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Tat Ming Sze

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HOW WELL DO HONG KONG GRADE 5 STUDENTS WITH DIFFERENT SOCIAL BACKGROUNDS SOLVE ROUTINE AND NON-ROUTINE MATHEMATICS WORD PROBLEMS

By

Tat Ming Sze

A DISSERTATION

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ABSTRACT

HOW WELL DO HONG KONG GRADE 5 STUDENTS
WITH DIFFERENT SOCIAL BACKGROUNDS SOLVE ROUTINE
AND NON-ROUTINE MATHEMATICS WORD PROBLEMS

By

Tat Ming Sze

The goal of this study is to explore and investigate whether Hong Kong elementary grade 5 students develop mathematical problem solving competence at school. Problem solving competence has become more and more important in developed countries such as Hong Kong. The job market has fewer and fewer routine job openings in the 21st Century. To survive, students must develop problem-solving competence in school and apply these skills in the workplace. But the existing Hong Kong curriculum and examination system focuses heavily on solving practiced problems. Students may not develop problem solving competence in schools.

In this study, grade 5 students (n = 123) worked a 10-item quiz comprised of routine and non-routine problems; a subset of 27 students participated in the second phase of this study in an interview setting. In the interviews, students participated in problem sorting activities and solved one to three other non-routine problems.

The results indicate that students from different social backgrounds have difficulty solving mathematical, non-routine problems in general. Their difficulties seem to be related to conceptual mistakes relative to two specific mathematical concepts—area and proportion. Comparing two groups of students who came from different social backgrounds, students from middle class families outperformed their counterparts on solving non-routine

problems. A more interesting result is the fact that even very successful students (grades and exams) struggled with non-routine problems. And, some students who were not strong performers in school showed surprising strength on some problems.

Dedicated to Dr. William M. Fitzgerald

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Second, I would like to thank and acknowledge my two high school teachers, Dr. Yat-shing Shaw and Mr. Kam-yung Choi. Dr. Shaw was my Chinese teacher and was the first person who awakened within me a hunger for philosophy. Dr. Shaw's philosophical stance and attitude towards truth had a great influence on me. He also encouraged me to read and understand my own culture. Mr. Choi was my biology teacher. I enjoyed his lessons and he fostered my interest in man as a physical being. His attitudes on teaching and ways of teaching impacted me on reflecting how to teach. Combined, these two teachers laid a path for me that drew me to psychology, which is a field combining philosophy and biology.

In my college days, two teachers had a great impact on me in education and psychology. Dr. Chun-hsing Chang was very enthusiastic about education and had an altruistic personality. His attitudes towards education and society impacted me greatly. He was one of my role models as an educator. Dr. Sieh-Hwa Lin was another teacher who changed my life and pushed me to think about the importance of providing an exciting learning environment and opportunities for students to explore on their own. He encouraged me to explore and to learn about computers. I will always remember what he once told me in his computer laboratory. He said that the equipment he brought to the department was for students to explore. He did not want students to worrying about breaking the machines. Dr. Lin felt that schools are the place for students to explore and learn. He believed it was his duty to provide an exciting environment in which students could explore. Dr. Lin graciously offered his services as a consultant on the statistical analysis in this study.

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TABLE OF CONTENTS

LIST OF TABLES xiii
LIST OF FIGURES xv
INTRODUCTION 1
Chapter 1
MOTIVATION OF THE RESEARCH 3
Psychological and Educational Concern 3
Teaching and Learning Concern 3
Social and Economic Concern 4
Personal Experience 5
Research Questions 6
Chapter 2
CONTEXTUAL BACKGROUND AND SCHOOLING In HONG KONG 7
Geographical Information 7
Historical Context 7
Schooling in Hong Kong 9
General Education and Student Population 9
Primary Schools 11
Secondary Schools 13
Junior Secondary Schools 13
Senior Secondary Schools 13
Pre-University Sectors 14
Chapter 3
LITERATURE REVIEW 16
Non-Routine Mathematics Word Problems 16
Philosophical Perspective 17
Psychological Perspective 18
Visibility of Goal State 19
Problem Space 20
Ill-defined Problems and Non-Routine Problems 21
Educational Perspective 22
Understanding 25
Perceptual Understanding 27
Mental Models 28
Johnson-Laird's Mental Models 29
Mayer and His Colleagues' on Conceptual Models 31
Mental Models Suggested by Hegarty, Mayer and Monk 32
Schema 33
Problem-Schema 34

Pragmatic Schema....35
Principle-based Understanding....37
Counting Principles....37
Subtraction Principles....39
Multiplicative Reasoning....43
Proportion....44
Speed....45
Area and Perimeter....47
Students from Low SES Families....49
Chapter 4
METHOD 53
Subjects....53
Tools....54
Problem Development....54
Quiz Section....55
Interview Section....56
Design and Procedures....57
Quiz Section....57
Interview Section....58
Scoring....60
Quiz Section....60
Scoring Scheme....60
Inter-rater Reliability....64
First Round Inter-rater Reliability....64
Second Round Inter-rater Reliability....66
Observations with Disagreement....68
Instance 1....69
Instance 2....70
Chapter 5
RESULT 71
Quiz Result....71
Table Description....75
A Special Example....93
Example....93
Interview Result....95
Card-Sorting....95
Students' Performance on the Interview Problems....96
The Train Problem....96
Classroom Board Problem....102
Chapter 6
NATURE OF CONCEPTUAL MISTAKES AND THEIR IMPACT ON PERFORMANCE 106
Two Focus Problems....107

The "Extended Square" Problem....108
Summary....118
The "Ratio" Problems....124
Summary....130
General Summary....133
Other Non-Routine Problems....134
Chapter 7
STUDENTS' CONCEPTUAL STRENGTHS AS EVIDENCED IN NOVEL SOLUTIONS....136
Students' Typical Solutions on the Two Ratio Problems....137
Students' Unexpected Solutions on the Two Ratio Problems....140
Intuitive Function Strategy....140
Summary and Conclusion....143
A Special Case—Bobby....145
The Routine Problems in the Quiz....146
The "Multiplication" and "Unknown Digit" Problem....146
The Non-routine Problems in the Quiz....147
The "Lake" Problem....146
The "Balance" Problem....149
The "Glass House" Problem....151
The "Coins" Problem....156
Non-routine Problems in the Interview....157
The "Train" Problem....157
The "Classroom Board" Problem....163
Summary and Implications....171
Chapter 8
CONCLUSION....174
APPENDICES....187
APPENDIX A....188
APPENDIX B....208
APPENDIX C....223
BIBLIOGRAPHY....225

LIST OF TABLES

Table 1.1 Comparison of Projected Manpower Supply & Demand in 1996 & 2001 5
Table 2.1 Hong Kong Student Population in Different Educational Institutions in 1999 9
Table 4.1 Students Distribution in Quiz Section 54
Table 4.2 Students Distribution in Interview Section 54
Table 4.3 Problem Distribution on Two Forms 56
Table 4.4 Problems for the Interview 57
Table 4.5 First Round Inter-rater Reliability in Form A 65
Table 4.6 First Round Inter-rater Reliability in Form B 65
Table 4.7 Second Round Inter-rater Reliability in Form A 67
Table 4.8 Second Round Inter-rater Reliability in Form B 68
Table 5.1 Performance from Both Schools on Form A 78
Table 5.2 Performance from Two Schools on Form B 79
Table 5.3 Students with Different Achievement Levels Distribution 80
Table 5.4 Performance from Low SES School on Form A 81
Table 5.5 Performance from Low SES School on Form B 82
Table 5.6 Performance from Middle Class School on Form A 83
Table 5.7 Performance from Middle Class School on Form B 84
Table 5.8 High Achievers Performance on Form A 86
Table 5.9 High Achievers Performance on Form B 87
Table 5.10 Mid Achievers Performance on Form A 88
Table 5.11 Mid Achievers Performance on Form B 89
Table 5.12 Low Achievers Performance on Form A 90
Table 5.13 Low Achievers Performance on Form B 91
Table 5.14 Frequencies from Students' Card-Sorting Results 95
Table 5.15: Brief Interview Performance on the Train Problem 98

Table 5.16: Brief Interview-Performance on the 4-Squares Problem 101

Table 5.17: Brief Interview Performance on the Classroom Board Problem 104

Table 6.1 Performance on the Extended Square Problem on Form A 120

Table 6.2 Performance on the Extended Square Problem on Form B 121

Table 6.3 Incorrect Responses Frequency on the Extended Square Problem 122

Table 6.4 Performance on the Photo-Robber Problem at Two Schools 131

Table 6.5 Performance on the Photo-You Problem at Two Schools 132

Table 6.6 Incorrect Responses Frequency on the Two Ratio Problems 132

Table 6.7 Incorrect Responses Frequency on the Cake Problem 135

Table 7.1 A Brief Summary of Bobby's Performance 146

Table 7.2 The First Case's Response on the "Glass House" Problem 153

Table 7.3 The Second Case's Response on the "Glass House" Problem 153

Table 8.1 Performance on Routine and Non-Routine Quiz Problems on Form A 175

Table 8.2 Performance on Routine and Non-Routine Quiz Problems on Form B 176

Table 8.3 Incorrect Responses Related to Conceptual and Computational Errors on the Two Ratio Problems 180

Table 8.4 Incorrect Responses Related to Conceptual and Computational Errors on the Extended Square Problems 181

Table 8.5 Two Schools Performance on the Non-Routine Problems 185

Table 8.6 Incorrect Responses Frequency on the Two Ratio Problems at Two Schools 186

LIST OF FIGURES

Figure 2.1 Hong Kong Schooling and Examination System	10
Figure 3.1 Tower of Hanoi Problem	19
Figure 3.2 Distance-centered Conception of Speed	46
Figure 3.3 Distance-Time Conception of Speed	47
Figure 4.1 Scoring Procedure	63
Figure 4.2 A Student's Work on the Balance Problem	69
Figure 4.3 A Student's Work on the Extended Square Problem	70
Figure 5.1 The Cake Problem on Form B	93
Figure 5.2 A Student's Response on the Cake Problem	94
Figure 5.3 The Train Problem	96
Figure 5.4 The 4-Square Problem	100
Figure 5.5 The Classroom Board Problem	102
Figure 6.1 The Extended Square Problem in Form A	108
Figure 6.2 The Extended Square Problem in Form B	107
Figure 6.3 The Square-root Confusion	109
Figure 6.4 Type A Conceptual Mistake on the Extended Square Problem	111
Figure 6.5 Type B Conceptual Mistake on the Extended Square Problem	112
Figure 6.6 Type C Conceptual Mistake on the Extended Square Problem	114
Figure 6.7 Type D Conceptual Mistake on the Extended Square Problem	115
Figure 6.8 Type E Conceptual Mistake 1 on the Extended Square Problem	116
Figure 6.9 Type E Conceptual Mistake 2 on the Extended Square Problem	117
Figure 6.10 The Photo-Robber Problem	125
Figure 6.11 The Photo-You Problem	125
Figure 6.12 Type A Mistake Case 1 on the Photo-Robber Problem	126
Figure 6.13 Type A Mistake Case 2 on the Photo-You Problem	127

Figure 6.14 Type B Conceptual Mistake on the Photo-Robber Problem 128
Figure 6.15 Type C Mistake Case 1 on the Photo-Robber Problem 129
Figure 6.16 Type C Mistake Case 2 on the Photo-Robber Problem 130
Figure 7.1 The Photo-Robber Problem on Form A 137
Figure 7.2 A Typical Solution for the Photo-Robber Problem on Form A 138
Figure 7.3 The Photo-You Problem on Form B 139
Figure 7.4 A Typical Solution for the Photo-You Problem on Form B 140
Figure 7.5 An Example of Functional Thinking on Proportion 141
Figure 7.6 An Example of Additive Thinking on Proportion 143
Figure 7.7 The "Unknown Digit" Problem 146
Figure 7.8 The Lake Problem and Its Solution on Form A 148
Figure 7.9 Bobby's Work on the "Lake" Problem 149
Figure 7.10 The "Balance" Problem and Its Solution on Form A 150
Figure 7.11 Bobby's Work on the "Balance" Problem 140
Figure 7.12 The "Glass-House" Problem and Its Solution on Form A 152
Figure 7.13 Bobby's Work on the "Glass House" Problem 155
Figure 7.14 The "Coin" Problem and Its Solution on Form A 157
Figure 7.15 Bobby's Work on the "Coins" Problem 157
Figure 7.16 Covarying Relationship Between Time and Distance 159
Figure 7.17 The "Train" Problem in the Interview 161
Figure 7.18 Bobby's Work on the "Train" Problem 162
Figure 7.19 Two Similar Speed Models for Two Speed Problems 163
Figure 7.20 The "Classroom Board" Problem in the Interview 164
Figure 7.21 Bobby's Work on the "Classroom Board" Problem 165

INTRODUCTION

Solving problems is a competence that we should help students develop in schools. The reasons for developing such competence are many. Nowadays, the most obvious and direct one is an economic one. Any developed country, like Hong Kong, needs more and more intelligent problem solvers in the workplace. The main focus of this study is to investigate whether Hong Kong students develop such competence in elementary schools.

Non-routine problems are considered to be true problems in this study; they are used to investigate whether students develop problem solving competence in schools. The underlying assumption about problem solving competence is that students can solve any non-routine problems under the constraint that they have the related knowledge and understand the concept(s) which are required for the problems. So, the competence is related to conceptual understanding. In this study, a special case, Bobby, revealed the validity of the assumption. Bobby had a very firm conceptualization on the concept, speed. He knew that the co-varying relationship between two entities was important to the concept. So, he had no difficulty solving the two speed problems, "Lake" and "Train," which had totally different structures and story contents. He also was the only students able to think about the "Lake" problem with a mathematical perspective.

This study focuses on the multiplicative reasoning among students in grade 5. Multiplicative reasoning is an important mathematical competence in upper elementary grades. It is related to many important mathematical concepts—area, proportion, speed, ratio, rate, and fraction—which students learn or develop in middle and upper elementary grades. This thinking is quite different from the additive thinking that students learn and develop in the lower elementary grades. However, it also poses some difficulties for students to develop understanding on the related concepts of area, proportion, and fraction. This study explores the possible reason(s) for the difficulties that students have in mathematical reasoning.

The difficulties in solving non-routine mathematical problems can be caused by many variables. It can be a developmental one, a linguistic one, a conceptual one, one related to social background, or an integrated one. This study found that the difficulties Hong Kong students have are related to the conceptual understanding of specific mathematical concept—area, perimeter, and proportion.

The social background is another focus of this study. Although there are lot of studies in the United States that reveal that social background and socio-economic status, are the main factors that influence students' learning of mathematics, Hong Kong educators are not interested in investigating the effect social background has on learning in any school subjects. This study will explore the influence of this factor on the way Hong Kong students learn mathematics.

There were 2 phases of this study. In the first phase, 123 Hong Kong grade 5 students participated. Students came from two different schools located in two different communities, one in a middle class community and one in low SES community. Students were asked to work on a quiz containing 10 problems, 4 were routine and 6 were non-routine problems, in one hour. In the second phase, a subset of 27 students participated in the interview setting. They were asked to participate in two activities, problem sorting and problem solving.

The result of this study revealed a very interesting finding. All of the self-developed models or rules for thinking about ratio were developed by the mid-achievers in two different schools. Most of the high achievers in both schools were able to solve the ratio problems with the cross-multiplication method, but none used any self-developed rules or models for solving the problems.

Chapter 1

MOTIVATION OF THE RESEARCH

Psychological and Educational Concern

A lot of educators and psychologists (Duncker, 1945; Schoenfeld, 1985; Thompson, 1985; Wertheimer, 1959) are interested in knowing how individuals or students solve problems. Although they may have different definitions for the term, problem, and may be concerned with different issues concerning problem-solving, there may be common reasons behind their concerns about human problem-solving. The most obvious reason may be that problem-solving is a crucial component of intelligence (Holyoak, 1995; Resnick & Glaser, 1976; Sternberg, 1982). An intelligent individual is one able to adjust oneself and make use of prior knowledge to deal with strange or new situations. My concern focuses on whether our schools or our educational systems help our students develop such competence, and whether they help or hinder the development of students' disposition to the use of their knowledge to tackle challenging problems. If we examine student assessments and observe classroom practices, we will find that schools put our students at a disadvantage relative to problem solving.

Teaching and Learning Concern

This study, like previous studies (diSessa 1982; Carpenter, Fennema, Peterson, Chiang & Leof, 1989), hopes to offer useful information to teachers and teacher educators. If teachers want to help their students learn a school subject with understanding, then teachers need to better understand student learning. In stating this, what I mean is how they learn, what are the cognitive difficulties they have when they learn mathematical concepts, and what are the possible cognitive structures and representations that help students understand various mathematical concepts. This study will inform teachers and teacher educators about some cognitive obstacles associated with mathematical concepts. For example, what are the difficulties that grade 4 and 5 students have relative to area and perimeter. This study will also target at issues about poor mathematical education practices at schools in Hong Kong.

Social and Economic Concern

While scholars and educators with idealistic goals and enthusiasm push toward educational improvement, education is also controlled by limited resources and societal demands. One societal demand is our labor market. In a capitalistic society, a strong influence on education is the need to educate workers to meet the demands of the workplace. Five years ago, the Education and Manpower Branch of the Hong Kong Government estimated the manpower supply and demand for the coming 7 years. In year the 2001, the job market will not need as many semi-skilled laborers; these are the workers that are high school graduates and below. But, the market will ask for more and more workers with technical knowledge and problem solving competence, most of which are graduates with post-secondary diplomas or degrees (Table 1.1). From the table, we can examine trends in the Hong Kong job-market in the 21st Century. There will be fewer and fewer labor-intensive job vacancies in the next century. More and more opportunities will open to workers who have a good technical knowledge-base and problem-solving competence.

Table 1.1 Comparison of Projected Manpower Supply & Demand in 1996 & 2001

Education Level/ Supply and Demand	Supply		Demand		Balance	
	1996	2001	1996	2001	1996	2001
Lower secondary & Below	1,226,900	1,108,600	1,177,300	1,068,300	+49,600	+40,300
Upper Secondary	708,900	647,600	715,400	633,200	-6,500	+14,000
Form Six	130,100	105,900	140,900	123,900	-10,800	-18,000
Craft	11,900	11,000	18,700	16,100	-6,800	-5,100
Technician	38,700	33,100	43,500	37,900	-4,800	-4,800
Sub-degree	87,800	74,000	92,300	77,500	-4,500	-3,200
First degree	118,200	89,600	132,300	111,600	-14,100	-22,000
Postgraduate	10,500	8,000	12,600	9,400	-2,100	-1,400

Source: Government of Hong Kong (1994). Manpower 2001 revisited projection of manpower supply and requirements for 2001. Education and Manpower Branch. Hong Kong, p. 44.

Personal Experience

In Hong Kong, my generation did not face issues related to social class in our schools. Most students came from low socioeconomic status (SES) families. Most of our parents left Mainland China in the early 1950's, and most of them were poor. That may be the reason why there have been no serious studies about how students from that social class learned and developed at school. Last year I spoke with one Hong Kong mathematics educator about my study. He told me that social class is an issue only for Americans; Hong Kong does not have such an issue in their system. As a result of his comments, I began to have a totally different idea about social class issues.

In the past 30 years, the Hong Kong economy has produced a new social class, the middle class. And, they are quickly growing and becoming the majority in the society. At the same time, this social group intentionally or unintentionally tried to scramble social resources. Such social change may put the low SES class in an even more disadvantaged position. Often, families in this social class are not as active in politics. Therefore, they can easily be neglected or sacrificed. Under such conditions, access to education becomes very important. The question is, "Can we lay out a plan so that all students have fair access to educational resources?" I think we need to rethink our old sabbatical-based educational system to push forward this social change. This study will highlight the issue of low socio-economic status in Hong Kong society, although I know that it cannot change the society or the educational system. Also, from a series of studies (Griffin, Case, & Siegler, 1994; Saxe, Guberman & Gearhart, 1987; Siegler, 1993), we know that students from low SES families do not perform as well as students from middle class families in school mathematics. This study may help teachers understand how they can help students learn and develop intellectually in school, regardless of social class.

All of these concerns and questions lead me to think about a study that would investigate them. My research questions are also based on those concerns and questions.

Research Questions

1. Do Hong Kong students have difficulties solving non-routine mathematical problems?

2. If students have difficulty, what are the general characteristics of their difficulties—conceptual, computational, linguistic, or developmental?

3. If students have difficulties, are these issues related to social classes?

Chapter 2

CONTEXTUAL BACKGROUND AND SCHOOLING IN HONG KONG

Geographical Information

Hong Kong is physically a small city. The total area is about 1,100 square kilometers (Phoenix, AZ, has about the same area, 1,125 sq. km), but there are more than 6.3 million people living in Hong Kong. So, the population density is 5,780 people per square kilometer. Obviously, the density of population is very high, especially in the urban districts. About half of the population (3.1 millions) live in public housing apartments. The average wage of a Hong Kong worker was about US $1,250 per month in 1996. About 95% of the residents of Hong Kong are Chinese. Most of them entered Hong Kong after the Chinese Communist Party took control of China in 1949. In the early fifties and late sixties, there were two major waves of immigrants who established the current population structure. In the coming years, Hong Kong will receive many new immigrants from Mainland China. The most conservative figure is that 60,000 new immigrants will come from the Mainland to Hong Kong in the coming five years. Most new immigrants will be school-age children and their working class parents. Also, Ming Pao reported in 1997 that the poor population was increasing and that about 13% of the total population was living in poverty.

Historical Context

Hong Kong was a British colony for more than 100 years. After the Opium War, 1840—1842, the British government focused their efforts into turning this small fishing village into an East Asian trade city. The British Government was very successful at sculpting this village into a modern city. The Government also imported a lot of western culture to this eastern land. The Hong Kong Chinese may be the most westernized Chinese in all of China (Hong Kong is part of China now). Although the British government did not establish a democratic system in the Hong Kong, they did establish a western legal system. This system has influenced the Hong Kong Chinese legal spirit. I think it is a fact that when we compare these three Chinese societies (Hong Kong, Mainland China, and Taiwan), we will find that the Hong Kong Chinese enjoy a fair legal system. I think no one will deny the British contribution to establishing a fair legal system. In 1997, the Hong Kong government switched from a colonial British Government to the current Chinese Government. Politically, there is no noticeable change. It is still a capitalistic, non-democratic, but free society. In fact, most of the Hong Kong people are more concerned about whether they will lose the legal system they had before than they are worried about any change in the capitalistic ideology or the political system overseeing the society.

Although there have been no big changes in the political system, the transition of sovereignty from Britain to China will undoubtedly bring a lot of changes to schooling in Hong Kong. First, the decolonization of education is taking place in Hong Kong. Certainly, no one will be able to guess the result of this shift from colonization to decolonization. On education, will the school administrators and teachers adjust themselves easily to this new age? Does the new government have enough foresighted policy makers in place to plan and carry the new policies forward for the new age? The answer to these questions of the future will only be able to be read in the 21st Century. Second, there will be some new curricula introduced into schools. We know that the Government will introduce a civil education curriculum into the schools in the very near future. Some educators and scholars have suggested that a general problem-solving skills curriculum, that is experimental, is used in some chosen schools. Third, there will be more than 60—100,000 new immigrants, most of them school-age, moving to Hong Kong within the next 5 to 10 years. How does the education system serve this population? These will be the big education issues of the society in the next 5 to 10 years.

Schooling in Hong Kong

General Education and Student Population

The education system in Hong Kong is composed of five sectors. They are, 1) pre-school, 2) general education, 3) technical education/vocational education, 4) higher education, and 5) adult education. The students who participated in this study all belonged to the general education sector. This section will provide some data on general education in Hong Kong. Most of the data will be based on the paper, The Education System, recently written by Dr. Cheng Kai Ming (1997). First, lets look at the student population (Table 2.1) in the general education and higher education sectors in 1999. Generally, primary school education in Hong Kong, as in the United States, comprised of grades 1 through 6. Secondary school education in Hong Kong, as in the United States, comprised of grades 7 through 12. The exception is that the secondary schools in Hong Kong cover 7 years of study. This general education scheme is a copy of the school system in England. Students only spend 3 years in college if they enroll in a full-time degree program. A description of general education in Hong Kong will be elaborated upon in the following sections.

Table 2.1 Hong Kong Student Population in Different Educational Institutions in 1999

	Number of Schools	Number of Enrollments	Pupil-Teacher Ratio
Kindergartens	744	175,073	13.0 : 1
Primary Schools	830	476,802 (17,799)*	22.7 : 1
Secondary Schools	471	455,872 (3,141)*	19.3 : 1
Special Schools	74	9513	5.9 : 1
Govt.-Funded Colleges	8	45,523**	N/A

source: Hong Kong Education Department Website, http://www.info.gov.hk/ed/
* These are the numbers of new immigrant students
** These are the enrollments for the 1st degree programs.

In addition to these general statistics, Figure 2.1 shows how the student population changes at different levels, and how examinations function at different levels as gatekeepers to track students as they venture down different educational paths.

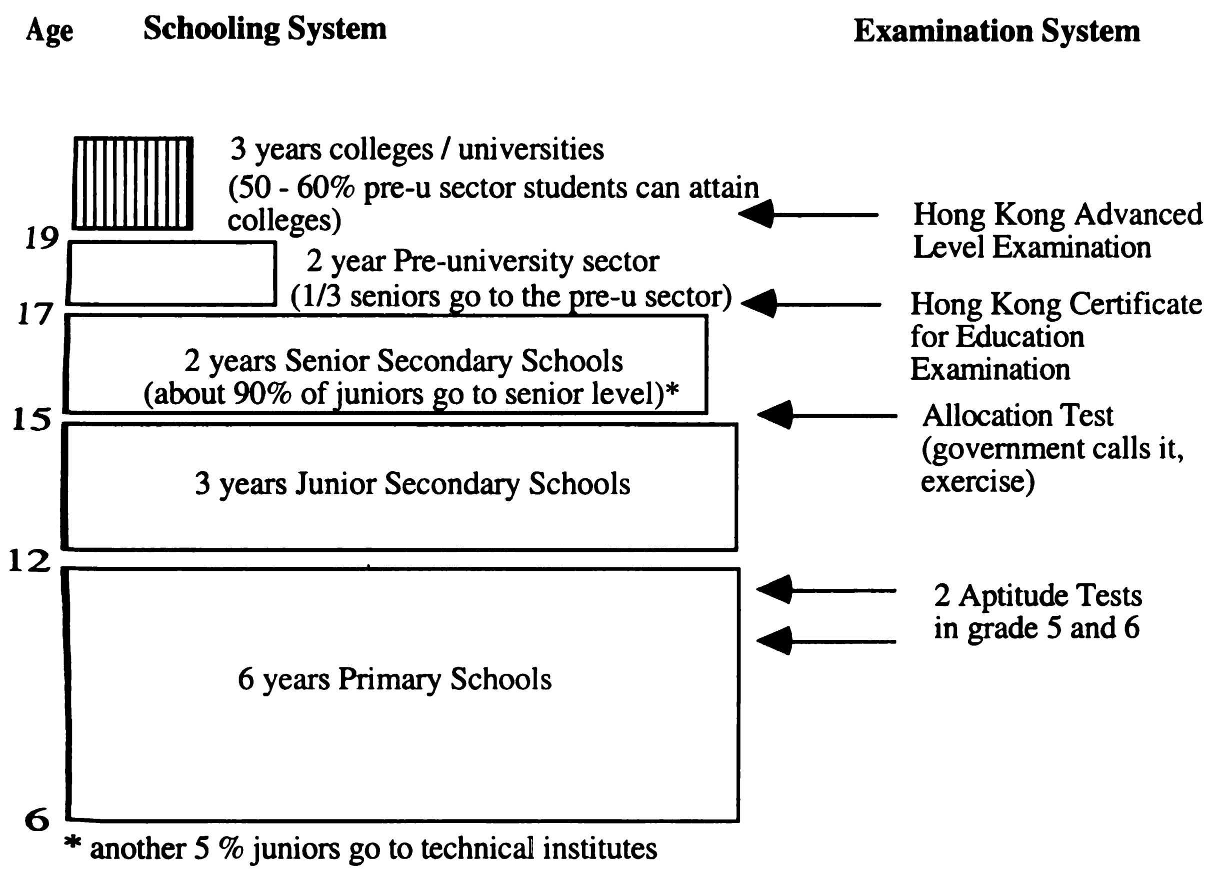


Figure 2.1 Hong Kong Schooling and Examination System

Formal schooling in Hong Kong starts at age 6. Nine years of school education, from primary 1 (P1) to junior secondary education (Form 3), which is equal to grades 1 through 9 in the United States, is compulsory. By law, all children are required to attend these 9 years of schooling. Home schooling is not accepted as an alternative. After 3 years of junior secondary education, there are two more sectors of secondary education. Following junior secondary education, there is a 2-year sector of senior secondary education. Between these two stages, there is an allocation exercise that screens and channels 5% of the junior secondary students to technical institutes for a technical education and another 5% is channeled directly to the workplace. After a 2-year senior secondary education, about one-third of the seniors will go on to attend another 2-year pre-university sector that will prepare them for a college education. Between these two senior secondary education sectors, there is a key examination in a Hong Kong students' life, the Hong Kong Certificate for Education Examination. This is the main gatekeeper examination that screens students for either further college education or the job market. After the 2-year pre-university sector, students need to take another examination for attending college, the Hong Kong Advanced Level Examination. About 50–60% of the students in the pre-university sector can attend colleges or the universities.

Primary Schools

Primary schools follow a uniform curriculum comprised of eight to nine subjects, including Chinese language, mathematics, English language, integrated science (integrating social and natural science into a school subject), Mandarin, art, music, physical education, religious studies (offered in schools sponsored by religious organizations), and civil education, which will be introduced in the schools very soon. Around 90% of the primary schools are running half-day sessions for economic and geographic (not enough land for building schools) reasons. So, two schools share the same location, one operates in the morning and another in the afternoon. But, the two schools are run by two different groups of school administrators and teachers. Although the government has promised that there will be a gradual conversion to whole-day schooling in primary schools, there are some unexpected obstacles—economic difficulties for government finances, and the sudden increase in the student population of new immigrant students from Mainland China. The class size is about 35 students per class. Some schools adopt the "activity approach" (learning is based mainly on designed activities in which students actively participate, a kind of learning-by-doing philosophy) for instruction instead of lecturing. Class size is about 30 students per class. In grades 5 and 6, students take a public examination called Aptitude Tests. The test results are the main point of reference for allocating students to different secondary schools after they finish their primary educations. The mechanism for allocating students to different secondary schools is quite complicated. Basically, it is a school-based referenced allocation mechanism. In other words, if your school has a lot of high achievers and they perform very well on the aptitude tests, then your school receives a higher priority in choosing secondary school seats for your students. Every year in Hong Kong, when it is time to apply to primary schools, a lot of parents actually line up outside some good primary schools in the very, very early morning (about 5 or 6 a.m.) to receive application forms. A good primary school will bring more guarantees that their children will be admitted to a good secondary school. Because the popularity of a school is determined by the students' performance on the aptitude tests, schools thus spend a lot of time coaching their students on how to achieve high scores in the examination.

Most of the primary schools use Cantonese as the instructional language. The Chinese written systems were united two thousand years ago. The modern written Chinese system was modified one hundred years ago. All Chinese still use the unified written system at the grammatical level (Although Mainland uses the simplified character system, and Hong Kong and Taiwan use the traditional system, their grammars are the same. The only difference is the shape of the characters). However, because the Chinese did not have a unified spoken system until a hundred years ago, Mandarin was used as the official spoken language. Cantonese is a dialect in southern China. In fact, Cantonese is an old Chinese language. One thousand and five hundred years ago, it was the main tongue in China. So, Cantonese is still the best dialect in which to read old and classical Chinese poems. The textbooks for different subjects are written in Chinese (the traditional Chinese written system), and Cantonese is the instructional spoken language. There are some exceptions, some elite and international schools use English as their instructional medium.

Secondary Schools

Junior Secondary Schools

Junior secondary schools also use a similar uniform curriculum comprised of 12 to 13 subjects, including Chinese language, English language, mathematics, Mandarin, integrated science, computer studies, Chinese history, world history, economic and public affairs, geography, home-economics, art, music, physical education. Religious studies are only offered in schools sponsored by religious groups. Students need not take all the subjects; most of them take 10 or 11. The class size is 40 students per class, although there is a current policy for reducing this to 35 students per class. In the third year of junior secondary education, there is another public examination—Allocation Exercise. About 90% of the students successfully complete this examination and attend the senior secondary schools. About 5% of the students will be tracked to technical institutes for craftsman education, and the other 5% will leave the general education system and go into the workplace.

Senior Secondary Schools

Senior secondary schools offer two different kinds of classes, science and arts. The different groups use different curricula and different schools offer their own subject combinations for their arts and science classes. Also, students can build their own combination of subjects. But, all the curricula for different subjects follow the same syllabus assigned by the Hong Kong Examination Authority. This government organization manages most of the public examinations in Hong Kong. Generally, subjects included in the arts classes are, Chinese language, Chinese literature, Chinese history, English language, English literature, some other foreign languages (French, German, ...etc.), mathematics, Chinese history, European history, American history, economics, public affairs, geography, biology, religious studies, and home-economics. For the science classes, most of the schools offer the following subjects: Chinese language, English language, mathematics, advanced mathematics, biology, chemistry, computer studies, economics, geography, geography, and religious studies. Some schools offer special subjects. For example, technical drawing is offered by most of the technical-oriented secondary schools. Most of the students will choose 6 to 9 subjects on which to focus. These 6 to 9 subjects are needed to get into the most important public examination of their lives, the Hong Kong Certificate for Education Examination, taken in their 2nd year of their senior high school. Most of the students take 6 to 9 papers based on school subjects in the examination. After this examination, only one-third of the students stay in the general education system and go on to attend the pre-university sector for another 2 years of general education. Another two-thirds take some non-degree technical courses offered by the technical institutes, or will go into the work-market directly. Most of the classrooms in the senior secondary schools have 40 students.

The language used for instruction is mixed in secondary schools. There were only a few Chinese-instruction schools in Hong Kong in the past. Most of the schools were known as Anglo-Chinese schools. The medium of instruction in those schools is a mix of English and Cantonese. Their textbooks are written in English and the spoken language in the classes is Cantonese. Beginning in 1999, the Hong Kong government began forcing all secondary schools to become Chinese-medium schools. There are about 100 schools are the exceptions to this rule. All of these 100 schools are elite schools in Hong Kong, and continue to use English as their instruction medium.

Pre-University Sectors

Most of the senior secondary schools have their own pre-university sector, but most of the students switch to other schools for their pre-university educations. The condition for admission for most pre-university sectors accepting students is based on the students' performance in the Hong Kong Certificate for Education Examination. Most of the pre-university sectors offer three different kinds of classes to their students—mathematics, biology, and the arts. Most of the students take 3 to 6 subjects in this 2-year sector. All the curricula in these 2 years are examination-oriented, too. Students in the pre-university sectors are preparing for the university-entrance assessment, Hong Kong Advanced Level Examination. The students in the mathematics classes are planning to attend a college of engineering or natural science. The students in the biology classes are planning to attend a college of medicine or natural science. Some students in the mathematics and biology programs takes classes offered by the arts program. An example would be taking economics in the pre-university sectors when they plan to apply to the college of business. Students in the arts programs are planning to attend the college of arts and business. The class size in this sector is 30 students per class. Only about 50 percent of the students will be able to attend the degree courses offered by the universities or colleges. The other students will attend non-degree programs. For example the teaching diploma offered by the Hong Kong Institute of Education. Or, some other non-degree programs offered by some universities and colleges. Or these students will go directly into the workplace.

There are a lot of non-degree courses offered by the technical and vocational education sectors. Students who cannot attend the senior secondary schools are likely to attend some craftsman courses offered by the technical institutes. Students who are not able to attend the pre-university sectors are likely to attend some technician courses offered by the technical colleges. Students who cannot attend the degree courses offered by the colleges or universities will likely attend non-degree courses or attend some part-time degree courses offered by Hong Kong Open University or other colleges and universities. But, this sector does not belong to the general education sector, so a detailed description will not be given here.

Chapter 3

LITERATURE REVIEW

There are three main sections in this chapter. In the first section, I will try to clarify what I mean by non-routine problems. Although I cannot offer you a final definition of the term, I will try to use different perspectives—philosophical, psychological, and educational—to characterize the concept. In the second section, I will talk about another vague, but important concept, understanding. Using different psychological constructs, mental models, mental schemes, and principle-based understanding, I will attempt to outline the characteristics of understanding. In the third section, I will attempt to give a review of the students' conceptions of several mathematical concepts—proportion, speed, area and perimeter. In the last section, I will address the mathematical competence growth of students from low social economic status (SES) families.

Non-Routine Mathematics Word Problems

First, let me briefly clarify the term, "non-routine problem." Some mathematics educators and psychologists (Schoenfeld, 1985; Voss, 1989) saw non-routine problems as true problems, and routine problems as exercises. They argued that routine problems cannot be counted as problems, psychologically. They further argued that a problem is presumed to exist when the steps needed to achieve the goal are not immediately apparent as when an individual confronts a problem situation. The term, "non-routine problem," used in this study corresponds to this "problem" idea. Second, the term, "mathematics problems," refers to designed problem situations that require students to bring their acquired mathematics knowledge to the problem situations in order to solve the problems. Mathematics problems are different from problems solved without any school mathematics knowledge base. For example, in the famous Tower of Hanoi Problem (Neves, 1977), subjects need not use any knowledge of mathematics as a basis for thinking about how they should move the 3 different size disks within a one-by-one format and how they can move the disks from the left peg to the right peg. Third, the term, "word problems," is used differently in computational problems. When word problems were presented to the students in the study, each of them faced a specific situation or story embedded by one or several different quantities. The Students needed to understand the situation or the story and the quantities in addition to the numerals and related mathematical operations. Numerical problems require students to use specific mathematical operation(s) to manipulate the numerals.

In the following paragraphs, 3 different perspectives—philosophical, psychological, and educational—will be used to outline the characteristics and the nature of the problems in this study.

Philosophical Perspective

The nature of "problem" has been a subject of debate in different disciplines, but mainly in philosophy and psychology (Agre, 1982; Voss, 1989). There is no general agreement among most investigators on this issue. From Dewey to Popper, the concept of problems, has not been well defined philosophically. Philosophers could only try to lay out the network of concepts that were closely tied to the ideas of the problem in order to describe the characteristics of the concept. Here, I would like to use the closely related construct, consciousness, to outline my philosophical stance on the concept. I need to clarify that it is only a theoretical stance, not the truth. And, I believe that there is still a long road ahead of us to before we will be able to achieve a true statement on the concept, problems.

Philosophers have argued with different perspectives as to the existence of the problem. Is the existence of the problem determined by our consciousness? Some, like Dewey (1986), preferred to think that problems are created through consciousness of or articulation about situations rather than being situations. Hence, problems cannot exist without prior consciousness. In a stronger tone, those same philosophers seemed to imply that the existence of a problem is determined by consciousness. However, some philosophers (Agre, 1982) maintained that there may be some fault in arguing that the existence of problems is determined by the human consciousness. There are many counter-examples revealing that this argument is not true. For example, a grade 7 or 8 student learning about the mathematical concept, infinity, is probably unaware (in her/his consciousness) that the concept is embedded in another mathematics problem, continuity. However, his unawareness does not make the problem, continuity, disappear.

Borrowing words from Merleau-Ponty (1962), "the world is always already out there before reflection begins," I philosophically admit that problems are not determined by our consciousness with a phenomenological stance. However, the focus of this study is educational and psychological, how students psychologically tackle school mathematics problems. Hence, it would be meaningless for me to study problems that students are unaware of from a psychological and educational sense. In other words, this study is focusing on a special kind of problem that humans are psychologically aware of, although I admit that the existence of such problems is not determined by our consciousness.

Psychological Perspective

At least since the cognitive revolution (Gardner, 1985), a group of educational psychologists and psychologists (Dunker, 1945; Hayes, 1989; Holyoak, 1995; Lester, 1983; Mayer, 1985; Mayer & Wittrock, 1996; Newell & Simon, 1972; Thorndike, 1950; Voss, 1989) have implicitly or explicitly suggested a similar psychological model for problems. Psychologically, a problem comes to one's mind when one has a goal to achieve, but the current psychological state does not have a very clear route or a ready solution for helping her/him to achieve the goal state. Although problems can generally be defined with this psychological sense, there are different kinds of problems under this umbrella definition. I would like to specify the characteristics of the problems in this study in the following paragraphs.

Visibility of Goal State

Tower of Hanoi (Neves, 1977) is one of the most frequently cited, classic psychological experimental problems. Let us take a look at what are the psychological characteristics of the problem, then I will describe the main difference between the experimental problems and the problems in the study, or most of the word-problems in school mathematics lessons. Psychologically, a problem appears in an individual's mind when there is an obstacle between the initial state and its goal state. Psychologists try to build a simple and easily defined experimental situation in which individuals can have such psychological experiences and they easily detect the possible psychological activities in an individual's mind. The Tower of Hanoi problem creates a kind of situation in which an individual will come up with such an experience. In the experiment, individuals will be shown the following situation (Figure 3.1)

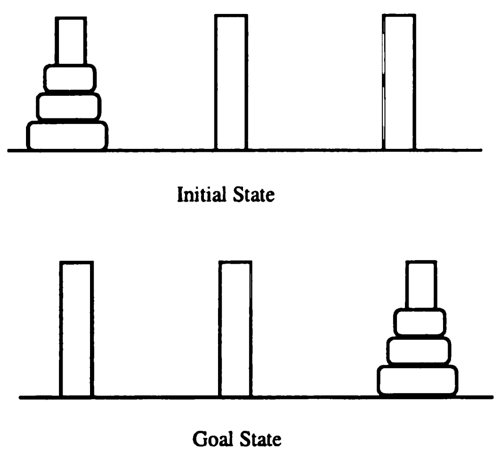


Figure 3.1 Tower of Hanoi Problem

and be given two rules: 1) they can only move one disk at a time, and 2) a large disk must never be placed on a small disk. Individuals who did not work on the puzzle before do not know immediately what series of actions must be performed to solve the puzzle. Psychologists consider the individual experience a problem, psychologically.

Comparing the Tower of Hanoi puzzle with the problems given to students in this study, I can say that the main obvious difference is that students in this study did not have a clear image about the goal state. In other words, students have a problem situation (initial state), the problem was written on the paper, but they did not have the final answer (goal state) as a criterion for evaluating their solution, though they could have the expectation that the answer should be a number. So, although the students in the study experienced a psychological obstacle between the initial state and the goal state when they read the problems, they did not have a sure answer to the problem (the goal state). This difference influenced how individuals in the experiment and students in the study solved the problems. The individuals in the experiment used the mean-end analysis to search for the solution. The students in the study did not use that strategy because they were better equipped with a deep understanding of some of the relevant mathematical concepts embedded in the problems. I want to clarify what "efficiency" means here. To me, "efficiency" is not only related to speed, and how much time students need to solve a problem; it is related to the generalizability of the knowledge. If students have knowledge that encompasses a wider generalizability, they can then apply their own theories to situations and they will be more efficient at solving the related problems.

Problem Space

In this section, I will discuss the internal representation, the problem space. Newell and Simon (1972) argued that there should be a place in which individuals can psychologically process problem-solving activities. They call the place, the problem space. In the problem space, individuals encode the problem components—defining goals, rules, and other aspects of the problem situation—in a space that represents the initial situation presented to them. The situations include the desired goal situation, various intermediate states (imagined or experienced), as well as any concepts they use to describe these situations to themselves. In the Tower of Hanoi puzzle, individuals visualize the very clear initial situation and the goal state of the problem. They also have two very clear constraints—large disk can never be placed on the small disk, move one disk at a time—which can be considered as the concepts they use to describe and understand the situation. Individuals will probably come up with various imagined, intermediate states, putting the 3 disks into to different pegs, while they are thinking through the problem and before they come up with the solution.

If we use this idea to describe the possible problem spaces of students in the study, we will find that there are differences in the content of the problem space between the students in the study and the subjects in the Tower of Hanoi study. First, students in the study cannot visualize the goal state, as mentioned in the last section. Second, students bring different concepts to describe and understand the problem situation in different problems. I did not give any rules or constraints to the students. So, there were a lot of individual differences among the students as to how they perceived and understood the problems. Third, these students had more complicated intermediate states than the states generated by the subjects in the Tower of Hanoi. Some of the intermediate states may have been more difficult for the researchers to detect because they were not as visual and were more abstract. Fourth, students sometimes experienced difficulties in reporting their psychological phenomena clearly to the researchers.

Ill-defined Problems and Non-Routine Problems

Some readers may ask why I use the term non-routine problem instead of ill-defined problem if I consider this study to be a psychological study. Some psychologists (Holyoak, 1995; Reitman, 1964; Simon, 1973; VanLehn, 1989) use the terms ill-defined and well-defined problems to categorize problems. I think that the categories, ill-defined and well-defined, were not well-suited for this study. The ill-well defined dimension is better suited for focusing on an individual's psychological problem space. If one knows the initial state, the goal state, and the constraints of the problem, then it is a well-defined problem, and vice versa. For instance, the Tower of Hanoi is typical of a well-defined problem. However, the problems in this study are concerned with students' past experiences. All the problems, no matter how routine or non-routine, are ill-defined because students do not know the goal state in advance and they often need to do some checking to ensure the correctness of their answers. In the coming section, I will talk more about the relationship between practice and problems within an educational context.

Educational Perspective

In the real world, there are a lot of unsolvable problems, but on the general education level (primary and secondary school educations), most of the problems for students are solvable. All problems in this study are solvable. Of course, "solvability" is related to the problem solver's competence, psychologically. In other words, while one problem is solvable for an individual, it may not be applicable to another person at the same time. Or, a problem that was not solvable in the past is solvable now, for the same individual. According to this relative and psychological sense, we know that the solvability of a problem must be highly related to an individual's knowledge background and competence. Problems designed for this study were selected based on the prior learning experience students had in school mathematics. So, I claim that all the problems were solvable to most ordinary students, although some were easily solved and some may have been solved with more difficulty. My argument follows. There are reasons for problems to be solvable at the elementary and secondary educational levels. First, problems designed in these educational settings should have a common goal of helping students develop a deeper understanding of the important concepts in the subjects. A problem situation offers a chance for students to think and reflect their understanding about the concepts. In what situations does a concept become applicable? What is its limitation? Or, which concepts have more generalizability power? All these questions may encourage students to think about their learned concepts when they face a solvable problem. Unsolvable problems cannot offer such a learning opportunity because students may not have the background knowledge to think and reflect about the concepts they have learned. Second, the goals of solvable problems offer the criteria for teachers to assess students' progress. If problems are unsolvable, it is hard for teachers to set up the assessment criteria. If the result is opened-ended and no one knows the resolution, then how can teachers know whether their students are headed in the right direction. Third, idealistically, one of the goals at the general education level is to offer students a chance to learn important ideas/theories/laws that have been developed systematically by scholars past and current. And, we believe that this knowledge is the basis for them to go further when they are ready. On the affective level, we hope students can value and appreciate similar ideas/theories/laws through their own problem-solving experiences.

In addition to using a dichotomy, solvable and unsolvable, we can use Polya's classification (1981) of mathematics problems with a pedagogical perspective for classifying the problems on a more continuous dimension, degree of difficulty. According to Polya, mathematics problems can be classified into the following types:

> 1. *One rule under your nose - the type of problem to be solved by mechanical application of a rule that has just been presented or discussed.*
> 2. *Application with some choice - a problem that can be solved by application of a rule or procedure given earlier in class so that the solver has to use some judgment.*
> 3. *Choice of a combination - a problem that requires the solver to combine two or more rules or examples given in class.*
> 4. *Approaching research level - a problem that also requires a novel combination of rules or examples but that has many ramifications and requires a high degree of independence and the use of plausible reasoning.*
>
> (Kilpatrick, 1985, pp. 4)

The term, " non-routine problems," used in this study corresponded to a hybrid of Polya's Type 2 (application with some choice) and Polya's Type 3 (choice of combination). And, the term, "routine problems" corresponded to a hybrid of Type 1 (one rule under your nose) and Type 2. Polya used the "difficulty" dimension to locate different kinds of mathematics problems in schools and the academic world. It has its own advantage, helping us to see a clearer relationship between individual efforts and the complexity of problems. More effort is needed when the complexity is increased. Polya seemed to suggest that the complexity of the problem nature, more rules are needed, makes the problem more difficult. To me, it is one of the sources for making a problem more difficult. Another source is familiarity. If students have a lot of practice with some problem situations, these experiences will help them evaluate the problem as an easy one (or a routine one). So, although some problems may need students to apply several rules to achieve the answer, they may find that they can easily solve the problems if their prior knowledge and understanding about those problems are quite well-established. In other words, the degree of difficulty has a combined effect on the problem structure (the complexity of a problem) and one's personal experiences.

To summarize what non-routine problems mean in this study, I will argue that problems are consciously detected by students in a philosophical and a psychological sense. Psychologically, students in this study did not have a clear idea about the goal states in different non-routine problems, and they had some very complicated problem spaces while they were solving the non-routine problems. Educationally, all problems are solvable, although with different degrees of difficulty. And, the degree of difficulty is determined by the complexity of the problem structure and the individual's prior experiences. For a problem to be qualified as a routine or non-routine problem is highly related to a student's prior experience and practice. For a problem to become routine is mostly attributed to practice. The process of routinalization is related to several psychological constructs, problem schemes and problem recognition (Hinsley, Hayes, & Simon, 1977). In the next section, understanding, I will talk more about, problem scheme, in detail.

Understanding

The term, understanding, is a crucial but slippery word in many disciplines: education, philosophy, psychology (Greeno, 1983; Greeno & Riley, 1987; Locke, 1706; Mandler, 1998; Resnick & Omanson, 1987; Thagard, 1992; Toulmin, 1972). Each discipline is concerned with its own interpretation. In education, educators are concerned about how teachers help their students achieve understanding; in philosophy, philosophers are concerned what understanding means; and in psychology, psychologists are concerned about how humans achieve this psychological state. This study is more concerned with the psychological state and the processes of understanding in an educational context, especially, mathematics understanding in elementary schools. In this work, the terms, "conceptualization" and "understanding," are used to refer to a similar psychological processes. More psychologists like to use the term, conceptualization, and more educators like to use the term, understanding. To me, they are using two different terms to describe how children and students develop or learn to use a more abstract entity to think and reason about their surroundings.

Recently, there have been more and more psychologists (Greeno & MSMTAP Group, 1998; Thelen & Smith, 1994) and educators (Cobb, Yackel, & Wood, 1992; Perkins, 1998) who have suggested that we need not have a representational mind to help us towards understanding how we think. In other words, those psychologists suggested that humans who conceptualize their outside world need not have an intermediate psychological entity, a mental representation of the outside world. Situations themselves offer enough power to build up conceptualization. However, I will only talk about the representational understanding in this paper. There are several reasons for me to do this.

First, the way in which I collected this data is not suitable for doing a non-representational analysis. In this study, I did not do any historical or cultural analysis. Second, I agree with Mandler (1998) that most of the non-representational models (Greeno, et al., 1998; Thelen & Smith, 1994) are based on perceptual theories and perceptual studies (Gibson, 1977; Thelen & Smith, 1994). They are more suitable for interpreting perception studies or categorization studies, than conceptual studies. Third, the non-representational theories (Perkins, 1998) are still developing, and they do not offer a clear explanation as to how students achieve understanding. For instance, Perkins' performance view on understanding, Perkins (1994, 1998) offered,

> *"In summary, understanding is being able to carry out a variety of 'performances' that show one's understanding of a topic and, at the same time, advance it. We call such performances, 'understanding performances'." (Educational Leadership, pp. 6, 1994)*
>
> *"..., understanding is the ability to think and act flexibly with what one knows."* (Teaching for Understanding, pp. 40, 1998)

Those definitions are very similar to the vague definition of intelligence: IQ scores or students' performances on the intelligence tests is intelligence. Philosophically or psychologically, these kinds of definitions do not help us understand what understanding is so I tried not to expend effort on these theories in this review section.

In the coming section, I will discuss two important concepts, conception and perception, by introducing non-symbolic representational kinds of understanding, perceptual understanding, and its relationship to conceptual understanding. Then, I will talk about two hybrids of mental representations (a mix of analogical and symbolic representations) that are used by many psychologists to describe understanding. They are the mental models (Johnson-Laird, 1983; Greeno, 1983), and the schema (Hinsley, Hayes, & Simon, 1977; Cheng & Holyoak, 1985). Finally, I will talk about another mental representation related to understanding, the principle-based understanding (Gelman & Gallistel, 1978; Resnick & Omanson, 1987).

Perceptual Understanding

From the history of philosophy and psychology we found that there is a great deal of debate about the differences between "conception" and "perception." Beginning with Pythagoreans, philosophers have always considered there to be a fundamental separation between conceptual abstraction and perceptual experience (Goldstone & Barsalou, 1998). Nowadays, psychologists (Chi, Feltovich, & Glaser, 1981; Murphy & Medin, 1985; Medin, 1989; Murphy, & Spalding, 1995) have suggested a similar version of distinction for the psychological theories of perception and conception. They have considered that concepts or abstract concepts are not organized around clusters of perceptual properties, but rather around organized systems of knowledge. All of these psychologists and other developmental psychologists (Carey, 1985; Gelman & Markman, 1986; Gelman, 1988; Keil, 1989; Mandler, 1998) tried to reveal that there are two different and separate kinds of processing and mechanisms for understanding perception and conception. And, conception is not based on perception. However, some psychologists (Goldstone & Barsalou, 1998; Medin & Ortony, 1989; Prinz & Barsalou, 1997) argued that perceptual properties are often good indicators of important, concept-defining properties and our perceptual systems have evolved so as to establish useful concepts.

My theoretical stance is not as radical as the eliminative view on human cognition (Barsalou, 1993; Barsalou & Prinz, 1997; Prinz & Barsalou, 1997). The eliminative view of perception argued that perceptual representations constitute all knowledge, and human knowledge contains no non-perceptual representation. I prefer to think that symbolic representation, the non-perceptual representation (like the rule-based cognition), and perceptual representation work hand-in-hand when we think. On the one hand, the object features (related to perception) sometimes do influence and are helpful and efficient when we conceptualize a situation and a problem because the perceptual and analogical representations preserve aspects of the represented objects in a relatively direct way. For example, it is always quite difficult for our elementary and junior high school students to conceptualize and understand the concept, function, with the algorithms and the equations. However, using a visual metaphor of selling machines or a pop machine to introduce the concept, it will be easier for students to capture some ideas of the concept, function. I will continue this argument with more concrete evidence in the discussion section when we discuss why students are occupied by the perceptions of the area of square when they confront an area and perimeter problem in the quiz. On the other hand, occasionally rule-based cognition does offer us efficiency. Some concepts are difficult to replace with an analogical representation (internal or external). For instance, students often use the speed formula to solve speed problems when they do their school work and solve the problem in an examination because the rule is efficient. Although we do not want to see our students simply memorize rules without any understanding about why the rules work, using rules to solve mathematical problems is easily observed in our classrooms. And, it has its part in understanding, although we may not be satisfied with rule-roting based understanding sometimes.

Mental Models

Many psychologists (Greeno, 1983; Johnson-Laird, 1983; Gentner & Gentner, 1983; Mayer, 1989; Norman, 1983) have investigated the properties of mental models. In this section I use ideas developed by Johnson-Laird (1983), Mayer (1989), and Hegarty, Mayer and Monk (1995) to talk about mental models and conceptual models, and characterize their characteristics. Before we go into detail, I would like to clarify first, that the mental models I will discuss in the following section are based on Johnson-Laird's ideas, not the ones suggested by Norman (1983), and Glaser, Lesgold and Lajoie (1985). The ideas suggested by Norman and Glaser, et al., are explained by how individuals construct mental representations of physical devices, and how the mental representations influence an individual's thinking. In contrast, the mental models I will discuss in the following section are not solely limited to mental representations of physical devices. Mental models can be analogical models and metaphorical models of abstract concepts. I will give a very concrete example, a pop-machine, as a mental and analogical model to mathematical function. A mental model helped me understand the concept, function, in the Johnson-Laird's mental model section.

Johnson-Laird's Mental Models

Johnson-Laird (1983) offered the most detailed psychological and historical information about mental models although his book, *Mental Models*, was concerned more about human reasoning. Theoretically, Johnson-Laird told us why we need another mental construct, mental models, to describe and explain our cognitive activities. The appearance of mental models aimed at solving the theoretical issues of mental logic. This is a big debate in psychology, whether we need to have mental logic in our psychological life. To lay out the debate in this thesis is impossible and unnecessary, however, it is important to know about the developmental history of mental models. It is especially important when we try to study students' mathematical learning. Because, several important figures, Inhelder and Piaget (1958), and Fischbein, Deri, Nello and Marino (1985), explicitly or implicitly suggested there is a mental logic in our mind.

Johnson-Laird (1983) mentioned several characteristics of mental models, like

> 1) *"There are no complete mental models for any empirical phenomena..., one does not necessarily increase the usefulness of a model by adding information to it beyond a certain level."*
> 2) *"[Mental models] have a content and form that fits them to their purpose, whether it be to explain, to predict, or to control. Their structure corresponds to the perceived or conceived structure of the world, and is accordingly more constrained than their contents."*

> 3) *"Mental models contain some elements, A', that correspond to physical elements, A, in the world. The elements constitute an ontology of the physical contents of the world. Certain abstract relations, however, hold between these entities ultimately in virtue of the contents of mental models. That is to say, people act towards the entities in the world in particular ways, and make judgments about them, because of certain relations that exist only in their models of the world."*
>
> (Johnson-Laird, 1983, pp. 4, 410, 418)

Although Johnson-Laird mentioned a lot of other characteristics, like the computational, economical, and representational features of mental models, I find the characteristics shared above to be important in explaining how and why students may be using mental models to help them understand their surroundings and their school work. The elements of mental models are based on daily concrete experiences and correspond to world elements. There are many cases mentioned by psychologists of how students use those concrete daily experiences as models to comprehend and understand some scientific concepts. The most famous one is the flowing water model for electricity flow (Gentner & Gentner, 1983). There were not many empirical studies revealing how students use mental models and analogies to understand some mathematical concepts although the term, analogy, was originally a mathematical term (Pimm, 1981). One personal experience impressed me. I remember learning the concept, function, in 8th grade. My teacher tried to help us think of "function" as a straight line and its equation on the Cartesian plane. I struggled for 2 years trying to capture the concept with this line model. Maybe the line model was still abstract and quite a distance from me. In high school, one of my classmates told me about another model for thinking about function. He told me functions were like vending-machines dispensing soft-drinks. He told me that when you put a 2 dollar coin into the machine and press a flavor button, then the machine will go through a series of procedures, and eventually you will get a can of your favorite soft drink. So, a function is just a set of procedures turning something, like a coin, into another thing, a can of soft-drink. And, you will get different flavors of soft-drinks when you press different buttons, although you put the same 2 dollar coin into the machine. The reason you get a different kind of soft-drink is that the machine performs different functions when you press different buttons. After that day, listening to his model explanation, I felt I really understood what a function was. The line and equation models are mental models, too. However, for young students, the line and equation model are not easily absorbed. The pop-machine model really helped me to capture the abstract mathematics concept, function. Although the model may not be true to any function situation, it helped me understand the concept at the high school mathematics level.

Mayer and His Colleagues' on Conceptual Models

There are different kinds of mental or conceptual models suggested by educators and psychologists that will help students learn better at school. First, consider the conceptual models suggested by Mayer (1989). Although not all the psychologists consider conceptual models as instructional models and teachable, I still use the conceptual models suggested by Mayer, Dyck, and Cook (1984) as mental models because students mentally carry these models and try to use them to understand new situations at the assessment stage. Mayer and his colleagues (Mayer, 1983; Mayer, Dyck, & Cook, 1984) tried to reveal how important good illustrations (pictures and diagrams) were for understanding. Mayer, et al., (1984) suggested that we use a complete and concise illustration for helping students capture and understand some scientific concepts, Ohm's Law, density, and nitrogen cycle. Mayer, et al., (Mayer, 1983; Mayer, Dyck, & Cook, 1984) found that concise and complete illustrations improved students' conceptual recall and problem-solving transfer on several scientific concepts. We often assume that conceptual retention and transfer are two criteria needed to assess students' understanding. Logically, we think that retention should be longer when students learn a concept with a coherent model. And, we consider that students who can transfer their knowledge to different situations likely understand the concepts better than students who cannot. The conceptual models suggested by Mayer (1989) are based on a psychological theory, structure-mapping (Gentner & Gentner, 1983). Structure-mapping theory tries to explain how we make use of analogical models to think and reason. The models convey the like-relational systems held within two different domains. The structure-mapping analogies hold the identical operations and relationships among non-identical things. For example, the pop-machine-function concept analogy, or solar system-atom analogy are those analogical models with structure-mapping. We started with a known base domain (pop-machine, solar system) to try to understand the target domain (function, atom). Structure-mapping is one of the characteristics that holds many mental models. That is, we solve problems with mental models by mapping the elements and relationship in the problem onto the corresponding elements and relationship in the model.

Mental Models Suggested by Hegarty, Mayer and Monk

Hegarty, Mayer and Monk (1995) suggested that successful mathematics word problem solvers were inclined to build object-based mental models for the problems on which they were working. They found that successful problem solvers might build number-line models for helping themselves understand and conceptualize problem situations. And, the unsuccessful problem solvers relied more on key word(s) searching and applying relative algorithms. This further reveals the evidence that students are using non-symbolic rules or principles to conceptualize a mathematical problem. There are two reasons that make this study special. First, it revealed how students worked on a very simple mathematics operation, arithmetic operation. Most of us think that students should rely more on numerical algorithms than models. In fact, those successful students might still prefer to construct object-based models when they confronted a simple mathematics word problem. Second, all these subjects were college students who should be skillful arithmetic problem solvers. Most of us logically think that skillful problem solvers should solve the simple problem with totally principle-based or symbolic-based rules. However, skillful problem solvers still relied on constructing object-based mental models when they confronted a simple arithmetic problem. This revealed how powerful the object-based model is to skillful problem solvers, and our intuition about skillful problem solvers who might rely more on skill/procedural knowledge than conceptualization when they face familiar problems.

The more interesting phenomenon for further investigation on the mental or conceptual models is why these illustrations or object-based models are so useful and powerful when students try to conceptualize the concept and problem situations which are not so easy to conceptualize with a propositional or symbolic format. A lot of people believe this phenomenon is true in a very common sense argument, a picture is worth a thousand words. In fact, we know very little about why this common sense argument appears to be true in some situations and not others. Larkin and Simon (1987) only concluded that pictures or diagrams may offer a good index of information that can support extremely useful and efficient computational processes. But, psychologists are still trying to explore how this index function and its mechanism looks.

Schema

Schema is a very broad concept in psychology. From Bartlett (1932) and Piaget (1952) to Anderson and Pearson (1984) and Rumelhart (1977), psychologists have been interested in using this psychological construct to explain how we understand our surroundings and descriptions in text. I see scheme as an experience-based psychological entity that helps us understand objects and events in our surroundings. Scheme itself is a complex psychological structure; it collects a lot of information and properties related to an object or a concept. The scheme is developed through the abstraction of many instances and experience. For example, Thompson (1994) identified a "quantity scheme," which is composed of object/concept, the features of the object, and related concepts about the object. In this section, I cannot conduct a complete review on this broad concept in detail, instead I will only focus on two specific instances, problem schema (Hinsley, Hayes, & Simon, 1977) and pragmatic reasoning schema (Cheng & Holyoak, 1985).

Problem-Schema

Many studies in the 70's and 80's (Bhaskar & Simon, 1977; Chi, Feltovich & Glaser, 1981; Hinsley, Hayes, & Simon, 1977; Larkin, 1983; Paige & Simon, 1966; Silver, 1981; Simon & Simon, 1978) explored the role of problem schema in students' learning of mathematics and science. This construct has had an influence on how we think about problem solving and understanding. Second, the concept, problem schema, has its interpretative power on the data in this study, although you may not agree that these students understood the problems and their relative concepts. Third, I think it offers a chance for us to rethink the relationship among "problem-solving," "searching" and "understanding."

Consider the study done by Hinsley, et al. (1977) to reveal what problem schema is. The study (Hinsley, et al., 1977) was theoretically based on the text comprehension studies done by Minsky (1975). In Minsky's view, schema approach is one in which individuals rely on their knowledge of specific semantic context in comprehending passages, and they make the main decisions on how to comprehend the passage very early on and make relatively minor decisions later. In a series of experiments to test these conjectures, they found that individuals did categorize problem types at the very beginning stage, and in some cases categorized the problem type in the initial noun phrase. The inference of problem type did lead individuals to formulate problem solutions. However, such problem solving strategies only happened when individuals recognized the problem type. Individuals went back to line-by-line comprehension strategy when they were unable to categorize the problems.

This study revealed a number of important points. First, routine or familiar problems are possibly solved by schema activation (recognizing the problem type at the beginning, using the schema relevant knowledge to solve the problem). Second, different kinds of problems lead to different kinds of problem solving strategies. Individuals cannot, by definition, possess problem schema for unfamiliar or non-routine problems because they have no relevant experience. Finally, familiar and routine problems cannot help us or teachers assess whether students understand the problem situations and the relevant conceptual knowledge we expect they need to develop. Students can only do the categorization and apply relevant procedures when they solve a familiar problem. There is a possibility that they do not conceptualize the problem situation and do not understand why the relevant procedures are useful and meaningful in that problem situation.

Pragmatic Schema

Do we develop our logical reasoning through our daily experiences? Or, do we have some in-born mental logic helping us see the world logically? Many psychologists (Evans, 1982; Griggs, 1983; Johnson-Laird, Legrenzi, & Legrenzi, 1972; Wason, 1983; Wason & Shapiro, 1971) have been very interested in investigating these issues. They want to know more about how individuals reason and how daily experiences influence their reasoning. The pragmatic reasoning schema (Cheng & Holyoak, 1985) is related to logical reasoning, especially the deductive reasoning. Why do I put this psychological construct here? The first reason is that pragmatic schema studies based on Wason's study (1966) highlighted the relationship between daily experiences and logical reasoning. Second, mathematical reasoning is a kind of logical reasoning and does relate to students' daily experiences. Students quite often use their daily experiences to learn the abstract and formal mathematical principles in school.

The study done by Cheng, et al. (1985) was simple. The experimenter presented four envelopes on the table: one envelope had a 20 cent stamp on it, one envelope had a 10 cent stamp on it, one envelope labeled "back of sealed envelope," and the last one labeled "back on unsealed envelope." The subjects were told that s/he was a postal clerk working in some foreign country. Part of her/his job was to go through letters to check the postage. The country's postal regulations required that if a letter was sealed, it must carry a 20 cent stamp. In order to verify that regulation was followed, which of the above four envelopes would s/he turn over? Turn over only those that s/he needed to check to be sure. Subjects were college students in Hong Kong and Michigan. Hong Kong has similar postage regulations, so Hong Kong subjects outperformed the Michigan subjects on this item. Nearly 90% of the Hong Kong subjects got the correct reasoning, and about 50% of the Michigan subjects got the correct reasoning.

In another logical problem situation, subjects were presented four forms. One carried the word "TRANSIT," one carried the word "ENTERING," one listed cholera, typhoid, and hepatitis, the last one listed typhoid and hepatitis. Subjects were told that they were immigration officers at the international airport in Manila, capital of the Philippines. Among the documents s/he had to check was a sheet called Form H. One side of this form indicated whether the passenger was entering the country or in transit, while the other side of the form listed the names of tropical diseases. S/he had to make sure that if the form said "ENTERING" on one side, then the other side included cholera among its list of diseases. Which of the above forms would s/he have had to turn over to check? Indicate only those that s/he needed to check to be sure. In this case, both subjects from Hong Kong and Michigan were not familiar with the situation. Their correct responses were nearly the same, about 50% of the subjects from both countries reasoned correctly. According to Cheng, et al. (1985), the subjects successfully solved the logic problem using pragmatic reasoning schema, but not formal logic rules (If P then Q; If $\sim P$ then $\sim Q$). The pragmatic reasoning schema was based on subjects' daily experiences. The Hong Kong subjects performed very differently in the two cases in which subjects could solve the problem with the same formal logic rules. So, we can infer that subjects did not use the formal logic rules.

I did not provide any detail relative to what formal logic reasoning is and how it is learned because it is not highly related to this study. The pragmatic reasoning schema is based on individual daily experiences, it may appear in different abstract thinking contexts. In learning abstract mathematical concepts or principles, students may not acquire abstract and formal principles, but they may develop schema based on their daily experiences in several problem situations. For instance, students may develop a very intuitive function schema for the concept, proportion. They may not acquire the cross-multiplicative principle taught by their teachers, but they may develop a "turn P to $\sim P$ and turn Q to $\sim Q$" schema through their daily experiences with vending machines, turning coins and dollars into tickets, opening-the-gates, food, and drinks. I will reveal such schema in the discussion section in detail.

Principle-based Understanding

Another line of psychological studies (Gelman & Gallistel, 1978; Greeno, Riley & Gelman, 1984; Resnick & Omanson, 1989) have argued that understanding is based on the acquisition of underlying principles in mathematics. Beginning with counting (Gelman, et al., 1978) to subtraction (Resnick, et al., 1989), these studies are based on the same theoretical background. I will give a brief review on these two studies in the following section. The reason I have chosen to have a section on principle-based understanding is that this line of studies reveal a very important characteristic of mathematical thinking, no matter how simple or procedural a mathematical task (counting, adding, or subtracting) is, it is constrained by some abstract principles in the background.

<u>Counting Principles</u>

Like many other cognitive studies, research on counting principles is related to a historical debate about perception and conception. Gelman (1984) wanted to show that the perceptual stance is false. Perceptual stance argued that young children who are about 3 years old initially develop a direct perceptual apprehension of direct quantities, at least for small numbers. Just like other animals (for example, birds), young children cannot abstract numerosity and have limited reasoning abilities, so they can only form a perception of quantity. The most obvious evidence is that young children have difficulty judging a set size with quantity more than 5. Gelman and other developmental psychologists (Beckmann, 1924; Gleman and Tucker, 1975) suggested another theoretical stance. Gelman thought that young children count to quantify a given small number of objects before taking advantage of a subitizing or perceptual grouping strategy. Gelman and Gallistel (1978) also found that children's counting performances were constrained by five counting principles. These principles constrain how children count.

1. The one-one principle: The use of this principle involves ticking off items in an array with distinct ticks (tags, numerons, numerlogs) in such a way that one and only one tick is used for each item in the array.

2. The stable-order principle: The tags children use to correspond to items in an array must be arranged or chosen in a stable and repeatable order. In other words, children have a stable tag-pattern for assigning tags to the items.

3. The cardinal principle: This principle says that the final tag in the series has a special significance. This tag, unlike any of the preceding tags, represents a property of the set as a whole.

4. The abstraction principle: This principle states that the preceding principles can be applied to any array or collection of entities.

5. The order-irrelevance principle: This principle states that the order of enumeration is irrelevant; the order in which the items are tagged and hence which item receives which tag, is irrelevant. In other words, the children know that it does not matter how and in order they count the items.

Furthermore, there is a developmental relationship among these principles. Gelman and Gallistel (1978) believe that children develop the one-one and stable-order principle first. Then, they develop the cardinal principle. When we say children have the competence to count, they should develop these principles in their mind, although the principles may be implicit to them. According to Gelman and Gallistel's conclusion (1978), these principles govern and constitute a scheme in that they both guide and motivate the development of proficiency at counting.

Subtraction Principles

With the theoretical base built by Gelman and Gallistel (1978) on counting principles, and the subtraction buggy algorithms found by Brown and VanLehn (1980, 1982), Resnick and Omanson (1989) tried to reveal how the number and quantity principles constrain students' work on subtraction problems. The main focus of the study was to test the effect of different kinds of instructional designs, mapping instruction and prohibition instruction. They tried to verify whether the conceptual understanding and learning of subtraction concepts can be explained by analogical mapping as suggested by Anderson, Greeno, Kline and Neves (1981). I am more concerned about the principles behind procedural and numerical manipulations in subtraction multi-digit. The subtraction principles identified by Resnick and Omanson (1989) were:

1. Additive composition of quantities: This is the principle that states that all quantities are compositions of other quantities. For example, 9 is not only a cardinal that describes a set of given size, it is also a composition of 3 and 6, of 2 and 7, and so forth.

2. Convention of decimal place value notation: In the decimal system, each position in a multi-digit number represents a successively higher power of ten. For example, digits in the right-most column have a unit value—that is, the digit is multiplied by 1 to find its value. Digits in the next column to the left have a tens value, digits in the next column after that have a hundreds value, and so on indefinitely.

3. Calculation through partitioning: This principle means recomposing the problem into a set of convenient sub-problems and cumulating partial results. For example, the

partition principle allows an individual to convert the problem 38 - 12 to perhaps a more easily soluble problem, (30 -10) + (8 - 2).

4. Recomposition and conservation of the minuend quantity: It is easier to illustrate this principle with an example:

832 - 267 = (800 - 200) + (30 - 60) + (2 - 7)

There are several within-columns yielding negative numbers. To avoid negative partial results, the minuend can be recomposed to:

832 - 267 = (700 + 130 + 2) - (200 + 60 + 7)

= (700 + 120 + 12) - (200 + 60 + 7)

= (700 - 200) + (120 - 60) + (12 - 7)

This recomposing is permitted by the additive composition principle, subject to the important constraint that the total quantity in the minuend be conserved. The basic idea of these principles is that they underlie and constrain the steps in students' procedures for subtraction. Resnick, et al. (1989) thought that students will use the subtraction buggy algorithms (Brown & Burton, 1978), smaller-from-larger, borrow-from-zero, etc..., when they face challenging subtraction problems if students do not develop the subtraction principles.

Unlike Gelman and Gallistel (1978), Resnick and Omanson (1989) did not offer a clear picture of how students develop these principles. They only revealed that their designed instruction sessions that emphasized these principles did not bring an impressive learning effect for correcting the buggy algorithms. So, they argued that the conceptual understanding was not attributed to the analogical mapping. They were more inclined to think that the connection between block, writing, and quantities, seemed to be more important for helping students develop understanding. They argued that understanding was related to the semantic understanding of quantities more so than the syntactic rules of subtraction. Students' errors in the post-test were related to the gap between semantic understanding of subtraction in quantities and their automatic buggy algorithms in syntactic subtraction.

Studies done by Gelman and Gallistel (1978) and Resnick and Omanson (1989) were concerned with the mathematical competence of young children, but the principles they found could not be applied to the data in this study. However, the studies highlighted several important characteristics of children's conceptual thinking in the mathematics domain. First, two studies revealed that no matter how simple and procedural a task is, there are some abstract principles or structures constraining how children perform. Although we may think counting and subtraction are simple and procedural manipulations or tasks, children need to develop some principles before they can perform the manipulation or task proficiently. Second, conceptual understanding may be easily developed through "quantities" understanding than automatic "syntactic" procedures.

To summarize the understanding section, there is evidence that perceptions, object-based mental models, and abstract principles (structures) have an impact on how children conceptualize and understand their surroundings and mathematics problems. Although I cannot offer a final definition of "understanding" based on the reviewed studies, I think the results helped me to outline its characteristics. First, all of these mental entities are representations. How I view representations is quite different from the commonly used definition of representation. The most commonly used definition sees representation as a symbol standing for another referent. I prefer to view representation as a complex concept, the first sense of representation brought up by Mandler (1983). This conception involves factual and experiential knowledge and how this knowledge is structured and organized. It is also related and similar to a Piaget's term. Piaget referred to this kind of representation as "conceptual representation" (Piaget, 1951). From the more sub-conscious level of mental activity, perception, to the higher level mental activity, principle-based thinking, different kinds of mental representations constrain how we think and conceptualize problem situations. Using a Gestalt's figure-and-ground idea, students' perception of a square will have an influential effect on how they solve the area and perimeter problem. It is possible that a student perceives the borders of the square as the figure and the surface as the ground. He may think that the sum of the length of the four borders is the area of a square. And, Gelman and Gallistel (1978) already showed us how the counting principles constrain how young children count items proficiently. In other words, I suggest that children should have two psychological assets, factual and experiential knowledge, and psychological structures, to help them organize this knowledge when we think children conceptualize or understand a situation or a problem. Second, nearly all these mental entities are experience-based. All of them are gradually built through students' daily or school experiences. But, they are not a copy of all the actual experiences. They are newly constructed or abstracted through a lot of experiences. Finally, Johnson-Laird (1983) suggested a very important characteristic for understanding; different individuals can achieve different levels of understanding. For example, when we think that an individual understands computers really well on the computer programmer level, we can assume that the individual understands computers well if s/he can use one or two computer programming languages skillfully. On the system expert level, not only can an individual program with one or two programming languages, s/he should also know how to heck the kernel of the system and build hardware drives for the system. I think it is the same when we think about students in primary or secondary schools. We should expect students to achieve different levels of understanding of mathematics according to their schooling level. In the school mathematics context, we consider that students in elementary schools understand the concept, proportion, with their own intuitive conceptual model—like an intuitive functional model for proportion that we will explore a sample in the results chapter.

Multiplicative Reasoning

In order to talk about multiplicative reasoning, I would like to talk about an important concept, quantity. In this study, quantity does not refer to the numbers given in the problems alone, it is a measurable aspect of some object in a problem situation. It is also a good criterion for teachers to use when checking to see whether their students are trying to understand the problem situation. If students do not try to understand the problem situation and instead use the key-word matching strategy to solve word problems they often use a wrong quantity to solve the problem. In some situations, quantities also embed mathematical concepts. The geometrical measurement units are the most common examples in elementary school mathematics. The quantity for length and quantity for surface are related to linear measurement and 2-dimensional measurement. Students can use an additive model to conceptualize the linear measurement, but they need a multiplicative model to conceptualize the 2-dimensional measurement.

Like Vergnaud (1994), I prefer to think that students conceptualize multiplication with many mathematical concepts and experiences: multiplication, division, linear and bilinear function, ratio, proportion, dimensional analysis, and linear mapping. So, I put proportion, speed, and area problems under the overall umbrella idea in this section.

In addition to these related concepts, students' multiplicative reasoning has a developmental relationship with addition. A number of studies and reports (Nesher, 1988; Peled & Nesher, 1988; Resnick, 1989) found that children often first understand multiplication with a kind of repeated addition model. For example, students will think about 4×5 as $4 + 4 + 4 + 4 + 4 = 20$. This additive model is helpful when the problems can be read as a complicated additive problems. However, this model can also be an obstacle for children learning more advanced mathematical concepts like, ratio, fraction, and area. In the following paragraphs, I will briefly talk about recent studies about students' work on proportion, speed, and area and perimeter.

Proportion

Proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and multiple comparison. Many educators and psychologists (Karplus, Pulos & Stage, 1983b; Lesh, Post, & Behr, 1988; Piaget & Inhelder, 1975; Schwartz & Moore 1998; Stigler, 1976) have tried to understand how students develop and learn to think proportionally. There are two main reasons why educators and psychologists are so concerned about students' proportional reasoning competence. First, proportional reasoning is the "conceptual watershed" (Lesh, et al., 1988). That is, proportional reasoning is both (1) one of the most elementary higher order understandings, and (2) one of the highest level elementary understandings. The same point can be made from a developmental perspective: proportional reasoning can be recognized as a capability that ushers in a significant concept shift from concrete operational levels of thought to formal operational levels of thought (Piaget & Beth, 1966). The second reason is related to the essential character of proportional reasoning, abstracting similarities among different systems—the relationship between two elements in each system that have the same variation pattern. The competence to abstract similarities among different systems has its important role in scientific discovery and creativity in different domains of knowledge (Vosniadou & Ortony, 1989).

Different models have proven how students reason proportionally in primary and secondary schools. According to Piaget and Inhelder (1975), the essential characteristic of proportional reasoning is that it must involve a relationships between two relationships. That is, rather than a simple relationship between two concrete objects, a second-order relationship is involved. And, Piagetians have also argued that an early phase in children's proportional reasoning often involves "additive reasoning" (Lesh, Post, & Behr, 1988). In other words, young children may see proportional relationships among the elements in a form like, $A - B = C - D$, instead of $A/B = C/D$. Another model suggested by Karplus, Pulos and Stage (1983a, 1983b) is that proportional reasoning must involve a linear relationship between two variables. Their main argument was an objection to proportional reasoning seen as a relationship between numbers. Instead, they argued that students must think about the relationship between two variables when considering students' development of proportional reasoning competence. In other words, students might think about the functional relationship among the elements in a proportional problem. The thinking model may look like, $Y = f(x)$ or $Y = Mx$, when students think about proportion. The last model is the most commonly observed model used by students in school settings. It is the algorithm for cross multiplication, $A/B = x/D$, x is the value for which students are looking for. Some educators (Lesh, Post, & Behr, 1988; Hart, 1984) argued that this is not a real proportional reasoning model if students do not understand why the algorithm works and do not naturally develop the algorithm by themselves. They also warned that using the rote cross multiplication algorithm without sense will impede students from thinking proportionally.

Speed

Speed is a kind of proportional reasoning. Like other kinds of proportional reasoning, it involves a sense of co-variation between units. However, the units in speed have different qualities, distance measurement and time measurement. Children develop the concept of speed through the experience of movement and motion (Thompson, 1994).

Piaget (1970) argued that the conception of speed is constructed as a proportional correspondence between distance moved and time of the movement—the elaboration first of concrete, and later of formal measurement operations. And, Thompson (1994) tried to elaborate this idea in a more empirical format. He revealed how a student developed the conception of speed, from a distance-centered conception (more concrete) to a distance-time co-varying relationship conception (more formal). In her early conception of speed, the student relied more on the distance quantity than time quantity to conceptualize the concept, speed. She saw time implicitly in how many "speed-lengths" were required to make a distance. So, her conception of speed was evaluated by partitioning the traveled distance by the rate-length (Figure 3.2).

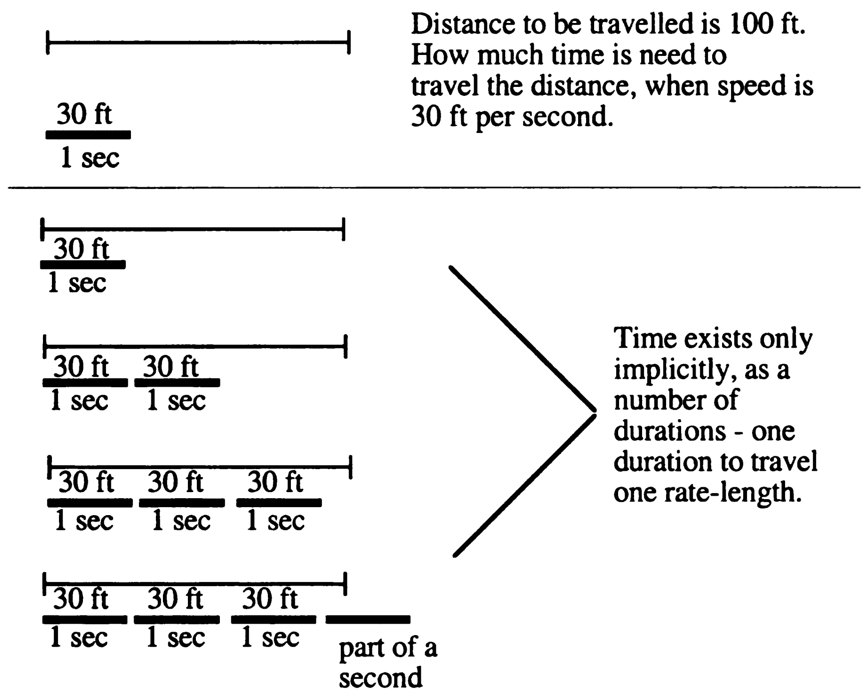


Figure 3.2 Distance-centered Conception of Speed

After several teaching experimental sessions, she began to conceive the co-varying accumulation of the quantities of distance and time (Figure 3.3).

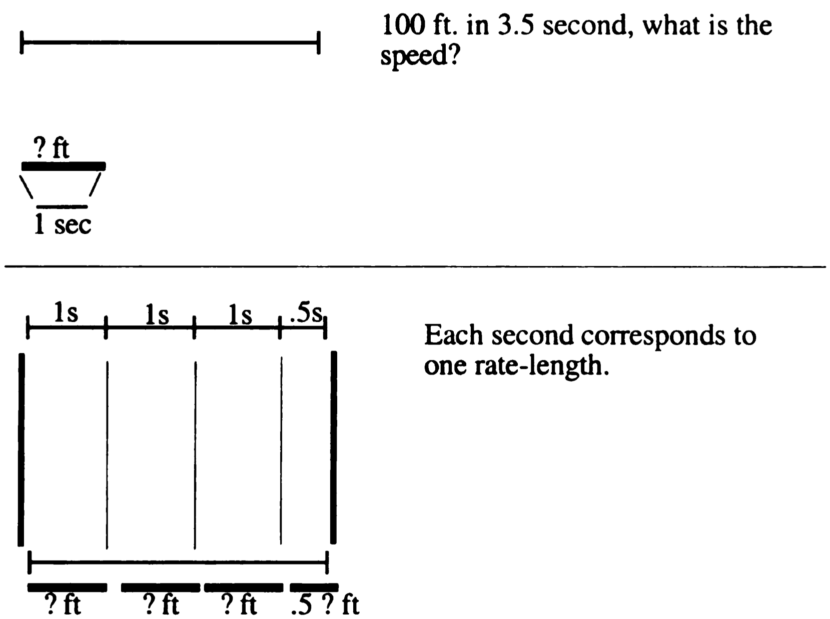


Figure 3.3 Distance-Time Conception of Speed

In the later stage, her conception of speed involved two dimensions, distance and time, and they were coordinated. In this new model, a time quantity was not an attachment to the distance quantity. Time and distance were separated, but had a co-varying relationship. Increasing a time quantity would cause an increase in distance quantity. The most obvious differences between the early and later stage of the conception are (1) the number of truly independent quantities in the student's model, and (2) the co-varying relationship between the quantities. In the early stage, the student only relied on one dimension (distance) to conceptualize speed. In the later stage, she integrated the two dimensions with the co-varying relationships to conceptualize the concept of speed.

Area and Perimeter

Area and perimeter are two additional important topics in elementary mathematics that involves additive and multiplicative reasoning. These concepts are essential in various measurement activities in everyday life. The concepts are also related to other mathematics topics, such as multiplication, fraction, and algebra (Freudenthal, 1983; Schultz, 1991). Although the concepts are used in daily life, students have difficulty conceptualizing the concepts, especially the concept "area" (Bell, Hughes, & Rogers, 1975; Hirstein, Lamb, & Osborne, 1978). There are several interpretations of students' difficulty in learning these geometrical concepts. A number of researchers (Simon & Blume, 1994; Tierney, Boyd, & Davis, 1990) attributed students' poor performance on the area problems to a tendency to learn the area formula by rote, instead of developing a conceptual basis for the formula. For instance, Carpenter, Lindquist, Brown, Kouba, Silver and Swafford (1988) reported the results of a large-scale assessment study in the United States. About 50% of the Grade 7 students could calculate the area of a rectangle when they were given both dimensions; only 13% of the students applied their knowledge of the area formula to a square when they were given the length of one side. In addition to the learning habit, students seemed to have some conceptual difficulties in learning the concept.

Some researchers (Dickson, 1989; Hart & Sinkinson, 1988) found that the instructional activities in which students experienced using concrete materials, like unit tiles, to cover the rectangular figures, were not very effective for helping students conceptualize the area concept. Mostly, students would find how many unit squares covered the rectangular figures, but they were unable to state the area of the figures. These results seem to suggest that students' difficulties might be related to failure to distinguish between additive knowledge from multiplicative knowledge and read their relationships. To summarize the two lines of studies, some researchers interpreted that the difficulties were related to the transfer issue; students learned about area by using manipulatives with an additive model, adding the unit squares. Those students may have experienced difficulties in transferring this additive model to a multiplicative model to understand the concept, area. Some other researchers thought that the conceptual obstacle was related to students' learning habits, students tried to rote-learn the area formula instead of building a conceptual base for the concept.

In addition to the line of studies focusing on instruction and learning that mathematics educators and educational psychologists are interested in, another line of studies done by developmental psychologists (Case, Marra, Bleiker, & Okamoto, 1996; Piaget & Inhelder, 1956) suggested another psychological reason for explaining why students have difficulties with area. According to Case (1998), by the age of 4, most children who live in developed countries and grow up in middle class families have developed a general schema for representing familiar 3-dimensional objects on a 2-dimensional surface. However, most children need to reach age 10 to reference a whole field of objects to two discrete mental reference axes that are orthogonal to each other. In other words, they can really represent objects in 2-dimensional models. Maybe this developmental evidence explains why students in elementary schools have so much difficulty understanding the 2-dimensional concept, area. Children need to reference two axes in a 2-dimensional model to really understand "area."

Although students' misconceptions about area are very common among students in elementary schools, there are only a few studies investigating why students have difficulties learning the concept. The above explanations are mostly cited. However, students' difficulties are more diversified than what researchers would suggest. For example, students may be confused by the concept, square root. Some subjects in this study tried to divide the area by 2. And, one of the scorers in this study suggested that this mistake can be explained. She thought students might confuse "square root" with dividing by 2. This common mistake cannot be interpreted by the above suggested explanations. And, I think that we should put more effort into this issue in the future.

Students from Low SES Families

Change is always taking place, but we sometimes neglect the changing. It is the case in Hong Kong. Hong Kong educators do not think that social economic status (SES) is an important factor influencing students' achievements in schools because they neglect the changes in the population of our students. It is a very serious issue in education, because the changing population of students is an important dimension in schooling

(Pallas, Natriello, & McDill, 1995). It is difficult to find any study related to students' achievement and SES in Hong Kong. To me, it might be related to an historical reason.

Fifty years ago, most of the population in Hong Kong was comprised of the new immigrants from Mainland China because the Chinese Communist Party took over the Mainland at that time. Most of new immigrants were not able to bring their assets or money to Hong Kong. So, most of them possessed lower SES status and corresponding resources upon settling in Hong Kong. That is also the reason why there was relatively no social class issues in the last generation; the SES structure was relatively "flat." However, there have been a number of changes in the last 50 years. At least, a new social class was and is emerging and growing. And, this new middle class also brings issues in education. Students from different social classes may need different styles of teaching and services at schools. For instance, more social workers are needed for the students from the low SES families. And, different kinds of instructional methods given by teachers may have different learning effects on students from different social classes.

Many studies and reports done in the United States (Duncan, Featherman & Dumcan, 1972; Entwisle, Alexander, & Olson, 1997; Ginsburg, Klein, & Starkey, 1998; Griffin, Case & Capodilupo, 1995; Natriello, McDill, & Pallas, 1990; Siegler, 1993; Zucker, 1995) revealed that social class does have an effect on students' achievements in mathematics in schools. Students from low SES families do not perform as well as students from those of upper and middle class families. Although not many studies focus on the SES issue at the elementary school level (Entwisle, Alexander, & Olson, 1997), some studies revealed that the SES effect on students' academic performance was an accumulative one that started in elementary schools or earlier (Broman, Nichols, & Kennedy, 1975; Entwisle, Alexander, & Olson, 1997). Recently, more and more studies (Entwisle & Hayduk, 1988; Ensminger & Slusarcick, 1992; Kerckhoff, 1993; Alexander, Entwisle, & Dauber, 1994) have revealed that elementary grades (Grades 1 to 6) have a great influence on children's future successes in and out of schools.

There is no doubt that social economic status is strongly correlated to children's schooling in different ways, from achievement and understanding to social and affective development. In the elementary school level, students' achievement is possibly related to a number of variables. The mostly cited, one may be related to students' starting competence. A number of studies (Huston, 1994; Saxe, Guberman & Gearhart, 1987; Siegler, 1993; Smith, 1972) revealed that students from low SES families arrive at the first grade with poorly developed verbal and mathematical skills compared to that of their counter-parts. In addition to this frequently cited family-related reason, Entwistle and Alexander (1992; 1994), Heyns (1978; 1987), and Murnane (1975) found that students from poor families gained little or no mathematics and reading competence over the summer. Entwistle, Alexander and Olson (1997) argued that poverty limited cognitive growth mainly by denying young children the resources they need to grow outside of school. Summer vacations had such an effect on growth. Entwistle, et al. (1997) also found that the resource effect only mattered in the summer, not over the winter holiday or during school days.

Students live in two worlds when they are attending school. One is with their families, another is in their schools. Students' learning and development is highly related to these two contexts. Some studies put more foci on the family side, and others try to explore the school effect. Oakes (1990) found that teachers who taught in schools serving large numbers of children from low SES families were more likely to focus on "computational basics" in their mathematics lessons. Zucker (1995) found that many teachers in those schools thought that their students appeared to have more deficiencies in their grasp of basic mathematical skills so they needed to put their effort in building the missing foundation. I had the same feeling when I talked to the head-teacher of mathematics at my low SES school during data collecting. The teacher told me that his students were not smart, and he needed to focus on enhancing the students' basic skills.

According to the previous studies, there were many interactive effects on "computational skills without understanding" and "summer vacation." Logically, when students learn a number of computational skills without understanding, they easily forget what they learned after a period of time. Entwistle, et al. (1997) found that when students from low SES families did not experience any growth in mathematical competence, it may have been related to how the students learned mathematics in schools. "Summer vacation effect" may be an auxiliary cause making the phenomenon easily observed, but the main cause may be related to "learning without understanding," so students do not experience retention of the knowledge they learned in their schools.

Chapter 4

METHOD

This study is a small-scale study. This study is not an experimental nor quasi-experimental study. Subjects and sites in this study were not randomly selected or assigned. The main goal of this study was to explore how Hong Kong students worked on non-routine mathematics word problems, and whether students from different social backgrounds had different kinds or levels of performance on routine and non-routine problems.

Subjects

Of the 123 subjects who voluntarily participated the first phase of this study, 27 of them voluntarily participated the second phase of the study as well. All of the subjects participated in the study with the permission of their parents. All of them were grade 5 students studying in two different schools in Hong Kong, although their chronographical ages were not the same. Most immigrant students were 1 to 3 years older than the local students.

The two schools were selected because of the socioeconomic status (SES) of the community. And the study were completed with the permission of the school principals. One of the schools was located in a low SES community, the other was in a middle-class community. In each school, there were two 5th grade classes that participated in the first phase. One was a high-achiever class and the other was a mid-to-low-achiever class. In the first phase, participants were asked to finish a 10 problem quiz within an hour in class. Of the 123 students who participated in the first phase of the study, 27 students were selected to participate in the second phase. Based on the school's achievement and social background (different SES background and whether they are new immigrants), students were selected to participate in the second section. Students were then categorized into 3 different groups, based on their performances on two recent tests given in mathematics. Subjects who had average scores equal to or higher than 90 (full mark was 100) were categorized as the high-achiever group; subjects who had average scores between 60 and 89 were categorized as the middle-achiever group; and subjects who had average scores less than 60 were categorized as the low-achiever group. In the second phase, selection criteria for the subjects were based on their social background, school achievement level, and their performance on the first phase. Some subjects were selected because their performance was quite impressive. Each participant was asked to sort some problem cards and solve one to three non-routine word problems within a one hour interview.

Tables 4.1 and 4.2 shows the frequency of student achievement level by school for the quiz and interview sections.

Table 4.1 Students Distribution in Quiz Section

School / Class	High-achiever Class	Mid-to-low Achiever Class
Middle Class School	28	27
Low SES School	34	34

Table 4.2 Students Distribution in Interview Section

Social Background / Achievement	High Achievers	Middle Achievers	Low Achievers
Hong Kong middle class	3	2	4
Hong Kong low SES	3	3	2
Immigrants from China	3	4	3

Tools

Problem Development

All the problems used in the quiz and interview were developed from two sources: 1) the textbooks and the exercise books used by the students. All the routine problems and the Balance problem were based on those books. 2) External resources included, "*Ideal Problem Solver (2nd ed.)*," Bransford and Stein (1993); *Connected Mathematics Project* (1998); "*Problem Solving: A Handbook for Elementary School Teachers*" Krulik and Rudnick (1988); the TIMSS report; and "A Study on the Mathematics Problem-solving Processes of Grade 5 and Grade 6 Students with Different Achievement Levels," Yang (1994). For example, the idea of the Glass House problem is based on a Connected Mathematics Project module, *Covering and Surrounding*, in the curriculum developed by Lappan, Fey, Fitzgerald, Friel, and Phillips (1998). The idea for the Lake problem comes from the book, *Ideal Problem Solver*, written by Bransford and Stein (1993). The reason for using external resources is that most of the students had no prior experience with these problems.

Quiz Section

Two forms (FORM A and FORM B) of a 10 problem quiz were developed to collect data. The routine problems in the two forms were parallel. However, the non-routine problems were not. The reason for having different non-routine problems in the two forms was to provide data on a variety of non-routine problems, which yields more information about how students tried to solve the problems. In each form, there were 4 routine problems and 6 non-routine problems (all the problems in the two forms can be found in Appendix A). The following table (Table 4.3) lists the order and names of problems on the quiz.

Table 4.3 Problem Distribution on Two Forms

Form & Order	Problem Name	Expected Problem Type
Form A 1	Multiplication	Routine
Form A 2	Unknown Digit	Routine
Form A 3	Balance	Non-routine
Form A 4	Glass House	Non-routine
Form A 5	Coins	Non-routine
Form A 6	Lake	Non-routine
Form A 7	Photo-Robber (Ratio)	Non-routine
Form A 8	Extended Square	Non-routine
Form A 9	Saving	Routine
Form A 10	Rectangle Width	Routine
Form B 1	Multiplication	Routine
Form B 2	Unknown Digit	Routine
Form B 3	Number Pattern	Non-routine
Form B 4	Lake	Non-routine
Form B 5	Cakes	Non-routine
Form B 6	Photo-You (Ratio)	Non-routine
Form B 7	Balance	Non-routine
Form B 8	Magazine	Routine
Form B 9	Extended Square	Non-routine
Form B 10	Rectangle Length	Routine

Interview Section

Problem-Cards: There are 15 problems—4 routine problems and 11 non-routine problems—printed on individual cards (all problems for the interview can be found in Appendix B). These problems were used in a problem-sorting task.

Table 4.4 Problems for the Interview

Problem Name	Expected Problem Type
4 Squares	Non-routine
Balance	Non-routine
Buying Magazine	Routine
Classroom Board	Non-routine
Fraction Division	Routine
Lake	Non-routine
Moon	Non-routine
Number-triangle	Routine
Paper Recycle	Non-routine
Poster Designer	Non-routine
Rope & Rectangle	Routine
Table-tennis	Non-routine
Tiles	Non-routine
Train	Non-routine
Wallpaper	Non-routine

Design and Procedures

Quiz Section

Students were given one hour to take the quiz individually in class. They were asked to try their best to solve the problems on the quiz and told that they would have an hour to finish the problems. The testing was done on 3 different days, one day in December 1997 in the low SES school, and two days in January 1998 in the middle class school. The days were arranged by the two principals in the two different schools. The arrangement was made so that no major subject lessons (language and mathematics) in their schools were disturbed. All the quizzes were given in the students' physical education or art class time. In the briefing, I spoke with the students about why I came to their school and what the study was investigating. Finally, I asked them to try their best to finish the quiz, although it is not a test.

Interview Section

Each student worked for an hour with me. There were two activities in this section. First, students were asked to read and sort the problem cards into three categories. Second, they were asked to solve one to three problems individually. Because each interview was restricted to one hour, the number of problem attempted differed. As students finished the first, I gave them another problem if time permitted.

Problem Sort: Students were asked to read the 15 problems aloud, one by one, and sort them into 3 categories depending on the level of their confidence in being able to solve the problem. The categories were "Not Know," "Not Sure," and "Know." This activity was used to assess whether students had difficulty reading the problem. It is possible that reading could be the source of a student's difficulty in solving word problems. Also, the activity was used to distinguish which problems were judged routine (the problems categorized as "Known" were routine) and non-routine (the problems categorized as "Not Sure" or "Not Known" were non-routine) by each student as routine. The instruction given was,

> "Now, I want you to read aloud all these problems. After reading each problem, you need to categorize the problem into one of the 3 categories. The first one is "Not Know": After the reading, you think that they do not know how to solve the problem. The second one is "Not Sure": After the reading, you are not very sure whether they can solve the problem. The Third one is "Know": After the reading, you are sure that they know how to solve the problem. Do you have any question on what we need to do on those problems?"

Problem Solving: Students were asked to solve one to three problems from the following problem set, "Train" problem, "4 Square" problem, "Board" problem, and "Wallpaper" problem (All the problem cards are shown in Appendix B). All the students were given the "Train" problem first. If students performed well on the "Train" problem," they were then assigned the "Board" problem. If time allowed them to do one more problem, students were asked to try the "Wallpaper" problem. Students were assigned the "4 Square" problem, if they had difficulty solving the "Train" problem. Each student was given the following instruction:

> "Later, I will pick some problems for you to solve. These problems are a little bit different with what you usually do. I would like you tell me what you are thinking while you are solving the problem. In other words, I would like you to report everything that comes up to your mind when you solve the problem. If I do not hear any report from you, I will ask, "What are you thinking?." But, I want you to know that I am not pushing you to finish the problem, I just want to remind you to tell me what you are thinking. Do you have any questions?"

There were two reasons for assigning the "Train" problem as the first trial problem. It is a speed problem, one of the focus problem types in the study. Second, I hoped that by using a story problem with a simple structure like the "Train" problem, students would be able to solve at least one problem in the interview. The "Board" and "4 Square" problems are area and perimeter problems. If the students had trouble solving the "Train" problem, I chose a simpler problem, the "4-Square" problem on which for them to work. If the students did not have trouble solving the "Train" problem, I chose the more complicated problem, the "Board" problem, for them to solve. I was more interested in collecting data on students' conceptualization of the concepts, area and perimeter, rather than text comprehension and unit transformation. Although all of these data (text comprehension and unit transformation) are important, I needed to make a priority decision in completing this study.

I picked a problem card from the stack and students were asked if they wanted to re-read the problem to refresh their memory. After the re-reading, students were asked which element(s) they found important in solving the problem. By asking students to report the important elements in the problems, I expected to gain a general sense of their competence in comprehending the text, and how they conceptualized a problem.

Scoring

Quiz Section

Two different rounds were used in scoring the quizes. The graders for the first round of scoring were operating under a time constraint and did not finish scoring all the problems. The outcome of this scoring trial did not result in a good inter-rater reliability. I will explain this observation in the following section. As a consequence, a second round of scoring was arranged and the results of the second round of scoring became the focus of this study. Although there were two different trials for scoring, the scoring schemes used were the same in each trial. The only differences were the instructions for the scorers and the time spent in scoring. In the second round, I intentionally reminded the scorer of the possibility of more than one conceptual or computational error being embedded in the students' work, and that answers were to be categorized into three groups (correct, numerically correct, or incorrect), with numerically correct not counting as correct. In the first trial, scorers had one night, about 8 hours for scoring. In the second trial, the scorers had a week for scoring the students' work. There were 3 scorers, including myself, in the first trial. The three scorers were graduate students majoring in educational psychology and who had completed their elementary and secondary schooling in Hong Kong. All of them were fluent in reading Chinese. There were 2 scorers, including myself, in the second trial. Both of them were graduate students majoring in educational psychology. The new scorer in the trial was a graduate student from Taiwan who was fluent in reading Chinese.

Scoring Scheme

The scoring scheme was developed using several scoring pilots and an examination of the students' work on the quiz and in interviews. First, I decided which problems should be scored for conceptual understanding. In this part of the scoring, I asked scorers to identify what sorts of conceptual errors the students made. Based on the written work given by the students and the importance of the embedded mathematical concepts (ratio, area and perimeter) in the upper grades of primary school, I decided which problems should be the main focus. Although all the problems were related to multiplicative reasoning, based on their written work, students seemed to be most confused by ratio, and area and perimeter. "Glass House" and "Extended Square" both focused on the mathematical concepts of area and perimeter. And, work done by students revealed their difficulty in solving these problems. However, there was more written work offered by the students on the "Extended Square" problem, thus giving me more information with which to investigate their conceptual understanding. So, the "Extended Square" problem was chosen as the area problem for conceptual scoring. In the two forms (Form A and B), a conceptual analysis was done on 5 problems—a pair of "Ratio" problems, a pair of "Extended Square" problems, and the "Cake" problem. For the other problems, scorers assessed whether the students' answers were—correct, numerically correct, incorrect, no answer, or no attempt. Next, I tried to categorize the error types. I found many mistakes related to students' neglect of the quantity given in problem situations. The issue of unit confusion seems to be common across all problems. Another frequently used code for conceptual error was the "Other" category. Errors made by students that I could not understand were categorized as "Other." All the other error types were limited to the specific problems. For example, in the ratio problem, a conceptual error is the misconceptualization of proportion as an additive relationship between entities. In the "Extended Square" problem, there were three other types of conceptual errors. One was related to the confusion between area and length. Students thought that they could get length when they divided the area by 4. The second type was related to confusion about the relationship between area and perimeter. Students gave the same number for both area and perimeter. The last type was related to adding a small square to the original square. (A detailed example can be found in the Appendix A)

Scorers were given procedures to use in assessing students' work. Figure 4.1 shows the flowchart for assessing students' work. In the half-hour scorers' training session, I read through the flowchart and explained every step of the assessment to the scorers with an example. The scoring sheet (Appendix C) offered the option for scorers to check each step. Scorers made an evaluation decision at each step, and then circled his/her decision on the scoring sheet. For instance, in the first step, scorers decided whether the student attempted the problem based on the student's written work on the quiz. The first column of the scoring sheet was the variable, Attempt. If a scorer decided that the student did not attempt the problem, s/he circled the sign, NA. If a scorer determined that the student did try to solve the problem, then s/he went to the 2nd step and examined whether the student provided procedures that led to the given answer. The scoring format used was a problem-by-problem format. In other words, scorers assessed the same problem done by all students, then moved on to the next problem.

For the problems that required conceptual analysis, scorers followed the scoring key (Appendix A) for making the scoring decision. Before scorers proceeded to the scoring, I did an introduction to the scoring key on each conceptual analysis problem.

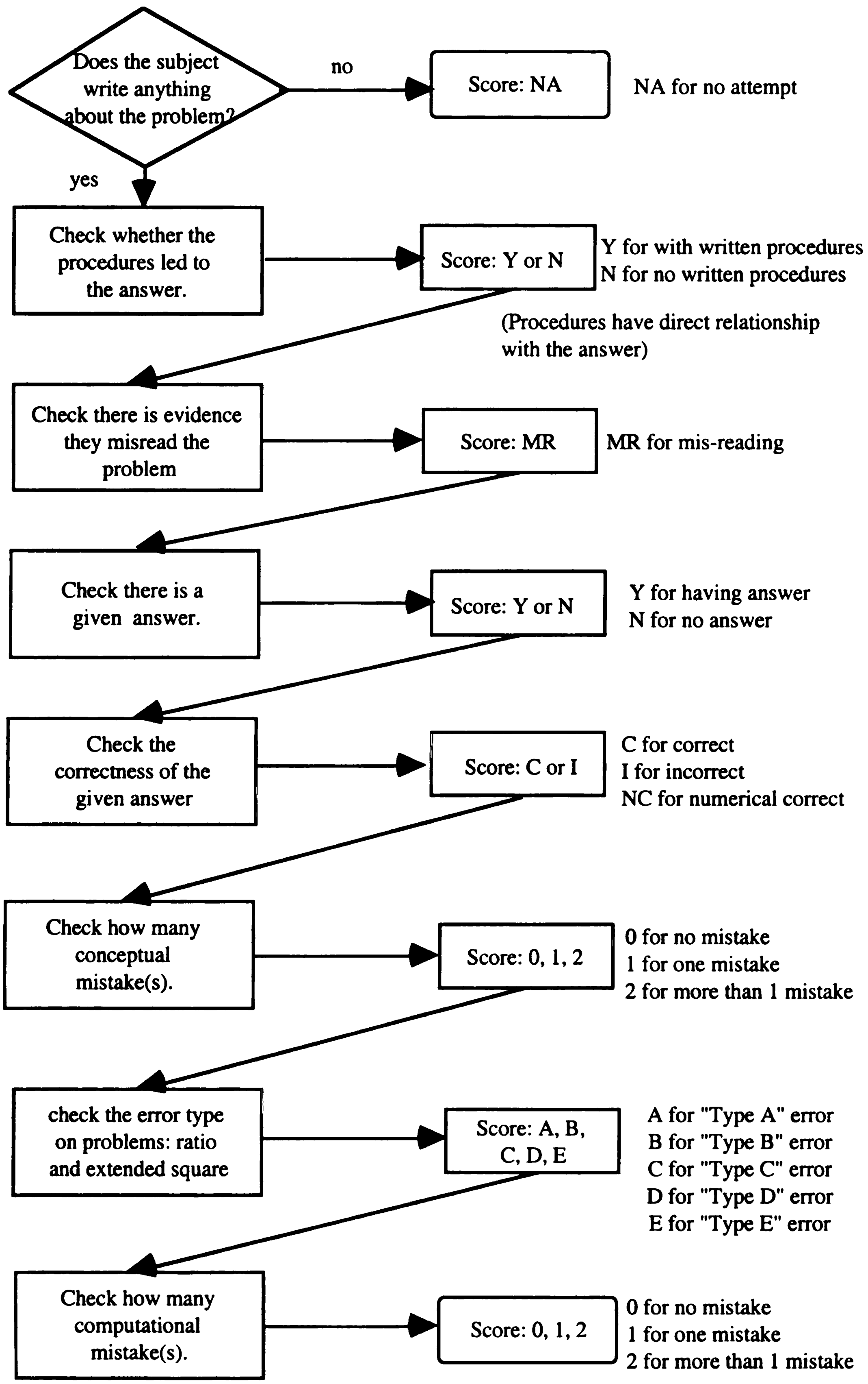


Figure 4.1 Scoring Procedure

Inter-rater Reliability

While some parts of the quiz data (for example, attempt or not) were not relatively trivial to score, other parts were. To ensure the objectivity of the scoring, other scorers were trained to use the scoring scheme to score the students' performances on the quiz in two rounds of scoring.

Kappa coefficient and Spearman correlation were used to measure the inter-rater reliability. One reason for using these non-parametric statistics, Kappa coefficient and Spearman correlation, was that the data I collected may not match with the population distribution (Siegel and Castellan, 1988). In fact, I do not know the exact nature of population distribution. In addition, all of the scoring categories were ordinal and categorical, and non-parametric tests are more suitable for analyzing these kinds of measurement. Because the error type is a categorical variable, the Kappa coefficient was used for assessing the reliability among raters on scores for the conceptual mistakes. For example, there were 3 different types of conceptual mistakes, 1) unit confusion; 2) additive relationship between variables; 3) others, to categorize students' conceptual mistakes in the "Ratio" problem (the Photo-Robber and the Photo-You problem). Because the correctness is an ordinal variable, the Spearman correlation was used for assessing the reliability among raters on scores of the correctness. For the correctness variable, students' performances were categorized into 3 ordinal categories, 1) Correct; 2) Numerical Correct; 3) Incorrect.

First Round Inter-rater Reliability

In the first round, 3 raters, including myself, scored the papers. Scorers did not discuss their scoring after each session. I underestimated how much time was needed for scorers who were new to the students' work. The two additional scorers only evaluated 3 problems on the quiz in one night. Under the original plan, I estimated one night would be enough for scoring all the of problems. The time arrangement pressed scorers and they made more mistakes when they scored the third problem, the "Cake" problem. As a result of these issues, the inter-rater reliability coefficients were not as robust. Also, I did not emphasize the possibility that students might make double or triple errors in a problem. The result was that the two new scorers easily neglected the second and third errors that some students made. Based on the unsatisfactory results of this trial, I decide to have another go at the scoring.

Table 4.5 First Round Inter-rater Reliability in Form A (N = 58)

Problem Name	Problem Type	Conceptual Mistake (k=inter-rater reliability)
Photo-Robber	Non-Routine	Type A Error: k = .75* Type B Error: k = .73* Type C Error: k = .55*
Extended Square (B)	Non-Routine	Type A Error: k = .46* Type B Error: k = .74* Type C Error: k = .53* Type D Error: k = .71* Type E Error: k = .77*
Extended Square (C)	Non-Routine	Type A Error: k = .58* Type B Error: k = .75* Type C Error: k = .57* Type D Error: k = .69* Type E Error: k = .70*

Table 4.6 First Round Inter-rater Reliability in Form B (N = 65)

Problem Name	Problem Type	Conceptual Mistake (k=inter-rater reliability)
Cake (A)	Non-Routine	Type A Error: k = .02* Type B Error: k = .64*
Cake (B)	Non-Routine	Type A Error: k = .05* Type B Error: k = .60*
Photo-You	Non-Routine	Type A Error: k = .76* Type B Error: k = .77* Type C Error: k = .75*
Extended Square (B)	Non-Routine	Type A Error: k = .65* Type B Error: k = .76* Type C Error: k = .62* Type D Error: k = .73* Type E Error: k = .68*
Extended Square (C)	Non-Routine	Type A Error: k = .76* Type B Error: k = .77* Type C Error: k = .75* Type D Error: k = .82* Type E Error: k = .74*

* Key: k = Kappa Coefficient

Second Round Inter-rater Reliability

In the second trial, I had two raters for scoring, including myself. Two raters discussed their differences on scoring after they finished the scoring on all of the students' performances. The scorers took sufficient time to score the papers—one week. The scorers scored 2 problems per day. Also, the additional scorer was reminded of the possibility that students might have double or triple errors in their work. There were 45 instances of disagreement between the two scorers, but the scorers reached an agreement on all the scores after discussion—except for two scores that will be described and reported in the coming paragraphs.

Table 4.7 Second Round Inter-rater Reliability in Form A (N = 58)

Problem Name	Answer Correctness					Conceptual Mistake					Computing Error		
	C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Multiplication [R]	$r_S=1$		$r_S=1$										
Unknown Digit [R]	$r_S=1$		$r_S=1$		$r_S=1$								
Balance [N]	$.95 < r_S < .99$	$.95 < r_S < .99$	$.95 < r_S < .99$	$.95 < r_S < .99$	$.95 < r_S < .99$								
Glass House (A) [N]			$r_S=1$	$r_S=1$	$r_S=1$								
Glass House (B) [N]	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$								
Coins [N]	$r_S=1$		$r_S=1$	$r_S=1$									
Lake [N]	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$								
Photo-Robber [N]	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$	k=1	k=1	k=1			$r_S=1$	$r_S=1$	$r_S=1$
Extended Square (B) [N]	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$	$r_S=1$	k=1	k=1	k=1	k=1	k=1	$r_S=1$	$r_S=1$	$r_S=1$
Extended Square (C) [N]	$r_S=1$		$r_S=1$	$r_S=1$	$r_S=1$	k=1	k=1	k=1	k=1	k=1	$r_S=1$	$r_S=1$	$r_S=1$
Saving [R]	$r_S=1$	$r_S=1$	$r_S=1$		$r_S=1$								
Rectangle Width [R]	$r_S=1$	$r_S=1$	$r_S=1$		$r_S=1$								

Keys:
C: Correct
NC: Numerically Correct
I: Incorrect
UC: Unclassified
UA: Unattempted

k: Kappa coefficient
r_S: Spearman correlation
[R]: Routine Problem
[N]: Non-routine Problem

Table 4.8 Second Round Inter-rater Reliability in Form B (N = 65)

Problem Name	Answer Correctness					Conceptual Mistake					Computing Error		
	C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Multiplication [R]	r_s=1		r_s=1										
Unknown Digit [R]	r_s=1		r_s=1	r_s=1									
Number Pattern (B) [N]	r_s=1		r_s=1		r_s=1								
Number Pattern (C) [N]	r_s=1		r_s=1		r_s=1								
Number Pattern (D) [N]	r_s=1		r_s=1		r_s=1								
Lake [N]	r_s=1		r_s=1	r_s=1	r_s=1								
Cake (A) [N]	r_s=1	r_s=1	r_s=1			k=1	k=1				r_s=1	r_s=1	r_s=1
Cake (B) [N]	r_s=1	r_s=1	r_s=1			k=1	k=1				r_s=1	r_s=1	r_s=1
Photo-You [N]	r_s=1	r_s=1	r_s=1	r_s=1	r_s=1	k=1	k=1	k=1			r_s=1	r_s=1	r_s=1
Balance [N]	r_s=1	r_s=1	r_s=1	r_s=1	r_s=1								
Magazine [R]	r_s=1	r_s=1	r_s=1	r_s=1	r_s=1								
Extended Square (B)[N]	r_s=1	r_s=1	r_s=1	r_s=1	r_s=1	k=1	k=1	k=1	k=1	k=.97	r_s=.98	r_s=.98	r_s=.98
Extended Square (C) [N]	r_s=1	r_s=1	r_s=1		r_s=1	k=1	k=1	k=1	k=1	k=.97	r_s=.98	r_s=.98	r_s=.98
Rectangle Width [R]	r_s=1	r_s=1	r_s=1										

Keys:

C:	Correct	k:	Kappa coefficient
NC:	Numerically Correct	r_s:	Spearman correlation
I:	Incorrect	[R]:	Routine Problem
UC:	Unclassified	[N]:	Non-routine Problem
UA:	Unattempted		

Observations with Disagreement

Although there were discussions between the two scorers after scoring in the second scoring trial, there were still two unresolved scores. The two instances appeared on different problems in two different forms. One was the "Balance" problem in Form A and another was the "Extended Square" problem in Form B. These two results made the Spearman correlation and Kappa coefficient lower than the other scores. In the following paragraphs, I will discuss the disagreements between the two scorers on these two problems.

<u>Instance 1</u>

The "Balance" problem was one in which the two scorers did not agree. One of the scorers considered the answers, 9 and 5 written on the balance stand as incorrect. The other scorer judged that the student tried the problem, but no answer was given.

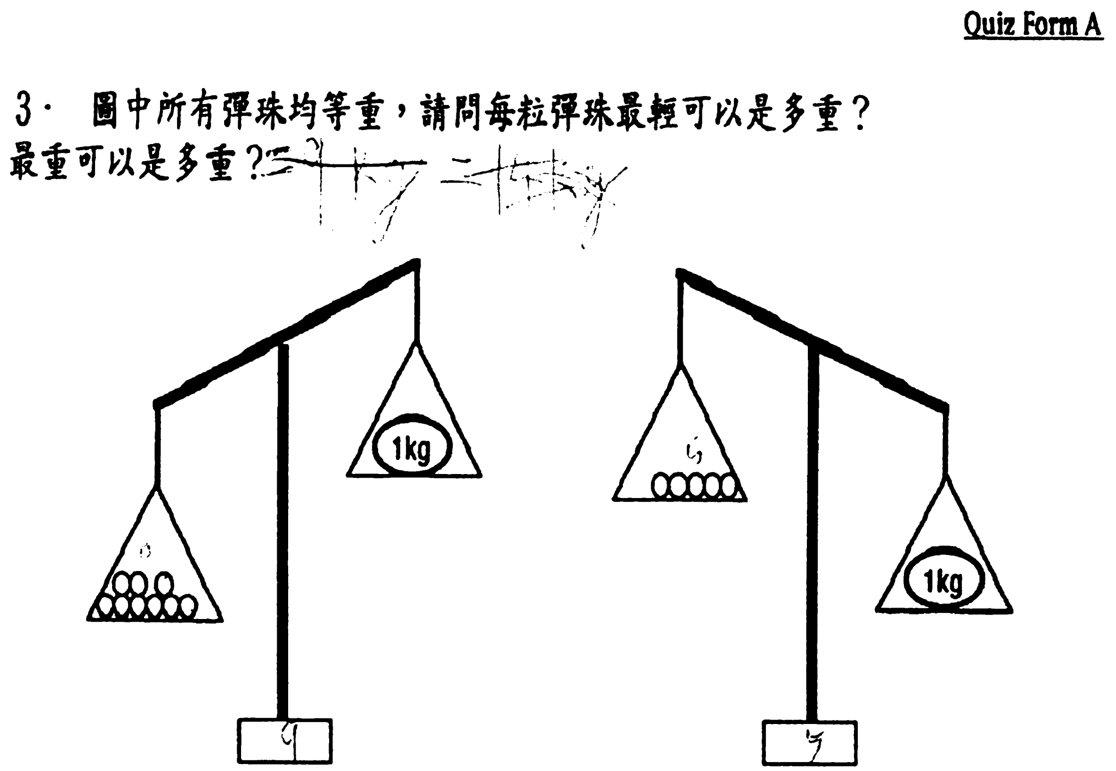
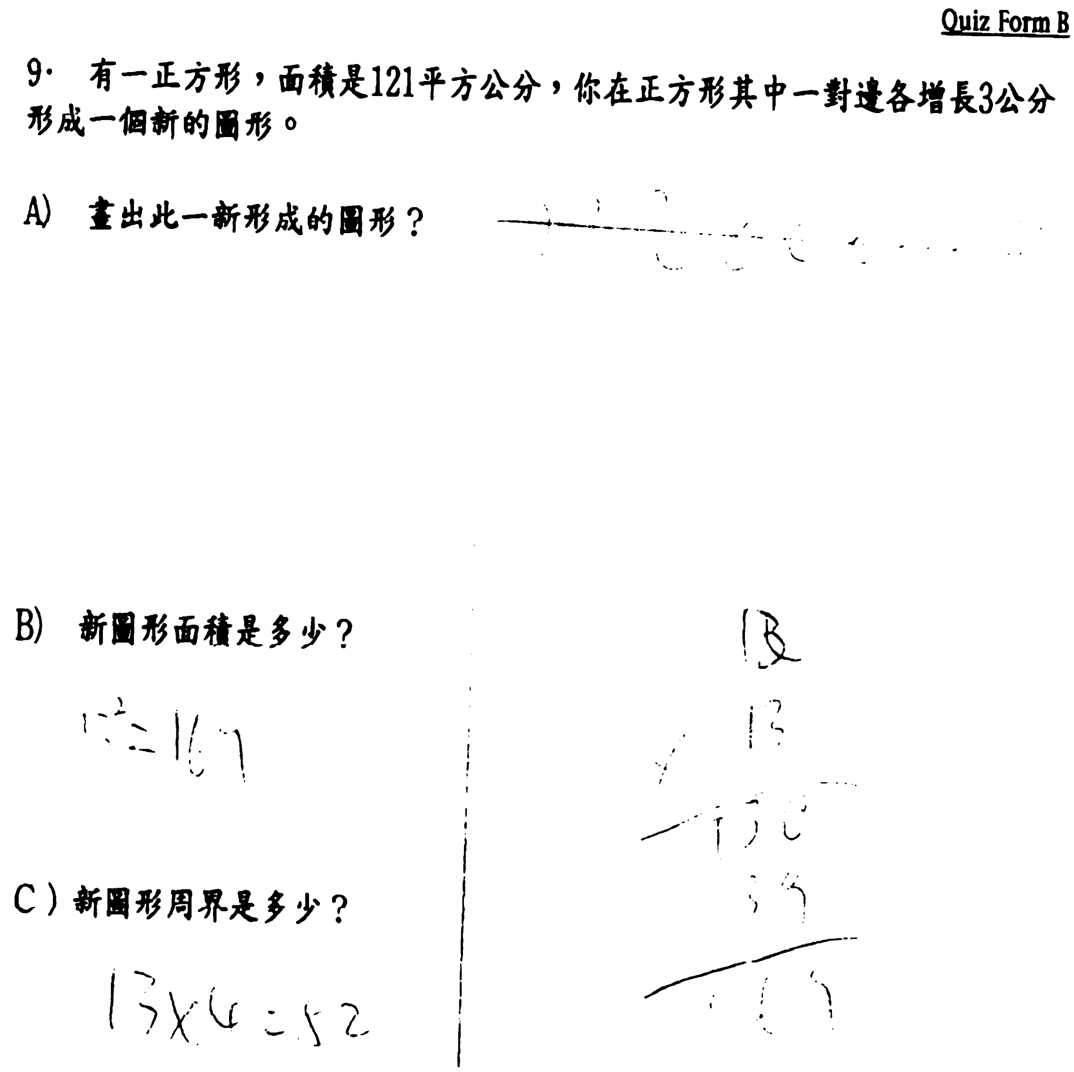


Figure 4.2 A Student's Work on the Balance Problem

Instance 2

On the "Extended Square" problem, the scorers also disagreed. The students did not provide the complete procedures showing how s/he figured out that the new area was 169 and the new perimeter was 52. One scorer maintained that the student committed a Type E error. The Type E error was the "Others" type that the scoring scheme did not classify. Another scorer preferred to judge that the student made a computational error; the student tried to find the square root of 121 cm^2 and added this length to 3 cm, but the student made a computing error in adding two number and got the sum, 13.


Quiz Form B

9· 有一正方形，面積是121平方公分，你在正方形其中一對邊各增長3公分形成一個新的圖形。

A) 畫出此一新形成的圖形？

B) 新圖形面積是多少？

13² = 169

C）新圖形周界是多少？

13×4 = 52

Figure 4.3 A Student's Work on the Extended Square Problem

Chapter 5

RESULT

In this chapter, results from two kinds of data are reported: 1) an analysis of all 123 student performances on the paper-and-pencil quiz, and 2) a summary of the performances of 27 students from two different schools who worked on one to three word problems in individual interviews. The main foci of the quiz analysis were the answers' correctness, the conceptual mistakes made by students, and their computing errors. In the second part, the main focus was the conceptual analysis of their performance.

Quiz Result

The quiz scoring scheme was designed to assess 3 different dimensions of students' performances. The first dimension was the correctness of the answer. Raters judged the answers given by the students; they did not try to assess their thinking processes. In other words, we classified students' performances into 5 different categories,

1) correct: the answer is correct,

2) numerically correct: the number is correct, but a wrong or no unit was given. For example, the correct answer should be 196 square centimeters, but a student wrote 196 cm or 196 without any unit as her/his answer. As a result, the answer will be judged as numerically correct,

3) incorrect: the answer is numerically incorrect,

4) unclassified: students tried to work on the problem, but no answer was given;

5) unattempted: there is no evidence that the students tried the problem was given.

The second dimension was a scheme to assess possible conceptual mistakes. From the students' performances on the problems, several conceptual error categories were developed. All the conceptual error categories were based on common mistakes made by students in which the error(s) could be explained and were related to one or more mathematical concepts. All uncommon errors and unexplainable errors were categorized as "Other." There were 3 types of errors —"unit error," "additive error," and "other"—in the "Ratio-Photo" problem. There were 5 types of errors—"unit error," "area & length confusion," "area & perimeter confusion," "adding a new area," and "other"—in the "Extended Square" problem. And, there were 2 types of errors—"unit error" and "other"—in the "Cake" problem.

The third dimension assessed students' computational errors. All computational mistakes in using the wrong numbers for computation were categorized as an error in computation. Scorers also counted how many computational mistake(s) were made by students in each problem after they assessed the students' performances. So, there were three numerical codes for computational mistakes in this category, 1) no mistake, 2) one computational mistake, 3) more than one computational mistake.

To summarize the results in the second round of scoring, I will present the general result first. In both schools, most of the students performed quite well on the routine problems—more than 65% of the students got the correct or numerically correct answer on those problems. In Form A, students generally performed quite well on the 3 routine problems—"Multiplication," "Unknown Digit," and "Rectangle Width"—more than 74% of the students gave correct responses. Only the "Saving" problem was an outlier, only 43% of the students completed this problem with the correct answer. Many did not give the unit in their answers. In Form B, students performed quite well on the 3 routine problems—"Multiplication," "Unknown Digit," and " Rectangle Width"—more than 69% of the students were able to finish the problems with correct responses. The students did not perform as well on the routine problem, the "Magazine". Only 55% of the students finished the problem with correct responses, and 12% of those students gave a wrong unit or missed the unit in their answer. The students' performances on the two forms were quite consistent. Their correctness on the routine problems were consistent on the four parallel problems. "Saving" and "Magazine" were two parallel problems for which students needed similar mathematical competence—understanding the text and relationship (addition, subtraction, multiplication, and division) between the quantities in the problems. Two problems, "Saving" and "Magazine," had similar story structures, but different story contents. And, students seemed to have more difficulty solving these problems in contrast to the other 3 routine problems.

On the non-routine problems, students' performances were not as good as those on the routine problems. Except in the "Cake" problem, more than 63% of the students solved this non-routine problems correctly. There was one interesting phenomenon on the two parallel "Ratio" problems; students who worked on Form B out-performed students who worked on Form A. The percent of correctness was 44.6% and 19% respectively.

There were several general characteristics of students' performances on the quiz. First, the simpler the language and the fewer the computations required in the problem, the higher the correctness of performance was on the routine problems. For example, "Multiplication," "Rectangle Length," and the "Rectangle Width" were the problems with simplest language content and fewest computational requirements. Their percentages of correctness were the highest, with more than 80% of the students getting the correct or numerically correct answers. Second, except for the "Cake" and "Number Pattern" problems, students did not perform as well on non-routine problems. Less than 45 percent of the students finished the non-routine problems with the correct answers. Third, the non-routine problems, the "Cake" and "Glass House" problems, were problems with similar story structure and with complicated stories that needed to be comprehended. The problems were unfamiliar to students. However, the students' performances on these two problems were quite different. Less than 4% of the students were able to finish the "Glass House" problem, while more than 63% of the students could solve the "Cake" problem. Fourth, among the non-routine problems, students' performances on the two "Ratio" problems in the two different Forms (A/B) were quite surprising. The two problems were very similar in structure and language content. The only difference was the story situation. Less than

20% of the students who worked on Form A could solve the problem. However, more than 44% of the students who worked on Form B could solve the problem. Finally, the conceptual analysis among the three non-routine problems ("Cake," "Ratio," and "Extended Square") revealed that unit confusion was a major difficulty for students.

Comparing the two schools from two communities, students' performances in the low SES school were weaker than that of the students' performances in the middle class school. Except in the cases of the "Multiplication," "Cake" (Part A), and "Balance" problems, students from the middle class school out-performed the students from the low SES school. On 14 out of 24 problem-items, there was more than a 10% difference between the two schools, favoring the middle class school. The problems that magnified the differences (more than 20% on both forms) between the two groups were the "Ratio" and the "Extended Square" problems. The conceptual analysis of these problems revealed that students in the low SES community experienced more difficulty in understanding the context of the problems, and experienced greater confusion with the units given in these problems. More than 25% of the students in the low SES schools confused the units on the "Ratio" (Photo-Robber and Photo-You) and "Extended Square" problems. In addition to their achievement level, students in the low SES school were quite different than their counterparts in their level of motivation. Students in the low SES school seemed to give up more easily on difficult problems. For example, more than 75% of the students did not attempt the "Glass House, part b" problem; and more than 25% of the students did not attempt the "Extended Square, part c" problem. Only 52% of the students from the middle class school did not attempt the "Glass House, part b" problem, and 3% of them did not attempted the "Extended Square, part c" problem.

Over the different achievement levels, low achievers performed poorly, in general, but quite well on the "Multiplication" and "Cake" problems. Their performances on the "Cake" problem were surprising as most of them did not have conceptual difficulty with the problem. The differences between the high achievers and the mid achievers were generally not very great. The only exception was the "Coins" problem, for which more than 55% of the high achievers solved the problem. Only about 5% of the middle achievers gave correct solutions. On a conceptual level, low achievers had more difficulty on the "Extended Square" problem. Many of their errors were classified as "Other" type, which means I could not interpret their errors. Between 30 to 53% of the student responses reflected an "Other" type error on the "Extended Square" problem in both forms. However, less than 16% of the other two groups (high and middle achievers) showed this type of error. Low achievers showed less motivation and seemed to give up easily on difficult problems. For instance, more than 92% of the low achievers did not attempted the "Glass House" problem, compared to other groups whose highest unattempted percent was 63% .

In the following tables, students' performances on both forms of the quiz will be revealed. The first set of tables present the general performance of the students from both schools. The second set of tables present the performance data based on students from different schools. These tables let us examine the differences in performance between the two groups. The last set of tables present the performance data based on students' achievement levels. These tables allow us to examine whether students with different achievement levels performed differently on two different kinds of problems.

Table Description

These tables have a unique format; they reveal the students' performances on three dimensions—correctness, conceptual, and computational. Under the correctness dimension, students' performances were categorized into 5 types:

1) "C" for a correct answer was given,

2) "NC" for a numerically correct answer was given,

3) "I" for an incorrect answer was given,

4) "UC" for an unclassified trial was given, which means that students tried to solve the problem, but did not given an answer, based on what they wrote on the quiz sheet,

5) "UA" for unattempt, which means no written information was provided on the quiz sheet.

Under the computational dimension, students' performances were categorized into 3 types: 1) 0 for no computational mistake, 2) 1 for one computational mistake, 3) 2 for more than one computational mistake.

The conceptual dimension was more complicated. Three different problems had their own scoring schemes for conceptual mistakes. The following paragraphs briefly describe the mistake categories on each problem:

Cake Problem

Type A Mistake is a unit confusion mistake. For example, a student wrote, \$963 - 11 x \$45 = 468 pieces. This algorithm can only offer the dollar unit, not pieces. Because, 11 cake multiplied by 45 dollars equals the total price of 11 cakes.

Type B Mistake is the "Other" type of mistake. This means that I could not understand or interpret the written work given by the students. For example, a student wrote the following algorithms and answer, 12 x 6 = 72 => 62 x 6 = 372 => 372 - 72 = \$300. This students seemed to put all the given numbers in the problem and manipulated them with arithmetic operations. And, all the arithmetic operations did not make sense within the context of the problem.

Ratio Problem (the Photo-Robber and the Photo-You problem)

Type A Mistake is a unit confusion mistake. For example, a student wrote 10 cm x 4 cm = 40 cm (it should be 40 cm^2). Although there is a multiplicative relationship between the variables in the problem, the relationship should not consist of picking up 2 numbers and multiplying them together. And, some students wrote, 1.4 m x 4 cm = 5.6 cm.

Type B Mistake is an additive model mistake. For example, a student wrote 1.4 + 6 = 2 m (6 is equal to 10 cm - 4 cm). the student thought that ratio was an additive relationship between the variables. In addition to the additive model mistake, this student violated another conceptual mistake (unit confusion) and a computational error in this case.

Type C Mistake is the "Other" type of mistake. For example, a student wrote on the quiz, 1.4 / 2 + 1.4 / 2 = 1.4. I could not interpret this algorithm. I really did not know what and how the student thought about ratio and the problem.

Extended Square Problem

Type A Mistake is a unit confusion mistake. There were many types of confusion in this unit. The most common one was adding cm^2 with cm. For example, a student wrote, (121 + 3)(121), on the quiz. The number 121 was an area, its unit was cm^2. The number 3 was a length and its unit was cm.

Type B Mistake is an area-length confusion mistake. In other words, the students confused the two different measurements for two different dimensions (area and length). For example, many students thought that dividing the area of a square by 4 would give them the length of a side.

Type C Mistake is an area-perimeter confusion mistake. Few students could not distinguish between area and perimeter, they gave the same answer to two different questions in this problem set.

Type D Mistake is an additive type mistake. A small square was added to the original square. For example, a student wrote, $121\ cm^2 + 3\ cm \times 3\ cm = 130\ cm^2$, on the quiz sheet.

Type E Mistake is the "Other" type of mistake. This type of error was more diverse and most of the mistakes could not be interpreted within the problem context and mathematically. For example, a student wrote 6.25 x 4 = 3200 as the answer for perimeter. Or, students just tried to use all the given numbers with different mathematical operations, but without any contextual and mathematical sense.

Table 5.1 Performance from Both Schools on Form A (N = 58)

Problem	R	Answer Correctness					Conceptual Mistake					Computing Error		
[Type]		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Multiplication [RO]	1	84.5 (49)		15.5 (9)										
Rectangle Width [RO]	2	79.3 (46)	10.3 (6)	8.6 (5)		1.7 (1)								
Unknown Digit [RO]	3	74.1 (43)		24.1 (14)		1.7 (1)								
Saving [RO]	4	43.1 (25)	22.4 (13)	32.8 (19)		1.7 (1)								
Balance [NR]	11	3.4 (2)	1.7 (1)	77.6 (45)	13.8 (8)	3.4 (2)								
Glass House (A) [NR]	12			77.6 (45)	1.7 (1)	20.7 (12)								
Glass House (B) [NR]	10	3.4 (2)	3.4 (2)	25.9 (15)	1.7 (1)	65.5 (38)								
Coins [NR]	7	12.1 (7)		77.6 (45)	10.3 (6)									
Lake [NR]	9	10.3 (6)	1.7 (1)	67.2 (39)	12.1 (7)	8.6 (5)								
Photo-Robber [NR]	5	19.0 (11)	5.2 (3)	63.8 (38)	5.2 (2)	6.9 (4)	39.7 (23)	36.2 (21)	13.8 (8)			60.3 (35)	17.2 (10)	0.0 (0)
Extended Square (B) [NR]	8	10.3 (6)	5.2 (3)	72.4 (42)	6.9 (4)	5.2 (3)	27.6 (16)	24.1 (14)	5.2 (3)	6.9 (4)	19.0 (11)	56.9 (33)	20.7 (12)	0.0 (0)
Extended Square (C) [NR]	6	19.0 (11)		69.0 (40)	1.7 (1)	10.3 (6)	27.6 (16)	27.6 (16)	6.9 (4)	0.0 (0)	20.7 (12)	74.1 (42)	5.2 (3)	0.0 (0)

Keys:
- C: Correct
- NC: Numerically Correct
- I: Incorrect
- UC: Unclassified
- UA: Unattempted
- R: Rank of the Correctness
- RO: Routine Problem
- NR: Non-Routine Problem

Table 5.2 Performance from Two Schools on Form B ($N = 65$)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Multiplication [RO]	1	81.5 (53)		18.5 (12)										
Rectangle Width [RO]	3	74.4 (49)	15.4 (10)	9.2 (6)										
Unknown Digit [RO]	4	69.2 (45)		26.2 (17)	4.6 (3)									
Magazine [RO]	6	55.4 (36)	12.3 (8)	27.7 (18)	3.1 (2)	1.5 (1)								
Number Pattern (B) [NR]	7	55.4 (36)		43.1 (28)		1.5 (1)								
Number Pattern (C) [NR]	9	41.5 (27)		47.7 (31)		10.8 (7)								
Number Pattern (D) [NR]	12	23.1 (15)		72.3 (47)		4.6 (3)								
Lake [NR]	13	13.8 (9)		75.4 (49)	1.5 (1)	9.2 (6)								
Cake (A) [NR]	2	76.9 (50)	1.5 (1)	20.0 (13)	1.5 (1)		10.8 (7)	6.2 (4)				93.8 (61)	4.6 (3)	0.0 (0)
Cake (B) [NR]	5	63.1 (41)	6.2 (4)	30.8 (20)			16.9 (11)	12.3 (8)				81.5 (53)	15.4 (10)	3.1 (2)
Photo-You [NR]	8	44.6 (29)	3.1 (2)	44.6 (29)	3.1 (2)	4.6 (3)	24.6 (16)	10.8 (7)	21.5 (14)			73.8 (48)	6.2 (4)	0.0 (0)
Balance [NR]	14	1.5 (1)	1.5 (1)	83.1 (54)	9.2 (6)	4.6 (3)								
Extended Square (B) [NR]	11	23.1 (15)	1.5 (1)	67.7 (44)	4.6 (3)	3.1 (2)	26.2 (17)	15.4 (10)	3.1 (2)	3.1 (2)	21.5 (14)	69.2 (45)	12.3 (8)	4.6 (3)
Extended Square (C) [NR]	10	23.1 (15)	3.1 (2)	58.5 (38)		15.4 (10)	20.0 (13)	15.4 (10)	1.5 (1)	0.0 (0)	15.4 (10)	70.8 (46)	9.2 (6)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

On the correctness dimension, students in both schools performed well on the routine problems on both forms. On Form A, the routine problems received the highest percentage of correct responses. On Form B, the three routine problems ("Multiplication," "Rectangle Width," and "Unknown Digit") received the highest percentage of correct responses. The "Cake" problem was the only outlier in this study. It was a non-routine problem, but its percentage of correct responses was higher than that of the routine problem, "Saving".

On the conceptual dimension, the major mistake in the 5 problems (4 of them were 2 sets of parallel problems) was related to unit confusion. On both forms, unit confusion had the highest percentage of conceptual errors on each item. This result revealed that students had more difficulty conceptualizing or understanding the units, or did not pay attention to the units when they tried to solve the problems.

On the computational dimension, using the 5 analyzed problems to examine the students' computational errors, the results revealed that the students in both schools did not experience many difficulties doing the computations. The highest percentage of error was 20%. On both forms, students seemed to make more computational errors on the "Extended Square" problem.

Table 5.3 Students with Different Achievement Levels Distribution

	High Achievers	Mid Achievers	Low Achievers
Low SES School*	4 (Form A) 3 (Form B) Total = 7	22 (Form A) 23 (Form B) Total = 45	6 (Form A) 8 (Form B) Total = 14
Middle Class School	5 (Form A) 9 (Form B) Total = 14	13 (Form A) 16 (Form B) Total = 29	7 (Form A) 5 (Form B) Total = 12

* Two students in the low SES school were new to the school; their performance records were not available when I collected the data, so both were not in these achievement level categories. Because of these two missing students, the total number of students were 121 in this table.

Table 5.4 Performance from Low SES School on Form A (N = 33)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Multiplication [RO]	1	87.9 (29)		12.1 (4)										
Rectangle Width [RO]	2	75.8 (25)	12.1 (4)	9.1 (3)		3.0 (1)								
Unknown Digit [RO]	3	66.7 (22)		30.3 (10)		3.0 (1)								
Saving [RO]	4	36.4 (12)	30.3 (10)	30.3 (10)		3.0 (1)								
Balance [NR]	8		3.0 (1)	72.7 (24)	18.2 (6)	6.1 (2)								
Glass House (A) [NR]	9			66.7 (22)	3.0 (1)	30.3 (10)								
Glass House (B) [NR]	8		3.0 (1)	18.2 (6)	3.0 (1)	75.8 (25)								
Coins [NR]	6	9.1 (3)		78.8 (26)	12.1 (4)									
Lake [NR]	5	9.1 (3)	3.0 (1)	69.7 (24)	9.1 (3)	9.1 (3)								
Photo-Robber [NR]	5	9.1 (3)	3.0 (1)	72.7 (23)	6.1 (2)	9.1 (3)	42.4 (14)	39.4 (13)	15.2 (5)			54.5 (18)	15.2 (5)	0.0 (0)
Extended Square (B) [NR]	7	6.1 (2)	3.0 (1)	69.7 (23)	12.1 (4)	9.1 (3)	36.4 (12)	30.3 (10)	6.1 (2)	9.1 (3)	21.2 (7)	48.5 (16)	21.2 (7)	0.0 (0)
Extended Square (C) [NR]	6	9.1 (3)		72.7 (24)	3.0 (1)	15.2 (5)	36.4 (12)	36.4 (12)	9.1 (3)	0.0 (0)	24.2 (8)	72.7 (24)	0.0 (0)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

Table 5.5 Performance from Low SES School on Form B (N = 35)

Problem	R	Answer Correctness					Conceptual Mistake					Computing Error		
[Type]		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Multiplication [RO]	1	82.9 (29)		17.1 (6)										
Rectangle Width [RO]	3	68.6 (24)	20.0 (7)	11.4 (4)										
Unknown Digit [RO]	5	57.1 (20)		37.1 (13)	5.7 (2)									
Magazine [RO]	6	51.4 (18)	20.0(7)	22.9 (8)	5.7 (2)									
Number Pattern (B) [NR]	7	48.6 (17)		48.6 (17)		2.9 (1)								
Number Pattern (C) [NR]	8	31.4 (11)		48.6 (17)		20.0 (7)								
Number Pattern (D) [NR]	10	14.3 (5)		80.0 (20)		5.7 (2)								
Lake [NR]	13	8.6 (3)		80.0 (28)		11.4 (4)								
Cake (A) [NR]	2	77.1 (27)	2.9 (1)	20.0 (7)			11.4 (4)	2.9 (1)				91.4 (32)	8.6 (3)	
Cake (B) [NR]	4	60.0 (21)	2.9 (1)	37.1 (13)			22.9 (8)	17.1 (6)				80.0 (28)	17.1 (6)	2.9 (1)
Photo-You [NR]	9	28.6 (10)	2.9 (1)	57.1 (20)	5.7 (2)	5.7 (2)	37.1 (13)	11.4 (4)	31.4 (11)			71.4 (25)	8.6 (3)	0.0 (0)
Balance [NR]	14			82.9 (29)	11.4 (4)	5.7 (2)								
Extended Square (B) [NR]	12	8.6 (3)		80.0 (28)	5.7 (2)	5.7 (2)	40.0 (14)	22.9 (8)	5.7 (2)	5.7 (2)	28.6 (10)	51.4 (18)	20.0 (7)	5.7 (2)
Extended Square (C) [NR]	11	8.6 (3)	2.9 (1)	62.9 (22)		25.7 (9)	28.6 (10)	22.9 (8)	2.9 (1)	0.0 (0)	20.0 (7)	48.6 (17)	17.1 (6)	0.0 (0)

Keys:
C: Correct
NC: Numerically Correct
I: Incorrect
UC: Unclassified
UA: Unattempted
R: Rank of the Correctness
RO: Routine Problem
NR: Non-Routine Problem

Table 5.6 Performance from Middle Class School on Form A (*N* = 25)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Rectangle Width [RO]	1	84.0 (21)	8.0 (2)	8.0 (2)										
Unknown Digit [RO]	2	84.0 (21)		16.0 (4)										
Multipli-cation [RO]	3	80.0 (20)		20.0 (5)										
Saving [RO]	4	52.0 (13)	12.0 (3)	36.0 (9)										
Balance [NR]	11	8.0 (2)		84.0 (21)	8.0 (2)									
Glass House (A) [NR]	12			92.0 (23)		8.0 (2)								
Glass House (B) [NR]	10	8.0 (2)	4.0 (1)	36.0 (9)		52.0 (13)								
Coins [NR]	8	16.0 (4)		76.0 (19)	8.0 (2)									
Lake [NR]	9	12.0 (3)		64.0 (16)	16.0 (4)	8.0 (2)								
Photo-Robber [NR]	5	32.0 (8)	8.0 (2)	56.0 (14)		4.0 (1)	36.0 (9)	32.0 (8)	12.0 (3)			68.0 (17)	20.0 (5)	0.0 (0)
Extended Square (B) [NR]	7	16.0 (4)	8.0 (2)	76.0 (19)			16.0 (4)	16.0 (4)	4.0 (1)	4.0 (1)	16.0 (4)	68.0 (20)	20.0 (5)	0.0 (0)
Extended Square (C) [NR]	6	32.0 (8)		64.0 (16)		4.0 (1)	16.0 (4)	16.0 (4)	4.0 (1)	0.0 (0)	16.0 (4)	72.0 (18)	12.0 (3)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

Table 5.7 Performance from Middle Class School on Form B (*N* = 30)

Problem [Type]	R	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Rectangle Width [RO]	1	83.3 (25)	10.0 (3)	6.7 (2)										
Unknown Digit [RO]	2	83.3 (25)		13.3 (4)	3.3 (1)									
Multipli-cation [RO]	3	80.0 (24)		20.0 (6)										
Magazine [RO]	8	60.0 (18)	3.3 (1)	33.3 (10)		3.3 (1)								
Number Pattern (B) [NR]	7	63.3 (19)		36.7 (11)										
Number Pattern (C) [NR]	9	53.3 (16)		46.7 (14)										
Number Pattern (D) [NR]	11	33.3 (10)		63.3 (19)		3.3 (1)								
Lake [NR]	12	20.0 (6)		70.0 (21)	3.3 (1)	6.7 (2)								
Cake (A) [NR]	4	76.7 (23)		23.3 (6)	3.3 (1)		10.0 (3)	10.0 (3)				96.7 (29)	0.0 (0)	0.0 (0)
Cake (B) [NR]	5	66.7 (20)	10.0 (3)	23.3 (7)			10.0 (3)	6.7 (2)				83.3 (25)	13.3 (4)	3.3 (1)
Photo-You [NR]	6	63.3 (19)	3.3 (1)	30.0 (9)		3.3 (1)	10.0 (3)	10.0 (3)	10.0 (3)			76.7 (23)	3.3 (1)	0.0 (0)
Balance [NR]	13	3.3 (1)	3.3 (1)	83.3 (25)	6.7 (2)	3.3 (1)								
Extended Square (B) [NR]	10	40.0 (12)	3.3 (1)	53.3 (16)	3.3 (1)		10.0 (3)	6.7 (2)	0.0 (0)	0.0 (0)	13.3 (4)	90.0 (27)	3.3 (1)	3.3 (1)
Extended Square (C) [NR]	10	40.0 (12)	3.3 (1)	53.3 (16)		3.3 (1)	10.0 (3)	6.7 (2)	0.0 (0)	0.0 (0)	10.0 (3)	96.7 (29)	0.0 (0)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

Comparing students' performances across two different schools, the results revealed that students from different schools did not perform differently on most routine problems, except in the case of the "Unknown Digit" problem on Form B. Students in the middle class school outperformed their counterparts. When we consider "Correct" and "Numerically Correct" as correct answers, there is no obvious difference between the two groups on the "Saving" problem. If we see the two categories of correctness as different levels of achievement, students from the middle class school made fewer mistakes (mostly unit mistakes) on the "Saving" problem. The performance of the two groups on the non-routine problems revealed another story. Except in the cases of the 4 problems ("Balance," "Glass House," "Coins," and "Lake") on Form A and 2 problems ("Cake" and "Balance") on Form B, students in the middle class school outperformed their counterparts by more than 10% on the dimension of correctness.

On the conceptual dimension, students in the low SES school seemed to experience more difficulty on the "Extended Square" problem on Form A. They made more conceptual mistakes on unit and area-length. Their error percentage was double that of those made by their counterparts—Extended Square [B]: unit confusion (36% versus 16%) and area-length confusion (36% versus 16%); Extended Square [C]: unit confusion (36% versus 16%) and area-length confusion (36% versus 16%). On Form B, students in the low SES school performed even more poorly. Except in the case of the "Cake" problem Part A, the conceptual mistakes were related to unit confusion. The other non-routine problems ("Ratio" and "Extended Square") completed by the students in the low SES school were double that of the mistakes made by their counterparts (Cake (B): 37% versus 10%; Ratio: 37% versus 10%; Extended Square (B): 40% versus 10%; Extended Square (C): 28% versus 10%). On the "Extended Square" problem, the mistakes made by the students in the low SES school were 3 times that of their counterparts (22% versus 6%) on area-length confusion category.

On the computational dimension, the two groups of students did quite well. The only obvious differences between the two groups were that students in the low SES school made more mistakes on the "Extended Square" problem Part (A). Of the students in the low

SES school, 20% made one computational mistake on this problem, compared to only 3% of the students in the middle class school.

Table 5.8 High Achievers Performance on Form A (N = 9)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Multiplication [RO]	1	100 (9)												
Rectangle Width [RO]	2	88.9 (8)	11.1 (1)											
Unknown Digit [RO]	3	88.9 (8)		11.1 (1)										
Saving [RO]	4	66.7 (6)	11.1 (1)	22.2 (2)										
Balance [NR]	9	22.2 (2)	11.1 (1)	44.4 (4)	22.2 (2)									
Glass House (A) [NR]	11			77.8 (7)		22.2 (2)								
Glass House (B) [NR]	10	11.1 (1)	11.1 (1)	33.3 (3)		44.4 (4)								
Coins [NR]	7	55.6 (5)		33.3 (3)	11.1 (1)									
Lake [NR]	8	33.3 (3)		22.2 (2)	44.4 (4)									
Photo-Robber [NR]	8	33.3 (3)		55.6 (5)	11.1 (1)		44.4 (4)	44.4 (4)	33.3 (3)			88.9 (8)	0.0 (0)	0.0 (0)
Extended Square (B) [NR]	6	55.6 (5)	11.1 (1)	33.3 (3)			0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	0.0 (0)	77.8 (7)	0.0 (0)	0.0 (0)
Extended Square (C) [NR]	5	66.7 (6)		33.3 (3)			11.1 (1)	11.1 (1)	11.1 (1)	0.0 (0)	0.0 (0)	100 (9)	0.0 (0)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

Table 5.9 High Achievers Performance on Form B (N = 12)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Multiplication [RO]	1	91.7 (11)		8.3 (1)										
Rectangle Width [RO]	1	91.7 (11)		8.3 (1)										
Unknown Digit [RO]	2	83.3 (10)		16.7 (2)										
Magazine [RO]	5	66.7 (8)		33.3 (4)										
Number Pattern (B) [NR]	7	50.0 (6)		50.0 (6)										
Number Pattern (C) [NR]	6	58.3 (7)		33.3 (4)		8.3 (1)								
Number Pattern (D) [NR]	8	25.0 (3)		75.0 (9)										
Lake [NR]	10			91.7 (11)	8.3 (1)									
Cake (A) [NR]	2	83.3 (10)		16.7 (2)			8.3 (1)	8.3 (1)				100 (12)	0.0 (0)	0.0 (0)
Cake (B) [NR]	4	66.7 (8)	8.3 (1)	25.0 (3)			16.7 (2)	16.7 (2)				83.3 (10)	8.3 (1)	8.3 (1)
Photo-You [NR]	3	75.0 (9)		25.0 (3)			8.3 (1)	0.0 (0)	16.7 (2)			83.3 (10)	8.3 (1)	0.0 (0)
Balance [NR]	10			91.7 (11)	8.3 (1)									
Extended Square (B) [NR]	9	16.7 (2)		83.3 (10)			16.7 (2)	8.3 (1)	0.0 (0)	0.0 (0)	16.7 (2)	83.3 (10)	16.7 (2)	0.0 (0)
Extended Square (C) [NR]	9	16.7 (2)		83.3 (10)			16.7 (2)	8.3 (1)	0.0 (0)	0.0 (0)	16.7 (2)	0.0 (0)	0.0 (0)	0.0 (0)

Keys:
C: Correct
NC: Numerically Correct
I: Incorrect
UC: Unclassified
UA: Unattempted
R: Rank of the Correctness
RO: Routine Problem
NR: Non-Routine Problem

Table 5.10 Mid Achievers Performance on Form A (N = 35)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Rectangle Width [RO]	1	88.6 (31)	5.7 (2)	2.9 (1)		2.9 (1)								
Multiplication [RO]	2	82.9 (29)		17.1 (6)										
Unknown Digit [RO]	3	77.1 (27)		20.0 (7)		2.9 (1)								
Saving [RO]	4	37.1 (13)	31.4 (11)	28.6 (10)		2.9 (1)								
Balance [NR]	10			88.6 (31)	8.6 (3)	2.9 (1)								
Glass House (A) [NR]	10			82.9 (29)		17.1 (6)								
Glass House (B) [NR]	9	2.9 (1)	2.9 (1)	31.4 (11)		62.9 (22)								
Coins [NR]	7	5.7 (2)		85.7 (30)	8.6 (3)									
Lake [NR]	7	5.7 (2)		74.3 (26)	8.6 (3)	11.4 (4)								
Photo-Robber [NR]	5	22.9 (8)	8.6 (3)	51.4 (18)	5.7 (2)	11.4 (4)	31.4 (11)	28.6 (10)	5.7 (2)			62.9 (22)	11.4 (4)	0.0 (0)
Extended Square (B) [NR]	8	2.9 (1)	5.7 (2)	74.3 (26)	8.6 (3)	8.6 (3)	28.6 (10)	34.3 (12)	0.0 (0)	5.7 (2)	14.3 (5)	60.0 (21)	20.0 (7)	0.0 (0)
Extended Square (C) [NR]	6	14.3 (5)		68.6 (24)	2.9 (1)	14.3 (5)	28.6 (10)	34.3 (12)	0.0 (0)	0.0 (0)	11.4 (4)	74.3 (26)	2.9 (1)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

Table 5.11 Mid Achievers Performance on Form B (N = 39)

Problem [Type]	R	Answer Correctness C	NC	I	UC	UA	Conceptual Mistake A	B	C	D	E	Computing Error 0	1	2
Rectangle Width [RO]	1	76.9 (30)	17.9 (7)	5.1 (2)										
Multiplication [RO]	2	76.9 (30)		23.1 (9)										
Unknown Digit [RO]	4	69.2 (27)		28.2 (11)	2.6 (1)									
Magazine [RO]	7	59.0 (23)	10.3 (4)	25.6 (10)	5.1 (2)									
Number Pattern (B) [NR]	6	61.5 (24)		35.9 (14)		2.6 (1)								
Number Pattern (C) [NR]	9	38.5 (15)		51.3 (20)		10.3 (4)								
Number Pattern (D) [NR]	12	28.2 (11)		64.1 (25)		7.7 (3)								
Lake [NR]	13	23.1 (9)		61.5 (24)		15.4 (6)								
Cake (A) [NR]	3	74.4 (29)		23.1 (9)	2.6 (1)		12.8 (5)	5.1 (2)				92.3 (36)	5.1 (2)	0.0 (0)
Cake (B) [NR]	5	66.7 (26)	2.6 (1)	30.8 (12)			20.5 (8)	7.7 (3)				82.1 (32)	15.4 (6)	2.6 (1)
Photo-You [NR]	8	43.6 (17)	5.1 (2)	41.0 (16)	5.1 (2)	5.1 (2)	25.6 (10)	12.8 (5)	23.1 (9)			74.4 (29)	7.7 (3)	0.0 (0)
Balance [NR]	14	2.6 (1)	2.6 (1)	79.5 (31)	10.3 (4)	5.1 (2)								
Extended Square (B) [NR]	11	28.2 (11)	2.6 (1)	59.0 (23)	5.1 (2)	5.1 (2)	28.2 (11)	20.5 (8)	5.1 (2)	5.1 (2)	12.8 (5)	66.7 (26)	10.3 (4)	7.7 (3)
Extended Square (C) [NR]	10	28.2 (11)	5.1 (2)	51.3 (20)		15.4 (6)	23.1 (9)	20.5 (8)	0.0 (0)	0.0 (0)	10.3 (4)	69.2 (27)	10.3 (4)	0.0 (0)

Keys:

- C: Correct
- NC: Numerically Correct
- I: Incorrect
- UC: Unclassified
- UA: Unattempted
- R: Rank of the Correctness
- RO: Routine Problem
- NR: Non-Routine Problem

Table 5.12 Low Achievers Performance on Form A (N = 13)

Problem [Type]	R	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Multiplication [RO]	1	84.6 (11)		15.4 (2)										
Unknown Digit [RO]	2	53.8 (7)		46.2 (6)										
Rectangle Width [RO]	3	46.2 (6)	23.1 (3)	30.8 (4)										
Saving [RO]	4	46.2 (6)		53.8 (7)										
Balance [NR]	6			69.2 (9)	23.1 (3)	7.7 (1)								
Glass House (A) [NR]	6			61.5 (8)	7.7 (1)	30.8 (4)								
Glass House (B) [NR]	6			7.7 (1)		92.3 (12)								
Coins [NR]	6			84.6 (11)	15.4 (2)									
Lake [NR]	5		7.7 (1)	84.6 (11)		7.7 (1)								
Photo-Robber [NR]	6			100 (13)			53.8 (7)	46.2 (6)	23.1 (3)			30.8 (4)	46.2 (6)	0.0 (0)
Extended Square (B) [NR]	6			92.3 (12)	7.7 (1)		46.2 (6)	15.4 (2)	23.1 (3)	7.7 (1)	46.2 (6)	23.1 (3)	38.5 (5)	0.0 (0)
Extended Square (C) [NR]	6			92.3 (12)		7.7 (1)	38.5 (5)	15.4 (2)	23.1 (3)	0.0 (0)	53.8 (7)	53.8 (7)	7.7 (1)	0.0 (0)

Keys:

C:	Correct	UA:	Unattempted
NC:	Numerically Correct	R:	Rank of the Correctness
I:	Incorrect	RO:	Routine Problem
UC:	Unclassified	NR:	Non-Routine Problem

Table 5.13 Low Achievers Performance on Form B (N = 13)

Problem [Type]	R	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Multiplication [RO]	1	84.6 (11)		15.4 (2)										
Rectangle Width [RO]	3	53.8 (7)	23.1 (3)	23.1 (3)										
Unknown Digit [RO]	4	53.8 (7)		30.8 (4)	15.4 (2)									
Magazine [RO]	8	30.8 (4)	30.8 (4)	30.8 (4)		7.7 (1)								
Number Pattern (B) [NR]	6	46.2 (6)		53.8 (7)										
Number Pattern (C) [NR]	7	38.5 (5)		46.2 (6)		15.4 (2)								
Number Pattern (D) [NR]	10	7.7 (1)		92.3 (12)										
Lake [NR]	11			100 (13)										
Cake (A) [NR]	2	76.9 (10)	7.7 (1)	15.4 (2)			7.7 (1)	7.7 (1)				92.3 (12)	7.7 (1)	0.0 (0)
Cake (B) [NR]	5	46.2 (6)	15.4 (2)	38.5 (5)			7.7 (1)	23.1 (3)				76.9 (10)	23.1 (3)	0.0 (0)
Photo-You [NR]	9	15.4 (2)		76.9 (10)		7.7 (1)	38.5 (5)	15.4 (2)	23.1 (3)			61.5 (8)	0.0 (0)	0.0 (0)
Balance [NR]	11			84.6 (11)	7.7 (1)	7.7 (1)								
Extended Square (B) [NR]	10	7.7 (1)		84.6 (11)	7.7 (1)		30.8 (4)	7.7 (1)	0.0 (0)	0.0 (0)	53.8 (7)	61.5 (8)	15.4 (2)	0.0 (0)
Extended Square (C) [NR]	10	7.7 (1)		61.5 (8)		30.8 (4)	15.4 (2)	7.7 (1)	7.7 (1)	0.0 (0)	30.8 (4)	53.8 (7)	7.7 (1)	0.0 (0)

Keys:
C: Correct
NC: Numerically Correct
I: Incorrect
UC: Unclassified
UA: Unattempted
R: Rank of the Correctness
RO: Routine Problem
NR: Non-Routine Problem

On the dimension of correctness, there were no obvious differences among 3 groups at the general level of pattern. Routine problems received the highest percentage of correctness among the 3 groups. The "Cake" problem was the only exceptional non-routine problem. The students received a high percentage of correctness on this problem. On the routine problems, the high achiever group received the highest percentages on all of these problems. The middle achiever group performed at a level between the two extreme groups, high and low achiever groups. The only exception to this case was the "Multiplication" problem; the middle achiever group received the lowest percentage of correctness on both forms.

The results on the non-routine problems were mixed. The high achiever group received the highest percentages of correctness on most of these problems. However, they performed poorly on the "Extended Square" problem on Form B. It seemed strange, the other group of high achievers did quite well on the parallel problem on Form A.

On Form A, the high achiever group outperformed the other two groups on all non-routine problems. And, the low achiever group did not respond correctly on these problems.

On Form B, the differences among the 3 groups were not so obvious when it came to dimension of correctness. The middle achiever group performed better than other two groups on several non-routine problems ("Lake," "Balance," "Extended Square,"), although the differences were not as great.

On the conceptual dimension, unit confusion mistakes were the most common ones made by the 3 groups across the 5 non-routine problems. In general, high achievers made fewer mistake on these 5 problems. The only exception was the "Ratio" problem on Form A. The "Extended Square" problems on both forms distinguished the high achievers from the other two groups, on the dimension of conception. The total percentage of mistakes made by the high achievers were far fewer than that of the other two groups.

On the computational dimension, 3 groups did quite well on the 5 non-routine problems. The only exception was the low achiever group who worked on the From A. They made a number of computational mistakes on the "Ratio" and "Extended Square (B)" problems, 46% and 38% respectively.

A Special Example

In addition to the general summary of the students' performances on the quiz, a special example will be presented in this section. This example will reveal a student who had a wrong idea about the problem, but got the correct answer. It was the only case in the data where an incorrect way of thinking lead to a correct answer. This example may have a special implication for assessment. Is the final answer the only criterion teachers should use to evaluate students work and performance? In this case, the researcher considered the student's answer to be a correct answer. I also considered the fact that the student violated the "other type" conceptual error.

Example

This response is the only example in which a student misconceptualized the problem, but got the correct answer. The student worked on the "Cake" problem section b. The student did not correctly conceptualize the problem, but got the correct answer. The correct algorithm should be $(\$963 - \$45 \times 11) \div 6$. The student thought the algorithm was $(\$963 \div 11 \times \$45) \div 6$, and after dividing 963 by 495 (the product of 11 and 45), s/he took the remainder and divided it by 6, and got 78. The answer is correct, but his/her thinking did not make sense.

Problem (from Form B):
A cake-shop owner found that some customers like to buy a slice of cakes and others like to buy whole cakes. So he cuts some of his cakes into 8 equal slices and sells them for \$6 per slice. He sells the whole cakes for \$45.

a) One day, he sold 12 whole cakes and 62 slices. How much money he got that day?
Answer: \$ 912
b) Another day, he had \$963 in his cash-register at the end of the day. He knew he sold 11 whole cakes. How many slices did he sell on that day?

Answer: 78 pieces

Figure 5.1 The Cake Problem on Form B

b)另一天，老闆賺了963元，他知道他賣了11個蛋糕，請你幫他算出他在那一天除了賣出11個蛋糕外，還賣出幾片蛋糕？

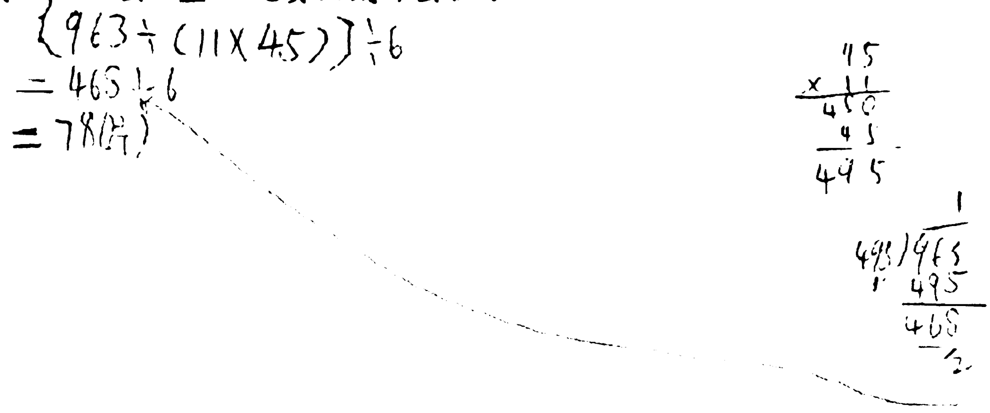


Figure 5.2 A Student's Response on the Cake Problem

Interview Result

Card-Sorting

In the card-sorting activity, the card sorting result matched well with my characterization of an item as a routine problem or a non-routine problem. My reasoning was that students would categorize the items into "Not Know" and "Not Sure" when no problem scheme came to their mind instantly. These items should be seen as non-routine problems. The result revealed that more than 50% of the students categorized the non-routine problems into "Not Know" or "Not Sure" categories. On the two non-routine problems ("Moon" and "Paper Recycle"), my characterization did not match with the students' categorization. Although it is a possibility that students over-estimated their own competence or misread the problems, the results still revealed that the problem categorization (routine and non-routine) that I did matched quite well with that of the students' categorization.

Table 5.14 Frequencies from Students' Card-Sorting Results

Problem Name	Responses		
	Not Know	Not Sure	Know
Balance	12	13	2
Board	6	15	6
Division <Routine>	0	0	27
Lake	14	12	1
Magazine <Routine>	2	8	17
Moon	5	3	19*
Number Triangle <Routine>	0	3	24
Poster	4	12	11
Paper Recycle	0	7	20*
Rope & Triangle <Routine>	1	3	23
4 Squares <Routine>	3	6	18
Table Tennis	3	18	6
Tiles	5	12	10
Train	2	14	11
Wallpaper	11	10	6

* Problems did not match with my characterization and student's categorization

Students' Performance on the Interview Problems

This interview section was designed to collect more information about how students solved non-routine problems and to let students experience less pressure of time to finish the non-routine problems. Two kinds of problems were selected. One was related to the concept of speed, the "Train" problem. The other two were related to the concepts, area and perimeter, the "Classroom Board" and "4 Squares" problems. These problems were selected because students seemed to have more difficulty conceptualizing the concepts—rate and area.

<u>The Train Problem:</u>

Two trains (A and B) are starting out from Mongkok Station in two different directions at the same time with the same speed, 66 km per hour. 2 minutes and 30 second later, how far apart distance are the trains?

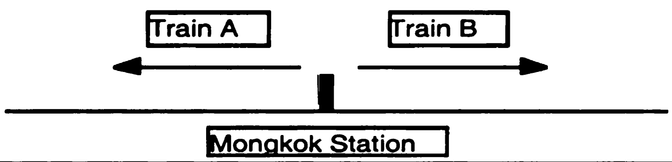


Figure 5.3 The Train Problem

In this problem, we examined the students' performances from two perspectives, a) conceptual correctness: the train's speed is based on the ratio between time elapsed and distance covered; b) computational correctness: all the computation between given and generated numbers are correct. By 'conceptually correct' we mean that the students' work indicated that he/she understood the essential relationships between the quantities stated in the problem. For example, in the Train problem, clear evidence was given that the students grasped the need to convert the speed in km/hr. to km/minute, and that speed were the result of a multiplicative relationship between distance and time. Many students committed a conceptual mistake on this problem, thinking it was a time-conversion problem. Other students made computational mistakes related to forgetting that the problem asked for the distance between two trains. They sought the distance traveled by a train and thought that they had found the answer.

There are 3 separate sections in Table 5.16. Each section represents students with different social backgrounds. The original plan of this study attempted to include students from 3 different social backgrounds—middle class, low SES, and new immigrant from Mainland China. Unfortunately, there were only a few students who belonged to the third category so I decide not to consider this category when I did the quantitative analysis. However, I would like to keep this group separately in the interview analysis.

In General, the result revealed that most high achievers did not have any conceptual or computational difficulty on the speed problem. The middle achievers gave a mixed result. About half of them experienced conceptual difficulty, although most of them did not have any computational difficulty. In the low achiever group, most had conceptual difficulties and did not have any serious difficulties on computation.

Table 5.15: Brief Interview Performance on the Train Problem

School / Achievement	High Achievers	Middle Achievers	Low Achievers
School 2 (Middle Class) Native H.K.. students	Aaron (2104): Conceptual x Computational √ (Know)	Alan (2207): Conceptual √ Computational x (Not Sure)	Angela (2206): Conceptual x Computational √ (Not Sure)
	Adam (2110): Conceptual √ Computational √ (Not Sure)	Alice (2112): Conceptual x Computational x (Know)	Andy (2203): Conceptual x Computational √ (Not Sure)
	Adrian (2133) Conceptual √ Computational √ (Not Sure)		Ashley (2220): Conceptual √ Computational √ (Not Know)
			Austin (2212): Conceptual x Computational √ (Know)
School 1 (Low SES) Native H.K. students	Barry (1118): Conceptual √ Computational √ (Know)	Blenda (1116): Conceptual x Computational √ (Know)	Brandon (1207): Conceptual x Computational √ (Not Sure)
	Ben (1108): Conceptual √ Computational √ (Know)	Billy (1121): Conceptual x Computational √ (Not Sure)	Brian (1231): Did not engage in the task.
	Betsy (1122): Conceptual x Computational √ (Know)	Bobby (1217): Conceptual √ Computational √ (Not Sure)	

Keys:
√: correct
x: Incorrect

Table 5.15 (cont.): Brief Interview Performance on the Train Problem

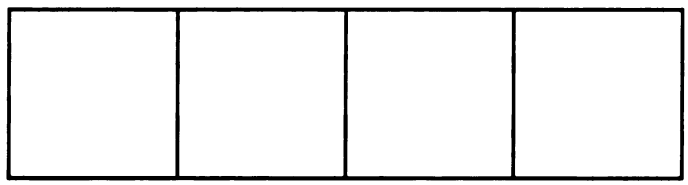
School / Achievement	High Achievers	Middle Achievers	Low Achievers
School 1 (Low SES) new immigrant students	Carl (1111): Conceptual √ Computational √ (Not Sure)	Cindy (1205): Conceptual x Computational √ (Know)	Crystal (1220): Conceptual x Computational x (Not Sure)
	Danny (1223): Conceptual √ Computational √ (Know)	Corey (1226) Conceptual √ Computational √ (Know)	Doug (1202): Conceptual x Computational √ (Not Know)
		Debby (1229): Conceptual x Computational √ (Not Sure)	Dorothy (1225): Conceptual x Computational √ (Not Sure)
		Diane (1228): Conceptual √ Computational √ (Not Sure)	
		Dion (1230): Conceptual x Computational √ (Not Sure)	

Keys:
√: correct
x: Incorrect

On this interview problem, the high achievers performed better than the other two groups. There were two exceptions, Bobby and Ashley. Bobby was a middle achiever, but his conceptualization on the speed problem was quite different than that of the other students. He used a more advanced proportional concept to see the relationship between the quantities in the problem. He wrote down a formula, 66/(60/2.5) + 66/(60/2.5), for the train problem after he thought about the problem for about 3 minutes. Ashley seemed to be an under-achiever; she under-estimated her own competence. She lacked self-confidence in doing mathematics. In the interview, she was asked if she liked her mathematics classes. She reported that she felt sleepy when she saw mathematics. In fact, her performance revealed that she understood the problem and the concept of speed.

4-Squares Problem

The following figure shows a rectangle consisting of 4 equal squares. If the area of the rectangle is 196 cm^2,



a) What is the area of each square?

b) What is the perimeter of each square?

c) What is the perimeter of the rectangle showed in the figure?

(NR)

Figure 5.4 The 4-Square Problem

Only 5 students were given the 4-Square Problem in the interview. All of them were low achievers at the two schools. I examined the students' performances from two perspectives, a) conceptual correctness: concepts on the square and perimeter of a square; b) computational correctness. By "conceptually correct," I mean that students' observed work indicated that he/she understood the relationship between the area of a square and the length of a side and the fact that area and length are in two different dimensions. The mistakes made by students on conceptual dimension were related to the concept of area. Those students thought that they could determine the length of a side of a square by dividing the area by 4. However, most of the students did not have any problem with the resulting computation. Ashley was still a special case. She did not have any difficulty with the concept of area and its relationship to the length of its side.

Table 5.16: Brief Interview-Performance on the 4-Squares Problem

School / Achievement Level	High Achievers	Middle Achievers	Low Achievers
School 2 (Middle Class) Native H.K. students			Andy (2203): Conceptual x Computational √ (Know) Ashley (2220): Conceptual √ Computational √ (Not Know)
School 1 (Low SES) Native H.K. students			Brandon (1207): Conceptual x Computational √ (Not Sure)
School 1 (Low SES) new immigrant students			Crystal (1220): Conceptual x Computational √ (Not Sure) Doug (1202): Conceptual x Computational √ (Know)

Keys:
√: correct
x: Incorrect

Table 5.17 shows that all the low achievers had conceptual difficulties relative to the concept, area. The only exception was Ashley. However, all low achievers in the interview did not make any computational errors.

Classroom Board Problem

You and 3 other classmates are assigned to design your classroom bulletin board this semester. The board is a rectangular shape, 2.5 meters by 1 meter. First you think about the background. 1) You choose the background color (light blue), and find there are two sizes of that color paper. One is 30 cm by 50 cm, and another is 40 cm by 50 cm. You want to buy the size that wastes the least paper a) Which will you buy? b) How many pieces of your chosen size do you need to buy to cover the whole board? (NR)

Figure 5.5 The Classroom Board Problem

This was the most complicated problem in the interview. Because the problem did not offer a visual model, students needed to understand the context of the problem. Also, there were a lot of unit-transformations students needed to handle in solving the problem. In addition to the concepts of area and perimeter, students needed to track the meaning of the numbers they generated. The investigators examined the students' performances from four perspectives, a) area formula, b) unit transformation, c) problem interpretation, d) number meaning. By "conceptually correct," I mean that the students' work I observed indicated that he/she understood the concept of area. For example, they knew how to transform 2.5 m^2 to 25000 cm^2, and they could assign a correct unit and meaning to the number they generated. For example, they knew that the unit of the remainder (1000) could be generated by dividing the area of the board (25000 cm^2) by area of the paper (1500 cm^2 or 2000 cm^2). None of the students experienced any difficulty giving the area formula of a rectangle. On unit transformation dimension, the students' mistakes were related to the misconception of the relationship between m^2 and cm^2. Students thought that they could turn 2.5 m^2 to 250 cm^2 by multiplying 2.5 by 100. On the problem interpretation dimension, some students interpreted the word, "waste," in several different unexpected ways. Some students thought that the more sheets one bought, the more waste would be created. Others thought that bigger paper would create more waste. On the number meaning dimension, a lot of students did not know what a number represented after they completed some conversions or operations on the numbers. For instance, students divided the board area (25000 cm^2) by the paper area (1500 cm^2 or 2000 cm^2), and then were confused by the remainder (1000 cm^2). They did not know what that remainder represented. Some students thought that the "1000 cm^2" was the left-over paper. They concluded that there was no difference in "waste" between the two different sizes of paper.

For this problem, none of the student gave a completely correct answer. The table 5.18 showed that all experienced one or more conceptual difficulties with the problem. However, most of the high achievers showed only one conceptual difficulty with the problem, and most of the mid and low achievers showed two or more conceptual difficulties. As on the train problem, Ashley and Bobby showed a similar pattern to that of the high achievers.

Table 5.17: Brief Interview Performance on the Classroom Board Problem

School / Achievement Level	High Achievers	Middle Achievers	Low Achievers
School 2 (Middle Class) Native H.K. students	Aaron (2104): Area formula: √ Unit Transformation: x Interpretation: √ Number Meaning: √ (Not Sure)	Alan (2207): Area formula: √ Unit Transformation: x Interpretation: √ Number Meaning: x (Not Sure)	Ashley (2220): Area formula: √ Unit Transformation: √ Interpretation: x Number Meaning: √ (Not Know)
	Adam (2110): Area formula: √ Unit Transformation: √ Interpretation: √ Number Meaning: x (Know)	Alice (2112): Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: x (Know)	
	Adrian (2133) Area formula: √ Unit Transformation: √ Interpretation: √ Number Meaning: x (Not Sure)		
School 1 (Low SES) Native H.K. students	Barry (1118): Area formula: √ Unit Transformation: √ Interpretation: x Number Meaning: √ (Not Know)	Blenda (1116): Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: √ (Know)	
	Ben (1108): Area formula: √ Unit Transformation: √ Interpretation: √ Number Meaning: x (Not Sure)	Bobby (1217): Area formula: √ Unit Transformation: √ Interpretation: √ Number Meaning: x (Not Sure)	
	Betsy (1122): Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: x (Not Sure)		

Keys:
√: correct
x: Incorrect

Table 5.17 (cont.): Brief Interview Performance on the Classroom Board Problem

School / Achievement Level	High Achievers	Middle Achievers	Low Achievers
School 1 (Low SES) new immigrant students	Carl (1111): Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: x (Not Know)	Corey (1226) Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: √ (Not Sure)	Dorothy (1225): Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: x (Not Sure)
	Danny (1223): Area formula: √ Unit Transformation: x Interpretation: √ Number Meaning: x (Not Sure)	Debby (1229): Area formula: √ Unit Transformation: x Interpretation: x Number Meaning: x (Not Sure)	
		Diane (1228): Area formula: √ Unit Transformation: √ Interpretation: x Number Meaning: x (Not Sure)	

Keys:
√: correct
x: Incorrect

Generally, the performance of high achievers in the conceptual domain was better than that of the two other groups except for the two special cases, Bobby and Ashley. On the computational domain, students from the three groups performed quite well; only 3 students on the "Train" problem made computational mistakes. So, students' difficulties on the non-routine mathematics word problems seemed to be related to their conceptual difficulties. Students relied quite heavily on mathematics formulas they had learned when they solved the problems. There were no examples of a student trying to draw a model for the "Classroom Board" problem. Based on the data for the "4 Square" problem and the "Classroom Board Problem," students had difficulty in understanding the concept of "area." The mistakes they made on unit transformation revealed their difficulty with the concept.

Chapter 6

NATURE OF CONCEPTUAL MISTAKES AND THEIR IMPACT ON PERFORMANCE

The results presented in Chapter 5 revealed that students had more difficulty solving non-routine problems than routine problems. However, there are many elements of non-routine problems (story structure, embedded mathematics concepts, answer format, number of sentences and words, different kinds of numbers—integers and decimals, transformation of measurement units, etc...) that could contribute to their difficulty. Some elements, like the embedded mathematics concepts, ratio, and the meaning of different measurement units, make non-routine problems more conceptually challenging. Some elements, like the length of the problem text, contribute to the difficulty without increasing the conceptual or mathematical challenge. For example, the "Cake" problem revealed that a story structure does not always make non-routine problems more difficult conceptually. Although this problem had a very complicated and unfamiliar story structure, most students performed quite well on this problem. More than 60% of the students were able to solve the problem correctly. A non-routine problem can be conceptually difficult when the mathematical concept(s) embedded in the problem is not well understood by the students. For example, the "Ratio" and "Extended Square" problems revealed such difficulty in conceptual understanding. Students did not understand the concepts, proportion and area, very well. They performed poorly on these two problems and committed several sorts of conceptual errors. For instance, many students used an additive model to think about proportion in the "Ratio" problem, and thought that dividing the area of a square by 4 could give them the length of a side in the "Extended Square" problem. If the problem structure is unfamiliar and complicated and the embedded concept is not well understood by students, the problem may be extremely difficult for students. For example, no student was able to solve the "Glass House" problem with two correct answers in two sections. In this chapter, I will give a deeper conceptual analysis of students' performances on non-routine problems. In particular, I will analyze the kinds of errors students made on some non-routine problems. In the following sections, the "Ratio" and the "Extended Square" problems will be the main focus. In each problem, I will show the students' conceptual difficulties with ratio and area and the kinds of errors they made.

Two Focus Problems

There were 12 non-routine problems distributed between two different forms in the quiz. Each form contained 6 non-routine problems. Among these problems, there were 4 pairs of parallel problems (Balance, Extended Square, Lake, Ratio), and 4 other non-parallel problems (Cake, Coin, Glass House, Number Pattern). Most parallel problems contained the same story-content, but the given numbers on each problem were different on the two different forms. The only exception was the "Ratio" problem, in which the two parallel problems had two different story-contents. However, the mathematical concept, proportional reasoning, and the story structure was the same. The two problems used the ratios between an object on a photograph and the actual object as the main theme, and asked students to figure the actual height of one object when they knew its represented height on the photograph.

My reasons for choosing the "Extended Square" and "Ratio" problems were three-fold. First, there were more written responses from students on these problems. If the conceptual analysis was to be the main focus, more students' written protocols were needed in order for an inference about their thinking. Second, the concepts embedded in these two problems introduced two important mathematical concepts (proportion and area) that were taught in grades 4 and 5. Proportion and area were two special multiplicative concepts and marked a new level of multiplicative understanding in the curriculum. Before these new multiplicative concepts were introduced, students treated multiplication as another arithmetic computation. Proportion and area required that they think about and see multiplication differently. The additive model is not a helpful or efficient one for students to use to think about multiplication in the context of proportion and area. Third, "Extended Square" and "Ratio" are two pairs of parallel problems so they offered us a more complete picture of the performances of all students.

The "Extended Square" Problem

Two versions of this problem were presented in two different forms (Form A and Form B). The two problems had the same story content; the only difference was the given numbers. In Form A, the given area was 196 cm^2 and the dimension of the square was lengthened by 5 cm. In Form B, the given area was 121 cm^2 and the extended length was 3 cm.

> <u>The Extended Square Problem in Form A</u>:
> The area of a square is 196 cm^2, now you lengthen one dimension by 5 cm to form a new shape.
>
> a. Draw the new shape.
> **Answer**: a rectangle
> b. What is its area?
> **Answer**: 14 cm x 19 cm = 266 cm^2
> c. What is its perimeter?
>
> **Answer**: (14+19)cm x 2 = 66 cm

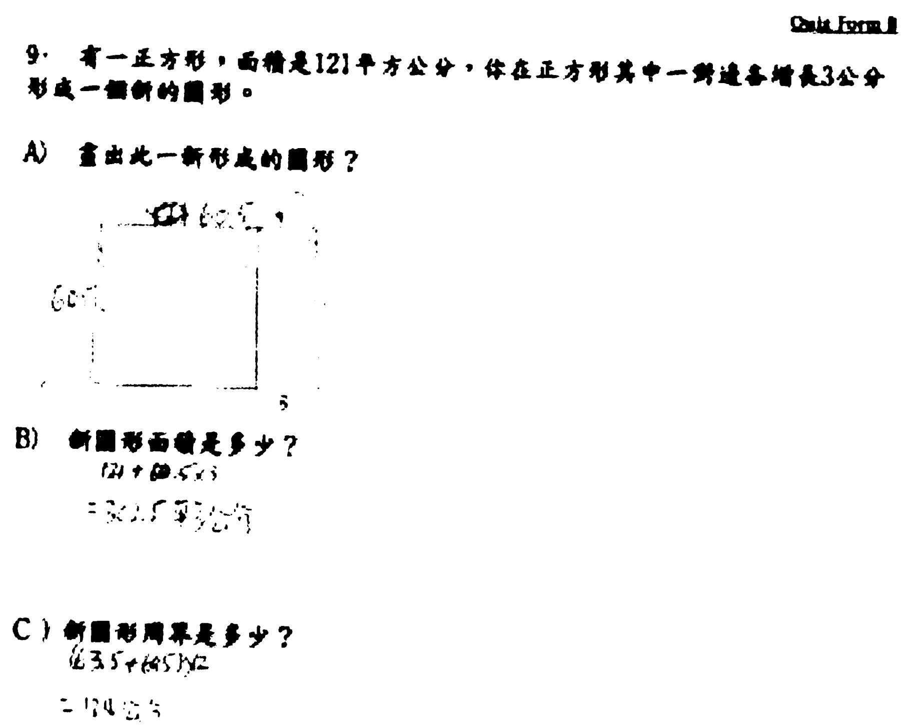
Figure 6.1 The Extended Square Problem in Form A

> <u>The Extended Square Problem in Form B</u>:
> The area of a square is 121 cm^2, now you lengthen one dimension by 3 cm to form a new shape.
>
> a. Draw the new shape.
> **Answer**: a rectangle
> b. What is its area?
> **Answer**: 11 cm x 14 cm = 154 cm^2
> c. What is its perimeter?
>
> **Answer**: (11+14)cm x 2 = 50 cm

Figure 6.2 The Extended Square Problem in Form B

The main focus of this problem was to investigate how well students conceptualized the concepts, area and perimeter. Students were given the area of a square; they needed to extend the two opposite sides of this given square to build a rectangle. They were asked to figure out the area and the perimeter of the new rectangle. Based on the collected written protocols, five types of conceptual errors were identified. The first three types were related to students' confusion between 1-dimensional and 2-dimensional measurements. The fourth type may have been related to difficulty in conceptualizing the problem situation. All the different kinds of errors that I could not logically and contextually interpret were placed into an "Other" class.

In the second round of scoring, one of the scorers discovered a sixth type of conceptual error not found in the original scoring scheme. There were 4 students who thought that dividing the area by 2 would give them the length of a side (see Figure 6.3).


Quiz Form I

9· 有一正方形，面積是121平方公分，你在正方形其中一對邊各增長3公分形成一個新的圖形。

A) 畫出此一新形成的圖形？

B) 新圖形面積是多少？

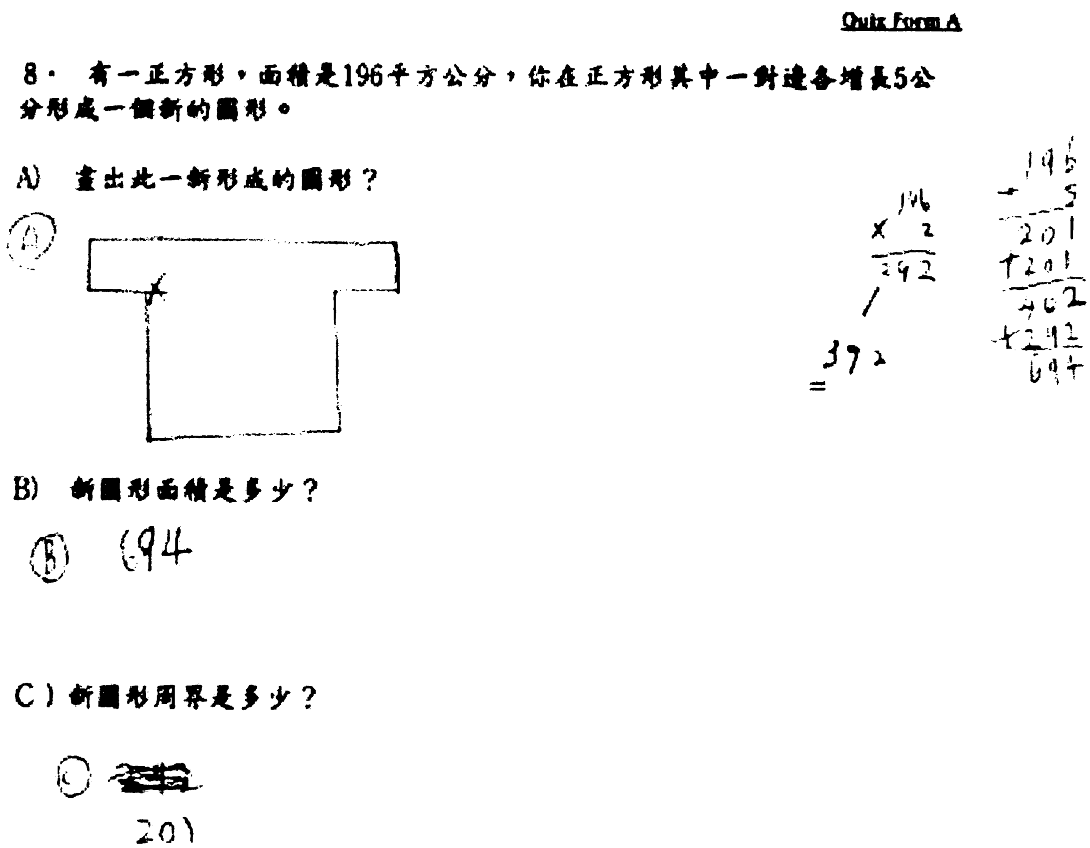
C）新圖形周長是多少？

Figure 6.3 The Square-root Confusion

This thinking might be related to the confusion between dividing by 2 and taking the square root. In this case, the student tried to get the length of a side by dividing the area by 2, in one case, 121 $cm^2 \div 2$, which is 60.5 cm. S/he kept the original area 121 cm^2 and added it to a new area, 60.5×3 cm^2. These four cases were originally scored as examples of "Other" errors.

In the next section, the first five different types of conceptual mistakes that appeared on the "Extended Square" problem will be described and illustrated. The student work revealed several conceptual mistakes in this one problem. This is important because these mistakes were related to the core concept, measurement in 1 and 2 dimensions. In other words, they confused the length and area measurements. They used the length measurement to think about area and operated on area quantity, for example, 121cm^2, as if they were lengths.

Type A is a unit confusion mistake. Of these students, 26.8% (33 students) committed this mistake. In Form A, there were 36.4% (12 students) in the low SES school, and 16% (4 students) of those in the middle class school made this mistake. In Form B, there were 40% (14 students) in the low SES school, and 10% (3 students) in the middle class school who committed this mistake. In this problem, the most common unit confusion was adding a quantity in cm^2 to a quantity in cm. For example, in the following case (Figure 6.4), the student added 5 cm to 196 cm^2. However, there were several students who multiplied 196 cm^2 by 5 cm. And, the second most common case of unit confusion was done by dividing the area by 4 and adding the quotient to 5 cm or 3 cm. In fact, dividing the area by 4 did not change the unit of the area. The division just partitioned the area into four equal sectors. All of these cases were categorized as a unit confusion mistake. These mistakes revealed that the students could not distinguish between the two different kinds of measurement, linear measurement (cm) and area measurement (cm^2). Students might not understand the difference between two different kinds of measurement, one for length and one for surfaces. Or, their knowledge of measuring length still had an influence on how they thought about area. Some students might still think about the additive function in the context of surface. This conceptual difficulty is quite common among grade 4 and 5 students, because they often learn the area concept by adding small tiles.


Quiz Form A

8· 有一正方形，面積是196平方公分，你在正方形其中一對邊各增長5公分形成一個新的圖形。

A) 畫出此一新形成的圖形？

B) 新圖形面積是多少？

C) 新圖形周界是多少？

Figure 6.4 Type A Conceptual Mistake on the Extended Square Problem

The illustrated case (Figure 6.4) included multiple conceptual and computational mistakes. In addition to the unit confusion, the student might not comprehend the problem correctly. First, s/he might be confused with length and area. S/he appeared to see 196 cm^2 as the length of a side; s/he added 5 cm to 196 cm^2 revealing length and area confusion. Her/his drawing revealed that s/he did not comprehend the problem correctly, the correct drawing should be a rectangle. The product of 196 and 2 is 392, and not 292. Her/his answer indicated that a computational mistake had been made. Finally, perimeter cannot be the product of the area (196 cm^2) and 2. This was another conceptual mistake that s/he made.

Type B is also a confusion of area and length, but in a slightly different way. Of these students, 21% (26 students) made this conceptual mistake. In Form A, there were 36.4% (12 students) in the low SES school and 16% (4 students) in the middle class school who committed this mistake. In Form B, there were 22.9% (8 students) in the low SES school and 6.7% (2 students) in the middle class school who committed this mistake. They thought that dividing the area by 4 would give them the length of a side. A lot of students who made this mistake were also confused about the units for different kinds of measurement. For instance, the student work in Figure 6.5 showed the same unit, cm, for area and perimeter.

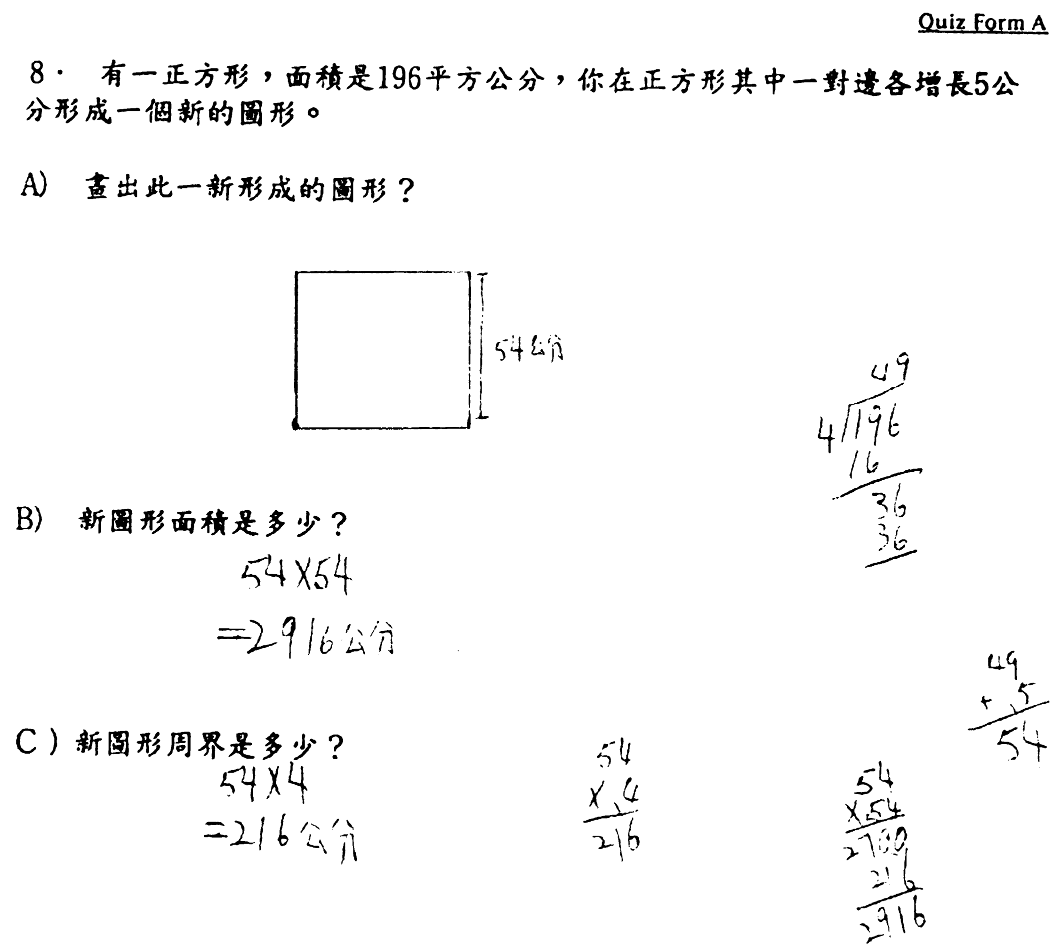
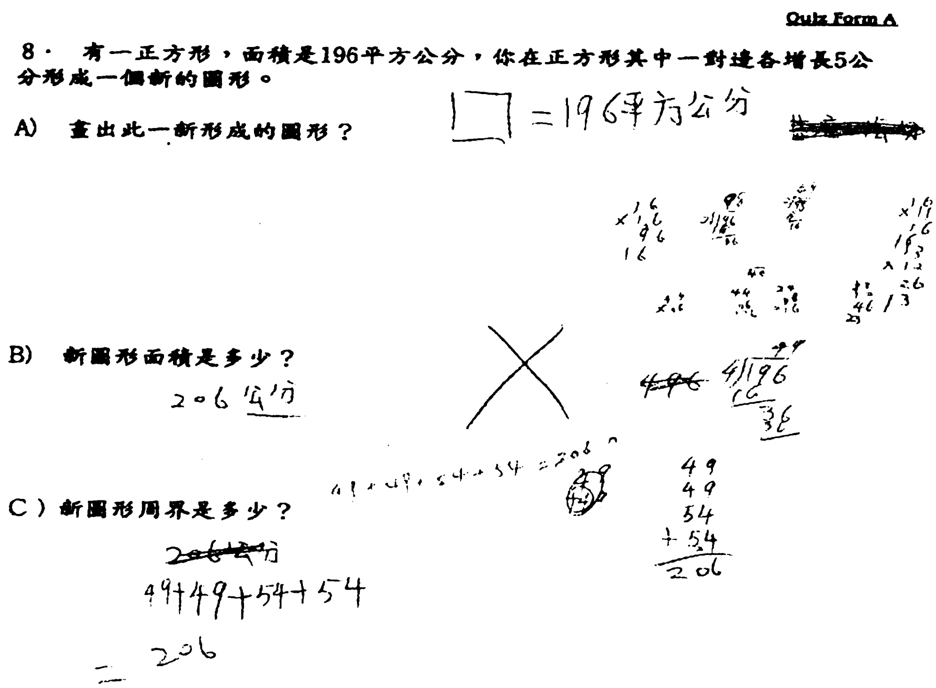


Figure 6.5 Type B Conceptual Mistake on the Extended Square Problem

This student thought that dividing 196 cm^2 by 4 would give her/him the length of a side. And, it was interesting that s/he thought that the product of sides (54 cm × 54 cm) would give her/him the new area. The student did not realize that s/he measured area in two totally different methods and procedures. One was done by multiplying the length of one side by 4 (dividing the area by 4 was revealed in her/his thinking about how s/he thought she could find the area). Another method was to multiply one side with its adjacent side (54 cm × 54 cm). The methods and procedures were not reversible. It revealed that the student had some idea about area, but her/his concepts about area was not firm. S/he might remember the formula for rectangular shapes, but s/he did not understand how and why the formula worked. There are several possible reasons for students making this conceptual mistake, dividing area by 4 to get the side. Students could be misled by perception. They saw the perimeter as the figure and the area as the ground. So, they did not realize that the focus should be on the surface and not the sides. Students may also have been confused by the concepts of area and perimeter. They may see area as the total length of four sides. That is, they might not clearly be able to distinguish between area and perimeter.

Type C is a complete confusion of area and perimeter. Of these students, only 0.5% (6 students) made this mistake. In Form A, there were 9.1% (3 students) in the low SES school and 4% (1 student) in the middle class school who made this mistake. In Form B, there were 5.7% (2 students) in the low SES school and 0% (no students) in the middle class school who made this mistake. So, only a few students could not distinguish between area and perimeter. The students who made this mistake gave the same answer to both questions. The student's work in Figure 6.6 revealed that s/he could not distinguish between area and perimeter. S/he might think that the sum of the length of 4 sides was area and it was also the perimeter.


Quiz Form A

8・ 有一正方形，面積是196平方公分，你在正方形其中一對邊各增長5公分形成一個新的圖形。

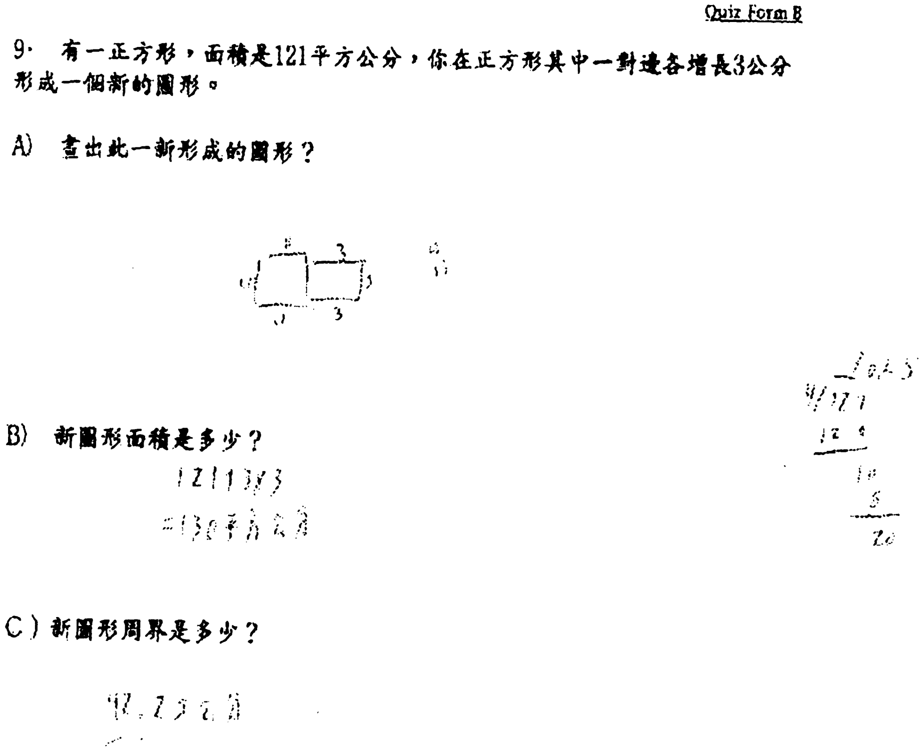
A) 畫出此一新形成的圖形？

B) 新圖形面積是多少？

C）新圖形周界是多少？

Figure 6.6 Type C Conceptual Mistake on the Extended Square Problem

Type D is the mistake of adding a small square to the original square. In this case, the students struggled more with their understanding of the problem statement than with the concept of area and perimeter. There were only 0.5% (6 students) who made this mistake. In Form A, there were 9.1% (3 students) in the low SES school and 4% (1 student) in the middle class school who made the mistake. In Form B, there were 5.7% (2 students) in the low SES school and 0% (no students) in the middle class school who made the mistake. In the illustrated case (Figure 6.7), a student wrote, 121 cm^2 + 3 cm × 3 cm = 130 cm^2, for the new area.


Quiz Form B

9. 有一正方形，面積是121平方公分，你在正方形其中一對邊各增長3公分形成一個新的圖形。

A) 畫出此一新形成的圖形？

B) 新圖形面積是多少？

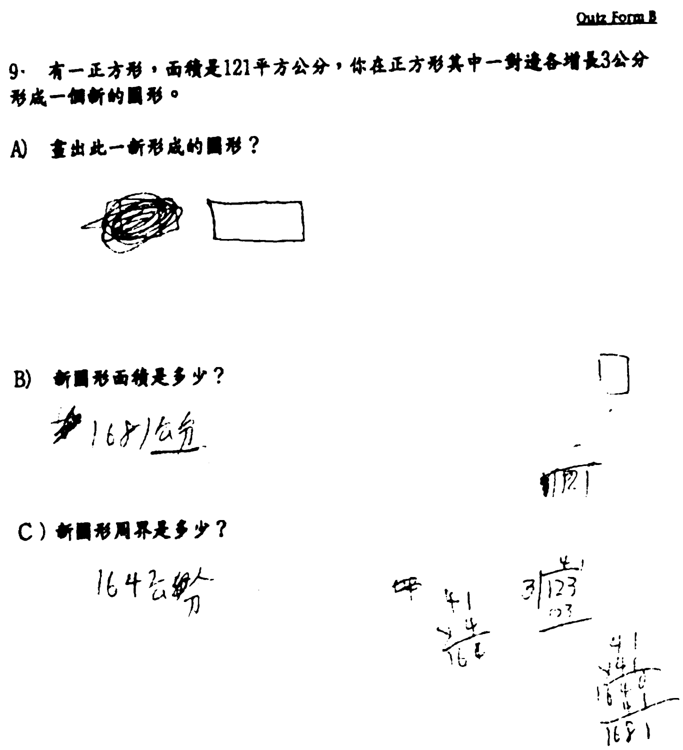
C) 新圖形周界是多少？

Figure 6.7 Type D Conceptual Mistake on the Extended Square Problem

In this case, the student seemed to experience a conceptual difficulty in addition to the additive mistake. Her/his work on section C revealed that s/he had a Type B conceptual mistake as well. S/he thought that dividing the area by 4 would give her/him the length of a side. The additive mistake seemed to be related to her/his difficulty in comprehending the context of the problem.

Type E is "Other" conceptual mistakes. This type of error was more diverse and these mistakes could not be easily interpreted relative to the context of the problem. In Form A, there were 7 students (21.2%) from the low SES school who committed this type of mistake on Part B, and 8 students (24.2%) on Part C. In the middle class school, 4 students (16%) committed this mistake on Part B, and 4 students (16%) committed this mistake on Part C. In Form B, 10 students (28.6%) from the low SES school committed this mistake in Part B, and 7 students (20%) committed this mistake in Part C. In the middle class school, 4 students (13.3%) committed this mistake in Part B, and 3 students

(10%) committed this mistake in Part C. For example, a student wrote, $6.25 \times 4 = 3200$, as the answer for perimeter. I could not interpret the source of the number, 6.25—how s/he figured out this number. Or, some students tried to use all the given numbers with different mathematical operations, but without any contextual and mathematical sense. Two different illustrative examples will be presented; they will suggest how diverse this type of conceptual mistake was.


Quiz Form B

9· 有一正方形，面積是121平方公分，你在正方形其中一對邊各增長3公分形成一個新的圖形。

A) 畫出此一新形成的圖形？

B) 新圖形面積是多少？

1681公分

C) 新圖形周界是多少？

164公分

Figure 6.8 Type E Conceptual Mistake 1 on the Extended Square Problem

In the case 1 (Figure 6.8), the student tried to use all the given numbers in the problem to solve the problem. In the area section, s/he divided 123 cm^2 (I presumed s/he misread the given area 123 cm^2 for 121 cm^2) by 3, and got 41. Then, s/he multiplied 41 by 41 and got 1681 cm. In the perimeter section, s/he multiplied 41 by 4 and got 164 cm. Obviously, this student did know about area and perimeter. At least, s/he remembered the formula for the area of a rectangular square (length times width) and the perimeter (one length of a side times 4). However, s/he seemed to have difficulty transforming or integrating this knowledge into this problem situation. And, s/he manipulated all the given numbers without a full understanding of the context of the problem. For example, she divided the misread area (123 cm^2) by 3, which did not make any contextual sense.

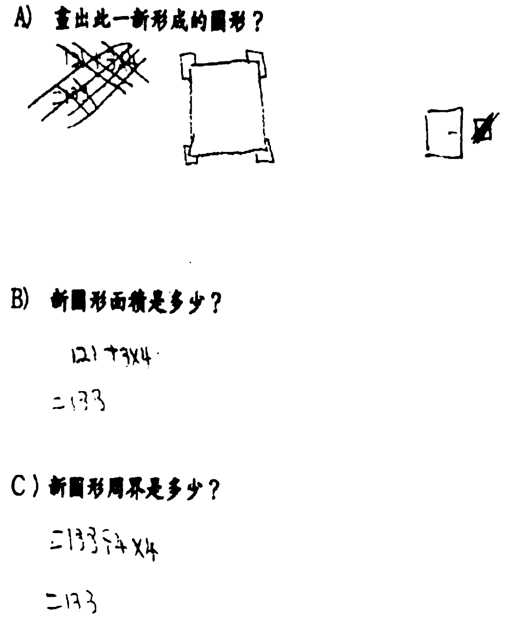


Figure 6.9 Type E Conceptual Mistake 2 on the Extended Square Problem

In case 2 (Figure 6.9), the student drew a very strange diagram. The diagram seemed to indicate that s/he did not comprehend the problem well. In the area section, s/he wrote 121 + 3 × 4 = 133. S/he did not give any unit to this number, and her/his written response did not make any contextual and mathematical sense in this problem situation. I could not explain how and why s/he came up with this resolution— adding the original area, 121 cm^2, to 3 multiplied by 4. If we examined this solution relative to her/his drawing, it would still be difficult to interpret the writing, 3 × 4. In the perimeter section, the written response revealed that the student had some idea about perimeter. S/he knew the formula for the perimeter of a square (one side length times 4). However, s/he made a Type B mistake—length-area confusion. S/he thought that dividing the area by 4 would give her/him the length of a side.

Summary

If we take a closer look at the five types of mistakes, these mistakes can be categorized into two higher level categories—measurement confusion, and comprehension and knowledge integration difficulty. The first three types, A, B, and C, and the square root confusion are related to measurement confusion. Students were confused by two different kinds of measurement, length and surface area. The confusion may be related to three sources. One source may be related to prior knowledge about linear measurement (length measurement). The students still used linear measurement to think about surface measurement. The unit confusion (Type A) might be related to a careless mistake, using a wrong unit, but it might also be related to confusion between two different types of measurement. Type B might be related to understanding of the concepts. Students did not really understand the differences between these measurements, and why and how the formula for area and perimeter worked. However, most of them knew the formula for area and perimeter. The case presented in Figure 6.5 showed that a student knew the formula, but did not understand that s/he used two incompatible algorithms in reasoning about area. In Part B, s/he divided the area by 4 to find the length of a side. In Part C, s/he used the "correct" method of multiplying the adjacent side lengths to find the area. The Type C error might have been related to perception. Some students may not see surface as the figure, but as the ground. They might see the perimeter as the figure as perceptually salient because the surface is blank, but the perimeter lines are obvious. So, they might easily perceive the sides and concentrate on how to measure them first. Finally, students' confusion could quite possibly be related to several sources. Memorizing the formula without deep understanding, and interacting this partial knowledge with the perceptual effect might easily trap students in confusion about measurement.

The second higher level category was comprehension and knowledge integration difficulties. Type D and E mistakes were related to this category. Students might not be able to comprehend the problem context, so they established a wrong model for the situation. For instance, students who added another small square beside the big square showed a miscomprehension of the problem. Students who tried to use all the given numbers to solve the problem without any contextual and mathematical sense showed that they might have difficulty comprehending the problem situation and in integrating prior knowledge they learned in school. Case 1 in the Type E mistake (Figure 6.8) showed this kind of phenomenon.

In general, students' work showed that their mistakes were related to their level of understanding of the concept of area. Most students had some sense of area and perimeter, but their ideas were strongly tied to the formula and less so to a conceptual model. Many knew that the product of adjacent sides gave the area, and 4 times the length of the side gave the perimeter of a square. However, they did not seem to clearly know why and how these formulas become the measurements of area and perimeter.

After studying these different types of mistakes, I examined students from different social classes according to the type of conceptual mistakes they made on this problem. In the following tables (Table 6.1, 6.2, 6.4, and 6.5), The abbreviations represent the students' performances on three dimensions—correctness, conceptual, and computational. Under the correctness dimension, students' performances are categorized into 5 types: 1) C for a correct answer was given; 2) NC for a numerically correct answer was given; 3) I for an incorrect answer was given; 4) UC for an unclassified trial was given, that means students tried to solve the problem, but no answer was given, based on what they wrote on the quiz sheet; 5) UA for unattempted, that means no written information was provided on the quiz sheet. Under the conceptual dimension, there were 5 different types of mistakes that were described in the last section. Under the computational dimension, students' performances were categorized into 3 types: 1) 0 for no computational mistake, 2) 1 for one computational mistake, 3) 2 for more than one computational mistake. And, "R" stands for the rank of correctness; "t" stands for the total rank of the problems on different forms.

Table 6.1 Performance on the Extended Square Problem on Form A

School	Problem	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Low SES School (N=33)	Extended Square (Part B)	6.1 (2)	3.0 (1)	69.7 (23)	12.1 (4)	9.1 (3)	36.4 (12)	30.3 (10)	6.1 (2)	9.1 (3)	21.2 (7)	48.5 (16)	21.2 (7)	0.0 (0)
	Extended Square (Part C)	9.1 (3)		72.7 (24)	3.0 (1)	15.2 (5)	36.4 (12)	36.4 (12)	9.1 (3)	0.0 (0)	24.2 (8)	72.7 (24)	0.0 (0)	0.0 (0)
Middle Class School (N=25)	Extended Square (Part B)	16.0 (4)	8.0 (2)	76.0 (19)			16.0 (4)	16.0 (4)	4.0 (1)	4.0 (1)	16.0 (4)	68.0 (20)	20.0 (5)	0.0 (0)
	Extended Square (Part C)	32.0 (8)		64.0 (16)		4.0 (1)	16.0 (4)	16.0 (4)	4.0 (1)	0.0 (0)	16.0 (4)	72.0 (18)	12.0 (3)	0.0 (0)

Based on the results in the Table 6.1, students in the low SES school had more difficulty with measurement confusion (Type A and Type B) in Form A. The students from the low SES school made, on the average, nearly twice as many mistakes across all types of mistakes, compared to their counterparts. However, the patterns of the mistakes were the same. Students who made these mistakes revealed that they could not distinguish between the two different kinds of units of measurement, and were conceptually confused with the relationship between length and area. The conceptual confusion of the relationship may have been caused by students who experienced confusion about units. Nearly one third of the students (30%) in the school thought that dividing the area by 4 would give them the length of a side. Yet, only 16% of the students in the middle class school showed this confusion.

Table 6.2 Performance on the Extended Square Problem on Form B

School	Problem	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Low SES School (N=35)	Extended Square (Part B)	8.6 (3)		80.0 (28)	5.7 (2)	5.7 (2)	40.0 (14)	22.9 (8)	5.7 (2)	5.7 (2)	28.6 (10)	51.4 (18)	20.0 (7)	5.7 (2)
	Extended Square (Part C)	8.6 (3)	2.9 (1)	62.9 (22)		25.7 (9)	28.6 (10)	22.9 (8)	2.9 (1)	0.0 (0)	20.0 (7)	48.6 (17)	17.1 (6)	0.0 (0)
Middle Class School (N=30)	Extended Square (Part B)	40.0 (12)	3.3 (1)	53.3 (16)	3.3 (1)		10.0 (3)	6.7 (2)	0.0 (0)	0.0 (0)	13.3 (4)	90.0 (27)	3.3 (1)	3.3 (1)
	Extended Square (Part C)	40.0 (12)	3.3 (1)	53.3 (16)		3.3 (1)	10.0 (3)	6.7 (2)	0.0 (0)	0.0 (0)	10.0 (3)	96.7 (29)	0.0 (0)	0.0 (0)

Based on the results in Table 6.2, students in the low SES school had more difficulty with the measurement confusion (Type A and Type B) and conceptualization of the problem situation (Type E) in Form B. In these three types of mistakes, the percentage of students making such mistakes in the low SES school were more than double that of their counterparts in the middle class school. In addition to measurement confusion, students in the low SES school had difficulty with comprehension in the problem situation, or integrating their learned mathematical knowledge into the problem situation. However, the students in both schools showed the same patterns of mistake.

Table 6.3 Incorrect Responses Frequency on the Extended Square Problem

School (*N* =)	Form	Problem	Frequency, Incorrect Response	Minimum Incorrect Response Frequency related to Conceptual Mistakes	Incorrect Response Frequency related to Computational Errors
Low SES School (33)	A	Extended Square (Part B)	23	16	7
	A	Extended Square (Part C)	24	24	0
Middle Class School (25)	A	Extended Square (Part B)	19	14	5
	A	Extended Square (Part C)	16	13	3
Low SES School (35)	B	Extended Square (Part B)	28	19	9
	B	Extended Square (Part C)	22	16	6
Middle Class School (30)	B	Extended Square (Part B)	16	14	2
	B	Extended Square (Part C)	16	16	0

Based on the results in Table 6.3, students in both schools who responded to the problems incorrectly, seemed to have more difficulty conceptually than computationally. In Form A, 23 out of 33 students in the low SES school responded incorrectly in Part B, and 24 students in Part C. Among the 23 students who responded incorrectly in Part B, there were only 7 students who committed one computational error. This means that at least 16 students committed one or more conceptual mistakes in this part. Among the 24 students who responded incorrectly in Part C, all of them committed at least one conceptual mistake. Students who responded incorrectly in the middle class school also showed the same pattern—most of them had more difficulty conceptually than computationally. In Form A, 19 out of 25 students in the middle class school responded incorrectly in Part B, and 16 students in Part C. Among the 19 students who responded incorrectly in Part B, only 5 committed one computational error. This means that at least 14 students committed one or more conceptual mistakes. Among the 16 students who responded incorrectly in Part C, only 3 of them committed one computational error. This means that at least 13 students committed one or more conceptual mistakes. In Form B, 28 out of 35 students in the low SES school responded incorrectly in Part B, and 22 students in Part C. Among the 28 students who responded incorrectly in Part B, only 9 students committed one or two computational errors. This means that at least 19 students committed one or more conceptual mistakes. Among the 22 students who responded incorrectly in Part C, only 6 students committed one computational mistake. This means that at least 16 students committed one or more conceptual mistakes. In Form B, 16 out 30 students in the middle class school responded incorrectly in Part B, and 16 students in Part C. Among the 16 students in Part B, only 2 of them committed one or two computational mistakes. This means that at least 14 students committed one or more conceptual mistakes. Among the 16 students in Part C who responded incorrectly, all of them committed at least one conceptual mistake.

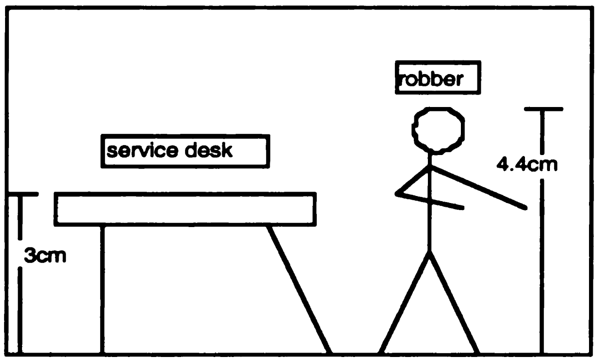
Based on the data collected in this study, I cannot explain why there was such a big difference in the numbers of conceptual mistakes made between students from the two schools. There are many possible reasons that could be used in interpreting the results. First, the middle class school had a screening system that picked the smarter students and rejected the students with learning difficulties. But, the school in the low SES community did not have such a screening system and they accepted any student who applied for a seat. In fact, there were a lot of new immigrant students in the low SES school. This phenomenon implies that this particular school did not have a screening system for picking their students. In Hong Kong, most good schools that have a lot of applicants will establish a screening system for picking their students. The schools that have a lot of new immigrant students are the ones without a screening system. So, the middle class school may have smarter students from middle class families. Second, students from the middle class may have several learning support resources—parents who know how to educate their children, and private tutors may be employed. On the other hand, students from the low SES families do not have access to resources for helping themselves learn. Third, the middle class schools may have good teachers who motivate their students to work hard. These teachers may also offer their students better classroom practice. Of course, the differences may result in an interactive effect between these reasons. Clearly, no matter what the reasons are for the differences, we need to worry about students from low SES families. They seemed to began experiencing difficulty with learning important upper grade mathematical concepts at the primary school level.

The "Ratio" Problems

There were two versions of the "Ratio" problems on the quiz (Photo-Robber on Form A and Photo-You on Form B). They were not perfectly parallel. The two problems were intentionally designed to ask students to use the same mathematical knowledge (proportion and ratio) to solve the problems, but they had their own story content in addition to different given numbers. Both problems used a photograph as the medium for asking students to think about the proportional relationship between actual objects and the represented objects on the photographs. However, the two photographs were taken in two different situations creating two different story contexts. One photograph was taken of a bank-robbery, the other in a leisure situation. The given numbers in these problems may have affected students' performances. The Photo-Robber problem had one more decimal than the Photo-You problem. The percentage of correctness for the Photo-Robber problem was 19%, whereas 44% of the students solved the Photo-You problem correctly.

Photo-Robber Problem (from Form A):

A hidden camera took the picture below during a robbery. In the picture, you can see the robber and a service desk. The height of the desk is 3 cm, and the height of the robber is 4.4 cm. The real height of the desk is 1.2 meter. Can you figure out the height of the robber? How?

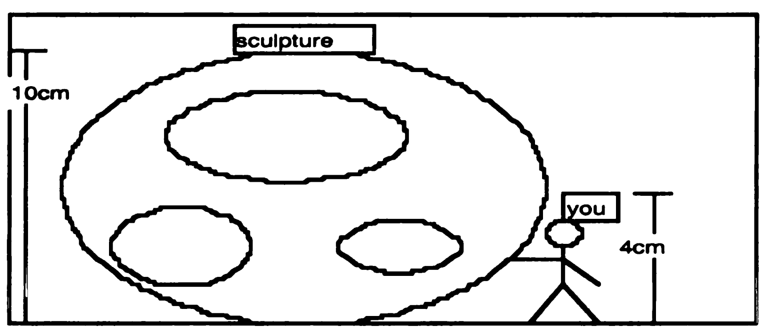


Answer: 1.76 m or 176 cm

Figure 6.10 The Photo-Robber Problem

Photo-You Problem (from Form B):

Last Sunday, you stood beside a sculpture and had your picture taken. Your height on the photo is 4 cm, and the height of the sculpture is 10 cm. You know your real height is 1.4 meters. Can you figure out the real height of the sculpture? How?



Answer: 3.5 m, or 350 cm

Figure 6.11 The Photo-You Problem

The main focus of the ratio problems was to investigate how students conceptualize and reason about the concept, ratio. Based on students' responses, 3 types of conceptual mistakes were categorized. They were: 1) mistakes related to the confusion of measuring the units, meter and centimeter (Type A), 2) mistakes related to inappropriate additive reasoning about ratio (Type B), and 3) mistakes which could not be interpreted in the problem context (Type C). The Type B and Type C mistakes interacted with the Type A mistake. Most students who made Type B and Type C mistakes also made Type A mistakes. In Form A, 39.7% (23 cases) of the students committed Type A and Type B or C mistakes, and only 5.2% (3 cases) of the students committed a single mistake. In Form B, 12% (15 cases) of the students committed Type A and Type B or C mistakes, and only 6.2% (4 cases) of the students committed a single mistake.

In the following paragraphs, three different types of conceptual mistakes will be described with one or more illustrated examples.

Type A is a unit confusion mistake. For example, a student wrote 10 cm × 4 cm = 40 cm (it should be 40 cm^2). Or, when a student wrote 3 – 1.2 = 1.8, it indicated that the student confused or neglected the units for 3 and 1.2. In the problem, 3 is for 3 centimeters and 1.2 is for 1.2 meters. This was the most common mistake made by the students; one third of the students made this mistake (33% or 41 students). The students experienced two types of unit confusion. They are illustrated in Figure 6.12 and Figure 6.13.

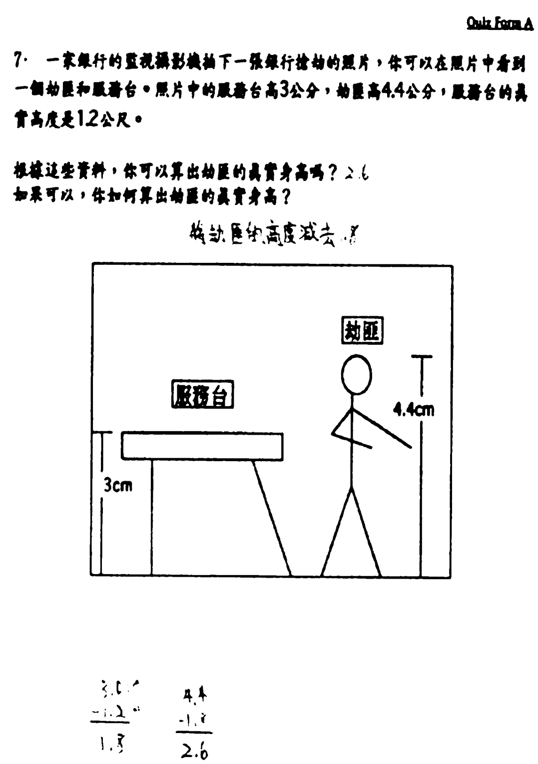


Figure 6.12 Type A Mistake Case 1 on the Photo-Robber Problem

In Figure 6.12, students subtracted two different units of measure, centimeter and meter, without noticing the difference. In the illustrated case, the student subtracted 1.2 m from 3 cm. The difference could not be equal to 1.8 m, but the student did not notice that the subtraction did not make sense within the problem context. The mistake that the student made involved confusion with unit measurement, and also included incorrect additive reasoning.

Quiz Form B

6·上個星期日，你站在一座雕塑品旁照了一張照片，你在照片裡的高度是4公分，雕塑品在照片裡的高度是10公分，你知道你現在的身高是1.4公尺。

根據這些資料，你可以算出雕塑品的真實高度嗎？可以.
如果可以，你如何算出雕塑品的真實高度？1.4x6
雕塑品的真實高度有多高？8.4

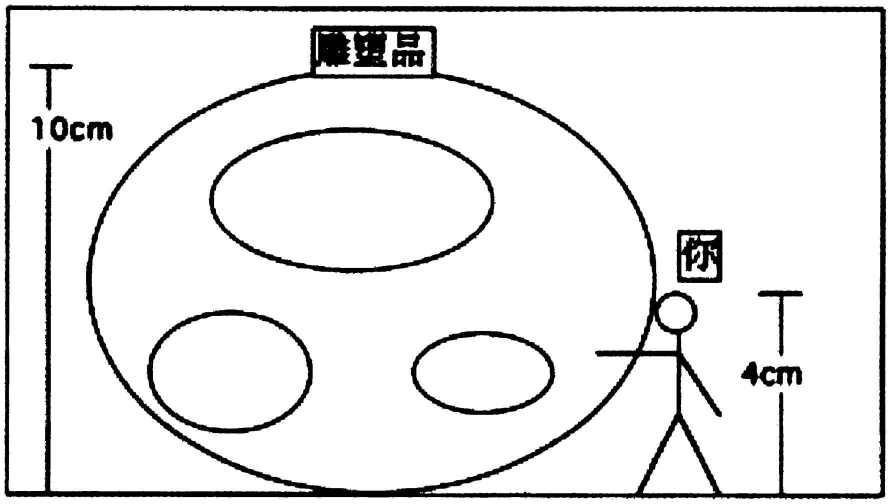
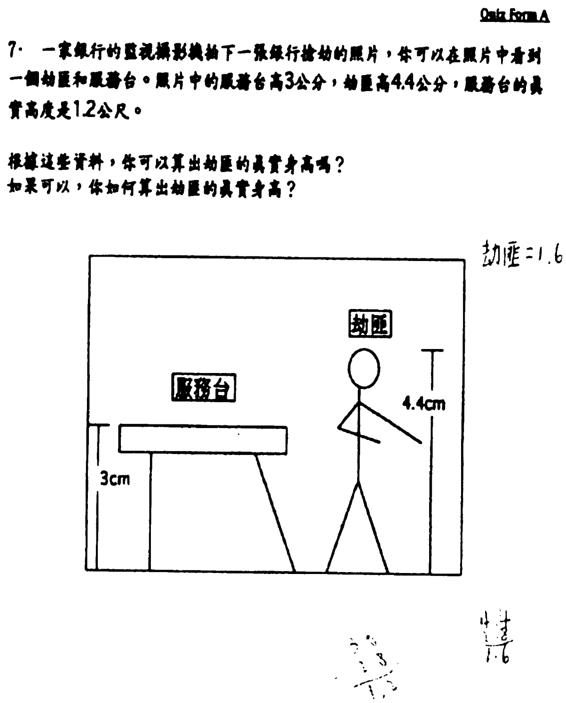


Figure 6.13 Type A Mistake Case 2 on the Photo-You Problem

In Figure 6.13, the student multiplied quantities given in two different units of measurement (6 cm and 1.4 m), without noticing their differences. S/he first subtracted 4 cm from 10 cm and got 6 cm. Then, s/he multiplied the difference times 1.4 m. The product should not be equal to 8.4 (centimeters or meters). The unit should be in "square" units. S/he did not know that the product of 6 cm and 1.4 m should yield square units.

Type B is an additive reasoning mistake. In the last section, Figure 6.12 showed that a student thought that 3 cm minus 1.2 m corresponded to 4.4 cm minus the difference between 3 cm and 1.2 m. The student seemed to think about ratio as an additive or subtractive relationship between two related numbers. The most common additive mistake made by students was to (1) interpret the relationship between quantities in the photo additively (Q1 - Q2) and then (2) apply that difference to increase the size of the missing real object.


Quiz Form A

7· 一家銀行的監視攝影機拍下一張銀行搶劫的照片，你可以在照片中看到一個劫匪和服務台。照片中的服務台高3公分，劫匪高4.4公分，服務台的真實高度是1.2公尺。

根據這些資料，你可以算出劫匪的真實身高嗎？
如果可以，你如何算出劫匪的真實身高？

Figure 6.14 Type B Conceptual Mistake on the Photo-Robber Problem

A more logical additive mistake will be presented in Figure 6.14. In this case, the student thought about how s/he could turn 3 cm into 1.2 m in an additive relationship. S/he reasoned that 3 minus 2.8 equaled to 1.2 (s/he made a computational mistake: 3 minus 2.8 actually equals 0.2). S/he then carried out the same subtraction pattern on 4.4 *cm* (4.4 - 2.8) to give her/him the robber's height. This student seemed to have an additive model of ratio in her/his mind. Compared to the case illustrated in Figure 6.12, this student had a functional idea about ratio, but her/her functional idea about ratio was an incorrect additive.

<u>Type C</u> mistakes could not be interpreted in relationship to the context of the problem. Students might not have comprehended the problem situation or been unable to integrate their ideas about ratio and proportion into the problem situation. So, many just tried to use all the given numbers without any contextual sense.

Quiz Form A

7· 一家銀行的監視攝影機拍下一張銀行搶劫的照片，你可以在照片中看到一個劫匪和服務台。照片中的服務台高3公分，劫匪高4.4公分，服務台的真實高度是1.2公尺。

根據這些資料，你可以算出劫匪的真實身高嗎？可以
如果可以，你如何算出劫匪的真實身高？6.2公尺

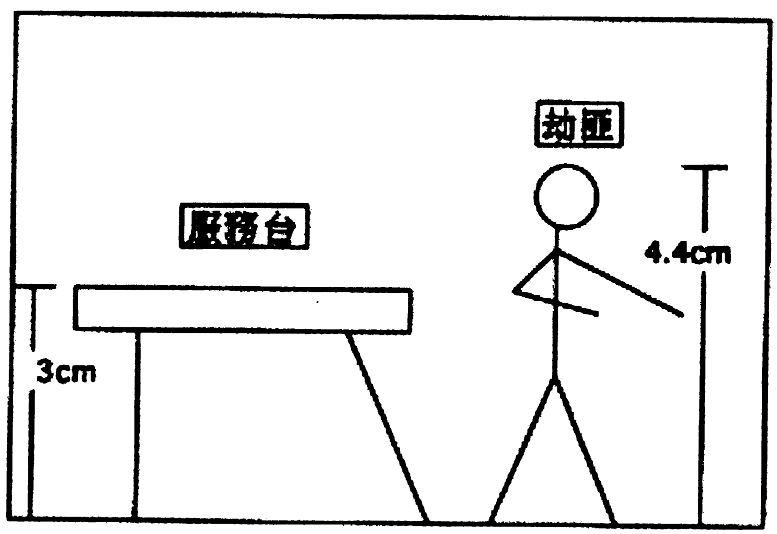


Figure 6.15 Type C Mistake Case 1 on the Photo-Robber Problem

In Figure 6.15, the student seemed to use all the numbers with additive and subtractive operations to solve the problem. S/he first subtracted 1.2 m from 3 cm, and got 1.8 m, then she added 1.8 m with 4.4 cm. S/he thought the answer should be 6.2 m. Based on her/his responses, s/he made an additive reasoning mistake on the concept of ratio, and also neglected the unit of measure among the given quantities in the quiz. However, it was very difficult to explain why s/he used the operations among quantities in this problem context.

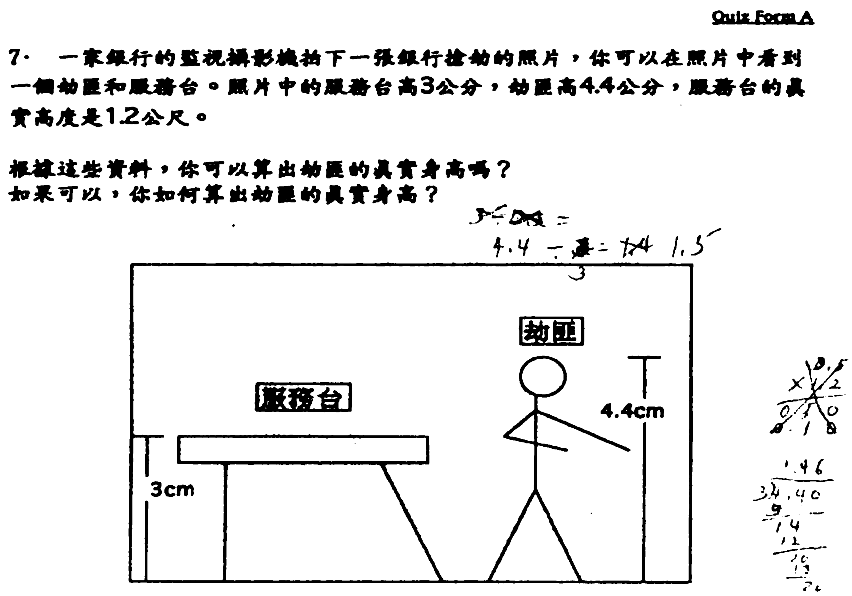


Figure 6.16 Type C Mistake Case 2 on the Photo-Robber Problem

In Figure 6.16, the student seemed to use another arithmetic operation, division, without a deep contextual sense. S/he used the given number "3" to divide another given number "4.4", and thought the answer was 1.5. Based on her/his responses, it was revealed that the student was confused by the unit measurement, but it is difficult to interpret why s/he did the division operation on those quantities.

<u>Summary</u>

From the examples given above, we can see that the unit confusion mistake often co-occurred with the other two types of mistakes. These two types of mistakes (B and C) represented two different levels of understanding, although both of them were categorized as conceptual mistakes in the scoring scheme. The "Additive" mistake (Type B) is related to misconceptions or false models for thinking about ratio. Students who made this mistake seemed to have been confused by the concept, ratio, as an additive relationship between the quantities, and not as a multiplicative relationship. The "Other" mistake was related to comprehension or recognition of the problem situation. Students who made this type of mistake may have been unable to comprehend the problem or relate the problem situation to the learned concept, ratio.

In this section, I would like to examine the differences between students at the two schools through the lens of conceptual mistakes. On Form A (see Table 6.4), there were no obvious differences between the two groups. They executed the same patterns of mistake, with unit confusion (Type A). It was the most common mistake made among all of the students. Although students from the middle class school seemed to perform better, the differences between the two groups on the 3 types of mistake were less than 8%. In contrast to the "Extended Square" problem, the frequency of these errors between the two schools was much closer.

Table 6.4 Performance on the Photo-Robber Problem at Two Schools

School	Problem	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Low SES School (N=33)	Photo-Robber	9.1 (3)	3.0 (1)	69.7 (23)	9.1 (3)	9.1 (3)	42.4 (14)	39.4 (13)	15.2 (5)			54.5 (18)	15.2 (5)	0.0 (0)
Middle Class School (N=25)	Photo-Robber	32.0 (8)	8.0 (2)	56.0 (14)		4.0 (1)	36.0 (9)	32.0 (8)	12.0 (3)			68.0 (17)	20.0 (5)	0.0 (0)

On Form B (see Table 6.5), students from the middle class school outperformed their counterparts. Only one tenth of the students at the middle class school made this mistake on each type. In general, these two groups of students made these mistakes in different patterns. In the low SES schools, more students made unit confusion mistakes (Type A) than other mistakes (Type C). More than one third of the students in this group made mistakes of these two types. This seems to imply that many students in this group did not understand the problem situation or were unable to recognize the problem as related to ratio and proportion. About the same number of students in the two schools reasoned additively about ratio (Type B mistake).

Table 6.5 Performance on the Photo-You Problem at Two Schools

School	Problem	Answer Correctness					Conceptual Mistake					Computing Error		
		C	NC	I	UC	UA	A	B	C	D	E	0	1	2
Low SES School (N=35)	Photo-You	28.6 (10)	2.9 (1)	57.1 (20)	5.7 (2)	5.7 (2)	37.1 (13)	11.4 (4)	31.4 (11)			71.4 (25)	8.6 (3)	0.0 (0)
Middle Class School (N=30)	Photo-You	63.3 (19)	3.3 (1)	30.0 (9)		3.3 (1)	10.0 (3)	10.0 (3)	10.0 (3)			76.7 (23)	3.3 (1)	0.0 (0)

Based on the results in Table 6.6, we can conclude that the incorrect responses were related to conceptual difficulties that students had, rather than the computational errors they made.

Table 6.6 Incorrect Responses Frequency on the Two Ratio Problems

School (*N*=)	Form	Problem	Frequency, Incorrect Response	Minimum Incorrect Response Frequency related to Conceptual Mistakes	Incorrect Response Frequency related to Computational Errors
Low SES School (33)	A	Photo-Robber	24	19	5
Middle Class School (25)	A	Photo-Robber	14	9	5
Low SES School (35)	B	Photo-You	20	17	3
Middle Class School (30)	B	Photo-You	9	8	1

Among the 24 out of 33 students who responded to the "Photo-Robber" problem incorrectly from the low SES school, at least 19 of them committed one or more conceptual mistakes. Only 5 of them responded incorrectly, and their incorrect responses could possibly be attributed to computational errors. And, among the 14 out of 25 students who responded to this problem incorrectly from the middle class school, at least 9 of them committed one or more conceptual mistakes. Only 5 of them responded incorrectly and their incorrect responses may have been attributable to computational errors. On the

"Photo-You" problem, 20 out of 35 students in the low SES school responded incorrectly. Among the 20 students, 17 of them committed one or more conceptual mistakes. Only 3 of them responded incorrectly, and their responses may have been caused by computational errors. Also, 9 out of 30 students from the middle class school responded to the problem incorrectly. Among the 9 students, 8 of them committed one or more conceptual mistakes. Only 1 of them responded incorrectly, and her/his response might have been caused by a computational error.

General Summary

In general, the students who responded to these two problem incorrectly seemed to experience more conceptual difficulty, rather than lack of computational skills or careless computational errors. Based on the mistakes made by students in the two problems, we can find several common characteristics. First, the most obvious one is that unit confusion was the most common mistake made by students across the two problems. And, this mistake seemed to be related to other conceptual or problem comprehension mistakes. Students who did not understand the concepts well made a unit confusion mistake. Most of these students were also confused by the concepts of area and ratio. They may have acquired an additive rather than multiplicative model for these concepts. Students who did not comprehend the problem situation also made the unit mistake. It would be interesting to learn why students who had conceptual and comprehension difficulties would neglect the quantities given. I do not have any answer, but I think it would be a good question for further investigation. Second, if we take out the comprehension problem that some students had and just focus on the conceptual mistakes, then we will find that the conceptual mistakes made by the students were related to the students' prior experiences. The mistake does not emerge suddenly without history. The common conceptual mistake in the "Extended Square" problem was related to confusion between length measurement and area measurement. Students used their length measurement knowledge in the area situation. In the "Photo-Ratio" problem, students used their knowledge of addition to think about multiplication. In this chapter, we could see that many conceptual mistakes were related to additive thinking. In the next chapter, we will see that several students made correct use of their additive knowledge to solve these ratio problems.

Other Non-Routine Problems

There were 6 more non-routine problems in the quiz and they were arranged between the two different forms. Among these problems, most of the answers given on 4 problems (Glass House, Coins, Lake, and Balance) by the students were incorrect, and most students did not outline the procedures that revealed their thinking. So, it is difficult to make any inference about their reasoning. Based on the students' responses, two problems (Number Pattern and Cake) were quite different from the four we mentioned. In the "Number Pattern" problem, students did very well on Parts A, B, and C, however, their answers for Part D were mostly incorrect. Neither students wrote their procedures in Part D. So, I cannot make any inference about their reasoning.

The "Cake" problem is the only one on which we can make some inference about students' reasoning. In general, students performed very well on this non-routine problem. But, the students who responded incorrectly seemed to have more conceptual difficulty, rather than making computational errors. Based on the results on Table 6.7,

Table 6.7 Incorrect Responses Frequency on the Cake Problem

School (*N* =)	Form	Problem	Frequency, Incorrect Response	Minimum Incorrect Response Frequency related to Conceptual Mistakes	Incorrect Response Frequency related to Computational Errors
Low SES School (35)	B	Cake Part A	7	4	3
	B	Cake Part B	13	8	7
Middle class School (30)	B	Cake Part A	6	6	0
	B	Cake Part B	7	3	5

we found that students committed more conceptual mistakes than computational errors. The only exception was the middle class school students' performance. Among the 7 students who gave the incorrect responses in Part B, only 3 of them committed conceptual mistakes; and the other 4 only made computational errors.

Chapter 7

STUDENTS' CONCEPTUAL STRENGTHS AS EVIDENCED IN NOVEL SOLUTIONS

This chapter will describe some unexpected student strengths in understanding and solving mathematical problems. Before I go into a brief introduction to this chapter, let me explain why I decided to include a chapter about these unexpected solutions. This chapter is written for two audiences—teachers and educational psychologists. In particular, I would like to reach primary school teachers. Sometimes, these novel student solutions are difficult for the teachers to detect and discover in their students' reasoning. To present my inferences about student reasoning to primary school teachers is the goal of this chapter. I anticipate that the teachers would like to know more about the students' actual thinking, rather than the statistics I presented in the previous chapters. Hopefully, teachers will find that this information is useful in their classrooms. The second audience I would like to address is educational psychologists. The data that will be presented in this chapter will reveal several characteristics of student learning in which educators and psychologists are interested. The students in this study showed the power of prior knowledge and intuition in learning mathematics. And, one student, Bobby, showed the potential that students have for learning the concept, proportion.

Several students in this study offered some unexpected solutions to quiz and interview problems including "Photo-Robber," "Photo-You," Lake," and "Train." Their solutions were closely related to students' intuitive ideas about several mathematical concepts, rather than ideas they learned in school. The basis of this inference is (1) that there were only a few students who offered such unexpected solutions, (2) they learned the mathematical topics from different mathematics teachers at two different schools, and most importantly, (3) these solutions did not easily flow into the solutions that the researcher expected. The solutions were less likely to be taught in schools.

There are two main sections in this chapter. In the first section, a common solution to the "Ratio" problem will be presented, then several unexpected solution to student work on the two "Ratio" problems will be presented. In the second section, I will focus on a special case, Bobby. All of his work on the quiz and in the interview will be analyzed in detail. At the end of the chapter, I will summarize the unexpected solutions by students and discuss their implications in the learning and teaching of mathematics in elementary schools. I would argue that such unexpected solutions offer us a chance to think about how students construct their mathematical knowledge in schools and consider what the possible conditions are for helping them learn mathematics with meaning. And, finally, I will focus on how teachers can build such learning environments in their classes.

Students' Typical Solutions on the Two Ratio Problems

Case 1

Problem (from Form A):
A hidden camera took the picture below during a robbery. In the picture, you can see the robber and a service desk. The height of the desk is 3 cm, and the height of the robber is 4.4 cm. The real height of the desk is 1.2 meter. Can you figure out the height of the robber? How?

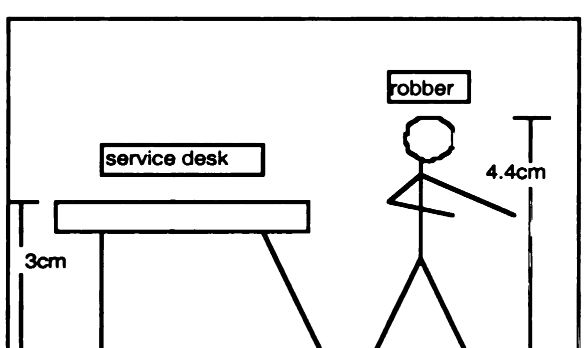


Figure 7.1 The Photo-Robber Problem on Form A

Of the students who gave a correct answer, 35 produced the following solutions on the two "Ratio" problems on the two different forms. On Form A, most students with a correct answer wrote the formula (see Figure 7.2), 120 cm / 3 cm × 4.4 or 1.2 / 3 × 4.4. Given only written formulas and correct answers given by the students, I cannot infer how they thought about the concept of ratio, but this was the most typical written protocol collected in this study. It was impossible for me to know whether they might be solving the problems by using a cross-multiplication strategy, or some other strategy. These students might be using the formula, Ratio x = Ratio y. Using the "Photo-Robber" problem as an example, these students may have had the formula, 120 *cm* / 3 *cm* = y / 4.4 *cm*, on their mind when they worked on the "Photo-Robber" problem.

Quiz Form A

7· 一家銀行的監視攝影機拍下一張銀行搶劫的照片，你可以在照片中看到一個劫匪和服務台。照片中的服務台高3公分，劫匪高4.4公分，服務台的真實高度是1.2公尺。

根據這些資料，你可以算出劫匪的真實身高嗎？176cm
如果可以，你如何算出劫匪的真實身高？120cm÷3cm×4.4cm

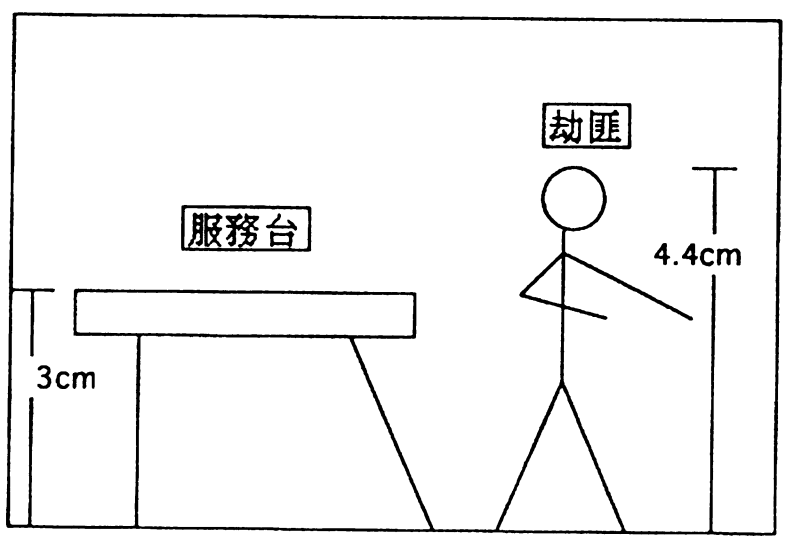


Figure 7.2 A Typical Solution for the Photo-Robber Problem on Form A

Case 2

Problem (from Form B):
Last Sunday, you stood beside a sculpture and had your picture taken. Your height on the photo is 4 cm, and the height of the sculpture is 10 cm. You know your real height is 1.4 meter. Can you figure out the real height of the sculpture? How?

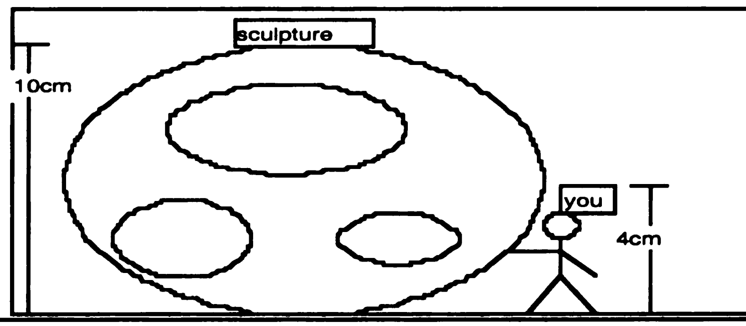


Figure 7.3 The Photo-You Problem on Form B

On Form B, the most typical written protocol collected was the following scanned example (Figure 7.4). Most students wrote a similar formula, 1.4 / 4 × 10. As with Photo-Robber, this protocol did not offer sufficient evidence to infer how these students thought about the concept of ratio. But, these responses seemed to suggest that these students might be solving the problem using a cross-multiplication strategy because this is the strategy most students learn at school.

Quiz Form B

6.上个星期日，你站在一座雕塑品旁照了一张照片，你在照片里的高度是4公分，雕塑品在照片里的高度是10公分，你知道你现在的身高是1.4公尺。

根据这些资料，你可以算出雕塑品的真实高度吗？
如果可以，你如何算出雕塑品的真实高度？
雕塑品的真实高度有多高？

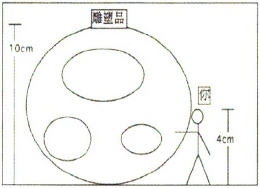


Figure 7.4 A Typical Solution for the Photo-You Problem on Form B

Students' Unexpected Solutions on the Two Ratio Problems

Five out of the 123 students gave two kinds of unexpected solutions on the two ratio problems. Thirty-five students solved these ratio problems using the learned algorithms mentioned in the last section. Among the five students, two of them used an intuitive functional model to think about ratio, and the other three used an additive model to think about the concept.

Intuitive Function Strategy

This answer is of special interest because the students used a model that was not directly related to the "cross-multiplication" algorithm that they probably learned in school to think about a proportional problem. There were two students who used this idea (I have chosen to call it, intuitive functional thinking) to think about proportion. I infer that the students thought about a rule to turn 3 centimeters into 1.2 meters. They may intuitively have used an implicit formula similar to the following:

$$1.2\ m = f(x)\ 3\ cm$$

They found that the functional rule, $f(x)$, is equal to dividing 3 by 5 and multiplying times 2, ($3 \div 5 \times 2 = 1.2$), which equals 1.2. Then, they applied the function rule to 4.4 cm and got 1.76 m, ($4.4 \div 5 \times 2 = 1.76$). Figure 7.5 shows the written response given by one student. Both students did similar work on their quiz sheet. Both students gave the correct unit, meter, in their answers. To me, the search for a rule was a primitive and intuitive functional idea—searching for an algorithm that helps them turn "*a*" into "*b*." The two students were from two different schools so it did not appear to be an instructional effect of their classes or schools.

Quiz Form A

7· 一家銀行的監視攝影機拍下一張銀行搶劫的照片，你可以在照片中看到一個劫匪和服務台。照片中的服務台高3公分，劫匪高4.4公分，服務台的真實高度是1.2公尺。

根據這些資料，你可以算出劫匪的真實身高嗎？
如果可以，你如何算出劫匪的真實身高？

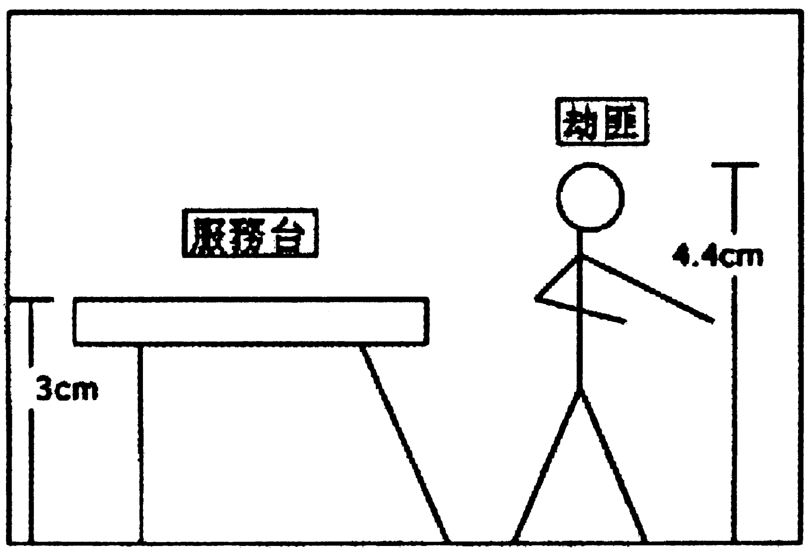


Figure 7.5 An Example of Functional Thinking on Proportion

Karplus, Pulos, and Stage (1983a, 1983b) suggested students should develop a similar model, as that given by these two students, when they learn about the concept, proportion. Karplus et al., (1983a, 1983b) suggested that students should read "proportion" as a relationship between two variables, and not as a relationship between two numbers. Students should see "proportion" as a functional relationship between two variables.

Additive Strategy

Three students used sort of an additive, "building it up," thinking (see Figure 7.6) to solve the ratio problem on Form B. Students reasoned that 10 cm is equal to 4 cm + 4 cm + 2 cm, and inferred that 1.4 m + 1.4 m + 1.4 m ÷ 2 = 3.5 m, because each 4 *cm* was corresponding to 1.4 m. Figure 7.6 shows that the written responses given by one student; were also duplicated by three students who gave similar written work on their quiz sheets. All three students who used this reasoning worked on the "Ratio-Photo" problem on Form B. No student used this reasoning in Form A.

Quiz Form B

6·上個星期日，你站在一座雕塑品旁照了一張照片，你在照片裡的高度是4公分，雕塑品在照片裡的高度是10公分，你知道你現在的身高是1．4公尺。

根據這些資料，你可以算出雕塑品的眞實高度嗎？
如果可以，你如何算出雕塑品的眞實高度？
雕塑品的眞實高度有多高？

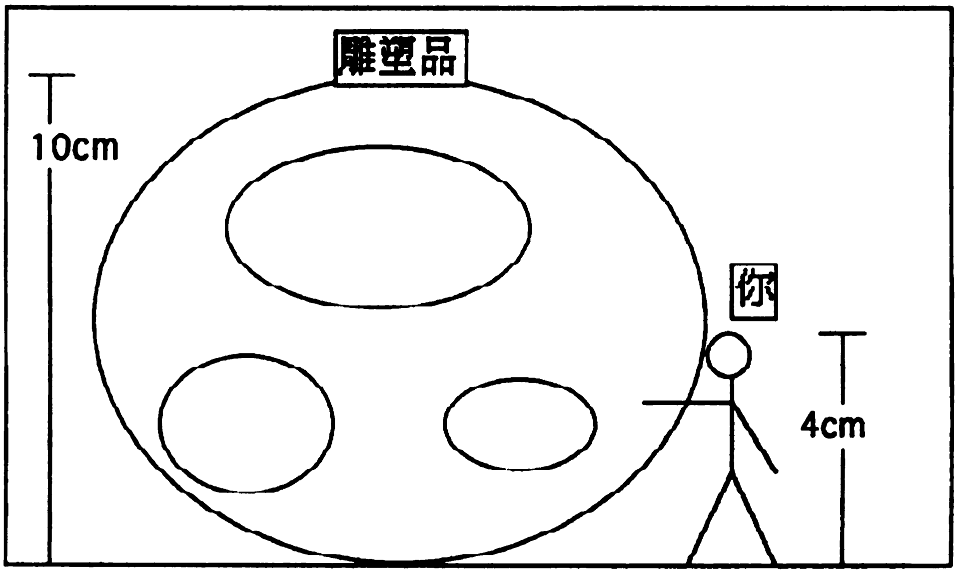


Figure 7.6 An Example of Additive Thinking on Proportion

This "building up" model is very common among students when they develop the concept, proportion, in their early phase (Lesh, Post, & Behr, 1988, Nesher, 1988; Peled and Nesher, 1988; Resnick, 1989). It is quite common for students to carry their additive knowledge to a multiplicative context. Proportional reasoning is a multiplicative relationship between two variables, and students try to use their additive knowledge to understand the proportional relationship between two variables.

Summary and Conclusion

There are two interesting characteristics of these unexpected solutions. First, the intuitive functional model only happened on the ratio problem on Form A, and the additive model only appeared on the ratio problem on Form B. So, these solutions seem to be tied in some way to the problem context. The relationship among given numbers in the situations seemed to have influenced how students constructed model(s) to conceptualize the problem situation and its related mathematical concepts. In the "Photo-Robber" problem, the functional model, 1.2 m $= f(x)$ 3 cm, may be more suitable in this context. The relationship between the represented quantities, 3 cm and 4.4 cm, makes it more difficult for students to apply an additive model. Unlike the "Photo-You" problem, the relationship between the two represented quantities, 4 cm and 10 cm, seemed to make it easier for several students to construct an additive model to solve the problem because 10 cm is equal to 4 cm plus 4 cm plus 2 cm. The relationship between these two quantities is 2.5 times, 4(2.5) = 10. Second, it is interesting that all of these unexpected solutions were given by the middle achievers in two different schools.

The main difference between these five cases and the other students who finished the problem using a cross-multiplication algorithm is that these five special cases revealed their understanding of the concept of proportion. Although some of their understandings of the concept was formed in an early phase of development (additive and building-up models), they still captured the co-variant relationship between the two variables when they conceptualize the proportional relationship. Many mathematics educators (Lesh, Post, & Behr, 1984; Hart, 1984) worried that students only developed the cross-multiplication algorithm to conceptualize the concept of proportion. These educators thought that most of these students do not develop a real understanding of the concept. They may never develop a co-variant or functional relationship between two variables.

A Special Case—Bobby

Bobby was a student who came from a low SES family. He studied at a low SES school and was assigned to a mid-to-low achiever class. He was a mid-achiever at the school. In the interview, I asked him about his favorite school subject and he told me science was his favorite. During the interview, Bobby was very quiet. In an informal talk with his teacher, she told me that Bobby's behavior was not like an average grade 5 student at times. He often behaved like a grade 2 or 3 student, and she did not find anything special about him. But, Bobby's former teacher told his current teacher that Bobby was really good in mathematics.

Bobby was a special case in this study because he showed a consistent strength in solving non-routine problems. Although Bobby did not solve all the non-routine problems correctly, he was on the right track on every problem. Bobby correctly conceptualized most problem situations, but he made computational mistakes in several problems. Bobby only tried 7 out of 10 quiz problems. As a result, I inferred that he did not have enough time to finish the last 3 problems in the quiz. Of the 6 finished problems, 2 were routine and 4 were non-routine. He solved 3 of them correctly; two of them were routine and one was non-routine. He made some computational mistakes on the other 3 problems, but he approached the problems with the right idea. On the "Lake" and the "Glass House" problems, he revealed his strength in being able to tackle non-routine and difficult problems. Although he only solved the "Glass House" problem correctly, on the "Lake" problem he displayed a correct concept about speed—the co-variant relationship between distance and time. In the interview, Bobby solved the two assigned non-routine problems. On the "Train" problem, he used an unexpected model to conceptualize the concept of speed. On the "Classroom Board" problem, he knew the correct measurement of units for pieces of paper, and for area and length. He only had difficulty explaining the remainder, 1000 cm^2.

In the next section, I will present his work on the non-routine problems, one problem at a time. Before I go into the detail about Bobby's performance, I would like to use Table 7.1 to briefly summarize Bobby's performance on the two different sections.

Table 7.1 A Brief Summary of Bobby's Performance

Performance Context	Problem	Problem Type	Solution Conceptual	Solution Computational
Quiz	Multiplication	Routine	correct	correct
	Unknown Digit	Routine	correct	correct
	Balance	Non-routine	correct	partly correct
	Glass House	Non-routine	correct	correct
	Coins	Non-routine	correct	mistake
	Lake	Non-routine	correct	mistake
Interview	Train	Non-routine	correct	correct
	Classroom Board	Non-routine	mostly correct	correct

The Routine Problems in the Quiz

The "Multiplication" and "Unknown Digit" Problem

"Multiplication" and "Unknown Digit" are the only two routine problems for which Bobby gave written responses. In the "Multiplication" problem, Bobby multiplied "187" times "31" correctly, and gave "5797" as his answer.

Problem Name: Unknown Digit	Problem Type: routine
Problem (from Form A): The 3-digit number "57?" is exactly divisible by 16. Find the value of "?" **Answer**: 6 or 576	

Figure 7.7 The "Unknown Digit" Problem

In the "Unknown Digit" problem, Bobby's written responses implied that he initially tried to divide "57" by "16," and got 9, 57 - 48 = 9. Then, Bobby seemed to try to see whether "90" was divisible by 16. Then, he tried "96" and he found that "96" was divisible by "16." So, he gave the answer, "576."

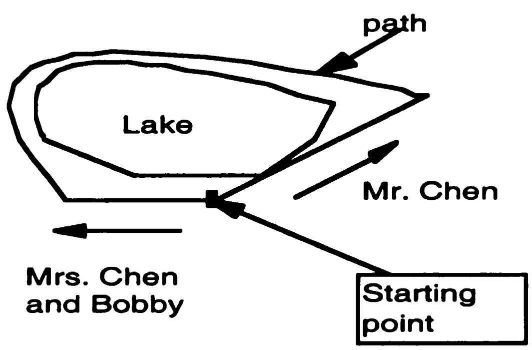
The Non-routine Problems in the Quiz

The "Lake" Problem

Only 6 out of 123 students correctly solved the "Lake" problem. Bobby did not finish the problem because there was no given answers on the sheet. However, Bobby was the only student who showed in detail what he thought about the "Lake" problem logically and mathematically. He tried to figure out how far the couple walked together in a minute. He found the couple walked 83.33 m a minute. So, he thought he could subtract 83.33 m from 10 km continuously to see how many times 83.33 m were used. He would then know how many minutes it took the couple to walk around the lake-track. The walking time of the couple also gave the dog's running time. He then multiplied the time used by the couple times the running speed of the dog (466.6 m/min). Mr. and Mrs. Chen did not walk 10 km individually; each of them only walked 5 km on the lake-track. So, using 10 km of total distance was a mistake. I do not have proof as to whether it was a conceptual or a careless mistake. Additionally, Bobby made two computational mistakes. The first one resulted when he subtracted 83.33 from 10000 and got 916.67. Bobby missed a digit, the correct answer should have been 9016.67. The second one occurred when he tried to compute the dog's running speed per minute. The correct speed should have been 466.6 m/min, not 46.6 m/min, but Bobby lost track of his decimal point. Figure 7.9 shows Bobby's work. Bobby's strength in the problem was that he grasped the co-varying relationship between distance and time and the co-varying relationship between the distance walked by the couples and that of the dog. This co-varying relationship is one of the core concepts of rate.

Lake Problem (from Form A):

Mr. and Mrs. Chen walk with their dog, Bobby, every morning. They walk at the same speed, 5 km per hour. The distance of the walking path around the lake is 10 km. They start and walk in opposite directions at the starting point and meet each other midway, then go to have their breakfast. Bobby starts his morning run with Mrs. Chen and runs toward Mr. Chen. When he meets Mr. Chen, he runs back to Mrs. Chen. He continues like this until Mr. Chen and Mrs. Chen meet. His running speed is 28 km per hour. How far does Bobby run, by the time the couple meet?



Answer: 28 km

Figure 7.8 The Lake Problem and Its Solution on Form A

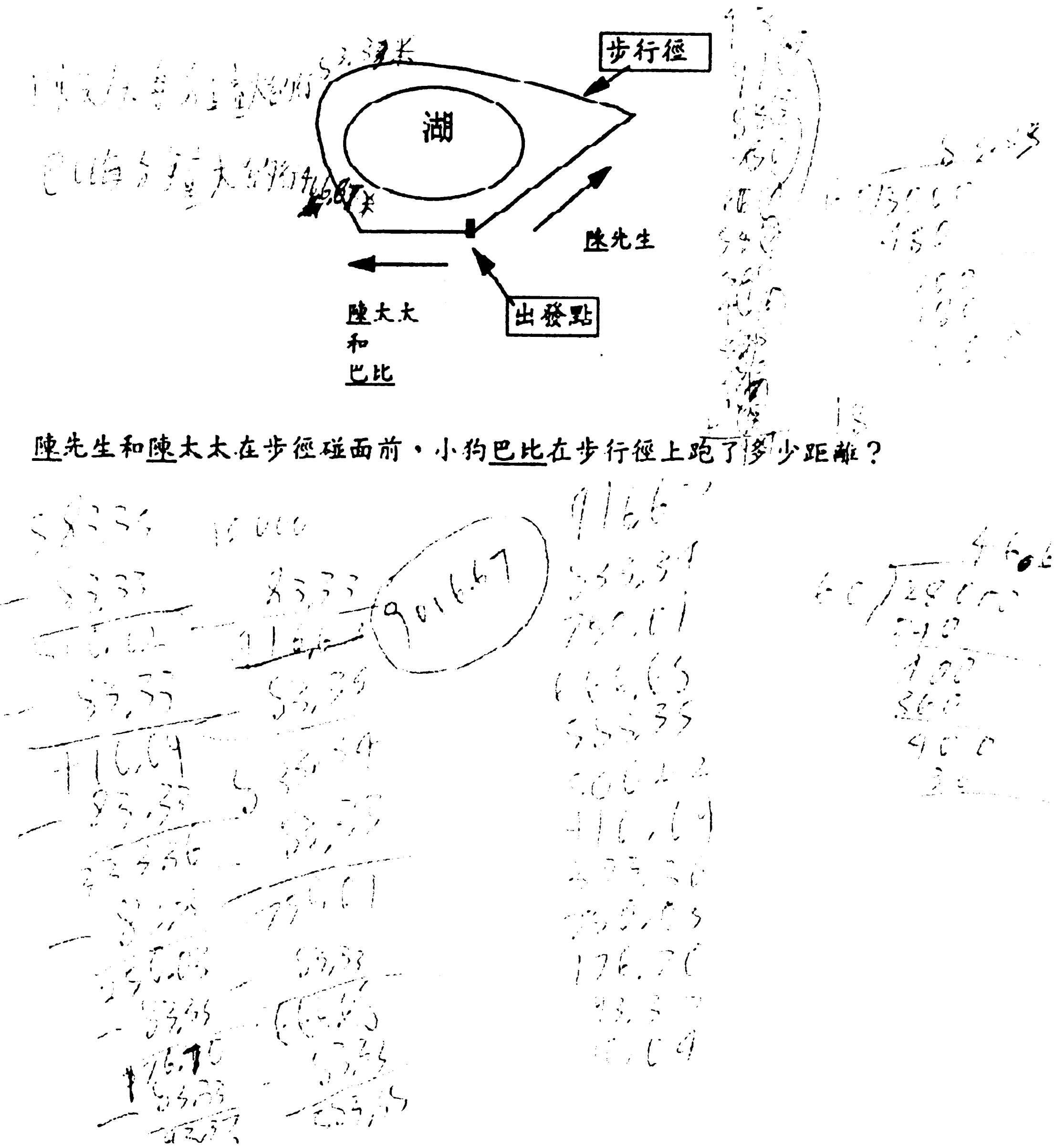


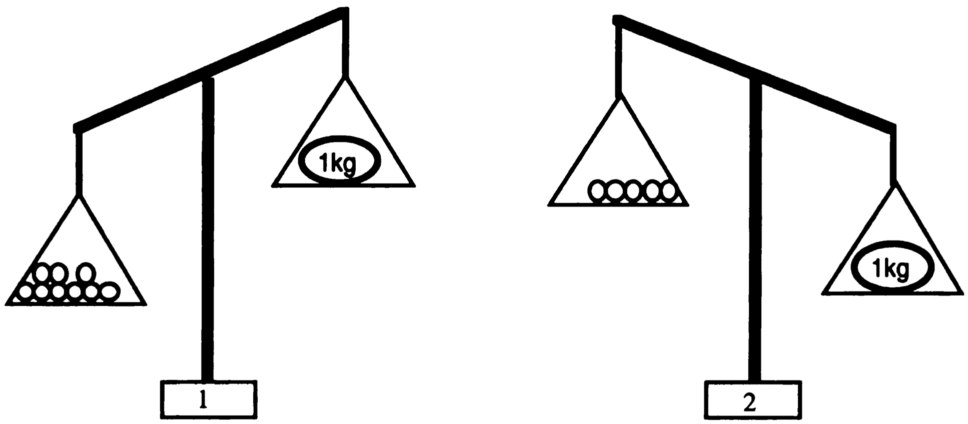
Figure 7.9 Bobby's Work on the "Lake" Problem

<u>The "Balance" Problem</u>

The "Balance" problem was one of the most difficult problems in the quiz. Only 3 students correctly solved the problem. Although there was no explicit evidence to explain Bobby's solution, Bobby seemed to only make a computational mistake, rather than experiencing conceptual difficulties in solving the problem.

Balance Problem (from Form A):

All marbles are the same weight, what are the heaviest and the lightest weight to a marble?



Answer: the lightest is 112g or 111.12g; the heaviest is 199g

Figure 7.10 The "Balance" Problem and Its Solution on Form A

Quiz Form A

3· 圖中所有彈珠均等重，請問每粒彈珠最輕可以是多重？
最重可以是多重？

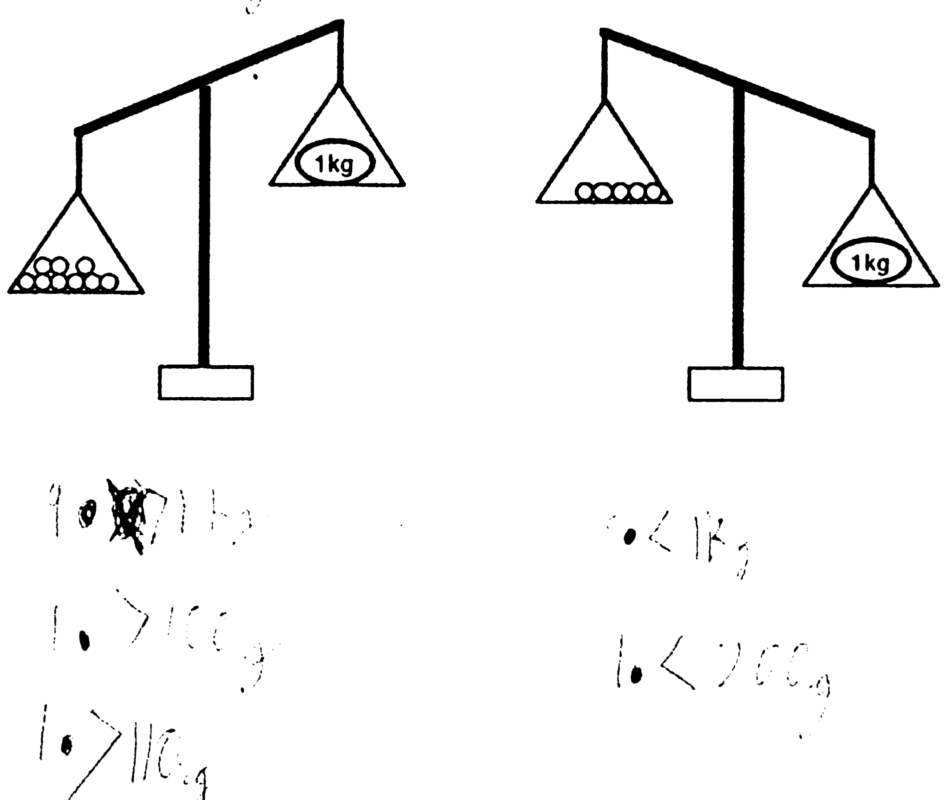


Figure 7.11 Bobby's Work on the "Balance" Problem

Several aspects of Bobby's written responses showed that he might have only made a computational mistake. First, Bobby used the signs, "<" and ">" correctly. This implied that he had a clear idea of the problem, knowing that the weight of a marble can be expressed in a range of weights. Second, Bobby added 1 gram to 110 grams and thought that it was the lightest weight of a marble, and that adding an additional gram would make the 9 marbles heavier than the 1000 gram weight. Using the same logic, he subtracted 1 gram from 200 gram and thought it was the heaviest weight of a marble. This subtraction made the weight of the 5 marbles lighter than the 1000 gram weight. Adding and subtracting gave more information about his thinking and supported the inference that he only made a computational mistake in the problem. Conceptually, Bobby was on the right track in thinking that he should find the range of the weight of a marble. The only mistake he seemed to make might have been related to dividing 1000 grams by 9 marbles. The quotient of the division should have given him 111 grams, not 110. And, since each marble weighed 111 grams, combined, the nine of them made the total weight equal to 999 grams. So, their total weight was less than one kilogram and the balance would not be like the demonstrated picture on Figure 7.8. Unfortunately, he did not write down the computational procedures on the quiz sheet, so I do not have a valid verification of this computational mistake. However, his mathematical sign usage and his logic of using addition and subtraction to the quotients again revealed that he might have only made a computational mistake.

The "Glass House" Problem

The "Glass House" problem was the most difficult problem on Form A of the quiz. None of the students gave a correct solution to "Part a," listing all the possible rectangles with different perimeters and with the same area of 24 m^2. Only 2 out of 123 students gave the correct answer on "Part b," the floor plan that costs the least to build the glass house.

Glass House Problem (from Form A):
Your school is planning to build a glass-house for the science classes. The area of the floor must be 24 square meters. The building will be 3 meters in height. The builder estimates the house costs $1200 for every 1 meter of the perimeter.

a. List out all the possible lengths, widths, perimeters, areas, and the cost of walls for different rectangular shapes. (You only need to consider rectangle with length and width equal to whole numbers of meter)

Answer:

Width	Length	Perimeter	Area	Cost of the wall
1	24	50	24	60000
2	12	28	24	33600
3	8	22	24	26400
4	6	20	24	24000

b. Based on the cost of walls, which rectangular shape would cost least to build?

Answer: 4 mx 6m

Figure 7.12 The "Glass-House" Problem and Its Solution on Form A

In Part a, most students did not understand what was being asked in the problem. These students filled in some numbers in the table, but the numbers did not make sense within the context of the problem. For instance, one student wrote 5 meters for width, 6 meters for length, 22 meters for perimeter, 30 square meter for area, and $36,000 for the cost. More than 20% of the students gave up on this problem. In part b, more than 65% of the students did not give any answer, and about 25% of the students gave non-contextual answers. Most of the cases with non-contextual answers showed that students might have had difficulty understanding the context of the problem, but might experience no difficulty memorizing the formula for area and perimeter. Here are the two typical examples of students' work:

Table 7.2 The First Case's Response on the "Glass House" Problem

Width	Length	Perimeter	Area	Cost of the wall
4 meter	2 meter	12 meter	8 meter^2	$400
2 meter	1 meter	6 meter	4 meter^2	$200

Table 7.3 The Second Case's Response on the "Glass House" Problem

Width	Length	Perimeter	Area	Cost of the wall
24	3	54	72	$86400

The first case revealed that the student did not understand the context of the problem. Her/his response to the "Area" column revealed her/his lack of understanding of the problem. The area should be a constant quantity, 24 meter^2, but s/he did not complete the cells with the constant quantity. Also, the student seemed to have no idea about how s/he could figure out the cost of the wall. The numbers s/he added to the cells did not make any contextual sense. However, s/he do not have any difficulty figuring the area and perimeter when s/he knew the width and length. The values given in the Perimeter and Area columns corresponded to the values of width and length. The second case revealed similar difficulty by that student. The student had no difficulty in using the formula for area and perimeter, as evidenced by the numbers that were filled in the "Width," "Length," "Perimeter," and "Area" columns. However, the numerals s/he completed in the cells did not make any contextual sense. For example, the quantities filled in the Width and the Length columns could not result in an area of 24 meters^2. This evidence revealed that s/he did not understand the problem context. Also, the quantity in the "Cost of the Wall" did not make any contextual sense. The answer, $86400 [$1200 × 72], was the product of the area and the building cost, $1200 per meter.

For Part a, Bobby wrote down the information for three different rectangular shapes—width, length, perimeter, area, and the cost of the wall. The only missing rectangular shape was the one with a 2 meter width and a 12 meter length. The information did not help me make an inference as to whether he understood that a 2 meter by 12 meter rectangle was also a candidate. I do not know whether he mentally missed this possibility, or he intentionally skipped this rectangular shape because he knew this shape could not offer the lowest building cost. Bobby also made a computational mistake on Part a. In the 3 meter by 8 meter rectangle row, he wrote 24 meters in the perimeter column. In part b, Bobby gave a numerically correct answer, he wrote width was 4 (without the unit label, "meter") and length was 6 (also without the unit label).

Bobby's response revealed his understanding of the problem situation. Most students did not reveal their understanding of the problem. Bobby also showed that he had no difficulty in distinguishing between the two geometrical concepts, area and perimeter.

4· 你的學校準備要蓋一座玻璃屋，供學生上自然課使用，玻璃屋的底部面積是24平方公尺，玻璃屋高度爲3公尺。建築商估計每蓋一面長1公尺高3公尺的牆，需要花費1200元。

A) 你可以設計不同形狀的長方形做爲玻璃屋的底面積，請寫一個表格列出各種長方形的長、寬、周界、面積、建牆成本。
（你只需要考慮整數做爲長方形的長跟寬）

長	寬	周界	面積	建牆成本
1	24	50	24	60000
3	8	22	24	26400
4	6	20	24	24000

B) 考慮建牆成本，選擇一個長方形爲玻璃屋的底面積是最省錢。

長4 寬6

Figure 7.13 Bobby's Work on the "Glass House" Problem

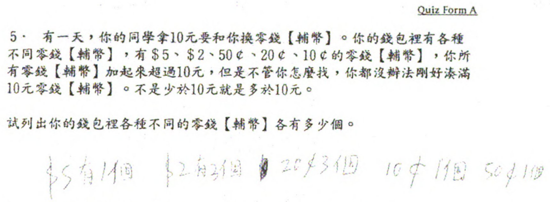
The "Coins" Problem

Only 7 out of 123 students were able to solve the "Coins" problem. Bobby did not produce a correct answer on this problem. There were two constraints to this problem. The first one was that the total should be more than 10 dollars. The second one was that no subset of the coins should be equal to 10 dollars. Most students violated the two constraints simultaneously, their answers were coin-combinations equal to 10 dollars. Bobby's answer, one 5-dollar coin, three 2-dollar coins, one 50-cent coin, three 20-cent coins, and one 10-cent coin, violated the second constraint. The coin-combination given by Bobby was more than 10 dollars, but one combination could be built to equal 10 dollars. From the written protocol, it was quite difficult to infer what the cause was of Bobby's mistake. On the one hand, he might have made a computational error. He did not discover that one of his coin combinations was equal to 10 dollars. On the other hand, he might not have understood the second constraint for the problem. Bobby only violated one constraint for the problem, rather than two.

Coins Problem (from Form A):
You know you have more than ten dollars in you pocket (all are coins, $5, $2, 50 ¢, 20¢, or 10¢) but you find you are unable to give someone a change for a ten dollar note. List out the combination of coins that you have.
Answer: any combination with all different coins, and the combination is not equal to $10 and is more than $10
Possible Combinations are:

Type\Quantity	option 1	option2	option3
$ 5	1	1	1
$2	3	4	4
50 cent	1	1	1
20 cent	1	1	1
10 cent	1	1	2

Figure 7.14 The "Coin" Problem and Its Solution on Form A


Quiz Form A

5· 有一天，你的同學拿10元要和你換零錢【輔幣】。你的錢包裡有各種不同零錢【輔幣】，有 $5、$2、50¢、20¢、10¢的零錢【輔幣】，你所有零錢【輔幣】加起來超過10元，但是不管你怎麼找，你都沒辦法剛好湊滿10元零錢【輔幣】。不是少於10元就是多於10元。

試列出你的錢包裡各種不同的零錢【輔幣】各有多少個。

$5有1個 $2有3個 20¢3個 10¢1個 50¢1個

Figure 7.15 Bobby's Work on the "Coins" Problem

Non-routine Problems in the Interview

The "Train" Problem

Most students who solved this problem successfully found the speed per minute by dividing 66 km by 60 minutes. Then, they multiplied 1.1 km per minute by 2.5 minutes. Finally, they multiplied 2.75 km by 2 because 2 trains were traveling at 2.75 km. This resulted in the correct answer, 5.5 km. Bobby solved this problem with a novel strategy. Although he did not give a good explanation (the verbal protocol he reported) as to why he did it, he seemed to have a special implicit model (Figure 7.16) for interpreting the concept of speed. Bobby might have a co-varying relationship model between time (minutes) and distance (km) in his mind. According to the verbal protocol, Bobby thought first about the ratio relationship between 2 minutes and 30 seconds (2.5 minutes) and 60 minutes (1 hour). He partitioned 60 minutes into 2.5 minute segments. Originally, he thought that the quotient (60 minutes ÷ 2.5 minutes) could be divided by 66 km, and this would give him the distance run by a train. He then reasoned that by doubling it, he would be able to determine the distance between two trains. His formula, $60 \div 2.5 \div 66 + 60 \div 2.5 \div 66$, confirmed his original thinking. However, he changed his mind when he was asked about the number, 66. He immediately reported that he made a mistake on the formula. He wrote a new formula,

$$66 \div (60 \div 2.5) + 66 \div (60 \div 2.5).$$

From the new formula, I inferred that Bobby thought he had some number of partitions (in this case, 24) in the time ratio by dividing 60 minutes by 2.5 minutes, he then reasoned that he could partition the 66 km into the same number of partitions. Each partition in the distance ratio represents 2.75 km. Moreover, the central idea of his thinking was that each partition of the two ratio lines (time ratio and distance ratio) corresponded to each other. So, he knew that a train could run 2.75 km in 2.5 minutes. Doubling the distance run by a train gave him the distance between two trains. Figure 7.16 shows diagrammatically how I interpreted Bobby's use of ratio models (time ratio and distance ratio) to think about the concept of rate (speed).

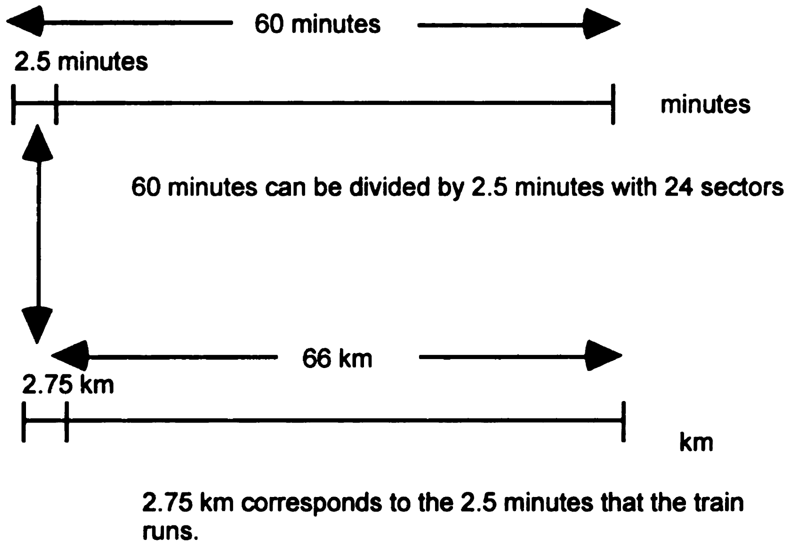


Figure 7.16 Covarying Relationship Between Time and Distance

The following section of the interview protocol gives us more information about how Bobby thought about the problem and the concept of speed. He spent quite a long time thinking about the problem situation before he wrote his formula. His extended thinking on this problem and his novel approach might explain why he did not have enough time to finish all the quiz problems.

(Bobby thought about the problem for about a minute, after reading)

Interviewer: What are you thinking?

Bobby: Think about how to write a formula.

(Bobby did not say anything for more than a minute)

Interviewer: Then, what do you think first?

Bobby: First, think about the numbers.

(Bobby did not say anything for about 1.5 minutes)

Bobby: I think two numbers at the same time.

Bobby: What is the relation?

(Bobby did not say anything for minutes)

Bobby: I am thinking about 60, now.

Interviewer: What is it, 60?

Bobby: I think about 60 minutes and 2 minutes and 30 seconds, their relationship.

Bobby's talk about the 2 numbers relationship (60 divided by 2.5) showed that he tried to think about the time ratio—the ratio relationship between one hour and 2.5 minutes. He transformed one hour to 60 minutes and thought about the ratio relationship between 60 minutes and 2.5 minutes.

(For about 2 minutes, Bobby did not say anything)

Interviewer: What are you thinking now?

Bobby: I think about the formula, but not sure.

Interviewer: Can you tell me more about your unsure formula?

Bobby: Can I write it down?

(Bobby wrote, 60 ÷ 2.5 ÷ 66 + 60 ÷ 2.5 ÷ 66)

The form of this expression (quantity ÷ quantity ÷ quantity) and the time it took Bobby to produce it strongly suggests that he did not retrieve it from memory. He chose to elaborate his thinking gradually.

Interviewer: What is 60, here?

Bobby: Minutes in an hour.

Interviewer: What is 2.5?

Bobby: 2 minutes and 30 seconds.

Interviewer: Why do you divide 60 by 2.5?

Bobby: I don't know.

Interviewer: Why do you divide 66 here?

Bobby: Oh! I made a mistake.

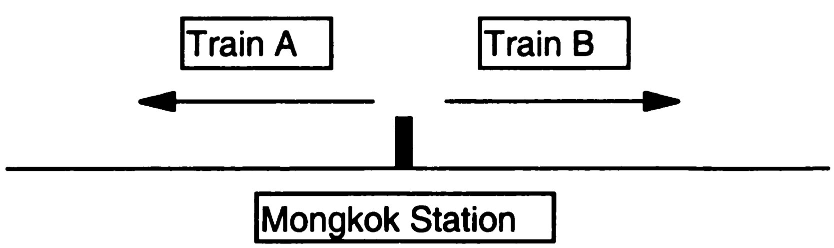
(Bobby rewrote his formula, 66 ÷ (60 ÷ 2.5) + 66 ÷ (60 ÷ 25)

Interviewer: What is the addition?

Bobby: Distance between two trains.

Bobby could not articulate the ratio concept when he was asked why he did a division between 60 and 2.5. He realized that the formula was not correct when the interviewer asked him about the number "66" in the formula. Although he did not report why he changed his mind to create a new formula, his instant response about the mistake he made in the original formula revealed that he had a clear idea of what he was doing. In general, there was a watershed point where Bobby behaved quite differently in two stages. When he generated his first formula, he spent a lot of time thinking about the problem. He needed one or two minutes to think before he reported his ideas to the interviewer. However, after Bobby was better able to conceptualize the situation (the quantities and how they relate), he replied to the interviewer's questions instantly. That pattern revealed that Bobby could answer the questions easily when he built a mental model to represent or to understand a problem situation.

Two trains (A and B) are starting out in two different directions at the same time with the same speed, 66 km per hour, from Mongkok Station. 2 minutes and 30 second later, how far apart distance are the trains?



[Train]
(NR)

Figure 7.17 The "Train" Problem in the Interview

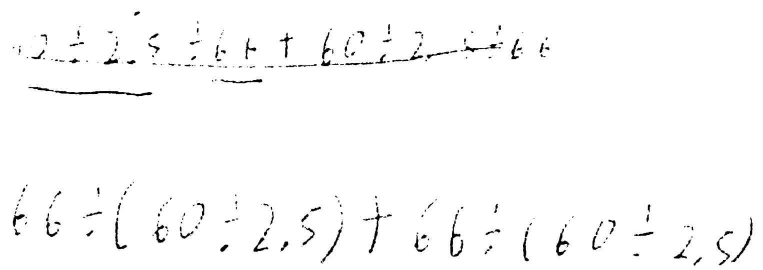
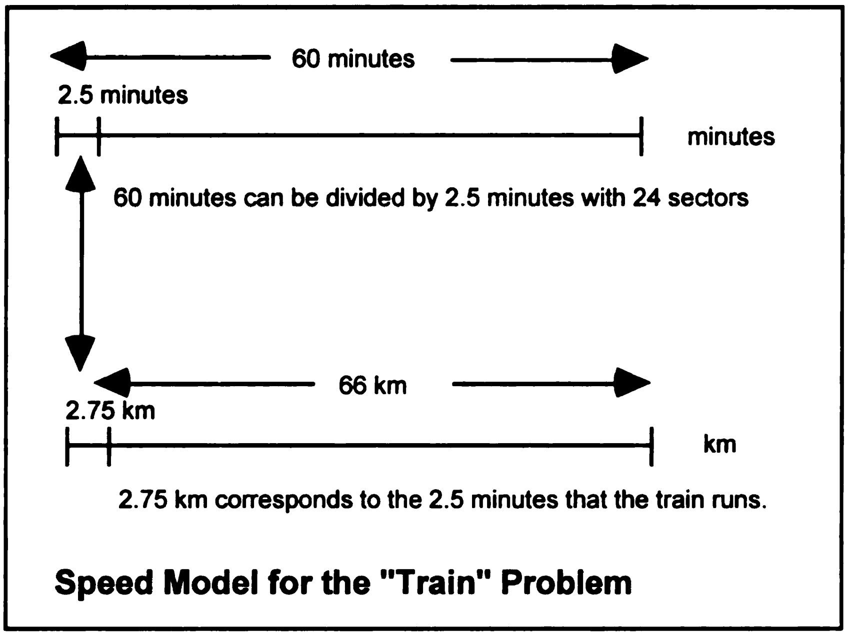


Figure 7.18 Bobby's Work on the "Train" Problem

If we compare the two problems involving speed (the Train and the Lake) solved by Bobby, we might find a common mental model about speed that Bobby has created in his mind, although he did not report the model verbally in the quiz or the interview. In the "Lake" problem, Bobby might also have been thinking about distance and time (Figure 7.19) using a model he created in the "Train" problem. According to what he wrote down on the quiz, he tried to build two corresponding lines. He used the known quantity, 5 km per hour, to figure out how many minutes were needed by the couple to finish their walk on the trail. He subtracted 83.3 meters continuously from 10 km. He then multiplied this quantity (minutes used by the couple in walking) times the dog's running speed of 466.6 meters per minute to find out how far the dog ran in the same amount of time walked by the couple. This co-varying relationship between distance and time appears to be the central idea in Bobby's work in the two problems. Although Bobby did not explain directly how he conceptualized these problems, his performance supports this interpretation.



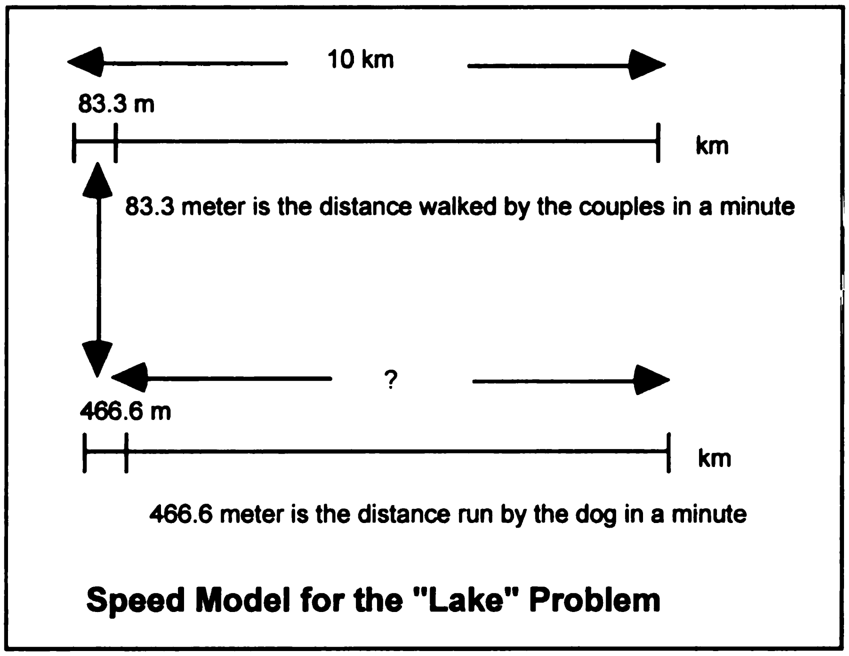


Figure 7.19 Two Similar Speed Models for Two Speed Problems

The "Classroom Board" Problem

In solving this problem, students needed to go through three main procedures, (1) find the area of each paper sheet, (2) divide the area of the board by the area of each paper sheet, (3) round up and subtract the remainders from the last whole paper sheet to find the waste. Bobby solved this problems like most of the high achievers. First, he thought that he needed to know how many pieces of poster paper were needed for covering the board when there were two different kinds of poster paper. He figured out he could find out how many pieces were needed by dividing the board area by the area of two different kinds of paper [$2.5\ m^2 \div (30 \times 50)\ cm^2$ and $2.5\ m^2 \div (40 \times 50)\ cm^2$]. Second, he tried to find the area of the paper and the board. Then, he thought that by subtracting the number of pieces of paper (17 pieces - 16.666 pieces and 13 pieces - 12.5 pieces) needed from the quotient he got by dividing the board area by the area of the papers he would know which kind of poster paper wasted less.

You and 3 other classmates are assigned to design your classroom board this semester. The board is a rectangular shape, 2.5 meters by 1 meter. First you think about the background.

1) You decide the background color (light blue), and find there are two sizes of that color paper. One is 30 cm by 50 cm, and another is 40 cm by 50 cm. You want to buy the size that waste the least paper

a) Which will you buy?

b) How many pieces do you need to buy to cover the whole board?

[Classroom Board]
(NR)

Figure 7.20 The "Classroom Board" Problem in the Interview

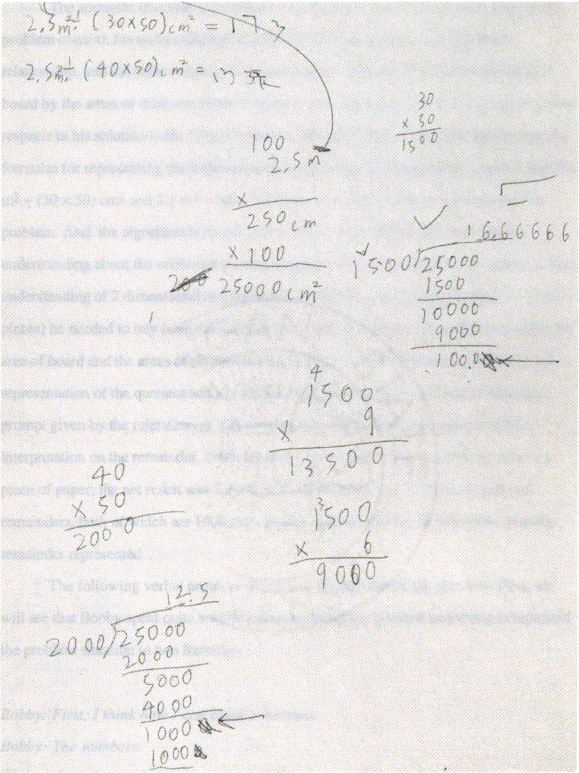


Figure 7.21 Bobby's Work on the "Classroom Board" Problem

The strengths that Bobby revealed in this problem were his understandings of the problem context, his understanding of units (m^2, cm^2, and, meter, cm) and their relationship, and his understanding of the quotient provided by dividing the area of the board by the areas of different kinds of poster paper. Bobby's solution was similar in some respects to his solution to the "Train" problem. He spent quite a long time figuring out the formulas for representing the situations at the beginning. The formulas he wrote down ($2.5\ m^2 \div (30 \times 50)\ cm^2$ and $2.5\ m^2 \div (40 \times 50)\ cm^2$) revealed how he conceptualized the problem. And, the algorithm he wrote, $2.5 \times 100 \times 100 = 25000\ cm^2$, revealed his understanding about the relationship between square meters and square centimeters, and his understanding of 2 dimensional measurement. Subtracting the quantity of paper (17 and 13 pieces) he needed to buy from the quotient (16.6 and 12.5 pieces) provided by dividing the area of board and the areas of different kinds of paper revealed his understanding of the representation of the quotient and his understanding of the meaning of waste (after the prompt given by the interviewer). The only weakness that Bobby revealed was his interpretation on the remainder, 1000. By dividing the area of the board by the area of a piece of paper, the net result was 2 quotients—16.6 pieces and 12.5 pieces, and two remainders, both of which are 1000 cm^2. Bobby reported that he did not know what the remainder represented.

The following verbal protocol shows how Bobby solved the problem. First, we will see that Bobby spent quite a while conceptualizing the problem and trying to represent the problem situation in two formulas.

Bobby: First, I think how I can build a formula.

Bobby: The numbers.

(For about a minute, he did not say anything.)

Interviewer: What are you thinking?

Bobby: Can I write?

Bobby wrote, $2.5 \div (30 \times 50)$ *and* $2.5 \div (40 \times 50)$

Interviewer: What do you get from them?

Bobby changed his formulas to, $2.5m \div (30 \times 50)$ *cm and* $2.5m \div (40 \times 50)$ *cm*

Interviewer: Do you know that all of them are areas? What unit for area?

(I introduced the idea, area here, and he did not say anything about area previously. So, I inferred that he might not have the correct unit for thinking about the problem at that time.)

Bobby: I forgot.

Bobby changed his formulas to, $2.5\ m^2 \div (30 \times 50)\ cm^2$ *and* $2.5\ m^2 \div (40 \times 50)\ cm^2$

Interviewer: Can you tell me what $2.5\ m^2$ *is?*

Bobby: The area of the board.

Interviewer: So, the others are the areas of the paper?

Interviewer: Can you tell what you get when you finish all the computation? I mean when you finish the computation of your formula, $2.5\ m^2 \div (30 \times 50)\ cm^2$*?*

Bobby: cm.

About 10 seconds.

Bobby: Pieces.

Bobby began to write on the working sheet. He wrote,

$$\begin{array}{r} 40 \\ \times\ 50 \\ \hline 2000 \end{array} \qquad \begin{array}{r} 30 \\ \times\ 50 \\ \hline 1500 \end{array}$$

$$\begin{array}{r} 100 \\ \times\ 2.5 \\ \hline 250 \end{array}$$

Interviewer: Can you tell me what it is?

Interviewer pointed at $100 \times 2.5 = 250$

Bobby: Area.

(At that time, Bobby was still working on 100×2.5*, but, the interviewer did not notice.)*

Interviewer: Do you think multiplying by 100 will give you the area?

Bobby continued his work, he wrote,

```
  100
x  2.5
------
  250 cm
  100
------
25000 cm sq
```

Interviewer: Oh! I made a mistake, you wanted to do the transformation here.

Transforming the unit from meters to centimeters revealed that Bobby understood the units very well. This suggested that he simply forgot to add the units to the two formulas he wrote at the beginning. Many middle and low achieving students confused linear and square measurement and their units. Some of them were confused with the transformation relationship between meter and cm, and some of them were confused by the units for linear measurement and the 2-dimensional measurement. Bobby's performance revealed his firm understanding about the unit transformation and unit measurement.

Bobby continued to work and wrote,

```
        16.666
1500 /25000
      1500              1500      1500
      -----                9         6
      10000            -----     -----
       9000            13500      9000
      -----
       1000
```

Interviewer: What are you thinking? Why do you stop when you get the 1000?

Bobby: Checking.

Bobby continued his work, and wrote,

```
          12.5
2000 /25000
      2000
      -----
       5000
       4000
       -----
       10000
       10000
       -----
```

Interviewer: What are you thinking now?

Bobby: Which one saves more?

Interviewer: How do you think which one saves more?

Bobby: From different views.

Interviewer: Can you tell me about your views?

Bobby: Think what it means, "do not waste any paper?"

Interviewer: Can you tell me what you think about, waste, what is waste?

Bobby: Si (a Cantonese dialect mean waste).

Interviewer: What does "si" mean?

Bobby: It is not used.

Interviewer: Not used.

Bobby: And, buy more.

Interviewer: Buy more. Let's see, how many pieces you buy in this size?

Bobby: 17 pieces.

Interviewer: This one.

Bobby: 13 pieces.

Interviewer: You know, you need to buy more of this one (17 pieces). But, if you use most of them and very less is left. Or say, the first 16 or the first 12 pieces put on the board. Do you waste any of those 16 and 12 pieces?

Most students were confused about the meaning of "waste;" they thought that if more was purchased, then more was wasted. Bobby also interpreted "waste" in that way, so he thought the class would waste more when they bought more pieces. I tried to clarify what I meant by "waste" to him in order to help him re-think about how to solve the problem.

Bobby: No.

Interviewer: So, how many pieces are used that do not tell you about the waste?

Bobby: Yes.

Interviewer: Now, how do you think?

Bobby: Use this (pointed at 17 pieces). Subtract and get the approximation.

When he used the word, approximation, Bobby might have been referring to the result of subtracting the quotient by dividing of the area of the board and the area of the poster paper (16.6 and 12.5 pieces) from the whole numbers of paper (17 and 13 pieces) that were purchased. This was a correct way to infer which kind of poster paper wasted less. The smaller the difference, the smaller the amount that was wasted. Bobby's talk of subtraction and approximation revealed his understanding of what the quotient represented in that situation. However, since I provided direct support for this way of thinking about "waste," I cannot claim that he reached this understanding completely on his own.

Interviewer: Get the approximation, hmm. Then, which one saves more?

Bobby pointed at 16.666.

Interviewer: What is left-over from the subtraction? The approximation?

Bobby: The paper that was wasted.

Interviewer: Ok! Bobby, do you think about this (the remainder 1000). What is it?

Bobby: No.

Interviewer: Do you draw picture when you solve a problem like this one?

Bobby: Seldom.

Although Bobby had no problem making sense of the quotients, he could not conceptualize the remainder. He could not figure out what 1000 as the remainder of 25000/1500 represented. This difficulty was very common among the students who participated in the interview. This result might reveal that Bobby and many students did not have a pictorial representation for the problem situation. Most of them seemed to rely on logical propositions to help them to infer the answer. But Bobby and some students had a sensible approach: The quotients ($2.5 \text{ m}^2 \div 30 \text{ cm} \times 50 \text{ cm}$ and $2.5 \text{ m}^2 \div 40 \text{ cm} \times 50 \text{ cm}$) were equal to how many pieces were used. Using this argument, they inferred that the decimals represented how large the partition was used on the last piece of paper. So, they figured out which poster wasted less, the one with a larger decimal.

Summary and Implications

First, I would like to clarify why I considered the solutions from Bobby and other students to be novel representatives of students' strengths. I do not deny that students who solved the problems with traditional and school-taught methods understood mathematical concepts, such as speed, ratio, and proportion. However, I believe that students who can think hard about the situation and construct their own models display greater understanding of the concepts than other students. If we agree that understanding is important to the quantity and quality of different perspectives in a situation, students who develop novel solutions may develop a deeper understanding of the specific concept.

There are several characteristics of novel solutions worth mentioning. First, on the most general level, all students who provided those solutions were middle achievers, based on their school's mathematics performances. Second, the students used their knowledge to develop such solutions in constructive ways. I have talked about the reasons why I think these solutions were not learned from traditional schooling. Only a few students solved the problems using such unexpected solutions and the novel solutions were distributed across different classes and different schools. This distribution pattern suggests that it was highly unlikely that students learned such models or rules in school. Third, the solutions were related to students' prior knowledge. The additive model and the intuitive functional model of proportional reasoning were typical examples. In the case of the additive model, students used their knowledge of addition to think about the problem situation. They found that there was a replicate additive relationship between two entities ($10 = 4 + 4 + 2$), the image height of the sculpture and the image height of him/her on the photograph. Fourth, the functional model used by several students to solve the problems was intuitive. They had not learned the function concept in their classrooms, but they developed an intuition about the concept through their daily experiences. I think students have a lot of experiences that foster the development of such intuitive ideas. The most obvious case is the vending machines they use every day and every where to buy snacks and tickets. Finally, students who constructed the novel solutions were trying to conceptualize the problem situation rather than just focusing on the numbers or key words in the problems when they solved them. I did not interview all the students who generated novel solutions, but Bobby's case revealed that he spent quite a good deal of time conceptualizing the problem situation before providing a solution.

Based on these characteristics, I propose some implications for teaching. If we agree that mathematical knowledge in elementary school can be constructive, by which I mean students can use their daily knowledge and experiences to generate model(s) or rules for the mathematical concepts they learn at schools, we need to rethink how we teach mathematics in elementary school. This study shows that some students can construct their own models to solve proportional problems; they do not need to rely on algorithms taught by the teachers in school. If we agree that this statement about the constructive nature of students' arguments are advantageous, we can see three implications for teaching. First, listening is important in classroom teaching. If students construct their own models for understanding and interpreting the problem situations, their models may be quite different from the models their teacher had in mind. So, teachers need listening skills to hear what students are saying. This is quite different from standard approaches. They should not deny the non-standard solutions, because they do not understand them. These listening skills should be based on the findings of developmental studies about mathematics learning. Second, teachers should provide enough time for students to think about problems. Based on Bobby's performance, we see that students need time to think and conceptualize problems before they find constructive solutions. Teachers should not rush through solutions or strategies quickly in order to move on. If classroom work and discussion are rushed, teachers will eliminate chances for their students to conceptualize and construct their own understanding of the concepts embedded in the problems. Third, evaluation should not be based solely on standard solution and final answers. If teachers want to help their students develop understanding or overcome conceptual obstacles, they need to hear and understand the procedures given by students. Also, they need to explore the non-standard solutions given by some students. How do the students construct those non-standard solutions? Do those solutions show a deep understanding of basic concepts? Or, Are they flawed and/or do they represent obstacles for future development? For example, the additive model for proportion may not be efficient and generalizable for students in the long-run. The model may only be applicable in specific problem situations where numbers can be easily related additively. Likewise, the intuitive function model may be inefficient for students to use to solve all proportion problems because the form of the function may be difficult to determine by examining the input and output values.

Chapter 8

CONCLUSION

In this final chapter, I return to my research questions, consider how my results address them, and discuss some issues related to the findings of this study. I will compare the performances on two different kinds of problems, routine and non-routine. In particular, I will focus on exploring the possible difficulties that the students experienced on three non-routine problems, the "Photo-Robber," "Photo-You," and "Extended Square". I selected these 3 problems because they were where the students provided the most complete data on their reasoning, allowing me to consider and weigh the effects of different sources of difficulty. I am concerned with these possible source(s) of difficulty. Were their poor performances on the non-routine quiz problems related more to conceptual, computational, linguistic, or developmental difficulties? I will try to use the collect data to interpret their difficulties. I will also consider whether or not the students' social background influenced their performances on the quiz.

In this section, I would like to answer my research questions by briefly summarizing the results which were presented in Chapter 5 to 7. Then, I will present a more detailed interpretation of the students' performance in this study.

1. Do Hong Kong students have difficulties solving non-routine mathematical problems?

In general, the students from both schools performed well on the routine problems, but poorly on the non-routine problems. Tables 8.1 and 8.2 reveal their difficulties solving the non-routine problems. The difficulty rankings (column 2) were based on the frequency of correct and numerically correct responses. In determining these rankings, the first criterion was the correct frequency. If the correct frequencies on two problems were the same, I considered the numerically correct frequency. If two problems had the same correct frequency, the one with more numerical correct responses had a lower difficulty rank. In column 1, [N] indicates non-routine problems, and [R] indicates routine problems.

Table 8.1 Performance on Routine and Non-Routine Quiz Problems Form A (*N*=58)

Problem	**Difficulty Rank** (*Total*=12)	**Percent Correct** *(Frequency)*	**Frequency, Correct and Numerically Correct**
Glass House (Part A)[N]	12	0% (0)	0
Balance [N]	11	3.4% (2)	3
Glass House (Part B)[N]	10	3.4% (2)	4
Lake [N]	9	10.3% (6)	7
Extended Square (Part B)[N]	8	10.3% (6)	9
Coins [N]	7	12.1% (7)	7
Extended Square (Part C)[N]	6	19% (11)	11
Photo-Robber [N]	5	19% (11)	14
Saving [R]	4	43.1% (25)	38
Unknown Digit [R]	3	74.1% (43)	43
Rectangle Width [R]	2	79.3% (46)	52
Multiplication [R]	1	84.5% (49)	49

On Form A, the non-routine problems were the most difficult. In addition, there was a large performance difference between the non-routine problems (considered as a group) and routine problems. Among the non-routine problems, the highest percentage correct was 19%, where the lowest percentage correct among the routine problems was 43%. Clearly, on Form A the non-routine problems were much more difficult for the 5th graders than the routine problems.

Table 8.2 Performance on Routine and Non-Routine Quiz Problems Form B (*N*=65)

Problem	Difficulty Rank (*Total*=14)	Percent Correct (*Frequency*)	Frequency, Correct and Numerically Correct
Balance [N]	14	1.5% (1)	2
Lake	13	13.8% (9)	9
Number Pattern (Part D)[N]	12	23.1% (15)	15
Extended Square (Part B)[N]	11	23.1% (15)	16
Extended Square (Part C)[N]	10	23.1% (15)	17
Number Pattern (Part C)[N]	9	41.5% (27)	27
Photo-You [N]	8	44.6% (29)	31
Number Pattern (Part B)[N]	7	55.4% (36)	36
Magazine [R]	6	55.4% (36)	47
Cake (Part B) [N]	5	63.1% (41)	45
Unknown Digit [R]	4	69.2% (45)	45
Rectangle Width [R]	3	74.4% (49)	49
Cake (Part A) [N]	2	76.9% (50)	51
Multiplication [R]	1	81.5% (53)	53

On Form B, the students also found the non-routine problems more difficult. The only two exceptions were the "Cake" and "Number Pattern (Part B)" problems. More than an half of the students solved these problems correctly. However, the "Cake" problem was actually more “routine” than the other non-routine problems. Although it had a very complicated story structure and a lot of words, the problem situation was familiar to the students. It asked the students to partition the whole cake into pieces. This multiplicative concept of division, partitioning, was a familiar story problem to the students. On the other non-routine problems, the students' performances revealed a difficulty pattern similar to that on Form A. The non-routine problems had the highest difficulty rankings, but—on average—were easier for students than the Form B non-routine problems. One possible reason for this overall difference is that 3 non-routine problems—"Cake," "Number Pattern," and "Photo-You"—were comparatively easier for students than their counterparts—"Glass House," "Coins," and "Photo-Robber"—on Form A.. “Cake” presented a familiar situation, and "Number Pattern" did not require thinking about as many possibilities as "Coins." It is unclear to me why "Photo-Robber" was more difficult than "Photo-You."

All these data revealed that the students had difficulties solving the non-routine problems. Form the Table 8.1 and 8.2, we can see that the non-routine problems earned the lower correct percentages and frequencies. Why did the students find the non-routine problem more difficult? I will try to investigate this question in the next section.

2. What are the general characteristics of their difficulties—conceptual, computational, linguistic, or developmental?

In general, the students' performances revealed their difficulties solving the non-routine problems. In this section, I will consider some possible sources of their difficulties. I consider four general factors—conceptual, computational, linguistic, and developmental challenges. Although the students' incorrect answers can be interpreted in different ways, and different sources of difficulty could have their impact on student's mistakes, I will try to use the data to make my argument—in general, the conceptual difficulties with several mathematical concepts had a strong impact on performance. However, I do not deny that other factors had their effects.

Before I start to discuss the effects of four factors, I would like to introduce them briefly. I hope that this introduction will provide a common ground to explore and investigate the sources of the students' difficulties.

1. Conceptual Factor: Different psychological traditions—Empiricist, Rationalist, and Sociohistoric—have their own interpretation to the psychological construct, conception (Case, 1998). In my discussion, I refer the conceptual sources as conceptual entities—abstract rules and mental models—are used by children to interpret their surroundings and solve problems. For example, Bobby (Chapter 7), seemed to acquire an abstract rule or mental model about the mathematical

concept, speed. He seemed to see speed as a co-varying relationship between two variables, distance and time. This co-varying relationship is a conceptual entity that a student can use to solve the relative unfamiliar non-routine problems. Students can solve non-routine problems if they can correctly conceptualize the mathematical concepts which they study in school. Though they might not conceptualize them in the same way that those concepts are taught. The models or rules for area, perimeter, and proportion are considered under this dimension.

2. Computational Factor: This is the most straightforward factor. It refers to students' ability to correctly complete the four arithmetic operations—addition, subtraction, multiplication, and division—on the numbers given in the problems. In other words, could the children correctly manipulate the numbers with different operations? I want to consider whether the students did not develop this competence well, so they performed poorly in the quiz.
3. Linguistic Factor: This is a more complicated dimension, combining 3 different, but related elements. The first one is the number of words in the problem text. The students had more experiences with simple, one or two sentence story problems. They seldom solved problems with long and complicated stories. The second is the familiarity of the story structure. The routine problems were taken directly from the students' textbooks, and some non-routine problems (e.g., "Cake") were adapted from textbook problems. These problems were more likely to be familiar to students from their school work. The third is the number of quantities (variables) in the problem. This element lies in the gray area between the conceptual and linguistic dimensions. More quantities in a problem meant higher cognitive load so it is related to the cognitive domain. I include number of quantities under the linguistic factor because I want to consider the effect of specific multiplicative concepts under the conceptual factor.

4. Developmental Factor: Developmental psychologists, educators, and curriculum designers (Gelman & Brown, 1986; Mumbauer & Odom, 1967; Olser & Kofsky, 1966; Renninger, 1998) have focused on the match between children's development and the materials taught in school. A common consideration has been the match between the cognitive/psychological maturity of children and the learning tasks that they work on in school. Have children matured enough (psychologically) to learn the abstract knowledge and apply the knowledge? The developmental factor refers to the students' psychological age when they participated in this study. I consider whether the students were psychologically ready to carry the abstract knowledge (mathematical concepts) that they learned in class to apply to unfamiliar situations (non-routine problems).

First, I would like to discuss the developmental dimension. The weak performance of these students could be taken as a developmental effect. Maybe teachers cannot do anything for students, because they are not ready to abstract the knowledge they learned and transfer it to a totally unfamiliar situation or new problem with too many new conceptual dimensions. I think we do not need to worry too much about the developmental factor among the students, except two cases who were identified as the mildly mentally retarded. According to the studies done by Mumbauer and Odom (1967), Olser and Kofsky (1966), students older the 7 years old should be capable to transfer their knowledge to different situations if they understand the knowledge (or rules) they learned. Only children less than 7 years old have problem to abstract the rule(s) from what they learned and apply these rules to different but applicable situations. All subjects in this study are older than 10 years old, I think they are developmentally ready for solving the non-routine problems.

Second, I would like to explore the conceptual and computational factors together. I did conceptual analysis on 2 non-routine problems—"Ratio" (Photo-Robber and Photo-You), "Extended Square." I chose these problems because they were the only problems that offer me enough data to compare the relative effects of conceptual and computational difficulty. I will examine the students' performances on the "Ratio" and "Extended Square" problems in next paragraph.

On the two "Ratio" problems, more than 60% of the students solved the problems incorrectly on Form A, and 44% on Form B (Table 8.3). Among the 67 students who gave the incorrect answer to the problems, at least 53 of them made one or more conceptual mistake. Only 14 incorrect responses were due to computational errors.

Table 8.3 Incorrect Responses Related to Conceptual and Computational Errors on the Two Ratio Problems

Form (N=)	Problem	Frequency, Incorrect Response	Minimum Incorrect Response Frequency related to Conceptual Mistakes	Incorrect Response Frequency related to Computational Errors
A (58)	Photo-Robber	38	28	10
B (65)	Photo-You	29	25	4

On the "Extended Square" problems, 72.4% of the students solved the problem incorrectly in Part B, and 69% in Part C on Form A (Table 8.4). In Part B, among the 42 students who gave an incorrect answer, at least 30 of them made one or more conceptual mistake. In Part C, among the 40 students who gave an incorrect answer, at least 37 of them made one or more conceptual mistake. On Form B, 67.7% of the students solved the problem incorrectly in Part B, and 58.5% in Part C. In Part B, among the 44 students who gave an incorrect answer, at least 33 of them made one or more conceptual mistake. In Part C, among the 38 students who gave an incorrect answer, at least 32 of them made one or more conceptual mistake.

Table 8.4 Incorrect Responses Related to Conceptual and Computational Errors on the Extended Square Problems

Form (*N*=)	Problem	Frequency, Incorrect Response	Minimum Incorrect Response Frequency related to Conceptual Mistakes	Incorrect Response Frequency related to Computational Errors
A (58)	Extended Square (Part B)	42	30	12
	Extended Square (Part C)	40	37	3
B (65)	Extended Square (Part B)	44	33	11
	Extended Square (Part C)	38	32	6

On these two problems, students' incorrect answers were clearly related to conceptual mistakes they made. The main factor for their poor performance was more likely related to their understanding of the main concepts embedded in the problems—area and proportionality. Many students thought about ratio with the additive model. They seemed to carry an inefficient or inappropriate abstract rule to the problem situations. On the problem related to area and perimeter, many students were confused by measurements for two different dimensions, area and length. They thought that they could divide the area by 4 to get the length of a side. They applied the one-dimensional measurement rule on the problem that needed the rule for two-dimensional measurement. Students were able to carry and transfer abstract knowledge from one situation to another. However, they seemed to conceptualize an inefficient or inappropriate rule.

Finally, I consider the linguistic factor. We can easily link the conceptual difficulty to another psychological construct—linguistic ability. Different psychologists (Cummins, Kintsch, Reusser, & Weimer, 1988; Cummins, 1991; Okamoto, 1996; Riley & Greeno, 1988) have their own theoretical stances on the importance of linguistic or conceptual influence on solving mathematics word problem. Cummins, et al. (1988) and Cummins

(1991) were inclined to consider the linguistic factor as the main one associated with students' difficulties in solving the arithmetic word problems. Many psychologists with the same stance cite Hudson's (1983) study to support their position: When the problem such as, "How many more birds are there than worms?" is recoded as "How may birds won't get worm?" is presented to the kindergartners, their performance improved from 25% to 96% correct. Cummins and others psychologists argued that children found the first version more difficult because they could not interpret key words and phrases in the problem text. Okamoto (1996) and other psychologists argued that this kind of change in problem statements might possibly lead students to construct different problem representations from which students needed to activate their mathematical knowledge to solve the problem. So, there are two possible difficulties, the first one may be related to students' ability to construct the representation for the situations. The second possibility is related to a lack of relative mathematics knowledge. The second one is more related to conceptual than linguistic competence.

There is evidence that linguistic factors had an influence on problem solving. If we examine the data carefully, we find that there was a higher percent correct on problems with fewer words. In other words, students performed better on problems (Multiplication, Unknown Digit, Rectangular Width, Saving, and Magazine) with fewer words. More than 62% of the students gave correct or numerically correct answers to these problems. However, language and problem type interacted with each other. The simpler story structures and fewer words were associated with routine problems. In part, the word problems that were familiar to the students were the problems in their textbook—that is, the routine problems. Most of these problems were with simple story structure and fewer words. Under this condition, the non-routine word problems I designed naturally contained more words and had more complicated structures. Among the non-routine problem, there was a different story. The story structure and the number of words were not always the significant factor.

Was the linguistic factor the main obstacle to students' successful performances? The Cake problem and the Glass House problem were the two most complicated story problems on the quiz. They had a lot of words and a complicated story structure. However, more than 65% of the students solved the "Cake" problem while less than 3.5% of the students solved the "Glass House" problem. The main difference between the two problems seems to be the mathematical concepts needed to solve the problems. The "Cake" problem asked students to apply arithmetic operations (addition, subtraction, multiplication, and division), to the partitioning and aggregation of familiar objects—cakes and pieces of cakes. The story structure and the number of words did not have an obvious impact on performance when the students had well-developed conceptions on the mathematical concepts in the problem. The "Glass House" problem asked for an understanding of area and perimeter. And, we know that some of the weakest conceptions were these concepts. This weakness did contribute to the poor performance on this problem. The difficulty also showed up on another problem, "Extended Square," that required thinking about area and length together. If linguistic difficulties were primary, students should have difficulties with both of these complicated story problems—"Cake" and "Glass House"—and they did not.

This does not mean that the linguistic factor was unimportant. As we can see the students' performances on the two non-routine problems—"Glass House" and "Extended Square"— which students needed to understand the same mathematical concept of area, the students performed slightly better on the problem that had a simpler story structure and fewer words—"Extended Square." On the "Extended Square" problem, the students achieved a higher percentage correct—10%, compared to the percentage correct, 3.4%, of the "Glass House" problem. So, I prefer to argue that conceptual and linguistic elements had an interactive effect on difficulty.

To conclude, the difficulties that students had were more likely related to their conceptual difficulties with the multiplicative concepts, area and perimeter, and ratio. Most of the students did not have computational difficulties, and were developmentally ready for learning the concepts. Although linguistic factors, like the story structure and number of words, had influence on students' performance, it did not appear to be the most significant cause for the students' performances.

3. Are these difficulties related to social classes?

Students from both schools, serving the low SES and middle class communities, performed well on the routine problems. The students solved the routine problems correctly about 65% the time (combining both correct and numerically correct responses). The students from the low SES school solved the routine problems correctly more than 57% the time. This result is consistent with the study done by Oakes (1990). Teachers who taught in schools in low SES communities were more likely focus on "computational basics" in their lessons. This may be the reason why the students from the low SES school did well on the non-routine problems. However, students from middle class outperformed their counterparts on the non-routine problems. This performance difference may be related to students' social background. Although the relation was revealed in this study, there is no any conclusive explanation, based on the data. I did not carry out any observations of classroom instruction, nor collect the data about parents' educational background, so it is difficult for me to make any conclusive statement about how differences in social background exerted an influence.

Students in the middle class school outperformed their counterparts on most of the non-routine problems. The only exception was the "Cake" problem. The differences between the two groups on performance in the two sets of parallel problems, "Ratio" and "Extended Square" problems, were the most obvious. Among these problems, the difference on the percentage correct between two groups were more than 20%. The only exception was the "Extended Square" Part B on Form A. You can read the differences in

Table 8.5. When the differences in students' performance between two schools were greater than 20%, the percentage correct is given in italics.

Table 8.5 Two Schools Performance on the Non-Routine Problems

Problems	Form	Low SES School	Middle Class School
Ratio	A	*9.1%*	*32%*
	B	*28.6%*	*63.3%*
Balance	A	0%	8%
	B	0%	3.3%
Lake	A	9.1%	12%
	B	8.6%	20%
Extended Square	A	6.1% *(B), 9.1% (C)*	16% *(B), 32% (C)*
	B	*8.6% (B), 8.6% (C)*	*40% (B), 40% (C)*
Glass House	A	0% (A), 0% (B)	0% (A), 8% (B)
Coins	A	9.1%	16%
Number Pattern	B	48.6% (B), *31.4%* (C), 14.3% (D)	63.3% (B), *53.3%* (C), 33.3% (D)
Cake	B	77.1% (A), 60% (B)	76.7% (A), 66.7%(B)

In addition to the level of general correctness, the students from the low SES school made more conceptual mistakes than the students from the middle class school on the "Ratio" problems. Tables 8.6 shows that the students from the low SES school experienced more conceptual difficulty on both "Ratio" problems. The incorrect responses given by the students from the low SES school and related to conceptual mistakes were greater than that of their counterparts. On the "Extended Square" problem on the two forms, the students from the low SES school might have experienced more conceptual difficulty than their counterparts. However, I cannot make a valid inference on their performance, because at least 6 of the students out of a total 68 gave unclassified written responses on one part of the problem, and at least 14 of the students gave up one part of the problems. With these kinds of responses, it is difficult for me to make any valid inference about their thinking.

Table 8.6 Students' Incorrect Frequency on Two Ratio Problems

School (*N*=)	Form	Problem	Frequency, Incorrect Response	Minimum Incorrect Response Frequency related to Conceptual Mistakes	Incorrect Response Frequency related to Computational Errors
Low SES School (33)	A	Photo-Robber	24	*19*	5
Middle Class School (25)	A	Photo-Robber	14	*9*	5
Low SES School (35)	B	Photo-You	20	*17*	3
Middle Class School (30)	B	Photo-You	9	*8*	1

In general, the performance between of these two groups were substantively different on the non-routine problems. Although the students from the middle class school did not do an excellent job on these non-routine problems, they out-performed their counterparts in the low SES school. And, the mistakes made by the students from the low SES school were highly related to conceptual difficulty. These results seemed to reveal that these two groups of students might have learned differently in two different schools, or they had different kind of support after schooling. In particular, their experiences in learning and understanding the multiplicative concepts—area and perimeter, and ratio—might be enlarging their performance differences on the quiz. This study cannot explain why there was this obvious difference between the two groups, but it raise the educators' and social services providers' attention to the social class issue that influences how students learn in and out schools.

APPENDICES

APPENDIX A

QUIZ PROBLEMS IN FORM A & FORM B

Form A Problems and Their Conceptual Analysis

<table>
<tr><td>Problem Name: Multiplication</td><td>Problem Type: routine</td></tr>
<tr><td colspan="2">Problem (from Form A):
187 x 31 =
Answer: 5797</td></tr>
<tr><td colspan="2">Mathematics Concepts Embedded:
multiplication
Quantities:

Expected Algorithms:</td></tr>
<tr><td colspan="2">Conceptual Analysis</td></tr>
<tr><td colspan="2">Conceptual Error

Computational Error</td></tr>
</table>

Problem Name: Unknown Digit	Problem Type: routine

Problem (from Form A):

The 3-digit number "57?" is exactly divisible by 16. Find the value of "?"

Answer: 6 or 576

Mathematics Concepts Embedded:

multiplication, division

Quantities:

Expected Algorithms:

1. Try to put the even numbers, 2, 4, 6, 8, 0 in to the unknown digit position, and try to divide it with 16. Or, try all digits, 1, 2, 3, ...9, 0 in to the last digit position.
2. Divide 57 by 16, and get the remainder 9, then think about what number is the product of 16 in a 2-digit number beginning with 9.

Conceptual Analysis

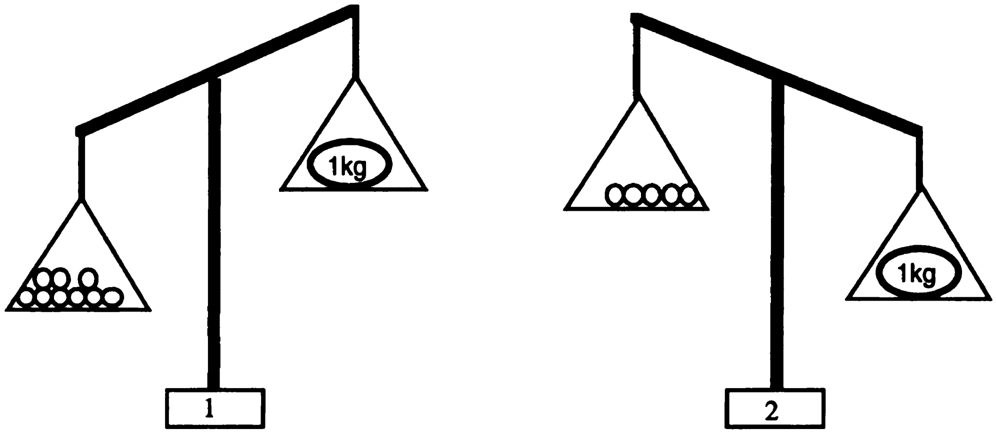
Conceptual Error

Computational Error

Problem Name: Balance	Problem Type: non-routine

Problem (from Form A):

All marbles are the same weight, what are the range of the possible weight that each marble can be?



Answer: the lightest is 112g or 111.12g; the heaviest is 199g

Mathematics Concepts Embedded:

multiplication: unit weight x number of units = total weight

order: many total weights less than 1 kg... < i kg. < many total weights greater than 1 kg.

Quantities:

unit weight, number of marbles, total weight of marbles in Balance 1, total weight of marbles in Balance 2.

Expected Algorithms:

1. Turn 1 kg to 1000 g
2. Divide 1000g by 9 and by 5
3. Think about the concepts, less and larger.

Conceptual Analysis

- Understand how the balance works, the heavier side will be down.
- Relationship between quantity of marbles and 1 kg; 1 kg / # of marbles can estimate the weight of a marble.
- Understand the unit, kg.
- The concept of the heaviest and the lightest weight.

Conceptual Error

- unit confusion: kg = g
- using 7 to divide 1 kg, because they thought 7 is between 5 and 9.
- multiply 5 or/and 9 with 1 kg
- divide 1 kg with 9 - 1. and 5 + 1
- divide 1 kg with 9 + 5

Computational Error

- Miscount the quantity of marbles

Problem Name: Glass House | Problem Type: non-routine

Problem (from Form A):

Your school is planning to build a glass-house for science classes. The area of the floor must be 24 square meters. The building will be 3 meters in height. The builder estimates the house costs $1200 for every 1 meter of the perimeter.

a. List all the possible lengths, widths, perimeters, areas, and the cost of walls for different rectangular shapes. (You only need to consider rectangle with length and width equal to whole numbers of the meter)

Answer:

Width	Length	Perimeter	Area	Cost of the wall
1	24	50	24	60000
2	12	28	24	33600
3	8	22	24	26400
4	6	20	24	24000

b. Based on the cost of walls, which rectangular shape would cost least to build?

Answer: 4 m x 6m

Mathematics Concepts Embedded:

3-dimensional, 2-dimensional and linear thinking (to have a model about the building in 3-d, understand that 2-d information, area, is fixed in the problem, and linear unit, perimeter, is needed to solve the problem), the relationship between area and perimeter.

Quantities:

cost for the length of the wall, square meters, meter

Expected Algorithms:

1. Factors of 24
2. Perimeters of different rectangles
3. Costs for building the wall with different perimeter

Conceptual Analysis

- Relationship between a fixed area and a perimeter with several dynamics.
- Understand that the 3rd dimension (height) is not useful for solving the problem.
- Understand different units (m^3, m^2, m) for measuring different dimensions.

Conceptual Error

- the area is not fixed at 24 m^2.
- using area to calculate the cost
- incorrect way to get the area
- incorrect way to get the perimeter
- multiply the cost $1200 with the length of two sides.

Computational Error

Problem Name: Coins	Problem Type: non-routine

Problem (from Form A):

You know you have more than ten dollars in you pocket (all are coins, $5, $2, 50 ¢, 20¢, or 10¢) but you find you are unable to give someone a change for a ten dollar note. List the combination of coins that you have.

Answer: any combination with all different coins, and the combination is not equal to $10 and is bigger than $10

Possible Combinations:

Type\Quantity	option 1	option2	option3
$ 5	1	1	1
$2	3	4	4
50 cent	1	1	1
20 cent	1	1	1
10 cent	1	1	2

Mathematics Concepts Embedded:

multiplication, addition

Quantities:

number of each coin; value of each coin

Expected Algorithms:

Starting from coin for $5, establishing all the possible combinations.

Conceptual Analysis

- understand there are different kinds of coins in the pocket
- understand you have more than 10 dollars
- understand the sum of the coins cannot be equal to 10 dollars

Conceptual Error

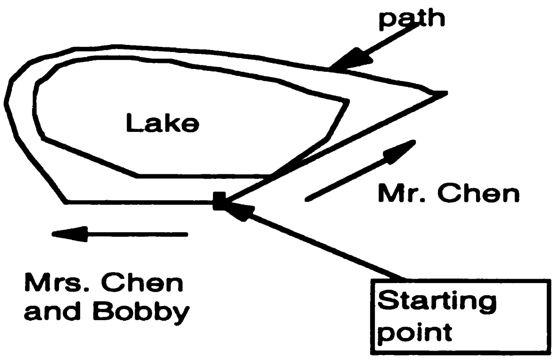
- the sum of the coins is less than 10 dollars
- one of the coin combinations is equal to 10 dollars
- do not list out the different kinds of coins
- no unit given

Computational Error

Problem Name: Lake	Problem Type: non-routine

Problem (from Form A):

Mr. and Mrs. Chen walk with their dog, Bobby, every morning. They walk at the same speed, 5 km per hour. The distance of the walking path around the lake is 10 km. They start and walk in opposite directions at the starting point and meet each other midway, then go to have their breakfast. Bobby starts his morning run with Mrs. Chen and runs toward Mr. Chen. When he meets Mr. Chen, he runs back to Mrs. Chen. He continues like this until Mr. Chen and Mrs. Chen meet. His running speed is 28 km per hour. How far does Bobby run, by the time the couple meet?



Answer: 28 km

Mathematics Concepts Embedded:

ratio

Quantities:

distance, time-spending

Expected Algorithms:

1. Read the relationship about the time spent by the couples
2. Using the time-spending by the couple to get the distance run by the dog.

Conceptual Analysis

- understand the couple need one hour to finish the morning walk
- the dog will run an hour on the path

Conceptual Error

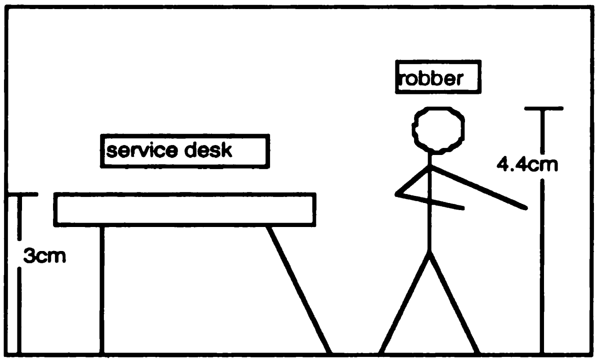
- multiply 28 km with 2. In this case, I suspect that the students think 10/5 = 2, and 30 km multiply by 2, then they used all the numbers in the problem.
- no unit is given
-

Computational Error

Problem Name: Photo-ratio (robber)	Problem Type: non-routine

Problem (from Form A):

A hidden camera took the picture below during a robbery. In the picture, you can see the robber and a service desk. The height of the desk is 3 cm, and the height of the robber is 4.4 cm. The real height of the desk is 1.2 meter. Can you figure out the height of the robber? How?



Answer: 1.76 m or 176 cm

Mathematics Concepts Embedded:

ratio, equal ratio, shrinking and stretching

Quantities:

Actual height of the desk, height of the desk on the photo; height of the robber on the photo, actual height of the robber

Expected Algorithms:

1. Get the relative actual height to 1 cm on the picture, and then multiply 4.4 cm to the relative actual height of 1 cm.
2. Cross-multiplication

$$\frac{1.2}{3} = \frac{x}{4.4}$$

3. Functional thinking, find the function by turning 3 to 1.2 or 3 to 120, then applying that function to 4.4.

Conceptual Analysis

- Proportional relationship between actual height and the height on the picture
- Unit sense; how long is a cm and a m?

Conceptual Error

Type A: unit confusion,

Example 1: 4.4 cm - 1.2 cm (it should be 1.2 m) = 3.2 cm; 3.0 - 1.2 = 1.8

Example 2: 3 m - 1.2 m = 1.8m (3 should be cm, not m)

Example 3: 1.2m x 4.4 cm = 52.8 cm (it should be $0.528m^2$)

Type B: seeing ratio as an additional relationship between two entities

Example 1: 4.4 - 3 + 1.2 = 2.6 cm

Type C: others

Example 1: 1.2 / 3 = 0.4, => (4.4 - 3) x 0.4 = 0.56 m;

Example 2: 4.4 cm - 3 cm = 1.4, 3 x 0.4 =1.2 m => 1.2 + 1.4 = 2.6 m

Example 3: 4.4 / 3 = 1.46

Computational Error

Example 1: 4.4 / 5 x 2 = 17.6 (should be 1.76)

Problem Name: Extended Square	Problem Type: non-routine

Problem (from Form A):

The area of a square is 196 cm^2, now you lengthen one dimension by 5 cm to form a new shape.

a. Draw the new shape.
Answer: a rectangle
b. What is its area?
Answer: 14 cm x 19 cm = 266 cm^2
c. What is its perimeter?
Answer: (14+19)cm x 2 = 66 cm

Mathematics Concepts Embedded:
area: the unit for measuring surface, perimeter: the unit for measuring the length
Quantities:
initial length; new length, perimeter of initial and new shape, area of initial and new shape
Expected Algorithms:
1. Get the length of one side of the square
2. Add 5 cm to two opposite sides
3. Add the length of two adjacent sides and multiply the sum by 2

Conceptual Analysis
- Relationship between the area of a square and its length on a side
- Area concept
- length concept
- Formula of area (width x length)
- Formula of perimeter [(width x length)x2]

Conceptual Error
Type A: unit confusion
wrong unit is used, for instance, cm for area
Example 1: 19cm x 19cm = 361cm
Example 2: 196 + 5 + 5 = 206cm^2 (196cm^2 cannot be added to 5cm)
Type B: confusion on area and length,
Example 1: 196/4 = 49 => 49 x 54 = 2646 => (49 + 54) x 2 = 206;
Example 2: (196/4 + 5 x 2) x (196/4) = 59 x 49 = 2891 => (59 + 49) x 2 = 216
Example 3: (196/4 x 5) x (196/4 x 5)
Type C: confusion between area and perimeter
Example 1: 196/4 = 49 => 49 + 49 + 54 + 54 = 206 (area), 206 (perimeter).
Type D: add a new area to the given area
Example 1: 196 + 5 x 5 = 221cm^2
Type E : others
Example 1: 5 x 5 = 25 => 5 + 5 = 10
Example 2: 5/4 = 1.25cm
Example 3: 196/2 = 98cm
Example 4: 16 x 12 = 196, 16 + 12 + 16 + 12 = 56 (re-assemble a product for 196)
Example 5: 196/4 x 5 = 245cm square => 196/2 + 5 = 103cm
Example 6: (196 x 5)cm = 980cm (area), (196 x 5)cm = 980cm (perimeter)
Example 7: (196 + 5) x 2 + 196 x 2 = 694 (area), 196 + 5 = 201 (perimeter)

Computational Error
- unable to find the square root of 196

Mis-reading
- extending 4 sides
Example 1: 19 x 19 = 361cm square, 19 x 4 = 76cm
- extending one side
Example 1: a trapezoid was drawn; (196 + 201) x 196/2 = 4962.5 => 4962.5 x 2/196 = 50.6

Problem Name: Saving	Problem Type: routine
Problem (from Form A): Every day, you save $2.20, and your brother saves $1.70. Four weeks later, how much money do you and your brother save? **Answer**: 109.2 dollars	
Mathematics Concepts Embedded: multiplication, addition Quantities: Expected Algorithms: (2.2 + 1.7) x 7 x 4 or 2.2 x 7 x 4 + 1.7 x 7 x 4	
Conceptual Analysis	
Conceptual Error Computational Error	

Problem Name: Rectangle Width	Problem Type: routine

Problem (from Form A):

The area of a rectangle is 72 cm^2, its length 9 cm, what is width?

Answer: 8 cm

Mathematics Concepts Embedded:

division, area, length

Quantities:

cm, cm^2

Expected Algorithms:

72 / 9

Conceptual Analysis

Conceptual Error

Computational Error

Form B Problems and Their Conceptual Analysis

Problem Name: Multiplication	Problem Type: routine
Problem (from Form B): 167 x 27 = **Answer**: 4509	
Mathematics Concepts Embedded: multiplication Quantities: Expected Algorithms:	
Conceptual Analysis	
Conceptual Error Computational Error	

Problem Name: Unknown Digit	Problem Type: routine
Problem (from Form B): The 3-digit number "55?" is exactly divisible by 12. Find the value of "?". **Answer**: 2, or 552	
Mathematics Concepts Embedded: multiplication, division Quantities: Expected Algorithms: 1. Try to put the even numbers, 2, 4, 6, 8, 0 in to the unknown digit position, and try to divide it by 12. Or, try all digits, 1,2, 3, ...9, 0 in to the last digit position. 2. Divide 55 by 12, and get the remainder 7, then think about what number is the product of 12 in a 2-digit number beginning with 7.	
Conceptual Analysis	
Conceptual Error Computational Error	

Problem Name: Number Pattern	Problem Type: non-routine

Problem (from Form B):

a) Fill the following empty cells with the pattern you find,

1st row	1	2
2nd row	4	3
3rd row	5	6
4th row	8	7
5th row	9	10
6th row		
7th row		
8th row		
9th row		
10th row		

b) What are the two numbers on 21st row?

Answer: 41, 42

c) What are the two numbers on the 73th row?

Answer: 145, 146

d) With the above table, we say 8 is in Row 4 and Column 1. What do we say about 87 relative to Row and Column?

Answer: Row 44 , Column 2

Mathematics Concepts Embedded:

multiplication, division, function, odd/even numbers

Quantities:

N/A

Expected Algorithms:

1. Complete all the cells, until they get the answer
2. Find out the pattern of number distribution
   * In odd number rows, numbers are ascending (col-2-numbers are one larger than Col-1-numbers)
   * In even number rows, number are descending (col-2-numbers are one less than Col-1-numbers)
   * Multiply 2 with the n of nth row, the product is one number on that row.
3. List the numbers in 20th Row and its following two rows, list the 70th Row and its following 3- 4 rows, list 40th Row and its following 5 rows.

Conceptual Analysis

- follow the given pattern, list all the number in the cells and extend the pattern until the answer is achieved
- understand the pattern, use multiplication to achieve the answer

Conceptual Error

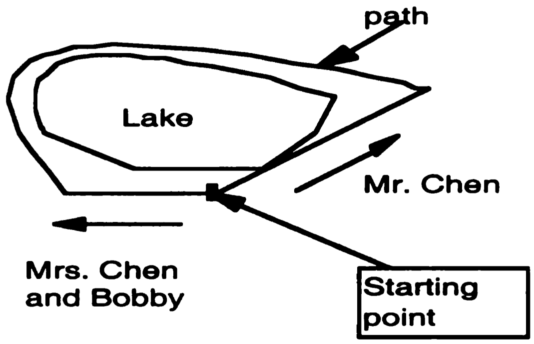
- have a wrong number arrangement pattern

Computational Error

Problem Name: Lake	Problem Type: non-routine

Problem (from Form B):

Mr. and Mrs. Chen walk with their dog, Bobby, every morning. They walk at the same speed, 4 km per hour. The distance of the walking path around the lake is 8 km. They start and walk in opposite directions at the starting point and meet each other midway, then go to have their breakfast. Bobby starts his morning run with Mrs. Chen and runs toward Mr. Chen. When he meets Mr. Chen, he runs back to Mrs. Chen. He continues like this until Mr. Chen and Mrs. Chen meet. His running speed is 30 km per hour. How far does Bobby run, by the time the couple meet?



Answer: 30 km

Mathematics Concepts Embedded:

ratio

Quantities:

distance, time-spending

Expected Algorithms:

1. Read the relationship about the time spent by the couples
2. Use the time-spending unit to get the distance run by the dog.

Conceptual Analysis

- understand that the couple needed one hour to finish their morning walk
- then, the dog will run an hour on the path

Conceptual Error

- multiply 30 km by 2. In this case I suspect the students think 8/4 = 2, and 30 km multiply by 2, then they used all the numbers in the problem.
- try to adding from 30 to 1
- no unit is given

Computational Error

Problem Name: Cake	Problem Type: non-routine

Problem (from Form B):

A cake-shop owner found that some customers like to buy a slice of cakes and others like to buy whole cakes. So he cuts some of his cakes into 8 equal slices and sells them for $6 per slice. He sells the whole cakes for $45.

a) One day, he sold 12 whole cakes and 62 slices. How much money did he earn that day?
Answer: $ 912
b) On other occasion, he had $963 in his cash-register at the end of the day. He knew he sold 11 whole cakes. How many slices did he sell on that day?
Answer: 78 pieces

Mathematics Concepts Embedded:
multiplication, addition, division, subtraction
Quantities:
price for a whole cake, price for a slice, total sales, sales for sold cakes, sales for sold slices
Expected Algorithms:
A. Multiply 12 with $45 and multiply 62 with $6, adding two products
B. Multiply 11 with $45, subtract the product by $963, then divide it by $6

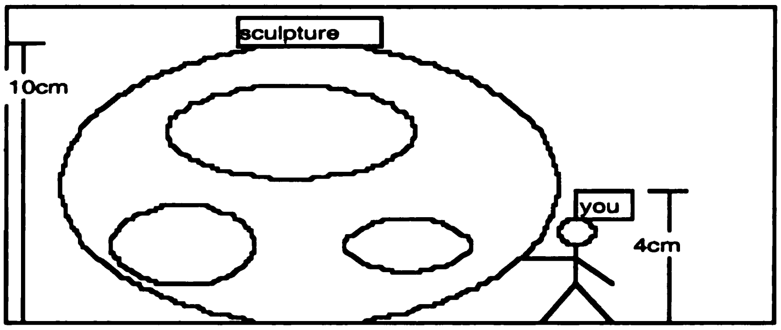
Conceptual Analysis
- unit price x quantities = sales

Conceptual Error
Type A: confused with the quantities or unit
Example 1: $963 - 11x $45 = 468 pieces (this algorithm can only offer the $ unit, not pieces)
Example 2: 963/11/6
Example 3: 963/11 = 87… 6
Type B: others
Example 1: 12 x 6 = 72 => 62 x 6 = 372 => 372 -72 = $300.
Computational Error
Example 1: 963- 495 = 568 (should be 468)
Misreading

Problem Name: Photo-ratio (you)	Problem Type: non-routine

Problem (from Form B):

Last Sunday, you stood beside a sculpture and had your picture taken. Your height on the photo is 4 cm, and the height of the sculpture is 10 cm. You know your real height is 1.4 meters. Can you figure out the real height of the sculpture? How?



Answer: 3.5 m, or 350 cm

Mathematics Concepts Embedded:

ratio

Quantities:

Actual height of the sculpture, height of the sculpture in the photo; your height in the photo, your actual height.

Expected Algorithms:

1. Get the relative actual height to 1 cm on the picture, and then multiply 10 cm to the relative actual height of 1 cm.
2. Cross-multiplication

$$\frac{1.4}{4} = \frac{x}{10}$$

3. Functional thinking, find the function turning 4 to 1.4 or 4 to 140, then applying that function to 4.4.

Conceptual Analysis

- Proportional relationship between actual height and the length on the picture
- Unit sense; how long is a cm and a m?
- 10 cm is relative to 1.4 m + 1.4 m + 1.4 m / 2 (10 cm = 4 cm + 4 cm +2 cm)

Conceptual Error

Type A: unit confusion

Example 1: 1.4 m x 4 cm = 5.6 cm

Example 2: 10 cm x 4 cm = 40 cm (it should be 40 cm^2)

Example 3: 10 cm x 4 cm = 40 cm

Type B: seeing ratio as an addition relationship between two entities

Example 1: 1.4 + 6 = 2 m

Example 2: 1.4 m + 1.4 m + 0.2 = 3 m

Example 3: 1.4 x 6 = 8.4 (6 is produced by subtracting 4 from 10)

Type C: others

Example 1: 10 + 4 / 4 = 11

Example 2: 1.4 / 2 + 1.4 / 2 =

Example 3: 10 cm x 4 cm = 40 cm

Example 4: 1.4 + 1.4 + 0.2 = 3 m

Computational Error

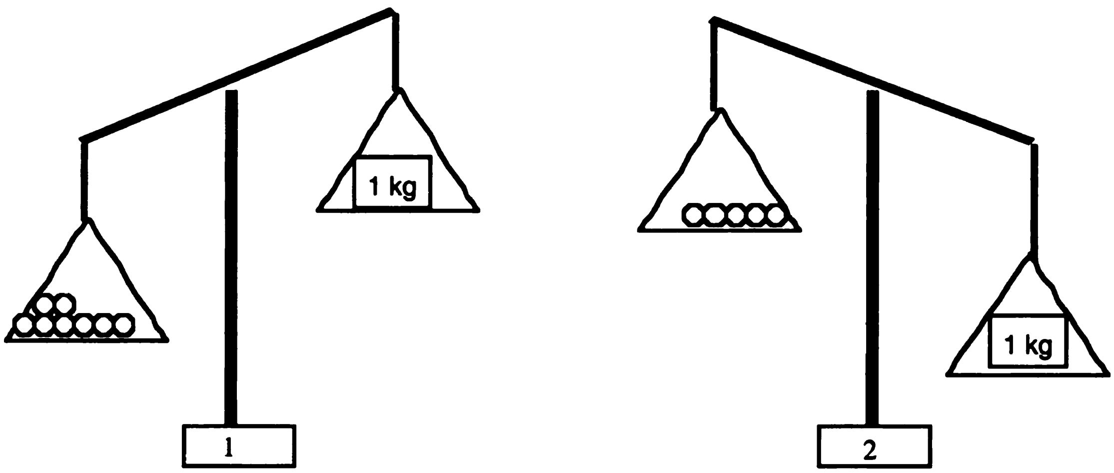
Example 1: 1.4 / 4 = 0.7 (should be 0.35)

Example 2: 1.4 m + 1.4 m + 0.2 = 3 m

Problem Name: Balance	Problem Type: non-routine

Problem (from Form B):

All marbles are the same weight, what are the range of the possible weight that each marble can be?



Answer: the lightest is 126 g and the heaviest is 199 g

Mathematics Concepts Embedded:

multiplication: unit weight x number of units = total weight

order: many total weights less than 1 kg < i kg. < many total weights greater than 1 kg.

Quantities:

unit weight, number of marbles, total weight of marbles in Balance 1, total weight of marbles in Balance 2.

Expected Algorithms:

1. Turn 1 kg to 1000 g
2. Divide 1000g by 8 and by 5
3. Think about the concepts, less and larger.

Conceptual Analysis

- - Understand how the balance works, the heavier side will be down.
- Relationship between quantity of marbles and 1 kg; 1 kg / # of marbles can estimate the weight of a marble.
- understand the unit, kg.
- The concept of the heaviest and the lightest weight.

Conceptual Error

- unit confusion, kg = g; or no unit is given
- multiply 5 or/and 8 with 1 kg
- confused with what is the heaviest and the lightest, Heaviest = 125 and Lightest = 142.
- take a number between 125 and 200.

Computational Error

- Miscount the quantity of marbles

Problem Name: Magazine	Problem Type: routine
Problem (from Form B): The listed price for the magazine, Electronic Games, is $14 a copy. You can also pay $114.20 for a 12-issue subscription. How much do you save by purchasing a subscription? **Answer**: 53.8 dollars	
Mathematics Concepts Embedded: multiplication, subtraction Quantities: dollars, issues Expected Algorithms: 12 x 14 - 114.20	
Conceptual Analysis	
Conceptual Error Computational Error	

Problem Name: Extended Square	Problem Type: non-routine

Problem (from Form B):

The area of a square is 121 cm^2, now you lengthen one dimension by 3 cm to form a new shape.

a. Draw the new shape.
Answer: a rectangle
b. What is its area?
Answer: 11 cm x 14 cm = 154 cm^2
c. What is its perimeter?
Answer: (11+14)cm x 2 = 50 cm

Mathematics Concepts Embedded:
area: the unit for measuring surface , perimeter: the unit for measuring the length
Quantities:
initial length; new length, perimeter of initial and new shape, area of initial and new shape
Expected Algorithms:
1. Get the length of one side of the square
2. Add 5 cm to two opposite sides
3. Add the length of two adjacent sides and multiply the sum by 2

Conceptual Analysis
- Relationship between the area of a square and its length of a side
- Area concept
- length concept
- Formula of area (width x length)
- Formula of perimeter [(width x length)x2]

Conceptual Error
Type A: unit confusion & mistake
  Example 1: (121/2 + 3) x (121/2 + 3) (121/2 = 60.5cm^2, cannot be added to a length)
  Example 2: (121 + 3)(121)
Type B: confusion on area and/or length
  Example 1: (121/4 + 2) x (121/4)= 9983.3125cm^2 (the correct computation should be 1005.8125 cm^2)
  Example 2: dividing the area by 4 and think they can get the length of a side.
Type C: confusion about area and perimeter
  Example 1: (121/4 + 3) x 4 = 133 (for area)
  Example 2: 121 + 3 x 4 = 133 (for area), 133/4 x 4 = 133 (for perimeter)
Type D: Add a new area to the given area
  Example 1: 121 + 3 x 3 = 130cm^2
  Example 2: 121 + 60.5 x 3= 302.5cm^2, (63.5 + 60.5) x 2 = 124 cm
Type E: others
  Example 1: 3 x 3 = 9cm square => 3 x 4 = 12 cm
  Example 2: 123/3 = 41 => 41 x 41 =1681cm square => 41 x 4 = 164 cm
  Example 3: 121 x 3 =363cm^2, 121 + 3=124 cm
  Example 4: 60.5 x 3 + 121 = 302.5cm square => (63.5 + 60.5) x 2 = 124 cm

Computational Error
  Example 1: 121/4 = 3.25
  Example 2: 6.25 x 4 = 2500

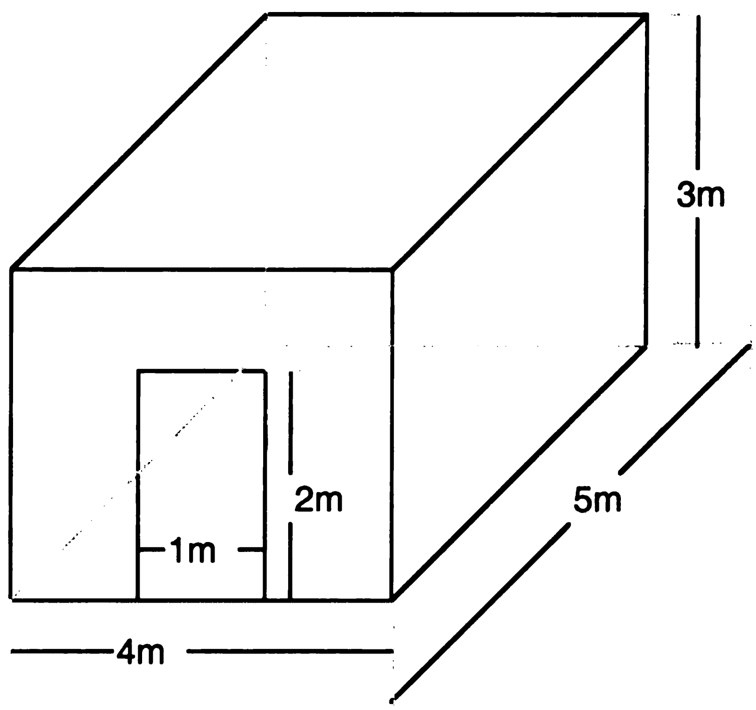
Misreading
- extending one side
  Example 1: a trapezoid was drawn: (11 + 14) x 11/2 = 137.5cm square => 11 x 3 + 14 = 47cm
- extending four sides
  Example 1: 14 x 14 = 196 cm square => 14 x 4 = 56 cm

Problem Name: Rectangle Length	Problem Type: routine
Problem (from Form B): The area of a rectangle is 72 cm^2, its width is 6 cm, what is the length? **Answer**: 12 cm	
Mathematics Concepts Embedded: division, area, length Quantities: cm, cm^2 Expected Algorithms: 72/6	
Conceptual Analysis	
Conceptual Error Computational Error	

APPENDIX B

INTERVIEW PROBLEMS

Johnny wants to decorate his new apartment.



a) He wants to buy wallpaper and put it onto the walls (he is not going to put wallpaper on the ceiling). What does he need to find out, the perimeter or the area?

b) He goes to a construction-material shop, and finds that the wallpaper packed in rolls, has 50 cm x 600 cm of material. How many rolls does he need to cover all the walls, except for the one with door?

[Apartment]
(NR)

You and 3 other classmates are assigned to design your classroom bulletin board this semester. The board is a rectangular shape, 2.5 meters by 1 meter. First you think about the background.

1) You decide the background color (light blue), and find there are two sizes of that color paper. One is 30 cm by 50 cm, and another is 40 cm by 50 cm. You want to buy the size that waste the least paper

a) Which will you buy?

b) How many pieces do you need to buy to cover the whole board?

2) The small paper costs $3 per piece, and the larger paper costs $5 per piece. One of your classmates suggests comparing the cost because your budget is very tight.

a) Compare the cost of using each size paper and decide which is cheaper.

[Classroom Board]
(NR)

Recycling newspaper is one way to save trees. For every 2 meter high stack of newspaper that is recycled, about one tree 14 meter tall is saved.

a) Your family can accumulate a 2 meter high stack in a month. How many trees 14 meter tall do your family save in a year?

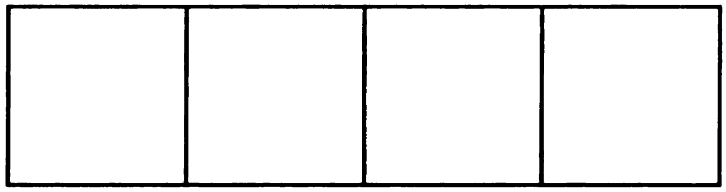
b) Your school subscribes to several different kinds of newspaper, it only takes 5 days to accumulate 2 meter high of newspaper in a stack. How many tree 14 meter tall does your school save in a year?

[Newspaper Recycling]
(NR)

A rope 30 centimeter long is formed into a rectangle. If the width of the rectangle is 6 centimeter, what is its length?

[Rope]
(R)

Following figure shows a rectangle consists of 4 equal squares. If the area of the rectangle is ,



a) What is the area of each square?

b) What is the perimeter of each square?

c) What is the perimeter of the rectangle showed in the figure?

[4 Squares]
(NR)

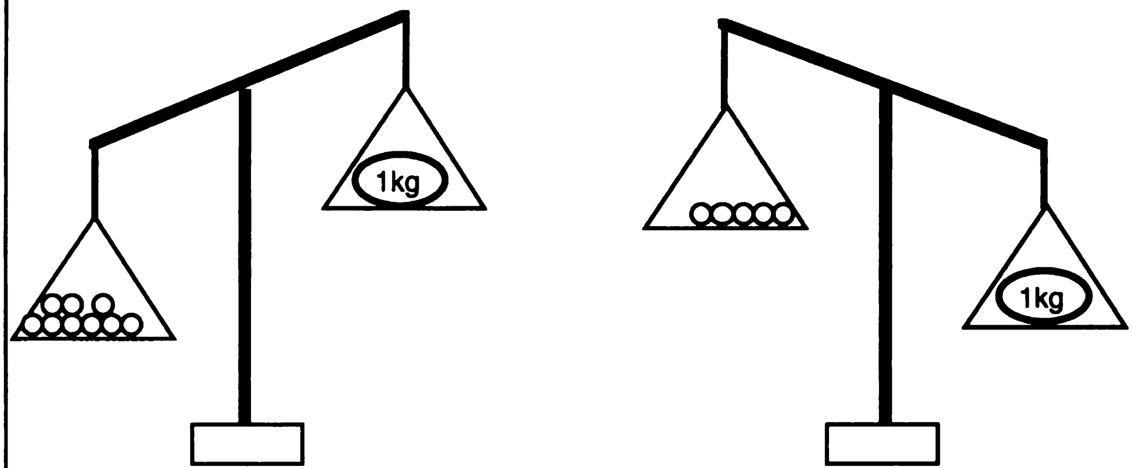
Your school plans to hold a table-tennis competition

a) There are 64 students applying for the singles competition, how many pieces record-paper do you need to prepare, from the starting to the final of that competition?

b) If you want to set up a table-tennis competition, can you have any number of participants? Or, will only some number of participants work? (Which number(s)?

[Table Tennis]
(NR)

All marbles are the same weight, what are the range of the possible weight that each marble can be?

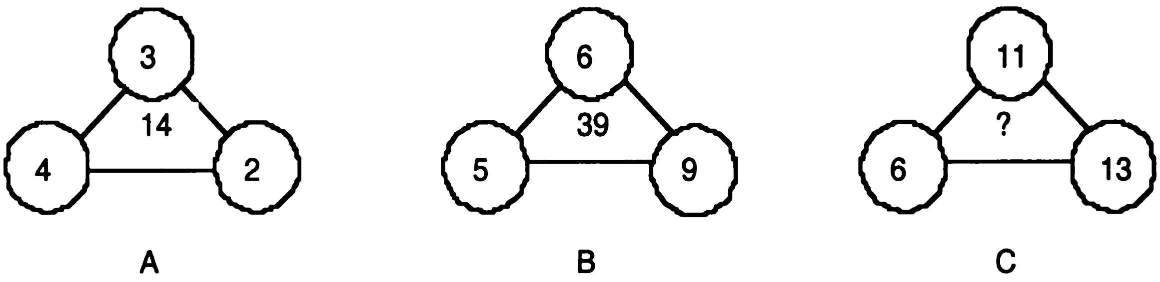


[Balance]
(NR)

The listed price for the magazine, Electronic Games, is $14 a copy. You can pay $288.88 for a 24-issue subscription. How much do you save by purchasing the magazine by subscription?

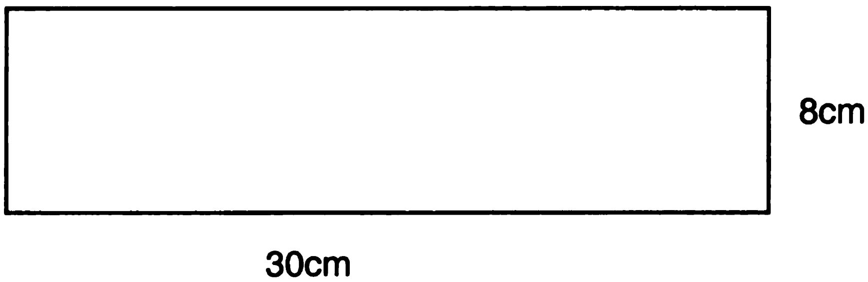
[Magazine]
(R)

Using the pattern given in Figure A and B, try to figure out the value of "?" is in the following Figure C.



[Numbers Relation]
(R)

You have 150 rectangular tiles. The length of each tile is 30 cm, and the width is 8 cm. You want to build a square platform from that set of rectangular tiles. How many tiles will you use?



(This picture is not the actual size of the tile)

[Tiles]
(NR)

Table of Moon Facts

The moon is smaller than the earth.
People weigh 6 times as much as on Earth as on the moon.
The moon goes around the earth once in 28 days.
The moon is about 240,000 miles from the earth

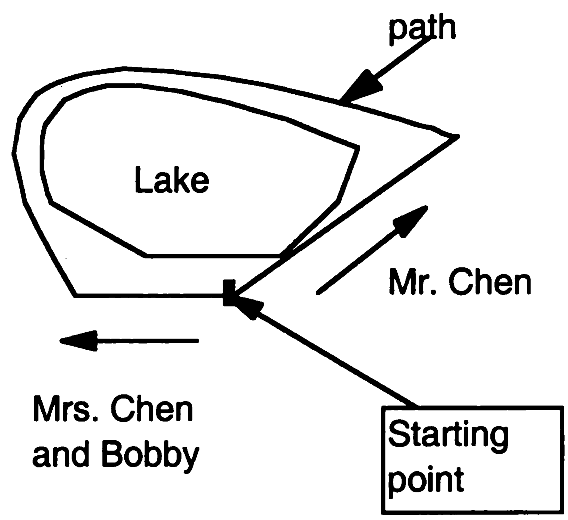
a. Peter figures that he would weigh 14 pounds on the moon. What does Peter weigh on Earth?

b. Peter's mother weighs 108 pounds on Earth. How much would she weigh on the moon?

c. About how long does it take the moon to go around the earth four times?

[Moon]
(NR)

Mr. and Mrs. Chen walk with their dog, Bobby, every morning. They are walk at the same speed, 5 km per hour. The distance of the walking path around the lake is 10 km. They start and walk in opposite directions at the starting point and meet each other midway, then go to have their breakfast. Bobby starts his morning run with Mrs. Chen and runs toward Mr. When he meets Mr. Chen, he runs back to Mrs. Chen. He continues like this until Mr. Chen and Mrs. Chen meet. His running speed is 28 km per hour. How far does Bobby run, by the time the couple meet?



[Lake]
(NR)

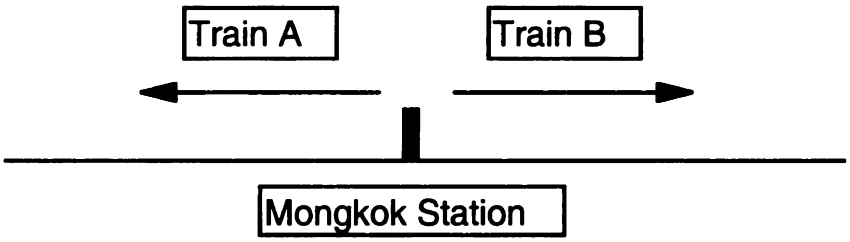
A poster designer just finished a poster on her computer. The size of the poster on her computer is 27.5 cm by 21 cm. She wants to make the length and the width 3 times larger and print the poster on a plotter to see how it works.

1) What is the area of the poster she prints?

2) The printed poster is larger than the poster on the computer screen, how many time larger?

[Poster]
(NR)

Two trains (A and B) are starting out from Mongkok Station in two different directions at the same time with the same speed, 66 km per hour. 2 minutes and 30 second later, how far apart distance-wise are the trains?



[Train]
(NR)

$$\frac{7}{9} \div 1\frac{3}{4} =$$

[Fraction Division]
(R)

APPENDIX C

SCORING SHEETS

Scoring Sheet for Non-routine Problems

Student ID: Rater:

Item	Att	Procedures provided	Read	Answer Provided	Answer correctness	Conceptual Error	Type of Error	Computing Error
A3 balance	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A4a glass house	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A4b glass house	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A5 coins	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A6 lake	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A7 ratio	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B C	0 1 2
A8b extended sq	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B C D E	0 1 2
A8c extended sq	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B C D E	0 1 2
B3b # pattern	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B3c # pattern	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B3d # pattern	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B4 lake	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B5a cakes	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B	0 1 2
B5b cake	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B	0 1 2
B6 ratio	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B C	0 1 2
B7 balance	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B9b extended sq	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B C D E	0 1 2
B9c extended sq	NA	Yes No	MR	Yes No	C I NC	0 1 2	A B C D E	0 1 2

Scoring Sheet for Routine Problems

Student ID: Rater:

Item	Att	Procedures provided	Read	Answer Provided	Answer correctness	Conceptual Error	Type of Error	Computing Error
A1 multiplication	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A2 unknown digit	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A9 saving	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
A10 rectangle width	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B1 multiplication	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B2 unknown digit	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B8 magazine	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2
B10 rectangle length	NA	Yes No	MR	Yes No	C I NC	0 1 2		0 1 2

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