

LIBRARY
Michigan State
University

PLACE IN RETURN BOX to remove this checkout from your record.
TO AVOID FINES return on or before date due.
MAY BE RECALLED with earlier due date if requested.

DATE DUE	DATE DUE	DATE DUE

The Theory of Function Spaces with Matrix Weights

By

Svetlana Roudenko

A DISSERTATION

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

2002

ABSTRACT

The Theory of Function Spaces with Matrix Weights

By

Svetlana Roudenko

Nazarov, Treil and Volberg defined matrix A_p weights and extended the theory of weighted norm inequalities on L^p to the case of vector-valued functions. We develop some aspects of Littlewood-Paley function space theory in the *matrix weight setting*. In particular, we introduce matrix-weighted homogeneous Besov spaces $\dot{B}_p^{\alpha q}(W)$ and matrix-weighted sequence Besov spaces $\dot{b}_p^{\alpha q}(W)$, as well as $\dot{b}_p^{\alpha q}(\{A_Q\})$, where the A_Q 's are reducing operators for W . Under any of three different conditions on the weight W , we prove the norm equivalences $\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}$, where $\{\vec{s}_Q\}_Q$ is the vector-valued sequence of φ -transform coefficients of $\vec{f}$. In the process, we note and use an alternate, more explicit characterization of the matrix A_p class. Furthermore, we introduce a weighted version of almost diagonality and prove that an almost diagonal matrix is bounded on $\dot{b}_p^{\alpha q}(W)$ if W is doubling. We also obtain the boundedness of almost diagonal operators on $\dot{B}_p^{\alpha q}(W)$ under any of the three conditions on W . This leads to the boundedness of convolution and non-

convolution type Calderón-Zygmund operators (CZO) on $\dot{B}_p^{\alpha q}(W)$, in particular, the Hilbert transform. We apply these results to wavelets to show that the above norm equivalence holds if the φ -transform coefficients are replaced by the wavelet coefficients. Next we determine the duals of the homogeneous matrix-weighted Besov spaces $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$. If W is a matrix A_p weight, then the dual of $\dot{B}_p^{\alpha q}(W)$ can be identified with $\dot{B}_{p'}^{-\alpha q'}(W^{-p'/p})$ and, similarly, $[\dot{b}_p^{\alpha q}(W)]^* \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$. Moreover, for certain W which may not be in A_p class, the duals of $\dot{B}_p^{\alpha q}(W)$ and $\dot{b}_p^{\alpha q}(W)$ are determined and expressed in terms of the Besov spaces $\dot{B}_p^{-\alpha q'}(\{A_Q^{-1}\})$ and $\dot{b}_p^{-\alpha q'}(\{A_Q^{-1}\})$, which we define in terms of reducing operators $\{A_Q\}_Q$ associated with W . We also develop the basic theory of these reducing operator Besov spaces. Finally, we construct inhomogeneous matrix-weighted Besov spaces $B_p^{\alpha q}(W)$ and show that results corresponding to those above are true also for the inhomogeneous case.

ACKNOWLEDGMENTS

I would like to express my deep gratitude to my advisor *Prof. Michael Frazier* for introducing the subject matter to me, for his encouragement and belief in me, for the uncountable number of hours spent sharing his knowledge and discussing various ideas, and for many useful comments and suggestions while examining my work. I would also like to express my thanks to *Prof. Alexander Volberg* for helpful suggestions and for sharing with me his enthusiasm for and appreciation of mathematics; to *Prof. Peter Yuditskii* for fruitful discussions, his availability to help me and his respect; and to *Prof. Fedor Nazarov* for conveying his sense of mathematical insight. I thank the other members of my thesis committee: *Prof. William Sledd* and *Prof. Joel Shapiro*. I would also like to thank *Prof. Clifford Weil* for introducing me to graduate analysis and teaching me how to write it well and rigorously.

I would like to say a special thank you to *D. Selahi Durusoy* for motivation, help and late night fruit snacks; to *Dr. Mark McCormick* for his valuable advice at the beginning of my professional career; to *Leo Larsson* for helpful discussions as well as suggestions in technical writing. I want to thank *Rebecca Grill* and *Amy Himes* for their tremendous support throughout my graduate studies. I would also like to thank *Dr. Burak Ozbagci* for his support and professional advice. Last but not least, I want

to thank *Dr. HsingChi Wang* and *Dr. David Gebhard* for their moral support, time and help throughout various stages of my doctorate.

TABLE OF CONTENTS

1	Introduction	1
1.1	History and Motivation	1
1.2	Overview of the results	2
2	Notation and Definitions	17
3	Matrix Weights	19
3.1	The A_p metric and reducing operators	19
3.2	Matrix A_p condition	21
3.3	Two properties of operators	22
3.4	The class $\mathcal{B}_p$	23
3.5	δ -doubling measures	26
3.6	δ -layers of the $\mathcal{B}_p$ class	27
3.7	Doubling measures	29
3.8	$\mathcal{B}_p$ implies the doubling property	32
3.9	An alternative characterization of the matrix A_p class	33
4	Boundedness of the φ-transform and Its Inverse on Matrix-Weighted Besov Spaces	36
4.1	Boundedness of the inverse φ -transform	36
4.2	Decompositions of an exponential type function	43
4.3	Boundedness of the φ -transform	45
4.4	Connection with reducing operators	55
5	Calderón-Zygmund Operators on Matrix-Weighted Besov Spaces	58
5.1	Almost diagonal operators	58
5.2	Calderón-Zygmund operators	69
6	Application to Wavelets	81
7	Duality	84
7.1	General facts on duality	84
7.2	Duality of sequence Besov spaces	85
7.3	Equivalence of sequence and discrete averaging Besov spaces	90
7.4	Properties of averaging L^p spaces	95

7.5	Convolution estimates	99
7.6	Duality of continuous Besov spaces	104
7.7	Application of Duality	109
8	Inhomogeneous Besov Spaces	111
8.1	Norm equivalence	111
8.2	Almost diagonality and Calderón-Zygmund Operators	116
8.3	Duality	117
9	Weighted Triebel-Lizorkin Spaces	121
9.1	Motivation	121
9.2	Equivalence of $\dot{f}_p^{\alpha q}(w)$ and $\dot{f}_p^{\alpha q}(\{w_Q\})$	122
10	Open Questions	126
A	Density and convergence	129
	BIBLIOGRAPHY	134

CHAPTER 1

Introduction

1.1 History and Motivation

Littlewood-Paley theory gives a unified perspective to the theory of function spaces. Well-known spaces such as Lebesgue, Hardy, Sobolev, Lipschitz spaces, etc. are special cases of either Besov spaces $\dot{B}_p^{\alpha q}$ (homogeneous), $B_p^{\alpha q}$ (inhomogeneous) or Triebel-Lizorkin spaces $\dot{F}_p^{\alpha q}$ (homogeneous), $F_p^{\alpha q}$ (inhomogeneous) (e.g., see [T]). These spaces are closely related to their discrete analogues: the sequence Besov spaces $\dot{b}_p^{\alpha q}$, $b_p^{\alpha q}$ and sequence Triebel-Lizorkin spaces $\dot{f}_p^{\alpha q}$, $f_p^{\alpha q}$ ([FJ1], [FJW]). Among other things, Littlewood-Paley theory provides alternate methods for studying singular integrals. The Hilbert transform, the classical example of a singular integral operator, led to the extensive modern theory of Calderón-Zygmund operators, mostly studied on the Lebesgue L^p spaces.

Motivated by the fundamental result of M. Riesz in the 1920s that the Hilbert transform preserves L^p for $1 < p < \infty$, Hunt, Muckenhoupt and Wheeden showed

that the famous A_p condition on a weight w is the necessary and sufficient condition for the Hilbert transform to be bounded on $L^p(w)$ (1973, [HMW]). More recent developments deal with matrix-weighted spaces where scalar methods simply could not be applied. In 1996 Treil and Volberg obtained the analogue of the Hunt-Muckenhoupt-Wheeden condition for the matrix case when $p = 2$ ([TV1]). Soon afterwards, Nazarov and Treil introduced in [NT] a new “Bellman function” method to extend the theory to $1 < p < \infty$. In 1997 Volberg presented a different solution to the matrix weighted L^p boundedness of the Hilbert transform via techniques related to classical Littlewood-Paley theory ([V]).

The purpose of this dissertation is to extend some aspects of Littlewood-Paley function space theory, previously obtained with no weights and partially for scalar weights, to the *matrix weight setting*.

1.2 Overview of the results

We define a new generalized function space: the vector-valued homogeneous Besov space $\dot{B}_p^{\alpha q}(W)$ with matrix weight W . Let $\mathcal{M}$ be the cone of nonnegative definite operators on a Hilbert space $\mathcal{H}$ of dimension m (consider $\mathcal{H} = \mathbb{C}^m$ or $\mathbb{R}^m$), i.e., for $M \in \mathcal{M}$ we have $(Mx, x)_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$. By definition, a *matrix weight* W is an a.e. invertible map $W : \mathbb{R}^n \rightarrow \mathcal{M}$. For a measurable $\vec{g} = (g_1, \dots, g_m)^T : \mathbb{R}^n \rightarrow \mathcal{H}$, let $\|\vec{g}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} \|W^{1/p}(t)\vec{g}(t)\|_{\mathcal{H}}^p dt \right)^{1/p}$. If the previous norm is finite, then $\vec{g} \in L^p(W)$. We say that a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ belongs to the class $\mathcal{A}$ of admissible kernels if $\text{supp } \hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\hat{\varphi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$. Set

$$\varphi_\nu(x) = 2^{\nu n} \varphi(2^\nu x) \text{ for } \nu \in \mathbb{Z}.$$

Definition 1.1 (Matrix-weighted Besov space $\dot{B}_p^{\alpha q}(W)$) For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$, $\varphi \in \mathcal{A}$ and W a matrix weight, the Besov space $\dot{B}_p^{\alpha q}(W)$ is the collection of all vector-valued distributions $\vec{f} = (f_1, \dots, f_m)^T$ with $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, $1 \leq i \leq m$ (the space of tempered distributions modulo polynomials) such that

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu \alpha} \|\varphi_\nu * \vec{f}\|_{L^p(W)} \right\}_\nu \right\|_{l_q} = \left\| \left\{ \|W^{1/p} \cdot (\varphi_\nu * \vec{f})\|_{L^p} \right\}_\nu \right\|_{l_q} < \infty,$$

where $\varphi_\nu * \vec{f} = (\varphi_\nu * f_1, \dots, \varphi_\nu * f_m)^T$.

The case $p = \infty$ is not of interest to us, since $\dot{B}_\infty^{\alpha q}(W) = \dot{B}_\infty^{\alpha q}$ because of the fact that $L^\infty(W) = L^\infty$. Since φ is directly involved in the definition of $\dot{B}_p^{\alpha q}(W)$, there seems to be a dependence on the choice of φ : $\dot{B}_p^{\alpha q}(W) = \dot{B}_p^{\alpha q}(W, \varphi)$. Under appropriate conditions on W , Theorem 1.8 below shows that this is not the case. The space $\dot{B}_p^{\alpha q}(W)$ is complete, as is discussed at the end of Section 4.4.

We also introduce the corresponding weighted sequence (discrete) Besov space $\dot{b}_p^{\alpha q}(W)$:

Definition 1.2 (Matrix-weighted sequence Besov space $\dot{b}_p^{\alpha q}(W)$) For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ and W a matrix weight, the space $\dot{b}_p^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_Q$, where $\vec{s}_Q = (s_Q^{(1)}, \dots, s_Q^{(m)})^T$, enumerated by the dyadic cubes Q contained in $\mathbb{R}^n$, such that

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu \alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right\}_\nu \right\|_{l_q}$$

$$= \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} (\|W^{1/p}(t)\vec{s}_Q\|_{\mathcal{H}}) \chi_Q(t) \right\|_{L^p(dt)} \right\}_{\nu} \right\|_{l_q^2} < \infty,$$

where $|Q|$ is the Lebesgue measure of Q and $l(Q)$ is the side length of Q .

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{\nu k}$ be the dyadic cube $\{(x_1, \dots, x_n) \in \mathbb{R}^n : k_i \leq 2^\nu x_i < k_i + 1, i = 1, \dots, n\}$ and $x_Q = 2^{-\nu}k$ is the lower left corner of $Q_{\nu k}$. Set $\varphi_Q(x) = |Q|^{-1/2} \varphi(2^\nu x - k) = |Q|^{1/2} \varphi_\nu(x - x_Q)$ for $Q = Q_{\nu k}$. For each $\vec{f}$ with $f_i \in \mathcal{S}'(\mathbb{R}^n)$ we define the φ -transform S_φ as the map taking $\vec{f}$ to the vector-valued sequence $S_\varphi(\vec{f}) = \left\{ \langle \vec{f}, \varphi_Q \rangle \right\}_Q = \left\{ (\langle f_1, \varphi_Q \rangle, \dots, \langle f_m, \varphi_Q \rangle)^T \right\}_Q$ for Q dyadic. We call $\vec{s}_Q(\vec{f}) := \langle \vec{f}, \varphi_Q \rangle$ the φ -transform coefficients of $\vec{f}$.

The next question is motivated by the following results:

(i) Frazier and Jawerth ([FJ1], 1985) showed that, in the unweighted scalar case,

$$\|f\|_{\dot{B}_p^{\alpha q}} \approx \|\{s_Q(f)\}_Q\|_{l_p^{\alpha q}},$$

where $\{s_Q(f)\}_Q$ are the φ -transform coefficients. A similar equivalence holds if $\{s_Q(f)\}_Q$ are the wavelet coefficients $\{\langle f, \psi_Q \rangle\}_Q$ of f with ψ_Q being smooth, say, Meyer's wavelets (see [M2]).

(ii) Nazarov, Treil and Volberg ([NT], 1996, [V], 1997) obtained

$$\|\vec{f}\|_{L^p(W)} \approx \left\| \left\{ \langle \vec{f}, h_Q \rangle \right\} \right\|_{f_p^{02}(W)} \quad \text{if } W \in A_p, \quad (1.1)$$

where $\{h_Q\}_Q$ is the Haar system and $f_p^{02}(W)$ is the coefficient (sequence Triebel-Lizorkin) space for $L^p(W)$. A particular case of (1.1), when $m = 1$ and w is a scalar weight, is

$$\|f\|_{\dot{B}_2^{02}(w)} = \|f\|_{L^2(w)} \approx \|\{\langle f, h_Q \rangle\}\|_{f_2^{02}(w)} = \|\{\langle f, h_Q \rangle\}\|_{b_2^{02}(w)},$$

where the first equality and the second equivalence hold if $w \in A_2$.

For our purposes we will use a condition on W that is equivalent to the matrix A_p condition of [NT] (for the proof, refer to Section 3.2):

Lemma 1.3 *Let W be a matrix weight, $1 < p < \infty$, and let p' be the conjugate of p ($1/p + 1/p' = 1$). Then*

$$\int_B \left(\int_B \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \leq c_{p,n} \quad \text{for every ball } B \subseteq \mathbb{R}^n \quad (1.2)$$

if and only if $W \in A_p$.

In (1.2), $\|W^{1/p}(x)W^{-1/p}(t)\|$ refers to the matrix (operator) norm.

The advantage of condition (1.2) is that it allows us to understand the A_p condition in terms of matrices, avoiding metrics ρ , ρ^* and their averagings as well as reducing operators (for definitions and details refer to Section 3.2).

Our first result is the norm equivalence between the continuous matrix-weighted Besov space $\dot{B}_p^{\alpha q}(W)$ and the discrete matrix-weighted Besov space $\dot{b}_p^{\alpha q}(W)$ under the A_p condition:

Theorem 1.4 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 < p < \infty$ and $W \in A_p$. Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)}. \quad (1.3)$$

In some cases, the A_p requirement on W can be relaxed. Recall that a scalar measure μ is called *doubling* if there exists $c > 0$ such that for any $\delta > 0$ and any

$z \in \mathbb{R}^n$,

$$\mu(B_{2\delta}(z)) \leq c \mu(B_\delta(z)), \quad (1.4)$$

where $B_\delta(z) = \{x \in \mathbb{R}^n : |z - x| < \delta\}$.

Definition 1.5 (*Doubling matrix*) *A matrix weight W is called a doubling matrix (of order p , $1 \leq p < \infty$), if there exists a constant $c = c_{p,n}$ such that for any $y \in \mathcal{H}$, any $\delta > 0$ and any $z \in \mathbb{R}^n$,*

$$\int_{B_{2\delta}(z)} \|W^{1/p}(t) y\|_{\mathcal{H}}^p dt \leq c \int_{B_\delta(z)} \|W^{1/p}(t) y\|_{\mathcal{H}}^p dt, \quad (1.5)$$

i.e., the scalar measure $w_y(t) = \|W^{1/p}(t) y\|_{\mathcal{H}}^p$ is uniformly doubling and not identically zero (a.e.). If $c = 2^\beta$ is the smallest constant for which (1.5) holds, then β is called the doubling exponent of W .

It is known that if $W \in A_p$, then w_y is a scalar A_p weight for any $y \in \mathcal{H}$ and the A_p constant is independent of y (for example, see [V]). This, in turn, implies that w_y is a scalar doubling measure (e.g., see [St2]) and the doubling constant is also independent of y . Using decomposition techniques, we prove the equivalence (1.3) under the doubling assumption on W with the restriction that p is large, and with no restriction on p in the case when W is a diagonal matrix:

Theorem 1.6 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let W be a doubling matrix of order p with doubling exponent β . Suppose $p > \beta$. Then the norm equivalence (1.3) holds. If W is diagonal, then (1.3) holds for all $1 \leq p < \infty$.*

The case of a scalar weight is a particular case of the diagonal matrix weight case, and thus, the equivalence (1.3) holds just under the doubling condition. This fact is essentially known (see [FJ2] for the case of $\dot{F}_p^{\alpha q}$); it is proved here for purposes of comparison and generalization to the diagonal matrix case.

Remark 1.7 *One of the directions of the norm equivalence uses only the doubling property of W with no restrictions (see Corollary 4.6), but the other direction requires the stated assumptions on W (see Theorem 4.15). Furthermore, the first direction is obtained from a more general norm estimate involving families of “smooth molecules” (see Theorem 4.2).*

Summarizing Theorems 1.4 and 1.6, the norm equivalence (1.3) holds under any of the following conditions:

- (A1) $W \in A_p$ with $1 < p < \infty$,
- (A2) W is a doubling matrix of order p , $1 \leq p < \infty$, with $p > \beta$, where β is the doubling exponent of W ,
- (A3) W is a diagonal doubling matrix of order p with $1 \leq p < \infty$.

Now we will state the independence of the space $\dot{B}_p^{\alpha q}(W, \varphi)$ from φ :

Theorem 1.8 *Let $\vec{f} \in \dot{B}_p^{\alpha q}(W, \varphi^{(1)})$, $\varphi^{(1)} \in \mathcal{A}$, $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and suppose any of (A1)-(A3) hold. Then for any $\varphi^{(2)} \in \mathcal{A}$,*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \varphi^{(1)})} \approx \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \varphi^{(2)})}.$$

If we use the language of reducing operators (see [V] or Section 3.2), we extend the norm equivalence (1.3) to a different sequence space, namely $\dot{b}_p^{\alpha q}(\{A_Q\})$. For each dyadic cube Q , consider a reducing operator A_Q corresponding to the L^p average over Q of the norm $\|W^{1/p} \cdot\|_{\mathcal{H}}$, i.e.,

$$\|A_Q \vec{u}\|_{\mathcal{H}} \approx \left(\frac{1}{|Q|} \int_Q \|W^{1/p}(t) \vec{u}\|_{\mathcal{H}}^p dt \right)^{1/p}$$

for all vector-valued sequences $\vec{u}$. Note that the assumption that W is a.e. invertible guarantees that each A_Q is invertible. Define the sequence space $\dot{b}_p^{\alpha q}(\{A_Q\})$ for $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ as the space containing all vector-valued sequences $\{\vec{s}_Q\}_Q$ with

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} (\|A_Q \vec{s}_Q\|_{\mathcal{H}}) \chi_Q \right\|_{L^p(dt)} \right\}_{\nu} \right\|_{l_q} < \infty.$$

Theorem 1.9 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$. Suppose W satisfies any of (A1)-(A3). Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}. \quad (1.6)$$

In Chapter 5 we study operators on $\dot{B}_p^{\alpha q}(W)$ by considering corresponding operators on $\dot{b}_p^{\alpha q}(W)$. In [FJW] it was shown that almost diagonal operators are bounded on $\dot{b}_p^{\alpha q}$ and, thus, on $\dot{B}_p^{\alpha q}$. In Section 5.1 we define a class of almost diagonal matrices $\mathbf{ad}_p^{\alpha q}(\beta)$ for the weighted case and show the boundedness of these matrices on $\dot{b}_p^{\alpha q}(W)$ if W is a doubling matrix weight:

Theorem 1.10 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let W be a doubling matrix of order p with doubling exponent β . Consider $A \in \mathbf{ad}_p^{\alpha q}(\beta)$. Then $A : \dot{b}_p^{\alpha q}(W) \rightarrow \dot{b}_p^{\alpha q}(W)$ is bounded.*

We say that a continuous linear operator $T : \mathcal{S} \rightarrow \mathcal{S}'$ is *almost diagonal*, $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, if for some pair of mutually admissible kernels (φ, ψ) (see (2.1), Section 2) the matrix $(\langle T\psi_P, \varphi_Q \rangle_{QP})_{Q,P \text{ dyadic}} \in \mathbf{ad}_p^{\alpha q}(\beta)$ (see Section 5.1). Combining the boundedness of an almost diagonal matrix with the norm equivalence, we obtain the boundedness of an almost diagonal operator on $\dot{B}_p^{\alpha q}(W)$ under any of (A1)-(A3):

Corollary 1.11 *Let $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. Then T is a bounded operator on $\dot{B}_p^{\alpha q}(W)$ if W satisfies any of (A1)-(A3).*

In Section 5.2 we consider classical convolution and generalized non-convolution Calderón-Zygmund operators (CZO). The following criterion is used: if an operator T maps “smooth atoms” into “smooth molecules” (see Sections 4.1 and 5.2 for definitions), then T is almost diagonal (Lemma 5.13) and, therefore, bounded on $\dot{B}_p^{\alpha q}(W)$. To show this property for a CZO, the definition of a “smooth molecule” is modified in order to compensate for the growth of the weight W (note the dependence of the decay rate of the molecule on the doubling exponent β), and, thus, more smoothness of a CZO kernel is required (see Theorems 5.25 and 5.19). In particular, for example, we obtain the boundedness of the Hilbert transform (when the underlying dimension is $n = 1$) and the Riesz transforms ($n \geq 2$) on $\dot{B}_p^{\alpha q}(W)$ under any of the conditions (A1)-(A3).

In Chapter 6 we apply the previous results to Meyer’s wavelets and Daubechies’ DN wavelets with N sufficiently large, to show that, instead of the φ -transform coefficients, one can use the wavelet coefficients for the norm equivalence:

Theorem 1.12 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ and let W satisfy any of (A1)-(A3), then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)},$$

where $\left\{ \vec{s}_Q(\vec{f}) \right\}_Q$ are the wavelet coefficients of $\vec{f}$.

The next goal (Chapter 7) is to determine the duals of the Besov function spaces $\dot{B}_p^{\alpha q}(W)$ and the corresponding sequence spaces $\dot{b}_p^{\alpha q}(W)$ for $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $1 < p < \infty$.

To understand what properties of W are needed to identify dual spaces, we will heavily use the technique of *reducing operators* (for definitions refer to Section 3.2 or [V]). In fact, instead of dealing with matrix weights, we consider a sequence of matrices enumerated by dyadic cubes and establish properties of Besov spaces with such sequences of matrix weights. Then, given a matrix W , its reducing operators constitute such a sequence.

Denote by $\mathcal{D}$ the collection of dyadic cubes in $\mathbb{R}^n$ and for each $Q \in \mathcal{D}$ let A_Q be a positive-definite (thus, self-adjoint) operator on $\mathcal{H}$. Also denote by $\mathcal{RS}_{\mathcal{D}}$ (reducing sequences) the collection of all sequences $\{A_Q\}_{Q \in \mathcal{D}}$ of positive-definite operators on $\mathcal{H}$.

In Chapter 7 as a main tool and a useful object by itself, we define the space $\dot{B}_p^{\alpha q}(\{A_Q\})$ with a sequence of discrete weights $\{A_Q\}_Q$:

Definition 1.13 (Averaging matrix-weighted Besov space $\dot{B}_p^{\alpha q}(\{A_Q\})$) *For $\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$, $0 < q \leq \infty$, $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$ and $\varphi \in \mathcal{A}$, the Besov space*

$\dot{B}_p^{\alpha q}(\{A_Q\})$ is the collection of all vector-valued distributions $\vec{f} = (f_1, \dots, f_m)^T$ with $f_i \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, $1 \leq i \leq m$ such that

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} \|A_Q \cdot (\varphi_\nu * \vec{f})\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\}_\nu \right\|_{l^q} < \infty.$$

This space is well-defined (i.e., independent of $\varphi \in \mathcal{A}$), see Corollary 7.10, if $\{A_Q\}_Q$ is a doubling matrix sequence defined as follows.

Definition 1.14 (Doubling sequence) We say $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$ is a (dyadic) doubling sequence (of order p , $1 \leq p < \infty$), if there exists $\beta \geq n$ and $c \geq 1$ such that for all P, Q dyadic

$$\|A_Q A_P^{-1}\|^p \leq c \frac{|P|}{|Q|} \max \left(1, \left[\frac{l(Q)}{l(P)} \right]^\beta \right) \left(1 + \frac{\text{dist}(P, Q)}{\max(l(P), l(Q))} \right)^\beta. \quad (1.7)$$

Observe that if (1.7) holds for some p , then it holds for $1 \leq q < p$, since the right-hand side is ≥ 1 .

Our main result of this chapter identifies the dual space of $\dot{B}_p^{\alpha q}(W)$. For $W \in A_p$ the result can be expressed in terms of matrix weights. However, even for $W \notin A_p$ but satisfying (A2) or (A3), we are able to characterize $[\dot{B}_p^{\alpha q}(W)]^*$ in terms of reducing operators. Set $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 < p < \infty$ and $p' = \infty$ if $p = 1$; $\frac{1}{q} + \frac{1}{q'} = 1$ if $1 < q < \infty$ and $q' = \infty$ if $0 < q \leq 1$. It is important to emphasize our convention for the duality pairing. In what follows, we say that a function space Y is a dual of a function space X , $X^* \approx Y$, in the sense that each $y \in Y$ defines an element l_y of X^* via the pairing $l_y(x) = (x, y) = \int_{\mathbb{R}^n} \langle x(t), y(t) \rangle dt$ and every element of X^* is of the kind l_y for some $y \in Y$ with $\|l_y\| \approx \|y\|_Y$. (For example, $[L^p(W)]^* \approx L^{p'}(W^{-p'/p})$, $1 < p < \infty$, with the pairing $(\vec{f}, \vec{g}) = \int_{\mathbb{R}^n} \langle \vec{f}(t), \vec{g}(t) \rangle dt$; refer to Section 7.2 for more details.)

Theorem 1.15 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q < \infty$ and let $\{A_Q\}_Q$ be reducing operators of a matrix weight W .*

$$\text{If } W \in A_p, 1 < p < \infty, \text{ then } \left[\dot{B}_p^{\alpha q}(W) \right]^* \approx \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}). \quad (1.8)$$

$$\text{If } W \text{ satisfies any of (A1)-(A3), then } \left[\dot{B}_p^{\alpha q}(W) \right]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \quad (1.9)$$

(For the proof refer to Section 7.4.)

Next we identify the dual space of the sequence (discrete) Besov space $\dot{b}_p^{\alpha q}(W)$. Recall that the connection between $\dot{b}_p^{\alpha q}(W)$ and $\dot{B}_p^{\alpha q}(W)$ is that $\vec{f} \in \dot{B}_p^{\alpha q}(W)$ if and only if the appropriate wavelet coefficient sequence of $\vec{f}$ belongs to $\dot{b}_p^{\alpha q}(W)$. Analogously to $\dot{B}_p^{\alpha q}(\{A_Q\})$ we introduce $\dot{b}_p^{\alpha q}(\{A_Q\})$.

Definition 1.16 (Averaging matrix-weighted discrete Besov space $\dot{b}_p^{\alpha q}(\{A_Q\})$.) For $\alpha \in \mathbb{R}$, $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, the space $\dot{b}_p^{\alpha q}(\{A_Q\})$ consists of all vector-valued sequences $\{\vec{s}_Q\}_{Q \in \mathcal{D}}$ such that

$$\begin{aligned} \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} &= \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} (\|A_Q \vec{s}_Q\|_{\mathcal{H}}) \chi_Q(t) \right\|_{L^p(dt)} \right\}_{\nu} \right\|_{l^q} \\ &= \left\| \{A_Q \vec{s}_Q\}_Q \right\|_{\dot{b}_p^{\alpha q}} < \infty. \end{aligned}$$

If $\{A_Q\}_Q$ is a sequence of reducing operators for a matrix weight W , then the norm equivalence

$$\dot{b}_p^{\alpha q}(W) \approx \dot{b}_p^{\alpha q}(\{A_Q\}) \quad (1.10)$$

holds for any matrix weight W , $\alpha \in \mathbb{R}$, $1 \leq p < \infty$ and $0 < q \leq \infty$ by Lemma 4.18.

Theorem 1.17 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q < \infty$ and let $\{A_Q\}_Q$ be reducing operators of a matrix weight W . Then*

$$\left[\dot{b}_p^{\alpha q}(W) \right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \quad (1.11)$$

Moreover, if $W \in A_p$, $1 < p < \infty$, then

$$\left[\dot{b}_p^{\alpha q}(W) \right]^* \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}). \quad (1.12)$$

The chapter 7 is organized as follows. In Section 7.2 we discuss the discrete Besov space $\dot{b}_p^{\alpha q}(W)$. We use a “one at a time reduction” approach meaning we reduce the space $\dot{b}_p^{\alpha q}(W)$ in the following order:

$$\dot{b}_p^{\alpha q}(W) \longrightarrow \dot{b}_p^{\alpha q}(\{A_Q\}) \longrightarrow \dot{b}_p^{\alpha q}(\mathbb{R}^m) \longrightarrow \dot{b}_p^{\alpha q}(\mathbb{R}^1),$$

where the last two spaces are unweighted vector-valued and scalar-valued discrete Besov spaces, and then identify the duals in the opposite order. A similar approach is used for $\dot{B}_p^{\alpha q}(W)$.

The fact that each A_Q is constant on each dyadic cube Q allows us establish

$$\left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \quad (1.13)$$

for any $\{A_Q\}_Q \in \mathcal{RS}_D$, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. If A_Q 's are generated by a matrix weight W , then combining (1.10) and (1.13), we get (1.11) of Theorem 1.17.

In order to connect $\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ with $\dot{b}_{p'}^{-\alpha q'}(\{A_Q^\#\}) \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$ the matrix A_p condition is needed, though only for one direction of the embedding; the other direction is automatic. Thus, the following chain of the embeddings holds for $\dot{b}_p^{\alpha q}(W)$:

$$\left[\dot{b}_p^{\alpha q}(W) \right]^* \stackrel{\text{any } W}{\approx} \left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \stackrel{A_p}{\approx} \dot{b}_{p'}^{-\alpha q'}(\{A_Q^\#\})$$

$$\stackrel{\text{any } W}{\approx} \dot{b}_p^{-\alpha q'}(W^{-p'/p}). \quad (1.14)$$

This completes the proof of Theorem 1.17.

In Section 7.3 we prove the norm equivalence between $\dot{B}_p^{\alpha q}(\{A_Q\})$ and $\dot{b}_p^{\alpha q}(\{A_Q\})$ for any doubling sequence $\{A_Q\}_Q$. Note that if A_Q 's are generated by a matrix weight W , then all that is required from the weight is the doubling condition. Compare this with (A1)-(A3) conditions for the norm equivalence between the original matrix-weighted spaces.

Theorem 1.18 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ and $\{A_Q\}_Q$ be a doubling sequence (of order p). Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \approx \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.$$

In Section 7.4 we establish the correspondence between the continuous Besov spaces $\dot{B}_p^{\alpha q}(W)$ and $\dot{B}_p^{\alpha q}(\{A_Q\})$.

Lemma 1.19 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 \leq p < \infty$. If W satisfies any of (A1)-(A3) and $\{A_Q\}_Q$ is a sequence of reducing operators generated by W , then*

$$\dot{B}_p^{\alpha q}(W) \approx \dot{B}_p^{\alpha q}(\{A_Q\}).$$

For one direction of the above equivalence it suffices to have W doubling.

In Section 7.6 it is shown that if $\{A_Q\}_Q$ is a doubling sequence of order p , $1 \leq p < \infty$, then

$$\left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^* \approx \dot{B}_p^{-\alpha q'}(\{A_Q^{-1}\}). \quad (1.15)$$

Using the above duality and equivalence, we get the following chain:

$$\begin{aligned}
[\dot{B}_p^{\alpha q}(W)]^* &\stackrel{(1)}{\approx} [\dot{B}_p^{\alpha q}(\{A_Q\})]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \stackrel{A_p}{\approx} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^\#\}) \\
&\stackrel{(4)}{\approx} \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}),
\end{aligned} \tag{1.16}$$

where the equivalences (1) and (4) hold if W and $W^{-p'/p}$, respectively, satisfy any of (A1)-(A3). The third equivalence holds under the A_p condition, however, the A_p condition is needed only for one direction of the embedding. This proves Theorem 1.15.

So far we have dealt only with homogeneous spaces. However, for a number of applications it is necessary to consider the inhomogeneous distribution spaces (e.g., localized Hardy spaces $H_{loc}^p = F_p^{02}, 0 < p < \infty$, in particular, $H_{loc}^2 = B_2^{02}$, see [Go]). In Chapter 8 we “transfer” the theory developed up until now to the inhomogeneous Besov spaces. The main difference is that instead of considering all dyadic cubes, we consider only the ones with side length $l(Q) \leq 1$, and the properties of functions corresponding to $l(Q) = 1$ are slightly changed. Modifying the definitions of the φ -transform and smooth molecules, we show that all the statements from the homogeneous case are essentially the same for the inhomogeneous spaces.

In Chapter 9 we study another class of function spaces - scalar weighted Triebel-Lizorkin spaces. As a starting point of this part we establish the norm equivalence between the scalar weighted Triebel-Lizorkin space $\dot{F}_p^{\alpha q}(w)$ and the averaging scalar weighted sequence Triebel-Lizorkin space $\dot{f}_p^{\alpha q}(\{w_Q\})$ (see definitions below) if $w \in A_\infty$ (see Chapter 3).

Definition 1.20 (scalar-weighted Triebel-Lizorkin space $\dot{F}_p^{\alpha q}(w)$) For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $\varphi \in \mathcal{A}$ and w a scalar weight, the Triebel-Lizorkin space $\dot{F}_p^{\alpha q}(w)$ is the collection of all distributions $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}_p^{\alpha q}(w)} = \left\| \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu \alpha} |\varphi_\nu * f|)^q \right)^{1/q} \right\|_{L^p(w)} < \infty,$$

where the l^q -norm is replaced by the supremum on ν if $q = \infty$.

This space is well-defined if w is a doubling measure (see [FJ2]).

Definition 1.21 (scalar weighted sequence Triebel-Lizorkin space $\dot{f}_p^{\alpha q}(w)$)

For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and w a scalar weight, the Triebel-Lizorkin space $\dot{f}_p^{\alpha q}(w)$ is the collection of all sequences $\{s_Q\}_{Q \in \mathcal{D}}$ such that

$$\|\{s_Q\}_Q\|_{\dot{f}_p^{\alpha q}(w)} = \left\| \left(\sum_{Q \in \mathcal{D}} \left(|Q|^{-\frac{\alpha}{n} - \frac{1}{2}} s_Q \chi_Q \right)^q \right)^{1/q} \right\|_{L^p(w)} < \infty,$$

where the l^q -norm is again replaced by the supremum on ν if $q = \infty$.

Definition 1.22 (averaging scalar weighted sequence Triebel-Lizorkin

space $\dot{f}_p^{\alpha q}(\{w_Q\})$.) For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $\{w_Q\}_{Q \in \mathcal{D}}$ a sequence of non-negative numbers, the Triebel-Lizorkin space $\dot{f}_p^{\alpha q}(\{w_Q\})$ is the collection of all sequences $\{s_Q\}_{Q \in \mathcal{D}}$ such that

$$\|\{s_Q\}_Q\|_{\dot{f}_p^{\alpha q}(\{w_Q\})} = \|\{w_Q^{1/p} s_Q\}_Q\|_{\dot{f}_p^{\alpha q}} = \left\| \left(\sum_{Q \in \mathcal{D}} \left(|Q|^{-\frac{\alpha}{n} - \frac{1}{2}} w_Q^{1/p} s_Q \chi_Q \right)^q \right)^{1/q} \right\|_{L^p} < \infty,$$

where the l^q -norm is again replaced by the supremum on ν if $q = \infty$.

Appendix contains several proofs on convergence and density.

CHAPTER 2

Notation and Definitions

Let $z \in \mathbb{R}^n$. Recall that $B(z, \delta) = \{x \in \mathbb{R}^n : |z - x| < \delta\} \equiv B_\delta(z)$. If the center z of the ball is not essential, we will write B_δ for simplicity. In further notation, $\langle V \rangle_B$ means the average of V over the set B : $\frac{1}{|B|} \int_B V(t) dt$. Denote $W_\nu(t) = W(2^{-\nu}t)$ for $\nu \in \mathbb{Z}$.

For each admissible $\varphi \in \mathcal{A}$, there exists $\psi \in \mathcal{A}$ (see e.g. [FJW, p.54]) such that

$$\sum_{\nu \in \mathbb{Z}} \overline{\hat{\varphi}(2^\nu \xi)} \cdot \hat{\psi}(2^\nu \xi) = 1, \quad \text{if } \xi \neq 0. \quad (2.1)$$

A pair (φ, ψ) with $\varphi, \psi \in \mathcal{A}$ and the property (2.1) will be referred to as a pair of mutually admissible kernels.

Similarly to φ_Q , define $\psi_Q(x) = |Q|^{-1/2} \psi(2^\nu x - k)$ for $Q = Q_{\nu k}$. The *inverse* φ -transform T_ψ is the map taking a sequence $s = \{s_Q\}_Q$ to $T_\psi s = \sum_Q s_Q \psi_Q$. In the vector case, $T_\psi \vec{s} = \sum_Q \vec{s}_Q \psi_Q$, where $\vec{s}_Q \psi_Q = (s_Q^{(1)} \psi_Q, \dots, s_Q^{(m)} \psi_Q)^T$. The φ -transform decomposition (see [FJ2] for more details) states that for all $f \in \mathcal{S}'/\mathcal{P}$,

$$f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q =: \sum_Q s_Q \psi_Q. \quad (2.2)$$

In other words, $T_\psi \circ S_\varphi$ is the identity on $\mathcal{S}'/\mathcal{P}$. Observe that if $\tilde{\varphi}(x) = \overline{\varphi(-x)}$ (note that $\tilde{\varphi} \in \mathcal{A}$), then $s_Q = \langle f, \varphi_Q \rangle = |Q|^{1/2}(\tilde{\varphi}_\nu * f)(2^{-\nu}k)$.

In order to establish the connection between matrix weighted Besov spaces and averaging Besov spaces in Chapter 7, we introduce an auxiliary L^p -space:

Definition 2.1 (Averaging space $L^p(\{A_Q\}, \nu)$) For $\nu \in \mathbb{Z}$, $1 \leq p \leq \infty$ and $\{A_Q\}_Q \in \mathcal{RS}_\mathcal{D}$, the space $L^p(\{A_Q\}, \nu)$ consists of all vector-valued locally integrable functions $\vec{f}$ such that

$$\|\vec{f}\|_{L^p(\{A_Q\}, \nu)} = \left\| \sum_{l(Q)=2^{-\nu}} \chi_Q(t) A_Q \vec{f}(t) \right\|_{L^p(dt)} < \infty.$$

Note that $\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \left\| \left\{ 2^{\nu\alpha} \left\| \varphi_\nu * \vec{f} \right\|_{L^p(\{A_Q\}, \nu)} \right\}_\nu \right\|_{l^q}$.

To make notation short, define $Q_\nu = \{Q \in \mathcal{D} : l(Q) = 2^{-\nu}\}$.

CHAPTER 3

Matrix Weights

3.1 The A_p metric and reducing operators

Let $t \in \mathbb{R}^n$. Consider the family of norms $\rho_t : \mathcal{H} \rightarrow \mathbb{R}^+$. Then the dual norm ρ^* is given by

$$\rho_t^*(x) = \sup_{y \neq 0} \frac{|(x, y)|}{\rho_t(y)}.$$

Following [V] (or [NT], [TV1]), we introduce the norms $\rho_{p,B}$ through the averagings of the metrics ρ_t over a ball B

$$\rho_{p,B}(x) = \left(\frac{1}{|B|} \int_B [\rho_t(x)]^p dt \right)^{1/p}.$$

Similarly, for the dual norm

$$\rho_{p',B}^*(x) = \left(\frac{1}{|B|} \int_B [\rho_t^*(x)]^{p'} dt \right)^{1/p'}.$$

Definition 3.1 (A_p - **metric**) *The metric ρ is an A_p -metric, $1 < p < \infty$, if*

$$\rho_{p',B}^* \leq C (\rho_{p,B})^* \quad \text{for every ball } B \subseteq \mathbb{R}^n. \quad (3.1)$$

The condition (3.1) is equivalent to

$$\rho_{p,B} \leq C (\rho_{p',B}^*)^* \quad \text{for every ball } B \subseteq \mathbb{R}^n,$$

which means that ρ^* is an $A_{p'}$ -metric.

If ρ is a norm on $\mathcal{H}$, then there exists a positive operator A , which is called a *reducing operator* of ρ , such that

$$\rho(x) \approx \|Ax\| \quad \text{for all } x \in \mathcal{H}.$$

For details we refer the reader to [V]. Let A_B be a reducing operator for $\rho_{p,B}$, and $A_B^\#$ for $\rho_{p',B}^*$. Then, in the language of the reducing operators, the condition (3.1) for the A_p class is

$$\|A_B^\# A_B\| \leq C < \infty \quad \text{for every ball } B \subseteq \mathbb{R}^n. \quad (3.2)$$

PROOF. Since $\rho_{p',B}^*(x) \approx \|A_B^\# x\|$ and $(\rho_{p,B})^*(x) = \sup_{y \neq 0} \frac{|(x, y)|}{\rho_{p,B}(y)}$, (3.3) implies

$$\|A_B^\# x\| \leq c \sup_{y \neq 0} \frac{|(x, y)|}{\|A_B y\|} = c \sup_{z \neq 0} \frac{|(x, A_B^{-1} z)|}{\|z\|}, \quad \text{where } z = A_B y.$$

Since A_B^{-1} is self-adjoint,

$$\|A_B^\# x\| \leq c \sup_{z \neq 0} \frac{|(A_B^{-1} x, z)|}{\|z\|} = c \|A_B^{-1} x\|.$$

With $u = A_B^{-1} x$, we obtain

$$\|A_B^\# A_B u\| \leq c \|u\|, \quad \text{or} \quad \|A_B^\# A_B\| \leq c.$$

■

Note that the opposite inequality $\|(A_Q^\# A_Q)^{-1}\| \leq c$ holds always as a simple consequence of Hölder's inequality: for any $x, y \in \mathcal{H}$ we have

$$|(x, y)| \leq \left(\int_Q \|W^{1/p}(t) x\|^p \frac{dt}{|Q|} \right)^{1/p} \left(\int_Q \|W^{-1/p}(t) y\|^{p'} \frac{dt}{|Q|} \right)^{1/p'} \approx \|A_Q x\| \|A_Q^\# y\|,$$

which implies $\|A_Q x\| \geq c \|(A_Q^\#)^{-1} x\|$ for any $x \in \mathcal{H}$ and, thus, the above statement follows.

3.2 Matrix A_p condition

The particular case of norms ρ_t , we will be interested from now on, is

$$\rho_t(x) = \|W^{1/p}(t) x\|, \quad x \in \mathcal{H}, \quad t \in \mathbb{R}^n.$$

Then the dual metric ρ_t^* is given by

$$\rho_t^*(x) = \sup_{y \neq 0} \frac{|(x, y)|}{\rho_t(y)} = \|W^{-1/p}(t) x\|.$$

Definition 3.2 (Matrix A_p weight) For $1 < p < \infty$, we say that a matrix weight W is an A_p matrix weight if there exists $C < \infty$ such that for every ball $B \subseteq \mathbb{R}^n$

$$\rho_{p', B}^* \leq C (\rho_{p, B})^*, \tag{3.3}$$

where both averaging metrics are generated by W , i.e.,

$$\rho_{p, B}(x) = \left(\frac{1}{|B|} \int_B \|W^{1/p}(t) x\|^p dt \right)^{1/p}$$

and

$$\rho_{p', B}^*(x) = \left(\frac{1}{|B|} \int_B \|W^{-1/p}(t) x\|^{p'} dt \right)^{1/p'}.$$

Remark 3.3 *If $p = 2$, the condition A_2 simplifies significantly:*

$$\| \langle W \rangle_B^{1/2} \langle W^{-1} \rangle_B^{1/2} \| \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n. \quad (3.4)$$

PROOF.

$$\begin{aligned} [\rho_{p,B}(x)]^2 &= \int_B \|W^{1/2}(t)x\|^2 \frac{dt}{|B|} = \int_B (W(t)x, x) \frac{dt}{|B|} \\ &= (\langle W \rangle_B x, x) = \| \langle W \rangle_B^{1/2} x \|^2. \end{aligned}$$

This means that a reducing operator A_B can be chosen explicitly as $\langle W \rangle_B^{1/2}$.

Similarly, $\rho_{p',B}^*(x) = \| \langle W^{-1} \rangle_B^{1/2} x \|^2$ and, thus, $A_B^\# \approx \langle W^{-1} \rangle_B^{1/2}$. Therefore, (3.4) follows from (3.2). ■

Remark 3.4 *If w is a scalar weight, the condition A_p is the celebrated Muckenhoupt A_p condition:*

$$\left(\int_B w(t) dt \right)^{1/p} \left(\int_B w^{-p'/p}(t) dt \right)^{1/p'} \leq c \quad \text{for every ball } B \subseteq \mathbb{R}^n. \quad (3.5)$$

Denote $w_x(t) = \|W^{1/p}(t)x\|^p$ and $w_x^*(t) = \|W^{-1/p}(t)x\|^{p'}$. Similarly, $w(t) = \|W^{1/p}(t)\|^p$ and $w^*(t) = \|W^{-1/p}(t)\|^{p'}$. Sometimes it is more convenient to work with these families of scalar-valued measures.

3.3 Two properties of operators

Observe the following two useful facts. First, if P and Q are two selfadjoint operators in a normed space, then

$$\|PQ\| = \|(PQ)^*\| = \|Q^*P^*\| = \|QP\|. \quad (3.6)$$

Thus, the operators can be commuted as long as we deal with norms.

Second, we need the following lemma:

Lemma 3.5 (NORM LEMMA) *If $\{e_1, \dots, e_m\}$ is any orthonormal basis in a Hilbert space $\mathcal{H}$, then for any linear operator $V : \mathcal{H} \rightarrow \mathcal{H}$ and $r > 0$,*

$$\|V\|^r \underset{(r,m)}{\approx} \sum_{i=1}^m \|Ve_i\|_{\mathcal{H}}^r.$$

PROOF. With $x_i = (x, e_i)_{\mathcal{H}}$, we get $\|V\|^r = \sup_{\|x\| \leq 1} \|V \sum_{i=1}^m x_i e_i\|_{\mathcal{H}}^r$

$$\leq c_r \sup_{\|x\| \leq 1} \sum_{i=1}^m |x_i|^r \|Ve_i\|_{\mathcal{H}}^r \leq c_r \sum_{i=1}^m \|Ve_i\|_{\mathcal{H}}^r \leq c_r m \|V\|^r. \quad \blacksquare$$

3.4 The class $\mathcal{B}_p$

Definition 3.6 *For $1 < p < \infty$ the class $\mathcal{B}_p$ is the collection of all matrix weights W so that for a given fixed $0 < \delta < 1$ there exists a constant $c = c_{\delta,p,n}$ such that for any $z \in \mathbb{R}^n$ and any $\nu \in \mathbb{Z}$ the following inequality holds*

$$\int_{B_{\delta}(z)} \left(\int_{B_{\delta}(z)} \|W_{\nu}^{1/p}(x) W_{\nu}^{-1/p}(t)\|^{p'} \frac{dt}{|B_{\delta}|} \right)^{p/p'} \frac{dx}{|B_{\delta}|} \leq c_{\delta,p,n}, \quad (3.7)$$

where $W_{\nu}(t) = W(2^{-\nu}t)$.

This condition seems to be dependent on the choice of δ , though it is not the fact.

PROOF. By changing variables we write (3.7) as

$$\int_{B_{2^{-\nu}\delta}(2^{-\nu}z)} \left(\int_{B_{2^{-\nu}\delta}(2^{-\nu}z)} \|W^{1/p}(x) W^{-1/p}(t)\|^{p'} \frac{dt}{|B_{2^{-\nu}\delta}|} \right)^{\frac{p}{p'}} \frac{dx}{|B_{2^{-\nu}\delta}|} \leq c_{\delta,p,n}. \quad (3.8)$$

Let $\epsilon > 0$. Then there exists $\nu_0 \in \mathbb{Z}$ such that $2^{-(\nu_0+1)}\delta \leq \epsilon < 2^{-\nu_0}\delta$. The following three simple observations will show that (3.7) is independent of δ .

1. $B_{2^{-(\nu_0+1)}\delta}(z) \subseteq B_\epsilon(z) \subseteq B_{2^{-\nu_0}\delta}(z)$,
2. $|B_{2^{-\nu_0}\delta}| = 2^n |B_{2^{-(\nu_0+1)}\delta}|$,
3. $\frac{1}{2^n} \int_{B_{2^{-(\nu_0+1)}\delta}} \cdots \frac{dt}{|B_{2^{-(\nu_0+1)}\delta}|} \leq \int_{B_\epsilon} \cdots \frac{dt}{|B_\epsilon|} \leq 2^n \int_{B_{2^{-\nu_0}\delta}} \cdots \frac{dt}{|B_{2^{-\nu_0}\delta}|}$.

■

Hence, the definition of the $\mathcal{B}_p$ class is equivalent to the following one:

Definition 3.7 *The class $\mathcal{B}_p$, $1 < p < \infty$, is the collection of all matrix weights W so that there exists a constant $c = c_{p,n}$ such that for any $z \in \mathbb{R}^n$ and any $\epsilon > 0$ the inequality (1.2) holds, i.e.,*

$$\int_{B_\epsilon(z)} \left(\int_{B_\epsilon(z)} \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} \frac{dt}{|B_\epsilon|} \right)^{p/p'} \frac{dx}{|B_\epsilon|} \leq c_{p,n}. \quad (3.9)$$

Remark 3.8 *It is also convenient to write condition (3.9) in terms of metrics ρ and ρ^* :*

$$\int_{B_\epsilon(z)} \left(\int_{B_\epsilon(z)} \sup_{y \neq 0} \left[\frac{\rho_t^*(y)}{\rho_x^*(y)} \right]^{p'} \frac{dt}{|B_\epsilon|} \right)^{p/p'} \frac{dx}{|B_\epsilon|} \leq c_{p,n}, \quad (3.10)$$

or

$$\int_{B_\epsilon(z)} \left(\int_{B_\epsilon(z)} \sup_{y \neq 0} \left[\frac{\rho_x(y)}{\rho_t(y)} \right]^{p'} \frac{dt}{|B_\epsilon|} \right)^{p/p'} \frac{dx}{|B_\epsilon|} \leq c_{p,n}. \quad (3.11)$$

PROOF. We will show only (3.10), since (3.11) uses the same argument. The left-hand side of (3.10) is equal to

$$\int_{B_\epsilon(z)} \left(\int_{B_\epsilon(z)} \sup_{y \neq 0} \frac{\|W^{-1/p}(t)y\|^{p'}}{\|W^{-1/p}(x)y\|^{p'}} \frac{dt}{|B_\epsilon|} \right)^{p/p'} \frac{dx}{|B_\epsilon|}.$$

Let $u = W^{-1/p}(x)y$, then the last expression is

$$\begin{aligned} & \int_{B_\epsilon(z)} \left(\int_{B_\epsilon(z)} \sup_{u \neq 0} \frac{\|W^{-1/p}(t)W^{1/p}(x)u\|^{p'}}{\|u\|^{p'}} \frac{dt}{|B_\epsilon|} \right)^{p/p'} \frac{dx}{|B_\epsilon|} \\ &= \int_{B_\epsilon(z)} \left(\int_{B_\epsilon(z)} \|W^{-1/p}(t)W^{1/p}(x)\|^{p'} \frac{dt}{|B_\epsilon|} \right)^{p/p'} \frac{dx}{|B_\epsilon|}, \end{aligned}$$

which is (3.9), by (3.6). $\blacksquare$

Remark 3.9 Similarly, (3.7) can be written in terms of metrics ρ and ρ^* :

$$\int_{B_\delta(z)} \left(\int_{B_\delta(z)} \sup_{y \neq 0} \left[\frac{\rho_{(2^{-\nu}t)}^*(y)}{\rho_{(2^{-\nu}x)}^*(y)} \right]^{p'} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|} \leq c_{\delta,p,n} \quad \text{for any } \nu \in \mathbb{Z}, \quad (3.12)$$

or

$$\int_{B_\delta(z)} \left(\int_{B_\delta(z)} \sup_{y \neq 0} \left[\frac{\rho_{(2^{-\nu}x)}(y)}{\rho_{(2^{-\nu}t)}(y)} \right]^{p'} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|} \leq c_{\delta,p,n} \quad \text{for any } \nu \in \mathbb{Z}. \quad (3.13)$$

PROOF. We will show only (3.12), since (3.13) uses the same argument. The condition (3.12) is equal to

$$\int_{B_\delta(z)} \left(\int_{B_\delta(z)} \sup_{y \neq 0} \frac{\|W_\nu^{-1/p}(t)y\|^{p'}}{\|W_\nu^{-1/p}(x)y\|^{p'}} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|}.$$

Let $u = W_\nu^{-1/p}(x)y$, then the last expression is

$$\begin{aligned} & \int_{B_\delta(z)} \left(\int_{B_\delta(z)} \sup_{u \neq 0} \frac{\|W_\nu^{-1/p}(t)W_\nu^{1/p}(x)u\|^{p'}}{\|u\|^{p'}} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|} \\ &= \int_{B_\delta(z)} \left(\int_{B_\delta(z)} \|W_\nu^{-1/p}(t)W_\nu^{1/p}(x)\|^{p'} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|}, \end{aligned}$$

which is (3.7), by (3.6). $\blacksquare$

With the help of the Norm Lemma, we observe

$$\begin{aligned} \langle w \rangle_B &= \int_B \|W^{1/p}(t)\|^p \frac{dt}{|B|} \approx \int_B \max_{1 \leq i \leq m} \|W^{1/p}(t)e_i\|^p \frac{dt}{|B|} \\ &\approx \max_{1 \leq i \leq m} \int_B [\rho_i(e_i)]^p \frac{dt}{|B|} = \max_{1 \leq i \leq m} [\rho_{p,B}(e_i)]^p \approx \|\rho_{p,B}\|^p. \end{aligned}$$

The last equivalence can also be viewed in terms of reducing operators

$$\langle w^* \rangle_B = \int_B \|W^{1/p}(t)\|^p \frac{dt}{|B|} \approx \max_{1 \leq i \leq m} \|A_B e_i\|^p \approx \|A_B\|^p.$$

Similarly, the dual metrics

$$\int_B \|W^{-1/p}(t)\|^{p'} \frac{dt}{|B|} \approx \max_{1 \leq i \leq m} \|\rho_{p',B}^*(e_i)\|^{p'} \approx \|\rho_{p',B}^*\|^{p'} \approx \|A_B^*\|^{p'}.$$

3.5 δ -doubling measures

First, recall that a scalar measure μ is called *doubling*, if there exists $c > 0$ such that for any $\delta > 0$ and any $z \in \mathbb{R}^n$ the inequality (1.4) holds, i.e.,

$$\mu(B_{2\delta}(z)) \leq c \mu(B_\delta(z)).$$

If the above inequality holds only for a specific $\delta > 0$, then we say μ is δ -doubling.

Definition 3.10 Fix $y \in \mathcal{H}$. Then $w_y(t) = \|W^{1/p}(t)y\|^p$ is a scalar valued δ -doubling measure (of order p), $1 \leq p < \infty$, if there exists $\delta > 0$ and a constant $c = c_{\delta,p,n}$ such that for any $z \in \mathbb{R}^n$

$$w_y(B_{2\delta}(z)) \leq c w_y(B_\delta(z)). \quad (3.14)$$

Remark 3.11 Note that if $w_y(t)$ is δ -doubling of order p for any $y \in \mathcal{H}$, then $w(t) = \|W^{1/p}(t)\|^p$ is also a scalar-valued δ -doubling measure of order p .

PROOF. Since (3.14) is true for any e_i - an orthonormal basis vector of $\mathcal{H}$, we have

$$\sum_{i=1}^m \int_{B_{2\delta}} \|W^{1/p}(t)e_i\|^p dt \leq c \sum_{i=1}^m \int_{B_\delta} \|W^{1/p}(t)e_i\|^p dt.$$

By the Norm Lemma, this inequality is equivalent to

$$\int_{B_{2\delta}} \|W^{1/p}(t)\|^p dt \leq c \int_{B_\delta} \|W^{1/p}(t)\|^p dt.$$

■

The reverse of the previous remark is not always true.

3.6 δ -layers of the $\mathcal{B}_p$ class

Lemma 3.12 *Fix $\delta > 0$. Suppose that the condition (3.7) or, equivalently, (3.12) is true for $\nu = -1$. Then $w_y^*(t) = \|W^{-1/p}(t)y\|^{p'}$ is a δ -doubling measure for any $y \in \mathcal{H}$.*

PROOF. By Hölder's inequality

$$\begin{aligned} 2^{-n} &= \frac{|B_\delta|}{|B_{2\delta}|} = \int_{B_{2\delta}} \chi_{B_\delta}(t) \frac{dt}{|B_{2\delta}|} = \int_{B_{2\delta}} \chi_{B_\delta}(t) \frac{\rho_t^*(y)}{\rho_t^*(y)} \frac{dt}{|B_{2\delta}|} \\ &\leq \left(\int_{B_{2\delta}} \chi_{B_\delta}(t) [\rho_t^*(y)]^{p'} \frac{dt}{|B_{2\delta}|} \right)^{1/p'} \left(\int_{B_{2\delta}} \frac{1}{[\rho_x^*(y)]^p} \frac{dx}{|B_{2\delta}|} \right)^{1/p} \\ &= \left[\frac{w_y^*(B_\delta)}{|B_{2\delta}|} \right]^{1/p'} \left(\int_{B_{2\delta}} \frac{1}{[\rho_x^*(y)]^p} \frac{dx}{|B_{2\delta}|} \right)^{1/p} \\ &= \left[\frac{w_y^*(B_\delta)}{w_y^*(B_{2\delta})} \right]^{1/p'} \left(\int_{B_{2\delta}} [\rho_t^*(y)]^{p'} \frac{dt}{|B_{2\delta}|} \right)^{1/p'} \left(\int_{B_{2\delta}} \frac{1}{[\rho_x^*(y)]^p} \frac{dx}{|B_{2\delta}|} \right)^{1/p}. \end{aligned}$$

Raising to the p^{th} power both sides of the previous chain and specifying z as a center of both balls B_δ and $B_{2\delta}$, we get

$$\begin{aligned}
2^{-np} &= \left(\frac{|B_\delta|}{|B_{2\delta}|} \right)^p \leq \left[\frac{w_y^*(B_\delta(z))}{w_y^*(B_{2\delta}(z))} \right]^{p/p'} \cdot \int_{B_{2\delta}(z)} \left(\int_{B_{2\delta}(z)} \left[\frac{\rho_t^*(y)}{\rho_x^*(y)} \right]^{p'} \frac{dt}{|B_{2\delta}|} \right)^{p/p'} \frac{dx}{|B_{2\delta}|} \\
&= \left[\frac{w_y^*(B_\delta(z))}{w_y^*(B_{2\delta}(z))} \right]^{p/p'} \cdot \int_{B_\delta(z/2)} \left(\int_{B_\delta(z/2)} \left[\frac{\rho_{2t}^*(y)}{\rho_{2x}^*(y)} \right]^{p'} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|} \\
&\leq c_{\delta,p,n} \left[\frac{w_y^*(B_\delta(z))}{w_y^*(B_{2\delta}(z))} \right]^{p/p'},
\end{aligned}$$

by the $\mathcal{B}_p$ condition in terms of metrics (3.12) with $\nu = -1$.

Simplifying the last chain, we get

$$w_y^*(B_{2\delta}(z)) \leq (2^{np'} \cdot c_{\delta,p,n}^{p'/p}) w_y^*(B_\delta(z)), \quad (3.15)$$

i.e., w_y^* is a δ -doubling measure. $\blacksquare$

Remark 3.13 *Repeating the same argument, it can be shown that w_y is also a δ -doubling measure.*

PROOF. The proof is similar to the previous one, though the splitting of the initial equality is slightly tricky. So, by Hölder's inequality

$$\begin{aligned}
2^{-np} &= \left(\frac{|B_\delta|}{|B_{2\delta}|} \right)^p = \left(\int_{B_{2\delta}(z)} \chi_{B_\delta(z)}(t) \frac{dt}{|B_{2\delta}|} \right)^p = \left(\int_{B_{2\delta}(z)} \chi_{B_\delta(z)}(t) \frac{\rho_t(y)}{\rho_t(y)} \frac{dt}{|B_{2\delta}|} \right)^p \\
&\leq \left(\int_{B_{2\delta}(z)} \chi_{B_\delta(z)}(t) [\rho_x(y)]^p \frac{dx}{|B_{2\delta}|} \right) \left(\int_{B_{2\delta}(z)} \frac{1}{[\rho_t(y)]^{p'}} \frac{dt}{|B_{2\delta}|} \right)^{p/p'} \\
&= \left[\frac{w_y(B_\delta(z))}{|B_{2\delta}|} \right] \left(\int_{B_{2\delta}(z)} \frac{1}{[\rho_t(y)]^{p'}} \frac{dt}{|B_{2\delta}|} \right)^{p/p'} \\
&= \left[\frac{w_y(B_\delta(z))}{w_y(B_{2\delta}(z))} \right] \left(\int_{B_{2\delta}(z)} [\rho_x(y)]^p \frac{dx}{|B_{2\delta}|} \right) \left(\int_{B_{2\delta}(z)} \frac{1}{[\rho_t(y)]^{p'}} \frac{dt}{|B_{2\delta}|} \right)^{p/p'}
\end{aligned}$$

$$= \left[\frac{w_y(B_\delta(z))}{w_y(B_{2\delta}(z))} \right] \int_{B_\delta(z/2)} \left(\int_{B_\delta(z/2)} \left[\frac{\rho_{2x}(y)}{\rho_{2t}(y)} \right]^{p'} \frac{dt}{|B_\delta|} \right)^{p/p'} \frac{dx}{|B_\delta|} \leq c_{\delta,p,n} \left[\frac{w_y(B_\delta(z))}{w_y(B_{2\delta}(z))} \right],$$

by the $\mathcal{B}_p$ condition in terms of metrics (3.13) with $\nu = -1$.

Simplifying, we get

$$w_y(B_{2\delta}(z)) \leq (2^{np} \cdot c_{\delta,p,n}) w_y(B_\delta(z)), \quad (3.16)$$

i.e., w_y is a δ -doubling measure. ■

Generalizing the previous lemmas, we get

Corollary 3.14 *Fix $\delta > 0$. Then $w_{\nu,y}^*(t) := \|W_\nu^{-1/p}(t)y\|^{p'}$ and $w_{\nu,y}(t) := \|W_\nu^{1/p}(t)y\|^p$ are δ -doubling measures for any $y \in \mathcal{H}$, if the condition (3.7) or, equivalently, (3.12) holds for $\nu - 1$.*

PROOF. Let $V(t) = W_\nu(t)$, then $V_{-1}(t) = W_{\nu-1}(t)$, and so (3.7) holds for V with $\nu = -1$. Applying previous lemma (3.12) to $u_y^*(t) := \|V^{-1/p}(t)y\|^{p'}$, we get u_y^* is δ -doubling, or, $u_y^*(t) = \|V^{-1/p}(t)y\|^{p'} = \|W_\nu^{-1/p}(t)y\|^{p'} = w_{\nu,y}^*(t)$ is δ -doubling. Analogous proof applies to $w_{\nu,y}$. ■

So each “layer” of the $\mathcal{B}_p$ condition implies δ -doubling property of the scalar-valued measures generated by the matrix weight W . Anticipating further results, one can predict that the whole $\mathcal{B}_p$ class will imply a standard doubling property.

3.7 Doubling measures

Let W be a doubling matrix of order p , i.e., (1.5) holds for any $y \in \mathcal{H}$, $\delta > 0$ and $z \in \mathbb{R}^n$. For $p = 2$ this simplifies to

$$\int_{B_{2\delta}} W(t) dt \leq c \int_{B_\delta} W(t) dt \quad (3.17)$$

for a given δ , where the inequality is understood in the sense of selfadjoint operators.

Remark 3.15 *Note that $\|W^{1/p}(t)\|^p$ is independent of p . If $w_y(t) = \|W^{1/p}(t)y\|_{\mathcal{H}}^p$ is doubling of order p for any $y \in \mathcal{H}$, then $w(t) = \|W^{1/p}(t)\|^p$ is also a scalar-valued doubling measure.*

PROOF. Fix $t \in \mathbb{R}^n$. Then there exist a unitary matrix U and a diagonal matrix Λ such that $W(t) = U \Lambda U^{-1}$, and so $W^{1/p}(t) = U \Lambda^{1/p} U^{-1}$. Moreover, since the norm of a positive diagonal matrix is the largest eigenvalue, say λ_0 , $\|W^{1/p}(t)\| = \lambda_0^{1/p}$ and, hence, $\|W^{1/p}(t)\|^p = \lambda_0$, regardless of what p is.

Now, since (1.5) is true with $y = e_i$ - any orthonormal basis vector of $\mathcal{H}$, by the Norm Lemma we get the second assertion:

$$\begin{aligned} \int_{B_{2\delta}} \|W^{1/p}(t)\|^p dt &\approx \sum_{i=1}^m \int_{B_{2\delta}} \|W^{1/p}(t)e_i\|^p dt \\ &\leq c \sum_{i=1}^m \int_{B_\delta} \|W^{1/p}(t)e_i\|^p dt \approx c \int_{B_\delta} \|W^{1/p}(t)\|^p dt. \end{aligned}$$

■

The doubling property of $w(t) = \|W^{1/p}(t)\|^p$ is not very helpful if one wants to understand the nature of W ; it only tells us how large the weight is, not how it is distributed in different directions. Therefore, we use the definition of doubling matrix in (1.5), which involves different directions of $y \in \mathcal{H}$.

Remark 3.16 *In the scalar case, (1.5) gives the standard doubling measure:*

$$\int_{B_{2\delta}} w(t)|y|^p dt \leq c \int_{B_\delta} w(t)|y|^p dt,$$

and if $y \neq 0$, then $w(B_{2\delta}) \leq c w(B_\delta)$. In particular, there is no dependence on p in the scalar situation.

Similar definitions for doubling weights (of order p') can be analogously given for the “dual” measure $w_y^*(t) = \|W^{-1/p}(t)y\|^{p'}$.

Remark 3.17 *The doubling property (1.4) is equivalent to*

$$\frac{\mu(F)}{\mu(E)} \leq c \left(\frac{|F|}{|E|} \right)^{\beta/n}, \quad (3.18)$$

where F is a ball (or a cube) and $E \subseteq F$ is a sub-ball (or a sub-cube) (not any subset of F ; any subset would be equivalent to the A_∞ condition, see the end of Section 3.9, also [St2]).

PROOF. Since $E \subseteq F$, there exists $j \in \mathbb{N}$ such that $2^j E \approx F$, i.e., $l(F) \approx 2^j l(E)$. Since μ is doubling, by (1.4) we have $\frac{\mu(F)}{\mu(E)} \leq c^j \approx c^{\log_2 \frac{l(F)}{l(E)}}$. Noticing that $\frac{|F|}{|E|} = \left[\frac{l(F)}{l(E)} \right]^n$, we get (3.18). ■

In further estimates, it is more convenient to use (3.18) instead of (1.4).

Observe that the doubling exponent of the Lebesgue measure in $\mathbb{R}^n$ is $\beta = n$; moreover, if μ is any nonzero doubling measure in $\mathbb{R}^n$, then $\beta(\mu) \geq n$.

It is a trivial fact that if W is a doubling matrix weight (of order p), then a reducing operator sequence $\{A_Q\}_Q$, generated by W , is a doubling sequence (of order p). (Recall the definition 1.14.)

3.8 $\mathcal{B}_p$ implies the doubling property

Corollary 3.18 *Let $W \in \mathcal{B}_p$. Then $w_{\nu,y}^*(t) = \|W_\nu^{-1/p}(t)y\|^{p'}$ and $w_{\nu,y}(t) = \|W_\nu^{1/p}(t)y\|^p$ are doubling measures for any $y \in \mathcal{H}$ and any $\nu \in \mathbb{Z}$.*

PROOF. First, by the Lemma (3.14) $w_{\nu,y}^*(t)$ and $w_{\nu,y}(t)$ are δ -doubling for any ν . Second, if $W \in \mathcal{B}_p$, then W_ν satisfies (3.7) for all $\nu \in \mathbb{Z}$ and a given $0 < \delta < 1$. But we know that the $\mathcal{B}_p$ class is independent of the choice of δ , which means $w_{\nu,y}^*(t)$ and $w_{\nu,y}(t)$ are δ -doubling measures for any δ . Therefore, the corollary follows trivially. ■

Lemma 3.19 *Let $x \in \mathcal{H}$ and $W \in A_p$. Then $v_x(t) := \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} = \|W^{-1/p}(t)W^{1/p}(x)\|^{p'}$ is a doubling measure, i.e., there exists a constant c such that for any $\delta > 0$*

$$\int_{B_{2\delta}} \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} dt \leq c \int_{B_\delta} \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} dt. \quad (3.19)$$

PROOF. Applying the Norm Lemma to the operator norm in the left-hand side, we obtain

$$v_x(t) \approx \sum_{i=1}^m \|W^{-1/p}(t)W^{1/p}(x)e_i\|^{p'} = \sum_{i=1}^m \|W^{-1/p}(t)y_i(x)\|^{p'} = \sum_{i=1}^m w_{y_i(x)}^*(t),$$

where $y_i(x) = W^{1/p}(x)e_i$. Then

$$v_x(B_{2\delta}) \approx \sum_{i=1}^m \int_{B_{2\delta}} w_{y_i(x)}^*(t) dt \leq \sum_{i=1}^m c \int_{B_\delta} w_{y_i(x)}^*(t) dt \leq c v_x(B_\delta),$$

since w_y^* is doubling ($W^{-p'/p} \in A_{p'}$). ■

3.9 An alternative characterization of the matrix

A_p class

or WHAT IS THE $\mathcal{B}_p$ CONDITION INDEED?

Now we are ready to reveal what the class $\mathcal{B}_p$ really is, or, in other words, we give a proof of the equivalence of condition (3.9), or (1.2), to the A_p condition.

PROOF OF LEMMA 1.3. By property (3.6) and the Norm Lemma

$$\begin{aligned}
 & \int_B \left(\int_B \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \\
 &= \int_B \left(\int_B \|W^{-1/p}(t)W^{1/p}(x)\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \\
 &\approx \int_B \left(\int_B \sum_{i=1}^m \|W^{-1/p}(t)W^{1/p}(x)e_i\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \\
 &\approx \sum_{i=1}^m \int_B \left(\int_B [\rho_i^*(W^{1/p}(x)e_i)]^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} = \sum_{i=1}^m \int_B [\rho_{p',B}^*(W^{1/p}(x)e_i)]^p \frac{dx}{|B|}.
 \end{aligned}$$

Now, in terms of the reducing operators, the last expression is equivalent to

$$\begin{aligned}
 & \sum_{i=1}^m \int_B \|A_B^\#(W^{1/p}(x)e_i)\|^p \frac{dx}{|B|} \approx \int_B \|A_B^\# W^{1/p}(x)\|^p \frac{dx}{|B|} \\
 &\approx \sum_{i=1}^m \int_B \|W^{1/p}(x)(A_B^\# e_i)\|^p \frac{dx}{|B|} \approx \sum_{i=1}^m [\rho_{p,B}(A_B^\# e_i)]^p \\
 &\approx \sum_{i=1}^m \|A_B(A_B^\# e_i)\|^p \approx \|A_B A_B^\#\|^p.
 \end{aligned}$$

Therefore, (1.2) is equivalent to $\|A_B^\# A_B\|^p \leq c$, i.e., the A_p condition. $\blacksquare$

Thus, the $\mathcal{B}_p$ class is nothing else but the matrix A_p class. Therefore, we will not use the notation $\mathcal{B}_p$ anymore, though it was useful to understand what layers

this class consists of (as well as A_p) and that each layer implies a certain doubling property.

Remark 3.20 *Rephrasing Corollary 3.18, we obtain that A_p implies doubling.*

Moreover, if $W \in A_p$, then by (3.16) W is the doubling matrix weight of order p and the doubling exponent $\beta \leq np + \log_2 c_{p,n} = p \left(n + \frac{\log_2 c_{p,n}}{p} \right)$, where $c_{p,n}$ is the constant in (1.2).

Also $W \in A_p$ implies that the “dual” weight $W^{-p'/p}$ is a doubling matrix of order p' with the doubling exponent $\beta' \leq p' \left(n + \frac{\log_2 c_{p,n}}{p} \right)$ by using (3.15), where again $c_{p,n}$ is the constant from (1.2).

Corollary 3.21 (SYMMETRY OF MATRIX A_p CONDITION) *The following statements are equivalent:*

- (i) $W \in A_p$;
- (ii) $W^{-p'/p} \in A_{p'}$;
- (iii) $\int_B \left(\int_B \|W^{1/p}(x)W^{-1/p}(t)\|^{p'} \frac{dt}{|B|} \right)^{p/p'} \frac{dx}{|B|} \leq c$ for every ball $B \subseteq \mathbb{R}^n$;
- (iv) $\int_B \left(\int_B \|W^{1/p}(x)W^{-1/p}(t)\|^p \frac{dx}{|B|} \right)^{p'/p} \frac{dt}{|B|} \leq c$ for every ball $B \subseteq \mathbb{R}^n$.

PROOF. Recall that $\rho \in A_p$ if and only if $\rho^* \in A_{p'}$. In terms of matrix weights, $W \in A_p$ if and only if $W^{-p'/p} \in A_{p'}$ (note that $\rho_t^*(x) = \|(W^{-p'/p})^{1/p'}(t)x\|$). By Lemma 1.3, the third statement is equivalent to $W \in A_p$, whereas the fourth is equivalent to $W^{-p'/p} \in A_{p'}$. ■

Observe that the scalar classes A_p are increasing in p , i.e., $A_p \subseteq A_q$ if $p \leq q$.

This observation brings us to the definition of the scalar A_∞ class.

Definition 3.22 (*Scalar A_∞ class*) Let $w \geq 0$. Then $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

Equivalently, $w \in A_\infty$ if there exists $\delta > 0$ such that given a cube (or a ball) F and any subset $E \subseteq F$,

$$\frac{w(F)}{w(E)} \leq c \left(\frac{|F|}{|E|} \right)^\delta.$$

(See [St2] for equivalence and other details.)

This property of scalar weights will be used in Chapter 9.

CHAPTER 4

Boundedness of the φ -transform and Its Inverse on Matrix-Weighted Besov Spaces

4.1 Boundedness of the inverse φ -transform

Consider $\dot{B}_p^{\alpha q}(W)$ with parameters $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ fixed. For $0 < \delta \leq 1$, $M > 0$ and $N \in \mathbb{Z}$ define (as in [FJ2]) m_Q to be a *smooth* (δ, M, N) -*molecule* for $Q \in \mathcal{D}$ if:

$$(M1) \quad \int x^\gamma m_Q(x) dx = 0, \quad \text{for } |\gamma| \leq N,$$

$$(M2) \quad |m_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)} \right)^{-\max(M, M-\alpha)},$$

$$(M3) \quad |D^\gamma m_Q(x)| \leq |Q|^{-1/2-|\gamma|/n} \left(1 + \frac{|x - x_Q|}{l(Q)} \right)^{-M} \quad \text{if } |\gamma| \leq [\alpha],$$

$$(M4) \quad |D^\gamma m_Q(x) - D^\gamma m_Q(y)| \leq |Q|^{-\frac{1}{2}-\frac{|\gamma|}{n}-\frac{\delta}{n}} |x - y|^\delta$$

$$\times \sup_{|z| \leq |x-y|} \left(1 + \frac{|x-z-x_Q|}{l(Q)} \right)^{-M} \text{ if } |\gamma| = [\alpha].$$

It is understood that (M1) is void if $N < 0$; and (M3), (M4) are void if $\alpha < 0$.

Also, $[\alpha]$ stands for the greatest integer $\leq \alpha$; γ is a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_i \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq n$, and the standard notation is used.

We say $\{m_Q\}_Q$ is a *family of smooth molecules* for $\dot{B}_p^{\alpha q}(W)$ if each m_Q is a (δ, M, N) -molecule with

$$(M.i) \quad \alpha - [\alpha] < \delta \leq 1,$$

$$(M.ii) \quad M > J, \text{ where } J = \frac{\beta}{p} + \frac{n}{p'} \text{ (if } p = 1, \text{ then } n/p' := 0 \text{ and } J = \beta),$$

$$(M.iii) \quad N = \max([J - n - \alpha], -1).$$

Remark 4.1 *Note that, in contrast to the case in [FJ2], there is a dependence of the family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$ on the weight W (more precisely, on the doubling exponent β).*

Theorem 4.2 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$, and let W be a doubling matrix weight of order p . Suppose $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$.*

Then

$$\left\| \sum_Q \bar{s}_Q m_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \leq c \|\{\bar{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \quad (4.1)$$

The proof uses the following estimates for Q dyadic with $l(Q) = 2^{-\mu}$, $\mu \in \mathbb{Z}$, and

φ_ν , $\nu \in \mathbb{Z}$, with $\varphi \in \mathcal{A}$:

if $\mu > \nu$, then for some $\sigma > J - \alpha$

$$|\varphi_\nu * m_Q(x)| \leq c |Q|^{-1/2} 2^{-(\mu-\nu)\sigma} (1 + 2^\nu |x - x_Q|)^{-M}; \quad (4.2)$$

if $\mu \leq \nu$, then for some $\tau > \alpha$

$$|\varphi_\nu * m_Q(x)| \leq c |Q|^{-1/2} 2^{-(\nu-\mu)\tau} (1 + 2^\mu |x - x_Q|)^{-M}. \quad (4.3)$$

The proofs are entirely elementary, but quite tedious (see [FJ2, Appendix B]). Note that in the statement of Lemma B.1 in [FJ2], it should say $j \leq k$. For (4.2), for $N \neq -1$, apply Lemma B.1 with $j = \nu$, $k = \mu$, $L = N$, $R = M$, $S = M - \alpha$, $g = 2^{-\nu n/2} \varphi_\nu$, $h = m_Q$ with $l(Q) = 2^{-\mu}$, $x_1 = x_Q$, $J - n - \alpha - [J - n - \alpha] < \theta \leq 1$. Letting $\sigma = N + n + \theta > J - \alpha$, we obtain (4.2). For $N = -1$, apply Lemma B.2 in [FJ2] with $\sigma = n > J - \alpha$ to get (4.2). Now for (4.3), for $\alpha > 0$, apply Lemma B.1 with $k = \nu$, $j = \mu$, $L = [\alpha]$, $R = M$, $\delta = \theta$, $S = [\alpha] + n + \delta$, $x_1 = 0$, $g(x) = m_Q(x + x_Q)$, $h = 2^{-\nu n/2} \varphi_\nu$, and observe that $\varphi_\nu * m_Q(x) = 2^{\nu n/2} g * h(x - x_Q)$ to get (4.3) with $\tau = \delta + [\alpha] > \alpha$. For $\alpha < 0$, Lemma B.2 in [FJ2] gives (4.3) with $\tau = 0 > \alpha$.

Lemma 4.3 (SQUEEZE LEMMA) *Fix a dyadic cube Q and let $w : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be a scalar doubling measure with the doubling exponent β . If $L > \beta$, then for $r \geq l(Q)$,*

$$\int_{\mathbb{R}^n} w(x) \left(1 + \frac{|x - x_Q|}{r}\right)^{-L} dx \leq c_\beta \left[\frac{r}{l(Q)}\right]^\beta \int_Q w(x) dx. \quad (4.4)$$

PROOF. Decompose $\mathbb{R}^n$ into the annuli $\mathcal{R}_m$:

$$\mathbb{R}^n = \bigcup_{m=1}^{\infty} \{x : 2^{m-1}r \leq |x - x_Q| < 2^m r\} \cup \{x : |x - x_Q| < r\} =: \bigcup_{m=0}^{\infty} \mathcal{R}_m.$$

Then the left-hand side of (4.4) is bounded by

$$\sum_{m=1}^{\infty} (1 + 2^{m-1})^{-L} w(\mathcal{R}_m) + w(\mathcal{R}_0). \quad (4.5)$$

Using the doubling property of w , we get

$$w(\mathcal{R}_m) \leq w(B(x_Q, 2^m r)) \leq c \left(\frac{|B(x_Q, 2^m r)|}{|\mathcal{R}_0|} \right)^{\beta/n} w(\mathcal{R}_0) = c 2^{m\beta} w(\mathcal{R}_0).$$

Thus, (4.5) is bounded by

$$c \sum_{m=0}^{\infty} 2^{m\beta - mL} w(\mathcal{R}_0) \leq c_{\beta} w(\mathcal{R}_0),$$

since $L > \beta$. Note that $B(x_Q, l(Q)) \subseteq 3Q$ and so $w(B(x_Q, l(Q))) \leq c_{\beta} w(Q)$. If $r > l(Q)$, then

$$w(\mathcal{R}_0) \leq c \left(\frac{|\mathcal{R}_0|}{|B(x_Q, l(Q))|} \right)^{\beta/n} w(B(x_Q, l(Q))) \leq c_{\beta} \left[\frac{r}{l(Q)} \right]^{\beta} w(Q),$$

which is (4.4). $\blacksquare$

Lemma 4.4 (SUMMATION LEMMA) *Let $\mu, \nu \in \mathbb{Z}$ and $y \in \mathbb{R}^n$. Then for $M > n$,*

$$\sum_{l(Q)=2^{-\mu}} \left(1 + \frac{|y - x_Q|}{2^{-\nu}} \right)^{-M} \leq c_{n,M} 2^{(\mu-\nu)n}, \quad \text{if } \mu \geq \nu. \quad (4.6)$$

PROOF. If $\mu \geq \nu$, i.e., $2^{-\nu} \geq 2^{-\mu}$, there are $2^{(\mu-\nu)n}$ dyadic cubes of size $2^{-\mu}$ in a dyadic cube of size $2^{-\nu}$. Fix $l \in \mathbb{Z}^n$ such that $y \in Q_{\nu l}$. Then the left-hand side of (4.6) is

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^n} (1 + 2^{\nu} |y - x_{Q_{\mu k}}|)^{-M} \\ &= \sum_{i \in \mathbb{Z}^n} \sum_{k: Q_{\mu k} \subseteq Q_{\nu(l+i)}} (1 + 2^{\nu} |y - x_{Q_{\mu k}}|)^{-M} \\ &\leq \sum_{i \in \mathbb{Z}^n} (1 + |i|)^{-M} \times 2^{(\mu-\nu)n} \leq c_n 2^{(\mu-\nu)n}, \end{aligned}$$

again since $M > n$. $\blacksquare$

PROOF OF THEOREM 4.2. By definition,

$$\begin{aligned} \left\| \sum_Q \vec{s}_Q m_Q \right\|_{\dot{B}_p^{\alpha q}(W)} &= \left\| \left\{ \left\| W^{1/p} \sum_Q \vec{s}_Q (\varphi_\nu * m_Q) \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha} \\ &= \left\| \left\{ \left\| \sum_{\mu \in \mathbb{Z}} \left[\sum_{l(Q)=2^{-\mu}} (W^{1/p} \vec{s}_Q)(\varphi_\nu * m_Q) \right] \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha}. \end{aligned}$$

By Minkowski's (or the triangle) inequality, the last expression is bounded by

$$\begin{aligned} &\left\| \left\{ \sum_{\mu \in \mathbb{Z}} \left\| \sum_{l(Q)=2^{-\mu}} (W^{1/p} \vec{s}_Q)(\varphi_\nu * m_Q) \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha} \\ &\leq \left\| \left\{ \sum_{\mu \in \mathbb{Z}} \left(\int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^{-\mu}} \|W^{1/p}(x) \vec{s}_Q\| |(\varphi_\nu * m_Q)(x)| \right)^p dx \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha} \\ &=: \left\| \left\{ \sum_{\mu > \nu} J_1^{1/p} + \sum_{\mu \leq \nu} J_2^{1/p} \right\}_\nu \right\|_{l_q^\alpha}. \end{aligned} \tag{4.7}$$

Using estimates (4.2) and (4.3) with $\theta_1 = -(\mu - \nu)\sigma$, $\theta_2 = -(\nu - \mu)\tau$ and $r_1 = 2^{-\nu}$,

$r_2 = 2^{-\mu}$, we bound each J_i , $i = 1, 2$:

$$J_i \leq c \int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^{-\mu}} \|W^{1/p}(x) \vec{s}_Q\|_{\mathcal{H}} |Q|^{-1/2} 2^{\theta_i} \left(1 + \frac{|x - x_Q|}{r_i} \right)^{-M} \right)^p dx.$$

If $p > 1$, split $M = M_1 + M_2$, where $M_1 > \beta/p$ and $M_2 > n/p'$ (this is possible since $M > J$). If $p = 1$, $M = M_1 > \beta$ (and $n/p' = 0$ in further calculations). Then by

the discrete Hölder inequality with $w_Q(x) = \|W^{1/p}(x) \vec{s}_Q\|_{\mathcal{H}}^p$, we get

$$\begin{aligned} J_i &\leq c_p \int_{\mathbb{R}^n} \left(\sum_{l(Q)=2^{-\mu}} w_Q(x) |Q|^{-p/2} 2^{\theta_i p} \left(1 + \frac{|x - x_Q|}{r_i} \right)^{-M_1 p} \right) \\ &\quad \times \left[\sum_{l(Q)=2^{-\mu}} \left(1 + \frac{|x - x_Q|}{r_i} \right)^{-M_2 p'} \right]^{p/p'} dx. \end{aligned}$$

By the Summation Lemma 4.4 (with $\nu = \mu$ in (4.6)), we have

$$J_2 \leq c_{p,n} 2^{\theta_2 p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} \int_{\mathbb{R}^n} w_Q(x) (1 + 2^\mu |x - x_Q|)^{-M_1 p} dx,$$

since $M_2 > n/p'$. Applying the Squeeze Lemma 4.3 with $r = 2^{-\mu} = l(Q)$ and $L = M_1 p$ (and so $L > \beta$), we get

$$J_2 \leq c_{p,n,\beta} 2^{-(\nu-\mu)\tau p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} w_Q(Q).$$

By the Summation Lemma 4.4 (with $\mu > \nu$ in (4.6)), we have

$$J_1 \leq c_{p,n} 2^{(\nu-\mu)(\sigma-n/p')p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} \int_{\mathbb{R}^n} w_Q(x) (1 + 2^\nu |x - x_Q|)^{-M_1 p} dx,$$

again since $M_2 > n/p'$. Applying the Squeeze Lemma 4.3 again with $r = 2^{-\nu} > 2^{-\mu} = l(Q)$ and $L = M_1 p$, we get

$$J_1 \leq c_{p,n,\beta} 2^{(\nu-\mu)(\sigma-n/p'-\beta/p)p} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} w_Q(Q).$$

Observe that the last sum is equal to $\left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(W)}^p$. Combining the estimates for J_1 and J_2 (recall that $J = \frac{n}{p'} + \frac{\beta}{p}$), we have

$$\begin{aligned} 2^{\nu\alpha} \left(\sum_{\mu > \nu} J_1^{1/p} + \sum_{\mu \leq \nu} J_2^{1/p} \right) &\leq c_{p,n,\beta} \sum_{\mu \in \mathbb{Z}} 2^{(\nu-\mu)\alpha} \left(2^{(\nu-\mu)(\sigma-J)} \chi_{\{\nu-\mu < 0\}} \right. \\ &\quad \left. + 2^{-(\nu-\mu)\tau} \chi_{\{\nu-\mu \geq 0\}} \right) \times 2^{\mu\alpha} \left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(W)}. \end{aligned} \quad (4.8)$$

Denote

$$a_i = 2^{i\alpha} \left(2^{i(\sigma-J)} \chi_{\{i < 0\}} + 2^{-i\tau} \chi_{\{i \geq 0\}} \right)$$

and

$$b_\mu = 2^{\mu\alpha} \left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(W)}.$$

Then the right side of (4.8) is nothing else but $c \sum_{\mu \in \mathbb{Z}} a_{\nu-\mu} b_{\mu} = c(a*b)(\nu)$. Substituting this into (4.7), we get

$$\left\| \sum_Q \vec{s}_Q m_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \leq \left\| \left\{ \sum_{\mu \in \mathbb{Z}} \sum_{i=1,2} J_i^{1/p} \right\}_{\nu} \right\|_{l_q^{\alpha}} \leq c_{p,n,\beta} \|a * b\|_{l_q}. \quad (4.9)$$

Observe that

$$\|a * b\|_{l_q} \leq \|a\|_{l^1} \|b\|_{l_q} \quad \text{for } q \geq 1 \quad (4.10)$$

and

$$\|a * b\|_{l_q} \leq \|a\|_{l_q} \|b\|_{l_q} \quad \text{for } q < 1 \quad (4.11)$$

(to get the last inequality, apply the q -triangle inequality followed by $\|a * b\|_{l^1} \leq \|a\|_{l^1} \|b\|_{l^1}$). For any $0 < q < \infty$, $\|a\|_{l_q}^q = \sum_{i < 0} 2^{i(\sigma+\alpha-J)q} + \sum_{i \geq 0} 2^{-i(\tau-\alpha)q}$. Both sums converge, since $\tau > \alpha$ and $\sigma + \alpha > J$ by (4.2) and (4.3). Hence, $\|a\|_{l_q} \leq c_q$ for any $q > 0$. (In fact, here we only need $0 < q \leq 1$.) Combining all the estimates together into (4.9), we obtain

$$\begin{aligned} \left\| \sum_Q \vec{s}_Q m_Q \right\|_{\dot{B}_p^{\alpha q}(W)} &\leq c \|b\|_{l_q} = c \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right\}_{\nu} \right\|_{l_q} \\ &= c \|\{\vec{s}_Q\}\|_{\dot{b}_p^{\alpha q}(W)}, \end{aligned}$$

where $c = c_{n,p,q,\beta}$. ■

Remark 4.5 Since $\psi \in \mathcal{A}$, observe the following properties of ψ_Q :

1. $0 \notin \text{supp } \hat{\psi}_Q$ for any dyadic Q , and, therefore, $\int x^\gamma \psi_Q(x) dx = 0$ for any multi-index γ ;

2. $|D^\gamma \psi_Q| \leq c_{\gamma,L} |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-L - |\gamma|}$ for each $L \in \mathbb{N} \cup \{0\}$ and γ as before.

Hence, $\{\psi_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, and for $\vec{f} = \sum_Q \vec{s}_Q \psi_Q$, we obtain the boundedness of the inverse φ -transform T_ψ :

Corollary 4.6 *Let W be a doubling matrix of order p , and consider the sequence $\vec{s} = \{\vec{s}_Q\}_Q \in \dot{b}_p^{\alpha q}(W)$. Then for all $1 \leq p < \infty$, $0 < q \leq \infty$ and $\alpha \in \mathbb{R}$,*

$$\|T_\psi \vec{s}\|_{\dot{B}_p^{\alpha q}(W)} = \left\| \sum_Q \vec{s}_Q \psi_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \leq c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \quad (4.12)$$

In particular, given $\vec{f} \in \dot{B}_p^{\alpha q}(W)$, consider $\vec{s} = S_\varphi \vec{f}$. Then by (2.2)

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} = \left\| \sum_Q \vec{s}_Q \psi_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \leq c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = c \|S_\varphi \vec{f}\|_{\dot{b}_p^{\alpha q}(W)}.$$

4.2 Decompositions of an exponential type function

Definition 4.7 *For $\nu \in \mathbb{Z}$, let $E_\nu = \{\vec{f} : f_i \in \mathcal{S}' \text{ and } \text{supp } \hat{f}_i \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{\nu+1}\}, i = 1, \dots, m\}$. Then we say that E_ν consists of vector functions of exponential type $2^{\nu+1}$.*

Consider the following lemma on the decomposition of an exponential type function (for the proof the reader is referred to [FJW, p.55]):

Lemma 4.8 Suppose $g \in \mathcal{S}'(\mathbb{R}^n)$, $h \in \mathcal{S}(\mathbb{R}^n)$ and $\text{supp } \hat{g}, \text{supp } \hat{h} \subseteq \{|\xi| < 2^\nu \pi\}$ for some $\nu \in \mathbb{Z}$. Then

$$(g * h)(x) = \sum_{k \in \mathbb{Z}^n} 2^{-\nu n} g(2^{-\nu} k) h(x - 2^{-\nu} k). \quad (4.13)$$

Note: if $\hat{h}(\xi) = 1$ on $\text{supp } \hat{g}$, then $g * h = g$.

Let $\Gamma = \{\gamma \in \mathcal{S} : \hat{\gamma} = 1 \text{ on } \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \text{ and } \text{supp } \hat{\gamma} \subseteq \{\xi \in \mathbb{R}^n : |\xi| < \pi\}\}$.

Define $\gamma_\nu(x) = 2^{\nu n} \gamma(2^\nu x)$ for $\nu \in \mathbb{Z}$. Since $\hat{\gamma}_\nu = \hat{\gamma}(2^\nu \xi)$, $\text{supp } \hat{\gamma}_\nu \subseteq \{\xi \in \mathbb{R}^n : |\xi| < 2^\nu \pi\}$.

Lemma 4.9 For $\nu \in \mathbb{Z}$ let $\tilde{g} \in E_\nu$ and fix $x \in Q_{\nu k}$ where $k \in \mathbb{Z}^n$. Then for any $y \in \mathbb{R}^n$ and $\gamma \in \Gamma$

$$\tilde{g}(y) = \sum_{l \in \mathbb{Z}^n} 2^{-\nu n} \tilde{g}(2^{-\nu} l + x) \gamma_\nu(y - (2^{-\nu} l + x)). \quad (4.14)$$

PROOF. Denote $\tilde{g}^x(y) = \tilde{g}(y + x)$. Trivially, $\tilde{g}(y) = \tilde{g}^x(y - x)$. Note that $(\tilde{g}^x)^\wedge(\xi) = e^{ix\xi} \hat{\tilde{g}}(\xi)$, and so $\text{supp } (\tilde{g}^x)^\wedge = \text{supp } \hat{\tilde{g}}$. Therefore, by (4.13) applied to $\tilde{g}^x$:

$$\tilde{g}(y) = \tilde{g}^x(y - x) = \sum_{l \in \mathbb{Z}^n} 2^{-\nu n} \tilde{g}^x(2^{-\nu} l) \gamma_\nu(y - x - 2^{-\nu} l),$$

which is (4.14). ■

Lemma 4.10 Let $g \in \mathcal{S}'$ with $\text{supp } \hat{g} \subseteq \{|\xi| \leq 3\}$ and $\gamma \in \Gamma$. Then for any $x, s \in \mathbb{R}^n$ and $j \in \mathbb{N} \cup \{0\}$

$$D_j g(x + s) = \sum_{k \in \mathbb{Z}^n} g(x + k) D_j \gamma(s - k). \quad (4.15)$$

PROOF. Let $g_x(s) = g(x + s)$. Observe that $\text{supp } \hat{g}_x \subseteq \{|\xi| \leq 3\}$, since $\hat{g}_x(\xi) = e^{ix\xi} \hat{g}(\xi)$ and $\text{supp } \hat{g} \subseteq \{|\xi| \leq 3\}$. Applying the decomposition from Lemma (4.8)

(with $\nu = 0$) $g(s) = \sum_{k \in \mathbb{Z}^n} g(k) \gamma(s - k)$ to $g_x(s)$, we get

$$g_x(s) = \sum_{k \in \mathbb{Z}^n} g_x(k) \gamma(s - k) = \sum_{k \in \mathbb{Z}^n} g(x + k) \gamma(s - k).$$

Note the following two implications:

1. $g_x = g_x * \gamma \implies D_j g_x = g_x * D_j \gamma$
2. $(D_j g_x)^\wedge = \hat{g}_x \cdot (D_j \gamma)^\wedge \implies \text{supp } (D_j g_x)^\wedge \subseteq \text{supp } \hat{g}_x \cap \text{supp } (D_j \gamma)^\wedge \subseteq \{|\xi| \leq 3\}$

Therefore,

$$D_j g(x + s) = D_j g_x(s) = \sum_{k \in \mathbb{Z}^n} g_x(k) D_j \gamma(s - k) = \sum_{k \in \mathbb{Z}^n} g(x + k) D_j \gamma(s - k).$$

■

Remark 4.11 Let $f \in \mathcal{S}'$. Recall the dilations of φ : $\varphi_\nu(x) = 2^{\nu n} \varphi(2^\nu x)$. Since $\text{supp } \hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$, $\text{supp } \hat{\varphi}_\nu \subseteq \{\xi \in \mathbb{R}^n : 2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\}$ as well as $\text{supp } (\varphi_\nu * f)^\wedge \subseteq \{\xi \in \mathbb{R}^n : 2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\}$. Observe that $(\varphi_\nu * f) \in \mathcal{S}'$ and so $(\varphi_\nu * f) \in E_\nu$. Thus, all previous lemmas apply to $\varphi_\nu * f$.

4.3 Boundedness of the φ -transform

Before we talk about the boundedness of the φ -transform, we develop two “maximal operator” type inequalities:

Lemma 4.12 *Let $1 < p < \infty$, $W \in A_p$ and $\vec{g} \in E_0$. Then*

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x) \vec{g}(k)\|^p dx \leq c_{p,n} \|\vec{g}\|_{L^p(W)}^p. \quad (4.16)$$

Remark 4.13 *Note that in terms of reducing operators, (4.16) is equivalent to*

$$\|\{A_{Q_{0k}} \vec{g}(k)\}_{k \in \mathbb{Z}^n}\|_{l^p} = \left(\sum_{k \in \mathbb{Z}^n} \|A_{Q_{0k}} \vec{g}(k)\|^p \right)^{1/p} \leq c_{p,n} \|\vec{g}\|_{L^p(W)}. \quad (4.17)$$

PROOF. Let $\gamma \in \Gamma$. Then for $\vec{g} \in E_0$, we have $\vec{g} = \gamma * \vec{g}$, and the left-hand side of (4.16) is

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \left\| W^{1/p}(x) \int_{\mathbb{R}^n} \vec{g}(y) \gamma(k-y) dy \right\|^p dx \\ & \leq c_M \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \left(\int_{\mathbb{R}^n} \frac{\|W^{1/p}(x) \vec{g}(y)\|}{(1+|k-y|)^M} dy \right)^p dx, \end{aligned}$$

for some $M > n + \beta p/p'$, where β is the doubling exponent of W , since $\gamma \in \mathcal{S}$. Since $\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}^n} Q_{0m}$ and $m_i \leq y_i < m_i + 1$, $i = 1, \dots, n$, on each Q_{0m} , the last sum is bounded by

$$c \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \left(\sum_{m \in \mathbb{Z}^n} \frac{\int_{Q_{0m}} \|W^{1/p}(x) \vec{g}(y)\| dy}{(1+|k-m|)^M} \right)^p dx.$$

Writing $M = M/p + M/p'$ and using the discrete Hölder inequality (note that $M > n$), we bound the last expression by

$$c \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \sum_{m \in \mathbb{Z}^n} \frac{\left(\int_{Q_{0m}} \|W^{1/p}(x) \vec{g}(y)\| dy \right)^p}{(1+|k-m|)^M} dx. \quad (4.18)$$

Observe that

$$\begin{aligned} & \left(\int_{Q_{0m}} \|W^{1/p}(x) \vec{g}(y)\| dy \right)^p \leq \left(\int_{Q_{0m}} \|W^{1/p}(x) W^{-1/p}(y)\| \|W^{1/p}(y) \vec{g}(y)\| dy \right)^p \\ & \leq \left(\int_{Q_{0m}} \|W^{1/p}(x) W^{-1/p}(y)\|^{p'} dy \right)^{p/p'} \left(\int_{Q_{0m}} \|W^{1/p}(y) \vec{g}(y)\|^p dy \right), \end{aligned}$$

again by Hölder's inequality. By Lemma 3.19, $v_x(y) = \|W^{1/p}(x)W^{-1/p}(y)\|^{p'}$ is a doubling measure with the doubling exponent β :

$$v_x(Q_{0m}) \leq v_x(B(m, |k - m| + \sqrt{n})) \leq c(1 + |k - m|)^\beta v_x(Q_{0k}).$$

Thus, (4.18) is bounded by

$$c \sum_{k, m \in \mathbb{Z}^n} (1 + |k - m|)^{\beta p/p' - M} \left[\int_{Q_{0k}} \left(\int_{Q_{0k}} \|W^{1/p}(x)W^{-1/p}(y)\|^{p'} dy \right)^{p/p'} dx \right] \quad (4.19)$$

$$\times \int_{Q_{0m}} \|W^{1/p}(y)\tilde{g}(y)\|^p dy.$$

By Lemma 1.3, the expression in the square brackets of (4.19) is bounded by a constant independent of k . Since $M > \beta p/p' + n$, the sum on k converges and, therefore, (4.19) is estimated above by

$$c \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}} \|W^{1/p}(y)\tilde{g}(y)\|^p dy = c \int_{\mathbb{R}^n} \|W^{1/p}(y)\tilde{g}(y)\|^p dy = c \|\tilde{g}\|_{L^p(W)}^p.$$

■

Lemma 4.14 *Let W be a doubling matrix of order p , $1 \leq p < \infty$, with doubling exponent β such that $p > \beta$, and let $\tilde{g} \in E_0$. Then (4.16) holds. Furthermore, if W is a diagonal matrix, then (4.16) holds for any $1 \leq p < \infty$.*

PROOF. First, assume $(\tilde{g})_i \in \mathcal{S}$ with $\text{supp } (\tilde{g})_i^\wedge \subseteq \{|\xi| < \pi\}$, $i = 1, \dots, m$.

We want to show that for such $\tilde{g}$, the sum on the left-hand side of (4.16) is finite.

Choosing $r > \beta + n$, we have

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x)\tilde{g}(k)\|^p dx \leq \sum_{k \in \mathbb{Z}^n} \frac{c}{(1 + |k|)^r} \int_{Q_{0k}} \|W^{1/p}(x)\|^p dx.$$

Since $w(x) = \|W^{1/p}(x)\|^p$ is a scalar doubling measure, $w(Q_{0k}) \leq c(1 + |k|)^\beta w(Q_{00})$.

Hence,

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x) \tilde{g}(k)\|^p dx \leq \sum_{k \in \mathbb{Z}^n} \frac{c w(Q_{00})}{(1 + |k|)^{r-\beta}} \leq c w(Q_{00}) < \infty,$$

since $r - \beta > n$.

Now we will prove (4.16) for $\tilde{g}$ with $(\tilde{g})_i \in S$ and $\text{supp } (\tilde{g})_i^\wedge \subseteq \{|\xi| \leq 3\}$, and then generalize it to $(\tilde{g})_i \in S'$. Let $0 < \delta < 1$. Then $B_\delta(k) \subseteq 3Q_{0k}$. Using the doubling property of $w_k(x) = \|W^{1/p}(x) \tilde{g}(k)\|^p$, we “squeeze” each Q_{0k} into $B_\delta(k)$:

$$w_k(Q_{0k}) \leq w_k(3Q_{0k}) \leq c \left[\frac{|3Q_{0k}|}{|B_\delta(k)|} \right]^{\beta/n} w_k(B_\delta(k)) \leq c_\beta \delta^{-\beta} w_k(B_\delta(k)).$$

Hence, the left-hand side of (4.16) is bounded by

$$c_\beta \delta^{-\beta} \sum_{k \in \mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(k)\|^p dx. \quad (4.20)$$

To estimate the integral, we will use the trivial identity $\tilde{g}(k) = \tilde{g}(x) + [\tilde{g}(k) - \tilde{g}(x)]$

for $x \in B_\delta(k)$. Apply the decomposition from Lemma 4.8 with $\gamma \in \Gamma$:

$$\tilde{g}(k) = \sum_{m \in \mathbb{Z}^n} \tilde{g}(m) \gamma(k - m) \quad \text{and} \quad \tilde{g}(x) = \sum_{m \in \mathbb{Z}^n} \tilde{g}(m) \gamma(x - m).$$

Using the Mean Value Theorem for $[\gamma(k - m) - \gamma(x - m)]$ and the properties of $\gamma \in \mathcal{S}$

(note that $|x - k| < \delta$), we have

$$\|W^{1/p}(x) \tilde{g}(k)\|^p \leq c_p \|W^{1/p}(x) \tilde{g}(x)\|^p + c_{p,M} \delta^p \sum_{m \in \mathbb{Z}^n} \frac{\|W^{1/p}(x) \tilde{g}(m)\|^p}{(1 + |k - m|)^M}, \quad (4.21)$$

for some $M > \beta + n$. Integrating (4.21) over $B_\delta(k)$, we get

$$\int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(k)\|^p dx \leq c_p \int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(x)\|^p dx$$

$$+c\delta^p \sum_{m \in \mathbb{Z}^n} \frac{\int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(m)\|^p dx}{(1+|k-m|)^M}. \quad (4.22)$$

Apply the doubling property of $w_m(x) = \|W^{1/p}(x) \tilde{g}(m)\|^p$ again:

$$\begin{aligned} w_m(B_\delta(k)) &\leq w_m(B(m, |k-m| + \delta)) \leq c \left[\frac{(\delta + |k-m|)^n}{\delta^n} \right]^{\beta/n} w_m(B_\delta(m)) \\ &= c \delta^{-\beta} (1 + |k-m|)^\beta w_m(B_\delta(m)). \end{aligned}$$

Substituting this estimate into (4.22) and summing over $k \in \mathbb{Z}^n$, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(k)\|^p dx &\leq c_p \sum_{k \in \mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(x)\|^p dx \\ &+ c \delta^{p-\beta} \sum_{m \in \mathbb{Z}^n} \int_{B_\delta(m)} \|W^{1/p}(x) \tilde{g}(m)\|^p dx \left(\sum_{k \in \mathbb{Z}^n} (1 + |k-m|)^{\beta-M} \right), \end{aligned}$$

where the last sum converges since $M > \beta + n$. If $p > \beta$, by choosing $0 < \delta < 1/2$ such that $1 - c \delta^{p-\beta} > 0$, we subtract the last term from both sides (note that it is finite because of our estimates above for $\tilde{g}_i \in \mathcal{S}$), substitute it into (4.20) and get the estimate of the left-hand side of (4.16) (note that $\sum_{k \in \mathbb{Z}^n} \int_{B_\delta(k)} \dots \leq \int_{\mathbb{R}^n} \dots$):

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x) \tilde{g}(k)\|^p dx &\leq \frac{c_\beta \delta^{-\beta} c_p}{(1 - c \delta^{p-\beta})} \sum_{k \in \mathbb{Z}^n} \int_{B_\delta(k)} \|W^{1/p}(x) \tilde{g}(x)\|^p dx \\ &\leq c_{n,\beta,p} \int_{\mathbb{R}^n} \|W^{1/p}(x) \tilde{g}(x)\|^p dx = c_{n,\beta,p} \|\tilde{g}\|_{L^p(W)}^p. \end{aligned} \quad (4.23)$$

Now let $(\tilde{g})_i \in S', i = 1, \dots, m$. Since $\tilde{g} \in E_0$, it follows that $(\tilde{g})_i \in C^\infty$, and $\tilde{g}$ and all its derivatives are slowly increasing. Pick a scalar-valued $\gamma \in \mathcal{S}$ such that $\gamma(0) = 1$ and $\text{supp } \hat{\gamma} \subseteq B(0, 1)$. Then for $0 < \epsilon < 1$, the function $\tilde{g}^\epsilon(x) := \tilde{g}(x) \gamma(\epsilon x)$ has its components in $\mathcal{S}$. Observe that $(\tilde{g}^\epsilon)^\wedge = (\tilde{g})^\wedge * [\gamma(\epsilon x)]^\wedge$, with $[\gamma(\epsilon x)]^\wedge(\xi) = (1/\epsilon) \hat{\gamma}(\xi/\epsilon)$, and, therefore,

$$\text{supp } (\tilde{g}^\epsilon)^\wedge \subseteq \text{supp } (\tilde{g})^\wedge + \text{supp } (1/\epsilon) \hat{\gamma}(\cdot/\epsilon) \subseteq \{\xi : |\xi| < 3\}.$$

We can apply the result (4.23) to $\tilde{g}^\epsilon$:

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x) \tilde{g}^\epsilon(k)\|^p dx \leq c \|\tilde{g}^\epsilon\|_{L^p(W)}^p,$$

or

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x) \tilde{g}(k) \gamma(\epsilon k)\|^p dx \leq c \int_{\mathbb{R}^n} \|W^{1/p}(x) \tilde{g}(x)\|^p |\gamma(\epsilon x)|^p dx.$$

Taking $\liminf$ as $\epsilon \rightarrow 0$ of both sides and using Fatou's Lemma on the left-hand side (with a discrete measure for the sum) and the Dominated Convergence Theorem on the right-hand side, we obtain

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^n} \liminf_{\epsilon \rightarrow 0} |\gamma(\epsilon k)|^p \int_{Q_{0k}} \|W^{1/p}(x) \tilde{g}(k)\|^p dx \\ & \leq c \int_{\mathbb{R}^n} \|W^{1/p}(x) \tilde{g}(x)\|^p \lim_{\epsilon \rightarrow 0} |\gamma(\epsilon x)|^p dx. \end{aligned}$$

Since $\gamma(\epsilon x) \xrightarrow{\epsilon \rightarrow 0} \gamma(0)$, we obtain (4.16) for all $\tilde{g} \in E_0$.

To get the second assertion of the Lemma, we consider the scalar case with w a scalar doubling measure. Then (4.22) becomes

$$\begin{aligned} w(B_\delta(k)) |g(k)|^p & \leq c_p \int_{B_\delta(k)} w(x) |g(x)|^p dx \\ & + c_p \delta^p w(B_\delta(k)) \sum_{m \in \mathbb{Z}^n} \frac{|g(m)|^p}{(1 + |k - m|)^M}, \end{aligned} \tag{4.24}$$

or

$$|g(k)|^p \leq c_p \frac{1}{w(B_\delta(k))} \int_{B_\delta(k)} w(x) |g(x)|^p dx + c \delta^p \sum_{m \in \mathbb{Z}^n} \frac{|g(m)|^p}{(1 + |k - m|)^M}.$$

We want to estimate the last sum on m . Fix $l \in \mathbb{Z}^n$. Dividing everything by $(1 + |k - l|)^M$ and summing on $k \in \mathbb{Z}^n$, we get

$$\sum_{k \in \mathbb{Z}^n} \frac{|g(k)|^p}{(1 + |k - l|)^M} \leq c_p \sum_{k \in \mathbb{Z}^n} \frac{\int_{B_\delta(k)} w(x) |g(x)|^p dx}{(1 + |k - l|)^M w(B_\delta(k))}$$

$$+c\delta^p \sum_{k \in \mathbb{Z}^n} \frac{1}{(1+|k-l|)^M} \sum_{m \in \mathbb{Z}^n} \frac{|g(m)|^p}{(1+|k-m|)^M}.$$

Note that in the last term

$$\sum_{k \in \mathbb{Z}^n} \frac{1}{(1+|k-l|)^M(1+|k-m|)^M} \leq \frac{c}{(1+|l-m|)^M},$$

since $M > n$. Therefore,

$$\sum_{k \in \mathbb{Z}^n} \frac{|g(k)|^p}{(1+|k-l|)^M} \leq c_p \sum_{k \in \mathbb{Z}^n} \frac{\int_{B_\delta(k)} w(x)|g(x)|^p dx}{(1+|k-l|)^M w(B_\delta(k))} + c\delta^p \sum_{m \in \mathbb{Z}^n} \frac{|g(m)|^p}{(1+|l-m|)^M}.$$

Choose $0 < \delta < 1/2$ such that $1 - c\delta^p > 0$. Then

$$\sum_{m \in \mathbb{Z}^n} \frac{|g(m)|^p}{(1+|l-m|)^M} \leq \frac{c_p}{1-c\delta^p} \sum_{m \in \mathbb{Z}^n} \frac{\int_{B_\delta(m)} w(x)|g(x)|^p dx}{(1+|l-m|)^M w(B_\delta(m))}.$$

Substituting this into (4.24) and summing on $k \in \mathbb{Z}^n$ (again using $\sum_{k \in \mathbb{Z}^n} \int_{B_\delta(k)} \dots \leq \int_{\mathbb{R}^n} \dots$), we obtain

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^n} w(B_\delta(k)) |g(k)|^p \\ & \leq c_p \|g\|_{L^p(w)}^p + c\delta^p \sum_{k \in \mathbb{Z}^n} w(B_\delta(k)) \sum_{m \in \mathbb{Z}^n} \frac{\int_{B_\delta(m)} w(x)|g(x)|^p dx}{(1+|k-m|)^M w(B_\delta(m))}. \end{aligned}$$

Use the doubling property of w to shift $B_\delta(k)$ to $B_\delta(m)$. Since δ is fixed, $w(B_\delta(k)) \leq c_{\delta,n} (1+|k-m|)^\beta w(B_\delta(m))$, and thus, the last term is dominated by

$$c\delta^p \sum_{m \in \mathbb{Z}^n} \int_{B_\delta(m)} w(x)|g(x)|^p dx \times \left(\sum_{k \in \mathbb{Z}^n} (1+|k-m|)^{\beta-M} \right), \quad (4.25)$$

where the sum on k converges, since $M > \beta + n$. Thus, (4.25) is estimated by

$c_{p,n,\beta} \|g\|_{L^p(w)}^p$. Hence,

$$\sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} w(x)|g(k)|^p dx \leq c_{p,n,\beta} \sum_{k \in \mathbb{Z}^n} w(B_\delta(k)) |g(k)|^p \leq c_{p,n,\beta} \|g\|_{L^p(w)}^p.$$

Now if W is a diagonal matrix, then

$$\|W^{1/p}(x) \vec{u}\|^p \approx \sum_{i=1}^m w_{ii}(x) |\vec{u}_i|^p,$$

and thus, applying the scalar case, we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W^{1/p}(x) \vec{g}(k)\|^p dx &\approx \sum_{i=1}^m \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} w_{ii}(x) |\vec{g}_i(k)|^p dx \\ &\leq \sum_{i=1}^m c \|\vec{g}_i\|_{L^p(w_{ii})}^p \approx c_{p,n,\beta,m} \|\vec{g}\|_{L^p(W)}^p. \end{aligned}$$

■

Theorem 4.15 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let W satisfy any of (A1)-(A3). Then*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)}, \quad (4.26)$$

where $\vec{s}_Q = S_\varphi \vec{f} = \langle \vec{f}, \varphi_Q \rangle$ for a given $\vec{f}$.

PROOF. By definition,

$$\begin{aligned} \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} &= \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-1/2} \|W^{1/p} \cdot \vec{s}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\}_\nu \right\|_{l_q^\alpha} \\ &=: \|\{J_\nu\}_\nu\|_{l_q^\alpha}. \end{aligned} \quad (4.27)$$

Fix $\nu \in \mathbb{Z}$. Then $Q = Q_{\nu k} = \prod_{i=1}^n \left[\frac{k_i}{2^\nu}, \frac{k_i+1}{2^\nu} \right]$, $k \in \mathbb{Z}^n$, $|Q| = 2^{-\nu n}$, $\vec{s}_Q = |Q|^{1/2}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)$ and

$$\begin{aligned} J_\nu^p &= \sum_{l(Q)=2^{-\nu}} |Q|^{-p/2} \int_Q \|W^{1/p}(t) \vec{s}_Q\|^p dt \\ &= \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \|W^{1/p}(t) (\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)\|^p dt. \end{aligned}$$

Let $\vec{f}_\nu(x) = \vec{f}(2^{-\nu}x)$. Then $(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k) = (\tilde{\varphi} * \vec{f}_\nu)(k)$. We substitute this in the last integral and note that the change of variables $y = 2^\nu t$ (with $W_\nu(t) := W(2^{-\nu}t)$) will yield

$$J_\nu^p = 2^{-\nu n} \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \|W_\nu^{1/p}(t)(\tilde{\varphi} * \vec{f}_\nu)(k)\|^p dt. \quad (4.28)$$

Observe that $(\tilde{\varphi} * \vec{f}_\nu)_i \in \mathcal{S}'$, $i = 1, \dots, m$, and $\tilde{\varphi} * \vec{f}_\nu \in E_0$, since $\text{supp } \hat{\tilde{\varphi}} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$. Using either Lemma 4.12 or Lemma 4.14 with $\vec{g} = \tilde{\varphi} * \vec{f}_\nu$ and W_ν instead of W (both the A_p condition and the doubling condition are invariant with respect to dilation), we obtain

$$J_\nu^p \leq c 2^{-\nu n} \int_{\mathbb{R}^n} \|W_\nu^{1/p}(t)(\tilde{\varphi} * \vec{f}_\nu)(t)\|^p dt.$$

Changing variables, we get

$$J_\nu^p \leq c \int_{\mathbb{R}^n} \|W^{1/p}(t)(\tilde{\varphi}_\nu * \vec{f})(t)\|^p dt = c \|(\tilde{\varphi}_\nu * \vec{f})\|_{L^p(W)}^p.$$

Combining the estimates of J_ν for all ν into (4.27), we get

$$\begin{aligned} & \left\| \left\{ \left\langle \vec{f}, \varphi_Q \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} = \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} = \|\{J_\nu\}_\nu\|_{l_q^\alpha} \\ & \leq c \left\| \left\{ \left\| (\tilde{\varphi}_\nu * \vec{f}) \right\|_{L^p(W)} \right\}_\nu \right\|_{l_q^\alpha} = c \left\| \vec{f} \right\|_{\dot{B}_p^{\alpha q}(W, \tilde{\varphi})}, \end{aligned} \quad (4.29)$$

where $c = c(p, \beta, n)$.

To finish the proof of the theorem, we have to establish the equivalence between $\dot{B}_p^{\alpha q}(W, \varphi)$ and $\dot{B}_p^{\alpha q}(W, \tilde{\varphi})$. As we mentioned in Section 2, $\tilde{\varphi} \in \mathcal{A}$, and so the pair $(\tilde{\varphi}, \tilde{\psi})$ satisfies (2.1), since $\hat{\tilde{\varphi}} = \bar{\tilde{\varphi}}$ and $\hat{\tilde{\psi}} = \bar{\tilde{\psi}}$. By (2.2), $\vec{f} = \sum_Q \left\langle \vec{f}, \tilde{\varphi}_Q \right\rangle \tilde{\psi}_Q$. Since

$\{\tilde{\psi}_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$ (see Remark 4.5), by Theorem 4.2 we have

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \tilde{\varphi})} \leq c \left\| \left\{ \left\langle \vec{f}, \tilde{\varphi}_Q \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)}. \quad (4.30)$$

Applying (4.29) to the right-hand side of the last inequality, we bound it by

$$c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \tilde{\varphi})} = c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \varphi)}. \quad (4.31)$$

Finally, combining (4.29) with (4.30) and (4.31), we obtain

$$\left\| \left\{ \left\langle \vec{f}, \varphi_Q \right\rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \equiv \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \varphi)}.$$

■

Remark 4.16 *The fact that φ and $\tilde{\varphi}$ were interchanged in the last step of the previous theorem can be generalized into Theorem 1.8 about the independence of the space $\dot{B}_p^{\alpha q}(W)$ from the choice of φ :*

PROOF OF THEOREM 1.8. Let $\{\varphi^{(1)}, \psi^{(1)}\}$ and $\{\varphi^{(2)}, \psi^{(2)}\}$ be two different sets of mutually admissible kernels. Decompose $\vec{f}$ in the second system:

$$\vec{f} = \sum_Q \left\langle \vec{f}, \varphi_Q^{(2)} \right\rangle \psi_Q^{(2)} = \sum_Q \vec{s}_Q^{(2)} \psi_Q^{(2)}.$$

Observe that $\psi_Q^{(2)}$ is a molecule for Q and, therefore, by Theorem 4.2,

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \varphi^{(1)})} \leq c \|\{\vec{s}_Q^{(2)}\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W, \varphi^{(2)})},$$

where the last inequality holds by Theorem 4.15. Interchanging $\varphi^{(1)}$ with $\varphi^{(2)}$, we get the norm equivalence between $\dot{B}_p^{\alpha q}(W, \varphi^{(1)})$ and $\dot{B}_p^{\alpha q}(W, \varphi^{(2)})$. In other words, the

space $\dot{B}_p^{\alpha q}(W)$ is independent of the choice of φ under any of the three assumptions on W . ■

Remark 4.17 *Combining boundedness of the φ -transform (Theorem 4.15) and that of the inverse φ -transform (Corollary 4.6), we get the norm equivalence claimed in Theorems 1.4 and 1.6.*

4.4 Connection with reducing operators

Now we connect the weighted sequence Besov space with its reducing operator equivalent. Recall that for each matrix weight W , we can find a sequence of reducing operators $\{A_Q\}_Q$ such that for all $\vec{u} \in \mathcal{H}$,

$$\rho_{p,Q}(\vec{u}) = \left(\frac{1}{|Q|} \int \|W^{1/p}(t) \cdot \vec{u}\|_{\mathcal{H}}^p \chi_Q(t) dt \right)^{1/p} \approx \|A_Q \vec{u}\|_{\mathcal{H}}. \quad (4.32)$$

Lemma 4.18 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let $\{A_Q\}_Q$ be reducing operators for W . Then*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \approx \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}. \quad (4.33)$$

PROOF. Using (4.32), we get the equivalence

$$\begin{aligned} \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} &= \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \|W^{1/p} \cdot \vec{s}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\}_{\nu} \right\|_{l_q^{\alpha}} \\ &= \left\| \left\{ \left(\sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{p}{2}} [\rho_{p,Q}(\vec{s}_Q)]^p |Q| \right)^{\frac{1}{p}} \right\}_{\nu} \right\|_{l_q^{\alpha}} \end{aligned}$$

$$\begin{aligned}
& \approx \left\| \left\{ \left(\sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{p}{2}} \|A_Q \tilde{s}_Q\|_{\mathcal{H}}^p \int \chi_Q(t) dt \right)^{\frac{1}{p}} \right\} \right\|_{\nu, l_q^\alpha} \\
& = \left\| \left\{ \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \|A_Q \tilde{s}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^p} \right\} \right\|_{\nu, l_q^\alpha} = \|\{\tilde{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.
\end{aligned}$$

■

Finally, combining Theorems 1.4 and 1.6 with (4.33), we get Theorem 1.9.

Corollary 4.19 *The space $\dot{B}_p^{\alpha q}(W)$ is complete when $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ and W satisfies any of (A1)-(A3).*

PROOF. If $\{\tilde{f}_n\}_{n \in \mathbb{N}}$ is Cauchy in $\dot{B}_p^{\alpha q}(W)$, then $\left\{ \left\{ \tilde{s}_Q(\tilde{f}_n) \right\}_Q \right\}_{n \in \mathbb{N}}$ is Cauchy in $\dot{b}_p^{\alpha q}(\{A_Q\})$ by Theorem 4.15 and Lemma 4.18 (or just Theorem 1.9). This implies that

$$\begin{aligned}
& \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \|A_Q [\tilde{s}_Q(\tilde{f}_n) - \tilde{s}_Q(\tilde{f}_m)]\|_{\mathcal{H}} \chi_Q \right\|_{L^p}^p \\
& = 2^{\nu n(p/2-1)} \sum_{l(Q)=2^{-\nu}} \|A_Q [\tilde{s}_Q(\tilde{f}_n) - \tilde{s}_Q(\tilde{f}_m)]\|_{\mathcal{H}}^p \xrightarrow{n, m \rightarrow \infty} 0,
\end{aligned}$$

for each $\nu \in \mathbb{Z}$. Hence, $\|A_Q [\tilde{s}_Q(\tilde{f}_n) - \tilde{s}_Q(\tilde{f}_m)]\|_{\mathcal{H}} \xrightarrow{n, m \rightarrow \infty} 0$ for each Q . Since the A_Q 's are invertible, $\{\tilde{s}_Q(\tilde{f}_n)\}_{n \in \mathbb{N}}$ is a vector-valued Cauchy sequence in $\mathcal{H}$ for each Q . Therefore, we can define $\tilde{s}_Q = \lim_{n \rightarrow \infty} \tilde{s}_Q(\tilde{f}_n)$. Set $\tilde{f} = \sum_Q \tilde{s}_Q \psi_Q$. Observe that

$$\begin{aligned}
\|\tilde{f}_n - \tilde{f}\|_{\dot{B}_p^{\alpha q}(W)} & = \left\| \sum_Q [\tilde{s}_Q(\tilde{f}_n) - \tilde{s}_Q] \psi_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \\
& \leq c \left\| \left\{ \tilde{s}_Q(\tilde{f}_n) - \tilde{s}_Q \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \\
& \leq c \liminf_{m \rightarrow \infty} \left\| \left\{ \tilde{s}_Q(\tilde{f}_n) - \tilde{s}_Q(\tilde{f}_m) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \xrightarrow{n \rightarrow \infty} 0,
\end{aligned}$$

by Corollary 4.6 and Lemma 4.18, the discrete version of Fatou's Lemma and the fact that $\left\{ \left\{ \tilde{s}_Q \left(\tilde{f}_n \right) \right\}_Q \right\}_{n \in \mathbb{N}}$ is Cauchy in $\dot{b}_p^{\alpha q}(\{A_Q\})$. Furthermore, $\vec{f} = (\vec{f} - \vec{f}_n) + \vec{f}_n \in \dot{B}_p^{\alpha q}(W)$. Thus, $\dot{B}_p^{\alpha q}(W)$ is complete. ■

Recall (Chapter 3) the A_p condition in terms of reducing operators: $\|A_Q A_Q^\# \| \leq c$ for any cube $Q \in \mathbb{R}^n$; in other words, $\|A_Q y\| \leq c \|(A_Q^\#)^{-1} y\|$ holds for any $y \in \mathcal{H}$. Also, the inverse inequality $\|(A_Q A_Q^\#)^{-1}\| \leq c$ (or, equivalently, $\|(A_Q^\#)^{-1} y\| \leq c \|A_Q y\|$ for any $y \in \mathcal{H}$) holds automatically. This implies the following imbeddings of the sequence Besov spaces:

Corollary 4.20 *For $\alpha \in \mathbb{R}$, $1 < p < \infty$, $0 < q \leq \infty$, and W a matrix weight with corresponding reducing operators A_Q and $A_Q^\#$,*

1. $\dot{b}_p^{\alpha q}(\{A_Q\}) \subseteq \dot{b}_p^{\alpha q}(\{(A_Q^\#)^{-1}\})$ *always,*
2. $\dot{b}_p^{\alpha q}(\{(A_Q^\#)^{-1}\}) \subseteq \dot{b}_p^{\alpha q}(\{A_Q\})$ *if $W \in A_p$.*

CHAPTER 5

Calderón-Zygmund Operators on Matrix-Weighted Besov Spaces

5.1 Almost diagonal operators

Consider $\dot{b}_p^{\alpha q}(W)$ with parameters α, p, q fixed ($\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$) and W a doubling matrix of order p with doubling exponent β . Also, if $p = 1$, then the convention is that $1/p' = 0$.

Definition 5.1 A matrix $A = (a_{QP})_{Q,P \in \mathcal{D}}$ is almost diagonal, $A \in \mathbf{ad}_p^{\alpha q}(\beta)$, if there exist $M > J = \frac{n}{p'} + \frac{\beta}{p}$ and $c > 0$ such that for all Q, P ,

$$|a_{QP}| \leq c \min \left(\left[\frac{l(Q)}{l(P)} \right]^{\alpha_1}, \left[\frac{l(P)}{l(Q)} \right]^{\alpha_2} \right) \left(1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))} \right)^{-M}, \quad (5.1)$$

with $\alpha_1 > \alpha + \frac{n}{2}$ and $\alpha_2 > J - (\alpha + \frac{n}{2})$.

Remark 5.2 This definition differs from the definition of almost diagonality in [FJW], since both α_2 and M depend on the doubling exponent β .

To simplify notation for the matrix A above, we will only write (a_{QP}) without specifying indices Q, P .

Example 5.3 (AN ALMOST DIAGONAL MATRIX) *Let $\varphi \in \mathcal{A}$. If $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, then*

$$(a_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta), \quad (5.2)$$

where $a_{QP} = \langle m_P, \varphi_Q \rangle$, by (4.2) and (4.3), since $\langle m_P, \varphi_Q \rangle = |Q|^{1/2}(\tilde{\varphi}_\nu * m_P)(x_Q)$ if $l(Q) = 2^{-\nu}$.

Now we show that almost diagonal matrices are bounded on $\dot{b}_p^{\alpha q}(W)$, i.e., Theorem 1.10. First we need the following approximation lemma:

Lemma 5.4 *Let P, Q be dyadic cubes and $t \in Q$. Then*

$$1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))} \underset{(n)}{\approx} 1 + \frac{|t - x_P|}{\max(l(Q), l(P))}. \quad (5.3)$$

PROOF. First suppose that $l(Q) \geq l(P)$. If $P \subseteq 3Q$, then $0 \leq |x_P - x_Q| \leq 2\sqrt{n}l(Q) = cl(Q)$ and so

$$1 \leq 1 + \frac{|x_Q - x_P|}{l(Q)} \leq 1 + c \iff 1 + \frac{|x_Q - x_P|}{l(Q)} \approx 1.$$

Also $0 \leq |x_P - t| \leq 2\sqrt{n}l(Q) = cl(Q)$ and thus

$$1 \leq 1 + \frac{|t - x_P|}{l(Q)} \leq 1 + c \iff 1 + \frac{|t - x_P|}{l(Q)} \approx 1,$$

and (5.3) follows.

If $P \cap 3Q = \emptyset$, then $|x_P - t| \geq l(Q)$ and $|x_P - x_Q| \geq l(Q)$. Since $|x_Q - t| \leq cl(Q)$, by the triangle inequality we get both

$$|x_P - x_Q| \leq |x_P - t| + |t - x_Q| \leq |x_P - t| + cl(Q) \leq |x_P - t| + c|x_P - t| \leq (1+c)|x_P - t|$$

and

$$|x_P - t| \leq |x_P - x_Q| + |x_Q - t| \leq |x_P - x_Q| + cl(Q) \leq (1+c)|x_P - x_Q|.$$

Therefore, $|x_P - x_Q| \approx |x_P - t|$.

Now assume that $l(Q) < l(P)$. Choose $\tilde{P}$ dyadic with $l(\tilde{P}) = l(P)$ and $Q \subseteq \tilde{P}$.

If $P \cap 3\tilde{P} = \emptyset$, then $|x_P - t| \geq cl(Q)$ and $|x_P - x_Q| \geq cl(Q)$. Hence,

$$|x_P - x_Q| \leq |x_P - t| + |t - x_Q| \leq |x_P - t| + cl(Q) \leq |x_P - t| + c|x_P - t| \leq (1+c)|x_P - t|,$$

and

$$|x_P - t| \leq |x_P - x_Q| + |x_Q - t| \leq |x_P - x_Q| + cl(Q) \leq (1+c)|x_P - x_Q|,$$

and we again get $|x_P - x_Q| \approx |x_P - t|$.

If $P \subseteq 3\tilde{P}$, then $0 \leq |x_Q - x_P| \leq c_1 l(\tilde{P}) = c_1 l(P)$ and $0 \leq |t - x_P| \leq c_2 l(\tilde{P}) = c_2 l(P)$; thus,

$$1 \leq 1 + \frac{|x_Q - x_P|}{l(P)} \leq 1 + c_1 \quad \text{and} \quad 1 \leq 1 + \frac{|t - x_P|}{l(P)} \leq 1 + c_2$$

which means

$$1 + \frac{|x_Q - x_P|}{l(P)} \approx 1 \approx 1 + \frac{|t - x_P|}{l(P)}.$$

■

PROOF OF THEOREM 1.10. Let $A = (a_{QP})$ with $A \in \mathbf{ad}_p^{\alpha q}(\beta)$. We want to show

that

$$\left\| \left\{ \sum_P a_{QP} \vec{s}_P \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \leq c_{n,p,q,\beta} \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \quad (5.4)$$

By definition,

$$\begin{aligned} & \left\| \left\{ \sum_P a_{QP} \vec{s}_P \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \\ & \leq \left\| \left\{ \left(\sum_{l(Q)=2^{-\nu}} |Q|^{-p/2} \int_Q \left(\sum_P |a_{QP}| \|W^{1/p}(t) \vec{s}_P\| \right)^p dt \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha} \\ & =: \left\| \left\{ 2^{\nu\alpha} 2^{\nu n/2} \left(\sum_{l(Q)=2^{-\nu}} J_Q \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha}. \end{aligned} \quad (5.5)$$

Substituting the estimate (5.1) for a_{QP} in J_Q , we get

$$\begin{aligned} J_Q & \leq c_{p,M} \int_Q \left(\sum_{j \geq 0} 2^{-j\alpha_2} \sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t) \vec{s}_P\| (1 + 2^\nu |x_Q - x_P|)^{-M} \right)^p dt \\ & + c_{p,M} \int_Q \left(\sum_{j < 0} 2^{j\alpha_1} \sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t) \vec{s}_P\| (1 + 2^{(\nu+j)} |x_Q - x_P|)^{-M} \right)^p dt. \end{aligned}$$

Pick $\epsilon > 0$ sufficiently small such that (i) $\alpha_1 - \epsilon > \alpha + n/2$, (ii) $\alpha_2 - \epsilon > J - \alpha - n/2$ and (iii) $M > \beta/p + (n + \epsilon)/p'$. Apply the discrete Hölder inequality twice, first with $\alpha_i = \epsilon + (\alpha_i - \epsilon)$ for the sum on j (note that $\alpha_1, \alpha_2 > 0$) and second with $M = \frac{n+\epsilon}{p'} + \left(M - \frac{n+\epsilon}{p'}\right)$ for the sum on P (if $p' = \infty$, then the $L^{p'}$ -norm is replaced by the supremum):

$$\begin{aligned} J_Q & \leq c_{p,M} \int_Q \left(\sum_{j \geq 0} 2^{-j\epsilon p'} \right)^{p/p'} \left[\sum_{j \geq 0} 2^{-j(\alpha_2 - \epsilon)p} \right. \\ & \times \left. \left(\sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t) \vec{s}_P\| (1 + 2^\nu |x_Q - x_P|)^{-M} \right)^p \right] dt \end{aligned}$$

$$\begin{aligned}
& + c_{p,M} \int_Q \left(\sum_{j<0} 2^{j\epsilon p'} \right)^{p/p'} \left[\sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \right. \\
& \times \left. \left(\sum_{l(P)=2^{-(\nu+j)}} \|W^{1/p}(t) \vec{s}_P\| (1 + 2^{(\nu+j)} |x_Q - x_P|)^{-M} \right)^p \right] dt \\
& \leq c_{p,M,\epsilon} \sum_{j \geq 0} 2^{-j(\alpha_2-\epsilon)p} \left[\sum_{l(P)=2^{-(\nu+j)}} (1 + 2^\nu |x_Q - x_P|)^{-n-\epsilon} \right]^{p/p'} \\
& \times \sum_{l(P)=2^{-(\nu+j)}} \int_Q \|W^{1/p}(t) \vec{s}_P\|^p (1 + 2^\nu |x_Q - x_P|)^{-(M-\frac{n+\epsilon}{p'})p} dt \\
& + c_{p,M,\epsilon} \sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \left[\sum_{l(P)=2^{-(\nu+j)}} (1 + 2^{(\nu+j)} |x_Q - x_P|)^{-n-\epsilon} \right]^{p/p'} \\
& \times \sum_{l(P)=2^{-(\nu+j)}} \int_Q \|W^{1/p}(t) \vec{s}_P\|^p (1 + 2^{(\nu+j)} |x_Q - x_P|)^{-(M-\frac{n+\epsilon}{p'})p} dt.
\end{aligned}$$

Use the Summation Lemma 4.4 to estimate the square brackets and denote $w_P(t) = \|W^{1/p}(t) \vec{s}_P\|^p$. By Lemma 5.4, x_Q can be replaced by any $t \in Q$, and so we get

$$\begin{aligned}
J_Q & \leq c_{p,M} \sum_{j \geq 0} 2^{-j(\alpha_2-\epsilon)p+jnp/p'} \\
& \times \sum_{l(P)=2^{-(\nu+j)}} \int_Q w_P(t) \left(1 + \frac{|t - x_P|}{l(Q)} \right)^{-(M-\frac{n+\epsilon}{p'})p} dt \\
& + c_{p,M} \sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \sum_{l(P)=2^{-(\nu+j)}} \int_Q w_P(t) \left(1 + \frac{|t - x_P|}{l(P)} \right)^{-(M-\frac{n+\epsilon}{p'})p} dt.
\end{aligned}$$

Summing on Q and applying the Squeeze Lemma 4.3 (recall $M > \beta/p + (n + \epsilon)/p'$), we get

$$\begin{aligned}
& \sum_{l(Q)=2^{-\nu}} J_Q \leq c_{p,n} \sum_{j \geq 0} 2^{-j(\alpha_2-\epsilon)p+jnp/p'} \\
& \times \sum_{l(P)=2^{-(\nu+j)}} \sum_{l(Q)=2^{-\nu}} \int_Q w_P(t) (1 + 2^\nu |t - x_P|)^{-(M-\frac{n+\epsilon}{p'})p} dt \\
& + c_{p,n} \sum_{j<0} 2^{j(\alpha_1-\epsilon)p} \sum_{l(P)=2^{-(\nu+j)}} \sum_{l(Q)=2^{-\nu}} \int_Q w_P(t) (1 + 2^{\nu+j} |t - x_P|)^{-(M-\frac{n+\epsilon}{p'})p} dt
\end{aligned}$$

$$\leq c_{p,n,\beta} \sum_{j \in \mathbb{Z}} \left(2^{-j(\alpha_2 - \epsilon)p + jnp/p' + j\beta} \chi_{\{j \geq 0\}} + 2^{j(\alpha_1 - \epsilon)p} \chi_{\{j < 0\}} \right) \sum_{l(P)=2^{-(\nu+j)}} w_P(P).$$

Observe that $2^{\nu np/2} = |P|^{-p/2} 2^{-jnp/2}$ for $l(P) = 2^{-(\nu+j)}$, and

$$\sum_{l(P)=2^{-(\nu+j)}} |P|^{-p/2} w_P(P) = \left\| \sum_{l(P)=2^{-(\nu+j)}} |P|^{-1/2} \vec{s}_P \chi_P \right\|_{L^p(W)}^p.$$

Then, using $1 < p < \infty$ to take the power $1/p$ inside the sum on j , we get

$$\begin{aligned} 2^{\nu\alpha} 2^{\nu n/2} \left(\sum_{l(Q)=2^{-\nu}} J_Q \right)^{1/p} &\leq c \sum_{j \in \mathbb{Z}} \left[2^{-j\alpha} 2^{-jn/2} \left(2^{-j(\alpha_2 - \epsilon)p + jnp/p' + j\beta} \chi_{\{j \geq 0\}} \right. \right. \\ &\quad \left. \left. + 2^{j(\alpha_1 - \epsilon)p} \chi_{\{j < 0\}} \right)^{1/p} \right] \times 2^{(\nu+j)\alpha} \left\| \sum_{l(P)=2^{-(\nu+j)}} |P|^{-1/2} \vec{s}_P \chi_P \right\|_{L^p(W)} \\ &=: c \sum_{j \in \mathbb{Z}} a_{-j} \times b_{\nu+j} = c(a * b)(\nu). \end{aligned}$$

Use (4.10) and (4.11) to estimate the norm of the convolution $\|a * b\|_{l^q}$. Then for

$q \leq 1$,

$$\|a\|_{l^q}^q = \sum_{j \leq 0} 2^{j(\alpha + n/2 + (\alpha_2 - \epsilon) - J)q} + \sum_{j > 0} 2^{j(\alpha + n/2 - (\alpha_1 - \epsilon))q} \leq c_q,$$

since $\alpha_1 - \epsilon > \alpha + n/2$ and $\alpha_2 - \epsilon > J - (\alpha + n/2)$. Using the $\|a\|_{l^1}$ estimate for

$q \geq 1$ and the $\|a\|_{l^q}$ estimate for $q < 1$, and substituting into (5.5), we obtain

$$\begin{aligned} \left\| \left\{ \sum_P a_{QP} \vec{s}_P \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} &\leq c \|b\|_{l^q} = c \left\| \left\{ 2^{\mu\alpha} \left\| \sum_{l(P)=2^{-\mu}} |P|^{-1/2} \vec{s}_P \chi_P \right\|_{L^p(W)} \right\}_\mu \right\|_{l^q} \\ &= c \|\{\vec{s}_P\}_P\|_{\dot{b}_p^{\alpha q}(W)}, \end{aligned}$$

where $c = c_{n,p,q,\beta}$. $\blacksquare$

Now we will show that the class of almost diagonal matrices is closed under composition. For $\epsilon > 0, \delta > 0, J = \frac{n}{p'} + \frac{\beta}{p}$ and $P, Q \in \mathcal{D}$, denote

$$w_{QP}(\delta, \epsilon) = \left[\frac{l(Q)}{l(P)} \right]^{\alpha + \frac{n}{2}} \min \left(\left[\frac{l(Q)}{l(P)} \right]^{\frac{\epsilon}{2}}, \left[\frac{l(P)}{l(Q)} \right]^{\frac{\epsilon}{2} + J} \right) \left(1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))} \right)^{-J - \delta}.$$

Theorem 5.5 *Let $A, B \in \mathbf{ad}_p^{\alpha q}(\beta)$. Then $A \circ B \in \mathbf{ad}_p^{\alpha q}(\beta)$.*

We need the following lemma, which is a modification of [FJ2, Theorem D.2] adjusted to the weighted **ad** condition:

Lemma 5.6 *Let $\delta, \gamma_1, \gamma_2 > 0$, $\gamma_1 \neq \gamma_2$, and $2\delta < \gamma_1 + \gamma_2$. Then there exists a constant $c = c_{n, \delta, \gamma_1, \gamma_2, J}$ such that*

$$\sum_R w_{QR}(\delta, \gamma_1) w_{RP}(\delta, \gamma_2) \leq c w_{QP}(\delta, \min(\gamma_1, \gamma_2)). \quad (5.6)$$

PROOF. Without loss of generality, we may assume that $\alpha = -n/2$, since the terms $[l(R)]^{\alpha+n/2}$ cancel in the sum of (5.6), leaving $\left[\frac{l(Q)}{l(P)}\right]^{\alpha+n/2}$ for the right-hand side of the inequality. Denote $\gamma = \min(\gamma_1, \gamma_2)$. With $l(Q) = 2^{-q}, l(P) = 2^{-p}, l(R) = 2^{-r}$, first assume $l(P) \leq l(Q)$. Then the sum in (5.6) can be split into the following terms:

$$\sum_{l(R) < l(P) \leq l(Q)} + \sum_{l(P) \leq l(R) \leq l(Q)} + \sum_{l(P) \leq l(Q) < l(R)} = I + II + III.$$

Then

$$\begin{aligned} I &= \sum_R \left[\frac{l(R)}{l(Q)}\right]^{\gamma_1/2+J} \left(1 + \frac{|x_Q - x_R|}{l(Q)}\right)^{-J-\delta} \left[\frac{l(R)}{l(P)}\right]^{\gamma_2/2} \left(1 + \frac{|x_R - x_P|}{l(Q)}\right)^{-J-\delta} \\ &= [l(Q)]^{-(\gamma_1/2+J)} [l(P)]^{-\gamma_2/2} \sum_{r=p+1}^{\infty} 2^{-r(\frac{\gamma_1+\gamma_2}{2}+J)} g_{P,Q,J+\delta,r}(x_P) \\ &\leq c [l(Q)]^{-(\gamma_1/2+J)} [l(P)]^{-\gamma_2/2+n} \left(1 + \frac{|x_Q - x_P|}{l(Q)}\right)^{-J-\delta} \sum_{r=p+1}^{\infty} 2^{-r(\frac{\gamma_1+\gamma_2}{2}+J-n)}, \end{aligned}$$

by [FJ2, Lemma D.1]. Since $J > n$, the geometric progression sum is bounded by $c 2^{-p(\frac{\gamma_1+\gamma_2}{2}+J-n)} = c [l(P)]^{(\frac{\gamma_1+\gamma_2}{2}+J-n)}$. Thus,

$$I \leq c \left[\frac{l(P)}{l(Q)}\right]^{\gamma_1/2+J} \left(1 + \frac{|x_Q - x_P|}{l(Q)}\right)^{-J-\delta} \leq c \left[\frac{l(P)}{l(Q)}\right]^{\gamma_1/2+J} \left(1 + \frac{|x_Q - x_P|}{l(Q)}\right)^{-J-\delta},$$

substituting γ_1 with γ , since $l(P) \leq l(Q)$.

Similarly, using [FJ2, Lemma D.1], we have

$$\begin{aligned}
II &= \sum_R \left[\frac{l(R)}{l(Q)} \right]^{\gamma_1/2+J} \left(1 + \frac{|x_Q - x_R|}{l(Q)} \right)^{-J-\delta} \left[\frac{l(P)}{l(R)} \right]^{\gamma_2/2+J} \left(1 + \frac{|x_R - x_P|}{l(R)} \right)^{-J-\delta} \\
&\leq \frac{[l(P)]^{\gamma_2/2+J}}{[l(Q)]^{\gamma_1/2+J}} \sum_{r=q}^p 2^{-r(\frac{\gamma_1+\gamma_2}{2})} g_{P,Q,J+\delta,r}(x_P) \\
&\leq c \left(1 + \frac{|x_Q - x_P|}{l(Q)} \right)^{-J-\delta} \frac{[l(P)]^{\gamma_2/2+J}}{[l(Q)]^{\gamma_1/2+J}} [l(P)]^{\frac{\gamma_1+\gamma_2}{2}} \\
&= c \left[\frac{l(P)}{l(Q)} \right]^{\gamma_1/2+J} \left(1 + \frac{|x_Q - x_P|}{l(Q)} \right)^{-J-\delta},
\end{aligned}$$

and γ_1 can be replaced by γ , since $l(P) \leq l(Q)$.

The estimate of III is also similar:

$$\begin{aligned}
III &\leq [l(Q)]^{\gamma_1/2} [l(P)]^{\gamma_2/2+J} \sum_{r=-\infty}^{q-1} 2^{r(\frac{\gamma_1+\gamma_2}{2}+J)} g_{P,Q,J+\delta,r}(x_P) \\
&\leq [l(Q)]^{\gamma_1/2} [l(P)]^{\gamma_2/2+J} \sum_{r=-\infty}^{q-1} 2^{r(\frac{\gamma_1+\gamma_2}{2}+J)} \left(1 + \frac{|x_Q - x_P|}{l(R)} \right)^{-J-\delta}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\left(1 + \frac{|x_Q - x_P|}{l(R)} \right)^{J+\delta} &> \left(\frac{l(Q)}{l(R)} + \frac{l(Q)}{l(R)} \frac{|x_Q - x_P|}{l(Q)} \right)^{J+\delta} \\
&> \left[\frac{l(Q)}{l(R)} \right]^{J+\delta} \left(1 + \frac{|x_Q - x_P|}{l(Q)} \right)^{J+\delta}.
\end{aligned}$$

Then

$$III \leq c [l(Q)]^{\gamma_1/2-J-\delta} [l(P)]^{\gamma_2/2+J} \left(1 + \frac{|x_Q - x_P|}{l(Q)} \right)^{-J-\delta} \sum_{r=-\infty}^{q-1} 2^{r(\frac{\gamma_1+\gamma_2}{2}+J)} 2^{r(-J-\delta)},$$

where the last sum converges since $\frac{\gamma_1+\gamma_2}{2} > \delta$ and is bounded by $c 2^{q(\frac{\gamma_1+\gamma_2}{2}-\delta)}$. Sim-

plifying, we get

$$III \leq c \left[\frac{l(P)}{l(Q)} \right]^{\gamma_2/2+J} \left(1 + \frac{|x_Q - x_P|}{l(Q)} \right)^{-J-\delta}.$$

Combining *I*, *II* and *III*, we get the right-hand side estimate of (5.6), if $l(P) \leq l(Q)$.

The case $l(P) > l(Q)$ follows by exact repetition of the steps above. $\blacksquare$

PROOF OF THEOREM 5.5. Since $A = (a_{QP}), B = (b_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$, for each $i = A, B$ there exist $0 < \epsilon_i < \min(\alpha_1 - (\alpha + n/2), \alpha_2 - J + \alpha + n/2)$ and $0 < \delta < M - J$ such that $|a_{QP}| \leq c w_{QP}(\delta, \epsilon_A)$ and $|b_{QP}| \leq c w_{QP}(\delta, \epsilon_B)$. Without loss of generality, we may assume $\epsilon_A < \epsilon_B$ and $\delta < \frac{\epsilon_A + \epsilon_B}{2}$. Then

$$|(AB)_{QP}| \leq \left| \sum_R a_{QR} b_{RP} \right| \leq c \sum_R w_{QR}(\delta, \epsilon_A) w_{RP}(\delta, \epsilon_B) \leq c w_{QP}(\delta, \epsilon_A),$$

by Lemma 5.6, which means that $A \circ B \in \mathbf{ad}_p^{\alpha q}(\beta)$. $\blacksquare$

Definition 5.7 Let T be a continuous linear operator from $\mathcal{S}$ to $\mathcal{S}'$. We say that T is an almost diagonal operator for $\dot{B}_p^{\alpha q}(W)$, and write $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, if for some pair of mutually admissible kernels (φ, ψ) , the matrix $(a_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$, where $a_{QP} = \langle T\psi_P, \varphi_Q \rangle$.

Remark 5.8 The definition of $T \in \mathbf{AD}_p^{\alpha q}(\beta)$ is independent of the choice of the pair (φ, ψ) .

PROOF. Define $\mathcal{S}_0 = \{f \in \mathcal{S} : 0 \notin \text{supp } \hat{f}\}$. Observe that $\psi \in \mathcal{A}$ implies $\psi, \psi_\nu, \psi_Q \in \mathcal{S}_0$ for $\nu \in \mathbb{Z}$ and Q dyadic. Moreover, if $g \in \mathcal{S}_0$, then $g_N := \sum_{\nu=-N}^N \tilde{\varphi}_\nu * \psi_\nu * g$ converges to g as $N \rightarrow \infty$ in the $\mathcal{S}$ -topology (for proof refer to Appendix, Lemma A.1). Since T is continuous from $\mathcal{S}$ into $\mathcal{S}'$, we have $Tg = \sum_{\nu \in \mathbb{Z}} T(\tilde{\varphi}_\nu * \psi_\nu * g)$. Furthermore, for $g \in \mathcal{S}_0$ and fixed $\nu \in \mathbb{Z}$, we have $\sum_{|k| \leq M} \langle g, \varphi_{Q_{\nu k}} \rangle \psi_{Q_{\nu k}}$ converges to $\tilde{\varphi}_\nu * \psi_\nu * g$ as $M \rightarrow \infty$ in the $\mathcal{S}$ -topology (again refer to Appendix,

Lemma A.2). Hence, $Tg = \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle g, \varphi_{Q_{\nu k}} \rangle T\psi_{Q_{\nu k}} = \sum_Q \langle g, \varphi_Q \rangle T\psi_Q$. Now, suppose $(\langle T\psi_P, \varphi_Q \rangle_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$ for some fixed pair (φ, ψ) of mutually admissible kernels. Take any other such pair $(\tilde{\varphi}, \tilde{\psi})$. Then $\tilde{\psi}_P = \sum_L \langle \tilde{\psi}_P, \varphi_L \rangle \psi_L$ and $\tilde{\varphi}_Q = \sum_R \langle \tilde{\varphi}_Q, \psi_R \rangle \varphi_R$, which gives

$$\langle T\tilde{\psi}_P, \tilde{\varphi}_Q \rangle = \sum_{R,L} \langle \tilde{\psi}_P, \varphi_L \rangle \langle T\psi_L, \varphi_R \rangle \overline{\langle \tilde{\varphi}_Q, \psi_R \rangle}.$$

Since both $\{\psi_R\}_R$ and $\{\varphi_L\}_L$ constitute families of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, by (5.2) the matrices $(\langle \tilde{\psi}_P, \varphi_L \rangle_{LP})$, $(\langle \tilde{\varphi}_Q, \psi_R \rangle_{QR}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. By Theorem 5.5, $(\langle T\tilde{\psi}_P, \tilde{\varphi}_Q \rangle_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. ■

A straightforward consequence of Theorem 1.10 is the following statement:

Corollary 5.9 *Let $T \in \mathbf{AD}_p^{\alpha q}(\beta)$, $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q < \infty$. Then T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$ if W satisfies any of (A1)-(A3).*

PROOF. First, consider $\vec{f}$ with $(\vec{f})_i \in \mathcal{S}_0$. Let (φ, ψ) be a pair of mutually admissible kernels. Denote $\vec{t}_Q = \sum_P \langle T\psi_P, \varphi_Q \rangle \vec{s}_P(\vec{f})$ and observe that $(\langle T\psi_P, \varphi_Q \rangle_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. Using the φ -transform decomposition $\vec{f} = \sum_P \vec{s}_P(\vec{f}) \psi_P$ and taking T inside the sum as in the previous remark, we get

$$\begin{aligned} \|T\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} &= \left\| \sum_P \vec{s}_P(\vec{f}) T\psi_P \right\|_{\dot{B}_p^{\alpha q}(W)} \\ &= \left\| \sum_Q \left(\sum_P \langle T\psi_P, \varphi_Q \rangle \vec{s}_P(\vec{f}) \right) \psi_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \\ &= \left\| \sum_Q \vec{t}_Q \psi_Q \right\|_{\dot{B}_p^{\alpha q}(W)} \leq c \|\{\vec{t}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \\ &\leq c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)}, \end{aligned}$$

by Corollary 4.6, Theorem 1.10 and Theorem 4.15.

Note that $\mathcal{S}_0$ is dense in $\dot{B}_p^{\alpha q}(W)$ if $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$ and W satisfies any of (A1)-(A3) (for the proof, refer to Appendix). Thus, T extends to all of $\dot{B}_p^{\alpha q}(W)$. ■

Note that if $q = \infty$, then T extends to a bounded operator on the closure of $\mathcal{S}_0$ in $\dot{B}_p^{\alpha \infty}(W)$.

Remark 5.10 Let $\{m_Q\}_Q$ be a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$. Apply the φ -transform to $\sum_P \tilde{s}_P m_P$:

$$\vec{t}_Q := S_\varphi \left(\sum_P \tilde{s}_P m_P \right) = \left\langle \sum_P \tilde{s}_P m_P, \varphi_Q \right\rangle = \sum_P \langle m_P, \varphi_Q \rangle \tilde{s}_P.$$

Then $(\langle m_P, \varphi_Q \rangle_{QP})$ forms an almost diagonal matrix by (5.2), and therefore, by Theorem 1.10,

$$\|\{\vec{t}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \equiv \|S_\varphi(\sum_P \tilde{s}_P m_P)\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\{\tilde{s}_P\}_P\|_{\dot{b}_p^{\alpha q}(W)},$$

if W is doubling.

Corollary 5.11 Let $T, S \in \mathbf{AD}_p^{\alpha q}(\beta)$. Then $T \circ S \in \mathbf{AD}_p^{\alpha q}(\beta)$.

PROOF. Since $T, S \in \mathbf{AD}_p^{\alpha q}(\beta)$, it follows that $(t_{QP}) := (\langle T\psi_P, \varphi_Q \rangle_{QP})$ is in $\mathbf{ad}_p^{\alpha q}(\beta)$, and so is $(s_{QP}) := (\langle S\psi_P, \varphi_Q \rangle_{QP})$. Thus, for Q, P dyadic we have $S\psi_P = \sum_R \langle S\psi_P, \varphi_R \rangle \psi_R$, and so

$$\langle T \circ S\psi_P, \varphi_Q \rangle = \sum_R \langle S\psi_P, \varphi_R \rangle \langle T\psi_R, \varphi_Q \rangle = \sum_R t_{QR} s_{RP} \in \mathbf{ad}_p^{\alpha q}(\beta),$$

by Theorem 5.5 (composition of almost diagonal matrices). ■

5.2 Calderón-Zygmund operators

In this section we show that Calderón-Zygmund operators (CZO) are bounded on $\dot{B}_p^{\alpha q}(W)$ for certain parameters α, p, q, β . First we recall the definition of smooth atoms and the fact that a CZO maps smooth atoms into smooth molecules. Then we use a general criterion for boundedness of operators: if an operator T maps smooth atoms into molecules, then its matrix $(\langle T\psi_P, \varphi_Q \rangle_{Q_P})$ forms an almost diagonal operator on $\dot{b}_p^{\alpha q}(W)$, and therefore, T is bounded on $\dot{B}_p^{\alpha q}(W)$.

Definition 5.12 *Let $N \in \mathbb{N} \cup \{0\}$. A function $a_Q \in \mathcal{D}(\mathbb{R}^n)$ is a smooth N -atom for Q if*

1. $\text{supp } a_Q \subseteq 3Q$,
2. $\int x^\gamma a_Q(x) dx = 0$ for $|\gamma| \leq N$, and
3. $|D^\gamma a_Q(x)| \leq c_\gamma l(Q)^{-|\gamma|-n/2}$ for all $|\gamma| \geq 0$.

Let $0 < \delta \leq 1$, $M > 0$, $N \in \mathbb{N} \cup \{0, -1\}$, $N_0 \in \mathbb{N} \cup \{0\}$.

Lemma 5.13 (BOUNDEDNESS CRITERION) *Suppose a continuous linear operator $T : \mathcal{S} \rightarrow \mathcal{S}'$ maps any smooth N_0 -atom into a fixed multiple of a smooth (δ, M, N) -molecule for $\dot{B}_p^{\alpha q}(W)$, $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ with δ, M, N satisfying (M.i), (M.ii) and (M.iii) (see Section 4.1). Suppose W satisfies any of (A1)-(A3). Then $T \in \mathbf{AD}_p^{\alpha q}(\beta)$ and, if $q < \infty$, T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$.*

PROOF. By Corollary 5.9, it suffices to show that $(\langle T\psi_P, \varphi_Q \rangle_{Q_P}) \in \mathbf{ad}_p^{\alpha q}(\beta)$ for some $\varphi, \psi \in \mathcal{A}$ satisfying (2.1). Observe that if $\psi \in \mathcal{A}$, then there exists $\theta \in \mathcal{S}$ with

$\text{supp } \theta \subseteq B_1(0)$, $\int x^\gamma \theta(x) dx = 0$, if $|\gamma| \leq N_0$, and $\sum_{\nu \in \mathbb{Z}} \hat{\theta}(2^{-\nu} \xi) \hat{\varphi}(2^{-\nu} \xi) = 1$ for $\xi \neq 0$ ([FJW, Lemma 5.12]). Using $\psi_P = \sum_{\nu \in \mathbb{Z}} \theta_\nu * \varphi_\nu * \psi_P$ as in the atomic decomposition theorem ([FJW, Theorem 5.11]), we have

$$\psi_P(x) = \sum_Q t_{QP} a_Q^{(P)}(x) \quad (5.7)$$

with $t_{QP} = |Q|^{1/2} \sup_{y \in Q} |(\varphi_\nu * \psi_P)(y)|$ for $l(Q) = 2^{-\nu}$, and each $a_Q^{(P)}$ is an N_0 -atom defined by

$$a_Q^{(P)}(x) = \frac{1}{t_{QP}} \int_Q \theta_\nu(x - y) (\varphi_\nu * \psi_P)(y) dy \quad \text{if } t_{QP} \neq 0 \quad (5.8)$$

and $a_Q^{(P)} = 0$ if $t_{QP} = 0$. Using (4.2)-(4.3) (valid because $\{\psi_P\}_P$ is a family of molecules for $\dot{B}_p^{\alpha q}(W)$), we get

$$|(\varphi_\nu * \psi_P)(y)| \leq c |P|^{-1/2} \min \left(\left[\frac{l(Q)}{l(P)} \right]^\tau, \left[\frac{l(P)}{l(Q)} \right]^\sigma \right) \left(1 + \frac{|y - x_P|}{\max(l(Q), l(P))} \right)^{-M},$$

for some $\tau > \alpha$ and $\sigma > J - \alpha$. In fact, $\varphi_\nu * \psi_P = 0$ if $|\mu - \nu| > 1$ ($2^{-\mu} = l(P)$), since $\varphi, \psi \in \mathcal{A}$, but all we require is the previous estimate. Since $y \in Q$, y can be replaced by x_Q in the last expression by Lemma 5.4, and so

$$|t_{QP}| \leq c \left(\frac{|Q|}{|P|} \right)^{1/2} \min \left(\left[\frac{l(Q)}{l(P)} \right]^\tau, \left[\frac{l(P)}{l(Q)} \right]^\sigma \right) \left(1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))} \right)^{-M},$$

which is exactly (5.1). Thus $(t_{QP}) \in \mathbf{ad}_p^{\alpha q}(\beta)$. Using (5.7), we obtain

$$\langle T\psi_P, \varphi_Q \rangle = \left\langle \sum_R t_{RP} T a_R^{(P)}, \varphi_Q \right\rangle = \sum_R t_{RP} \langle T a_R^{(P)}, \varphi_Q \rangle.$$

Since T maps any N_0 -atom $a_R^{(P)}$ into a fixed multiple of a smooth (δ, M, N) -molecule m_R : $T a_R^{(P)} = c m_R$ and c depends neither on R nor on Q , we get

$$\langle T a_R^{(P)}, \varphi_Q \rangle = c \langle m_R, \varphi_Q \rangle =: c \tilde{t}_{QR},$$

and by (5.2), since m_R is a smooth (δ, M, N) -molecule for $\dot{B}_p^{\alpha q}(W)$, $(\tilde{t}_{QR}) \in \mathbf{ad}_p^{\alpha q}(\beta)$.

Hence,

$$\left(\langle T\psi_P, \varphi_Q \rangle_{QP} \right) = \left(c \sum_R \tilde{t}_{QR} t_{RP} \right) \in \mathbf{ad}_p^{\alpha q}(\beta),$$

since the composition of two almost diagonal operators is again almost diagonal by Theorem 5.11. $\blacksquare$

Let T be a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, and let $K = K(x, y)$ be its distributional kernel defined on $\mathbb{R}^{2n} \setminus \Delta$, where $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ (for definitions refer to [FJW], Chapter 8). Then $T \in CZO(\epsilon)$, $0 < \epsilon \leq 1$, if K has the following properties:

$$(I) \quad |K(x, y)| \leq \frac{c}{|x - y|^n},$$

$$(II_\epsilon) \quad |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq c \frac{|x - x'|^\epsilon}{|x - y|^{n+\epsilon}} \quad \text{if } 2|x - x'| \leq |x - y|.$$

To show that a CZO maps atoms into molecules we start with the following result from [FJW]:

Theorem 5.14 ([FJW], Theorem 8.13) *Let $0 < \epsilon \leq 1$ and $0 < \alpha < 1$. If $T \in CZO(\epsilon) \cap WBP$ and $T1 = 0$, then T maps any smooth 0-atom a_Q into a fixed multiple of a smooth $(\epsilon, n + \epsilon, -1)$ -molecule m_Q .*

Thus, if a_Q is a smooth 0-atom for Q , then $Ta_Q = cm_Q$, where m_Q satisfies

1. $|m_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)} \right)^{-(n+\epsilon)},$
2. $|m_Q(x) - m_Q(y)| \leq |Q|^{-\frac{1}{2}} \left(\frac{|x - y|}{l(Q)} \right)^\epsilon \sup_{|z| \leq |x - y|} \left(1 + \frac{|x - z - x_Q|}{l(Q)} \right)^{-(n+\epsilon)},$

and c is uniform for all Q . Moreover, an $(\epsilon, n + \epsilon, -1)$ -molecule is a smooth molecule for $\dot{B}_p^{\alpha q}(W)$ (see Section 4.1) if $1 \leq p < \infty$, $0 < q \leq \infty$, $0 < \alpha < \epsilon$ and $\beta < n + p\alpha$:

$$(i) \quad \delta = \epsilon \text{ and } 0 < \alpha < \epsilon \leq 1,$$

$$(ii) \quad J = \frac{n}{p'} + \frac{\beta}{p} < n + \alpha < n + \epsilon = M,$$

$$(iii) \quad J - n - \alpha = \frac{\beta - n}{p} - \alpha < 0 \implies N = \max([J - n - \alpha], -1) = -1.$$

The next theorem follows by combining the two statements mentioned above, and gives the boundedness of certain Calderón-Zygmund operators on $\dot{B}_p^{\alpha q}(W)$ with some restriction on the weight W :

Theorem 5.15 *Suppose $0 < \epsilon \leq 1$, $0 < \alpha < \epsilon$, $1 \leq p < \infty$, $0 < q < \infty$, and let W satisfies any of (A1)-(A3). Assume $\beta < n + p\alpha$. If $T \in CZO(\epsilon) \cap WBP$ and $T1 = 0$, then T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$.*

Remark 5.16 *If also $T^*1 = 0$ in Theorem 5.14, then $\alpha = 0$ can be included in the range, since $\int Ta(x) dx = \langle Ta, 1 \rangle = \langle a, T^*1 \rangle = 0$ and so T maps any smooth 0-atom into a smooth $(\epsilon, n + \epsilon, 0)$ -molecule (see also [FJW, Cor. 8.21]).*

Corollary 5.17 *Let $1 \leq p < \infty$, $0 < q < \infty$, $0 < \epsilon \leq 1$ and $0 \leq \alpha < \epsilon$. Assume $\beta < n + p\epsilon$. If $T \in CZO(\epsilon) \cap WBP$ and $T1 = T^*1 = 0$, then T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$, in particular, for $\alpha = 0$.*

PROOF. Since $N = 0$, the bound on β from the previous theorem can be relaxed to $\beta < n + p\epsilon$. ■

Remark 5.18 *The condition $T^*(y^\gamma) = 0$ for $|\gamma| \leq N, N \geq 1$, produces more vanishing moments of a molecule Ta , so it is not difficult to satisfy (M.iii). But (M.ii) $M = n + \epsilon > J = n + \frac{\beta-n}{p} \iff \beta < n + p\epsilon$ creates a major restriction on the doubling exponent of W . Note that in this case, we get that T maps any smooth 0-atom into a smooth $(\epsilon, n + \epsilon, N)$ -molecule, but this molecule is not a smooth molecule for $\dot{B}_p^{\alpha q}(W)$.*

From now on $N \geq 0$, since the case $N = -1$ is completely covered by Theorem 5.15.

Next we want to show that the restriction on the weight W (to be more precise the restriction on the doubling exponent β) can be removed in some cases by requiring more smoothness than (II_ϵ) on the kernel K .

We say that $T \in CZO(N + \epsilon)$, $N \in \mathbb{N} \cup \{0\}$, $0 < \epsilon \leq 1$, if T is a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ and K , its distributional kernel defined on $\mathbb{R}^{2n} \setminus \Delta$, has the following properties:

$$\begin{aligned}
(I) \quad & |K(x, y)| \leq \frac{c}{|x - y|^n}, \\
(II_N) \quad & |D_{(2)}^\gamma K(x, y)| \leq \frac{c}{|x - y|^{n+|\gamma|}}, \text{ for } |\gamma| \leq N, \\
(II_{N+\epsilon}) \quad & |D_{(2)}^\gamma K(x, y) - D_{(2)}^\gamma K(x', y)| + |D_{(2)}^\gamma K(y, x) - D_{(2)}^\gamma K(y, x')| \\
& \leq c \frac{|x - x'|^\epsilon}{|x - y|^{n+|\gamma|+\epsilon}}, \text{ for } 2|x - x'| \leq |x - y| \text{ and } |\gamma| = N,
\end{aligned}$$

where the subscript 2 in $D_{(2)}^\gamma$ refers to differentiation with respect to the second argument of $K(x, y)$.

Note that $CZO(\epsilon) \supseteq CZO(N + \epsilon)$ for $N \geq 0$.

Theorem 5.19 *Let $0 \leq \alpha < 1$, $0 < \epsilon \leq 1$, $N_0 \in \mathbb{N} \cup \{0\}$. Suppose $T \in CZO(N_0 + \epsilon) \cap WBP$, $T1 = 0$ and $T^*(y^\gamma) = 0$ for $|\gamma| \leq N_0$. Then T maps any N_0 -atom a_Q into a fixed multiple of a smooth $(\epsilon, N_0 + n + \epsilon, N_0)$ -molecule.*

More precisely, we will show that $Ta_Q = c m_Q$ with c independent of Q and

$$(i) \quad \int x^\gamma Ta_Q(x) dx = 0, \quad \text{for } |\gamma| \leq N_0,$$

$$(ii) \quad |Ta_Q(x)| \leq c |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-(N_0 + n + \epsilon)},$$

$$(iii) \quad |Ta_Q(x) - Ta_Q(y)| \leq c |Q|^{-\frac{1}{2}} \left[\frac{|x - y|}{l(Q)} \right]^\epsilon \sup_{|z| \leq |x - y|} \left(1 + \frac{|z - (x - x_Q)|}{l(Q)}\right)^{-(N_0 + n + \epsilon)}.$$

Before we start the proof, we quote the following estimate, due to Meyer:

Lemma 5.20 ([M1]) *Let $T : \mathcal{D} \rightarrow \mathcal{D}'$ be a continuous linear operator with $T \in CZO(\epsilon) \cap WBP$, $0 < \epsilon \leq 1$ and $T1 = 0$. Then T maps $\mathcal{D}$ into L^∞ and there exists a constant c such that for any fixed $z \in \mathbb{R}^n$, $t > 0$, $\varphi \in \mathcal{D}$ with $\text{supp } \varphi \in B_t(z)$*

$$\|T\varphi\|_{L^\infty} \leq c (\|\varphi\|_{L^\infty} + t \|\nabla \varphi\|_{L^\infty}).$$

PROOF OF THEOREM 5.19. For simplicity, we give the proofs of (i), (ii) and (iii) for $Q = Q_{00}$. The same methods apply to the general cube because of the dilation-translation nature of the estimates. Thus, consider the unit atom $a = a_{Q_{00}}$ with $x_{Q_{00}} = 0$ and $l(Q_{00}) = 1$. First property (i) immediately follows from the fact that $T^*(y^\gamma) = 0$ for $|\gamma| \leq N_0$. To get (ii) we consider two cases: $|x| \leq 6\sqrt{n}$ and $|x| > 6\sqrt{n}$. For $|x| \leq 6\sqrt{n}$, use Lemma 5.20 to obtain

$$|Ta(x)| \leq \|Ta\|_{L^\infty} \leq c (\|a\|_{L^\infty} + \|\nabla a\|_{L^\infty}) \leq c.$$

If $|x| > 6\sqrt{n}$, we get

$$\begin{aligned} |Ta(x)| &= \left| \int K(x, y) a(y) dy \right| \\ &= \left| \int_{3Q_{00}} \left[K(x, y) - \sum_{|\gamma| \leq N_0} \frac{D_{(y)}^\gamma K(x, 0)}{\gamma!} y^\gamma \right] a(y) dy \right|, \end{aligned} \quad (5.9)$$

since a_Q is an N_0 -atom, and thus, has N_0 vanishing moments $\int y^\gamma a_Q(y) dy = 0$ for $|\gamma| \leq N_0$. Then (5.9) is bounded by

$$\int_{3Q_{00}} \sum_{|\gamma| \leq N_0} \left| \left[D_{(y)}^\gamma K(x, \theta(y)) - D_{(y)}^\gamma K(x, 0) \right] \frac{y^\gamma}{\gamma!} \right| |a(y)| dy.$$

Note that if $y \in \text{supp } a$, then $2|\theta(y)| \leq 2|y| \leq 2 \cdot 3\sqrt{n} < |x|$, and, using the property $(II_{N+\epsilon})$ of the kernel K to estimate the difference, we get

$$\sum_{|\gamma| \leq N_0} |D_{(y)}^\gamma K(x, \theta(y)) - D_{(y)}^\gamma K(x, 0)| \leq c \frac{|\theta(y)|^\epsilon}{|x|^{n+\gamma+\epsilon}} \leq c \frac{|y|^\epsilon}{|x|^{n+N_0+\epsilon}}.$$

Thus,

$$|Ta(x)| \leq \frac{c_{n, N_0}}{|x|^{n+N_0+\epsilon}} \int_{3Q_{00}} |y|^{N_0+\epsilon} |a(y)| dy \leq \frac{c}{|x|^{n+N_0+\epsilon}}.$$

In order to show (iii), we prove that

$$|Ta(x) - Ta(x')| \leq c |x - x'|^\epsilon \left(\frac{1}{(1 + |x|)^{n+N_0+\epsilon}} + \frac{1}{(1 + |x'|)^{n+N_0+\epsilon}} \right). \quad (5.10)$$

In the case $|x - x'| \geq 1$, the estimate (5.10) follows trivially from (ii) and the triangle inequality. For $|x - x'| < 1$ and $|x| > 10\sqrt{n}$, we use vanishing moments of $a(x)$ and the integral form of the remainder to get

$$\begin{aligned} |Ta(x) - Ta(x')| &= \left| \int (K(x, y) - K(x', y)) a(y) dy \right| = \left| \int_{3Q_{00}} [K(x, y) \right. \\ &\quad \left. - \sum_{|\gamma| \leq N_0-1} \frac{D_{(y)}^\gamma K(x, 0)}{\gamma!} y^\gamma - K(x', y) + \sum_{|\gamma| \leq N_0-1} \frac{D_{(y)}^\gamma K(x', 0)}{\gamma!} y^\gamma] a(y) dy \right| \end{aligned}$$

$$\leq \int_{3Q_{00}} \int_0^1 \frac{(1-s)^{N_0-1}}{(N_0-1)!} \sum_{|\gamma|=N_0} \left| D_{(y)}^\gamma K(x, sy) - D_{(y)}^\gamma K(x', sy) \right| \frac{|y|^\gamma}{\gamma!} |a(y)| ds dy.$$

If $|x| \geq 10\sqrt{n}$ and $y \in \text{supp } a$, then $|x - sy| \geq |x| - s|y| \geq 10\sqrt{n} - 3\sqrt{n} \geq 2|x - x'|$ and also $|x - sy| \geq |x| - s|y| \geq |x| - 3\sqrt{n} \geq |x| - \frac{|x|}{2} \geq \frac{|x|}{2}$. By $(II_{N+\epsilon})$ the last integral is bounded by

$$c \int_{3Q_{00}} \int_0^1 \frac{|x - x'|^\epsilon}{|x - sy|^{n+N_0+\epsilon}} |y|^{N_0} ds dy \leq c \frac{|x - x'|^\epsilon}{|x|^{n+N_0+\epsilon}}.$$

In case $|x - x'| < 1$ and $|x| \leq 10\sqrt{n}$, an exact repetition of the argument on p. 85 of [FJW] or part (c) on p.62 of [FTW] shows that

$$|Ta(x) - Ta(x')| \leq c|x - x'|^\epsilon$$

by using the decay property (I) and the Lipschitz condition $(II_{0+\epsilon})$ of the kernel K , which holds for any $CZO(N_0 + \epsilon)$, $N_0 \geq 0$. This completes the proof. ■

Corollary 5.21 *Let $1 \leq p < \infty$, $0 < q < \infty$, and let W satisfy any of (A1)-(A3). Suppose $0 \leq \alpha \leq \frac{\beta-n}{p} - [\frac{\beta-n}{p}]$, where β is the doubling exponent of W . Let $N = [\frac{\beta-n}{p} - \alpha]$ and $\frac{\beta-n}{p} - [\frac{\beta-n}{p}] < \epsilon \leq 1$. If $T \in CZO(N + \epsilon) \cap WBP$, $T1 = 0$ and $T^*(y^\gamma) = 0$ for $|\gamma| \leq N$, then T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$.*

PROOF. By the previous theorem T maps any smooth N -atom into a smooth $(\epsilon, N + n + \epsilon, N)$ -molecule. This molecule is a smooth molecule for $\dot{B}_p^{\alpha q}(W)$ if

- (i) $\alpha < \epsilon \leq 1$,
- (ii) $M = N + n + \epsilon > J = n + \frac{\beta-n}{p} \iff [\frac{\beta-n}{p} - \alpha] = [\frac{\beta-n}{p}] > \frac{\beta-n}{p} - \epsilon$ and
- (iii) $N = \max([J - n - \alpha], -1) = [\frac{\beta-n}{p} - \alpha]$, which are all true.

By the boundedness criterion (Lemma 5.13), T is bounded on $\dot{B}_p^{\alpha q}(W)$. ■

Corollary 5.22 *Let $1 \leq p < \infty$, $0 < q < \infty$, and let W satisfy any of (A1)-(A3).*

Suppose $0 \leq \frac{\beta-n}{p} - [\frac{\beta-n}{p}] < \alpha < 1$, where β is the doubling exponent of W . Let $N = [\frac{\beta-n}{p} - \alpha]$ and $\alpha < \epsilon \leq 1$. If $T \in CZO(N+1+\epsilon) \cap WBP$, $T1 = 0$ and $T^(y^\gamma) = 0$ for $|\gamma| \leq N+1$, then T is bounded on $\dot{B}_p^{\alpha q}(W)$.*

PROOF. By Theorem 5.19, T maps any smooth $(N+1)$ -atom into a smooth $(\epsilon, N+1+n+\epsilon, N+1)$ -molecule, which is also a smooth $(\epsilon, N+1+n+\epsilon, N)$ -molecule. This one, in its turn, is a smooth molecule for $\dot{B}_p^{\alpha q}(W)$, since

$$(i) \quad \alpha < \epsilon \leq 1,$$

$$(ii) \quad M = N+1+n+\epsilon > J = n + \frac{\beta-n}{p} \iff [\frac{\beta-n}{p} - \alpha] + 1 > \frac{\beta-n}{p} - \epsilon \text{ and}$$

$$(iii) \quad N = \max([J-n-\alpha], -1) = [\frac{\beta-n}{p} - \alpha].$$

By the boundedness criterion (Lemma 5.13), T extends to a bounded operator on $\dot{B}_p^{\alpha q}(W)$. ■

Remark 5.23 *Note that the condition $T^*(y^\gamma) = 0, |\gamma| \leq N$, can be very restrictive; for example, the Hilbert transform does not satisfy this condition for $|\gamma| > 0$. On the other hand, we have considered a general class of CZOs, not necessarily of convolution type. Utilizing the convolution structure will let us drop the above condition.*

Let $N \in \mathbb{N} \cup \{0\}$. Let T be a convolution operator, i.e., the kernel $K(x, y) = K(x-y)$ is defined on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$(C.1) \quad |K(x)| \leq \frac{c}{|x|^n},$$

$$(C.2) \quad |D^\gamma K(x)| \leq \frac{c}{|x|^{n+|\gamma|}}, \text{ for } |\gamma| \leq N+1,$$

$$(C.3) \quad \int_{R_1 < |x| < R_2} K(x) dx = 0, \text{ for all } 0 < R_1 < R_2 < \infty.$$

Remark 5.24 We replace (II_N) and $(II_{N+\epsilon})$ of the general CZO kernel with the slightly stronger smoothness condition (C.2) to make the proof below more concise. The reader can check that conditions corresponding to (II_N) and $(II_{N+\epsilon})$ in the convolution case would suffice for the statements below.

Now we obtain an analog of Theorem 5.19 saying that T maps smooth atoms into smooth molecules, and then we show the boundedness of T .

Theorem 5.25 Let $0 \leq \alpha < 1$, $0 < \epsilon \leq 1$, $N \in \mathbb{N} \cup \{0\}$. Let T be a convolution operator with a kernel K satisfying (C.1)-(C.3). Then T maps any smooth N -atom a_Q into a fixed multiple of a smooth $(\epsilon, N + 1 + n, N)$ -molecule.

More precisely, we will show that

$$(i) \quad \int x^\gamma T a_Q(x) dx = 0, \text{ for } |\gamma| \leq N, \text{ and}$$

$$(ii) \quad |D^\gamma T a_Q(x)| \leq c |Q|^{-1/2-|\gamma|/n} \left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-(N+n+1)} \text{ for } |\gamma| = 0, 1.$$

By the Mean Value Theorem, (ii) with $|\gamma| = 1$ implies the Lipschitz condition (M4) for $|\gamma| = 0$.

PROOF. To obtain (ii) we first consider $x \notin 10\sqrt{n}Q$. Then

$$\begin{aligned} |T a_Q(x)|, |\nabla T a_Q(x)| &= \left| \sum_{|\gamma_0|=0 \text{ or } 1} \int_{3Q} D^{\gamma_0} K(x-y) a_Q(y) dy \right| \\ &= \left| \sum_{|\gamma_0|=0 \text{ or } 1} \int_{3Q} [D^{\gamma_0} K(x-y)] \right| \end{aligned}$$

$$- \sum_{|\gamma| \leq N-|\gamma_0|} \left[\frac{D^\gamma D^{\gamma_0} K(x-x_Q)}{\gamma!} (x_Q-y)^\gamma \right] a_Q(y) dy \Bigg| \quad (5.11)$$

since a_Q is an N -atom, and thus, has N vanishing moments $\int y^\gamma a_Q(y) dy = 0$ for $|\gamma| \leq N$. Then (5.11) is bounded by

$$\sum_{|\gamma_0|=0 \text{ or } 1} \int_{3Q} \sum_{|\gamma|=N+1-|\gamma_0|} \frac{|D^{\gamma+\gamma_0} K(x-x_Q+\theta(y-x_Q))|}{\gamma!} |x_Q-y|^{N-|\gamma_0|+1} |a_Q(y)| dy,$$

for some $0 \leq \theta \leq 1$. Since $x \notin 10\sqrt{n}Q$ and $y \in 3Q$, $|y-x_Q| \leq 2\sqrt{n}l(Q) \leq \frac{1}{2}|x-x_Q|$.

Using property (C.2) of the kernel K , we get

$$|D^\gamma K(x-x_Q+\theta(y-x_Q))| \leq \frac{c}{|x-x_Q+\theta(y-x_Q)|^{n+|\gamma|}} \approx \frac{c}{|x-x_Q|^{n+|\gamma|}}.$$

So,

$$\begin{aligned} |Ta_Q(x)|, |\nabla Ta_Q(x)| &\leq \frac{c_{n,N}}{|x-x_Q|^{n+N+1}} \int_{3Q} |x_Q-y|^{N-|\gamma_0|+1} |a_Q(y)| dy \\ &\leq c \frac{[l(Q)]^{N-|\gamma_0|+1}}{|x-x_Q|^{n+N+1}} |Q|^{-1/2} |Q| = c |Q|^{-1/2-|\gamma_0|/n} \left[\frac{l(Q)}{|x-x_Q|} \right]^{n+N+1}, \end{aligned}$$

by the properties of a_Q .

If $x \in 10\sqrt{n}Q$ and $y \in 3Q$, then $|x-y| \leq 13nl(Q)$, so by the cancellation property (C.3) of K (using $D^\gamma(K * a_Q) = K * (D^\gamma a_Q)$), we obtain

$$\begin{aligned} |Ta_Q(x)|, |\nabla Ta_Q(x)| &\leq \sum_{|\gamma_0|=0 \text{ or } 1} \left| \int_{3Q} K(x-y) D^{\gamma_0} a_Q(y) dy \right| \\ &= \sum_{|\gamma_0|=0 \text{ or } 1} \left| \int_{3Q} K(x-y) [D^{\gamma_0} a_Q(y) - D^{\gamma_0} a_Q(x)] dy \right| \\ &\leq c \int_{|y-x| \leq 13nl(Q)} \frac{1}{|x-y|^n} |Q|^{-1/2-|\gamma_0|/n-1/n} |x-y| dy \\ &\leq c |Q|^{-1/2-|\gamma_0|/n-1/n} \int_0^{13nl(Q)} r^{-n} r r^{n-1} dr = c |Q|^{-1/2-|\gamma_0|/n}. \end{aligned}$$

This concludes the proof of (ii).

Property (i) comes from the fact that T is a convolution operator and a_Q has vanishing moments up to order N . Property (ii) guarantees the absolute convergence of the integral in (i). ■

Corollary 5.26 *Convolution operators with kernels satisfying (C.1)-(C.3) are bounded on $\dot{B}_p^{\alpha q}(W)$ if W satisfies any of (A1)-(A3) and $0 \leq \alpha < \epsilon \leq 1$, $0 < q < \infty$, $1 < p < \infty$. In particular, the Hilbert transform $\mathbb{H}$ ($n = 1$) is bounded on $\dot{B}_p^{\alpha q}(W)$ and the Riesz transforms $\mathcal{R}_j$, $j = 1, \dots, n$ ($n \geq 2$), are bounded on $\dot{B}_p^{\alpha q}(W)$.*

PROOF. This is an immediate consequence of Theorem 5.25 and Lemma 5.13: choose $N = \lceil \frac{\beta-n}{p} - \alpha \rceil$ in Theorem 5.25; then T maps any smooth N -atom into a smooth $(\epsilon, N+1+n, N)$ -molecule, which is either a smooth $(\epsilon, N+1+n, N)$ -molecule for $\dot{B}_p^{\alpha q}(W)$, if $\alpha \leq \frac{\beta-n}{p} - \lceil \frac{\beta-n}{p} \rceil$ or an $(\epsilon, N+1+n, N-1)$ -molecule for $\dot{B}_p^{\alpha q}(W)$, if $1 > \alpha > \frac{\beta-n}{p} - \lceil \frac{\beta-n}{p} \rceil$. Note that both Hilbert and Riesz transforms are convolution type operators with kernels satisfying (C.1)-(C.3). ■

CHAPTER 6

Application to Wavelets

Consider a pair (φ, ψ) from $\mathcal{A}$ with the mutual property (2.1). Then the family $\{\varphi_Q, \psi_Q\}$ behaves similarly to an orthonormal system because of the property

$$f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q = \sum_Q s_Q \psi_Q \quad \text{for all } f \in \mathcal{S}'/\mathcal{P}.$$

However, this system does not constitute an orthonormal basis. This can be achieved by the Meyer and Lemarié construction of a wavelet basis with the generating function $\theta \in \mathcal{S}$ (see [LM] and [M1]):

Theorem 6.1 *There exist real-valued functions $\theta^{(i)} \in \mathcal{S}(\mathbb{R}^n)$, $i = 1, \dots, 2^n - 1$, such that the collection $\{\theta_{\nu k}^{(i)}\} = \{2^{\nu n/2} \theta^{(i)}(2^\nu x - k)\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.*

The functions $\theta^{(i)}$ satisfy

$$\text{supp } \hat{\theta}^{(i)} \subseteq \left\{ \left[-\frac{8}{3}\pi, \frac{8}{3}\pi \right]^n \setminus \left[-\frac{2}{3}\pi, \frac{2}{3}\pi \right]^n \right\}$$

and, hence,

$$\int_{\mathbb{R}} x^\gamma \theta(x) dx = 0 \quad \text{for all multi-indices } \gamma.$$

Thus, we have $f = \sum_{i=1}^{2^n-1} \sum_Q \langle f, \theta_Q^{(i)} \rangle \theta_Q^{(i)}$ for all $f \in L^2(\mathbb{R}^n)$. This identity extends to all $f \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$.

Theorem 6.2 *Let $\alpha \in \mathbb{R}, 0 < q \leq \infty, 1 \leq p < \infty$, and let W satisfy any of (A1)-(A3). Let $\theta^{(i)}, i = 1, \dots, 2^n - 1$, be generating wavelet functions as in Theorem 6.1. Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \sum_{i=1}^{2^n-1} \left\| \left\{ \langle \vec{f}, \theta_Q^{(i)} \rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)}.$$

PROOF. Assume $i = 1, \dots, 2^n - 1$. Since $\{\theta_Q^{(i)}\}_{Q,i}$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$, the inequality

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \equiv \left\| \sum_{Q,i} \langle \vec{f}, \theta_Q^{(i)} \rangle \theta_Q^{(i)} \right\|_{\dot{B}_p^{\alpha q}(W)} \leq c \sum_i \left\| \left\{ \langle \vec{f}, \theta_Q^{(i)} \rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \quad (6.1)$$

follows immediately from Theorem 4.2. Therefore, we need to focus only on the opposite direction.

Let $\varphi \in \mathcal{A}$ be such that $\sum_{\nu \in \mathbb{Z}} |\hat{\varphi}(2^\nu \xi)|^2 = 1$ for $\xi \neq 0$. Let $\vec{s}_Q = \langle \vec{f}, \varphi_Q \rangle$. Applying the boundedness of the φ -transform (Theorem 4.15), we obtain

$$\left\| \{\vec{s}_Q\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)}. \quad (6.2)$$

Now for each i and Q , define $\vec{t}_Q^{(i)} = \langle \vec{f}, \theta_Q^{(i)} \rangle$. Since $\theta_Q^{(i)} \in \mathcal{S}$, by the φ -transform decomposition with $\psi = \varphi$, we have $\theta_Q^{(i)} = \sum_P \langle \theta_Q^{(i)}, \varphi_P \rangle \varphi_P$, which gives

$$\vec{t}_Q^{(i)} = \sum_P \overline{\langle \theta_Q^{(i)}, \varphi_P \rangle} \langle \vec{f}, \varphi_P \rangle = \sum_P a_{QP}^{(i)} \vec{s}_P.$$

Since $\text{supp } \hat{\varphi}_P \cap \text{supp } \hat{\theta}_Q^{(i)} \neq \{\emptyset\}$ only if $l(Q) = 2^j l(P)$ with $j = 1, 2, 3, 4$ (recall that $\text{supp } \hat{\varphi}_P \subseteq \{\xi \in \mathbb{R}^n : 2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\}$ when $l(P) = 2^{-\mu}$), we see that

$a_{QP}^{(i)} = \langle \theta_Q^{(i)}, \varphi_P \rangle = 0$ unless $2 \leq \frac{l(Q)}{l(P)} \leq 16$, in which case

$$|a_{QP}| \leq c_M \left(1 + \frac{|x_Q - x_P|}{l(Q)} \right)^{-M} \text{ for each } M > 0,$$

as was shown in [FJW], p. 72. Let $M > \frac{n}{p'} + \frac{\beta}{p}$. Then $A^{(i)} := (a_{QP}^{(i)})$ is an almost diagonal matrix for each i , and, by Theorem 1.10,

$$\|\{\vec{t}_Q^{(i)}\}_Q\|_{\dot{b}_p^{\alpha q}(W)} \leq c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}. \quad (6.3)$$

Combining (6.3) with (6.2) we get the opposite direction of (6.1). $\blacksquare$

Corollary 6.3 *Let $\{N\psi^{(i)}\}$, $i = 1, \dots, 2^n - 1$, be a collection of Daubechies DN generating wavelet functions for $L^2(\mathbb{R}^n)$ with compact supports linearly dependent on N (for more details, see [D]). Then for any $\vec{f}$ with $f_j \in \mathcal{S}'/\mathcal{P}(\mathbb{R}^n)$, $j = 1, \dots, m$,*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} \approx \sum_{i=1}^{2^n-1} \left\| \left\{ \langle \vec{f}, N\psi_Q^{(i)} \rangle \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(W)} \quad (6.4)$$

for sufficiently large N .

PROOF. First, observe that there exists a constant c such that for all $i = 1, \dots, 2^n - 1$, the functions $\frac{N\psi^{(i)}}{c}$ are smooth molecules, and so $\left\{ \frac{N\psi_Q^{(i)}}{c} \right\}_Q$ is a family of smooth molecules for $\dot{B}_p^{\alpha q}(W)$ if we choose N sufficiently large to have the necessary smoothness and vanishing moments. Second, if $\varphi \in \mathcal{A}$, then $\left(\langle N\psi_Q^{(i)}, \varphi_P \rangle_{QP} \right) \in \mathbf{ad}_p^{\alpha q}(\beta)$ by (5.2). Applying these two facts in the proof of the previous theorem, we get (6.4). $\blacksquare$

CHAPTER 7

Duality

7.1 General facts on duality

An important tool that we need is the duality on $l^q(X)$ with X being a Banach space.

By definition $l^q(X)$, $0 < q < \infty$ is the set of all sequences $\{f_\nu\}_{\nu \in \mathbb{Z}}$ with $f_\nu \in X$, $\nu \in \mathbb{Z}$ such that $\left(\sum_{\nu \in \mathbb{Z}} \|f_\nu\|_X^q \right)^{1/q} < \infty$. If $1 \leq q < \infty$, then $(l^q(X))^* = l^{q'}(X^*)$ (see [D, Chapter 8]), and if g is a continuous linear functional on $l^q(X)$ identified with $\{g_\nu\}_{\nu \in \mathbb{Z}} \in l^{q'}(X^*)$, then the duality is represented as

$$g(f) = (f, g) = \sum_{\nu \in \mathbb{Z}} \langle f_\nu, g_\nu \rangle_X,$$

where $\langle f_\nu, g_\nu \rangle_X = g_\nu(f_\nu)$ is the pairing between X and X^* . We will mainly be concerned with $X = L^p$, $1 \leq p < \infty$, or $L^p(W)$, $1 < p < \infty$, and, thus, $X^* = L^{p'}$ or $L^{p'}(W^{-p'/p})$, respectively, with the pairing $\langle f, g \rangle_X = \int \langle f(x), g(x) \rangle_{\mathcal{H}} dx$.

If $0 < q < 1$, and $X = L^p$, $1 \leq p < \infty$, then $(l^q(L^p))^* = l^\infty(L^{p'})$ (see [T, p.177]) and the pairing is defined as above.

7.2 Duality of sequence Besov spaces

Theorem 7.1 *Let W be a matrix weight, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 < p < \infty$. Then*

- (i) $\dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}) \subseteq \left[\dot{b}_p^{\alpha q}(W) \right]^*$ *always*
- (ii) $\left[\dot{b}_p^{\alpha q}(W) \right]^* \subseteq \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$ *if $W \in A_p$.*

We will prove this theorem, which implies (1.12) of Theorem 1.17, in several steps. The use of reducing operators is essential and helps to understand why certain conditions on the weight W are necessary.

PROOF OF (I) OF THEOREM 7.1. For each $\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$ define a functional l_t on $\dot{b}_p^{\alpha q}(W)$ by

$$l_t(\vec{s}) = (\vec{s}, \vec{t}) = \sum_Q \langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}} \quad \text{for any } \vec{s} = \{\vec{s}_Q\}_Q \in \dot{b}_p^{\alpha q}(W).$$

The calculations below show that this sum converges and $l_t \in \left[\dot{b}_p^{\alpha q}(W) \right]^*$:

$$\begin{aligned} \left| \sum_Q \langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}} \right| &\leq \sum_{\nu \in \mathbb{Z}} \sum_{Q \in Q_\nu} |\langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}}| = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \sum_{Q \in Q_\nu} |\langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}}| |Q|^{-1} \chi_Q(t) dt \\ &\quad \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \sum_{Q \in Q_\nu} |Q|^{-1} |\langle W^{-1/p}(x) W^{1/p}(x) \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}}| \chi_Q(x) dx. \end{aligned} \quad (7.1)$$

Using the self-adjointness of W and the Cauchy-Schwarz inequality, we bound (7.1)

by

$$\sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \sum_{Q \in Q_\nu} \left(|Q|^{-\frac{\alpha}{n} - \frac{1}{2}} \chi_Q(x) \|W^{1/p}(x) \vec{s}_Q\|_{\mathcal{H}} \right) \left(|Q|^{\frac{\alpha}{n} - \frac{1}{2}} \chi_Q(x) \|W^{-1/p}(x) \vec{t}_Q\|_{\mathcal{H}} \right) dx.$$

Applying Hölder's inequality several times, we estimate $l_t(\vec{s})$ by

$$\sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \left(\sum_{Q \in Q_\nu} \left(|Q|^{-\frac{\alpha}{n} - \frac{1}{2}} \chi_Q(x) \|W^{1/p}(x) \vec{s}_Q\|_{\mathcal{H}} \right)^p \right)^{\frac{1}{p}}$$

$$\begin{aligned}
& \times \left(\sum_{Q \in Q_\nu} \left(|Q|^{\frac{\alpha}{n}-\frac{1}{2}} \chi_Q(x) \|W^{-1/p}(x) \vec{t}_Q\|_{\mathcal{H}} \right)^{p'} \right)^{\frac{1}{p'}} dx \\
& \leq \sum_{\nu \in \mathbb{Z}} \left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{\alpha}{n}-\frac{1}{2}} \chi_Q \vec{s}_Q \right\|_{L^p(W)} \left\| \sum_{Q \in Q_\nu} |Q|^{\frac{\alpha}{n}-\frac{1}{2}} \chi_Q \vec{t}_Q \right\|_{L^{p'}(W^{-p'/p})} \\
& \leq \|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})},
\end{aligned} \tag{7.2}$$

for $1 < q < \infty$. In case of $0 < q \leq 1$, we bound (7.2) by $\|\vec{s}\|_{\dot{b}_p^{\alpha 1}(W)} \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha \infty}(W^{-p'/p})}$.

Since l^q is embedded into l^1 when $0 < q \leq 1$, we estimate the previous product by

$$\|\vec{s}\|_{\dot{b}_p^{\alpha q}(W)} \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha \infty}(W^{-p'/p})}. \quad \blacksquare$$

In terms of reducing operators (or using (1.10)) the previous lemma states

$$\dot{b}_{p'}^{-\alpha q'}(\{A_Q^\#\}) \subseteq \left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^*. \tag{7.3}$$

If we follow along the lines of the proof again but instead of $W^{-1/p}(t) W^{1/p}(t)$ in (7.1)

use $A_Q^{-1} A_Q$, then we obtain the following statement.

Lemma 7.2 *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$ and $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$. Then*

$$\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \subseteq \left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^*. \tag{7.4}$$

In fact, if we have only proven (7.4), then (7.3) (and equivalently part (i) of Theorem

7.1) could have been obtained as a consequence of (7.4) and Corollary 4.20, i.e.,

$$\dot{b}_{p'}^{-\alpha q'}(\{A_Q^\#\}) \subseteq \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \subseteq \left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^*. \tag{7.5}$$

Observe that (7.4) holds for any $\{A_Q\}_Q \in \mathcal{RS}_{\mathcal{D}}$, not necessarily generated by W .

Now we will study the opposite embeddings. By Lemma 7.3 below, we will get

$$\left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^* \subseteq \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \tag{7.6}$$

without any additional assumptions on the sequence $\{A_Q\}_Q$. Note that combining (7.4) and (7.6), we obtain (1.13). Applying Corollary 4.20 again, (7.6) is continued as

$$\left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^* \subseteq \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \stackrel{A_p}{\subseteq} \dot{b}_{p'}^{-\alpha q'}(\{A_Q^\# \}) \approx \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p})$$

with the second embedding being held under A_p condition. Thus, the embedding (ii) of Theorem 7.1 holds if $W \in A_p$.

Lemma 7.3 *Let $\{A_Q\}_Q \in \mathcal{RS}_D$, $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. Then (7.6) holds.*

PROOF. Let $l \in \left[\dot{b}_p^{\alpha q}(\{A_Q\}) \right]^*$. We show that there exists $\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ such that for any $\vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\})$

$$l(\vec{s}) = (\vec{s}, \vec{t}) = \sum_Q \langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}} \quad \text{and} \quad \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|l\|.$$

Let $\vec{e}_J^{(k)}$ denote a vector-valued sequence enumerated by dyadic cubes such that in the k^{th} component (k^{th} row) of this vector the J^{th} entry (corresponding to the dyadic cube J) is equal to 1 and all other entries are zero:

$$\vec{e}_J^{(k)} = (\dots, \{0\}_Q, \dots, \{\dots 0 \dots 1_{J^{th} \text{ entry}} \dots 0 \dots\}_Q - k^{th} \text{ row}, \dots, \{0\}_Q, \dots)^T.$$

Now if $\vec{s}$ has only finitely many non-zero entries, i.e.,

$$\vec{s} = \sum_{Q \text{ finite}} \sum_{k=1}^m s_Q^{(k)} \vec{e}_Q^{(k)},$$

then by linearity

$$l(\vec{s}) = \sum_{Q \text{ finite}} \sum_{k=1}^m s_Q^{(k)} l(\vec{e}_Q^{(k)}) =: \sum_{Q \text{ finite}} \sum_{k=1}^m s_Q^{(k)} t_Q^{(k)}.$$

By continuity, since finitely non-zero sequences are dense ($p, q < \infty$), we get

$$l(\vec{s}) = \sum_{Q \in \mathcal{D}} \sum_{k=1}^m s_Q^{(k)} t_Q^{(k)} = \sum_{Q \in \mathcal{D}} \langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}} \quad \text{for any } \vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\}).$$

Now everything is set up to show that $\vec{t} := (\{t_Q^{(1)}\}_Q, \{t_Q^{(2)}\}_Q, \dots, \{t_Q^{(m)}\}_Q)^T \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$. For $\vec{s} \in \dot{b}_p^{\alpha q}(\mathbb{R}^m)$, set $\vec{s}_Q = A_Q^{-1} \vec{s}$ and define

$$\begin{aligned} \tilde{l}(\vec{s}) &:= l(\{A_Q^{-1} \vec{s}_Q\}_Q) = l(\{\vec{s}_Q\}_Q) = \sum_Q \langle \vec{s}_Q, \vec{t}_Q \rangle_{\mathcal{H}} = \sum_Q \langle A_Q \vec{s}_Q, A_Q^{-1} \vec{t}_Q \rangle_{\mathcal{H}} \\ &= \sum_Q \langle \vec{s}_Q, \vec{\tilde{t}}_Q \rangle_{\mathcal{H}}, \end{aligned}$$

where $\vec{\tilde{t}}_Q = A_Q^{-1} \vec{t}_Q$. By above,

$$|\tilde{l}(\vec{s})| \leq c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = c \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\mathbb{R}^m)},$$

i.e., l induces a continuous linear functional $\tilde{l}$ on $\dot{b}_p^{\alpha q}(\mathbb{R}^m)$. By Lemma 7.4 below $\{\vec{\tilde{t}}_Q\}_Q \in \dot{b}_{p'}^{-\alpha q'}(\mathbb{R}^m)$. Since the inside $L^{p'}$ -norm of the $\dot{b}_{p'}^{-\alpha q'}(\mathbb{R}^m)$ -norm of $\vec{\tilde{t}}$ is

$$\left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{1}{2}} \vec{\tilde{t}}_Q \chi_Q \right\|_{L^{p'}} = \left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{1}{2}} \|A_Q^{-1} \vec{t}_Q\|_{\mathcal{H}} \chi_Q \right\|_{L^{p'}} = \left\| \sum_{Q \in Q_\nu} |Q|^{-\frac{1}{2}} \vec{t}_Q \chi_Q \right\|_{L^{p'}(\{A_Q^{-1}\})},$$

$\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ and the lemma is proved. $\blacksquare$

Lemma 7.4 *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. Then*

$$\left[\dot{b}_p^{\alpha q}(\mathbb{R}^m) \right]^* \approx \dot{b}_{p'}^{-\alpha q'}(\mathbb{R}^m). \quad (7.7)$$

PROOF. It suffices to show only the scalar case ($m = 1$) of (7.7), since $\vec{s} \in \dot{b}_p^{\alpha q}(\mathbb{R}^m)$ means that each component $s^{(i)}$ belongs to $\dot{b}_p^{\alpha q}$ and by making zero all but one of the components of an arbitrary $\vec{s}$ we obtain (7.7).

The embedding $\left[\dot{b}_p^{\alpha q}\right]^* \supseteq \dot{b}_{p'}^{-\alpha q'}$ is a trivial application of Hölder's inequality plus the embedding $\dot{b}_p^{\alpha q} \rightarrow \dot{b}_p^{\alpha 1}$ for $q < 1$, so we concentrate only on the opposite embedding.

Suppose $l \in \left[\dot{b}_p^{\alpha q}\right]^*$. Using linearity and continuity, l can be represented by some sequence $\{t_Q\}_Q$ as $l(s) = \sum_Q s_Q \bar{t}_Q$ for any $s = \{s_Q\} \in \dot{b}_p^{\alpha q}$ and

$$|l(s)| = \left| \sum_Q s_Q \bar{t}_Q \right| \leq \|l\| \|s\|_{\dot{b}_p^{\alpha q}}. \quad (7.8)$$

Case $q \geq 1$: For each $\nu \in \mathbb{Z}$ let $f_\nu(s)(x) = \sum_{Q \in Q_\nu} |Q|^{-\frac{\alpha}{n} - \frac{1}{2}} s_Q \chi_Q(x)$. Define a map $I : \dot{b}_p^{\alpha q} \rightarrow l^q(L^p)$ by $I(s) = \{f_\nu(s)\}_{\nu \in \mathbb{Z}}$. Observe that $\|I(s)\|_{l^q(L^p)} = \|s\|_{\dot{b}_p^{\alpha q}}$, in other words, by the natural construction I is a linear isometry onto the subspace $I(\dot{b}_p^{\alpha q})$ of $l^q(L^p)$. Then l induces a continuous linear functional $\tilde{l}$ on $I(\dot{b}_p^{\alpha q}) \subseteq l^q(L^p)$ (continuous in $l^q(L^p)$ -norm) by $\tilde{l}(I(s)) = l(s)$. Since $l^q(L^p)$ is a Banach space, by the Hahn-Banach Theorem $\tilde{l}$ extends to a continuous linear functional $\tilde{l}_{ext}$ on all of $l^q(L^p)$ by $\tilde{l}_{ext}|_{I(\dot{b}_p^{\alpha q})} = \tilde{l}$ with $\|\tilde{l}_{ext}\| = \|\tilde{l}\| \leq \|l\|$. Since $[l^q(L^p)]^* = l^{q'}(L^{p'})$, $\tilde{l}_{ext}$ is represented by a sequence $g = \{g_\nu\}_{\nu \in \mathbb{Z}} \in l^{q'}(L^{p'})$ with $\|g\| = \|\{g_\nu\}_\nu\|_{l^{q'}(L^{p'})} \leq \|l\|$ and

$$\sum_Q s_Q \bar{t}_Q = l(s) = \tilde{l}(\{f_\nu(s)\}) = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} f_\nu(s)(x) \bar{g}_\nu(x) dx, \quad \text{for any } \vec{s} \in \dot{b}_p^{\alpha q},$$

or

$$\sum_Q s_Q \bar{t}_Q = \sum_{\nu \in \mathbb{Z}} \sum_{Q \in Q_\nu} |Q|^{-\frac{\alpha}{n} - \frac{1}{2}} s_Q \int_Q \bar{g}_\nu(x) dx.$$

Taking $s_Q = 0$ for all but one cube, we get $t_Q = |Q|^{-\frac{\alpha}{n} + \frac{1}{2}} \langle g_\nu \rangle_Q$. Using Hölder's inequality, we have

$$\|t\|_{\dot{b}_{p'}^{-\alpha q'}} = \left\| \left\{ \left\| \sum_{Q \in Q_\nu} \langle g_\nu \rangle_Q \chi_Q \right\|_{L^{p'}} \right\}_\nu \right\|_{l^{q'}} \leq \|\{\bar{g}_\nu\}_\nu\|_{l^{q'}(L^{p'})} \leq \|l\|.$$

Case $0 < q < 1$: Suppose $1 < p < \infty$. Fix $\nu \in \mathbb{Z}$ and let F_ν denote a finite collection of cubes from Q_ν . Set $\tau_\nu = \sum_{Q \in F_\nu} \left(|Q|^{\frac{\alpha}{n} - \frac{1}{2} + \frac{1}{p'}} |t_Q| \right)^{p'}$. Since the sum is finite, $\tau_\nu < \infty$. Let $s_Q = |Q|^{\left(\frac{\alpha}{n} - \frac{1}{2} + \frac{1}{p'}\right)p'} |t_Q|^{p'-2} t_Q$, if $Q \in F_\nu$ and $t_Q \neq 0$; otherwise let $s_Q = 0$. Note that $\|\{s_Q\}_Q\|_{\dot{b}_p^{\alpha q}} = \tau_\nu^{1/p}$. Observe that $\sum_Q s_Q \bar{t}_Q = \tau_\nu$ and by (7.8) $\tau_\nu \leq \|l\| \|s\|_{\dot{b}_p^{\alpha q}} = \|l\| \tau_\nu^{1/p}$. Since τ_ν is finite, we get $\tau_\nu^{1/p'} \leq \|l\|$ and the estimate holds independently of the collection F_ν taken. Hence, we can pass to the limit from F_ν to Q_ν . Then,

$$\|t\|_{\dot{b}_{p'}^{-\alpha\infty}} = \sup_{\nu \in \mathbb{Z}} \left(\sum_{Q \in Q_\nu} \left(|Q|^{\frac{\alpha}{n} - \frac{1}{2} + \frac{1}{p'}} |t_Q| \right)^{p'} \right)^{1/p'} = \sup_{\nu \in \mathbb{Z}} \tau_\nu^{1/p'} \leq \|l\| \quad \text{or} \quad t \in \dot{b}_{p'}^{-\alpha\infty}.$$

Now assume $p = 1$. Fix $P \in \mathcal{D}$ and set $s^{(P)} = \{s_Q^{(P)}\}_Q$ by $s_Q^{(P)} = |Q|^{\frac{\alpha}{n} - \frac{1}{2}} \text{sgn } \bar{t}_Q$ if $Q = P$ and $s_Q^{(P)} = 0$ otherwise. Then $\left\| \{s_Q^{(P)}\}_Q \right\|_{\dot{b}_1^{\alpha q}} = 1$ and $|P|^{\frac{\alpha}{n} - \frac{1}{2}} |t_P| = \sum_Q s_Q^{(P)} \bar{t}_Q = l(s^{(P)}) \leq \|l\| \left\| \{s_Q^{(P)}\}_Q \right\|_{\dot{b}_1^{\alpha q}} = \|l\|$ for any $P \in \mathcal{D}$. Hence,

$$\|t\|_{\dot{b}_\infty^{-\alpha\infty}} = \sup_{P \in \mathcal{D}} |P|^{\frac{\alpha}{n} - \frac{1}{2}} |t_P| \leq \|l\| \quad \text{or} \quad t \in \dot{b}_\infty^{-\alpha\infty}.$$

■

7.3 Equivalence of sequence and discrete averaging Besov spaces

In this section we discuss norm equivalence between $\dot{B}_p^{\alpha q}(\{A_Q\})$ and $\dot{b}_p^{\alpha q}(\{A_Q\})$. We suppose $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 \leq p < \infty$ for all statements in this section. If $q = \infty$, then set $q' = 1$.

Lemma 7.5 *If $\{A_Q\}$ is a doubling sequence of order p , then for $\vec{s}_Q = \langle \vec{f}, \varphi_Q \rangle$*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}. \quad (7.9)$$

PROOF. Note that $\vec{s}_Q = |Q|^{1/2}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)$ for $Q = Q_{\nu k}$, where $\tilde{\varphi}(x) = \overline{\varphi(-x)}$.

Let $\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} =: \|\{J_\nu^{1/p}\}_\nu\|_{l_q^\alpha}$, where

$$J_\nu = \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \|A_{Q_{\nu k}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)\|^p dx. \quad (7.10)$$

Since $\tilde{\varphi}_\nu * \vec{f} \in E_\nu$, Lemma 4.9 implies

$$(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k) = \sum_{l \in \mathbb{Z}^n} (\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x) \gamma(k - l - 2^\nu x), \quad x \in Q_{\nu k}$$

for some $\gamma \in \Gamma$. Then

$$\begin{aligned} J_\nu &\leq \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \left(\sum_{l \in \mathbb{Z}^n} \|A_{Q_{\nu k}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\| |\gamma(k - l - 2^\nu x)| \right)^p dx \\ &\leq c \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \left(\sum_{l \in \mathbb{Z}^n} \frac{\|A_{Q_{\nu k}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\|}{(1 + |k - l - 2^\nu x|)^M} \right)^p dx \quad \text{for some } M > \beta + n. \end{aligned}$$

Using the discrete Hölder inequality and the fact that $M > n$, we bring the p^{th} power inside the sum on l . Furthermore, since $\{A_Q\}_Q$ is doubling, (1.7) implies

$$\|A_{Q_{\nu k}} \vec{u}\|^p \leq c(1 + |l|)^\beta \|A_{Q_{\nu(k+l)}} \vec{u}\|^p, \quad \text{for any } \vec{u} \in \mathcal{H}. \quad (7.11)$$

Thus,

$$J_\nu \leq c \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \sum_{l \in \mathbb{Z}^n} \frac{(1 + |l|)^\beta \|A_{Q_{\nu(k+l)}}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}l + x)\|^p}{(1 + |k - l - 2^\nu x|)^M} dx.$$

Changing variable ($t = x + 2^{-\nu}l$) and reindexing the sum on l , we get

$$J_\nu \leq c \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu l}} \sum_{l \in \mathbb{Z}^n} (1 + |k - l|)^{\beta - M} \|A_{Q_{\nu l}}(\tilde{\varphi}_\nu * \vec{f})(t)\|^p dt$$

$$\leq c \sum_{l \in \mathbb{Z}^n} \int_{Q_{\nu l}} \|A_{Q_{\nu l}}(\tilde{\varphi}_{\nu} * \vec{f})(t)\|^p dt = c \|\tilde{\varphi}_{\nu} * \vec{f}\|_{L^p(\{A_Q\}, \nu)}$$

(the sum on k converges since $M - \beta > n$). Thus,

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\}, \tilde{\varphi})}.$$

Now we need an independence of the space $\dot{B}_p^{\alpha q}(\{A_Q\})$ on the choice of φ (or $\tilde{\varphi}$). We apply the same strategy as in Theorem 4.15, namely, we use the proof of Corollary 7.10 below, which will imply that the last expression is equivalent to $c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\}, \varphi)}$ and, thus, (7.9) is proved. ■

Corollary 7.6 *If $\{A_Q\}$ is a doubling sequence of order p , then for $\vec{s}_Q = \langle \vec{f}, \varphi_Q \rangle$*

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(\{A_Q^{-1}\})} \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})}. \quad (7.12)$$

and

$$\|\{\vec{s}_Q\}_Q\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq c \|\vec{f}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}. \quad (7.13)$$

PROOF. For (7.12) repeat the previous proof with each A_Q replaced by A_Q^{-1} and instead of the estimate (7.11) use

$$\|A_{Q_{\nu k}}^{-1} \vec{u}\|^p \leq c(1 + |l|)^{\beta} \|A_{Q_{\nu(k+l)}}^{-1} \vec{u}\|^p, \quad \text{for any } u \in \mathcal{H}, \quad (7.14)$$

which follows from the doubling property (1.7) and duality $\|A_Q^{-1} \vec{u}\| = \sup_{\vec{v} \neq 0} \frac{|(\vec{u}, \vec{v})|}{\|A_Q \vec{v}\|}$.

For (7.13) use the obvious replacements for α , p , q and A_Q . If $1 < p < \infty$, choose $M > \beta p'/p + n$ and replace (7.11) by

$$\|A_{Q_{\nu k}}^{-1} \vec{u}\|^{p'} \leq c(1 + |l|)^{\beta p'/p} \|A_{Q_{\nu(k+l)}}^{-1} \vec{u}\|^{p'}, \quad \text{for any } \vec{u} \in \mathcal{H}, \quad (7.15)$$

which is obtained from (7.14) by raising to the power p'/p . If $p = 1$ ($p' = \infty$), then replace (7.10) with the L^∞ -norm:

$$J_\nu = \sup_{x \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} \|A_{Q_{\nu k}}^{-1}(\tilde{\varphi}_\nu * \vec{f})(2^{-\nu}k)\| \chi_{Q_{\nu k}}(x)$$

and use (7.14) instead of (7.11) to get

$$J_\nu \leq c \sup_{t \in \mathbb{R}^n} \sum_{l \in \mathbb{Z}^n} \|A_{Q_{\nu l}}^{-1}(\tilde{\varphi}_\nu * \vec{f})(t)\| \chi_{Q_{\nu l}}(t) = c \|\tilde{\varphi}_\nu * \vec{f}\|_{L^\infty(\{A_Q^{-1}\}, \nu)}.$$

■

Lemma 7.7 *Suppose $\{A_Q\}_Q$ is a doubling sequence of order p . Then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \leq c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}. \quad (7.16)$$

PROOF. Using $\vec{f} = \sum_Q \vec{s}_Q(\vec{f})\psi_Q$, we get

$$\begin{aligned} & \left\| \sum_Q \vec{s}_Q(\vec{f})\psi_Q \right\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \\ & \leq \left\| \left\{ \sum_{\mu \in \mathbb{Z}} \left(\sum_{l(P)=2^{-\nu}} \int_P \left(\sum_{l(Q)=2^{-\mu}} \|A_P \vec{s}_Q\| |(\varphi_\nu * \psi_Q)(x)| \right)^p dx \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha} \\ & = \left\| \left\{ \sum_{\mu=\nu-1}^{\nu+1} \left(\sum_{l(P)=2^{-\nu}} \int_P \left(\sum_{l(Q)=2^{-\mu}} \|A_P \vec{s}_Q\| |(\varphi_\nu * \psi_Q)(x)| \right)^p dx \right)^{1/p} \right\}_\nu \right\|_{l_q^\alpha} \\ & =: \left\| \{J_\nu^{1/p}\}_\nu \right\|_{l_q^\alpha}, \end{aligned}$$

since $\varphi_\nu * \psi_Q = 0$ if $|\mu - \nu| > 1$. Using the convolution estimates (4.2) and (4.3), we get (for any $M > 0$)

$$|(\varphi_\nu * \psi_Q)(x)| \leq c_M |Q|^{-1/2} (1 + 2^\nu |x - x_Q|)^{-M}, \quad \text{if } \mu = \nu - 1, \nu, \nu + 1. \quad (7.17)$$

If $1 < p < \infty$, choose $M = M_1 + M_2$ with $M_1 > \beta/p + n/p$ and $M_2 > n/p'$; if $p = 1$, let $M = M_1 > \beta + n$. Then applying the above estimate and Hölder's inequality, we obtain

$$J_\nu \leq c \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(P)=2^{-\nu}} \sum_{l(Q)=2^{-\mu}} \|A_P \vec{s}_Q\|^p |P| |Q|^{-p/2} (1 + 2^\nu |x_P - x_Q|)^{-M_1 p}.$$

Shifting A_P to A_Q by doubling, we get

$$J_\nu \leq c \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} \|A_Q \vec{s}_Q\|^p |Q| \sum_{l(P)=2^{-\nu}} c_\beta (1 + 2^\nu |x_P - x_Q|)^{-M_1 p + \beta}.$$

Applying Lemma 4.4 (Summation Lemma) to the sum on P , we have

$$J_\nu \leq c \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(Q)=2^{-\mu}} |Q|^{-p/2} \|A_Q \vec{s}_Q\|^p |Q| = c \sum_{\mu=\nu-1}^{\nu+1} \left\| \sum_{l(Q)=2^{-\mu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(\{A_Q\}, \mu)}.$$

Combining the estimates for all J_ν and reindexing when necessary, we get

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \leq 3c \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(\{A_Q\}, \nu)} \right\}_{\nu} \right\|_{l_q} = c \|\{\vec{s}_Q\}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.$$

■

Remark 7.8 *Theorem 1.18 is obtained by combining Lemmas 7.5 and 7.7.*

Corollary 7.9 *If $\{A_Q\}_Q$ is doubling (of order p), then*

$$\|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})} \leq c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_p^{\alpha q}(\{A_Q^{-1}\})} \quad (7.18)$$

and

$$\|\vec{f}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_Q \right\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}. \quad (7.19)$$

PROOF. For (7.18) use the previous proof with the following shifting of A_P to A_Q (similar to (7.14)):

$$\|A_P^{-1}\tilde{s}_Q\|^p \leq c_{n,\beta,p} (1 + 2^\nu |x_P - x_Q|)^\beta \|A_Q^{-1}\tilde{s}_Q\|^p, \quad (7.20)$$

where $l(P) = 2^{-\nu}$ and $l(Q) = 2^{-\mu}$ with $\mu = \nu - 1, \nu$ or $\nu + 1$; for (7.19) use the above proof with the indices $-\alpha, q', p'$; if $1 < p < \infty$, take $M > \beta p'/p + n$ and apply (7.20) raised to the power p'/p ; if $p' = \infty$, then

$$J_\nu \leq \sup_{x \in \mathbb{R}^n} \sum_{\mu=\nu-1}^{\nu+1} \sum_{l(P)=2^{-\nu}} \sum_{l(Q)=2^{-\mu}} \|A_P^{-1}\tilde{s}_Q\| |(\varphi_\nu * \psi_Q)(x)| \chi_P(x).$$

Using the convolution estimate (7.17) (with $M = M_1 > \beta + n$) and (7.20) for shifting A_P^{-1} to A_Q^{-1} , we get

$$J_\nu \leq c \sum_{\mu=\nu-1}^{\nu+1} \left\| \sum_{l(Q)=2^{-\mu}} \|A_Q^{-1}\tilde{s}_Q\| \chi_Q \right\|_{L^\infty},$$

which gives (7.19). ■

Corollary 7.10 *The spaces $\dot{B}_p^{\alpha q}(\{A_Q\})$, $\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})$ and $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ are independent of the choice of the admissible kernel, if $\{A_Q\}_Q$ is doubling (of order p).*

PROOF. Repeat the proof of Theorem 1.8 with W replaced by A_Q and use Lemmas 7.5 and 7.7 for the space $\dot{B}_p^{\alpha q}(\{A_Q\})$; for the space $\dot{B}_p^{\alpha q}(\{A_Q^{-1}\})$ apply (7.12) and (7.18); and for the space $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ use (7.13) and (7.19). ■

7.4 Properties of averaging L^p spaces

In this section we study the connection between $L^p(\{A_Q\}, \nu)$ and $L^p(W)$ and the dual of $L^p(\{A_Q\}, \nu)$.

Lemma 7.11 *Let W be a doubling matrix weight of order p , $1 \leq p < \infty$. Then for $\vec{f} \in E_\nu$, $\nu \in \mathbb{Z}$*

$$\|\vec{f}\|_{L^p(W)} \leq c \|\vec{f}\|_{L^p(\{A_Q\}, \nu)}, \quad (7.21)$$

where $\{A_Q\}_Q$ is a sequence of reducing operators generated by W and c is independent of ν .

PROOF. Using the notation $W_\nu(t) = W(2^{-\nu}t)$ and $\vec{f}_\nu(t) = \vec{f}(2^{-\nu}t)$, we write

$$\|\vec{f}\|_{L^p(W)}^p = \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \|W^{1/p}(t) \vec{f}(t)\|^p dt = \sum_{k \in \mathbb{Z}^n} 2^{-\nu n} \int_{Q_{0k}} \|W_\nu^{1/p}(t) \vec{f}_\nu(t)\|^p dt.$$

Since $\vec{f}_\nu \in E_0$, there exists $\gamma \in \Gamma$ such that $\vec{f}_\nu = \vec{f}_\nu * \gamma$. Using the decay of γ and Hölder's inequality, we get

$$\|\vec{f}\|_{L^p(W)}^p \leq \sum_{k \in \mathbb{Z}^n} 2^{-\nu n} \int_{Q_{0k}} \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}} \frac{\|W^{1/p}(t) \vec{f}_\nu(y)\|^p}{(1 + |m - k|)^M} dy dt,$$

for some $M > \beta + n$. Observe that $\|A_{Q_{\nu k}} \vec{f}_\nu(y)\|^p \approx \int_{Q_{0k}} \|W_\nu^{1/p}(t) \vec{f}_\nu(y)\|^p dt$. Using the doubling property of W to shift $A_{Q_{\nu k}}$ to $A_{Q_{\nu m}}$ (see (7.11)), we bound the previous line by

$$\begin{aligned} & c \sum_{m \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} 2^{-\nu n} \int_{Q_{0m}} (1 + |m - k|)^{-(M-\beta)} \|A_{Q_{\nu m}} \vec{f}_\nu(y)\|^p dy \\ & \leq c \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}} \|A_{Q_{\nu m}} \vec{f}_\nu(y)\|^p dy, \end{aligned}$$

where the sum on k converges, since $M > \beta + n$. Changing variables $x = 2^{-\nu}y$, we get the desired inequality (7.21). ■

Corollary 7.12 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 \leq p < \infty$. If W is doubling (of order p) and $\{A_Q\}_Q$ is a sequence of reducing operators generated by W , then*

$$\dot{B}_p^{\alpha q}(\{A_Q\}) \subseteq \dot{B}_p^{\alpha q}(W).$$

PROOF. Since $\varphi_\nu * \vec{f} \in E_\nu$, the previous lemma implies

$$\begin{aligned} \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)} &= \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * \vec{f}\|_{L^p(W)} \right\}_\nu \right\|_{l^q} \\ &\leq c \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * \vec{f}\|_{L^p(\{A_Q\}, \nu)} \right\}_\nu \right\|_{l^q} = c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}. \end{aligned}$$

■

Lemma 7.13 *Let $1 < p < \infty$ and W satisfies any of (A1)-(A3). Suppose $\vec{f} \in E_\nu$, $\nu \in \mathbb{Z}$. Then*

$$\|\vec{f}\|_{L^p(\{A_Q\}, \nu)} \leq c \|\vec{f}\|_{L^p(W)}, \quad (7.22)$$

where $\{A_Q\}_Q$ is a sequence of reducing operators produced by W and c is independent of ν .

PROOF. Using the definition of reducing operators, we write

$$\begin{aligned} \|\vec{f}\|_{L^p(\{A_Q\}, \nu)} &\approx \sum_{k \in \mathbb{Z}^n} \int_{Q_{\nu k}} \frac{1}{|Q_{\nu k}|} \int_{Q_{\nu k}} \|W^{1/p}(t) \vec{f}(x)\|^p dt dx \\ &= \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \int_{Q_{\nu k}} \|W^{1/p}(t) \vec{f}_\nu(y)\|^p dt dy, \end{aligned}$$

by changing variables $x = 2^{-\nu}y$ and denoting $\vec{f}_\nu(y) = \vec{f}(2^{-\nu}y)$. Note that $\vec{f}_\nu \in E_0$.

Applying the decomposition of an exponential type function (Lemma 4.8) to $\vec{f}_\nu = \vec{f}_\nu * \gamma$ for $\gamma \in \Gamma$ and Hölder's inequality (choose $M > \beta + n$), the last expression is bounded by

$$\begin{aligned} &c \sum_{k \in \mathbb{Z}^n} \int_{Q_{0k}} \int_{Q_{\nu k}} \sum_{m \in \mathbb{Z}^n} \frac{\|W^{1/p}(t) \vec{f}_\nu(m)\|^p}{(1 + |y - m|)^M} dt dy \\ &\leq c \sum_{m \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \frac{1}{(1 + |k - m|)^{M-\beta}} \int_{Q_{0k}} \int_{Q_{\nu m}} \|W^{1/p}(t) \vec{f}_\nu(m)\|^p dt dy, \end{aligned}$$

by applying the doubling property of W (any of (A1)-(A3) imply that W is doubling).

Integrating on y and summing on k ($M > \beta + n$), we obtain

$$\begin{aligned} \|\vec{f}\|_{L^p(\{A_Q\}, \nu)}^p &\leq c \sum_{m \in \mathbb{Z}^n} \int_{Q_{\nu m}} \|W^{1/p}(t) \vec{f}_{\nu}(m)\|^p dt \\ &= c 2^{-\nu n} \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}} \|W_{\nu}^{1/p}(t) \vec{f}_{\nu}(m)\|^p dt, \end{aligned}$$

again by changing variables. Now applying Lemma 4.12 and Lemma 4.14 (this is where (A1)-(A3) come into play), we bound the above by $c 2^{-\nu n} \|\vec{f}_{\nu}\|_{L^p(W_{\nu})}^p = c \|\vec{f}\|_{L^p(W)}^p$, which gives (7.22). ■

Corollary 7.14 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 < p < \infty$. If W satisfies any of (A1)-(A3) and $\{A_Q\}_Q$ is a sequence of reducing operators generated by W , then*

$$\dot{B}_p^{\alpha q}(W) \subseteq \dot{B}_p^{\alpha q}(\{A_Q\}).$$

PROOF. As in the proof of Corollary 7.12, use the fact that $\varphi_{\nu} * \vec{f} \in E_{\nu}$ and Lemma 7.13. ■

Remark 7.15 *Combining Corollaries 7.12 and 7.14, we have Lemma 1.19.*

In order to establish the dual of $L^p(\{A_Q\}, \nu)$, $1 < p < \infty$, we consider the following idea:

$$\begin{aligned} \|\vec{f}\|_{L^p(\{A_Q\}, \nu)}^p &= \sum_{Q \in Q_{\nu}} \int_Q \|A_Q \vec{f}(x)\|_{\mathcal{H}}^p dx = \int_{\mathbb{R}^n} \left(\sum_{Q \in Q_{\nu}} \|A_Q \vec{f}(x)\|_{\mathcal{H}} \chi_Q(x) \right)^p dx \\ &= \int_{\mathbb{R}^n} \left\| \sum_{Q \in Q_{\nu}} A_Q \chi_Q(x) \vec{f}(x) \right\|_{\mathcal{H}}^p dx =: \int_{\mathbb{R}^n} \|U_{\nu}^{1/p}(x) \vec{f}(x)\|_{\mathcal{H}}^p dx = \|\vec{f}\|_{L^p(U_{\nu})}^p, \end{aligned}$$

i.e., $L^p(\{A_Q\}, \nu) = L^p(U_\nu)$, where $U_\nu(x) = \sum_{Q \in Q_\nu} A_Q^p \chi_Q(x)$ is a matrix weight. Since the dual $[L^p(U_\nu)]^*$ can be identified with $L^{p'}(U_\nu^*)$ with $U_\nu^{1/p}(x) = (U_\nu^*)^{-1/p'}(x)$ (e.g. see [NT] or [V]), i.e., $U_\nu^*(x) = \sum_{Q \in Q_\nu} A_Q^{-p'} \chi_Q(x)$, we obtain

$$\begin{aligned} \|\vec{f}\|_{L^{p'}(U_\nu^*)}^{p'} &= \int_{\mathbb{R}^n} \left\| \sum_{Q \in Q_\nu} A_Q^{-1} \chi_Q(x) \vec{f}(x) \right\|_{\mathcal{H}}^{p'} dx \\ &= \sum_{Q \in Q_\nu} \int_Q \|A_Q^{-1} \vec{f}(x)\|_{\mathcal{H}}^{p'} dx = \|\vec{f}\|_{L^{p'}(\{A_Q^{-1}\}, \nu)}^{p'}, \end{aligned}$$

or

$$[L^p(\{A_Q\}, \nu)]^* \approx L^{p'}(\{A_Q^{-1}\}, \nu). \quad (7.23)$$

If $p = 1$, then the standard duality argument gives $[L^1(\{A_Q\}, \nu)]^* \approx L^\infty(\{A_Q^{-1}\}, \nu)$.

The details are left to the reader.

7.5 Convolution estimates

The boundedness of the convolution operator with a decaying kernel on averaging $L^p(\{A_Q\}, \nu)$ will be helpful in the next section. We establish it here.

Lemma 7.16 *Let $|\Phi(t)| \leq \frac{c}{(1+|t|)^M}$ for some $M > \beta/p + n$ and for $\nu \in \mathbb{Z}$ define*

$\Phi_\nu(t) = 2^{\nu n} \Phi(2^\nu t)$. Let $\{A_Q\}_Q$ be a doubling matrix sequence of order p , $1 \leq p < \infty$.

Fix $\lambda, \mu, \nu \in \mathbb{Z}$. Then

$$(i) \quad \|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q\}, \lambda)} \leq c_0 (c_1)^{\lambda-\nu} (c_2)^{\mu-\nu} \|\vec{f}\|_{L^p(\{A_Q\}, \nu)},$$

$$(ii) \quad \|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q^{-1}\}, \lambda)} \leq c_0 (c_3)^{\lambda-\nu} (c_2)^{\mu-\nu} \|\vec{f}\|_{L^p(\{A_Q^{-1}\}, \nu)},$$

where $c_1 = 2^{n/p} \chi_{\{\lambda > \nu\}} + 2^{(n-\beta)/p} \chi_{\{\lambda \leq \nu\}}$, $c_2 = 2^n \chi_{\{\mu > \nu\}} + 2^{n-M} \chi_{\{\mu \leq \nu\}}$, $c_3 = 2^{(\beta-n)/p} \chi_{\{\lambda > \nu\}} + 2^{-n/p} \chi_{\{\lambda \leq \nu\}}$, and c_0 is independent of λ , μ and ν .

PROOF. Using the decay of Φ , namely, $|\Phi_\mu(x-y)| \leq c k_2 \frac{2^{\nu n}}{(1+2^\nu|x-y|)^M}$, where $k_2 = 2^{(\mu-\nu)n} \chi_{\{\mu > \nu\}} + 2^{(\nu-\mu)(M-n)} \chi_{\{\mu \leq \nu\}}$, we have

$$\begin{aligned} \|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q\}, \lambda)}^p &= \sum_{Q \in Q_\lambda} \int_Q \|A_Q(\Phi_\mu * \vec{f})(x)\|^p dx \\ &\leq \sum_{Q \in Q_\lambda} \int_Q \left(\int_{\mathbb{R}^n} \|A_Q \vec{f}(y)\| |\Phi_\mu(x-y)| dy \right)^p dx \\ &\leq c \sum_{Q \in Q_\lambda} \int_Q \left(\int_{\mathbb{R}^n} \frac{k_2 2^{\nu n} \|A_Q \vec{f}(y)\|}{(1+2^\nu|x-y|)^M} dy \right)^p dx \\ &\approx c \sum_{k \in \mathbb{Z}^n} \int_{Q_{\lambda k}} \left(\sum_{m \in \mathbb{Z}^n} \int_{Q_{\nu m}} \frac{k_2 2^{\nu n} \|A_{Q_{\lambda k}} \vec{f}(y)\|}{(1+2^\nu|x-x_{Q_{\nu m}}|)^M} dy \right)^p dx. \end{aligned}$$

Since $\{A_Q\}_Q$ is doubling, we “shift” $A_{Q_{\lambda k}}$ to $A_{Q_{\nu m}}$:

$$\|A_{Q_{\lambda k}} \vec{f}(y)\| \leq c k_1 (1+2^\nu|x-x_{Q_{\nu m}}|)^{\beta/p} \|A_{Q_{\nu m}} \vec{f}(y)\|, \quad \text{for } x \in Q_{\lambda k}, \quad (7.24)$$

where $k_1 = 2^{(\lambda-\nu)n/p} \chi_{\{\lambda > \nu\}} + 2^{(\nu-\lambda)(\beta-n)/p} \chi_{\{\lambda \leq \nu\}}$. Substituting (7.24) into the convolution estimate, we get

$$\|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q\}, \lambda)}^p \leq c \int_{\mathbb{R}^n} \left(\sum_{m \in \mathbb{Z}^n} \int_{Q_{\nu m}} \frac{k_1 k_2 2^{\nu n} \|A_{Q_{\nu m}} \vec{f}(y)\|}{(1+|2^\nu x - m|)^{M-\beta/p}} dy \right)^p dx.$$

Using the discrete Hölder inequality on the sum inside and then Jensen’s inequality to bring p^{th} power inside of the integral (if $p > 1$), the last line is bounded above by

$$\begin{aligned} &c k_1^p k_2^p \int_{\mathbb{R}^n} \left(\sum_{l \in \mathbb{Z}^n} \frac{1}{(1+|2^\nu x - l|)^{M-\beta/p}} \right)^{p/p'} \left(\sum_{m \in \mathbb{Z}^n} \int_{Q_{\nu m}} \frac{2^{\nu n} \|A_{Q_{\nu m}} \vec{f}(y)\|^p}{(1+|2^\nu x - m|)^{M-\beta/p}} dy \right) dx \\ &\leq c k_1^p k_2^p \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \frac{2^{\nu n}}{(1+|2^\nu x - m|)^{M-\beta/p}} \int_{Q_{\nu m}} \|A_{Q_{\nu m}} \vec{f}(y)\|^p dy dx, \end{aligned}$$

since $M - \beta/p > n$, the sum on l converges (independently of x). Changing variables ($t = 2^\nu x$) and observing that the integral on t converges (independently of m), again since $M - \beta/p > n$, we obtain

$$\|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q\}, \mu)}^p \leq c k_1^p k_2^p \sum_{m \in \mathbb{Z}^n} \int_{Q_{\nu m}} \|A_{Q_{\nu m}} \vec{f}(y)\|^p dy.$$

Put $c_1 = k_1^{1/(\lambda-\nu)}$ and $c_2 = k_2^{1/(\mu-\nu)}$. Then part (i) is proved.

For the second part observe that (1.7) (“shift” $A_{Q_{\nu m}}$ to $A_{Q_{\lambda k}}$) together with $\|A_Q^{-1} \vec{v}\| = \sup_{\vec{u} \neq 0} \frac{|(\vec{v}, \vec{u})|}{\|A_Q \vec{u}\|}$ imply

$$\|A_{Q_{\lambda k}}^{-1} \vec{f}(y)\| \leq c k_3 (1 + 2^\nu |x - x_{Q_{\nu m}}|)^{\beta/p} \|A_{Q_{\nu m}}^{-1} \vec{f}(y)\|, \quad x \in Q_{\lambda k}, \quad (7.25)$$

where $k_3 = 2^{(\lambda-\nu)(\beta-n)/p} \chi_{\{\lambda > \nu\}} + 2^{(\nu-\lambda)n/p} \chi_{\{\lambda \leq \nu\}}$. Note that (7.25) is similar to (7.24), so previous estimates with each A_Q replaced by A_Q^{-1} prove (ii) with $c_3 = k_3^{1/(\lambda-\nu)}$.

■

Remark 7.17 Recall that $\|A_Q^{-1} \vec{u}\| \leq c \|A_Q^\# \vec{u}\|$ for any $\vec{u} \in \mathcal{H}$ (since $\|(A_Q^\# A_Q)^{-1}\| \leq c$). Suppose that $W^{-p'/p}$ is a doubling matrix of order p' , $1 < p' < \infty$, with the doubling exponent β^* (instead of the assumption that W is doubling of order p).

Then

$$\|A_{Q_{\lambda k}}^{-1} \vec{f}(y)\| \leq c \|A_{Q_{\lambda k}}^\# \vec{f}(y)\| \leq c k_1^* (1 + 2^\nu |x - x_{Q_{\nu m}}|)^{\beta^*/p'} \|A_{Q_{\nu m}}^\# \vec{f}(y)\|,$$

(where $k_1^* = 2^{(\lambda-\nu)n/p'} \chi_{\{\lambda > \nu\}} + 2^{(\nu-\lambda)(\beta^*-n)/p'} \chi_{\{\lambda \leq \nu\}}$, i.e., k_1 with β replaced by β^* and p by p') holds instead of (7.24). Choosing $M > \beta^*/p' + n$ in the previous lemma, we get

$$(iii) \quad \|\Phi_\mu * \vec{f}\|_{L^p(\{A_Q^{-1}\}, \lambda)} \leq c_0 (c_1^*)^{\lambda-\nu} (c_2)^{\mu-\nu} \|\vec{f}\|_{L^p(\{A_Q^\#\}, \nu)}, \quad 1 < p < \infty.$$

Remark 7.18 *A similar convolution estimate can be proved for $L^p(W)$ spaces, $1 < p < \infty$:*

$$\|\Phi * \vec{f}\|_{L^p(W)} \leq c \|\vec{f}\|_{L^p(W)}. \quad (7.26)$$

*Recall that if Φ were to be a Calderón-Zygmund singular kernel K , then $\|K * \vec{f}\|_{L^p(W)} \leq c \|\vec{f}\|_{L^p(W)}$ if $W \in A_p$ (see [NT], [TV1], [V]). Conversely, if (7.26) holds for every $\Phi \in \mathcal{S}$, then $W \in A_p$ is necessary (see the scalar case below).*

Before we show the necessity of the A_p condition, we demonstrate that, for example, having just a doubling weight is not sufficient for (7.26).

Example 7.19 *Suppose w is a doubling scalar measure such that there exist $E \subset [0, 1]$ with $0 \neq |E| = \alpha < 1$ but $w(E) = 0$. Such a measure exists (cf. [St2] or [FM]).*

Let $\varphi = \chi_{[-1,1]}$. Choose $f = \chi_E$. Then $\|f\|_{L^p(w)} = 0$.

*Now $(\varphi * f)(x) = \int_{[x-1, x+1]} \chi_E(t) dt = |[x-1, x+1] \cap E|$. So if $x \in [0, 1]$, then $(\varphi * f)(x) = |E| = \alpha$ and if $x \notin [-1, 2]$, then $(\varphi * f)(x) = 0$.*

*Thus, $\|\varphi * f\|_{L^p(w)}^p = \int_{[-1,0] \cup [1,2]} |(\varphi * f)(x)|^p w(x) dx + \int_{[0,1] \setminus E} \alpha^p w(x) dx > 0$, since the second term is for sure positive.*

Proposition 7.20 *Let w be a scalar weight and $1 < p < \infty$. Suppose that for every $\Phi \in \mathcal{S}$ the inequality*

$$\|\Phi * f\|_{L^p(w)} \leq C_\Phi \|f\|_{L^p(w)} \quad (7.27)$$

holds for any $f \in L^p(w)$. Then

$$\left(\frac{1}{|I|} \int_I w(y) dy \right) \left(\frac{1}{|I|} \int_I w^{-p'/p}(y) dy \right)^{p/p'} \leq c \quad (7.28)$$

holds for any interval of side length $l(I) \leq c_\Phi$, where c_Φ is determined by Φ .

Remark 7.21 Observe that if (7.28) holds for any interval, then $w \in A_p$.

PROOF OF PROPOSITION 7.20. Choose $\Phi \in \mathcal{S}$ such that $\Phi(x) \geq 1$ for $x \in [-2, 2]$.

Take $f \geq 0$. Then $(\Phi * f)(x) \geq \int_{[x-2, x+2]} f(y) dy$ for $x \in [-2, 2]$.

Consider $I_1, I_2 \in \mathcal{D}$ such that $l(I_i) = 1$, $i = 1, 2$, and I_2 is right adjacent to I_1 . Let $f = \chi_{I_1}$. Then for $x \in I_2$, we have $(\Phi * f)(x) \geq \int_{[x-2, x+2]} \chi_{I_1}(y) dy = 1$, and so $\|\Phi * f\|_{L^p(w)}^p \geq \int_{I_2} w(y) dy$. Also $\|f\|_{L^p(w)}^p = \int_{I_1} w(y) dy$. By (7.27) we get $\int_{I_2} w(y) dy \leq (C_\Phi)^p \int_{I_1} w(y) dy$. By symmetry (suppose $x \in I_1$), we get

$$\int_{I_1} w(y) dy \leq (C_\Phi)^p \int_{I_2} w(y) dy. \quad (7.29)$$

Note that the above two inequalities say that w is at least doubling.

Next, let $f(y) = w^{-p'/p}(y) \chi_{I_1}(y)$. Then for $x \in I_2$, we have $(\Phi * f)(x) = \int_{I_1} w^{-p'/p}(y) dy$, and so $\|\Phi * f\|_{L^p(w)}^p \geq \int_{I_2} w(y) dy \left(\int_{I_1} w^{-p'/p}(y) dy \right)^p$. Also $\|f\|_{L^p(w)}^p = \int_{I_1} w^{-p'/p}(y) dy$. Again by (7.27),

$$\int_{I_2} w(y) dy \left(\int_{I_1} w^{-p'/p}(y) dy \right)^p \leq (C_\Phi)^p \int_{I_1} w^{-p'/p}(y) dy.$$

Substituting (7.29) and simplifying, we get

$$\left(\int_{I_1} w(y) dy \right) \left(\int_{I_1} w^{-p'/p}(y) dy \right)^{p/p'} \leq (C_\Phi)^{2p},$$

which is the scalar A_p condition for intervals of side length 1 (since I_1 was arbitrary).

Now, let $l(I_1) = l(I_2) = 2^{-\nu}$, $\nu \geq 0$. Repeating the same argument as above with $f = \chi_{I_1}$ and using symmetry, we get $\int_{I_1} w(y) dy \leq (2^\nu C_\Phi)^p \int_{I_2} w(y) dy$. Using $f = w^{-p'/p} \chi_{I_1}$ again as before, we obtain $\int_{I_2} w(y) dy \left(\int_{I_1} w^{-p'/p}(y) dy \right)^p \leq$

$(2^\nu C_\Phi)^p \int_{I_1} w^{-p'/p}(y) dy$, and thus,

$$\left(\frac{1}{|I_1|} \int_{I_1} w(y) dy \right) \left(\frac{1}{|I_1|} \int_{I_1} w^{-p'/p}(y) dy \right)^{p/p'} \leq (C_\Phi)^{2p},$$

which gives (7.28) for intervals of side length ≤ 1 ($c_\Phi = 1$ in this case). $\blacksquare$

7.6 Duality of continuous Besov spaces

Now we shift our attention to continuous Besov spaces and our task is to construct

$$\left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^* \text{ and eventually } \left[\dot{B}_p^{\alpha q}(W) \right]^*.$$

Lemma 7.22 *Let $\{A_Q\}_Q$ be a doubling matrix sequence of order p , $1 \leq p < \infty$. Let*

$\alpha \in \mathbb{R}$ and $0 < q < \infty$. Then

$$\dot{B}_p^{-\alpha q'}(\{A_Q^{-1}\}) \subseteq \left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^*. \quad (7.30)$$

PROOF. Take $\varphi, \psi \in \mathcal{A}$ with the mutual property (2.1). Let $\tilde{\psi}(x) = \overline{\psi(-x)}$. Note that $\hat{\tilde{\psi}}(\xi) = \overline{\hat{\psi}(\xi)}$. Let $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$ and $\vec{g} \in \dot{B}_p^{-\alpha q'}(\{A_Q^{-1}\})$. First consider $\mathcal{S}_0 = \{f \in \mathcal{S} : 0 \notin \text{supp } \hat{f}\}$ a dense subspace of $\dot{B}_p^{\alpha q}(\{A_Q\})$ (see Appendix) and take $\vec{f}$ with $(\vec{f})_i \in \mathcal{S}_0$, $i = 1, \dots, m$ (and $\vec{g}$ with $(\vec{g})_i \in \mathcal{S}'$). Then

$$\vec{g} = \sum_{\nu \in \mathbb{Z}} \vec{g} * (\varphi_\nu * \tilde{\psi}_\nu), \quad \text{since} \quad \sum_{\nu \in \mathbb{Z}} (\varphi_\nu * \tilde{\psi}_\nu)(\xi) = 1, \quad \text{by (2.1),}$$

and

$$\begin{aligned} \vec{g}(\vec{f}) &= \sum_{\nu \in \mathbb{Z}} \left(\vec{g} * (\varphi_\nu * \tilde{\psi}_\nu) \right) (\vec{f}) = \sum_{\nu \in \mathbb{Z}} (\vec{g} * (\varphi_\nu * \tilde{\psi}_\nu), \vec{f}) \\ &= \sum_{\nu \in \mathbb{Z}} \left((\vec{g} * \varphi_\nu), (\vec{f} * \psi_\nu) \right) = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \left\langle (\vec{g} * \varphi_\nu)(x), (\vec{f} * \psi_\nu)(x) \right\rangle_{\mathcal{H}} dx \end{aligned}$$

$$\begin{aligned}
&= \sum_{\nu \in \mathbb{Z}} \sum_{Q \in Q_\nu} \int_Q \left\langle A_Q A_Q^{-1} (\vec{g} * \varphi_\nu)(x), (\vec{f} * \psi_\nu)(x) \right\rangle_{\mathcal{H}} dx \\
&\leq \sum_{\nu \in \mathbb{Z}} \sum_{Q \in Q_\nu} \int_{\mathbb{R}^n} \|A_Q^{-1} (\vec{g} * \varphi_\nu)(x)\|_{\mathcal{H}} \|A_Q (\vec{f} * \psi_\nu)(x)\|_{\mathcal{H}} \chi_Q(x) dx,
\end{aligned}$$

by the self-adjointness of each A_Q and the Cauchy-Schwarz inequality. Using Hölder's inequality several times, we obtain

$$\begin{aligned}
|\vec{g}(\vec{f})| &\leq \sum_{\nu \in \mathbb{Z}} 2^{\nu\alpha} \left\| (\vec{f} * \psi_\nu) \right\|_{L^p(\{A_Q\}, \nu)} \cdot 2^{-\nu\alpha} \|(\vec{g} * \varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} \quad (7.31) \\
&\leq \left\| \left\{ 2^{\nu\alpha} \|(\vec{f} * \psi_\nu)\|_{L^p(\{A_Q\}, \nu)} \right\}_\nu \right\|_{l^q} \left\| \left\{ 2^{-\nu\alpha} \|(\vec{g} * \varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} \right\}_\nu \right\|_{l^{q'}}, \quad \text{if } 1 < q < \infty,
\end{aligned}$$

and if $0 < q \leq 1$, we bound (7.31) by

$$\begin{aligned}
&\left\| \left\{ 2^{\nu\alpha} \|(\vec{f} * \psi_\nu)\|_{L^p(\{A_Q\}, \nu)} \right\}_\nu \right\|_{l^1} \left\| \left\{ 2^{-\nu\alpha} \|(\vec{g} * \varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} \right\}_\nu \right\|_{l^\infty} \\
&\leq \left\| \left\{ 2^{\nu\alpha} \|(\vec{f} * \psi_\nu)\|_{L^p(\{A_Q\}, \nu)} \right\}_\nu \right\|_{l^q} \left\| \left\{ 2^{-\nu\alpha} \|(\vec{g} * \varphi_\nu)\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} \right\}_\nu \right\|_{l^\infty}.
\end{aligned}$$

Combining cases and using the independence of $\dot{B}_p^{\alpha q}(\{A_Q\})$ and $\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ on the choice of the admissible kernel if $\{A_Q\}_Q$ is doubling (Corollary 7.10), we get

$$|\vec{g}(\vec{f})| \leq \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \|\vec{g}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}.$$

Since $\mathcal{S}_0$ is dense in $\dot{B}_p^{\alpha q}(\{A_Q\})$, we get the above inequality for any $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$.

Thus, $\vec{g} \in \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ belongs to $\left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^*$ and $\|\vec{g}\|_{oper} \leq \|\vec{g}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}$.

■

Lemma 7.23 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q < \infty$ and $\{A_Q\}_Q$ be a doubling sequence of order p . Then*

$$\left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^* \subseteq \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}). \quad (7.32)$$

PROOF. Let $l \in \left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^*$. We show that there exists $\vec{g} \in \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ such that $l(\vec{f}) = \vec{g}(\vec{f}) = (\vec{f}, \vec{g})$ for any $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$.

Case $1 \leq q < \infty$. Take $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$, and for any $\nu \in \mathbb{Z}$ denote $\vec{f}_\nu = \vec{f} * \varphi_\nu$. Set T by $T(\{\vec{f}_\nu\}_\nu) = l(\vec{f})$, so T is defined on a subspace of $l_q^\alpha(L^p(\{A_Q\}, \nu))$. Since l is bounded, so is T :

$$|T(\{\vec{f}_\nu\}_\nu)| = |l(\vec{f})| \leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = c \|\{\vec{f}_\nu\}_\nu\|_{l_q^\alpha(L^p(\{A_Q\}, \nu))}.$$

Extend T , denote the extension $\tilde{T}$, onto all of $l_q^\alpha(L^p(\{A_Q\}, \nu))$ (note: $q \geq 1$). Since $[l^q(X)]^* \approx l^{q'}(X^*)$ (refer to Section 7.2 or [D, Ch.8]), $[l_q^\alpha(L^p(\{A_Q\}, \nu))]^* \approx l_{q'}^{-\alpha}([L^p(\{A_Q\}, \nu)]^*) \approx l_{q'}^{-\alpha}(L^{p'}(\{A_Q^{-1}\}, \nu))$ by (7.23). Thus, there exists a vector-valued sequence $\{\vec{g}_\nu\}_{\nu \in \mathbb{Z}} \in l_{q'}^{-\alpha}(L^{p'}(\{A_Q^{-1}\}, \nu))$ such that $\|\{\vec{g}_\nu\}_{\nu \in \mathbb{Z}}\|_{l_{q'}^{-\alpha}(L^{p'}(\{A_Q^{-1}\}, \nu))} \leq \|l\|$ and for any $\vec{f} \in \dot{B}_p^{\alpha q}(\{A_Q\})$

$$\begin{aligned} l(\vec{f}) &= \tilde{T}(\{\vec{f}_\nu\}) = T(\{\vec{f}_\nu\}) = \{\vec{g}_\nu\}(\{\vec{f}_\nu\}) = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \langle \vec{f}_\nu(x), \vec{g}_\nu(x) \rangle_{\mathcal{H}} dx \\ &= \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \langle (f * \varphi_\nu)(x), \vec{g}_\nu(x) \rangle_{\mathcal{H}} dx = \sum_{\nu \in \mathbb{Z}} \int_{\mathbb{R}^n} \langle \vec{f}(x), (\vec{g}_\nu * \tilde{\varphi}_\nu)(x) \rangle_{\mathcal{H}} dx. \end{aligned}$$

Define $\vec{g}(x) = \sum_{\nu \in \mathbb{Z}} (\vec{g}_\nu * \tilde{\varphi}_\nu)(x)$. Then $l(\vec{f}) = (\vec{f}, \vec{g})$. Moreover, for any $\psi \in \mathcal{A}$ (by Corollary 7.10)

$$\begin{aligned} \|\vec{g}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} &\approx \left\| \left\{ \left\| \sum_{\nu \in \mathbb{Z}} \vec{g}_\nu * \tilde{\varphi}_\nu * \psi_\mu \right\|_{L^{p'}(\{A_Q^{-1}\}, \mu)} \right\}_\mu \right\|_{l_{q'}^{-\alpha}} \\ &\leq \left\| \left\{ \sum_{\nu \in \mathbb{Z}} \|\vec{g}_\nu * \tilde{\varphi}_\nu * \psi_\mu\|_{L^{p'}(\{A_Q^{-1}\}, \mu)} \right\}_\mu \right\|_{l_{q'}^{-\alpha}} \\ &= \left\| \left\{ \sum_{\nu=\mu-1}^{\mu+1} \|\vec{g}_\nu * \tilde{\varphi}_\nu * \psi_\mu\|_{L^{p'}(\{A_Q^{-1}\}, \mu)} \right\}_\mu \right\|_{l_{q'}^{-\alpha}}, \end{aligned}$$

since $\text{supp } \hat{\psi}_\mu \subseteq \{\xi : 2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\}$ and so $\tilde{\varphi}_\nu * \psi_\mu = 0$ if $|\mu - \nu| > 1$. Reindexing the inner sum, we get

$$\|\vec{g}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})}^{q'} \leq c \sum_{\mu \in \mathbb{Z}} 2^{-\mu \alpha q'} \sum_{j=-1}^1 \|\vec{g}_{\mu+j} * \tilde{\varphi}_\mu * \psi_{\mu+j}\|_{L^{p'}(\{A_Q^{-1}\}, \mu)}^{q'}.$$

Since $\{A_Q\}_Q$ is doubling and sum on j is finite, we apply Lemma 7.16 (ii) to get

$$\|\vec{g}\|_{\dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq c' \left\| \left\{ 2^{-\mu \alpha} \|\vec{g}_\mu\|_{L^{p'}(\{A_Q^{-1}\}, \mu)} \right\}_\mu \right\|_{l^{q'}} \leq \|l\|.$$

Case $0 < q < 1$. Take $\vec{f}$ with $(\vec{f})_i \in \mathcal{S}_0$. Since $\varphi \in \mathcal{S}_0$, for $\nu \in \mathbb{Z}$ by definition of convolution and then boundedness of l , we have

$$|(l * \varphi_\nu)(\vec{f})| = |l(\vec{f} * \tilde{\varphi}_\nu)| \leq \|l\| \|\vec{f} * \tilde{\varphi}_\nu\|_{\dot{B}_p^{\alpha q}(\{A_Q\})}. \quad (7.33)$$

Note that each component of $l * \varphi_\nu$ is a $\mathcal{C}^\infty$ -function and also $\|\vec{f} * \tilde{\varphi}_\nu\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} \leq 2^{\nu \alpha} \sum_{\mu=\nu-1}^{\nu+1} \|\vec{f} * \tilde{\varphi}_\mu\|_{L^p(\{A_Q\}, \mu)} \leq c 2^{\nu \alpha} \|\vec{f}\|_{L^p(\{A_Q\}, \nu)}$ by Lemma 7.16 (i). Substituting this estimate into (7.33), we get $|(l * \varphi_\nu)(\vec{f})| \leq c 2^{\nu \alpha} \|l\| \|\vec{f}\|_{L^p(\{A_Q\}, \nu)}$. By duality,

$$2^{-\nu \alpha} \|l * \varphi_\nu\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} = 2^{-\nu \alpha} \sup_{\vec{f} \in \mathcal{S}_0} \frac{|(l * \varphi_\nu)(\vec{f})|}{\|\vec{f}\|_{L^p(\{A_Q\}, \nu)}} \leq c \|l\|,$$

i.e., the functional $l * \varphi_\nu$ can be associated with a function $\vec{g}_\nu \in L^{p'}(\{A_Q^{-1}\}, \nu)$ such that $2^{-\nu \alpha} \|\vec{g}_\nu\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} \leq c \|l\|$. Let $\vec{g} = \sum_{\nu \in \mathbb{Z}} \vec{g}_\nu * \theta_\nu$, where θ is as in the atomic decomposition theorem [FJW, Lemma 5.12], which implies $\hat{\vec{g}} = \sum_{\nu \in \mathbb{Z}} \hat{l} \hat{\varphi}_\nu \hat{\theta}_\nu = \hat{l} \cdot 1$ and so $g = l$. Observe that $\vec{g} \in \dot{B}_{p'}^{-\alpha \infty}(\{A_Q^{-1}\})$:

$$\|g\|_{\dot{B}_{p'}^{-\alpha \infty}(\{A_Q^{-1}\})} = \sup_{\nu \in \mathbb{Z}} 2^{-\nu \alpha} \|g * \varphi_\nu\|_{L^{p'}(\{A_Q^{-1}\}, \nu)} \leq c \|l\|.$$

Thus, the functional $l \in \left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^*$ can be associated with $\vec{g} \in \dot{B}_{p'}^{-\alpha \infty}(\{A_Q^{-1}\})$ and $l(\vec{f}) = (\vec{f}, \vec{g})$. This completes the proof. $\blacksquare$

Summarizing the results of the previous two sections we get the following embeddings of B -spaces:

Corollary 7.24 *Let W be a matrix weight and $\{A_Q\}_Q$ its reducing operators. Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$. Then*

$$\begin{aligned} \left[\dot{B}_p^{\alpha q}(W) \right]^* &\stackrel{(1)}{\subseteq} \left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^* \stackrel{(2)}{\subseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \stackrel{(3)}{\subseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^\#\}) \\ &\stackrel{(4)}{\subseteq} \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}), \end{aligned} \quad (7.34)$$

where

(1) holds if W is doubling of order p , $1 < p < \infty$,

(2) holds if W is doubling of order p , $1 \leq p < \infty$,

(3) holds if $W \in A_p$, $1 < p < \infty$,

(4) holds if $W^{-p'/p}$ is doubling of order p' , $1 < p < \infty$.

Also,

$$\begin{aligned} \left[\dot{B}_p^{\alpha q}(W) \right]^* &\stackrel{(1^*)}{\supseteq} \left[\dot{B}_p^{\alpha q}(\{A_Q\}) \right]^* \stackrel{(2^*)}{\supseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\}) \stackrel{(3^*)}{\supseteq} \dot{B}_{p'}^{-\alpha q'}(\{A_Q^\#\}) \\ &\stackrel{(4^*)}{\supseteq} \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}), \end{aligned} \quad (7.35)$$

where

(1^{*}) holds if W satisfies any of (A1)-(A3),

(2^{*}) holds if W is doubling of order p , $1 \leq p < \infty$,

(3^{*}) holds for any matrix weight W ,

(4^*) holds if $W^{-p'/p}$ satisfies any of (A1)-(A3).

In terms of a matrix weight W only, (7.34) and (7.35) are

$$\left[\dot{B}_p^{\alpha q}(W)\right]^* \subseteq \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}) \quad \text{if } W \in A_p, 1 < p < \infty$$

and

$$\left[\dot{B}_p^{\alpha q}(W)\right]^* \supseteq \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}) \quad \text{if } W, W^{-p'/p} \text{ satisfy any of (A1)-(A3).}$$

In particular, if $W \in A_p$ (and so $W^{-p'/p} \in A_{p'}$), then $\left[\dot{B}_p^{\alpha q}(W)\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p})$, otherwise, (W still satisfies any of (A1)-(A3), or otherwise there is a dependance on φ) $\left[\dot{B}_p^{\alpha q}(W)\right]^* \approx \dot{B}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$, which completes the proof of Theorem 1.15.

7.7 Application of Duality

In this section we will study T_ψ^* and S_φ^* and we will briefly show how duality can be used to prove boundedness of operators T_ψ and S_φ . Recall that $T_\psi : \dot{b}_p^{\alpha q}(W) \longrightarrow \dot{B}_p^{\alpha q}(W)$ by

$$s = \{\tilde{s}_Q\}_Q \longmapsto \sum_Q \tilde{s}_Q \psi_Q.$$

Moreover, T_ψ is bounded if W is a doubling matrix of order p .

Let $\tilde{s}_Q \in \dot{b}_p^{\alpha q}(W)$ and $\vec{g} \in \dot{B}_p^{\alpha q}(W)$. Then

$$\begin{aligned} (\vec{s}, T_\psi^* \vec{g}) &= (T_\psi \vec{s}, \vec{g}) = \left(\sum_Q \tilde{s}_Q \psi_Q, \vec{g} \right) = \int \left\langle \sum_Q \tilde{s}_Q \psi_Q(x), \vec{g}(x) \right\rangle_{\mathcal{H}} dx \\ &= \sum_Q \left\langle \tilde{s}_Q, \int \overline{\psi_Q}(x) \vec{g}(x) dx \right\rangle_{\mathcal{H}} = \sum_Q \langle \tilde{s}_Q, (\vec{g}, \psi_Q) \rangle_{\mathcal{H}} = \sum_Q \langle \tilde{s}_Q, S_\psi \vec{g} \rangle_{\mathcal{H}} = (\vec{s}, S_\psi \vec{g}). \end{aligned}$$

Therefore, $T_\psi^* = S_\psi$ and, similarly, $S_\varphi^* = T_\varphi$. So we have

$$T_\psi^* : \left[\dot{B}_p^{\alpha q}(W) \right]^* \longrightarrow \left[\dot{b}_p^{\alpha q}(W) \right]^*$$

or, another words,

$$S_\psi : \dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}) \longrightarrow \dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}),$$

and so S_ψ is bounded if $W \in A_p$. Reformulating this by changing indices, we get that under the A_p condition the following operators are bounded:

$$T_\psi^* : \left[\dot{B}_{p'}^{-\alpha q'}(W^{-p'/p}) \right]^* \longrightarrow \left[\dot{b}_{p'}^{-\alpha q'}(W^{-p'/p}) \right]^*$$

or

$$S_\psi : \dot{B}_p^{\alpha q}(W) \longrightarrow \dot{b}_p^{\alpha q}(W).$$

(This is another proof of Theorem 1.4.)

CHAPTER 8

Inhomogeneous Besov Spaces

8.1 Norm equivalence

In this section we discuss the inhomogeneous spaces. Before we define the vector-valued inhomogeneous Besov space $B_p^{\alpha q}(W)$ with matrix weight W , we introduce a class of functions $\mathcal{A}^{(I)}$ with properties similar to those of an admissible kernel: we say $\Phi \in \mathcal{A}^{(I)}$ if $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $\text{supp } \hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $|\hat{\Phi}(\xi)| \geq c > 0$ if $|\xi| \leq \frac{5}{3}$.

Definition 8.1 (*Inhomogeneous matrix-weighted Besov space $B_p^{\alpha q}(W)$*)

For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$, W a matrix weight, $\varphi \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{(I)}$, we define the Besov space $B_p^{\alpha q}(W)$ as the collection of all vector-valued distributions $\vec{f} = (f_1, \dots, f_m)^T$ with $f_i \in \mathcal{S}'(\mathbb{R}^n)$, $1 \leq i \leq m$, such that

$$\|\vec{f}\|_{B_p^{\alpha q}(W)} = \|\Phi * \vec{f}\|_{L^p(W)} + \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * \vec{f}\|_{L^p(W)} \right\}_{\nu \geq 1} \right\|_{l^q} < \infty.$$

Note that now we consider all vector-valued distributions in $\mathcal{S}'(\mathbb{R}^n)$ (rather than $\mathcal{S}'/\mathcal{P}$ as in the homogeneous case), since $\hat{\Phi}(0) \neq 0$.

The corresponding inhomogeneous weighted sequence Besov space $b_p^{\alpha q}(W)$ is defined for the vector sequences enumerated by the dyadic cubes Q with $l(Q) \leq 1$.

Definition 8.2 (*Inhomogeneous weighted sequence Besov space $b_p^{\alpha q}(W)$*)

For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ and W a matrix weight, the space $b_p^{\alpha q}(W)$ consists of all vector-valued sequences $\vec{s} = \{\vec{s}_Q\}_{l(Q) \leq 1}$ such that

$$\|\vec{s}\|_{b_p^{\alpha q}(W)} = \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right\}_{\nu \geq 1} \right\|_q < \infty.$$

Following [FJ2], given $\varphi \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{(I)}$, we select $\psi \in \mathcal{A}$ and $\Psi \in \mathcal{A}^{(I)}$ such that

$$\hat{\Phi}(\xi) \cdot \hat{\Psi}(\xi) + \sum_{\nu \geq 1} \hat{\varphi}(2^{-\nu}\xi) \cdot \hat{\psi}(2^{-\nu}\xi) = 1 \quad \text{for all } \xi, \quad (8.1)$$

where $\tilde{\Phi}(x) = \overline{\Phi(-x)}$. Analogously to the φ -transform decomposition (2.2), we have the identity for $f \in \mathcal{S}'(\mathbb{R}^n)$

$$f = \sum_{l(Q)=1} \langle f, \Phi_Q \rangle \Psi_Q + \sum_{\nu=1}^{\infty} \sum_{l(Q)=2^{-\nu}} \langle f, \varphi_Q \rangle \psi_Q, \quad (8.2)$$

where $\Phi_Q(x) = |Q|^{-1/2} \Phi(2^\nu x - k)$ for $Q = Q_{\nu k}$ and Ψ_Q is defined similarly.

For each $\vec{f}$ with $f_i \in \mathcal{S}'(\mathbb{R}^n)$ we define the *inhomogeneous φ -transform* $S_\varphi^{(I)} : B_p^{\alpha q}(W) \longrightarrow b_p^{\alpha q}(W)$ by setting $(S_\varphi^{(I)} \vec{f})_Q = \langle \vec{f}, \varphi_Q \rangle$ if $l(Q) < 1$, and $(S_\varphi^{(I)} \vec{f})_Q = \langle \vec{f}, \Phi_Q \rangle$ if $l(Q) = 1$.

The *inverse inhomogeneous φ -transform* $T_\psi^{(I)}$ is the map taking a sequence $s = \{s_Q\}_{l(Q) \leq 1}$ to $T_\psi^{(I)} s = \sum_{l(Q)=1} s_Q \Psi_Q + \sum_{l(Q) < 1} s_Q \psi_Q$. In the vector case, $T_\psi^{(I)} \vec{s} = \sum_{l(Q)=1} \vec{s}_Q \Psi_Q + \sum_{l(Q) < 1} \vec{s}_Q \psi_Q$. By (8.2), $T_\psi^{(I)} \circ S_\varphi^{(I)}$ is the identity on $\mathcal{S}'(\mathbb{R}^n)$.

Next we show that the relation between $B_p^{\alpha q}(W)$ and $b_p^{\alpha q}(W)$ is the same as for the homogeneous spaces.

Theorem 8.3 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$, and let W satisfy any of (A1)-(A3). Then*

$$\|\vec{f}\|_{B_p^{\alpha q}(W)} \approx \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(W)}. \quad (8.3)$$

Before we outline the proof we need to adjust the notation of smooth molecules for the inhomogeneous case. Define a *family of smooth molecules* $\{m_Q\}_{l(Q) \leq 1}$ for $B_p^{\alpha q}(W)$ as a collection of functions with the properties:

1. for dyadic Q with $l(Q) < 1$, each m_Q is a smooth (δ, M, N) -molecule with (M.i)-(M.iii) as for the homogeneous space $\dot{B}_p^{\alpha q}(W)$ (see Section 4.1);
2. for dyadic Q with $l(Q) = 1$, each m_Q (sometimes we denote it as M_Q to emphasize the difference) satisfies (M3), (M4) and a modification of (M2) (which makes it a particular case of (M3) when $\gamma = 0$):

$$(M2^*) \quad |m_Q(x)| \leq |Q|^{-1/2} \left(1 + \frac{|x - x_Q|}{l(Q)} \right)^{-M}.$$

Note that M_Q does not necessarily have vanishing moments. Now one direction of the norm equivalence (8.3) comes from the modified version of Theorem 4.2:

Theorem 8.4 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ and W be a doubling matrix weight of order p . Suppose $\{m_Q\}_{l(Q) \leq 1}$ is a family of smooth molecules for $B_p^{\alpha q}(W)$. Then*

$$\left\| \sum_{l(Q) \leq 1} \vec{s}_Q m_Q \right\|_{B_p^{\alpha q}(W)} \leq c \left\| \{\vec{s}_Q\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(W)}. \quad (8.4)$$

SKETCH OF THE PROOF. We have

$$\begin{aligned} & \left\| \sum_{l(Q) \leq 1} \tilde{s}_Q m_Q \right\|_{B_p^{\alpha q}(W)} = \left\| \sum_{l(Q) \leq 1} \tilde{s}_Q (\Phi * m_Q) \right\|_{L^p(W)} \\ & + \left\| \left\{ 2^{\nu \alpha} \left\| \sum_{l(Q) \leq 1} \tilde{s}_Q (\varphi_\nu * m_Q) \right\|_{L^p(W)} \right\}_{\nu \geq 1} \right\|_{l^q} = I + II. \end{aligned}$$

As in Theorem 4.2, which uses the convolution estimates (4.2) and (4.3), we need similar inequalities for modified molecules (the proofs are routine applications of Lemmas B.1 and B.2 from [FJ2]):

if $l(Q) = 1$, then

$$|\Phi * M_Q(x)| \leq c (1 + |x - x_Q|)^{-M}, \quad (8.5)$$

if $l(Q) = 2^{-\mu}$, $\mu \geq 1$, then for some $\sigma > J - \alpha$

$$|\Phi * m_Q(x)| \leq c |Q|^{-\frac{1}{2}} 2^{-\mu \sigma} (1 + |x - x_Q|)^{-M}, \quad (8.6)$$

if $\nu \geq 1$ and $l(Q) = 1$, then for some $\tau > \alpha$

$$|\varphi_\nu * M_Q(x)| \leq c 2^{-\nu \tau} (1 + |x - x_Q|)^{-M}, \quad (8.7)$$

if $\nu \geq 1$ and $l(Q) < 1$, the estimate of $|(\varphi_\nu * m_Q)(x)|$ comes from either (4.2)

or (4.3).

To estimate I we use (8.5) and (8.6) (note that (8.5) is a special case of (8.6) for $\mu = 0$) and follow the steps of Theorem 1.10 by using Hölder's inequality twice to

bring the p^{th} power inside of the sum, and the Squeeze and the Summation Lemmas from Section 4.1 (it is essential that $\sigma > J - \alpha$ for convergence purposes) to get

$$I \leq c \|\{\vec{s}_Q\}_{l(Q) \leq 1}\|_{b_p^{\alpha q}(W)}.$$

The second term II is also estimated by $\|\{\vec{s}_Q\}_{l(Q) \leq 1}\|_{b_p^{\alpha q}(W)}$, which is obtained by exact repetition of the proof of Theorem 4.2, only restricting the sum over $\mu \in \mathbb{Z}$ to the sum over $\mu \geq 0$. Also note that (8.7) is a particular case of (4.3) when $\mu = 0$ and, thus, $l(Q) = 1$. Therefore, (8.4) is proved. ■

In particular, since Φ and Ψ generate families of smooth molecules for $B_p^{\alpha q}(W)$, we get

$$\|\vec{f}\|_{B_p^{\alpha q}(W)} \leq c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(W)},$$

which gives one direction of the norm equivalence (8.3). To show the other direction, i.e., that the (inhomogeneous) φ -transform is bounded, we simply observe that $\tilde{\Phi} * \vec{f} \in E_0$, which is true since $\left(\tilde{\Phi} * \vec{f} \right)_i^\wedge \in \mathcal{S}'$ and $\text{supp } \hat{\tilde{\Phi}} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$. Hence, Lemmas 4.12 and 4.14 apply to $\vec{g} = \tilde{\Phi} * \vec{f}$ as stated. We have

$$\begin{aligned} & \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(W)} \approx \left\| \sum_{k \in \mathbb{Z}^n} \left(\tilde{\Phi} * \vec{f} \right)(k) \chi_{Q_{0k}} \right\|_{L^p(W)} \\ & + \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \chi_Q \langle \vec{f}, \varphi_Q \rangle \right\|_{L^p(W)} \right\}_{\nu \geq 1} \right\|_{l^q} = I + II. \end{aligned}$$

Using $\tilde{\Phi} * \vec{f} \in E_0$ and repeating the proof of Theorem 4.15 for both terms (in the second term we take the l^q norm only over $\nu \in \mathbb{N}$), we get the desired estimate:

$$\left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(W)} \leq c \|\vec{f}\|_{B_p^{\alpha q}(W)}.$$

Note that as a consequence we also get independence of $B_p^{\alpha q}(W)$ from the choices of Φ and φ .

8.2 Almost diagonality and Calderón-Zygmund Operators

Now we will briefly discuss operators on the inhomogeneous spaces. An *almost diagonal matrix* on $b_p^{\alpha q}(W)$ is the matrix $A = (a_{QP})_{l(Q), l(P) \leq 1}$ whose entries satisfy (5.1), i.e., $|a_{QP}|$ is bounded by (5.1) only for dyadic Q, P with $l(Q), l(P) \leq 1$. Such a matrix A is a bounded operator on $b_p^{\alpha q}(W)$ for the following reasons: let $\vec{s} \in b_p^{\alpha q}(W)$ and then define $\vec{\tilde{s}} = \{\vec{\tilde{s}}_Q\}_{Q \in \mathcal{D}}$ by setting $\vec{\tilde{s}}_Q = \vec{s}_Q$ if $l(Q) \leq 1$ and $\vec{\tilde{s}}_Q = 0$ if $l(Q) > 1$. Note that $\vec{s}$ is a restriction of $\vec{\tilde{s}}$ on $b_p^{\alpha q}(W)$. Also set $\dot{A} = (\dot{a}_{QP})_{Q, P \in \mathcal{D}}$ putting $\dot{a}_{QP} = a_{QP}$ if $l(Q), l(P) \leq 1$ and $\dot{a}_{QP} = 0$ otherwise. Then

$$\begin{aligned} \|A \vec{s}\|_{b_p^{\alpha q}(W)} &= \left\| \left\{ \sum_{l(P) \leq 1} a_{QP} \vec{s}_P \right\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(W)} \\ &= \left\| \left\{ \sum_{P \text{ dyadic}} \dot{a}_{QP} \vec{\tilde{s}}_P \right\}_Q \right\|_{b_p^{\alpha q}(W)} \leq c \|\vec{\tilde{s}}\|_{b_p^{\alpha q}(W)}, \end{aligned}$$

by Theorem 1.10. By the construction, $\|\vec{\tilde{s}}\|_{b_p^{\alpha q}(W)} = \|\vec{s}\|_{b_p^{\alpha q}(W)}$, and so we get boundedness of A on $b_p^{\alpha q}(W)$.

It is easy to see that the class of almost diagonal matrices on $b_p^{\alpha q}(W)$ is closed under composition. The same statements (boundedness and being closed under composition) are true for the corresponding almost diagonal operators on $B_p^{\alpha q}(W)$ by combining the norm equivalence (8.3) and the above results about almost diagonal

matrices on $b_p^{\alpha q}(W)$. For Calderón-Zygmund operators on inhomogeneous matrix-weighted Besov spaces, some minor notational changes should be made. The collection of smooth N -atoms $\{a_Q\}_{Q \in \mathcal{D}}$ in the homogeneous case ought to be replaced by the set of atoms $\{a_Q\}_{l(Q) < 1} \cup \{A_Q\}_{l(Q)=1}$, where the a_Q 's have the same properties as before and the A_Q 's are such that $\text{supp } A_Q \subseteq 3Q$ and $|D^\gamma A_Q(x)| \leq 1$ for $\gamma \in \mathbb{Z}_+^n$. This leads to a slight change of the smooth atomic decomposition (see [FJ2, p. 132]):

$$f = \sum_{l(Q) < 1} s_Q a_Q + \sum_{l(Q)=1} s_Q A_Q.$$

With these adjustments, all corresponding statements about CZOs hold with essentially the same formulations for the inhomogeneous spaces.

8.3 Duality

Let $\mathcal{RS}^{(I)}$ be the collection of all sequences $\{A_Q\}_{l(Q) \leq 1}$ of positive-definite operators on $\mathcal{H}$. Similar to the homogeneous case, we introduce the averaging space $b_p^{\alpha q}(\{A_Q\})$.

Definition 8.5 (*Inhomogeneous averaging matrix-weighted sequence Besov space $b_p^{\alpha q}(\{A_Q\})$.*) For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p \leq \infty$ and $\{A_Q\}_{l(Q) \leq 1} \in \mathcal{RS}^{(I)}$, let

$$b_p^{\alpha q}(\{A_Q\}) = \left\{ \vec{s} = \{\{\vec{s}_Q\}_{l(Q) \leq 1}\} : \left\| \left\{ 2^{\nu\alpha} \left\| \sum_{l(Q)=2^{-\nu}} |Q|^{-\frac{1}{2}} \vec{s}_Q \chi_Q \right\|_{L^p(\{A_Q\}, \nu)} \right\}_{\nu \geq 0} \right\|_{l^q} < \infty \right\}.$$

Let $\vec{s} \in b_p^{\alpha q}(W)$. Define $\vec{s} = \{\vec{s}_Q\}_{Q \in \mathcal{D}}$ as in the previous section. Applying (1.10), we get

$$\|\vec{s}\|_{b_p^{\alpha q}(W)} = \left\| \vec{s} \right\|_{\dot{b}_p^{\alpha q}(W)} \approx \left\| \vec{s} \right\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} = \|\vec{s}\|_{b_p^{\alpha q}(\{A_Q\})},$$

which proves the following proposition.

Proposition 8.6 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q \leq \infty$ and let W be a matrix weight with reducing operators $\{A_Q\}_Q$. Then*

$$b_p^{\alpha q}(W) \approx b_p^{\alpha q}(\{A_Q\}),$$

in the sense of the norm equivalence.

Note that it's enough to consider reducing operators A_Q generated by a matrix weight W only for dyadic cubes of side length $l(Q) \leq 1$, i.e., $\{A_Q\}_{l(Q) \leq 1}$.

Now we establish the duality.

Theorem 8.7 *Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$, $0 < q < \infty$ and let W be a matrix weight with reducing operators $\{A_Q\}_{l(Q) \leq 1}$. Then*

$$[b_p^{\alpha q}(W)]^* \approx b_{p'}^{-\alpha q'}(\{A_Q^{-1}\}).$$

Moreover, if $W \in A_p$, $1 < p < \infty$, then

$$[b_p^{\alpha q}(W)]^* \approx b_{p'}^{-\alpha q'}(W^{-p'/p}).$$

To prove this theorem, one can simply repeat the arguments from Section 7.2 with proper adjustments (for example, consider sums on ν taken only over $\nu \geq 0$). However, we would like to give a simple proof for the embedding

$$[b_p^{\alpha q}(\{A_Q\})]^* \subseteq b_{p'}^{-\alpha q'}(\{A_Q^{-1}\}).$$

PROOF. Let $l \in [b_p^{\alpha q}(\{A_Q\})]^*$. Let P be the projection from $\dot{b}_p^{\alpha q}(\{A_Q\})$ to $b_p^{\alpha q}(\{A_Q\})$ defined by restricting a sequence $\{\vec{s}_Q\}_{Q \in \mathcal{D}}$ to $\{\vec{s}_Q\}_{l(Q) \leq 1}$. Set $\tilde{l}$ by $\tilde{l}(\vec{s}) = l(P\vec{s})$ for each $\vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\})$. Then $\tilde{l} \in [\dot{b}_p^{\alpha q}(\{A_Q\})]^*$, since

$$|\tilde{l}(\vec{s})| = |l(P\vec{s})| \leq \|l\| \|P\vec{s}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})} \leq \|l\| \|\vec{s}\|_{\dot{b}_p^{\alpha q}(\{A_Q\})}.$$

Then by Lemma 7.3 (or, equivalently, by (7.6)), $\tilde{l}$ is represented by $\vec{t} \in \dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ such that $\tilde{l}(\vec{s}) = (\vec{s}, \vec{t})$ and $\|\vec{t}\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|\tilde{l}\| \leq \|l\|$. Let $\vec{t} = P\vec{t}$. For $\vec{s} \in \dot{b}_p^{\alpha q}(\{A_Q\})$ define $\vec{\tilde{s}} \in \dot{b}_p^{\alpha q}(\{A_Q\})$ as above (i.e., $P\vec{\tilde{s}} = \vec{s}$). Then

$$l(\vec{s}) = \tilde{l}(\vec{\tilde{s}}) = (\vec{\tilde{s}}, \vec{t}) = \sum_{l(Q) \leq 1} \vec{\tilde{s}}_Q \vec{t}_Q + \sum_{l(Q) > 1} \vec{\tilde{s}}_Q \vec{t}_Q = \sum_{l(Q) \leq 1} \vec{\tilde{s}}_Q \vec{t}_Q = (\vec{s}, \vec{t}),$$

since $\vec{\tilde{s}}_Q = 0$ for $l(Q) > 1$. Moreover, $\|\vec{t}\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|\vec{t}\|_{\dot{b}_{p'}^{-\alpha q'}(\{A_Q^{-1}\})} \leq \|l\|$. ■

Analogously, we introduce the averaging space $B_p^{\alpha q}(\{A_Q\})$.

Definition 8.8 (Averaging matrix-weighted Besov space $B_p^{\alpha q}(\{A_Q\})$) For

$\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p \leq \infty$, $\varphi \in \mathcal{A}$, $\Phi \in \mathcal{A}^{(I)}$ and $\{A_Q\}_{l(Q) \leq 1} \in \mathcal{RS}^{(I)}$, let

$$B_p^{\alpha q}(\{A_Q\}) = \left\{ \vec{f} = (f_1, \dots, f_m)^T \text{ with } f_i \in \mathcal{S}'(\mathbb{R}^n), 1 \leq i \leq m : \right.$$

$$\left. \|\vec{f}\|_{\dot{B}_p^{\alpha q}(\{A_Q\})} = \|\Phi * \vec{f}\|_{L^p(\{A_Q\}, 0)} + \left\| \left\{ 2^{\nu\alpha} \|\varphi_\nu * \vec{f}\|_{L^p(\{A_Q\}, \nu)} \right\}_{\nu \geq 1} \right\|_{l_q} < \infty \right\}.$$

Now the remaining results from Sections 7.3, 7.4 and 7.6 transfer easily to the inhomogeneous Besov spaces by using the properties (discussed earlier in this section) such as replacing a family $\{\varphi_\nu\}_{\nu \in \mathbb{Z}}$ with $\{\varphi_\nu\}_{\nu \in \mathbb{N} \cup \Phi}$; observing that $\Phi * \vec{f} \in E_0$ and summing over $\nu \geq 0$ (or $l(Q) \leq 1$) in all sums. In particular, we get

Theorem 8.9 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, $1 \leq p < \infty$ and let $\{A_Q\}_{l(Q) \leq 1}$ be a doubling sequence (of order p). Then for $\bar{s}_Q(\vec{f}) = \langle \vec{f}, \varphi_Q \rangle$*

$$\|\vec{f}\|_{B_p^{\alpha q}(\{A_Q\})} \approx \left\| \left\{ \bar{s}_Q(\vec{f}) \right\}_{l(Q) \leq 1} \right\|_{b_p^{\alpha q}(\{A_Q\})}.$$

Corollary 8.10 *The spaces $B_p^{\alpha q}(\{A_Q\})$, $B_p^{\alpha q}(\{A_Q^{-1}\})$ and $B_{p'}^{-\alpha q'}(\{A_Q^{-1}\})$ are independent of the choice of the pair of admissible kernels (φ, Φ) , if $\{A_Q\}_{l(Q) \leq 1}$ is doubling (of order p), $1 \leq p < \infty$, $\alpha \in \mathbb{R}$, $0 < q \leq \infty$.*

Lemma 8.11 *Let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $1 \leq p < \infty$. If W satisfies any of (A1)-(A3) and $\{A_Q\}_{l(Q) \leq 1}$ is a sequence of reducing operators generated by W , then*

$$B_p^{\alpha q}(W) \approx B_p^{\alpha q}(\{A_Q\}).$$

Theorem 8.12 *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$ and let $\{A_Q\}_{l(Q) \leq 1}$ be reducing operators of a matrix weight W . If $W \in A_p$, $1 < p < \infty$, then*

$$[B_p^{\alpha q}(W)]^* \approx B_{p'}^{-\alpha q'}(W^{-p'/p}).$$

If W satisfies any of (A1)-(A3), then

$$[B_p^{\alpha q}(W)]^* \approx B_{p'}^{-\alpha q'}(\{A_Q^{-1}\}).$$

Thus, all results obtained for the matrix-weighted homogeneous Besov spaces are essentially the same for the inhomogeneous case.

CHAPTER 9

Weighted Triebel-Lizorkin Spaces

9.1 Motivation

The study in this chapter was stimulated by the question posed by A. Volberg in [V].

He proved that if $W \in A_p$, $1 < p < \infty$, then the following equivalence takes place

$$\left\| \left\{ s_Q(\vec{f}) \right\}_Q \right\|_{j_p^{02}(W)} \approx \left\| \left\{ s_Q(\vec{f}) \right\}_Q \right\|_{j_p^{02}(\{A_Q\})}, \quad (9.1)$$

where $s_Q(\vec{f}) = \langle \vec{f}, h_Q \rangle$ with $\{h_Q\}_Q$ being the well-known Haar system and $\{A_Q\}_Q$ the reducing operators for W . Moreover, he pointed out that the equivalence does not necessarily require $W \in A_p$. For example, it holds always for $p = 2$. He conjectured that for $p \geq 2$, the condition on the metric ρ generated by W , which is similar to a scalar A_∞ condition, $\rho \in A_{p,\infty}$ might be sufficient. The criterion for (9.1) was asked.

In the light of our studies of function spaces, we rephrase (and partially answer) the question of Volberg in familiar terms: what conditions on W are needed for the

equivalence below to hold?

$$\left\| \{\vec{s}_Q\}_Q \right\|_{\dot{f}_p^{\alpha q}(W)} \approx \left\| \{\vec{s}_Q\}_Q \right\|_{\dot{f}_p^{\alpha q}(\{A_Q\})} \quad (9.2)$$

Our main result of this chapter deals with scalar weights and the matrix case is left for future research. We show in Theorem 9.3 that if a scalar weight $w \in A_\infty$, then (9.2) holds for $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $0 < p < \infty$. Furthermore, by using the result of Frazier and Jawerth (see [FJ2, Proposition 10.14]), we connect the reducing operators sequence space $\dot{f}_p^{\alpha q}(\{w_Q\})$ with the continuous Triebel-Lizorkin space $\dot{F}_p^{\alpha q}(w)$, and therefore, obtain the following norm equivalence

$$\|f\|_{\dot{F}_p^{\alpha q}(w)} \approx \|\{\langle f, \varphi_Q \rangle\}_Q\|_{\dot{f}_p^{\alpha q}(\{w_Q\})}.$$

9.2 Equivalence of $\dot{f}_p^{\alpha q}(w)$ and $\dot{f}_p^{\alpha q}(\{w_Q\})$

Before we prove the main result, we establish two lemmas for the weighted and unweighted maximal functions. Denote $w_Q = \frac{1}{|Q|} \int_Q w(x) dx$.

Lemma 9.1 *Let $E_Q = \{x \in Q : w(x) \leq 2w_Q\}$. If $w \in A_\infty$, then $M_w(\chi_{E_Q}) \geq c \chi_Q$.*

PROOF. Using the definition of the maximal function, for $x \in Q$ we have

$$M_w(\chi_{E_Q})(x) = \sup_{I: x \in I} \frac{1}{w(I)} \int_I \chi_{E_Q}(y) w(y) dy \geq \frac{1}{w(Q)} \int_Q \chi_{E_Q}(y) w(y) dy = \frac{w(E_Q)}{w(Q)}.$$

The condition $w \in A_\infty$ implies that $\frac{w(E_Q)}{w(Q)} \geq c \left(\frac{|E_Q|}{|Q|} \right)^\beta$ for some $\beta > 0$. Since $E_Q = \{x \in Q : w(x) \leq 2w_Q\}$, the compliment $E_Q^c = \{x \in Q : w(x) > 2w_Q\}$. This gives us the following chain of inequalities:

$$w(Q) = \int_Q w(x) dx \geq \int_{E_Q^c} w(x) dx > \int_{E_Q^c} 2w_Q dx = 2w_Q |E_Q^c|.$$

In other words, $|Q|w_Q = w(Q) > 2w_Q|E_Q^c|$, or $|Q| > 2|E_Q^c|$, which implies $|E_Q| > \frac{1}{2}|Q|$. Hence, for $x \in Q$ we have

$$M_w(\chi_{E_Q})(x) \geq c \left(\frac{1}{2}\right)^\beta = c' \chi_Q(x),$$

which finishes the proof. $\blacksquare$

Lemma 9.2 *There exists $0 < \delta < 1$ such that if $E_Q = \{x \in Q : w(x) \geq \delta w_Q\}$ and $w \in A_\infty$, then $M(\chi_{E_Q}) \geq c \chi_Q$.*

PROOF. The proof goes similarly to the proof of the first lemma except we will apply the A_∞ condition in a slightly different way. If $x \in Q$, then

$$M(\chi_{E_Q})(x) = \sup_{I: x \in I} \frac{1}{|I|} \int_I \chi_{E_Q}(y) dy \geq \frac{1}{|Q|} \int_Q \chi_{E_Q}(y) dy \geq \frac{|E_Q|}{|Q|}.$$

Considering the compliment of E_Q , we have $E_Q^c = \{x \in Q : w(x) < \delta w_Q\}$. Then,

$$w(E_Q^c) = \int_{E_Q^c} w(x) dx < \int_{E_Q^c} \delta w_Q dx \leq \delta w_Q \int_Q dx = \delta w_Q |Q| = \delta w(Q).$$

So $w(E_Q^c) \leq \delta w(Q)$ implies $w(E_Q) \geq (1 - \delta) w(Q)$. Since $w \in A_\infty$ and $\frac{w(E_Q^c)}{w(Q)} \leq \delta$,

there exists $\epsilon < 1$ such that $\frac{|E_Q^c|}{|Q|} \leq \epsilon$ or $\frac{|E_Q|}{|Q|} \geq 1 - \epsilon$. Hence, for $x \in Q$ we have

$$M(\chi_{E_Q})(x) \geq (1 - \epsilon), \text{ or}$$

$$M(\chi_{E_Q})(x) \geq (1 - \epsilon)\chi_Q(x).$$

$\blacksquare$

Theorem 9.3 *Suppose $w \in A_\infty$ and let $\alpha \in \mathbb{R}$, $0 < q \leq \infty$ and $0 < p < \infty$. Then*

$$\|\{s_Q\}_Q\|_{j_p^{\alpha q}(\{w_Q\})} \approx \|\{s_Q\}_Q\|_{j_p^{\alpha q}(w)}.$$

Moreover, if $f \in \dot{F}_p^{\alpha q}(w)$, then for $s_Q(f) = \langle f, \varphi_Q \rangle$,

$$\|f\|_{\dot{F}_p^{\alpha q}(w)} \approx \|\{s_Q(f)\}_Q\|_{j_p^{\alpha q}(\{w_Q\})}. \quad (9.3)$$

PROOF. By definition,

$$\|\{s_Q(f)\}_Q\|_{j_p^{\alpha q}(w)} = \left\| \left(\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w \chi_Q)^q \right)^{1/q} \right\|_{L^p}.$$

We need to show that the last norm is equivalent to

$$\left\| \{s_Q \cdot w_Q^{1/p}\}_Q \right\|_{j_p^{\alpha q}} = \left\| \left(\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w_Q^{1/p} \chi_Q)^q \right)^{1/q} \right\|_{L^p}.$$

Let $E_Q = \{x \in Q : w(x) \leq 2w_Q\}$. Choose $A > 0$ such that $p/A > 1$ and $q/A > 1$.

By Lemma 9.1, $\chi_Q(x) \leq c \left(M_w(\chi_{E_Q}^A)(x) \right)^{1/A}$. Therefore,

$$\begin{aligned} \|\{s_Q\}_Q\|_{j_p^{\alpha q}(w)} &= \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \chi_Q(x))^q \right]^{p/q} w(x) dx \right)^{1/p} \\ &\leq c \left(\int \left[\sum_Q \left(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \left(M_w(\chi_{E_Q}^A)(x) \right)^{1/A} \right)^q \right]^{p/q} w(x) dx \right)^{1/p} \\ &\leq c \left(\int \left[\sum_Q \left(M_w(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \chi_{E_Q}(x))^A \right)^{q/A} \right]^{p/q} w(x) dx \right)^{1/p} \\ &\leq c \left\| \left[\sum_Q \left(M_w(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \chi_{E_Q}(x))^A \right)^{q/A} \right]^{A/q} \right\|_{L^{p/A}(w)}^{1/A}. \end{aligned}$$

Since w is a doubling measure and the weighted maximal function M_w satisfies the vector-valued maximal inequality (see [St1] or [St2]), the last expression is bounded above by

$$c_{p,q} \left\| \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \chi_{E_Q})^q \right]^{A/q} \right\|_{L^{p/A}(w)}^{1/A}$$

$$\begin{aligned}
&= c_{p,q} \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \chi_{E_Q}(x) w^{1/p}(x))^q \right]^{p/q} dx \right)^{1/p} \\
&\leq 2^{1/p} c_{p,q} \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w_Q^{1/p} \chi_{E_Q}(x))^q \right]^{p/q} dx \right)^{1/p} \\
&\leq c \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w_Q^{1/p} \chi_Q(x))^q \right]^{p/q} dx \right)^{1/p} = c \left\| \{s_Q w_Q^{1/p}\}_Q \right\|_{f_p^{\alpha q}}.
\end{aligned}$$

In the last inequality we used $E_Q \subseteq Q$.

For the opposite direction, set $E_Q = \{x \in Q : w(x) \geq \delta w_Q\}$ and again choose $A > 0$ such that $p/A > 1$ and $q/A > 1$. Then by Lemma 9.2,

$$\begin{aligned}
&\left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w_Q^{1/p} \chi_Q(x))^q \right]^{p/q} dx \right)^{1/p} \\
&\leq c \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w_Q^{1/p} M(\chi_{E_Q}^A)(x))^q \right]^{p/q} dx \right)^{1/p} \\
&= c \left(\int \left[\sum_Q \left(M(|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w_Q^{1/p} \chi_{E_Q}(x))^A \right)^{q/A} dx \right]^{p/q} dx \right)^{1/p} \\
&\leq \delta^{1/p} c \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| w^{1/p}(x) \chi_{E_Q}(x))^q \right]^{p/q} dx \right)^{1/p} \\
&\leq c \left(\int \left[\sum_Q (|Q|^{-\frac{1}{2}-\frac{\alpha}{n}} |s_Q| \chi_Q(x))^q \right]^{p/q} w(x) dx \right)^{1/p} = c \left\| \{s_Q\}_Q \right\|_{f_p^{\alpha q}(w)}.
\end{aligned}$$

It is easy to show the second assertion. By [FJ1, Proposition 10.14]

$$\|f\|_{\dot{F}_p^{\alpha q}(w)} \approx \|\{s_Q(f)\}_Q\|_{f_p^{\alpha q}(w)}.$$

Combining this equivalence with the first result, we get (9.3). The proof is complete.

■

CHAPTER 10

Open Questions

1. In the unweighted theory of function spaces, special cases of the Besov and Triebel-Lizorkin spaces are the Lebesgue spaces: e.g. $\dot{B}_2^{02} = L^2$ and $\dot{F}_p^{02} = L^p$, $1 < p < +\infty$. In the scalar weighted situation it is known that $\dot{B}_2^{02}(w) = L^2(w)$ and $\dot{F}_p^{02}(w) = L^p(w)$ if and only if $w \in A_p$. In the matrix case it is expected that the A_p condition is the minimal condition on W needed for this equivalence to hold. Using vector-valued square function operators might be one of the approaches to this problem.
2. The crucial step for our theory of Besov spaces is the norm equivalence between continuous $\dot{B}_p^{\alpha q}(W)$ and discrete $\dot{b}_p^{\alpha q}(W)$ spaces. We were able to show that it holds for any doubling matrix weights if the order p is greater than the doubling exponent of W . In the special case of diagonal matrices (equivalently, in the scalar case) this restriction is removed. A conjecture is that the equivalence holds for any doubling matrix weight W .

3. One major goal is to answer the same “equivalence of norms” questions for the matrix-weighted Triebel-Lizorkin spaces $\dot{F}_p^{\alpha q}(W)$ and $\dot{f}_p^{\alpha q}(W)$. The scalar weighted case is known (see [FJ2], [FJW]) and the matrix-weighted case is to be studied. This will also lead to the question of the boundedness of singular integral operators on $\dot{F}_p^{\alpha q}(W)$. Possible approaches include a variation of the exponential type estimates used in the Besov space case or Volberg’s factorization method mentioned in the introduction.

4. All previous research was done on function spaces with the parameter p being between 1 and ∞ , quite often not including the end point $p = 1$, which requires more careful consideration. Moreover, this raises the question of matrix “ A_1 ” weights and their factorization, and whether this can be developed further into an extrapolation theory of matrix-weighted distribution spaces (similar to Rubio de Francia results in scalar case). Furthermore, it would be interesting to study a case when $0 < p < 1$. Nothing is known in this area, except for certain scalar cases.

5. The motivation for the norm equivalence studied above came from the fact that $\|\vec{f}\|_{L^p(W)} \approx \|\{\langle \vec{f}, h_I \rangle\}_I\|_{\dot{f}_p^{02}(W)}$, where $\{h_I\}_I$ is a Haar system and $\{\langle \vec{f}, h_I \rangle\}_I$ constitutes a sequence of the Haar coefficients of $\vec{f}$. Recall that we obtained the norm equivalence when the generators of the expansion (either the φ -transform or *wavelet* functions) have some degree of smoothness. This property is lacking for the Haar system. Nevertheless, the Haar system is widely

used in applications. This creates the open question (for certain indices α , q , p) of the norm equivalence between continuous and discrete function spaces with the Haar coefficients.

6. A very difficult problem of modern Fourier analysis is to obtain weighted norm inequalities on the function spaces (at least on L^p spaces) with different weights. Complete answers to the *scalar* two-weight problem is known only for the Hardy-Littlewood maximal function (by Muckenhoupt and Wheeden in [MW] and by Sawyer in [Sa]). For the Hilbert Transform the necessary condition is given by Muckenhoupt and Wheeden and the dyadic version is studied by Nazarov, Treil and Volberg in [NTV]. Furthermore, the necessary and sufficient conditions for the case $p = 2$ are obtained by Cotlar and Sadosky in [CS]. Since the Lebesgue spaces are special cases of Besov and Triebel-Lizorkin function spaces, the same two-weight questions should be asked in the light of Littlewood-Paley theory. The hope is to consider at least the scalar case and to obtain the conditions on the two weights for the boundedness of the φ -transform, almost diagonal operators, maximal function operators (such as Peetre's maximal operator), the Hilbert Transform and possibly other singular integral operators.

Appendix A

Density and convergence

Define $\mathcal{S}_0 = \{f \in \mathcal{S} : 0 \notin \text{supp } \hat{f}\}$. Observe that $\psi \in \mathcal{A}$ implies $\psi, \psi_\nu, \psi_Q \in \mathcal{S}_0$ for $\nu \in \mathbb{Z}$ and Q dyadic.

Lemma A.1 *Let $f \in \mathcal{S}_0$. Then $f_N := \sum_{|\nu| \leq N} \tilde{\varphi}_\nu * \psi_\nu * f = \sum_{|\nu| \leq N} \langle f, \varphi_{Q_{\nu k}} \rangle \psi_{Q_{\nu k}}$ converges to f in the $\mathcal{S}$ -topology as $N \rightarrow \infty$.*

PROOF. For $\nu \in \mathbb{Z}$, define $f_{(\nu)} = \tilde{\varphi}_\nu * \psi_\nu * f = \sum_{Q \in Q_\nu} \langle f, \varphi_Q \rangle \psi_Q$. Then $\hat{f}_{(\nu)}(\xi) = \hat{\tilde{\varphi}}_\nu(\xi) \hat{\psi}_\nu(\xi) \hat{f}(\xi)$ and $\tilde{\varphi}_\nu, \psi_\nu, f \in \mathcal{S}_0 \implies f_{(\nu)} \in \mathcal{S}_0$. Observe another fact: since $f \in \mathcal{S}_0$, there exists $N_0 \in \mathbb{N}$ large such that $f = \sum_{\nu=-N_0}^{\infty} f_{(\nu)}$. Indeed, $0 \notin \text{supp } \hat{f}$ implies that $\hat{f}(x) = 0$ if $|x| \leq 2^{-N_0}$ for some $N_0 > 0$. Thus, for large N , $f_N = \sum_{\nu=-N_0}^N f_{(\nu)} = \sum_{\nu \leq N} f_{(\nu)}$.

To prove the lemma, it suffices to show that $\rho_\gamma(\hat{f}_N - \hat{f}) \rightarrow 0$ as $N \rightarrow \infty$. Denote $m_N(\xi) = 1 - \sum_{\nu=-N_0}^N \hat{\tilde{\varphi}}_\nu(\xi) \hat{\psi}_\nu(\xi)$. Because of the support of $\hat{\varphi}_\nu$ and $\hat{\psi}_\nu$ and the mutual property (2.1), $m_N(\xi) = 0$ for $2^{-N_0} < |\xi| < 2^N$. Moreover, $0 \leq m_N(\xi) \leq 1$ for

$2^N < |\xi| < 2^{N+1}$ and $m_N(\xi) = 1$ for $|\xi| \geq 2^{N+1}$. Using these facts, we obtain

$$\begin{aligned} \rho_\gamma(\hat{f}_N - \hat{f}) &= \sup_{|\xi| \geq 2^N, |\alpha| \leq \gamma} (1 + |\xi|^2)^\gamma |D^\alpha(m_N \hat{f})(\xi)| \\ &\leq c \sup_{|\xi| \geq 2^N, |\alpha| \leq \gamma} (1 + |\xi|^2)^\gamma \sum_{\gamma_1 + \gamma_2 = \alpha} |D^{\gamma_1} m_N(\xi)| |D^{\gamma_2} \hat{f}(\xi)|. \end{aligned}$$

Observe that $|D^{\gamma_1} m_N(\xi)| \sim 1/|\xi|^{|\gamma_1|}$ if $|\xi| \sim 2^N$ and that $f \in \mathcal{S}$ implies $|D^{\gamma_2} \hat{f}(\xi)| \leq \frac{c_L}{(1 + |\xi|)^{L + |\gamma_2|}}$ for any $L > 0$. Take $L > 2\gamma$. Then

$$\rho_\gamma(\hat{f}_N - \hat{f}) \leq c_L \sup_{|\xi| \geq 2^N, |\alpha| \leq \gamma} \sum_{\gamma_1 + \gamma_2 = \alpha} \frac{(1 + |\xi|)^{2\gamma}}{(1 + |\xi|)^{|\gamma_1|} (1 + |\xi|)^{L + |\gamma_2|}} \leq \frac{c_L}{(1 + 2^N)^{L - 2\gamma}} \xrightarrow{N \rightarrow \infty} 0.$$

Thus, $f_N \xrightarrow{N \rightarrow \infty} f$ in $\mathcal{S}$ -topology. ■

Lemma A.2 *Let $f \in \mathcal{S}_0$ and fix $\nu \in \mathbb{Z}$. Then $\sum_{|k| \leq M} \langle f, \varphi_{Q_{\nu k}} \rangle \psi_{Q_{\nu k}}$ converges to $f_\nu = \tilde{\varphi}_\nu * \psi_\nu * f = \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{Q_{\nu k}} \rangle \psi_{Q_{\nu k}}$ in $\mathcal{S}$ -topology as $M \rightarrow \infty$.*

PROOF. Denote $f_{\nu, M} = \sum_{k \leq M} \langle f, \varphi_{Q_{\nu k}} \rangle \psi_{Q_{\nu k}}$. Obviously, $f_{\nu, M} \in \mathcal{S}_0$. Then

$$\begin{aligned} \rho_\gamma(f_{\nu, M} - f_\nu) &= \sup_{x \in \mathbb{R}^n, |\alpha| \leq \gamma} (1 + |x|^2)^\gamma |D^\alpha(f_{\nu, M} - f_\nu)(x)| \\ &= \sup_{x \in \mathbb{R}^n, |\alpha| \leq \gamma} (1 + |x|^2)^\gamma \left| D^\alpha \sum_{k > M} \langle f, \varphi_Q \rangle \psi_Q(x) \right|. \end{aligned}$$

Observe that we can bring D^α inside of the sum. (A similar argument as below proves this claim.) Then

$$\rho_\gamma(f_{\nu, M} - f_\nu) \leq \sup_{x \in \mathbb{R}^n, |\alpha| \leq \gamma} (1 + |x|)^{2\gamma} \sum_{|k| > M} |\tilde{\varphi}_\nu * f(2^{-\nu} k)| |D^\alpha(\psi(2^\nu x - k))| \quad (\text{A.1})$$

Choose $L_1 > 2\gamma - |\alpha|$, $L_2 > 2\gamma + n$ and $L_3 > \max(L_1, \gamma)$. Using properties of $\mathcal{S}$ functions, we have

$$|D^\alpha(\psi(2^\nu x - k))| = 2^{\nu|\alpha|} |(D^\alpha \psi)(2^\nu x - k)| \leq \frac{c_{L_1, \alpha} 2^{\nu|\alpha|}}{(1 + |2^\nu x - k|)^{L_1 + |\alpha|}}.$$

Applying the convolution estimates (4.2) and (4.3), we bound $(\tilde{\varphi}_\nu * f)(k)$:

$$|(\tilde{\varphi}_\nu * f)(k)| \leq \frac{c_{L_2} 2^{-|\nu|L_3}}{(1 + |k|)^{L_2}}.$$

Substituting the above estimates into (A.1), we obtain

$$\begin{aligned} \rho_\gamma(f_{\nu,M} - f_\nu) &\leq \sup_{x \in \mathbb{R}^n, |\alpha| \leq \gamma} \sum_{|k| > M} \frac{c_{L_1, L_2, \alpha} (1 + |x|)^{2\gamma} 2^{\nu|\alpha| - |\nu|L_3}}{(1 + |k|)^{L_2} (1 + |2^\nu x - k|)^{L_1 + |\alpha|}} \\ &\leq c_{L_1, L_2, \alpha} \left(\sup_{x \in \mathbb{R}^n, |\alpha| \leq \gamma} \frac{(1 + |x|)^{2\gamma} 2^{\nu|\alpha| - |\nu|L_3}}{(1 + 2^\nu |x|)^{L_1 + |\alpha|}} \right) \sum_{|k| > M} \frac{1}{(1 + |k|)^{L_2 - 2\gamma}}, \end{aligned}$$

by using $(1 + 2^\nu |x|) \leq (1 + |2^\nu x - k|) (1 + |k|)$. The supremum on x and α is bounded by $c_\nu = 2^{\nu(\gamma - L_3)} \chi_{\{\nu \geq 0\}} + 2^{\nu(L_3 - L_1)} \chi_{\{\nu < 0\}}$, since $L_1 > 2\gamma - |\alpha|$. Thus, we get

$$\rho_\gamma(f_{\nu,M} - f_\nu) \leq c_{L_1, L_2, \gamma} c_\nu \sum_{|k| > M} \frac{1}{(1 + |k|)^{L_2 - 2\gamma}} \longrightarrow 0 \text{ as } M \longrightarrow \infty$$

as a tail of a convergent series, since $L_2 - 2\gamma > n$. Thus, $f_{\nu,M} \xrightarrow{M \rightarrow \infty} f_\nu$ in $\mathcal{S}$ -topology.

■

Remark A.3 If T is a continuous linear operator from $\mathcal{S}$ into $\mathcal{S}'$ and $f \in S_0$, then

$$Tf = \sum_{\nu \in \mathbb{Z}} T(\tilde{\varphi}_\nu * \psi_\nu * f) = \sum_{\nu \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{Q_{\nu k}} \rangle T\psi_{Q_{\nu k}} = \sum_Q \langle f, \varphi_Q \rangle T\psi_Q.$$

Lemma A.4 S_0 is dense in $\dot{B}_p^{\alpha q}(W)$ for $\alpha \in \mathbb{R}$, $0 < q < \infty$, $1 \leq p < \infty$ and if W satisfies any of (A1)-(A3).

PROOF. Let $\vec{f} \in \dot{B}_p^{\alpha q}(W)$. For $N \in \mathbb{N}$ denote $\vec{f}^N = \sum_{\nu=-N}^N \sum_{Q \in Q_\nu} \langle \vec{f}, \varphi_Q \rangle \psi_Q$. Then by Corollary 4.6

$$\|\vec{f} - \vec{f}^N\|_{\dot{B}_p^{\alpha q}(W)}^q = \left\| \sum_{|\nu| > N} \sum_{Q \in Q_\nu} \vec{s}_Q(\vec{f}) \right\|_{\dot{B}_p^{\alpha q}(W)}^q \leq c \left\| \left\{ \vec{s}_Q(\vec{f}) \right\}_{Q \in Q_\nu, |\nu| > N} \right\|_{\dot{b}_p^{\alpha q}(W)}^q$$

$$= c \sum_{|\nu| > N} \left(2^{\nu\alpha} \left\| \sum_{Q \in Q_\nu} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right)^q \xrightarrow{N \rightarrow \infty} 0$$

as a tail of the convergent series

$$\begin{aligned} \sum_{\nu \in \mathbb{Z}} A_\nu &:= \sum_{\nu \in \mathbb{Z}} \left(2^{\nu\alpha} \left\| \sum_{Q \in Q_\nu} |Q|^{-1/2} \vec{s}_Q \chi_Q \right\|_{L^p(W)} \right)^q = \|\{\vec{s}_Q\}_Q\|_{\dot{b}_p^{\alpha q}(W)}^q \quad (\text{A.2}) \\ &\leq c \|\vec{f}\|_{\dot{B}_p^{\alpha q}(W)}^q < \infty, \end{aligned}$$

if W satisfies any of (A1)-(A3) by Theorem 4.15.

As in the previous proposition for each $\nu \in \mathbb{Z}$ and $M \in \mathbb{N}$ define $\vec{f}_{\nu,M} = \sum_{k \leq M} \langle \vec{f}, \varphi_{Q_{\nu k}} \rangle \psi_{Q_{\nu k}} \in \mathcal{S}_0$ and recall $\vec{f}_\nu = \sum_{Q \in Q_\nu} \langle \vec{f}, \varphi_Q \rangle \psi_Q$. Note that $\vec{f}^N = \sum_{|\nu| \leq N} \vec{f}_\nu$. Then

$$\begin{aligned} \|\vec{f}_\nu - \vec{f}_{\nu,M}\|_{\dot{B}_p^{\alpha q}(W)} &\leq c \left\| \left\{ \vec{s}_{Q_{\nu k}}(\vec{f}) \right\}_{|k| \geq M} \right\|_{\dot{b}_p^{\alpha q}(W)} = c 2^{\nu\alpha} \left\| \sum_{|k| \geq M} 2^{\nu n/2} \vec{s}_{Q_{\nu k}} \chi_{Q_{\nu k}} \right\|_{L^p(W)} \\ &= c 2^{\nu\alpha} \left(\sum_{|k| \geq M} 2^{\nu n p/2} \int_{Q_{\nu k}} \|W^{1/p}(t) \vec{s}_{Q_{\nu k}}\|^p dt \right)^{1/p} \xrightarrow{M \rightarrow \infty} 0 \end{aligned}$$

again as a tail of the convergent series $\sum_{|k| \in \mathbb{Z}^n} 2^{\nu n p/2} \int_{Q_{\nu k}} \|W^{1/p}(t) \vec{s}_{Q_{\nu k}}\|^p dt = (2^{-\nu\alpha} A_\nu^{1/q})^p < \infty$, since each $A_\nu < \infty$ (see (A.2)). Thus, each $\vec{f} \in \dot{B}_p^{\alpha q}(W)$ is a limit (in $\dot{B}_p^{\alpha q}(W)$ -norm) of $\mathcal{S}_0$ functions and so $\mathcal{S}_0$ is a dense subset of $\dot{B}_p^{\alpha q}(W)$.

■

Proposition A.5 $\mathcal{S}_0$ is dense in $\dot{B}_p^{\alpha q}(\{A_Q\})$ if $\{A_Q\}_Q$ is doubling of order p , $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$, $0 < q < \infty$.

PROOF. Repeat the previous proof with W replaced by $\{A_Q\}_Q$ and refer to Lemma 7.7 instead of Corollary 4.6 and to Lemma 7.5 instead of Theorem 4.15. Both require $\{A_Q\}_Q$ to be only doubling. ■

BIBLIOGRAPHY

BIBLIOGRAPHY

- [CG] M. CHRIST AND M. GOLDBERG, *Vector A_2 weights and a Hardy-Littlewood Maximal Function*, Trans. Amer. Math. Soc., v. 353, no. 5, pp. 1995-2002, 2001.
- [CS] M. COTLAR AND C. SADOSKY, *On the Helson-Szegö Theorem and a Related Class of Modified Toeplitz Kernels*, Harmonic Analysis in Euclidean Spaces, G. Weiss and S. Waigner, eds., vol. 1, Proc. Sympos. Pure Math., no. 35, Amer. Math. Soc., Providence, pp. 383-407, 1979.
- [D] I. DAUBECHIES, *Orthonormal Bases of Compactly Supported Wavelets*, Comm. Pure Appl. Math, v. 41, pp. 909-996, 1988.
- [D] R. EDWARDS, *Functional Analysis: Theory and Applications*, Holt, Rinehart and Winston, Inc., 1965.
- [FJ1] M. FRAZIER AND B. JAWERTH, *Decomposition of Besov Spaces*, Indiana Univ. Math. J., v. 34, pp. 777-799, 1985.
- [FJ2] M. FRAZIER AND B. JAWERTH, *A Discrete Transform and Decompositions of Distribution Spaces*, J. Funct. Anal., v. 93, pp. 34-170, 1990.
- [FJW] M. FRAZIER, B. JAWERTH AND G. WEISS, *Littlewood-Paley Theory and Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, v. 79, AMS, Providence, RI, 1991.
- [FM] C. FEFFERMAN AND B. MUCKENHOUT, *Two Nonequivalent Conditions for Weight Functions*, Proc. Amer. Math. Soc., v. 45, no.1, pp. 99-104, 1974.
- [FTW] M. FRAZIER, R. TORRES AND G. WEISS, *The Boundedness of Calderón-Zygmund Operators on the Spaces $\dot{F}_p^{\alpha q}(W)$* , Rev. Mat. Iberoamericana, v. 4, no.1, pp. 41-72, 1988.
- [Ga] J.B. GARNETT, *Bounded Analytic Functions*, Acad. Press, NY, 1981.

- [Go] D. GOLDBERG, *A Local Version of Real Hardy Spaces*, Duke Math. J., v. 46, pp. 27-42, 1979.
- [HMW] R. HUNT, B. MUCKENHOUT AND R. WHEEDEN, *Weighted Norm Inequalities for Conjugate Function and Hilbert Transform*, Trans. Amer. Math. Soc., v. 176, pp. 227-251, 1973.
- [LM] P.G. LEMARIÉ AND Y. MEYER, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana, v. 2, pp. 1-18, 1986.
- [M1] Y. MEYER, *Principe D'incertitude, Bases Hilbertiennes et Algèbres D'opérateurs*, Séminaire Bourbaki, v. 662, pp. 1-15, 1985-1986.
- [M2] Y. MEYER, *Wavelets and Operators*, Cambridge University Press, 1992.
- [MW] B. MUCKENHOUT AND R. WHEEDEN, *Two Weight Function Norm Inequalities for the Hardy-Littlewood Maximal Function and the Hilbert Transform*, Studia Math., v.60, pp. 279-294, 1976.
- [NT] F. NAZAROV, S. TREIL, *The Hunt for a Bellman Function: Applications to Estimates for Singular Integral Operators and to Other Classical Problems of Harmonic Analysis*, Algebra i Analiz (in russian) 8, no. 5, pp. 32-162, 1996.
- [NTV] F. NAZAROV, S. TREIL, A. VOLBERG, *The Bellman Functions and Two Weight Inequalities for Haar Multipliers*, J. Amer. Math. Soc, v.12, N4, pp. 909-928, 1999.
- [P] J. PEETRE, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, Durham, NC, 1976.
- [St1] E. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [St2] E. STEIN, *Harmonic Analysis*, Princeton University Press, 1993.
- [Sa] E. SAWYER, *A Characterization of a Two-Weight Norm Inequality for Maximal Operators*, Studia Math., v. 75, pp. 1-11, 1982.
- [T] H. TRIEBEL, *Theory of Function Spaces*, Monographs in Math., vol. 78, Birkhäuser Verlag, Basel, 1983.
- [TV1] S. TREIL, A. VOLBERG, *Wavelets and the Angle between Past and Future*, J. Funct. Anal. 143, pp. 269-308, 1997.

- [TV2] S. TREIL, A. VOLBERG, *Continuous Frame Decomposition and Matrix Hunt-Muckenhoupt-Wheeden Theorem*, Ark. Mat. 35, pp. 363-386, 1997.
- [V] A. VOLBERG, *Matrix A_p Weights via S -functions*, J. Amer. Math. Soc., v.10, no.2, pp. 445-466, 1997.