HEEGAARD FLOER HOMOLOGY AND L-SPACE KNOTS

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ABSTRACT

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Heegaard Floer theory consists of a set of invariants of three- and four-dimensional manifolds. Three-manifolds with the simplest Heegaard Floer invariants are called L-spaces, and the name stems from the fact that lens spaces are L-spaces. The overarching goal of the dissertation is to understand L-spaces better. More specifically, this dissertation could be considered as a step towards finding topological characterizations of L-spaces and L-space knots without referencing Heegaard Floer homology. We study knots in S^3 that admit positive L-space Dehn surgeries. In particular, we give new examples of knots in S^3 within both the families of hyperbolic and satellite knots admitting L-space surgeries. It should be pointed out that for satellite knot examples, we use Berge-Gabai knots (i.e. knots in $S^1 \times D^2$ with non-trivial solid torus Dehn surgeries) as the pattern. Moreover, we study the relationship between satellite knots and L-space surgeries in the general setting, i.e. when the pattern is an arbitrary knot in $S^1 \times D^2$.

To my parents, Fatemeh Vakili and Hossein Vafaee

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parents, and also my brother Farhad. Their unending love, encouragement, and support have been a constant source of comfort and counsel for me during the last six years.

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Introduction

In [OS04d], Oszváth and Szabó introduced Heegaard Floer theory, which produces a set of invariants of three- and four-dimensional manifolds. One example of such invariants is $\widehat{HF}(Y)$, which associates a graded abelian group to a closed 3-manifold Y. When Y is a rational homology three-sphere, rk $\widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$ [OS04c]. If equality is achieved, then Y is called an *L-space*. Examples include lens spaces, and more generally, all connected sums of manifolds with elliptic geometry [OS05b] (or equivalently, with finite fundamental group by the Geometrization Theorem [KL08]). L-spaces are of interest for various reasons. For instance, such manifolds do not admit co-orientable taut foliations [OS04a, Theorem 1.4].

A knot $K \subset S^3$ is called an *L-space knot* if it admits a positive L-space Dehn surgery. Any knot with a positive lens space surgery is then an L-space knot. In [Ber], Berge gave a conjecturally complete list of knots that admit lens space surgeries. Therefore it is natural to look beyond Berge's list for L-space knots. Examples include the $(-2, 3, 2n+1)$ pretzel knots (for positive integers n) [BH96, FS80, OS05b], which are known to live outside of Berge's collection when $n \geq 5$ [Mat00]. We should first note that the pretzel knots $(2, 3, 1)$, $(2, 3, 3)$, and $(2, 3, 5)$ are isotopic to the $(2, 5)$, $(3, 4)$, and $(3, 5)$ torus knots, respectively. Torus knots, a proper subfamily of Berge knots, are well-known to admit lens space surgeries [Mos71]. The hyperbolic pretzel knot $(2, 3, 7)$ is also known to have two lens space surgeries [FS80]. The knot $(2, 3, 9)$ has two finite, non-cyclic surgeries [BH96]. Finally, the remaining knots, $(2, 3, 2n + 1)$ for $n \geq 5$, are known to have Seifert fibered L-space surgeries with infinite fundamental group [OS05b]. It is also proved in [LM13] that these 3-strand pretzel knots are the only pretzel knots with L-space surgeries. Another source of L-space knots is within the set of cable knots. By combining work of Hedden [Hed09] and Hom [Hom11a], the (m, n) cable of a knot, K, is an L-space knot if and only if K is an L-space knot and $n/m \geq 2g(K) - 1$. Also, if K is a quasi-alternating knot with unknotting number one, then the preimage of an unknotting arc in the branched double cover of K is a knot in an L-space with an S^3 surgery (see [OS05a, Section 8.3]). The dual to this curve is therefore a knot in S^3 with an L-space surgery, so either it or its mirror image is an L-space knot. However, at present, there is no explicit parametrization of the knots that arise in this way. Quasi-alternating knots were first introduced by Ozsváth and Szabó in [OS05c]. This class of knots appeared in the context of the Floer homology of links as a natural generalization of alternating links. This comes from the fact that the branched double cover of an alternating link, and more generally a quasi-alternating link, is an L-space [OS05c, Proposition 3.3]. We remind the reader that the branched double cover $\Sigma(K)$ of a knot K with unknotting number one can be obtained by doing Dehn surgery on a knot, $\widetilde{K}$, in S^3 [Mon73] (the Montesinos trick). Therefore, for the case that K is quasi-alternating with unknotting number one, if the surgery coefficient in the branched double cover is positive, then $\widetilde{K}$ will be an L-space knot.

L-space knots are those knots that admit positive Dehn surgeries so that the resulting three-manifolds have the same Heegaard Floer homology as lens spaces. One of the most prominent problems in relating Heegaard Floer homology to low-dimensional topology is to find topological characterizations of L-spaces and L-space knots. Ghiggini proved in [Ghi08] that genus one L-space knots are fibered (see also [OS05b]). Later, Ni [Ni07] showed that all L-space knots are fibered. It was proved by Hedden [Hed10] that knots in S^3 with positive

L-space surgeries are *strongly quasipositive*. A strongly quasipositive knot is a knot that has a particular type of Seifert surface, called a *quasipositive surface*. Quasipositive surfaces are those surfaces obtained from n parallel disks by attaching positive bands. A knot or link is strongly quasipositive if it can be realized as the boundary of such a surface. Note that for a fibered knot $K \subset S^3$, being strongly quasipositive is equivalent to having the corresponding open book decomposition associated to (F, K), where F is the fiber surface, inducing the unique tight contact structure on S^3 [Hed10, Proposition 2.1]. In particular, L-space knots induce the standard tight contact structure on S^3. (We refer the reader to [Etn06, OS04] for a review of contact geometry.) Unfortunately, it is not the case that every strongly quasipositive fibered knot admits a non-trivial L-space Dehn surgery. An example of such a knot is the $(2, 1)$ cable of the right-handed trefoil. It is strongly quasipositive by [Hed10, Corollary 1.3]. However, an exploration of its knot Floer homology groups (found in [Hed05]) reveal that it cannot admit an L-space surgery.

In [Sch49], Schubert showed that every knot in S^3 decomposes as a connected sum of prime knots in a unique way (up to reordering). We also recall that every knot in S^3 is either hyperbolic, a satellite, or a torus knot. It is well-known that positive torus knots admit positive lens space surgeries [Mos71], and therefore are L-space knots. In this work we demonstrate new examples of L-space knots within both the families of hyperbolic and satellite knots. (We should point out that L-space knots are prime [Krc13, Theorem 1.2].)

1.1 Hyperbolic L-space knots

The primary purpose of this section is to investigate L-space knots in the family of *twisted torus knots*, $K(p, q; s, r)$, which are defined to be, roughly speaking, the (p, q) torus knots with

r full twists on s adjacent strands where $0 < s < p$. See Figure 1.2. Twisted torus knots are an interesting class of knots that have been studied in several contexts. These knots represent many different knot types. Morimoto and Yamada [MY] and Lee [Lee12] have constructed twisted torus knots which are cables. Morimoto has also shown that infinitely many twisted torus knots are composite [Mor94]. Guntel has shown that infinitely many twisted torus knots are torus knots [Gun12]. Also, a complete characterization of twisted torus knots which are isotopic to the unknot has been done by Lee in [Lee13]. Moriah and Sedgwick have shown that certain hyperbolic twisted torus knots have minimal genus Heegaard splittings which are unique up to isotopy [MS09]. In terms of the relationship between twisted torus knots and their Floer homology, however, not much was previously known.

Watson proved in [Wat09] that the knots $K(3, 3k+2; 2, 1)$ are L-space knots ($k > 0$). We generalize this result in Corollary 2.2.3 by showing that all twisted torus knots $K(3, q; 2, s)$, for all $q, s > 0$, admit L-space surgeries (in order for a twisted $(3, q)$ torus link to be a knot, q must be an integer that does not divide 3).

In Chapter 2, we classify all the L-space twisted (p, q) torus knots with $q = kp \pm 1$ (see [Vaf14]). The question of what happens when $q \neq kp \pm 1$ remains unanswered. Our examples include the L-space pretzel knots as a proper subfamily since the $(-2, 3, 2m+3)$ pretzel knot is isotopic to $K(3, 4; 2, m)$ for $m \geq 1$.

We now state the main result of this section. For $p \geq 2$, $k \geq 1$, $r > 0$ and $0 < s < p$:

Theorem 1.1.1. *The twisted torus knot, $K(p, kp \pm 1; s, r)$, is an L-space knot if and only if either $s = p - 1$ or $s \in \{2, p-2\}$ and $r = 1$.*

A key ingredient of the proof is the observation that all of the twisted torus knots being

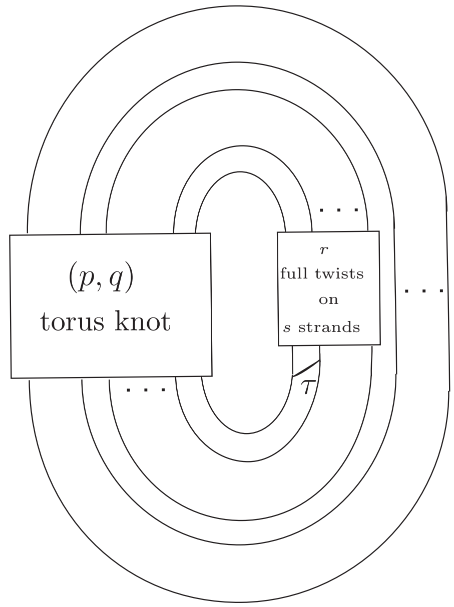


Figure 1.1 A (p, q) torus knot with r positive full twists on s adjacent strands. (Here, p denotes the longitudinal winding.) The arc τ is a one-bridge, i.e. it divides the knot into two arcs, where one arc is unknotted and the other arc can be trivialized (unknotted) by sliding one or both of its endpoints along the *a priori* unknotted arc. In order to make sense of adjacency of strands, we need to have the standard presentation of a torus knot. Note that where the twist occurs is irrelevant.

studied in Theorem 1.1.1 are $(1, 1)$ *knots*, that is, knots that can be placed in one-bridge position with respect to a genus one Heegaard splitting of S^3. Thus, the knot is comprised of two properly embedded unknotted arcs, one in each of the two solid tori of the Heegaard splitting. These arcs meet along their endpoints so that their union is equal to the knot.

From the perspective of knot Floer homology, (1,1) knots are particularly appealing. It was first observed by Goda, Morifuji, and Matsuda [GMM05] that $(1, 1)$ knots are exactly those knots that can be presented by a *doubly-pointed Heegaard diagram* of genus one. The chain complex for knot Floer homology is defined in terms of a doubly-pointed Heegaard diagram. As shown by Ozsváth and Szabó [OS04b], for knots admitting a genus one diagram, knot Floer homology can be computed combinatorially and efficiently.

Recall that the tunnel number of a knot K in S^3 is the minimum number of mutually disjoint arcs with endpoints on K so that the exterior of the resulting 1-complex is homotopy equivalent to a handlebody. It is known that all of the twisted torus knots of Theorem 1.1.1, as well as all of the Berge knots, are $(1,1)$ knots (see, for instance, [MSY96] for the case of twisted torus knots). It is also known that all $(1,1)$ knots have tunnel number one. In [Mot14] Motegi shows that there exist infinitely many hyperbolic L-space knots with tunnel number two (see also [BM14, Question 24]). We propose the following question:

Question 1.1.2. *Is there a non-satellite L-space knot with tunnel number greater than two?*

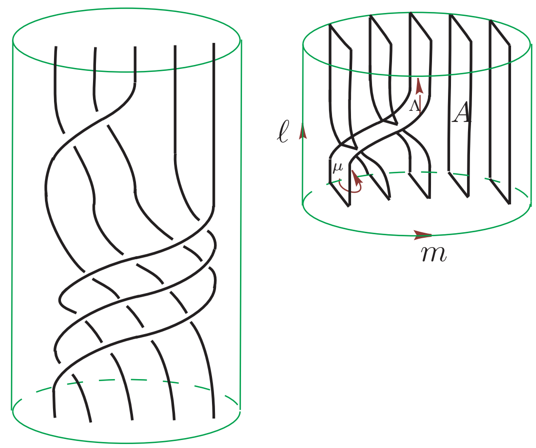
1.2 Satellite knots and L-space surgery

As stated previously, cabling an L-space knot, when the ratio n/m of the cabling parameters is large enough, is an L-space satellite operation. (We recall that a cabled knot is a special case of a satellite knot; namely, the pattern is an (m,n)-torus knot.) We generalize this result by introducing a new L-space satellite operation using Berge-Gabai knots [Gab90] as the pattern.

Definition 1.2.1. A knot $P \subset S^1 \times D^2$ is called a *Berge-Gabai knot* if it admits a non-trivial solid torus filling.[1]

To see that this satellite operation is a generalization of cabling, it should be noted that any torus knot with the obvious solid torus embedding is a Berge-Gabai knot [Sei33]. Note also that any Berge-Gabai knot P which is isotopic to a positive braid, when considered as a knot in S^3, admits a positive lens space surgery; for if performing appropriate surgery on P in one of the solid tori in the genus one Heegaard splitting of S^3 returns a solid torus, then

[1] Berge-Gabai knots, in the literature, are defined to be 1-bridge braids in solid tori with non-trivial solid tori fillings. We relax that definition to include torus knots as a proper subfamily.



(a) (b)

Figure 1.2 Berge-Gabai knots are knots in $S^1 \times D^2$ with non-trivial solid tori fillings. Such knots are always the closure of the braid $(\sigma_b\sigma_{b-1}\ldots\sigma_1)(\sigma_{w-1}\sigma_{w-2}\ldots\sigma_1)^t$ where $0 \le b \le w-2$, and $|t| \ge 1$. (a) An example of a braid in a solid cylinder $I \times D^2$ that closes to form a Berge-Gabai knot with $b = 2$, $t = 3$, and $w = 5$. (The fact that the picture depicted above represents a Berge-Gabai knot is verified in [Gab90, Example 3.8].) Recall that we write $t = t_0 + qw$, where here $t_0 = 3$ and $q = 0$. (b) An immersed annulus A that can be arranged to be an embedded surface in $V = S^1 \times D^2$ joining P to $T = \partial V$ by performing oriented cut and paste and adding a $2\pi t/w$ twist. Note that the embedded surface A provides, in the exterior of P, a homology from $w\ell + tm$ in T to Λ in $J = \partial \mathrm{nb}(P)$.

the corresponding surgery on the knot in S^3 will result in a lens space. For positive braids, this surgery is positive by Lemma 3.1.1 and [Mos71, Proposition 3.2].

It is shown in [Gab89] that any Berge-Gabai knot must be either a torus knot or a 1-bridge braid in $S^1 \times D^2$. More precisely, every Berge-Gabai knot $P \subset V = S^1 \times D^2$ is necessarily of the following form. (For a sufficient condition determining when a knot of this form is a Berge-Gabai knot, see [Gab90, Lemma 3.2].) Let w denote the braid index of P. In the braid group B_w let σ_i denote the generator of B_w that performs a positive half twist on strands i and $i+1$. Let $\sigma = \sigma_b\sigma_{b-1}\ldots\sigma_1$ be a braid in B_w with $0 \le b \le w-2$ and let t be a nonzero integer. Place σ in a solid cylinder and glue the ends by a $2\pi t/w$ twist, i.e., form the closure of the braid word $(\sigma_b\sigma_{b-1}\dots\sigma_1)(\sigma_{w-1}\sigma_{w-2}\dots\sigma_1)^t$. We only consider the case where this construction produces a knot, rather than a link. This construction forms a torus knot if $b=0$ and a 1-bridge representation of P in V if $1 \le b \le w-2$. We call b the *bridge width*, and t the *twist number* of P. Note that the twist number can be written as $t = t_0 + qw$ for some integers t_0 and q where t_0 can be chosen so that $1 \le t_0 \le w-1$.[2] See Figure 1.2(a). Also, note that if $b \neq 0$ then the possibility of $t_0 = w-1$ is disallowed as otherwise we would obtain a link with at least two components [Gab90].

Remark 1.2.2. Note that if $t < 0$, then the braid $\sigma = (\sigma_b\sigma_{b-1}\dots\sigma_1)(\sigma_{w-1}\sigma_{w-2}\dots\sigma_1)^t$ is isotopic to a negative braid:

$$\begin{aligned}\sigma &\sim (\sigma_b\sigma_{b-1}\dots\sigma_1)(\sigma_{w-1}\sigma_{w-2}\dots\sigma_1)^t \\ &\sim (\sigma_{w-1}\sigma_{w-2}\dots\sigma_{b+1})^{-1}(\sigma_{w-1}\sigma_{w-2}\dots\sigma_1)^{t+1}.\end{aligned}$$

We are now ready to state the main result. Let $P(K)$ denote a satellite knot with pattern P and companion K.

Theorem 1.2.3. *Let P be a Berge-Gabai knot with bridge width b, twist number t, and winding number w, and let K be a non-trivial knot in S^3. Then the satellite $P(K)$ is an L-space knot if and only if K is an L-space knot and $\frac{b+tw}{w^2} \ge 2g(K)-1$.*

Note that when $b=0$, we can take $w=m$ and $t=n$, and Theorem 1.2.3 reduces to the cabling result of [Hed09, Hom11a]. A version of the "if" direction of Theorem 1.2.3 appears

[2] Our construction of Berge-Gabai knots, which enables us to define them up to isotopy of the knot in $S^1 \times D^2$, is slightly different than that of Gabai [Gab90]. In Gabai's original construction, he always took $q = 0$ and considered knots in the solid torus up to homeomorphism of $S^1 \times D^2$ taking one knot to the other.

in [Mot14, Proposition 7.2].

The outline of the proof of Theorem 1.2.3 is as follows. By applying techniques developed in [Gab90, Gor83] to carefully explore the framing change of the solid torus surgered along P, we prove the "if" direction of the theorem. More precisely, surgery on $P(K)$ corresponds to first doing surgery on P (namely removing a neighborhood of P from $S^1 \times D^2$ and Dehn filling along the new toroidal boundary component) and, second, attaching this to the exterior of K. Therefore, if one chooses the filling on P such that the result is a solid torus (using that P is a Berge-Gabai knot), then the overarching surgery on $P(K)$ corresponds to attaching a solid torus to the exterior of K (performing surgery on K). Moreover, note that by positively twisting P by performing a positive Dehn twist on $S^1 \times D^2$ (i.e., increasing q), we can obtain an infinite family of Berge-Gabai knots. Fixing an L-space knot K, for sufficiently large q, the satellite $P(K)$ will admit a positive L-space surgery. Finally, the "only if" direction is proved by methods similar to those used in [Hom11a].

In order to prove Theorem 1.2.3, we establish the following lemma, which may be of independent interest.

Lemma 1.2.4. *Let $P \subset S^1 \times D^2$ be a negative braid and $K \subset S^3$ be an arbitrary knot. Then the satellite knot $P(K)$ is never an L-space knot.*

We point out that Lemma 1.2.4 can be extended more generally to the case that P is a homogeneous braid which is not isotopic to a positive braid [Sta78, Theorem 2]. The proof of Lemma 1.2.4 was inspired by the arguments in [BM14].

We have the following corollary concerning the Ozsváth-Szabó concordance invariant τ and the smooth 4-ball genus.

Corollary 1.2.5. *Let $P \subset S^1 \times D^2$ be a Berge-Gabai knot and $K \subset S^3$ be an L-space knot.*

If $\frac{b+tw}{w^2} \geq 2g(K) - 1$*, then*

$$\tau(P(K)) = \tau(P) + w\tau(K),$$

and

$$g_4(P(K)) = g_4(P) + wg_4(K),$$

where $\tau(P)$*, respectively* $g_4(P)$*, denotes* τ*, respectively the* 4*-ball genus, of the knot obtained from the standard embedding of* $S^1 \times D^2$ *into* S^3*.*

Proof. If J is an L-space knot, then $\tau(J) = g_4(J) = g(J)$ by [OS05b, Corollary 1.6] and [Ni07, Corollary 1.3]. Furthermore, by Lemma 3.2.4,

$$g(P(K)) = g(P) + wg(K).$$

By assumption, K is an L-space knot. Since P is a Berge-Gabai knot with a positive twist number, it follows that P is isotopic to a positive braid. Therefore, by the discussion following Definition 1.2.1, P has a positive lens space surgery, and thus is an L-space knot. Furthermore, by Theorem 1.2.3, we also have that $P(K)$ is an L-space knot, and the result follows. □

1.3 Further results and discussion

Theorem 1.2.3 allows one to construct new examples of L-spaces as follows. First, begin with any L-space knot and then satellite with a Berge-Gabai knot satisfying the conditions in Theorem 1.2.3. Sufficiently large positive surgery will then result in an L-space. Using this technique, we will construct L-spaces with any number of hyperbolic and Seifert fibered pieces in the JSJ decomposition.

Theorem 1.3.1. *Let r and s be non-negative integers such that at least one is non-zero. Then there exist infinitely many irreducible L-spaces whose JSJ decompositions consist of exactly r hyperbolic pieces and s Seifert fibered pieces.*

As discussed, an L-space cannot admit a co-orientable taut foliation. Therefore, Theorem 1.3.1 will yield irreducible rational homology spheres without co-orientable taut foliations whose JSJ decompositions consist of any numbers of hyperbolic and Seifert fibered pieces. We remark that all rational homology spheres with Sol geometry are L-spaces [BGW13].

It is also natural to ask in what sense Theorem 1.2.3 generalizes; in particular, given a satellite knot which is an L-space knot, what must hold for the pattern or the companion? We propose the following conjecture (see also [BM14, Question 22]).

Conjecture 1.3.2. *If $P(K)$ is an L-space knot, then so are K and P.*

Similarly, we conjecture that the converse holds as well, contingent on the pattern being embedded "nicely" in the solid torus (e.g., as a strongly quasipositive braid closure) and sufficiently "positively twisted" (akin to the condition in Theorem 1.2.3). We will not attempt to make these notions precise in the dissertation.

As supporting evidence for Conjecture 1.3.2, we will study it from the viewpoint of left-orderability. Recall that a non-trivial group G is *left-orderable* if there exists a left-invariant total order on G (see Section 3.5 for a more detailed discussion). We recall the conjecture of Boyer, Gordon, and Watson relating Heegaard Floer homology to the left-orderability of three-manifold groups.

Conjecture 1.3.3 (Boyer-Gordon-Watson [BGW13])**.** *Let Y be an irreducible rational homology sphere. Then Y is an L-space if and only if $\pi_1(Y)$ is not left-orderable.*

We point out that the computational strengths of Heegaard Floer homology and left-orderability tend to be fairly different. It is hopeful that if Conjecture 1.3.3 is true then the strengths of each theory could be combined to derive new topological consequences. We utilize this philosophy to establish Conjecture 1.3.2 under the assumption of Conjecture 1.3.3.

Proposition 1.3.4. *Assuming Conjecture 1.3.3, if $P(K)$ is an L-space knot, then so are P and K.*

Conjecture 1.3.2 can be also viewed from the perspective of strongly quasipositive knots. Recall that L-space knots are strongly quasipositive [Hed10, Proposition 2.1]. Therefore it makes sense to ask that for a strongly quasipositive fibered satellite knot $P(K)$, what must hold for the pattern P and companion K? We propose the following:

Question 1.3.5. *For a fibered satellite knot $P(K)$ with pattern a non-trivial knot $P \subset S^1 \times D^2$ (i.e., not isotopic to the unknot when considered as a knot in S^3) and companion $K \subset S^3$, is it true that strongly quasipositiveness of any of the two implies the strongly quasipositiveness of the third.*

Chapter 2

Twisted torus knots and L-space surgery

This chapter is mainly based on [Vaf14]. The focus of Chapter 2 is to prove Theorem 1.1.1. Section 2.1 introduces the theory of $(1, 1)$ knots and presents how to draw a genus one Heegaard diagram for $(1, 1)$ knots via an explicit example. Section 2.2 contains the proof of the main result (Theorem 1.1.1), as well as the corollaries. In the final section, we state some questions that address future research.

2.1 Background and preliminary lemmas

We start this section by showing that the knots $K(p, kp \pm 1; s, r)$ are $(1, 1)$ knots. Next, we explain an algorithm which produces genus one Heegaard diagrams for the twisted torus knots with a $(1, 1)$ decomposition. Finally, we assemble some preliminary facts needed to prove Theorem 1.1.1.

2.1.1 (1, 1) knots and genus one Heegaard diagrams

Let p and q be relatively prime positive integers and let r and s be integers. We denote the knot illustrated in Figure 1.2 by $K(p, q; s, r)$. Let τ be the arc indicated in Figure 1.2. By untying the crossings of the r full twists above the arc through edge slides along the arc, we

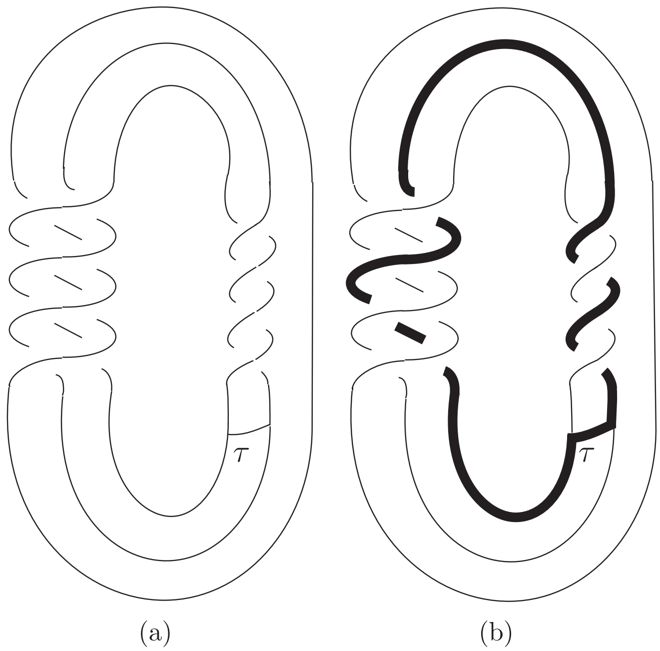


Figure 2.1 A $(3, 4)$ torus knot with two positive full twists on two adjacent strands. The one-bridge is indicated by τ.

will show that τ becomes a one-bridge for $K(p, q; s, r)$ provided that $q = kp \pm 1$. See Figure 2.1 for an explicit example. It has been a long standing question of whether or not any twisted torus knot, with q that is not of the form $kp \pm 1$, is a $(1, 1)$ knot. In 1991, Morimoto, Sakuma, and Yokota conjectured that the answer is negative:

Conjecture 2.1.1 ([MSY96], Conjecture 1.3)**.** *$K(p, q; 2, r)$ admits no $(1, 1)$ decomposition unless either $p \equiv \pm 1 \pmod q$, or $q \equiv \pm 1 \pmod p$, or $r = 0, \pm 1$.*

Having $s = 2$ does not seem to play an important role in the conjecture and, in fact, we expect a similar conjecture to hold when the twisting is on any number of strands. Bowman, Taylor, and Zupan have proved this conjecture when the number of twists is large [BTZ14, Theorem 1.1].

In the rest of this subsection, we give an explicit construction of a genus one doubly-pointed Heegaard diagram via a specific example, namely $K = K(3, 4; 2, 2)$. See Figure 2.1. This example should help clarify the strategy we use for our calculations.

We now describe a procedure to see that the arc τ (indicated in Figure 2.1) is a *one-bridge*, i.e. it divides the knot K into two arcs, where one arc is *a priori* unknotted and the other arc can be trivialized (unknotted) by sliding one or both endpoints of this arc along the bold curve in Figure 2.1(b). (See [Ord06] for a detailed discussion on how to produce a genus one Heegaard diagram for a certain family of $(1, 1)$ knots.) The closed curve indicated in bold is the union of the one-bridge, τ, and the *a priori* unknotted arc. Therefore, its neighborhood is an unknotted torus. In Figure 2.2 we show, diagrammatically, how to use the one-bridge and the unknotting process to obtain a Heegaard diagram for the knot K. (The red and blue curves in Figure 2.2 (α and β curves respectively) are the boundaries of the meridional disks corresponding to the two solid tori of the genus one Heegaard splitting of S^3.) We do this by trivializing the arc living in the complement of the torus. To begin, move the z base point in the counterclockwise direction, making sure that the z base point passes to the left of the w base point, as otherwise we would create more crossings rather than simplify the arc. See Figure 2.2(b). Now move the w base point in the clockwise direction, passing to the left of the z base point. See Figure 2.2(c). That completes the construction of the genus one Heegaard diagram. See Figure 2.2(d).

This construction can be generalized to an algorithm with three steps to produce a genus one Heegaard diagram for $K(p, kp \pm 1; s, r)$. Note that the number of longitudinal and meridional windings is dictated by the arc living in the torus complement:

Step 1: Wind the z base point once around the torus in the counter clockwise direction. Note that z traverses the torus $(k + r)$ times meridionally.

Step 2: Wind the w base point $(s-2)$ times in the clockwise direction. Note that each time w traverses the torus $(k+r)$ times meridionally.

Step 3: Finally, wind the w base point $(p-s)$ times, longitudinally, to completely trivialize the arc (in the sense that the planar projection of the arc no longer has any self-intersection). Note that each longitudinal winding goes through k meridional moves.

Remark 2.1.2. To trivialize the part of the knot that lives outside of the torus, we isotope the base points, z and w, on the torus which forces the α curve to be perturbed. Specifically, in a neighborhood of the base points, the isotopy drags one (or more) sub-arc(s) of α.

Note that the Heegaard diagram in Figure 2.2(d) may be represented by a rectangle with canonical identification implicit. See Figure 2.4(a).

2.1.2 Lifted Heegaard diagrams, L-space knots, and CFK^-

For $K \subset S^3$ a knot, let $CFK^-(K)$ denote the knot Floer complex associated to K [OS04b]. Fortunately, computing $CFK^-(K)$ for a $(1,1)$ knot K is purely combinatorial. We refer the interested reader to [OS04b, p.89] and [GMM05] for further details. To analyze holomorphic disks in the torus, it is convenient to pass to the universal covering space $\pi : \mathbb{C} \to T$. Given the base points z and w in T, $\pi^{-1}(z)$ and $\pi^{-1}(w)$ lift to affine lattices Z and W, respectively. Also let $\{\tilde{\alpha}_i\}$ and $\{\tilde{\beta}_j\}$ be the connected components of $\pi^{-1}(\alpha)$ and $\pi^{-1}(\beta)$, respectively. Now, given two intersection points x and y between α and β, the element $\phi \in \pi_2(x,y)$ is a Whitney disk that has Maslov index one and admits a holomorphic representative if and only if there is a bigon $\tilde{\phi} \in \pi_2(\tilde{x},\tilde{y})$ with Maslov index one, where $\tilde{x}$ and $\tilde{y}$ are lifts of x and y, intersection points between $\tilde{\alpha}_i$ and $\tilde{\beta}_j$ (for some i and j). In particular, $\mathcal{M}(\tilde{\phi}) \cong \mathcal{M}(\phi)$. See [OS04b] for the notation

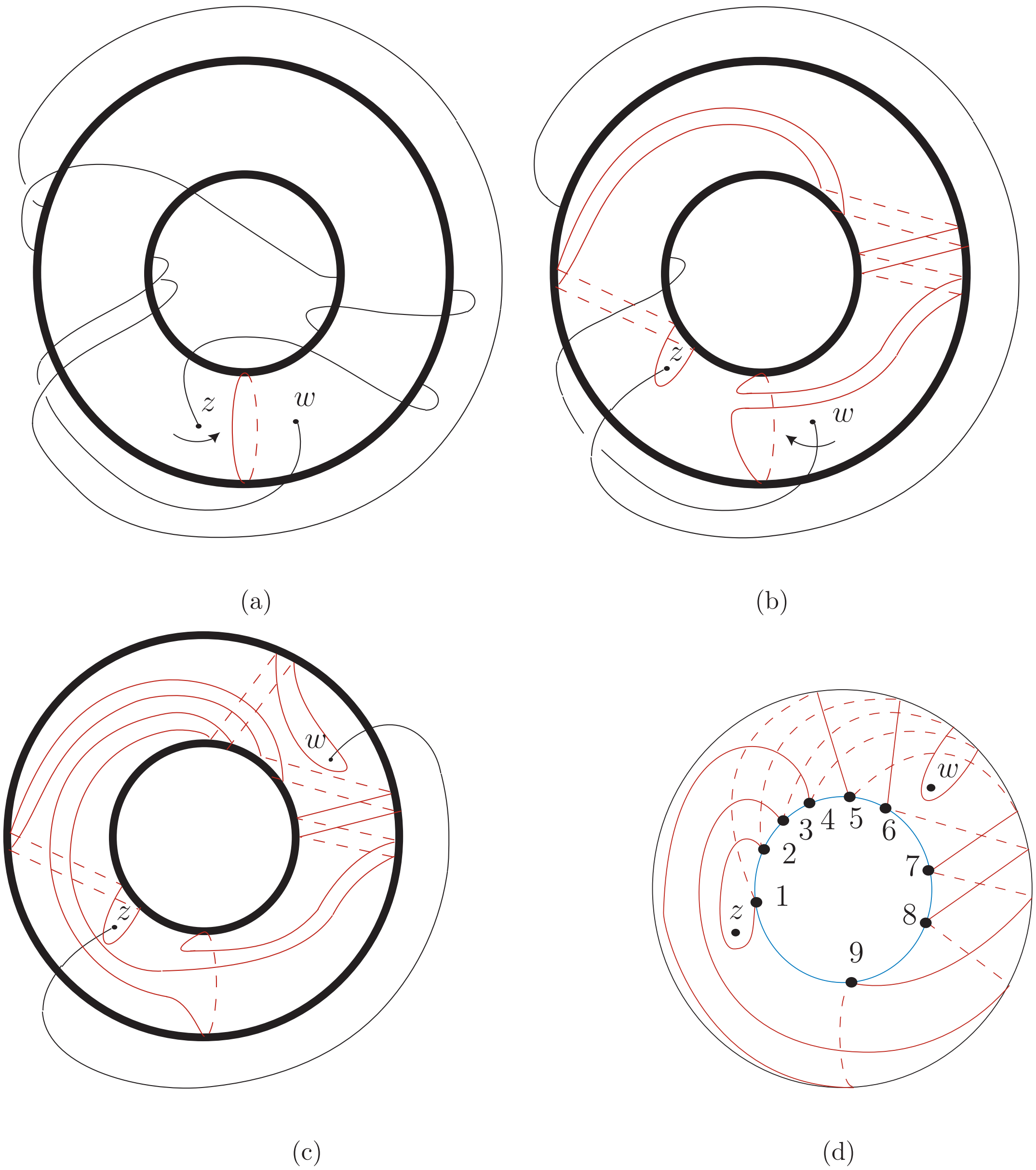


Figure 2.2 The process of obtaining a genus one Heegaard diagram for the $(3,4)$ torus knot with two positive full twists on two adjacent strands. In the algorithm discussed in Section 2.1.1, Figure 2.2(b) corresponds to Step 1, and also Figure 2.2(c) corresponds to, simultaneously, implementing Step 2 and Step 3. Note that the torus (in bold) corresponds to a neighborhood of the bold curve of Figure 2.1(b). Note also that the α curve is drawn in red and the β curve is drawn in blue.

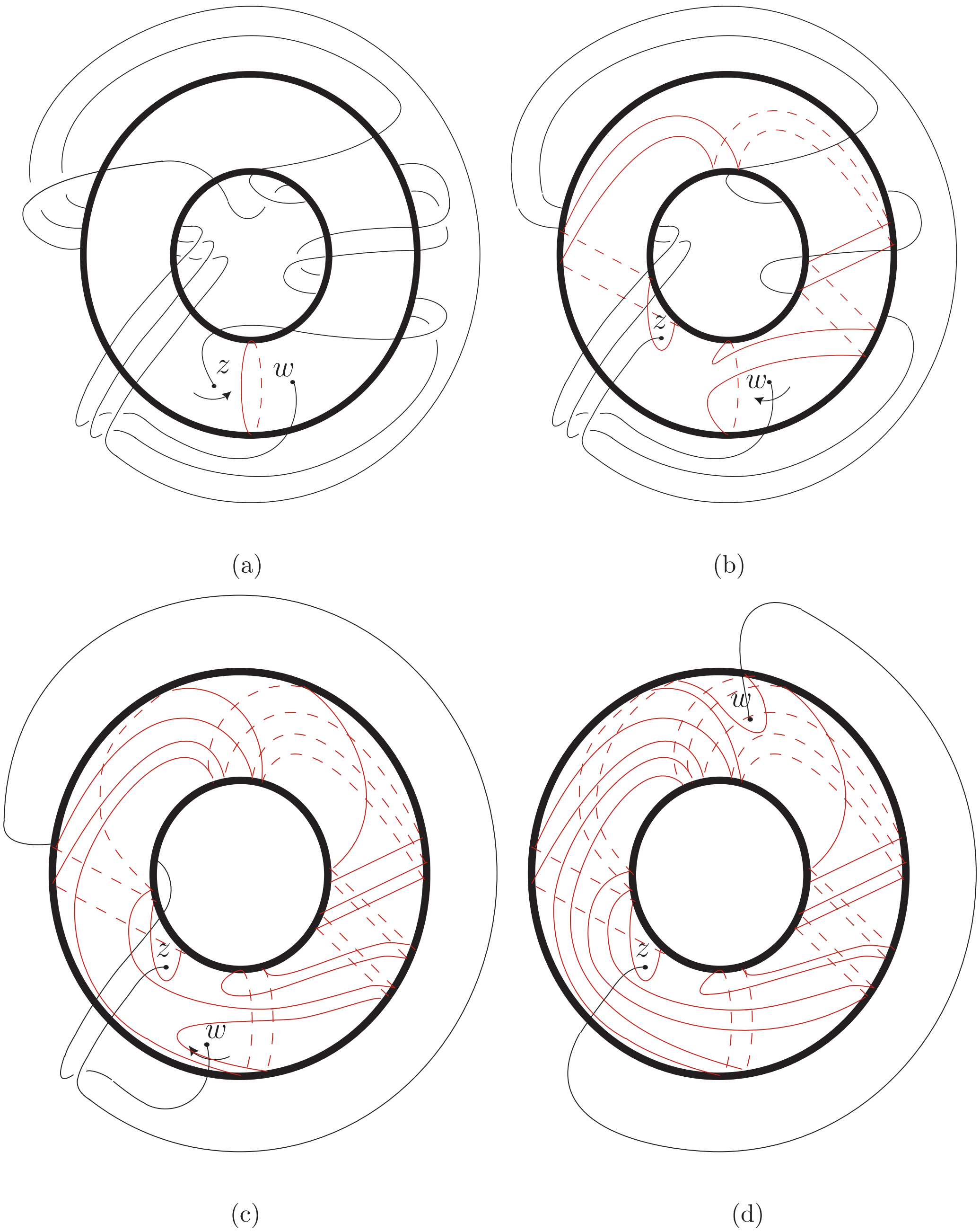


Figure 2.3 The process of drawing a genus one Heegaard diagram for the $(4, 5)$ torus knot with two positive full twists on three adjacent strands. Figure 2.3(b), Figure 2.3(c), and Figure 2.3(d) correspond to Step 1, Step 2, and Step 3, respectively, in the algorithm discussed in Section 2.1.1. The α curve is drawn in red. The base points must pass to the left of each other, as otherwise we would create more crossings rather than simplify the arc living in the torus complement.

used above. Figure 2.6(b) shows a Heegaard diagram for $K = K(3,4;2,2)$ that has been lifted to $\mathbb{C}$. Also, Figure 2.7 represents $CFK^{-}(K)$. An L-space knot K can be thought of as a knot with the simplest knot Floer invariants. To make sense of this fact, note that [OS04c]

$$\Delta_K(T) = \sum_{m,\mathfrak{s}} (-1)^m \mathrm{rk}\ \widehat{HFK}_m(K,\mathfrak{s}) T^{\mathfrak{s}}, \tag{2.1.2.1}$$

where $\Delta_K(T)$ is the symmetrized Alexander polynomial of K. We observe that the total rank of $\widehat{HFK}(K)$ is bounded below by the sum of the absolute values of the coefficients of the Alexander polynomial of K. A necessary condition for K to be an L-space knot is for this bound to be sharp. The following lemma turns out to be useful during the course of proving Part (c) of Theorem 2.2.1. See [OS05b, Theorem 1.2] for the complete statement.

Lemma 2.1.3. *Assume that $K \subset S^3$ is a knot for which there is an integer p such that $S^3_p(K)$ is an L-space. Then*

$$\mathrm{rk}\ \widehat{HFK}(K,\mathfrak{s}) \leq 1 \quad \forall \mathfrak{s} \in \mathbb{Z}.$$

In particular, all of the non-zero coefficients of $\Delta_K(T)$ are ± 1.

Therefore, if the absolute value of one of the coefficients of $\Delta_K(T)$ is greater than one, then K is not an L-space knot. We end this subsection by noting that a knot Floer complex with a staircase-shape (as in Figure 2.7) represents an L-space knot. Such a complex has a basis

$\{x_1, x_2, ..., x_m\}$ for $CFK^\infty(K)$ (defined in [OS04b]) such that

$$\begin{aligned} \partial x_i = &\quad x_{i-1} + x_{i+1} \quad \text{for } i \text{ even} \\ \partial x_i = &\quad 0 \quad \text{otherwise,} \end{aligned} \tag{2.1.2.2}$$

where the arrow from x_i to x_{i-1} is horizontal and the arrow from x_i to x_{i+1} is vertical. (We refer the reader to [Hom11b, Section 6] for the concept of a knot Floer complex with a staircase-shape.) The following corollary is a consequence of [Hom11b, Remark 6.6].

Corollary 2.1.4. *For a knot $K \subset S^3$, if $CFK^-(K)$ has a staircase-shape, then K is an L-space knot.*

2.2 Proof of the main theorem

This section is devoted to the proof of the main result of this chapter. For the sake of the proof, it will convenient to restate Theorem 1.1.1 in the following equivalent form:

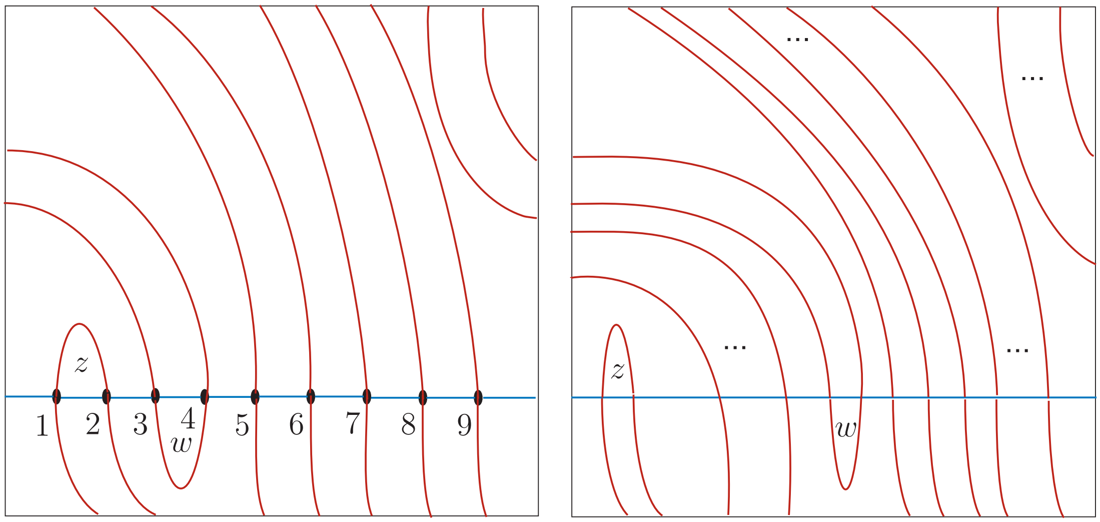
Theorem 2.2.1. *For $p \geq 2$, $k \geq 1$, $r > 0$ and $0 < s < p$, we have that $K(p, kp \pm 1; s, r)$:*

(a) is an L-space knot if $s = p - 1$,

(b) is an L-space knot if $r = 1$ and $s \in \{2, p-2\}$, and

(c) does not admit any L-space surgeries otherwise.

We prove part (a) by explicitly computing the knot Floer complex of $K(p, kp \pm 1; p - 1, r)$. Parts (b) and (c) are proved by focusing on the similarities and differences of the corresponding complexes to those of $K(p, kp \pm 1; p-1, r)$. The key to the proof is in identifying whether or not the knot Floer complex associated to $K(p, kp \pm 1; s, r)$ has a staircase-shape (Corollary 2.1.4).



(a) A Heegaard diagram for the $(3, 4)$ torus knot with two positive full twists on two adjacent strands

(b) The general form of a Heegaard diagram for $K(p, kp \pm 1; p-1, r)$, where r is an arbitrary integer

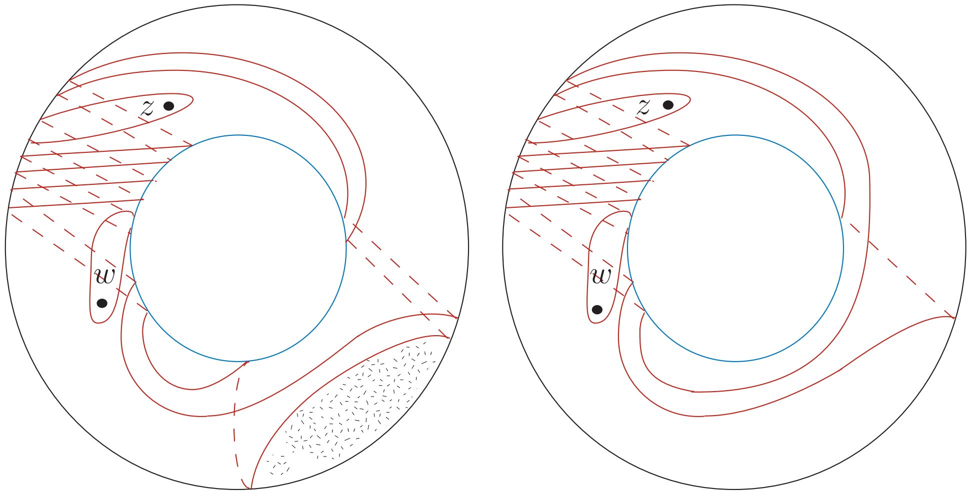
Figure 2.4 Heegaard diagrams on the torus, represented by a rectangle with opposite sides identified

Proof of Theorem 2.2.1(a). It will help to break the proof into two steps:

Proof Step 1: We show that $K(p, kp \pm 1; p-1, r)$ can be presented by a genus one Heegaard diagram with the general form given in Figure 2.4(b).

Case 1: We first consider the case $K(p, kp+1; p-1, r)$. The case $p = 2$ is trivial. The construction of a Heegaard diagram in the case when $p = 3$ was given in Section 2.1. Also Figure 2.3 shows the process for $K = K(4, 5; 3, 2)$.

To obtain a Heegaard diagram when $p \geq 5$ we can follow a similar procedure. Note that the w base point winds around the longitude of the torus once in the case $p = 3$, twice in the case $p = 4$, and $p - 2$ times in general. Moreover, in each longitudinal winding, the w base point traverses the torus $k + r$ times meridionally, except for the last longitudinal winding where α traverses the torus only k times meridionally. The latter fact holds since we are twisting $p - 1$ strands of the $(p, kp + 1)$ torus knot (set $s = p - 1$ in Step 3 of the algorithm given in Section 2.1). Note that as a result of $s = p - 1$, we always drag only one



(a) (b)

Figure 2.5 By an isotopy, the shaded region disappears and the Heegaard diagram will have two less intersection points.

sub-arc of α around the torus (Remark 2.1.2). Translating the resulting Heegaard diagram obtained this way into the rectangular representation of the torus, we get the general form of Figure 2.4(b).

Case 2: For the case $q = kp - 1$ we will have a similar setup, though the base points have to pass to the right of each other, not to the left. In this case, there will always be two intersection points of α and β that can be removed by an isotopy (see Figure 2.5(a)). To indicate the general case, we consider $K = K(3, 5; 2, 1)$. The resulting Heegaard diagram is isotopic to a Heegaard diagram for $K(3, 4; 2, 2)$ shown in Figure 2.5(b). As in Case 1, the Heegaard diagram will have the general form of Figure 2.4(b).

Proof Step 2: In this step, the goal is to calculate the filtered chain complex $CFK^-(K)$ for $K = K(p, kp \pm 1; p - 1, r)$. Figure 2.7 shows $CFK^-(K(3, 4, 2, 2))$. We claim that, in general, $CFK^-(K)$ has the same staircase-shape.

As in Section 2.1.2 we lift the diagrams, obtained in Step 1, to $\mathbb{C}$. Fix a connected component $\tilde{\alpha}$ of $\pi^{-1}(\alpha)$. We claim that such a component is a union of "N"-shapes (Figure 2.6(a)). To see this fact, we notice that the lift of a genus one Heegaard diagram can be obtained by gluing together infinitely many copies of the rectangular form of the Heegaard diagram in the plane (gluing from the sides of the rectangles). Figure 2.6(b) represents a portion of such a lift for a specific example. Pick an intersection point and start moving it along the $\tilde{\alpha}$ curve. (For example, pick the intersection point 9 on $\tilde{\alpha}$ in Figure 2.6(b) and start moving it upward.) The direction of the motion will reverse by turning around either of the z or w base points. (In Figure 2.6(b), the direction of the motion will change from upward to downward, and also from downward to upward, by going from 1 to 2, and from 3 to 4, respectively.) Note that the rectangular form of the genus one Heegaard diagram of K, as depicted in Figure 2.4(b), consists of a single β arc, together with α arcs having endpoints on the edge(s) of the rectangle. Note also that there are only two α arcs with both of their endpoints lying on one edge of the rectangle (namely the arcs that turn around the base points). Therefore, by thinking of the lift of the diagram in $\mathbb{C}$ as coming from infinitely many rectangles glued together along the sides and fixing a connected component of $\pi^{-1}(\alpha)$, the change in the direction of the motion (equivalently, turning around either the z or w base point) never happens twice in a single rectangle.[1] Moreover, to recover all the intersection points in the lift, only two changes of direction are needed. As a result, we get the shape of the lifted digram as claimed.

Let us first consider the example, $CFK^{-}(K(3,4;2,2))$ whose Heegaard diagram is given in Figure 2.6(b). Given a pair of intersection points x and y, the moduli space of holomorphic representatives of Whitney disks $\phi \in \pi_2(x,y)$ with Maslov index one, modulo reparametrization, is either empty or consists of one map. In what follows, we write $x \to y$ if the moduli space consists of one such map, and if so, we record how many times it passes over the z

[1] Note that we do not distinguish between the z and w base points downstairs, and their lifts in $\mathbb{C}$.

and w base points:

- $2 \to 1$, $6 \to 5$, $8 \to 7$ using one z base point,
- $3 \to 9$ using two z base points,
- $6 \to 7$, $8 \to 9$, $3 \to 4$ using one w base point, and
- $2 \to 5$ using two w base points.

From Figure 2.6(b), it is easy to see that we need four $\tilde{\beta}$ lines to generate the whole nine intersection points in the lifted Heegaard diagram, i.e. fixing $\tilde{\alpha}$, by using only four connected components of the lift of β we can obtain a lift of all the intersection points between α and β. Starting from $\tilde{\beta}_4$, See [OS04b] for the notation. By a similar method, we can find the remaining Whitney disks in the list above and use them to complete the ordering of the Alexander gradings. At this point, we can obtain the staircase-shape of Figure 2.7.

For the general case of Figure 2.6(a), it is straightforward to observe that our strategy can be extended. Assume that $\{x_1, x_2, ..., x_m\}$ is the set of intersection points between α and β curves coming from the genus one Heegaard diagram of $K = K(p, kp+1; p-1, r)$ (see Figure 2.4(b) and Figure 2.6(a)). Assume also that, fixing $\tilde{\alpha}$ a connected component of $\pi^{-1}(\alpha)$, we need n connected components of $\pi^{-1}(\beta)$ to recover all the m intersection points downstairs between α and β (Figure 2.6(a)). Our strategy is first ordering the generators based on their Alexander gradings and, second, finding all the differentials. Using the "N"-shape of Figure 2.6(a) and starting from $\tilde{\beta}_n$, there are three intersection points (x_m, x_{m-1} and x_{m-2}) with one disk $x_{m-1} \to x_m$ using one w base point and one other disk $x_{m-1} \to x_{m-2}$ using the z base point(s). Note that there exists no other non-trivial Whitney disk with Maslov index one connecting x_{m-1} to another intersection point of Figure 2.6(a). Also on $\tilde{\beta}_{n-1}$, there is one disk $x_{m-3} \to x_{m-2}$ using the w base point(s). Continuing this process,

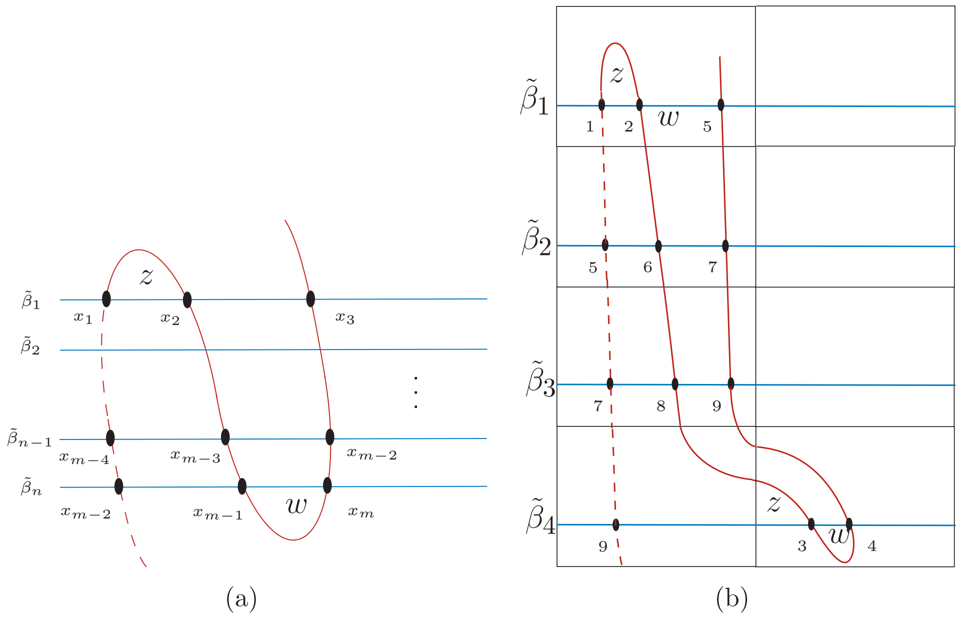


Figure 2.6 (a) A portion of the Heegaard diagram for $K = K(p, kp \pm 1; p-1, r)$ lifted to $\mathbb{C}$, where r is an arbitrary integer. Note that m is the number of intersection points in the genus one Heegaard diagram of K. It is assumed, fixing $\tilde{\alpha}$ a connected component of $\pi^{-1}(\alpha)$, that we need n connected components of $\pi^{-1}(\beta)$ to obtain a complete list of all the m intersection points between α and β downstairs. (b) A portion of the Heegaard diagram for the $(3, 4)$ torus knot with two positive full twists on two adjacent strands, lifted to $\mathbb{C}$. Note that the base points specified in the picture depicted above are the only relevant base points needed to compute CFK^-.

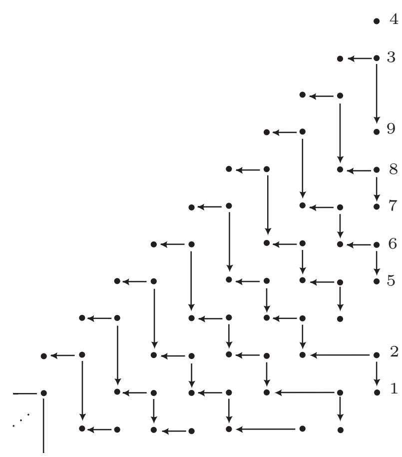


Figure 2.7 $CFK^-(K(3, 4; 2, 2))$

we deduce that

$$A(x_m) > A(x_{m-1}) > A(x_{m-2}) > A(x_{m-3}) > ... > A(x_1).$$

By noting that there is no other non-trivial Whitney disk with Maslov index one, we see that the set $\{x_1, x_2, ..., x_m\}$ forms a basis for $CFK^-(K)$ such that there are three intersection points (3, 4 and 9) with one disk $4 \to 3$ using one w base point and one other disk $9 \to 3$ using two z base points. Thus, in terms of the Alexander gradings $A(i)$ of the intersection points, $i \in \{1, 2, ..., 9\}$, we have that:

- $A(3) - A(4) = n_z(\tilde{\phi}) - n_w(\tilde{\phi}) = -1$, and
- $A(3) - A(9) = n_z(\tilde{\phi}) - n_w(\tilde{\phi}) = 2$.

$$\begin{aligned} \partial x_i = & \quad x_{i-1} + x_{i+1} && \text{for } i \text{ even} \\ \partial x_i = & \quad 0 && \text{otherwise.} \end{aligned}$$

This formula for the differentials (which is the same as (2.1.2.2)), together with the existence of three intersection points on each $\tilde{\beta}_j$ line of Figure 2.6(a) with exactly two disks using different base point types (i.e. z and w), gives the staircase-shape of $CFK^-(K)$ (see the discussion about a knot Floer complex with a staircase-shape in Section 2.1.2). Now, Corollary 2.1.4 completes the proof.

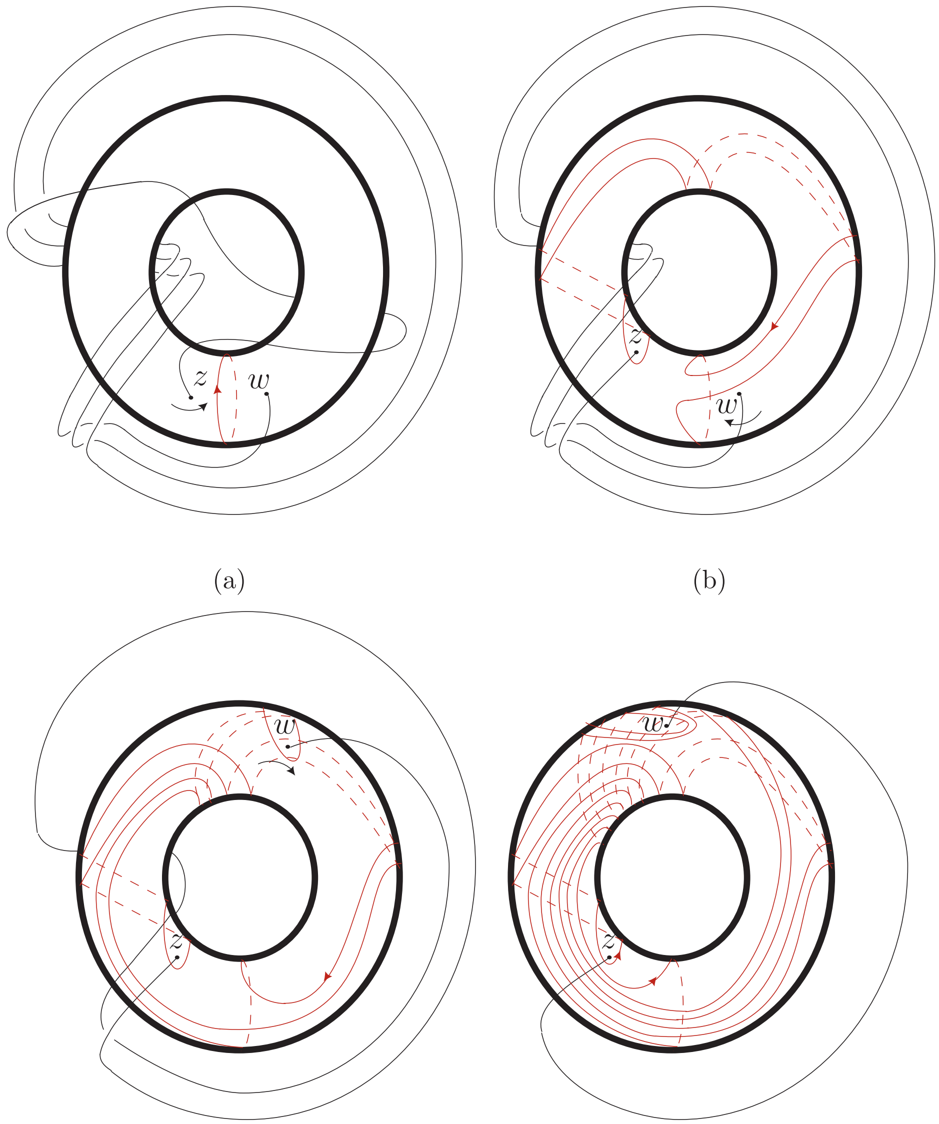
□

Proof of (b) and (c). Let $K(p, q; s, r)$ be a twisted torus knot where $2 \leq s \leq p - 2$. We discuss the case when $q = kp + 1$ and leave the case $q = kp - 1$ to the reader. Since we apply the same algorithm, as used in Part (a), to obtain a Heegaard diagram, we will only highlight the differences in this case. Recalling the algorithm explained in Section 2.1.1, we first wind z once in the counterclockwise direction (Step 1). Then we wind the w base point $(s-2)$ times in the clockwise direction, traversing the torus $(k+r)$ times meridionally in each winding (Step 2). Finally, we wind the w base point $(p-s)$ more times around the torus longitudinally (Step 3). Note that in the latter step, w goes through only k meridional moves in each winding.

It will be convenient to pick an arbitrary orientation for the α curve. Note that, unlike Part (a), more than one sub-arc will be dragged since $2 \leq s \leq p-2$ (Remark 2.1.2). With the α curve oriented, either these sub-arcs will have all the same orientation or there will be at least one pair of sub-arcs with opposite orientations. The case for only two sub-arcs can be seen in Figure 2.9. Figure 2.8 shows the process of constructing a Heegaard diagram for $K(4,5,2,1)$, which indicates the pattern, particularly in the case when $s \in \{2, p-2\}$.

Claim. Unless $s \in \{2, p-2\}$ and $r = 1$, the trivializing process will drag oppositely oriented sub-arcs.

Proof. Suppose $r = 1$. The first longitudinal traversal of Step 3 drags no additional sub-arcs. The second traversal of Step 3, however, drags $(s-1)$ sub-arcs, all oriented in the same direction. The next winding drags $(s-2)$ additional sub-arcs, all oriented in the same direction but opposite to those of the first $(s-1)$ sub-arcs. This opposite orientation will clearly not occur if $s = 2$. Suppose $s = p-2$. Then in Step 3 the w base point is wound longitudinally around the torus $p-(p-2) = 2$ times (twice). Hence, only sub-arcs with the same direction will be dragged. If $r \geq 2$ the full twists of Step 1 create future oppositely oriented sub-arcs in Step 3, i.e. the w base point will be dragging sub-arcs with opposite orientations, starting the second longitudinal traverse of Step 3. More specifically, if the



(a) (b) (c) (d)

Figure 2.8 The process of drawing a genus one Heegaard diagram for $K(4,5;2,1)$. The α curve in each step is oriented. This example indicates the pattern when $s \in \{2, p-2\}$ and $r = 1$. In general when $r = 1$, to go from (c) to (d), w first drags $(s-1)$ sub-arcs, all oriented in the same direction. In the next winding it drags $(s-2)$ additional sub-arcs, all oriented in the same direction but opposite to those of the first $(s-1)$ sub-arcs. Dragging oppositely oriented sub-arcs does not occur in this example since $s = 2$. Note that the orientation is irrelevant once the Heegaard diagram is completed.

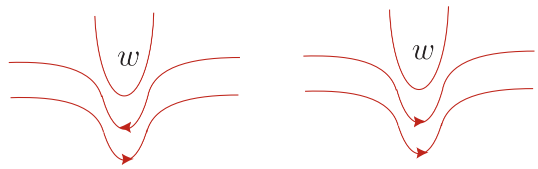


Figure 2.9 The base point w drags more than one sub-arc of α. The picture depicted above is schematic.

number of full twists is greater than one, each additional twist will create two oppositely oriented sub-arcs and the w base point will drag both of these sub-arcs after the first $(s-1)$ longitudinal windings. $\square$

Since the hypotheses of Part (b) imply that the sub-arcs have the same orientation, a similar argument to Part (a), once we lift the diagram to $\mathbb{C}$, shows that the ordering of the Alexander gradings of the intersection points will follow the same manner as in the case $s = p-1$. More precisely, if we think of the lift of the Heegaard diagram as coming from infinitely many rectangles glued together, by picking an intersection point and moving it along a fixed connected component $\tilde{\alpha}$ of $\pi^{-1}(\alpha)$, we see that the picked point, during its motion, never turns around the z (or w) base point twice in a single rectangle. Therefore, although the lifted diagrams are not looking the same as Part (a), we claim that the corresponding complexes have the staircase-shape. In particular, for the case $s = 2$ (respectively $s = p-2$), we need four (respectively $2p-4$) changes of direction[2] to recover all the intersection points of downstairs. For the specific example of $K(4,5;2,1)$ depicted in Figure 2.13(b):

$$A(6) > A(5) > A(9) > A(8) > A(7) > A(4) > A(1) > A(11) > A(10) > A(3) > A(2).$$

Exploring the Whitney disks in the lifted diagram will give a staircase-shape for the asso-

[2]We remind the reader that by changing direction we mean turning around one of the base points (z or w).

ciated complex. To see this in the general case, note that the set of all intersection points between α and β curves forms a basis for CFK^-. Moreover, for every intersection point x_i, either the differential vanishes, or there exist two Whitney disks with Maslov index one connecting x_i to another two distinct intersection points, using z and w base points alternatively. (This shows that the differentials are of the form of (2.1.2.2).) That is, for each intersection point x_i, either there is no arrow joining it to another intersection point, or there are two arrows joining x_i to two distinct intersection points such that one arrow is horizontal and the other is vertical. This gives us the staircase-shape of the knot Floer complex. Finally, Corollary 2.1.4 completes the proof of Part (b).

To prove Part (c), note that if the arcs dragged by w have different orientations, then, after lifting the diagram to $\mathbb{C}$, the following phenomenon occurs:

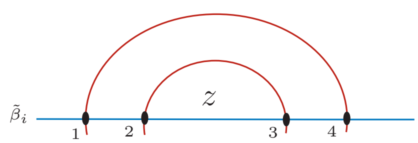


Figure 2.10 A portion of the knot complex lifted to $\mathbb{C}$, when the sub-arcs with different orientations have been dragged.

Claim: The associated complex does not represent an L-space knot.

Proof. As in the proof of Part (a), we can order the Alexander gradings of the intersection points from the Whitney disks in the lifted Heegaard diagram. Let $\tilde{\beta}_1$, ..., $\tilde{\beta}_k$ denote the lifts of β needed to find all of the Whitney disks. Work from $\tilde{\beta}_k$ to $\tilde{\beta}_1$ and stop at the first $\tilde{\beta}_i$ that exhibits the phenomenon in Figure above. Then part of the diagram is as Figure 2.11.

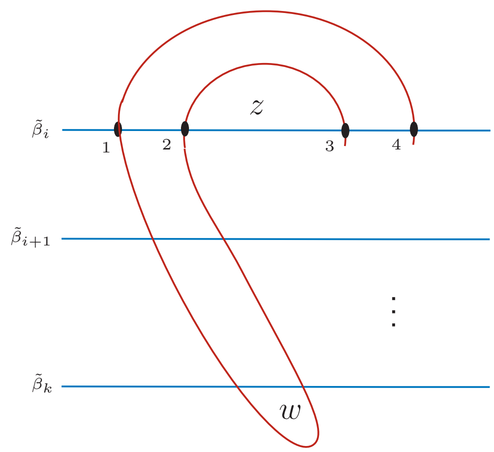


Figure 2.11 A sub-diagram of a lifted Heegaard diagram, fixing one connected component of $\tilde{\alpha}$

We analyze this by looking at the Whitney disks:

- $4 \to 1$, $3 \to 2$ using one z base point, and
- $1 \to 2$, $4 \to 3$ using one w base point.

As a result, the part of CFK^- involving the intersection points, $\{1, 2, 3, 4\}$, on $\tilde{\beta}_i$ will look like

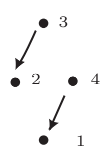
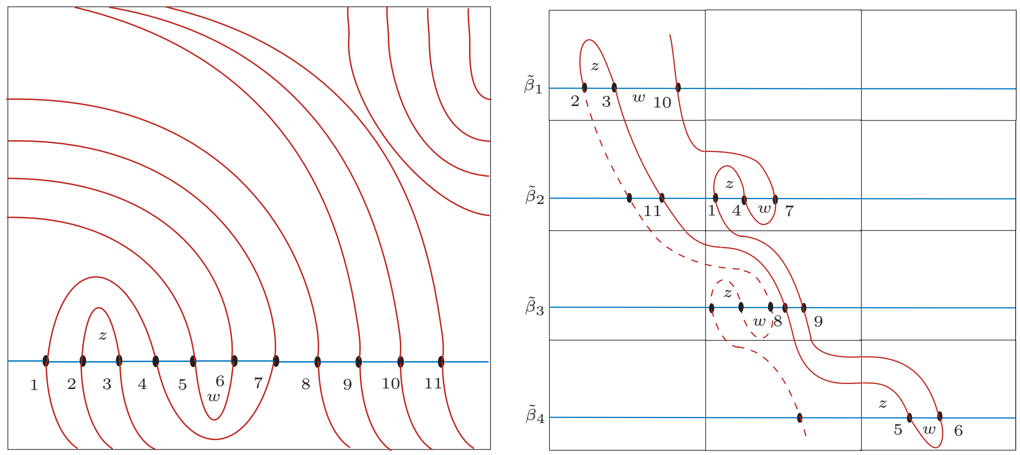


Figure 2.12 Part of a knot complex, representing the phenomenon of having two generators in one Alexander grading.

Note that the boundary map decreases the Maslov grading by one, and the U-action decreases the grading by two. Combining these facts with the existence of the disks $1 \to 2$ and $4 \to 3$, we find that the intersection points 2 and 4 both have the same Maslov gradings as well as the same Alexander gradings. (We are assuming that there are no trivial Whitney disks connecting two intersection points; if there is a bigon that does not pass over any of the base



(a) (b)

Figure 2.13 A genus one Heegaard diagram for $K(4,5;2,1)$, as well as its lift to $\mathbb{C}$

points, we can isotop it away.) Thus,

$$\text{rk } \widehat{HFK}(K,\mathfrak{s}) \geq \text{rk } \widehat{HFK}_m(K,\mathfrak{s}) = \text{rk } \widehat{CFK}_m(K,\mathfrak{s}) \geq 2,$$

where $\mathfrak{s}$ is the Alexander grading of the intersection points 2 and 4. Now, Lemma 2.1.3 completes the proof of the claim and Part (c). $\square$

The Heegaard diagrammatic observation in Figure 2.5 can be generalized. The author suspects that the following corollary could have been proved differently, using braid words for instance:

Corollary 2.2.2. *The twisted torus knot, $K(p,kp+1;p-1,r)$, is isotopic to $K(p,(k+1)p-1;p-1,r-1)$.*

Proof. We start from the genus one Heegaard diagram of $K_1 = K(p,(k+1)p+1;p-1,r-1)$, obtained from implementing the algorithm explained in Section 2.1.2. The proof is done by first doing an isotopy to get rid of the two extra generators in the genus one Heegaard diagram of K_1[3] and, second, tracking back the drag of the w and z base points in the torus. More precisely, after removing the extra generators, if we track back the w base point, we see that it passes, during its $p-2$ longitudinal windings, to the right of z. Now, by tracking back the z base point once around the torus, we see that it also passes to the right of w. These facts can be verified in the example depicted in Figure 2.5(b). (Thus, while implementing the algorithm to obtain the diagram in the first place, the base points must have passed by the left of each other). During this process, except for the first winding of w that goes through k meridional moves, the rest of windings traverse the torus $k+r$ times meridionally. Therefore, by noting that only one sub-arc of α has been dragged by the base points, we get that the diagram obtained after doing the isotopy is a genus one Heegaard diagram for $K_2 = K(p, kp+1; p-1, r)$. □

When $p = 3$ in Theorem 1.1.1, we obtain a generalization of [Wat09, Theorem 1.2]:

Corollary 2.2.3. *All twisted* $(3, q)$ *torus knots admit L-space surgeries.*

2.3 Directions for future research

Closely related to the main result of this chapter, one can ask the question of which operations on knots produce L-space knots. Satellite operations are the first in line. As pointed out in Chapter 1, the (p, q) cabling is an L-space satellite operation [Hom11a]. More generally, Hom, Lidman and the author introduced an L-space satellite operation, using *Berge-Gabai knots* as the pattern [HLV14]. By definition, a knot $P \subset S^1 \times D^2$ is called a Berge-Gabai knot if it admits a non-trivial solid torus surgery. We also suspect that one can obtain

[3] Note that the phenomenon (of having two removable intersection points) in Figure 2.5, once we implement the algorithm explained in Section 2.1.2, will always occur in the genus one Heegaard diagram of $K(p, kp-1; p-1, r)$.

more L-space satellite operations, choosing the patterns from the the list of L-space knots of Theorem 1.1.1. Although classifying such operations does not seem to be an easy task to do, there is an obstruction to obtaining L-space satellite knots (Lemma 2.1.3) which can be appealing. Let $P(K)$ be a satellite knot with pattern $P \subset V = S^1 \times D^2$ and companion K. We recall the behavior of the Alexander polynomial of a satellite knot:

$$\Delta_{P(K)}(T) = \Delta_P(T)\Delta_K(T^w)$$

where w is the geometric intersection number of the pattern P with a fixed meridional disk of V (see for instance [Lic97]). So one can attack the following question by first examining the obstruction of Lemma 2.1.3, using algebraic methods.

Question 1: Is there a classification of L-space satellite operations?

Another interesting direction one can pursue, encouraged by the computations done in this chapter, is to calculate the Alexander polynomials $\Delta_K(T)$ of twisted (p, q) torus knots with $q = kp \pm 1$ or more generally with q an arbitrary non-zero integer. In [Mor06], Morton gives a closed formula for $\Delta_K(T)$ where $K = K(p, q; 2, r)$ and $p > q > 0$.

Finally, notice that the $(2, 2n + 1)$ torus knots have the particular property that they admit lens space surgeries and also have branched double covers that are lens spaces. It seems reasonable to ask what class of knots have this property. The following question was first brought to the author's attention by Allison Moore:

Question 2: What class of knots have surgeries and branched double covers that are lens spaces (or L-spaces)?

Chapter 3

Berge-Gabai knots and L-space satellite operations

This chapter is mainly based on [HLV14]. We start by providing background on 1-bridge braids in solid tori and Dehn surgery on satellite knots. See [Ber91, Gab90, Gor83] for further details. We then present the rank formula, found in [OS11], for the Heegaard Floer homology of the surgered manifold obtained by doing Dehn surgery on a knot in S^3. Theorem 1.2.3 is proved in Section 3.4. We end this chapter by clarifying the discussions in Section 1.3 and proving Theorem 1.3.1 and Proposition 1.3.4. Throughout the rest of the chapter, we assume that P is a Berge-Gabai knot in $V = S^1 \times D^2$ (i.e., P admits a non-trivial solid torus surgery) unless otherwise stated. We also consider the standard embedding of $S^1 \times D^2$ into S^3 such that $S^1 \times \{*\}$ bounds an embedded disk in S^3. When it is clear from context, we will not distinguish between the Berge-Gabai knot $P \subset V$ and $P \subset S^3$.

3.1 Berge-Gabai knots

The primary goal of this subsection is to highlight the Dehn surgeries on $P \subset V$ that will return a solid torus. In what follows, we provide a setup similar to that of [Gab90].

An arbitrary knot P in V is called a *1-bridge braid* if P can be isotoped to be a braid in V that lies in $S^1 \times \partial D^2$ except for one arc that is properly embedded in V, and P is not a torus knot. Gabai [Gab89] showed that any knot in a solid torus with a non-trivial solid torus surgery must be either a torus knot or a 1-bridge braid in $S^1 \times D^2$, and Berge [Ber91] classified all 1-bridge braids in $S^1 \times D^2$ with non-trivial solid tori fillings. We denote the braid index of P by w.

We will consider $\widehat{V}$, the exterior of $P \subset V$. Let $T = \partial V$ and $J = \partial \mathrm{nb}(P)$. We equip T with the homological generators (m, ℓ) where ℓ is the longitude $S^1 \times \{*\}$ of T and m is $\{*\} \times \partial D^2$; therefore, ℓ becomes null-homologous after standardly embedding V in S^3 and removing $\mathrm{nb}(P)$. We equip J with homological generators (μ, Λ) as follows. The generator μ is the meridian of P. Note that m is homologous to $w\mu$ in $\widehat{V}$. To define Λ, consider the immersed annulus A connecting J to T with b arcs of self-intersection in Figure 1.2(b). By doing oriented cut and paste to the arcs of self-intersection we can arrange A to be an embedded surface in $\widehat{V}$ joining J to T. Define Λ to be $A \cap J$. Orient m, ℓ, μ, and Λ as in Figure 1.2(b). Note that $A \cap T = w\ell + tm$, and so $w\ell + tm$ is homologous to Λ in $\widehat{V}$.

Let λ be the simple closed curve on J that is homologous to $\Lambda - wt\mu \in H_1(J; \mathbb{Z})$. Thus, we have the following equalities in $H_1(\widehat{V}; \mathbb{Z})$:

$$\begin{aligned}
[\lambda] &= [\Lambda - wt\mu] \\
&= [w\ell + tm - wt\mu] \\
&= [w\ell],
\end{aligned}$$

where the last equality follows from the fact that m is homologous to $w\mu$. In particular, λ becomes null-homologous after standardly embedding V in S^3 and removing $\mathrm{nb}(P)$. Now the equation $[\lambda] = [\Lambda - wt\mu]$ can be used to switch from (μ, Λ)- to (μ, λ)-coordinates, where (μ, λ) are the usual meridian-longitude coordinates on P when V is standardly embedded in

S^3.

We recall that a 1-bridge braid in $S^1 \times D^2$ with winding number w, bridge width b, and twist number t can be represented via the braid word $\sigma = (\sigma_b \sigma_{b-1} ... \sigma_1)(\sigma_{w-1}\sigma_{w-2}...\sigma_1)^t$ where $|t| \geq 1$, and $1 \leq b \leq w-2$. The following lemma is a consequence of [Gab90, Lemma 3.2]:

Lemma 3.1.1. *Let P be a 1-bridge braid in V and s a positive integer. If filling $\widehat{V}$ along a curve $\alpha = d\mu + s\Lambda$ in J yields $S^1 \times D^2$, then $s = 1, d \in \{b, b+1\}$, and* $\gcd(w, d) = 1$.

In (μ, λ)-coordinates these possible exceptional surgeries are $\alpha = (tw + d)\mu + \lambda$ where $d \in \{b, b+1\}$.[1]

Note that when P is an (m, n)-torus knot in V, there are infinitely many surgeries on P that will return a solid torus, including $mn + 1 = tw + b + 1$; this follows, for instance, from the proof of [Mos71, Proposition 3.2].

Let $(P; n_1/n_2)$ denote the result of filling $\widehat{V}$ along the curve $n_1\mu + n_2\lambda$. Lemma 3.1.1 shows that if P is a Berge-Gabai knot, then $(P; p_d)$ will be homeomorphic to $S^1 \times D^2$ for at least one of the coefficients $p_d = tw + d, d \in \{b, b+1\}$.

Note that adding a positive full-twist to all of the w strands of P results in a new knot P' where t changes into $t + w$. Correspondingly, there exists a homeomorphism of the solid torus (doing a positive meridional twist), which takes P to P'. Iterating this process q times, we get the following:

Proposition 3.1.2. *Let P be a Berge-Gabai knot in $S^1 \times D^2$, standardly embedded in S^3, so that $(P; p)$ is homeomorphic to a solid torus. Let P' be the knot obtained from P by adding*

[1] We have stated Lemma 3.1.1 so that the orientation of (μ, λ) agrees with the standard convention that $\mu \cdot \lambda = 1$. In Gabai's paper [Gab90], μ is oriented opposite to that of Figure 1.2(b).

q positive Dehn twists. Then

$$(P'; p + qw^2) \cong S^1 \times D^2.$$

Hence if we have a Berge-Gabai knot P with twist number t, adding q full twists to all w strands of P will produce a Berge-Gabai knot with twist number $t + qw$.

3.2 Surgery on $P(K)$

Let $P(K)$ be a satellite knot with pattern $P \subset V$ and companion K. Let $f : V \to \mathrm{nb}(K)$ be a homeomorphism that determines the zero framing of K, i.e., $[f(S^1 \times \{*\})] = 0 \in H_1(X; \mathbb{Z})$ where $X = S^3 - \mathrm{nb}(K)$. Thus $P(K) = f(P)$.

Recall that $m, \ell \in H_1(T; \mathbb{Z})$ are the natural meridian and longitude coordinates of $T = \partial V$, oriented such that $m \cdot \ell = 1$. Recall also that $\widehat{V} = V - \mathrm{nb}(P)$. Note that $H_1(\widehat{V}) = \mathbb{Z}\langle \ell \rangle \oplus \mathbb{Z}\langle \mu \rangle$ where μ is the class of the meridian of $\mathrm{nb}(P)$. When P is viewed as a knot in S^3, let $\lambda \subset \partial\mathrm{nb}(P)$ be the unique curve on $\partial\mathrm{nb}(P)$ which is null-homologous in $S^3 - \mathrm{nb}(P)$ (i.e., the zero framing of P). That is, if f is as above, then $f(\lambda)$ is the zero framing of $P(K)$. Thus, $S^3_{p_1/p_2}(P(K)) \cong X \cup_f (P; p_1/p_2)$, where the notation means ∂X and $\partial(P; p_1/p_2)$ are identified via the restriction of f to $\partial(P; p_1/p_2) = \partial V$. With the above notation:

Lemma 3.2.1 ([Gor83, Lemma 3.3]). *For relatively prime integers p_1, p_2, and $P \subset V$ with winding number w:*

(a) $H_1((P; p_1/p_2); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_{\gcd(w,p_1)}$.

(b) If $w \neq 0$, the kernel of $H_1(\partial(P; p_1/p_2); \mathbb{Z}) \to H_1((P; p_1/p_2); \mathbb{Z})$ is the cyclic group generated by

$$\frac{p_1}{\gcd(w, p_1)} m + \frac{p_2 w^2}{\gcd(w, p_1)} \ell.$$

Note that Lemma 3.2.1 is valid regardless of whether or not P is a Berge-Gabai knot. However, when P is a Berge-Gabai knot, we can use Lemma 3.2.1 to relate surgeries on K and $P(K)$ in the following sense.

Corollary 3.2.2. *Let P be a Berge-Gabai knot in V with winding number w so that $(P; p) \cong S^1 \times D^2$. Then*

$$S^3_p(P(K)) \cong S^3_{p/w^2}(K).$$

Proof. The result essentially follows from the fact that

$$S^3_p(P(K)) \cong X \cup_f (P; p).$$

By assumption, $(P; p)$ is homeomorphic to a solid torus. Therefore, in order to find the corresponding surgery coefficient on K, one needs to determine the slope of the meridian of $\partial(P; p)$ under the canonical identification with ∂V, and where it is sent under f.

Note that the slope of the meridian of $(P; p)$ is precisely the generator of

$$\ker\Big(H_1(\partial(P; p); \mathbb{Z}) \to H_1((P; p); \mathbb{Z})\Big).$$

Using the identification of ∂V and $\partial(P; p)$, we have that the slope of the meridian, in (m, ℓ)-coordinates, is given by (p, w^2) by Lemma 3.2.1. Since f sends m (respectively ℓ) to the meridian (respectively longitude) of K, the result follows. $\square$

Combining Lemma 3.1.1 with Corollary 3.2.2, we deduce the following:

Proposition 3.2.3. *Let P be a Berge-Gabai knot with bridge width $b \neq 0$, winding number w, and twist number t, and let K be an arbitrary knot in S^3. Then for at least one $d \in \{b, b+1\}$,*

$$S^3_{d+tw}(P(K)) \cong S^3_{\frac{d+tw}{w^2}}(K).$$

Note that $\gcd(d+tw, w^2) = 1$ (see Lemma 3.1.1). We end this subsection by stating the following lemma, which will be useful in the proof of Theorem 1.2.3. Let $\Delta_K(T)$ denote the symmetrized Alexander polynomial of K. Recall the behavior of the Alexander polynomial for satellites (see for instance [Lic97]):

$$\Delta_{P(K)}(T) = \Delta_P(T)\Delta_K(T^w). \tag{3.2.0.1}$$

Lemma 3.2.4. *Let $P(K)$ be a fibered satellite knot where P has winding number w. Then*

$$g(P(K)) = g(P) + wg(K).$$

Furthermore, if P is a Berge-Gabai knot as above with $t > 0$, then

$$g(P) = \frac{(t-1)(w-1)+b}{2}. \tag{3.2.0.2}$$

Proof. Since $P(K)$ is a fibered knot, we deduce that $\deg \Delta_{P(K)}(T) = g(P(K))$. It also follows that K and P are both fibered [HMS08]. Combining these two facts with (3.2.0.1), we see that $g(P(K)) = g(P) + wg(K)$.

In order to calculate $g(P)$, notice that P is a positive braid if $t > 0$. Hence, the Seifert surface R obtained from Seifert's algorithm is a minimal genus Seifert surface for P [Sta78]. Then

$$\chi(R) = 1 - 2g(P) \Rightarrow w - b - t(w-1) = 1 - 2g(P). \qquad \square$$

3.3 Input from Heegaard Floer theory

In this subsection we mainly use the notation of [Hom11a]. Recall that an *L-space* Y is a rational homology sphere with the simplest possible Heegaard Floer homology, i.e., $\mathrm{rk}\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. We say that a knot K in S^3 is an *L-space knot* if it admits a positive L-space surgery.

We let $\tau(K)$ denote the integer-valued concordance invariant from [OS03]. Let $\mathcal{P}$ denote the set of all knots K for which $g(K) = \tau(K)$. (Recall from [Hed10] that for fibered knots, $g(K) = \tau(K)$ is equivalent to being strongly quasipositive.) If K is an L-space knot, then $K \in \mathcal{P}$. This follows from [OS05b, Corollary 1.6] and the fact that L-space knots are fibered [Ni07, Corollary 1.3].

Let

$$s_K = \sum_{i \in \mathbb{Z}} \left(\mathrm{rk} H_*(\widehat{A}_i^K) - 1 \right),$$

where $\widehat{A}_i^K$ is the subquotient complex of $CFK^\infty(K)$ defined in [OS08]. It is proved in [Hom11a] that $\mathrm{rk} H_*(\widehat{A}_i^K)$ is always odd, and so s_K is always a non-negative even integer.

For a pair of relatively prime non-zero integers m and n, $n > 0$, let

$$t_K^{m/n} = 2\max(0, n(2\nu(K) - 1) - m). \tag{3.3.0.3}$$

Observe that

$$t_K^{m/n} = 0 \quad \text{if and only if} \quad m/n \geq 2\nu(K) - 1. \tag{3.3.0.4}$$

The term $\nu(K)$ is another integer-valued invariant of K, defined in [OS11, Definition 9.1], which is bounded below by $\tau(K)$ and above by $g(K)$. In particular, if $K \in \mathcal{P}$, then $\nu(K) = g(K)$.

Let m and n be as above, and suppose that $\nu(K) \geq \nu(\overline{K})$ where $\overline{K}$ denotes the mirror of K. (This condition is automatically satisfied for $K \in \mathcal{P}$.) If $\nu(K) > 0$ or $m > 0$, then

$$\mathrm{rk}\widehat{HF}(S^3_{m/n}(K)) = m + ns_K + t_K^{m/n} \tag{3.3.0.5}$$

by [OS11, Proposition 9.6].

By (3.3.0.5), when $m > 0$ we have that

$$S^3_{m/n}(K) \text{ is an L-space if and only if } t_K^{m/n} = 0 \text{ and } s_K = 0. \tag{3.3.0.6}$$

By [OS04b, Theorem 4.4], the group $H_*(\widehat{A}_i^K)$ is isomorphic to $\widehat{HF}(S^3_N(K), [i])$ for $N \gg 0$ and $|i| \leq N/2$. Thus, we have that

$$K \text{ is an L-space knot if and only if } s_K = 0. \tag{3.3.0.7}$$

In fact, if K is a non-trivial L-space knot, $S^3_{m/n}(K)$ is an L-space if and only if $m/n \geq$

$2g(K)-1$. This follows from (3.3.0.4), (3.3.0.5), and the fact that for K a non-trivial L-space knot, $\nu(K)=g(K)>0$. (The original argument for the forward direction is given in [KMOS07].)

3.4 Proof of Theorem 1.2.3

This subsection is devoted to the proof of Theorem 1.2.3. We begin with the proof of Lemma 1.2.4. We do not review the concept of a quasipositive Seifert surface but instead refer the reader to [Hed10, Rud98].

Proof of Lemma 1.2.4. Suppose for contradiction that $P(K)$ is an L-space knot. Recall that L-space knots are fibered [Ni07, OS05b]. It is also a well-known fact that a minimal genus Seifert surface for a negative braid can be expressed as a plumbing of negative Hopf bands [Sta78, Theorem 2]. (See also [AO01, Theorem 1] for an explicit construction in the case of torus knots.) Since $P(K)$ is fibered, this implies that K is fibered and P is fibered in the solid torus [HMS08], so the fiber for $P(K)$ is constructed by patching the fiber for P in the solid torus to w copies of the fiber for K. As a result, when P is a negative braid, the fiber surface for $P(K)$ contains (at least) as many negative Hopf bands as the one for P.

By the above description of the fiber surface, we can deplumb a negative Hopf band. This means we can decompose the fiber surface for $P(K)$ as a Murasugi sum, where one of the summands is not a quasipositive surface. By [Rud98], if a Seifert surface is a Murasugi sum, it is quasipositive if and only if all of the summands are quasipositive. Thus, the fiber surface for $P(K)$ is not a quasipositive surface. However, since $P(K)$ is an L-space knot, it is strongly quasipositive [Hed10], which gives a contradiction. $\square$

We prove Theorem 1.2.3 only for the cases where $b \neq 0$ (consequently $1 \leq t_0 \leq w-2$)

and refer the reader to [Hed09, Hom11a] for the case $b = 0$.

Proof of Theorem 1.2.3. ($\Leftarrow$) The proof of this direction follows from Proposition 3.2.3, which tells us that

$$S^3_{d+tw}(P(K)) \cong S^3_{\frac{d+tw}{w^2}}(K).$$

Since K is a non-trivial L-space knot and $\frac{b+tw}{w^2} \geq 2g(K) - 1 > 0$, it follows that $S^3_{\frac{d+tw}{w^2}}(K)$ is an L-space. Here we are using that $d \geq b$. Therefore, $P(K)$ is an L-space knot.

($\Rightarrow$) For the case that $t < 0$ (see Remark 2.1.2), we apply Lemma 1.2.4 to see that $P(K)$ cannot be an L-space knot. Therefore, we can assume that $t > 0$ and $P(K)$ is an L-space knot. For simplicity of notation, we set $m = d + t_0 w + qw^2$ where $d \in \{b, b+1\}$ is such that $(P; m) \cong S^1 \times D^2$. Again from Proposition 3.2.3 we have

$$\mathrm{rk}\widehat{HF}(S^3_m(P(K))) = \mathrm{rk}\widehat{HF}(S^3_{m/w^2}(K)). \tag{3.4.0.8}$$

Since $P(K)$ is an L-space knot, it follows that $g(P(K)) = \tau(P(K))$, and we see that

$$t^m_{P(K)} = 2\max(0, 2g(P(K)) - 1 - m). \tag{3.4.0.9}$$

We first suppose that $\nu(K) \geq \nu(\overline{K})$. Since $m > 0$, we may combine (3.3.0.5), (3.3.0.7), and (3.4.0.8) to obtain

$$m + t^m_{P(K)} = m + w^2 s_K + t^{m/w^2}_K,$$

or equivalently

$$t^m_{P(K)} = w^2 s_K + t^{m/w^2}_K. \tag{3.4.0.10}$$

Note that by Lemma 3.2.4, (3.2.0.2), and (3.4.0.9), we have that

$$t^m_{P(K)} = \max(0, 4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d). \tag{3.4.0.11}$$

Claim. The equality in (3.4.0.10) does not hold unless both sides are identically zero.

Proof of the Claim. If $t^m_{P(K)} \neq 0$ then we have two cases:

Case 1. Suppose $t_K^{m/w^2} = 0$. Using (3.4.0.11), we see (3.4.0.10) is equivalent to

$$4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d = w^2 s_K.$$

It follows that w divides $2t_0 + 2d - 2b$. Since $d - b \in \{0, 1\}$ and $1 \leq t_0 \leq w - 2$, we conclude that $w = 2t_0 + 2d - 2b$. Since

$$4wg(K) - 2w - w - 2qw = w^2 s_K,$$

then

$$4g(K) - 3 - 2q = ws_K.$$

The right side is an even number and the left side is odd which is a contradiction.

Case 2. Suppose $t_K^{m/w^2} \neq 0$. By expanding both sides of (3.4.0.10) and again using (3.4.0.11), we see that

$$4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d = w^2 s_K + 4w^2\nu(K) - 2w^2 - 2d - 2t_0 w - 2qw^2.$$

By rearranging terms, we get

$$4wg(K) - 2w + 2(b - t_0) - 2qw + 2t_0w = w^2(4\nu(K) - 2 - 2q + s_K).$$

Therefore w divides $2(b - t_0)$. Since b and t_0 are both bounded above by $w - 2$, we have either $2(b - t_0) = \pm w$ or $b = t_0$.

Recall that we described P as a braid closure in Chapter 1. Viewing this braid as a mapping class of the disk with w punctures, it is straightforward to verify that if $b = t_0$, the $(t_0 + 1)^{\text{th}}$ puncture is fixed. Therefore, in this case P has at least two components, which contradicts P being a knot. Thus, we must have $2(b - t_0) = \pm w$.

Substituting and dividing by w gives:

$$4g(K) - 2 \pm 1 - 2q + 2t_0 = w(4\nu(K) - 2 - 2q + s_K).$$

As in Case 1, comparing the parities of each side gives a contradiction. ☐

Having proved the claim, all the terms in (3.4.0.10) are identically zero. Since $s_K = 0$, (3.3.0.7) gives that K is an L-space knot. Also, $t^m_{P(K)} = 0$ together with (3.4.0.11) implies

$$\frac{t_0 + qw + d - b}{w} \geq 2g(K) - 1. \tag{3.4.0.12}$$

Since $1 \leq t_0 \leq w - 2$ and $(d - b) \in \{0, 1\}$, we have that $0 \leq t_0 + d - b < w$. Note that $2g(K) - 1$ is an integer, so we deduce that (3.4.0.12) holds if and only if

$$q \geq 2g(K) - 1,$$

which implies that

$$\frac{b + t_0 w + q w^2}{w^2} \geq 2g(K) - 1,$$

as desired.

Now suppose that $\nu(K) < \nu(\overline{K})$. We claim that in this case, $P(K)$ is not an L-space knot, which is a contradiction. Recall from [OS11, Equation (34)] that $\nu(K)$ is equal to either $\tau(K)$ or $\tau(K)+1$, and from [OS03, Lemma 3.3] that $\tau(\overline{K}) = -\tau(K)$. Thus, when $\nu(K) < \nu(\overline{K})$, it follows that $\nu(\overline{K}) > 0$. By [OS04c, Proposition 2.5], the total rank of $\widehat{HF}(Y)$, for a closed three-manifold Y, is independent of the orientation of Y, i.e.,

$$\mathrm{rk}\widehat{HF}(Y) = \mathrm{rk}\widehat{HF}(-Y). \tag{3.4.0.13}$$

By combining (3.4.0.13), Proposition 3.2.3, and the fact that

$$S^3_{m/n}(K) \cong -S^3_{-m/n}(\overline{K}), \tag{3.4.0.14}$$

we deduce that

$$\mathrm{rk}\widehat{HF}(S^3_m(P(K))) = \mathrm{rk}\widehat{HF}(S^3_{-m/w^2}(\overline{K})). \tag{3.4.0.15}$$

By combining (3.3.0.5), (3.3.0.7), and (3.4.0.15), since $P(K)$ is an L-space knot, we have

$$m + t^m_{P(K)} = -m + w^2 s_{\overline{K}} + t^{-m/w^2}_{\overline{K}}. \tag{3.4.0.16}$$

Using (3.3.0.3) and the fact that $\nu(\overline{K}) > 0$, we observe that $t^{-m/w^2}_{\overline{K}} \neq 0$.

Claim. The equality in (3.4.0.16) never holds.

Proof of the Claim. We prove the claim by considering the following two cases:

Case 1. Suppose $t^m_{P(K)} \neq 0$. Using (3.4.0.11), by expanding both sides of (3.4.0.16) we get that

$$\begin{aligned} &d + t_0 w + qw^2 + 4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d \\ &= -d - t_0 w - qw^2 + w^2 s_{\overline{K}} + 4w^2\nu(\overline{K}) - 2w^2 + 2d + 2t_0 w + 2qw^2. \end{aligned}$$

A similar reasoning as in Case 1 of the previous part of the proof shows that this equality gives a contradiction.

Case 2. Suppose $t^m_{P(K)} = 0$. Using (3.4.0.11), we see that (3.4.0.16) is equivalent to

$$d + t_0 w + qw^2 = -d - t_0 w - qw^2 + w^2 s_{\overline{K}} + 4w^2\nu(\overline{K}) - 2w^2 + 2d + 2t_0 w + 2qw^2.$$

This equation reduces to $2w^2 = w^2 s_{\overline{K}} + 4w^2\nu(\overline{K})$. However, this equation has no solutions, since $\nu(\overline{K}) > 0$ and $s_{\overline{K}} \geq 0$. □

Having proved the claim, it follows that if $\nu(K) < \nu(\overline{K})$, then $P(K)$ could not have been an L-space knot. This completes the proof. □

3.5 Left-orderability

Recall that a non-trivial group G is *left-orderable* if there exists a left-invariant total order on G. Examples of left-orderable groups include $\mathbb{Z}$ and $Homeo_+(\mathbb{R})$, while any group with torsion (e.g., a finite group) is not left-orderable. It is natural to ask which three-manifold groups can be left-ordered. Such groups are well-suited for this study due to the following theorem.

Theorem 3.5.1 (Boyer-Rolfsen-Wiest [BRW05])**.** *Let Y be a compact, connected, irreducible, P^2-irreducible three-manifold. If there exists a non-trivial homomorphism $f : \pi_1(Y) \to G$ where G is left-orderable, then $\pi_1(Y)$ is left-orderable. In particular, if there exists a non-zero degree map from Y to Y', where $\pi_1(Y')$ is left-orderable, then $\pi_1(Y)$ is left-orderable.*

Rather than define P^2-irreducible, we simply point out that if Y is orientable, then irreducibility implies P^2-irreducibility. For compact, orientable, irreducible three-manifolds with $b_1 > 0$, it then follows that their fundamental groups are always left-orderable. However, there are more interesting phenomena for rational homology spheres; for example $+3/2$-surgery on the left-handed trefoil has left-orderable fundamental group, while $-3/2$-surgery has torsion-free, non-left-orderable fundamental group (this can be deduced for instance from [BRW05, Theorem 1.3]). Surprisingly, the left-orderability of the fundamental groups of three-manifolds is conjecturally characterized by Heegaard Floer homology. The following conjecture was made in [BGW13]:

Conjecture 1.3.3 (Boyer-Gordon-Watson)**.** *Let Y be an irreducible rational homology sphere. Then Y is an L-space if and only if $\pi_1(Y)$ is not left-orderable.*

There exists a large amount of support for this conjecture, as it is known to be true for manifolds with Seifert or Sol geometry, branched double covers of non-split alternating links, graph manifold integer homology spheres, and many other families of examples (see for instance [BB13, BGW13, Pet09]). We also remark that irreducibility is necessary, as $\Sigma(2,3,7)\#\Sigma(2,3,5)$ has non-left-orderable fundamental group, but is not an L-space.

In the proof of Proposition 1.3.4 below, we remind the reader that we will be assuming Conjecture 1.3.3.

Proof of Proposition 1.3.4. Suppose that $P(K)$ is an L-space knot. Then for all $\alpha \in \mathbb{Q}$

with $\alpha \geq 2g(P(K)) - 1$, we have $S^3_\alpha(P(K))$ is an L-space. For all but finitely many such α, we have that $S^3_\alpha(P(K))$ is irreducible as well. Thus, by Conjecture 1.3.3, we have that $\pi_1(S^3_\alpha(P(K)))$ is not left-orderable for $\alpha \gg 2g(P(K)) - 1$.

We first study the pattern P. By [CW11, Proposition 13], for such α, $\pi_1(S^3_\alpha(P))$ is not left-orderable. Furthermore, for all but finitely many α, we have that $S^3_\alpha(P)$ is irreducible. Therefore, we appeal to Conjecture 1.3.3 to conclude that P is an L-space knot.

We modify the argument of [CW11, Proposition 13] to study the companion K. Recall that w represents the winding number of P in the solid torus V. We also consider the basis (m, ℓ) for $H_1(\partial V; \mathbb{Z})$ as given in Section 3.1. We choose $n \in \mathbb{Z}$ such that $\gcd(w, n) = 1$ and $n \gg 2g(P(K)) - 1$. As discussed, we have $S^3_n(P(K))$ is irreducible and $\pi_1(S^3_n(P(K))$ is not left-orderable. We consider the manifold $(P; n)$. We have that the kernel of $i_* : H_1(\partial(P; n); \mathbb{Z}) \to H_1((P; n); \mathbb{Z})$ is generated by $nm + w^2\ell$ by Lemma 3.2.1. Since $\gcd(w, n) = 1$ by assumption, we have that the element $nm + w^2\ell$ is represented by a simple closed curve on $\partial(P; n)$ which bounds in $(P; n)$. It then follows that there exists a degree one map $\phi : (P; n) \to S^1 \times D^2$, which restricts to a homeomorphism on the boundary (see for instance [Ron95, Lemma 2.2]). Since $nm + w^2\ell$ bounds in $(P; n)$, we must have that $\phi(nm + w^2\ell)$ is isotopic to $\{*\} \times D^2$.

By extending ϕ to be the identity on the exterior of K, one obtains a degree one map from $S^3_n(P(K))$ to $S^3_{n/w^2}(K)$. Since $S^3_n(P(K))$ is irreducible and $\pi_1(S^3_n(P(K)))$ is not left-orderable, we have that $\pi_1(S^3_{n/w^2}(K))$ is not left-orderable by Theorem 3.5.1. Since w is fixed, by choosing sufficiently large n with $\gcd(w, n) = 1$, we can arrange that $S^3_{n/w^2}(K)$ is irreducible as well. Again, by Conjecture 1.3.3, K is an L-space knot. $\square$

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