

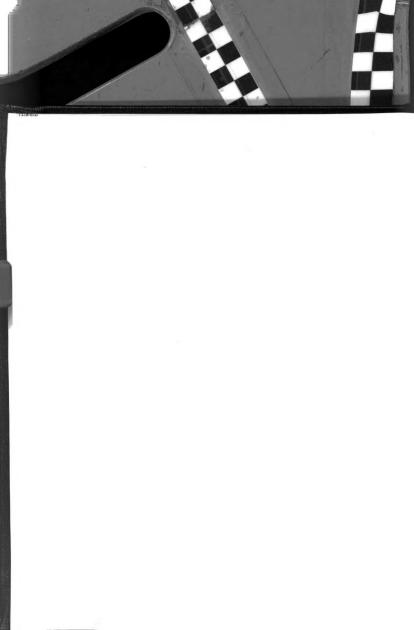
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COMPARATIVE STATICS UNDER RISK AVERSION AND PRUDENCE

Ву

Suyeol Ryu

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Economics

2001



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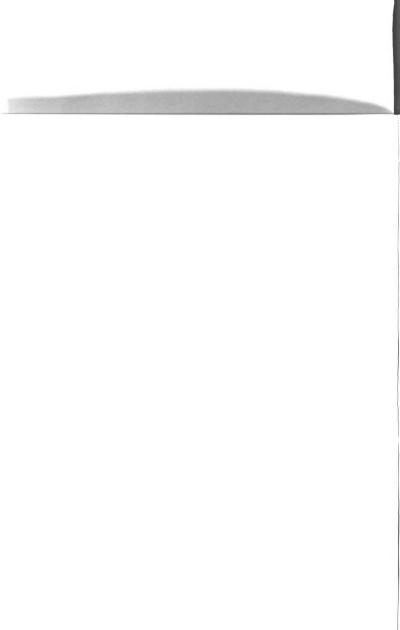
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ABSTRACT

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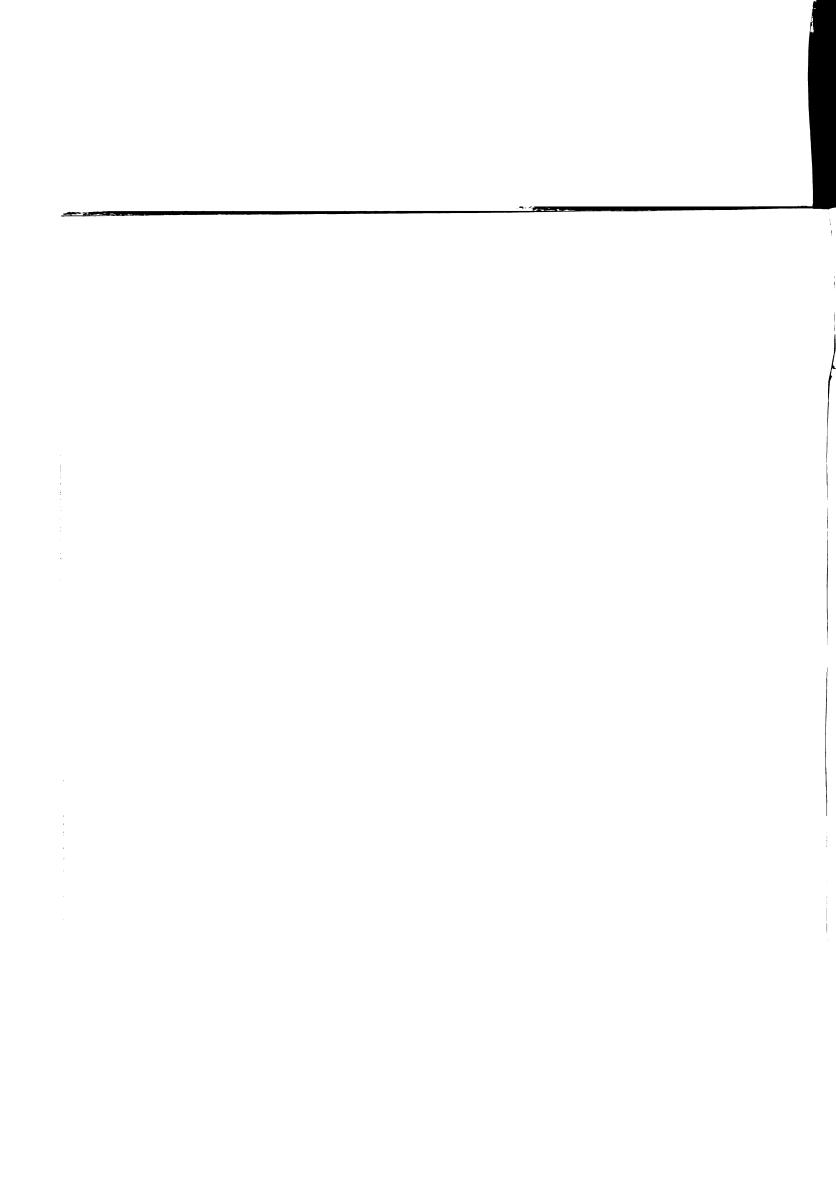
By

Suyeol Ryu

An economic decision model with randomness consists of the following four components: a set of decision makers, an objective function, random exogenous parameters and choice variables. An important comparative static question in the study of decision under uncertainty is how to predict the direction of change for a choice variable selected by the decision maker when a given random parameter changes. This general comparative static analysis is carried out by restricting the following three components:

(i) the changes in probability distribution function (PDF) or cumulative distribution function (CDF) of the random parameter, and/or (iii) the set of decision makers, and/or (iii) the structure of the economic decision model.

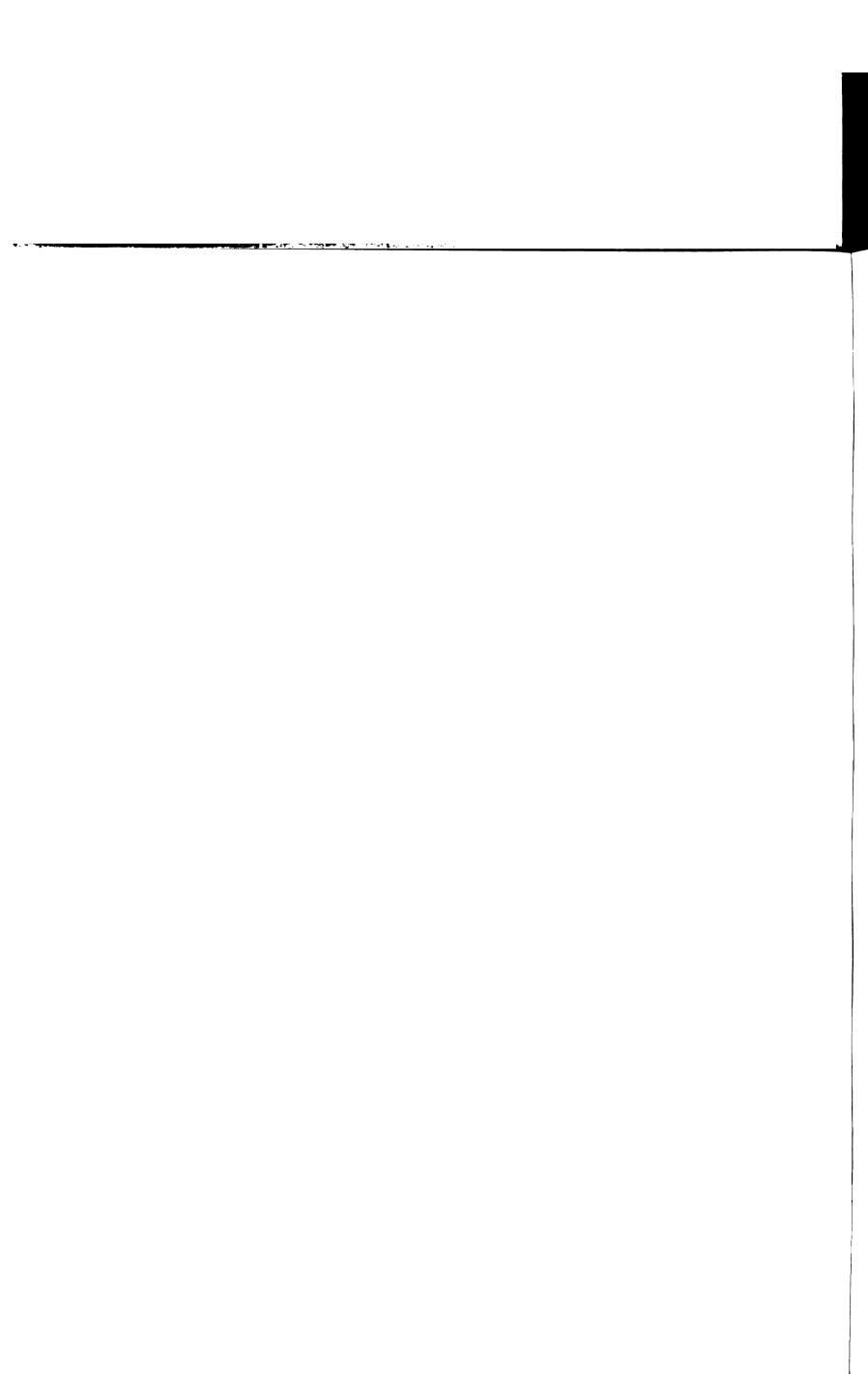
Our study focuses on finding sufficient conditions (or a necessary condition) on the change in distribution of the random parameter that cause risk averse decision makers with $u^{-n} \ge 0$ to adjust their choice variable in the same direction in a general decision model. Therefore all the comparative statics results obtained in this dissertation are associated with the set of risk averse individuals with a non-negative third derivative of their utility function. This set includes utility functions representing quite plausible preferences, such as the ones exhibiting decreasing absolute risk aversion (DARA)





generally accepted as a reasonable attitude toward risk. This class of utility functions also includes the concept of 'prudence' $(\eta = -u''/u'')$ introduced by Kimball (1992), which denotes a precautionary saving motive.

This study deals with two particular types of R-S increases in risk with single crossing and three types of R-S increases in risk with multiple crossing in chapter 3, and three types of K-L increases in risk in chapter 4. For each given type of change in the random parameter, we developed conditions on the class of decision makers and the structure of the decision model that are sufficient for making a general comparative static statement. In these chapters, we use the traditional approach that restricts separately the changes in PDF or CDF and the structure of the given decision model for comparative static purposes. However Gollier (1995) restricts two components jointly with a single restriction to obtain a general comparative static statement. In chapter 5, following his technique, we deal with the problem of determining the conditions under which a change in distribution of the random parameter increases the optimal value of a decision variable for the set of risk averse individuals with $u^m > 0$, which was done before by Gollier (1995) for all risk averse individuals.





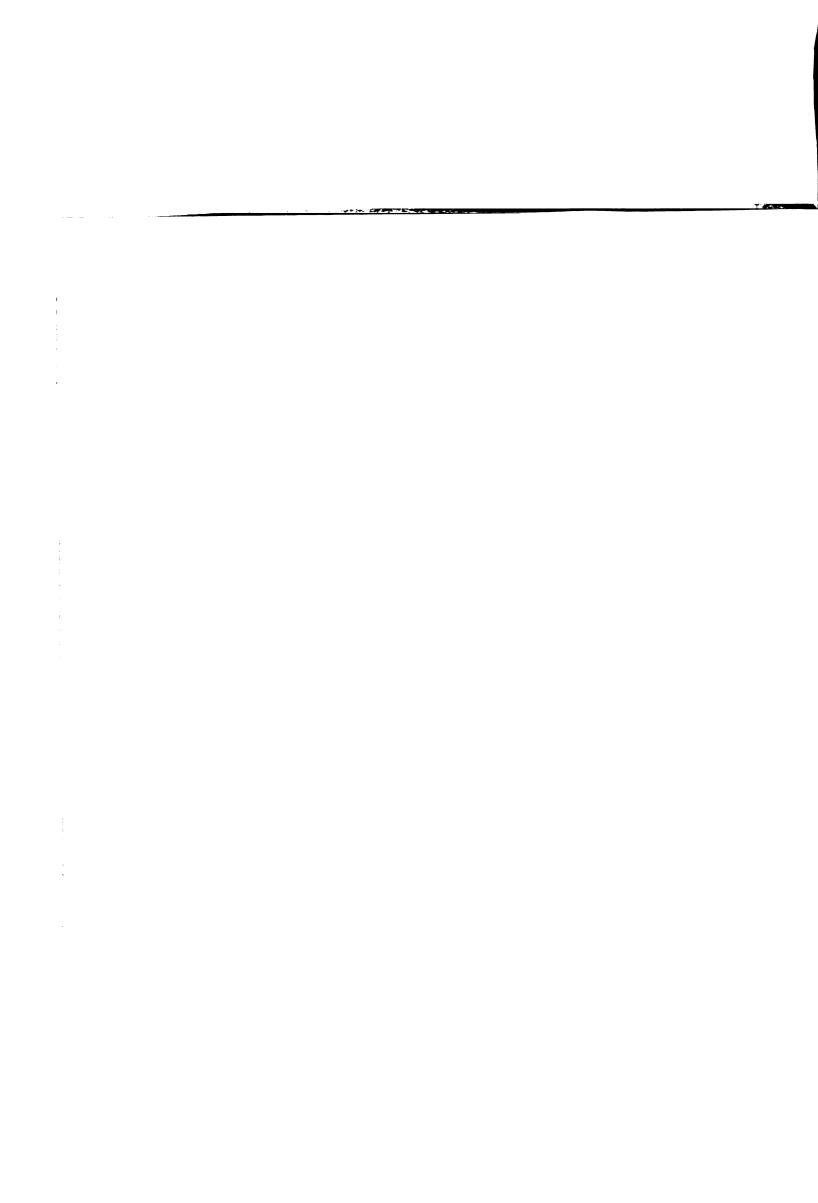
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Dedicated to My Parents



Dedicated to My Parents





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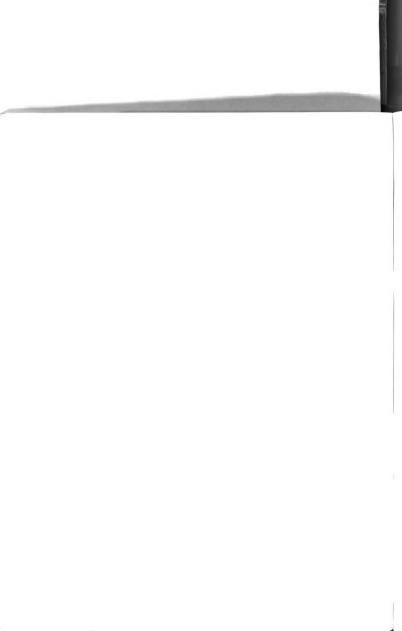


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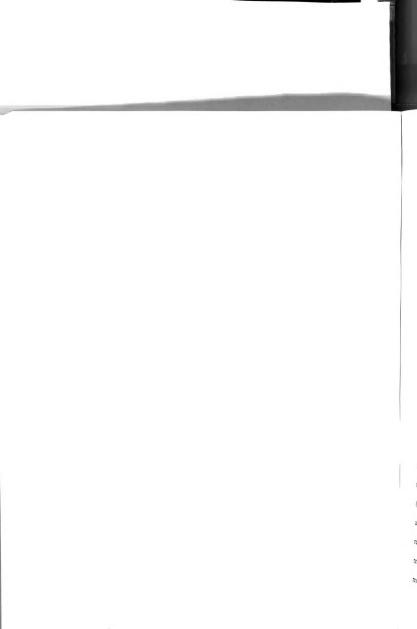


Chapter 1

INTRODUCTION

The effects of uncertainty on an individual's choices are theoretical interesting and have significant policy implications. In fact, the attention paid to this aspect of economic decision-making has a long tradition in the history of economics. Since its introduction by von Neumann and Morgenstern (1944), expected utility theory has been the dominant framework for the economic analysis of uncertainty and there has been much progress in the theoretical and applied analysis of choice under uncertainty. This risk analysis is included in many economic fields of study such as insurance, futures markets, stock markets, international trade and finance.

For the decision maker who is faced with a specific pair of risky prospects and whose utility function is a member of a specified class U, the concept of stochastic dominance (SD) or stochastic ordering of risky prospects is an important tool to identify efficient or undominated prospects among a given choice set. The stochastic dominance (SD) rules that have been developed to date include rules for decision makers who prefer more wealth to less, those who are also risk averse, and those who also have a nonnegative third derivative of their utility function $(u'' \ge 0)$. These rules are the 'first degree stochastic dominance' (FSD) rule, the 'second degree stochastic dominance' (SSD) rule,

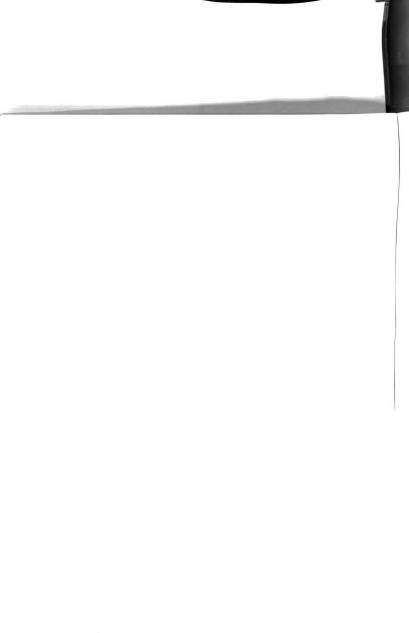


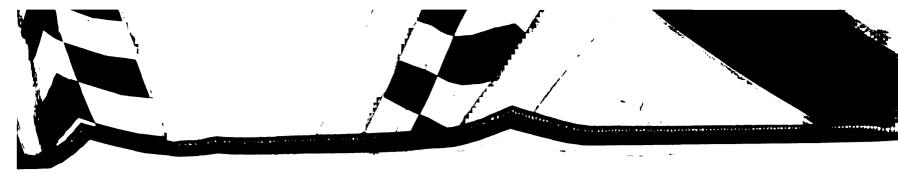


and the 'third degree stochastic dominance' (TSD) rule, respectively.

Since the seminal and simultaneous publications of Hadar and Russell (1969) and Hanoch and Levy (1969) there has been a virtual explosion of papers investigating implications of stochastic dominance rules for decisions under uncertainty. Dominance principles have important applications to portfolio choice, capital budgeting and financial intermediation decisions. Stochastic dominance can also be applied to investment and production decision problems and others under uncertainty. In these areas, overall impacts of uncertainty have received wide attention.

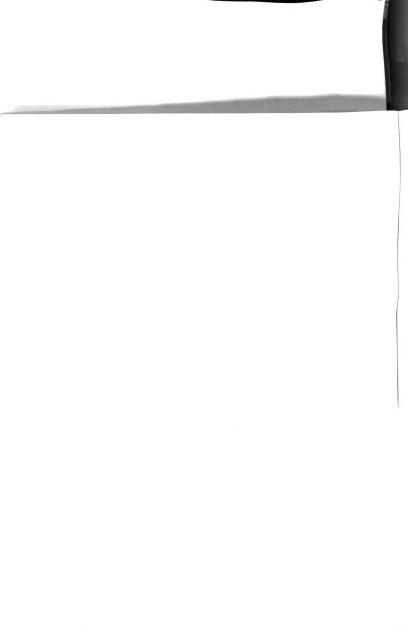
An important comparative static question in the study of decisions under uncertainty is how to predict the direction of change for a choice variable selected by the decision maker when a given random parameter changes. This general comparative static analysis is usually carried out by restricting the following components; (i) the changes in probability distribution function (PDF) or cumulative distribution function (CDF) of the random parameter, and/or (ii) the set of decision makers, and/or (iii) the structure of the given economic decision model. When the first and the third components are restricted separately or jointly, many authors obtain the comparative statics results that are associated with either the set of all individuals with non-decreasing utility functions or the set of all risk averse agents. These examples are included in Meyer and Ormiston (1985), Black and Bulkley (1989), Landsberger and Meilijson (1990), Dionne, Eeckhoudt and Gollier (1993), Eeckhoudt and Gollier (1995), Gollier (1995) and others. Note that relatively strong restrictions on one component are usually related to relatively weak restrictions on the other two components. In this study we impose somewhat stronger restrictions on the risk preference of decision makers.





All the comparative statics results obtained in this dissertation are associated with the set of risk averse individuals with a non-negative third derivative of their utility function. This set includes utility functions representing quite plausible preferences, such as the ones exhibiting decreasing absolute risk aversion (DARA) generally accepted as a reasonable attitude toward risk. This class of utility functions also includes the concept of 'prudence' $(\eta = -u'''/u'')$ introduced by Kimball (1990), which denotes a precautionary saving motive. Note that the term 'prudence' is meant to suggest the propensity to prepare and forearm oneself in the face of uncertainty. Therefore our study focuses on finding sufficient conditions (or a necessary condition) on the change in distribution of the random parameter that cause risk averse decision makers with $u''' \ge 0$ to adjust their choice variable in the same direction in a general decision model.

In order to generate interesting comparative statics results, the common restrictions to impose on the changes in PDF or CDF are general stochastic dominance orders. It implies that these SD rules play an important role in this comparative static analysis. There are three approaches specifying these restrictions. First, the CDF difference approach directly imposes restrictions on the difference between the initial and the final CDFs or PDFs. This approach is used to define a 'mean-preserving truncation' in Eeckhoudt and Hansen (1981) and a 'strong increase in risk' (SIR) in Meyer and Ormiston (1985). Second, the ratio approach imposes restrictions on the ratio of a pair of PDFs or of a pair of CDFs. Examples using this approach are included in Black and Bulkley (1989) who introduce the concept of a 'relatively strong increase in risk' (RSIR), Landsberger and Meilijson (1990) who define a 'monotone likelihood ratio' (MLR) and Eeckhoudt and Gollier (1995) who consider a 'monotone probability ratio' (MPR) among

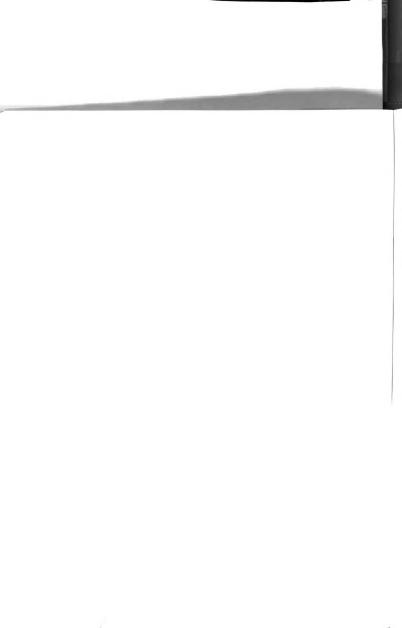


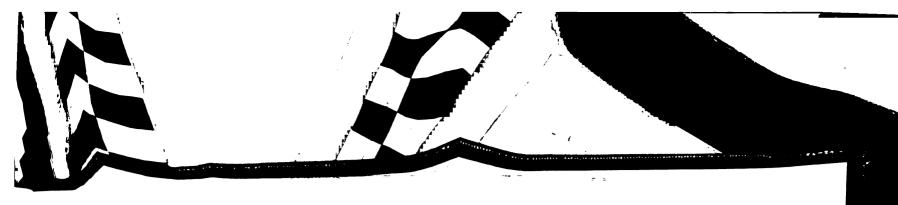


others. Finally, the deterministic transformation approach also restricts the change in the distribution by placing conditions on the transformation, which transforms an initial random variable into another random variable. This approach is popularized by Sandmo (1971) and extended by Meyer and Ormiston (1989) who define a 'simple increase in risk' (sIR). These three approaches are used in this study to specify restrictions imposed on the changes in PDF or CDF.

This dissertation is organized as follows. In the next chapter, we will give a short review of the literature concerning the SD selection rules and the previous work concerning comparative static analysis under uncertainty. We introduce a special 'increase in risk' defined by Kroll et. al (1995) and call it a 'K-L increase in risk', that is, third degree stochastic dominance (TSD) change with equal means. Since SSD implies a TSD, the set of K-L increases in risk includes the set of R-S increases in risk. This implies that K-L increases in risk extend the R-S definition of risk to a larger set of CDFs that could not be classified as 'more risky' before. We also provide terminology, notation, definitions and the decision model used in this study. The terminology and notation used here follows that in the established literature. When defining an increase in the riskiness of a random variable in the R-S sense or the K-L sense, we use F(x) and G(x) to denote the initial, less risky, and the final, more risky, CDFs. We use the general decision model previously employed by Kraus (1979) and Katz (1981) in their work.

Chapter 3 treats several subsets of R-S increases in risk. In addition to the existing subsets of R-S increases in risk, we define three subsets of R-S increases in risk called a 'left-side relatively weak increase in risk' (L-RWIR), a 'left-side strong increase in risk' (L-SIR) and a 'monotone strong increase in risk' (MSIR). Each subset generalizes a





definition in the published literature. This implies that a 'strong increase in risk' (SIR) in Meyer and Ormiston (1985), a 'relatively weak increase in risk' (RWIR) in Dionne, Eeckhoudt and Gollier (1993) and an 'extended strong increase in risk' (ESIR) in Kim (1998) are extended to a L-SIR, a L-RWIR and a MSIR, respectively. The restrictions used to define these subsets are discussed and graphical examples are given to illustrate the basic relations among the subsets. For basic relations, it is shown that the set of L-SIR shifts is included in the set of 'left-side relatively strong increases in risk' (L-RSIR) shifts in Kim (1998), which is, in turn, included in the set of L-RWIR shifts. We propose two more subsets of R-S increases in risk with multiple crossing called an 'outside strong increase in risk' (OSIR) and an 'outside relatively strong increase in risk' (ORSIR). The conditions in these two shifts imply that there exists a R-S decrease in risk in the given interval and a R-S increase in risk outside this interval. We also provide other characterizations of the MSIR ranking in order to provide an interpretation and compare the MSIR order with the ESIR one. We show that the MSIR order implies that the conditional expectation of a random variable under F is greater than and equal to that under G, and the converse is also true. Restricting the payoff function to be linear in the random variable, we show that the effects of these shifts can be determined for all risk averse decision makers with non-negative third derivative of utility functions. We also show that the MSIR condition is less restrictive than the Gollier (1995) condition presented in chapter 2 when the payoff function is linear in the random variable.

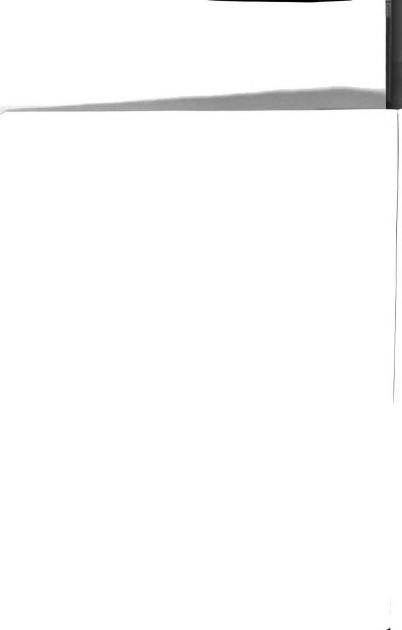
Chapter 4 provides general comparative static statements regarding several subsets of K-L increases in risk. In order to obtain comparative statics results, many researchers have specified particular types of CDF changes which are subsets of FSD





shifts, SSD shifts, or R-S increases in risk, So far, no one has examined subsets of TSD shifts for comparative static purposes. We introduce three subsets of K-L increases in risk. A 'strong increase in risk in the K-L sense' (SIRK) is defined by imposing restrictions on the difference between two cumulative of CDFs (C-CDFs). These restrictions replace the restrictions on the difference between two CDFs used by Mever and Ormiston (1985) who define the SIR order. A 'simple increase in risk across r in the K-L sense' ($sIR_K(r)$) is formally defined in this chapter. This concept also can be characterized by using a deterministic transformation of a random variable. A 'relatively strong increase in risk in the K-L sense' (RSIRK) consists of three parts that are a 'leftside monotone probability ratio' (L-MPR), a 'right-side monotone probability ratio' (R-MPR) and FSD shifts, or two parts which are 'left-side extended strong increases in risk' (L-ESIR) and 'right-side extended strong decreases in risk' (R-ESDR). A L-ESIR (R-ESDR) requires the monotonicity of the probability ratio only on the left-side (right-side) of the given pair of CDFs. These restrictions on the ratio for CDFs replace the restrictions on the ratio for PDFs imposed by Black and Bulkley (1989) who introduce the RSIR order. We have the basic relationship between the SIRK and the RSIRK orders as: the RSIR_K ranking includes the SIR_K one. All the comparative statics results obtained in this chapter are concerned with the set of risk averse individuals with $u''' \ge 0$.

In chapter 5, we deal with the problem of determining the conditions under which a change in distribution of the random parameter increases the optimal value of a decision variable for the set of risk-averse individuals with a positive third derivative of their utility function $(u^{\pi} > 0)$, which was done before by Gollier (1995) for all risk averse individuals. Given the set of decision makers, the traditional approach restricts separately





the changes in PDF or CDF and the structure of the given decision model. However he restricts two components jointly with a single restriction to obtain a general comparative static statement. His condition is quite simple and generalizes all previous restrictions on changes in risk such as a SIR and a RSIR imposed for all concave utility functions. Following the technique used in Gollier (1995), we obtain a necessary condition under which all risk-averse individuals with u''' > 0 react in the same direction when facing a given shift in distribution. Our condition is less restrictive than the Gollier one, but is difficult to interpret. We also obtain a sufficient condition for the same economic problem. We define a different subset of SSD that satisfies our sufficient condition and includes a R-S increase in risk as a special case.





Chapter 2

LITERATURE REVIEW

For many years the expected utility approach to decision making under uncertainty has gained increasing attention among economic researchers. An important question in the study of economic decision models involving randomness is how a particular type of a change in the random variable affects the level of the choice variable selected by a decision maker. General stochastic dominance (SD) orders play an important role in this comparative static analysis. Section 2.1 provides a brief review of general stochastic dominance rules in the literature. Section 2.2 gives a general decision model formulation used in this study. Section 2.3 presents a review of the literature concerning comparative static analysis under uncertainty. For notational convenience, the random variable x is assumed to have the initial and the final distribution characterized by CDFs F(x) and G(x), respectively and if not stated otherwise, the supports of these CDFs are assumed to be in the finite interval [a,b] where F(a) = G(a) = 0 and F(b) = G(b) = 1.

2.1 Stochastic Dominance Criteria





When decision makers compare uncertain prospects, it is assumed here they follow the rules of expected utility maximization. This section provides various formulations of SD rules and various ways of defining preference categories. Each formulation has advantage in solving certain issues. These rules apply to pairs of random variables, and indicate when one is to be ordered higher than the other by specifying a condition which the difference between their CDFs must satisfy. Given two risky prospects with CDFs F and G, the establishment of an order of preference between F and G is determined.

There are the following three main SD rules for ordering a pair of uncertain prospects F and G: first, second and third-degree stochastic dominance, denoted respectively by FSD, SSD, and TSD which are introduced by Quirk and Saposnik (1962), Schneeweiss (1969), Hanoch and Levy (1969), Hadar and Russell (1969) and Whitmore (1970). Each SD rule is formally defined as follows:

Definition 2.1. The following three main rules are introduced:

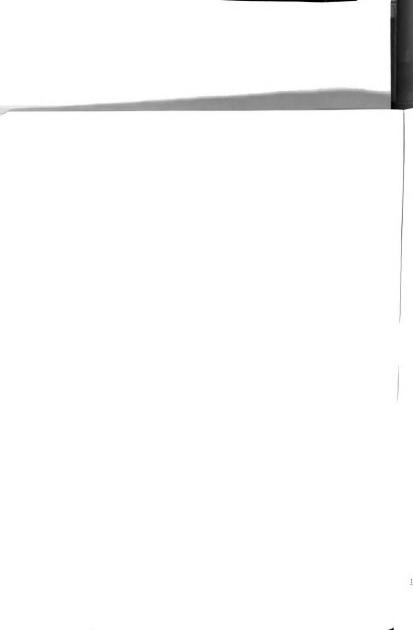
- (a) F(x) is said to stochastically dominate G(x) in the first-degree (denoted by F FSD
 - G) if and only if

$$G(x)-F(x) \ge 0$$
, for all $x \in [a,b]$,

(b) F(x) is said to stochastically dominate G(x) in the second-degree (denoted by F SSD G) if and only if

$$\int_{a}^{t} [G(x) - F(x)] dx \ge 0, \text{ for all } t \in [a, b],$$

- (c) F(x) is said to stochastically dominate G(x) in the third-degree (denoted by F TSD
 - G) if and only if





$$\int_a^s \int_a^t [G(x) - F(x)] dx dt \ge 0 \text{ for all } s \in [a, b] \text{ and}$$

$$\int_a^b [G(x) - F(x)] dx \ge 0.$$

Each rule defines a partial ranking on the set of all probability distributions such that the set of CDFs can be ordered by FSD is included in the set that can be ordered by SSD, and the latter set is included in the set that can be ordered by TSD. Each stochastic dominance ranking has the property of transitivity such that considering CDFs F_1 , F_2 and F_3 , if F_1 FSD (SSD or TSD) F_2 and F_2 FSD (SSD or TSD) F_3 , then F_1 FSD (SSD or TSD) F_3 .

Rothschild and Stiglitz (R-S) (1970) propose several definitions of "more risky" or "more variable" random variables and show that they are equivalent to one another:

A random variable y is riskier than a random variable x if y is equal to x plus a
noise term ε:

$$y = x + \varepsilon$$

where "=" means "has the same as distribution as" and ε is a random variable with the property that $E(\varepsilon|x) = 0$ for all x.

A random variable y is riskier than a random variable x if and only if all risk averse
individuals prefer x over y, namely

$$EU(x) \ge EU(y)$$
 for all concave utility functions.

3. A random variable y is riskier if it has more weight in the tails than a random





variable r.

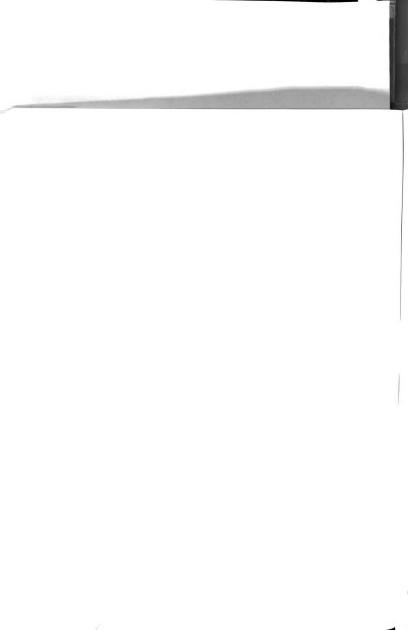
Rothschild and Stiglitz analyze these three definitions and conclude that the three definitions are equivalent and lead to a single definition of 'increased risk'. An important aspect of R-S risk analysis is included in their third definition which leads to the definition of a mean preserving spread (MPS). This definition is well known as 'integral conditions' being the restrictions imposed on the difference between two CDFs. Assuming x and y are random variables with CDFs F and G, respectively, R-S give a general definition of an 'increase in risk' in the following:

Definition 2.2. G(x) is said to be riskier than F(x) in the Rothschild-Stiglitz sense (denoted by G MPS F) if and only if

(a)
$$\int_{a}^{b} [G(x) - F(x)] dx = 0$$

(b)
$$\int_{a}^{t} [G(x) - F(x)] dx \ge 0$$
, for all $t \in [a, b]$.

The MPS (or R-S increases in risk) is a shift of probability mass when probability is moved from the center of the distribution to the tails without affecting the mean. This implies that probability mass is taken from a certain set of points and redistributed to points to the left and the right in such a way that the mean value of the random variable is kept unchanged. Note that a R-S increase in risk is an SSD change with equal means and also gives a partial ranking with the property of transitivity on a set of probability distributions like the three previous rules.





By analogy to the R-S definition of MPS, Kroll et al. (1995) introduce a new concept of probability mass shifts and call it a 'mean preserving spread-anitspread' (MPSA). It is assumed that Y and Z are two random variables with a cumulative distribution function of F(x) and G(x), respectively. S(x) and A(x) denote the mean preserving spread (MPS) function and the mean preserving antispread (MPA) function, respectively. The antispread shifts in probability mass function are the exact opposite to those imposed by the well-known MPS suggested by R-S.

The 'mean preserving spread-antispread' (MPSA) function SA is defined as follows:

$$SA(x) = S(x) + A(x)$$
.

Similar to R-S's MPS, if F(x) = G(x) + SA(x), F(x) differs from G(x) by a single MPSA step.

Definition 2.3. A MPSA function SA = S + A is said to satisfy the TSD criterion if

$$\int_{a}^{x} \int_{a}^{v} S(t) dt dv \ge - \int_{a}^{x} \int_{a}^{v} A(t) dt dv$$

for all x in $\left[a,b\right]$ and with a strict strong inequality for at least one x .

Note that the MPA improves a distribution while the MPS does the opposite.

Observe that the MPA is smaller but located to the right of the MPS in Definition 2.3.

According to Kroll et al., they specify the conditions on the MPSA functions that enable the classification of one random variable as 'more risky' than another random variable for different sets of utility functions. In their first theorem, 'more risky' random



variables are defined by employing MPS in the same way as R-S. For all risk averters' utility functions Theorem 2.1 is established.

Theorem 2.1 (SSD). Let Y and Z be two random variables with equal means with support bounded by [a,b]. Let F(x) and G(x) be the CDFs of Y and Z, respectively. Then F(x) dominates G(x) by the SSD criterion if and only if there exists a sequence of $\{SA\}_{i=1}^{\infty}$ of MPSA such that

$$F + \sum_{i=1}^{\infty} SA_i = G,$$

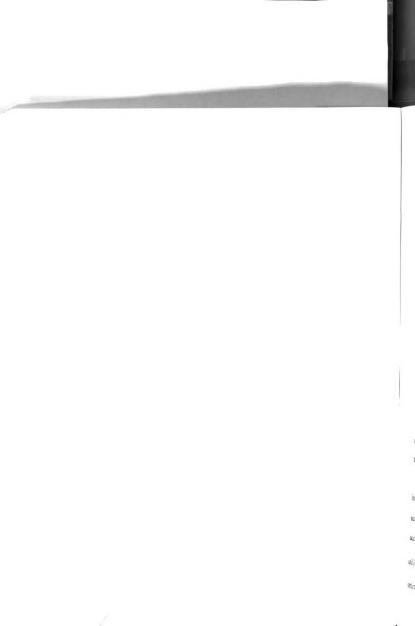
where for each $i = 1, 2, ..., A_i = 0$ (in SA_i) and the convergence is in the weak sense.

Note that this is the original theorem of R-S (1970).

For the TSD rule, R-S's original MPS is not sufficient and the antispread function is needed as well (that is, $A \neq 0$ in SA). In their second theorem, they specify the conditions that ensure that G can only be constructed from F using MPSA when F dominates G by the TSD criterion. For all risk averse individuals with DARA utility functions Theorem 2.2 is established.

Theorem 2.2 (TSD). Let F and G be the CDFs of two equal mean random variables, Y and Z respectively, with support bounded by [a,b]. Then F(x) dominates G(x) by TSD if and only if there exists a sequence of $\{SA_i\}_{i=1}^{\infty}$ of MPSA satisfying the TSD criterion such that

$$F + \sum_{i=1}^{\infty} SA_i = G.$$





The MPSA that satisfies the third degree stochastic dominance (TSD) criterion generates a riskier distribution for all DARA utilities but not all risk averters. This new definition extends R-S's definition of risk to a larger set of CDFs that could not be classified as 'more risky' before. We call it an 'increase in risk in the K-L sense' and give its formal definition as:

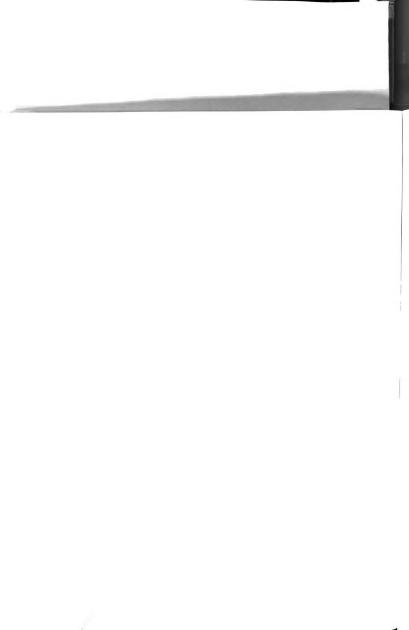
Definition 2.4. G(x) is said to be riskier than F(x) in the K-L sense if and only if

(a)
$$\int_{a}^{b} [G(x) - F(x)] dx = 0$$

(b)
$$\int_{a}^{s} \int_{a}^{t} \left[G(x) - F(x) \right] dx dt \ge 0 \text{ for all } s \in [a, b].$$

Condition (a) implies that two distributions have equal means. Condition (b) implies that a MPSA function SA = S + A satisfies the TSD criterion. These conditions imply that an increase in risk in the K-L sense is a TSD change with equal means. Since SSD implies TSD, the set of K-L increases in risk includes the set of R-S increases in risk. Observe that, for random variables with equal means, Definition 2.4 is equivalent to the TSD rule.

Assume that all risk averse individuals have DARA utility functions. As analyzed in Vickson (1975) and Bawa (1975), in the case of E(Y) = E(Z), a necessary and sufficient condition for all such individuals to prefer Y to Z is for Y to dominate Z according to the TSD rule. That is, if E(Y) = E(Z), Z is riskier than Y for all DARA utility functions if and only if $\int_a^x \int_a^x [G(t) - F(t)] dt dv \ge 0$ for all x in [a,b] and with a strict strong inequality for at least one x.





Each of the above stochastic dominance rules for unanimous choice is associated with a well-defined set of utility functions. These utility function classes are defined as follows:

Definition 2.5. Assuming that a utility function $u(\cdot)$ is continuous, bounded and three times differentiable for the support [a,b], five utility function classes are defined in the following ways:

(a)
$$U_1 = \{u(\cdot) | u' \ge 0\}$$

(b)
$$U_2 = \{u(\cdot) | u' \ge 0 \text{ and } u'' \le 0\}$$

(c)
$$U_3 = \{u(\cdot) | u'' \le 0\}$$

(d)
$$U_4 = \{u(\cdot) | u' \ge 0, u'' \le 0 \text{ and } u''' \ge 0\}$$

(e)
$$U_5 = \{u(\cdot) | u(\cdot) \text{ displays decreasing absolute risk aversion (DARA)} \}$$
.

Note that generally a utility function that satisfies the DARA condition $\left(A'(x) = \left(-u''(x)/u'(x)\right)' < 0, \forall x \in R\right) \text{ implies that } u' \geq 0, \ u'' \leq 0 \text{ and } u''' \geq 0; \text{ however vice versa this is not always valid. We assume that, for a utility function } u, EU_F \text{ and } EU_G \text{ represent the expected utilities when the CDFs are given by } F \text{ and } G, \text{ respectively.}$ The SD rules and the relevant class of preferences are related in the following way:

Theorem 2.3. Let F and G be the cumulative distribution of two distinct uncertain prospects. The followings are established:





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Theorem 2.3. Let F and G be the cumulative distribution of two distinct uncertain prospects. The followings are established:





- (a) $EU_F \ge EU_G$ for every u in U_1 if and only if F FSD G
- (b) $EU_F \ge EU_G$ for every u in U_2 , if and only if F SSD G
- (c) $EU_F \ge EU_G$ for every u in U_3 if and only if G is riskier in the R-S sense than F
- (d) $EU_F \ge EU_G$ for every u in U_4 if and only if F TSD G
- (e) $EU_F \ge EU_G$ for every u in U_5 if and only if G is riskier in the K-L sense than F.

The proofs of Theorem 2.3 are found in the literature: Quirk and Saposnik (1962) and Schneeweiss (1969) for the FSD rule, Hanoch and Levy (1969) and Hadar and Russell (1969) for the SSD rule, Rothschild and Stiglitz (1970) for the MPS rule, Whitmore (1970) for the TSD rule and Vickson (1975), Bawa (1975) and Kroll et al. (1995) for the condition (e) in Theorem 2.3.

2.2 Decision Model

An economic decision model with randomness consists of the following four components: a set of decision makers, an objective function, random exogenous parameters and choice variables. Many economic decision models including randomness can be usually divided into two types; specific decision models and general decision models. While the former are constructed to represent specific economic situations where the model structure and variables have specific interpretations, the latter are formulated to include many specific models as special cases.

We use the general decision model in this study introduced by Kraus (1979) and





Katz (1981) in their work. The decision maker is assumed to choose the optimal value for a choice variable α taking the random variable x as given. He chooses α so as to maximize expected utility, where utility u depends on a scalar valued function of the choice variable and the random variable, $z(x,\alpha)$. Formally, the economic agent's decision problem is to select α to maximize $E[u(z(x,\alpha))]$. That is,

$$\max_{\alpha} E[u(z(x,\alpha))]. \tag{2.1}$$

In this decision framework, utility depends only on the outcome variable z, that is, the objective function is single dimensional. Thus, problems involving multidimensionality are avoided.

In this study, we assume that utility function u(z) is thrice differentiable with respect to its argument with $u'(z) \ge 0$, $u''(z) \le 0$ and $u'''(z) \ge 0$; thus, the decision maker is a risk averter with $u'''(z) \ge 0$. The function $z(x,\alpha)$ is assumed three times differentiable with $z_{aa}(x,\alpha) < 0$. This condition insures that the second order condition for the maximization problem is satisfied. To simplify the discussion, we follow the literature and focus on the case where $z_x(x,b) \ge 0$. This assumption, combined with $u'(z) \ge 0$, indicates that higher values of the random variable are preferred to lower values. The case where $z_x(x,\alpha) \le 0$ can be handled with appropriate modifications. To focus on interior solutions to the maximization problem, it is assumed that $z_\alpha(x,\alpha) = 0$ is satisfied for some finite α for all relevant values of x.

Many researchers have used this general decision model in the study of choice under uncertainty and it includes a variety of economic decision problems. When we assume that the outcome variable is linear in the random variable, the simple form of





 $z(x,\alpha)$ may be expressed as $z(x,\alpha) = \alpha(x-c) + z_0$ where z_0 and c are exogenous constants. As analyzed by Sandmo (1971), Rothschild and Stiglitz (1971), Fishburn and Porter (1976), Dionne, Eeckhoudt and Gollier (1993) and Eeckhoudt and Gollier (1995), the applications of this simple form of a decision model are numerous: the standard portfolio problem, the problem of the competitive firm with constant marginal cost under output price uncertainty, the coinsurance problem and others.

In the standard portfolio model, the payoff function can be written as $z(x,\alpha)=z_0+bW_0(x-c)$ where b is the fraction of the initial wealth W_0 allocated to the risky asset, x the random rate of return of the risky asset and $z_0\equiv W_0(1+c)$ with c being the sure interest rate. This payoff function is equivalent to the simple form of $z(x,\alpha)$ when $\alpha\equiv bW_0$. For the competitive firm, the linear function is $z(x,\alpha)=\alpha(x-c)+z_0$, where x is the uncertain output price, c marginal cost, $-z_0$ the fixed cost and α the output level. In the standard coinsurance problem, the payoff function is given by the final wealth $z(x,\alpha)=W_0-\lambda\mu-(1-b)(x-\lambda\mu)$ where x is the amount of random loss, μ the expected loss, b coinsurance rate, $b\lambda\mu$ the insurance premium, and W_0 the initial wealth. This payoff function is equivalent to the simple form of $z(x,\alpha)$ when $z_0\equiv W_0-\lambda\mu$, $\alpha\equiv -(1-b)$ and $c\equiv \lambda\mu$. If we limit the discussion to private insurance contracts, the coinsurance rate b belongs to the interval [0,1]. Then, by definition, α is non-positive and belongs to the interval [-1,0]. Other examples of this simple form with appropriate modifications are included in Fedar (1977) who examines the problem of hiring workers and in Paroush and Kahana (1980) who investigate the cooperative firm model.

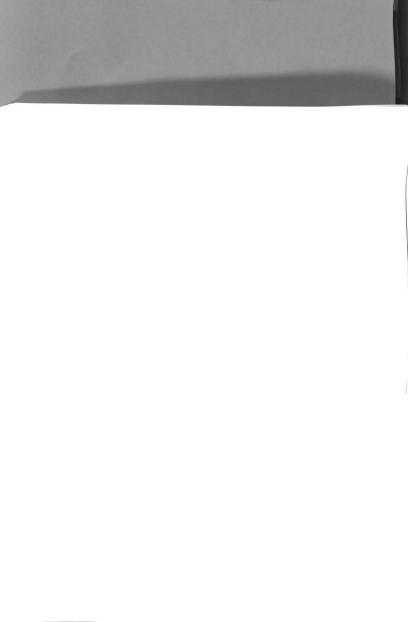
While most decision models include only one random and one choice variable,





some include more than one random and one choice variable. For decision models with one random and more than one choice variable, it is more difficult to make determinate comparative static statements than for the case with one random and one choice variable because of the interactions among choice variables. There are some examples of specific decision models with one random and two choice variables. Batra and Ullah (1974) investigate a competitive firm's input decisions with two inputs under output price uncertainty. Feder, Just and Schmitz (1977) examine an international trade model. Katz, Paroush and Kahana (1982) deal with the optimal policy of a price discriminating firm which operates under price uncertainty in one of two markets. Eeckhoudt, Meyer and Ormiston (1997) investigate the decision to insure and the portfolio composition decision and analyze the interaction between the demands for insurance and for the risky asset. All of these cited papers deal with a specific form of a decision model. Considering a general form of a decision model with one random and two choice variables, Choi (1992) investigates a special case of a corner solution.

For a decision model with more than one source of randomness, it is generally difficult to do the comparative static analysis for a change in any one random variable without restricting the correlations among random variables. To solve this problem, some papers assume that the random variables are independent of one another, allowing one to change while the others are held fixed. Hadar and Seo (1990) investigate the optimal proportions of the assets when the distribution of one of the assets undergoes some general type of shift which uses the general SD orders as restrictions on the change in the random variable. When random parameters are not independently distributed, we face the difficulty of defining a shift in risk of one random parameter without altering the



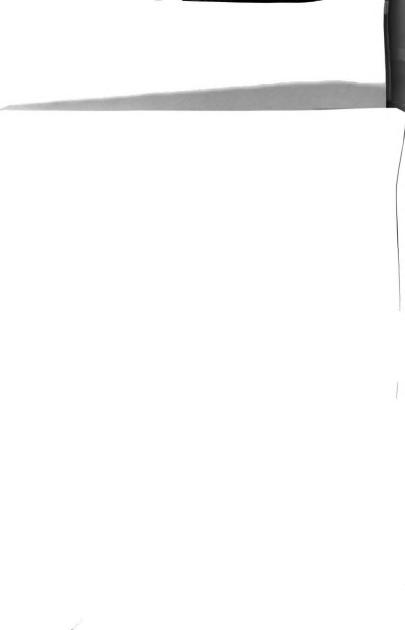
riskiness of the other random parameters; that is, without altering the marginal distribution of the other parameters. Meyer (1992) proposes the use of deterministic transformations to solve this problem. Other examples with multiple sources of risk are analyzed in the articles of Meyer and Ormiston (1994), Dionne and Gollier (1992, 1996) and others. For independently distributed other risks, referred to as background risk, Meyer and Meyer (1998) examine the effect of changes in the distribution function for this background risk on the decision to insure. Other examples for a change in background risk are included in Eeckhoudt, Gollier and Schlesinger (1996), Eeckhoudt and Kimball (1992) and others.

2.3 Comparative Static Analysis

Faced with uncertainty concerning the economic environment, an interesting question for comparative statics is to investigate necessary and/or sufficient conditions for determining the direction of change in the decision variable when a given random parameter changes. Our study focuses on a general decision model $z(x,\alpha)$ that includes one random, one choice and one outcome variable. Therefore, we review the comparative statics results in the literature concerning a general one-argument decision model in more detail

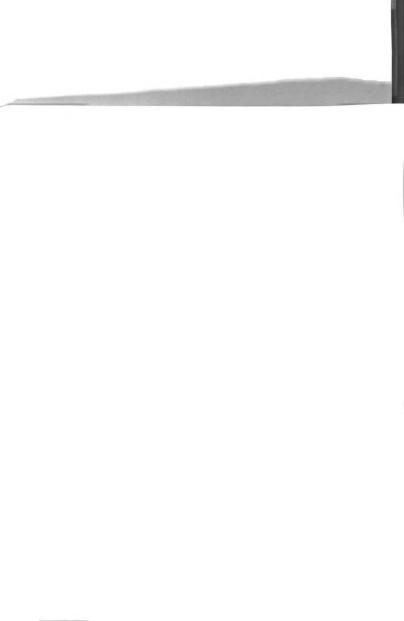
2.3.1 An Overview

It is generally known that the standard SD rules such as FSD, SSD and R-S



increases in risk are not sufficient to allow one to make general comparative static statements concerning the effect of a change in random variable on the choice made by an arbitrary decision maker with a non-decreasing and/or concave utility function. Using a general decision model Meyer and Ormiston (1983) demonstrate that the class of SSD shifts in distribution or R-S increases in risk causes all risk averse decision makers to adjust their level of the choice variable in the same direction if and only if its optimal level under certainty is independent of the value of the random exogenous variable. clearly not an interesting economic problem. Therefore, general comparative static analysis is carried out by imposing restrictions on the following components; the set of decision makers, the structure of the decision model and the set of changes in the random variable. Generally, when relative strong restrictions are imposed on one component, the derived comparative static statements are usually related to relatively weak restrictions on the other components. Many examples showing the above relationship are included in the following literature: Rothschild and Stiglitz (1971), Hadar and Russell (1978), Hadar and Seo (1990), Meyer and Ormiston (1983, 1985, 1989), Dionne, Eeckhoudt and Gollier (1993), Black and Bulkley (1989), Ormiston and Schlee (1993) and others.

In order to generate interesting comparative statics results, the common restrictions to impose on the changes in the random variable are general stochastic dominance orders. It implies that these SD rules play an important role in this comparative static analysis. There are three approaches specifying these restrictions. The CDF difference approach imposes restrictions on the initial and the final CDFs F and G, that is, restrictions on the difference between two CDFs G - F. Some examples using this approach are included in Sandmo (1971), Kraus (1979), Katz (1981), Eeckhoudt and



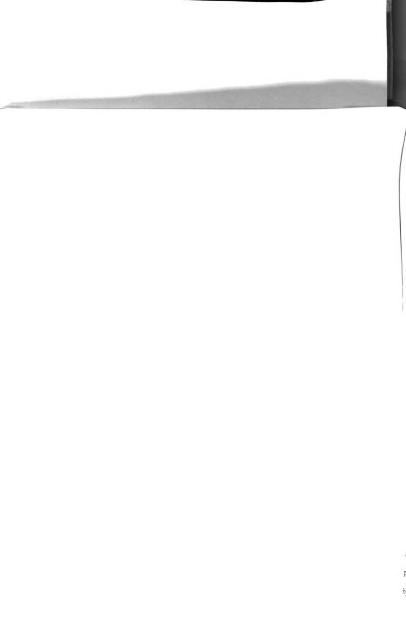


Hansen (1981) and Meyer and Ormiston (1985). The ratio approach imposes restrictions on the ratio for PDFs or the ratio for CDFs. Some examples using this approach are included in Black and Bulkley (1989), Landsberger and Meilijson (1990), Dionne, Eeckhoudt and Gollier (1993) and Kim (1998).

The deterministic transformation approach proposed by Meyer and Ormiston (1989) imposes some specific restrictions on the transformation function, which transforms an initial random variable into another random variable. The emphasis in this approach is on pairs of random variables which are related one another by means of a transformation, or on sets of random variables where each are related to a common random variable by means of a transformation. Assuming that the initial random variable x is characterized by CDF F(x), the transformed random variable y = t(x) is obtained from x by means of a transformation. The transformation t(x) is assumed to be non-decreasing, continuous and piecewise differentiable. Combined with the monotonic preferences for outcomes, the non-decreasing assumption ensures that the transformation does not reverse the preference ranking over the various possible outcomes of the original random variable. This assumption is necessary to make interesting statements about the effects of transformations of random variables on expected utility. Sandmo (1971), Meyer (1989), Meyer and Ormiston (1989) and Ormiston (1992) give examples on this approach.

2.3.2 Comparative Statics Results with Subsets of FSD Shifts

Before we give a short review for comparative statics results with subsets of FSD shifts, we assume that the support of G(x) is a finite interval $[x_1, x_3]$ and the support of





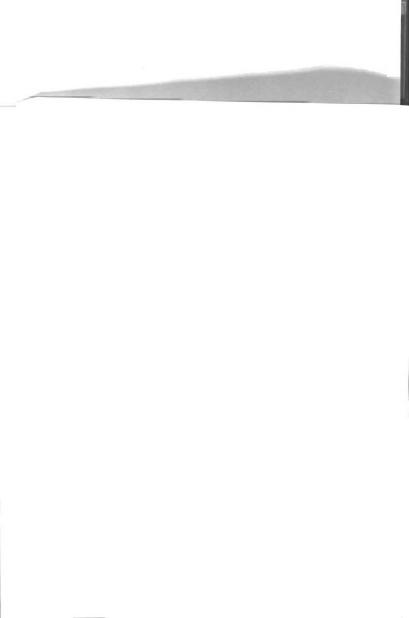
F(x) is another finite interval $[x_2, x_4]$ where $x_1 \le x_2$ and $x_3 \le x_4$.

Fishburn and Porter (1976) demonstrate that the FSD order does not allow a determinate general comparative static statement for all risk averse agents, even in the simplest case of portfolio problem. Landsberger and Meilijson (1990) introduce the concept of a 'monotone likelihood ratio' (MLR) order that is defined by imposing a monotonicity restriction on the ratio of a pair of PDFs. This restriction is widely used in the statistical literature.

Definition 2.6. F(x) represents a monotone likelihood ratio FSD shift from G(x) (denoted by F MLR G) if there exists a non-decreasing function $I:[x_2,x_3] \to [0,\infty)$ such that f(x)=I(x)g(x) for all $x \in [x_1,x_3]$.

Definition 2.6 implies that $g(x) \ge f(x)$ when $l \le 1$, $g(x) \le f(x)$ when $l \ge 1$, and the PDFs f and g cross only once. An MLR order is a FSD shift. Using the standard portfolio choice problem, a MLR shift induces all individuals with non-decreasing utility functions to increase their demand for the risky asset. Landsberger and Meilijson's analysis depends on a weak optimality condition of the economic decision problem where the optimal level of the decision variable is determinable, including an unbounded or a corner solution.

Eeckhoudt and Gollier (1995) introduce the concept of a 'monotone probability ratio' (MPR) order that is defined by imposing monotonicity restriction on the ratio of a pair of CDFs. This restriction replaces the restriction on the ratio of a pair of PDFs used by Landsberger and Meilijson who define a MLR order.



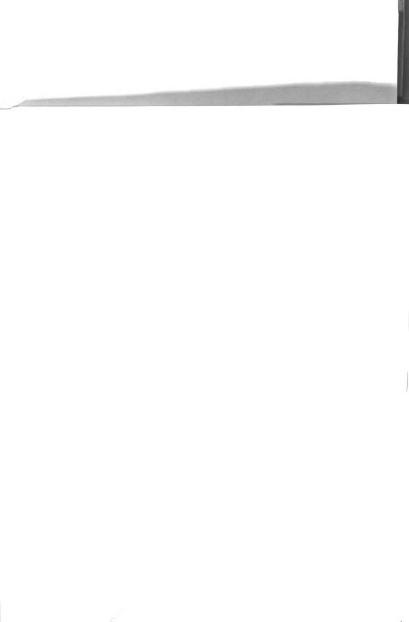
Definition 2.7. F(x) represents a monotone probability ratio FSD shift from G(x) (denoted by F MPR G) if there exists a non-decreasing function $h: [x_2, x_3] \rightarrow [0, 1]$ such that F(x) = h(x)G(x) for all $x \in [x_2, x_3]$.

A MPR order is a FSD shift from Definition 2.7. A MPR order is less restrictive than a MLR order since the former does not restrict the number of times of crossing between the PDFs f and g. Note that the MLR ranking implies the MPR one. All risk averse decision makers are considered in their paper.

Kim (1998) finds more subsets of FSD shifts by weakening the restrictions imposed on changes in PDF or CDF. Among them, he considers the concept of a 'left-side monotone likelihood ratio' (L-MLR) order that extends the MLR order. The L-MLR shifts are obtained from the relaxing the monotonicity requirement for points to the right of the crossing point.

Definition 2.8. F(x) represents a left-side monotone likelihood ratio FSD shift from G(x) (denoted by F L-MLR G) if there exists a point $m \in [x_2, x_3]$ and a non-decreasing function $l: [x_2, m] \rightarrow [0, 1]$ such that f(x) = l(x)g(x) for all $x \in [x_2, m)$ and $g(x) \le f(x)$ for all $x \in [m, x_3]$.

The L-MLR condition requires that two PDFs cross only once at the point m and that $g(x) \ge f(x)$ for all points to the left of m and $g(x) \le f(x)$ for all points to the right of m. Compared with the result in MLR shifts, the comparative statics result in L-MLR



shifts includes a larger set of FSD changes and a smaller set of decision makers. That is, all risk averse decision makers are considered in his analysis. He also obtains the property such that the L-MLR order lies between the MLR and the MPR one. This implies that F MLR $G \Rightarrow F$ L-MLR $G \Rightarrow F$ MPR G.

Note that the above three papers use the ratio approach in the comparative static analysis under uncertainty. Turning to the deterministic transformation approach,

Ormiston (1992) defines a simple FSD transformation as a class of FSD shifts and provides a general comparative static statement.

Definition 2.9. The random variable described by a transformation I(x) represents a simple FSD shift from the initial random variable x given by F(x) if $k(x) = I(x) - x \ge 0$ and $k'(x) \le 0$, for all $x \in [a,b]$.

Theorem 2.4. The optimal value of the choice variable increases for any simple FSD shift if

- (a) u(z) displays decreasing absolute risk aversion (DARA)
- (b) $z_x \ge 0$, $z_{xx} \le 0$ and $z_{\alpha x} \ge 0$.

Remember that generally a utility function satisfying the DARA condition implies that $u' \ge 0$, $u'' \le 0$ and $u''' \ge 0$; however vice versa this is not always valid. Theorem 2.4 contains the Sandmo (1971) result concerning the effect of a linear risk-altering transformation on the competitive firm's choice of output level as a special case. The transformation function used in his analysis is a linear form as:





$$t(x) = \gamma(x - \overline{x}) + \theta + \overline{x}$$

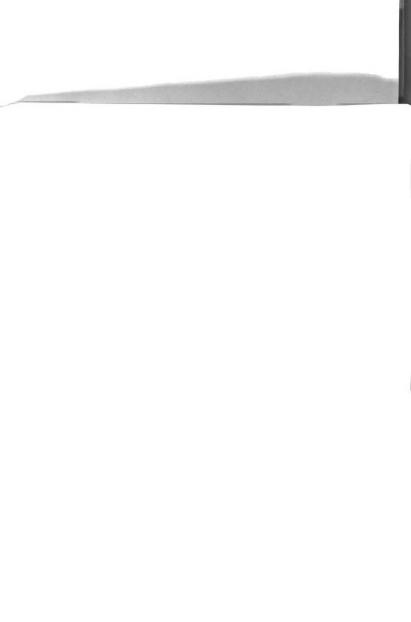
where γ is a multiplicative shift parameter, θ an additive shift parameter and \bar{x} the mean of the random variable x. An increase in θ (at $\gamma = 1$ and $\theta = 0$) is equivalent to moving the probability distribution to the right without changing its shape. That is, it defines a type of FSD shift.

2.3.3 Comparative Statics Results with Subsets of R-S increases in risk

This subsection reviews some important subsets of R-S increases in risk which provide general comparative static statements. Examination of literature shows that many authors have investigated an increase in risk from an initial nonrandom situation to yield interesting comparative static theorems for the set of risk averse decision makers. Some call it an 'introduction of risk' and others a 'global increase in risk', which is a particular type of R-S increases in risk. Examples of this concept are included in Sandmo (1971), Leland (1972), Batra and Ullah (1974), Kraus (1979), Katz (1981) and others.

While a global increase in risk yields interesting comparative statics results for all risk averse agents, its restriction is rather severe and limits significantly the situation to which those results can be applied. Eeckhoudt and Hansen (1981) propose an alternative definition of marginal change in randomness, that is, a 'mean-preserving truncation' which is less restrictive than a global increase in risk.

Definition 2.10. F(x) represents a mean-preserving truncation from G(x) if their difference, G(x) - F(x), satisfies





(a)
$$\int_{x_{-}}^{x_{+}} [G(x) - F(x)] dx = 0$$

(b)
$$\int_{a}^{y} [G(x) - F(x)] dx \ge 0$$
 for all $y \in [x_1, x_4]$

(c)
$$F(x) = 0$$
 for all $x \in [x_1, x_2)$, $F(x) = 1$ for all $x \in (x_3, x_4]$ and $F(x) = G(x)$ for all $x \in [x_3, x_4]$.

Definition 2.10 is a R-S increase in risk and includes a global increase in risk as a special case. Compared with the Sandmo result, their theorem is rather robust because the same comparative statics result is obtained for a more general set of R-S increases in risk without additional assumptions required. A mean-preserving truncation can be applied to real economic phenomena such as the existence of guaranteed minimum and/or imposed maximum prices.

Meyer and Ormiston (1985) introduce the concept of a 'strong increase in risk'

(SIR) as a subset of R-S increases in risk, which is a direct generalization of probability

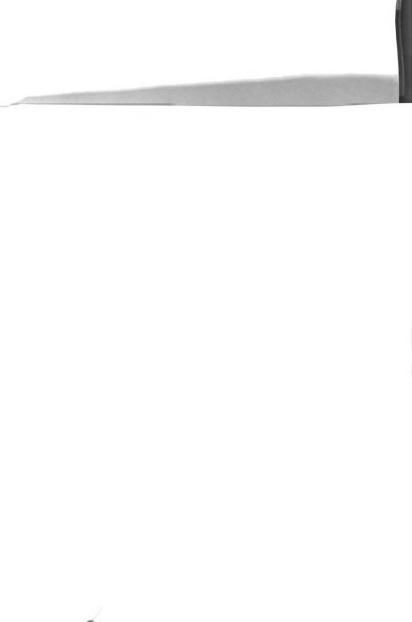
mass transfers involved in the introduction of risk. A SIR is formally defined as:

Definition 2.11. G(x) represents a strong increase in risk from F(x) (denoted by G SIR F) if their difference, G(x) - F(x), satisfies

(a)
$$\int_{x_{-}}^{x_{4}} [G(x) - F(x)] dx = 0$$

(b)
$$\int_{x}^{t} [G(x) - F(x)] dx \ge 0$$
 for all $t \in [x_1, x_4]$

(c) G(x) - F(x) is non-increasing on (x₂, x₃), where the support of F is contained in [x₂, x₁], the support of G is contained in [x₁, x₄].





A SIR is carried out by transferring probability mass from the interval (x_2, x_3) to the left tail interval $[x_2, x_3]$ and the right tail interval $[x_3, x_4]$. Compared with the result in Eeckhoudt and Hansen (1981), the comparative statics result in SIR shifts includes a larger set of R-S increases in risk and a general decision model. As a result, the SIR order represents a net improvement over the mean-preserving truncation order without any cost of additional assumptions. Note that the above subsets of R-S increases in risk are defined by imposing restrictions on the difference between the initial and the final CDFs. That is, the CDF difference approach is used as restrictions on changes in CDF.

A further generalization of a SIR is given by Black and Bulkley (1989) who introduce the concept of a 'relatively strong increase in risk' (RSIR). They use a ratio approach as restrictions on changes in PDF and their comparative static analysis is carried out for the set of risk averse agents.

Definition 2.12. G(x) represents a relatively strong increase in risk from F(x) (denoted by G RSIR F) if

- (a) $\int_{x}^{x_0} [G(x) F(x)] dx = 0$
- (b) For all points in the interval $[x_3, x_4]$, $f(x) \ge g(x)$ and for all points outside this interval $f(x) \le g(x)$ where $x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6$, $[x_1, x_6]$ being the supports of x under G(x) and $[x_2, x_5]$ being the supports under F(x)
- (c) f(x)/g(x) is non-decreasing in the interval $[x_2, x_3]$
- (d) f(x)/g(x) is non-increasing in the interval $(x_4, x_5]$.



Conditions (a) and (b) are sufficient for G(x) to represent a R-S increase in risk from F(x). That is, these conditions impose the restrictions that the two distributions have the equal mean, two PDFs cross only twice and probability mass is transferred from points within the interval (x_3, x_4) to points lying outside this interval. Conditions (c) and (d) restrict the extent to which probability mass can be transferred to any one value in the tails of F(x) relative to any other.

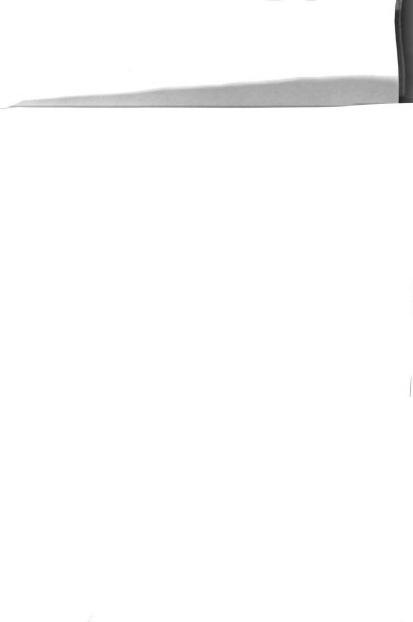
According to Black and Bulkley, the RSIR conditions are satisfied in many decision models if f(x) and g(x) are both normal distributions with $f(x) = N(\mu, \sigma_1^2)$ and $g(x) = N(\mu, \sigma_2^2)$ where $\sigma_2^2 > \sigma_1^2$. Conditions (a) and (b) are obviously satisfied. Since

$$\frac{\partial [f(x)/g(x)]}{\partial x} = -(x-\mu) \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right) \frac{f(x)}{g(x)}$$

f(x)/g(x) is increasing for $x < \mu$ and decreasing for $x > \mu$. Hence (c) and (d) are also satisfied. Note also that the conditions for a RSIR order are met if f(x) and g(x) are both gamma distributions with the same mean.

By relaxing the restrictions imposed to the right of the point m, Kim (1998) defines a 'left-side relatively strong increases in risk' (L-RSIR) that is a less stringent type of R-S increases in risk than a RSIR order. We assume that the support of G(x) is a finite interval $[x_1, x_5]$ and the support of F(x) is another finite interval $[x_2, x_4]$ where $x_1 \le x_2 \le x_4 \le x_5$.

Definition 2.13. G(x) represents a left-side relatively strong increase in risk from



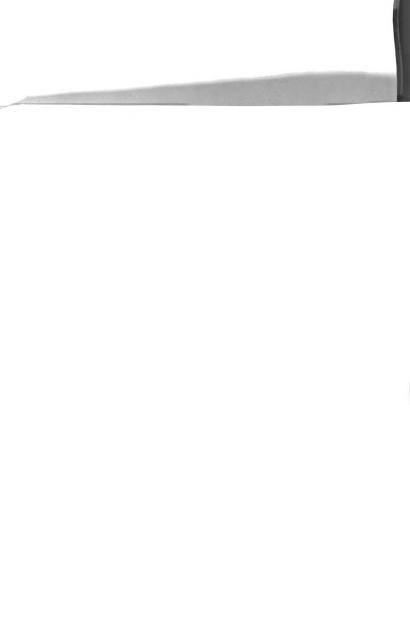


F(x) (denoted by G L-RSIR F) if

- (a) $\int_{x}^{x_{0}} [G(x) F(x)] dx = 0$
- (b) $\int_{0}^{y} [G(x) F(x)] dx \ge 0$ for all $y \in [x_1, x_5]$
- (c) There exists a point $m \in [x_2, x_4]$ such that $F(x) \le G(x)$ for all $x \in [x_2, m)$ and $F(x) \ge G(x)$ for all $x \in [m, x_4]$
- (d) There exists a point $x_3 \in [x_2, m]$ such that f(x)/g(x) is non-decreasing for all $x \in [x_2, x_3)$ and $f(x) \ge g(x)$ for all $x \in [x_3, m]$.

Conditions (a) and (b) define R-S increases in risk. Condition (c) imposes the restriction that the two CDFs cross only once at the point m. Condition (d) implies that, to the left of the point m, a L-RSIR order requires the same restriction used by Black and Bulkley who define a RSIR one. When we assume that the payoff function is linear in the random variable, the following result shows a trade-off between the restrictions on the set of decision makers and the set of changes in distribution. Compared with the result in RSIR shifts, the comparative statics result in L-RSIR shifts contains a larger set of changes in distribution and a smaller set of decision maker with an additional assumption such as $u^m \ge 0$.

Let us turn to a deterministic transformation approach. Meyer and Ormiston (1989) provide a fourth characterization of a R-S increase in risk and introduce the concept of a "simple increase in risk" which is obtained by further restricting the k(x) functions to be monotonic.





Definition 2.14. The transformation I(x) represents a simple increase in risk for a random variable given by F(x) if the function $k(x) \equiv t(x) - x$ satisfies

- (a) $\int_{a}^{b} k(x) dF(x) = 0$
- (b) $\int_{0}^{t} k(x) dF(x) \le 0$, for all $t \in [a, b]$
- (c) $k'(x) \ge 0$.

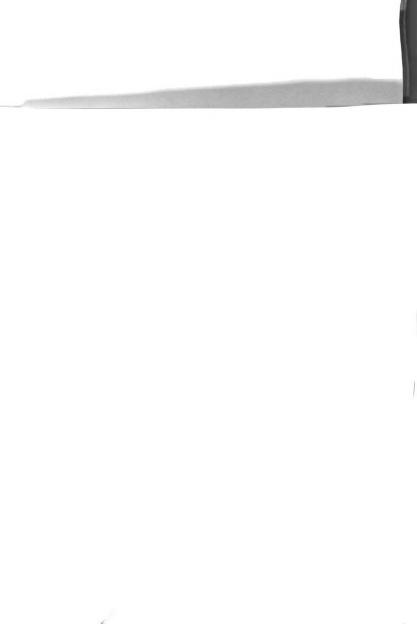
Condition (a) guarantees that the mean of the random variable is preserved.

Condition (b) guarantees that for an increase in risk, the initial random variable dominates the transformed random variable in the second-degree. These conditions provide a fourth characterization of a R-S increase in risk. Condition (c) is the added condition which identifies this particular type of an increase in risk and allows general comparative static statements to be made.

Theorem 2.5. Facing a simple R-S increase in risk, a decision maker will decrease the optimal value of α if

- (a) u(z) displays DARA
- (b) $z_x \ge 0$, $z_{xx} \le 0$, $z_{\alpha\alpha} \ge 0$ and $z_{\alpha\alpha\alpha} \le 0$.

Theorem 2.5 generalizes the Sandmo-Ishii result in some aspects: while the latter uses a linear transformation of the random variable and a specific form of a decision model such as the competitive firm model, the former adopts a more general class of transformation and the general decision model which includes a specific form as a special





case.

2.3.4 Gollier's Work

Examining the restrictions used to do comparative static analysis gives a general notion that given the set of decision makers, many authors restrict separately the changes in distribution of the random variable (PDFs or CDFs) and the structure of the given decision model to obtain the intuitively appealing comparative statics results. Gollier (1995), however, restricts them jointly with a single restriction to obtain the necessary and sufficient condition under which all risk averse individuals adjust the decision variable in the same direction when faced with a given shift in a random parameter. This implies that he obtains the least constraining condition on changes in risk that yields the general comparative static statements for a given economic model and for the class of risk averse agents. He presents the condition using the location-weighted probability mass functions as follows:

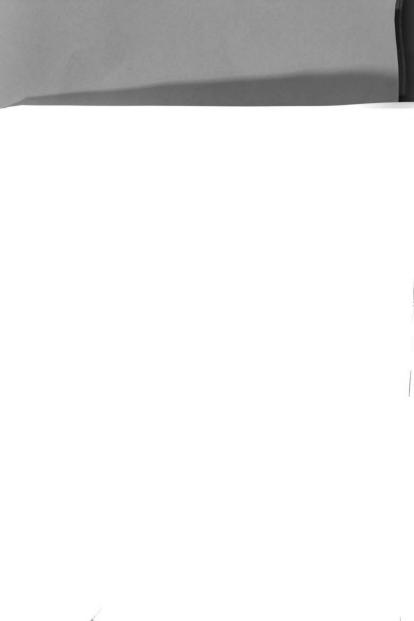
$$T(x,\alpha;G,z) \le \gamma T(x,\alpha;F,z), \ \forall x \in [a,b]$$

where
$$\gamma > 0$$
, $T(x, \alpha; F, z) = \int_{\alpha}^{x} z_{\alpha} dF(s)$ and $T(x, \alpha; G, z) = \int_{\alpha}^{x} z_{\alpha} dG(s)$.

He restates the necessary and sufficient condition as requiring that

$$\inf_{\{x\mid T(x,\alpha;F,z)>0\}} \frac{T(x,\alpha;G,z)}{T(x,\alpha;F,z)} \ge \sup_{\{x\mid T(x,\alpha;F,z)>0\}} \frac{T(x,\alpha;G,z)}{T(x,\alpha;F,z)}.$$

The above condition shows the importance of the ratios for two T functions. This implies that the infimum of the left-side ratio (when $T(x,\alpha;F,z) < 0$) is greater than or equal to the supremum of the right-side ratio (when $T(x,\alpha;F,z) > 0$). The reverse case of his condition yields ambiguous comparative statics.





He also introduces the concept of the greater central riskiness order when the payoff function is linear in the random variable, $z(x,\alpha) = \alpha x + z_0$, where z_0 is an exogenous parameter. His condition is quite simple and generalizes all previous restrictions on changes in risk such as a SIR and a RSIR imposed for all concave utility functions. His another result shows that the SSD order is neither sufficient nor necessary to get the result. By using the same technique in Gollier, we deal with this economic problem for the class of risk averse individuals with a positive third derivative of their utility functions that will be presented in chapter 5.



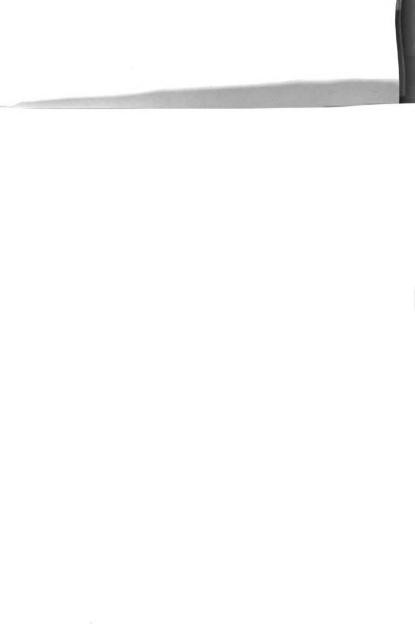


Chapter 3

COMPARATIVE STATIC ANALYSIS FOR SUBSETS OF R-S INCREASES RISK

This chapter provides definitions and general comparative static statements regarding several subsets of R-S increases in risk. We define two subsets of R-S increases in risk called a 'left-side strong increase in risk' (L-SIR) and a 'left-side relatively weak increase in risk' (L-RWIR). Each subset generalizes a definition in the established literature. Specifically, a 'strong increase in risk' (SIR) in Meyer and Ormiston (1985) and a 'relatively weak increase in risk' (RWIR) in Dionne, Eeckhoudt and Gollier (1993) are extended to a L-SIR and a L-RWIR, respectively. For basic relations among these subsets, the set of L-SIR shifts is included in the set of 'left-side relatively strong increases in risk' (L-RSIR) shifts in Kim (1998), and the latter set is included in the set of L-RWIR shifts. Whereas the L-SIR and the L-RSIR orders impose monotonicity restrictions on the difference between two CDFs and on the ratio of a pair of PDFs, respectively, the L-RWIR condition imposes a bound on the likelihood ratio in a specific interval of the support under the initial distribution. These shifts allow only single crossing between two CDFs for comparative static purposes.

When multiple crossing between two CDFs is allowed, we define a 'monotone





strong increase in risk' (MSIR) that imposes monotonicity restrictions on the ratio of the two cumulative of cumulative distribution functions (C-CDFs). This shift also generalizes a definition in the literature. That is, an 'extended strong increase in risk' (ESIR) in Kim (1998) is extended to a MSIR. The MSIR order implies that the conditional expectation of a random variable under F is greater than or equal to that under G in the interval under the initial distribution.

We also define two more subsets of R-S increases in risk with multiple crossing called an 'outside strong increase in risk' (OSIR) and an 'outside relatively strong increase in risk' (ORSIR). The conditions in these two shifts imply that there exists a R-S decrease in risk in the given interval and a R-S increase in risk outside this interval. Note that the OSIR ranking can be decomposed into two L-SIR shifts, and the ORSIR ranking can be decomposed into a L-SIR shift and a 'left-side extended strong increase in risk' (L-ESIR) shift. The basic relationship between these two shifts shows that the set of OSIR shifts is included in the set of ORSIR shifts.

Restricting the payoff function to be linear in the random variable $(z_{xx} = 0)$, we show that the effect of these shifts can be determined for all risk averse decision makers with non-negative third derivative of utility functions $u''' \ge 0$. This implies that we extend the subsets of R-S increases in risk, but use somewhat stronger restrictions on the structure of the decision model and the set of decision makers.

Section 3.1 provides definitions and comparative statics results for two subsets of R-S increases in risk with single crossing. Section 3.2 gives definitions and comparative statics results for several subsets of R-S increases in risk with multiple crossing. We also provide other characterizations of the MSIR ranking in order to provide an interpretation





and to compare the MSIR ranking with the ESIR ranking. Section 3.3 contains some remarks specific to this chapter.

3.1 Subsets of R-S Increases in Risk with Single Crossing

In this section, we begin by identifying a special category of risk increases that is a subset of R-S increases in risk, called a 'left-side strong increase in risk' (L-SIR). The L-SIR order is a less stringent type of R-S increases in risk than the SIR order proposed by Meyer and Ormiston (1985), who imposes the restriction on the difference between the two CDFs. We assume that the supports of x under G(x) are $[x_1, x_4]$ and under F(x) are $[x_2, x_3]$ where $x_1 \le x_2 \le x_3 \le x_4$.

Definition 3.1. G(x) represents a left-side strong increase in risk from F(x)F(x) (denoted by G L-SIR F) if

(a)
$$\int_{x_1}^{y} [G(x) - F(x)] dx \ge 0$$
 for all $y \in [x_1, x_4]$

(b)
$$\int_{x}^{x_4} [G(x) - F(x)] dx = 0$$

(c) There exists a point $m \in [x_2, x_3]$ such that G(x) - F(x) is non-increasing on $x \in (x_2, m)$ and $F(x) \ge G(x)$ for all $x \in [m, x_4]$.

Conditions (a) and (b) imply that the L-SIR order is a R-S increase in risk. That is, F dominates G in the second degree and the mean of the random variable is kept



fi



constant. Condition (c) imposes the restriction that the two CDFs cross only once at a point m. This condition implies that, to the left of the point m, the L-SIR order requires the same restriction used by Meyer and Ormiston to define the SIR order. Note that, to the right of the point m, restrictions ($F \ge G$) imposed on L-SIR shifts and L-RWIR shifts are same, and they are less restrictive than those on SIR shifts. Therefore the set of SIR shifts is a subset of the set of L-SIR shifts.

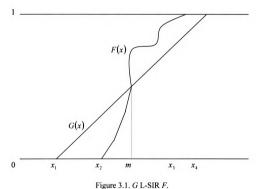
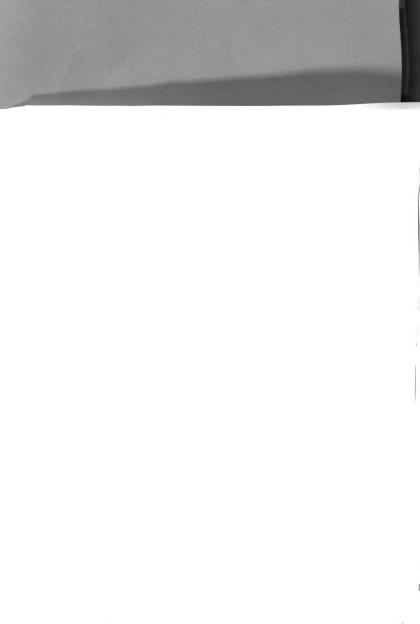


Figure 3.1 illustrates an example of a left-side strong increase in risk and a case where restrictions on the difference between the two CDFs in the interval $x \in [m, x_3)$ to obtain a strong increase in risk are not met. Note that the L-SIR order can be obtained from the SIR one by relaxing the restrictions imposed to the right of the point m.

Dionne, Eeckhoudt and Gollier (1993) introduce the concept of a 'relatively weak





increases in risk' (RWIR). In the second definition, we propose the 'left-side relatively weak increase in risk' (L-RWIR) order that extends the RWIR order.

Definition 3.2. G(x) represents a left-side relatively weak increase in risk from F(x) (denoted by G L-RWIR F) if

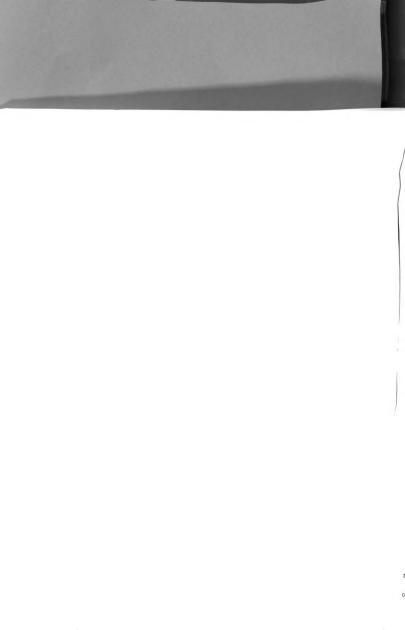
- (a) $\int_{x}^{y} [G(x) F(x)] dx \ge 0$ for all $y \in [x_1, x_5]$
- (b) $\int_{x}^{x_5} [G(x) F(x)] dx = 0$
- (c) There exists a point $m \in [x_2, x_4]$ such that $F(x) \le G(x)$ for all $x \in [x_2, m]$ and $F(x) \ge G(x)$ for all $x \in [m, x_4]$
- (d) There exists a point $x_3 \in [x_2, m]$ such that $f(x) \le g(x)$ for all $x \in [x_2, x_3)$ and $f(x) \ge g(x)$ for all $x \in [x_3, m]$
- (e) When $x^* \in [x_2, x_3)$, the following condition is satisfied:

$$\frac{g(x)}{f(x)} \ge \frac{g(x^*)}{f(x^*)}, \qquad x_2 \le x \le x^*,$$

$$\frac{g(x)}{f(x)} \le \frac{g(x^*)}{f(x^*)}, \qquad x^* \le x \le x_3,$$

where x^* is the value of x satisfying $z_{\alpha}(x, \alpha_F) = 0$ in the interval $[x_2, x_4]$.

Conditions (a) and (b) define R-S increases in risk. Condition (c) imposes the restriction that the two CDFs cross only once at a point m. Conditions (d) and (e) imply that, to the left of the point m, the L-RWIR order requires the same restriction used by Dionne. Eeckhoudt and Gollier to define the RWIR order. Note that, to the right of the





point m, there is no restriction on the number of times of crossing between the two PDFs, nor is there any restriction on the ratio of a pair of PDFs. The restriction on the right of the point m is that $F \le G$. Therefore the set of L-RWIR shifts includes the set of RWIR shifts. Note also that the above condition (e) is less restrictive than that proposed by Kim (1998), who imposes a restriction on the sign of the derivative of the likelihood ratio when defining the L-RSIR order.

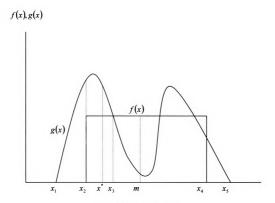
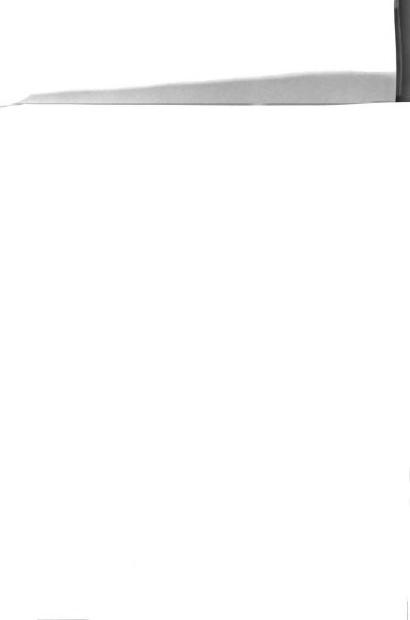


Figure 3.2. G L-RWIR F.

Figure 3.2 illustrates an example of a left-side relatively weak increase in risk and a case where a monotonicity restriction on the interval $x \in [x_2, x_3)$ to obtain a left-side relatively strong increase in risk is not met. Compared with the RWIR order, the L-RWIR order is obtained by relaxing the restrictions imposed to the right of the point m.





Observe finally that the set of L-SIR shifts is included in the set of L-RSIR shifts that, in turn, is included in the set of L-RWIR shifts. That is, G L-SIR $F \Rightarrow G$ L-RSIR F $\Rightarrow G$ L-RWIR F.

3.1.1 Comparative Static Analysis

In this subsection, we provide general comparative static statements concerning the L-SIR and the L-RWIR orders. In what follows, the payoff function is restricted to be linear in the random variable. Using the simple form of the general decision model such as $z(x,\alpha) \equiv z_0 + \alpha x$, the necessary and sufficient condition for α_F is written as

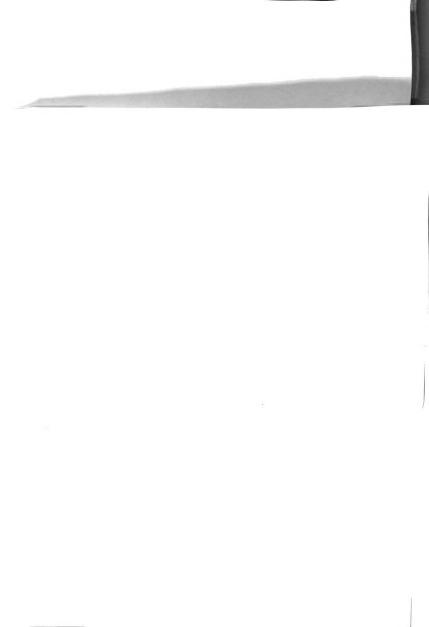
$$\int_{0}^{b} u'(z_{0} + \alpha_{F}x) x dF(x) = 0.$$
(3.1)

It is well known that α_F has the same sign as $E_F(b) = \int_c^b x dF(x)$ (see Dionne, Eeckhoudt and Gollier (1993)). Therefore, we assume that $E_F(b)$ is positive, as it is the case in the applications presented in chapter 2. In order to prove $\alpha_F \ge \alpha_G$ for a specified change in PDF (or CDF) from f (or F) to g (or G), it is sufficient to show that for all $x \in [a,b]$,

$$Q(\alpha_F) = \int_0^b u'(z_0 + \alpha_F x) x d[F(x) - G(x)] \ge 0.$$
(3.2)

The following comparative statics results indicate that, when $z_{xx} = 0$, one can further extend the subset of R-S increases in risk with the cost of adding additional restriction on the risk preferences of decision makers. Since the set of L-SIR shifts is included in the set of L-RWIR shifts, we only consider the L-RWIR shifts.

Theorem 3.1. If G L-RWIR F and $z_{xx} = 0$, then $\alpha_F \ge \alpha_G$ for all risk averse





decision makers with $u''' \ge 0$.

Proof: Using the general decision model $z(x,\alpha)$ in (3.1), let x^* be the value of x satisfying $z_{\alpha}(x,\alpha_F)=0$, and then x^* exists in the interval $[x_2,x_4]$. We consider the following three cases:

Case (i): $x_2 \le x^* \le x_3$.

We consider the sign of the expression $\int_{x_1}^{x_2} z_a(f-g) dx$. First, assume that

$$\int_{r_{i}}^{x_{i}} z_{\alpha}(f-g) dx$$
. Rewriting $Q(\alpha_{F})$ in (3.2) as

$$Q(\alpha_F) = \int_{x_1}^{x_2} u'(z) z_a(f-g) dx + \int_{x_2}^{m} u'(z) z_a(f-g) dx + \int_{m}^{x_2} u'(z) z_a(f-g) dx.$$

Using the given assumptions and the L-RWIR condition, we have

$$Q(\alpha_{F}) \ge u' [z(x',\alpha_{F})] \int_{z_{1}}^{z_{1}} z_{\alpha}(f-g) dx + u' [z(m,\alpha_{F})] \int_{z_{1}}^{m} z_{\alpha}(f-g) dx$$

$$+ \int_{m}^{z_{1}} u'(z) z_{\alpha}(f-g) dx.$$
(3.3)

Adding and subtracting $u'[z(m,\alpha_F)]\int_{x_1}^{x_2} z_{\alpha}(f-g)dx$ in the RHS of (3.3) gives

$$Q(\alpha_{F}) \ge \left\{ u' [z(x^{*}, \alpha_{F})] - u' [z(m, \alpha_{F})] \right\}_{\alpha_{F}}^{s_{F}} z_{\alpha}(f - g) dx$$

$$+ u' [z(m, \alpha_{F})] \int_{\alpha_{F}}^{m} z_{\alpha}(f - g) dx + \int_{\alpha_{F}}^{s_{F}} u'(z) z_{\alpha}(f - g) dx. \quad (3.4)$$

Since u'(z) is non-increasing and $\int_{z_i}^{x_i} z_{\alpha}(f-g) dx$, the first term in the RHS of (3.4) is non-negative. Integrating the second term in (3.4) by parts and using the assumption





 $z_{xx} = 0$, we obtain

$$u'[z(m,\alpha_F)]\int_{r_0}^{m} z_{\alpha\alpha}[G(x)-F(x)]dx = u'[z(m,\alpha_F)]z_{\alpha\alpha}\int_{r_0}^{m}[G(x)-F(x)]dx.$$
 (3.5)

Also using integration by parts, the third term in (3.4) is equal to

$$q(\alpha_F) = \int_{m}^{x_s} u''(z) z_x z_\alpha [G(x) - F(x)] dx + \int_{m}^{x_s} u'(z) z_{\alpha x} [G(x) - F(x)] dx.$$
 (3.6)

Since z_a is positive and [G(x) - F(x)] is non-positive for all $x \in [m, x_s]$, the first term in (3.6) is non-negative. Thus, we have

$$q(\alpha_F) \ge \int_m^{x_S} u'(z) z_{\alpha x} [G(x) - F(x)] dx.$$
 (3.7)

Because $u'(z)z_{ax}$ is non-increasing and [G(x)-F(x)] is non-positive for all $x \in [m,x_5]$, we have

$$q(\alpha_F) \ge u'[z(m,\alpha_F)] z_{\alpha x} \int_{-\infty}^{x_F} [G(x) - F(x)] dx.$$
(3.8)

Hence, from (3.5) and (3.8), the second and third term in (3.4) can be written as

$$u'[z(m,\alpha_F)]\int_{x_1}^m z_\alpha(f-g)dx + \int_m^{x_2} u'(z)z_\alpha(f-g)dx$$

$$\geq u'[z(m,\alpha_F)]z_{\alpha\alpha}\int_{x_F}^{x_F}[G(x)-F(x)]dx = 0.$$
 (3.9)

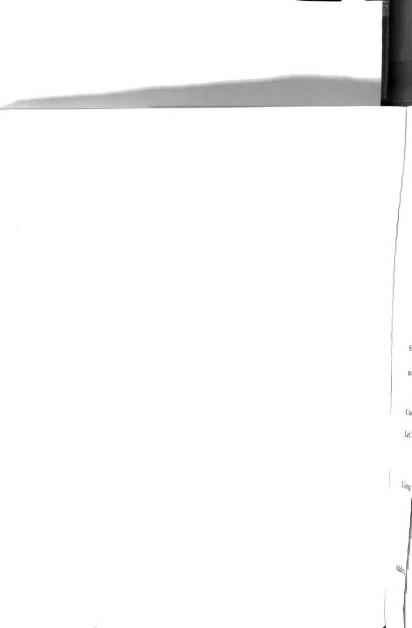
Therefore, $Q(\alpha_F)$ is non-negative.

Second, assuming that $\int_{x_1}^{x_2} z_{\alpha}(f-g) dx \le 0$, let's rewrite $Q(\alpha_F)$ in (3.2) as

$$Q(\alpha_F) \ge \int_{z_1}^{z_1} u'(z) z_\alpha (f - g) dx + u'[z(m, \alpha_F)] \int_{z_1}^{m} z_\alpha (f - g) dx$$

$$+ \int_{z_1}^{z_1} u'(z) z_\alpha (f - g) dx. \qquad (3.10)$$

Adding and subtracting $u'[z(m,\alpha_F)]\int_{x_L}^{x_L} z_{\alpha}(f-g)dx$ in the RHS of (3.10) gives





$$\begin{split} Q(\alpha_F) &\geq \int_{z_1}^{z_1} u'(z) z_\alpha(f-g) dx - u'[z(m,\alpha_F)] \int_{z_1}^{z_2} z_\alpha(f-g) dx \\ &+ u'[z(m,\alpha_F)] \int_{z_1}^{m} z_\alpha(f-g) dx + \int_{m}^{z_2} u'(z) z_\alpha(f-g) dx \,. \end{split}$$

From (3.9) and the assumption $\int_{x_1}^{x_2} z_{\alpha} (f - g) dx \le 0$,

$$Q(\alpha_F) \ge \int_{x_1}^{x_2} u'(z) z_{\alpha}(f-g) dx = -\int_{x_1}^{x_2} u'(z) z_{\alpha} g dx + \int_{x_2}^{x_2} u'(z) z_{\alpha}(f-g) dx. \quad (3.11)$$

The first term with minus sign in the RHS of (3.11) is non-negative, and by applying the condition (d) in Definition 3.2, the second term can be written as

$$\int_{z_1}^{z_1} u'(z) z_a(f-g) dx = \int_{z_1}^{z_1} u'(z) z_a \left(1 - \frac{g}{f}\right) f dx \ge \left[1 - \frac{g(x')}{f(x')}\right] \int_{z_1}^{z_1} u'(z) z_a f dx.$$

Since $g(x^*)/f(x^*) \ge 1$ and $\int_{z_1}^{z_1} u'(z) z_\alpha f \, dx \le 0$ by the first-order condition, $Q(\alpha_F)$ is non-negative.

Case (ii): $x_3 \le x^* \le m$

Let's rewrite $Q(\alpha_F)$ in (3.2) as

$$Q(\alpha_F) = \int_{x_1}^{x_2} u'(z) z_a(f-g) dx + \int_{x_2}^{m} u'(z) z_a(f-g) dx + \int_{m}^{x_2} u'(z) z_a(f-g) dx.$$

Using the given assumptions and the L-RWIR condition, we have

$$Q(\alpha_{F}) \ge u'[z(x_{3}, \alpha_{F})] \int_{x_{1}}^{x_{2}} z_{\alpha}(f - g) dx + u'[z(m, \alpha_{F})] \int_{x_{1}}^{\infty} z_{\alpha}(f - g) dx$$

$$+ \int_{x_{1}}^{x_{2}} u'(z) z_{\alpha}(f - g) dx.$$
(3.12)

Adding and subtracting $u'[z(m,\alpha_F)]\int_{z_i}^{x} z_{\alpha}(f-g)dx$ in the RHS of (3.12) gives





$$Q(\alpha_F) \ge \{ u'[z(x_3, \alpha_F)] - u'[z(m, \alpha_F)] \} \int_{x_1}^{x_2} z_\alpha(f - g) dx$$

$$+ u'[z(m, \alpha_F)] \int_{x_2}^{x_2} z_\alpha(f - g) dx + \int_{x_2}^{x_2} u'(z) z_\alpha(f - g) dx . \quad (3.13)$$

From (3.9), the second and third term in the RHS of (3.13) is non-negative. Thus, we have

$$Q(\alpha_F) \ge \left\{ u'[z(x_3,\alpha_F)] - u'[z(m,\alpha_F)] \right\} \int_{t_0}^{x} z_\alpha(f-g) dx.$$

Integrating by parts,

$$\int_{x_{1}}^{x^{2}} z_{\alpha}(f-g) dx = \int_{x_{1}}^{x^{2}} z_{\alpha x} [G(x) - F(x)] dx = z_{\alpha x} \int_{x_{1}}^{x^{2}} [G(x) - F(x)] dx \ge 0$$

because $z_{\alpha x}$ is non-negative and does not depend on x, and $\int_{x_1}^x [G(x) - F(x)] dx \ge 0$ for all $t \in [x_1, x_5]$. Thus, by the assumption $u''(z) \le 0$, $Q(\alpha_F)$ is non-negative.

Case (iii): $m \le x^* \le x_A$.

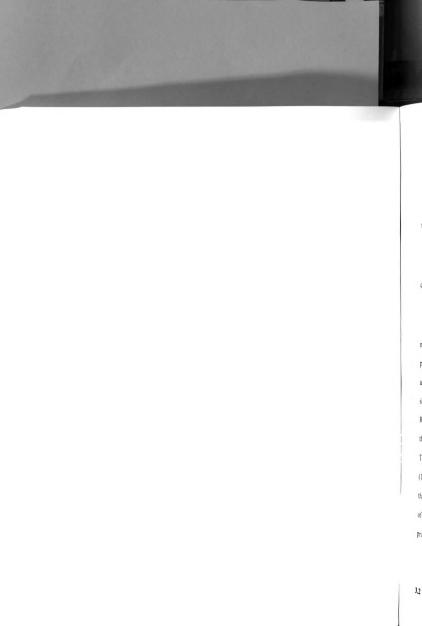
Integrating by parts, $Q(\alpha_F)$ can be written as

$$Q(\alpha_F) = \int_{r}^{x} [u''(z)z_x z_\alpha + u'(z)z_{\alpha x}] [G(x) - F(x)] dx.$$

Note that $u^*(z)z_xz_\alpha + u'(z)z_{\alpha x}$ is positive and non-increasing in x in the interval $[x_1, x^*]$, and it has its maximum at $x = x^*$ in the interval $[x^*, x_s]$ because $u''(z)z_xz_\alpha$ is always non-positive and $u'(z)z_{\alpha x}$ is non-increasing in x. Since $m \le x^*$, this implies that

$$\left[u''(z) z_x z_\alpha + u'(z) z_{\alpha x} \right]_{x \le m} \ge \left[u''(z) z_x z_\alpha + u'(z) z_{\alpha x} \right]_{x = m} \ge \left[u''(z) z_x z_\alpha + u'(z) z_{\alpha x} \right]_{x \ge m} .$$

By the L-RWIR condition which implies $G(x) - F(x) \ge 0$ for all $x \in [x_1, m]$ and $G(x) - F(x) \le 0$ for all $x \in [m, x_5]$, we have the following inequality,





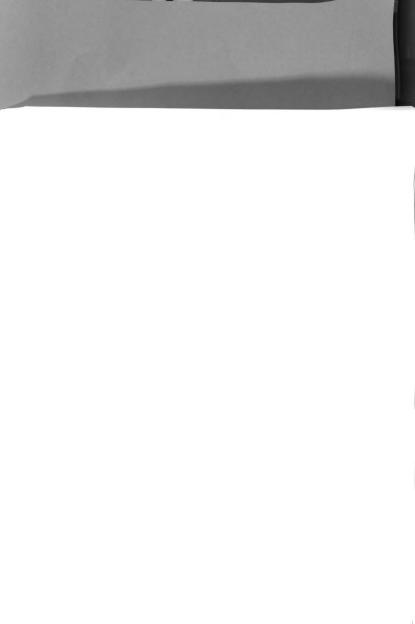
$$Q(\alpha_{\scriptscriptstyle F}) \geq \left[u''(z) z_{\scriptscriptstyle x} z_{\scriptscriptstyle \alpha} + u'(z) z_{\scriptscriptstyle \alpha x} \right]_{x=m} \int_{x_1}^{x_5} [G(x) - F(x)] dx = 0 \,. \tag{Q.E.D.}$$

Remember that the L-SIR order implies the L-RWIR order. Therefore we obtain the following result.

Corollary 3.1. If G L-SIR F and $z_{xx} = 0$, then $\alpha_F \ge \alpha_G$ for all risk averse decision makers with $u''' \ge 0$.

While $z_{xx}=0$ restricts the set of decision problems, linear payoffs prevail in many economic environments; as indicated in chapter 2, these include the standard portfolio model, the optimal behavior of a competitive firm with constant marginal costs, and the coinsurance problem. Compared with the comparative statics results in L-SIR shifts and L-RSIR shifts in Kim (1998), the result in Theorem 3.1 includes a larger set of R-S increases in risk. As a result, the L-RWIR order represents a net improvement over the L-SIR and the L-RSIR orders without any cost of additional assumptions. That is, Theorem 3.1 improves the robustness of the results in Corollary 3.1 and Theorem in Kim (1998). Compared with the comparative statics results in SIR shifts and RWIR shifts in the published literature, the results in Corollary 3.1 and Theorem 3.1 contain a larger set of changes in distribution, a smaller set of decision makers and a smaller set of decision problem.

3.2 Subsets of R-S Increases in Risk with Multiple Crossing





Many researchers have obtained intuitively appealing comparative statics results by restricting the changes in distribution of the random parameter when two CDFs cross only once. When multiple crossing between the initial and the final CDF F and G is allowed, this section provides definitions and general comparative static statements regarding several subsets of R-S increases in risk with multiple crossing.

Kim (1998) defines an 'extended strong increase in risk' (ESIR) that imposes monotonicity restrictions on the ratio of a pair of CDFs. We introduce the less stringent type of R-S increases in risk and name it a 'monotone strong increase in risk' (MSIR). Instead of using monotonicity restrictions on the ratio for CDFs, the MSIR order is defined by imposing monotonicity restrictions on the ratio of the two cumulative of CDFs (C-CDFs).

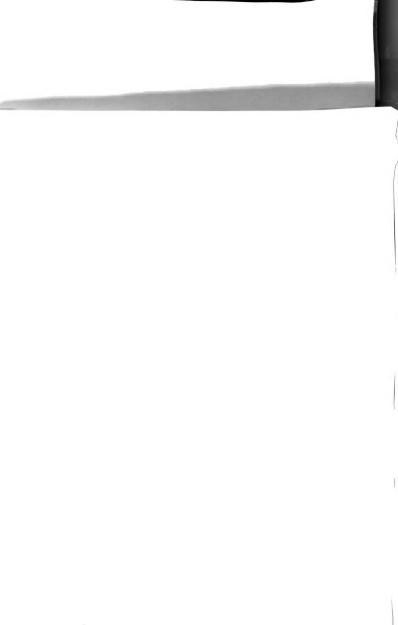
Definition 3.3. G(x) represents a monotone strong increase in risk from F(x) (denoted by G MSIR F) if

(a)
$$\int_{a}^{y} [G(x) - F(x)] dx \ge 0 \text{ for all } y \in [a, b]$$

(b)
$$\int_{a}^{b} [G(x) - F(x)] dx = 0$$

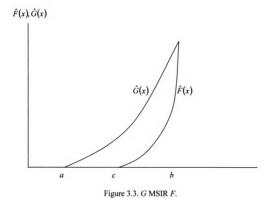
(c) There exists a non-negative and non-increasing function $H(x):(c,b] \to [1,\infty)$ such that $F(x) = \hat{F}(x) = 0$ for all $x \in [a,c]$ and $\hat{G}(x)/\hat{F}(x) = H(x)$ for all $x \in [c,b]$ where $\hat{F}(x) = \int_a^x F(t) dt = \int_c^x F(t) dt$, $\hat{G}(x) = \int_a^x G(t) dt$ and $a \le c \le b$.

The first two conditions guarantee that G represents a R-S increase in risk from F. Condition (c) does not restrict the number of times of crossing between the two CDFs.

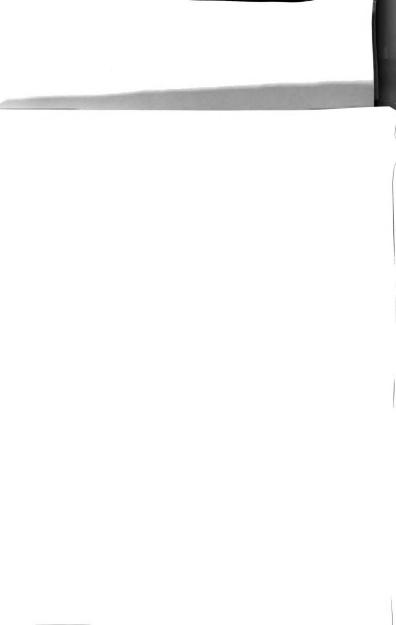


This comes from replacing a monotonicity restriction on the ratio of a pair of CDFs with a monotonicity restriction on the ratio of a pair of cumulative of CDFs (C-CDFs). Note that the ESIR order defined in Kim (1998) requires that two CDFs cross only once.

Therefore the MSIR ranking is less demanding than the ESIR ranking. In next subsection, we will provide an equivalent definition of the MSIR order with a nice interpretation, and show that an ESIR order implies a MSIR one. Figure 3.3 shows an example of a monotone strong increase in risk that is not an ESIR.



We also propose two more subsets of R-S increases in risk called an 'outside strong increase in risk' (OSIR) and an 'outside relatively strong increase in risk' (ORSIR), where the two CDFs have multiple crossing. We assume that the support of

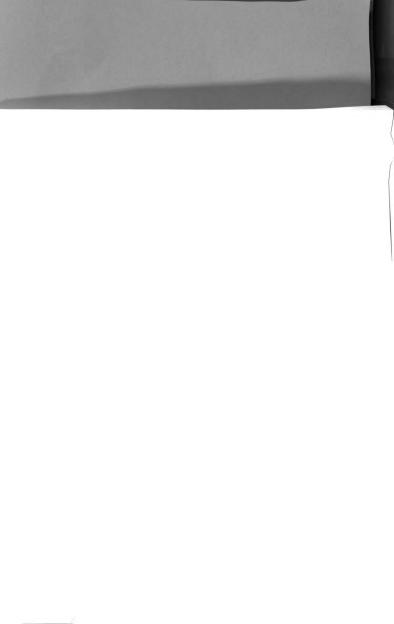


F(x) is contained in some finite interval $[x_2, x_6]$ and the support of G(x) is contained in another finite interval $[x_1, x_7]$ where $x_1 \le x_2 \le x_3 \le x_4 \le x_5 \le x_6 \le x_7$. First, we introduce an OSIR shift that is defined as:

Definition 3.4. G(x) represents an outside strong increase in risk from F(x) (denoted by G OSIR F) if

- (a) $\int_{x_0}^{x_1} [G(x) F(x)] dx = 0$
- (b) $\int_{x_1}^{y} [G(x) F(x)] dx > 0$ for all $y \in [x_1, x_2]$ and $\int_{y}^{x_1} [G(x) F(x)] dx < 0$ for all $y \in (x_1, x_2]$
- (c) G(x) F(x) is non-increasing on $x \in (x_2, x_3)$, $F(x) \ge G(x)$ for all $x \in [x_3, x_4]$, $F(x) \le G(x)$ for all $x \in [x_4, x_5]$, $\int_{x_5}^{x_5} [F(x) G(x)] dx = 0$ and $F(x) \ge G(x)$ for all $x \in (x_5, x_7]$.

Conditions (a) and (b) define a R-S increase in risk. Condition (c) implies that the two CDFs have multiple crossing and to the left of the point x_3 , the OSIR ranking requires the same restriction used by Meyer and Ormiston to define the SIR order. These conditions imply that there exists a R-S decrease in risk in the interval $[x_3, x_3]$ and a R-S increase in risk outside this interval. Thus, we name this increase in risk the OSIR order. Figure 3.4 illustrates an example of the OSIR order. The OSIR ranking can be decomposed into two left-side strong increases in risk (L-SIR). That is, if G L-SIR G_1 and G_1 L-SIR F, then G OSIR F.





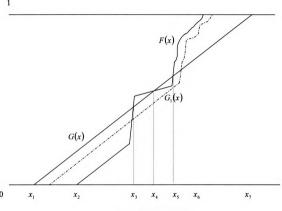


Figure 3.4. G OSIR F.

We also define an 'outside relatively strong increase in risk' (ORSIR) that relaxes the restriction imposed to the left of the point x_3 in OSIR shifts.

Definition 3.5. G(x) represents an outside relatively strong increase in risk from F(x) (denoted by G ORSIR F) if

(a)
$$\int_{x_1}^{x_2} [G(x) - F(x)] dx = 0$$

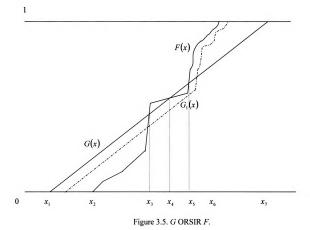
(b)
$$\int_{x_1}^{y} [G(x) - F(x)] dx > 0$$
 for all $y \in [x_1, x_2]$ and $\int_{y}^{x_2} [G(x) - F(x)] dx < 0$ for all





$$y\in \left(x_{1},x_{7}\right]$$

(c) G(x)/F(x) is non-increasing on $x \in (x_2, x_3)$, $F(x) \ge G(x)$ for all $x \in [x_3, x_4]$, $F(x) \le G(x)$ for all $x \in [x_4, x_5]$, $\int_{x_5}^{x_5} [F(x) - G(x)] dx = 0$ and $F(x) \ge G(x)$ for all $x \in (x_5, x_7]$.



The conditions in ORSIR shifts are same as those in OSIR shifts except that, to the left of the point x_3 , G(x)/F(x) is non-increasing. Note that the OSIR order implies the ORSIR order. Figure 3.5 illustrates an example of the ORSIR order that cannot be the OSIR order. The ORSIR ranking can be decomposed into a 'left-side strong increases in





risk' (L-SIR) and a 'left-side extended strong increase in risk' (L-ESIR). That is, if G L-SIR G_1 and G_1 L-ESIR F, then G ORSIR F.

3.2.1 Other Characterizations of the MSIR Order and Comparison with the ESIR Order In this subsection, we provide other characterizations of the MSIR ranking in order to provide an interpretation and to compare the MSIR order with the ESIR one.
First, we demonstrate that the following three conditions are equivalent.

Lemma 3.1. The following three conditions are equivalent.

(a) G MSIR F for all $x \in (c,b]$.

(b)
$$\frac{\hat{F}(x)}{F(x)} \le \frac{\hat{G}(x)}{G(x)}$$
 for all $x \in (c, b]$

where $\hat{F}(x) = \int_a^x F(t) dt = \int_c^x F(t) dt$ since F(x) = 0 for $x \le c$ and $\hat{G}(x) = \int_a^x G(t) dt$.

(c)
$$E_F[t | t \le x] \ge E_G[t | t \le x]$$
 for all $x \in (c, b]$.

Proof: We rewrite a MSIR condition as: For all $x \in (c, b]$

$$\hat{G}(x)/\hat{F}(x) = H(x) \tag{3.14}$$

where H(x) is a non-negative and non-increasing function.

(3.14) is equivalent to

$$\frac{d}{dx}H(x) = \frac{d}{dx}\left(\frac{\hat{G}(x)}{\hat{F}(x)}\right) = \frac{G(x)\hat{F}(x) - F(x)\hat{G}(x)}{\hat{F}^2(x)} \le 0.$$

This is in turn equivalent to





$$G(x)\hat{F}(x) \le F(x)\hat{G}(x) \Leftrightarrow \frac{\hat{F}(x)}{F(x)} \le \frac{\hat{G}(x)}{G(x)}$$
 (3.15)

Thus, conditions (a) and (b) are equivalent.

Condition (c) can be written as follows: since $E_F\left[t\,\middle|\, t\leq x\right]$ is not defined for $x\leq c$, for all x>c,

$$\frac{\int_{a}^{x} t dF(t)}{F(x)} \ge \frac{\int_{a}^{x} t dG(t)}{G(x)}.$$
(3.16)

By integrating the numerator of (3.16) by parts, we obtain

$$\frac{xF(x)-\int_a^xF(t)dt}{F(x)}\geq\frac{xG(x)-\int_a^xG(t)dt}{G(x)},$$

or, equivalently,

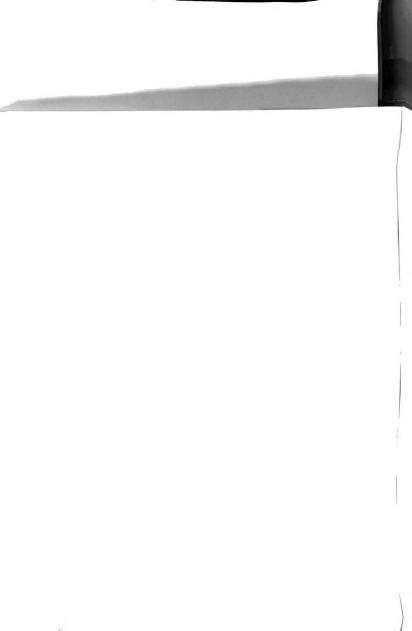
$$\frac{\hat{F}(x)}{F(x)} \le \frac{\hat{G}(x)}{G(x)}.$$

Thus, conditions (b) and (c) are equivalent. Since condition (a) is equivalent to condition
(b) which is also equivalent to condition (c), condition (a) is equivalent to condition (c).

This concludes the proof.

Q.E.D.

Condition (a) implies that the MSIR order imposes a monotonicity restriction on the ratio of a pair of cumulative of CDFs (C-CDFs). Since condition (b) can be written as $F(x)/\hat{F}(x) \ge G(x)/\hat{G}(x)$, it implies that $\hat{F}(x)$ is smaller than $\hat{G}(x)$, but the percentage rate of increase for $\hat{F}(x)$ is always greater than or equal to that for $\hat{G}(x)$. That is, $d \log \hat{F}(x)/dx$ is always greater than or equal to $d \log \hat{G}(x)/dx$. Condition (c) implies that the conditional expectation of a random variable under F is greater than or equal to





that under G for all $x \in (c,b]$. Note that in the condition (c) $f_c(t) = f(t)/F(x)$ is a conditional PDF with its support [a,x] and $\int_a^x t \frac{dF(t)}{F(x)}$ is a mean value of the given interval.

When the payoff function is linear in the random variable, we compare the MSIR condition with the Gollier (1995) condition. As indicated in chapter 2, the Gollier condition can be written as

$$\gamma \int_{a}^{x} t dF(t) \ge \int_{a}^{x} t dG(t)$$
 for all x

where γ is a scalar. The MSIR condition in (3.16) indicates that a scalar γ is replaced with G(x)/F(x) that is a function of the random variable x. The Gollier condition can be written as $\gamma = \max \frac{G(x)}{F(x)}$. If γ is big enough (for example $\gamma \to \infty$) and $\int_a^x tdF(t)$ is negative for some x, the above condition is violated. Therefore the MSIR condition is less restrictive than the Gollier condition.

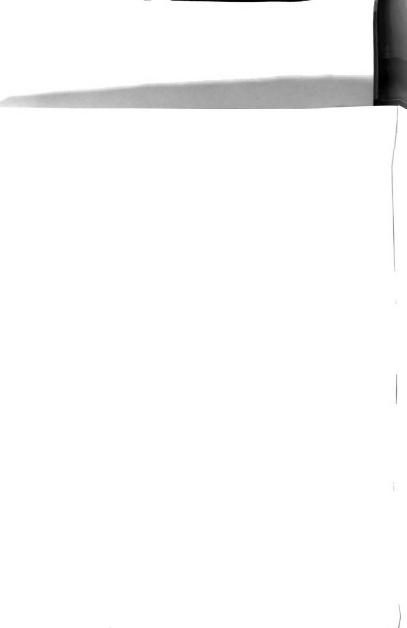
In order to compare a MSIR order with an ESIR one, recall that Kim (1998) formally defines an ESIR order as:

Definition 3.6. G(x) represents an extended strong increase in risk from F(x) (denoted by G ESIR F) if

(a)
$$\int_{0}^{y} [G(x) - F(x)] dx \ge 0$$
 for all $y \in [a, b]$

(b)
$$\int_{a}^{b} [G(x) - F(x)] dx = 0$$

(c) There exists a non-negative and non-increasing function $h(x):(c,b] \to (0,\infty)$ such





that
$$F(x) = 0$$
 for all $x \in [a, c]$ and $G(x) = h(x)F(x)$ for all $x \in (c, b]$ where $a \le c \le b$.

Conditions (a) and (b) simply require that an extended strong increase in risk be a R-S increase in risk. Condition (c) implies that the two CDFs cross only once, but it does not restrict the number of times of crossing between the two PDFs.

Whereas the ESIR order is linked to the derivative of the ratio of a pair of CDFs, the MSIR order is linked to the derivative of the ratio of a pair of C-CDFs. The assertion that the ESIR order implies the MSIR order can be made by showing that the condition that the ratio of a pair of CDFs, h, is non-increasing implies the condition that the ratio of a pair of C-CDFs, H, is non-increasing. We prove the following lemma.

Lemma 3.2. The ESIR order implies the MSIR order.

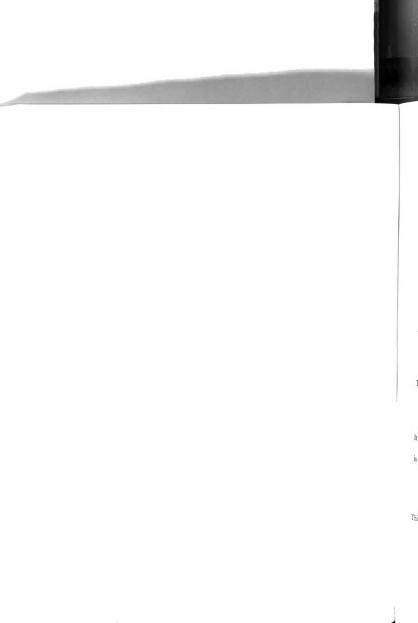
Proof: Before proving this lemma, we define $\int_a^t I(t) dt = \hat{I}(x)$ where I = F or G. First, notice that by the definition of ESIR, we have for $m \ge c$

$$\hat{G}(m) = \hat{G}(c) + \int_{c}^{m} h(x) F(x) dx$$

with h(x) being non-negative and non-increasing. Define function H(m) as

$$H(m) = \frac{\hat{G}(c) + \int_{c}^{m} h(x) F(x) dx}{\hat{F}(m)} \quad \text{for all } m \ge c.$$

Combining the above two equations yields $\hat{G}(m) = H(m)\hat{F}(m)$ for all m > c. Therefore, we have to prove that $H(m') \le H(m)$ for all c < m < m'. We have





$$H(m') = \frac{\hat{G}(c) + \int_{c}^{m} h(x) F(x) dx}{\hat{F}(m')}$$

$$= \frac{\hat{G}(c) + \int_{c}^{m} h(x) F(x) dx + \int_{m}^{m'} h(x) F(x) dx}{\hat{F}(m')}$$

$$= \frac{H(m)\hat{F}(m) + \int_{m}^{m'} h(x) F(x) dx}{\hat{F}(m')}.$$

Because h(x) is non-increasing, we obtain

$$H(m') \leq \frac{H(m)\hat{F}(m) + h(m)[\hat{F}(m') - \hat{F}(m)]}{\hat{F}(m')}$$

Adding and subtracting $H(m)\hat{F}(m')$ in the above inequality gives

$$H(m') \le \frac{H(m)\hat{F}(m) - H(m)\hat{F}(m') + H(m)\hat{F}(m') + h(m)[\hat{F}(m') - \hat{F}(m)]}{\hat{F}(m')}$$

This implies in turn that

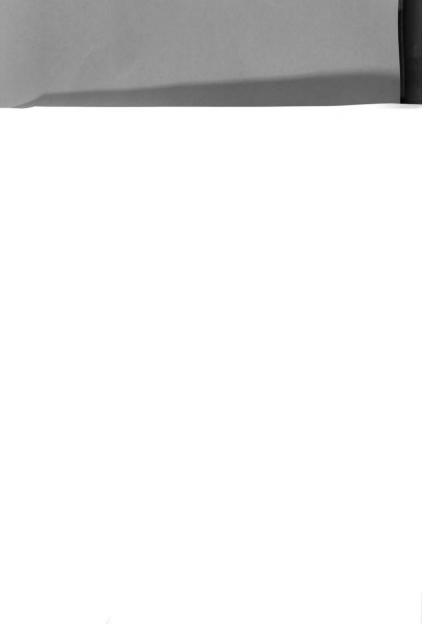
$$H(m') \leq \frac{[\hat{F}(m') - \hat{F}(m)][h(m) - H(m)]}{\hat{F}(m')} + H(m).$$

It follows that $H(m') \le H(m)$ if $[\hat{F}(m') - \hat{F}(m)][h(m) - H(m)]$ is non-positive. Since m' is larger than m, this is equivalent to $h(m) \le H(m)$, or

$$h(m) \leq \frac{\hat{G}(c) + \int_{c}^{m} h(x) F(x) dx}{\hat{F}(m)}.$$

This is always true since

$$\frac{\hat{G}(c) + \int_{c}^{m} h(x) F(x) dx}{\hat{F}(m)} \ge \frac{\int_{c}^{m} h(x) F(x) dx}{\hat{F}(m)} = h(m) - \frac{\int_{c}^{m} h'(x) \int_{c}^{x} F(t) dt dx}{\hat{F}(m)}$$
$$\ge h(m).$$





Since $h'(x) \le 0$ for all $x \in (c, b]$, this concludes the proof.

Q.E.D.

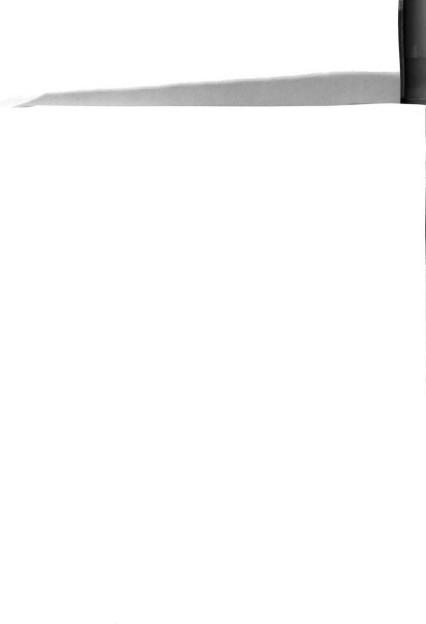
Observe that a MSIR shift imposes less stringent restrictions on the change in the random variable than an ESIR shift does. Lemma 3.2 states that as subsets of R-S increases in risk, the set of all CDF changes satisfying the MSIR order includes the set of all CDF changes satisfying the ESIR order.

Kim (1998) also introduces the concept of a 'left-side extended strong increase in risk' (L-ESIR) that relaxes the restrictions imposed on ESIR shifts. That is, this class of R-S increases in risk includes, as a special case, the ESIR order. The L-ESIR order is formally defined as:

Definition 3.7. G(x) represents a left-side extended strong increase in risk from F(x) (denoted by G L-ESIR F) if

- (a) $\int_{a}^{b} [G(x) F(x)] dx \ge 0$ for all $y \in [a, b]$
- (b) $\int_{a}^{b} [G(x) F(x)] dx = 0$
- (c) There exists a point $m \in [c, b]$ and a non-increasing function $h(x) : (c, m] \to [1, \infty)$ such that G(x) = h(x) F(x) for all $x \in (c, m]$ and $G(x) \le F(x)$ for all $x \in (m, b]$ where $a \le c \le b$.

The L-ESIR order has the same restriction as the ESIR one in that the two CDFs cross only once at the point *m*, but it is less restrictive than the ESIR order since the L-ESIR order requires only the left-side monotonicity of the CDF ratio.





The following lemma shows that the set of L-ESIR shifts is also included in the set of MSIR shifts.

Lemma 3.3. G L-ESIR F implies $E_F[t | t \le x] \ge E_G[t | t \le x]$ for all $x \in (c, b]$.

That is, the L-ESIR order implies the MSIR order.

Proof: Since $E_F[t \mid t \le x]$ is not defined for $x \le c$, we have to prove that for all x > c,

$$\frac{\int_a^x t dF(t)}{F(x)} \ge \frac{\int_a^x t dG(t)}{G(x)}.$$

By integrating the numerator of the above equation by parts, we obtain

$$\frac{\int_{u}^{s} F(t) dt}{F(x)} \le \frac{\int_{u}^{s} G(t) dt}{G(x)},$$
(3.17)

or, equivalently,

$$h(x)F(x)\int_a^x F(t)\,dt \le F(x)\int_a^x h(t)\,F(t)\,dt$$

This is in turn equivalent to

$$h(x) \leq \int_{0}^{x} h(t) \omega(t) dt$$
,

where $\omega(t) = F(t) / \int_c^t F(y) dy$. For all $x \in (c, m]$, the fact that h(t) is non-increasing establishes this lemma. For all $x \in (m, b]$, from (3.17),

$$\inf \frac{\int_a^x G(t) dt}{\int_a^x F(t) dt} \ge \sup \frac{G(x)}{F(x)} = 1.$$

This concludes the proof.

Q.E.D.





3.2.2 Comparative Static Analysis

In this subsection, we provide general comparative static statements regarding the MSIR, the OSIR, and the ORSIR orders. Before we present the comparative statics result in a MSIR shift, Kim (1998) obtains the following result.

Theorem 3.2. For all risk averse decision makers, $\alpha_F \ge \alpha_G$ if

- (a) G ESIR F
- (b) $z_x \ge 0$, $z_{\alpha x} \ge 0$, $z_{\alpha \alpha} \le 0$ and $z_{\alpha x x} \le 0$.

Kim shows that if G ESIR F, then there exists a series of RSIR shifts. That is, any ESIR shift can be decomposed into a series of RSIR shifts. Remember that Theorem 3.2 shows a net improvement over the result in Black and Bulklev's analysis.

Before proving Theorem 3.3, we need the following lemmas. First, we define

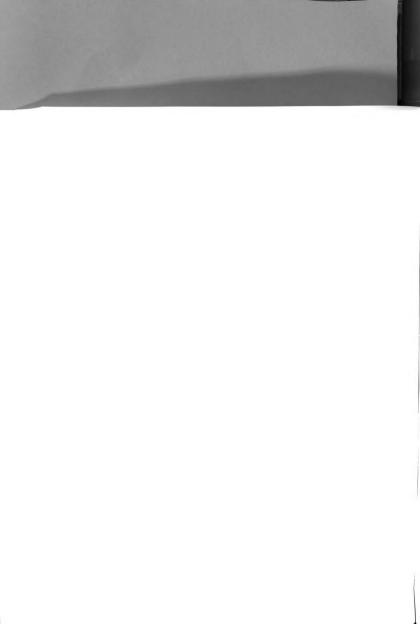
$$k_F(x) = \frac{\int_a^x t dF(t)}{F(x)} = \frac{\int_a^x t dF(t)}{F(x)} \text{ for all } x > c \text{ since } \int_a^x t dF(t) = xF(x) - \hat{F}(x) \text{ and } F(x) = 0$$

for all $x \le c$. Note that $k_F'(x) = \frac{\hat{F}(x)f(x)}{F^2(x)} \ge 0$ for all x > c. We assume that $\alpha_F = 1$ and

 $z_0 = 0$ for notational convenience.

Lemma 3.4. $\delta(r) = \int_{0}^{r} u''(x) k_{F}(x) F(x) dx$ is always non-negative for all $r \in [c, b]$.

Proof: Note that $\delta(c) = 0$. From (3.1), integrating by parts of $\delta(r)$ gives



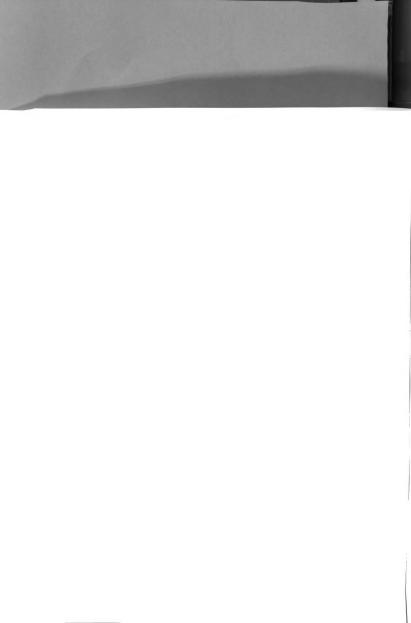


 $\delta(r)=u'(r)k_F(r)F(r)-\int_c^r u'(x)xdF(x)$. We know that $\int_c^r u'(x)xdF(x)\leq 0$ for all $r\in [c,b]$. Let x_0 denote the point where $k_F(x)$ changes sign from negative to positive. For all $r\geq x_0$, $\delta(r)\geq 0$ since the first part is positive and the second one with minus is also. For all $r\leq x_0$, $\delta(r)$ is an increasing function since $u''(x)k_F(x)F(x)\geq 0$. Given that $\delta(c)=0$, $\delta(r)\geq 0$ for all $r\leq x_0$. This concludes the proof. Q.E.D.

Lemma 3.5. $\varphi(r) = -\int_c^r \left[u'(x)k_F'(x) + u'''(x)k_F(x) \right] \hat{F}(x) dx$ is always non-negative for all $r \in [c,b]$.

Proof: Note that $\varphi(c) = 0$. From (3.1), integrating by parts of $\varphi(r)$ gives $\varphi(r) = u'(r)k_F(r)F(r) - u''(r)k_F(r)\hat{F}(r) - \int_c^r u'(x)xdF(x).$ We know that $\int_c^r u'(x)xdF(x) \le 0 \text{ for all } r \in [c,b]. \text{ Let } x_0 \text{ denote the point where } k_F(x) \text{ changes sign}$ from negative to positive. For all $r \ge x_0$, $\varphi(r) \ge 0$ since the first part is positive, and the second and the third one with minus are also. For all $r \le x_0$, $\varphi(r)$ is an increasing function since $-[u''(x)k_F'(x) + u'''(x)k_F(x)]\hat{F}(x) \ge 0$. Given that $\varphi(c) = 0$, $\varphi(r) \ge 0$ for all $r \le x_0$. This concludes the proof. Q.E.D.

Theorem 3.3. If G MSIR F, $z_{xx} = 0$ and $z_{aa} = 0$, then $\alpha_F \ge \alpha_G$ for all risk averse decision makers with $u''' \ge 0$.





Proof: The case with an unbounded α_F yields an obvious comparative statics property (see Dionne, Eeckhoudt and Gollier (1993)). Therefore, without loss of generality, we assume that $\alpha_F = 1$ and $z_0 = 0$ (for notational convenience). We define $E_I(x) = \int_a^x t dI(t)$ and $k_I(x) = E_I(x)/F(x)$ where I = F or G. From (3.2),

$$Q(\alpha_{F}) = u'(b) \int_{a}^{b} x dF(x) - \int_{a}^{b} u''(x) \int_{a}^{x} t dF(t) dx$$

$$- u'(b) \int_{a}^{b} x dG(x) + \int_{a}^{b} u''(x) \int_{a}^{x} t dG(t) dx$$

$$= u'(b) [E_{F}(b) - E_{G}(b)] - \int_{a}^{b} u''(x) [E_{F}(x) - E_{G}(x)] dx. \qquad (3.18)$$

Since $\int_a^b F(x) dx = \int_a^b G(x) dx$, $E_F(b) = E_G(b)$. Therefore, (3.18) becomes

$$Q(\alpha_F) = -\int_a^b u''(x) [E_F(x) - E_G(x)] dx$$

$$= \int_a^b u''(x) k_G(x) G(x) dx - \int_a^b u''(x) [k_F(x) F(x) - k_G(x) G(x)] dx. \qquad (3.19)$$

Since α_F is finite, the lower bound of the support of F(x)- which is denoted c- must be negative and $k_G(x) \le (c \le)0$ for all $x \le c$. Therefore, the first term in the RHS of (3.19) is non-negative. Adding and subtracting $\int_{-\infty}^{\infty} u''(x) k_F(x) G(x) dx$ in the RHS of (3.19) gives

$$Q(\alpha_F) \ge -\int_c^b u''(x)k_F(x)[F(x) - G(x)]dx - \int_c^b u''(x)[k_F(x) - k_G(x)]G(x)dx. \quad (3.20)$$

Since, by the condition (c) in Lemma 3.1, $k_G(x)$ is less than or equal to $k_F(x)$ for all x in the support of F(x), the second term with minus sign in the RHS of (3.20) is non-negative. Therefore, (3.20) implies that

$$Q(\alpha_F) \ge -\int_0^b u''(x) k_F(x) [F(x) - G(x)] dx. \tag{3.21}$$





Integrating the RHS of (3.21) by parts yields

$$Q(\alpha_{F}) \ge -u''(b)k_{F}(b) \int_{c}^{b} [F(x) - G(x)] dx$$

$$+ \int_{c}^{b} [u''(x)k'_{F}(x) + u'''(x)k_{F}(x)] \left[\int_{c}^{x} F(t) dt - \int_{c}^{x} G(t) dt \right] dx . \quad (3.22)$$

We assume that $k_F(c) = 0$ since $k_F(c) = \frac{\int_a^c t dF(t)}{F(c)}$, F(c) = 0 and $\hat{F}(c) = 0$. Since $u''(b)k_F(b)\int_a^c G(x)dx = \int_a^c G(t)dt \int_c^b \left[u''(x)k_F'(x) + u'''(x)k_F(x)\right]dx$ and $\int_c^b F(x)dx = \int_a^b F(x)dx$, adding and subtracting $u''(b)k_F(b)\int_a^c G(x)dx$ in the RHS of (3.22) implies that

$$Q(\alpha_{F}) \ge -u''(b)k_{F}(b) \int_{a}^{b} [F(x) - G(x)] dx$$

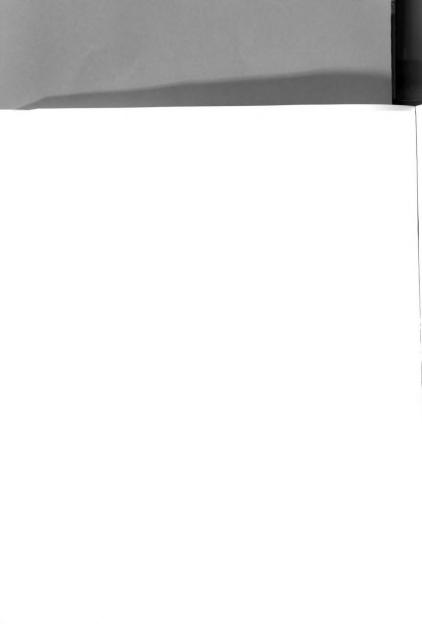
$$+ \int_{c}^{b} [u''(x)k_{F}'(x) + u'''(x)k_{F}(x)] \Big[\int_{c}^{x} F(t) dt - \int_{a}^{x} G(t) dt \Big] dx . \quad (3.23)$$

Since $\int_a^b [F(x) - G(x)] dx = 0$, the first term in the RHS of (3.23) is zero. Since $\int_a^x F(t) dt = \int_a^x F(t) dt = \hat{F}(x)$, integrating the second term in the RHS of (3.23) by parts yields

$$Q(\alpha_F) \ge [1 - H(b)] \int_c^b [u^*(x)k_F'(x) + u^*(x)k_F(x)] \hat{F}(x) dx$$

 $+ \int_c^b H'(x) \int_c^x [u^*(t)k_F'(t) + u^*(t)k_F(t)] \hat{F}(t) dt dx$. (3.24)

Since H(b) = 1, the first term in the RHS of (3.24) is zero. Since $H'(x) \le 0$ for all $x \in (c,b]$ and $\varphi(x) = \int_{c}^{x} [u''(t)k_{F}'(t) + u'''(t)k_{F}(t)]\hat{F}(t)dt \le 0$ for all $x \in [c,b]$ by Lemma 3.5, the second term in the RHS of (3.24) is non-negative. Therefore, $Q(\alpha_{F}) \ge 0$. This



concludes the proof. Q.E.D.

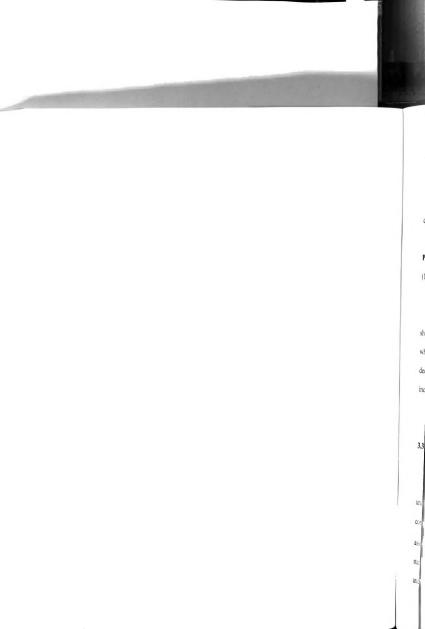
Remember that from Lemma 3.3, the L-ESIR order implies the MSIR order. Therefore we obtain the following result.

Corollary 3.2. If G L-ESIR F, $z_{xx}=0$ and $z_{\alpha\alpha}=0$, then $\alpha_F \geq \alpha_G$ for all risk averse decision makers with $u''' \geq 0$.

Since the class of MSIR shifts contains the class of L-ESIR shifts, the result in Theorem 3.3 improves the robustness of the comparative statics result in L-ESIR shifts. The efficiency gain is obtained from extending the admissible set of R-S increases in risk yielding the result, without any cost of additional restrictions. Note also that, compared with the result in ESIR shifts, our result in Theorem 3.3 extends the set of R-S increases in risk with the cost of adding additional restrictions ($z_{xx} = 0$ and $u^{\pi} \ge 0$) on the structure of the decision model and the risk preferences of decision makers.

Next theorems concern the OSIR and the ORSIR orders. As indicated in the beginning of this chapter, the OSIR ranking can be decomposed into two L-SIR shifts and the ORSIR ranking can be decomposed into a L-SIR shift and a L-ESIR shift. We use these properties to prove the following theorems.

Theorem 3.4. If G OSIR F and $z_{xx}=0$, then $\alpha_F \geq \alpha_G$ for all risk averse decision makers with $u''' \geq 0$.





Proof: It is clear by Corollary 3.1 in L-SIR shifts.

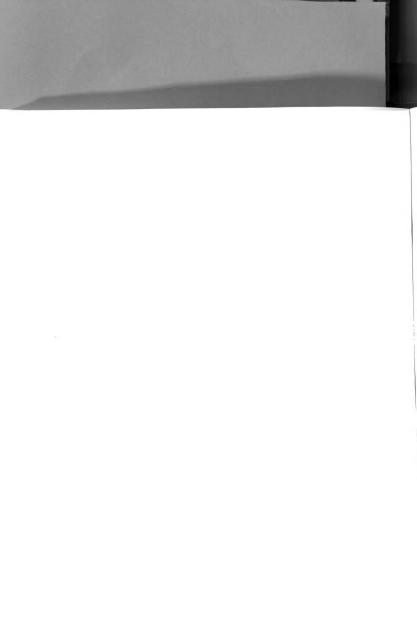
Theorem 3.5. If G ORSIR F and $z_{xx}=0$, then $\alpha_F \geq \alpha_G$ for all risk averse decision makers with $u''' \geq 0$.

Proof: It is clear by Corollary 3.1 in L-SIR shifts and the proof in L-ESIR shifts in Kim (1998).

The basic relationship between the comparative statics results in these two shifts shows that the ORSIR ranking represents a net improvement over the OSIR one. That is, when the same restrictions on the structure of the decision model and on the set of the decision makers are used in the analysis of these two shifts, the set of ORSIR shifts includes the set of OSIR shifts.

3.3 Concluding Remarks

While linear payoffs turn out to be a special case in the set of decision problems under uncertainty, they appear to prevail in many economic environments. Reviewing the comparative static analysis in this chapter shows that, when $z_{xx} = 0$ and u''' > 0 are assumed, more general subsets of R-S increases in risk are allowed in this comparative static analysis. Notice that our study in this chapter follows the CDF difference approach and the ratio approach in order to generate interesting comparative static statements.





From the results in this chapter, it seems that the effect of adding $u^{-}>0$ is to focus on imposing restrictions on the left-hand side of two PDFs or two CDFs. These properties are found in the analysis of the L-SIR, the L-RSIR, the L-RWIR, the OSIR and the ORSIR orders.

Most comparative statics results in the published literature and this study are associated with whether the two CDFs cross only once. This single crossing property is not a restrictive assumption since members of some of well-known families of distributions have this property. There is a list of these distributions; uniform, normal, log normal, binomial, exponential and others. This list only applies if two random variables are each from the same family; for example both must be normally distributed or both must be uniformly distributed. Note also that a pair of random variables, each from a different family may not guarantee single crossing.

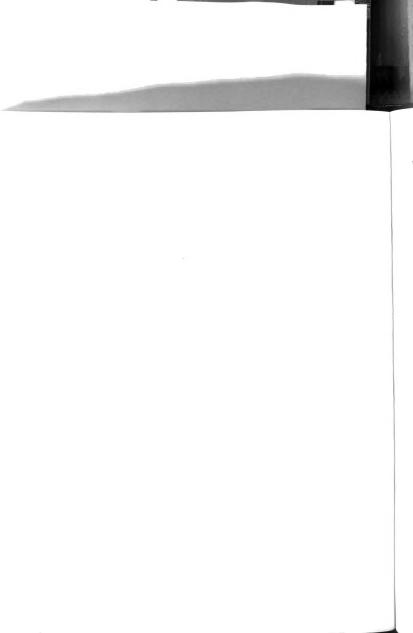


Chapter 4

COMPARATIVE STATIC ANALYSIS FOR SUBSETS OF K-L INCREASES IN RISK

This chapter provides general comparative static statements regarding several subsets of increases in risk in the K-L sense. In order to obtain comparative statics results, many researchers have specified particular types of CDF changes which are subsets of FSD shifts, SSD shifts, or R-S increases in risk. So far, no one has examined subsets of TSD shifts for comparative static purposes. Kroll et al. (1995) define a special 'increase in risk' as a TSD change with equal means. Note that whereas R-S increases in risk are SSD changes with equal means, K-L increases in risk are TSD changes with equal means. Since SSD implies TSD, the set of K-L increases in risk includes the set of R-S increases in risk.

In this chapter, we consider three subsets of K-L increases in risk. These are a 'strong increase in risk in the K-L sense' (SIR_K), a 'simple increase in risk across r in the K-L sense' (sIR_K(r)) and a 'relatively strong increase in risk in the K-L sense' (RSIR_K). Notice that, since a K-L increase in risk is a particular type of TSD change, the derived comparative static statements are associated with a risk averse decision maker with $u''' \ge 0$. In section 4.1, we provide definitions for subsets of K-L increases in risk used in



this study. In section 4.2, we do comparative static analysis concerning the three types of K-L increases in risk defined in section 4.1.

4.1 Subsets of Increases in Risk in the K-L Sense

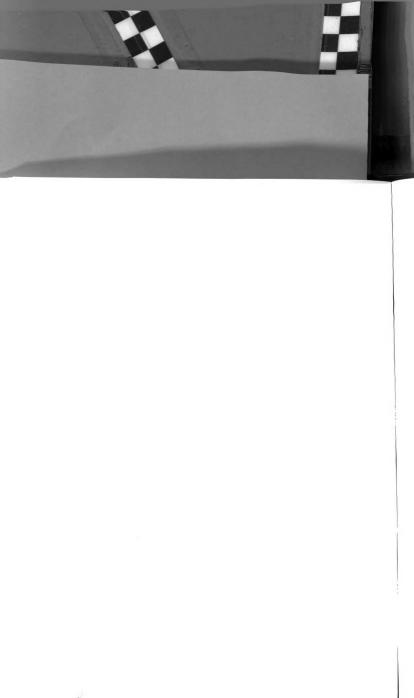
This section gives the definitions for three subsets of increases in risk in the K-L sense. Meyer and Ormiston (1985) consider the concept of a 'strong increase in risk' (SIR), a particular type of R-S increases in risk defined by additional restriction on the difference between the two CDFs. Using the similar restriction, we introduce a 'strong increases in risk in the K-L sense' (SIR_K) which is defined by imposing restrictions on the difference between the two cumulative of CDFs (C-CDFs). We assume that the supports of F(x) and G(x) are located in the interval (x_2, x_3) and (x_1, x_4) , respectively, where $x_1 \le x_2 \le x_3 \le x_4$. We define that $\hat{F}(x) = \int_x^x F(t) dt$ and $\hat{G}(x) = \int_x^x G(t) dt$.

Definition 4.1. G(x) represents a strong increase in risk in the K-L sense with respect to F(x) (denoted by $G SIR_K F$) if

(a)
$$\int_{x_1}^{x_4} [G(x) - F(x)] dx = 0$$

(b)
$$\int_{x}^{s} [\hat{G}(x) - \hat{F}(x)] dx \ge 0$$
 for all $s \in [x_1, x_4]$

(c)
$$\hat{G}(x) - \hat{F}(x)$$
 is non-increasing on (x_2, x_3) .





Condition (a) implies that the two distributions have the equal mean. Condition (b) expresses the third-degree stochastic dominance (TSD) between F(x) and G(x). These conditions are sufficient for G(x) to represent an increase in risk in the K-L sense from F(x). Condition (c) states that G(x) never exceeds F(x) in the interval $[x_2, x_3]$. This implies that $G \le F$. This is the added condition which allows general statements to be made concerning the effect of a SIR_K on the choice made by a risk averse decision maker with $u''' \ge 0$. These three conditions transfer area under $\hat{F}(x)$ from locations where it is initially distributed, to areas at or to the left and to the right of the endpoints of the interval over which the original distribution is defined; that is, this is to areas to the right of and to the left of the interval containing the support of F(x). These restrictions imposed on the difference between two C-CDFs to define the SIR_K order are similar to the restrictions on the difference between two CDFs used by Meyer and Ormiston (1985) who define the SIR order.

Black and Bulkley (1989) use the concept of a 'relatively strong increase in risk' (RSIR) that is a particular type of R-S increases in risk by imposing the restriction on the ratio of a pair of PDFs. By replacing the restriction on the ratio of a pair of PDFs with the restriction on the ratio of a pair of CDFs, we introduce the second definition of K-L increases in risk and call it a 'relatively strong increase in risk in the K-L sense' (RSIR_K). Note that, as the SIR order implies the RSIR order, the SIR_K ranking also implies the RSIR_K ranking. That is, the set of SIR_K shifts is included in the set of RSIR_K shifts.

Definition 4.2. G(x) represents a relatively strong increase in risk in the K-L sense with respect to F(x) (denoted by $G RSIR_K F$) if





(a)
$$\int_{x}^{x_4} [G(x) - F(x)] dx = 0$$

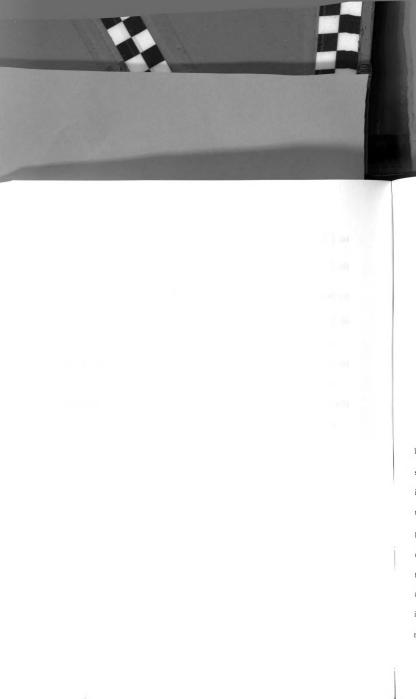
(b)
$$\int_{a}^{s} [\hat{G}(x) - \hat{F}(x)] dx \ge 0$$
 for all $s \in [x_1, x_4]$

(c) There exists a point
$$m^0$$
 satisfying $\int_{x_0}^{m^0} [G(x) - F(x)] dx = \int_{m^0}^{x_0} [G(x) - F(x)] dx = 0$

- (d) There exists a pair of points $m_1, m_2 \in [x_2, x_3]$ where $m_1 \le m_2$, such that $F(x) \ge G(x)$ for all $x \in [m_1, m_2]$ and $F(x) \le G(x)$ for all x in $[x_2, m_1]$ and $[m_2, x_3]$
- (e) For all $x \in [x_2, m_1]$, there exists a non-decreasing function $H_1 : [x_2, m_1) \to [0, 1]$ such that $F(x) = H_1(x)G(x)$
- (f) For all $x \in [m_2, x_3]$, there exists a non-increasing function $H_2: (m_2, x_3] \rightarrow [0, 1]$ such that $F(x) = H_2(x)G(x)$.

Conditions (a) and (b) define a K-L increase in risk. Condition (c) means that a relative strong increase in risk in the K-L sense (RSIR_K) satisfies second-degree stochastic dominance (SSD) on both side of m_0 with equal means. Condition (d) imposes the restriction that the two CDFs cross only twice. Conditions (e) and (f) restrict the extent to which cumulative probability mass can be transferred to any one value in the tails of F(x) relative to any other and impose monotonicity restrictions on the ratio between a pair of CDFs in the intervals $x \in [x_2, m_1]$ and $x \in [m_2, x_3]$.

Figure 4.1 satisfies the RSIR_K conditions given in Definition 4.2. H_1 is non-decreasing and H_2 is non-increasing for each corresponding interval $x \in [x_2, m_1]$ and $x \in [m_2, x_3]$, respectively. If, for any point in the interval $[x_2, m_1]$ ($(m_2, x_3]$), the tangent



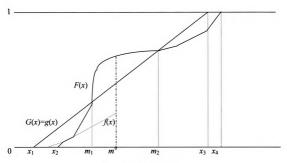


Figure 4.1. G RSIR_K F.

lines f(x) and g(x) (= G(x)) do not meet to the left (right) direction in the probability space between zero and one, the ratio of a pair of CDFs $H_1(H_2)$ is non-decreasing (non-increasing). A RSIR_K order consists of the sum of three shifts, which are a 'left-side monotone probability ratio' (L-MPR) in the interval $x \in [x_1, m_1]$, a 'right-side monotone probability ratio' (R-MPR) in the interval $x \in [m_2, x_4]$ and a first-order stochastically dominated shift (the opposite of a FSD shift) in the interval $x \in [m_1, m_2]$. Or, it consists of the sum of two shifts, which are 'left-side extended strong increases in risk' (L-ESIR) and 'right-side extended strong decreases in risk' (R-ESDR) for each corresponding interval $x \in [x_1, m^0]$ and $[m^0, x_4]$, respectively. A L-ESIR (R-ESDR) requires the monotonicity of the ratio only on the left side (right side) of the given pair of CDFs.

We introduce the last specialized concept of K-L increases in risk and name it a





'simple increase in risk across r in the K-L sense' ($sIR_K(r)$) that is formally defined as:

Definition 4.3. G(x) represents a simple increase in risk across r in the K-L sense with respect to F(x) (denoted by $G\operatorname{sIR}_{K}(r)F$) if

(a)
$$\int_{x_1}^{x_4} [G(x) - F(x)] dx = 0$$

(b)
$$\int_{x_1}^{s} [\hat{G}(x) - \hat{F}(x)] dx \ge 0$$
 for all $s \in [x_1, x_4]$

(c)
$$(\hat{F}(x) - \hat{G}(x))(x - r) \ge 0$$
 for all $x \in [x_1, x_4]$,
where r is the point in the interval $[x_1, x_4]$.

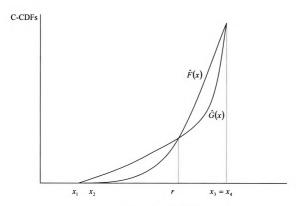


Figure 4.2. $G \operatorname{sIR}_{K}(r) F$.

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Conditions (a) and (b) imply that the sIRK(r) order is a K-L increase in risk.

Condition (c) means that the sIR_K(r) ranking satisfies the second-degree stochastic dominance (SSD) on both side of r. That is, this condition expresses the SSD of F(x) over G(x) in the interval $x \in [x_1, r)$ and of G(x) over F(x) in the interval $x \in [r, x_4]$. This implies that $\hat{F}(x)$ is less than $\hat{G}(x)$ whenever x is less than r, and $\hat{F}(x)$ is larger than $\hat{G}(x)$ whenever x is larger than r. A sIR_K(r) is a particular case of TSD. Its particularity comes from the fact that the curves $\hat{F}(x)$ and $\hat{G}(x)$ are forced to intersect at a single point x = r. Such sIR_K(r) is presented in Figure 4.2. Note that a simple increase in risk in the K-L sense allows for cumulative probability mass transfers to points inside the initial support, but a strong increase in risk in the K-L sense limits the redistribution of cumulative probability mass to points outside the initial support.

It can also be convenient to represent a $sIR_K(r)$ by using a deterministic transformation of returns for asset x. As popularized by Sandmo (1971) and extended by Meyer and Ormiston (1989), a deterministic transformation of a lottery replaces every payoff x of the lottery by a new payoff t(x). A so-called deterministic transformation approach imposes some specific restrictions on the function t(x).

Definition 4.4. The function t(x) represents a simple deterministic (risk-increasing in the K-L sense) transformation across r for a random variable given by F(x) if the function $k(x) \equiv t(x) - x$ satisfies

(a)
$$\int_{x_1}^{x_4} k(x) dF(x) = 0$$

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(b)
$$\int_{x_{1}}^{x} \int_{x_{2}}^{t} k(s) dF(s) dt \le 0$$
 for $x \in [x_{1}, x_{4}]$

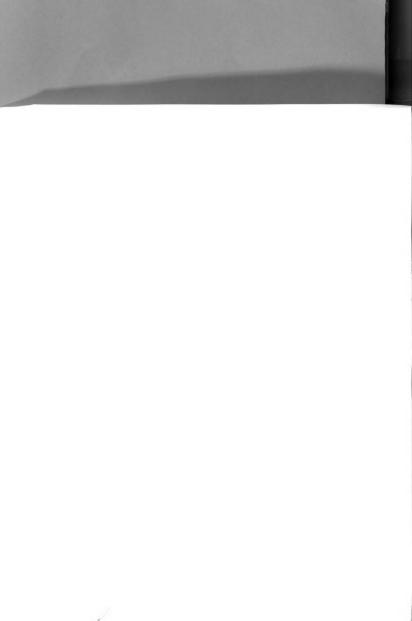
(c)
$$\int_{x_i}^x k(s) dF(s) \le 0$$
 whenever $x < r$, and $\int_{x_i}^x k(s) dF(s) \ge 0$ whenever $x > r$.

Condition (a) guarantees that the mean of the random variable is preserved.

Condition (b) implies that the initial random variable third-order stochastically dominates the transformed random variable. These conditions characterize increases in risk in the K-L sense. Condition (c) implies that a simple deterministic transformation across r satisfies the second-order stochastic dominance (SSD) transformation on both sides of r. That is, this condition expresses the SSD transformation of x over t(x) in the interval $x \in [x_1, r)$ and of t(x) over x in the interval $x \in [r, x_4]$. A simple deterministic transformation across r differs from the notion of simple deterministic transformation by Meyer and Ormiston (1989) who impose k(x) = t(x) - x to be monotonically increasing. Recall that we do not consider any other restrictions on $\int_{r_1}^x k(s) dF(s)$ than nonpositivity for x < r and nonnegativity for x > r.

4.2 Comparative Static Analysis

In this section, we provide general comparative static statements regarding the SIR_K , the $sIR_K(r)$, and the $RSIR_K$ order defined in section 4.1. Doing comparative static analysis, we assume that the first and the second-order condition are satisfied to





guarantee a unique interior solution. Faced by the CDF of the random variable, F(x), the first-order condition defining the optimal value for α is

$$\int_{a}^{x_{4}} u'[z(x,\alpha)] z_{\alpha}(x,\alpha) dF(x) = 0$$

$$(4.1)$$

The optimal solution satisfying (4.1) is guaranteed to be a global optimum by the secondorder condition that

$$\int_{x}^{x_{i}} \left\{ u''[z(x,\alpha)] z_{\alpha}^{2} + u'[z(x,\alpha)] z_{\alpha\alpha} \right\} dF(x) < 0.$$
 (4.2)

Under risk aversion, restriction $z_{aa} < 0$ insures that the condition (4.2) is satisfied. Thus, in order to see the result that $\alpha_F \ge \alpha_G$ for a specified change in CDF from F to G, it is sufficient to show that

$$Q(\alpha_F) = \int_a^{x_4} u'[z(x,\alpha_F)] z_\alpha(x,\alpha_F) d[G(x) - F(x)] \le 0.$$

$$(4.3)$$

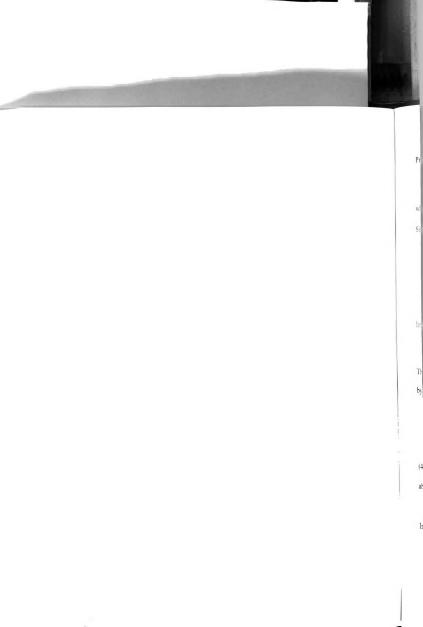
Before proving Theorem 4.1, the following result from Yitzhaki (1983) is needed.

Lemma 4.1. Let s(x) be any function and $s(y) = \int_a^y s(x) dx \ge 0$ for all $y \in [a,b]$. If another function l(x) is non-negative and non-increasing for all $x \in [a,b]$, then

$$\int_{a}^{y} s(x) l(x) dx \ge 0 \quad \text{for all } y \in [a, b].$$

Theorem 4.1. For a class of risk-averse decision makers with $u''' \ge 0$, $\alpha_F \ge \alpha_G$ if (a) $G SIR_K F$

(b)
$$z_x \ge 0$$
, $z_{xx} \le 0$, $z_{\alpha x} \ge 0$, $z_{\alpha xx} \le 0$ and $z_{\alpha xxx} \ge 0$.





Proof: From (4.3), $Q(\alpha_F)$ can be written as

$$Q(\alpha_F) = \int_{r_0}^{r_4} u'(z) z_\alpha(x, \alpha_F) dH(x)$$
(4.4)

where H(x) = G(x) - F(x).

Since $H(x_1) = H(x_4) = 0$, integration by parts of (4.4) yields

$$Q(\alpha_{F}) = -\int_{z_{i}}^{z_{i}} \{u'(z)z_{\alpha x}(x,\alpha_{F}) + u''(z)z_{x}z_{\alpha}(x,\alpha_{F})\}H(x)dx$$

$$= -\int_{z_{i}}^{z_{i}} u'(z)z_{\alpha x}(x,\alpha_{F})H(x)dx - \int_{z_{i}}^{z_{i}} u''(z)z_{x}z_{\alpha}(x,\alpha_{F})H(x)dx. \tag{4.5}$$

First, let's consider the first term in (4.5).

Integration by parts of the first term in (4.5) yields

$$-u'(z)z_{\alpha\alpha}(x,\alpha_F)\hat{H}(x)\big|_{x=x_1}^{x_1}+\int_{x}^{x_4}\left\{u'(z)z_{\alpha\alpha\alpha}(x,\alpha_F)+u''(z)z_{x}z_{\alpha\alpha}(x,\alpha_F)\right\}\hat{H}(x)dx. \quad (4.6)$$

The first term of (4.6) is zero because of the condition (a) in Definition 4.1. Integration by parts of the second term in (4.6) yields

$$\begin{aligned} &\{u'(z)z_{axx} + u''(z)z_{x}z_{ax}\} \int_{z_{1}}^{z_{1}} \hat{H}(t)dt \Big|_{z=z_{1}}^{z_{1}} \\ &- \int_{z_{1}}^{z_{1}} \{u'(z)z_{axxx} + 2u''(z)z_{x}z_{axx} + u''(z)z_{xx}z_{ax} + u'''(z)z_{x}^{2}z_{ax}\} \int_{z_{1}}^{z_{1}} \hat{H}(t)dt dx . \end{aligned}$$
(4.7)

(4.7) is non-positive because of the condition (b) in Definition 4.1 and assumptions made about z_{α} .

Second, let's consider the second term in (4.5).

Integration by parts of the second term in (4.5) yields

$$-u''(z)z_{x} \int_{z_{i}}^{y} z_{\alpha}(x, \alpha_{F})H(x) dx \Big|_{y=z_{i}}^{z_{i}}$$

$$+ \int_{z_{i}}^{z_{i}} \{u''(z)z_{x} + u'''(z)z_{x}^{2}\} \int_{z_{i}}^{y} z_{\alpha}(x, \alpha_{F})H(x) dx dy.$$
(4.8)





A sufficient condition for (4.8) is

$$\int_{r}^{y} z_{\alpha}(x, \alpha_{F}) H(x) dx \le 0 \quad \text{for all } y \in [x_{1}, x_{4}].$$

$$\tag{4.9}$$

Integration by parts of (4.9) yields

$$z_a(x,\alpha_F)\hat{H}(x)|_{x=x_1}^y - \int_{x_1}^y z_{ax}(x,\alpha_F)\hat{H}(x)dx$$
. (4.10)

Using Lemma 4.1 shows that the integral in (4.10) is non-negative because of the condition (b) in Definition 4.1 and assumptions made about z_a . The first term in (4.10) is equal to

$$z_{\alpha}(x,\alpha_F)\hat{H}(y) \tag{4.11}$$

which is zero when $y=x_F$. Hence (4.9) is satisfied at x_F . Taking into account Definition 4.1, (4.11) is non-positive for y belonging to $[x_1,x_2]$ and because of $z_a(y,\alpha_F)$ is non-positive for $y \le x_F$, (4.9) is also satisfied for y belonging to (x_2,x_F) . Indeed.

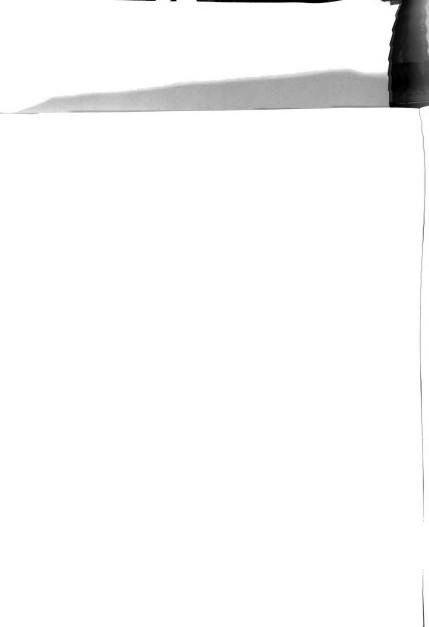
$$\int_{r_{\bullet}}^{y} z_{\alpha}(x,\alpha_{F})H(x)dx = \int_{r_{\bullet}}^{x_{F}} z_{\alpha}(x,\alpha)H(x)dx - \int_{r_{\bullet}}^{x_{F}} z_{\alpha}(x,\alpha_{F})H(x)dx. \tag{4.12}$$

Clearly, the first term on the right hand side of (4.12) is non-positive, as shown above, and the second integral is positive because of the condition (c) in Definition 4.1 and the assumptions made about z_{α} .

For y in (x_F, x_3) one gets

$$\int_{r_{i}}^{y} z_{\alpha}(x,\alpha_{F})H(x)dx = \int_{r_{i}}^{x_{F}} z_{\alpha}(x,\alpha_{F})H(x)dx + \int_{r_{i}}^{y} z_{\alpha}(x,\alpha_{F})H(x)dx. \tag{4.13}$$

Because of the condition (c) in Definition 4.1, (4.13) is non-positive. Finally, for $y \in [x_3, x_4]$, (4.10) is non-positive from Definition 4.1. Hence (4.9) holds for all





$$y \in [x_1, x_4].$$
 Q.E.D.

Our result in Theorem 4.1 uses somewhat stronger restrictions on the structure of the decision model and the risk preferences of decision makers than does Meyer and Ormiston (1985) for SIR.

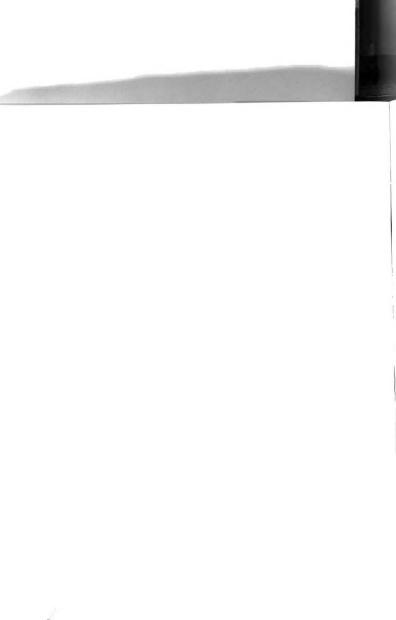
Next theorem concerns the class of a simple increase in risk across r in the K-L sense (sIR_K(r)) presented in section 4.1. Using the simple form of the general decision model $z(x,\alpha) \equiv z_0 + \alpha x$, we assume that the random variable x changes sign from negative to positive at the point r. Because a sIR_K(r) is a particular case of TSD changes, we know that such a change in risk in the K-L sense reduces the expected utility of decision makers. Therefore, it is sufficient to show that

$$Q(\alpha_F) = \int_{x}^{x_4} u'(z)(x-r)d[G(x)-F(x)] \le 0.$$
 (4.14)

Theorem 4.2. If $G \operatorname{sIR}_K(r) F$ and $z_{xx} = 0$, then $\alpha_F \ge \alpha_G$ for all risk averse decision makers with $u''' \ge 0$.

Proof: The case with an unbounded α_F yields an unambiguous comparative static property (see Dionne and Gollier (1992) and Dionne, Eeckhoudt and Gollier (1993)). Therefore, we limit the analysis to the case with a bounded solution. Without loss of generality, we assume that $\alpha_F = 1$. We have to prove that (4.14) is true. Integrating (4.14) by parts yields

$$Q(\alpha_F) = u'(z(x_4,\alpha)) \int_{x_1}^{x_4} (x-r) d[G(x) - F(x)] - \int_{x_1}^{x_4} u''(z) \int_{x_1}^{x} (t-r) d[G(t) - F(t)] dx.$$



Integrating the first term of $Q(\alpha_F)$ by parts and using the sIR_K(r) conditions in Definition 4.3, the first term of $Q(\alpha_F)$ is non-positive. Thus, we have

$$Q(\alpha_F) \le -\int_{r_0}^{x_1} u''(z) \int_{r_0}^{x} (t-r) d[G(t) - F(t)] dx$$
 (4.15)

Integrating (4.15) by parts yields

$$Q(\alpha_F) \le -u''[z(x_4, \alpha)] \int_{x_1}^{x_4} \int_{x_1}^{x_4} (t-r) d[G(t) - F(t)] dx$$

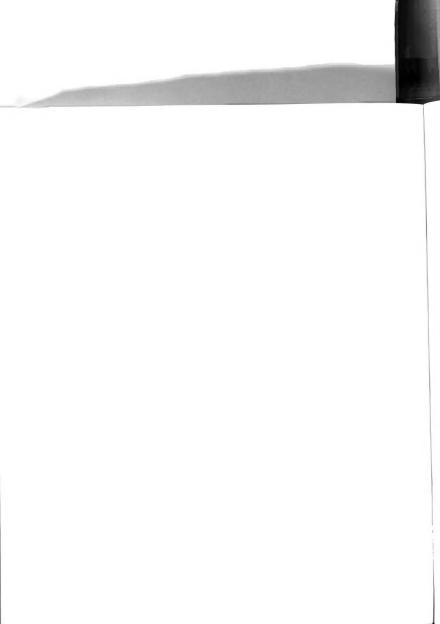
$$+ \int_{x_1}^{x_4} u'''(z) \int_{x_1}^{x} \int_{x_1}^{x_2} (t-r) d[G(t) - F(t)] ds dx. \qquad (4.16)$$

We denote that $W(x) = \int_{x_1}^x \int_{x_1}^x (t-r) d[G(t) - F(t)] ds$ for all $x \in [x_1, x_4]$. Given the assumption on u, the left-hand side of equality in (4.16) would be negative if W(x) is non-positive for all $x \in [x_1, x_4]$. Integrating W(x) by parts two times for all $x \in [x_1, x_4]$ yields

$$W(x) = (x - r)[\hat{G}(x) - \hat{F}(x)] - 2\int_{x}^{x} [\hat{G}(s) - \hat{F}(s)] ds.$$
 (4.17)

Observe that the first term in the right-hand side of (4.17) is negative by the condition (c) in Definition 4.3 presented in section 4.1, since (x-r) and $\left[\hat{G}(x) - \hat{F}(x)\right]$ always exhibit opposite signs. The second term in (4.17) is always positive since it satisfies TSD. Therefore, the difference of these two terms is always negative, that is, $W(x) \le 0$ for all $x \in [x_1, x_4]$. This concludes the proof. Q.E.D.

As the last result developed in this section, the set of K-L increases in risk for the comparative statics result in Theorem 4.1 is extended to the set of RSIR_K shifts which is larger than the set of SIR_K shifts. We assume that $z_x > 0$, $z_{xx} \ge 0$, and $z_{xx} = 0$. Given the





assumptions about $z(x, \alpha)$ to prove $\alpha_F \ge \alpha_G$ it is sufficient to show

$$Q(\alpha_F) = \int_{-\infty}^{x_4} u'(z) z_{\alpha}(x, \alpha_F) d(F(x) - G(x)) \ge 0.$$
 (4.18)

Integration by parts of the right-hand side (RHS) of $Q(\alpha_F)$ in (4.18) yields

$$Q(\alpha_{F}) = u'(z(x_{4}, \alpha_{F}))[T(x_{4}, \alpha_{F}; F, z) - T(x_{4}, \alpha_{F}; G, z)]$$

$$- \int_{z_{4}}^{z_{4}} u''(z) z_{s}[T(x, \alpha_{F}; F, z) - T(x, \alpha_{F}; G, z)] dx \ge 0$$
(4.19)

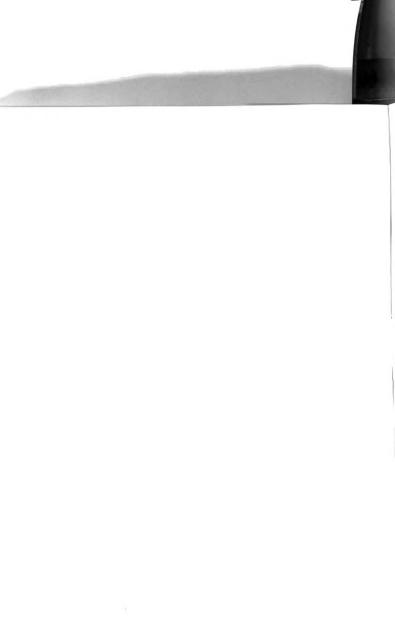
where $T(\cdot,\cdot;I,z):[x_1,x_4]\times R\to R$ is defined as

$$T(x,\alpha_F;I,z) \equiv \int_x^x z_\alpha(t,\alpha_F) dI(t)$$
 where $I = F, G$.

where T denotes location-weighted probability mass function used in Gollier (1995). In order to prove Theorem 4.3 we need first to prove the following lemma.

Lemma 4.2. If $z_{\alpha x} \ge 0$, then $\varphi(s) = \int_{z_2}^s u^n(z) z_x T(x, \alpha; F, z) dx \ge 0$ for all s in $[x_2, x_4]$.

Proof: Note that $\varphi(x_2) = 0$. By integrating the first-order condition by parts we get $\varphi(s) = u'(z)T(s,\alpha;F,z) - \int_{s_2}^s u'(z)z_\alpha dF(x)$. From the first-order condition $\int_{s_2}^s u'(z)z_\alpha dF(x) \le 0 \text{ for all } s \text{ in } [x_2,x_4]. \text{ Since the sign of } \frac{\partial T}{\partial x}(x,\alpha;F,z) \text{ is equal to the sign of } z_\alpha$, T must alternate in sign. Let x^c denote the value of x where T changes sign from negative to positive. For all $s \ge x^c$, $\varphi(s) \ge 0$ because the first term is positive and the second one with minus sign is also. For all $s \le x^c$, $\varphi(s)$ is an increasing function





because $u''(z)z_xT(x,\alpha;F,z) \ge 0$. Given that $\varphi(x_2) = 0$, $\varphi(s) \ge 0$ for all $s \le x^c$. Q.E.D.

Before proving Lemma 4.3, we need to know the following things. The integrand in expression (4.19), $u''(z)z_x[T(x,\alpha_F;F,z)-T(x,\alpha_F;G,z)]$, has a sign depending on the difference between two T functions.

$$T(x, \alpha_F; F, z) - T(x, \alpha_F; G, z)$$

= $z_{\alpha}[F(x) - G(x)] - \int_{0}^{x} z_{\alpha}[F(t) - G(t)]dt$. (4.20)

When z_{α} has a sign change over the intervals $[x_1, m_1]$, $[m_1, m^0]$, $[m^0, m_2]$ and $[m_2, x_4]$, (4.20) changes sign at most two times from positive to negative. When (4.20) has a sign change only once, it is trivial. Over four intervals let x^* denote the point where (4.20) changes sign when z_{α} has a sign change for each interval. Let us define that $S(x, \alpha) = z_{\alpha}[F(x) - G(x)]$ and $I(x, \alpha) = \int_{x_1}^{x} z_{\alpha t}[F(t) - G(t)]dt$. We consider the following four cases in (4.20); (i) $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ (ii-a) and (ii-b) $k_1 = a$, $k_2 = x^*$ and $k_3 = b$, and (iii) $k_1 = a$, $k_2 = b$ and $k_3 = x^*$, where k_2 is the end point when the first sign change occurs for each interval, and k_1 and k_3 are the points which the first and the second sign change occurs from positive to negative, respectively. Now, we draw the figures to get a clear demonstration for each case (see Figure 4.3-Figure 4.6).

Lemma 4.3. The value of α which maximizes $E[u(z(x,\alpha))]$ is lower for a class of risk-averse decision makers with $u^{-} \ge 0$ under G(x) than F(x) where G(x) and F(x) satisfy conditions (a), (b) and (c) for G(x) to represent a RSIR_K presented in Definition





because $u''(z)z_xT(x,\alpha;F,z) \ge 0$. Given that $\varphi(x_2) = 0$, $\varphi(s) \ge 0$ for all $s \le x^c$. Q.E.D.

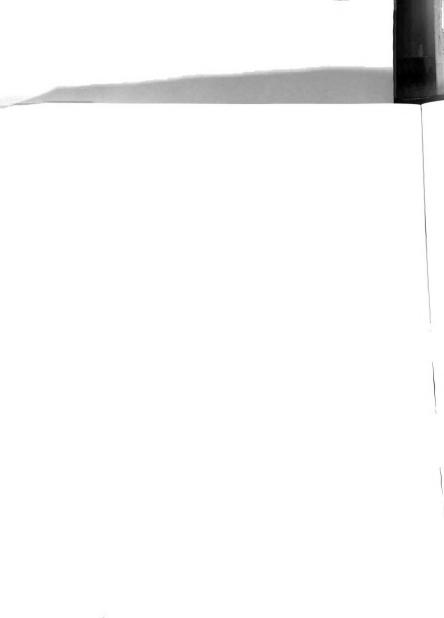
Before proving Lemma 4.3, we need to know the following things. The integrand in expression (4.19), $u''(z)z_x[T(x,\alpha_F;F,z)-T(x,\alpha_F;G,z)]$, has a sign depending on the difference between two T functions.

$$T(x, \alpha_F; F, z) - T(x, \alpha_F; G, z)$$

$$= z_a [F(x) - G(x)] - \int_0^x z_{at} [F(t) - G(t)] dt.$$
(4.20)

When z_{α} has a sign change over the intervals $[x_1, m_1]$, $[m_1, m^0]$, $[m^0, m_2]$ and $[m_2, x_4]$, (4.20) changes sign at most two times from positive to negative. When (4.20) has a sign change only once, it is trivial. Over four intervals let x^* denote the point where (4.20) changes sign when z_{α} has a sign change for each interval. Let us define that $S(x,\alpha) = z_{\alpha}[F(x) - G(x)]$ and $I(x,\alpha) = \int_{x_1}^{x} z_{nt}[F(t) - G(t)]dt$. We consider the following four cases in (4.20); (i) $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ (ii-a) and (ii-b) $k_1 = a$, $k_2 = x^*$ and $k_3 = b$, and (iii) $k_1 = a$, $k_2 = b$ and $k_3 = x^*$, where k_2 is the end point when the first sign change occurs for each interval, and k_1 and k_3 are the points which the first and the second sign change occurs from positive to negative, respectively. Now, we draw the figures to get a clear demonstration for each case (see Figure 4.3-Figure 4.6).

Lemma 4.3. The value of α which maximizes $E[u(z(x,\alpha))]$ is lower for a class of risk-averse decision makers with $u^{-} \ge 0$ under G(x) than F(x) where G(x) and F(x) satisfy conditions (a), (b) and (c) for G(x) to represent a RSIR_K presented in Definition





Case (i): $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ when $x_1 \le x^* \le m_1$.

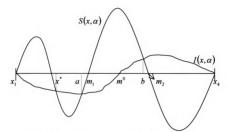


Figure 4.3. $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ when $x_1 \le x^* \le m_1$.

Case (ii-a): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \le x^* \le m^0$.

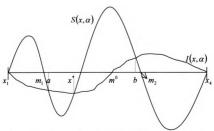
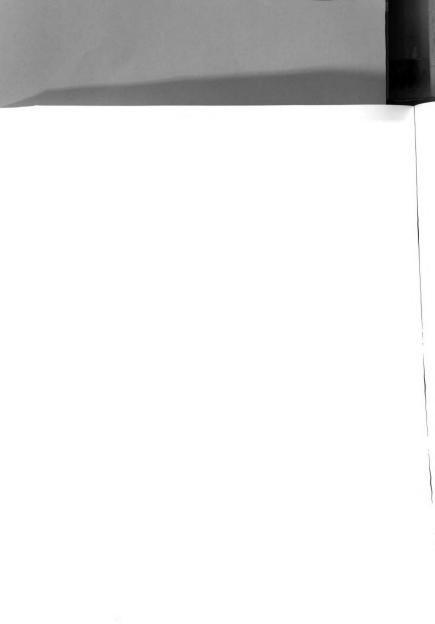


Figure 4.4. $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \le x^* \le m^0$.





Case (ii-b): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m^0 \le x^* \le m_2$.

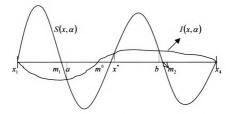


Figure 4.5. $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m^0 \le x^* \le m_2$.

Case (iii): $k_1 = a$, $k_2 = b$ and $k_3 = x^*$ when $m_2 \le x^* \le x_4$.

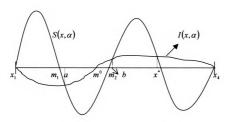
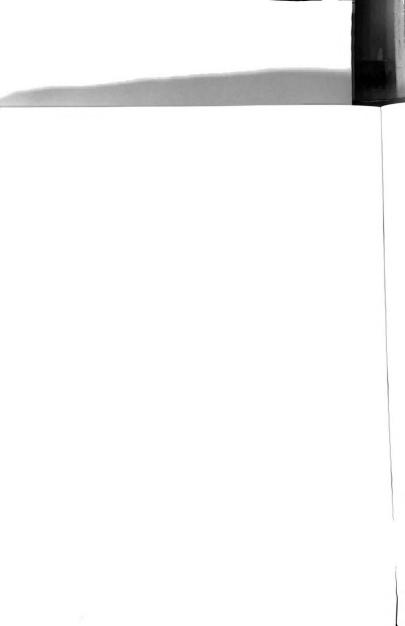


Figure 4.6. $k_1 = a$, $k_2 = b$ and $k_3 = x^*$ when $m_2 \le x^* \le x_4$.





4.2 if

$$\int_{r_0}^{k_2} \left[T(x, \alpha_F; F, z) - T(x, \alpha_F; G, z) \right] dx > 0$$

and if the given assumptions about $z(x, \alpha)$ are satisfied, where k_2 is the end point when the first sign change occurs for each interval, and k_1 and k_3 are the points which the first and the second sign change occurs from positive to negative, respectively in (4.20).

Proof: For notational convenience, we define that $T(x, \alpha_F; F, z) = T_F(x, \alpha_F)$ and $T(x, \alpha_F; G, z) = T_G(x, \alpha_F)$.

$$Q(\alpha_F) = u'(z(x_4, \alpha_F))[T_F(x_4, \alpha_F) - T_G(x_4, \alpha_F)]$$
$$- \int_{a_0}^{x_4} u''(z)z_x[T_F(x, \alpha_F) - T_G(x, \alpha_F)]dx.$$

Using the given assumptions about $z(x,\alpha)$ and the RSIR_K conditions, the first term of $Q(\alpha_F)$ is non-negative. Therefore,

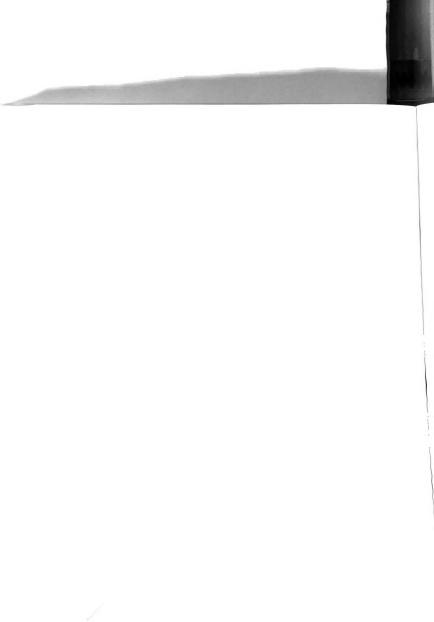
$$Q(\alpha_F) \ge -\int_{z_1}^{z_1} u^*(z) z_s [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx .$$

$$- \left[\int_{z_1}^{z_1} u^*(z) z_s [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx . \right]$$

Since $-u''(z)z_x$ is non-negative and decreasing, we have the following four cases:

(i)
$$k_1 = x^*$$
, $k_2 = a$ and $k_3 = b$ (ii-a) and (ii-b) $k_1 = a$, $k_2 = x^*$ and $k_3 = b$, and (iii) $k_1 = a$, $k_2 = b$ and $k_3 = b^*$.

$$Q(\alpha_F) \ge -u''(z(k_1,\alpha_F))z_x \int_{z_1}^{k_2} [T_F(x,\alpha_F) - T_G(x,\alpha_F)] dx$$
$$-u''(z(k_3,\alpha_F))z_x \int_{k_2}^{k_3} [T_F(x,\alpha_F) - T_G(x,\alpha_F)] dx.$$





Adding and subtracting $-u''(z(k_3,\alpha_F))z_x\int_{z_1}^{k_3} \left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx$ in the RHS of the above inequality gives

$$\begin{split} Q(\alpha_F) &\geq \left\{ -u''(z(k_1, \alpha_F)) z_z + u''(z(k_2, \alpha_F)) z_z \right\}_{i_1}^{i_2} \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx \\ &- u''(z(k_2, \alpha_F)) z_z \int_{i_1}^{i_2} \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx \; . \end{split}$$

Since $-u''(z)z_x$ is non-negative and decreasing, the bracket of the first term in the RHS of the above inequality is non-negative.

If
$$\int_{x_1}^{k_2} \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx \ge 0$$
,

$$Q(\alpha_F) \ge -u''(z(k_3,\alpha_F))z_x \int_{x_0}^{x_4} [T_F(x,\alpha_F) - T_G(x,\alpha_F)] dx.$$

The given assumptions about $z(x, \alpha)$ and the RSIR_K conditions imply

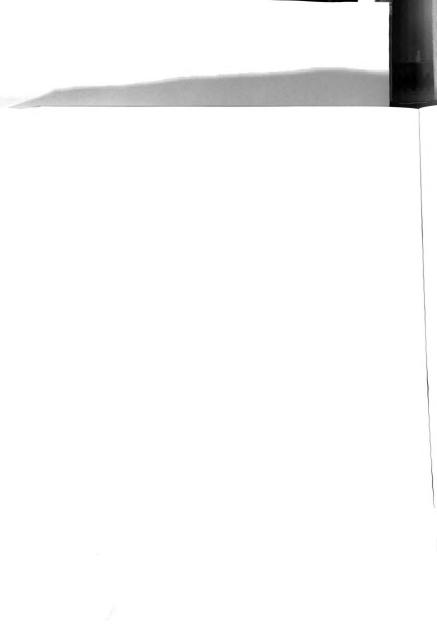
$$\int_{x_{i}}^{x_{i}} \left[T_{F}(x, \alpha_{F}) - T_{G}(x, \alpha_{F}) \right] dx \ge 0 \text{ and hence } Q(\alpha_{F}) \ge 0.$$
 Q.E.D.

Lemmas 4.2 and 4.3 are required in the proof of Theorem 4.3.

Theorem 4.3. For a class of risk-averse decision makers with $u''' \ge 0$, $\alpha_F \ge \alpha_G$ if (a) $G \operatorname{RSIR}_K F$

(b)
$$z_x \ge 0$$
, $z_{ax} \ge 0$, $z_{aa} \le 0$ and $z_{xx} = 0$.

Proof: Let k_1 , k_2 , k_3 be the ordered values of x^* , a, b. With the points m_1 , m^0 and m_2 used in Definition 4.3 where $x_2 \le m_1 \le m^0 \le m_2 \le x_4$, we consider the following four





cases:

Case (i): $k_1 = x^*$, $k_2 = a$ and $k_3 = b$ when $x_1 \le x^* \le m_1$ (see Figure 4.3).

Let's rewrite $Q(\alpha_F)$ in (4.19) as

$$Q(\alpha_F) = u'(z(x_4, \alpha_F))[T_F(x_4, \alpha_F) - T_G(x_4, \alpha_F)]$$

$$- \int_{z_4}^{z_4} u'(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx$$

$$- \int_{z_4}^{z_4} u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx.$$

Using the given assumptions about $z(x,\alpha)$ and the RSIR_K conditions, the first term of $Q(\alpha_F)$ is non-negative. Therefore,

$$\begin{split} Q(\alpha_F) &\geq -\int_{z_i}^{z_i} u^{\bullet}(z) \, z_x \big[T_F(x, \alpha_F) - T_G(x, \alpha_F) \big] dx \\ &\qquad - \int_{z_i}^{z_i} u^{\bullet}(z) \, z_x \big[T_F(x, \alpha_F) - T_G(x, \alpha_F) \big] dx \; . \end{split}$$

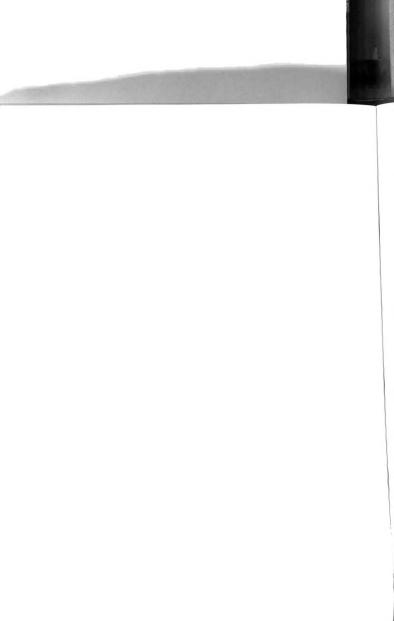
Consider the sign of the expression $\int_{x_i}^a \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx$. First, if

 $\int_{x_i}^a \left[T_F(x,\alpha_F) - T_G(x,\alpha_F) \right] dx \ge 0 \text{, then Theorem 4.3 follows from Lemma 4.3.}$

Second, assuming that $\int_{x_i}^a \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx \le 0$, $Q(\alpha_F)$ becomes

$$Q(\alpha_F) \ge -\int_{z_i}^a u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx$$
$$-u''(z(b, \alpha_F)) z_x \int_{z_i}^{z_i} [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx.$$

Adding and subtracting $-u^*(z(b,\alpha_F))z_x\int_{z_i}^u \left[T_F(x,\alpha_F)-T_G(x,\alpha_F)\right]dx$ in the RHS of the above inequality gives



 $Q(\alpha_F) \ge - \int_0^x u^*(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx$

$$\begin{aligned} \mathcal{Q}(\alpha_F) \geq & - \int_{z_i}^{\alpha} u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx \\ & + u''(z(b, \alpha_F)) z_x \int_{z_i}^{\alpha} [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx \\ & - u''(z(b, \alpha_F)) z_x \int_{z_i}^{z_i} [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx \end{aligned}$$

From the RSIR_K conditions and $\int_{x_i}^a [T_F(x,\alpha_F) - T_G(x,\alpha_F)] dx \le 0$,

$$Q(\alpha_F) \ge -\int_{x_1}^a u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx, \qquad (4.21)$$

where $-\int_{x_1}^a u''(z) z_x [T_F(x,\alpha_F) - T_G(x,\alpha_F)] dx$

$$= \int_{x_1}^{x_2} u''(z) z_x T_G(x_2, \alpha_F) dx - \int_{x_2}^{\alpha} u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx.$$

Let x^0 be the value of x that satisfies $z_\alpha(x,\alpha_F) = 0$. $T_G(x_2,\alpha_F) = z_\alpha G(x_2) - \int_{x_1}^{x_2} z_{\alpha r} G(t) dt$

is non-positive since $z_{\alpha} \le 0$ for $x \in [x_1, x_2]$. Therefore, (4.21) becomes

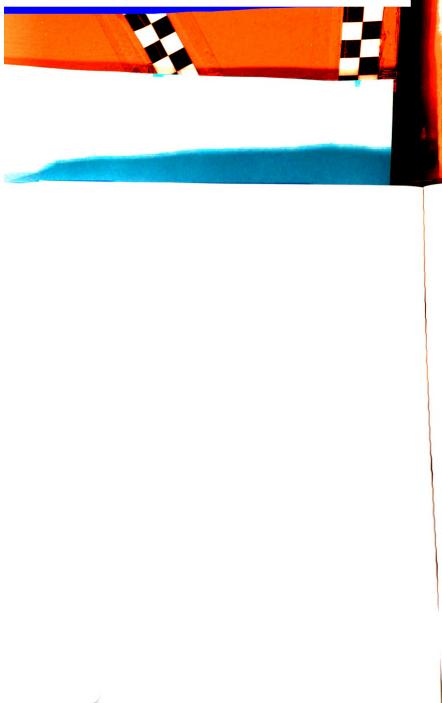
$$Q(\alpha_F) \ge -\int_{x_1}^a u''(z) z_x [T_F(x, \alpha_F) - T_G(x, \alpha_F)] dx.$$

$$(4.22)$$

Consider the term $[T_F(x,\alpha_F)-T_G(x,\alpha_F)]$ for $x \in [x_2,a)$.

$$\begin{split} T_{F}(x,\alpha_{F}) - T_{G}(x,\alpha_{F}) \\ &= z_{\alpha}[F(x) - G(x)] - \int_{x_{1}}^{x} z_{\alpha}[F(t) - G(t)] dt \\ &= z_{\alpha} \left[1 - \frac{1}{H_{1}(x)}\right] F(x) - \int_{x_{1}}^{x} z_{\alpha} \left[1 - \frac{1}{H_{1}(t)}\right] F(t) dt \quad \left(\text{Let } \frac{1}{H_{1}(x)} = h_{1}(x)\right) \\ &= z_{\alpha} \left[1 - h_{1}(x)\right] F(x) - \left[1 - h_{1}(x)\right] \int_{x_{1}}^{x} z_{\alpha} F(t) dt - \int_{x_{1}}^{x} h_{1}'(t) \int_{x_{1}}^{x} z_{\alpha k} F(k) dk dt \right]. \end{split}$$

The last equality of the above equation can be obtained by integration by parts. Since,



according to the condition (e) in Definition 4.2, $H_1(x)$ is non-decreasing and $H_1(x) \le 1$ for all $x \in [x_2, m_1)$, $h_1(x)$ is non-increasing and $h_1(x) \ge 1$. Therefore,

$$T_{F}(x,\alpha_{F}) - T_{G}(x,\alpha_{F}) \ge z_{\alpha} [1 - h_{1}(x)] F(x) - [1 - h_{1}(x)] \int_{z_{\alpha}}^{x} z_{\alpha r} F(t) dt$$

$$\ge [1 - h_{1}(x)] T_{F}(x,\alpha_{F}).$$

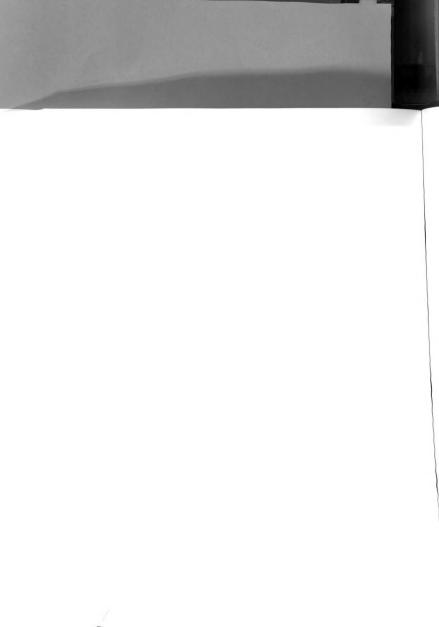
Thus (4.22) becomes

$$Q(\alpha_F) \ge -\int_{x_1}^{\alpha} u''(z) z_x [1 - h_1(x)] T_F(x, \alpha_F) dx \ge -\left[1 - h_1(x^c)\right] \int_{x_2}^{\alpha} u''(z) z_x T_F(x, \alpha_F) dx$$

Since $h_1(x)$ is non-increasing, $h_1(x^c) \ge 1$ and $\int_{z_1}^u u^u(z) z_x T_F(x, \alpha_F) dx \ge 0$ from Lemma 4.2, $Q(\alpha_F) \ge 0$.

Case (ii): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \le x^* \le m_2$ (see Figures 4.4 and 4.5). Before proving this case, we need to know the sign of $z_\alpha(x^*,\alpha_F) \int_{t_1}^{t_1} \left[F(x) - G(x) \right] dx$. Let x^0 be the value of x satisfying $z_\alpha(x,\alpha_F) = 0$ and m^0 be the point satisfying $\int_{t_1}^{m^0} \left[F(x) - G(x) \right] dx = 0$. From the definition of RSIR_K $\int_{t_1}^{t_1} \left[F(t) - G(t) \right] dt$ changes sign from negative to positive at the point m^0 in the interval (m_1, m_2) . We define that $\int_{t_1}^{t_2} F(t) dt = \hat{F}(x)$ and $\int_{t_1}^{t_2} G(t) dt = \hat{G}(x)$. Let x^* denote the point where $\left[T_F(x,\alpha_F) - T_G(x,\alpha_F) \right]$ changes its sign when z_α has a sign change for each interval, that is, for x^*

$$z_{\alpha}[F(x) - G(x)] = \int_{r}^{x} z_{\alpha t}[F(t) - G(t)]dt. \tag{4.23}$$





We consider the following two sub-cases where $S(x, \alpha) = z_{\alpha}[F(x) - G(x)]$ and $I(x, \alpha) = \int_{0}^{x} z_{\alpha t}[F(t) - G(t)]dt.$

Case (ii-a): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m_1 \le x^* \le m^0$ (see Figures 4.4 and 4.7).

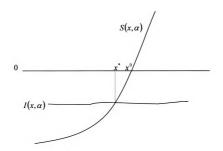


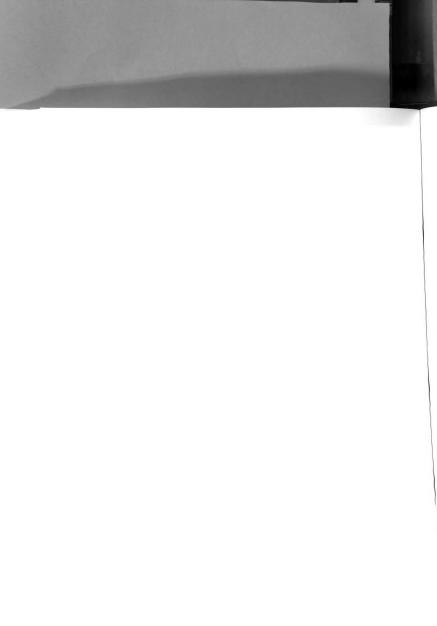
Figure 4.7. $m_1 \le x^* \le m^0$.

In Figure 4.7 $z_{\alpha}(x^{0}, \alpha_{F}) = 0$ and $z_{\alpha}(x^{\bullet}, \alpha_{F}) < 0$ since $[F(x^{\bullet}) - G(x^{\bullet})]$ is positive from Definition 4.2 and $S(x^{\bullet}, \alpha)$ is negative. From Lemma 4.3,

$$\int_{z_i}^{z'} \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx$$

$$= \int_{z_i}^{z'} z_\alpha \left[F(x) - G(x) \right] dx - \int_{z_i}^{z'} \int_{z_i}^{x} z_{\alpha t} \left[F(t) - G(t) \right] dt dx$$

$$= z_\alpha(x^*, \alpha_F) \left[\hat{F}(x^*) - \hat{G}(x^*) \right] - \int_{z_i}^{z'} z_{\alpha t} \left[\hat{F}(x) - \hat{G}(x) \right] dx$$





$$-\int_{x_1}^{x_1^*} \int_{x_1}^{x} z_{ct} [F(t) - G(t)] dt dx.$$
 (4.24)

The RHS of (4.24) is non-negative since $z_{\sigma}(x^{\star}, \alpha_F) < 0$, $\left[\hat{F}(x^{\star}) - \hat{G}(x^{\star})\right]$ is non-positive and the condition (b) in Definition 4.2 is satisfied. Therefore,

$$\int_{x_{-}}^{x^{*}} \left[T_{F}(x,\alpha_{F}) - T_{G}(x,\alpha_{F}) \right] dx \geq 0.$$

This is sufficient for $Q(\alpha_F) \ge 0$.

Case (ii-b): $k_1 = a$, $k_2 = x^*$ and $k_3 = b$ when $m^0 \le x^* \le m_2$ (see Figures 4.5 and 4.8).

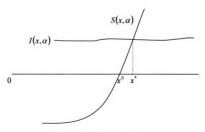
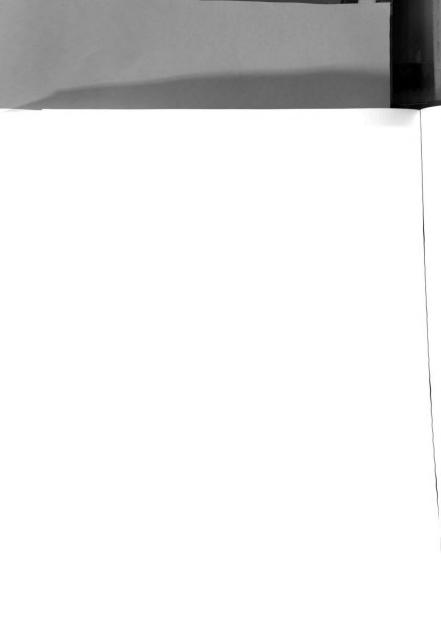


Figure 4.8. $m^0 \le x^* \le m_2$.

In Figure 4.8 $z_{a}(x^{0},\alpha_{F})=0$ and $z_{a}(x^{*},\alpha_{F})>0$ since $\left[F(x^{*})-G(x^{*})\right]$ is positive from Definition 4.2 and $S(x^{*},\alpha)$ in positive. The RHS of (4.24) is non-negative since $z_{a}(x^{*},\alpha_{F})>0$ and $\left[\hat{F}(x^{*})-\hat{G}(x^{*})\right]$ is non-negative and the condition (b) in Definition 4.2 is satisfied. Therefore,





$$\int_{x_r}^{x^*} \left[T_F(x, \alpha_F) - T_G(x, \alpha_F) \right] dx \ge 0.$$

This is sufficient for $Q(\alpha_F) \ge 0$.

Case (iii): $k_1 = a$, $k_2 = b$ and $k_3 = x^*$ when $m_2 \le x^* \le x_4$ (see Figure 4.6).

From the definition of α_F

$$\int_{x_1}^{x_4} u'(z) z_{\alpha}(x, \alpha_F) dF(x) = 0.$$

Integrating by parts gives

$$\begin{split} &\int_{z_1}^{z_4} u'(z) \, z_\alpha dF(x) = u'(z(x_4, \alpha_F)) T_F(x_4, \alpha_F) - \int_{z_1}^{z_4} u'(z) \, z_x T_F(x, \alpha_F) dx \\ &\leq u'(z(x_4, \alpha_F)) T_F(x_4, \alpha_F) - u'(z(x_F, \alpha_F)) z_x \left[\overset{z_4}{\sim} T_F(x, \alpha_F) dx \right]. \end{split}$$

The above inequality holds since $-u''(z)z_x$ is non-negative and decreasing in x, and $T_F(x,\alpha_F)$ changes sign from negative to positive at the point x^c . Therefore,

$$\int_{x_1}^{x_4} T_F(x,\alpha_F) dx \ge 0,$$

but this implies $\int_s^{x_4} T_F(x, \alpha_F) dx \ge 0$ for all $s \in [x_2, x_4]$. From Lemma 4.3 where $k_2 = b$,

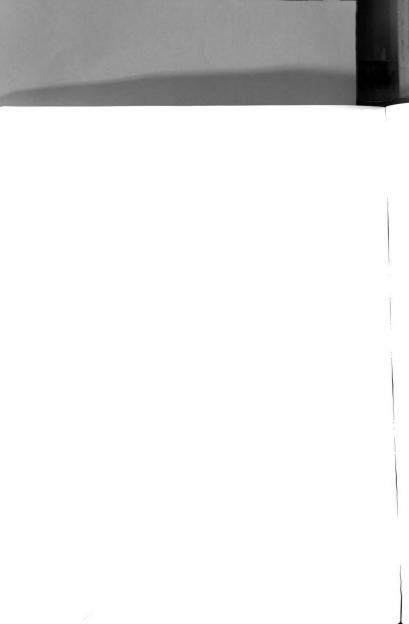
$$\int_{b}^{x_{k}} \left[T_{F}(x, \alpha_{F}) - T_{G}(x, \alpha_{F}) \right] dx$$

$$= \int_{a}^{x_{k}} \left\{ z_{\alpha} [F(x) - G(x)] - \int_{a}^{x_{k}} z_{\alpha \alpha} [F(t) - G(t)] dt \right\} dx. \tag{4.25}$$

Consider the second term with minus sign in the bracket in the RHS of (4.25).

$$-\int_{x_{1}}^{x} z_{at} [F(t) - G(t)] dt = -\int_{x_{1}}^{b} z_{at} [F(t) - G(t)] dt - \int_{b}^{x} z_{at} [F(t) - G(t)] dt.$$
 (4.26)

The first term with minus sign in the RHS of (4.26) is non-positive since b is located at



the right side of m^0 , where m^0 is the point satisfying $\int_{x_1}^{m^0} \big[F(x) - G(x)\big] dx = 0$.

Therefore.

$$-\int_{s_{t}}^{s} z_{\alpha t} [F(t) - G(t)] dt \le -\int_{s_{t}}^{s} z_{\alpha t} \left[1 - \frac{1}{H_{2}(t)} \right] F(t) dt \qquad \left[Let \frac{1}{H_{2}(t)} = h_{2}(t) \right]$$

$$\le -\left[1 - h_{2}(x) \right] \int_{s_{t}}^{s} z_{\alpha t} F(t) dt - \int_{s_{t}}^{s} h_{2}'(t) \int_{s_{t}}^{s} z_{\alpha t} F(k) dk dt$$

$$\le -\left[1 - h_{2}(x) \right] \int_{s_{t}}^{s} z_{\alpha t} F(t) dt . \qquad (4.27)$$

The last inequality in (4.27) can be obtained by integration by parts. Since, according to the condition (f) in Definition 4.2, $H_2(x)$ is non-increasing and $H_2(x) \le 1$ for all $x \in (m_2, x_3]$, $h_2(x)$ is non-decreasing and $h_2(x) \ge 1$ for all $x \in (m_2, x_3]$.

Therefore, adding and subtracting $-\left[1-h_2(x)\right]\int_{x_1}^b z_{at}F(t)\,dt$ in the RHS of (4.27) gives

$$-\int_{z_{t}}^{x} z_{aa} [F(t) - G(t)] dt \le -[1 - h_{2}(x)] \int_{z_{t}}^{x} z_{aa} F(t) dt + [1 - h_{2}(x)] \int_{z_{t}}^{b} z_{aa} F(t) dt$$

$$\le -[1 - h_{2}(x)] \int_{z_{t}}^{x} z_{aa} F(t) dt. \qquad (4.28)$$

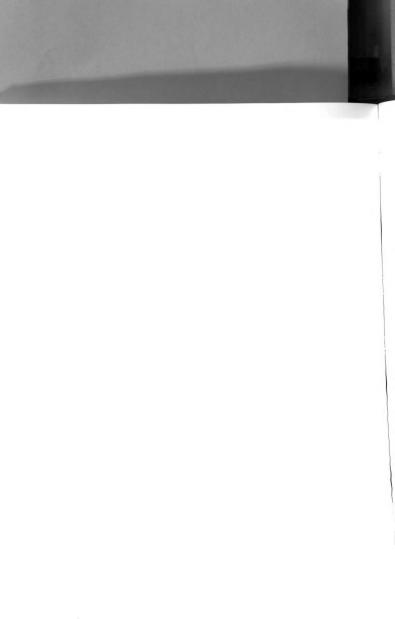
Since $[1-h_2(x)]$ is non-positive and $\int_{z_i}^b z_{ca} F(t) dt$ is non-negative, the last inequality in (4.28) holds. Thus,

$$T_F(x, \alpha_F) - T_G(x, \alpha_F)$$

$$\leq z_o [1 - h_2(x)] F(x) - [1 - h_2(x)] \int_x^x z_{ot} F(t) dt = [1 - h_2(x)] T_F(x, \alpha_F). \quad (4.29)$$

Therefore (4.25) becomes

$$\int_{b}^{x_{4}} \left[T_{F}(x,\alpha_{F}) - T_{G}(x,\alpha_{F}) \right] dx \leq \int_{b}^{x_{4}} \left[1 - h_{2}(x) \right] T_{F}(x,\alpha_{F}) dx.$$





$$\leq \left[1 - h_2(x^c)\right] \int_b^{x_t} T_F(x, \alpha_F) dx \leq 0.$$
 (4.30)

The second inequality in (4.30) holds since $h_2(x)$ is non-decreasing and $T_F(x,\alpha_F)$ changes sign from negative to positive at x^c . Thus, the left-hand side of (4.30) is non-positive because $\left[1-h_2(x^c)\right]$ is non-positive and $\int_b^{x_t} T_F(x,\alpha_F) dx$ is non-negative. Since

$$\int_{x_1}^{x_4} \left[T_F(x,\alpha_F) - T_G(x,\alpha_F) \right] dx \ge 0,$$

this implies

$$\int_{x_{-}}^{k_{2}(=b)} \left[T_{F}(x,\alpha_{F}) - T_{G}(x,\alpha_{F}) \right] dx \ge 0,$$

and hence from Lemma 4.3, $Q(\alpha_F) \ge 0$.

Q.E.D.

Since the SIR $_{\rm K}$ ranking implies the RSIR $_{\rm K}$ ranking, the comparative statics result in Theorem 4.3 includes a larger set of K-L increases in risk but shows somewhat stronger restriction on the structure of the decision model than that in Theorem 4.1. When the structural restriction $z_{xx}=0$ is added, the relationship between the results in Black and Bulkley's analysis and in Theorem 4.3 shows a trade-off between the restrictions on the set of decision makers and the set of changes in distribution.





Chapter 5

THE COMPARATIVE STATICS OF CHANGES IN DISTRIBUTION OF THE RANDOM VARIABLES

This chapter deals with the problem of determining the conditions under which a change in distribution of the random parameter decreases the optimal value of a decision variable for the set of risk averse individuals with a positive third derivative of their utility function (u'' > 0), which was done before by Gollier (1995) for all risk averse decision makers. Unlike the traditional approach that restricts separately the changes in PDFs or CDFs and the structure of the given decision model, he restricts these two components jointly with a single restriction to obtain a general comparative statement. Following the technique used in Gollier (1995), we obtain the least constraining conditions under which all risk averse individuals with u''' > 0 react in the same direction when facing a given shift in distribution. We also define a different subset of SSD that satisfies our sufficient condition and includes a R-S increase in risk as a special case. In section 5.1, we provide the least constraining conditions on the change in distribution of random parameter for signing its effect on utility function with $u' \ge 0$, $u'' \le 0$ and u''' > 0. Section 5.2 contains some concluding remarks specific to this study.





5.1 Comparative Statics of Change in Distribution of the Random Parameter

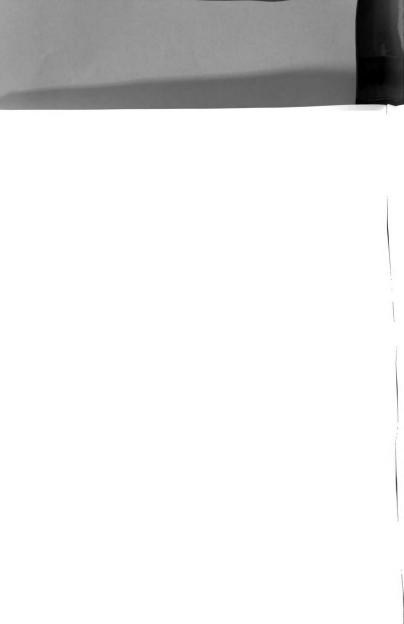
Since the seminal papers by Rothschild and Stiglitz (R-S) (1970, 1972), many researchers have investigated the comparative statics effect of change in distribution of a random variable such as subsets of FSD shifts, SSD shifts or R-S increases in risk. When they use the stochastic dominance orders (SD) to get the desired comparative statics results, they impose some restrictions on a pair of PDFs or CDFs. These examples are included in Meyer and Ormiston (1985), Black and Bulkley (1989), Landsberger and Meilijson (1990), Dionne, Eeckhoudt, and Gollier (1993), Eeckhoudt and Gollier (1995), Kim (1998) and others. As indicated in chapter 2, as an alternative to this traditional approach, Gollier (1995) obtains a quite simple condition for risk averse decision makers by restricting the changes in PDFs or CDFs and the structure of the decision model jointly. His condition entails all existing sufficient conditions as particular cases, including a SIR and a RSIR, imposed for all concave utility functions. We follow his approach to obtain a general comparative static statement for the case where u''' > 0 is also assumed.

Before presenting a sufficient condition in Theorem 5.2, we introduce the following definition.

Definition 5.1. G(x) is said to be riskier than F(x) if there exists a scalar $\lambda \ge 1$ such that

(a)
$$\int_a^b [G(x) - \lambda F(x)] dx = 0$$

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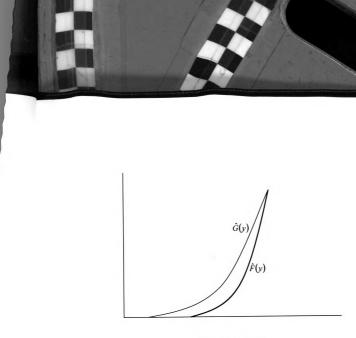


Figure 5.1. G MPS F.

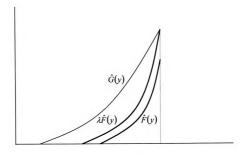
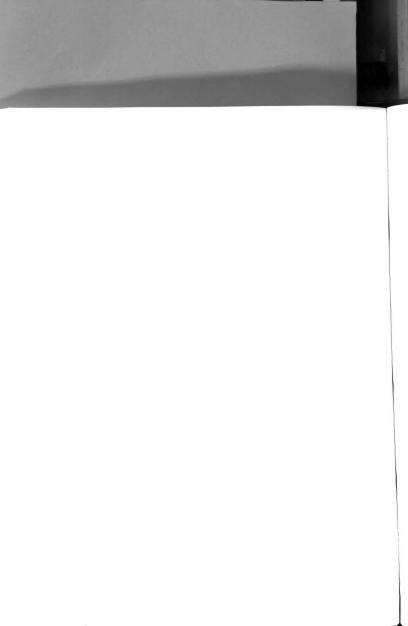


Figure 5.2. F SSD G.





(b)
$$\int_{a}^{y} [G(x) - \lambda F(x)] dx \ge 0$$
, for all $y \in [a, b]$.

This definition introduces another subset of SSD that includes a R-S increase in risk as a special case. Note that λ adjusts a probability so that in the interval [a,b] the areas under two distribution functions, F and G, are equal. This implies that λ is a scaling factor that makes $\lambda \int_a^b F(x) dx$ and $\int_a^b G(x) dx$ equal. From condition (b) in Definition 5.1,

$$\int_{0}^{\infty} [G(x) - F(x) + F(x) - \lambda F(x)] dx \ge 0,$$

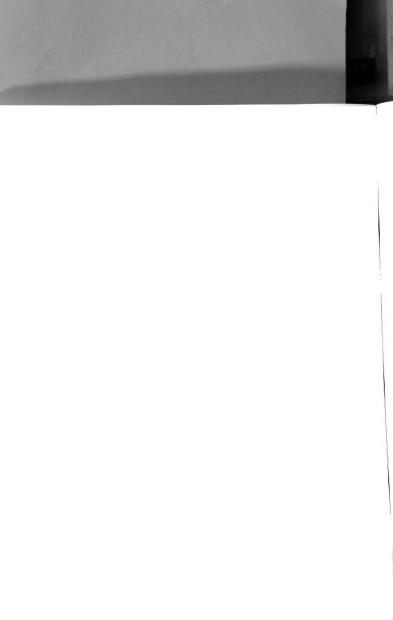
which is equivalent to

$$\int_{a}^{y} [G(x) - F(x)] dx \ge (\lambda - 1) \int_{a}^{y} F(x) dx \ge 0.$$

Since $\lambda \ge 1$, SSD condition, $\int_a^y [G(x) - F(x)] dx \ge 0$, is always satisfied. That is, this condition is stronger than SSD condition but not MPS condition. If $\lambda = 1$, Definition 5.1 is exactly same as R-S increases in risk presented in chapter 2. If $\lambda > 1$, Definition 5.1 defines a different subset of SSD. To explain more clearly, we define $\int_a^y F(x) dx = \hat{F}(y)$, $\int_a^y G(x) dx = \hat{G}(y)$ and draw Figures 5.1 and 5.2. Figure 5.1 ($\lambda = 1$) shows the definition of R-S increases in risk from F(x) to G(x). Figure 5.2 does not satisfy R-S increases in risk from F(x) to G(x) but satisfies Definition 5.1.

Remember that a decision maker maximizes the expected utility of his final payoff. The economic decision model in (2.1) can be rewritten as

$$\alpha(u, F, z) \in arg \max_{\alpha} H(\alpha; u, F, z) = E_F[u(z(x, \alpha))]$$





$$= \int_a^b u(z(x,\alpha)) dF(x), \tag{5.1}$$

where the subscript F for the expectation symbol E indicates that random variable x is distributed according to CDF F whose support is in [a,b]. The first-order condition is

$$H'(\alpha(u, F, z); u, F, z) = \int_{0}^{b} u'(z(x, \alpha(u, F, z))) z_{\alpha}(x, \alpha(u, F, z)) dF(x) = 0$$
 (5.2)

It is assumed that $z_{\alpha\alpha}$ is non-positive so that condition (5.2) is necessary and sufficient condition for maximization problem (5.1), since it makes H concave in α . Note that $z_{\alpha}(x,\alpha(u,F,z))$ must alternate in sign in order to satisfy condition (5.2).

An integration by parts of the first-order condition yields

$$H'(\alpha; u, F, z) = u'(z(b, \alpha))T(b, \alpha; F, z) - \int_{a}^{b} u''(z)z_{x}T(x, \alpha; F, z)dx, \qquad (5.3)$$

where $T(\cdot,\cdot;F,z):[a,b]\times R\to R$ is defined as

$$T(x,\alpha;F,z) \equiv \int_{-\infty}^{\infty} z_{\alpha}(t,\alpha) dF(t). \tag{5.4}$$

According to Gollier (1995), this location-weighed probability mass function has some properties which are worth noting:

$$T(a,\alpha;F,z) = 0 (5.5)$$

$$sgn\frac{\partial T}{\partial x}(x,\alpha;F,z) = sgnz_{\alpha}(x,\alpha). \tag{5.6}$$

The sign of the derivative of T with respect to x is independent of the distribution function. Note also that T is right continuous in x. Since z_{α} can be either positive or negative, T has an ambiguous sign. However, from (5.2) and (5.3), it is apparent that $T(x,\alpha(u,F,z);F,z)$ must alternate in sign.

Multiplying (5.2) and (5.3) by any scalar λ does not change the first-order





condition. To prove $\alpha_F \ge \alpha_G$, it is sufficient to show that

$$Q(\alpha_F) = u'(z(b,\alpha))[\lambda T(b,\alpha;F,z) - T(b,\alpha;G,z)]$$

$$- \int_0^b u''(z) z_x [\lambda T(x,\alpha;F,z) - T(x,\alpha;G,z)] dx. \qquad (5.7)$$

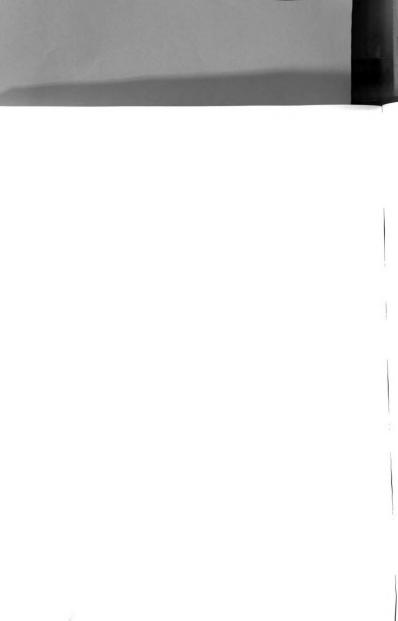
Gollier (1995) presents the necessary and sufficient condition under which all risk averse individuals react in the same direction when facing a given shift in distribution.

Theorem 5.1. Take the economic model represented by the payoff function z and the initial and final distribution functions F and G as given. All risk-averse individuals selecting α under F and z reduce their optimal level of α after the change in CDF from F to G if and only if there exists a scalar $\gamma \in R$ such that

$$T(x,\alpha;G,z) \le \gamma T(x,\alpha;F,z), \qquad \forall x \in [a,b].$$
 (5.8)

The Gollier's condition generalizes all previous restrictions on changes in risk such as 'strong increases in risk' (SIR) defined by Meyer and Ormiston (1985) and 'relatively strong increases in risk' (RSIR) introduced by Black and Bulkley (1989). When the payoff function is linear in the random variable, $z(x,\alpha) = z_0 + \alpha x$, where z_0 is an exogenous parameter, he introduces the concept of the greater central riskiness (CR) order by using his condition. He also shows that second stochastic dominance (SSD) is neither sufficient nor necessary for centrally riskier dominance.

Before proving Theorem 5.2 for risk averse individuals with a positive third derivative of their utility function, we need the following lemma.





Lemma 5.1. If $z_{ax} > 0$ and $z_{axx} \le 0$, Definition 5.1 implies $\lambda T(b,\alpha;F,z) \ge T(b,\alpha;G,z).$

Proof: $\lambda T(b,\alpha;F,z) \ge T(b,\alpha;G,z)$ can be written as

$$\lambda \int_{a}^{b} z_{\alpha}(x,\alpha) dF(x) \ge \int_{a}^{b} z_{\alpha}(x,\alpha) dG(x). \tag{5.9}$$

Integration by parts of (5.9) yields

$$z_{\alpha}(b,\alpha)(\lambda-1) - \int_{a}^{b} z_{\alpha x}(x,\alpha)(\lambda F(x) - G(x)) dx \ge 0.$$
 (5.10)

Since $z_{\alpha}(x,\alpha)$ changes sign from negative to positive, $z_{\alpha}(b,\alpha)$ is positive. Note that from Definition 5.1, $\lambda \ge 1$. Therefore, the first term in (5.10) is positive. Integrating the second term in (5.10) by parts gives

$$-z_{ax}(b,\alpha)\int_{a}^{b}(\lambda F(x)-G(x))dx+\int_{a}^{b}z_{axx}\int_{a}^{y}(\lambda F(x)-G(x))dxdy\geq0.$$
 (5.11)

Since conditions (a) and (b) in Definition 5.1 are satisfied, and z_{α} is concave in x, (5.11)

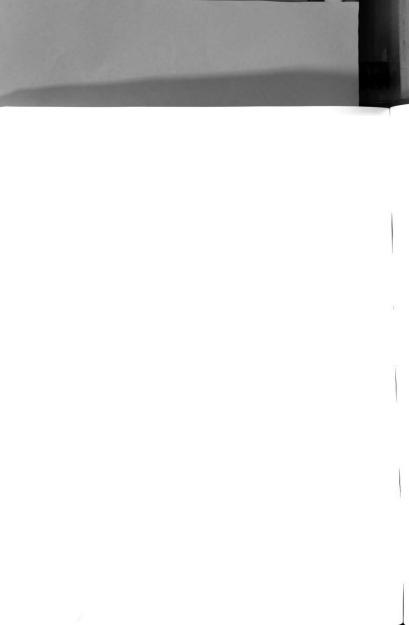
is positive. This concludes the proof. Q.E.D.

Theorem 5.2. Let F(x) and G(x) be the initial and the final distribution function. All risk-averse individuals with u''' > 0 reduce their optimal level of α after the change in CDF from F(x) to G(x) if there exists a scalar $\lambda \ge 1$ such that

$$\lambda \Psi(x,\alpha;F,z) - \Psi(x,\alpha;G,z) \ge 0$$
 with $\lambda T(b,\alpha;F,z) - T(b,\alpha;G,z) \ge 0$, (5.12)

where $\Psi(x,\alpha;F,z) = \int_a^x z_x(y,\alpha) T(y,\alpha;F,z) dy$ and

 $\Psi(x,\alpha;G,z) = \int_{-\infty}^{x} z_x(y,\alpha) T(y,\alpha;G,z) dy \text{ for all } x \in [a,b].$





Proof: First, from Lemma 5.1, the first term of the right-hand side in (5.7) is nonnegative. Second, consider the second term of the right-hand side in (5.7). Its integration by parts gives

$$-u^{*}(z(b,\alpha))\int_{0}^{b}z_{x}[\lambda T(x,\alpha;F,z)-T(x,\alpha;G,z)]dx$$

$$+\int_{0}^{b}u^{*}(z)z_{x}\int_{0}^{z}z_{x}(y,\alpha)[\lambda T(y,\alpha;F,z)-T(y,\alpha;G,z)]dydx.$$

Since u''' > 0 and (5.12) is assumed to hold, $Q(\alpha_F) \ge 0$.

Q.E.D.

Note that if $\lambda = 1$ or $\lambda > 1$, our condition in (5.12) must be true for the subsets of R-S increases in risk or the subsets of SSD shifts, respectively.

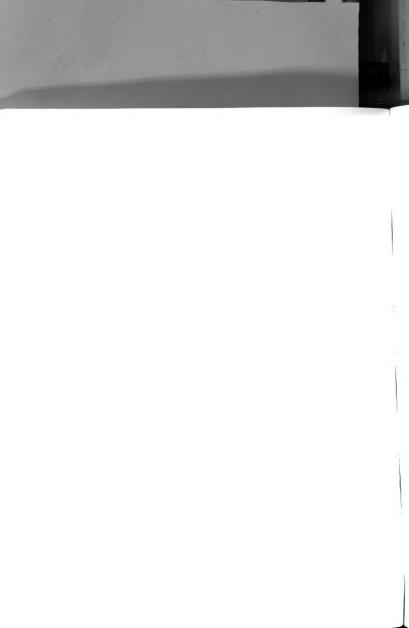
Now, consider our condition in (5.12) to compare with the Gollier's condition. It is obvious that the Gollier's condition in (5.8) implies our condition. Therefore, our condition is less restrictive than the Gollier one. A 'strong increase in risk' (SIR) and a 'simple increase in risk' (sIR) satisfy the Gollier's condition with $\gamma=1$, and a 'relatively strong increase in risk' (RSIR) also satisfies his condition, but with a γ that may differ from unity. Since SIR, sIR and RSIR are the subsets of R-S increases in risk with $\lambda=1$ in Definition 5.1, they also satisfy our condition.

We have found a sufficient condition for a risk-averse individual with u''' > 0 when facing the change in CDF from F to G. If so, what is the necessary condition for this economic problem? Integration by parts of the first-order condition in (5.3) yields

$$H'(\alpha; u, F, z) = u'(z(b, \alpha))T(b, \alpha; F, z) - u''(z(b, \alpha))\Psi(b, \alpha; F, z)$$

 $+ \int_{-\infty}^{b} u''(z)z \cdot \Psi(x, \alpha; F, z) dx = 0$ (5.13)

where $\Psi(x, \alpha; F, z)$ is defined in (5.12).



Similar to T in (5.5) and (5.6), Ψ has the following properties:

$$\Psi(a,\alpha;F,z) = 0 \tag{5.14}$$

$$sgn \frac{\partial \Psi}{\partial x}(x,\alpha;F,z) = sgn z_x(y,\alpha) T(y,\alpha;F,z). \tag{5.15}$$

Since T can be either positive or negative, Ψ has an ambiguous sign. From (5.13), if u''' = 0, $\Psi(b, \alpha; F, z)$ must be negative. But if u''' > 0, $\Psi(x, \alpha; F, z)$ must alternate in sign under risk aversion with u''' > 0.

Theorem 5.3. If all risk-averse individuals with u''' > 0 selecting α under F and z reduce their optimal level of α after the change in CDF from F(x) to G(x), then there exists a positive scalar λ such that

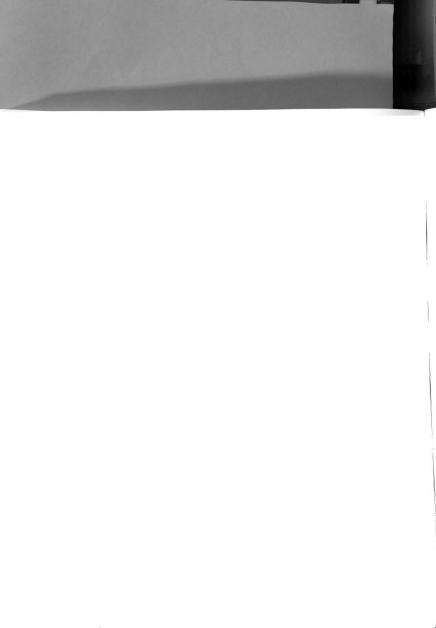
$$\lambda \Psi(x,\alpha;F,z) \ge \Psi(x,\alpha;G,z)$$
 for all $x \in [a,b]$.

Proof: Define $X_- = \{x \in [a,b] | \Psi(x,\alpha;F,z) < 0\}$ and $X_+ = \{x \in [a,b] | \Psi(x,\alpha;F,z) > 0\}$. Neither X_- nor X_+ is empty. Suppose by contradiction that for all λ , there exists $y \in [a,b]$ such that $\lambda \Psi(x,\alpha;F,z) < \Psi(x,\alpha;G,z)$. Because Ψ is right continuous in x, this is possible only if there exist non-empty subsets Ω_- and Ω_+ in X_- and X_+ respectively such that

$$\frac{\Psi(x_{-},\alpha;G,z)}{\Psi(x_{-},\alpha;F,z)} < \frac{\Psi(x_{+},\alpha;G,z)}{\Psi(x_{+},\alpha;F,z)}, \quad \forall x_{-} \in \Omega_{-}, \quad \forall x_{+} \in \Omega_{+},$$
 (5.16)

or equivalently,

$$\frac{z_{x}(x_{-},\alpha)\Psi(x_{-},\alpha;G,z)}{z_{x}(x_{-},\alpha)\Psi(x_{-},\alpha;F,z)} < \frac{z_{x}(x_{+},\alpha)\Psi(x_{+},\alpha;G,z)}{z_{x}(x_{+},\alpha)\Psi(x_{+},\alpha;F,z)}, \ \forall x_{-} \in \Omega_{-}, \ \forall x_{+} \in \Omega_{+}. (5.17)$$





Define

$$\begin{split} \omega_{-F} &= \int_{\Omega_{-}} z_{x}(x,\alpha) \Psi(x,\alpha;F,z) dx \,, \qquad \omega_{+F} &= \int_{\Omega_{-}} z_{x}(x,\alpha) \Psi(x,\alpha;F,z) dx \,, \\ \\ \omega_{-G} &= \int_{\Omega_{-}} z_{x}(x,\alpha) \Psi(x,\alpha;G,z) dx \,, \qquad \omega_{+G} &= \int_{\Omega_{-}} z_{x}(x,\alpha) \Psi(x,\alpha;G,z) dx \,. \end{split}$$

Since $z_x > 0$ and $\Psi(x, \alpha; F, z) < 0$ in Ω_- , we have $\omega_{-F} < 0$. Similarly, $\omega_{*F} > 0$. After some algebraic manipulations, (5.17) implies that

$$\frac{\omega_{-G}}{\omega_{-E}} < \frac{\omega_{*G}}{\omega_{*E}}. \tag{5.18}$$

Consider an individual with utility v, where $v(z(x,\alpha)) = v_1(z(x,\alpha)) - kz(x,\alpha)$ and k is a positive scalar, defined as

$$v'''(z(x,\alpha)) = \begin{cases} 1 & \text{if } x \in \Omega_{-} \\ \eta & \text{if } x \in \Omega_{+} \\ 0 & \text{otherwise,} \end{cases}$$

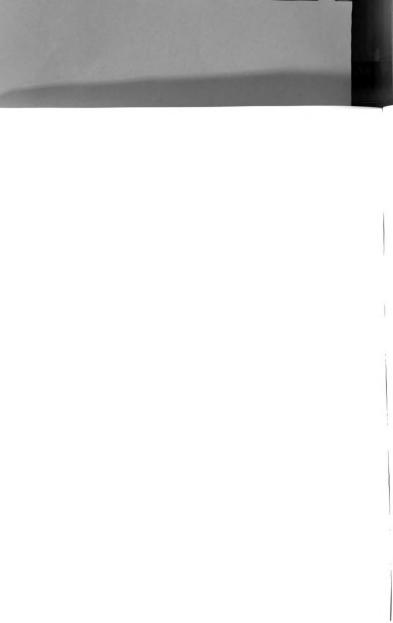
where η is some positive scalar. Function v is fully specified by assuming that $v_1'(z(b,\alpha)) = k$ and $v_1''(z(b,\alpha)) = 0$. This is possible since $z(x,\alpha) \le z(b,\alpha)$ for all $x \in [a,b]$. In order to force the optimal solution for individual v to be α , we select $\eta = -\omega_{-F}/\omega_{+F} > 0$. Therefore, we can verify that

$$H'(\alpha; v, F, z) = \int_a^b v'''(z(x, \alpha)) z_x(x, \alpha) \Psi(x, \alpha; F, z) dx$$

= $\omega_{-F} + \eta \omega_{-F} = 0$.

We now verify that an individual with v increases the optimal value of α when facing the change in CDF:

$$H'(\alpha; \nu, G, z) = \omega_{-G} + \eta \omega_{+G} = \omega_{-F} \left(\frac{\omega_{-G}}{\omega_{-F}} - \frac{\omega_{+G}}{\omega_{+F}} \right) > 0.$$

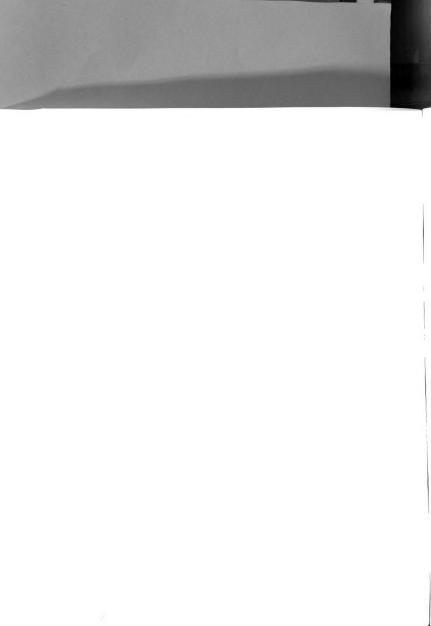


The last inequality is the consequence of ω_{-F} < 0 and inequality in (5.18). Therefore, contradiction is established. Q.E.D.

5.2 Concluding Remarks

All the comparative statics results obtained in this dissertation are associated with the set of risk averse decision makers with a non-negative third derivative of their utility function $(u''' \ge 0)$. Using the simple form of the general one-argument decision model, general comparative static statements are developed for the subsets of R-S increases in risk presented in chapter 3 and for the subsets of K-L increases in risk given in chapter 4. In this comparative analysis, we use three approaches found in the published literature, which specify restrictions between the initial and the final CDFs or PDFs: the CDF difference approach, the ratio approach and the deterministic transformation approach. As an alternative to these traditional approaches, we follow the Gollier's technique to obtain a general comparative static statement presented in section 5.1.

For the subsets of R-S increases in risk or K-L increases in risk, each definition specifies a particular type of change from a CDF F(x) to another G(x) or, correspondingly, from a PDF f(x) to another g(x). Therefore there exist basic relationships among subsets. For the subsets of R-S increases in risk, we have (i) G L-SIR $F \Rightarrow G$ L-RSIR $F \Rightarrow G$ L-RWIR F, (ii) G ESIR $F \Rightarrow G$ L-ESIR $F \Rightarrow G$ MSIR F, and (iii) G OSIR $F \Rightarrow G$ ORSIR F. Another relationship between the subsets of K-L increases in risk shows that G SIR_K $F \Rightarrow G$ RSIR_K F.





When we assume that the payoff function is linear in the random variable x, we discuss the sufficient conditions obtained in this dissertation. Assuming that $\alpha_F = 1$ and $z_0 = 0$ for notational convenience, we consider our sufficient condition in (5.12) which can be rewritten as

$$\lambda \int_{a}^{x} \int_{a}^{t} y dF(y) dt \ge \int_{a}^{x} \int_{a}^{t} y dG(y) dt.$$
 (5.19)

Remember that this condition is less restrictive than the Gollier condition in Theorem 5.1. Integration by parts of (5.19) yields

$$\lambda \int_{a}^{x} tF(t)dt - \int_{a}^{x} tG(t)dt + \int_{a}^{x} \left[\hat{G}(t) - \lambda \hat{F}(t)\right]dt \ge 0.$$
 (5.20)

The L-SIR, the L-RWIR and the MSIR orders presented in chapter 3 satisfy Definition 5.1 with $\lambda = 1$. Therefore (5.20) becomes

$$\int_{a}^{x} tF(t)dt - \int_{a}^{x} tG(t)dt \ge 0.$$

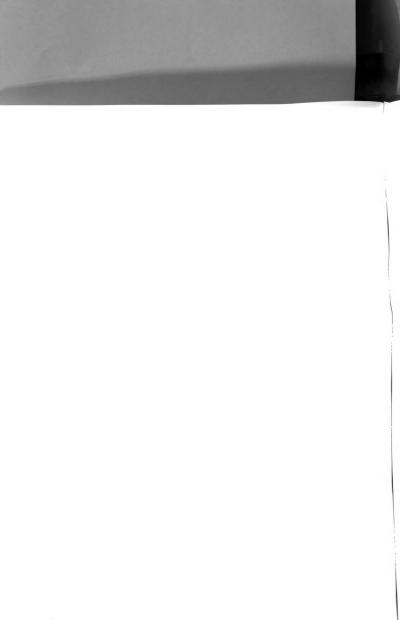
$$(5.21)$$

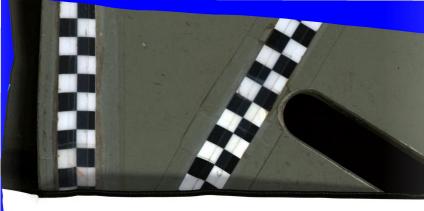
The sufficient condition for L-SIR shifts or L-RWIR shifts satisfies (5.21) that implies our condition in (5.12). Note that, after some algebraic manipulation of (5.12), if (5.21) is satisfied, the MSIR condition also implies our condition. As indicated in chapter 3, the Gollier condition is violated for the MSIR order.

Definition 5.1 can be modified for the subsets of TSD that include the subsets of K-L increases in risk as special cases.

Definition 5.2. G(x) is said to be greater riskier than F(x) if there exists $\lambda \ge 1$ such that

(a)
$$\int_a^b [G(x) - \lambda F(x)] dx = 0$$





(b)
$$\int_a^y \left[\hat{G}(x) - \lambda \hat{F}(x) \right] dx \ge 0$$
, for all $y \in [a, b]$.

This definition introduces another subset of TSD with a scaling factor λ where λ makes the areas under F and G equal. From condition (b) in Definition 5.2,

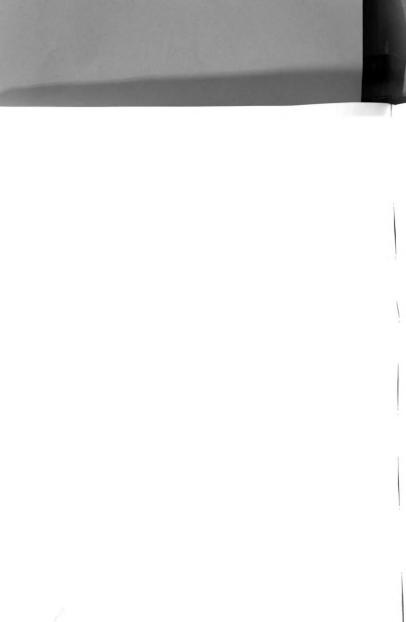
$$\int_{0}^{x} \left[\hat{G}(x) - \hat{F}(x) + \hat{F}(x) - \lambda \hat{F}(x) \right] dx \ge 0,$$

which is equivalent to

$$\int_a^y \left[\hat{G}(x) - \hat{F}(x) \right] dx \ge (\lambda - 1) \int_a^y \hat{F}(x) dx \ge 0.$$

Since $\lambda \ge 1$, TSD condition, $\int_{a}^{r} [\hat{G}(x) - \hat{F}(x)] dx \ge 0$, is always satisfied. That is, this condition is stronger than TSD condition but not a K-L increase in risk. If $\lambda = 1$, Definition 5.2 is exactly same as K-L increases in risk presented in chapter 2. If $\lambda > 1$, Definition 5.2 defines a different subset of TSD. Thus, with a linearity of the payoff function, Lemma 5.1 can also be applied for Definition 5.2.

The SIR_K , the $RSIR_K$ and the $sIR_K(r)$ orders presented in chapter 4 satisfy Definition 5.2 with $\lambda=1$. The sufficient condition for SIR_K shifts or $RSIR_K$ shifts satisfies (5.21) that implies our condition in (5.12). Integration by parts of (5.21) and a TSD condition give a sufficient condition for the $sIR_K(r)$ order that also implies our condition.





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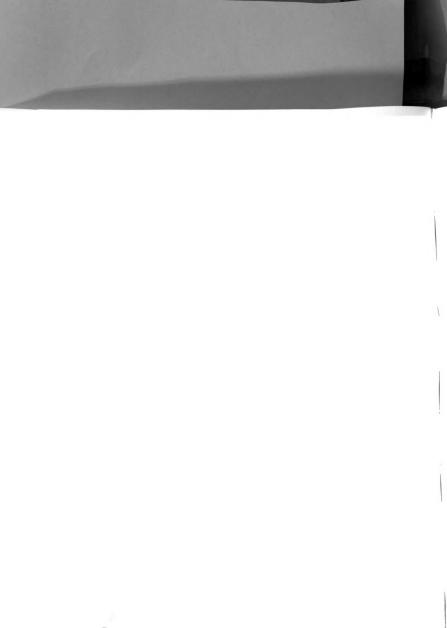
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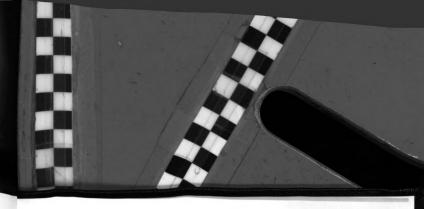
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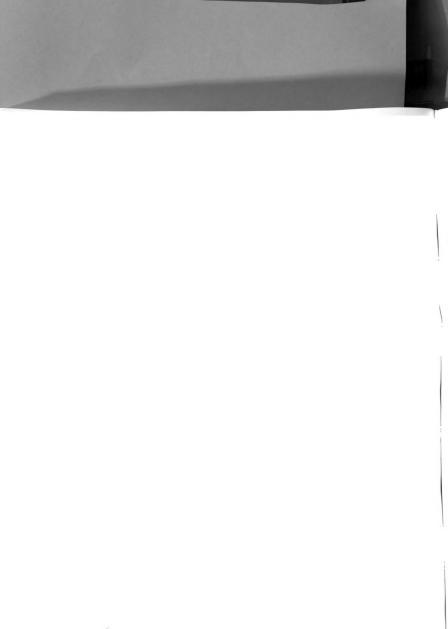
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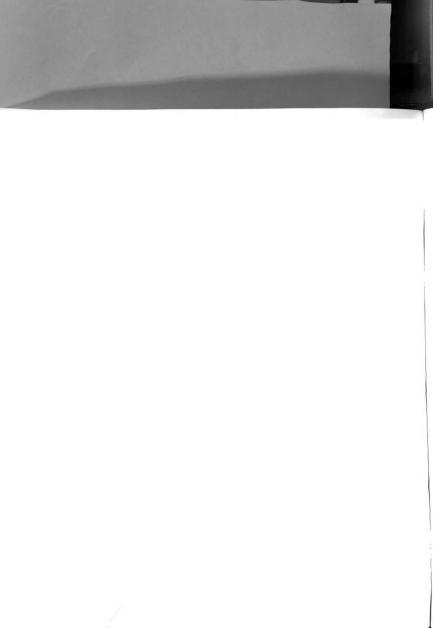
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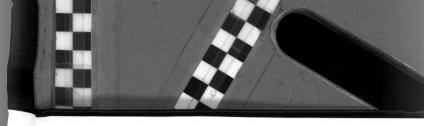
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