$ of $h \in \overline{H}_*(X)$ as the limit $\lim_{j\to\infty} \|h_j\|_1(\theta)$, where $h_j \in H_*(X, U_j)$ is the homomorphic image of $h \in \overline{H}_*(X)$ for a sequence of subsets $U_j \subset X$ which are large, i.e., the closure of $X - U_j$ is compact, and such that only finitely many U_j intersects any given compact subset of X. It seems plausible that the groups $H_*^{\infty}(X)$ and $\overline{H}_*(X)$ are isomorphic and the norms $\|\cdot\|_s$ and

$\|\cdot\|_l(\omega)$ are equivalent to the norms $\|\cdot\|_1$ and $\|\cdot\|_1(\theta)$ for $\omega=\theta$ respectively but we leave it as an open question. We consider an amenable covering of X. A subset Y of X is called amenable if for every path connected component Y' of Y the image of the inclusion homomorphism $\pi_1 Y' \to \pi_1 X$ is an amenable group. A covering of a space X is amenable if all its elements are amenable (see Section 1.3.2). We prove Equivalence theorem, which is only stated in the Gromov's paper [7], to the effect that on $H_*^\infty(X)$ the norm $\|\cdot\|_s$ is equal to the norms $\|\cdot\|_l(\omega)$ for every $\omega\in[0,\infty]$ if a space X is amenable at infinity, i.e., every large set $U\subset X$ contains another large amenable subset $U'\subset U$ [Theorem 7.5]. Finally, let a space X admit an amenable covering. Then we prove the Gromov's vanishing theorem in [7] for $H_*^{\ell_1}(X)$ and for $H_*^\infty(X)$ to the effect that the norm $\|\cdot\|_1$ on $H_n^{\ell_1}(X)$ and the norm $\|\cdot\|_s$ on $H_n^\infty(X)$ are equal to zero for $n\geq m$ if every point of X is contained in at most m elements of this covering [Theorem 7.7 and Corollary 7.8]. We also prove the Gromov's finite theorem for $H_*^\infty(X)$ to the effect that the norm $\|\cdot\|_s$ on $H_n^\infty(X)$ is finite for $n\geq m$ if there is a large set every point of which is contained in at most m elements of this covering [Theorem 7.9]. In [7] Gromov proved Vanishing-Finite theorem for the group $\overline{H}_*(X)$ using locally finite diffusion operators $\mu *$, where μ is a non-negative real valued function on a group G such that $\|\mu\|_1=\sum|\mu(g)|=1$. Then for an ℓ_1 function f on X and a group G acting on X, the diffusion operators are defined as $(\mu * f)(x)=\sum_{g\in G}\mu(g)f(g^{-1}x)$ for $x\in X$.

CHAPTER 1

Absolute bounded cohomology groups

In this chapter, we review the basic definitions and results of the theory of bounded cohomology in [8].

Throughout this chapter G denotes a discrete group.

1.1 Bounded cohomology of groups

1.1.1 Bounded G-modules

By a *bounded left G module* we mean a real Banach space V together with a left action of G on V such that $\|g \cdot v\| \leq \|v\|$ for all $g \in G$ and $v \in V$. We define a bounded right G-module analogously. We shall call a bounded left G-module simply G-module. For two G-modules V and W, a bounded linear operator $V \to W$ which commutes with the action of G is called a G-morphism. The simplest and also an important example of G-module is $\mathbf{R}$, considered together with the trivial action of G. Another important example of G-module is $B(G^n)$ the set of all bounded functions $f\colon G^n \to \mathbf{R}$, where $G^n = \underbrace{G \times G \times \cdots \times G}_{n}$ is considered together with the action:

$g \cdot f(g_1, \cdots, g_{n-1}, g_n) = f(g_1, \cdots, g_{n-1}, g_n g)$.

More generally, for any Banach space V, we consider the space $B(G, V)$ of functions $f: G \to V$ such that $\|f\| = \sup\{ |f(g)| \mid g \in G\} < \infty$. Then $B(G, V)$ is a Banach space with the norm $\| \cdot \|$ and the action defined by $g \cdot f(h) = f(hg)$ turns it into a G-module. It is clear that the space $B(G^{n+1})$ is isomorphic with $B(G, B(G^n))$, where $B(G^n)$ is considered simply as a Banach space.

1.1.2 Relatively injective G-modules

An injective G-morphism of G-modules $i: V \to W$ is said to be *strongly injective* if there exists a bounded linear operator $\sigma: W \to V$ such that $\sigma \circ i = id$ and $\|\sigma\| \leq 1$. We call a G-module U *relatively injective* if, for any strongly injective G-morphism of G-modules $i: V \to W$ and any G-morphism of G-modules $\alpha: V \to U$, there exists a G-morphism $\beta: W \to U$ such that $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\alpha\|$. For example, for any Banach space V, the G-module $B(G, V)$ is relatively injective. In particular, the G-modules $B(G^n)$ are relatively injective (see Lemma 3.2.2 in [8]).

1.1.3 Resolutions

By a *strong relatively injective G-resolution* of a G-module V we mean a sequence of G-modules and G-morphisms of the form

$$0 \to V \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots$$

which is exact as a sequence of vector spaces and satisfies the following two conditions:

i) the sequence is provided with a contracting homotopy, i.e., a sequence of bounded linear operators $k_n: V_n \to V_{n-1}$ such that $d_{n-1} \circ k_n + k_{n+1} \circ d_n = id$ for every $n \geq 0$ and $k_0 \circ d_{-1} = id$ and also $\| k_n \| \leq 1$;

ii) every G-module V_n is relatively injective.

We consider a G-resolution of the trivial G-module $\mathbf{R}$ of the form

$$0 \to \mathbf{R} \xrightarrow{d_{-1}} B(G) \xrightarrow{d_0} B(G^2) \xrightarrow{d_1} B(G^3) \xrightarrow{d_2} \cdots ,$$

where the boundary operators are defined by the formulas $d_{-1}(c)(g) = c$ and

$$d_n(f)(g_1, \cdots, g_{n+2}) = (-1)^{n+1} f(g_2, \cdots, g_{n+2}) + \sum_{i=1}^{n+1} (-1)^{n+1-i} f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+2}).$$

This G-resolution becomes strong relatively injective with the contracting homotopy

$$\mathbf{R} \underset{k_0}{\leftarrow} B(G) \underset{k_1}{\leftarrow} B(G^2) \underset{k_2}{\leftarrow} \cdots ,$$

where $k_n(f)(g_1, \cdots, g_n) = f(g_1, \cdots, g_n, 1)$. We call this resolution the *standard G-resolution.* It will play an important role in the theory of bounded cohomology.

1.1.4 Bounded cohomology of groups

For any G-module V, we denote by V^G the space of *G-invariant elements* in V, i.e.,

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}.$$

For any strong relatively injective G- resolution of the trivial G-module $\mathbf{R}$

$$0 \to \mathbf{R} \to V_0 \to V_1 \to V_2 \to \cdots$$

the induced sequence

$$0 \to V_0^G \to V_1^G \to V_2^G \to \cdots$$

is a complex and the cohomology of this complex depends only on G. Its n-th cohomology group is called the n-th *bounded cohomology group* of G and is denoted by $\widehat{H}^n(G)$. Note that in $\widehat{H}^*(G)$ there is a natural seminorm which induces a natural topological vector space structure on it. Also note that this seminorm depends on the choice of resolution.

We define the canonical seminorm on $\widehat{H}^*(G)$ as the infimum of the seminorms which arise from all resolutions. In fact, this infimum is achieved by the standard resolution, i.e., the seminorm on $\widehat{H}^*(G)$, defined by the standard G-resolution, coincides with the canonical seminorm (see Theorem 3.6 in [8]). Remark that $\widehat{H}^*(G)$ is a contravariant functor of G.

1.2 Bounded cohomology of topological spaces

Let X be a topological space. For every $n \geq 0$, we denote by $S_n(X)$ the set of n-dimensional singular simplices in X and by $C^n(X)$ the real n-dimensional singular cochain group, i.e., the set of arbitrary functions $S_n(X) \to \mathbf{R}$. As is well known, the sequence

$$0 \to C^0(X) \xrightarrow{d_0} C^1(X) \xrightarrow{d_1} C^2(X) \xrightarrow{d_2} \cdots$$

is a complex, where d is defined by $d_n f(\sigma) = \sum_{i=0}^{n+1}(-1)^i f(\partial_i \sigma)$ and $\partial_i \sigma$ is the i-th face of the singular simplex σ.

The cohomology of this complex is $H^*(X)$ the real singular cohomology group of X. Let $B^n(X) \subset C^n(X)$ be the space of bounded functions $S_n(X) \to \mathbf{R}$. Its elements are called bounded cochains. There is a natural norm $\|\cdot\|$ in the space $B^n(X)$ given by $\|f\| = \sup\{|f(\sigma)| \mid \sigma \in S_n(X)\}$ which obviously turns it into a Banach space. It is clear that $d_n B^n(X) \subset B^{n+1}(X)$.

The cohomology of the complex

$$0 \to B^0(X) \xrightarrow{d_0} B^1(X) \xrightarrow{d_1} B^2(X) \xrightarrow{d_2} \cdots$$

is called the bounded cohomology of X and is denoted by $\widehat{H}^*(X)$.

In $\widehat{H}^*(X)$ there is a natural seminorm $\|\cdot\|$ defined by $\|[c]\| = \inf \|f\|$ for a cohomology class $[c]$ of $\widehat{H}^*(G)$, where the infimum is taken over all bounded cochains f in $B^*(G)$ lying in the cohomology class $[c]$. Remark that the inclusions $B^n(X) \hookrightarrow$

$C^n(X)$ induce a canonical map $\widehat{H}^*(X) \to H^*(X)$, which in general is neither injective nor surjective.

The first basic result of the theory is that the bounded cohomology of a simply connected space is equal to zero (see Theorem 2.4 in [8]). Moreover, $\widehat{H}^*(X)$ depends only on the fundamental group $\pi_1(X)$ (see Theorem 4.1 in [8]).

1.3 Amenable groups and amenable coverings

Amenable groups play a special role in the theory of bounded cohomology. The technical aspect of this role is, roughly speaking, that for bounded functions on amenable groups one can define their mean value in a natural way. One of the important facts is that the bounded cohomology of an amenable group is zero (see Theorem 3.8.4 in [8]).

1.3.1 Amenable groups

Let S be a set. As is well known, the space $B(S)$ of all bounded functions on S is a Banach space with the norm $\|f\| = \sup\{\, |f(x)| \mid x \in S\,\}$. A linear functional $m\colon B(S) \to \mathbf{R}$ is called a *mean* if

$$\inf\{f(x) \mid x \in S\} \le m(f) \le \sup\{f(x) \mid x \in S\} \quad \text{for all } f \in B(S).$$

Let the group G act on S on the right. Then G acts on $B(S)$ on the left by the formula $g \cdot f(s) = f(s \cdot g)$, where $g \in G$, $f \in B(S)$, and $s \in S$. The mean m on $B(S)$ is called right-invariant if $m(g \cdot f) = m(f)$ for all $g \in G, f \in B(S)$. If there is a right-invariant mean on $B(G)$, then the group G is called *amenable.* The important facts are that abelian groups and the homomorphic images of amenable groups are amenable.

1.3.2 Amenable covering

A connected subset Y of the space X is called amenable if the image of the inclusion homomorphism $\pi_1(Y) \to \pi_1(X)$ is an amenable group. An arbitrary subset of the space X is called amenable if all its components are amenable. Finally, a covering of the space X is called amenable if all its elements are amenable and in addition it satisfies the following conditions i) and ii):

i) the space X, all elements of the covering, and all their finite intersections, are homotopy equivalent to countable cellular spaces;

ii) either the covering is open or it is closed and locally finite and the space X is paracompact.

For example, if $\pi_1(Y)$ is amenable, then Y is amenable.

1.4 Main properties of absolute bounded cohomology

Now, we state the main results of the theory of *absolute* bounded cohomology. We refer the proofs to [8].

Theorem 1.1. *Let X be a countable cellular space. If X is simply connected, then $\widehat{H}^n(X) = 0$ for all $n \geq 1$.*

Theorem 1.2. *Let A be an amenable normal subgroup of G. Then the map $\varphi^*\colon \widehat{H}^*(G/A) \to \widehat{H}^*(G)$, induced by the canonical homomorphism $\varphi\colon G \to G/A$, is an isometric isomorphism, i.e., it preserves the canonical seminorm.*

Corollary 1.3. *If G is amenable, then $\widehat{H}^n(G) = 0$ for all $n \geq 1$.*

Theorem 1.4. *Let X be a connected countable cellular space. Then $\widehat{H}^*(X)$ is canonically isomorphic with $\widehat{H}^*(\pi_1 X)$. The seminorm in $\widehat{H}^*(X)$ is carried to the canonical seminorm in $\widehat{H}^*(\pi_1 X)$ through this isomorphism.*

Proof. For our further use, we sketch the proof of this theorem.

Let $\pi\colon \mathcal{X} \to X$ be a universal covering of X.

Let $\|\cdot\|$ denote the canonical seminorm on $\widehat{H}^*(\boldsymbol{\pi}_1 X)$ and let $\|\cdot\|_s$ the seminorm on $\widehat{H}^*(X)$.

First, it is shown that the sequence

$$0 \to \mathbf{R} \to B^0(\mathcal{X}) \to B^1(\mathcal{X}) \to B^2(\mathcal{X}) \to \cdots \tag{1.4.1}$$

is a strong relatively injective $\boldsymbol{\pi}_1 X$-resolution of the trivial $\boldsymbol{\pi}_1 X$-module $\mathbf{R}$. Since the map $\pi^*\colon B^*(X) \to B^*(\mathcal{X})$ establishes an isometric isomorphism $B^*(X) \to B^*(\mathcal{X})^{\boldsymbol{\pi}_1 X}$ and commutes with the boundary operators, the bounded cohomology $\widehat{H}^*(\boldsymbol{\pi}_1 X)$ induced from the complex

$$0 \to B^0(\mathcal{X})^{\boldsymbol{\pi}_1 X} \to B^1(\mathcal{X})^{\boldsymbol{\pi}_1 X} \to B^2(\mathcal{X})^{\boldsymbol{\pi}_1 X} \to B^3(\mathcal{X})^{\boldsymbol{\pi}_1 X} \to \cdots$$

coincides with $\widehat{H}^*(X)$ as topological vector spaces.

Remark that the seminorm on $\widehat{H}^*(X)$ coincides with the seminorm on $\widehat{H}^*(\boldsymbol{\pi}_1 X)$ induced by the resolution in (1.4.1). So we have $\|\cdot\| \le \|\cdot\|_s$.

On the other hand, for every $n \ge 0$, it is shown that there is a $\boldsymbol{\pi}_1 X$-morphism $\zeta^n\colon B\big((\boldsymbol{\pi}_1 X)^{n+1}\big) \to B^n(\mathcal{X})$ of the resolutions

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbf{R} & \longrightarrow & B(\boldsymbol{\pi}_1 X) & \longrightarrow & B\big((\boldsymbol{\pi}_1 X)^2\big) & \longrightarrow & B\big((\boldsymbol{\pi}_1 X)^3\big) & \longrightarrow & \cdots \\
\big\downarrow & & {\scriptstyle id_{\mathbf{R}}}\big\downarrow & & {\scriptstyle \zeta^0}\big\downarrow & & {\scriptstyle \zeta^1}\big\downarrow & & {\scriptstyle \zeta^2}\big\downarrow & & \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & B^0(\mathcal{X}) & \longrightarrow & B^1(\mathcal{X}) & \longrightarrow & B^2(\mathcal{X}) & \longrightarrow & \cdots
\end{array}$$

extending $id_{\mathbf{R}}$ and such that $\|\zeta^n\| \le 1$ for $n \ge 0$. Namely, let $\mathcal{X}_0 \subset \mathcal{X}$ consist of one element from each $\boldsymbol{\pi}_1 X$-orbit. For each singular simplex $\sigma\colon \Delta_n \to \mathcal{X}$ we set $\{\sigma\} = (g_0, \cdots, g_n)$, where $g_i \in \boldsymbol{\pi}_1 X$ such that $\sigma(v_i) \in g_{n-i} \cdots g_n \mathcal{X}_0$ and v_i is the i-th vertex of the simplex Δ_n. Then we define ζ^n by the formula $\zeta^n(f)(\sigma) = f(\{\sigma\})$ for every $f \in B\big((\boldsymbol{\pi}_1 X)^{n+1}\big)$. Since $\|\zeta^n\| \le 1$, this shows that we have $\|\cdot\|_s \le \|\cdot\|$. Hence the seminorm in $\widehat{H}^*(X)$ is equal to the canonical seminorm in $\widehat{H}^*(\boldsymbol{\pi}_1 X)$. □

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Corollary 1.5. *The group* $\widehat{H}^*(X)$ *is zero if* $\pi_1 X$ *is amenable.*

Theorem 1.6. *Let* X_1 *and* X_2 *be connected countable cellular spaces and let* $f\colon X_1 \to X_2$ *be a continuous map. If the homomorphism* $f_*\colon \pi_1 X_1 \to \pi_1 X_2$ *is a surjection with an amenable kernel, then the homomorphism* $\widehat{f}^*\colon \widehat{H}^*(X_2) \to \widehat{H}^*(X_1)$ *is an isometric isomorphism.*

Theorem 1.7. *Let* X *be a topological space,* $\mathcal{U}$ *be an amenable covering of the space* X, $\mathcal{N}$ *be the nerve of this covering and* $|\mathcal{N}|$ *be the geometric realization of the nerve. Then the canonical map* $\widehat{H}^*(X) \to H^*(X)$ *factors through the map* $H^*(|\mathcal{N}|) \to H^*(X)$.

Corollary 1.8. *If* X *admits an amenable covering such that each point of* X *is contained in no more than m elements of the covering, then the canonical homomorphism* $\widehat{H}^n(X) \to H^n(X)$ *vanishes for* $n \geq m$.

CHAPTER 2

Relative bounded cohomology of groups

Throughout this chapter, G and A denote the discrete groups.

Let $\varphi\colon A \to G$ be a group homomorphism. Then any G-module U can be made into an A-module by defining the action of A to be $a \cdot u = \varphi(a) \cdot u$ for $a \in A$ and $u \in U$.

We recall that $\widehat{H}^*(G)$ is a contravariant functor of G. This functoriality can also be described in terms of arbitrary resolutions, as follows: Let

$$0 \to \mathbf{R} \to U_0 \to U_1 \to \cdots \qquad \text{and} \qquad 0 \to \mathbf{R} \to V_0 \to V_1 \to \cdots$$

be strong relatively injective G- and A-resolutions of the trivial G- and A-module $\mathbf{R}$, respectively. Then the map $id_{\mathbf{R}}$ extends to an A-morphism of resolutions, i.e., to a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R} & \longrightarrow & U_0 & \longrightarrow & U_1 & \longrightarrow & \cdots \\ \downarrow & & \downarrow{\scriptstyle id_{\mathbf{R}}} & & \downarrow{\scriptstyle \lambda^0} & & \downarrow{\scriptstyle \lambda^1} & & \\ 0 & \longrightarrow & \mathbf{R} & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & \cdots \end{array}$$

where $\lambda^i(\varphi(a) \cdot u) = a \cdot \lambda^i(u)$ for $a \in A$ and $u \in U$.

It follows from the last formula that $\lambda^i(U_i^G) \subset V_i^A$, and hence there is an induced homomorphism $\varphi^*\colon \widehat{H}^*(G) \to \widehat{H}^*(A)$, which depends only on φ (see Lemma 3.3.2 in

[8]). Also remark that $\|\varphi^*\| \leq 1$.

Definition 2.1. *Let $\varphi\colon A \to G$ be a group homomorphism. A strong relatively injective G-resolution of a G-module U*

$$0 \to U \underset{k_0}{\overset{\partial_{-1}}{\rightleftarrows}} U_0 \underset{k_1}{\overset{\partial_0}{\rightleftarrows}} U_1 \underset{k_2}{\overset{\partial_1}{\rightleftarrows}} U_2 \underset{k_3}{\overset{\partial_2}{\rightleftarrows}} \cdots$$

and a strong relatively injective A-resolution of an A-module U

$$0 \to U \underset{t_0}{\overset{\partial'_{-1}}{\rightleftarrows}} V_0 \underset{t_1}{\overset{\partial'_0}{\rightleftarrows}} V_1 \underset{t_2}{\overset{\partial'_1}{\rightleftarrows}} V_2 \underset{t_3}{\overset{\partial'_2}{\rightleftarrows}} \cdots$$

are called an allowable pair of resolutions for $(G, A; U)$ if id_U can be extended to an A-morphism of resolutions $\lambda^n\colon U_n \to V_n$ such that λ^n commutes with the contracting homotopies k_n and t_n for all $n \geq 0$.

Proposition 2.1. *Let $\varphi\colon A \to G$ be a group homomorphism. The standard G- and A- resolutions of the trivial G- and A-module $\mathbf{R}$ are an allowable pair of resolutions for $(G, A; \mathbf{R})$.*

Proof. Recall that the standard G- and A-resolutions

$$0 \to \mathbf{R} \underset{k_0}{\overset{d_{-1}}{\rightleftarrows}} B(G) \underset{k_1}{\overset{d_0}{\rightleftarrows}} B(G^2) \underset{k_2}{\overset{d_1}{\rightleftarrows}} B(G^3) \underset{k_3}{\overset{d_2}{\rightleftarrows}} \cdots,$$

$$0 \to \mathbf{R} \underset{t_0}{\overset{\bar{d}_{-1}}{\rightleftarrows}} B(A) \underset{t_1}{\overset{\bar{d}_0}{\rightleftarrows}} B(A^2) \underset{t_2}{\overset{\bar{d}_1}{\rightleftarrows}} B(A^3) \underset{t_3}{\overset{\bar{d}_2}{\rightleftarrows}} \cdots$$

of the trivial G- and A-module $\mathbf{R}$ are strong relatively injective. Also recall that the contracting homotopy $k_n\colon B(G^{n+1}) \to B(G^n)$ is defined by the formula

$$k_n(f)(g_1, \cdots, g_n) = f(g_1, \cdots, g_n, 1)$$

and that G acts on $B(G^n)$ by $g \cdot f(g_1, \cdots, g_n) = f(g_1, \cdots, g_n g)$.

It suffices to show that there is an A-morphism of the standard G-resolution to the standard A-resolution

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbf{R} & \xrightarrow{d_{-1}} & B(G) & \xrightarrow{d_0} & B(G^2) & \xrightarrow{d_1} & B(G^3) & \xrightarrow{d_2} & \cdots \\
\downarrow & & \downarrow{\scriptstyle id_{\mathbf{R}}} & & \downarrow{\scriptstyle p^0} & & \downarrow{\scriptstyle p^1} & & \downarrow{\scriptstyle p^2} & & \\
0 & \longrightarrow & \mathbf{R} & \xrightarrow{\bar{d}_{-1}} & B(A) & \xrightarrow{\bar{d}_0} & B(A^2) & \xrightarrow{\bar{d}_1} & B(A^3) & \xrightarrow{\bar{d}_2} & \cdots
\end{array}$$

extending $id_{\mathbf{R}}$ and such that p^n commutes with the contracting homotopies k_n and t_n for every $n \geq 0$.

We define p^n by the formula

$$p^n(f)(a_1, \cdots, a_{n+1}) = f(\varphi(a_1), \cdots, \varphi(a_{n+1}))$$

for $f \in B(G^{n+1})$ and $(a_1, \cdots a_{n+1}) \in A^{n+1}$. It is easy to check that p^n commutes with the boundary operators and has the norm $\|p^n\| \leq 1$.

Note that, for $a \in A$ and $f \in B(G^{n+1})$, we have

$$\begin{aligned}
&p^n(a \cdot f)(a_1, \cdots, a_{n+1}) \\
&= p^n(\varphi(a) \cdot f)(a_1, \cdots, a_{n+1}) = (\varphi(a) \cdot f)(\varphi(a_1), \cdots, \varphi(a_{n+1})) \\
&= f(\varphi(a_1), \cdots, \varphi(a_{n+1})\varphi(a)) = f(\varphi(a_1), \cdots, \varphi(a_{n+1}a)) \\
&= p^n f(a_1, \cdots, a_{n+1}a) = a \cdot p^n f(a_1, \cdots, a_{n+1})
\end{aligned}$$

and so the map p^n commutes with the action of A. Thus p^n is an A-morphism.

Now, for every $f \in B(G)$ and $r \in \mathbf{R}$, we have

$$(id_{\mathbf{R}} k_0 - t_0 p^0)(f)(r) = id_{\mathbf{R}} k_0(f)(r) - t_0 p^0(f)(r) = f(1) - f(1) = 0.$$

Also, by noting that $\varphi(1) = 1$, we have for every $n \geq 0$

$$\begin{aligned}
&(p^{n-1}k_n - t_n p^n)(f)(a_1, \cdots, a_n) \\
&= p^{n-1}k_n(f)(a_1, \cdots, a_n) - t_n p^n(f)(a_1, \cdots, a_n) \\
&= k_n(f)(\varphi(a_1), \cdots, \varphi(a_n)) - p^n(f)(a_1, \cdots, a_n, 1) \\
&= f(\varphi(a_1), \cdots, \varphi(a_n), 1) - f(\varphi(a_1), \cdots, \varphi(a_n), \varphi(1)) \\
&= 0.
\end{aligned}$$

Thus the A-morphism p^n commutes with the contracting homotopies k_n and t_n. This finishes the proof. □

Definition 2.2. *Let $\varphi\colon A \to G$ be a group homomorphism. Let*

$$0 \to \mathbf{R} \to U_0 \to U_1 \to \cdots \quad \text{and} \quad 0 \to \mathbf{R} \to V_0 \to V_1 \to \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$. For every $n \geq 0$, the mapping cone $M^n(A \xrightarrow{\varphi} G)$ and the mapping cylinder $EM^n(A \xrightarrow{\varphi} G)$ of the cochain complexes induced by φ are defined as follows:

$$M^n(A \xrightarrow{\varphi} G) = U_n^G \bigoplus V_{n-1}^A$$
$$EM^n(A \xrightarrow{\varphi} G) = V_n^A \bigoplus U_n^G \bigoplus V_{n-1}^A,$$

for every $n \geq 0$ and where $V_{-1}^A = 0$.

Lemma 2.2. *Let $\varphi\colon A \to G$ be a group homomorphism. Let*

$$0 \to \mathbf{R} \xrightarrow{\partial_{-1}} U_0 \xrightarrow{\partial_0} U_1 \xrightarrow{\partial_1} \cdots \quad \text{and} \quad 0 \to \mathbf{R} \xrightarrow{\partial'_{-1}} V_0 \xrightarrow{\partial'_0} V_1 \xrightarrow{\partial'_1} \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$, and let $\lambda^n\colon U_n \to V_n$ be an A-morphism of resolutions commuting with the contracting homotopies. Then the sequences

$$0 \to M^0(A \xrightarrow{\varphi} G) \xrightarrow{d_0} M^1(A \xrightarrow{\varphi} G) \xrightarrow{d_1} M^2(A \xrightarrow{\varphi} G) \xrightarrow{d_2} \cdots \tag{2.2.1}$$

$$0 \to EM^0(A \xrightarrow{\varphi} G) \xrightarrow{d_0} EM^1(A \xrightarrow{\varphi} G) \xrightarrow{d_1} EM^2(A \xrightarrow{\varphi} G) \xrightarrow{d_2} \cdots \tag{2.2.2}$$

are complexes, where the boundary operators d_n are defined by the formulas

$$d_n(u_n, v_{n-1}) = (\partial_n u_n,\ -\lambda^n u_n - \partial'_{n-1} v_{n-1}) \qquad \text{on } M^n(A \xrightarrow{\varphi} G)$$

$$d_n(v_n, u_n, v_{n-1}) = (\partial'_n v_n,\ \partial_n u_n,\ v_n - \lambda^n u_n - \partial'_{n-1} v_{n-1}) \qquad \text{on } EM^n(A \xrightarrow{\varphi} G).$$

Proof. We check $d_{n+1}d_n = 0$ for $EM^n(A \xrightarrow{\varphi} G)$.

$$\begin{aligned}
&d_{n+1}d_n(v_n,\, u_n,\, v_{n-1}) \\
&= d_{n+1}(\partial'_n v_n,\, \partial_n u_n,\, v_n - \lambda^n u_n - \partial'_{n-1}v_{n-1}) \\
&= (\partial'_{n+1}\partial'_n v_n,\, \partial_{n+1}\partial_n u_n,\, \partial'_n v_n - \lambda^{n+1}(\partial_n u_n) - \partial'_n(v_n - \lambda^n u_n - \partial'_{n-1}v_{n-1})) \\
&= (0,\, 0,\, \partial'_n v_n - \lambda^{n+1}\partial_n u_n - \partial'_n v_n + \partial'_n\lambda^n u_n + \partial'_n\partial'_{n-1}v_{n-1}) \\
&= (0,\, 0,\, 0).
\end{aligned}$$

By the same way, it is easy to check $d_{n+1}d_n = 0$ for $M^n(A \xrightarrow{\varphi} G)$. □

Definition 2.3. *The n-th cohomology of the complex in (2.2.1) is called the n-th relative bounded cohomology of G modulo A and is denoted by $\widehat{H}^n(A \xrightarrow{\varphi} G)$. Also the n-th cohomology of the complex in (2.2.2) is denoted by $\widehat{H}^n(EM(A \xrightarrow{\varphi} G))$.*

We define the norm $\|\cdot\|$ on $EM^n(A \xrightarrow{\varphi} G) = V_n^A \bigoplus U_n^G \bigoplus V_{n-1}^A$ by setting

$$\|(v_n, u_n, v_{n-1})\| = \max\{\|v_n\|,\, \|u_n\|,\, \|v_{n-1}\|\},$$

and similarly on $M^n(A \xrightarrow{\varphi} G)$ by setting $\|(u_n, v_{n-1})\| = \max\{\|u_n\|,\, \|v_{n-1}\|\}$.

Remark that these norms define the seminorms $\|\cdot\|$ on $\widehat{H}^*(A \xrightarrow{\varphi} G)$ and $\widehat{H}^*(EM(A \xrightarrow{\varphi} G))$ respectively.

Furthermore, for every $\omega \geq 0$, we define a norm $\|\cdot\|(\omega)$ on $M^n(A \xrightarrow{\varphi} G)$ by putting

$$\|(u_n, v_{n-1})\|(\omega) = \max\{\|u_n\|,\, (1+\omega)^{-1}\|v_{n-1}\|\}.$$

Observe that all norms $\|\cdot\|(\omega)$ are equivalent to the norm $\|\cdot\| = \|\cdot\|(0)$. Now with this norm on $M^n(A \xrightarrow{\varphi} G)$ we have the corresponding norm $\|\cdot\|(\omega)$ on $\widehat{H}^*(A \xrightarrow{\varphi} G)$. Finally we define this norm $\|\cdot\|(\omega)$ on $\widehat{H}^*(A \xrightarrow{\varphi} G)$ for all ω in the closed interval $[0, \infty]$ by passing to the limits.

Proposition 2.3. *Let $\varphi\colon A \to G$ be a group homomorphism. Let*

$$0 \to \mathbf{R} \xrightarrow{\partial_{-1}} U_0 \xrightarrow{\partial_0} U_1 \xrightarrow{\partial_1} \cdots \quad \textit{and} \quad 0 \to \mathbf{R} \xrightarrow{\partial'_{-1}} V_0 \xrightarrow{\partial'_0} V_1 \xrightarrow{\partial'_1} \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$, and let $\lambda^n \colon U_n \to V_n$ be an A-morphism commuting with the contracting homotopies. Then the natural projection map $\rho^n \colon EM^n(A \xrightarrow{\varphi} G) \to U_n^G$ induces an isometric isomorphism

$$H^n(\rho) \colon \widehat{H}^n(EM(A \xrightarrow{\varphi} G)) \to \widehat{H}^n(G).$$

Proof. We consider the exact sequence

$$0 \to V_n^A \bigoplus V_{n-1}^A \hookrightarrow EM^n(A \xrightarrow{\varphi} G) = V_n^A \bigoplus U_n^G \bigoplus V_{n-1}^A \xrightarrow{\rho^n} U_n^G \to 0.$$

It is easy to check $V_n^A \bigoplus V_{n-1}^A$ is a complex. If (v_n, v_{n-1}) is a cocycle of the complex $V_n^A \bigoplus V_{n-1}^A$, then we have $0 = d(v_n, v_{n-1}) = (\partial'_n v_n, v_n - \partial'_{n-1} v_{n-1})$ and so $v_n = \partial'_{n-1} v_{n-1}$. Thus $(v_n, v_{n-1}) = (\partial'_{n-1} v_{n-1}, v_{n-1}) = d_{n-1}(v_{n-1}, 0)$ and thus (v_n, v_{n-1}) is a coboundary. This shows that the cohomology of the complex $V_n^A \bigoplus V_{n-1}^A$ vanishes so that the map $H^n(\rho)$ is an isomorphism.

For $(v_n, u_n, v_{n-1}) \in EM^n(A \xrightarrow{\varphi} G) = V_n^A \bigoplus U_n^G \bigoplus V_{n-1}^A$, we have

$$\|\rho^n(v_n, u_n, v_{n-1})\| = \|u_n\| \leq \max\{\|v_n\|, \|u_n\|, \|v_{n-1}\|\} = \|(v_n, u_n, v_{n-1})\|.$$

This shows that $\|\rho^n\| \leq 1$ and so $\|H^n(\rho)\| \leq 1$.

On the other hand, we define a map $\widetilde{\rho^n} \colon U_n^G \to V_n^A \bigoplus U_n^G \bigoplus V_{n-1}^A$ by the equation $\widetilde{\rho^n}(u_n) = (\lambda^n u_n,\ u_n,\ 0)$. Then we have

$$d_n \widetilde{\rho^n} u_n = (\partial'_n \lambda^n u_n, \partial_n u_n, 0) = (\lambda^{n+1} \partial_n u_n, \partial_n u_n, 0) = \widetilde{\rho}^{n+1} \partial_n u_n$$

and so the map $\widetilde{\rho^n}$ commutes with the boundary operators. It is clear that $\rho^n \widetilde{\rho^n} = id$. Note that, for every $u_n \in U_n^G$, we have

$$\|\widetilde{\rho^n}(u_n)\| = \|(\lambda^n u_n, u_n, 0)\| = \max\{\|\lambda^n u_n\|, \|u_n\|, 0\} \leq \|u_n\|$$

and so $\|\widetilde{\rho^n}\| \leq 1$. This shows that the inverse map $(H^n(\rho))^{-1}$ of $H^n(\rho)$ also has the norm $\|(H^n(\rho))^{-1}\| \leq 1$. Hence the isomorphism $H^n(\rho)$ is an isometry. □

Let $\varphi\colon A \to G$ be a group homomorphism, and let

$$0 \to \mathbf{R} \xrightarrow{\partial_{-1}} U_0 \xrightarrow{\partial_0} U_1 \xrightarrow{\partial_1} \cdots \quad \text{and} \quad 0 \to \mathbf{R} \xrightarrow{\partial'_{-1}} V_0 \xrightarrow{\partial'_0} V_1 \xrightarrow{\partial'_1} \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$.

Remark that there is an exact sequence of complexes

$$0 \to M^n(A \xrightarrow{\varphi} G) \xrightarrow{i^n} EM^n(A \xrightarrow{\varphi} G) \xrightarrow{p^n} V_n^A \to 0, \tag{2.1}$$

where i^n and p^n are natural inclusion and projection maps respectively. This exact sequence in (2.1) induces a long exact sequence

$$\cdots \to \widehat{H}^{n-1}(A) \to \widehat{H}^n(A \xrightarrow{\varphi} G) \to \widehat{H}^n(G) \to \widehat{H}^n(A) \to \cdots. \tag{2.2}$$

Recall that the canonical seminorm on $\widehat{H}^*(G)$ is defined as the infimum of the seminorms which arise from all strong relatively injective G-resolutions of the trivial G-module $\mathbf{R}$.

Theorem 2.4. *The canonical seminorm on $\widehat{H}^*(G)$ is induced by the standard G-resolution.*

Proof. Let

$$0 \to \mathbf{R} \underset{k_0}{\overset{\partial_{-1}}{\rightleftarrows}} U_0 \underset{k_1}{\overset{\partial_0}{\rightleftarrows}} U_1 \underset{k_2}{\overset{\partial_1}{\rightleftarrows}} U_2 \underset{k_3}{\overset{\partial_2}{\rightleftarrows}} \cdots$$

be a strong relatively injective G-resolutions of the trivial G-module $\mathbf{R}$.

From Theorem 3.6 in [8], it is proved that there is a morphism α_n from this resolution to the standard resolution

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbf{R} & \longrightarrow & U_0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow{\scriptstyle id_{\mathbf{R}}} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \\ 0 & \longrightarrow & \mathbf{R} & \longrightarrow & B(G) & \longrightarrow & B(G^2) & \longrightarrow & B(G^3) & \longrightarrow & \cdots \end{array}$$

extending $id_{\mathbf{R}}$ and such that $\|\alpha_n\| \le 1$ for every $n \ge 0$, where the morphism α_n is defined by the formula

$$\alpha_n(f)(g_1, \cdots, g_n, g_{n+1}) = k_0(g_1 \cdot k_1(\cdots (k_{n-1}(g_n \cdot k_n(g_{n+1} \cdot f))) \cdots)).$$

This finishes the proof. □

Corollary 2.5. *The seminorm on $\widehat{H}^*(EM(A \overset{\varphi}{\to} G))$, induced by the standard resolutions, coincides with the canonical seminorm on $\widehat{H}^*(G)$.*

Proof. Remark that the standard G- and A- resolutions define the complex

$$EM^n(A \overset{\varphi}{\to} G) = B(A^{n+1})^A \bigoplus B(G^{n+1})^G \bigoplus B(A^n)^A$$

and the cohomology of which is $\widehat{H}^*(EM(A \overset{\varphi}{\to} G))$. Hence it follows from Proposition 2.3 and Theorem 2.4. □

Remark that, for a group homomorphism $\varphi\colon A \to G$, a seminorm on $\widehat{H}^*(A \overset{\varphi}{\to} G)$ depends on the choice of an allowable pair of resolutions for $(G, A; \mathbf{R})$.

Definition 2.4. *We define the canonical seminorm $\|\cdot\|(\omega)$ for every $\omega \in [0, \infty]$ on $\widehat{H}^*(A \overset{\varphi}{\to} G)$ as the infimum of the seminorms which arise from every allowable pair of resolutions for $(G, A; \mathbf{R})$.*

As in $\widehat{H}^*(G)$ and $\widehat{H}^*(EM(A \overset{\varphi}{\to} G))$, we shall see the canonical seminorm on $\widehat{H}^*(A \overset{\varphi}{\to} G)$ is also induced by the standard resolutions.

Lemma 2.6. *Let $\varphi\colon A \to G$ be a group homomorphism. Let*

$$0 \to \mathbf{R} \underset{k_0}{\overset{\partial_{-1}}{\rightleftarrows}} U_0 \underset{k_1}{\overset{\partial_0}{\rightleftarrows}} U_1 \underset{k_2}{\overset{\partial_1}{\rightleftarrows}} \cdots \quad \text{and} \quad 0 \to \mathbf{R} \underset{t_0}{\overset{\partial'_{-1}}{\rightleftarrows}} V_0 \underset{t_1}{\overset{\partial'_0}{\rightleftarrows}} V_1 \underset{t_2}{\overset{\partial'_1}{\rightleftarrows}} \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$, and let $\lambda^n\colon U_n \to V_n$ be an A-morphism of resolutions commuting with the contracting homotopies. Then there is a commutative diagram

$$\begin{array}{ccc} U_n^G & \xrightarrow{\lambda^n} & V_n^A \\ \downarrow & & \downarrow \\ B(G^{n+1})^G & \xrightarrow{p^n} & B(A^{n+1})^A, \end{array}$$

where p^n is defined by $p^n f(a_1, \cdots, a_{n+1}) = f(\varphi(a_1), \cdots, \varphi(a_{n+1}))$.

Proof. In the proof of Theorem 2.4, there is a cochain map $\alpha_n\colon U_n^G \to B(G^{n+1})^G$ defined by the formula

$$\alpha_n(f)(g_1, \cdots, g_n, g_{n+1}) = k_0(g_1 \cdot k_1(\cdots(k_{n-1}(g_n \cdot k_n(g_{n+1} \cdot f)))\cdots)).$$

Similarly, we define a cochain map $\gamma_n\colon V_n^G \to B(A^{n+1})^A$. It is clear that they have the norms $\|\alpha_n\| \le 1$ and $\|\gamma_n\| \le 1$.

We prove that $\gamma_n\lambda^n = p^n\alpha_n$. Let $f \in U_n^G$ and $(a_1, \cdots, a_n, a_{n+1}) \in A^{n+1}$. Then

$$\begin{aligned}
&\gamma_n\lambda^n(f)(a_1, \cdots, a_n, a_{n+1}) \\
&= t_0(a_1 \cdot t_1(\cdots(t_{n-1}(a_n \cdot t_n(a_{n+1} \cdot \lambda^n(f)))))\cdots) \\
&= t_0(a_1 \cdot t_1(\cdots(t_{n-1}(a_n \cdot t_n(\lambda^n(\varphi(a_{n+1}) \cdot f))))\cdots) \\
&= t_0(a_1 \cdot t_1(\cdots(t_{n-1}(a_n \cdot \lambda^{n-1}k_n(\varphi(a_{n+1}) \cdot f))))\cdots) \\
&= t_0(a_1 \cdot t_1(\cdots(t_{n-1}\lambda^{n-1}(\varphi(a_n) \cdot k_n(\varphi(a_{n+1}) \cdot f))))\cdots) \\
&= t_0(a_1 \cdot t_1(\cdots(\lambda^{n-2}k_{n-1}(\varphi(a_n) \cdot k_n(\varphi(a_{n+1}) \cdot f))))\cdots) \\
&= \cdots \\
&= id_{\mathbf{R}}k_0(\varphi(a_1) \cdot k_1(\cdots(k_{n-1}(\varphi(a_n) \cdot k_n(\varphi(a_{n+1}) \cdot f))))\cdots)
\end{aligned}$$

and also

$$\begin{aligned}
&p^n\alpha_n(f)(a_1, \cdots, a_n, a_{n+1}) \\
&= \alpha_n(f)(\varphi(a_1), \cdots, \varphi(a_n), \varphi(a_{n+1})) \\
&= k_0(\varphi(a_1) \cdot k_1(\cdots(k_{n-1}(\varphi(a_n) \cdot k_n(\varphi(a_{n+1}) \cdot f)))\cdots).
\end{aligned}$$

Thus $p^n\alpha_n = \gamma_n\lambda^n$ for every $n \ge 0$. □

Theorem 2.7. *The seminorm $\|\cdot\|(\omega)$ on $\widehat{H}^*(A \overset{\varphi}{\to} G)$, induced by the standard G- and A-resolutions, coincides with the canonical seminorm for every $\omega \in [0, \infty]$.*

Proof. Let

$$0 \to \mathbf{R} \to U_0 \to U_1 \to \cdots \quad \text{and} \quad 0 \to \mathbf{R} \to V_0 \to V_1 \to \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$, and let $\lambda^n\colon U_n \to V_n$ be an A-morphism of resolutions commuting with the contracting homotopies.

Note that we have two complexes $U_n^G \bigoplus V_{n-1}^A$ and $B(G^{n+1})^G \bigoplus B(A^n)^A$ which are induced from the resolutions above and the standard resolutions respectively. It is enough for us to prove that there is a cochain map

$$\beta_n\colon U_n^G \bigoplus V_{n-1}^A \to B(G^{n+1})^G \bigoplus B(A^n)^A$$

such that it has the norm $\|\beta_n\| \le 1$.

For simplicity, we denote all boundary operators by the same notation d.

Recall that, from Lemma 2.6, there is a commutative diagram

$$\begin{array}{ccc} U_n^G & \xrightarrow{\lambda^n} & V_n^A \\ {\scriptstyle \alpha_n}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_n} \\ B(G^{n+1})^G & \xrightarrow{p^n} & B(A^{n+1})^A \end{array}$$

in which the maps have the norms $\|\alpha_n\| \le 1$ and $\|\gamma_n\| \le 1$.

We define β_n by $\beta_n(u_n, v_{n-1}) = (\alpha_n u_n,\ \gamma_{n-1} v_{n-1})$. Then we have

$$\begin{aligned} &\beta_{n+1} d_n(u_n,\ v_{n-1}) \\ &= \beta_{n+1}(d_n u_n,\ -\lambda^n u_n - d_{n-1} v_{n-1}) = (\alpha_{n+1} d_n u_n,\ -\gamma_n \lambda^n u_n - \gamma_n d_{n-1} v_{n-1}) \\ &= (d_n \alpha_n u_n,\ -p^n \alpha_n u_n - d_{n-1}\gamma_{n-1} v_{n-1}) = d_n(\alpha_n u_n,\ \gamma_{n-1} v_{n-1}) \\ &= d_n \beta_n(u_n,\ v_{n-1}) \end{aligned}$$

and so β_n commutes with the boundary operators.

Now let $\omega \ge 0$. Then we have

$$\begin{aligned} &\|\beta_n(u_n,\ v_{n-1})\|(\omega) \\ &= \|(\alpha_n u_n,\ \gamma_{n-1} v_{n-1})\|(\omega) = \max\{\|\alpha_n u_n\|,\ (1+\omega)^{-1}\|\gamma_{n-1} v_{n-1}\|\} \\ &\le \max\{\|u_n\|,\ (1+\omega)^{-1}\|v_{n-1}\|\} = \|(u_n,\ v_{n-1})\|(\omega) \end{aligned}$$

and so the map β_n has the norm $\|\beta_n\| \le 1$ for the norm $\|\cdot\|(\omega)$. □

Corollary 2.8. *The seminorm $\|\cdot\|$ on $\widehat{H}^*(A \xrightarrow{\varphi} G)$, induced by the standard G- and A-resolutions, coincides with the canonical seminorm.*

Proof. This follows from Theorem 2.7 by setting up $\omega = 0$. □

One of the important examples of $\widehat{H}^*(A \xrightarrow{\varphi} G)$ arises from a subgroup A of G. Recall that $B(S)$, the set of all bounded functions on a set S, is a Banach space with the norm $\|f\| = \sup\{|f(s)| \mid s \in S\}$.

In the following, we introduce another strong relatively injective G-resolution which also provides for the canonical seminorm in $\widehat{H}^*(G)$.

Let A be a subgroup of G. Then G/A, the set of all (right) cosets Ag of A in G, has a right G-action given by $Ag \cdot g' = Agg'$. We note that the space $B((G/A)^n)$ is a G-module with the action:

$$g' \cdot f(Ag_1, \cdots, Ag_n) = f(Ag_1, \cdots, Ag_n g').$$

The canonical map $i^n\colon B((G/A)^n) \to B(G^n)$ is a G-morphism and it has the norm $\|i^n\| = 1$.

Lemma 2.9. *Let A be an amenable subgroup of G. Then there is a G-morphism $\pi^n\colon B(G^n) \to B((G/A)^n)$ such that $\pi^n i^n = id$ and $\|\pi^n\| \leq 1$.*

Proof. From Lemma 3.8.1 in [8], it is proved that there exists a G-morphism $\pi\colon B(G) \to B(G/A)$ such that $\pi \circ i = id$ and $\|\pi\| \leq 1$. In fact, π is defined by the formula $\pi(f)(Ag) = m_g(f|Ag)$, where m_g is a mean on $B(Ag)$ induced from a right invariant mean on $B(A)$. Also from Corollary 3.8.2 in [8], it is proved that this G-morphism π provides a G-morphism $\pi^n\colon B(G^n) \to B((G/A)^n)$ such that $\pi^n \circ i^n = id$ and $\|\pi^n\| \leq 1$ for every $n \geq 1$. □

Proposition 2.10. *Let A be an amenable subgroup of G. Then the sequence*

$$0 \to \mathbf{R} \underset{k_0}{\overset{d_{-1}}{\rightleftarrows}} B(G/A) \underset{k_1}{\overset{d_0}{\rightleftarrows}} B((G/A)^2) \underset{k_2}{\overset{d_1}{\rightleftarrows}} B((G/A)^3) \underset{k_3}{\overset{d_2}{\rightleftarrows}} \cdots \tag{2.10.1}$$

is a strong relatively injective G-resolution of the trivial G-module $\mathbf{R}$, where the boundary operators d_n and the contracting homotopy k_n are defined as the same ways with the standard resolution.

Proof. It is easy to check that the sequence in (2.10.1) is a strong G-resolution.

By using the G-morphism $\pi^n\colon B(G^n) \to B((G/A)^n)$ in Lemma 2.9 and the fact that $B(G^n)$ is the relatively injective G-module, it is proved that $B((G/A)^n)$ is the relatively injective G-module from Lemma 3.8.3 in [8]. □

Note that the resolution in (2.10.1) induces the complex

$$0 \to B(G/A)^G \to B((G/A)^2)^G \to B((G/A)^3)^G \to \cdots \tag{2.3}$$

and the cohomology of which is $\widehat{H}^*(G)$.

Corollary 2.11. *Let A be an amenable subgroup of G. Then the seminorm on $\widehat{H}^*(G)$, induced by the complex in (2.3), coincides with the canonical seminorm.*

Proof. In $\widehat{H}^*(G)$, let $\|\cdot\|_c$ denote the canonical seminorm induced by the standard G-resolution and let $\|\cdot\|$ denote the seminorm induced by the complex in (2.3).

Recall that there is the canonical G-morphism $i^n\colon B((G/A)^n) \to B(G^n)$ such that it has the norm $\|i^n\| = 1$. This shows that $\|\cdot\|_c \leq \|\cdot\|$.

Also, from Lemma 2.9, there is a G-morphism $\pi^n\colon B(G^n) \to B((G/A)^n)$ such that $\pi^n \circ i^n = id$ and $\|\pi^n\| \leq 1$. This shows that $\|\cdot\| \leq \|\cdot\|_c$. □

Note that A/A, the set of cosets of A in A, consists of only one element which we will denote by $\{A\}$. Hence $B(\{A\}^n)$ consists of all bounded functions on one element $\{A\}^n$ and so it is isomorphic with $\mathbf{R}$.

If A is an amenable group, then there is a strong relatively injective A-resolution of the trivial A-module $\mathbf{R}$

$$0 \to \mathbf{R} \xrightarrow{\bar{d}_{-1}} B(\{A\}) \xrightarrow{\bar{d}_0} B(\{A\}^2) \xrightarrow{\bar{d}_1} B(\{A\}^3) \xrightarrow{\bar{d}_2} B(\{A\}^4) \xrightarrow{\bar{d}_3} \cdots .$$

This resolution induces the complex

$$0 \to B(\{A\})^A \xrightarrow{\bar{d}_0} B(\{A\}^2)^A \xrightarrow{\bar{d}_1} B(\{A\}^3)^A \xrightarrow{\bar{d}_2} B(\{A\}^4)^A \xrightarrow{\bar{d}_3} \cdots \tag{2.4}$$

and the cohomology of which is $\widehat{H}^*(A)$.

From definition of the boundary operators, $\bar{d}_n(f)(\underbrace{\{A\},\cdots,\{A\}}_{n+2})$ is the $n+2$ alternating sum of $f(\underbrace{\{A\},\cdots,\{A\}}_{n+1})$ and so

$$\bar{d}_n = \begin{cases} id & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Note that this gives another proof of the bounded cohomology of an amenable group is zero.

Corollary 2.12. *Let A be an amenable subgroup of G. Then the sequences*

$$0 \to \mathbf{R} \to B(G/A) \to B((G/A)^2) \to B((G/A)^3) \to \cdots \quad \textit{and}$$
$$0 \to \mathbf{R} \to B(\{A\}) \to B(\{A\}^2) \to B(\{A\}^3) \to \cdots$$

are an allowable pair of resolutions for $(G, A; \mathbf{R})$.

Proof. We define a map $p^n\colon B((G/A)^{n+1}) \to B(\{A\}^{n+1})$ by the formula $p^n(f)(\underbrace{\{A\},\cdots,\{A\}}_{n+1}) = f(\underbrace{A,\cdots,A}_{n+1})$.

It is clear that p^n is an A-morphism of resolutions extending $id_{\mathbf{R}}$. Also, as the same way we proved in the standard resolutions, it is easy to check that p^n commutes with the contracting homotopies. □

Proposition 2.13. *Let A be an amenable subgroup of G. Let $\varphi\colon A \hookrightarrow G$ be a natural inclusion map. The seminorm $\|\cdot\|(\omega)$ in $\widehat{H}^*(A \xrightarrow{\varphi} G)$, induced by the complex $B((G/A)^{n+1})^G \bigoplus B(\{A\}^n)^A$, coincides with the canonical seminorm for every $\omega \in [0,\infty]$.*

Proof. For all $\omega \geq 0$, let $\|\cdot\|_c(\omega)$ denote the canonical seminorm in $\widehat{H}^*(A \overset{\varphi}{\to} G)$ and let $\|\cdot\|(\omega)$ denote the seminorm in $\widehat{H}^*(A \overset{\varphi}{\to} G)$ induced by the complex $B((G/A)^{n+1})^G \bigoplus B(\{A\}^n)^A$. By definition of the canonical seminorm, we have $\|\cdot\|_c(\omega) \leq \|\cdot\|(\omega)$. So, it suffices for us to show that $\|\cdot\|(\omega) \leq \|\cdot\|_c(\omega)$.

From Theorem 2.7, the canonical seminorm in $\widehat{H}^*(A \overset{\varphi}{\to} G)$ is induced by the complex $B(G^{n+1})^G \bigoplus B(A^n)^A$. Recall that there are canonical chain maps

$$\alpha^n\colon B((G/A)^{n+1})^G \to B(G^{n+1})^G \quad \text{and} \quad \gamma^n\colon B(\{A\}^{n+1})^A \to B(A^{n+1})^A$$

such that $\|\alpha^n\| = 1$ and $\|\gamma^n\| = 1$. Also, from Lemma 2.9, there are chain maps

$$\pi^n\colon B(G^{n+1})^G \to B((G/A)^{n+1})^G \quad \text{and} \quad \lambda^n\colon B(A^{n+1})^A \to B(\{A\}^{n+1})^A$$

such that $\pi^n\alpha^n = id$ and $\lambda^n\gamma^n = id$ and they have the norms $\|\pi^n\| \leq 1$ and $\|\lambda^n\| \leq 1$. We consider the following diagram

$$\begin{array}{ccc} B(G^{n+1})^G & \xrightarrow{q^n} & B(A^{n+1})^A \\ {\scriptstyle \pi^n}\big\downarrow & & \big\downarrow{\scriptstyle \lambda^n} \\ B((G/A)^{n+1})^G & \xrightarrow{p^n} & B(\{A\}^{n+1})^A \end{array}$$

in which q^n and p^n are defined as the restriction maps so that they have the norms $\|q^n\| \leq 1$ and $\|p^n\| \leq 1$. We prove that this diagram is commutative. Note that

$$\begin{aligned} \lambda^n q^n f(\{A\}, \cdots, \{A\}) &= m(q^n f) = m(f|_{A^{n+1}}) \\ &= \pi^n f(A, \cdots, A) \\ &= p^n \pi^n f(\{A\}, \cdots, \{A\}), \end{aligned}$$

where m is a mean on $B(A^{n+1})$. Thus we have $\lambda^n q^n = p^n \pi^n$.

Now we define a map

$$\beta^n\colon B(G^{n+1})^G \bigoplus B(A^n)^A \to B((G/A)^{n+1})^G \bigoplus B(\{A\}^n)^A$$

by the formula $\beta^n(f, f') = (\pi^n f, \lambda^{n-1} f')$. Then we have

$$\begin{aligned} d\beta^n(f, f') &= d(\pi^n f,\ \lambda^{n-1} f') = (d\pi^n f,\ -p^n \pi^n f - d\lambda^{n-1} f') \\ &= (\pi^{n+1} df,\ -\lambda^n q^n f - \lambda^n df') \\ &= \beta^{n+1}(df,\ -q^n f - df') = \beta^{n+1} d(f, f') \end{aligned}$$

and so β^n commutes with the boundary operators. Finally, note that

$$\begin{aligned} \|\beta^n(f, f')\|(\omega) &= \max\{\|\pi^n f\|,\ (1+\omega)^{-1}\|\lambda^n f'\|\} \\ &\leq \max\{\|f\|,\ (1+\omega)^{-1}\|f'\|\} = \|(f, f')\|(\omega) \end{aligned}$$

and so the map β^n has the norm $\|\beta^n\| \leq 1$ for the norm $\|\cdot\|(\omega)$. This shows that $\|\cdot\|(\omega) \leq \|\cdot\|_c(\omega)$. Thus we have $\|\cdot\|(\omega) = \|\cdot\|_c(\omega)$ on $\widehat{H}^*(A \overset{\varphi}{\to} G)$ for every $\omega \in [0, \infty]$ by passing to the limits. □

Notation: We always distinguish a (co)homology class from a (co)chain by using brackets: for example, $[f]$ stands for a (co)homology class while f stands for a (co)cycle.

Theorem 2.14. *Let A be an amenable subgroup of G, and let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then, for every $n \geq 2$, the induced homomorphism $H^n(i)\colon \widehat{H}^n(A \overset{\varphi}{\to} G) \to \widehat{H}^n(G)$ is an isometric isomorphism for the norm $\|\cdot\|$, i.e., $H^n(i)$ preserves the canonical seminorms.*

Proof. By Proposition 2.3 and Proposition 2.13, it is enough for us to consider the complexes in the sequences (2.3) and (2.4). We define the complexes

$$\begin{aligned} M^n(A \overset{\varphi}{\to} G) &= B\big((G/A)^{n+1}\big)^G \bigoplus B(\{A\}^n)^A \\ EM^n(A \overset{\varphi}{\to} G) &= B(\{A\}^{n+1})^A \bigoplus B\big((G/A)^{n+1}\big)^G \bigoplus B(\{A\}^n)^A. \end{aligned}$$

Then the exact sequence

$$0 \to M^n(A \overset{\varphi}{\to} G) \overset{i}{\to} EM^n(A \overset{\varphi}{\to} G) \to B(\{A\}^{n+1})^A \to 0$$

induces a long exact sequence

$$\cdots \to \widehat{H}^{n-1}(A) \to \widehat{H}^n(A \overset{\varphi}{\to} G) \xrightarrow{H^n(i)} \widehat{H}^n(G) \to \widehat{H}^n(A) \to \cdots .$$

Since A is amenable, the map $H^*(i)$ is an isomorphism. Also it is clear that the map $H^*(i)$ has the norm $\|H^*(i)\| \leq 1$. We denote by ∂_* and ∂'_* the boundary operators on $B\big((G/A)^*\big)^G$ and $B(\{A\}^*)^A$ respectively.

Let $(f'', f, f') \in B(\{A\}^{n+1})^A \bigoplus B\big((G/A)^{n+1}\big)^G \bigoplus B(\{A\}^n)^A$ be a cocycle. Then, by definition of the boundary operator, we have

$$\partial'_n f'' = 0, \quad \partial_n f = 0, \quad \text{and} \quad f'' - p_n f - \partial'_{n-1} f' = 0.$$

Let n be odd, so that $\partial'_n = id$. Then we have $f'' = 0$ and so

$$d(f, f') = (\partial_n f, -p_n f - \partial'_{n-1} f') = 0.$$

It is easy to check that $(H^n(i))^{-1}([f'', f, f'])$ is represented by a cocycle $(f, f') \in M^n(A \overset{\varphi}{\to} G)$ and $\|(H^n(i))^{-1}\| \leq 1$.

If n is even, then $\partial'_{n-1} = id$. So there is an element $f''_0 \in B(\{A\}^n)^A$ such that $\partial'_{n-1} f''_0 = f''$ and $\|f''\| = \|f''_0\|$. Then we have

$$(f'', f, f') - d(f''_0, 0, 0) = (0, f, f' - f''_0)$$

and also $d(f, f' - f''_0) = (\partial_n f, -p^n f - \partial'_{n-1} f' + \partial'_{n-1} f''_0) = (0, 0)$. Now it is easy to check that $(H^n(i))^{-1}([f'', f, f'])$ is represented by a cocycle $(f, f' - f''_0)$. Remark that

$$\|f' - f''_0\| = \|\partial'_{n-1}(f' - f''_0)\| = \|f'' - p_n f - \partial'_{n-1} f''_0\| = \|p_n f\| \leq \|f\|.$$

Thus we have

$$\begin{aligned} \|(H^n(i))^{-1}([f'', f, f'])\| &\leq \|(f, f' - f''_0)\| = \max\{\|f\|, \|f' - f''_0\|\} = \|f\| \\ &\leq \max\{\|f''\|, \|f\|, \|f'\|\} = \|(f'', f, f')\|. \end{aligned}$$

This shows that $\|(H^n(i))^{-1}\| \leq 1$ is also true for every even n.

Thus the isomorphism $H^n(i)$ is also an isometry. □

From Theorem 3.8.4 in [8], it is proved that, if A is an amenable normal subgroup of G, then the groups $\widehat{H}^*(G/A)$ and $\widehat{H}^*(G)$ are isometrically isomorphic. Hence, by Theorem 2.14, the groups $\widehat{H}^*(A \overset{\varphi}{\to} G)$ and $\widehat{H}^*(G/A)$ are also isometrically isomorphic.

Theorem 2.15. *Let A be an amenable subgroup of G and let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then the norms $\|\cdot\|(\omega)$ on the group $\widehat{H}^n(A \overset{\varphi}{\to} G)$ are equal for $n \geq 2$ and for every $\omega \in [0,\infty]$.*

Proof. Let $\omega > 0$. Since it is clear that $\|\cdot\|(\omega) \leq \|\cdot\| = \|\cdot\|(0)$ for every $\omega \in [0,\infty]$ from definition, we only show that $\|\cdot\|(\omega) \geq \|\cdot\|$.

By Proposition 2.3 and Proposition 2.13, it is enough for us to consider the complex $B\big((G/A)^{n+1}\big)^G \bigoplus B(\{A\}^n)^A$.

If (f, f') is a cocycle of the complex $B\big((G/A)^{n+1}\big)^G \bigoplus B(\{A\}^n)^A$, then we have $0 = d_n(f, f') = (\partial_n f, -p^n f - \partial'_{n-1} f')$, where $p^n\colon B\big((G/A)^{n+1}\big)^G \to B(\{A\}^n)^A$ is defined as a restriction map and so it has the norm $\|p^n\| \leq 1$. Also ∂ and ∂' are the boundary operators on $B((G/A)^*)^G$ and $B(\{A\}^*)^A$ respectively.

Recall that, since the group A is amenable, if $n-1$ is odd, then $\partial'_{n-1} = id$ and so $-p^n f - \partial'_{n-1} f' = -p^n f - f' = 0$. Thus $f' = -p^n f$. This shows that

$$\begin{aligned}\|(f,\, f')\| &= \max\{\|f\|,\, \|f'\|\} = \max\{\|f\|,\, \|p^n f\|\} = \|f\|\\ &= \max\{\|f\|,\, (1+\omega)^{-1}\|p^n f\|\} = \max\{\|f\|,\, (1+\omega)^{-1}\|f'\|\}\\ &= \|(f, f')\|(\omega)\end{aligned}$$

and so $\|[f,\, f']\| \leq \|[f,\, f']\|(\omega)$.

On the other hand, if $n-1$ is even, then $\partial'_{n-1} = 0$. So $\partial'_{n-1} f' = 0$ and so $f' \in \ker(\partial'_{n-1}) = \operatorname{Im}(\partial'_{n-2})$. Note that $\partial'_{n-2} = id$. Thus there is an element $f'' \in B(A^{n-1})^A$ such that $\partial'_{n-2} f'' = f'$ and $\|f''\| = \|f'\|$. Note that

$$(f,\, f') + d_{n-1}(0,\, f'') = (f,\, f') + (0,\, -\partial'_{n-2} f'') = (f, 0).$$

Thus we have $\|[f, f']\| = \|[f, 0]\| \leq \max\{\|f\|, 0\} = \|f\| = \|(f, 0)\|(\omega)$ and so $\|[f, f']\| \leq \|[f, 0]\|(\omega) = \|[f, f']\|(\omega)$.

By passing to the limits, we have $\|[f, f']\| \leq \|[f, f']\|(\omega)$ for every $\omega \in [0, \infty]$. This finishes the proof. □

Theorem 2.16. *Let $\varphi\colon A \to G$ and $\varphi'\colon A' \to G'$ be the group homomorphisms respectively. Let $\alpha\colon G \to G'$ and $\gamma\colon A \to A'$ be the surjective homomorphisms with the amenable kernels respectively and such that $\alpha \circ \varphi = \varphi' \circ \gamma$. Then the groups $\widehat{H}^*(A' \xrightarrow{\varphi'} G')$ and $\widehat{H}^*(A \xrightarrow{\varphi} G)$ are isometrically isomorphic for the norm $\|\cdot\|(\omega)$ for every $\omega \in [0, \infty]$. This isomorphism preserves the canonical seminorms.*

Proof. Denote $\ker(\alpha)$ and $\ker(\gamma)$ by K and N respectively. We identify the groups G' and A' with G/K and A/N respectively and denote the homomorphism $A/N \to G/K$ by ρ. Then we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & A/N \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\rho}\downarrow \\ G & \xrightarrow{\alpha} & G/K. \end{array}$$

Remark that ρ is defined by the formula $\rho(Na) = K\varphi(a)$.

It suffices for us to show that $\widehat{H}^*(A/N \xrightarrow{\rho} G/K)$ and $\widehat{H}^*(A \xrightarrow{\varphi} G)$ are isometrically isomorphic. Since K and N are amenable normal subgroups of G and A respectively, the groups $\widehat{H}^*(G)$ and $\widehat{H}^*(A)$ are isometrically isomorphic with $\widehat{H}^*(G/K)$ and $\widehat{H}^*(A/N)$ respectively. We consider the standard G/K- and A/N-resolutions of the trivial G/K- and A/N-module $\mathbf{R}$. Note that there is a diagram

$$\begin{array}{ccccc} B\big((G/K)^{n+1}\big)^G & \xrightarrow{i^n} & B(G^{n+1})^G & \xrightarrow{\pi^n} & B\big((G/K)^{n+1}\big)^G \\ {\scriptstyle q^n}\downarrow & & {\scriptstyle p^n}\downarrow & & {\scriptstyle q^n}\downarrow \\ B\big((A/N)^{n+1}\big)^A & \xrightarrow{j^n} & B(A^{n+1})^A & \xrightarrow{\lambda^n} & B\big((A/N)^{n+1}\big)^A \end{array}$$

where each row consists of the maps in Lemma 2.9 such that $\pi^n i^n = id$ and $\lambda^n j^n = id$ and also they have the norms $\|i^n\| = 1$, $\|j^n\| = 1$, $\|\pi^n\| \leq 1$, and $\|\lambda^n\| \leq 1$. The maps p^n and q^n are defined by the formulas

$$p^n f(a_1, \cdots, a_{n+1}) = f(\varphi(a_1), \cdots, \varphi(a_{n+1}))$$

$$q^n f'(Na_1, \cdots, Na_{n+1}) = f'(K\varphi(a_1), \cdots, K\varphi(a_{n+1})).$$

We prove that this diagram is commutative. It is easy to check that the first square is commutative. For the second square, we note that

$$\begin{aligned} q^n \pi^n f(Na_1, \cdots, Na_{n+1}) &= \pi^n f(K\varphi(a_1), \cdots, K\varphi(a_{n+1})) \\ &= \text{mean of } f \text{ on } (K\varphi(a_1), \cdots, K\varphi(a_{n+1})) \\ &= \text{mean of } p^n f \text{ on } (Na_1, \cdots, Na_{n+1}) \\ &= \lambda^n p^n f(a_1, \cdots, a_{n+1}) \end{aligned}$$

and so we have $q^n \pi^n = \lambda^n p^n$.

From definitions, we have the following complexes

$$\begin{aligned} M^n(A/N \xrightarrow{\rho} G/K) &= B\big((G/K)^{n+1}\big)^G \bigoplus B\big((A/N)^n\big)^A \\ EM^n(A/N \xrightarrow{\rho} G/K) &= B\big((A/N)^{n+1}\big)^A \bigoplus B\big((G/K)^{n+1}\big)^G \bigoplus B\big((A/N)^n\big)^A \\ M^n(A \xrightarrow{\varphi} G) &= B(G^{n+1})^G \bigoplus B(A^n)^A \\ EM^n(A \xrightarrow{\varphi} G) &= B(A^{n+1})^A \bigoplus B(G^{n+1})^G \bigoplus B(A^n)^A. \end{aligned}$$

We consider the following diagram

$$\begin{array}{ccccc} 0 \to M^n(A/N \xrightarrow{\rho} G/K) & \longrightarrow & EM^n(A/N \xrightarrow{\rho} G/K) & \longrightarrow & B\big((A/N)^n\big)^A \to 0 \\ \beta^n \big\downarrow & & \alpha^n \big\downarrow & & \gamma^n \big\downarrow \\ 0 \to M^n(A \xrightarrow{\varphi} G) & \longrightarrow & EM^n(A \xrightarrow{\varphi} G) & \longrightarrow & B(A^n)^A \to 0 \end{array}$$

in which each row is exact and each column is defined by the formulas

$$\alpha^n(f'', f, f') = (j^n f'', i^n f, j^{n-1} f'), \quad \beta^n(f, f') = (i^n f, j^{n-1} f'), \quad \gamma^n f'' = j^n f''.$$

It is easy to check that these maps commute with the boundary operators and the diagram is commutative. Also this diagram induces the following commutative diagram

$$\begin{array}{ccccccc}
\widehat{H}^{n-1}(A/N) & \longrightarrow & \widehat{H}^n(A/N \xrightarrow{\rho} G/K) & \longrightarrow & \widehat{H}^n(G/K) & \longrightarrow & \widehat{H}^n(A/N) \\
{\scriptstyle H^{n-1}(\gamma)}\downarrow & & {\scriptstyle H^n(\beta)}\downarrow & & {\scriptstyle H^n(\alpha)}\downarrow & & {\scriptstyle H^n(\gamma)}\downarrow \\
\widehat{H}^{n-1}(A) & \longrightarrow & \widehat{H}^n(A \xrightarrow{\varphi} G) & \longrightarrow & \widehat{H}^n(G) & \longrightarrow & \widehat{H}^n(A).
\end{array}$$

Since $H^*(\alpha)$ and $H^*(\gamma)$ are isomorphisms, the map $H^*(\beta)$ induced from β^* is also an isomorphism. Also note that we have $\|\beta^n\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \geq 0$. So the induced map $H^n(\beta)$ has the norm $\|H^n(\beta)\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \in [0, \infty]$.

On the other hand, we define $\widetilde{\beta}^n \colon M^n(A \xrightarrow{\varphi} G) \to M^n(A/N \xrightarrow{\rho} G/K)$ by the formula $\widetilde{\beta}^n(\zeta, \zeta') = (\pi^n\zeta, \lambda^{n-1}\zeta')$. Then we have

$$\begin{aligned}
d^n\widetilde{\beta}^n(\zeta,\zeta') &= d^n(\pi^n\zeta,\ \lambda^{n-1}\zeta') = (d^n\pi^n\zeta,\ -q^n\pi^n\zeta - d^{n-1}\lambda^{n-1}\zeta') \\
&= (\pi^{n+1}d^n\zeta,\ -\lambda^n p^n\zeta - \lambda^n d^{n-1}\zeta') = \widetilde{\beta}^{n+1}(d^n\zeta,\ -p^n\zeta - d^{n-1}\zeta') \\
&= \widetilde{\beta}^{n+1}d^n(\zeta,\zeta')
\end{aligned}$$

and so $\widetilde{\beta}^n$ commutes with the boundary operator. It is easy to check that $\beta^n\widetilde{\beta}^n = id$. Since we have $\|\pi^n\| \leq 1$ and $\|\lambda^{n-1}\| \leq 1$, it is clear that the map $\widetilde{\beta}^n$ has the norm $\|\widetilde{\beta}^n\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \geq 0$. Hence the induced map $H^n(\widetilde{\beta}) \colon \widehat{H}^n(A \xrightarrow{\varphi} G) \to \widehat{H}^n(A/N \xrightarrow{\rho} G/K)$ is the inverse of $H^n(\beta)$ and also has the norm $\|H^n(\widetilde{\beta})\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \in [0, \infty]$. Thus the isomorphism $H^n(\beta)$ is also an isometry. □

Corollary 2.17. *Let A be an amenable group, and let $\varphi \colon A \to G$ be a group homomorphism. Then the groups $\widehat{H}^n(A \xrightarrow{\varphi} G)$ and $\widehat{H}^n(G)$ are isomorphic. Furthermore, the norms $\|\cdot\|(\omega)$ in $\widehat{H}^n(A \xrightarrow{\varphi} G)$ are equal to the norm $\|\cdot\|$ in $\widehat{H}^n(G)$ for every $\omega \in [0, \infty]$.*

Proof. We note that the image $\varphi(A)$ is an amenable subgroup of G and also note that $\ker(\varphi)$ is an amenable subgroup of A.

We denote by $\rho\colon \varphi(A) \hookrightarrow G$ an inclusion map and consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \varphi(A) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\rho} \\ G & \xrightarrow{id} & G. \end{array}$$

It is clear the diagram is commutative and the horizontal maps are the surjective maps with the amenable kernels respectively. Then, by Theorem 2.16, the groups $\widehat{H}^n(A \xrightarrow{\varphi} G)$ and $\widehat{H}^n(\varphi(A) \xrightarrow{\rho} G)$ are isometrically isomorphic for the norm $\|\cdot\|(\omega)$.

Now, by Theorem 2.14, the groups $\widehat{H}^n(\varphi(A) \xrightarrow{\rho} G)$ and $\widehat{H}^n(G)$ are isometrically isomorphic for the norm $\|\cdot\|$. Also, by Theorem 2.15, the norms $\|\cdot\|(\omega)$ on the group $\widehat{H}^n(\varphi(A) \xrightarrow{\rho} G)$ are equal to the norm $\|\cdot\|(0) = \|\cdot\|$. □

In the rest of this chapter, we let A be a subgroup of G and let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then we give another description of relative bounded cohomology of G modulo A.

Definition 2.5. *Let*

$$0 \to \mathbf{R} \xrightarrow{\partial_{-1}} U_0 \xrightarrow{\partial_0} U_1 \xrightarrow{\partial_1} \cdots \quad \textit{and} \quad 0 \to \mathbf{R} \xrightarrow{\partial'_{-1}} V_0 \xrightarrow{\partial'_0} V_1 \xrightarrow{\partial'_1} \cdots$$

be an allowable pair of resolutions for $(G, A; \mathbf{R})$, and let $\lambda^n\colon U_n \to V_n$ be an A-morphism of resolutions commuting with the contracting homotopies as in Definition 2.1. If λ^n induces a surjective map $\lambda^n\colon U_n^G \to V_n^A$ as the restriction map of λ^n for every $n \geq 0$, this pair of resolutions is said to be proper.

Proposition 2.18. *The pair of standard G- and A-resolutions is proper for $(G, A; \mathbf{R})$.*

Proof. From Proposition 2.1, the standard G- and A-resolutions are an allowable pair for $(G, A; \mathbf{R})$. Note that the map $p^n\colon B(G^{n+1}) \to B(A^{n+1})$ is defined by the formula $p^n(f)(a_1, \cdots, a_{n+1}) = f(a_1, \cdots, a_{n+1})$.

It suffices to show that the restriction map $p^n\colon B(G^{n+1})^G \to B(A^{n+1})^A$ is surjective. Note that for every G-invariant element f in $B(G^{n+1})$ the value of f at every

$(g_1, g_2, \cdots, g_n, g_{n+1}) \in G^{n+1}$ is independent of g_{n+1}. So we can identify a function $f \in B(G^{n+1})^G$ with the function $f' \in B(G^n)$ defined by the formula $f'(g_1, \cdots, g_n) = f(g_1, \cdots, g_n, 1)$. Conversely a function $f' \in B(G^n)$ is identified with the function $f \in B(G^{n+1})^G$ defined by the formula $f(g_1, \cdots, g_n, g_{n+1}) = f'(g_1, \cdots, g_n)$. Thus the subspace $B(G^{n+1})^G$ of G-invariant elements in $B(G^{n+1})$ can be identified naturally with $B(G^n)$, and similarly $B(A^{n+1})$ with $B(A^n)$. Hence the surjectivity of the restriction map, $B(G^{n+1})^G \to B(A^{n+1})^A$, follows from the fact that the maps $p^{n-1}\colon B(G^n) \to B(A^n)$ are surjective for all $n \geq 1$. This finishes the proof. $\square$

Let

$$0 \to \mathbf{R} \xrightarrow{\partial_{-1}} U_0 \xrightarrow{\partial_0} U_1 \xrightarrow{\partial_1} \cdots \quad \text{and} \quad 0 \to \mathbf{R} \xrightarrow{\partial'_{-1}} V_0 \xrightarrow{\partial'_0} V_1 \xrightarrow{\partial'_1} \cdots$$

be a proper pair of resolutions for $(G, A; \mathbf{R})$. Then there is an exact sequence

$$0 \to \ker(p^n) \to U_n^G \to V_n^A \to 0.$$

It is easy to check that the sequence

$$0 \to \ker(p^0) \to \ker(p^1) \to \ker(p^2) \to \cdots \tag{2.5}$$

is a complex.

Definition 2.6. *The n-th cohomology of the complex in (2.5) is denoted by $\widehat{H}^n(G, A)$.*

As an important example, the standard G- and A-resolutions induces an exact sequence

$$0 \to P^n(G, A) \to B(G^{n+1})^G \xrightarrow{p^n} B(A^{n+1})^A \to 0,$$

where $P^n(G, A) = \ker(p^n)$. Also this exact sequence induces a long exact sequence

$$\cdots \to \widehat{H}^{n-1}(A) \to \widehat{H}^n(G, A) \to \widehat{H}^n(G) \to \widehat{H}^n(A) \to \cdots.$$

Theorem 2.19. *The groups* $\widehat{H}^n(G,A)$ *and* $\widehat{H}^n(A \overset{\varphi}{\to} G)$ *are isomorphic. This isomorphism carries the seminorm on* $\widehat{H}^*(G,A)$ *to a seminorm equivalent to the canonical seminorm on* $\widehat{H}^*(A \overset{\varphi}{\to} G)$.

Proof. Recall that the complexes induced from the standard resolutions

$$M^n(A \overset{\varphi}{\to} G) = B(G^{n+1})^G \bigoplus B(A^n)^A$$
$$EM^n(A \overset{\varphi}{\to} G) = B(A^{n+1})^A \bigoplus B(G^{n+1})^G \bigoplus B(A^n)^A.$$

We consider the following diagram

$$\begin{array}{ccccc} 0 \to P^n(G,A) & \xrightarrow{i_n} & B(G^{n+1})^G & \xrightarrow{p_n} & B(A^{n+1})^A \to 0 \\ \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \gamma_n} \\ 0 \to M^n(A \overset{\varphi}{\to} G) & \xrightarrow{j_n} & EM^n(A \overset{\varphi}{\to} G) & \xrightarrow{q_n} & B(A^{n+1})^A \to 0, \end{array}$$

where $\alpha_n(f) = (p_n f, f, 0)$, $\gamma_n(f'') = f''$, and $\beta_n(f) = (f, 0)$. It is clear that the diagram is commutative and so there is an induced commutative diagram

$$\begin{array}{ccccccc} \to \widehat{H}^{n-1}(A) & \longrightarrow & \widehat{H}^n(G,A) & \longrightarrow & \widehat{H}^n(G) & \longrightarrow & \widehat{H}^n(A) \to \\ \downarrow{\scriptstyle H^{n-1}(\gamma)} & & \downarrow{\scriptstyle H^n(\beta)} & & \downarrow{\scriptstyle H^n(\alpha)} & & \downarrow{\scriptstyle H^n(\gamma)} \\ \to \widehat{H}^{n-1}(A) & \longrightarrow & \widehat{H}^n(A \overset{\varphi}{\to} G) & \longrightarrow & \widehat{H}^n(G) & \longrightarrow & \widehat{H}^n(A) \to \end{array}$$

Note that the maps $H^*(\alpha)$ and $H^*(\gamma)$ are the (isometric) isomorphisms. So the map $H^*(\beta)$ is an isomorphism. Also, since $\|\beta_n(f)\| = \|(f,0)\| = \|f\|$, the map $H^*(\beta)$ has the norm $\|H^*(\beta)\| \leq 1$.

Let $(f, f') \in M^n(A \overset{\varphi}{\to} G) = B(G^{n+1})^G \bigoplus B(A^n)^A$ be a cocycle. Then $\partial_n f = 0$ and also $\partial'_{n-1} f' = -p_n f$. Since p_{n-1} is surjective, we can choose an element $f_1 \in B(G^n)^G$ such that $p_{n-1} f_1 = f'$ and $\|f_1\| = \|f'\|$. Then

$$p_n f = -\partial'_{n-1} f' = -\partial'_{n-1} p_{n-1} f_1 = -p_n \partial_{n-1} f_1$$

so that $f + \partial_{n-1} f_1 \in P^n(G,A)$ and $\partial_n(f + \partial_{n-1} f_1) = 0$. Now it is easy to check that $(H^n(\beta))^{-1}([(f,f')])$ is represented by a cocycle $f + \partial_{n-1} f_1 \in P^n(G,A)$. Then

$$\begin{aligned} \|f + \partial_{n-1} f_1\| &\leq \|f\| + \|\partial_{n-1}\|\|f_1\| = \|f\| + (n+1)\|f'\| \\ &\leq (n+2)\max\{\|f\|, \|f'\|\} = (n+2)\|(f,f')\| \end{aligned}$$

This show that, for $[f] \in \widehat{H}^n(G, A)$, we have

$$\frac{1}{n+2}\|[f]\| \leq \|H^n(\beta)[f]\| \leq \|[f]\|.$$

□

CHAPTER 3

Relative bounded cohomology of spaces

Throughout this chapter, we assume all spaces are connected countable cellular spaces.

Recall that the bounded cohomology of a space X, denoted by $\widehat{H}^*(X)$, is defined by the cohomology of the complex

$$0 \to B^0(X) \to B^1(X) \to B^2(X) \to \cdots ,$$

where $B^n(X)$ is a space of real bounded functions on $S_n(X)$ the set of all singular n-simplices. There is a natural norm $\|\cdot\|$ in $B^n(X)$:

$$\|f\| = \sup\{|f(\sigma)| \big| \sigma \in S_n(X)\}$$

which turns it into a Banach space. Thus in $\widehat{H}^n(X)$ there is a seminorm $\|[f]\| = \inf \|f\|$, where the infimum is taken over all cochains f lying in the cohomology class $[f] \in \widehat{H}^*(X)$. A continuous map $\alpha\colon U \to X$ induces a homomorphism $\alpha^*\colon B^*(X) \to B^*(U)$ and the norm of α^* is bounded by one as $\|\alpha^*(f)\| \leq \|f\|$ for all $f \in B^*(X)$.

Definition 3.1. *Let $\varphi\colon Y \to X$ be a continuous map of spaces. The mapping cone $M^n(Y \xrightarrow{\varphi} X)$ and the mapping cylinder $EM^n(Y \xrightarrow{\varphi} X)$ of cochain complexes induced* *by φ are defined as follows:*

$$M^n(Y \xrightarrow{\varphi} X) = B^n(X) \bigoplus B^{n-1}(Y)$$
$$EM^n(Y \xrightarrow{\varphi} X) = B^n(Y) \bigoplus B^n(X) \bigoplus B^{n-1}(Y).$$

We define the boundary operators on $M^n(Y \xrightarrow{\varphi} X)$ and $EM^n(Y \xrightarrow{\varphi} X)$ by the same formulas as in Lemma 2.2, i.e., on $EM^n(Y \xrightarrow{\varphi} X)$

$$d(v_n, u_n, v_{n-1}) = (dv_n, du_n, v_n - \lambda^n u_n - dv_{n-1}),$$

where $\lambda^n \colon B^n(X) \to B^n(Y)$ is a cochain map induced by a continuous map $\varphi \colon Y \to X$. Then we have the complexes

$$0 \to M^0(Y \xrightarrow{\varphi} X) \to M^1(Y \xrightarrow{\varphi} X) \to M^2(Y \xrightarrow{\varphi} X) \to \cdots \tag{3.1}$$
$$0 \to EM^0(Y \xrightarrow{\varphi} X) \to EM^1(Y \xrightarrow{\varphi} X) \to EM^2(Y \xrightarrow{\varphi} X) \to \cdots . \tag{3.2}$$

Definition 3.2. *The n-th cohomology of the complex in (3.1) is called the n-th relative bounded cohomology of X modulo Y and is denoted by $\widehat{H}^n(Y \xrightarrow{\varphi} X)$. Also the n-th cohomology of the complex in (3.2) is denoted by $\widehat{H}^n(EM(Y \xrightarrow{\varphi} X))$.*

We define the norm $\|\cdot\|$ on $EM^n(Y \xrightarrow{\varphi} X)$ and on $M^n(Y \xrightarrow{\varphi} X)$ by setting

$$\|(v_n, u_n, v_{n-1})\| = \max\{\|v_n\|, \|u_n\|, \|v_{n-1}\|\}$$
$$\|(u_n, v_{n-1})\| = \max\{\|u_n\|, \|v_{n-1}\|\}.$$

Also, for every $\omega \geq 0$, we define the norm $\|\cdot\|(\omega)$ on $M^n(Y \xrightarrow{\varphi} X)$ by setting

$$\|(u_n, v_{n-1})\| = \max\{\|u_n\|, (1+\omega)^{-1}\|v_{n-1}\|\}.$$

Note that there are corresponding seminorms $\|\cdot\|$ on the groups $\widehat{H}^*(EM(Y \xrightarrow{\varphi} X))$ and $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ respectively. Also there is corresponding seminorms $\|\cdot\|(\omega)$ on $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ for every $\omega \geq 0$. Finally we define these norms $\|\cdot\|(\omega)$ on $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ for all ω in the closed interval $[0, \infty]$ by passing to the limits.

Proposition 3.1. *Let $\varphi\colon Y \to X$ be a continuous map of spaces. Then the groups $\widehat{H}^n(EM(Y \overset{\varphi}{\to} X))$ and $\widehat{H}^n(X)$ are isometrically isomorphic.*

Proof. Note that there is a map $\lambda^n\colon B^n(X) \to B^n(Y)$ induced by the map $\varphi\colon Y \to X$. Then, as in Proposition 2.3, the natural projection map $EM^n(Y \overset{\varphi}{\to} X) \to B^n(X)$ induces the isometrically isomorphic groups $\widehat{H}^n(EM(Y \overset{\varphi}{\to} X))$ and $\widehat{H}^n(X)$. □

Remark that there is an exact sequence

$$0 \to M^n(Y \overset{\varphi}{\to} X) \to EM^n(Y \overset{\varphi}{\to} X) \to B^n(Y) \to 0$$

and it induces a long exact sequence

$$\cdots \to \widehat{H}^n(Y \overset{\varphi}{\to} X) \to \widehat{H}^n(X) \to \widehat{H}^n(Y) \to \widehat{H}^{n+1}(Y \overset{\varphi}{\to} X) \to \cdots . \tag{3.3}$$

Recall that, as shown in Theorem 1.4, the group $\widehat{H}^*(X)$ is canonically isomorphic with $\widehat{H}^*(\boldsymbol{\pi}_1 X)$ and this isomorphism carries the seminorm in $\widehat{H}^*(X)$ to the canonical seminorm in $\widehat{H}^*(\boldsymbol{\pi}_1 X)$. Thus it is natural to consider the relationship between $\widehat{H}^*(Y \overset{\varphi}{\to} X)$ and the fundamental groups $\boldsymbol{\pi}_1 X$ and $\boldsymbol{\pi}_1 Y$. Note that from the induced homomorphism $\varphi_*\colon \boldsymbol{\pi}_1 Y \to \boldsymbol{\pi}_1 X$, we can define the relative bounded cohomology $\widehat{H}^*(\boldsymbol{\pi}_1 Y \overset{\varphi_*}{\longrightarrow} \boldsymbol{\pi}_1 X)$.

Remark 3.1. Let $\pi\colon \mathcal{X} \to X$ be a universal covering of X. As shown in Theorem 1.4, the sequence

$$0 \to \mathbf{R} \to B^0(\mathcal{X}) \to B^1(\mathcal{X}) \to B^2(\mathcal{X}) \to \cdots \tag{3.4}$$

is a strong relatively injective $\boldsymbol{\pi}_1 X$-resolution of the trivial $\boldsymbol{\pi}_1 X$-module $\mathbf{R}$. Also the induced map $\pi^*\colon B^*(X) \to B^*(\mathcal{X})$ establishes an isometric isomorphism $B^*(X) \to B^*(\mathcal{X})^{\boldsymbol{\pi}_1 X}$, so that the bounded cohomology $\widehat{H}^*(\boldsymbol{\pi}_1 X)$ induced from the resolution in (3.4) coincides with $\widehat{H}^*(X)$ as topological vector spaces.

Lemma 3.2. *Let $\varphi\colon Y \to X$ be a continuous map of spaces. Let $\mathcal{X} \to X$ and $\mathcal{Y} \to Y$ be the universal coverings of X and Y respectively. Then the sequences*

$$0 \to \mathbf{R} \to B^0(\mathcal{X}) \to B^1(\mathcal{X}) \to B^2(\mathcal{X}) \to \cdots$$
$$0 \to \mathbf{R} \to B^0(\mathcal{Y}) \to B^1(\mathcal{Y}) \to B^2(\mathcal{Y}) \to \cdots \tag{3.2.1}$$

are an allowable pair of resolutions for $(\pi_1 X, \pi_1 Y; \mathbf{R})$. Furthermore, there is a commutative diagram (3.2.2)

$$\begin{array}{ccccc} B^n(\mathcal{X})^{\pi_1 X} & \longrightarrow & B((\pi_1 X)^{n+1})^{\pi_1 X} & \longrightarrow & B^n(\mathcal{X})^{\pi_1 X} \\ \lambda^n \downarrow & & p^n \downarrow & & \lambda^n \downarrow \\ B^n(\mathcal{Y})^{\pi_1 Y} & \longrightarrow & B((\pi_1 Y)^{n+1})^{\pi_1 Y} & \longrightarrow & B^n(\mathcal{Y})^{\pi_1 Y}, \end{array}$$

where the maps λ^n and p^n are induced by a lifting map $\lambda\colon \mathcal{Y} \to \mathcal{X}$ and $\varphi_*\colon \pi_1 Y \to \pi_1 X$ respectively.

Proof. We denote by G and A the fundamental groups $\pi_1 X$ and $\pi_1 Y$ respectively. As explained in Remark 3.1, the sequences in (3.2.1) are strong relatively injective G- and A-resolutions respectively. From Theorem 2.4 in [8], the contracting homotopy

$$0 \leftarrow \mathbf{R} \leftarrow B^0(\mathcal{X}) \leftarrow B^1(\mathcal{X}) \leftarrow \cdots$$

is defined by using the cone construction $S_n(\mathcal{X}) \to S_{n+1}(\mathcal{X})$.

By standard calculation, it is easy to check the map $\lambda^n\colon B^n(\mathcal{X}) \to B^n(\mathcal{Y})$ is an A-morphism and it commutes with the contracting homotopies. Thus the sequences in (3.2.1) are an allowable pair of resolutions for $(G, A; \mathbf{R})$.

Now we consider the diagram in (3.2.2)

$$\begin{array}{ccccc} B^n(\mathcal{X})^G & \xrightarrow{\alpha^n} & B(G^{n+1})^G & \xrightarrow{\zeta^n} & B^n(\mathcal{X})^G \\ \lambda^n \downarrow & & p^n \downarrow & & \lambda^n \downarrow \\ B^n(\mathcal{Y})^A & \xrightarrow{\gamma^n} & B(A^{n+1})^A & \xrightarrow{\eta^n} & B^n(\mathcal{Y})^A. \end{array}$$

The maps α^n and γ^n are defined by the same formulas in Lemma 2.6. Also the maps ζ^n and η^n are defined by the same formulas in Theorem 1.4.

Since the resolutions in (3.2.1) are an allowable pair for $(G, A; \mathbf{R})$, it follows from Lemma 2.6 that the first square is commutative.

On the other hand, let $\sigma\colon \Delta_n \to \mathcal{Y}$ be a singular simplex. Also let $(a_0, \cdots, a_n) \in A^{n+1}$ be such that $\sigma(v_i) \in a_{n-i} \cdots a_n \mathcal{Y}_0$, where $\mathcal{Y}_0$ is a fundamental set for the action of A on $\mathcal{Y}$(see Theorem 4.1 in [8]).

Note that, if $\sigma(v_i) = a_{n-i} \cdots a_n y$ for some $y \in \mathcal{Y}_0$, then

$$\lambda(\sigma(v_i)) = \lambda(a_{n-i} \cdots a_n y) = \varphi_*(a_{n-i}) \cdots \varphi_*(a_n)\lambda(y). \tag{3.2.3}$$

Also note that we have $\eta^n p^n f(\sigma) = p^n f(a_0, \cdots, a_n) = f(\varphi_*(a_0), \cdots, \varphi_*(a_n))$ and $\lambda^n \zeta^n f(\sigma) = \zeta^n f(\lambda \circ \sigma) = f(\varphi_*(a_0), \cdots, \varphi_*(a_n))$, where the second equality follows from the equation (3.2.3). This shows that the second square in (3.2.2) is commutative, so that $\eta^n p^n = \lambda^n \zeta^n$. This finishes the proof. □

Theorem 3.3. *Let $\varphi\colon Y \to X$ be a continuous map of spaces, and let $\varphi_*\colon \boldsymbol{\pi}_1 Y \to \boldsymbol{\pi}_1 X$ be an induced homomorphism. Then the groups $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^n(\boldsymbol{\pi}_1 Y \xrightarrow{\varphi_*} \boldsymbol{\pi}_1 X)$ are isometrically isomorphic for the norm $\|\cdot\|(\omega)$ for every $\omega \in [0, \infty]$. This isomorphism carries the seminorm in $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ to the canonical seminorm in $\widehat{H}^n(\boldsymbol{\pi}_1 Y \xrightarrow{\varphi_*} \boldsymbol{\pi}_1 X)$.*

Proof. Let G and A denote the groups $\boldsymbol{\pi}_1 X$ and $\boldsymbol{\pi}_1 Y$ respectively.

Recall that the canonical seminorm in $\widehat{H}^n(A \xrightarrow{\varphi} G)$ is induced by the complex $B(G^{n+1})^G \bigoplus B(A^n)^A$.

Let $\pi_1\colon \mathcal{X} \to X$ and $\pi_2\colon \mathcal{Y} \to Y$ be the universal coverings of X and Y respectively. By Remark 3.1, we can identify

$$B^n(X) \bigoplus B^{n-1}(Y) = B^n(\mathcal{X})^G \bigoplus B^{n-1}(\mathcal{Y})^A.$$

We prove that there are cochain maps

$$\Phi^n\colon B^n(\mathcal{X})^G \bigoplus B^{n-1}(\mathcal{Y})^A \to B(G^{n+1})^G \bigoplus B(A^n)^A \qquad \text{and}$$

$$\Psi^n\colon B(G^{n+1})^G \bigoplus B(A^n)^A \to B^n(\mathcal{X})^G \bigoplus B^{n-1}(\mathcal{Y})^A$$

such that $\Psi^n\Phi^n$ is chain homotopic to id and they have the norms $\|\Phi^n\| \leq 1$ and $\|\Psi^n\| \leq 1$. Recall that there is the commutative diagram in (3.2.2)

$$\begin{array}{ccccc} B^n(\mathcal{X})^G & \xrightarrow{\alpha^n} & B(G^{n+1})^G & \xrightarrow{\zeta^n} & B^n(\mathcal{X})^G \\ \lambda^n \downarrow & & p^n \downarrow & & \lambda^n \downarrow \\ B^n(\mathcal{Y})^A & \xrightarrow{\gamma^n} & B(A^{n+1})^A & \xrightarrow{\eta^n} & B^n(\mathcal{Y})^A \end{array}$$

so that $p^n\alpha^n = \gamma^n\lambda^n$ and $\lambda^n\zeta^n = \eta^n p^n$. Note that, from definitions, the maps $\zeta^n\alpha^n$ and $\eta^n\gamma^n$ are chain homotopic to $id_{B^n(\mathcal{X})^G}$ and $id_{B^n(\mathcal{Y})^A}$ respectively. Also they have the norms $\|\alpha^n\| \leq 1$, $\|\zeta^n\| \leq 1$, $\|\gamma^n\| \leq 1$, and $\|\eta^n\| \leq 1$.

We define Φ^n and Ψ^n by the formulas

$$\Phi^n(f, f') = (\alpha^n f, \gamma^{n-1} f') \quad \text{and} \quad \Psi^n(u, u') = (\zeta^n u, \eta^{n-1} u').$$

For simplicity, we denote all boundary operators by the same notation d. Then

$$\begin{aligned} \Phi^{n+1} d^n(f, f') &= \Phi^{n+1}(df, -\lambda^n f - d^{n-1} f') = (\alpha^{n+1} d^n f, -\gamma^n\lambda^n f - \gamma^n d^{n-1} f') \\ &= (d^n\alpha^n f, -p^n\alpha^n f - d^{n-1}\gamma^{n-1} f') \\ &= d^n(\alpha^n f, \gamma^{n-1} f') = d^n\Phi^n(f, f') \end{aligned}$$

and so Φ^n commutes with the boundary operators. Also for every $\omega \geq 0$

$$\begin{aligned} \|\Phi^n(f, f')\|(\omega) &= \|(\alpha^n f, \gamma^{n-1} f')\|(\omega) \\ &= \max\{\|\alpha^n f\|, (1+\omega)^{-1}\|\gamma^{n-1} f'\|\} \\ &\leq \max\{\|f\|, (1+\omega)^{-1}\|f'\|\} = \|(f, f')\|(\omega). \end{aligned}$$

Thus we have $\|\Phi^n\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \geq 0$.

By the same way, we can prove that Ψ^n commutes with the boundary operators and it has the norm $\|\Psi^n\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \geq 0$.

Finally, from definitions, the map $\Phi^n\Psi^n$ is chain homotopic to the identity and $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^n(\boldsymbol{\pi}_1 Y \xrightarrow{\varphi} \boldsymbol{\pi}_1 X)$ are isomorphic. Also, since we have $\|\Phi^n\| \leq 1$ and $\|\Psi^n\| \leq 1$ for the norm $\|\cdot\|(\omega)$ for every $\omega \geq 0$, these groups are isometric for the norm $\|\cdot\|(\omega)$ for every $\omega \geq 0$ and so for all $\omega \in [0, \infty]$ by passing to the limits. □

We recall that the group $\widehat{H}^*(X)$ is zero if $\boldsymbol{\pi}_1 X$ is amenable.

Corollary 3.4. *Let $\varphi\colon Y \to X$ be a continuous map of spaces such that the fundamental group $\boldsymbol{\pi}_1 Y$ is amenable. Then the groups $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^n(X)$ are isometrically isomorphic for the norm $\|\cdot\|$.*

Proof. We have the following sequence of isometrically isomorphic groups

$$\begin{aligned} \widehat{H}^n(Y \xrightarrow{\varphi} X) &\cong \widehat{H}^n(\boldsymbol{\pi}_1 Y \xrightarrow{\varphi_*} \boldsymbol{\pi}_1 X) && \text{by Theorem 3.3} \\ &\cong \widehat{H}^n(\boldsymbol{\pi}_1 X) && \text{by Corollary 2.17} \\ &\cong \widehat{H}^n(X) && \text{by Theorem 1.4.} \end{aligned}$$

This finishes the proof. □

Theorem 3.5. *Let $\varphi\colon Y_1 \to X_1$ and $\rho\colon Y_2 \to X_2$ be the continuous maps of spaces respectively. Let $\alpha\colon X_1 \to X_2$ and $\gamma\colon Y_1 \to Y_2$ be the continuous maps of spaces such that $\alpha \circ \varphi = \rho \circ \gamma$. Let the induced homomorphisms $\alpha_*\colon \boldsymbol{\pi}_1 X_1 \to \boldsymbol{\pi}_1 X_2$ and $\gamma_*\colon \boldsymbol{\pi}_1 Y_1 \to \boldsymbol{\pi}_1 Y_2$ be the surjective maps with the amenable kernels respectively. Then, for every $\omega \in [0,\infty]$, the groups $\widehat{H}^n(Y_2 \xrightarrow{\rho} X_2)$ and $\widehat{H}^n(Y_1 \xrightarrow{\varphi} X_1)$ are isometrically isomorphic for the norm $\|\cdot\|(\omega)$.*

Proof. Let $\varphi_*\colon \boldsymbol{\pi}_1 Y_1 \to \boldsymbol{\pi}_1 X_1$ and $\rho_*\colon \boldsymbol{\pi}_1 Y_2 \to \boldsymbol{\pi}_1 X_2$ be the induced homomorphism by φ and ρ respectively. We consider the following diagram

$$\begin{array}{ccc} \boldsymbol{\pi}_1 Y_1 & \xrightarrow{\gamma_*} & \boldsymbol{\pi}_1 Y_2 \\ {\scriptstyle\varphi_*}\downarrow & & {\scriptstyle\rho_*}\downarrow \\ \boldsymbol{\pi}_1 X_1 & \xrightarrow{\alpha_*} & \boldsymbol{\pi}_1 X_2. \end{array}$$

It is clear that this diagram is commutative and the horizontal maps γ_* and α_* are surjective maps with the amenable kernels. Thus we have the following sequences of isometrically isomorphic groups

$$\begin{aligned}\widehat{H}^n(Y_2 \xrightarrow{\rho} X_2) &\cong \widehat{H}^n(\boldsymbol{\pi}_1 Y_2 \xrightarrow{\rho_*} \boldsymbol{\pi}_1 X_2) && \text{by Theorem 3.3}\\ &\cong \widehat{H}^n(\boldsymbol{\pi}_1 Y_1 \xrightarrow{\varphi_*} \boldsymbol{\pi}_1 X_1) && \text{by Theorem 2.16}\\ &\cong \widehat{H}^n(Y_1 \xrightarrow{\varphi} X_1) && \text{by Theorem 3.3.}\end{aligned}$$

This finishes the proof. □

In the rest of this chapter, we consider a pair of spaces X and $Y \subset X$ and denote it by (X, Y).

Remark that, as in the ordinary cohomology, there is an exact sequence

$$0 \to \ker(p^n) \hookrightarrow B^n(X) \xrightarrow{p^n} B^n(Y) \to 0, \tag{3.5}$$

where p^n is defined as the restriction to $S_n(Y)$. We denote $\ker(p^n)$ by $P^n(X, Y)$. Then there is an induced sequence

$$0 \to P^0(X,Y) \to P^1(X,Y) \to P^2(X,Y) \to \cdots \tag{3.6}$$

which is obviously a complex.

Definition 3.3. *The n-th cohomology of the complex in (3.6) is denoted by $\widehat{H}^n(X, Y)$.*

For a pair of spaces (X, Y), there is a natural norm $\|\cdot\|$ on $P^*(X, Y)$ induced from the norm $\|\cdot\|$ in $B^*(X)$ by the inclusion $P^*(X,Y) \hookrightarrow B^*(X)$. Thus there is a natural seminorm $\|\cdot\|$ in $\widehat{H}^*(X, Y)$.

Note that the exact sequence in (3.5) induces a long exact sequence

$$\cdots \to \widehat{H}^n(X,Y) \to \widehat{H}^n(X) \to \widehat{H}^n(Y) \to \widehat{H}^{n+1}(X,Y) \to \cdots . \tag{3.7}$$

As we will see in the next theorem, if $\varphi\colon Y \hookrightarrow X$ is an inclusion of a subspace Y into X, then the groups $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^*(X, Y)$ are canonically isomorphic. Moreover, as the same theorem shows, the natural seminorms on these groups are equivalent. Nevertheless, there is no reason to expect that these norms are equal. From our point of view, the group $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ with its seminorm is more natural invariant of a pair of spaces (X, Y) and we consider it to be primary invariant.

Theorem 3.6. *Let (X, Y) be a pair of spaces and $\varphi\colon Y \to X$ be an inclusion map. Then the groups $\widehat{H}^n(X, Y)$ and $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ are isomorphic and the seminorms are equivalent.*

Proof. From definition, the group $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ is the cohomology of the complex

$$M^n(Y \xrightarrow{\varphi} X) = B^n(X) \bigoplus B^{n-1}(Y).$$

Also from Proposition 3.1, we can define the group $\widehat{H}^n(X)$ as the cohomology of the complex

$$EM^n(Y \xrightarrow{\varphi} X) = B^n(Y) \bigoplus B^n(X) \bigoplus B^{n-1}(Y).$$

We consider the following diagram (3.6.1)

$$\begin{array}{ccccc}
0 \to P^n(X,Y) & \xrightarrow{i^n} & B^n(X) & \xrightarrow{p^n} & B^n(Y) \to 0 \\
\beta^n \downarrow & & \alpha^n \downarrow & & \gamma^n \downarrow \\
0 \to M^n(Y \xrightarrow{\varphi} X) & \xrightarrow{j^n} & EM^n(Y \xrightarrow{\varphi} X) & \xrightarrow{q^n} & B^n(Y) \to 0
\end{array}$$

where $\alpha^n(f) = (p^n f, f, 0)$ and $\gamma^n = id$ and $\beta^n(f) = (f, 0)$. Then it is clear that this diagram is commutative and it induces the following diagram

$$\begin{array}{ccccccc}
\to \widehat{H}^{n-1}(Y) & \longrightarrow & \widehat{H}^n(X,Y) & \longrightarrow & \widehat{H}^n(X) & \longrightarrow & \widehat{H}^n(Y) \to \\
H^{n-1}(\gamma) \downarrow & & H^n(\beta) \downarrow & & H^n(\alpha) \downarrow & & H^n(\gamma) \downarrow \\
\to \widehat{H}^{n-1}(Y) & \longrightarrow & \widehat{H}^n(Y \xrightarrow{\varphi} X) & \longrightarrow & \widehat{H}^n(X) & \longrightarrow & \widehat{H}^n(Y) \to .
\end{array}$$

Note that the maps $H^*(\gamma)$ and $H^*(\alpha)$ are (isometric) isomorphism. So the map $H^*(\beta)$ is an isomorphism.

Since $\|\beta^n(f)\| = \|(f, 0)\| = \max\{\|f\|, 0\} = \|f\|$, we have $\|\beta^n\| = 1$ and so the map $H^n(\beta)$ has the norm $\|H^n(\beta)\| \leq 1$.

Let $[f, f'] \in \widehat{H}^n(Y \xrightarrow{\varphi} X)$ and we represent it by a cocycle

$$(f, f') \in M^n(Y \xrightarrow{\varphi} X) = B^n(X) \bigoplus B^{n-1}(Y).$$

For simplicity, we denote every boundary operator by the same notation d. From the definition of the boundary operator, we have

$$0 = d^n(f, f') = (d^n f, -p^n f - d^{n-1} f'), \text{ so that } d^n f = 0 \text{ and } p^n f = -d^{n-1} f'.$$

Since the map p^{n-1} is surjective, we can choose an element $f'' \in B^{n-1}(X)$ such that $p^{n-1} f'' = f'$ and $\|f''\| = \|f'\|$. Then

$$p^n(f + d^{n-1} f'') = p^n f + p^n d^{n-1} f'' = p^n f + d^{n-1} p^{n-1} f'' = p^n f + d^{n-1} f' = 0$$

and so $f + d^{n-1} f'' \in P^n(X, Y)$. Also $d^n(f + d^{n-1} f'') = 0$ and so $f + d^{n-1} f''$ is a relative cocycle. It is easy to check $(H^n(\beta))^{-1}[f, f']$ is represented by this relative cocycle $f + d^{n-1} f''$. Remark that

$$\begin{aligned} \|f + d^{n-1} f''\| &\le \|f\| + \|d^{n-1}\| \|f''\| \le \|f\| + (n+1)\|f'\| \\ &\le (n+2) \max\{\|f\|, \|f'\|\} = (n+2)\|(f, f')\| \end{aligned}$$

and so $\|(H^n(\beta))^{-1}[f, f']\| \le (n+2)\|[f, f']\|$.

Thus, for $[f] \in \widehat{H}^n(X, Y)$, we have

$$\frac{1}{n+2}\|[f]\| \le \|H^n(\beta)[f]\| \le \|[f]\|.$$

This finishes the proof. □

Corollary 3.7. *The groups $\widehat{H}^*(X, Y)$ and $\widehat{H}^*(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ are isomorphic and this isomorphism carries the seminorm in $\widehat{H}^*(X, Y)$ to a seminorm, which is equivalent to the canonical seminorm, in $\widehat{H}^*(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$.*

Proof. From Theorem 3.3, the groups $\widehat{H}^*(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^*(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ are isometrically isomorphic. Hence it follows from Theorem 3.6. □

Corollary 3.8. *Let (X, Y) be a pair of spaces and let the fundamental group $\pi_1 Y$ be amenable. Then the groups $\widehat{H}^n(X, Y)$ and $\widehat{H}^n(X)$ are isomorphic and the seminorms are equivalent.*

Proof. Let $\varphi\colon Y \hookrightarrow X$ be an inclusion map. From Theorem 3.6, the groups $\widehat{H}^n(X,Y)$ and $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ are isomorphic and the seminorms are equivalent.

Since $\pi_1 Y$ is amenable, the groups $\widehat{H}^n(Y \xrightarrow{\varphi} X)$ and $\widehat{H}^n(X)$ are isometrically isomorphic by Corollary 3.4. Hence the groups $\widehat{H}^n(X,Y)$ and $\widehat{H}^n(X)$ are isomorphic and the seminorms are equivalent. □

CHAPTER 4

ℓ_1 homology of groups

We now dualize the notion of relatively injectivity to define ℓ_1 homology group of groups.

Throughout this chapter, G denotes a discrete group. Recall that $\mathbf{R}$ and the $B(G^n)$ are the important examples of G-modules for the theory of bounded cohomology.

Now we introduce another G-module which will be useful for computing the ℓ-homology of groups. Let $C'_n(G)$ be a free $\mathbf{R}$-module generated by the $(n+1)$-tuples $(g_0, \cdots, g_n)$ of elements of G, with the G-action given by $g\cdot(g_0, \cdots, g_n) = (g\cdot g_0, \cdots, g\cdot g_n)$. We take the $(n+1)$-tuples whose first element is 1 which represent the G-orbits of $(n+1)$-tuples. We write such an $(n+1)$-tuple in the form $(1, g_1, g_1g_2, \cdots, g_1g_2\cdots g_n)$ and introduce the bar notation

$$[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, \cdots, g_1g_2\cdots g_n),$$

and define $C_n(G)$ as the free $\mathbf{R}$-module generated by the n-tuples $[g_1|g_2|\cdots|g_n]$ with the G-action. Since the operation on a basis with an element of $g \in G$ yields an element $g[g_1|\cdots|g_n]$ in $C_n(G)$, we may describe $C_n(G)$ as the free $\mathbf{R}$-module generated by all $g[g_1|\cdots|g_n]$ so that an element of $C_n(G)$ can be written as a finite sum of the form $\sum r_i g_i[g_{i_1}|\cdots|g_{i_n}]$ where $r_i \in \mathbf{R}$, $g_i \in G$.

In particular, $C_0(G)$ has one generator, denoted by $[\,]$, so its element is a finite sum of the form $\sum r_i g_i[\,]$. We define the ℓ_1 norm $\|\cdot\|_1$ in $C_n(G)$ by putting

$$\|\sum r_i g_i[g_{i_1}|\cdots|g_{i_n}]\|_1 = \sum |r_i|.$$

Now let $C_n^{\ell_1}(G)$ be the norm completion of $C_n(G)$. Thus

$$C_n^{\ell_1}(G) = \left\{ \sum_{i=1}^{\infty} r_i g_i[g_{i_1}|\cdots|g_{i_n}] \mid \sum_{i=1}^{\infty} |r_i| < \infty \right\}$$

is a Banach space with the G-action such that $\|g\cdot c\|_1 \leq \|c\|_1$ for every $g \in G$, $c \in C_n^{\ell_1}(G)$. Hence $C_n^{\ell_1}(G)$ is a G-module.

Definition 4.1. *A surjective G-morphism of G-modules $\pi\colon V \to W$ is said to be strongly projective if there exists a bounded linear operator $\sigma\colon W \to V$ such that $\pi\circ\sigma = id$ and $\|\sigma\| \leq 1$. Also a G-module U is said to be relatively projective, if for any strongly projective G-morphism of G-modules $\pi\colon V \to W$ and any G-morphism of G-modules $\alpha\colon U \to W$ there exists a G-morphism $\beta\colon U \to V$ such that $\pi\circ\beta = \alpha$ and $\|\beta\| \leq \|\alpha\|$. The definition is illustrated by the following diagram (4.1.1):*

$$\begin{array}{ccc} U & = & U \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ V & \xrightarrow{\pi} & W \end{array}$$

Lemma 4.1. *The G-modules $C_n^{\ell_1}(G)$ are relatively projective for all $n \geq 0$.*

Proof. Let $\pi\colon V \to W$ be strongly projective G-morphism of G-modules. We consider the situation pictured in diagram (4.1.1), in which $U = C_n^{\ell_1}(G)$ and all the rest are given.

Let $x = \sum_{i=1}^{\infty} r_i g_i u_i \in U$, where $u_i = [g_{i_1}|\cdots|g_{i_n}]$ and $g_{i_j} \in G$. We define β by the formula

$$\beta(\sum_{i=1}^{\infty} r_i g_i u_i) = \sum_{i=1}^{\infty} r_i g_i \sigma\alpha(u_i).$$

Then $\pi\beta = \alpha$ follows from the following:

$$\begin{aligned} \pi\beta(\sum_{i=1}^{\infty} r_i g_i u_i) &= \pi(\sum_{i=1}^{\infty} r_i g_i \sigma\alpha(u_i)) = \sum_{i=1}^{\infty} r_i g_i \pi\big(\sigma\alpha(u_i)\big) \\ &= \sum_{i=1}^{\infty} r_i g_i \alpha(u_i) = \alpha\big(\sum_{i=1}^{\infty} r_i g_i u_i\big). \end{aligned}$$

Also note that for every $g' \in G$

$$g'\beta(\sum_{i=1}^{\infty} r_i g_i u_i) = g' \sum_{i=1}^{\infty} r_i g_i \sigma\alpha(u_i) = \sum_{i=1}^{\infty} r_i g' g_i \sigma\alpha(u_i)$$
$$= \beta(\sum_{i=1}^{\infty} r_i g' g_i u_i) = \beta(g' \sum_{i=1}^{\infty} r_i g_i u_i)$$

and so β commutes with the action of G. Finally,

$$\|\beta(\sum_{i=1}^{\infty} r_i g_i u_i)\|_1 = \|\sum_{i=1}^{\infty} r_i g_i \sigma\alpha(u_i)\|_1$$
$$\leq \sum_{i=1}^{\infty} | r_i | \, \|g_i\|_1 \|\sigma\| \|\alpha\| \leq \sum_{i=1}^{\infty} | r_i | \, \|\alpha\|,$$

i.e., $\|\beta(x)\|_1 \leq \|x\|_1 \|\alpha\|$, for any $x = \sum_{i=1}^{\infty} r_i g_i u_i \in U$.

Hence β is a G-morphism such that $\pi \circ \beta = \alpha$ and $\|\beta\| \leq \|\alpha\|$. □

Recall that a G-resolution is said to be strong if it is an exact sequence (as a vector space) of G-modules and G-morphisms which is provided with a contracting homotopy whose norm is less than or equal to 1.

Definition 4.2. *A strong G-resolution of a G-module V*

$$\cdots \to V_2 \to V_1 \to V_0 \to V \to 0$$

is said to be relatively projective if all G-modules V_n are relatively projective.

Now we consider the sequence of G-modules and G-morphisms

$$\cdots \to C_3^{\ell_1}(G) \to C_2^{\ell_1}(G) \to C_1^{\ell_1}(G) \to C_0^{\ell_1}(G) \to \mathbf{R} \to 0, \tag{4.1}$$

where the boundary operator $\partial_n \colon C_n^{\ell_1}(G) \to C_{n-1}^{\ell_1}(G)$ for every $n \geq 0$ is defined by

$$\partial_n[g_1|\cdots|g_n] = (-1)^n g_1[g_2|\cdots|g_n]$$
$$+ \sum_{i=1}^{n-1}(-1)^{n-i}[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] + [g_1|\cdots|g_{n-1}],$$

while $\varepsilon[\,] = 1$ is a G-morphism $\varepsilon \colon C_0^{\ell_1}(G) \to \mathbf{R}$.

Also we define $s_{-1}\colon \mathbf{R} \to C_0^{\ell_1}(G)$ and $s_n\colon C_n^{\ell_1}(G) \to C_{n+1}^{\ell_1}(G)$ by the formulas respectively:

$$s_{-1}1 = [\,] \quad \text{and} \quad s_n(g[g_1|\cdots|g_n]) = (-1)^{n+1}[g|g_1|\cdots|g_n].$$

By the standard calculation and by Lemma 4.1, it is clear that the sequence in (4.1) is a strong relatively projective G-resolution of the trivial G-module $\mathbf{R}$.

Definition 4.3. *The sequence in (4.1) is called the bar resolution of G.*

Definition 4.4. *For any G-module V the space of co-invariants of V, which is denoted by V_G, is defined to be the quotient of V by the additive submodule generated by the elements of the form $gv - v$ for all $g \in G$ and $v \in V$.*

For any strong relatively projective G-resolution

$$\cdots \to V_2 \to V_1 \to V_0 \to \mathbf{R} \to 0$$

of the trivial G-module $\mathbf{R}$, it is easy to see that the induced sequence

$$\cdots \to (V_2)_G \to (V_1)_G \to (V_0)_G \to 0 \tag{4.2}$$

is a complex and the homology of this complex depends only on G.

Definition 4.5. *The n-th homology group of the complex in (4.2) is called the n-th ℓ_1 homology group of G and is denoted by $H_n^{\ell_1}(G)$.*

Note that the homology of the complex in (4.2) has a natural seminorm which induces a topological vector space structure. Also note that this seminorm depends on the choice of a resolution.

Definition 4.6. *We define the canonical seminorm in $H_*^{\ell_1}(G)$ as the supremum of the seminorms which arise from all strong relatively projective G-resolutions of the trivial G-module $\mathbf{R}$.*

We shall see the canonical seminorms on $H_*^{\ell_1}(G)$ can be achieved by the bar resolution of G from the following theorem.

Theorem 4.2. *Let*

$$\cdots \underset{t_2}{\overset{\partial_3'}{\rightleftarrows}} V_2 \underset{t_1}{\overset{\partial_2'}{\rightleftarrows}} V_1 \underset{t_0}{\overset{\partial_1'}{\rightleftarrows}} V_0 \underset{t_{-1}}{\overset{\varepsilon'}{\rightleftarrows}} \mathbf{R} \to 0$$

be a strong relatively projective G-resolution of trivial G-module $\mathbf{R}$. There exists a G-morphism of the bar resolution of G to this resolution

$$\begin{array}{ccccccccccc} \longrightarrow & C_2^{\ell_1}(G) & \xrightarrow{\partial_2} & C_1^{\ell_1}(G) & \xrightarrow{\partial_1} & C_0^{\ell_1}(G) & \xrightarrow{\varepsilon} & \mathbf{R} & \longrightarrow & 0 \\ \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow id_{\mathbf{R}} & \downarrow \\ \longrightarrow & V_2 & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & \mathbf{R} & \longrightarrow & 0 \end{array}$$

extending $id_{\mathbf{R}}$ and such that $\|f_n\| \leq 1$ for every $n \geq 0$.

Proof. We define f_n by the formula

$$f_n(g[g_1|\cdots|g_n]) = (-1)^n g t_{n-1}(g_1 t_{n-2}(g_2 \cdots (g_{n-1} t_0(g_n t_{-1}(1))\cdots).$$

It is clear $g \cdot f_n([g_1|\cdots|g_n]) = f_n(g[g_1|\cdots|g_n])$ so that f_n commutes with action of G. Moreover, since $\|t_*\| \leq 1$ and $\|g \cdot x\|_1 \leq \|x\|_1$ for all $g \in G, x \in V_*$, the map f_n is a G-morphism and has the norm $\|f_n\| \leq 1$.

Note that, for $n = -1$, we have $\varepsilon' f_0[] = 1 = \varepsilon[]$. It remains for us to verify that $f_n \partial_{n+1} = \partial'_{n+1} f_{n+1}$ for every $n \geq 0$. First, note that

$$\begin{aligned} f_{n+1}([g_1|\cdots|g_{n+1}]) &= (-1)^{n+1} t_n(g_1 t_{n-1}(g_2 \cdots t_0(g_{n+1} t_{-1}(1))\cdots) \\ &= (-1)^{n+1} t_n\big(g_1 f_n([g_2|\cdots|g_{n+1}])\big). \end{aligned}$$

Then we assume $f_{n-1}\partial_n = \partial'_n f_n$ and we prove that $f_n \partial_{n+1} = \partial'_{n+1} f_{n+1}$ by the induction on n. Since $\partial'_{n+1}t_n + t_{n-1}\partial'_n = id$, we have

$$\partial'_{n+1}f_{n+1}([g_1|\cdots|g_{n+1}]) = \partial'_{n+1}(-1)^{n+1}\big(t_n(g_1f_n([g_2|\cdots|g_{n+1}])\big)$$

$$= (-1)^{n+1}\partial'_{n+1}t_n\big(g_1f_n([g_2|\cdots|g_{n+1}])\big)$$

$$= (-1)^{n+1}(id - t_{n-1}\partial'_n)\big(g_1f_n([g_2|\cdots|g_{n+1}])\big)$$

$$= (-1)^{n+1}g_1f_n([g_2|\cdots|g_{n+1}]) - (-1)^{n+1}t_{n-1}\partial'_n\big(g_1f_n([g_2|\cdots|g_{n+1}])\big)$$

$$= (-1)^{n+1}f_n(g_1[g_2|\dots|g_{n+1}]) + (-1)^n t_{n-1}\big(g_1\partial'_nf_n([g_2|\cdots|g_{n+1}])\big)$$

$$= (-1)^{n+1}f_n(g_1[g_2|\dots|g_{n+1}]) + (-1)^n t_{n-1}\big(g_1f_{n-1}\partial_n([g_2|\cdots|g_{n+1}])\big)$$

$$= (-1)^{n+1}f_n(g_1[g_2|\cdots|g_{n+1}]) + (-1)^n t_{n-1}\big(g_1f_{n-1}((-1)^n g_2[g_3|\cdots|g_{n+1}]$$

$$+\sum_{i=2}^{n}(-1)^{n+1-i}[g_2|\cdots|g_ig_{i+1}|\cdots|g_{n+1}] + [g_2|\cdots|g_n])\big)$$

$$= (-1)^{n+1}f_n(g_1[g_2|\cdots|g_{n+1}]) + (-1)^n(-1)^n t_{n-1}\big(g_1g_2f_{n-1}([g_3|\cdots|g_{n+1}])\big)$$

$$+\sum_{i=2}^{n}(-1)^{n+1-i}(-1)^n t_{n-1}g_1f_{n-1}([g_2|\cdots|g_ig_{i+1}|\cdots|g_{n+1}])$$

$$+(-1)^n t_{n-1}g_1f_{n-1}([g_2|\cdots|g_n])$$

$$= (-1)^{n+1}f_n(g_1[g_2|\cdots|g_{n+1}]) + (-1)^n f_n([g_1g_2|g_3|\cdots|g_{n+1}])$$

$$+\sum_{i=2}^{n}(-1)^{n+1-i}f_n([g_1|g_2|\cdots|g_ig_{i+1}|\cdots|g_{n+1}]) + f_n([g_1|\cdots|g_n])$$

$$= (-1)^{n+1}f_n(g_1[g_2|\cdots|g_{n+1}])$$

$$+\sum_{i=1}^{n}(-1)^{n-i}f_n([g_1|g_2|\cdots|g_ig_{i+1}|\cdots|g_{n+1}]) + f_n([g_1|\cdots|g_n])$$

$$= f_n\partial_{n+1}([g_1|\cdots|g_{n+1}]).$$

Thus we have $f_n\partial_{n+1} = \partial'_{n+1}f_{n+1}$. □

Corollary 4.3. *In $H_*^{\ell_1}(G)$ the seminorm induced by the bar resolution of G coincides with the canonical seminorm.*

Proof. It follows from Theorem 4.2. □

Recall that the ordinary cochain groups of a space are defined as the algebraic dual space of the chain groups. Now we describe bounded cochain groups as the dual space of ℓ_1 chain groups.

Proposition 4.4. *The space $B(G^{n+1})^G$ is the dual space of $C_n^{\ell_1}(G)_G$. Moreover the boundary operator $d_{n-1}\colon B(G^n)^G \to B(G^{n+1})^G$ is the adjoint of the operator $\partial_n\colon C_n^{\ell_1}(G)_G \to C_{n-1}^{\ell_1}(G)_G$.*

Proof. If $x = \sum r_i[g_{i_1}|\cdots|g_{i_n}] \in C_n^{\ell_1}(G)_G$ and $f \in B(G^{n+1})^G$, then

$$< x, f >= \sum r_i f(g_{i_1}, \cdots, g_{i_n}, 1).$$

Also for every $f \in B(G^n)^G$

$$\begin{aligned}
&< x, d_{n-1}f >= \sum r_i(d_{n-1}f)(g_{i_1}, \cdots, g_{i_n}, 1)\\
&= \sum r_i((-1)^n f(g_{i_2}, \cdots, g_{i_n}, 1)\\
&+ \sum_{j=1}^{n-1}(-1)^{n-j} f(g_{i_1}, \cdots, g_{i_j}g_{i_{j+1}}, \cdots, g_{i_n}, 1)) + f(g_{i_1}, \cdots, g_{i_{n-1}}, g_{i_n} \cdot 1)\\
&= \sum r_i((-1)^n f(g_{i_2}, \cdots, g_{i_n}, 1) + \sum_{j=1}^{n-1}(-1)^{n-j} f(g_{i_1}, \cdots, g_{i_j}g_{i_{j+1}}, \cdots, g_{i_n}, 1))\\
&+ f(g_{i_1}, \cdots, g_{i_{n-1}}, 1)\\
&=< r_i((-1)^n[g_{i_2}|\cdots|g_{i_n}] + \sum_{j=1}^{n-1}[g_{i_1}|\cdots|g_{i_j}g_{i_{j+1}}|\cdots|g_{i_n}] + [g_{i_1}|\cdots|g_{i_{n-1}}]), f >\\
&=< \partial_n x, f > .
\end{aligned}$$

This finishes the proof. ∎

Remark 4.1. In [10] Theorem 2.3 shows that if $\mathrm{Im}\{C_{n+1}^{\ell_1}(G) \xrightarrow{\partial_{n+1}} C_n^{\ell_1}(G)\}$ is closed in $C_n^{\ell_1}(G)$, then $\widehat{H}^{n+1}(G)$ is isomorphic with the dual Banach space of $H_{n+1}^{\ell_1}(G)$. Also it is known that $H_1^{\ell_1}(G) = 0$. In fact, by using the bar resolution of G, it is easy to see that $\partial_1 = 0$. Also for any $[g] \in C_1^{\ell_1}(G)_G$, it is constructed $S([g]) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}}[g^{2^k} \mid g^{2^k}]$ which is clearly $\|S([g])\|_1 = 1$ and $\partial_2 S([g]) = [g]$ showing that $H_1^{\ell_1}(G) = 0$.

Note that $H_*^{\ell_1}(G)$ is a covariant functor of G. As in the bounded cohomology, this functoriality can also be described in terms of arbitrary resolutions as follows: given a group homomorphism $\alpha\colon G \to H$ and strong relatively projective resolutions $\mathcal{U}$ and $\mathcal{V}$ of the trivial G- and H-module $\mathbf{R}$, respectively, we can regard $\mathcal{V}$ as a strong G-resolution via α. So we have an augmentation preserving G-morphism of resolutions $\tau\colon \mathcal{U} \to \mathcal{V}$, well-defined up to homotopy. The condition that τ be a G-morphism is expressed by the formula $\tau(gx) = \alpha(g)\tau(x)$ for $g \in G$ and $x \in \mathcal{U}$. Clearly τ induces a map $\mathcal{U}_G \to \mathcal{V}_H$, well-defined up to homotopy, hence we obtain a well-defined map $\alpha_*\colon H_*^{\ell_1}(G) \to H_*^{\ell_1}(H)$ which depend only on α. Note that $\|\alpha_*\| \le 1$.

Now we shall see the relationship between amenable groups and ℓ_1 homology. Let A be an amenable subgroup of G. We consider G/A, the set of (right) cosets Ag of A in G. Since the set of cosets Ag has the G-action by right translation, we can define $C_*^{\ell_1}(G/A)$ as the same manner with $C_*^{\ell_1}(G)$. Namely, we can take $C_n^{\ell_1}(G/A)$ as the free $\mathbf{R}$-module generated by the n-tuples of the form $[Ag_1|\cdots|Ag_n]$. The action of a G-module is given by the formula $g'[Ag_1|\cdots|Ag_n] = Ag'[Ag_1|\cdots|Ag_n]$. The canonical map $p_n\colon C_n^{\ell_1}(G) \to C_n^{\ell_1}(G/A)$ is a G-morphism and has the norm $\|p_n\| \le 1$.

Lemma 4.5. *Let A be an amenable subgroup of G. Then there exists a G-morphism $q_1\colon C_1^{\ell_1}(G/A) \to C_1^{\ell_1}(G)$ such that $p_1 \circ q_1 = id$ and $\|q_1\| = 1$.*

Proof. Recall that there is a right invariant mean on $B(A)$, i.e., the linear functional $m\colon B(A) \to \mathbf{R}$ so that $m(a \cdot f) = m(f)$ where $a \cdot f(a') = f(a'a)$ for $a, a' \in A$ and $f \in B(A)$. Also recall that, on any coset Ag, the map m defines a mean $m_g\colon B(Ag) \to \mathbf{R}$ by $m_g(\varphi) = m(f)$, where $f(a) = \varphi(ag)$.

For each $x \in G$, consider the characteristic function $\delta_x\colon G \to \mathbf{R}$, i.e.,

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

For every $Ag'[Ag] \in C_1^{\ell_1}(G/A)$, we define q_1 by the formula

$$q_1(Ag'[Ag]) = \sum_{x \in G} m_g(\delta_x|_{Ag}) g'[x].$$

Since $0 \le \delta_x|_{Ag} \le 1$, we have $0 \le m_g(\delta_x|_{Ag}) \le 1$ and also

$$\sum_{x\in G} \left| m_g(\delta_x|_{Ag}) \right| = \sum_{x\in G} m_g(\delta_x|_{Ag}) = m_g(\sum_{x\in G} \delta_x|_{Ag}) = m_g(\bar{1}_{Ag}) = 1, \tag{4.5.1}$$

where $\bar{1}_{Ag}$ is a constant function on Ag with value 1. Thus we have $\|q_1\| = 1$.

Note that

$$\begin{aligned} p_1 \circ q_1([Ag]) &= p_1\big(\sum_{x\in G} m_g(\delta_x|_{Ag})[x]\big) \\ &= \sum_{x\in G} m_g(\delta_x|_{Ag}) p_1([x]) = \sum_{x\in Ag} m_g(\delta_x|_{Ag}) p_1([x]) \\ &= \sum_{x\in Ag} m_g(\delta_x|_{Ag})[Ag] = \big(\sum_{x\in G} m_g(\delta_x|_{Ag})\big)[Ag] \\ &= [Ag], \end{aligned}$$

where the last equality follows from (4.5.1). This shows $p_1 \circ q_1 = id$.

Finally note that

$$\begin{aligned} g'q_1([Ag]) &= g' \sum_{x\in G} m_g(\delta_x|_{Ag})[x] = \sum_{x\in G} m_g(\delta_x|_{Ag}) g'[x] \\ &= q(Ag'[Ag]) = q(g'[Ag]) \end{aligned}$$

and so q_1 commutes with the action of G. □

Corollary 4.6. *Let A be an amenable subgroup of G. Then for every $n \ge 0$ there exists a G-morphism $q_n \colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ such that $p_n q_n = 1$ and $\|q_n\| = 1$, where $p_n \colon C_n^{\ell_1}(G) \to C_n^{\ell_1}(G/A)$ is the canonical map.*

Proof. Since the spaces $C_0^{\ell_1}(G)$ and $C_0^{\ell_1}(G/A)$ have only one basis element denoted by $[\,]$, we define q_0 by the formula $q_0([\,]) = [\,]$.

Note that $(G/A)^n = G^n/A^n$ and A^n is an amenable normal subgroup of G^n. We may consider $C_n^{\ell_1}(G/A)$ as $C_1^{\ell_1}(G^n/A^n)$ by setting up each basis $[Ag_1|\cdots|Ag_n]$ of $C_n^{\ell_1}(G/A)$ by

$$[Ag_1|\cdots|Ag_n] = A^n[(g_1,\cdots,g_n)].$$

Then Lemma 4.5 provides a G^n-morphism $q_n\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ such that $p_n q_n = 1$ and $\|q_n\| = 1$. Especially, since the G-module structure on $C_n^{\ell_1}(G)$ and on $C_n^{\ell_1}(G/A)$ are the restriction of the canonical G^n-module structure, p_n is also a G-morphism. □

Lemma 4.7. *Let A be an amenable subgroup of G. Then every G-module $C_n^{\ell_1}(G/A)$ is relatively projective for every $n \geq 0$.*

Proof. Consider the diagram

$$\begin{array}{ccc} C_n^{\ell_1}(G) & \xrightarrow{p} & C_n^{\ell_1}(G/A) \\ {\scriptstyle\beta'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ V & \xrightarrow{\pi} & W \end{array}$$

where a G-morphism α and a strongly projective G-morphism π are given.

We need to construct a G-morphism $\beta\colon C_n^{\ell_1}(G/A) \to V$ such that $\pi\beta = \alpha$ and $\|\beta\| \leq \|\alpha\|$. Since $C_n^{\ell_1}(G)$ is a relatively projective G-module, there exists a G-morphism $\beta'\colon C_n^{\ell_1}(G) \to V$ such that $\alpha p = \pi\beta'$ and $\|\beta'\| \leq \|\alpha p\| \leq \|\alpha\|$. Moreover, there is a G-morphism $q\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ constructed in Corollary 4.6. We define $\beta = \beta' q$. Then $\pi\beta = \pi\beta' q = \alpha p q = \alpha$ and also $\|\beta\| = \|\beta' q\| \leq \|\beta'\|\|q\| \leq \|\alpha\|\|q\| \leq \|\alpha\|$. □

We introduce another useful strong relatively projective G-resolution.

Let A be an amenable subgroup of G. From Lemma 4.7, the G-module $C_*^{\ell_1}(G/A)$ is relatively projective. Moreover the sequence

$$\to C_3^{\ell_1}(G/A) \xrightarrow{\partial_3} C_2^{\ell_1}(G/A) \xrightarrow{\partial_2} C_1^{\ell_1}(G/A) \xrightarrow{\partial_1} C_0^{\ell_1}(G/A) \xrightarrow{\varepsilon} \mathbf{R} \to 0 \qquad (4.3)$$

is a strong relatively projective G-resolution of the trivial G-module $\mathbf{R}$, where the boundary and contracting operators are defined by the same formulas in the sequence

(4.1). Also it is easy to see that the induced sequence

$$\to C_3^{\ell_1}(G/A)_G \xrightarrow{\partial_3} C_2^{\ell_1}(G/A)_G \xrightarrow{\partial_2} C_1^{\ell_1}(G/A)_G \xrightarrow{\partial_1} C_0^{\ell_1}(G/A)_G \xrightarrow{\partial_0} 0 \tag{4.4}$$

is a complex and the homology of this complex is $H_*^{\ell_1}(G)$.

Proposition 4.8. *Let A be an amenable subgroup of G. Then the seminorm in $H_*^{\ell_1}(G)$ induced by the resolution in (4.3) coincides with the canonical seminorm.*

Proof. Let $\|\cdot\|_1$ denote the canonical seminorm in $H_*^{\ell_1}(G)$ and $\|\cdot\|_1^s$ the seminorm in $H_*^{\ell_1}(G)$ induced by the resolution (4.3). By definition of the canonical seminorm in $H_*^{\ell_1}(G)$, we have $\|\cdot\|_1^s \leq \|\cdot\|_1$.

Note that, as we proved in Corollary 4.6, for every $n \geq 0$ there exist a G-morphism $q_n\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ such that $p_n q_n = id$ and $\|q_n\| = 1$, where $p_n\colon C_n^{\ell_1}(G) \to C_n^{\ell_1}(G/A)$ is the canonical map. Thus the seminorm achieved by the resolution (4.3) is not less than the canonical seminorm, i.e., $\|\cdot\|_1 \leq \|\cdot\|_1^s$. Hence $\|\cdot\|_1 = \|\cdot\|_1^s$. □

Theorem 4.9. *Let A be an amenable normal subgroup of G. Then the map $\varphi_*\colon H_*^{\ell_1}(G) \to H_*^{\ell_1}(G/A)$, which is induced by the canonical map $\varphi\colon G \to G/A$, is an isometric isomorphism, i.e., the isomorphism preserves the canonical seminorm.*

Proof. Note that the sequence (4.3) is the bar resolution of G/A. So the homology of the complex in (4.4) is $H_*^{\ell_1}(G/A)$ and the induced seminorm in which is the canonical one. Hence it follows from Proposition 4.8. □

As before, we denote the coset A in A by $\{A\}$. If A is an amenable group, we have a complex

$$\to C_3^{\ell_1}(\{A\})_A \xrightarrow{\partial_3'} C_2^{\ell_1}(\{A\})_A \xrightarrow{\partial_2'} C_1^{\ell_1}(\{A\})_A \xrightarrow{\partial_1'} C_0^{\ell_1}(\{A\})_A \xrightarrow{\partial_0'} 0 \tag{4.5}$$

induced from the strong relatively projective A-resolution of the trivial A-module $\mathbf{R}$

$$\to C_3^{\ell_1}(\{A\}) \xrightarrow{\partial_3'} C_2^{\ell_1}(\{A\}) \xrightarrow{\partial_2'} C_1^{\ell_1}(\{A\}) \xrightarrow{\partial_1'} C_0^{\ell_1}(\{A\}) \xrightarrow{\varepsilon} \mathbf{R} \to 0.$$

It is clear that the homology of the complex in (4.5) is $H_*^{\ell_1}(A)$ and the norm induced by this complex coincides with the canonical seminorm.

Remark that the boundary operators in the complex (4.5) are in fact

$$\partial'_n = \begin{cases} id & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4.10. *If the group G is amenable, then the group $H_*^{\ell_1}(G)$ is zero.*

Proof. It follows from the complex in (4.5) by setting up $G = A$. □

CHAPTER 5

Relative ℓ_1 homology of groups

Let $\varphi\colon A \to G$ be a group homomorphism. As in the relative bounded cohomology, we define the relative ℓ_1 homology of G modulo A. Remark that there is an induced homomorphism $\varphi_*\colon H_*^{\ell_1}(A) \to H_*^{\ell_1}(G)$ which depends only on φ. Also remark that $\|\varphi_*\| \leq 1$.

Definition 5.1. *Let $\varphi\colon A \to G$ be a group homomorphism. A strong relatively projective G-resolution of a G-module U*

$$\cdots \underset{k_2}{\overset{\partial_3}{\rightleftarrows}} U_2 \underset{k_1}{\overset{\partial_2}{\rightleftarrows}} U_1 \underset{k_0}{\overset{\partial_1}{\rightleftarrows}} U_0 \underset{k_{-1}}{\overset{\partial_0}{\rightleftarrows}} U \to 0$$

and a strong relatively projective A-resolution of an A-module U

$$\cdots \underset{t_2}{\overset{\partial'_3}{\rightleftarrows}} V_2 \underset{t_1}{\overset{\partial'_2}{\rightleftarrows}} V_1 \underset{t_0}{\overset{\partial'_1}{\rightleftarrows}} V_0 \underset{t_{-1}}{\overset{\partial'_0}{\rightleftarrows}} U \to 0$$

are called a co-allowable pair of resolutions for $(G, A; U)$ if id_U can be extended to an A-morphism of resolutions $\lambda_n\colon V_n \to U_n$ such that λ_n commutes with the contracting homotopies k_n and t_n for all $n \geq 0$.

Proposition 5.1. *Let $\varphi\colon A \to G$ be a group homomorphism. The bar resolutions of G and A are a co-allowable pair of resolutions for $(G, A; \mathbf{R})$.*

Proof. Since the proof is very similar to an allowable pair of resolutions in Proposition 2.1, we refer to it in detail. Recall that the bar resolutions of G and A

$$\cdots \underset{k_2}{\overset{\partial_3}{\rightleftarrows}} C_2^{\ell_1}(G) \underset{k_1}{\overset{\partial_2}{\rightleftarrows}} C_1^{\ell_1}(G) \underset{k_0}{\overset{\partial_1}{\rightleftarrows}} C_0^{\ell_1}(G) \underset{k_{-1}}{\overset{\partial_0}{\rightleftarrows}} \mathbf{R} \to 0$$

$$\cdots \underset{t_2}{\overset{\partial_3'}{\rightleftarrows}} C_2^{\ell_1}(A) \underset{t_1}{\overset{\partial_2'}{\rightleftarrows}} C_1^{\ell_1}(A) \underset{t_0}{\overset{\partial_1'}{\rightleftarrows}} C_0^{\ell_1}(A) \underset{t_{-1}}{\overset{\partial_0'}{\rightleftarrows}} \mathbf{R} \to 0$$

are strong relatively projective.

We define a map $\lambda_n \colon C_n^{\ell_1}(A) \to C_n^{\ell_1}(G)$ by the formula

$$\lambda_n([a_1|\cdots|a_n]) = [\varphi(a_1)|\cdots|\varphi(a_n)].$$

Then, by the standard calculation, it is easy to check that λ_n is an A-morphism extending $id_{\mathbf{R}}$ and such that it commutes with the contracting homotopies. □

Definition 5.2. *Let $\varphi \colon A \to G$ be a group homomorphism. Let*

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \textit{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be the G- and A-resolutions respectively such that they are a co-allowable pair for $(G, A; \mathbf{R})$. The mapping cone and mapping cylinder of chain complexes induced by φ, respectively, are defined as follows:

$$C_n(A \overset{\varphi}{\to} G) = (U_n)_G \bigoplus (V_{n-1})_A$$
$$EC_n(A \overset{\varphi}{\to} G) = (V_n)_A \bigoplus (U_n)_G \bigoplus (V_{n-1})_A$$

Lemma 5.2. *Let*

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \textit{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair for $(G, A; \mathbf{R})$ as in Definition 5.2. Also let $\lambda_n \colon V_n \to U_n$ be an A-morphism of resolutions commuting with the contracting homotopies. Then the sequences

$$\cdots \xrightarrow{d_3} C_2(A \xrightarrow{\varphi} G) \xrightarrow{d_2} C_1(A \xrightarrow{\varphi} G) \xrightarrow{d_1} C_0(A \xrightarrow{\varphi} G) \to 0 \quad (5.2.1)$$

$$\cdots \xrightarrow{d_3} EC_2(A \xrightarrow{\varphi} G) \xrightarrow{d_2} EC_1(A \xrightarrow{\varphi} G) \xrightarrow{d_1} EC_0(A \xrightarrow{\varphi} G) \to 0 \quad (5.2.2)$$

are complexes, where the boundary operators are defined by the formulas

$$d_n(x_n, a_{n-1}) = (\partial_n x_n + \lambda_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1})$$

$$d_n(a_n, x_n, a_{n-1}) = (\partial'_n a_n - a_{n-1}, \partial_n x_n + \lambda_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1}).$$

Proof. We check $d_{n-1}d_n = 0$ for $C_n(A \xrightarrow{\varphi} G)$.

$$\begin{aligned} &d_{n-1}d_n(x_n, a_{n-1}) \\ &= d_{n-1}(\partial_n x_n + \lambda_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1}) \\ &= (\partial_{n-1}\partial_n x_n + \partial_{n-1}\lambda_{n-1} a_{n-1} - \lambda_{n-1}\partial'_{n-1} a_{n-1}, \partial'_{n-2}\partial'_{n-1} a_{n-1}) \\ &= (0, 0). \end{aligned}$$

By the same way, we can check $d_{n-1}d_n = 0$ for $EC_n(A \xrightarrow{\varphi} G)$. □

Definition 5.3. *The n-th homology of the complex in (5.2.1) is called the n-th relative ℓ_1 homology of G modulo A and is denoted by $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$. The n-th homology of the complex in (5.2.2) is denoted by $H_n^{\ell_1}(EC(A \xrightarrow{\varphi} G))$.*

We define the norm $\|\cdot\|_1$ on $C_n(A \xrightarrow{\varphi} G)$ by setting

$$\|(x_n, a_{n-1})\|_1 = \|x_n\|_1 + \|a_{n-1}\|_1$$

and similarly on $EC_n(A \xrightarrow{\varphi} G)$. Remark that these norms define the seminorms $\|\cdot\|_1$ in $H_*^{\ell_1}(EC(A \xrightarrow{\varphi} G))$ and in $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ respectively.

Furthermore, for every $\omega \geq 0$, we define a norm $\|\cdot\|_1(\omega)$ on $C_n(A \xrightarrow{\varphi} G)$ by putting

$$\|(x_n, a_{n-1})\|_1(\omega) = \|x_n\|_1 + (1+\omega)\|a_{n-1}\|_1.$$

Then we have the corresponding seminorm $\|\cdot\|_1(\omega)$ in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$. Finally we define these norms in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ for all ω in the closed interval $[0,\infty]$ by passing to the limits. Remark that, for $0 \le \omega_1 \le \omega_2$, we have

$$\|\cdot\|_1 = \|\cdot\|_1(0) \le \|\cdot\|_1(\omega_1) \le \|\cdot\|_1(\omega_2).$$

Theorem 5.3. *Let $\varphi\colon A \to G$ be a group homomorphism. Then the natural inclusion map $\rho_n\colon C_n^{\ell_1}(G)_G \to EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ induces an isometric isomorphism $H_n(\rho)\colon H_n^{\ell_1}(G) \to H_n^{\ell_1}(EC(A \xrightarrow{\varphi} G))$ for the norm $\|\cdot\|_1$.*

Proof. We consider the exact sequence

$$\begin{aligned} 0 &\to C_n^{\ell_1}(G)_G \\ &\xrightarrow{\rho_n} EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A \\ &\to C_n^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(A)_A \to 0. \end{aligned}$$

It is easy to check that $C_n^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(A)_A$ is a complex. If (a,b) is a cycle of the complex $C_n^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(A)_A$, then we have $0 = d(a,b) = (da - b, -db)$ and so $b = da$. Thus $(a,b) = d(0,-a)$ is a boundary. This shows that the groups $H_*^{\ell_1}(G)$ and $H_*^{\ell_1}(EC(A \xrightarrow{\varphi} G))$ are isomorphic.

We denote every boundary operator by the same notation d. We consider the diagram

$$\begin{array}{ccccc} C_n^{\ell_1}(G)_G & \xrightarrow{\rho_n} & C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A & \xrightarrow{\widetilde{\rho}_n} & C_n^{\ell_1}(G)_G \\ d\downarrow & & d\downarrow & & d\downarrow \\ C_{n-1}^{\ell_1}(G)_G & \xrightarrow{\rho_{n-1}} & C_{n-1}^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(G)_G \bigoplus C_{n-2}^{\ell_1}(A)_A & \xrightarrow{\widetilde{\rho}_{n-1}} & C_{n-1}^{\ell_1}(G)_G \end{array}$$

where $\widetilde{\rho}_n(a,x,b) = x + \lambda_n a$ and $\lambda_n\colon C_n^{\ell_1}(A)_A \to C_n^{\ell_1}(G)_G$ is an induced homomorphism from φ. It is clear that $\widetilde{\rho}_n \rho_n = id$ and that the first square is commutative. Since we have $\|\rho_n(x)\|_1 = \|(0,x,0)\|_1 = \|x\|_1$, the map $H_n(\rho)$ has the norm $\|H_n(\rho)\| \le 1$.

Note that

$$\widetilde{\rho}_n d(a,x,b) = \widetilde{\rho}_n(da-b, dx+\lambda b, -db) = dx+\lambda b+\lambda(da-b)$$
$$= dx+\lambda da = d(x+\lambda a) = d\widetilde{\rho}_n(a,x,b),$$

so that $\widetilde{\rho}_n$ commutes with the boundary operators. Also note that

$$\|\widetilde{\rho}_n(a,x,b)\|_1 = \|x+\lambda_n a\|_1 \leq \|x\|_1 + \|a\|_1$$
$$\leq \|a\|_1 + \|x\|_1 + \|b\|_1 = \|(a,x,b)\|_1$$

and so the map $\widetilde{\rho}_n$ has the norm $\|\widetilde{\rho}_n\| \leq 1$. This shows that the inverse map $(H_n(\rho))^{-1}$ has the norm $\|(H_n(\rho))^{-1}\| \leq 1$. Hence the isomorphism $H_n(\rho)$ is also an isometry. □

Let $\varphi\colon A \to G$ be a group homomorphism. Let the sequences

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair of resolutions for $(G, A; \mathbf{R})$. Then there is an exact sequence

$$0 \to (V_n)_A \xrightarrow{i_n} EC_n(A \xrightarrow{\varphi} G) \xrightarrow{p_n} C_n(A \xrightarrow{\varphi} G) \to 0,$$

where i_n and p_n are natural inclusion and projection maps respectively. Also this sequence induces a long exact sequence

$$\cdots \to H_n^{\ell_1}(A) \to H_n^{\ell_1}(G) \to H_n^{\ell_1}(A \xrightarrow{\varphi} G) \to H_{n-1}^{\ell_1}(A) \to \cdots.$$

Remark that the seminorm in $H_*^{\ell_1}(EC(A \xrightarrow{\varphi} G))$, which is induced by the bar resolutions, coincides with the canonical seminorm. Also remark that a seminorm in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ depends on the choice of a co-allowable pair of resolutions for $(G, A; \mathbf{R})$. We define the canonical seminorm in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ by the supremum of the seminorms which arise from every co-allowable pair of resolutions for $(G, A; \mathbf{R})$.

Theorem 5.4. *The seminorm $\|\cdot\|_1(\omega)$ in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$, which is induced by the bar resolutions of G and A, coincides with the canonical seminorm for every $\omega \in [0,\infty]$.*

Proof. Let

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair for $(G, A; \mathbf{R})$ as in Definition 5.2. Also let $\lambda_n \colon V_n \to U_n$ be an A-morphism of resolutions commuting with the contracting homotopies. Let $\alpha_n \colon C_n^{\ell_1}(G)_G \to (U_n)_G$ and $\gamma_{n-1} \colon C_{n-1}^{\ell_1}(A)_A \to (V_{n-1})_A$ be the maps defined in Theorem 4.2. We define a map

$$\beta_n \colon C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A \to (U_n)_G \bigoplus (V_{n-1})_A$$

by the formula $\beta_n(x, a) = (\alpha_n x,\ \gamma_{n-1} a)$.

Since we can prove that the map β_n is a chain map as the same way in Theorem 2.7, we refer this proof to it. Now note that, for every $\omega \geq 0$,

$$\begin{aligned} \|\beta_n(x,\ a)\|_1(\omega) &= \|(\alpha_n x,\ \gamma_{n-1} a)\|_1(\omega) \\ &= \|\alpha_n x\|_1 + (1+\omega)\|\gamma_{n-1} a\|_1 \\ &\leq \|x\|_1 + (1+\omega)\|a\|_1 = \|(x,\ a)\|_1(\omega) \end{aligned}$$

and so $\|\beta_n\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$. □

Lemma 5.5. *Let A be an amenable subgroup of G. Then the sequences*

$$\to C_3^{\ell_1}(G/A) \xrightarrow{\partial_3} C_2^{\ell_1}(G/A) \xrightarrow{\partial_2} C_1^{\ell_1}(G/A) \xrightarrow{\partial_1} C_0^{\ell_1}(G/A) \xrightarrow{\varepsilon} \mathbf{R} \to 0$$

$$\to C_3^{\ell_1}(\{A\}) \xrightarrow{\partial_3'} C_2^{\ell_1}(\{A\}) \xrightarrow{\partial_2'} C_1^{\ell_1}(\{A\}) \xrightarrow{\partial_1'} C_0^{\ell_1}(\{A\}) \xrightarrow{\varepsilon} \mathbf{R} \to 0$$

are a co-allowable pair of resolutions for $(G, A; \mathbf{R})$.

Proof. We define a map $\lambda_n \colon C_n^{\ell_1}(\{A\}) \to C_n^{\ell_1}(G/A)$ by the formula

$$\lambda_n([\underbrace{\{A\}|\cdots|\{A\}}_{n}]) = [\underbrace{A|\cdots|A}_{n}].$$

The rest of the proof is exactly same as the proof of Proposition 5.1. □

Theorem 5.6. *Let A be an amenable subgroup of G, and let $\varphi\colon A \hookrightarrow G$ be an inclusion map. Then the seminorm $\|\cdot\|_1(\omega)$ in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$, induced by the complex*

$$C_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A,$$

coincides with the canonical seminorm for every $\omega \in [0, \infty]$.

Proof. Let $\|\cdot\|_1(\omega)$ denote the canonical seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ and let $\|\cdot\|_1^s(\omega)$ the seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ induced by the complex $C_n(A \xrightarrow{\varphi} G)$.

By definition of the canonical seminorm, we have $\|\cdot\|_1^s(\omega) \leq \|\cdot\|_1(\omega)$. So it is enough for us to show that $\|\cdot\|_1(\omega) \leq \|\cdot\|_1^s(\omega)$.

From Theorem 5.4, the canonical seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ is induced by the bar resolutions of G and A, i.e., by the complex $C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$.

Recall that, from Corollary 4.6, there are maps

$$q_n\colon C_n^{\ell_1}(G/A)_G \to C_n^{\ell_1}(G)_G \quad \text{and} \quad q_n'\colon C_n^{\ell_1}(\{A\})_A \to C_{n-1}^{\ell_1}(A)_A.$$

We define the map

$$\beta_n\colon C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A \to C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$$

by the formula $\beta_n(x, a) = (q_n x,\ q_{n-1}' a)$.

By the standard calculation, it is easy to check the map β_n commutes with the boundary operators. Also, since we have $\|q_n\| \leq 1$ and $\|q_{n-1}'\| \leq 1$, it is easy to check that β_n has the norm $\|\beta_n\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$. This shows that $\|\cdot\|_1(\omega) \leq \|\cdot\|_1^s(\omega)$. □

Theorem 5.7. *Let A be an amenable subgroup of G and let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then, for every $n \geq 2$, the groups $H_n^{\ell_1}(G)$ and $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ are isometrically isomorphic for the norm $\|\cdot\|_1$.*

Proof. It is enough for us to consider the following co-allowable pair of resolutions for $(G, A; \mathbf{R})$:

$$\to C_3^{\ell_1}(G/A) \xrightarrow{\partial_3} C_2^{\ell_1}(G/A) \xrightarrow{\partial_2} C_1^{\ell_1}(G/A) \xrightarrow{\partial_1} C_0^{\ell_1}(G/A) \xrightarrow{\varepsilon} \mathbf{R} \to 0$$

$$\to C_3^{\ell_1}(\{A\}) \xrightarrow{\partial_3'} C_2^{\ell_1}(\{A\}) \xrightarrow{\partial_2'} C_1^{\ell_1}(\{A\}) \xrightarrow{\partial_1'} C_0^{\ell_1}(\{A\}) \xrightarrow{\varepsilon} \mathbf{R} \to 0.$$

Let

$$C_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A$$

$$EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(\{A\})_A \bigoplus C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A.$$

Then there is an exact sequence

$$0 \to C_n^{\ell_1}(\{A\})_A \xrightarrow{i_n} EC_n(A \xrightarrow{\varphi} G) \xrightarrow{p_n} C_n(A \xrightarrow{\varphi} G) \to 0$$

and it induces the following exact sequence

$$\cdots \to H_n^{\ell_1}(A) \to H_n^{\ell_1}(G) \xrightarrow{H_n(p)} H_n^{\ell_1}(A \xrightarrow{\varphi} G) \to H_{n-1}^{\ell_1}(A) \to \cdots.$$

Since the group $H_*^{\ell_1}(A) = 0$, the groups $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ and $H_n^{\ell_1}(G)$ are isomorphic. We denote by ∂_* and ∂_*' the boundary operators on $C_n^{\ell_1}(G/A)_G$ and $C_n^{\ell_1}(\{A\})_A$ respectively. Note that the induced map $\lambda_n \colon C_n^{\ell_1}(\{A\})_A \to C_n^{\ell_1}(G/A)_G$ is defined as an inclusion map.

Let $(b, x, a) \in C_n^{\ell_1}(\{A\})_A \bigoplus C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A$ be a cycle. Then, by definition of boundary operator, we have

$$\partial_n' b - a = 0, \quad \partial_n x + a = 0, \quad \text{and} \quad \partial_{n-1}' a = 0.$$

Let n be odd. Then $\partial_{n-1}' = id$ and so $a = 0$. Also since $\partial_n' = 0$ and $\partial_n' b - a = \partial_n' b = 0$, there is an element $c \in C_n^{\ell_1}(\{A\})_A$ such that $\partial_n' c = b$. So we have $(b, x, a) = (b, x, 0) = (0, x, 0) + d(c, 0, 0)$ and $d(0, x, 0) = 0$. Thus

$$\begin{aligned}(H_n(p))^{-1} H_n(p)([b, x, a]) &= (H_n(p))^{-1}([x, a]) \\ &= (H_n(p))^{-1}([x, 0]) = [0, x, 0] = [b, x, a].\end{aligned}$$

Then $\|(H_n(p))^{-1}[x,\, a]\|_1 = \|[0,\, x,\, 0]\| \le \|[x,\, a]\|_1$ and so $\|(H_n(p))^{-1}\| \le 1$ for every odd n.

Let n be even. Then $\partial'_n = id$ and $\partial'_{n-1} = 0$. So $a \in \ker(\partial'_{n-1}) = \mathrm{Im}(\partial'_n)$ and so there is an element $c \in C_n^{\ell_1}(\{A\})_A$ such that $\partial'_n c = a$ and $\|c\|_1 = \|a\|_1$. Note that $d(0,\, x + c,\, 0) = (0,\, \partial_n x + \partial'_n c,\, 0) = (0,\, 0,\, 0)$ and also

$$(0,\, x + c,\, 0) + d(0,\, 0,\, -c) = (c,\, x,\, \partial'_n c) = (b,\, x,\, a).$$

Thus $(H_n(p))^{-1}([x,\, a])$ is represented by a cycle $(0,\, x+c,\, 0) \in EC_n(A \overset{\varphi}{\to} G)$ and also

$$\begin{aligned}\|(H_n(p))^{-1}([x,\, a])\|_1 &\le \|(0,\, x + c,\, 0)\|_1 = \|x + c\|_1 \\ &\le \|x\|_1 + \|c\|_1 = \|x\|_1 + \|a\|_1 = \|(x,\, a)\|_1.\end{aligned}$$

This shows that $\|(H_n(p))^{-1}\| \le 1$ for every even n. □

Theorem 5.8. *Let A be an amenable subgroup of G and let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then the norms $\|\cdot\|_1(\omega)$ in $H_n^{\ell_1}(A \overset{\varphi}{\to} G)$ are equal for all $n \ge 2$ and for all $\omega \in [0, \infty]$.*

Proof. Let $\omega \ge 0$. Since we have the inequality $\|\cdot\|_1 = \|\cdot\|_1(0) \le \|\cdot\|_1(\omega)$, it is enough for us to show that $\|\cdot\|_1(\omega) \le \|\cdot\|_1$.

Let $(x, a) \in C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A$ be a cycle. Then we have $0 = d(x, a) = (\partial_n x + \lambda_{n-1} a,\, -\partial'_{n-1} a)$ and so $\partial'_{n-1} a = 0$, where $\lambda_n\colon C_n^{\ell_1}(\{A\})_A \to C_n^{\ell_1}(G/A)_G$ is an induced map from φ and ∂_n and ∂'_n are the boundary operators on $C_n^{\ell_1}(G/A)_G$ and $C_n^{\ell_1}(\{A\})_A$ respectively.

Recall that, if $n-1$ is even, then $\partial'_{n-1} = id$ and so $\partial'_{n-1} a = 0$ gives $a = 0$. Thus

$$\|(x, a)\|_1(\omega) = \|x\|_1 + (1 + \omega)\|a\|_1 = \|x\|_1 = \|x\|_1 + \|a\|_1 = \|(x, a)\|_1.$$

If $n-1$ is odd, then $\partial'_n = id$ and $\partial'_{n-1} a = 0$ gives $a \in \ker(\partial'_{n-1}) = \mathrm{Im}(\partial'_n)$. So there is an element $a_n \in C_n^{\ell_1}(\{A\})_A$ such that $\partial'_n a_n = a$ and $\|a_n\|_1 = \|\partial'_n a_n\|_1 = \|a\|_1$.

Then we have

$$(x,a)+d(0,a_n)=(x,a)+(\lambda_n a_n,\,-\partial'_n a_n)$$
$$=(x+\lambda_n a_n,\;a-\partial'_n a_n)=(x+\lambda_n a_n,\;0).$$

So we have

$$\|[x,a]\|_1(\omega)\le\|(x+\lambda_n a_n,\;0)\|_1(\omega)=\|x+\lambda_n a_n\|_1$$
$$\le\|x\|_1+\|\lambda_n a_n\|_1\le\|x\|_1+\|a_n\|_1$$
$$=\|x\|_1+\|a\|_1=\|(x,\;a)\|_1.$$

In the both cases we have $\|[x,\,a]\|_1(\omega)\le\|[x,\,a]\|_1$. Then, by passing to the limits, we have $\|[x,\,a]\|_1(\omega)\le\|[x,\,a]\|_1$ for all $\omega\in[0,\infty]$. □

Theorem 5.9. *Let $\varphi\colon A\to G$ and $\varphi'\colon A'\to G'$ be the group homomorphisms respectively. Let $\alpha\colon G\to G'$ and $\gamma\colon A\to A'$ be the surjective homomorphisms with the amenable kernels respectively and such that $\alpha\circ\varphi=\varphi'\circ\gamma$. Then the groups $H_*^{\ell_1}(A'\xrightarrow{\varphi'}G')$ and $H_*^{\ell_1}(A\xrightarrow{\varphi}G)$ are isometrically isomorphic for the norm $\|\cdot\|_1(\omega)$ for every $\omega\in[0,\infty]$. This isomorphism preserves the canonical seminorms.*

Proof. Denote $\ker(\alpha)$ and $\ker(\gamma)$ by K and N respectively. We identify the groups G' and A' with G/K and A/N respectively and denote the homomorphism $A/N\to G/K$ by ρ. By considering the following complexes

$$C_n(A/N\xrightarrow{\rho}G/K)=C_n^{\ell_1}(G/K)_G\bigoplus C_{n-1}^{\ell_1}(A/N)_A$$
$$EC_n(A/N\xrightarrow{\rho}G/K)=C_n^{\ell_1}(A/N)_A\bigoplus C_n^{\ell_1}(G/K)_G\bigoplus C_{n-1}^{\ell_1}(A/N)_A$$
$$C_n(A\xrightarrow{\varphi}G)=C_n^{\ell_1}(G)_G\bigoplus C_{n-1}^{\ell_1}(A)_A$$
$$EC_n(A\xrightarrow{\varphi}G)=C_n^{\ell_1}(A)_A\bigoplus C_n^{\ell_1}(G)_G\bigoplus C_{n-1}^{\ell_1}(A)_A,$$

since the proof is very similar to the one of Theorem 2.16, we leave it to the reader. □

Corollary 5.10. *Let A be an amenable group, and let $\varphi\colon A \to G$ be a group homomorphism. Then the groups $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ and $H_*^{\ell_1}(G)$ are isomorphic. Furthermore, the norms $\|\cdot\|_1(\omega)$ in $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ are equal to the norm $\|\cdot\|_1$ in $\widehat{H}^n(G)$ for every $\omega \in [0,\infty]$.*

Proof. As in Corollary 2.17, the image $\varphi(A)$ and $\ker(\varphi)$ are the amenable subgroups of G and A respectively. We denote by $\rho\colon \varphi(A) \hookrightarrow G$ an inclusion map and consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \varphi(A) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\rho} \\ G & \xrightarrow{id} & G. \end{array}$$

Then it follows from Theorem 5.9, Theorem 5.7, and Theorem 5.8. □

Now, for a subgroup A of G, we give another description of relative ℓ_1 homology of G mod A.

Definition 5.4. *Let the following G- and A-resolutions respectively*

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair for $(G, A; \mathbf{R})$. This pair of resolutions is said to be co-proper if an A-morphism $\lambda_n\colon V_n \to U_n$ induces an injective map $\lambda_n\colon (V_n)_A \to (U_n)_G$.

Proposition 5.11. *The bar resolutions of G and A are co-proper.*

Proof. Note that the injective homomorphism $A \hookrightarrow G$ induces the A-morphism $C_*^{\ell_1}(A) \to C_*^{\ell_1}(G)$ which is clearly injective. Also, it is easy to check that the induced map $C_*^{\ell_1}(A)_A \to C_*^{\ell_1}(G)_G$ is injective. □

Let

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be the sequences in Definition 5.4. Note that there is an exact sequence

$$0 \to (V_n)_A \hookrightarrow (U_n)_G \to (U_n)_G/(V_n)_A \to 0. \tag{5.1}$$

It is easy to check that the induced sequence

$$\cdots \to (U_2)_G/(V_2)_A \to (U_1)_G/(V_1)_A \to (U_0)_G/(V_0)_A \to 0 \qquad (5.2)$$

is a complex.

Definition 5.5. *We denote by $H_n^{\ell_1}(G,A)$ the n-th homology of the complex (5.2).*

Note that the sequence (5.1) induces an exact sequence

$$\to H_{n+1}^{\ell_1}(G,A) \to H_n^{\ell_1}(A) \to H_n^{\ell_1}(G) \to H_n^{\ell_1}(G,A) \to H_{n-1}^{\ell_1}(A) \to . \qquad (5.3)$$

As an important example, the bar resolutions of G and A induces an exact sequence

$$0 \to C_*^{\ell_1}(A)_A \xrightarrow{i_*} C_*^{\ell_1}(G)_G \xrightarrow{p_*} C_*^{\ell_1}(G)_G/C_*^{\ell_1}(A)_A \to 0. \qquad (5.4)$$

We denote the quotient space $C_*^{\ell_1}(G)_G/C_*^{\ell_1}(A)_A$ by $C_*^{\ell_1}(G,A)$. Remark that there is a complex

$$\cdots \to C_2^{\ell_1}(G,A) \to C_1^{\ell_1}(G,A) \to C_0^{\ell_1}(G,A) \to 0. \qquad (5.5)$$

In [7], for every $\theta \geq 0$, Gromov defines a norm $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(G,A)$ as follows: first we define a norm $\|\cdot\|_1(\theta)$ on $C_*^{\ell_1}(G)_G$ by putting

$$\|x\|_1(\theta) = \|x\|_1 + \theta\|\partial x\|_1.$$

Then, using the quotient homomorphism $p_*\colon C_*^{\ell_1}(G)_G \to C_*^{\ell_1}(G,A)$, we define the norm $\|\bar{c}\|_1(\theta)$ of $\bar{c} \in C_*^{\ell_1}(G,A)$ by taking all $c \in p_*^{-1}(\bar{c}) \subset C_*^{\ell_1}(G)$ and setting

$$\|\bar{c}\|_1(\theta) = \inf\{\, \|c\|_1(\theta) \mid p_*(c) = \bar{c}\,\}.$$

Then there is a corresponding seminorm $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(G,A)$. Finally we define this norm on $H_*^{\ell_1}(G,A)$ for all θ in the closed interval $[0,\infty]$ by passing to the limits.

Theorem 5.12. *Let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then the groups $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ and $H_n^{\ell_1}(G,A)$ are isomorphic and the norm $\|\cdot\|_1(\omega)$ in $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ and the norm $\|\cdot\|_1(\theta)$ in $H_n^{\ell_1}(G,A)$ are equivalent for $\omega = \theta \in [0,\infty]$.*

Proof. Recall that

$$C_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$$
$$EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$$

We consider the following diagram

$$\begin{array}{ccccccccc} 0 \to C_n^{\ell_1}(A)_A & \xrightarrow{i_n} & EC_n(A \xrightarrow{\varphi} G) & \xrightarrow{p_n} & C_n(A \xrightarrow{\varphi} G) \to 0 \\ \gamma_n \downarrow & & \alpha_n \downarrow & & \beta_n \downarrow \\ 0 \to C_n^{\ell_1}(A)_A & \xrightarrow{j_n} & C_n^{\ell_1}(G)_G & \xrightarrow{q_n} & C_n^{\ell_1}(G, A) \to 0 \end{array}$$

where $\gamma_n(a') = a'$, $\alpha_n(a', x, a) = x$, and $\beta_n(x, a) = x + C_n^{\ell_1}(A)_A$. It is clear that the diagram is commutative and so there is an induced commutative diagram

$$\begin{array}{ccccccc} \to H_n^{\ell_1}(A) & \longrightarrow & H_n^{\ell_1}(G) & \longrightarrow & H_n^{\ell_1}(A \xrightarrow{\varphi} G) & \longrightarrow & H_{n-1}^{\ell_1}(A) \to \\ H_n(\gamma) \downarrow & & H_n(\alpha) \downarrow & & H_n(\beta) \downarrow & & H_{n-1}(\gamma) \downarrow \\ \to H_n^{\ell_1}(A) & \longrightarrow & H_n^{\ell_1}(G) & \longrightarrow & H_n^{\ell_1}(G, A) & \longrightarrow & H_{n-1}^{\ell_1}(A) \to \end{array}$$

Note that the maps $H_*(\gamma)$ and $H_*(\alpha)$ are (isometric) isomorphisms. So the map $H_n(\beta)$ is an isomorphism.

Let $\omega = \theta \geq 0$. Let $(x, a) \in C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ be a cycle. Then $d(x, a) = (\partial x + a,\ -\partial' a) = 0$ and so $\partial x = -a$. Then we have

$$\begin{aligned} \|\beta_n(x, a)\|_1(\theta) &= \|x + C_n^{\ell_1}(A)_A\|_1(\theta) \\ &\leq \|x\|_1(\theta) = \|x\|_1 + \theta\|\partial x\|_1 \\ &\leq \|x\|_1 + (1 + \omega)\|a\|_1 = \|(x, a)\|_1(\omega). \end{aligned}$$

On the other hand, let $x \in C_n^{\ell_1}(G)_G$ be a relative cycle so that $\partial x \in C_{n-1}^{\ell_1}(A)_A$. Then we have $(x,\ -\partial x) \in C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ and also we have $d(x,\ -\partial x) = (\partial x - \partial x,\ \partial' \partial x) = (0,\ 0)$. It is easy to check that $(H_n^{\ell_1}(\beta))^{-1}[x]$ is represented by a cycle $(x,\ -\partial x)$. Also note that

$$\begin{aligned} \|(x, -\partial_n x)\|_1(\omega) &= \|x\|_1 + (1 + \omega)\|\partial_n x\|_1 = \|x\|_1 + \|\partial_n x\|_1 + \omega\|\partial_n x\|_1 \\ &\leq (n + 2)\|x\|_1 + \omega\|\partial_n x\|_1 \leq (n + 2)(\|x\|_1 + \theta\|\partial_n x\|_1) \\ &= (n + 2)\|x\|_1(\theta). \end{aligned}$$

Hence, for $[x, a] \in H_n^{\ell_1}(A \xrightarrow{\varphi} G)$, we have

$$\frac{1}{n+2}\|[x,a]\|_1(\omega) \leq \|H_n(\beta)([x,a])\|_1(\theta) \leq \|[x,a]\|_1(\omega).$$

□

CHAPTER 6

Relative ℓ_1 homology of spaces

Throughout this chapter we assume every space is a connected countable cellular space.

For a space X, we denote by $C_*(X)$ the real chain complex of X: a chain $c \in C_*(X)$ is a finite combination $\sum_i r_i \sigma_i$ of singular simplices σ_i in $X$with real coefficients r_i. We define the ℓ_1-norm, denoted by $\|\cdot\|_1$, in $C_*(X)$ by setting $\|c\|_1 = \sum_i |r_i|$.

Let $C_*^{\ell_1}(X)$ be the completion of $C_*(X)$ with respect to this norm, i.e.,

$$C_*^{\ell_1}(X) = \left\{ \sum_{i=1}^{\infty} r_i \sigma_i \middle| \sum_{i=1}^{\infty} |r_i| < \infty \right\}.$$

It is easy to check the sequence

$$\cdots \to C_3^{\ell_1}(X) \xrightarrow{\partial_3} C_2^{\ell_1}(X) \xrightarrow{\partial_2} C_1^{\ell_1}(X) \xrightarrow{\partial_1} C_0^{\ell_1}(X) \to 0 \tag{6.1}$$

is a complex of Banach spaces and bounded operators, where the boundary operator ∂_n is defined by extending linearly the boundary operator on the ordinary chain complex $C_*(X)$ and it has the norm $\|\partial_n\| \leq n+1$.

Definition 6.1. *The n-th homology of the complex in (6.1) is called the n-th ℓ_1 homology of X and is denoted by $H_n^{\ell_1}(X)$.*

We define a seminorm $\|[x]\|_1$ of $[x] \in H_*^{\ell_1}(X)$ by taking all cycles $c \in C_*^{\ell_1}(X)$ lying in the homology class corresponding to $[x]$ and setting

$$\|[x]\|_1 = \inf\{\|c\|_1 \mid [x] = [c]\}.$$

In [10], it is shown that the bounded cohomology $\widehat{H}^*(X)$ is zero if and only if the ℓ_1 homology $H_*^{\ell_1}(X)$ is also zero. Thus, if a space X is simply connected or its fundamental group $\boldsymbol{\pi}_1 X$ is amenable, then $H_*^{\ell_1}(X)$ is zero.

In the next theorem, we prove that the $H_*^{\ell_1}(X)$ also depends only on its fundamental group $\boldsymbol{\pi}_1 X$ as the group $\widehat{H}^*(X)$ does.

Theorem 6.1. *The group $H_*^{\ell_1}(\boldsymbol{\pi}_1 X)$ is canonically isomorphic with $H_*^{\ell_1}(X)$. This isomorphism carries the canonical seminorm in $H_*^{\ell_1}(\boldsymbol{\pi}_1 X)$ to the seminorm in $H_*^{\ell_1}(X)$.*

Proof. Let $\pi\colon \mathcal{X} \to X$ be the universal covering of X so that $\boldsymbol{\pi}_1 X$ acts freely on $\mathcal{X}$ and $\mathcal{X}/\boldsymbol{\pi}_1 X = X$. The action of $\boldsymbol{\pi}_1 X$ on $\mathcal{X}$ induces the action on the chain groups $C_*^{\ell_1}(\mathcal{X})$ and thus turns them into bounded $\boldsymbol{\pi}_1 X$-modules. We show that these $\boldsymbol{\pi}_1 X$-modules are relatively projective. Let $\mathcal{X}_0 \subset \mathcal{X}$ consist one element from each $\boldsymbol{\pi}_1 X$-orbit. The complex $C_*(\mathcal{X})$ is free on all simplexes the first vertices of which are in $\mathcal{X}_0$ (see Theorem 10.20 in [12]). Then by a standard calculation as in Lemma 4.1, it is easy to see that these $\boldsymbol{\pi}_1 X$-modules $C_*^{\ell_1}(\mathcal{X})$ are relatively projective.

Now we consider the sequence

$$\cdots \to C_2^{\ell_1}(\mathcal{X}) \to C_1^{\ell_1}(\mathcal{X}) \to C_0^{\ell_1}(\mathcal{X}) \to \mathbf{R} \to 0. \qquad (6.1.1)$$

Since $\mathcal{X}$ is simply connected, $H_*^{\ell_1}(\mathcal{X}) = 0$ and hence the sequence in (6.1.1) is exact. Thus the sequence (6.1.1) is a strong relatively projective $\boldsymbol{\pi}_1 X$-resolution of the trivial $\boldsymbol{\pi}_1 X$-module $\mathbf{R}$, where the fact that this resolution is strong is shown in Theorem 2.4 from [8]. Remark that the map $\pi_*\colon C_*^{\ell_1}(\mathcal{X}) \to C_*^{\ell_1}(X)$ establishes an isometric isomorphism between $\left(C_*^{\ell_1}(\mathcal{X})\right)_{\pi_1 X}$ and $C_*^{\ell_1}(X)$ and it commutes with the boundary operators. Thus the ℓ_1 homology group of $\pi_1 X$ coincides as topological vector spaces with the homology of the complex

$$\cdots \to C_2^{\ell_1}(X) \to C_1^{\ell_1}(X) \to C_0^{\ell_1}(X) \to 0.$$

It remains for us to prove that the isomorphism constructed between $H_*^{\ell_1}(\pi_1 X)$ and $H_*^{\ell_1}(X)$ is an isometry. Let $\|\cdot\|_1$ denote the canonical seminorm in $H_*^{\ell_1}(\pi_1 X)$ and $\|\cdot\|_1^s$ the seminorm in $H_*^{\ell_1}(X)$. By definition of the canonical seminorm, we have $\|\cdot\|_1 \geq \|\cdot\|_1^s$, so that it suffices for us to prove that $\|\cdot\|_1 \leq \|\cdot\|_1^s$. Since the canonical seminorm is achieved by the bar resolution, it suffices to construct a $\pi_1 X$-morphism of the resolution (6.1.1) into the bar resolution of $\pi_1 X$ consisting of maps of norm ≤ 1.

Let $\sigma\colon \Delta_n \to \mathcal{X}$ be a singular n-simplex the first vertex of which is in $\mathcal{X}_0$, where $\Delta_n = [v_0, \cdots, v_n]$. We define a map $f_n\colon C_n^{\ell_1}(\mathcal{X}) \to C_n^{\ell_1}(\pi_1 X)$ by $f_n(\sigma) = g_0[g_1|g_2|\cdots|g_n]$, where $g_i \in \pi_1 X$ such that $\sigma(v_i) = g_i \cdots g_0 \mathcal{X}_0$. It is easy to see that f_n commutes with the boundary operators and so it determines a $\pi_1 X$-morphism of the resolutions

$$\begin{array}{ccccccccccc}
\longrightarrow & C_2^{\ell_1}(\mathcal{X}) & \longrightarrow & C_1^{\ell_1}(\mathcal{X}) & \longrightarrow & C_0^{\ell_1}(\mathcal{X}) & \longrightarrow & \mathbf{R} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle id_{\mathbf{R}}} & & \downarrow \\
\longrightarrow & C_2^{\ell_1}(\pi_1 X) & \longrightarrow & C_1^{\ell_1}(\pi_1 X) & \longrightarrow & C_0^{\ell_1}(\pi_1 X) & \longrightarrow & \mathbf{R} & \longrightarrow & 0
\end{array}$$

extending $id_{\mathbf{R}}$. Also, from definition, it is clear that $\|f_*\| \leq 1$ and so we have $\|\cdot\|_1 \leq \|\cdot\|_1^s$. This finishes the proof. □

Corollary 6.2. *Let $\alpha\colon X_1 \to X_2$ be a continuous map such that the induced homomorphism $\alpha_*\colon \pi_1(X_1) \to \pi_1(X_2)$ is a surjection with an amenable kernel. Then for every $n \geq 0$ the homomorphism $H_n(\alpha)\colon H_n^{\ell_1}(X_1) \to H_n^{\ell_1}(X_2)$ is an isometric isomorphism for the norm $\|\cdot\|_1$.*

Proof. This follows from Theorem 4.9 and Theorem 6.1. □

Now let us define relative ℓ_1 homology of spaces.

Definition 6.2. *Let $\varphi\colon Y \to X$ be a continuous map of spaces. The mapping cone and the mapping cylinder of the chain complexes induced by φ are defined as follows:*

$$C_n(Y \xrightarrow{\varphi} X) = C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y)$$
$$EC_n(Y \xrightarrow{\varphi} X) = C_n^{\ell_1}(Y) \bigoplus C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y).$$

We define the boundary operators on $C_n(Y \xrightarrow{\varphi} X)$ and $EC_n(Y \xrightarrow{\varphi} X)$ respectively by the same formulas as in Lemma 5.2, i.e.,

$$d_n(x_n, a_{n-1}) = (\partial_n x_n + \lambda_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1})$$
$$d_n(a_n, x_n, a_{n-1}) = (\partial'_n a_n - a_{n-1}, \partial_n x_n + \lambda_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1}).$$

Then, as shown in Lemma 5.2, there are complexes

$$\cdots \xrightarrow{d_3} C_2(Y \xrightarrow{\varphi} X) \xrightarrow{d_2} C_1(Y \xrightarrow{\varphi} X) \xrightarrow{d_1} C_0(Y \xrightarrow{\varphi} X) \to 0 \tag{6.2}$$
$$\cdots \xrightarrow{d_3} EC_2(Y \xrightarrow{\varphi} X) \xrightarrow{d_2} EC_1(Y \xrightarrow{\varphi} X) \xrightarrow{d_1} EC_0(Y \xrightarrow{\varphi} X) \to 0. \tag{6.3}$$

Definition 6.3. *The n-th homology of the complex in (6.2) is called the n-th relative ℓ_1 homology of X modulo Y and is denoted by $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$. We denote the n-th homology of the complex in (6.3) by $H_n^{\ell_1}(EC(Y \xrightarrow{\varphi} X)$.*

We define the norm $\|\cdot\|_1$ on $EC_*(Y \xrightarrow{\varphi} X)$ by the formula

$$\|(b,\, x,\, a)\|_1 = \|b\|_1 + \|x\|_1 + \|a\|_1$$

and similarly on $C_*(Y \xrightarrow{\varphi} X)$.

As is well known, the norm $\|\cdot\|$ on $M^*(Y \xrightarrow{\varphi} X)$ in Definition 3.1 is the dual of the ℓ_1 norm $\|\cdot\|_1$ on $C_*(Y \xrightarrow{\varphi} X)$. Then, by applying Hahn-Banach theorem, we conclude that the seminorms $\|\cdot\|$ on $\hat{H}^*(Y \xrightarrow{\varphi} X)$ and $\|\cdot\|_1$ on $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ are also dual.

For every $\omega \geq 0$, we define a norm $\|\cdot\|_1(\omega)$ on $C_n(Y \xrightarrow{\varphi} X)$ by the formula

$$\|(x, a)\|_1(\omega) = \|x\|_1 + (1+\omega)\|a\|_1.$$

There is the corresponding seminorm $\|\cdot\|_1(\omega)$ in $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$. We define these norms in $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ for all $\omega \in [0, \infty]$ by passing to the limits.

Remark that the seminorms $\|\cdot\|(\omega)$ in $\hat{H}^*(Y \xrightarrow{\varphi} X)$ and $\|\cdot\|_1(\omega)$ in $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ are also dual for every $\omega \in [0, \infty]$.

Theorem 6.3. *Let $\varphi\colon Y \to X$ be a continuous map. Then the natural inclusion map $\rho_n\colon C_n^{\ell_1}(X) \to EC_n(Y \xrightarrow{\varphi} X)$ induces an isometric isomorphism $H_n(\rho)\colon H_n^{\ell_1}(X) \to H_n^{\ell_1}(EC(Y \xrightarrow{\varphi} X))$ for the norm $\|\cdot\|_1$.*

Proof. We can prove this as the same way as we proved Theorem 5.3. □

Note that there is an exact sequence

$$0 \to C_n^{\ell_1}(Y) \to EC_n(Y \xrightarrow{\varphi} X) \to C_n(Y \xrightarrow{\varphi} X) \to 0$$

and so there is an induced long exact sequence

$$\cdots \to H_n^{\ell_1}(Y) \to H_n^{\ell_1}(X) \to H_n^{\ell_1}(Y \xrightarrow{\varphi} X) \to H_{n-1}^{\ell_1}(Y) \to \cdots.$$

Theorem 6.4. *Let $\varphi\colon Y \to X$ be a continuous map and $\varphi_*\colon \pi_1 Y \to \pi_1 X$ be the induced homomorphism. Then the groups $H_n^{\ell_1}(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ and $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ are isometrically isomorphic. This isomorphism carries the canonical seminorm $\|\cdot\|_1(\omega)$ in $H_n^{\ell_1}(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ to the seminorm in $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ for every $\omega \in [0, \infty]$.*

Proof. Let G and A denote the groups $\pi_1 X$ and $\pi_1 Y$ respectively.

Recall that, from Theorem 5.4, the canonical seminorm on $H_n^{\ell_1}(A \xrightarrow{\varphi_*} G)$ is induced by the complex $C_n(A \xrightarrow{\varphi_*} G) = C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$.

Let $\pi_1\colon \mathcal{X} \to X$ and $\pi_2\colon \mathcal{Y} \to Y$ be the universal coverings. As we saw in Theorem 6.1, we can identify

$$C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y) = \left(C_n^{\ell_1}(\mathcal{X})\right)_{\pi_1 X} \bigoplus \left(C_{n-1}^{\ell_1}(\mathcal{Y})\right)_{\pi_1 Y}.$$

Then, as the same way as we proved Theorem 3.3, we can show that there are chain maps

$$C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A \underset{\Psi_n}{\overset{\Phi_n}{\rightleftarrows}} \left(C_n^{\ell_1}(\mathcal{X})\right)_{\pi_1 X} \bigoplus \left(C_{n-1}^{\ell_1}(\mathcal{Y})\right)_{\pi_1 Y}$$

such that $\Psi_n\Phi_n$ is chain homotopic to *id* and they have the norms $\|\Phi_n\| \leq 1$ and $\|\Psi_n\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0, \infty]$. □

Corollary 6.5. *Let $\varphi\colon Y \to X$ be a continuous map of spaces such that the fundamental group $\boldsymbol{\pi}_1 Y$ is amenable. Then $H_n^{\ell_1}(X)$ and $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ are isometrically isomorphic for the norm $\|\cdot\|_1$ for every $n \geq 2$.*

Furthermore, the norms $\|\cdot\|_1(\omega)$ in the group $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ are equal for all $\omega \in [0, \infty]$ and for all $n \geq 2$.

Proof. Let $\varphi_*\colon \boldsymbol{\pi}_1 Y \to \boldsymbol{\pi}_1 X$ be the homomorphism induced by $\varphi\colon Y \to X$. We have the following sequence of isometrically isomorphic groups

$$\begin{aligned} H_n^{\ell_1}(Y \xrightarrow{\varphi} X) &\cong H_n^{\ell_1}(\boldsymbol{\pi}_1 Y \xrightarrow{\varphi_*} \boldsymbol{\pi}_1 X) && \text{by Theorem 6.4} \\ &\cong H_n^{\ell_1}(\boldsymbol{\pi}_1 X) && \text{by Corollary 5.10} \\ &\cong H_n^{\ell_1}(X) && \text{by Theorem 6.1.} \end{aligned}$$

The second part also follows from Theorem 6.4 and Corollary 5.10. □

Let Y and $Y \subset X$ be a pair of spaces and let $\varphi\colon Y \to X$ be a natural inclusion map. Then the injective homomorphism $i_n\colon C_n^{\ell_1}(Y) \hookrightarrow C_n^{\ell_1}(X)$ induces an exact sequence

$$0 \to C_n^{\ell_1}(Y) \hookrightarrow C_n^{\ell_1}(X) \to C_n^{\ell_1}(X)/C_n^{\ell_1}(Y) \to 0. \tag{6.4}$$

We denote $C_n^{\ell_1}(X)/C_n^{\ell_1}(Y)$ by $C_n^{\ell_1}(X,Y)$. Remark that the induced sequence

$$\cdots \to C_3^{\ell_1}(X,Y) \to C_2^{\ell_1}(X,Y) \to C_1^{\ell_1}(X,Y) \to 0 \tag{6.5}$$

is a complex.

Definition 6.4. *The n-th homology of the complex in (6.5) is denoted by $H_n^{\ell_1}(X,Y)$.*

The exact sequence (6.4) induces a long exact sequence

$$\to H_{n+1}^{\ell_1}(X,Y) \to H_n^{\ell_1}(Y) \to H_n^{\ell_1}(X) \to H_n^{\ell_1}(X,Y) \to H_{n-1}^{\ell_1}(Y) \to$$

As in $H_*^{\ell_1}(G,A)$, for every $\theta \geq 0$, we define a norm $\|c\|_1(\theta)$ of $c \in C_*^{\ell_1}(X)$ by putting

$$\|c\|_1(\theta) = \|c\|_1 + \theta\|\partial c\|_1.$$

Then, by using the surjective homomorphism $p_*\colon C_*^{\ell_1}(X) \to C_*^{\ell_1}(X,Y)$, we define the norm $\|\bar{c}\|_1(\theta)$ of $\bar{c} \in C_*^{\ell_1}(X,Y)$ by setting

$$\|\bar{c}\|_1(\theta) = \inf\{\, \|c\|_1(\theta) \mid c \in p_*^{-1}(\bar{c}) \,\}.$$

With this norm $\|\cdot\|_1(\theta)$ on $C_*^{\ell_1}(X,Y)$, we have the corresponding seminorm $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(X,Y)$. Finally we define this norm on $H_*^{\ell_1}(X,Y)$ for all $\theta \in [0,\infty]$ by passing to the limits.

Theorem 6.6. *Let $\varphi\colon Y \hookrightarrow X$ be an inclusion map of spaces $Y \subset X$. Then the groups $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ and $H_n^{\ell_1}(X,Y)$ are isomorphic and the norm $\|\cdot\|_1(\omega)$ in $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ and the norm $\|\cdot\|_1(\theta)$ in $H_n^{\ell_1}(X,Y)$ are equivalent for every $\omega = \theta \in [0,\infty]$.*

Proof. We can prove this as the same way as we proved Theorem 5.12. □

CHAPTER 7

Locally finite ℓ_1 homology of spaces

Throughout this chapter, if $\varphi\colon X' \hookrightarrow X$ is a natural inclusion map for a pair of spaces X' and $X' \subset X$, then we denote $H_*^{\ell_1}(X' \overset{\varphi}{\to} X)$ by $H_*^{\ell_1}(X' \hookrightarrow X)$.

Now we let

$$\mathcal{I} = \{K_j \mid K_j \text{ is a compact subspace of } X,\ j = 0, 1, 2, \cdots\}.$$

We define a relation $\leq$ on the set $\mathcal{I}$ by $K_j \leq K_i$ if and only if $K_j \subset K_i$. It is easy to see that $(\mathcal{I}, \leq)$ forms a directed quasi-ordered set.

Then, whenever $K_j, K_i \in \mathcal{I}$ satisfy $K_j \leq K_i$, there is an injective homomorphism $\rho_n\colon C_n^{\ell_1}(X - K_i) \to C_n^{\ell_1}(X - K_j)$ induced by the inclusion $X - K_i \subset X - K_j$. Thus there is a chain map

$$\alpha_j^i\colon C_n^{\ell_1}(X) \bigoplus C_n^{\ell_1}(X - K_i) \to C_n^{\ell_1}(X) \bigoplus C_n^{\ell_1}(X - K_j)$$

defined by the formula $\alpha_n(x, a) = (x, \rho_n a) = (x, a)$. This induces a canonical homomorphism

$$H_n(\alpha_j^i)\colon H_*^{\ell_1}((X - K_i) \hookrightarrow X) \to H_*^{\ell_1}((X - K_j) \hookrightarrow X).$$

Thus the groups $H_*^{\ell_1}((X - K) \hookrightarrow X)$ form an inverse system indexed by the compact subspaces $K \subset X$. Note that $\|\alpha_j^i\| = 1$ for the norm $\|\cdot\|_1(\omega)$ and so we have $\|H_n(\alpha_j^i)\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0, \infty]$.

Definition 7.1. *Let $\mathcal{I}$ be the index set as above. For every $n \geq 0$ we define the n-th locally finite ℓ_1 homology of X, denoted by $H_n^\infty(X)$, as the inverse limit of an inverse system $\{H_n^{\ell_1}((X - K_j) \hookrightarrow X), H_n(\alpha_j^i)\}$, i.e.,*

$$H_n^\infty(X) = \varprojlim_{K_j \in \mathcal{I}} H_n^{\ell_1}((X - K_j) \hookrightarrow X).$$

Remark 7.1. A). If X is compact, $H_*^\infty(X) = H_*^{\ell_1}(X)$.

B). Let $\mathcal{I}' \subset \mathcal{I}$ consist of increasing sequence of $K_j' \in \mathcal{I}$ such that whose interiors, $\{(K_j')^\circ\}$, cover X, i.e.,

$$\mathcal{I}' = \{K_j' \in \mathcal{I} \mid K_j' \subset K_{j+1}' \text{ and } \bigcup_{j=0}^{\infty} (K_j')^\circ = X\}.$$

Then $\mathcal{I}'$ is cofinal in $\mathcal{I}$, i.e., for each $K \in \mathcal{I}$, there exists $K' \in \mathcal{I}'$ such that $K \leq K'$. Hence there is an isomorphism

$$\Phi\colon \varprojlim_{K \in \mathcal{I}} H_*^{\ell_1}((X - K) \hookrightarrow X) \to \varprojlim_{K' \in \mathcal{I}'} H_*^{\ell_1}((X - K') \hookrightarrow X)$$

called the injection of the system into the subsystem, induced by an index inclusion $\mathcal{I}' \hookrightarrow \mathcal{I}$.

C). The inverse limit $H_*^\infty(X)$ is a subgroup of a product of the relative ℓ_1 homology groups $H_*^{\ell_1}((X - K_j) \hookrightarrow X)$, i.e.,

$$\begin{aligned} & H_*^\infty(X) \\ &= \varprojlim_{K_j \in \mathcal{I}} H_*^{\ell_1}((X - K_j) \hookrightarrow X) \\ &= \{([x_j]) \in \prod_j H_*^{\ell_1}((X - K_j) \hookrightarrow X) \mid [x_j] = H_n(\alpha_j^i)[x_i], \text{ whenever } K_j \leq K_i\} \end{aligned}$$

From now on, we always represent an element $[x]$ of $H_*^\infty(X)$ as an element $([x_j])$ of a subgroup of the product $\prod_j H_*^{\ell_1}((X - K_j) \hookrightarrow X)$.

Definition 7.2. *We define the seminorm $\|\cdot\|_s$ in $H_*^\infty(X)$ by putting*

$$\|[x]\|_s = \sup_j \|[x_j]\|_1 \quad \textit{for every } [x] = ([x_j]) \in H_*^\infty(X).$$

Also, for every $\omega \geq 0$, we define the norms $\|\cdot\|_l(\omega)$ in $H_*^\infty(X)$ by

$$\|[x]\|_l(\omega) = \lim_j \|[x_j]\|_1(\omega) \quad \textit{for every } [x] = ([x_j]) \in H_*^\infty(X).$$

Then we define these norms in $H_*^\infty(X)$ for all $\omega \in [0,\infty]$ by passing to the limits.

As explained above, since the map $H_n(\alpha_j^i)$ satisfies $\|H_n(\alpha_j^i)\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$, the limit exists. Also the norm $\|\cdot\|_s$ may be infinite and $\|\cdot\|_l(\omega_1) \leq \|\cdot\|_l(\omega_2)$ for $\omega_1 \leq \omega_2$.

Proposition 7.1. *Let the index sets $\mathcal{I}$ and $\mathcal{I}'$ be as in Remark 7.1 B). Then the isomorphism Φ: $\varprojlim_{K\in\mathcal{I}} H_*^{\ell_1}((X-K) \hookrightarrow X) \to \varprojlim_{K'\in\mathcal{I}'} H_*^{\ell_1}((X-K') \hookrightarrow X)$ induced by the inclusion $\mathcal{I}' \hookrightarrow \mathcal{I}$ is isometric for the norm $\|\cdot\|_s$ and the norms $\|\cdot\|_l(\omega)$ for all $\omega \in [0,\infty]$.*

Proof. Let $[x] = ([x_K]) \in H_*^{\ell_1}(X) \subset \prod_{K\in\mathcal{I}} H_*^{\ell_1}((X-K) \hookrightarrow X)$. We denote $\Phi([x])$ by $([x_{K'}]) \in \prod_{K'\in\mathcal{I}'} H_*^{\ell_1}((X-K') \hookrightarrow X)$. Since $\mathcal{I}'$ is cofinal in $\mathcal{I}$, there exists $K' \in \mathcal{I}'$ for each $K \in \mathcal{I}$ such that $K \leq K'$ and so there is a homomorphism $H_*(\alpha_K)\colon H_*^{\ell_1}((X - K') \hookrightarrow X) \to H_*^{\ell_1}((X-K) \hookrightarrow X)$ induced by an inclusion $X - K' \subset X - K$.

Note that, since $[x_K] = H_*(\alpha_K)[x_{K'}]$ from definition of the inverse limit, we have $\|[x_K]\|_1 = \|H_*(\alpha_K)[x_{K'}]\|_1 \leq \|[x_{K'}]\|_1$. Thus the norm of every coordinate $[x_K]$ of $[x]$ is bounded by the norm of some coordinate of $\Phi([x])$ so that we have $\sup_{K\in\mathcal{I}} \|[x_K]\|_1 \leq \sup_{K'\in\mathcal{I}'} \|[x_{K'}]\|_1$. Hence $\|\Phi\| \geq 1$ for the norm $\|\cdot\|_s$. On the other hand, since the supremum taken over $\mathcal{I}$ is not less than the one taken over the subset $\mathcal{I}' \subset \mathcal{I}$, the map Φ has the norm $\|\Phi\| \leq 1$ for the norm $\|\cdot\|_s$. Thus the map Φ is an isometry for the norm $\|\cdot\|_s$.

Similarly, for the norm $\|\cdot\|_l(\omega)$, it is easy to see that

$$\lim_{K\in\mathcal{I}} \|[x_K]\|_1(\omega) = \lim_{K'\in\mathcal{I}'} \|[x_{K'}]\|_1(\omega) \quad \text{for all } \omega \in [0,\infty].$$

Hence Φ is also an isometry for the norm $\|\cdot\|_l(\omega)$ for every $\omega \in [0,\infty]$. □

Corollary 7.2. *For $[x] \in H_*^\infty(X)$, we have*

$$\|[x]\|_s \le \|[x]\|_l(\omega) \quad \text{for every } \omega \in [0,\infty].$$

Proof. We prove that $\|[x]\|_s = \|[x]\|_l(0)$. By Proposition 7.1, we consider the cofinal index set $\mathcal{I}' \subset \mathcal{I}$ in Remark 7.1.B. Then there is a sequence of groups induced by inclusions

$$\to H_*^{\ell_1}((X-K_2) \hookrightarrow X) \to H_*^{\ell_1}((X-K_1) \hookrightarrow X) \to H_*^{\ell_1}((X-K_0) \hookrightarrow X) \to 0.$$

Let $[x] = ([x_j]) \in H_*^\infty(X) \subset \prod_j H_*^{\ell_1}((X-K_j) \hookrightarrow X)$ be given. Then we have

$$H_*(\alpha_j^{j+1})[x_{j+1}] = [x_j] \text{ and so } \|[x_j]\|_1 = \|H_*(\alpha_j^{j+1})[x_{j+1}]\|_1 \le \|[x_{j+1}]\|_1.$$

Since $\|[x_j]\|_1 = \|[x_j]\|_1(0)$, we have an increasing sequence of norms

$$0 \le \|[x_0]\|_1(0) \le \|[x_1]\|_1(0) \le \cdots \le \|[x_j]\|_1(0) \le \|[x_{j+1}]\|_1(0) \le \cdots.$$

This shows that

$$\sup_j \|[x_j]\|_1 = \begin{cases} \lim_j \|[x_j]\|_1(0) & \text{if the sequence is bounded above,} \\ \lim_j \|[x_j]\|_1(0) = \infty & \text{if the sequence is unbounded.} \end{cases}$$

Hence we have $\|[x]\|_s = \sup_j \|[x_j]\|_1 = \lim_j \|[x_j]\|_1(0) = \|[x]\|_l(0) \le \|[x]\|_l(\omega)$. □

Definition 7.3. *For an orientable n-manifold M, we define its simplicial volume as the norm $\|[x]\|_s$ of $[x] = ([x_K]) \in H_n^\infty(M)$ and denoted it by $\|M\|$, where $[x_K]$ is the homology class of $H_n^{\ell_1}((M-K) \hookrightarrow M)$ for every compact $K \subset M$. When M is not orientable we pass to the double covering $\widetilde{M}$ and set $\|M\| = \frac{1}{2}\|\widetilde{M}\|$.*

Remark 7.2. Note that if M is a closed orientable n-manifold, its simplicial volume is the norm $\|\cdot\|_1$ of its fundamental class in $H_n^{\ell_1}(M)$.

The locally finite ℓ_1 homology $H_*^\infty(X)$ is corresponding to the homology $\overline{H}_*(X)$ in [7] which is defined with the locally finite cycles $c = \sum_{i=1}^\infty r_i\sigma_i$ such that each compact subset of X intersects only finitely many (images of) simplices σ_i. Note that the ℓ_1-norm $\|c\| = \sum_{i=1}^{\infty} |r_i|$ may be infinite. Also Gromov defined the simplicial volume of an open manifold M by the norm $\|\cdot\|_1$ of the fundamental class of $\overline{H}_*(M)$. It seems that our simplicial volume of an open manifold is equivalent to Gromov's simplicial volume but we leave it as an open question.

Definition 7.4. *A pair of spaces (X, Y) is said to be perfect if the inclusion homomorphism $\pi_1 Y \to \pi_1 X$ is injective.*

Remark 7.3. As it is well known, we can always construct a perfect pair of spaces from a given pair of spaces (X, Y) by killing the kernel of the inclusion homomorphism $\rho\colon \pi_1 Y \to \pi_1 X$. Namely, for each $[\omega] \in \ker(\rho)$, we represent it by a loop ω in Y which is null homotopic in X. To X we attach a two dimensional disk along the loop ω as boundary. Attaching disks in this way we can kill the entire kernel of the inclusion homomorphism $\rho\colon \pi_1 Y \to \pi_1 X$.

Let $\widetilde{X}$ and $\widetilde{Y}$ be the resulting spaces of attaching 2-dimensional disks to X and Y, respectively, to kill the entire kernel of ρ. Then it is clear that this pair of spaces $(\widetilde{X}, \widetilde{Y})$ is perfect.

Note that the fundamental groups $\pi_1 X$ and $\pi_1 \widetilde{X}$ are isomorphic and so $H_*^{\ell_1}(X)$ and $H_*^{\ell_1}(\widetilde{X})$ are isometrically isomorphic. Also note that the inclusion homomorphism $\pi_1 Y \to \pi_1 \widetilde{Y}$ is surjective and its kernel is equal to the kernel of $\{\rho\colon \pi_1 Y \to \pi_1 X\}$ so that the groups $\pi_1 \widetilde{Y}$ and $\rho(\pi_1 Y)$ are isomorphic.

Definition 7.5. *By the perfect pair of spaces induced from a given pair of spaces (X, Y) we understand a pair of spaces $(\widetilde{X}, \widetilde{Y})$ constructed from (X, Y) by attaching disks to kill the entire kernel of inclusion homomorphism $\pi_1 Y \to \pi_1 X$.*

Proposition 7.3. *Let X and Y be the connected countable cellular spaces. Let $(\widetilde{X}, \widetilde{Y})$ be the perfect pair induced from (X, Y). If Y is amenable, then the groups $H_*^{\ell_1}(Y \hookrightarrow$*

X) and $H_*^{\ell_1}(\widetilde{Y} \hookrightarrow \widetilde{X})$ are isometrically isomorphic for the norms $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$.

Proof. Recall that the group $\pi_1(\widetilde{Y})$ is isomorphic with the image of the homomorphism $\pi_1 Y \to \pi_1 X$. So the group $\pi_1(\widetilde{Y})$ is amenable and so the groups $H_*^{\ell_1}(\widetilde{Y})$ is trivial. Also remark that there is a commutative diagram

$$\begin{array}{ccccccccc}
H_n^{\ell_1}(Y) & \longrightarrow & H_n^{\ell_1}(X) & \xrightarrow{H_n(p)} & H_n^{\ell_1}(Y \hookrightarrow X) & \longrightarrow & H_{n-1}^{\ell_1}(Y) & \longrightarrow & H_{n-1}^{\ell_1}(X) \\
\downarrow{\scriptstyle H_n(\gamma)} & & {\scriptstyle H_n(\alpha)}\downarrow & & {\scriptstyle H_n(\beta)}\downarrow & & {\scriptstyle H_{n-1}(\gamma)}\downarrow & & {\scriptstyle H_{n-1}(\alpha)}\downarrow \\
H_n^{\ell_1}(\widetilde{Y}) & \longrightarrow & H_n^{\ell_1}(\widetilde{X}) & \xrightarrow{H_n(q)} & H_n^{\ell_1}(\widetilde{Y} \hookrightarrow \widetilde{X},) & \longrightarrow & H_{n-1}^{\ell_1}(\widetilde{Y}) & \longrightarrow & H_{n-1}^{\ell_1}(\widetilde{X})
\end{array}$$

Note that the map $H_*(\gamma)$ is surjective and the map $H_*(\alpha)$ is an isometric isomorphism for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$. So the map $H_n(\beta)$ is surjective by Five Lemma.

From Corollary 6.5, the map $H_n(q)$ is an isometric isomorphism for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$. Thus the composite $H_n(q)H_n(\alpha)$ is also an isometric isomorphism. Then, since $\|H_n(p)\| \leq 1$, we have

$$1 = \|H_n(q)H_n(\alpha)\| = \|H_n(\beta)H_n(p)\| \leq \|H_n(\beta)\|\|H_n(p)\| \leq \|H_n(\beta)\|$$

for the norms $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$. Hence we have $1 \leq \|H_n(\beta)\|$ for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$.

Since it is clear that we have $\|H_n(\beta)\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$, the map $H_n(\beta)$ is a surjective isometry. Since an isometry is injective, the map $H_n(\beta)$ is an isometric isomorphism for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$. □

Corollary 7.4. *Let X and Y be the connected countable cellular spaces. If Y is amenable subset of X, then the groups $H_n^{\ell_1}(X)$ and $H_n^{\ell_1}(Y \hookrightarrow X)$ are isometrically isomorphic for the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$.*

Proof. Let $(\widetilde{X}, \widetilde{Y})$ be the perfect pair induced from (X, Y). Then the following groups are isometrically isomorphic for the norms $\|\cdot\|(\omega)$ for all $\omega \in [0,\infty]$:

$$\begin{aligned} H_n^{\ell_1}(Y \hookrightarrow X) &\cong H_n^{\ell_1}(\widetilde{Y} \hookrightarrow \widetilde{X}) && \text{by Proposition 7.3} \\ &\cong H_n^{\ell_1}(\widetilde{X}) && \text{because } \pi_1\widetilde{Y} \text{ is amenable} \\ &\cong H_n^{\ell_1}(X) && \text{because } \pi_1\widetilde{X} \text{ and } \pi_1 X \text{ are isomorphic.} \end{aligned}$$

□

Definition 7.6. *A subset L of X is said to be large if the complement $X - L$ is relatively compact, i.e., the closure of $X - L$, denoted by $\overline{X - L}$, is compact subset of X.*

We call the space X amenable at infinity if every large set $L \subset X$ contains another large set, $L' \subset L$, such that L' is an amenable subset of L.

Theorem 7.5 (Equivalence theorem). *Let X be a connected countable cellular space. If X is amenable at infinity, then for every $n \geq 2$ the norms on $H_n^\infty(X)$ satisfy $\|\cdot\|_l(\omega) = \|\cdot\|_s$ for every $\omega \in [0,\infty]$.*

Proof. Let $n \geq 2$ and let $X = L_0 \supset L_1 \supset L_2 \supset \cdots$ be a sequence of large open subsets of X such that L_{j+1} is an amenable subset of L_j for $j \geq 0$.

Note that $X - L_j = \overline{X - L_j}$ is a closed compact subset of X. We set $K_j = X - L_j$ so that $L_j = X - K_j$. Then $\{K_j\}$ is an increasing sequence of compact subspaces of X and so there is a sequence of homomorphisms induced by inclusions:

$$\to H_n^{\ell_1}((X - K_3) \hookrightarrow X) \to H_n^{\ell_1}((X - K_2) \hookrightarrow X) \to H_n^{\ell_1}((X - K_1) \hookrightarrow X) \to 0.$$

By definition $H_n^\infty(X) = \varprojlim_j H_n^{\ell_1}((X - K_j) \hookrightarrow X)$ and for $[x] \in H_n^\infty(X)$ we have

$$\|[x]\|_s = \|[x]\|_l(0) \leq \|[x]\|_l(\omega) \quad \text{for every } \omega \in [0,\infty].$$

We claim that the norms $\|[x]\|_l(\omega)$ are equal for every $\omega \in [0,\infty]$. Then, especially, the norms $\|[x]\|_l(\omega)$ are equal to $\|[x]\|_l(0)$ for every $\omega \in [0,\infty]$ and so the theorem will be proved.

For each $j \geq 1$, let $\rho_j\colon \pi_1 L_{j+1} \to \pi_1 L_j$ and $\tau_j\colon \pi_1 L_j \to \pi_1 X$ be homomorphisms induced by inclusions. Then $\tau_1 \circ \rho_1 \circ \cdots \circ \rho_j = \tau_{j+1}$ and $\tau_j \circ \rho_j = \tau_{j+1}$.

Since $\tau_1(\pi_1 L_1)$ is an amenable subgroup of $\pi_1 X$ and $\rho_j(\pi_1 L_{j+1})$ is an amenable subgroup of $\pi_1 L_j$, the group $\tau_1 \circ \rho_1 \circ \cdots \circ \rho_j(\pi_1 L_{j+1}) = \tau_{j+1}(\pi_1 L_{j+1})$ is an amenable subgroup of $\pi_1 X$. Thus every large set $L_j = X - K_j$ is an amenable subset of X.

Then by Corollary 6.5 and Corollary 7.4, for every $\omega \in [0,\infty]$, the norms $\|[x_j]\|_1(\omega)$ of $[x_j] \in H_n^{\ell_1}(L_j \hookrightarrow X)$ are equal. Hence for every $\omega \in [0,\infty]$ the norms $\|[x]\|_l(\omega) = \lim_j \|[x_j]\|_1(\omega)$ are also equal. This proves our claim and so the proof is finished. □

In the following Vanishing and Finiteness theorems on $H_*^{\ell_1}(X)$, we are concerned with the amenable covering of X and so we assume the spaces and their coverings satisfy the conditions for amenable covering in Section 1.3.2.

First we state Theorem on Double complexes, which we will use technically for the proof of Vanishing theorem.

Theorem 7.6 (Theorem on Double Complexes). *Let $(C_{p,q})$ be a first quadrant double complex with differentials*

$$\partial_q\colon C_{p,q} \to C_{p,q-1} \quad \text{and} \quad \delta_p\colon C_{p,q} \to C_{p-1,q},$$

and let T_n be its total complex. Also, for each $p \geq 0$, let M_p denote the cokernel of the differential $\partial_1\colon C_{p,1} \to C_{p,0}$. Then M_* together with the differential δ is a subcomplex of the total complex T_*. Furthermore, if the complexes $(C_{p,*}, \partial)$ are exact, then the inclusion $M_* \to T_*$ induces a homology isomorphism $H_*(T_*) \to H_*(M_*)$.

This Theorem on double complexes is a special case of Theorem 4.8.1 of Chapter I of Algebraic Topology and Sheaf Theory by R.Godement. Its standard proof is based on the spectral sequences of a double complex. However it is also easy to prove directly, using a diagram chase.

Theorem 7.7 (Vanishing theorem on $H_*^{\ell_1}(X)$). *Let $\mathcal{U} = \{U_j\}$ be an amenable covering of X. If every point of X is contained in at most m subsets U_j for some $m = 1, 2, \cdots$, then $\|[x]\|_1 = 0$ for every $[x] \in H_n^{\ell_1}(X)$ and for every $n \geq m$.*

Proof. First, we show that the theorem reduces to the case that the fundamental groups of all components of elements of the covering and their finite intersections are amenable. Let U be some component. As in Remark 7.3, we attach to X the disks to kill the entire kernel of $\pi_1 U \to \pi_1 X$. Then we get a new space $X \cup \{\text{attached disks}\}$ and its new covering $\mathcal{U}'$ by including the attached disks. Since $\mathcal{U}'$ has the same nerve of $\mathcal{U}$, we may consider the space X with covering $\mathcal{U}$ as the new space with the covering $\mathcal{U}'$. So we can reduce the theorem to the case that the group $\pi_1 U_j$ is amenable for every $U_j \in \mathcal{U}$. In the proof we consider this case only.

Let $\mathcal{N}$ be the nerve of the covering $\mathcal{U}$. For every $\sigma \in \mathcal{N}$, we denote by $|\sigma|$ the intersection of elements of the covering corresponding to its vertices.

For $p, q \geq 0$, we set

$$C_p(\mathcal{N}, C_q^{\ell_1}) = \bigoplus_{\sigma \in N_p} C_q^{\ell_1}(|\sigma|),$$

where N_p is the set of p-dimensional simplices of the nerve $\mathcal{N}$. Taking the direct sum of the complexes

$$\cdots \to C_2^{\ell_1}(|\sigma|) \xrightarrow{\partial} C_1^{\ell_1}(|\sigma|) \xrightarrow{\partial} C_0^{\ell_1}(|\sigma|) \to \mathbf{R} \to 0$$

over $\sigma \in N_p$, we get a complex for each $p \geq 0$

$$\cdots \to C_p\left(\mathcal{N}, C_2^{\ell_1}\right) \xrightarrow{\partial} C_p\left(\mathcal{N}, C_1^{\ell_1}\right) \xrightarrow{\partial} C_p\left(\mathcal{N}, C_0^{\ell_1}\right) \xrightarrow{\eta} C_p(\mathcal{N}) \to 0,$$

where $C_p(\mathcal{N})$ are the real simplicial chain groups of the nerve $\mathcal{N}$.

If the simplex $\sigma \in N_p$ has the vertices $v_0 < v_1 < \cdots < v_p$, we denote by $\partial_i(\sigma)$, the $(p-1)$-simplex $\{v_0, \cdots, \hat{v}_i, \cdots, v_p\}$ for $i = 0, \cdots, p$. Then the inclusions $C_q^{\ell_1}(|\sigma|) \hookrightarrow C_q^{\ell_1}(|\partial_i \sigma|)$ induce a chain map

$$\delta_i : C_p\left(\mathcal{N}, C_q^{\ell_1}\right) \to C_{p-1}\left(\mathcal{N}, C_q^{\ell_1}\right) \qquad \text{for every } p \geq 1.$$

We set $\delta = \sum_{i=0}^{p}(-1)^i\delta_i$ which is a chain map. Similarly, together with the inclusions $C_q^{\ell_1}(|\sigma|) \hookrightarrow C_q^{\ell_1}(X)$, we define a chain map

$$\begin{array}{ccccccccc}
\to C_0(\mathcal{N}, C_2^{\ell_1}) & \xrightarrow{\partial} & C_0(\mathcal{N}, C_1^{\ell_1}) & \xrightarrow{\partial} & C_0(\mathcal{N}, C_0^{\ell_1}) & \xrightarrow{\eta} & C_0(\mathcal{N}) & \longrightarrow & 0 \\
\mu\big\downarrow & & \mu\big\downarrow & & \mu\big\downarrow & & \mu\big\downarrow & & \\
\to C_2^{\ell_1}(X) & \xrightarrow{\partial} & C_1^{\ell_1}(X) & \xrightarrow{\partial} & C_0^{\ell_1}(X) & \xrightarrow{\eta} & \mathbf{R} & \longrightarrow & 0.
\end{array}$$

Then we have the following commutative diagram (7.7.1)

$$\begin{array}{ccccccccc}
\to C_2(\mathcal{N}, C_2^{\ell_1}) & \xrightarrow{\partial} & C_2(\mathcal{N}, C_1^{\ell_1}) & \xrightarrow{\partial} & C_2(\mathcal{N}, C_0^{\ell_1}) & \xrightarrow{\eta} & C_2(\mathcal{N}) & \longrightarrow & 0 \\
\delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \\
\to C_1(\mathcal{N}, C_2^{\ell_1}) & \xrightarrow{\partial} & C_1(\mathcal{N}, C_1^{\ell_1}) & \xrightarrow{\partial} & C_1(\mathcal{N}, C_0^{\ell_1}) & \xrightarrow{\eta} & C_1(\mathcal{N}) & \longrightarrow & 0 \\
\delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \\
\to C_0(\mathcal{N}, C_2^{\ell_1}) & \xrightarrow{\partial} & C_0(\mathcal{N}, C_1^{\ell_1}) & \xrightarrow{\partial} & C_0(\mathcal{N}, C_0^{\ell_1}) & \xrightarrow{\eta} & C_0(\mathcal{N}) & \longrightarrow & 0 \\
\mu\big\downarrow & & \mu\big\downarrow & & \mu\big\downarrow & & \mu\big\downarrow & & \\
\to C_2^{\ell_1}(X) & \xrightarrow{\partial} & C_1^{\ell_1}(X) & \xrightarrow{\partial} & C_0^{\ell_1}(X) & \xrightarrow{\eta} & \mathbf{R} & \longrightarrow & 0.
\end{array}$$

Note that both rows and columns in the diagram (7.7.1) are complexes and the family $\left(C_p(\mathcal{N}, C_q^{\ell_1})\right)_{p,q\geq 0}$ together with the differentials ∂ and δ is a double complex.

Now, we consider the total complex T defined by

$$T_n = \bigoplus_{p+q=n} C_p(\mathcal{N}, C_q^{\ell_1}),$$

where the differential $d_n : T_n \to T_{n-1}$ is given by the formula $d_n|C_p(\mathcal{N}, C_q^{\ell_1}) = \partial + \delta$. Then the diagram (7.7.1) defines maps $T_* \to C_*^{\ell_1}(X)$ and $T_* \to C_*(\mathcal{N})$, where $C_*^{\ell_1}(X)$ and $C_*(\mathcal{N})$ are the lowest row and the right column of the diagram (7.7.1) without the group $\mathbf{R}$. Note that the homology of the complex T_* is the ℓ_1-homology $H_*^{\ell_1}(T)$.

Remark that for every $\sigma \in \mathcal{N}$ the group $\pi_1(|\sigma|)$ is amenable and so $H_*^{\ell_1}(|\sigma|) = 0$. Hence the rows in the diagram (7.7.1) except the lowest one are exact. Thus we have an isomorphism $H_*^{\ell_1}(T) \to H_*^{\ell_1}(X)$ by Theorem 7.6.

Now we claim that the columns in the diagram (7.7.1) except the right one are also exact. To prove our claim we give an alternative description of $C_p\left(\mathcal{N}, C_q^{\ell_1}\right)$.

Let S_q be the set of singular q-simplices of the space X. For every $c \in S_q$ we let N_c be the subcomplex of $\mathcal{N}$ consisting of the simplices $\sigma \in \mathcal{N}$ such that $c \subset |\sigma|$, i.e., for $c\colon \Delta_q \to X$, the image $c(\Delta_q) \subset |\sigma|$. Then the complex $C_p\left(\mathcal{N}, C_q^{\ell_1}\right) = \bigoplus_{\sigma \in N_p} C_q^{\ell_1}(|\sigma|)$ has one basis element for every pair $(\sigma, c) \in N_p \times S_q$ such that $c \subset |\sigma|$.

We consider the complex $\prod_{c \in S_q} C_p(N_c)$. We shall show that the complex $\bigoplus_{\sigma \in N_p} C_q^{\ell_1}(|\sigma|)$ is isomorphic to a subcomplex of $\prod_{c \in S_q} C_p(N_c)$.

For every $\sigma_i \in N_p$, we can write an element x_i of $C_q^{\ell_1}(|\sigma_i|)$ in the form

$$x_i = \sum_{j=1}^{\infty} r_{ij} c_{ij} \in C_q^{\ell_1}(|\sigma_i|),$$

where $c_{ij}\colon \Delta_q \to |\sigma_i| \subset X$ and $r_{ij} \in \mathbf{R}$. Let

$$x = \bigoplus(x_i) \in \bigoplus_{\sigma_i \in N_p} C_q^{\ell_1}(|\sigma_i|).$$

By reordering σ_i if necessary, we may assume that only the first k coordinates $x_1, \cdots, x_k$ are nonzero. Then, since $\sigma_i \in N_p$ and $c_{ij}(\Delta_q) \subset |\sigma_i|$, we have $r_{ij}\sigma_i \in C_p(N_{c_{ij}})$.

If $c_{ij} = c_{i'j'}$, by arranging them nicely, we can define a map

$$\Gamma_{pq}\colon \bigoplus_{\sigma \in N_p} C_q^{\ell_1}(|\sigma|) \to \prod_{c \in S_q} C_p(N_c).$$

It is clear that the map Γ_{pq} is well-defined and injective. Also note that

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{k} |r_{ij}|\right) = \sum_{i=1}^{k}\left(\sum_{j=1}^{\infty} |r_{ij}|\right) < \infty$$

We denote by $\prod_{c \in S_q} B_p(N_c)$ the image of Γ_{pq}. It is clear that the sequence

$$\xrightarrow{\delta'} \prod_{c \in S_q} B_2(N_c) \xrightarrow{\delta'} \prod_{c \in S_q} B_1(N_c) \xrightarrow{\delta'} \prod_{c \in S_q} B_0(N_c) \xrightarrow{\mu'} C_q^{\ell_1}(X) \to 0. \tag{7.7.2}$$

is a complex, where the differential $\delta'\colon \prod_{c \in S_q} B_p(N_c) \to \prod_{c \in S_q} B_{p-1}(N_c)$ is induced from the boundary operator $C_p(N_c) \to C_{p-1}(N_c)$. Also the complex $\bigoplus_{\sigma \in N_p} C_q^{\ell_1}(|\sigma|)$ is isomorphic to the complex $\prod_{c \in S_q} B_p(N_c)$.

Moreover, an examination of the definition of the maps δ and μ shows that the map $\delta\colon C_p(\mathcal{N}, C_q^{\ell_1}) \to C_{p-1}(\mathcal{N}, C_q^{\ell_1})$ is equal to the map $\prod_{c\in S_q} B_p(N_c) \to \prod_{c\in S_q} B_{p-1}(N_c)$, and similarly for μ.

Thus the columns except the right in the diagram (7.7.1):

$$\xrightarrow{\delta} C_2(\mathcal{N}, C_q^{\ell_1}) \xrightarrow{\delta} C_1(\mathcal{N}, C_q^{\ell_1}) \xrightarrow{\delta} C_0(\mathcal{N}, C_q^{\ell_1}) \xrightarrow{\mu} C_q^{\ell_1}(X) \to 0$$

is isomorphic to the sequence of complex in (7.7.2).

Note that the subcomplex N_c is acyclic so that $H_*(N_c) = 0$. So the sequence (7.7.2) is exact and so our claim is proved.

Then, by Theorem 7.6, we have an isomorphism $H_*^{\ell_1}(T) \to H_*(\mathcal{N})$.

Note that the homology of the complex $C_*(\mathcal{N})$ is actually the homology of the simplicial scheme of $\mathcal{N}$ which coincides with $H_*(|\mathcal{N}|)$, where $|\mathcal{N}|$ is the geometric realization of the nerve. Since every point of X is contained in at most m elements U_j of the covering $\mathcal{U}$ of X, $H_n(|\mathcal{N}|) = 0$ for every $n \geq m$.

Thus, for every $n \geq m$, we have

$$H_n^{\ell_1}(X) = H_n^{\ell_1}(T) = H_n(\mathcal{N}) = 0$$

and hence $\|[x]\|_1 = 0$ for every $[x] \in H_n^{\ell_1}(X)$. So the proof is finished. □

Corollary 7.8 (Vanishing theorem on $H_*^{\infty}(X)$). *Let $\mathcal{U} = \{U_j\}$ be an amenable covering of X. If every point of X is contained in at most m subsets U_j, for some $m = 1, 2, \cdots$, then $\|[x]\|_s = 0$ for every $[x] \in H_n^{\infty}(X)$ and for every $n \geq m$.*

Proof. As in Theorem 7.7, we can reduce the theorem to the case that the group $\pi_1 U_j$ is amenable for every $U_j \in \mathcal{U}$.

By Corollary 7.4, for every $U_j \in \mathcal{U}$, the groups $H_*^{\ell_1}(X)$ and $H_*^{\ell_1}(X, U_j)$ are isometrically isomorphic for the norm $\|\cdot\|_1$. Then by Theorem 7.7, the group $H_n^{\ell_1}(X)$ and so the group $H_n^{\ell_1}(X, U_j)$ are trivial for every $n \geq m$.

We consider $\mathcal{U}$-small singular simplices. Let $n \geq m$, and let

$$\mathcal{I} = \{K_i \mid K_i \text{ is a compact subspace of } X,\ i = 0, 1, 2, \cdots\}.$$

For given $[x] = ([x_i]) \in H_n^\infty(X) \subset \prod_{K_i \in \mathcal{I}} H_n^{\ell_1}((X - K_i) \hookrightarrow X)$, we represent each $[x_i] \in H_n^{\ell_1}((X - K_i) \hookrightarrow X)$ by a cycle $(z_i, a_i) \in C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(X - K_i)$. Note that $\partial z_i = -a_i$. We denote by z_i^j the linear combination of n-singular simplex σ in z_i such that $\partial_n \sigma$ is entirely contained in U_j. Then we can write z_i as a sum of z_i^j, i.e., $z_i = \sum_j z_i^j$. Note that $\|z_i\|_1 \leq \sum \|z_i^j\|_1$ and also note that $(z_i^j, -\partial z_i^j) \in C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(U_j)$ is a cycle. Thus

$$[z_i^j, -\partial z_i^j] \in H_n^{\ell_1}(U_j \hookrightarrow X) = 0 \quad \text{and so} \quad \|[z_i^j, -\partial z_i^j]\|_1 = 0.$$

Now we have

$$\|[x_i]\|_1 \leq \|(z_i, a_i)\|_1 \leq \sum_j \|(z_i^j, -\partial z_i^j)\|_1,$$

and so $\|[x_i]\|_1 = 0$.

Hence $\|[x]\|_s = \sup_{K_i} \|[x_i]\|_1 = 0$ and so the proof is finished. □

Theorem 7.9 (Finiteness theorem on $H_*^\infty(X)$). *Let $\mathcal{U} = \{U_j\}$ be an amenable open covering of X such that U_j is relatively compact. If there is a large open set every point of which is contained in at most m subsets U_j, for some $m = 1, 2, \cdots$, then the norm $\|[x]\|_s$ of $[x] \in H_n^\infty(X)$ is finite for every $n \geq m$.*

Proof. Let $n \geq m$. Let L be a large open set every point of which is contained in at most m subsets U_j. We set $K_0 = \overline{X - L}$. Then K_0 is a compact subset of X. By reordering the indices j if necessary, we may set for some index k_0

$$K_0 = \bigcup_{j \leq k_0} \overline{U_j} \quad \text{and} \quad L = \bigcup_{j > k_0} U_j.$$

As before we may reduce the theorem to the case that the group $\pi_1 U_j$ is amenable. We consider $\mathcal{U}$-small singular simplices. Note that the set $\{U_j \in \mathcal{U} \mid j > k_0\}$ is an amenable covering of L. Hence, by Corollary 7.8, we have $\|[y]\|_s = 0$ of $[y] \in H_n^\infty(L)$ for every $n \geq m$.

We define K_i by $K_i = K_{i-1} \cup \overline{U_{k_0+i}}$ inductively on $i \geq 1$. Then there is an increasing sequence of compact subsets $K_0 \subset K_1 \subset K_2 \subset \cdots$ such that $\bigcup_{i=0}^\infty (K_i)^\circ = X$. So, by Remark 7.1.C and Proposition 7.1, we can define $H_n^\infty(X) = \varprojlim_{K_i} H_n^{\ell_1}((X - K_i) \hookrightarrow X)$.

Let $[x] = ([x_i]) \in H_n^\infty(X) \subset \prod_{K_i} H_n^{\ell_1}((X - K_i) \hookrightarrow X)$ be given. We represent $[x_i]$ by a cycle $(z_i, c_i) \in C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(X - K_i)$. Note that $\partial z_i = -c_i$. Let a_i be a linear combination of n-singular simplices in z_i such that whose images are entirely contained in $L = \bigcup_{j > k_0} U_j$ and b_i be the rest of z_i after choosing a_i. Then z_i is written as the form $z_i = a_i + b_i$.

Note that $a_i \in C_n^{\ell_1}(L)$ such that $\partial a_i \in C_{n-1}^{\ell_1}(L - K_i)$. Thus $(a_i, -\partial a_i) \in C_n^{\ell_1}(L) \bigoplus C_{n-1}^{\ell_1}(L - K_i)$ is a cycle. By applying to $[a_i, -\partial a_i] \in H_n^{\ell_1}((L - K_i) \hookrightarrow L)$ the same argument on $[z_i, -\partial z_i] \in H_n^{\ell_1}((X - K_i) \hookrightarrow X)$ and $[z_i^j, -\partial z_i^j] \in H_n^{\ell_1}(U_j \hookrightarrow X)$ in the proof of Corollary 7.8, we have $\|[a_i, -\partial a_i]\|_1 = 0$.

Also note that $b_i \in C_n^{\ell_1}(K_0)$ and $\partial b_i \in C_{n-1}^{\ell_1}\big(K_0 \cap (X - K_i)\big)$. Since the spaces $K_0 \cap (X - K_i)$ is empty for all $i \geq 0$, we have $\partial b_i = 0$. Thus $b_i \in C_n^{\ell_1}(K_0)$ is in fact an n-cycle on K_0 for every i. So we have $[b_i] \in H_n^{\ell_1}(K_0)$ and $\|[b_i]\|_1 < \infty$ for every i. Recall that for every $i \geq 1$ the homomorphism

$$H_n(\alpha_0^i)\colon H_n^{\ell_1}((X - K_i) \hookrightarrow X) \to H_n^{\ell_1}((X - K_0) \hookrightarrow X),$$

induced from the inclusion $\alpha_0^i\colon C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(X - K_i) \to C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(X - K_0)$, satisfies $H_n(\alpha_0^i)[z_i, -\partial z_i] = [z_0, -\partial z_0]$. So we have $[z_i, -\partial z_i] = [z_0, -\partial z_0]$ in $H_n^{\ell_1}((X - K_0) \hookrightarrow X)$. Thus there is an element $(z_i', c_i') \in C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(X - K_0)$ such that

$$\begin{aligned}(z_i, -\partial z_i) - (z_0, -\partial z_0) &= (a_i + b_i, -\partial a_i - \partial b_i) - (a_0 + b_0, -\partial a_0 - \partial b_0) \\ &= (a_i - a_0 + b_i - b_0, -\partial a_i + \partial a_0) \\ &= \partial(z_i', c_i') = (\partial z_i' + c_i', -\partial c_i').\end{aligned}$$

As before, we rewrite $z_i' = a_i' + b_i'$ so that a_i' and b_i' are the linear combination of singular simplices in z_i' such that whose images are entirely contained in L and in K_0 respectively. It is easy to check $b_i - b_0 = \partial b_i'$. This shows that $[b_i] = [b_0]$ and so $\|[b_i]\|_1 = \|[b_0]\|_1 \leq \infty$.

Thus we have

$$\|[x]\|_s = \sup_{K_i} \|[x_i]\|_1 = \sup_{K_i} \|[z_i, -\partial z_i]\|_1 = \sup_{K_i} \|[a_i + b_i, -\partial a_i - \partial b_i]\|_1$$

$$\leq \sup_{K_i} \|[a_i, -\partial a_i]\|_1 + \sup_{K_i} \|[b_i]\|_1 = \|[b_0]\|_1 < \infty.$$

Hence the norm $\|[x]\|_s$ of $[x] \in H_n^\infty(X)$ is finite for $n \geq m$. □

Remark that if X is an m-dimensional manifold satisfying the assumption for Finiteness theorem, then the simplicial volume of X is finite.

Also if X satisfies the assumption for Vanishing Theorem on $H_*^\infty(X)$, then the simplicial volume of X is zero.

BIBLIOGRAPHY

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[1] G.Bredon, *Topology and Geometry*, Springer-Verlag, New-York, 1991.

[2] R.Benedetti, C.Petronio, *Lectures on hyperbolic geometry*, Springer-Verlag, New-York, 1991.

[3] R.Brooks, *Some remarks on bounded cohomology*, Ann. Math. Studies., **97**, 1981, 53-63.

[4] K.S.Brown, *Cohomology of Groups*, Springer-Verlag, New-York, 1982.

[5] M.Greenberg, J.Harper, *Algebraic topology-A first course*, Addison-Wesley, 1981.

[6] R.I.Grigorchuk, *Some results on bounded cohomology*, London Math.Soc., **202**, 1993, 111-163.

[7] M.Gromov, *Volume and bounded cohomology*, Publ. Math. IHES, **56**, 1982, 5-100.

[8] N.Ivanov, *Foundation of theory of bounded cohomology*, Zapiski Nauchnykh Seminarov Leningradskogo otdeleniya Matematicheskogo Instituta im. V.A.Steklova AN SSSR, **143**, 1985, 69-109.

[9] N.Ivanov, *The second bounded cohomology group*, Zapiski Nauchnykh Seminarov Leningradskogo otdeleniya Matematicheskogo Instituta im. V.A.Steklova AN SSSR, **167**, 1988, 117-120.

[10] S.Matsumoto, S.Morita, *Bounded cohmology of certain groups of homeomorphisms*, Proc. AMS, **94**, 1985, 539-544.

[11] J.Munkres, *Elements of algebraic topology*, Addison-Wesley, Menlo Park, 1984.

[12] J.Rotman, *An introduction to homological algebra*, Academic Press, San Diego, 1979.

[13] T.Soma, *The Gromov invaiants for links*, Invent.Math., **64**, 1981, 445-454.