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Dragos-Bogdan Suceava

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B. Y. Chen
Major professor

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NEW RIEMANNIAN AND KÄHLERIAN CURVATURE INVARIANTS AND STRONGLY MINIMAL SUBMANIFOLDS

By

Dragos-Bogdan Suceava

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ABSTRACT

NEW RIEMANNIAN AND KÄHLERIAN CURVATURE INVARIANTS AND STRONGLY MINIMAL SUBMANIFOLDS

By

Dragos-Bogdan Suceava

During the last decade, B.-Y.Chen's fundamental inequalities have been investigated by many authors from various viewpoints. In Section 2 we provide an alternate proof for Chen's fundamental inequality associated with classical invariants. In Subsection 2.5, we obtain an inequality for warped product manifolds as a consequence of the previous study. Section 3 is devoted to the study of applications of Chen's fundamental inequality. It is well-known that the classical obstruction to minimal isometric immersions into Euclidean space is $Ric \geq 0$. In this section, we present a method to construct examples of Riemannian manifolds with $Ric < 0$ which don't admit any minimal isometric immersion into Euclidean spaces for any codimension. The study of the relations between curvature invariants and the topology of the manifold yields in section 4 a Myers type theorem for almost Hermitian manifolds. Chen's fundamental inequality for Kähler submanifolds in complex space forms is discussed in Section 5. We provide an extension of the inequality and provide characterizations of strongly minimal complex surfaces in the complex three dimensional space. The last section is dedicated to the study of strong minimality through examples.

To my family

NEW RIEMANNIAN AND KÄHLERIAN CURVATURE INVARIANTS AND STRONGLY MINIMAL SUBMANIFOLDS

Contents

1 Introduction to Chen's Fundamental Inequalities

In the geometry of submanifolds, the following problem is fundamental:

Establish simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds.

The first result in this respect was Gauss' Theorema Egregium, which in 1827 asserted that the Gaussian curvature is an intrinsic invariant. Also concerning this fundamental problem, Chen's fundamental inequalities obtained in [14, 20] are the starting point for many recent papers done by various geometers during the last ten years or so, as for example one may see in [18], [29], [30], [31], [32], [44], [58], [59], [60].

We will discuss in this first section the context and the problems we have worked on in the present dissertation.

Let h denote the second fundamental form of an isometric immersion of a Riemannian n-manifold M^n into an ambient Riemannian space $\bar{M}^{n+m}$. Then the mean curvature vector field is $H = (1/n)\,\mathrm{trace}\,h$. The immersion is called minimal if its mean curvature vector field H vanishes identically.

The following is a classical basic problem in Riemannian geometry.

Problem: *When does a given Riemannian manifold M admit (or does not admit) a minimal immersion into a Euclidean space of arbitrary dimension ?*

For a minimal submanifold M in a Euclidean space the Gauss equation implies that the Ricci tensor of the minimal submanifold satisfies

$$Ric(X,X) = -\sum_{i=1}^{n} |h(X,e_i)|^2 \leq 0, \tag{1}$$

where $e_1, ..., e_n$ is an orthonormal local frame field on M. This gives rise to the first solution to the Problem above; namely, the Ricci tensor of a minimal submanifold M of a Euclidean space is negative semi-definite, and a Ricci-flat minimal submanifold of a Euclidean space is totally geodesic.

The second solution to the Problem mentioned above was obtained by B.Y. Chen as an immediate application of his fundamental inequality and his invariants [14, 20]. Based on these facts, it is interesting to construct precise examples of Riemannian manifolds with $Ric < 0$, but which do not admit any minimal isometric immersion into a Euclidean space for any codimension.

Let M^n be a Riemannian n-manifold. For any orthonormal basis $e_1, ..., e_n$ of the tangent space T_pM, the scalar curvature is defined to be $scal(p) = \sum_{i<j} sec(e_i \wedge e_j)$. For any r-dimensional subspace of T_pM denoted L with orthonormal basis $e_1, ..., e_r$ one may define

$$scal(L) = \sum_{1 \leq i < j \leq r} sec(e_i \wedge e_j). \tag{2}$$

In [20], Chen considered the finite set $S(n)$ of k-tuples $(n_1, ..., n_k)$ with $k \geq 0$ which satisfy the conditions: $n_1 < n$, $n_i \geq 2$, and $n_1 + ... + n_k \leq n$. For each $(n_1, ..., n_k) \in S(n)$ he introduced the following Riemannian invariants:

$$\delta(n_1, ..., n_k)(p) = scal(p) - \inf\left\{scal(L_1) + ... + scal(L_k)\right\}(p), \tag{3}$$

where the infimum is taken for all possible choices of orthogonal subspaces $L_1, ..., L_k$, satisfying n_j =dim L_j, $(j = 1, ..., k)$. Note that the Chen invariant with $k = 0$ is nothing but the scalar curvature.

As in [20], we put

$$c(n_1, ..., n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)},$$

$$b(n_1, ..., n_k) = \frac{1}{2}\Big\{(n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)\Big\}.$$

Chen's fundamental inequalities obtained in [20] can be stated as follows:

Theorem 1.1 *For any n-dimensional submanifold M of a Riemannian space form $R^{n+m}(\varepsilon)$ of constant sectional curvature ε and for any k-tuple $(n_1, ..., n_k) \in S(n)$, we have*

$$\delta(n_1, ..., n_k) \leq c(n_1, ..., n_k)|H|^2 + b(n_1, ..., n_k)\varepsilon. \tag{4}$$

The equality case of the inequality above holds at a point $p \in M$ if and only if there exists an orthonormal basis $e_1, ..., e_{n+m}$ at p such that the shape operators of M in $R^{n+m}(\varepsilon)$ at p take the following forms: $S_r = diag\,(A_1^r, ..., A_k^r, \mu_r, ..., \mu_r)$ for $r = n + 1, ..., m$, where each A_j^r is a symmetric $n_j \times n_j$ submatrix such that

$$trace(A_1^r) = ... = trace(A_k^r) = \mu_r.$$

The invariants $\delta(n_1, \dots, n_k)$ became known as the Chen invariants in literature and inequality (1.4) as Chen's fundamental inequality. Chen's fundamental inequality has many nice applications; for example, one has the following important result as an immediate consequence.

Theorem 1.2 *Let M be a Riemannian n-manifold. If there exists a k-tuple* $(n_1, ..., n_k)$ *in* $S(n)$ *and a point* $p \in M$ *such that*

$$\delta(n_1, ..., n_k)(p) > \frac{1}{2}\{n(n-1) - \sum n_j(n_j - 1)\}\varepsilon, \tag{5}$$

then M admits no minimal isometric immersion into any Riemannian space form $R^m(\varepsilon)$ *with arbitrary codimension.*

In particular, if $\delta(n_1, ..., n_k)(p) > 0$ *at a point for some k-tuple* $(n_1, ..., n_k) \in S(n)$, *then M admits no minimal isometric immersion into any Euclidean space for any codimension.*

We will use the second part of this theorem in our applications. Namely, in the context of Theorem 1.2, we are interested in the following problem.

Are there examples of manifolds with Ric < 0 *, but which have some positive Chen invariant ?*

This is similar to a classical problem 4 mentioned in Peter Petersen's list of problems in [2]:

Scalar versus Ricci curvature problem. *Are there examples of simply connected manifolds which admit Riemannian metrics of positive scalar curvature, but do not admit Riemannian metrics of positive Ricci curvature ?*

We will solve the Ricci vs. Chen invariant problem in the subsections 3.1 and 3.2. As far as we know, the scalar vs. Ricci curvature problem is still open.

In the section 2 we study Chen's fundamental inequality associated with classical invariants, and also consider a few of its algebraic implications. Specifically, we will provide an alternate proof of the following (see [17]).

Theorem 1.3 *Let $f : M^n \to R^{n+m}(\epsilon)$ be an isometric immersion of a Riemannian n-manifold M^n with normalized scalar curvature ρ into an $(n+m)$-dimensional Riemannian space form $R^{n+m}(\epsilon)$ of sectional curvature ϵ. Then*

$$\rho \leq |H|^2 + \epsilon. \tag{6}$$

The equality holds at a point $p \in M$ if and only if p is totally umbilical point.

The last two sections of the present work are dedicated to the study of Chen's fundamental inequalities for complex submanifolds. The context of our study is the following.

Let M^n be a Kähler manifold of complex dimension n. Let us denote by J its complex structure. We denote by $sec(X \wedge Y)$ and $scal(p)$ the sectional curvature of the plane determined by the vectors X and Y and respectively the scalar curvature at the point p. Consider U a coordinate chart on M and $e_1,..., e_n, e_1^* = Je_1,..., e_n^* = Je_n$ a local orthonormal frame on U. Then we have at $p \in U$:

$$scal(p) = \sum_{i<j} sec(e_i \wedge e_j), \;\; i, j = 1, ..., n, 1^*, ..., n^*. \tag{7}$$

Let $\pi \subset T_pM$ be a plane section. Then π is called totally real if $J\pi$ is perpendicular to π. For each real number k, B.-Y.Chen's Kählerian invariant of order 2 and coefficient k at $p \in M$ is defined by

$$\delta_k^r(p) = scal(p) - k \inf sec(\pi^r), \tag{8}$$

where $\inf sec(\pi^r)$ is taken over all totally real plane sections in T_pM.

In [22] the following theorem is proved:

Theorem 1.4 *For any Kähler submanifold M^n of complex dimension $n \geq 2$ in a complex space form $\bar{M}^{n+p}(4c)$, the following statements hold:*

(1) *For each $k \in (-\infty, 4]$ we have*

$$\delta_k^r \leq (2n^2 + 2n - k)c. \tag{9}$$

(2) *Inequality (9) fails for every $k > 4$.*

(3) *$\delta_k^r = (2n^2 + 2n - k)c$ holds identically for some $k \in (-\infty, 4)$ if and only if M^n is a totally geodesic Kähler submanifold of $\bar{M}^{n+p}(4c)$.*

The theorem describes completely, in a forth claim, the pointwise equality situation in the case $k = 4$.

In Theorem 5.4 we extend B.-Y.Chen's fundamental inequality for Kählerian curvature invariants. In Proposition 5.5 we give a characterization of strongly minimal surfaces in $\mathbb{C}^3$. The last section of the present work is dedicated to the study of strong minimality through examples. Namely, we prove that the Kähler surface $z_1 + z_2 + z_3^2 = \kappa$, with $\kappa \in \mathbb{C}$ is strongly minimal in $\mathbb{C}^3$, and we prove that on the Kähler surfaces $Az_1^2 + Bz_2^2 + Cz_3^2 = 0$ and $z_1^3 + z_2^3 + z_3^3 = 1$ there exist points where the strong minimality condition is satisfied. This study is inspired by the discussion on Chen's Kählerian curvature invariants from [22], in the context described above.

2 Chen's Fundamental Inequality with Classical Invariants

2.1 The Hypersurface Case

We will discuss in this chapter Chen's fundamental inequality associated with classical invariants. To clarify the geometrical interpretation in the equality case, we distinguish two situations: the hypersurface case and the general codimension case. The present section is dedicated to the codimension one case.

The main goal of this section is to prove the following:

Proposition 2.1 *Let M^n be a hypersurface in a Riemannian $(n+1)$-manifold $\bar{M}^{n+1}$. Then at every point $p \in M$ the following inequality holds:*

$$scal(p) \leq \frac{n(n-1)}{2} H^2 + \sum_{i<j} \overline{sec}(e_i \wedge e_j), \tag{10}$$

where scal is the scalar curvature of M at p, H is the mean curvature at p, and $\overline{sec}(e_i \wedge e_j)$ is the sectional curvature on the plane generated by vectors e_i and e_j tangent to the ambient space $\bar{M}$.

The equality holds at p if and only if p is an umbilical point.

We first need the following elementary lemma.

Lemma 2.2 *Let g be a real polynomial*

$$g(X) = a_0 X^n + a_1 X^{n-1} + a_2 X^{n-2} + \dots + a_n$$

with $a_0 \neq 0$ and $n > 1$. If all the roots of g are real, then

$$\Delta(g, n) \equiv \frac{2(n-1)}{n} a_1^2 - 4a_0 a_2 \geq 0. \tag{11}$$

Proof of the lemma: If g has only real roots, then g' has also only real roots. After $(n-2)$ steps, we know that $g^{(n-2)}$ has only real roots. Hence, we obtain $\Delta \geq 0$.

Proof: Now we can prove the proposition. Let $p \in M$ and $\eta \in T_p^{\perp}M$. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM in which $S_\eta = S$ is in diagonal form, i.e. $S(e_i) = \lambda_i e_i$, $i = 1, ..., n$, where $\lambda_1, ..., \lambda_n$ are the shape operator's eigenvalues. Then, taking g in the lemma to be the characteristic polynomial of S, we have

$$nH = \lambda_1 + ... + \lambda_n = -a_1, \tag{12}$$

$$\sum_{i<j} \lambda_i \lambda_j = a_2, \tag{13}$$

$$a_0 = 1. \tag{14}$$

On the other hand (see for example [28], pg.131) we have the following well-known fact

$$sec(e_i \wedge e_j) - \overline{sec}(e_i \wedge e_j) = \lambda_i \lambda_j, \tag{15}$$

for any $i < j$. Therefore the inequality in the lemma becomes

$$\frac{2(n-1)}{n}(nH)^2 - 4\sum_{i<j}\{sec(e_i \wedge e_j) - \overline{sec}(e_i \wedge e_j)\} \geq 0 \tag{16}$$

or

$$n(n-1)H^2 - 2\sum_{i<j} sec(e_i \wedge e_j) + 2\sum_{i<j} \overline{sec}(e_i \wedge e_j) \geq 0 \tag{17}$$

and since the sum in the second term is nothing but $scal(p)$ we obtain, after a division by 2, the claimed inequality. The equality case holds if and only if $\lambda_1 = ... = \lambda_n = H/n$, i.e. when the point p is an umbilical point.

The following fact comes from Lemma 1 of [17] for the hypersurface case:

Corollary 2.3 *Let $\rho = 2\,scal(p)/n(n-1)$ denote the normalized scalar curvature of a hypersurface M^n isometrically immersed in a Riemannian space form $R^{n+1}(\epsilon)$. Then we have the inequality:*

$$R \leq H^2 + \epsilon, \tag{18}$$

at every point $p \in M$.

The equality holds if and only if p is an umbilical point.

Proof: For any $i \neq j$, $i, j \in \{1, ..., n\}$, we use the fact that $\overline{sec}(e_i \wedge e_j) = \epsilon$ in the inequality in the previous proposition.

2.2 The General Codimension Case

We have discussed in the previous section the hypersurface case. We present in this section an alternate proof of Lemma 1 of [17] in the general codimension case, i.e. as it was obtained in [17]. One of the main points of this result is that the codimension is arbitrary. We emphasize that the term in the left hand side of (19) is an intrinsic quantity and the terms in the right hand side term are extrinsic quantities.

Proposition 2.4 *Let M^n be isometrically immersed in a Riemannian manifold $\bar{M}^{n+m}$. Let sec, $\overline{sec}$, and scal(p) be the sectional curvature of M, the sectional curvature of*

$\bar{M}$*, and the scalar curvature of* M *at* p*, respectively. Then the following inequality holds.*

$$scal(p) \leq \frac{n(n-1)}{2}|H|^2 + \sum_{i<j} \overline{sec}(e_i \wedge e_j) \tag{19}$$

Proof: The argument in the proof also uses Lemma 1. Let $\{\xi_1, ..., \xi_m\}$ be an orthonormal frame of $T_p^{\perp}M$ at p. Let us denote by $\lambda_1^r, ..., \lambda_n^r$ the eigenvalues of the shape operator $S_r = S_{\xi_r}$. Then $\lambda_1^r + ... + \lambda_n^r = trace(h^r)$ and if S_r is in diagonal form then the characteristic polynomial of S_r has the coefficient corresponding to λ^{n-2} equal to

$$a_2 = \sum_{1 \leq i < j \leq n} \lambda_i^r \lambda_j^r. \tag{20}$$

If S_r is not in diagonal form, then the quantity $\sum_{i,j} \lambda_i^r \lambda_j^r$ is the sum of the 2×2 minors in the matrix $(h_{ij}^r)_{1 \leq i < j \leq n}$, since two equivalent matrices have the same characteristic polynomial. In fact, if $g = \det(S_r - \lambda I_n) = 0$, is the real polynomial in lemma 2.2, then

$$a_2 = \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2), \tag{21}$$

$$-a_1 = \sum_{i=1}^{n} h_{ii}^r, \tag{22}$$

$$a_0 = 1, \tag{23}$$

$$\sum_{i,j} \lambda_i^r \lambda_j^r = \sum (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) \tag{24}$$

and the inequality $\Delta(g,n) \geq 0$ is, in fact, for our choice of g :

$$\frac{2(n-1)}{n}(\sum_{i=1}^{n} h_{ii}^{r})^2 - 4 \sum_{1\leq i<j\leq n} (h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^2) \geq 0 \tag{25}$$

or, summing for $r = n+1, ..., n+m$:

$$n(n-1)|H|^2 - 2\sum_{i<j} \sum_{r=n+1}^{n+m} (h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^2) \geq 0. \tag{26}$$

To express the last sum we need the Gauss' equation (and this is the major difference with respect to the hypersurface case). Let us denote by R the curvature tensor of M and by $\bar{R}$ the curvature tensor of $\bar{M}$. Then, for any $X, Y \in T_pM$, we have

$$< \bar{R}(X,Y)X, Y >=< R(X,Y)X, Y > - < h(X,X), h(Y,Y) > +|h(X,Y)|^2 \tag{27}$$

or, if $\{e_1, ..., e_n\}$ is an orthonormal frame at $p \in M$:

$$sec(e_i \wedge e_j) - \overline{sec}(e_i \wedge e_j) = \sum_{r=n+1}^{n+m} (h_{ii}^{r}h_{jj}^{r} - (h_{ij}^{r})^2) \tag{28}$$

and with this substitution the inequality become

$$2\sum_{i<j} sec(e_i \wedge e_j) \leq n(n-1)|H|^2 + 2\sum_{i<j} \overline{sec}(e_i \wedge e_j) \tag{29}$$

which is the inequality we had to prove, after a division by 2.

Lemma 1 of [17] is the following result, to which we refer as Chen's fundamental inequality with classical invariants.

Corollary 2.5 *Let $f : M^n \to R^{n+m}(\epsilon)$ be an isometric immersion of a Riemannian n-manifold M^n with normalized scalar curvature ρ into an $(n+m)$-dimensional Riemannian space form $R^{n+m}(\epsilon)$ of sectional curvature ϵ. Then*

$$\rho \leq |H|^2 + \epsilon. \tag{30}$$

The equality holds at a point $p \in M$ if and only if p is totally umbilical point.

Proof: For any $i \neq j$, $i, j \in \{1, ..., n\}$ we use that $\overline{sec}(e_i \wedge e_j) = \epsilon$ in the inequality proved in the proposition.

One can state the following immediate consequence which is more or less in the same spirit as the obstruction results obtained in [20].

Corollary 2.6 *Let M^n be a Riemannian n-manifold and $\bar{M}^{n+m}$ be a Riemannian $(n+m)$-manifold. If the scalar curvature of M is greater than the scalar curvature of every n-plane section L of $\bar{M}^{n+m}$, then M admits no minimal immersion into $\bar{M}$.*

2.3 A Remark on Totally Umbilical Points

We recall the fact that the inequality $\Delta(g, n) \geq 0$ has also been used in [63]. Let us define

$$\sigma_i = 2||S_i||^2 - \frac{2}{n}(trace(S_i))^2, \tag{31}$$

where S_i is the shape operator in the normal direction ξ_i and $||S_i||^2 = (\lambda_1^i)^2 + ... + (\lambda_n^i)^2$. Let us denote by L_i the length of shape operator's spectrum in the direction of the normal vector ξ_i, i.e. the distance on the real axis between the greatest and the smallest of the shape operator's eigenvalues in the normal direction ξ_i. Then the following result was proved in [63]:

Theorem 2.7 *Let M^n be a submanifold in a Riemannian manifold $\bar{M}^{n+m}$. For any $p \in M$ and for any normal basis $\xi_1, ..., \xi_m$ we have:*

$$\sqrt{\frac{2}{n(n-1)}\sigma_i(p)} \leq L_i(p) \leq \sqrt{\sigma_i(p)}. \tag{32}$$

The equality holds if and only if p is a totally umbilical point of M in $\bar{M}$.

One may get from the previous inequalities the following.

Corollary 2.8 *Let M^n, $n > 2$ be a submanifold of a Riemannian manifold $\bar{M}^{n+m}$. If for some $p \in M$ there exists a normal direction ξ such that $\sigma_\xi(p) > 0$, then the point p cannot be a totally umbilical point.*

Since the double inequality (32) was obtained by the same procedure as Chen's basic inequality involving the classic invariants, they have in common the proof basisd on the idea $\Delta(g, n) \geq 0$, as it is presented in the previous two sections. In fact, the main idea used in both cases is that the shape operator's characteristic polynomial has only real roots. The algebraic background of the next section is also related to the study of Chen's fundamental inequality with classical invariants.

2.4 A Conformal Invariant Related to Chen's Fundamental Inequality with Classical Invariants

In the classic matrix theory, *the spread of a matrix* has been defined by Mirsky in [47] and then mentioned in various references, as for example in [46].

Let $A \in M_n(\mathbb{C})$, $n \geq 3$, and let $\lambda_1, \ldots, \lambda_n$ be the characteristic roots of A. The *spread* of A is defined to be $s(A) = \max_{i,j} |\lambda_i - \lambda_j|$. We denote by $||A||$ the Euclidean norm of the matrix A, i.e.: $||A||^2 = \sum_{i,j=1}^{m,n} |a_{ij}|^2$. We also use the classical notation E_2 for the sum of all 2-square principal subdeterminants of A. If $A \in M_n(\mathbb{C})$, then we have the following inequalities (see for example [46])

$$s(A) \leq (2||A||^2 - \frac{2}{n}|trA|^2)^{1/2}, \tag{33}$$

$$s(A) \leq \sqrt{2}||A||. \tag{34}$$

If $A \in M_n(\mathbb{R})$, then

$$s(A) \leq \left[2\left(1 - \frac{1}{n}\right)(trA)^2 - 4E_2(A)\right]^{1/2}, \tag{35}$$

with equality holding if and only if $n-2$ of the characteristic roots of A are equal to the arithmetic mean of the remaining two.

Consider now an isometrically immersed submanifold M^n of dimension $n \geq 2$ in a Riemannian manifold $(\bar{M}^{n+s}, \bar{g})$. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$

$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for every $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$. Take a vector η in the normal space to M at the point p and consider the linear mapping $A_\eta : T_pM \to T_pM$. Consider the eigenvalues $\lambda_\eta^1, .., \lambda_\eta^n$ of A_η. We put

$$L_\eta(p) = \sup_{i=1,\ldots,n} (\lambda_\eta^i) - \inf_{i=1,\ldots,n} (\lambda_\eta^i). \tag{36}$$

Then L_η is the spread of the shape operator's in the direction η. We define *the spread of the shape operator at the point* p by

$$L(p) = \sup_{\eta \in \nu_p M} L_\eta(p). \tag{37}$$

Let us remark that when M^2 is a surface we have

$$L_\nu^2(p) = \left(\lambda_\nu^1(p) - \lambda_\nu^2(p)\right)^2 = 4\left(|H(p)|^2 - K(p)\right),$$

where ν is the normal vector at p, H is the mean curvature, and K is the Gaussian curvature. In [7] it is proved that, for a compact surface M^2 in $\mathbb{E}^{2+s}$, the geometric quantity $(|H|^2 - K)dV$ is a conformal invariant. As a consequence, one obtains that $L_\nu^2 dV$ is a conformal invariant for every compact orientable surface in $\mathbb{E}^{2+s}$.

Let $\xi_{n+1}, ..., \xi_{n+s}$ be a local orthonormal frame in the normal fibre bundle νM. Let us recall the definition of *the extrinsic scalar curvature* from [9]:

$$ext = \frac{2}{n(n-1)} \sum_{r=1}^{s} \sum_{i<j} \lambda_{n+r}^i \lambda_{n+r}^j.$$

In [9] it is proved that, for every submanifold M^n of a Riemannian manifold $(\bar{M}, \bar{g})$, the geometric quantity $(|H|^2 - ext)g$ is invariant under any conformal change of metric. When M is compact (see also [9]), this result implies that, for an n-dimensional compact submanifold M of a Riemannian manifold $(\bar{M}, \bar{g})$, the geometric quantity $\int (|H|^2 - ext)^{\frac{n}{2}} dV$ is a conformal invariant.

Let us prove the following fact.

Proposition 2.9 *Let M^n be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. Then spread of shape operator is a conformal invariant.*

Proof: The context and the idea of the proof are similar to the one given in [3, pp.204-205]. Consider a nowhere vanishing positive function ρ on $\bar{M}$. Suppose that we have a conformal change of metric in the ambient space $\bar{M}$ given by

$$\bar{g}^* = \rho^2 \bar{g}.$$

Let us denote by h and h^* the second fundamental forms of M in $(\bar{M}, \bar{g})$ and $(\bar{M}, \bar{g}^*)$, respectively. Then we have (see [13])

$$g(A^*_\xi X, Y) = g(A_\xi X, Y) + g(X, Y)\bar{g}(U, \xi),$$

where U is the vector field defined by $U = (d\rho)^\#$. Let $e_1, ..., e_n$ be a local basis of the principal normal directions of A_ξ with respect to g. Then $\rho^{-1}e_1, ..., \rho^{-1}e_n$, form a local orthonormal frame of M with respect to g^*, and they are the principal directions of A^*_ξ. Therefore

$$L^*(p) = \sup_{\xi^* \in \nu_p M; ||\xi^*||_* = 1} L^*_{\xi^*} = \sup_{\xi^* \in \nu_p M; ||\xi^*||_* = 1} \left(\sup_{i=1,\dots,n} (\lambda^i_\xi)^* - \inf_{j=1,\dots,n} (\lambda^j_\xi)^* \right)$$

$$= \sup_{\xi \in \nu_p M; ||\xi|| = 1} \left[\sup_{i=1,\dots,n} (\lambda^i_\xi + \bar{g}(U,\xi)) - \inf_{j=1,\dots,n} (\lambda^j_\xi + \bar{g}(U,\xi)) \right] =$$

$$= \sup_{\xi \in \nu_p M; ||\xi|| = 1} \left[\sup_{i=1,\dots,n} (\lambda^i_\xi) - \inf_{j=1,\dots,n} (\lambda^j_\xi) \right] = L(p).$$

This proves the proposition.

When M is a compact surface, both L and L^2dV are conformal invariants.

The *shape discriminant* of the submanifold M in $\bar{M}$ with respect to a normal direction η was discussed in [63]. Let A_η be the shape operator associated with an arbitrary normal vector η at p. The shape discriminant of η is defined by

$$D_\eta = 2||A_\eta||^2 - \frac{2}{n}(trace\, A_\eta)^2, \tag{38}$$

where $||A_\eta||^2 = (\lambda_\eta^1)^2 + \cdots + (\lambda_\eta^n)^2$, at every point $p \in M \subset \bar{M}$.

The following pointwise double inequality was proved in [63]:

$$\frac{D_\eta}{\binom{n}{2}} \leq L_\eta^2 \leq D_\eta, \tag{39}$$

We will use this inequality later on. The proof of this fact is algebraically related to the proof of Chen's fundamental inequality with classical curvature invariants (see [17]). The alternate proof of this result is presented in [64].

2.4.1 Geometric inequalities on compact submanifolds

In this section, we study the relationship between the spread of the shape operator's spectrum and the conformal invariant from [9]. The main result of the present section is Proposition 2.10. For its proof we need a few preliminary steps.

Proposition 2.10 *Let M^n be a compact submanifold of a Riemannian manifold $\bar{M}^{n+s}$. Then the following inequality holds:*

$$\left(\int_M L dV\right)^2 (vol(M))^{\frac{n}{2n-2}} \leq 2n(n-1)\left(\int_M (|H|^2 - ext)^{\frac{n}{2}} dV\right)^{\frac{2}{n}}. \tag{40}$$

Equality holds if and only if either $n = 2$ or M is a totally umbilical submanifold of dimension $n \geq 3$.

Before presenting the proof, let us describe what this inequality means. For any conformal diffeomorphism ϕ of the ambient space $\bar{M}$, the quantity

$$\left(\int_{\phi(M)} L dV_\phi\right)^2 (vol(\phi(M)))^{\frac{n}{2n-2}}$$

is bounded above by the conformal invariant geometric quantity expressed in (40).

First, let us prove the following.

Lemma 2.11 *Let $M^n \subset \bar{M}^{n+s}$ be a compact submanifold and p an arbitrary point in M. Consider an orthonormal normal frame $\xi_1, ..., \xi_s$ at p and let D_α be the shape discriminant corresponding to ξ_α, where $\alpha = 1, ..., s$. Then we have*

$$\frac{1}{2n(n-1)} \sum_{\alpha=1}^{s} D_\alpha = |H|^2 - ext. \tag{41}$$

Proof: Since

$$H = \frac{1}{n} \sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} \lambda_\alpha^i \right) \xi_\alpha,$$

$$ext = \frac{2}{n(n-1)} \sum_{\alpha=1}^{s} \sum_{i<j} \lambda_\alpha^i \lambda_\alpha^j,$$

we have

$$|H|^2 - ext = \frac{1}{n^2} \sum_{\alpha=1}^{s} \sum_{i=1}^{n} (\lambda_\alpha^i)^2 - \frac{2}{n^2(n-1)} \sum_{\alpha=1}^{s} \sum_{i<j} \lambda_\alpha^i \lambda_\alpha^j. \tag{42}$$

A direct computation yields

$$D_\alpha = \frac{2(n-1)}{n} \sum_{i=1}^{n} (\lambda_\alpha^i)^2 - \frac{4}{n} \sum_{i<j} \lambda_\alpha^i \lambda_\alpha^j. \tag{43}$$

Summing from $\alpha = 1$ to $\alpha = s$ in (43) and comparing the result with (42) one may get (41).

From the cited result in [9] and the previous lemma, we have

Corollary 2.12 *If M is a compact submanifold in the ambient space $\bar{M}$, then*

$$\int_M \left(\sum_{\alpha=1}^{s} D_\alpha\right)^{\frac{n}{2}} dV$$

is a conformal invariant.

Let us remark that for $n = 2$ this is a well-known fact.

Lemma 2.13 *Let M be a submanifold in the arbitrary ambient space $\bar{M}$. With the previous notations we have*

$$4(|H|^2 - ext) \leq \sum_{\alpha}^{s} L_\alpha^2(p) \leq 2n(n-1)(|H|^2 - ext)$$

at each point $p \in M$. The equalities holds if and only if p is an umbilical point.

Proof: This is a direct consequence of Lemma 2.11 and (39).

Proof of proposition 2.10 : Let p be an arbitrary point of M and let η_0 be a normal direction such that $L(p) = L_{\eta_0}(p)$. Consider the completion of η_0 up to a orthonormal normal basis $\eta_0 = \eta_1, \cdots, \eta_s$. Then we have

$$L^2(p) = L_{\eta_0}^2(p) \leq \sum_{\alpha=1}^{s} L_\alpha^2(p) \leq 2n(n-1)(|H|^2 - ext). \tag{44}$$

By applying Hölder's inequality, one has

$$\left(\int_M LdV\right)^2 \leq \left(\int_M L^2 dV\right)(vol(M)).$$

Applying Hölder's inequality one more time yields

$$\int_M (|H|^2 - ext)\, dV \leq \left(\int_M (|H|^2 - ext)^{\frac{n}{2}} dV\right)^{\frac{2}{n}} (vol(M))^{\frac{n-2}{n}}$$

Therefore, by using the inequality established in lemma 2.13, we have

$$\left(\int_M LdV\right)^2 \leq \left(\int_M L^2dV\right)(vol(M)) \leq 2n(n-1)vol(M)\int_M \left(|H|^2 - ext\right)dV \leq$$

$$\leq 2n(n-1)\left(vol(M)\right)^{\frac{2n-2}{n}}\left(\int_M \left(|H|^2 - ext\right)^{\frac{n}{2}} dV\right)^{\frac{2}{n}}.$$

Let us discuss when the equality case may occur. We have seen that we get identity if $n = 2$.

Now, let us assume $n \geq 3$. The first inequality in (44) is an equality at p if there exist $s-1$ umbilical directions (i.e. $L_\alpha(p) = 0$ for $s = 2, ..., n$). The second inequality in (44) is an equality if and only if p is umbilical point (see [63]). Finally, the two Hölder inequalities are indeed equalities if and only if there exist real numbers θ and μ satisfying $L(p) = \theta$ and $|H|^2 - ext = \mu$ at every $p \in M$. The first equality conditions impose pointwise $L(p) = 0$, which yields $\theta = \mu = 0$. This means that M is totally umbilical.

2.4.2 The noncompact case

Let M be an n-dimensional noncompact submanifold of an $(n+d)$-dimensional Riemannian manifold $(\bar{M}, g)$.

Proposition 2.14 *Let $M^n \subset \bar{M}^{n+d}$ be a complete noncompact submanifold and $\eta_1, ..., \eta_d$ a local orthonormal basis of the normal bundle. Suppose that $\sum \lambda^i_\alpha \lambda^j_\alpha \geq 0$ and $L_\alpha \in L^2(M)$. Then*

$$\int_M (|H|^2 - ext)dV < \infty.$$

Proof: We use the inequality (39). It is sufficient to prove locally the inequality:

$$|H|^2 - ext \leq \sum_{i=1}^{d} D_i$$

This is true since the following elementary inequality holds:

$$(\lambda_\alpha^1)^2 + ... + (\lambda_\alpha^d)^2 - \frac{2n}{n-1}\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j \leq 2[(\lambda_\alpha^1)^2 + ... + (\lambda_\alpha^d)^2] - \frac{2}{n}\Big\{\sum_{i=1}^{d}(\lambda_\alpha^i)\Big\}^2.$$

This is equivalent to

$$n(n-1)\sum_{i=1}^{d}(\lambda_\alpha^i)^2 - 2n^2\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j \leq 2(n-1)^2\sum_{i=1}^{d}(\lambda_\alpha^i)^2 - 4(n-1)\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j$$

or

$$(n^2 - 3n + 2)\{\sum_{i=1}^{d}(\lambda_\alpha^i)^2\} + 2(n^2 - 2n + 2)\sum_{i<j}\lambda_\alpha^i\lambda_\alpha^j \geq 0,$$

which holds by using the hypothesis and that $n \geq 2$.

The inequality is the α-component of the invariant inequality we are going to prove. By adding up d such inequalities and by considering the improper integral on M of the appropriate functions, the conclusion follows. This is due to

$$\int_M (|H|^2 - ext)dV \leq \int_M \sum_{i=1}^{d} D_i dV \leq \binom{n}{2}\sum_{i=1}^{d}\int_M L_i^2 dV$$

by the first inequality in (39).

In the next proposition we establish a relation between $\int_M [L(p)]^2 dV$ and the Willmore-Chen integral, $\int_M (|H|^2 - ext)dV$, studied in [9].

Proposition 2.15 *Let $M^n \subset \bar{M}^{n+d}$ be a complete noncompact orientable submanifold. If $L(p) \in L^2(M)$, then $\int_M (|H|^2 - ext)dV < \infty$.*

Proof: By a direct computation, we have

$$\int_M (|H|^2 - ext)dV = \frac{1}{n^2(n-1)} \int_M \sum_{\alpha=1}^{d} \sum_{i<j} (\lambda_\alpha^1 - \lambda_\alpha^j)^2 dV \leq \tag{45}$$

$$\frac{1}{n^2(n-1)} \int_M \sum_{\alpha=1}^{d} \sum_{i<j} L^2(p)dV = \frac{d}{2n} \int_M L^2(p)dV.$$

2.4.3 Examples

Let us look now at two examples. First, let us consider *the catenoid* defined by

$$f_c(u,v) = \left(c\cos u \cosh\frac{v}{c}, c \sin u \cosh\frac{v}{c}, v\right).$$

Using the classical formulas for example from [62] one finds

$$\lambda_1 = -\lambda_2 = \frac{1}{c}\cosh^{-2}\frac{v}{c}.$$

Therefore, we have

$$\int_{-\infty}^{\infty} L(p)dv = \int_{-\infty}^{\infty} \frac{2}{c}\cosh^{-2}\frac{v}{c}dv = 4\int_{-\infty}^{\infty} \frac{e^t dt}{e^{2t}+1} = 4\pi < \infty.$$

Let us consider *the pseudosphere* whose profile functions are given by (see, for example [37])

$$c_1(v) = ae^{-v/a}$$

$$c_2(v) = \int_0^v \sqrt{1 - e^{-2t/a}}dt$$

for $0 \leq v < \infty$. For simplicity, let us consider just the "upper" part of the pseudosphere. We have

$$\lambda_1 = \frac{e^{v/a}}{a}\sqrt{1 - e^{-2v/a}},$$

$$\lambda_2 = -\left(ae^{v/a}\sqrt{1 - e^{-2v/a}}\right)^{-1}.$$

Remark that

$$\int_M LdV = \int_0^\infty \frac{e^{t/a}}{a\sqrt{1 - e^{-2t/a}}}dt = \frac{1}{2}\int_1^\infty \frac{dy}{\sqrt{y-1}} = \infty.$$

A natural question is to find a characterization for surfaces of rotation that have finite integral of the spread of their shape operator.

Consider surfaces of revolution whose profile curves are described as $c(s) = (y(s), s)$ (see, for example, [62]). Then we have the following.

Proposition 2.16 *Let M be a surface of rotation in Euclidean 3-space defined by*

$$\psi(s,t) = (y(s)\cos t, y(s)\sin t, s)\,.$$

Then the integral of the spread of the shape operator on M is finite if and only if there exist an integrable $C^\infty(\mathbb{R})$ function $f > 0$ which satisfies the following second order differential equation:

$$-yy'' = 1 + (y')^2 \pm f(s)y(1 + (y')^2)^{\frac{3}{2}}$$

For the proof, we use the classical formulas from [37, p.228]. We have for $\lambda_1 = k_{meridian}$, and respectively for $\lambda_2 = k_{parallel}$:

$$\lambda_1 = \frac{-y''}{[1+(y')2]^{3/2}},$$
$$\lambda_2 = \frac{1}{y[1+(y')^2]^{1/2}}.$$

Then, the condition that the integral is finite means that there exists an integrable function $f > 0$ such that

$$\int_R |\lambda_1 - \lambda_2| ds = \int_R f(s) ds.$$

If we assume that $f \in C^\infty$, then the equality between the function under integral holds everywhere and a straightforward computation yields the claimed equality.

For example, for the catenoid $f(s) = 0$.

2.5 A Fundamental Inequality for Warped Product Manifolds

For a warped product $N_1 \times_f N_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathcal{D}_2$ obtained by tangent vectors of N_2 via the vertical lift. Let $\phi : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times_f N_2$ into a Riemannian manifold with constant sectional curvature c. Denote by h the second fundamental form of ϕ. The immersion ϕ is called *mixed totally geodesic* if $h(X, Z) = 0$ for any X in $\mathcal{D}_1$ and Z in $\mathcal{D}_2$.

The problem of suitable conditions on isometric immersions in space forms is analyzed and explained in [23]. Let us consider $N_1 \times_f N_2$ be the warped product of two Riemannian manifolds and let n_1 and n_2 respectively their dimensions. We will use the notation $n = n_1 + n_2$.

The following inequality for warped product spaces is proved in [23].

Theorem 2.17 *Let $\iota : N_1 \times_f N_2 \to R^m(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c, \tag{46}$$

where $n_i = \dim N_i$, $i = 1, 2$, H^2 is the squared mean curvature of ϕ, and Δ is the Laplacian operator of N_1.

The equality sign of (46) holds identically if and only if $\iota : N_1 \times_f N_2 \to R^m(c)$ is a mixed totally geodesic immersion with $\operatorname{trace} h_1 = \operatorname{trace} h_2$, where $\operatorname{trace} h_1$ and $\operatorname{trace} h_2$ denote the trace of h restricted to N_1 and N_2, respectively.

Several applications of this theorem are given in [23].

The classification of immersions from warped products into real space forms satisfying the equality case of (46) is obtained in [24].

Here, we prove the following inequality, in the same spirit, but whose proof will use a different argument, namely the idea from our proof to Chen's fundamental inequality with classical invariants.

Proposition 2.18 *Let $\iota : N_1 \times N_2 \to M^{n+m}$ be an isometric immersion of a warped product manifold into a Riemannian manifold M. Then at every point $p \in M$ the following inequality holds :*

$$n_2\frac{\Delta f}{f} + scal(T_pN_1) + scal(T_pN_2) \leq \frac{n(n-1)}{2}H^2 + \sum_{i<j} s\bar{e}c(e_i \wedge e_j), \qquad (47)$$

where scal is the scalar curvature corresponding to the indicated tangent space with respect to the warped product metric.

Proof: The following relation was proved in [64], it was also proved in section 2.2.

:

$$\sum_{i<j} sec(e_i \wedge e_j) \leq \frac{n(n-1)}{2}H^2 + \sum_{i<j} s\bar{e}c(e_i \wedge e_j). \qquad (48)$$

The left hand side term of the above inequality can be written in detail as

$$\sum_{i<j} sec(e_i \wedge e_j) = \sum_{s=1}^{n_2}\sum_{i=1}^{n_1} sec(e_i \wedge e_s) + \sum_{i<j} sec(e_i \wedge e_j) + \sum_{s<t} sec(e_s \wedge e_t) \qquad (49)$$

where $i, j = 1, ..., n_1$ and $s, t = n_1 + 1, ...n$ are the subscripts corresponding to the tangent spaces to N_1, respectively N_2, at every point $T_p(N_1 \times_f N_2)$.

We have (see, for example, [5] or [23]):

$$\sum_{s=1}^{n_2}\sum_{i=1}^{n_1} sec(e_i \wedge e_s) = n_2\frac{\Delta f}{f},$$

$$\sum_{i<j} sec(e_i \wedge e_j) = scal(T_pN_1),$$

$$\sum_{s<t} sec(e_s \wedge e_t) = scal(T_pN_2).$$

Replacing these quantities in (48) we get the claimed inequality.

The equality holds if and only if the following relation holds at every point:

$$n(n-1)H^2 = 2\sum_{i<j}[sec(e_i \wedge e_j) - s\bar{e}c(e_i \wedge e_j)]. \tag{50}$$

In the case when in relation (47) the ambient space is a space form, we get the following.

Corollary 2.19 *Let $\iota : N_1 \times N_2 \to R^{n+m}(c)$ be an isometric immersion of a warped product into a simply-connected space form. Then at every point $p \in M$:*

$$n_2\frac{\Delta f}{f} + scal(T_pN_1) + scal(T_pN_2) \leq \frac{n(n-1)}{2}(H^2 + c), \tag{51}$$

3 Applications of Chen's Fundamental Inequality: the General Case

3.1 Warped Product of Hyperbolic Planes

In the *Introduction* we have explained that we are interested to construct explicit examples of Riemannian manifolds with $Ric < 0$, but which have some positive Chen invariants.

For such construction, we use the notion of warped product metrics introduced by Kručkovič in 1957 and by Bishop and O'Neill in [5] in Sections 3.1 and 3.2. (A reference on warped product metrics is in [1], which is, in particular, useful in the calculation on Ricci curvature of a warped product metric. Another reference is [56]. A discussion in the context of manifolds with nonpositive curvature, based mainly on [5], can be found in [57].)

Let us consider two copies of the hyperbolic plane (H^2, g_0). The first has coordinates (x, y) with $y > 0$ and has metric $g_0 = (1/y^2)(dx^2 + dy^2)$. Let u and v denote the coordinates of the second copy of the hyperbolic plane with $v > 0$. We consider the open subset $U = \{(x, y) \in H^2 | y > \varepsilon/2\}$, for sufficiently small $\varepsilon > 0$. On the product manifold $(U \times_f H^2, g)$ we consider the warped product metric $g = g_0 + f^2 g_0$, i.e.,

$$g = \frac{1}{y^2}(dx^2 + dy^2) + \frac{f^2(x, y)}{v^2}(du^2 + dv^2), \tag{52}$$

where f is a positive differentiable function. We use the subscripts 1, 2, 3, 4 corresponding to the coordinates x, y, u, v, respectively. At every point $p \in M$, we use the following notation for the tangent vectors

$$\frac{\partial}{\partial x}=\partial_x,\frac{\partial}{\partial y}=\partial_y,\frac{\partial}{\partial u}=\partial_u,\frac{\partial}{\partial v}=\partial_v.$$

We claim the following: *There exist differentiable functions f on $(U\times_f H^2,g)$ such that $Ric<0$ and $\delta(2,2)>0$ everywhere.*

A straightforward computation gives

$$sec(\partial_x\wedge\partial_y)=-1, \tag{53}$$

$$sec(\partial_x\wedge\partial_u)=sec(\partial_x\wedge\partial_v)=\frac{y}{f(x,y)}\left(\frac{\partial f}{\partial y}-y\frac{\partial^2 f}{\partial x^2}\right) \tag{54}$$

$$sec(\partial_y\wedge\partial_u)=sec(\partial_y\wedge\partial_v)=-\frac{y}{f(x,y)}\left(\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}\right) \tag{55}$$

$$sec(\partial_u\wedge\partial_v)=-\frac{1}{f^2(x,y)}-\frac{y^2}{f^2(x,y)}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right] \tag{56}$$

Therefore, the half of scalar curvature at $p=(x,y,u,v)$ is given by

$$\begin{aligned}scal(p)=-1-\frac{1}{f^2(x,y)}-\frac{2y^2}{f(x,y)}\left[\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right]\\ -\frac{y^2}{f^2(x,y)}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]\end{aligned} \tag{57}$$

Using eventually Proposition 9.106 from [1] and the fact that the components of the Hessian of a function ϕ are given in general by :

$$(h_\phi)_{jk}=\frac{\partial^2\phi}{\partial x^j\partial x^k}-\frac{\partial\phi}{\partial x^r}\Gamma^r_{jk}$$

the values of the Ricci tensor are :

$$Ric(\partial_x, \partial_x) = -\frac{1}{y^2} + \frac{2}{yf(x,y)}\frac{\partial f}{\partial y} - \frac{2}{f(x,y)}\frac{\partial^2 f}{\partial x^2}, \tag{58}$$

$$Ric(\partial_y, \partial_y) = -\frac{1}{y^2} - \frac{2}{yf(x,y)}\left(\frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2}\right), \tag{59}$$

$$Ric(\partial_u, \partial_u) = Ric(\partial_v, \partial_v) = -\frac{1}{v^2} - \frac{y^2 f(x,y)}{v^2}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] \tag{60}$$

$$-\frac{y^2}{v^2}\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$$

$$Ric(\partial_x, \partial_y) = -\frac{2}{f(x,y)}\frac{\partial^2 f}{\partial y \partial x} - \frac{2}{yf(x,y)}\frac{\partial f}{\partial x} \tag{61}$$

$$Ric(\partial_x, \partial_u) = Ric(\partial_x, \partial_v) = Ric(\partial_y, \partial_u) = Ric(\partial_y, \partial_v) = Ric(\partial_u, \partial_v) = 0 \tag{62}$$

To complete our example, we choose a function "close" to 1 which has the desired properties: $Ric < 0$ at every point $p = (x, y, u, v)$, but at least one of Chen invariants is strictly positive.

Consider $f(x,y) = e^{\varepsilon \arctan y}$. For this specific function one gets by direct computation that

$$Ric(\partial_u, \partial_u) = Ric(\partial_v, \partial_v) = \tag{63}$$

$$\frac{-1}{v^2(1+y^2)^2}\left[(1+y^2)^2 + 2\varepsilon y^2(\varepsilon - y)e^{2\varepsilon \arctan y}\right] < 0.$$

This last conclusion shows us that the only minor we need to study is the one corresponding to subscripts 1 and 2.

The canonical basis we've considered is not an orthonormal one. To complete the computation on an orthonormal basis let us take $e_1 = y\partial_x$, $e_2 = y\partial_y$, $e_3 = (v/f(x,y))\partial_u$, $e_4 = (v/f(x,y))\partial_v$. Then

$$Ric(e_1, e_1) = y^2 Ric(\partial_x, \partial_x),$$

$$Ric(e_1, e_2) = y^2 Ric(\partial x, \partial y),$$

$$Ric(e_2, e_2) = y^2 Ric(\partial_y, \partial_y),$$

$$Ric(e_3, e_3) = (v^2/f^2) Ric(\partial_u, \partial_u) < 0,$$

$$Ric(e_4, e_4) = (v^2/f^2) Ric(\partial_v, \partial_v) < 0.$$

To see that $Ric < 0$, we have to study the 2×2 minor:

$$Ric(e_1, e_1) = -1 + \frac{2y}{f}\frac{\partial f}{\partial y} - \frac{2y^2}{f}\frac{\partial^2 f}{\partial x^2},$$
$$Ric(e_1, e_2) = Ric(e_2, e_1) = -\frac{2y^2}{f}\frac{\partial^2 f}{\partial y \partial x} - \frac{2y}{f}\frac{\partial f}{\partial x},$$
$$Ric(e_2, e_2) = -1 - \frac{2y}{f}\frac{\partial f}{\partial y} - \frac{2y^2}{f}\frac{\partial^2 f}{\partial y^2},$$

or, for the considered function:

$$Ric(e_1, e_1) = -1 + \frac{2\varepsilon y}{1 + y^2},$$

$$Ric(e_1, e_2) = Ric(e_2, e_1) = 0,$$

$$Ric(e_2, e_2) = -1 - \frac{2\varepsilon y}{1 + y^2} - \frac{2\varepsilon y^2(\varepsilon - 2y)}{(1 + y^2)^2}.$$

On the other hand, since on U we get $sec(\partial_x \wedge \partial_y) < sec(\partial_x \wedge \partial_u)$, $sec(\partial_x \wedge \partial_y) < sec(\partial_y \wedge \partial_u)$, $sec(\partial_u \wedge \partial_v) < sec(\partial_x \wedge \partial_u)$, $sec(\partial_u \wedge \partial_v) < sec(\partial_y \wedge \partial_v)$, the smallest values of $sec(e_i \wedge e_j)$ on the considered basis are $sec(\partial_x \wedge \partial_y)$ and $sec(\partial_u \wedge \partial_v)$, we have on U:

$$\delta(2,2) \geq 2sec(\partial_x \wedge \partial_u) + 2sec(\partial_y \wedge \partial_u) \tag{64}$$

$$= -\frac{y^2}{f(x,y)}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = -\frac{2\varepsilon y^2(\varepsilon - 2y)}{(1+y^2)^2} > 0.$$

The last inequality allows us to apply Theorem 1.2 to obtain the following :

Proposition 3.1 *For sufficiently small $\epsilon > 0$, the Riemannian manifold*

$$M = (U \times H^2, g_0 + (e^{2\varepsilon \arctan y})g_0)$$

cannot be isometrically immersed in any Euclidean ambient space E^m as a minimal submanifold for any codimension, even though $Ric < 0$.

One may obtain similar result by applying the same construction with some other warping functions on an appropriate open set $U \subset H^2$.

Let us notice that one doesn't need a specific computation for $\delta(2,2)$ to apply Theorem 1.2. An estimate as in the relation (64) is sufficient to obtain the obstruction to minimal immersions into a Euclidean space of any codimension.

3.2 Multiwarped Product Spaces

Let us now consider a *multiwarped product* of hyperbolic spaces defined as follows. Let us use a similar notation $U = \{(x,y) \in H^2 | y > \frac{n}{2}\varepsilon\}$ to the previous section. Consider the product manifold of U with n warped copies of the hyperbolic plane H^2, endowed with coordinates $(x, y, u_1, v_1, \ldots, u_n, v_n)$ with $y, v_1, \ldots, v_n > 0$. At an arbitrary point of the product manifold let $\eta_1, \eta_2, \ldots, \eta_{2n+2}$ denote respectively the tangent vectors:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}.$$

The multiwarped product metric on $(U \times_{f_1} H^2 \times_{f_2} .. \times_{f_n} H^2, g)$ is defined by

$$g = \frac{1}{y^2}(dx^2 + dy^2) + \sum_{i=1}^{n} \frac{f_i^2(x,y)}{v_i^2}(du_i^2 + dv_i^2) \tag{65}$$

where $f_1(x,y), \ldots, f_n(x,y)$ are positive differentiable functions.

We claim the following: *There are some choices of $f_1, \ldots, f_n$ which satisfy $Ric < 0$ everywhere and at least one of Chen invariants is positive.*

By direct computation we have, for $i = 1, \ldots, n$:

$$sec(\eta_1 \wedge \eta_2) = -1, \tag{66}$$

$$sec(\eta_1 \wedge \eta_{2i+1}) = sec(\eta_1 \wedge \eta_{2i+2}) = \frac{y}{f_i(x,y)}\left(\frac{\partial f_i}{\partial y} - y\frac{\partial^2 f_i}{\partial x^2}\right), \tag{67}$$

$$sec(\eta_2 \wedge \eta_{2i+1}) = sec(\eta_2 \wedge \eta_{2i+2}) = -\frac{y}{f_i(x,y)}\left(\frac{\partial f_i}{\partial y} + y\frac{\partial^2 f_i}{\partial y^2}\right), \tag{68}$$

$$sec(\eta_{2i+1} \wedge \eta_{2i+2}) = -\frac{1}{f_i^2(x,y)} - \frac{y^2}{f_i^2(x,y)}\left[\left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2\right], \tag{69}$$

$$sec(\eta_{2i+2} \wedge \eta_{2j+2}) = sec(\eta_{2i+2} \wedge \eta_{2j+1}) = sec(\eta_{2i+1} \wedge \eta_{2j+1}) = \tag{70}$$

$$-\frac{y^2}{f_i(x,y)\, f_j(x,y)}\left[\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x} + \frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y}\right],$$

$$scal(p) = -1 - \sum_{i=1}^{n}\left\{\frac{1}{f_i^2} + \frac{y^2}{f_i^2}\left[\left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2\right]\right\} \tag{71}$$

$$-2y^2\sum_{i=1}^{n}\left(\frac{\partial^2 f_i}{\partial x^2}+\frac{\partial^2 f_i}{\partial^2 y}\right)-4y^2\sum_{i,j=1;i\neq j}^{n}\frac{1}{f_i f_j}\left[\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x}+\frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial j}\right],$$

$$Ric(\eta_1,\eta_1)=-\frac{1}{y^2}+\frac{2}{y}\sum_{i=1}^{n}\frac{1}{f_i}\left(\frac{\partial f_i}{\partial y}-y\frac{\partial^2 f_i}{\partial x^2}\right) \tag{72}$$

$$Ric(\eta_2,\eta_2)=-\frac{1}{y^2}-\frac{2}{y}\sum_{i=1}^{n}\frac{1}{f_i}\left(\frac{\partial f_i}{\partial y}+y\frac{\partial^2 f_i}{\partial y^2}\right) \tag{73}$$

$$Ric(\eta_{2i+2},\eta_{2i+2})=Ric(\eta_{2i+1},\eta_{2i+1})=-\frac{1}{v_i^2}-\frac{y^2}{v_i^2}\left[\left(\frac{\partial f_i}{\partial x}\right)^2+\left(\frac{\partial f_i}{\partial y}\right)^2\right] \tag{74}$$

$$-\frac{y^2 f_i}{v_i^2}\left(\frac{\partial^2 f_i}{\partial x^2}+\frac{\partial^2 f_i}{\partial y^2}\right)-\frac{2y^2 f_i^2}{v_i^2}\sum_{i=1;i\neq j}^{n}\frac{1}{f_i f_j}\left[\frac{\partial f_i}{\partial x}\frac{\partial f_j}{\partial x}+\frac{\partial f_i}{\partial y}\frac{\partial f_j}{\partial y}\right].$$

A long computation yields the other terms of the matrix of Ric tensor. Let us explain how to compute $Ric(\eta_1,\eta_2)$. We need to compute terms of the type ${R_{k1}}^{k}{}_{2}$. We distinguish three cases: $k=1$, $k=2$ and $k\neq 1,2$. Then

$${R_{11}}^{1}{}_{2}=0,\quad {R_{21}}^{2}{}_{2}=0,\quad {R_{k1}}^{k}{}_{2}=-\frac{1}{f_k}\frac{\partial^2 f_k}{\partial y\partial x}-\frac{1}{yf_k}\frac{\partial f_k}{\partial x}.$$

A similar discussion is taking place for every element of the matrix of Ric tensor, to yield that all non-diagonal terms vanish everywhere, except

$$Ric(\eta_1,\eta_2)=-2\sum_{k=1}^{n}\left[\frac{1}{f_k}\frac{\partial^2 f_k}{\partial y\partial x}+\frac{1}{yf_k}\frac{\partial f_k}{\partial x}\right]. \tag{75}$$

For a specific example let us consider $f_i(x,y)=f(x,y)=e^{\varepsilon\arctan y}$ for $i=1,\dots,n$. To simplify the computations one may choose $0<\varepsilon\leq 1/n$. For the orthonormal basis we work with, let us denote as above $e_1=y\eta_1$, $e_2=y\eta_2$, and $e_{2k+1}=(v_k/f_k)\eta_{2k+1}$, $e_{2k+2}=(v_k/f_k)\eta_{2k+2}$, for $k=1,\dots,n$, respectively.

For the subscript 3 to $2n$, the Ric matrix is in diagonal form at every point. Through a direct computation, we obtain, for $i = 1, \ldots, n$, that

$$Ric(e_{2i+1}, e_{2i+1}) = Ric(e_{2i+2}, e_{2i+2}) = -\frac{1}{f^2} - \frac{(2n+1)y^2}{f^2}\left(\frac{\partial f}{\partial y}\right)^2 - \frac{y^2}{f}\frac{\partial^2 f}{\partial y^2} < 0. \tag{76}$$

In order to estimate Chen invariant, we compute the sectional curvatures as follows, for $i, j = 1, \ldots, n$, $i \neq j$:

$$sec(\eta_1 \wedge \eta_2) = -1, \tag{77}$$

$$sec(\eta_1 \wedge \eta_{2i+1}) = sec(\eta_1 \wedge \eta_{2i+2}) = \frac{\varepsilon y}{1+y^2} > 0, \tag{78}$$

$$sec(\eta_2 \wedge \eta_{2i+1}) = sec(\eta_2 \wedge \eta_{2i+2}) = \frac{\varepsilon y(y^2 - \varepsilon y - 1)}{(1+y^2)^2} > 0, \tag{79}$$

$$sec(\eta_{2i+1} \wedge \eta_{2i+2}) = -\frac{1}{f^2} - \frac{\varepsilon^2 y^2}{(1+y^2)^2} < 0, \tag{80}$$

$$sec(\eta_{2i+1} \wedge \eta_{2j+1}) = sec(\eta_{2i+2} \wedge \eta_{2j+2}) = sec(\eta_{2i+1} \wedge \eta_{2j+2}) = -\frac{\varepsilon^2 y^2}{(1+y^2)^2} < 0. \tag{81}$$

In fact, one can easily obtain that

$$sec(\eta_{2i+1} \wedge \eta_{2i+2}) < sec(\eta_{2i+2} \wedge \eta_{2j+2}). \tag{82}$$

This allows us to obtain the estimate of the $(2, 2, \ldots, 2)$-order Chen invariant (2 repeats $n+1$ times) such that

$$\delta(2, \ldots, 2) \geq \tau(p) - \left[sec(\eta_1 \wedge \eta_2) + \sum_{i=1}^{n} sec(\eta_{2i+1} \wedge \eta_{2i+2})\right] = \tag{83}$$

$$= \frac{2\varepsilon n y^2}{(1+y^2)^2}(2y - \varepsilon n) > 0$$

Thus, by applying Theorem 1.2, we have proved the following.

Proposition 3.2 *The Riemannian manifold* $(U \times H^2 \times ... \times H^2, g)$, *endowed with the metric given by* (65) *with* $f_i(x,y) = e^{\varepsilon \arctan y}$, $i = 1, \dots, n$, *cannot be isometrically immersed as a minimal submanifold into a Euclidean space for arbitrary dimension, even though* $Ric < 0$.

The same procedure with some other functions f_i may also give rise to other specific examples of Riemannian manifolds whose Chen's invariants obstruct minimal immersions via Theorem 1.2, although the classical invariants do not provide obstruction to minimal immersions.

4 Curvature and Topology: A Myers Type Theorem for Almost Hermitian Manifolds

The classic S.B.Myers' theorem (see [48]) asserts that a complete Riemannian manifold M that satisfies the condition $Ric_p(v,v) \geq r^{-2} > 0$, for every point $p \in M$ and for any unit vector $v \in T_pM$, is compact and its diameter is less than or at most equal to πr. The condition $Ric_p(v,v) \geq 0$ everywhere and a Ricci curvature condition along geodesic rays from a point $p_0 \in M$ has been studied by Calabi in [6]. For some other references on the topic one may see for example [28].

Let us consider $(M^{2n}, J, \langle \ , \ \rangle)$ an almost Hermitian manifold with curvature tensor R. To establish the notations, let us consider just for this section the following sign convention for the curvature tensor

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

for any tangent vector fields $X, Y, Z \in TM$. The Ricci tensor will be denoted by Ric and the sectional curvature by sec. The holomorphic sectional curvature is given by

$$H(X) = sec(X \wedge JX) = \frac{\langle R(JX,X)JX, X\rangle}{\langle X, X\rangle^2}.$$

The main result of this section is Theorem 4.1. Chronologically, the first result of Myers type in Kählerian context was established by Tsukamoto in [68]; his result states that a complete $2n$-dimensional Kählerian manifold M whose holomorphic sectional curvature is greater than or equal to $a > 0$ is compact and has the diameter less than or equal to $\pi/\sqrt{a}$. Furthermore, under the mentioned hypothesis, M is simply connected.

A result of Myers type for nearly Kähler manifolds, with the holomorphic curvature condition, has been proved by A.Gray in [35]. Gray also proved in [36] a corresponding result for almost Hermitian manifolds as follows.

Theorem A *Let M^{2n} be a complete almost Hermitian manifold. Assume that the holomorphic sectional curvature of M satisfies:*

$$H(X) - ||(\nabla_X J)X||^2 ||X||^{-4} \geq a > 0, \tag{84}$$

for all $X \in T_pM$ and all $p \in M$. Then M is compact and the diameter of M is not greater than $\pi/\sqrt{a}$. Furthermore, M is simply connected.

A theorem of Myers type for locally conformal Kähler manifolds has been proven by Vaisman in [69]. A generalization of Myers' theorem for contact manifolds has been proven by Blair and Sharma in [3].

Recall that in [33] Gallaway established the following fact, mentioned also in [28].

Theorem B *Let M^n be a Riemannian manifold. Suppose there exist constants $a > 0$ and $c > 0$ such that for every pair of points in M^{2n} and minimal geodesic γ joining these points having unit tangent $\gamma'(t)$, the Ricci curvature satisfies:*

$$Ric(\gamma'(t), \gamma'(t)) \geq a + \frac{df}{dt} \tag{85}$$

along γ, where f is some function of the arc length t satisfying $|f(t)| \leq c$ along γ. Then M^{2n} is compact and:

$$diam(M^{2n}) \leq \frac{\pi}{a}\left[c + \sqrt{(c^2 + a(n-1))}\right] \tag{86}$$

Furthermore, the universal covering of M^{2n} is compact, with diameter bound as in (86) and the fundamental group of M^{2n} is finite.

For $c = 0$ one may get the classic Myers theorem. It's natural to think about a result similar to Theorem B in the almost Hermitian context, i.e. the context from Theorem A. The curvature condition we study is inspired from A.Gray's Theorem A. We establish the following (see [66]).

Theorem 4.1 *Let M^{2n} be a complete almost Hermitian manifold. Suppose there exist constants $a > 0$ and $c > 0$ such that for every pair of points in M^{2n} and minimal geodesic γ joining these points having unit tangent $\gamma'(t)$, the holomorphic sectional curvature satisfies:*

$$H(\gamma'(t)) \geq a + \frac{df}{dt} \tag{87}$$

along γ, where f is some function of the arc length t, satisfying $|f(t)| \leq c$ along γ. Then M^{2n} is compact and

$$diam(M^{2n}) \leq \frac{\pi}{a}\left[c + \sqrt{(c^2 + a)}\right]. \tag{88}$$

Furthermore, the universal covering of M^{2n} is compact, with diameter bound as in (88)*, and M^{2n} is simply connected.*

Proof: Let us consider two points $p, q \in M$. Let γ be a minimizing geodesic parametrized by arc length t that joins p and q, $\gamma : [0, l] \to M$, the length of γ being l. Let us consider the vector field (as for example in [36]):

$$V(t) = \left(sin\frac{\pi t}{l}\right) J\gamma'(t), \tag{89}$$

for any $t \in [0,l]$. Then let us consider the proper variation of γ in direction of V. We denote by E the energy functional given by:

$$E(\gamma) = \int_0^l ||\gamma'(t)||^2 dt.$$

Synge's second variation formula (see for example [28]) yields:

$$\frac{1}{2}\frac{d^2E}{dt^2}(0) = -\int_0^l \left\langle V(t), \frac{D^2V}{dt^2} + R(\gamma'(t), V(t))\gamma'(t) \right\rangle dt \tag{90}$$

or, replacing the expression of $V(t)$ from relation (89):

$$\frac{1}{2}\frac{d^2E}{dt^2}(0) = -\int_0^l \left\langle \left(sin\frac{\pi t}{l}\right) J\gamma'(t), \left(-\frac{\pi^2}{l^2}\right)\left(sin\frac{\pi t}{l}\right) J\gamma'(t) \right\rangle - \tag{91}$$

$$-\int_0^l \left\langle \left(sin\frac{\pi t}{l}\right) J\gamma'(t), R(\gamma'(t), \left(sin\frac{\pi t}{l}\right) J\gamma'(t))\gamma'(t) \right\rangle =$$

$$= \frac{\pi^2}{l^2}\int_0^l sin^2\left(\frac{\pi t}{l}\right) dt - \int_0^l sin^2\left(\frac{\pi t}{l}\right) \langle \gamma', R(J\gamma', \gamma')J\gamma' \rangle \, dt.$$

The curvature term in the last equation is the holomorphic sectional curvature $sec(J\gamma' \wedge \gamma')$ and we may use the condition (87) to get:

$$\frac{1}{2}\frac{d^2E}{dt^2}(0) \le \frac{\pi^2}{2l} - \frac{al}{2} - \int_0^l sin^2\left(\frac{\pi t}{l}\right)\frac{df}{dt}dt. \tag{92}$$

Integrating by parts and using that $|f(t)| < c$ on γ one may get:

$$\frac{1}{2}\frac{d^2E}{dt^2}(0) \le \frac{\pi^2}{2l} - \frac{al}{2} + \int_0^l |sin\frac{2\pi t}{l}||f(t)|dt \le \tag{93}$$

$$\le \frac{\pi^2}{2l} - \frac{al}{2} + \pi c.$$

Thus, if $l > \pi(c + \sqrt{c^2 + a})/a$ then the variation would minimize the length of γ, contradicting the fact that γ is minimizing. Hence, the length of γ is bounded above by this quantity, therefore (88) holds.

To see the last claim of the theorem, let's assume the contrary (the argument is the same as in [36]). Then there exists a non-trivial free homotopy class of loops which contains a non-trivial minimal geodesic γ_0, defined on $[0, l]$. Assume that γ_0 has unit speed. The deformation of γ_0 in the direction of $V_0(t) = sin(\pi t/l)J\gamma_0'(t)$ yields, by the second variation formula, since the length of γ_0 is bounded above by $\pi(c + \sqrt{c^2 + a})/a$:

$$\frac{1}{2}\frac{d^2E}{dt^2}(0) < 0,$$

therefore γ_0 cannot be a minimal geodesic. Therefore M is simply connected.

Corollary 4.2 *Let M^{2n} be a complete Kähler submanifold in a complex space form $\bar{M}^{2(n+k)}(\varepsilon)$. Suppose there exist constants $a > 0$ and $c > 0$ such that for every pair of points in M^{2n} and minimal geodesic γ joining these points having unit tangent $\gamma'(t)$, the second fundamental form h satisfies along γ:*

$$2||h(\gamma'(t), \gamma'(t))||^2 + a + \frac{df}{dt} \leq \varepsilon, \tag{94}$$

where f is some function of the arc length t, satisfying $|f(t)| \leq c$ along γ. Then M^{2n} is compact and:

$$diam(M^{2n}) \leq \frac{\pi}{a}\left[c + \sqrt{(c^2 + a)}\right] \tag{95}$$

Furthermore, the universal covering of M^{2n} is compact, with diameter bound as in (95), and M^{2n} is simply connected.

Proof of Corollary 4.2 : It is known (see, for example, [51]) that

$$H(X) = \varepsilon - 2\sum g(A_\alpha X, X)^2 = \varepsilon - 2||h(\gamma'(t), \gamma'(t))||^2.$$

Then we apply Theorem 4.1.

Let us remark the Corollary's hypothesis cannot be relaxed to $a = c = 0$. For example, in the case $\varepsilon = 0$ there exist complex totally geodesic noncompact submanifolds.

Let us remark that Myers' Theorem can be stated in terms of Chen's invariants. In [18] B.-Y.Chen introduced also the following string of Riemannian curvature invariants.

$$\hat{\delta}(n_1, n_2, ..., n_k) = scal(p) - \sup\{scal\, L_1 + ... + scal\, L_{n_k}\}, \tag{96}$$

where L_1, L_2, ... L_{n_k} are mutually orthogonal linear spaces of dimension n_1, n_2, ... n_k. With this notation, we can state Myers' Theorem as follows.

Theorem 4.3 *Let (M, g) be a Riemannian manifold such that, at every point $p \in M$, the condition: $\hat{\delta}(n-1) \geq a^2 > 0$ holds. Then M is compact.*

Proof: One may write, for any unit vector,

$$Ric(v, v) \geq \hat{\delta}(n-1) \geq a^2 > 0.$$

Thus, the hypothesis from Myers' theorem is verified.

In general, the positivity of a certain Chen invariant doesn't imply compactness. For example, $\delta(2, 2, ..., 2) \geq a^2$ doesn't imply compactness, as one may see from the following example. Consider $M = S^2(1) \times \mathbb{R}^2 \times ... \times \mathbb{R}^2$, where $\mathbb{R}^2$ is taken n times. In this case $\delta(2, 2, ..., 2) = 1 > 0$ at every point, but M is not compact.

5 Chen's Fundamental Inequality for Complex Submanifolds

5.1 New Kählerian Invariants

Let M^n be a Kähler manifold of complex dimension n. Let us denote by J its complex structure. We denote by $sec(X \wedge Y)$ and $scal(p)$ the sectional curvature of the plane determined by the vectors X and Y and respectively the scalar curvature at the point p. Consider U a coordinate chart on M and $e_1,\ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n$ a local orthonormal frame on U. Then we have at $p \in U$:

$$scal(p) = \sum_{i<j} sec(e_i \wedge e_j), \quad i, j = 1, \ldots, n, 1^*, \ldots, n^*. \tag{97}$$

Let $\pi \subset T_pM$ be a plane section. Then π is called totally real if $J\pi$ is perpendicular to π. For each real number k, B.-Y.Chen's Kählerian invariant of order 2 and coefficient k at $p \in M$ is defined by

$$\delta_k^r(p) = scal(p) - k \inf sec(\pi^r), \tag{98}$$

where $\inf sec(\pi^r)$ is taken over all totally real plane sections in T_pM.

In [22] the following theorem is proved:

Theorem 5.1 *For any Kähler submanifold M^n of complex dimension $n \geq 2$ in a complex space form $\bar{M}^{n+p}(4c)$, the following statements hold:*

(1) *For each $k \in (-\infty, 4]$ we have*

$$\delta_k^r \leq (2n^2 + 2n - k)c. \tag{99}$$

(2) *Inequality (99) fails for every* $k > 4$.

(3) $\delta_k^r = (2n^2 + 2n - k)c$ *holds identically for some* $k \in (-\infty, 4)$ *if and only if* M^n *is a totally geodesic Kähler submanifold of* $\bar{M}^{n+p}(4c)$.

The theorem describes completely, in a forth claim, the pointwise equality situation in the case $k = 4$.

5.2 Strongly Minimal Submanifolds

It is known (see for example [51]) that the shape operator of a Kähler submanifold M^n in $\bar{M}^{n+p}$ satisfies:

$$A_{J\xi_r} = JA_r, \quad JA_r = -A_r J, \tag{100}$$

for $r = 1, ..., p, 1^*, ..., p^*$, and where we use the well-known convention $A_r = A_{\xi_r}$. Therefore the shape operator of M^n takes the form

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha \\ A''_\alpha & -A'_\alpha \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha \\ A'_\alpha & A''_\alpha \end{pmatrix}, \quad \alpha = 1, ..., p \tag{101}$$

where A'_α and A''_α are $n \times n$ matrices. The condition (101)implies that every Kähler submanifold M^n is minimal, i.e. $trace\ A_\alpha = trace\ A_{\alpha^*} = 0$, $\alpha = 1, ..., p$.

Definition: A Kähler submanifold M^n of a Kähler manifold $\bar{M}^{n+p}$ is called *strongly minimal* if at each point there exists an orthonormal frame e_1,..., e_n,$e_1^* = Je_1$,...,$e_n^* = Je_n$ such that the shape operator satisfies the conditions

$$trace\ A'_\alpha = trace\ A''_\alpha = 0, \quad \alpha = 1, ..., p.$$

This class of submanifolds was introduced and studied by B.-Y.Chen in [22]. From [22] we have the following two results.

Theorem 5.2 [22] *A complete Kähler submanifold* M^n $(n \geq 2)$ *in* $\mathbb{CP}^{n+p}(4c)$ *satisfies the equality*

$$\delta_4^r = 2(n^2 + n - 2)c \tag{102}$$

identically if and only if

(1) M^n *is a totally geodesic Kähler submanifold, or*

(2) $n = 2$ *and* M^2 *is a strongly minimal Kähler surface in* $\mathbb{CP}^{2+p}(4c)$.

Theorem 5.3 [22] *A complete Kähler submanifold* M^n $(n \geq 2)$ *of* $\mathbb{C}^{n+p}$ *satisfies* $\delta_4^r = 0$ *identically if and only if*

(1) M^n *is a complex n-plane of* $\mathbb{C}^{n+p}$, *or*

(2) M^n *is a complex cylinder over a strongly minimal Kähler surface* M^2 *in* $\mathbb{C}^{n+p}$ *(i.e.* M^n *is the product submanifold of a strongly minimal Kähler surface* M^2 *in* $\mathbb{C}^{p+2}$ *and the identity map of the complex Euclidean* $(n-2)$*-space* $\mathbb{C}^{n-2}$*).*

Among the examples studied in [22] let us mention a nontrivial example: The complex surface N^2 in $\mathbb{C}^3$ given by the equation $z_1^2 + z_2^2 + z_3^2 = 1$ is a strongly minimal Kähler surface.

The above mentioned results and examples motivate our present study of the strongly minimal submanifolds. One of the problems we discuss in the present dissertation is the characterization of strongly minimal surfaces in $\mathbb{C}^3$.

5.3 An extension of B.-Y. Chen's Fundamental Inequality with Kählerian Invariants

In the present section we extend the inequality (99) to orders higher than 2. Let us motivate first this generalization. As we have mentioned before, the first form of B.-Y.Chen's fundamental inequality in Riemannian context has been given in [14], in 1993, and the string of B.-Y.Chen's fundamental inequalities has been obtained in [20], in 2000. It is natural to ask what could be the most general statement one may get from the geometric idea of B.-Y.Chen's fundamental inequality for Kähler submanifolds in space forms, presented in [22].

An l-dimensional linear subspace $L_l \subset T_pM$ is called *totally real* if JL_l is orthogonal to L. For each real number k one may extend the above invariants to *Kählerian invariant of order l and coefficient k* by

$$\delta^r_{l,k}(p) = scal(p) - \frac{k}{l-1} \inf_{L^r_l \subset T_pM} [scal(L_l(p))], \tag{103}$$

where L^r_l runs over all totally real linear subspaces in T_pM, and the scalar curvature of a linear subspace is

$$scal[L_l(p)] = \sum_{1 \leq i < j \leq l} sec(\eta_i \wedge \eta_j),$$

for the orthonormal basis $\eta_1, ..., \eta_l$ in L_l.

Let us now assume that M^n is a Kähler submanifold of complex dimension n in a complex space form $\bar{M}^{n+p}(4c)$. We are closely following the notations from [22] unless stated otherwise.

Let us consider the orthonormal basis $e_1,..., e_n, e^*_1 = Je_1,..., e^*_n = Je_n$ such that

$L_l = \text{span}\ \{\ e_1,\ldots,\ e_l\}$, where L_l achieves the infimum for $scal[L]$, with $\dim L = l$. If R is the curvature tensor of M, then the Gauss equation is

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle + \tag{104}$$

$$+c\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle JY,Z\rangle\langle JX,W\rangle -$$

$$-\langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle,$$

where h is the second fundamental form.

The result we prove is the following.

Theorem 5.4 *Let M^n be a Kähler submanifold of complex dimension $n \geq 2$ in a complex space form $\bar{M}^{n+p}(4c)$. Let $\delta^r_{k,l}$ be the Chen's Kählerian invariant of order l and coefficient k. Then we have*

(1) *For any $2 \leq l \leq n$, the following inequality holds*

$$\delta^r_{4,l} \leq (2n^2 + 2n - \tbinom{l}{2})c. \tag{105}$$

The equality case for $l = 2$ has been described in [22]. Equality holds at every point for a fixed $l \geq 3$ if and only if M^n is a totally geodesic submanifold.

(2) *For any $k \in [0,4]$ the following inequality holds*

$$\delta^r_{k,l} \leq \left[2n^2 + 2n - \frac{k}{4}\tbinom{l}{2}\right]c. \tag{106}$$

Equality holds at every point for a fixed $l \geq 3$ if and only if M^n is a totally geodesic submanifold.

Proof: Let us discuss first claim (1) of the theorem. Taking $X = W = e_k$ and $Y = Z = e_s$ for $1 \leq k, s \leq t$ in the Gauss' equation one gets

$$sec(e_k \wedge e_s) = \sum_{\alpha=1}^{p} \left[h^{\alpha}_{kk}h^{\alpha}_{ss} - (h^{\alpha}_{ks})^2 + h^{\alpha*}_{kk}h^{\alpha*}_{ss} - (h^{\alpha*}_{ks})^2\right] + c. \tag{107}$$

Consider $l \in \{2, 3, ..., n\}$, the dimension of the totally real space L^r_l. Then

$$scal(L^r_l) = \sum_{1 \leq k < s \leq l} \langle R(e_k, e_s)e_s, e_k\rangle = \tag{108}$$

$$= \sum_{\alpha=1}^{p} \sum_{1 \leq k < s \leq l} \left\{\left[h^{\alpha}_{ii}h^{\alpha}_{jj} - (h^{\alpha}_{ij})^2\right] + \left[h^{\alpha*}_{ii}h^{\alpha*}_{jj} - (h^{\alpha*}_{ij})^2\right]\right\} + \binom{l}{2} c.$$

For the following computations, we use $(h^{\alpha}_{kk})^2 + (h^{\alpha}_{ss})^2 \geq -2h^{\alpha}_{kk}h^{\alpha}_{ss}$, the similar relations for α^* and that $l \geq 2$ implies

$$2\,(h^{\alpha}_{ks})^2 \geq \frac{2}{l-1}\,(h^{\alpha}_{ks})^2. \tag{109}$$

As in relation (3.5) from [22] one may compute the quantity

$$4n(n+1)c - 2scal = 4\sum_{\alpha=1}^{p}\left\{||A'_{\alpha}||^2 + ||A''_{\alpha}||^2\right\} \geq \tag{110}$$

$$\geq 4\sum_{\alpha=1}^{p}\left\{\sum_{i=1}^{l}(h^{\alpha}_{ii})^2 + 2\sum_{1 \leq i < j \leq l}(h^{\alpha}_{ij})^2 + \sum_{i=1}^{l}(h^{\alpha*}_{ii})^2 + 2\sum_{1 \leq i < j \leq l}(h^{\alpha*}_{ij})^2\right\} =$$

$$= 4\sum_{\alpha=1}^{p}\left\{\sum_{1 \leq i < j \leq l}\left[\frac{(h^{\alpha}_{ii})^2}{l-1} + \frac{(h^{\alpha}_{jj})^2}{l-1}\right] + 2\sum_{i<j}(h^{\alpha}_{ij})^2\right\} +$$

$$4\sum_{\alpha=1}^{p}\left\{\sum_{1 \leq i < j \leq l}\left[\frac{(h^{\alpha*}_{ii})^2}{l-1} + \frac{(h^{\alpha*}_{jj})^2}{l-1}\right] + 2\sum_{i<j}(h^{\alpha*}_{ij})^2\right\} \geq$$

$$\geq 4\sum_{\alpha=1}^{p}\left\{\sum_{1\leq i<j\leq l}\left[\frac{-2h_{ii}^{\alpha}h_{jj}^{\alpha}}{l-1}+2(h_{ij}^{\alpha})^2\right]\right\}+$$

$$+4\sum_{\alpha=1}^{p}\left\{\sum_{1\leq i<j\leq l}\left[\frac{-2h_{ii}^{\alpha*}h_{jj}^{\alpha*}}{l-1}+2(h_{ij}^{\alpha*})^2\right]\right\}\geq$$

$$4\sum_{\alpha=1}^{p}\sum_{1\leq i<j\leq l}\left\{\frac{-2}{l-1}[h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2]\right\}+$$

$$+\sum_{\alpha=1}^{p}\sum_{1\leq i<j\leq l}\left\{\frac{-2}{l-1}[h_{ii}^{\alpha*}h_{jj}^{\alpha*}-(h_{ij}^{\alpha*})^2]\right\}=$$

$$=-\frac{8}{l-1}\sum_{\alpha=1}^{p}\sum_{1\leq i<j\leq l}\left[h_{ii}^{\alpha}h_{jj}^{\alpha}-(h_{ij}^{\alpha})^2+h_{ii}^{\alpha*}h_{jj}^{\alpha*}-(h_{ij}^{\alpha*})^2\right]=$$

$$=-8\left[scal(L_l^r)+\tbinom{l}{2}c\right].$$

Therefore we have proved

$$2n(n+1)c-\tbinom{l}{2}c\geq scal-\frac{4}{l-1}scal(L_l^r) \tag{111}$$

for every totally real l-dimensional space L_l^r. Therefore

$$\delta_{4,l}^r\leq[2n(n+1)-\tbinom{l}{2}]c. \tag{112}$$

From the sequence of inequalities above, it is clear that equality holds everywhere if and only if $(h_{ii}^{\alpha})^2+(h_{jj}^{\alpha})^2=-2h_{ii}^{\alpha}h_{jj}^{\alpha}$, for any disjoint pair i,j from 1 to l and

$2 = \frac{2}{l-1}$. The equality case for $l = 2$ has been completely described by B.-Y.Chen in [22]. For $l \geq 3$ the fact that $h_{ii} = 0$ is immediate.

Now let's prove the claim (2) of the theorem 5.4. It has been proved in [22] that

$$scal \leq (2n^2 + 2n)c. \tag{113}$$

We multiply this inequality by $p > 0$ and we add it to (105) term by term. We get

$$(p+1)scal - \frac{4}{l-1} inf\ scal(L_l^r) \leq [(p+1)(2n^2 + 2n) - \binom{l}{2}]c \tag{114}$$

Dividing both sides by $p + 1 > 0$ we get

$$scal - \frac{4}{(l-1)(p+1)} inf\ scal(L_l^2) \leq [(2n^2 + 2n) - \frac{1}{p+1}\binom{l}{2}]c. \tag{115}$$

By denoting $\frac{4}{p+1} = k$, we get $p = \frac{4}{k} - 1 = \frac{4-k}{k}$. Using this in (115) we can write the result as

$$scal - \frac{k}{l-1} inf\ scal(L_l^2) \leq [(2n^2 + 2n) - \frac{k}{4}\binom{l}{2}]c, \tag{116}$$

which is the claimed inequality. From the equality case in relation (3.3) in [22], claim 3 of Theorem 2 from [22] and the claim (1) of present theorem, if the equality holds at every point for some $k \in [0, 4]$ and some l between 2 and n, then M^2 is a totally geodesic submanifold. In detail, the argument can be written as follows, for $k \in (0, 4)$. (The argument is practically the same as in [22].)

$$\left[2n^2 + 2n - \frac{k}{4}\binom{l}{2}\right] c = \left(1 - \frac{k}{4}\right) \delta_{0,l}^r + \frac{k}{4}\delta_{4,l}^r \leq \tag{117}$$

$$\leq \left(1 - \frac{k}{4}\right)(2n^2 + 2n)c + \frac{k}{4}\left[2n^2 + 2n - \binom{l}{2}\right] c = \left[2n^2 + 2n - \frac{k}{4}\binom{l}{2}\right] c.$$

From this equality, in particular one gets $\delta^r_{0,l} = (2n^2 + 2n)c$. It is shown in [22] that this equality at every point implies that the submanifold M^2 is totally geodesic.

Let us remark that the implication of being totally geodesic from the equality doesn't have a statement which is similar to (4) of Theorem 2 in [22]. In fact, for the case $l = 2$, the equality situation has already been completely discussed in [22]. In this sense, the equality case for $n \geq 3$, $3 \leq l \leq n$, is different from the situation for $l = 2$ where strongly minimal complex surfaces appear naturally. We have seen in the generalization that a natural cut off in the expression of $4n(n+1)c - 2scal$ matches an expression obtained from Gauss equation. This match points out the sharpness of inequality in the case $l = 2$ studied in [22].

5.4 Characterizations of Strongly Minimal Surfaces in the Complex Three Dimensional Space

In this section we consider a complex surface $M^2 \subset \mathbb{C}^3$. The coordinates of the ambient space are z_1, z_2, z_3; for $j = 1, 2, 3$. Put $z_j = x_j + iy_j$. We suppose M is embedded so that there exists $\phi \in hol(\mathbb{C}^3)$ such that $M = \{z \in \mathbb{C}^3 |\, \phi(z) = 0\} = V(\phi)$ and

$$\frac{\partial \phi}{\partial z} = \left(\frac{\partial \phi}{\partial z_1}, \frac{\partial \phi}{\partial z_2}, \frac{\partial \phi}{\partial z_3} \right)$$

never vanishes on M.

Let us assume that $p = (z_1^0, z_2^0, z_3^0)$ is a nonsingular point on M. Two unit normal vectors at p are ξ and $J\xi$, where

$$\xi = \frac{1}{||\partial\phi/\partial z||} \frac{\bar{\partial \phi}}{\partial z}.$$

By definition, M^2 is called strongly minimal in $\mathbb{C}^3$ if the second fundamental form can be written pointwise as follows. There exist an open neighborhood $U \subset M$ of p such that there exists two orthogonal unit length vector fields X and Y on U, such that the second fundamental form with respect to the orthonormal basis $\{X, Y, JX, JY\}$ can be written in the form:

$$A_\xi = \begin{pmatrix} a(z) & b(z) & c(z) & d(z) \\ b(z) & -a(z) & d(z) & -c(z) \\ c(z) & d(z) & -a(z) & -b(z) \\ d(z) & -c(z) & -b(z) & a(z) \end{pmatrix}$$

and, respectively,

$$A_{J\xi} = \begin{pmatrix} -c(z) & -d(z) & a(z) & b(z) \\ -d(z) & c(z) & b(z) & -a(z) \\ a(z) & b(z) & c(z) & d(z) \\ b(z) & -a(z) & d(z) & -c(z) \end{pmatrix}$$

where a, b, c, d are real analytic functions on U.

Suppose that the strongly minimal submanifold is realized on U as a graph manifold, i.e., $z_3 = f(z_1, z_2)$. Let us consider an open set $V \subset \mathbb{C}^2$ and $\omega : V \to \mathbb{C}^3$ such that $\omega(z_1, z_2) = (z_1, z_2, f(z_1, z_2))$. We also have $\phi(z_1, z_2, z_3) = f(z_1, z_2) - z_3$. Then

$$e_1 = \frac{\partial \omega}{\partial z_1} = \left(1, 0, \frac{\partial f}{\partial z_1}\right) \quad e_2 = \frac{\partial \omega}{\partial z_2} = \left(0, 1, \frac{\partial f}{\partial z_2}\right),$$

and $Je_j = ie_j$, where $j = 1, 2$. To express f as a function of $a(z), b(z), c(z), d(z)$, one may use relations of type:

$$a(z) = \langle A_\xi X, X \rangle = \langle h(X, X), \xi \rangle = \left\langle \bar{\nabla}_X X - \nabla_X X, \xi \right\rangle,$$

$$b(z) = \langle A_\xi Y, X \rangle = \langle h(Y, X), \xi \rangle = \left\langle \bar{\nabla}_Y X - \nabla_Y X, \xi \right\rangle,$$

$$c(z) = \langle A_\xi JX, X \rangle = \langle h(JX, X), \xi \rangle = \left\langle \bar{\nabla}_X JX - \nabla_X JX, \xi \right\rangle,$$

$$d(z) = \langle A_\xi JY, X \rangle = \langle h(JY, X), \xi \rangle = \left\langle \bar{\nabla}_X JY - \nabla_X JY, \xi \right\rangle.$$

We use below these equations in the proof of the parametric equations of a strongly minimal surface.

We consider the real and complex parts of the function f as follows.

$$z_3 = f(z_1, z_2) = u(x_1, x_2, y_1, y_2) + iv(x_1, x_2, y_1, y_2).$$

We use the notation:

$$\frac{\partial u}{\partial x_1} = u_{x_1},$$

and the other similar notations. In fact, we have

$$u_{x_j} = v_{y_j} \qquad v_{x_j} = -u_{y_j}, \tag{118}$$

since f is holomorphic with respect to both variables. Let us compute e_1 and e_2 in terms of function u and its derivatives:

$$e_1 = \frac{\partial \omega}{\partial z_1} = (1, 0, u_{x_1}, 0, 0, -u_{y_1}),$$

$$e_2 = \frac{\partial \omega}{\partial z_2} = (0, 1, u_{x_2}, 0, 0, -u_{y_2}),$$

where the first three components correspond to the real part, and the last three components correspond to the imaginary part of e_1 and e_2, regarded in $\mathbb{C}^3$. Let us compute

$$\bar{\nabla}_{e_1} e_1 = \left(0, 0, \frac{\partial^2 f}{\partial z_1^2}\right) \tag{119}$$

To compute the projection $\nabla_{e_1} e_1$ of $\bar{\nabla}_{e_1} e_1$ to T_pM, one needs to compute every term of the expression:

$$\nabla_{e_1} e_1 = \left\langle \bar{\nabla}_{e_1} e_1, \frac{e_1}{||e_1||} \right\rangle \frac{e_1}{||e_1||} + \left\langle \bar{\nabla}_{e_1} e_1, \frac{e_2}{||e_2||} \right\rangle \frac{e_2}{||e_2||} + \tag{120}$$
$$\left\langle \bar{\nabla}_{e_1} e_1, \frac{Je_1}{||Je_1||} \right\rangle \frac{Je_1}{||Je_1||} + \left\langle \bar{\nabla}_{e_1} e_1, \frac{Je_2}{||Je_2||} \right\rangle \frac{Je_2}{||Je_2||}$$

In the following considerations, one may use the Cauchy-Riemann equations and the fact that $u = Re\ f$ is harmonic, one may express everything in terms of u.

Let us remark that the harmonicity of u can be written as

$$u_{x_1x_1} + u_{y_1y_1} = 0, \qquad u_{x_2x_2} + u_{y_2y_2} = 0.$$

We get the following expressions for the covariant derivatives on the complex surface.

$$\nabla_{e_1} e_1 = \frac{u_{x_2} u_{x_1 x_1} + u_{y_1} u_{x_1 y_1}}{1 + u_{x_1}^2 + u_{y_1}^2}(1, 0, u_{x_1}, 0, 0, -u_{y_1}) + \tag{121}$$

$$\frac{u_{x_2} u_{x_1 x_1} + u_{y_2} u_{x_1 y_1}}{1 + u_{x_2}^2 + u_{y_2}^2}(1, 0, u_{x_2}, 0, 0, -u_{y_2}) + \frac{u_{y_1} u_{x_1 x_1} - u_{x_1} u_{x_1 y_1}}{1 + u_{x_1}^2 + u_{y_1}^2}(0, 0, u_{y_1}, 1, 0, u_{x_1}) +$$

$$\frac{u_{y_2} u_{x_1 x_1} - u_{x_2} u_{x_1 y_1}}{1 + u_{x_2}^2 + u_{y_2}^2}(0, 0, u_{y_2}, 1, 0, u_{x_2}),$$

$$\nabla_{e_1} e_2 = \frac{u_{x_1} u_{x_1 x_2} + u_{y_1} u_{y_1 x_2}}{2(1 + u_{x_1}^2 + u_{y_1}^2)}(1, 0, u_{x_1}, 0, 0, -u_{y_1}) + \tag{122}$$

$$\frac{u_{x_2} u_{x_1 x_2} + u_{y_2} u_{y_1 x_2}}{2(1 + u_{x_2}^2 + u_{y_2}^2)}(1, 0, u_{x_2}, 0, 0, -u_{y_2}) + \frac{u_{y_1} u_{x_1 x_2} - u_{x_1} u_{y_1 x_2}}{2(1 + u_{x_1}^2 + u_{y_1}^2)}(0, 0, u_{y_1}, 1, 0, u_{x_1}) +$$

$$\frac{u_{y_2} u_{x_1 x_2} - u_{x_2} u_{y_1 x_2}}{2(1 + u_{x_2}^2 + u_{y_2}^2)}(0, 0, u_{y_2}, 1, 0, u_{x_2}),$$

$$\nabla_{e_2} e_2 = \frac{u_{x_1} u_{x_2 x_2} + u_{y_1} u_{x_2 y_2}}{1 + u_{x_1}^2 + u_{y_1}^2}(1, 0, u_{x_1}, 0, 0, -u_{y_1}) + \tag{123}$$

$$\frac{u_{x_2} u_{x_2 x_2} + u_{y_2} u_{x_2 y_2}}{1 + u_{x_2}^2 + u_{y_2}^2}(1, 0, u_{x_2}, 0, 0, -u_{y_2}) +$$

$$\frac{u_{y_1} u_{x_2 x_2} - u_{x_1} u_{x_2 y_2}}{1 + u_{x_1}^2 + u_{y_1}^2}(0, 0, u_{y_1}, 1, 0, u_{x_1}) +$$

$$\frac{u_{y_2}u_{x_2x_2} - u_{x_2}u_{x_2y_2}}{1 + u_{x_2}^2 + u_{y_2}^2}(0, 0, u_{y_2}, 1, 0, u_{x_2}).$$

The other expressions for covariant derivative may be deduced as follows.

$$\nabla_{e_2}e_1 = \nabla_{e_1}e_2,$$

$$\nabla_{Je_1}e_1 = J\nabla_{e_1}e_1 + [Je_1, e_1] = J\nabla_{e_1}e_1,$$

$$\nabla_{Je_2}e_2 = J\nabla_{e_2}e_2 + [Je_2, e_2] = J\nabla_{e_2}e_2,$$

$$\nabla_{Je_1}e_2 = J\nabla_{e_2}e_1 + [Je_1, e_2],$$

$$\nabla_{Je_2}e_1 = J\nabla_{e_1}e_2 + [Je_2, e_1],$$

$$\nabla_{Je_1}Je_1 = J\nabla_{e_1}Je_1 + J[Je_1, e_1] = -\nabla_{e_1}e_1,$$

$$\nabla_{Je_1}Je_2 = J\nabla_{e_2}Je_1 + J[Je_1, e_2],$$

$$\nabla_{Je_2}Je_1 = \nabla_{Je_1}Je_2 + [Je_2, Je_1],$$

$$\nabla_{Je_2}Je_2 = J\nabla_{Je_2}e_2 + J[Je_2, e_2] = J\nabla_{Je_2}e_2.$$

Let us use the notation suggested by the example in the work [22] and let us consider two vector fields $X = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ and $Y = (\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3)$ on the open neighborhood U. We put

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \beta = (\beta_1, \beta_2, \beta_3), \quad \gamma = (\gamma_1, \gamma_2, \gamma_3), \quad \delta = (\delta_1, \delta_2, \delta_3).$$

The fact that X and Y are tangent vector fields can be expressed as

$$X = (\alpha_1 + i\beta_1)\Big(1, 0, \frac{\partial f}{\partial z_1}\Big) + (\alpha_2 + i\beta_2)\Big(0, 1, \frac{\partial f}{\partial z_2}\Big) \tag{124}$$

$$Y = (\gamma_1 + i\delta_1)\Big(1, 0, \frac{\partial f}{\partial z_1}\Big) + (\gamma_2 + i\delta_2)\Big(0, 1, \frac{\partial f}{\partial z_2}\Big) \tag{125}$$

and, for the third component, one may get by direct computations the following.

$$\alpha_3 = \alpha_1 u_{x_1} + \beta_1 u_{y_1} + \alpha_2 u_{x_2} + \beta_2 u_{y_2}, \tag{126}$$

$$\beta_3 = \beta_1 u_{x_1} - \alpha_1 u_{y_1} + \beta_2 u_{x_2} - \alpha_2 u_{y_2}. \tag{127}$$

The conditions $g(X,Y) = g(JX,Y) = 0$ can be written

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 + \beta_1\delta_1 + \beta_2\delta_2 + \beta_3\delta_3 = 0, \tag{128}$$

$$-\gamma_1\beta_1 - \gamma_2\beta_2 - \gamma_3\beta_3 + \alpha_1\delta_1 + \alpha_2\delta_2 + \alpha_3\delta_3 = 0, \tag{129}$$

where $\alpha_j, \beta_j, \gamma_j, \delta_j$, are real analytic functions on the open set $U \subset M$, for $j = 1, 2, 3$. (The conditions $g(X, JY) = g(JX, JY) = 0$ are insured by the relations above and the conditions $g(X, JX) = g(Y, JY) = 0$ yield trivially.)

The fact that X and Y are unit vector fields can be written as

$$\alpha_1^2+\alpha_2^2+\alpha_3^2+\beta_1^2+\beta_2^2+\beta_3^2=1, \tag{130}$$

$$\gamma_1^2+\gamma_2^2+\gamma_3^2+\delta_1^2+\delta_2^2+\delta_3^2=1. \tag{131}$$

We have the following *parametric equations of strongly minimal complex surfaces in* $\mathbb{C}^3$.

Proposition 5.5 *Let u be the real part of a holomorphic function $f(z_1, z_2, z_3)$. Then the complex surface $V(f) = M$ is strongly minimal if, for every point $p \in M$, there exist an open coordinate neighborhood $U \subset M$ of p such that on U there exist four real analytic functions a, b, c, d and two orthonormal tangent vector fields X and Y such that $g(X,X) = g(Y,Y) = 1$, $g(X,Y) = g(X,JY) = 0$*

$$X=(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3),\quad Y=(\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,\delta_3)$$

$$\alpha=(\alpha_1,\alpha_2,\alpha_3),\beta=(\beta_1,\beta_2,\beta_3),\gamma=(\gamma_1,\gamma_2,\gamma_3),\delta=(\delta_1,\delta_2,\delta_3),$$

such that on U we have

$$-a(z)(1+u_{x_1}^2+u_{x_2}^2+u_{y_1}^2+u_{y_2}^2)^{1/2}=(\alpha_1^2-\beta_1^2)u_{x_1x_1}+2\alpha_1\beta_1u_{x_1y_1}+ \tag{132}$$

$$(\alpha_1\alpha_2-\beta_1\beta_2)u_{x_1x_2}+(\alpha_1\beta_2+\alpha_2\beta_1)u_{y_1x_2}+(\alpha_2^2-\beta_2^2)u_{x_2x_2}+2\alpha_2\beta_2u_{x_2y_2}$$

$$=-(\gamma_1^2-\delta_1^2)u_{x_1x_1}-2\gamma_1\delta_1u_{x_1y_1}-(\gamma_1\gamma_2-\delta_1\delta_2)u_{x_1x_2}$$

$$-(\gamma_1\delta_2+\gamma_2\delta_1)u_{y_1x_2}-(\gamma_2^2-\delta_2^2)u_{x_2x_2}-2\gamma_2\delta_2u_{x_2y_2},$$

$$-b(z)(1+u_{x_1}^2+u_{x_2}^2+u_{y_1}^2+u_{y_2}^2)^{1/2}=(\alpha_1\gamma_1-\beta_1\delta_1)u_{x_1x_1}+ \tag{133}$$

$$(\beta_1\gamma_1+\alpha_1\delta_1)u_{x_1y_1}+\frac{1}{2}(\alpha_2\gamma_1+\alpha_1\gamma_2-\beta_2\delta_1-\beta_1\delta_2)u_{x_1x_2}+$$

$$\frac{1}{2}(\beta_2\gamma_1+\beta_1\gamma_2+\alpha_2\delta_1+\alpha_1\delta_2)u_{y_1x_2}+(\alpha_2\gamma_2-\beta_2\delta_2)u_{x_2x_2}+(\beta_2\gamma_2+\alpha_2\delta_2)u_{x_2y_2},$$

$$-c(z)(1+u_{x_1}^2+u_{x_2}^2+u_{y_1}^2+u_{y_2}^2)^{1/2}=-2\alpha_1\beta_1u_{x_1x_1}+(\alpha_1^2-\beta_1^2)u_{x_1y_1}- \tag{134}$$

$$(\alpha_1\beta_2+\alpha_2\beta_1)u_{x_1x_2}+(\alpha_1\alpha_2-\beta_1\beta_2)u_{y_1x_2}-2\alpha_2\beta_2u_{x_2x_2}+(\alpha_2^2-\beta_2^2)u_{x_2y_2}=$$

$$2\gamma_1\delta_1u_{x_1x_1}+(\delta_1^2-\gamma_1^2)u_{x_1y_1}+(\gamma_1\delta_2+\gamma_2\delta_1)u_{x_1x_2}+$$

$$(\delta_1\delta_2-\gamma_1\gamma_2)u_{y_1x_2}+2\gamma_2\delta_2u_{x_2x_2}+(\delta_2^2-\gamma_2^2)u_{x_2y_2},$$

$$d(z)(1+u_{x_1}^2+u_{x_2}^2+u_{y_1}^2+u_{y_2}^2)^{1/2}=(\alpha_1\delta_1+\beta_1\gamma_1)u_{x_1x_1}- \tag{135}$$

$$-(\alpha_1\gamma_1-\beta_1\delta_1)u_{x_1y_1}+\frac{1}{2}(\alpha_1\delta_2+\alpha_2\delta_1+\beta_1\gamma_2+\beta_2\gamma_1)u_{x_1x_2}+$$

$$\frac{1}{2}(\beta_2\delta_1+\beta_1\delta_2-\alpha_2\gamma_1-\alpha_1\gamma_2)u_{y_1x_2}+(\alpha_2\delta_2+\beta_2\gamma_2)u_{x_2x_2}+(\beta_2\delta_2-\alpha_2\gamma_2)u_{x_2y_2}.$$

Proof: Let us remark that the normal unit vector field ξ has the form (see for example [70])

$$\xi=(1+u_{x_1}^2+u_{x_2}^2+u_{y_1}^2+u_{y_2}^2)^{-1/2}(u_{x_1},u_{x_2},-1,u_{y_1},u_{y_2},0) \tag{136}$$

Applying this fact, let us consider the expression which yields the first entry in the second fundamental form operator: $g(A_\xi X, X) = a(z)$. Now, Let us use a computational idea presented as relation (3.2) in [70] respectively relation (5.12) in [22] to get

$$g\left(-||\frac{\partial f}{\partial z}||^{-1}\{\bar{X}\bar{H_{jk}}\}^{tan}, X\right) = a, \tag{137}$$

where

$$H_{jk} = \frac{\partial^2 \phi}{\partial z_j \partial z_k}.$$

This can be written in detail as

$$-||\frac{\partial \phi}{\partial z}||^{-1} g\left(\left[\begin{pmatrix} \alpha_1 - i\beta_1 \\ \alpha_2 - i\beta_2 \\ \alpha_3 - i\beta_3 \end{pmatrix}\begin{pmatrix} \frac{\partial^2 \bar{\phi}}{\partial \bar{z}_1^2} & \frac{\partial^2 \bar{\phi}}{\partial \bar{z}_1 \partial \bar{z}_2} & 0 \\ \frac{\partial^2 \bar{\phi}}{\partial \bar{z}_1 \partial \bar{z}_2} & \frac{\partial^2 \bar{\phi}}{\partial \bar{z}_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^{tan}, \begin{pmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \\ \alpha_3 + i\beta_3 \end{pmatrix}\right) = a(z).$$

Let us remark that in general one can use as basis of the tangent fibre bundle on U the orthonormal frame $\{X, Y, JX, JY\}$. This means, for our computations, that

$$v^{tan} = \langle v, X\rangle X + \langle v, Y\rangle Y + \langle v, JX\rangle JX + \langle v, JY\rangle JY.$$

In fact we need just $g(v^{tan}, X) = \langle v, X\rangle$. (We denote consistently by $\langle\ ,\ \rangle$ the scalar product in $\mathbb{R}^6$.)

In this context, we have

$$\bar{X}\bar{H_{jk}} = (\alpha_1 u_{x_1x_1} + \beta_1 u_{x_1y_1} + \frac{\alpha_2}{2}u_{x_1x_2} + \frac{\beta_2}{2}u_{y_1x_2}, \tag{138}$$

$$\frac{\alpha_1}{2} + \frac{\beta_1}{2}u_{y_1x_2} + \alpha_2 u_{x_2x_2} + \beta_2 u_{x_2y_2}, 0,$$

$$\alpha_1 u_{x_1y_1} - \beta_1 u_{x_1x_1} - \frac{\beta_2}{2}u_{x_1x_2} + \frac{\alpha_2}{2}u_{y_1x_2}, \frac{\alpha_1}{2}u_{y_1x_2} - \frac{\beta_1}{2}u_{x_1x_2} + \alpha_2 u_{x_2y_2} - \beta_2 u_{x_2x_2}, 0).$$

Computing the 6-dimensional scalar product we get the claimed equation. Similar computations prove the other analogous equalities.

Now, let us study the Gauss and Codazzi equations of a strongly minimal complex surface into $\mathbb{C}^3$. Following [61] and [50], the computational idea is to write explicitly relevant relations of the complex surface in $\mathbb{C}^3$. In [61] there are defined and studied the symmetric covariant tensors h and k and the tensor field s, of type (0,1), such that the Gauss and Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi + k(X,Y)J\xi, \tag{139}$$

$$\bar{\nabla}_X \xi = -AX + s(X)J\xi. \tag{140}$$

With these notations, the Gauss and Codazzi equations (see for example [61]) are

$$R(X,Y,Z,W) = g(AX,Z)g(AY,W) - g(AX,W)g(AY,Z) + \tag{141}$$

$$+g(JAX,W)g(JAY,Z) - g(JAX,Z)g(JAY,W),$$

$$(\nabla_X A)Y - (\nabla_Y A)X = s(X)JAY - s(Y)JAX \tag{142}$$

The Gauss equation has been used to prove Proposition 6 in [22]. It is also the main idea in the following.

Proposition 5.6 *Let $\{X,Y\}$ a pair of orthonormal tangent vectors $X,Y \in T_pM$, such that $g(X,Y) = g(X,JY) = 0$, with respect to which the shape operator of the* *manifold* $M = V(\phi)$ *on the open set* U *satisfies the strong minimality condition. Then we have*

$$Ric(X,X) = Ric(Y,Y) = -2(a^2 + b^2 + c^2 + d^2).$$

Proof: We compute, by the Gauss equation, that

$$sec(X \wedge Y) = g(A_\xi X, X)g(A_\xi Y, Y) - g(A_\xi X, Y)g(A_\xi Y, X) +$$

$$+g(JA_\xi X, Y)g(JA_\xi Y, X) - g(JA_\xi X, X)g(JA_\xi Y, Y).$$

Either using this relation or using relation (5) from [61], we get the claimed fact.

As a remark, the condition $[A'_\xi, A''_\xi] = 0$ proved also in Proposition 6 from the cited work is satisfied identically once we prescribe the shape operator in the form A_ξ, as we did.

Proposition 5.7 *Let* U *be an open neighborhood of a regular point* M *such that on* U *there exists a pair of orthonormal tangent vector fields* X *and* Y*, with the property that at every point* $g(X,Y) = g(X,JY) = 0$*, satisfying the strong minimality condition. If* s *is the tensor field of type* $(0,1)$ *defined by the Weingarten formula*

$$\bar{\nabla}_X \xi = -A_\xi X + s(X)J\xi,$$

then the following relations hold:

$$X(c(z)) - JX(a(z)) = s(X)a(z) + s(JX)c(z); \tag{143}$$

$$X(a(z)) + JX(c(z)) = -s(X)c(z) + s(JX)a(z); \tag{144}$$

$$Y(c(z)) - JY(a(z)) = s(Y)a(z) + s(JY)c(z); \tag{145}$$

$$Y(a(z)) + JY(c(z)) = -s(Y)c(z) + s(JY)a(z). \tag{146}$$

Proof: It is convenient to work with the following form of the Codazzi equation:

$$\nabla_X(A_\xi Y) - \nabla_Y(A_\xi X) + A_\xi([Y, X]) = s(x)JA_\xi Y - s(Y)JA_\xi X. \tag{147}$$

Let us prove for example the third equation from the ones stated above. We write the Codazzi equation in Y and JY and multiply on the right by Y (we understand by *multiply* the product given by the metric $\langle\ ,\ \rangle$ in $\mathbb{C}^3$). We get

$$\langle \nabla_Y(A_\xi JY), Y\rangle - \langle \nabla_{JY}(A_\xi Y), Y\rangle = s(Y)\langle JA_\xi JY, Y\rangle - s(JY)\langle JA_\xi Y, Y\rangle. \tag{148}$$

Using $JY = iY$ and the metric property of the Riemannian connection on the submanifold U, we have

$$Y\langle A_\xi Y, JY\rangle - \langle A_\xi Y, \nabla_Y JY\rangle - JY\langle A_\xi Y, Y\rangle + \langle A_\xi Y, \nabla_{JY} Y\rangle$$

$$= -s(Y)a(z) - s(JY)c(z).$$

Now, one can use the fact that: $J\nabla_Y Y = \nabla_Y JY = \nabla_Y iY = i\nabla_Y Y$ to simply the expression. Furthermore, the shape operator A_ξ has a prescribed form on the considered basis. Therefore, we find

$$Y(-c(z)) - \langle A_\xi Y, J\nabla_Y Y\rangle - JY(-a(z)) + \langle A_\xi Y, i\nabla_Y Y\rangle$$

$$= -s(Y)a(z) - s(JY)c(z).$$

This proves the relation (145).

Similarly one can prove by the same steps the other equations.

We have used so far four cases of the Codazzi equation: in X and JX multiplied by X and JX, and in Y and JY multiplied by Y and JY. Let us now use Codazzi equation in X, JX multiplied by Y, respectively JY, then Codazzi equation in Y, JY, multiplied by X, respectively JX.

Proposition 5.8 *Let U be an open neighborhood of a regular point M such that on U there exists a pair of orthonormal tangent vector fields X and Y, with the property that at every point $g(X,Y) = g(X,JY) = 0$, satisfying the strong minimality condition. If s is the tensor field of type $(0,1)$ defined by the Weingarten formula*

$$\bar{\nabla}_X \xi = -A_\xi X + s(X) J\xi,$$

then the following relations hold:

$$X(d(z)) - JX(b(z)) = s(X)b(z) + s(JX)d(z); \tag{149}$$

$$X(b(z)) + JX(d(z)) = -s(x)d(z) + s(JX)b(z); \tag{150}$$

$$Y(d(z)) - JY(b(z)) = s(Y)b(z) + s(JY)d(z); \tag{151}$$

$$Y(b(z)) + JY(d(z)) = -s(Y)d(z) + s(JY)b(z). \tag{152}$$

The proof is similar to the one given in the previous proposition.

Corollary 5.9 *Let U be an open neighborhood of a regular point M such that on U there exists a pair of orthonormal tangent vector fields X and Y, with the property that at every point $g(X,Y) = g(X,JY) = 0$, satisfying the strong minimality condition. If s is the tensor field of type $(0,1)$ defined by the Weingarten formula*

$$\bar{\nabla}_X\xi = -A_\xi X + s(X)J\xi,$$

then we have the following relations:

$$(X+Y)(d(z)+b(z)) + (JX+JY)(d(z)-b(z)) = \tag{153}$$

$$(s(X)+s(Y))[(b(z)-d(z)) + i(b(z)+d(z))]$$

$$(X+Y)(c(z)+a(z)) + (JX+JY)(c(z)-a(z)) = \tag{154}$$

$$(s(X)+s(Y))[(a(z)-c(z)) + i(c(z)+a(z))]$$

$$(X+Y)(a(z)+b(z)+c(z)+d(z)) + (JX+JY)(c(z)+d(z)-a(z)-b(z)) = \tag{155}$$

$$(s(X)+s(Y))[(a(z)+b(z)-c(z)-d(z)+i(a(z)+b(z)+c(z)+d(z))].$$

Proof: Straightforward linear computations from the previous two propositions.

Proposition 5.10 *Let U be an open neighborhood of a regular point M such that on U there exists a pair of orthonormal tangent vector fields X and Y, with the property that at every point $g(X,Y) = g(X,JY) = 0$, satisfying the strong minimality condition. Suppose that at least one of the analytic functions a, b, c, d is nonvanishing everywhere on U, say $a \neq 0$ on U. Then we have*

$$X(d(z) - ib(z)) = \frac{b(z) + id(z)}{a(z) + ic(z)} X(c(z) - ia(z)), \tag{156}$$

$$Y(d(z) - ib(z)) = \frac{b(z) + id(z)}{a(z) + ic(z)} Y(c(z) - ia(z)). \tag{157}$$

Proof: We have the relations:

$$X(d(z)) - JX(b(z)) = s(X)(b(z) + id(z)),$$

$$X(c(z)) - JX(a(z)) = s(X)(a(z) + ic(z)).$$

Solving the second equation for $s(X)$ and replacing in the first, then using $JX = iX$ and keeping in mind that $a \neq 0$ everywhere on U, we get the claimed result. Similar relations hold true in Y.

6 A Study of Strong Minimality Through Examples

6.1 $\delta_4^2 = 0$ on degree two complex surfaces

In the previous section we have seen a few characterizations of strongly minimal complex surfaces in $\mathbb{C}^3$, as for example the parametric equations, in Proposition 5.5. We keep the same notations in the present section, which is consistent with the notation of [22].

In [22] it was proved that $z_1^2 + z_2^2 + z_3^2 = 1$ is a strongly minimal surface. We compute here locally the functions a, b, c, d and see how the parametric equations look like in this case. First of all, we have

$$f(z_1, z_2) = (1 - z_1^2 - z_2^2)^{1/2},$$

where we work locally on the principal branch of the complex radical. Since $z_j = x_j + iy_j$, $j = 1, 2, 3$, we will denote ζ by

$$\zeta = 1 - z_1^2 - z_2^2 = (1 - x_1^2 - x_2^2 + y_1^2 + y_2^2) + i(-2x_1y_1 - 2x_2y_2).$$

If we denote by θ the polar angle of the complex number ζ then, on the principal branch of the complex radical, we get

$$\zeta^{(1/2)} = \sqrt{|\zeta|}\Big(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\Big),$$

or, by a direct computation, we have

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}(\sqrt{|\zeta| + Re\,\zeta} + i\sqrt{|\zeta| - Re\,\zeta}),$$

where $Re\,\zeta = 1 - x_1^2 - x_2^2 + y_1^2 + y_2^2$. Therefore, we find

$$u(x_1, x_2, y_1, y_2) = \frac{1}{\sqrt{2}}\sqrt{|\zeta| + Re\,\zeta}, \tag{158}$$

$$v(x_1, x_2, y_1, y_2) = \frac{1}{\sqrt{2}}\sqrt{|\zeta| - Re\,\zeta}. \tag{159}$$

It was established by Chen in [22] that, for a position vector $x = (a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \in V(f)$, the basis realizing the strong minimality condition is given by

$$e_1 = \frac{1}{\sqrt{2}}\left(\frac{b_2}{\sqrt{a_1^2 + b_2^2}}, \frac{ia_1}{\sqrt{a_1^2 + b_2^2}}, 1\right) \tag{160}$$

$$e_2 = \frac{1}{\sqrt{2}}\left(\frac{ib_2}{\sqrt{a_1^2 + b_2^2}}, \frac{a_1}{\sqrt{a_1^2 + b_2^2}}, i\right), \tag{161}$$

Therefore, by a direct computation, we get

$$a(z) = -||\frac{\partial f}{\partial z}||^{-1}\left(\frac{b_2^2 - a_1^2}{a_1^2 + b_2^2} + 1\right) \tag{162}$$

$$b(z) = -||\frac{\partial f}{\partial z}||^{-1} \tag{163}$$

$$c(z) = 0 \tag{164}$$

$$d(z) = -||\frac{\partial f}{\partial z}||^{-1}\left(\frac{b_2^2 - a_1^2}{a_1^2 + b_2^2} - 1\right) \tag{165}$$

at every point $x = (a_1 + ib_1, a_2 + ib_2, a_3 + ib_3) \in V(\phi)$.

Let us study here the following generalization of B.-Y.Chen's example presented in [22]: *what are the sufficient conditions for a complex surface given by* $Az_1^2 + Bz_2^2 +$

$Cz_3^2 = 1$ *to be strongly minimal ?* The general answer is stated below in Proposition 6.1. The existence of points where the strong minimality is observed is proved in Proposition 6.2.

Let M^2 be the complex surface in $\mathbb{C}^3$ defined by:

$$M^2 = \{z \in \mathbb{C}^3 / Az_1^2 + Bz_2^2 + Cz_3^2 = 1, A \neq 0, B \neq 0\}, \tag{166}$$

We have seen in the previous section that M^2 is strongly minimal if at every point $p \in M$ we can prove the existence of two vectors $X, Y \in T_pM$ such that the following system holds

$$g(X,Y) = g(X,JY) = 0, \tag{167}$$

$$g(X,X) = g(Y,Y) = 1, \tag{168}$$

$$g(X,\xi) = g(X,J\xi) = g(Y,\xi) = g(Y,J\xi) = 0, \tag{169}$$

$$g(A_\xi X, X) + g(A_\xi Y, Y) = 0, \tag{170}$$

$$g(A_{J\xi} X, X) + g(A_{J\xi} Y, Y) = 0. \tag{171}$$

From equations (167, 168), using the notation $X = \alpha + i\beta$ and $Y = \gamma + i\delta$, we get

$$\langle \alpha, \gamma \rangle + \langle \beta, \delta \rangle = 0, \quad \langle \alpha, \delta \rangle = \langle \beta, \gamma \rangle, \tag{172}$$

$$||\alpha||^2 + ||\beta||^2 = 1, \qquad ||\gamma||^2 + ||\delta||^2 = 1. \tag{173}$$

Let us remark that $\frac{\partial f}{\partial z} = 2(Az_1, Bz_2, Cz_3)$ never vanishes on M^2. The unit normal vector field is:

$$\xi = \frac{1}{||\frac{\partial f}{\partial z}||}\frac{\bar{\partial} f}{\partial z} = 2||\frac{\partial f}{\partial z}||^{-1}(A(a_1 - ib_1), B(a_2 - ib_2), C(a_3 - ib_3)), \tag{174}$$

where $||\frac{\partial f}{\partial z}|| = 2(A^2(a_1+b_1)^2 + B^2(a_2+b_2)^2 + C(a_3+b_3)^2)^{1/2}$. The conditions $g(X,\xi) = g(Y,\xi) = 0$ yield the equations

$$Aa_1\alpha_1 + Ba_2\alpha_2 + Ca_3\alpha_3 - Ab_1\beta_1 - Bb_2\beta_2 - Cb_3\beta_3 = 0, \tag{175}$$

$$Aa_1\gamma_1 + Ba_2\gamma_2 + Ca_3\gamma_3 - Ab_1\delta_1 - Bb_2\delta_2 - Cb_3d_3 = 0. \tag{176}$$

The conditions $g(X, J\xi) = g(Y, J\xi) = 0$, yield the equations:

$$Ab_1\alpha_1 + Bb_2\alpha_2 + Cb_3\alpha_3 + Aa_1\beta_1 + Ba_2\beta_2 + Ca_3\beta_3 = 0, \tag{177}$$

$$Ab_1\gamma_1 + Bb_2\gamma_2 + Cb_3\gamma_3 + Aa_1\delta_1 + Ba_2\delta_2 + Ca_3\delta_3 = 0. \tag{178}$$

Let us compute now the matrix:

$$\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k} = 2diag\{A, B, C\}. \tag{179}$$

By direct computation we see that $g(A_\xi X, X) + g(A_\xi Y, Y) = 0$ is equivalent to

$$A(\alpha_1^2 - \beta_1^2 + \gamma_1^2 - \delta_1^2) + B(\alpha_2^2 - \beta_2^2 + \gamma_2^2 - \delta_2^2) + C(\alpha_3^2 - \beta_3^2 + \gamma_3^2 - \delta_3^2) = 0. \tag{180}$$

Using the properties of the complex structure J, from

$$g(A_{J\xi}X, X) + g(A_{J\xi}Y, Y) = 0,$$

we get

$$g(A_\xi JX, X) + g(A_\xi JY, Y) = 0. \tag{181}$$

This equation can be written as

$$A\alpha_1\beta_1 + B\alpha_2\beta_2 + C\alpha_3\beta_3 + A\gamma_1\delta_1 + B\gamma_2\delta_2 + C\gamma_3\delta_3 = 0. \tag{182}$$

Besides the above presented equations, we have also the constraints that describes that $p = (a_1 + ib_1, a_2 + ib_2, a_3 + ib_3)$. These two relations are

$$A(a_1^2 - b_1^2) + B(a_2^2 - b_2^2) + C(a_3^2 - b_3^2) = 1, \tag{183}$$

$$Aa_1b_1 + Ba_2b_2 + Ca_3b_3 = 0. \tag{184}$$

Therefore we have proved the following.

Proposition 6.1 *The manifold $M^2 = \{z \in \mathbb{C}^3 / Az_1^2 + Bz_2^2 + Cz_3^2 = 1\}$ is strongly minimal in $\mathbb{C}^3$ if and only if the following system of equations admits a solution in α, β, γ, δ :*

$$\langle \alpha, \gamma \rangle + \langle \beta, \delta \rangle = 0, \tag{185}$$

$$\langle \alpha, \delta \rangle = \langle \beta, \gamma \rangle, \tag{186}$$

$$||\alpha||^2 + ||\beta||^2 = ||\gamma||^2 + ||\delta||^2 = 1, \tag{187}$$

$$Aa_1\alpha_1 + Ba_2\alpha_2 + Ca_3\alpha_3 - Ab_1\beta_1 - Bb_2\beta_2 - Cb_3\beta_3 = 0, \tag{188}$$

$$Aa_1\gamma_1 + Ba_2\gamma_2 + Ca_3\gamma_3 - Ab_1\delta_1 - Bb_2\delta_2 - Cb_3d_3 = 0. \tag{189}$$

$$Ab_1\alpha_1 + Bb_2\alpha_2 + Cb_3\alpha_3 + Aa_1\beta_1 + Ba_2\beta_2 + Ca_3\beta_3 = 0, \tag{190}$$

$$Ab_1\gamma_1 + Bb_2\gamma_2 + Cb_3\gamma_3 + Aa_1\delta_1 + Ba_2\delta_2 + Ca_3\delta_3 = 0. \tag{191}$$

$$A(\alpha_1^2 - \beta_1^2 + \gamma_1^2 - \delta_1^2) + B(\alpha_2^2 - \beta_2^2 + \gamma_2^2 - \delta_2^2) + C(\alpha_3^2 - \beta_3^2 + \gamma_3^2 - \delta_3^2) = 0. \tag{192}$$

$$A(\alpha_1\beta_1 + \gamma_1\delta_1) + B(\alpha_2\beta_2 + \gamma_2\delta_2) + C(\alpha_3\beta_3 + \gamma_3\delta_3) = 0 \tag{193}$$

This result is needed to prove the following.

Proposition 6.2 *On the complex manifold* $M^2 = \{z \in \mathbb{C}^3 / Az_1^2 + Bz_2^2 + Cz_3^2 = 1, A \neq 0, B \neq 0\}$ *there exists points of strong minimality. At these points* $\delta_4^r = 0$.

Proof: Inspired by the solution given by B.-Y.Chen in [22], we will look for solutions satisfying the conditions: $\alpha_2 = 0$, $\beta_1 = \beta_3 = 0$, $\gamma_1 = \gamma_3 = 0$, $\delta_2 = 0$. Two of the equations of the system vanish identically. The other eight equations are:

$$\alpha_1\delta_1 + \alpha_3\delta_3 = \beta_2\gamma_2, \tag{194}$$

$$\alpha_1^2 + \alpha_3^2 + \beta_2^2 = 1, \tag{195}$$

$$\gamma_2^2 + \delta_1^2 + \delta_3^2 = 1, \tag{196}$$

$$Aa_1\alpha_1 + Ca_3\alpha_3 - Bb_2\beta_2 = 0, \tag{197}$$

$$Ba_2\gamma_2 - Ab_1\delta_1 - Cb_3\delta_3 = 0, \tag{198}$$

$$Ab_1\alpha_1 + Cb_3\alpha_3 + Ba_2\beta_2 = 0, \tag{199}$$

$$Bb_2\gamma_2 + Aa_1\delta_1 + Ca_3\delta_3 = 0, \tag{200}$$

$$A(\alpha_1^2 - \delta_1^2) + B(\gamma_2^2 - \beta_2^2) + C(\alpha_3^2 - \delta_3^2) = 0. \tag{201}$$

Let us consider points with $a = (a_1, 0, 0)$, and $b = (0, b_2, 0)$. The constraint is $Aa_1^2 - Bb_2^2 = 1$. Let us assume further that $a_1 \neq 0$, $b_2 \neq 0$.

With this assumption, two of the equations vanish identically. The remaining system has six equations:

$$\alpha_1\delta_1 + \alpha_3\delta_3 = \beta_2\gamma_2, \tag{202}$$

$$\alpha_1^2 + \alpha_3^2 + \beta_2^2 = 1, \tag{203}$$

$$\gamma_2^2 + \delta_1^2 + \delta_3^2 = 1, \tag{204}$$

$$Aa_1\alpha_1 - Bb_2\beta_2 = 0, \tag{205}$$

$$Bb_2\gamma_2 + Aa_1\delta_1 = 0, \tag{206}$$

$$A(\alpha_1^2 - \delta_1^2) + B(\gamma_2^2 - \beta_2^2) + C(\alpha_3^2 - \delta_3^2) = 0. \tag{207}$$

Let us solve for α_1 in (205) and for δ_1 in (206):

$$\alpha_1 = \frac{Bb_2\beta_2}{Aa_1}, \tag{208}$$

$$\delta_1 = -\frac{Bb_2\gamma_2}{Aa_1}. \tag{209}$$

We obtain a system with four equations and with four unknowns, α_3, β_2, γ_2 and δ_3, as follows.

$$-\frac{B^2b_2^2\beta_2\gamma_2}{A^2a_1^2} + \alpha_3\delta_3 = \beta_2\gamma_2, \tag{210}$$

$$\frac{B^2b_2^2\beta_2^2}{A^2a_1^2} + \alpha_3^2 + \beta_2^2 = 1, \tag{211}$$

$$\gamma_2^2 + \frac{B^2b_2^2\gamma_2^2}{A^2a_1^2} + \delta_3^2 = 1, \tag{212}$$

$$\frac{B^2b_2^2}{Aa_1^2}(\beta_2^2 - \gamma_2^2) + B(\gamma_2^2 - \beta_2^2) + C(\alpha_3^2 - \delta_3^2) = 0. \tag{213}$$

This system can be written equivalently as:

$$\alpha_3\delta_3 = \left(1 + \frac{B^2b_2^2}{A^2a_1^2}\right)\beta_2\gamma_2, \tag{214}$$

$$\left(1 + \frac{B^2b_2^2}{A^2a_1^2}\right)\beta_2^2 + \alpha_3^2 = 1, \tag{215}$$

$$\left(1 + \frac{B^2b_2^2}{A^2a_1^2}\right)\gamma_2^2 + \delta_3^2 = 1, \tag{216}$$

$$(\beta_2^2 - \gamma_2^2)\left(\frac{B^2b_2^2}{Aa_1^2} - B\right) + C(\alpha_3^2 - \delta_3^2) = 0. \tag{217}$$

In this setting the constraint is $Bb_2^2 + 1 = Aa_1^2$. This system admits the solution (in particular for $A = 1$ and $B = 1$ the result is consistent to the one obtained by B.-Y.Chen in [22] for all the points of the surface with $A = B = C = 1$)

$$\alpha = \left(\frac{Bb_2}{\sqrt{2(A^2a_1^2 + B^2b_2^2)}}, 0, \frac{1}{\sqrt{2}} \right), \tag{218}$$

$$\beta = \left(0, \frac{Aa_1}{\sqrt{2(A^2a_1^2 + B^2b_2^2)}}, 0 \right), \tag{219}$$

$$\gamma = \left(0, \frac{Aa_1}{\sqrt{2(A^2a_1^2 + B^2b_2^2)}}, 0 \right), \tag{220}$$

$$\delta = \left(-\frac{Bb_2}{\sqrt{2(A^2a_1^2 + B^2b_2^2)}}, 0, \frac{1}{\sqrt{2}} \right). \tag{221}$$

This concludes the proof of Proposition 6.2.

If the previous result proved the existence of points satisfying the strong minimality condition in $\mathbb{C}^3$ for a specific class of surfaces, the next Theorem describes an example where the strong minimality holds at every point.

Theorem 6.3 *The Kähler submanifold given by*

$$M^2 = \{z \in \mathbb{C}^3 / z_1 + z_2 + z_3^2 = \kappa, \kappa \in \mathbb{C}\}$$

is strongly minimal in $\mathbb{C}^3$.

Proof: We use consistently the notations from [22], as well as everywhere in this current section. Let $f(z_1, z_2, z_3) = z_1 + z_2 + z_3^2$, then $\frac{\partial f}{\partial z} = (1, 1, 2z_3)$. For $z = (z_1, z_2, z_3) = (a_1 + ib_1, a_2 + ib2, a_3 + ib_3)$, we get $||\frac{\partial f}{\partial z}|| = (2 + 4(a_3^2 + b_3^2))^{1/2} > 0$.

We compute also $\frac{\partial^2 f}{\partial z_j \partial z_k} = diag(0, 0, 2)$.

The submanifold is strongly minimal if and only if at every point z there exists two vectors X and Y such that

$$g(X, X) = g(Y, Y) = 1, \tag{222}$$

$$g(X, Y) = g(X, JY) = 0, \tag{223}$$

$$g(X, \xi) = g(X, J\xi) = g(Y, \xi) = g(Y, J\xi) = 0, \tag{224}$$

$$g(A_\xi X, X) + g(A_\xi Y, Y) = 0, \tag{225}$$

$$g(A_{J\xi} X, X) + g(A_{J\xi} Y, Y) = 0. \tag{226}$$

From the first equation of the system we keep just $g(X, X) = g(Y, Y)$, and in the last step we will normalize the basis obtained. With this adjustment, the system becomes, using the same notation convention as before, i.e. $X = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$, $Y = (\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3)$

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \delta_1^2 + \delta_2^2 + \delta_3^2, \tag{227}$$

$$\langle \alpha, \gamma \rangle + \langle \beta, \delta \rangle = 0, \tag{228}$$

$$\langle \alpha, \delta \rangle = \langle \beta, \gamma \rangle, \tag{229}$$

$$\alpha_1 + \alpha_2 + 2a_3\alpha_3 - 2b_3\beta_3 = 0, \tag{230}$$

$$\beta_1 + \beta_2 + 2a_3\beta_3 + 2b_3\alpha_3 = 0, \tag{231}$$

$$\gamma_1 + \gamma_2 + 2a_3\gamma_3 - 2b_3\delta_3 = 0, \tag{232}$$

$$\delta_1 + \delta_2 + 2a_3\delta_3 + 2b_3\gamma_3 = 0, \tag{233}$$

$$\alpha_3^2 - \beta_3^2 + \gamma_3^2 - \delta_3^2 = 0, \tag{234}$$

$$\alpha_3\beta_3 + \gamma_3\delta_3 = 0. \tag{235}$$

A solution for this system is obtained if one is taking $\beta_3 = \gamma_3 = 0$. With this choice, one may elliminate the unknown

$$\alpha_1 = -\alpha_2 - 2a_3\alpha_3 \qquad \beta_1 = -\beta_2 - 2b_3\alpha_3, \tag{236}$$

$$\gamma_1 = -\gamma_2 + 2b_3\delta_3 \qquad \delta_1 = -\delta_2 - 2a_3\delta_3. \tag{237}$$

Setting $\alpha_2 = 0$, $\alpha_3 = 1$ and $\delta_3 = -1$ one may get, through a direct computation:

$$\alpha = (-2a_3, 0, 1), \tag{238}$$

$$\beta = (-b_3 - \frac{\sqrt{2 + 4b_3^2}}{2}, -b_3 + \frac{\sqrt{2 + 4b_3}}{2}, 0), \tag{239}$$

$$\gamma = (-b_3 + \frac{\sqrt{2 + 4b_3^2}}{2}, -b_3 - \frac{\sqrt{2 + 4b_3^2}}{2}, 0), \tag{240}$$

$$\delta = (0, 2a_3, -1). \tag{241}$$

One may verify directly that the system (227)-(235) is satisfied by the above solution. Therefore, after normalization, the basis satisfying the strong minimality condition is

$$X = (2 + 4a_3^2 + 4b_3^2)^{-1/2} \left(-2a_3 - i(b_3 + \frac{\sqrt{2 + 4b_3^2}}{2}), i(-b_3 + \frac{\sqrt{2 + 4b_3^2}}{2}), 1 \right), \quad (242)$$

$$Y = (2 + 4a_3^2 + 4b_3^2)^{-1/2} \left(-b_3 + \frac{\sqrt{2 + 4b_3^2}}{2}, -b_3 - \frac{\sqrt{2 + 4b_3^2}}{2} + 2a_3 i, -i \right). \quad (243)$$

6.2 $\delta_4^r = 0$ on degree three complex surfaces

The surfaces $z_1 + z_2 + z_3 = 1$ and $z_1^2 + z_2^2 + z_3^2 = 1$, as we have seen, are strongly minimal. We mentioned above sufficient conditions for $Az_1^2 + Bz_2^2 + Cz_3^2 = 1$ to be strongly minimal. Let us extend our discussion to complex hypersurfaces of higher algebraic degree; we prove the following.

Theorem 6.4 *On the complex surface M given by the algebraic equation $z_1^3+z_2^3+z_3^3 = 1$ there exists points where the strongly minimality condition in $\mathbb{C}^3$ is satisfied.*

Proof: Using the notations from [22], we get $f(z) = z_1^3 + z_2^3 + z_3^3 - 1$; therefore

$$\frac{\partial f}{\partial z} = (3z_1^2, 3z_2^2, 3z_3^2).$$

Let us prove first that $||\frac{\partial f}{\partial z}|| \neq 0$ at every point p. We have

$$||\frac{\partial f}{\partial z}|| = 3[(x_1^2 - y_1^2)^2 + (x_2^2 - y_2^2)^2 + (x_3^2 - y_3^2)^2 + 4x_1^2y_1^2 + 4x_2^2y_2^2 + 4x_3^2y_3^2] =$$

$$3[(x_1^2+y_1^2)^2+(x_2^2+y_2^2)^2+(x_3^2+y_3^2)^2>0.$$

The unit normal vector ξ (written as a real vector field) is given by

$$\xi=\left(\sum_{i=1}^{3}(x_i^2+y_i^2)^2\right)^{-1/2}(x_1^2-y_1^2,x_2^2-y_2^2,x_3^2-y_3^2,-2x_1y_1,-2x_2y_2,-2x_3y_3). \quad (244)$$

Let us consider a point p in $\mathbb{C}^3$ whose position vector is given by $(a_1+ib_1,a_2+ib_2,a_3+ib_3)$. The tangent space to our complex surface M at p is the set of all vectors of the form

$$Z=(u_1+iv_1,u_2+iv_2,u_3+iv_3)$$

which satisfy the following conditions:

$$g\left(Z,\frac{\bar{\partial} f}{\partial z}\right)=0 \quad g\left(Z,i\frac{\bar{\partial} f}{\partial z}\right)=0,$$

and these conditions yield the equations:

$$u_1(x_1^2-y_1^2)+u_2(x_2^2-y_2^2)+u_3(x_3^2-y_3^2)-2x_1y_1v_1-2x_2y_2v_2-2x_3y_3v_3=0, \quad (245)$$

$$2u_1x_1y_1+2u_2x_2y_2+2u_3x_3y_3+v_1(x_1^2-y_1^2)+v_2(x_2^2-y_2^2)+v_3(x_3^2-y_3^2)=0. \quad (246)$$

The condition that the point $p=(z_1,z_2,z_3)$ lies on the complex surface M is expressed by the following two equations:

$$\sum_{j=1}^{3}(a_j^3-3a_jb_j^2)=1, \quad (247)$$

$$\sum_{j=1}^{3}(3a_j^2b_j-b_j^3)=0. \quad (248)$$

Now let us study the shape operator, using the formula (discussed in [70] and applied in a similar setting in [22]):

$$A_\xi W = -||\frac{\partial f}{\partial z}||^{-1} \left\{ \bar{W} \left(\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k} \right) \right\}^{tan} \tag{249}$$

First note that

$$\frac{\partial^2 f}{\partial z_j \partial z_k} = \begin{pmatrix} 6z_1 & 0 & 0 \\ 0 & 6z_2 & 0 \\ 0 & 0 & 6z_3 \end{pmatrix} \tag{250}$$

The product in braces yields

$$\bar{X}\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k} = (6(\alpha_1 - i\beta_1)(x_1 - iy_1), 6(\alpha_2 - i\beta_2)(x_2 - iy_2), 6(\alpha_3 - i\beta_3)(x_3 - iy_3)) = \tag{251}$$

$$(\ 6(\alpha_1 x_1 - \beta_1 y_1) - 6i(\alpha_1 y_1 + \beta_1 x_1), 6(\alpha_2 x_2 - \beta_2 y_2) - 6i(\alpha_2 y_2 + \beta_2 x_2),$$

$$6(\alpha_3 x_3 - \beta_3 y_3) - 6i(\alpha_3 y_3 + \beta_3 x_3)\ ).$$

The first condition for strong minimality is $g(A_\xi X, X) + g(A_\xi Y, Y) = 0$:

$$g\left(\left\{\bar{X}\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k}\right\}^{tan}, X\right) + g\left(\left\{\bar{Y}\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k}\right\}^{tan}, Y\right) = 0 \tag{252}$$

This equation can be written also as

$$\sum_{j=1}^{3} \left[(\alpha_j^2 - \beta_j^2)a_j - 2\alpha_j\beta_j b_j + (\gamma_j^2 - \delta_j^2)a_j - 2\gamma_j\delta_j b_j\right] = 0 \tag{253}$$

The second condition for strong minimality is $g(A_{J\xi}X, X) + g(A_{J\xi}Y, Y) = 0$. Since $A_{J\xi} = JA_\xi = -JA_\xi$ (see for example [61]), this is the same as discussing: $g(A_\xi JX, X) + g(A_\xi JY, Y) = 0$. This can be written as

$$g\left(\left\{\bar{JX}\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k}\right\}^{tan}, X\right) + g\left(\left\{\bar{JY}\frac{\partial^{\bar{2}} f}{\partial z_j \partial z_k}\right\}^{tan}, Y\right) = 0 \tag{254}$$

or, breaking down the computation, as

$$\sum_{j=1}^{3}[\alpha_j(-a_j\beta_j - \alpha_j b_j) + \beta_j(\beta_j b_j - \alpha_j a_j) + \gamma_j(-a_j\delta_j - \gamma_j b_j) + \delta_j(\delta_j b_j - \gamma_j a_j)] = 0 \tag{255}$$

and, grouping terms as in the similar relation above, we get

$$\sum_{j=1}^{3}[-2a_j\alpha_j\beta_j + b_j(\beta_j^2 - \alpha_j^2) - 2a_j\gamma_j\delta_j + b_j(\delta_j^2 - \gamma_j^2)] = 0. \tag{256}$$

In fact, to prove the strongly minimality of M in $\mathbb{C}^3$ is equivalent to find an orthonormal basis $\{e_1, e_2\}$, $e_1 = (\alpha + i\beta)$, $e_2 = (\gamma + i\delta)$, satisfying the following system:

$$\alpha_1(a_1^2 - b_1^2) + \alpha_2(a_2^2 - b_2^2) + \alpha_3(a_3^2 - b_3^2) - 2a_1b_1\beta_1 - 2a_2b_2\beta_2 - 2a_3b_3\beta_3 = 0, \tag{257}$$

$$\gamma_1(a_1^2 - b_1^2) + \gamma_2(a_2^2 - b_2^2) + \gamma_3(a_3^2 - b_3^2) - 2a_1b_1\delta_1 - 2a_2b_2\delta_2 - 2a_3b_3\delta_3 = 0 \tag{258}$$

$$2\alpha_1a_1b_1 + 2\alpha_2a_2b_2 + 2\alpha_3a_3b_3 + \beta_1(a_1^2 - b_1^2) + \beta_2(a_2^2 - b_2^2) + \beta_3(a_3^2 - b_3^2) = 0, \tag{259}$$

$$2\gamma_1a_1b_1 + 2\gamma_2a_2b_2 + 2\gamma_3a_3b_3 + \delta_1(a_1^2 - b_1^2) + \delta_2(a_2^2 - b_2^2) + \delta_3(a_3^2 - b_3^2) = 0, \tag{260}$$

$$\sum_{j=1}^{3}\left[(\alpha_j^2 - \beta_j^2)a_j - 2\alpha_j\beta_j b_j + (\gamma_j^2 - \delta_j^2)a_j - 2\gamma_j\delta_j b_j\right] = 0, \tag{261}$$

$$\sum_{j=1}^{3}[-2a_j\alpha_j\beta_j + b_j(\beta_j^2 - \alpha_j^2) - 2a_j\gamma_j\delta_j + b_j(\delta_j^2 - \gamma_j^2)] = 0. \tag{262}$$

$$\langle\alpha, \gamma\rangle + \langle\beta, \delta\rangle = 0, \tag{263}$$

$$\langle\alpha, \delta\rangle = \langle\beta, \gamma\rangle, \tag{264}$$

$$\alpha_1^2+\alpha_2^2+\alpha_3^2+\beta_1^2+\beta_2^2+\beta_3^2=\gamma_1^2+\gamma_2^2+\gamma_3^2+\delta_1^2+\delta_2^2+\delta_3^2=1 \tag{265}$$

$$a_1^3-3a_1b_1^2+a_2^3-3a_2b_2+a_3^3-3a_3b_3^2=1, \tag{266}$$

$$3a_1^2b_1-b_1^3+3a_2^2b_2-b_2^3+3a_3^2b_3-b_3^3=0, \tag{267}$$

Given two vectors $a, b \in \mathbb{R}^3$ with constraints (266), (267), the equations above (257)-(265) are an undetermined system which admits some nontrivial solutions $\alpha, \beta, \gamma, \delta \in \mathbb{R}^3$.

At the point $(1,0,0)$ the system implies from relation (257) that $\alpha_1 = 0$. From this and the next three relations we deduce that: $\alpha = (0, \alpha_2, \alpha_3)$, $\beta = (0, \beta_2, \beta_3)$, $\gamma = (0, \gamma_2, \gamma_3)$, $\delta = (0, \delta_2, \delta_3)$.

The system we need to solve now is

$$||\alpha||^2+||\beta||^2=1, \tag{268}$$

$$||\gamma||^2+||\delta||^2=1, \tag{269}$$

$$\langle\alpha,\gamma\rangle+\langle\beta,\delta\rangle=0, \tag{270}$$

$$\langle\alpha,\delta\rangle=\langle\beta,\gamma\rangle\,. \tag{271}$$

This system admits the solution:

$$\alpha=\left(0,\frac{1}{\sqrt{2}},0\right), \qquad \beta=\left(0,0,\frac{1}{\sqrt{2}}\right) \tag{272}$$

$$\gamma = \left(0, \frac{1}{\sqrt{2}}, 0\right), \qquad \delta = \left(0, 0, -\frac{1}{\sqrt{2}}\right). \tag{273}$$

This means the strongly minimality condition is satisfied at $(1, 0, 0)$.

For the points of type $(a_1, 0, 0)$, $(0, a_2, 0)$, $(0, 0, a_3)$ a similar system admits the same solutions. Therefore along these curves the stronlgy minimality condition is satisfied on the orthonormal basis $e_1 = \alpha + i\beta$, $e_2 = \gamma + i\delta$. This concludes the proof of theorem 6.4.

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