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Stanislav Jabuka

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GRAFTING SEIBERG-WITTEN MONOPOLES

By

Stanislav Jabuka

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ABSTRACT

GRAFTING SEIBERG-WITTEN MONOPOLES

By

Stanislav Jabuka

We demonstrate that the operation of taking disjoint unions of J-holomorphic curves (and thus obtaining new J-holomorphic curves) has a Seiberg-Witten counterpart. The main theorem (theorem 5.10) asserts that, given two solutions (A_i, ψ_i), $i = 0, 1$ of the Seiberg-Witten equations for the $Spin^c$-structures $W_{E_i} = E_i \oplus (E_i \otimes K^{-1})$ (with certain restrictions), there is a solution (A, ψ) of the Seiberg-Witten equations for the $Spin^c$-structure W_E with $E = E_0 \otimes E_1$, obtained by "grafting" the two solutions (A_i, ψ_i).

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1 Introduction

In his series of groundbreaking works [12, 13, 14], Taubes showed that the Seiberg-Witten invariants and the Gromov-Witten invariants (as defined in [15]) for a symplectic 4-manifold (X, ω) are the same. His results opened the door to a whole new world of interactions between the two theories that had previously only been speculations. The most spectacular outcomes of this interplay were new results that in one theory were obvious but when translated into the other theory, became highly nontrivial. An example of such a phenomenon is the simple formula relating the Seiberg-Witten invariant of a $Spin^c$-structure W to the Seiberg-Witten invariant of its dual $Spin^c$-structure W^*, i.e. the one with $c_1(W^*) = -c_1(W)$. The formula reads

$$SW_X(W^*) = \pm SW_X(W)$$

When translated into the Gromov-Witten language, this duality becomes

$$Gr_X(E) = \pm Gr_X(K - E) \tag{1}$$

Here K is the canonical class of (X, ω) and $E \in H^2(X; \mathbb{Z})$ is related to W as $c_1(W^+) = 2E - K$. This is a highly nonobvious result about J-holomorphic curves, even in the simplest case when $E = 0$. In that case we obtain that $Gr_X(K) = \pm Gr_X(0) = \pm 1$, the latter equation simply being the definition of $Gr_X(0)$. This gives an existence result of a J-holomorphic representative for the class K, a result unknown prior to Taubes' theorem. The formula (1) has recently been proved by S. Donaldson and I. Smith [4] without any reference to Seiberg-Witten theory (but under slightly stronger restrictions on (X, ω) than in Taubes' theorem).

In the author's opinion, proving a result about Gromov-Witten theory which had only been known through its relation with Seiberg-Witten theory, without relying on the latter, has a number of benefits. One is to understand Gromov-Witten theory from within better. But also to possibly generalize the theorem to a broader class of manifolds. Recall that Taubes' theorem equates the two invariants only on symplectic 4-manifolds. Both Seiberg-Witten and Gromov-Witten theory are defined over larger sets of manifolds, namely all smooth 4-manifolds and all symplectic manifolds (of any dimension) respectively. On the other hand, even within the category of symplectic 4-manifolds, one can hope for more nonvanishing theorems i.e. theorems of the type $Gr_X(E) \neq 0$ for classes $E \neq 0,\ K$. The techniques used by Donaldson and Smith are promising in that direction.

The main result of this thesis is to prove an assertion in the same vein but going the opposite direction. Namely, on the Gromov-Witten side, given two classes $E_i \in H^2(X;\mathbb{Z})$, $i = 0, 1$ with $E_0 \cdot E_1 = 0$ and J-holomorphic curves Σ_i with $[\Sigma_i] = P.D.(E_i)$, one can define a new J-holomorphic curve $\Sigma = \Sigma_0 \sqcup \Sigma_1$. By the assumption $E_0 \cdot E_1 = 0$, the two curves Σ_i are either disjoint or share toroidal components (see [5]). In the former case, Σ is simply the disjoint union of Σ_0 and Σ_1 and in the latter case one needs to replace the shared tori with their appropriate multiple covers. This induces a map on moduli spaces

$$\mathcal{M}_X^{Gr}(E_0) \times \mathcal{M}_X^{Gr}(E_1) \xrightarrow{\sqcup} \mathcal{M}_X^{Gr}(E_0 + E_1) \tag{2}$$

This thesis describes the Seiberg-Witten counterpart of (2). That is, given two complex line bundles E_0 and E_1 (with certain restrictions, see theorem 5.10 for a precise statement) and two solutions (A_i, ψ_i) of the Seiberg-Witten equations for the $Spin^c$-structures $W_{E_i} = E_i \oplus (E_i \otimes K^{-1})$, $i = 0, 1$, with Taubes' large r perturbation, we show how to produce a solution $(A, \psi) = (A_0, \psi_0) \cdot (A_1, \psi_1)$ for the $Spin^c$-structure W_E with $E = E_0 \otimes E_1$. We say that (A, ψ) was obtained by *grafting* the two solutions (A_i, ψ_i), the choice of this term will be justified by the construction of (A, ψ) described in section 5. The operation of grafting induces the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_X^{SW}(E_0) \times \mathcal{M}_X^{SW}(E_1) & \xrightarrow{\cdot} & \mathcal{M}_X^{SW}(E_0 \otimes E_1) \\ \Theta \downarrow & & \downarrow \Theta \\ \mathcal{M}_X^{Gr}(E_0) \times \mathcal{M}_X^{Gr}(E_1) & \xrightarrow{\sqcup} & \mathcal{M}_X^{Gr}(E_0 + E_1) \end{array} \tag{3}$$

Here the map $\Theta : \mathcal{M}_X^{SW}(E) \to \mathcal{M}_X^{Gr}(E)$ is the map described in [12] that associates to each solution of the Seiberg-Witten equations an embedded J-holomorphic curve. We call the map in the top row of (3) the *grafting map*. The monopole (A, ψ) is constructed out of the the two monopoles (A_i, ψ_i). The key observation here is that for the large r version of Taubes' perturbation, a solution (B, ϕ) of the Seiberg-Witten equations for the $Spin^c$-structure W_E is "concentrated"near the zero set of $\sqrt{r}\,\alpha$, the E component of ϕ. That is, the restriction of (B, ϕ) to the complement of a regular neighborhood of $\alpha^{-1}(0)$ converges pointwise (under certain bundle identifications) to the unique solution $(\mathcal{A}_0, \sqrt{r}\,u_0)$ for the anticanonical $Spin^c$-structure $W_0 = \underline{\mathbb{C}} \oplus K^{-1}$. This is used to define a first approximation of ψ by declaring it to be equal to ψ_i in a regular neighborhood V_i of $\alpha_i^{-1}(0)$ and equal to $\sqrt{r}\,u_0$ on the complement of $V_0 \cup V_1$.

Bump functions are used to produce a smooth spinor. The first approximation of A is simply the product connection $A_0 \otimes A_1$. The contraction mapping principle is then evoked to deform this approximate solution to an honest solution of the Seiberg-Witten equations. The author has learned the techniques employed in this article from the inspiring work of Taubes on gauge theory of symplectic 4-manifolds, most notably from [13].

The thesis is organized as follows. In sections 2 and 3 we review the basics of Seiberg-Witten and Gromov-Witten theory. Since the emphasis of the thesis is mainly on Seiberg-Witten theory, most claims in section 2 come with proofs while in section 3 we refer the interested reader to the available literature. Section 4 points out the specifics of Seiberg-Witten theory on symplectic 4-manifolds. It explains important bounds that a Seiberg-Witten monopole satisfies and that will be used amply in the later chapters. It also explains Taubes' theorem equating the two invariants on symplectic manifolds and proves some important corollaries. Section 5 contains the bulk of the work presented here. It explains how to define an "almost"monopole (A', ψ') from a pair of monopoles (A_i, ψ_i), $i = 0, 1$. It analyzes the asymptotic (as $r \to \infty$) regularity theory for the linearized operators $L_{(A_i,\psi_i)}$ and deduces a corresponding result for $L_{(A',\psi')}$. The latter is used in combination with the contraction mapping principle to obtain an "honest"monopole (A, ψ). Section 6 compares the present method of grafting monopoles to the one used in exploring Seiberg-Witten theory on manifolds X which are obtained as a fiber sum: $X = X_1 \#_\Sigma X_2$. Finally, section 7 proves a converse to theorem 5.10. It explains which monopoles in the $Spin^c$-structure

W_E can be obtained as products of monopoles (A_i, ψ_i) in the $Spin^c$-structures W_{E_i}, $i = 0, 1$ with $E_0 \otimes E_1 = E$.

2 Seiberg-Witten Theory

This section is an introduction to Seiberg-Witten theory. If defines all basic concepts and provides the statements and proofs of the bare bone theorems needed to get the gauge theory machinery going. The author has learned most of the material in this section from professor Tom Parker during a one-semester course on Seiberg-Witten theory taught at Michigan State University, as well as from his excellent accompanying notes [10]. The exposition in this section relies heavily on these notes and I would like to take this opportunity to express my gratitude to him for having done a superb job.

2.1 $Spin^c$-structures on 4-manifolds

The Seiberg-Witten equations are a pair of coupled, partial differential elliptic equations for a pair consisting of a connection A and a *positive spinor* ψ. This subsection is concerned with defining the notion of a positive (as well as a negative) spinor.

Definition 2.1 *Let V be a (real) vector space of dimension n and let $g : Sym^2(V) \longrightarrow \mathbb{R}_0^+$ be a metric on V. The Clifford algebra $\mathcal{C}(V, g)$ associated to the pair (V, g) is the algebra whose underlying vector space is*

$$\mathcal{C}(V, g) = \oplus_{i=0}^{n} V^{\otimes i}$$

and whose multiplication law is subject to the relation $x \cdot y + y \cdot x = -2g(x, x)\,1$ for all $x, y \in V$ (here 1 is to be viewed as $1 \in \mathbb{R} = V^{\otimes 0} \subseteq \mathcal{C}(V, g)$).

In particular, if $e_1, ..., e_n$ is an orthonormal basis of V, then $\mathcal{C}(V,g)$ is generated by elements of the form $e_{i_1} \cdot ... \cdot e_{i_k}$ with $i_1 < ... < i_k$ and $0 \leq k \leq n$ (with $e_0 = 1$). This implies that the dimension of $\mathcal{C}(V,g)$ is 2^n. Notice also that the e_i satisfy the relations

$$e_i \cdot e_i = -1 \qquad \text{and} \qquad e_i \cdot e_j = -e_j \cdot e_i \quad \text{for } i \neq j \tag{4}$$

As vector spaces, $\mathcal{C}(V,g)$ and $\oplus_{i=0}^n \Lambda^i V$ are isomorphic but not so as algebras, the reason being the first relation in (4) which in $\oplus_{i=0}^n \Lambda^i V$ doesn't hold.

Example 2.2 *The Clifford algebras of Euclidean spaces are well known:*

n	$\mathcal{C}(\mathbb{R}^n, g)$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \times \mathbb{H}$
4	$M_{\mathbb{H}}(2)$
5	$M_{\mathbb{C}}(4)$
6	$M_{\mathbb{R}}(8)$
7	$M_{\mathbb{R}}(8) \oplus M_{\mathbb{R}}(8)$
8	$M_{\mathbb{R}}(16)$
$n+8$	$\mathcal{C}(\mathbb{R}^n, g) \otimes M_{\mathbb{R}}(16)$

In the above table, g denotes the Euclidean metric on $\mathbb{R}^n$ while $M_{\mathbb{F}}(k)$ are the $k \times k$ matrices with entries belonging to the field $\mathbb{F}$.

The definition of the Clifford algebra associated to a vector space extends without difficulty to vector bundles $V \to X$ over manifolds, giving rise to the Clifford algebra bundle (which we will still denote by $\mathcal{C}(V,g)$). The most important example for us will be that of the tangent bundle TX of a Riemannian manifold X. We will denote the associated Clifford algebra bundle simply by $\mathcal{C}(X)$ (suppressing the vector bundle and metric from the notation).

Recall that a Hermitian vector bundle is a complex vector bundle equipped with a Hermitian metric.

Definition 2.3 *A $Spin^c$-structure on a smooth 4-manifold X is a Hermitian, rank 4 vector bundle $W \to X$ together with a bundle map of algebras*

$$x \mapsto x. : \mathcal{C}(X) \to End(W) \tag{5}$$

called Clifford multiplication. We also demand that $(x.)^* = -x.$ for $x \in TX \subseteq \mathcal{C}(X)$ (here $*$ signifies taking the adjoint operator).

Since $\mathcal{C}(X)$ is generated by TX, it suffices to define Clifford multiplication on elements of TX subject to the relation $x.y. + y.x. = -2\langle x, y\rangle \,\mathrm{Id}$. This observation becomes particularly useful when trying to verify that a given bundle map $TX \to W$ defines a $Spin^c$-structure on X.

By abuse of terminology, we will often call the bundle $W \to X$ itself a $Spin^c$-structure, keeping the Clifford multiplication in the background.

Every $Spin^c$-structure $W \to X$ admits a splitting $W = W^+ \oplus W^-$ into two complex, rank 2 bundles called the *positive* and the *negative spinor bundles.* The splitting comes about as follows: for $x \in X$, let e_i, $i = 1, ..., 4$, be an orthonormal frame of TX in a neighborhood of x. It is easy to check that $e = (e_1.e_2.e_3.e_4.)$ is independent of the choice of orthonormal frame and satisfies the relation $e^2 = \mathrm{Id}$.

Thus, the eigenvalues of $e. \in \mathrm{End}(W_x)$ are ± 1 and $W_x^{\pm}$ are the associated eigenspaces, they fit together to give the bundles $W^{\pm}$.

In the special case when X admits an almost-complex structure J, there are two canonically defined *Spin*^{c}-structures called the *canonical* and *anti-canonical Spin*^{c}-structure of X. We will define below only the anti-canonical *Spin*^{c}-structure being the one we shall use in subsequent chapters. The definiton of the canonical *Spin*^{c}-structure is left as an (easy) exercise to the interested reader.

Definition 2.4 *The anti-canonical Spin*^{c}*-structure* $W_0 = W_0^+ \oplus W_0^- \to X$ *associated to an almost-complex structure* J *on* TX *compatible with the Riemannian metric* g *(i.e. such that* $g(v, J(v)) = 0$ *for all* $v \in TX$*), is defined to be*

$$W_0^+ = \Lambda^{0,0}(T^*X) \oplus \Lambda^{0,2}(T^*X) \qquad W_0^- = \Lambda^{0,1}(T^*X) \tag{6}$$

with Clifford multiplication given by

$$v.\alpha = \sqrt{2}\,\left(v^*_{0,1} \wedge \alpha - \iota_v\alpha\right) \qquad v \in \Gamma(TX),\ \alpha \in \Gamma(W_0) \tag{7}$$

In the above, $v^*_{0,1} = (v^* + iJ(v^*))/2 \in \Lambda^{0,1}(T^*X)$ *denotes the (0,1) projection of* $v^* \in T^*X$*, the dual of* $v \in TX$.

That (7) indeed defines a Clifford multiplication on W_0 follows from the remark after definition 2.3 and the following easy check:

$$\begin{aligned} v.v.\alpha &= \sqrt{2}v.(v^*_{0,1} \wedge \alpha - \iota_v\alpha) \\ &= -2\iota_v(v^*_{0,1} \wedge \alpha) - 2v^*_{0,1} \wedge (\iota_v\alpha) \\ &= -2[\iota_v(v^*_{0,1}) \wedge \alpha - v^*_{0,1} \wedge (\iota_v\alpha) + v^*_{0,1} \wedge (\iota_v\alpha) \end{aligned}$$

$$= -|v|^2 \alpha \qquad \forall\, v \in \Gamma(TX),\ \alpha \in \Gamma(W_0)$$

The anti-canonical $Spin^c$-structure receives its name from the fact that the determinant line bundle of W_0^+ is K^{-1}, the anti-canonical bundle of J. Notice also that definition 2.4 says that $W_0^+ = \underline{\mathbb{C}} \oplus K^{-1}$

The significance of W_0 is that all other $Spin^c$-structures of X can be obtained from W_0 by tensoring it with a complex line bundle E and extending Clifford multiplication trivially over the E factor, i.e.

$$W_E^{\pm} = E \otimes W_0^{\pm} \qquad \text{and} \qquad v.(\varphi \otimes \alpha) = \varphi \otimes (v.\alpha) \tag{8}$$

with $\varphi \in \Gamma(E)$, $v \in \Gamma(TX)$, $\alpha \in \Gamma(W_0)$. Since complex line bundles are classified by their first Chern class, it follows immediately that the space of $Spin^c$-structures on a manifold X admitting an almost-complex structure J, is in 1-1 correspondence with elements of $H^2(X;\mathbb{Z})$. This fact is still true even when such a J does not exist.

Another useful observation is that, for $v \in \Gamma(TX)$, the endomorphism $v.$ exchanges the parity of $W^{\pm}$, i.e. $v. \in \mathrm{End}(W^{\pm}, W^{\mp})$. In the presence of an almost-complex structure J, this is easily checked by direct computation for W_0 and it follows for all other $Spin^c$-structures W_E from (8). It still remains true even in the absence of an almost-complex structure.

2.2 The Seiberg-Witten equations

In this section we describe the Seiberg-Witten equations. The main ingredients are the Dirac operator and a certain quadratic map on a spinor bundle, both of which we now proceed to define.

To obtain the Dirac operator, we first need the following definition:

Definition 2.5 *Let $W \to X$ be a $Spin^c$-structure structure on X and let ∇^0 denote the Levi-Civita connection associated to the Riemannian metric g on X. A connection ∇ on W is called a $Spin^c$-connection if it is compatible with the Hermitian metric on W and if the product rule*

$$\nabla_w(v.\alpha) = \nabla^0_w(v).\alpha + v.\nabla_w(\alpha) \qquad \forall\, v, w \in \Gamma(TX),\ \alpha \in \Gamma(W) \tag{9}$$

holds.

It is a known fact that $Spin^c$-connections always exist (c.f. [3]). They are in 1-1 correspondence with connections on the determinant line bundle L of W, the correspondence being $\nabla \mapsto \nabla \wedge \nabla$ (as $L = \Lambda^2 W^+$). Given a connection A on L, we will label the corresponding $Spin^c$-connection by ∇^A. Observe that if A_1 and A_2 are two connections on L with $A_2 - A_1 = a \in i\Omega^1_X$, then $\nabla^{A_2} - \nabla^{A_1} = a/2$.

$Spin^c$-connections preserve the parity of the spinor bundles $W^\pm$, that is, if $\alpha \in \Gamma(W^\pm)$ and $v \in \Gamma(TX)$, then $\nabla_v(\alpha) \in \Gamma(W^\pm)$ as well. This can been seen as follows: recall the element $e = e_1.e_2.e_3.e_4. \in \mathcal{C}(X)$ defined for a local orthonormal frame e_i. Since the individual endomorphisms $e_i.$ change the parity of $W^\pm$, $e.$ will preserve the parity. Now, if $\alpha \in \Gamma(W^+)$, then $e.\alpha = \alpha$. On the other hand we have

$$e.\nabla_v(\alpha) - \nabla_v(e.\alpha) = e.\nabla_v(\alpha) - \nabla_v(\alpha) = \nabla_v(e).\alpha \in \Gamma(W^+)$$

Write $\nabla_v(\alpha) = \beta^+ + \beta^-$ with $\beta^\pm \in W^\pm$. Then $e.\beta^\pm = \pm\beta^\pm$ and so the above equation reads

$$\beta^+ - \beta^- - (\beta^+ + \beta^-) = -2\beta^- \in W^+$$

which immediately gives $\beta^- = 0$ and thus $\nabla_v(\alpha) = \beta^+ \in W^+$. The case $\alpha \in W^-$ is treated in the same way.

Definition 2.6 *Let $W \to X$ be a $Spin^c$-structure and let $x \in X$ be an arbitrary point. Let e_i, $i = 1, ..., 4$ be an orthonormal frame in a neighborhood U of x. For a connection A on L =det(W^+), we define the operator D_A for a section $\alpha \in \Gamma(W)$ with supp(α) $\subseteq U$, as:*

$$D_A(\alpha) = e_i.\nabla^A_{e_i}\alpha \qquad \alpha \in \Gamma(W) \tag{10}$$

As is easily checked, this definition is independent of the choice of the orthonormal basis e_i and thus defines a global differential operator $D_A : \Gamma(W) \to \Gamma(W)$ called the Dirac operator associated to A.

Since the $Spin^c$-connection ∇^A preserves the parity of the spinor bundles $W^\pm$ and the endomorphism $e_i.$ reverses it, it follows from the definition that the Dirac operator reverses parity, i.e. we get two operators $D^\pm_A : \Gamma(W^\pm) \to \Gamma(W^\mp)$. In most cases, where the chance of confusion is little, we will omit the superscript $\pm$ from these Dirac operators.

The Dirac operator is defined on a much broader domain than just the set of smooth sections $\Gamma(W)$. Since it is a first order differential operator, all that is required of a section, for it to lie in the domain of the Dirac operator, is that it should have one derivative. Thus, we get a whole panoply of Dirac operators (which will all still be denoted by D_A) acting on the various Sobolev spaces: $D_A : L^{p,q}(W) \to L^{p-1,q}(W)$, $p \geq 1$ (for a definition and basic properties of Sobolev spaces see for example [1, 2]). With this definition understood, the following proposition summarizes some important properties of the Dirac operator. Its proof can found in [3].

Proposition 2.7 *Let $W = W^+ \oplus W^-$ be a Spin^c-structure on X. Pick a connection A on L=det(W^+) and let $D_A^\pm : L^{p,q}(W^\pm) \longrightarrow L^{p-1,q}(W^\mp)$, $p \geq 1$, be its associated Dirac operator. Then the following hold:*

1. *The Weitzenböck formula:*

$$D_A^- D_A^+ \psi = \nabla^A \nabla^A \psi + \frac{1}{4} s\, \psi + \frac{1}{2} F_A^+ . \psi \tag{11}$$

where s denotes the scalar curvature of the Riemannian metric g and F_A^+ is the self-dual part of the curvature F_A of the connection A on L.

2. *The operators $D_A^\pm$ are elliptic operators. In particular, elliptic regularity applies to them:*

$$||\psi||_{p,q} \leq C\,(||D_A^\pm \psi||_{p-1,q} + ||\psi||_q) \tag{12}$$

The constant C only depends on the pair (p,q) and the Riemannian metric g, but not on ψ.

3. *The index of D_A^+ can be calculated by the Atiyah-Singer index theorem:*

$$Ind(D_A^+) = dim\ (Ker D_A^+) - dim\ Coker(D_A^+) = \frac{1}{4}\,(L \cdot L - \sigma_X) \tag{13}$$

with $\sigma_X = b^+ - b^-$ being the signature of X.

4. *The Dirac operator obeys the unique continuation theorem: If $D_A \psi = 0$ and ψ vanishes on an open set, then $\psi \equiv 0$.*

The second important ingredient of the Seiberg-Witten equations is the bilinear map $q : W^+ \otimes W^+ \longrightarrow i\,\Lambda^{2,+}(T^*X)$ which we now define. Let $v \in \Lambda^{2,+}_x(TX)$ for some $x \in X$.

Lemma 2.8 *The endomorphism $v. : W^+_x \longrightarrow W^+_x$ is traceless and skew-hermitian. Furthermore, the assignment $v \longmapsto v.$ is injective.*

Proof. We proof the lemma here for the case when X admits an almost-complex structure J, the general case can be found in [3]. It is a somewhat tedious but straightforward calculation to see that $v.$ is traceless in the case when $W = W_0$. The general case now follows from this special case together with the definition (8). Namely, suppose $v. = [v_{i,j}]$ in some basis α_i, $i = 1, ..., 4$, of $(W_0^+)_x$. Let E be any complex line bundle and $\varphi \in \Gamma(E)$ with $\varphi(x) \neq 0$. Then we still have that $v. = [v_{i,j}]$ in the basis $\varphi \otimes \alpha_i$ of $(E \otimes W_0^+)_x$. In particular, $\mathrm{tr}(v.) = 0$.

The fact that $v.$ is skewhermitian follows directly from the definition 2.3. To prove injectivity, suppose that $v. = 0$ for some $v \in T_xX$. Apply $v.$ to both sides to obtain $v.\,v. = 0$. But by definition 2.1 we know that $v.\,v. = -|v|^2\,\mathrm{Id}$. Thus $v. = 0$ implies $|v| = 0$ which in turn shows that $v = 0$. ∎

Denote the space of traceless, skewhermitian endomorphisms of W^+ by $\mathrm{End}_0^s(W^+)$. The above claim showed that $v \longmapsto v.$ defines a monomorphism

$$\Theta : i\,\Lambda^{2,+}(T^*X) \to \mathrm{End}_0^s(W^+) \tag{14}$$

It is easy to calculate that in fact both vector spaces have dimension 3. This leads to the conclusion that the above monomorphism Θ is actually an isomorphism

Let now $\psi \in \Gamma(W^+)$ with $W^+ = E \oplus E \otimes K^{-1}$ and define the quadratic map $q : \Gamma(W^+) \otimes \Gamma(W^+) \to i\,\Omega^{2,+}_X$ as

$$q(\psi,\psi) = \Theta^{-1}\left(\begin{bmatrix} \frac{1}{2}(\psi_1\,\psi_1^* - \psi_2\,\psi_2^*) & \psi_2\,\psi_1^* \\ \psi_1\,\psi_2^* & \frac{1}{2}(\psi_2\,\psi_2^* - \psi_1\,\psi_1^*) \end{bmatrix}\right) \tag{15}$$

In the above, $\psi = (\psi_1, \psi_2)$ with $\psi_1 \in \Gamma(E)$, $\psi_2 \in \Gamma(E \otimes K^{-1})$ and ψ_i^* is the dual of ψ_i (i.e.. $\psi_i^*(\varphi) = \langle \psi_i, \varphi \rangle$).

We are now ready to define the Seiberg-Witten equations. Let $W = W^+ \oplus W^- \to X$ be a *Spin*^{c}-structure and let $\mu \in i\,\Omega^{2,+}$ be an arbitrary self-dual two form. The Seiberg-Witten equations are a pair of coupled equations for a pair (A, ψ) where A is a connection on L =det(W^+) and $\psi \in \Gamma(W^+)$. The equations read:

$$\begin{aligned} D_A(\psi) &= 0 \\ F_A^+ &= q(\psi,\psi) + \mu \end{aligned} \tag{16}$$

A spinor ψ satisfying $D_A(\psi) = 0$ is called a *harmonic spinor*. In the second equation, F_A^+ is the self-dual part of the curvature form F_A of the connection A. The form μ serves as a perturbation parameter of the equations. A solution (A, ψ) of (16) is called *reducible* if $\psi \equiv 0$ and *irreducible* otherwise.

For later reference, we calculate the linearized operator

$$L_{(A,\psi)} : \Gamma(i\,\Lambda \oplus W^+) \to \Gamma(i\,\Lambda^{2,+} \oplus W^-) \tag{17}$$

associated to a solution (A, ψ) of (16). Recall that by definition, for a pair $(b, \varphi) \in$

$\Gamma(i\,\Lambda \oplus W^+)$ we have

$$L_{(A,\psi)}(b,\varphi) = \frac{d}{dt}\left(D_{A+t\,b}(\psi + t\,\varphi)\,,\,F^+_{A+t\,b} - q(\psi + t\,\varphi, \psi + t\,\varphi) - \mu\right)$$

The right-hand side expression is easily calculated using the formulas

$$\begin{aligned} D_{A+b}\,\xi =& D_A\xi + \frac{1}{2}\,b.\xi \\ F^+_{A+b} =& F^+_A + d^+b \end{aligned}$$

In conclusion, we find the following expression for $L_{(A,\psi)}$:

$$L_{(A,\psi)}(b,\varphi) = (D_A\varphi + \frac{1}{2}b.\psi, d^+b - q(\psi,\varphi) - q(\varphi,\psi)) \tag{18}$$

We conclude this section with two useful lemmas.

Lemma 2.9 *If (A,ψ) is a solution to the Seiberg-Witten equations* (16) *of class at least C^2, then ψ and F^+_A satisfy the pointwise bounds*

$$\begin{aligned} |\psi|^2_x \leq& max\,\{0, -\frac{s}{4} + ||\mu||_{C^0}\} \\ |F^+_A|_x \leq& max\,\{||\mu||_{C^0}, -\frac{s}{4} + 2\,||\mu||_{C^0}\} \qquad \forall x \in X \end{aligned} \tag{19}$$

Proof. If $\psi \equiv 0$ then (19) holds trivially. Thus, assume that $\psi \neq 0$. In that case, $|\psi|^2$ attains a global maximum at some point $x \in X$ with $|\psi|_x > 0$ and $\Delta|\psi|^2_x \geq 0$. On the other hand we have

$$\begin{aligned} \Delta|\psi|^2 = d^*\,d|\psi|^2 &= d^*\,2\,\mathrm{Re}\,(\langle\nabla^A\psi,\psi\rangle \\ &= 2\,\mathrm{Re}\,(\langle(\nabla^A)^*\nabla^A\psi,\psi\rangle) - 2|\nabla^A\psi|^2 \\ &= 2\,\mathrm{Re}\,\langle -\frac{s}{4}\psi - \frac{1}{2}F^+_A.\psi,\psi\rangle - 2|\nabla^A\psi|^2 \end{aligned}$$

$$= -\frac{s}{2}|\psi|^2 - \text{Re}\,((q(\psi,\psi)+\mu).\psi,\psi) - 2|\nabla^A\psi|^2$$
$$\leq -\frac{s}{2}|\psi|^2 - 2|\psi|^4 + |\mu|\cdot|\psi|^2$$
$$= |\psi|^2(-\frac{s}{2} - 2|\psi|^2 + |\mu|) \tag{20}$$

The calculation used the easy to check formula $\langle q(\psi,\psi).\psi,\psi\rangle = 2\,|\psi|^4$ as well as the Weitzenböck formula (11) in going from line 2 to line 3.

At the maximum point $x \in X$ we must have $-s/2 - 2|\psi|_x^2 + |\mu|_x \geq 0$ leading to the desired formula $|\psi|_x^2 \leq -s/4 + |\mu|_x/2$. Since x is the maximum of $|\psi|^2$, the first inequality in (19) follows.

For the second inequality, use the second Seiberg-Witten equation together with $|q(\psi,\psi)|^2 = 2\,|\psi|^4$:

$$|F_A^+|_x \leq |q(\psi,\psi)|_x + |\mu|_x = \sqrt{2}\,|\psi|_x^2 + |\mu|_x$$
$$\leq \sqrt{2}\max\,\{0, -\frac{s}{4} + ||\mu||_{C^0}\} + |\mu|_x$$
$$\leq \max\,\{||\mu||_{C^0}, -\frac{s}{4} + 2||\mu||_{C^0}\} \quad \blacksquare \tag{21}$$

Lemma 2.10 *If (A,ψ) is a solution of the Seiberg-Witten equations* (16) *of class at least $L^{1,2}$, then (A,ψ) is gauge equivalent to a C^∞ solution.*

Proof. This follows from the usual bootstrapping process and the fact that the Seiberg-Witten equations are elliptic. Elliptic regularity of the Dirac operator implies

$$||\psi||_{p,2} \leq C\,||\psi||_2 \tag{22}$$

showing immediately that $\psi \in L^{p,2}$ for all $p \geq 1$ and thus $\psi \in C^\infty$. As for A, choose first a gauge such that $d^*a = 0$ where $a = (A - A_0)$ and A_0 is some C^∞ base connection. Then elliptic regularity of $d^* + d^+$ gives

$$||a||_{p,2} \leq C\,(||a||_2 + ||d^+a||_{p-1,2}) \leq C\,(||a||_2 + ||\mu||_{p-1,2} + ||q(\psi,\psi)||_{p-1,2}) \qquad (23)$$

This last inequality together with (22) shows that $A \in L^{p,2}$ for $p \geq 1$ and thus also $A \in C^\infty$. ∎

2.3 The moduli space

In order to define invariants of the smooth structure of X from the Seiberg-Witten equations, one needs to ensure that the moduli space of the solutions of (16) has certain compactness properties. This and other properties of the moduli space is what we will study in the present section.

The gauge group underlying Seiberg-Witten theory is $\mathcal{G} =$Map(X, S^1) (since $S^1 = U(1)$ is the structure group of $L =$det(W^+)). The gauge group acts on pairs (A, ψ) of connections on L and sections of W, in the following manner:

$$\begin{aligned} g \cdot A =& A - 2g^{-1}\,dg \\ g \cdot \psi =& g\,\psi \qquad\qquad g \in \mathcal{G} \end{aligned} \qquad (24)$$

Notice that the first equation shows that the action of $\mathcal{G}$ on a $Spin^c$-connection ∇^A is given by $g \cdot \nabla^A := \nabla^{g\cdot A} = \nabla^A - g^{-1}\,dg$, implying immediately that $F_{g\cdot A} = F_A$. It follows directly from the definition that $q(g\cdot\psi, g\cdot\psi) = q(\psi,\psi)$ since $|g| = 1$.

That the Seiberg-Witten equations are invariant under this action of $\mathcal{G}$ can easily be checked. We give here the check for the first equation in (16), the invariance of the second equation is obvious:

$$D_{g\cdot A}(g\cdot\psi) = e_i.(\nabla^{g\cdot A}_{e_i}(g\,\psi))$$

$$= e_i.((\nabla^A_{e_i} - g^{-1}d_{e_i}g)(g\,\psi))$$

$$= e_i.(\nabla^A_{e_i}(g\,\psi) - d_{e_i}\psi)$$

$$= e_i.(d_{e_i}\psi + g\nabla^A_{e_i}\psi - d_{e_i}\psi)$$

$$= e_i.(g\nabla^A_{e_i}\psi)$$

$$= g\,D_A(\psi) \tag{25}$$

From this we see that if the Seiberg-Witten equations (16) have at least one solution, then they necessarily have infinitely many of them. This leads to a big redundancy of solutions: from a point of view of trying to formulate invariants of the smooth structure of X, the solutions (A, ψ) and $g \cdot (A, \psi)$ carry exactly the same information. Thus we are motivated to henceforth identify solutions that differ by a gauge transformation. The Seiberg-Witten moduli space is the set of these equivalence classes of solutions.

Before continuing the discussion, we first introduce some more notation. Let A_0 be a connection on L which we will referred to as a *base connection* (but which at this point is completely arbitrary). Assume throughout that $p \geq 1$. We denote by $\overline{\mathcal{B}}(L)$ the space $\mathcal{A}^{p,2}(L) \oplus L^{p,2}(W^+)$ with $\mathcal{A}^{p,2}(L)$ being the space of $L^{p,2}$ connections on L. Let $\overline{\mathcal{B}}^*(L) = \mathcal{A}^{p,2}(L) \oplus (L^{p,2}(W^+)\backslash\{0\})$. The configuration spaces of the theory are $\mathcal{B}(L) = \overline{\mathcal{B}}(L)/\mathcal{G}$ and $\mathcal{B}^*(L) = \overline{\mathcal{B}}^*(L)/\mathcal{G}$. The space $\widetilde{\mathcal{B}}(L)$ is an intermediate of sorts of the spaces $\overline{\mathcal{B}}(L)$ and $\mathcal{B}(L)$ (same is true of its $*$-analogue) and is defined as $\widetilde{\mathcal{B}}(L) = \{(A, \psi) \in \overline{\mathcal{B}}(L) \,|\, d^*(A - A_0) = 0\}$ (and $\widetilde{\mathcal{B}}^*(L) = \{(A, \psi) \in \overline{\mathcal{B}}^*(L) \,|\, d^*(A - A_0) = 0\}$).

We also introduce the moduli spaces

$$\begin{aligned}
\overline{\mathcal{M}}_X^{SW}(L) =& \{(A,\psi) \in \mathcal{B}^*(L) \mid (A,\psi) \text{ solves (16) } \} \\
\widetilde{\mathcal{M}}_X^{SW}(L) =& \{(A,\psi) \in \widetilde{\mathcal{B}}^*(L) \mid (A,\psi) \text{ solves (16) } \} \\
\mathcal{M}_X^{SW}(L) =& \overline{\mathcal{M}}_X^{SW}(L)/\mathcal{G}
\end{aligned} \tag{26}$$

(we suppress the perturbations form μ from our notation for the moduli spaces, but keep in mind that different choices of μ give rise to different moduli spaces). The condition $d^*(A - A_0) = 0$ is called the *Coulomb gauge fixing* condition. While this condition doesn't determine a unique gauge for A, it reduces the number of possible gauges considerably: if (A, ψ) is a solution of (16) with $d^*(A - A_0) = 0$, then the subgroup $\mathcal{G}_0 \subseteq \mathcal{G}$ of elements g for which $d^*(g \cdot A - A_0) = 0$, is homotopy equivalent to $H^1(X;\mathbb{Z}) \times S^1$ (the S^1 factor corresponds to the constant maps). In particular, the moduli space $\mathcal{M}_X^{SW}(L)$ can also be expressed as $\widetilde{\mathcal{M}}_X^{SW}(L)/\mathcal{G}_0$.

Proposition 2.11 *The moduli space $\mathcal{M}_X^{SW}(L)$ is compact.*

Proof. Without loss of generality, assume that $p \geq 5$ (every $L^{1,2}$ solution (A, ψ) of (16) is C^∞ according to lemma 2.10). Choose a sequence $(A_n, \psi_n) \in \mathcal{M}_X^{SW}(L) = \widetilde{\mathcal{M}}_X^{SW}(L)/\mathcal{G}_0$. Recall that A_0 denotes our base connection and set $a_n = A_n - A_0$. Clearly, the connections A_n are bounded if and only if the 1-forms a_n are bounded. Note that $d^* a_n = 0$ and $F_{A_n} = F_{A_0} + d^+ a_n$. The operator $d^* + d^+ : i\Omega_X^1 \to i\Omega_x^0 \oplus i\Omega_X^{2,+}$ is elliptic and Karen Uhlenbeck's theorem applies to it: there exist gauge transformations $g_n \in \mathcal{G}_0$ (chosen from the $H^1(X;\mathbb{Z})$ part of $\mathcal{G}_0$) such that $a_n' = g_n \cdot A_n - A_0 =$

$a_n - 2g_n^{-1}\, dg_n$ obey the inequality

$$||a'_n||_{p,2} \le C\,(1 + ||(d^* + d^+)a'_n\,||_{p-1,2}) = C\,(1 + ||d^+ a'_n\,||_{p-1,2})$$
$$\le C\,(1 + ||q(\psi'_n, \psi'_n)||_{p-1,2}) \tag{27}$$

In the above, C is a constant depending only on the Riemannian metric, the $Spin^c$-structure and A_0 and its precise value may change from line to line. The spinor ψ'_n is given by $\psi'_n = g_n \cdot \psi_n$. Agree to rename the (a'_n, ψ'_n) back to (a_n, ψ_n). Observe that $2\, D_{A_n} = a_n. + 2\, D_{A_0}$. Using elliptic regularity of D_{A_0} applied to ψ_n gives

$$||\psi_n||_{p,2} \le C\,(||D_{A_0}\psi_n||_{p-1,2} + ||\psi_n||_2)$$
$$\le C\,(||a_n.\psi_n||_{p-1,2} + ||\psi_n||_2) \tag{28}$$

Recall that the Sobolev space $L^{5,2}$ on a 4-manifold embeds (compactly) into the space of C^2 maps. Because of that, lemma 2.9 applies and we get the n-independent bounds: $||\psi_n||_q \le C$. In particular, the L^4-bounds induce $L^{1,2}$-bounds (these two norms are equivalent on a 4-manifold). We now repeatedly use equations (27) and (28) to improve on these bounds (this process is called bootstrapping):

$$(27) \Rightarrow ||a_n||_{2,2} \le C\,(1 + ||q(\psi_n, \psi_n)||_{1,2}) \le C\,(1 + ||\psi_n||_8^2) \le C$$
$$(28) \Rightarrow ||\psi_n||_{2,2} \le C\,(||a_n.\psi_n||_{1,2} + ||\psi_n||_2) \le C\,(||a_n||_{2,2} \cdot ||\psi_n||_4 + 1) \le C$$
$$(27) \Rightarrow ||a_n||_{3,2} \le C\,(1 + ||q(\psi_n, \psi_n)||_{2,2}) \le C\,(1 + ||\psi_n||_{2,2}^2) \le C$$
$$(28) \Rightarrow ||\psi_n||_{3,2} \le C\,(||a_n.\psi_n||_{2,2} + ||\psi_n||_2) \le C\,(||a_n||_{2,2} \cdot ||\psi_n||_{2,2} + 1) \le C \tag{29}$$

These inequalities serve as the base of the induction process that completes the proof of the proposition: from (27) and (28) we see that if (a_n, ψ_n) is a bounded sequence in $L^{p-1,2}$, then it is also a bounded sequence in $L^{p,2}$. However, the Sobolev embedding $L^{p-1,2} \hookrightarrow L^{p,2}$ is compact and so $(a_n, \psi_n) \in L^{p-1,2}$ has a convergent subsequence. By induction on p we conclude that the sequence (a_n, ψ_n) has a convergent subsequence in each $L^{p,2}$, $p \geq 5$. It still remains to take the quotient of the moduli space under the S^1 component of $\mathcal{G}_0$. This however preserves compactness as S^1 is a compact group.

■

Our next aim is to show that for "good "choices of μ in (16), the moduli space $\mathcal{M}_X^{SW}(L)$ is a smooth manifold of finite dimension. To see this, consider the map $SW : \mathcal{B}^*(L) \longrightarrow L^{p-1,2}(W^- \oplus \Lambda^{2,+}(T^*X))$ given by

$$SW(A, \psi) = (D_A(\psi), F_A^+) \tag{30}$$

Recall that a linear map between Banach spaces is called *Fredholm* if its image is closed and both its kernel and cokernel are finite-dimensional. A map between Banach spaces is called *Fredholm* if its differential is Fredholm at each point.

Lemma 2.12 *The map SW is a smooth Fredholm map, transverse to the linear subspace* $\{(0, \mu) \in L^{p,2}(W^- \oplus \Lambda^{2,+}(T^*X))\}$ *at all* (A, ψ) *solving* (16) *with* $\psi \neq 0$ *and with* $d^*(A - A_0) = 0$.

Proof. To prove the Fredholm property, we first consider the differential $d_{(A,\psi)}SW$ of SW at a point (A, ψ). We incorporate the gauge group action by choosing a gauge fixing condition: $d^*(A - A_0) = 0$ (recall that A_0 is a base connection on L). For

$(A,\psi) \in \widetilde{\mathcal{M}}_X^{SW}(L)$ we find the said differential

$$d_{(A,\psi)}SW(b,\varphi) = (D_A\varphi + \frac{1}{2}b.\psi, d^+b - q(\psi,\varphi) - q(\varphi,\psi)) \tag{31}$$

The gauge condition $d^*(A - A_0) = 0$ translates into $d^*b = 0$.

The first component of (30) is evidently smooth while the second component is smooth since both d^+ and $q(\psi,\psi)$ are linear, bounded maps (this fact uses the Sobolev multiplication theorem). Thus SW is smooth. The Fredholm property is established by observing that $d_{(A,\psi)}SW$ is a compact perturbation (such perturbations preserve the property of being Fredholm) of the operator $D_A \oplus d^+$. The Fredholmness of the later follows from two facts: one being part two of proposition (2.7) (elliptic operators on compact manifolds are Fredholm) and the second being an explicit calculation of the kernel and cokernel of $d^+|_{\text{Ker }(d^*)}$:

$$\begin{aligned} \text{Ker }\left(d^+|_{\text{Ker }(d^*)}\right) &\cong \mathcal{H}^1(X) \\ \text{Coker }\left(d^+|_{\text{Ker }(d^*)}\right) &\cong \mathcal{H}^{2,+}(X) \end{aligned} \tag{32}$$

Here $\mathcal{H}^1(X)$ are the harmonic 1-forms and $\mathcal{H}^{2,+}(X)$ are the harmonic, self-dual 2-forms on X.

The transversality of SW to the subspace $\{(0,\mu) \in L^{p,2}(W^- \oplus \Lambda^{2,+}(T^*X))\}$ is equivalent to the surjectivity of the first component of (31). Suppose that it is not onto, then there exists $\phi \in L^{p-1,2}(W^-)$ that is L^2 perpendicular to the image of the map $(b,\varphi) \mapsto D_A\varphi + \frac{1}{2}b.\psi$. Said differently, the equation

$$\langle D_A\varphi + \frac{1}{2}b.\psi, \phi\rangle_{L^2} = 0$$

holds for all b and φ. Choosing $b = 0$ and integrating by parts shows that $D_A\phi = 0$. A contradiction is now produced by constructing a special 1-form b: locally b is defined as $b = \text{Re}\ (\langle \phi, e_i.\psi \rangle\ e^i$ (where $\{e_i\}$ and $\{e^i\}$ are a dual pair of local orthonormal frames of TX and T^*X). Using $D_A\psi = 0$ and $D_A\phi = 0$ one can verify that $d^*b = 0$ and an easy calculation shows that $|b|^2 = |\phi|^2\,|\psi|^2$. Using this b, with $\varphi = 0$ leads to

$$0 = \langle b.\psi, \phi \rangle_{L^2} = ||a||_2 = \int_X |\phi|^2\,|\psi|^2$$

Since ψ is assumed to be nonzero, ϕ vanishes on at least an open set and thus everywhere by the unique continuation theorem for the Dirac operator. ∎

Invoke now an infinite-dimensional analogue of Sard's theorem (due to Smale [11]) guaranteeing that the set of regular values of a Fredholm map between Banach manifolds are of second category (i.e. the countable intersection of open, dense sets). Agree from now on to choose the perturbation parameter μ in the Seiberg-Witten equations to be such a regular value of the map SW. Under such a choice, the moduli space $\mathcal{M}_X^{SW}(L)$ is a smooth compact manifold.

To conclude this section, we calculate the dimension of the moduli space. Since $\widetilde{\mathcal{M}}_X^{SW}(L) = SW^{-1}(0, \mu)$, the dimension of $\widetilde{\mathcal{M}}_X^{SW}(L)$ is given by the index of the differential of SW and we calculate this dimension first. From (31) we see that

$$\text{Index}\ (d_{(A,\psi)}SW) = \text{Index}\ D_A + \text{Index}\ d^+$$

The first summand on the right hand side is provided by (13) while the second follows from (32) to be: $\text{Index}(d^+) = b_1 - b^+$. Putting these together we obtain

$$\text{Index}\ (d_{(A,\psi)}SW) = \frac{1}{4}(L \cdot L - \sigma_x) + b_1 - b^+ = \frac{1}{4}(L \cdot L - 5b^+ + b^- + 4b_1)$$

Since $\mathcal{M}_X^{SW}(L) = \widetilde{\mathcal{M}}_X^{SW}(L)/\mathcal{G}_0$ and dim $\mathcal{G}_0 = 1$, we find that

$$\dim\, \mathcal{M}_X^{SW}(L) = \frac{1}{4}(L \cdot L - 5b^+ + b^- + 4b_1) - 1 = \frac{1}{4}(L \cdot L - 3\sigma_X - 2e_X) \tag{33}$$

Here e_X and σ_X are the Euler characteristic and the signature of X.

2.4 The Seiberg-Witten invariant

This section defines the Seiberg-Witten invariant coming from the moduli space of gauge equivalent solutions of the Seiberg-Witten equations (16).

To begin with, let $W \to X$ be a *Spin*^{c}-structure of X and denote the determinant line bundle of W^+, as usual, by L. Recall that the moduli space $\mathcal{M}_X^{SW}(L)$ is a compact, smooth, finite-dimensional manifold and as such, it carries a fundamental class $[\mathcal{M}_X^{SW}(L)]$ in its top dimensional (non-zero) homology group. The Seiberg-Witten invariant will be obtained by pairing this homology class against certain cohomology classes of the configuration space $\mathcal{B}^*(L)$. Our next task is to understand the cohomology group of $\mathcal{B}^*(L)$.

Recall the definitions of $\widetilde{\mathcal{B}}^*(L)$ and $\mathcal{G}_0$:

$$\widetilde{\mathcal{B}}^*(L) = \{(A, \psi) \in \overline{\mathcal{B}}(L) \,|\, d^*(A - A_0) = 0\} \quad \text{and} \quad \mathcal{G}_0 \simeq H^1(X; \mathbb{Z}) \times S^1$$

Choose to view $\mathcal{B}^*(L)$ as the quotient $\widetilde{\mathcal{B}}^*(L)/\mathcal{G}_0$. The space $\overline{\mathcal{B}}(L)$ is an affine space and hence contractible. On the other hand, the set $\overline{\mathcal{B}}(L)\backslash\overline{\mathcal{B}}^*(L) = \{(A, 0) \in \overline{\mathcal{B}}(L)\}$ is an affine set of infinite codimension in $\overline{\mathcal{B}}(L)$ implying that the open set $\overline{\mathcal{B}}^*(L)$ is also contractible. But the action of $\mathcal{G}_0$ on $\overline{\mathcal{B}}^*(L)$ is free and thus the quotient $\widetilde{\mathcal{B}}^*(L)/\mathcal{G}_0$ is the classifying space for $\mathcal{G}_0$. This space is homotopy equivalent to the product of the

Eilenberg-MacLane space $K(H^1(X;\mathbb{Z}),1)$ with $\mathbb{CP}^\infty$. To summarize, we have proved the following

Lemma 2.13 *The space $\mathcal{B}^*(L)$ is homotopy equivalent to $T^{b_1} \times \mathbb{CP}^\infty$ where b_1 is the first Betti number of X.*

Denote the generator of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$ by h, thought of as an element of $H^2(\mathcal{B}^*(L);\mathbb{Z})$.

Definition 2.14 *Let $W \to X$ be a $Spin^c$-structure on X with determinant line bundle L. Let $d = (L^2 - 3\sigma_X - 2e_X)/4$ be the dimension of the moduli space $\mathcal{M}_X^{SW}(L)$. Then the Seiberg-Witten invariant $SW_X(L)$ is defined to be:*

1. $SW_X(L) = 0$; *if d is odd or $d < 0$.*
2. $SW_X(L) = \langle h^{d/2}, [\mathcal{M}_X^{SW}(L)]\rangle$; *if d is even.*

Remark 2.15 *The above definition pairs $[\mathcal{M}_X^{SW}(L)]$ only the with the correct power of $h \in H^2(\mathcal{B}^*(L);\mathbb{Z})$. In the case when X is not simply connected, this second cohomology group has a richer structure and other pairings with $[\mathcal{M}_X^{SW}(L)]$ are possible, leading to more invariants. However, we will restrict ourselves only to the case of definition 2.14.*

Our next objective is to justify the use of the word “invariant ”in the above definition. The Seiberg-Witten moduli space $\mathcal{M}_X^{SW}(L)$ depends, a priori, on the choice of the Riemannian metric g as well as the choice of a (generic) perturbation form μ. The dependence on the metric g manifests itself on several levels: it defines a Levi-Civita connection and thus determines what a $Spin^c$-connection is. This in turn gives rise to the Dirac operator. On the other hand, the metric determines the Hodge star operator $* : \Omega^i_X \to \Omega^{4-i}_X$ and so determines the projection $\pi : \Omega^2_X \to \Omega^{2,+}_X$ which is used via $F_A^+ = \pi(F_A)$ in the Seiberg-Witten equations. While the moduli space itself may change when altering those two choices, it turns out that these different moduli spaces carry the same homology class $[\mathcal{M}_X^{SW}(L)]$ and thus determine the same Seiberg-Witten invariant.

Lemma 2.16 *Assume that X is a 4-manifold with $b^+ \geq 2$. Let $W \to X$ be a Spin^c-structure on X with determinant line bundle L. Then the Seiberg-Witten invariant $SW_X(L)$ doesn't depend on the choice of (a generic, see proof below) Riemannian metric g and perturbation form μ.*

Proof. We first describe the set of "bad"pairs (g, μ), i.e. the ones for which the Seiberg-Witten equations admit reducible solutions. If $(A, 0)$ is such a solution, then the second equation in (16) gives $(F_A - \mu)^+ = 0$. Denote by $\pi^\pm(g)$ the orthogonal projection from Ω^2_X onto the (anti-) self-dual, harmonic 2-forms $\mathcal{H}^\pm(g)$. Likewise, denote the orthogonal projection from Ω^2_X onto the harmonic 2-forms $\mathcal{H}(g)$ by $\pi(g)$. Then we have

$$(F_A - \mu)^+ = 0 \quad \Leftrightarrow \quad \pi^+(g)(F_A - \mu) = 0 \quad \Leftrightarrow \quad \pi(g)(F_A - \mu) \in \mathcal{H}^-(g) \qquad (34)$$

On the other hand, $L_h := \pi(g)(i\, F_A/2\pi)$ is independent of the choice of the particular connection A since it is the unique harmonic representative of L. We conclude that (g, μ) is a bad pair if and only if $\pi(g)(i\,\mu/2\pi) \in L_h + \mathcal{H}^-(g)$. The space $\mathcal{H}^-(g)$ has codimension b^+ in $\mathcal{H}(g) = \text{Codomain } (\pi(g))$. This shows that the set

$$\mathcal{W}(L, g) = \{\mu \in i\,\Omega^{2,+}_X \mid \pi(g)\left(\frac{i}{2\pi}\mu\right) \in L_h + \mathcal{H}^-(g)\} \qquad (35)$$

has codimension b^+ inside of $i\Omega_X^{2,+}$.

Consider now two pairs (g_i, μ_i), $i = 1, 2$. Take a path (g_t, μ_t), $t \in [1, 2]$, connecting (g_1, μ_1) to (g_2, μ_2) and define the set $\mathcal{W}(L; (g_t, \mu_t))$ as

$$\mathcal{W}(L; (g_t, \mu_t)) = \{(g_t, \mu) \,|\, \mu \in \mathcal{W}(L, g_t), t \in [1, 2]\}$$

Since by assumption $b^+ \geq 2$, the set $\pi_2(\mathcal{W}(L; (g_t, \mu_t)))$ has codimension at least 1 inside of $i\,\Omega_X^{2,+}$ (π_2 is projection onto the second coordinate). It is shown in [11] that a generic path (g_t, μ_t) will intersect $\mathcal{W}(L; (g_t, \mu_t))$ in at most finitely many points and each of these is a transverse intersection point. It follows now, by invoking again the results from [11], that the set

$$\mathcal{M}_X^{SW}(L; (g_t, \mu_t)) = \{(A, \psi) \,|\, (A, \psi) \text{ solve (16) for any of the pairs } (g_t, \mu_t), t \in [1, 2]\}$$

is a compact, smooth, oriented manifold of dimension $d + 1$ ($d = \dim \mathcal{M}_X^{SW}(L)$), providing a cobordism between the Seiberg-Witten moduli spaces obtained from using (g_1, μ_1) and (g_2, μ_2). In particular, the homology class $[\mathcal{M}_X^{SW}(L)]$ is independent of the chosen "good" pair (g, μ). ■

In the case $b^+ = 1$, the Seiberg-Witten invariant does depend on the choice of the pair (g, μ). That dependence is well understood and completely described by the next lemma (cf. [9]), which we give without proof.

Lemma 2.17 *Let X be a 4-manifold with $b^+ = 1$ and let $\omega_g \in \Omega_X^{2,+}$ be a generator of the positive forward cone in $H^2(X; \mathbb{Z})$. Let $W \to X$ be a Spin^c-structure on X with determinant line bundle L. Then the Seiberg-Witten invariants $SW_X(L; (g_1, \mu_1))$*

and $SW_X(L;(g_2,\mu_2))$ *calculated from two pairs* (g_j,μ_j) *are the same, provided the two expressions*

$$\langle L\cdot\omega_{g_j},[X]\rangle - \frac{i}{2\pi}\int_X \mu_j\wedge\omega_{g_j} \qquad j=1,2$$

have the same sign (notice that this sign doesn't depend on the specific choice of ω_g*).*

Definition 2.18 *An element* $L\in H^2(X;\mathbb{Z})$ *is called a Seiberg-Witten basic class if* $SW_X(L)\neq 0$.

Proposition 2.19 *The Seiberg-Witten invariants of* X *have the following properties:*

1. *If* X *admits a metric of positive scalar curvature, then* $SW_X(L)=0$ *for all* $L\in H^2(X;\mathbb{Z})$.

2. *The number of basic classes of* X *is finite.*

3. *There is an inherent duality in Seiberg-Witten theory, namely that of replacing* L *by* $-L$. *The Seiberg-Witten invariants of a dual pair of* $Spin^c$*-structures are related by a simple relation:*

$$SW_X(L)=(-1)^{\frac{e_X+\sigma_X}{4}}SW_X(-L) \tag{36}$$

Proof. The first two claims follow readily from (19). We give a proof of claim 3 only in the case when X admits an almost-complex structure J and refer the reader to the general case to [7]. Let $L=2E-K$, then the associated $Spin^c$-structure has positive spinor bundle $W_E^+=E\oplus(E\otimes K^{-1})$ and we write spinors $\psi\in\Gamma(W^+)$ as $\psi=(\alpha,\beta)$. The duality $L\mapsto -L$ corresponds to replacing E by $E\otimes K^{-1}$. A direct check shows that if $(A,(\alpha,\beta))$ is a solution of (16) for the $Spin^c$-structure L (with perturbation form μ), then $(\overline{A},(\beta,\alpha))$ is a solutions of (16) for the $Spin^c$-structure $-L$ (with perturbation form $-\mu$). Here $\overline{A}$ is the dual connection of A. This gives a diffeomorphism between $\mathcal{M}_X^{SW}(L)$ and $\mathcal{M}_X^{SW}(-L)$ and thus $SW_X(L) = \pm\, SW_X(-L)$. The correct sign is calculated by a, somewhat tedious but straightforward, comparison of the orientations of $\mathcal{M}_X^{SW}(L)$ and $\mathcal{M}_X^{SW}(-L)$, and is omitted here. ■

We finish this section with a few examples:

Example 2.20 *1. The Seiberg-Witten invariants of* $\mathbb{CP}^2$ *and* $\Sigma_g \times S^2$ *calculated using a pair* (g,μ) *with* $|\mu|$ *small and with* $s_g > 0$*, are all zero.*

2. The Seiberg-Witten invariants of the simply-connected, elliptic surfaces $E(n)$ *without multiple fibers are:*

$$SW_X(L) = \begin{cases} \begin{pmatrix} n-2 \\ q \end{pmatrix} & ; L = \pm\, q\, F, \quad F = \text{class of regular fiber} \\ 0 & ; \text{otherwise} \end{cases}$$

3. The only basic classes of complex surfaces of general type, are $\pm K$ *(cf. [7]). The invarinat for each class is* ± 1*.*

3 Gromov-Witten Theory

In this section we give a very brief introduction to the theory of counting embedded J-holomorphic curves in symplectic 4-manifolds. We omit proofs and instead refer the interested reader to the vast and comprehensive literature available on the subject. At the onset we would like to point out, as there are many Gromov-Witten theories in existence today, differing from each other in the types of objects they are counting (e.g. embedded J-holomorphic curves, immersed J-holomorphic curves, J-holomorphic maps, etc.), that we will follow the approach of Taubes [15].

Definition 3.1 *Let (X, ω) be a symplectic 4-manifold. A triple (ω, g, J) consisting of the symplectic form ω, a Riemannian metric g and an almost-complex structure J, is called a compatible triple if*

$$g(u, v) = \omega(u, J(v)) \qquad u, v \in TX \tag{37}$$

It is easy to see that any two members of a compatible triple, uniquely determine the third member. An important consequence of (37) is that J becomes an orthogonal map, i.e.

$$g(J(u), J(v)) = g(u, v)$$

Also, observe that $(u, v) \mapsto g(u, v) + i\,\omega(u, v)$ defines a Hermitian metric on the complexified tangent space $T_{\mathbb{C}}X$. Given a symplectic form ω, an abundant supply of compatible triples always exists (cf. [6]). The canonical classes K_1 and K_2, associated to almost-complex structures J_1, J_2, each of which belongs to a compatible triple, are the same. The canonical class only depends on ω.

Definition 3.2 *Let $C \hookrightarrow X$ be an embedded symplectic submanifold of X. We say that C is a J-holomorphic curve if the tangent space T_xC is a complex subspace of T_xX at every point $x \in C$.*

The genus g of a connected J-holomorphic curve C is determined by its square and its pairing with the canonical class, as given by the adjunction formula

$$2g - 2 = [C]^2 + K \cdot [C] \tag{38}$$

Another important property that J-holomorphic curves share with holomorphic curves is that they intersect each other locally positively. Namely, if C_1 and C_2 are two distinct J-holomorphic curves, then $[C_1] \cdot [C_2] \geq 0$ and each point $x \in C_1 \cap C_2$ contributes positively to that intersection number. This is a result of Dusa McDuff and can be found in [5].

For a given $E \in H_2(X; \mathbb{Z})$, set

$$d = \frac{1}{2}(E^2 - E \cdot K) \tag{39}$$

where K is the canonical class associated to ω. Introduce $\mathcal{A}_d$ as the set of pairs (J, Ω) with J an almost-complex structure compatible with ω and Ω a set of d distinct points of X. It has the structure of a smooth manifold inherited from the Frechet manifold $C^\infty(\mathrm{End}(TX)) \times \mathrm{Sym}^d(X)$.

Each J-holomorphic curve C comes equipped with a linear operator $D_C : C^\infty(N_C) \longrightarrow C^\infty(N_C \otimes T^{0,1}C)$ obtained as the linearisation of the generalized Cauchy-Riemann operator $\overline{\partial_C}$. Here N_C is the normal bundle of C in X (which is also a complex subspace of TX). In the case when C contains all points of Ω, let $ev_\Omega : C^\infty(N_C) \to \oplus_{p\in\Omega} N_p$ be the evaluation map associated to Ω. If $d = 0$, we say that D_C is non-degenerate if $\text{Coker}(D_C) = \{0\}$. In the case $d > 0$, D_C is called non-degenerate if

$$D_C \oplus ev_\Omega : C^\infty(N_C) \to C^\infty(N_C \otimes T^{0,1}C) \oplus_{p\in\Omega} N_p$$

has trivial cokernel.

A pair $(J, \Omega) \in \mathcal{A}_m$, $m \geq 0$, is said to be generic if the following five conditions are met (see [15] for more details, especially on the definition of n-non-degenerate which is immaterial for the present discussion and we omit it):

1. For a fixed class $E \in H_2(X; \mathbb{Z})$, there are only finitely many embedded J-holomorphic curves representing E and containing d points of Ω.

2. For each J-holomorphic curve C, the operator D_C is non-degenerate.

3. There are no connected J-holomorphic curves representing the class $E \in H_2(X; \mathbb{Z})$ containing more than d points of Ω.

4. There is an open neighborhood of (J, Ω) in $\mathcal{A}_d$ such that each pair (J', Ω') from that neighborhood satisfies conditions 1-3 above. Furthermore, the number of J'-holomorphic curves containing d points of Ω', is constant as (J', Ω') varies trough the said neighborhood.

5. If $E^2 = K \cdot E = 0$ then each of the finitely many J-holomorphic curves in E containing d points of Ω, is n-non-degenerate for each positive integer n.

The set of generic pairs (J, Ω), which we denote by $\mathcal{J}^{reg}$, is a Baire subset of $\mathcal{A}_d$.

For the choice of a generic pair (J, Ω), each J-holomorphic curve containing all the points of Ω, is assigned a weight $\varepsilon(C)$. The weights for genus $g \geq 2$ curves are always ± 1, however, weights of J-holomorphic tori may be other integers as well. The definition of $\varepsilon(C)$ is not an all together simple matter and the reader is referred to the excellent account [15]. We only give here the definition of $\varepsilon(C)$ for the case $d = 0$ (and thus $\Omega = \emptyset$) and $g \geq 2$. In that case, the operator $D_C : C^\infty(N_C) \to C^\infty(N_C \otimes T^{0,1}C)$ has the form

$$D_C\, s = \overline{\partial}_C s + \nu s + \mu \overline{s} \qquad \nu \in \Gamma(T^{0,1}C),\ \mu \in \Gamma(T^{0,1}C \otimes N^{\otimes 2})$$

Here $\overline{s} \in \Gamma(\overline{N})$ is the dual section of s and the sections ν and μ are determined by the almost-complex structure J. Find a path of Fredholm, index zero, operators $D_t : C^\infty(N_C) \to C^\infty(N_C \otimes T^{0,1}C)$ connecting D_C to the complex linear operator $\overline{\partial}_C s + \nu s$. Such a path, if chosen from a suitable Baire set of generic paths, will have only finitely many singularities, i.e. there are points $t_1 < ... < t_n$ with dim $\mathrm{Ker}(D_{t_i}) = 1$ and dim $\mathrm{Ker}(D_t) = 0$ for $t \neq t_i$. The weight $\varepsilon(C)$ is defined as $(-1)^n$. While n may depend on the chosen path D_t, its parity does not, and thus $\varepsilon(C)$ is well defined.

The Gromov-Witten invariant is now defined in the following manner:

Definition 3.3 *Let $E \in H^2(X; \mathbb{Z})$ be a cohomology class and pick a pair $(J, \Omega) from \mathcal{J}^{reg}$. Let $\mathcal{M}_X^{Gr}(E)$ be the moduli space of all J-holomorphic curves passing through each point of Ω and homologous to the Poincaré dual of E. Then the Gromov-Witten* *invariant $Gr_X(E)$ of E is defined as*

$$Gr_X(E) = \sum_{C \in \mathcal{M}_X^{Gr}(E)} \varepsilon(C) \qquad (40)$$

An argument similar to the one used in section 2.4 to show the independence of the Seiberg-Witten invariant of the choice of the pair (g, μ), shows that the Gromov-Witten invariant is independent of the choice of the pair $(J, \Omega) \in \mathcal{J}^{reg}$.

4 Gauge theory on symplectic 4-manifolds

4.1 Introduction

While Seiberg-Witten theory is defined for all smooth, compact 4-manifolds with $b^+ \geq 1$, it has some additional features on 4-manifolds which possess a symplectic structure. The two most outstanding of these are the fact that there are always Seiberg-Witten basic classes on a symplectic 4-manifold and their spectacular relation with the Gromov-Witten invariants. Both of these results are due to Taubes.

Let (X, ω) be a symplectic, smooth, compact 4-manifold with symplectic form ω and pick a compatible triple (ω, g, J). The symplectic form ω induces a splitting of $\Lambda^{2,+} := \Lambda^{2,+}(T^*X)$ as

$$\Lambda^{2,+} \cong \mathbb{R} \cdot \omega \oplus \Lambda^{0,2} \tag{41}$$

which will be used below to write the curvature component of the Seiberg-Witten equations (16) as two equations, one for each of the summands on the right-hand side of (41).

It proves more convenient and natural for the purposes of this section, to denote the Seiberg-Witten invariant of the $Spin^c$-structure $W_E = E \oplus (E \otimes K^{-1})$ by $SW_X(W_E)$ rather than $SW_X(L)$ (with $L = 2E - K$). It also proves convenient to write the spinor $\psi \in \Gamma(W_E^+)$ in the form

$$\psi = \sqrt{r}(\alpha, \beta) \qquad \alpha \in \Gamma(E),\ \beta \in \Gamma(E \otimes K^{-1})$$

where $r \geq 1$ is a parameter whose significance will become clear later. With this notation, the map q from (15) can be calculated to be

$$q(\psi,\psi) = \frac{ir}{8}(|\alpha|^2 - |\beta|^2)\omega + \frac{ir}{4}(\bar{\alpha}\beta + \alpha\bar{\beta}) \tag{42}$$

4.2 The anticanonical $Spin^c$-structure

Among the first spectacular results in Seiberg-Witten theory was Taubes' theorem [16] saying that the Seiberg-Witten invariant of the anticanonical $Spin^c$-structure on a symplectic manifold is equal to ± 1. More is true: the equations have exactly one solution $(\mathcal{A}_0, \sqrt{r}\cdot u_0)$, $u_0 \in \Gamma(\underline{\mathbb{C}})$, for the choice of

$$\mu = F^+_{\mathcal{A}_0} - \frac{i\,r}{8}\,\omega \tag{43}$$

in (16) and for $r \gg 1$. The purpose of this section is to describe the solution $(\mathcal{A}_0, \sqrt{r}\,u_0)$ and its linearized operator.

The pair $(\mathcal{A}_0, \sqrt{r}\cdot u_0)$ is characterized (up to gauge) by the condition

$$\langle \nabla^0 u_0, u_0 \rangle = 0 \tag{44}$$

(where ∇^0 is the $Spin^c$-connection induced by $\mathcal{A}_0$) and can be obtained as follows: let u_0 be any section of $\underline{\mathbb{C}} \oplus K^{-1}$ with $|u_0| = 1$ and whose projection onto the second summand is zero. Likewise, let A be any connection on K^{-1} and let ∇^A be its induced $Spin^c$-connection on $W_0^+ = \underline{\mathbb{C}} \oplus K^{-1}$. Set $a = \langle u_0, \nabla^A u_0\rangle$. This defines an imaginary valued 1-form as can easily be seen:

$$a + \bar{a} = \langle \nabla^A u_0, u_0\rangle + \langle u_0, \nabla^A u_0 \rangle = d\,|u_0|^2 = 0$$

Define the connection $\mathcal{A}_0$ on K^{-1} by $\mathcal{A}_0 = A - a$ which induces the $Spin^c$-connection $\nabla^0 = \nabla^A - a$ on W_0^+. This connection clearly satisfies (44). With the choice of μ as in (43), the Seiberg-Witten equations (16) take the form

$$\begin{aligned} D_A\psi &= 0 \\ F_A^+ - F_{\mathcal{A}_0}^+ &= \frac{ir}{8}(|\alpha|^2 - 1 - |\beta|^2)\omega + \frac{ir}{4}(\bar{\alpha}\beta + \alpha\bar{\beta}) \end{aligned} \tag{45}$$

Since the β-component of u_0 is zero and since $|\alpha| = |u_0| = 1$, the pair $(\mathcal{A}_0, u_0)$ clearly solves the second equation of (45). The fact that is also solves the first equation relies on the closedness of ω as well as (44). Taubes [16] showed that there are, up to gauge, no other solutions to (45) and, as we shall presently see, that the solution $(\mathcal{A}_0, u_0)$ is a smooth solution in the sense that the linearisation of (45) at $(\mathcal{A}_0, u_0)$ has trivial cokernel. These two facts together prove the

Theorem 4.1 *Let (X, ω) be a symplectic manifold. Then $SW_X(W_0) = \pm 1$.*

Define $S : L^{1,2}(i\Lambda^1 \oplus W_0^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-)$ to be the linearized Seiberg-Witten operator for the solution $(\mathcal{A}_0, u_0)$. Thus, for $(b, (\xi_0, \xi_2)) \in L^{1,2}(i\Lambda^1 \oplus (\underline{\mathbb{C}} \oplus K^{-1}))$ we have

$$S(b, (\xi_0, \xi_2)) = \begin{array}{l} (D_{\mathcal{A}_0}(\xi_0, \xi_2) + \frac{\sqrt{r}}{2} b.u_0, \\ d^+ b - \sqrt{r} q(\xi, u_0) - \sqrt{r} q(u_0, \xi), \\ d^* b + i\frac{\sqrt{r}}{\sqrt{2}} \operatorname{Im}(\bar{u_0}\xi_2)) \end{array} \tag{46}$$

Let $S^* : L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-) \to L^{1,2}(i\Lambda^1 \oplus W_0^+)$ be the formal adjoint of S. The following proposition and corollary are proved in [13], section 4.

Proposition 4.2 *Let S and S^* be as above. Then the operator SS^* on $L^2(i\Lambda^0 \oplus$*

$i\Lambda^{2,+} \oplus W_0^-)$ is given by

$$SS^* = \frac{1}{4}\nabla^{0,*}\nabla^0 + \mathcal{R}_0 + \sqrt{r}\mathcal{R}_1 + \frac{r}{8} \tag{47}$$

where $\nabla^{0,*}$ is the adjoint of ∇^0 and where $\mathcal{R}_i$, $i = 0, 1$ are certain r-independent endomorphism on $L^2(i(\Lambda^0 \oplus \Lambda^{2,+}) \oplus W_0^-)$.

The proof is a straightforward calculation, terms of the form $D_{\mathcal{A}_0} D^*_{\mathcal{A}_0}$ are simplified using the Weitzenböck formula for the Dirac operator. An important consequence of (47) is the following:

Corollary 4.3 *With S and S^* as above, the smallest eigenvalue λ_1 of SS^* is bounded from below by $r/16$. In particular, S is invertible and S^{-1} satisfies the bounds*

$$||S^{-1}y||_2 \leq \frac{4}{\sqrt{r}}||y||_2 \quad \text{and} \quad ||S^{-1}y||_{1,2} \leq C\,||y||_2 \tag{48}$$

where C is r-independent.

4.3 The general case and $SW_X(W_E) = Gr_X(E)$

Consider now a $Spin^c$-structure $W_E = E \otimes W_0$ on X. The connection $\mathcal{A}_0$ on K^{-1} and a choice of a connection B_0 on E together induce a connection $B_0^{\otimes 2} \otimes \mathcal{A}_0$ on $E^{\otimes 2} \otimes K^{-1} = c_1(W_E^+)$ by the product rule

$$B_0^{\otimes 2} \otimes \mathcal{A}_0(\varphi_1 \otimes \varphi_2 \otimes \phi) = B_0(\varphi_1) \otimes \varphi_2 \otimes \phi + \varphi_1 \otimes B_0(\varphi_2) \otimes \phi + \varphi_1 \otimes \varphi_2 \otimes \mathcal{A}_0(\phi)$$

The space of connections on $E^{\otimes 2} \otimes K^{-1}$ is an affine space with associated vector space $i\Omega^1_X$. With the choice of a base connection $B_0^{\otimes 2} \otimes \mathcal{A}_0$ in place, we will from now on regard solutions to the Seiberg-Witten equations as pairs $(a,\psi) \in i\Omega^1_X \times \Gamma(W^+_E)$ rather than $(A,\psi) \in \mathrm{Conn}(E^{\otimes 2} \otimes K^{-1}) \times \Gamma(W^+_E)$, the relation between the two being

$$A = B_0^{\otimes 2} \otimes \mathcal{A}_0 + a$$

We will agree to use henceforth the choice of μ in (16) to be

$$\mu = -\frac{ir}{8} + F^+_{\mathcal{A}_0} \tag{49}$$

For $\psi \in \Gamma(E \otimes (\underline{\mathbb{C}} \oplus K^{-1}))$ we will write $\psi = \sqrt{r}\,(\alpha \otimes u_0, \beta)$ with $\alpha \in \Gamma(E)$ and $\beta \in \Gamma(E \otimes K^{-1})$ and u_0 as in the previous section.

With these conventions understood and with the use of (41), the Seiberg-Witten equations become

$$\begin{aligned} D_a\psi &= 0 \\ F_a^{1,1} &= \frac{ir}{8}(|\alpha|^2 - |\beta|^2 - 1)\,\omega \\ F_a^{0,2} &= \frac{ir}{4}\bar{\alpha}\beta \end{aligned} \tag{50}$$

Here $F_a^{i,j}$ is the orthogonal projection of $2\,F^+_{B_0} + d^+a$ onto $\Lambda^{i,j}$.

We also use this section to remind the reader of several useful bounds that a solution (a,ψ) of the Seiberg-Witten equations satisfies. These bounds are provided courtesy of [12] and their proofs rely solely on properties of the Seiberg-Witten equations.

A solution (a,ψ) of (50) satisfies the following bounds:

$$|\alpha| \leq 1 + \frac{C}{r}$$

$$|\beta|^2 \leq \frac{C}{r}(1-|\alpha|^2) + \frac{C'}{r^3} \tag{51}$$
$$|\nabla^A \alpha|_x^2 \leq C\sqrt{r}\, \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha^{-1}(0))\right), \quad x \in X$$
$$|1-|\alpha(x)|^2| \leq C \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha^{-1}(0))\right), \quad x \in X$$

The constants C and C' appearing above only depend on E and the Riemannian metric g but not on the particular choice of r.

The inequalities (51) (together with a monotonicity formula which we don't need for our discussion) are the basis of [12] where Taubes shows that every solution of the Seiberg-Witten equations gives rise to an embedded, possibly disconnected, J-holomorphic curve. The converse of this fact is also true. Namely, in [13] Taubes shows that every J-holomorphic curve with genus $g \geq 2$ can be used to construct a Seiberg-Witten monopole. Tori are special cases, not every torus gives rise to a Seiberg-Witten monopole, but certain collections of tori together do. In any case, the following theorem holds and is a magnificent culmination of the interplay between Seiberg-Witten and Gromov-Witten theory:

Theorem 4.4 (Taubes, 1996) *Let (X, ω) be a symplectic manifold with $b^+ \geq 2$ and $E \in H^2(X;\mathbb{Z})$. Then*

$$SW_X(W_E) = Gr_X(E) \tag{52}$$

Some of the most immediate consequences of theorem 4.4 are summarized in the following corollary. Recall that a manifold X is said to have Seiberg-Witten simple type if for all basic classes L of X, the dimensions of the corresponding moduli spaces $\mathcal{M}_X^{SW}(L)$, are all zero.

Corollary 4.5 *1. The Poincaré dual of the canonical class K of a symplectic manifold can be represented by an embedded J-holomorphic curve.*

2. There is a duality in Gromov-Witten theory relating the Gromov-Witten invariant of E to $K - E$ via

$$Gr_X(E) = \pm Gr_X(K - E) \tag{53}$$

3. Symplectic manifolds with $b^+ \geq 2$ have simple type.

Proof. The first claim is a direct consequence of theorems 4.1 and 4.4, while the second follows from theorem 4.4 and proposition 2.19.

The third point requires a bit more thought. If W_E is a Seiberg-Witten basic class, then by theorem 4.4, E is a Gromov-Witten basic class. Thus, its Poincaé dual can be represented by an embedded J-holomorphic curve Σ of genus g. Also, the dimension of the Gromov-Witten moduli space has to be non-negative:

$$\dim \mathcal{M}^{Gr}(E) = \frac{1}{2}(E^2 - K \cdot E) \geq 0 \tag{54}$$

Combining (54) and the adjunction formula (38), one obtains the following two inequalities:

$$E^2 \geq g - 1 \qquad \text{and} \qquad K \cdot E \leq g - 1 \tag{55}$$

Let $n \geq 0$ be the integer such that $E^2 = g - 1 + n$ and thus (via (38)) also $K \cdot E = g - 1 - n$. Now we use, the already proved, second point of the present corollary, namely $Gr(E) = \pm Gr(K - E)$. The conclusion is that the class $K - E$ is also a basic class and so its Poincaré dual is represented by another J-holomorphic curve Σ'.

Now one invokes the aforementioned result of Dusa McDuff [5] on the positivity of the local intersection of two J-holomorphic curves: $[\Sigma] \cap [\Sigma'] \geq 0$. Translated into cohomology classes this gives

$$0 \leq E \cdot (K - E) = E \cdot K - E^2 = g - 1 - n - (g - 1 + n) = -2n \leq 0$$

Thus $n = 0$ and $E^2 = K \cdot E = g - 1$. The Seiberg-Witten simple type is now an easy consequence:

$$\dim \mathcal{M}^{SW}(L) = \frac{1}{4}\left((2E - K)^2 - (3\sigma + 2e)\right) = \frac{1}{4}\left(K^2 - (3\sigma + 2e)\right) = 0$$

where $L = 2E - K$. ■

5 Grafting Seiberg-Witten Monopoles

This section is the heart of the thesis as it is here where we construct the grafted monopole (a, ψ) for the $Spin^c$-structure W_E from two monopoles (a_i, ψ_i) for the $Spin^c$-structures W_{E_i}, $i = 0, 1$ (where as usual $E = E_0 \otimes E_1$). Recall from the introduction that the term "grafing"refers to the map described by the top row in the commutative diagram (3).

5.1 Producing the approximate solution (a, ψ) from a pair (a_0, ψ_0), (a_1, ψ_1)

Let E_0 and E_1 be two complex line bundles over X. The aim of this section is to produce an approximate solution (a, ψ) of the Seiberg-Witten equations for the $Spin^c$-structure $W_{E_0 \otimes E_1}$ from two solutions (a_0, ψ_0) and (a_1, ψ_1) for the $Spin^c$-structures W_{E_0} and W_{E_1} respectively. Implicit to our discussion are the choices of two "base" connections B_0 and B_1 on E_0 and E_1 and the product connection $B_0 \otimes B_1$ they determine on $E_0 \otimes E_1$. As before, we will write $\psi_i = \sqrt{r}(\alpha_i \otimes u_0, \beta_i)$, $i = 0$, 1, and $\psi = \sqrt{r}(\alpha \otimes u_0, \beta)$. We define (a, ψ) as

$$\begin{aligned} a &= a_0 + a_1 \\ \alpha &= \alpha_0 \otimes \alpha_1 \\ \beta &= \alpha_0 \otimes \beta_1 + \alpha_1 \otimes \beta_0 \end{aligned} \tag{56}$$

The first task at hand is to check how close (a, ψ) comes to solving the Seiberg-Witten equations. We begin by calculating $D_a \psi$ locally at a point $x \in X$. Choose an orthonormal frame $\{e_i\}_i$ in a neighborhood of x and let $\{e^i\}_i$ be its dual frame.

$$\begin{aligned}
D_a(\psi) &= \sqrt{r} D_a(\alpha_0 \otimes \alpha_1 \otimes u_0 + \alpha_0 \otimes \beta_1 + \alpha_1 \otimes \beta_0) \\
&= \sqrt{r} D_{a_0}(\alpha_0 \otimes u_0) \otimes \alpha_1 + \sqrt{r}\alpha_0 \otimes D_{a_1}(\alpha_1 \otimes u_0) + \\
&\quad + \sqrt{r} e^i . \nabla^a_{e_i}(\alpha_0 \otimes \beta_1 + \alpha_1 \otimes \beta_0) \\
&= \sqrt{r} D_{a_0}(\alpha_0 \otimes u_0) \otimes \alpha_1 + \sqrt{r}\alpha_0 \otimes D_{a_1}(\alpha_1 \otimes u_0) + \\
&\quad + \sqrt{r}(\alpha_0 \otimes e^i.(\nabla^{a_1}_{e_i}\beta_1) + \alpha_1 \otimes e^i.(\nabla^{a_0}_{e_i}\beta_0) + \\
&\quad + (\nabla^{a_0}_{e_i}\alpha_0) \otimes e^i.\beta_1 + (\nabla^{a_1}_{e_i}\alpha_1) \otimes e^i.\beta_0) \\
&= \sqrt{r} D_{a_0}(\alpha_0 \otimes u_0) \otimes \alpha_1 + \sqrt{r}\alpha_0 \otimes D_{a_1}(\alpha_1 \otimes u_0) + \\
&\quad + \sqrt{r}(\alpha_0 \otimes D_{a_1}\beta_1 + \alpha_1 \otimes D_{a_0}\beta_0) + \\
&\quad + \sqrt{r}((\nabla^{a_0}_{e_i}\alpha_0) \otimes e^i.\beta_1 + (\nabla^{a_1}_{e_i}\alpha_1) \otimes e^i.\beta_0) \\
&= (D_{a_0}\psi_0) \otimes \alpha_1 + \alpha_0 \otimes (D_{a_1}\psi_1) + \\
&\quad + \sqrt{r}((\nabla^{a_0}_{e_i}\alpha_0) \otimes e^i.\beta_1 + (\nabla^{a_1}_{e_i}\alpha_1) \otimes e^i.\beta_0) \\
&= \sqrt{r}(\nabla^{a_0}_{e_i}\alpha_0) \otimes e^i.\beta_1 + \sqrt{r}(\nabla^{a_1}_{e_i}\alpha_1) \otimes e^i.\beta_0 \qquad (57)
\end{aligned}$$

It is easy to see, using the bounds in (51), that the first term in (57) satisfies the following pointwise estimate :

$$\begin{aligned}
& r\,|(\nabla^{a_0}_{e_i}\alpha_0) \otimes e^i.\beta_1|^2_x \leq \\
& \leq C r \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha_0^{-1}(0))\right) \cdot \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha_1^{-1}(0))\right) \qquad (58)
\end{aligned}$$

The second term in (57) satisfies the same bound. In order for the right hand side of (58) to pointwise converge to zero, it is sufficient and necessary that there exist some $r_0 \geq 1$ such that for all $r \geq r_0$, the distance from $\alpha_0^{-1}(0)$ to $\alpha_1^{-1}(0)$ be bounded from below by some r-independent $M > 0$. This condition, under the map Θ from (3), is the Seiberg-Witten equivalent of the condition that $\Sigma_i = \Theta(A_i, \psi_i)$ be disjoint curves. Thus, from now onward we will make the

Assumption: There exists an $r_0 \geq 1$ and $M > 0$ such that for all $r \geq r_0$ the inequality

$$\mathrm{dist}(\alpha_0^{-1}(0), \alpha_1^{-1}(0)) \geq M \tag{59}$$

holds.

We now proceed by looking at the second equation in (50):

$$\begin{aligned}
F_a^{1,1} - \frac{i}{8} r\,(|\alpha|^2 - 1 - |\beta|^2)\,\omega = \\
&= F_{a_0}^{1,1} + F_{a_1}^{1,1} - \frac{i}{8} r\,(|\alpha_0|^2 \cdot |\alpha_1|^2 - 1 - |\alpha_0|^2 \cdot |\beta_1|^2 - |\alpha_1|^2 \cdot |\beta_0|^2 - 2\langle \alpha_0\beta_1, \alpha_1\beta_0\rangle)\,\omega \\
&= F_{a_0}^{1,1} + F_{a_1}^{1,1} - \frac{i}{8} r\,|\alpha_1|^2(|\alpha_0|^2 - 1 - |\beta_0|^2)\,\omega - \frac{i}{8} r\,|\alpha_0|^2(|\alpha_1|^2 - 1 - |\beta_1|^2)\,\omega + \\
&\quad + \frac{i}{8} r\,(|\alpha_0|^2 - 1)(|\alpha_1|^2 - 1)\,\omega + \frac{i}{4} r\,\langle \alpha_0\beta_1, \alpha_1\beta_0\rangle\,\omega \\
&= \frac{i}{8} r\,(1 - |\alpha_1|^2)(|\alpha_0|^2 - 1 - |\beta_0|^2)\,\omega - \frac{i}{8} r\,(1 - |\alpha_0|^2)(|\alpha_1|^2 - 1 - |\beta_1|^2)\,\omega + \\
&\quad + \frac{i}{8} r\,(|\alpha_0|^2 - 1)(|\alpha_1|^2 - 1)\,\omega + \frac{i}{4} r\,\langle \alpha_0\beta_1, \alpha_1\beta_0\rangle\,\omega
\end{aligned}$$

From this last equation, and again using (51), one easily deduces that

$$\begin{aligned}
&|F_a^{1,1} - \frac{i}{8} r\,(|\alpha|^2 - 1 - |\beta|^2)\,\omega| \leq \\
&\quad \leq C r \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha_0^{-1}(0))\right) \cdot \exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x, \alpha_1^{-1}(0))\right) + \frac{C}{\sqrt{r}}
\end{aligned} \tag{60}$$

Finally, we consider the third equation in (50):

$$\begin{aligned}
F_a^{0,2} - \frac{i}{4} r\,\bar{\alpha}\beta &= F_{a_0}^{0,2} + F_{a_1}^{0,2} - \frac{i}{4} r\,\overline{\alpha_0\alpha_1}(\alpha_0\beta_1 + \alpha_1\beta_0) \\
&= \frac{i}{4} r\,\bar{\alpha}_0\beta_0 + \frac{i}{4} r\,\bar{\alpha}_1\beta_1 - \frac{i}{4} r\,|\alpha_0|^2\bar{\alpha}_1\beta_1 - \frac{i}{4} r\,|\alpha_1|^2\bar{\alpha}_0\beta_0
\end{aligned}$$

$$= \frac{i}{4} r\,(1-|\alpha_1|^2)\bar{\alpha_0}\beta_0 + \frac{i}{4} r\,(1-|\alpha_0|^2)\bar{\alpha_1}\beta_1$$

Once again using the bounds (51), we find from this last equation:

$$|F_a^{0,2} - \frac{i}{4} r\,\bar{\alpha}\beta| \leq \tag{61}$$
$$\leq Cr\exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x,\alpha_0^{-1}(0))\right)\cdot\exp\left(-\frac{\sqrt{r}}{C}\mathrm{dist}(x,\alpha_1^{-1}(0))\right) + \frac{C}{\sqrt{r}}$$

To summarize, we have proved the following

Proposition 5.1 *Let (a, ψ) be defined as in (56) and assume that there exists an $r_0 \geq 1$ and $M > 0$ such that for all $r \geq r_0$, the distance $dist(\alpha_0^{-1}(0), \alpha_1^{-1}(0))$ is bounded from below by M. Then for large enough r and any $x \in X$ the pointwise bound below holds:*

$$|(D_a(\psi), F_a^{1,1} - \frac{i}{8} r\,(|\alpha|^2 - 1 - |\beta|^2)\,\omega, F_a^{0,2} - \frac{i}{4} r\bar{\alpha}\beta)|_x \leq \frac{C}{\sqrt{r}} \tag{62}$$

5.2 Inverting the linearized operators of (a_i, ψ_i)

This section serves as a digression of sorts. The main result here is theorem 5.5, an asymptotic (as $r \to \infty$) regularity statement for the linear operators $L_{(a_i,\psi_i)}$ (as defined by (18)).

We start with two easy auxiliary lemmas:

Lemma 5.2 *Let $L : V \to W$ be a surjective Fredholm operator between Hilbert spaces. Then there exists a $\delta > 0$ such that for every linear operator $\ell : V \to W$ with $||\ell(x)||_W \leq \delta\,||x||_V$, the operator $L + \ell$ is still surjective.*

Proof. Since L is Fredholm, we can orthogonally decompose V as $V = \text{Ker}(L) \oplus \text{Im}(L^*)$. Let L_1 be the restriction of L to $\text{Im}(L^*)$. Then $L_1 : \text{Im}(L^*) \to W$ is an isomorphism with bounded inverse L_1^{-1}.

If the lemma were not true then we could find for all integers $n \geq 1$ an operator $\ell_n : V \to W$ with $||\ell_n x||_W \leq 1/n \cdot ||x||_V$ and with $\text{Coker}(L + \ell_n) \neq \{0\}$. Let $0 \neq y_n \in \text{Coker}(L + \ell_n)$ with $||y_n||_W = 1$ and $x_n = L_1^{-1}(y_n)$. Notice that the sequence $\{x_n\}_n$ is bounded by $||L_1^{-1}||$. Since $y_n \in \text{Coker}(L+\ell_n)$, y_n is orthogonal to $\text{Im}(L+\ell_n)$. In particular,

$$\langle (L + \ell_n)x_n, y_n \rangle = 0$$

This immediately leads to a contradiction for large enough n since $\langle L\, x_n, y_n \rangle = 1$ and $|\langle \ell_n x_n, y_n \rangle| \leq ||L_1^{-1}||/n$. ■

Lemma 5.3 *Let V and W be two finite rank vector bundles over X and $L_r : L^{1,2}(V) \to L^2(W)$ a smooth one-parameter family (indexed by $r \geq 1$) of elliptic, first order, differential operators of index zero. Assume further that there exists a $\delta > 0$ and $r_0 \geq 1$ such that for any zeroth order linear operator $\ell : L^{1,2}(V) \to L^2(W)$ with $||\ell(x)||_2 < \delta ||x||_{1,2}$, the operator $L_r + \ell$ is onto. Then there exists a $r_1 \geq r_0$ and a $M > 0$ such that for all $r \geq r_1$ the inverses of the operators L_r are uniformly bounded by M, i.e. $||L_r^{-1} y||_{1,2} \leq M ||y||_2$.*

Proof. Notice that a universal upper bound on L_r^{-1} is equivalent to a universal lower bound on L_r. Suppose the lemma is not true: then there exists a sequence $r_n \to \infty$ and $x_n \in L^{1,2}(V)$ with $||x_n||_{1,2} = 1$ and $||L_{r_n} x_n||_2 < 1/n$. Choose n

large enough so that $1/n < \delta$ and define the operator $\ell : L^{1,2}(V) \to L^2(W)$ by $\ell(x) = -\langle x_n, x\rangle_{1,2} \cdot L_{r_n}(x_n)$. For this ℓ the assumption of the lemma is met, namely

$$||\ell(x)||_2 \leq \frac{1}{n}||x||_{1,2} < \delta\,||x||_{1,2}$$

Thus the operator $L_{r_n} + \ell$ should be onto and into (since the index of $L_r + \ell$ is zero). But x_n is clearly a nonzero kernel element. This is a contradiction. ∎

Recall that the set $\mathcal{J}$ of almost-complex structures compatible with the symplectic form ω, contains a Baire subset $\mathcal{J}_0$ of generic almost-complex structures in the sense of Gromov-Witten theory (see [15]). Also, as in the introduction, let

$$\Theta : \mathcal{M}_X^{SW}(W_E) \to \mathcal{M}_X^{Gr}(E) \tag{63}$$

be the map introduced in [12] which associates an embedded J-holomorphic curve to a Seiberg-Witten monopole.

Proposition 5.4 *Let J be chosen from $\mathcal{J}_0$ and let (a,ψ) be a solution of the Seiberg-Witten equations* (50) *such that $\Theta(a,\psi)$ doesn't contain any multiply covered components. Then there exists a $\delta > 0$ and an $r_0 \geq 1$ such that for all linear operators $\ell : L^{1,2}(i\Lambda^1 \oplus E \otimes W_0^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus E \otimes W_0^-)$ with norm $||\ell(x)||_2 < \delta||x||_{1,2}$, the operator $L_{(a,\psi)} + \ell$ is surjective.*

Before proceeding, the reader interested in the proof of proposition 5.4 is advised to familiarize her/himself with the definitions and notation in [13] since our proof will heavily rely on results proved therein.

Proof. The proof is a bit technical and relies on the even more technical account from [13] on the connection between the deformation theory of the Seiberg-Witten equations on one hand and the Gromov-Witten equation on the other. The idea is however very simple: for large $r \gg 1$, a certain perturbation of the operator L (with the size of the perturbation getting smaller with larger r) has no cokernel if a certain perturbation of the linearisation of the generalized del-bar operator has no cokernel. The latter is ensured by the choice of a generic almost complex structure J from the Baire set $\mathcal{J}_0$ of almost complex structures compatible with ω.

For the convenience of the reader we restate here the parts of lemma 4.11 and a slightly modified version of lemma 6.7 from [13] relevant to our situation.

Lemma 4.11 *The equation $Lq + \eta q = g$ is solvable if and only if, for each k, the equation*

$$\Delta_{c_k} w^k + \gamma_0^k(w) + \eta_k(w) = x(g^k) + \gamma_1^k(g)$$

is solvable.

In the above, k indexes the set of components of $\Theta(a, \psi)$ and Δ_{c_k} represents, roughly speaking, the Cauchy-Riemann operator associated to the component of $\Theta(a, \psi)$ with index k. The terms η_k and $x(g^k)$ are constructed from η and g respectively while γ_0 and γ_1 are some auxiliary operators which depend on r (and whose norm gets smaller as r increases, they should be thought of as small correction terms). The assignment of η_k to η is linear i.e. for two operators η and η', we have $(\eta + \eta')_k = \eta_k + \eta'_k$.

Lemma 6.7′ *The equation $(L_{\Psi_r(y)} + \ell)p = g$ has an $L^{1,2}$ solution p if and only if there exists $u = (u^1, ..., u^k) \in \oplus_k L^{1,2}(N^{(k)})$ for which*

$$\Delta_y u^k + \phi_0^k(u) + \ell_k(u) = \Upsilon_1^{-1} x(g^k) + \phi_1^k(g)$$

holds for each k.

Here y is a J-holomorphic curve without multiply covered components and $\Psi_r(y)$ is its associated Seiberg-Witten monopole. As with the notation in lemma 4.11, the terms ℓ_k and $x(g^k)$ are constructed from ℓ and g respectively. Similar to the terms γ_i from lemma 4.11, the terms ϕ_i serve as correction terms whose size diminishes as r grows.

The proof of lemma 6.7$'$ is almost identical to that of the original lemma 6.7 in [13]. The only difference is in *Step 2* where Taubes shows that one can write the equation $L_{\Psi_r(y)}p = g$ in the form $Lp + \eta p = g$ with L as in lemma 4.11 and with $\eta = \sqrt{r}\, 2\,\omega(q', p)$. The difference here is that in our case one can write $(L_{\Psi_r(y)} + \ell)p = g$ as $Lp + \eta' p = g$ (with L again as in lemma 4.11) but with $\eta'(p) = \sqrt{r} 2\omega(q', p) + \ell(p)$. Since ℓ is assumed bounded, lemma 4.11 applies to η' in the exact same way as it applied to the original η and the proof of lemma 6.7 in [13] transfers verbatim to our case. Note also that the operators ϕ_i^k occurring in lemmas 6.7 and 6.7$'$ are identical so in particular they continue to satisfy the bounds asserted in lemma 6.7 of [13].

According to lemma 5.2 there exists a $\delta' > 0$ such that $\Delta_y + \ell'$ is still surjective if $||\ell'|| < \delta'$. Choose r large enough so that $||\phi_0^k|| < \delta'/2k$. On the other hand, since $\ell_k(v) = \pi(\chi_{25\delta,k}\ell(\sum_{k'} \chi_{100\delta,k'}\underline{v}^{k'}))$ (see (66) for a definition of $\chi_{\delta,k}$ in the present context) we find that $||\ell_k|| \leq C\,||\ell||$. Thus choosing $\delta = \delta'/2C$ ensures that $L_{\Psi_r(y)} + \ell$ is surjective provided that $||\ell|| < \delta$. This finishes to proof of proposition 5.4. ■

Together, the last lemma and proposition imply the following:

Theorem 5.5 *Choose $J \in \mathcal{J}_0$ and let (a, ψ) be a solution of the Seiberg-Witten* *equations for the Spin^c-structure W_E with parameter r. Assume that $\Theta(a,\psi)$ contains no multiply covered components. Then there exists a $r-$independent $M > 0$ and $r_0 \geq 1$ such that for all $r \geq r_0$*

$$||L_{(a,\psi)}^{-1}x||_{1,2} \leq M||x||_2 \tag{64}$$

5.3 The linearized operator at (a, ψ)

In order to use the contraction mapping principle to deform the approximate solution (a, ψ) to an honest solution of the Seiberg-Witten equations, we need to know that $L = L_{(a,\psi)}$ admits an inverse whose norm is bounded independently of r. We start by exploring when the equation

$$L\xi = g \tag{65}$$

has a solution ξ for a given g. Here

$$\xi \in L^{1,2}(i\Lambda^1 \oplus (E_0 \otimes E_1 \otimes W_0^+)) \quad \text{and} \quad g \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus (E_0 \otimes E_1 \otimes W_0^-))$$

The idea is to restrict equation (65) first to a neighborhood of $\alpha_0^{-1}(0)$. Over such a neighborhood the bundle E_1 is trivial and, under an isomorphism trivializing E_1, the equation (65) becomes a zero-th order perturbation of the equation $L_0\xi_0 = g_0$ (with ξ_0 and g_0 being appropriately defined in terms of ξ and g). This allows one to take advantage of the results of theorem 5.5 about the inverse of $L_0 = L_{(a_0,\psi_0)}$. Then one restricts (65) to a neighborhood of $\alpha_1^{-1}(0)$ where the bundle E_0 trivializes and once again uses theorem 5.5, this time for the inverse of $L_1 = L_{(a_1,\psi_1)}$. Finally, one restricts to the complement of a neighborhood of $\alpha_0^{-1}(0) \cup \alpha_1^{-1}(0)$ where both E_0 and

E_1 become trivial and L becomes close to S - the linearized operator of the unique solution $(\mathcal{A}_0, \sqrt{r}\, u_0)$ for the anticanonical $Spin^c$-structure W_0.

To begin this process, choose regular neighborhoods V_i of $\alpha_i^{-1}(0)$, $i = 0, 1$ subject to the condition

$$\text{dist}(V_0, V_1) \geq M \qquad \text{for some } M > 0$$

The existence of such neighborhoods V_i follows from our main assumption (59). A priori, as one chooses larger values of r, it seems that the sets V_i may need to be chosen anew as well. However, it was shown in [12], section 5c, that in fact this is not necessary. An initial "smart" choice of V_i for large enough r ensures that for $r' > r$, the zero sets $\alpha_i^{-1}(0)$ continue to lie inside of V_i. Choose an open set U such that $X = V_0 \cup V_1 \cup U$ and such that

$$U \cap (\alpha_0^{-1}(0) \cup \alpha_1^{-1}(0)) = \emptyset$$

Arrange the choices of V_i and U further so that ∂V_i is an embedded 3-manifold of X and so that $U \cap V_i$ contains a collar $\partial V_i \times I$. Here I is some segment $[0, d]$ and ∂V_i corresponds to $\partial V_i \times \{d\}$. For the sake of simplicity of notation, we shall make the assumption that for large values of r, the sets $\alpha_i^{-1}(0)$, $i = 0, 1$, are connected. The case of disconnected zero sets of the α_i's is treated much in the same way except for that in the following, one would have to choose a bump function $\chi_{\delta,i}$ (see below) for each connected component. This complicates notation to a certain degree but doesn't lead to new phenomena.

Fix once and for all a bump function $\chi : [0, \infty) \to [0, 1]$ which is 1 on [0,1] and 0 on $[2,\infty)$. For $0<\delta<d/1000$ define $\chi_{\delta,i}:X\to[0,1]$ by

$$\chi_{\delta,i}(x)=\begin{cases}1 & x\in V_i\backslash(\partial V_i\times I)\\ \chi(t/\delta) & x=(y,t)\in\partial V_i\times I\\ 0 & x\notin V_i\end{cases}\tag{66}$$

Set $V_0'=V_0\cup U$ and $V_1'=V_1\cup U$. Define the isomorphisms $\Upsilon_0:\mathbb{C}\times V_0'\to E_1|_{V_0'}$ and $\Upsilon_1:\mathbb{C}\times V_1'\to E_0|_{V_1'}$ as $\Upsilon_0(\lambda,x)=\alpha_1(x)\cdot\lambda$ and $\Upsilon_1(\lambda,x)=\alpha_0(x)\cdot\lambda$. The isomorphism Υ_1 defined here shouldn't be confused with the isomorphism of the same name appearing in lemma 6.7′. While both serve to identify a pair of bundles, the bundles in question in these two situations are not the same.

For $i=0$, 1 define the operators

$$M_i:L^{1,2}(i\Lambda^1\oplus(E_i\otimes W_0^+);V_i')\to L^2(i\Lambda^0\oplus i\Lambda^{2,+}\oplus(E_i\otimes W_0^-);V_i')$$

and

$$T:L^{1,2}(i\Lambda^1\oplus W_0^+;U)\to L^2(i\Lambda^0\oplus i\Lambda^{2,+}\oplus W_0^-;U)$$

by demanding the diagrams

$$\begin{array}{ccc}L^{1,2}(i\Lambda^1\oplus(E_0\otimes E_1\otimes W_0^+);V_i') & \xleftarrow{\Upsilon_i} & L^{1,2}(i\Lambda^1\oplus(E_i\otimes W_0^+);V_i')\\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle M_i}\\ L^2(i\Lambda^0\oplus i\Lambda^{2,+}\oplus(E_0\otimes E_1\otimes W_0^-);V_i') & \xleftarrow{\Upsilon_i} & L^2(i\Lambda^0\oplus i\Lambda^{2,+}\oplus(E_i\otimes W_0^-);V_i')\end{array}$$

and

$$\begin{array}{ccc}L^{1,2}(i\Lambda^1\oplus(E_0\otimes E_1\otimes W_0^+);U) & \xleftarrow{\Upsilon_0\circ\Upsilon_1} & L^{1,2}(i\Lambda^1\oplus W_0^+;U)\\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle T}\\ L^2(i\Lambda^0\oplus i\Lambda^{2,+}\oplus(E_0\otimes E_1\otimes W_0^-);U) & \xleftarrow{\Upsilon_0\circ\Upsilon_1} & L^2(i\Lambda^0\oplus i\Lambda^{2,+}\oplus W_0^-;U)\end{array}$$

to be commutative diagrams.

We now start our search for a solution ξ of (65) in the form

$$\xi=\Upsilon_0(\chi_{100\delta,0}\xi_0)+\Upsilon_1(\chi_{100\delta,1}\xi_1)+\Upsilon_0\Upsilon_1\left((1-\chi_{4\delta,0})(1-\chi_{4\delta,1})\eta\right)\tag{67}$$

Here $\xi_i \in L^{1,2}(i\Lambda^1 \oplus (E_i \otimes W_0^+))$ and $\eta \in L^{1,2}(i\Lambda^1 \oplus W_0^+)$. Given a $g \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus (E_0 \otimes E_1 \otimes W_0^-))$, define $g_i \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus (E_i \otimes W_0^-))$ and $\gamma \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-)$ as

$$g_i = \Upsilon_i^{-1}(\chi_{25\delta,i}\, g) \quad \text{and} \quad \gamma = (\Upsilon_0 \Upsilon_1)^{-1}\left((1-\chi_{25\delta,0})(1-\chi_{25\delta,1})g\right) \tag{68}$$

It is easy to check that g, g_i and γ satisfy a relation similar to (67), namely

$$g = \Upsilon_0(\chi_{100\delta,0}\, g_0) + \Upsilon_1(\chi_{100\delta,1}\, g_1) + \Upsilon_0\Upsilon_1\left((1-\chi_{4\delta,0})(1-\chi_{4\delta,1})\gamma\right) \tag{69}$$

Putting the form (67) of ξ and the form (69) of g into equation (65), after a few simple manipulations, yields the equation

$$\begin{aligned} &\Upsilon_0(\chi_{100\delta,0}(M_0(\xi_0) - \Upsilon_1 \mathcal{P}(d\chi_{4\delta,0}, \eta) - g_0)) + \\ &+\Upsilon_1(\chi_{100\delta,1}(M_1(\xi_1) - \Upsilon_0 \mathcal{P}(d\chi_{4\delta,1}, \eta) - g_1)) + \\ &+\Upsilon_0\Upsilon_1((1-\chi_{4\delta,0})(1-\chi_{4\delta,1})\,(T\eta + \mathcal{P}(d\chi_{100\delta,0}, \xi_0) + \mathcal{P}(d\chi_{100\delta,1}, \xi_1) - \gamma)) \quad = 0 \end{aligned} \tag{70}$$

In the above, $\mathcal{P}$ denotes the principal symbol of L. This last equation suggests a splitting into three equations (each corresponding to one line in (70)):

$$\begin{aligned} M_0(\xi_0) - \Upsilon_1 \mathcal{P}(d\chi_{4\delta,0}, \eta) &= g_0 \\ M_1(\xi_1) - \Upsilon_0 \mathcal{P}(d\chi_{4\delta,1}, \eta) &= g_1 \\ T\eta + \mathcal{P}(d\chi_{100\delta,0}, \xi_0) + \mathcal{P}(d\chi_{100\delta,1}, \xi_1) &= \gamma \end{aligned} \tag{71}$$

Equation (70) (and hence also equation (65)) can be recovered from (71) by multiplying the three equations by $\Upsilon_0 \cdot \chi_{100\delta,0}$, $\Upsilon_1 \cdot \chi_{100\delta,1}$ and $\Upsilon_0\Upsilon_1 \cdot ((1-\chi_{4\delta,0})(1-\chi_{4\delta,1})$ respectively and then adding them. Thus, given a g and with g_i and γ defined by

(68), solutions ξ_i and η of (71) lead to a solution ξ of (65) via (67). However, the problem with (71) is that the operators M_i and T are not defined over all of X. We remedy this in the next step.

Define new operators :

$$M_i' : L^{1,2}(i\Lambda^1 \oplus (E_i \otimes W_0^+)) \to L^2(i\Lambda^0 \oplus \Lambda^{2,+} \oplus (E_i \otimes W_0^-))$$

and

$$T' : L^{1,2}(i\Lambda^1 \oplus W_0^+) \to L^2(i\Lambda^0 \oplus \Lambda^{2,+} \oplus W_0^-)$$

by

$$\begin{aligned} M_i' &= \chi_{200\delta,i} M_i + (1 - \chi_{200\delta,i}) L_i \\ T' &= (1 - \chi_{\delta,0})(1 - \chi_{\delta,1}) T + (\chi_{\delta,0} + \chi_{\delta,1}) S \end{aligned} \tag{72}$$

Here $L_i = L_{(a_i, \psi_i)}$. Now replace the coupled equations (71) by the following system:

$$\begin{aligned} M_0'(\xi_0) - \Upsilon_0 \mathcal{P}(d\chi_{4\delta,0}, \eta) &= g_0 \\ M_1'(\xi_1) - \Upsilon_1 \mathcal{P}(d\chi_{4\delta,1}, \eta) &= g_1 \\ T'\eta + \mathcal{P}(d\chi_{100\delta,0}, \xi_0) + \mathcal{P}(d\chi_{100\delta,1}, \xi_1) &= \gamma \end{aligned} \tag{73}$$

The advantage of (73) over (71) is that the former is defined over all of X. On the other hand, solutions of (73) give rise to solutions of (65) in the same way as solutions of (71) did because

$$\chi_{100\delta,i} \cdot M_i' = \chi_{100\delta,i} \cdot M_i \qquad i = 0,\ 1$$

$$(1 - \chi_{4\delta,0})(1 - \chi_{4\delta,1}) T' = (1 - \chi_{4\delta,0})(1 - \chi_{4\delta,1}) T$$

Lemma 5.6 *For every $\epsilon > 0$ there exists an $r_\epsilon \geq 1$ such that for $r \geq r_\epsilon$ the following hold:*

$$||(M_i' - L_i)x_i||_2 \leq \epsilon ||x_i||_2$$

$$||(T' - S)y||_2 \leq \epsilon ||y||_2$$

Here $x_i \in L^{1,2}(i\Lambda^1 \oplus E_i \otimes W_0^+)$ and $y \in L^{1,2}(i\Lambda^1 \oplus W_0^+)$.

Proof. The above Sobolev inequalities are proved by first calculating pointwise bounds for $|(M_i' - L_i)x_i|_p$ and $|(T' - S)y|_p$, $p \in X$. Notice firstly that $|(M_i' - L_i)x_i|_p = 0$ if $p \in V_i$ and $|(T' - S)y|_p = 0$ if $p \notin U$. For $p \notin V_i$ and for $q \in U$, a straightforward but somewhat tedious calculation shows that

$$|(M_i' - L_i)x_i|_p \leq C\ \left(\sqrt{r}\,|1 - |\alpha_i|^2| + \sqrt{r}\,|\beta_i|\,|\alpha_i| + |\nabla^{a_i}\alpha_i|\right)|x_i|_p$$

$$|(T' - S)y|_q \leq C\,(\sqrt{r}\,|1 - |\alpha_0|^2| + \sqrt{r}\,|1 - |\alpha_1|^2| + \sqrt{r}\,|\beta_0| +$$

$$+ \sqrt{r}\,|\beta_1| + |\nabla^{a_0}\alpha_0| + |\nabla^{a_1}\alpha_1|)\,|y|_q$$

Squaring and then integrating both sides over X together with a reference to (51) gives the desired Sobolev inequalities. ■

The lemma suggests that the system (73) can be replaced by the system

$$\begin{aligned} L_0(\xi_0') - \Upsilon_0\mathcal{P}(d\chi_{4\delta,0}, \eta') &= g_0 \\ L_1(\xi_1') - \Upsilon_1\mathcal{P}(d\chi_{4\delta,1}, \eta') &= g_1 \\ S\eta' + \mathcal{P}(d\chi_{100\delta,0}, \xi_0') + \mathcal{P}(d\chi_{100\delta,1}, \xi_1') &= \gamma \end{aligned} \tag{74}$$

Lemmas 5.6 and 5.2 say that for $r \gg 0$, (73) has a solution (ξ_0, ξ_1, η) if (74) has a solution (ξ_0', ξ_1', η'). It is this latter set of equations that we now proceed to solve.

Since S is onto, we can solve the third equation in (74), regarding ξ_0' and ξ_1' as parameters. Thus

$$\eta' = \eta'(\xi_0', \xi_1') = S^{-1}(\gamma - \mathcal{P}(d\chi_{100\delta,0}, \xi_0') - \mathcal{P}(d\chi_{100\delta,1}, \xi_1')) \tag{75}$$

Recall that the inverse of S satisfies the bound (48)

$$||S^{-1}y||_2 \leq \frac{4}{\sqrt{r}}||y||_2 \quad \text{for} \quad y \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-)$$

We will solve the first two equations in (74) simultaneously by first rewriting them in the form:

$$\begin{aligned} \xi_0' =& L_0^{-1}(g_0 + \Upsilon_0 \mathcal{P}(d\chi_{4\delta,0}, \eta'(\xi_0', \xi_1'))) \\ \xi_1' =& L_1^{-1}(g_1 + \Upsilon_1 \mathcal{P}(d\chi_{4\delta,1}, \eta'(\xi_0', \xi_1'))) \end{aligned} \tag{77}$$

To solve (77) is the same as to find a fixed point of the map $Y : L^2(i\Lambda^1 \oplus W_{E_0}^+) \times L^2(i\Lambda^1 \oplus W_{E_1}^+) \to L^2(i\Lambda^1 \oplus W_{E_0}^+) \times L^2(i\Lambda^1 \oplus W_{E_1}^+)$ given by

$$\begin{aligned} &Y(\xi_0', \xi_1') = \\ &\quad =(L_0^{-1}(g_0 + \Upsilon_1^{-1}\Upsilon_0\Upsilon_1\mathcal{P}(d\chi_{4\delta,0}, \eta')), L_1^{-1}(g_1 + \Upsilon_1\mathcal{P}(d\chi_{4\delta,1}, \eta'))) \end{aligned} \tag{78}$$

with η' given by 75. The existence and uniqueness of such a fixed point will be guaranteed by the fixed point theorem for Banach spaces if we can show that Y is a contraction mapping. To see this, let x, $y \in L^2(i\Lambda^1 \oplus W_{E_0}^+) \times L^2(i\Lambda^1 \oplus W_{E_1}^+)$ be two arbitrary sections. Using the first bound of (48) and the result of theorem 5.5 to bound the norms of L_i^{-1}, one finds

$$||Y(x) - Y(y)||_2^2 =$$

$$
\begin{aligned}
&= ||L_0^{-1}(g_0 + \Upsilon_0 \mathcal{P}(d\chi_{4\delta,0}, \eta(x))) - L_0^{-1}(g_0 + \Upsilon_0 \mathcal{P}(d\chi_{4\delta,0}, \eta(y)))||_2^2 \\
&+ ||L_1^{-1}(g_1 + \Upsilon_1 \mathcal{P}(d\chi_{4\delta,1}, \eta(x))) - L_1^{-1}(g_1 + \Upsilon_1 \mathcal{P}(d\chi_{4\delta,1}, \eta(y)))||_2^2 \\
&\leq C_0||\eta(x) - \eta(y)||_2^2 + C_1||\eta(x) - \eta(y)||_2^2 \\
&\leq C||S^{-1}(\mathcal{P}(d\chi_{100\delta,0}, y) - \mathcal{P}(d\chi_{100\delta,0}, x) + \mathcal{P}(d\chi_{100\delta,1}, y) - \mathcal{P}(d\chi_{100\delta,1}, x))||_2^2 \\
&\leq \frac{C}{r}||x - y||_2^2
\end{aligned} \tag{79}
$$

Choosing $r > 2C$, where C is the constant in the last line of (79), makes Y a contraction mapping. Thus we finally arrive at an L^2 solution (ξ_0', ξ_1'). It is in fact an $L^{1,2}$ solution because of (77). This, together with equation (75) provides a solution (ξ_0', ξ_1', η') of (74). As explained above, this gives rise to a solution (ξ_0, ξ_1, η) of (73) and thus provides a solution $\xi \in L^{1,2} \in (i\Lambda^1 \oplus W_0^+)$ of (65). In particular, we have proved half of the following

Theorem 5.7 *Let* (a, ψ) *be constructed from* (a_i, ψ_i) *as in* (56). *Suppose that the* (a_i, ψ_i) *meet assumption* (59) *and that* J *has been chosen from the Baire set* $\mathcal{J}_0$ *of compatible almost complex structures. Then* $L_{(a,\psi)} : L^{1,2}(i\Lambda^1 \oplus E_0 \otimes E_1 \otimes W_0^+) \rightarrow L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus E_0 \otimes E_1 \otimes W_0^-)$ *is invertible with bounded inverse* $||L_{(a,\psi)}^{-1} y||_{1,2} \leq C\,||y||_2$ *for all sufficiently large* r. *Here* C *is independent of* r.

Proof. It remains to prove the inequality $||L_{(a,\psi)}^{-1} y||_{1,2} \leq C\,||y||_2$. Each of the two lines of (77), together with the bound (64) on L_i^{-1}, yields:

$$
||\xi_i'||_{1,2} \leq C\,(||g_i||_2 + ||\eta'(\xi_0', \xi_1')||_2) \tag{80}
$$

A bound for the second term on the right-hand side of (80) comes from (75) and the

L^2 bound in (48):

$$||\eta'(\xi_0', \xi_1')||_2 \le \frac{C}{\sqrt{r}}(||\gamma||_2 + ||\xi_0'||_2 + ||\xi_1'||_2) \tag{81}$$

Adding the two inequalities (80) for $i = 0$, 1 and using (81) gives

$$(1 - \frac{C}{\sqrt{r}})\,(||\xi_0'||_{1,2} + ||\xi_1'||_{1,2}) \le C\,(||g_0||_2 + ||g_1||_2 + \frac{1}{\sqrt{r}}||\gamma||_2) \tag{82}$$

For large enough r, this last inequality gives a bound on the $L^{1,2}$ norm of (ξ_0', ξ_1') in terms of an r-independent multiple of the L^2 norm of (g_0, g_1, γ). With this established, the missing piece, namely the $L^{1,2}$ bound of η', comes from (75) and the $L^{1,2}$ bound in (48):

$$||\eta'||_{1,2} \le C\,(||\gamma||_2 + ||\xi_0'||_2 + ||\xi_1'||_2) \le C\,(||\gamma||_2 + ||g_0||_2 + ||g_1||_2) \tag{83}$$

It remains to relate the now established bound on (ξ_0', ξ_1', η') to a bound for (ξ_0, ξ_1, η). To begin doing that, write the systems (74) and (73) schematically as

$$\mathcal{F}(\xi_0', \xi_1', \eta') = (g_0, g_1, \gamma) \quad \text{and} \quad \mathcal{G}(\xi_0, \xi_1, \eta) = (g_0, g_1, \gamma)$$

Lemma 5.6 implies that for any $\varepsilon > 0$ there exists a $r_\varepsilon \ge 1$ such that for all $r \ge r_\varepsilon$ the inequality $||(\mathcal{F} - \mathcal{G})\,x||_2 \le \varepsilon\,||x||_2$ holds. The established surjectivity of $\mathcal{F}$ guarantees (by means of lemma 5.2) that $\mathcal{G}$ is also surjective. The proof of theorem 5.7 thus far, also shows that $||\mathcal{F}^{-1}|| \le C$ where C is $r-$independent. Now the standard inequality

$$||\mathcal{G}^{-1}|| \le ||\mathcal{F}^{-1}|| + ||\mathcal{G}^{-1} - \mathcal{F}^{-1}|| \le ||\mathcal{F}^{-1}|| + ||\mathcal{F}^{-1}|| \cdot ||\mathcal{G}^{-1}|| \cdot ||\mathcal{G} - \mathcal{F}||$$

implies the $r-$independent bound for $||\mathcal{G}^{-1}||$

$$||\mathcal{G}^{-1}|| \le \frac{||\mathcal{F}^{-1}||}{1 - ||\mathcal{F}^{-1}|| \cdot ||\mathcal{G} - \mathcal{F}||} \le \frac{C}{1 - C\varepsilon}$$

This last inequality provides $L^{1,2}$ bounds on (ξ_0, ξ_1) and η in terms of the L^2 norms of (g_0, g_1) and γ which in turn imply an $r-$independent $L^{1,2}$ bound on $\xi = L^{-1}g$ in terms of the L^2 norm of g through (67) and (68). This finishes the proof of theorem 5.7. ■

5.4 Deforming (a, ψ) to an honest solution

The goal of this section is to show that the approximate solution (a, ψ) can be made into an honest solution of the Seiberg-Witten equations by a deformation whose size goes to zero as r goes to infinity.

To set the stage, let $SW : L^{1,2}(i\Lambda^1 \oplus W_E^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_E^-)$ denote the Seiberg-Witten operator

$$SW(b, \phi) = (D_b\phi, F_b^+ - F_{A_0}^+ - q(\phi, \phi) + \frac{ir}{8}\omega)$$

We will search for a zero of SW of the form $(a, \psi) + (a', \psi')$ with $(a', \psi') \in B(\delta)$. Here $B(\delta)$ is the closed ball in $L^{1,2}(i\Lambda^1 \oplus W_E^+)$ centered at zero and with radius $\delta > 0$ which we will choose later but which should be thought of as being small. The equation $SW((a, \psi) + (a', \psi')) = 0$ can be written as

$$0 = SW(a, \psi) + L_{(a,\psi)}(a', \psi') + Q(a', \psi') \tag{84}$$

Here $Q : L^{1,2}(i\Lambda^1 \oplus W_E^+) \to L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_E^-)$ is the quadratic map given by

$$Q(b, \phi_0, \phi_2) = (b.(\phi_0 + \phi_2), \frac{i}{8}(|\phi_0|^2 - |\phi_2|^2)\omega + \frac{i}{4}(\bar{\phi_0}\,\phi_2 + \phi_0\,\bar{\phi_2})) \tag{85}$$

Lemma 5.8 *For x, $y \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$, the map Q satisfies the inequality:*

$$||Q(x) - Q(y)||_2 \le C\,(||x||_{1,2} + ||y||_{1,2})\,||x - y||_{1,2} \tag{86}$$

Proof. This is a standard inequality for quadratic maps and it can be explicitly checked using the definition of Q and the multiplication theorem for Sobolev spaces. We give the calculation for the first component of the right hand side of (85). Let $x = (b, \phi)$ and $y = (c, \varphi)$, then we have

$$\begin{aligned} ||b.\phi - c.\varphi||_2 =& ||b.\phi - c.\phi + c.\phi - c.\varphi||_2 \leq ||(b-c).\phi||_2 + ||c.(\phi - \varphi)||_2 \\ &\leq C\,||b-c||_{1,2}\,||\phi||_{1,2} + C\,||c||_{1,2}\,||\phi - \varphi||_{1,2} \\ &\leq C\,(||(b,\phi) - ,\varphi)||_{1,2})\,(||(b,\phi)||_{1,2} + ||,\varphi)||_{1,2}) \end{aligned}$$

The other components are checked similarly. ■

Solving equation (84) for $(a', \psi') \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$ is equivalent to finding a fixed point of the map $Y : B(\delta) \to B(\delta)$ given by

$$Y(b, \phi) = -L_{(a,\psi)}^{-1}(SW(a, \psi) + Q(b, \phi)) \tag{87}$$

In order for the image of Y to lie in $B(\delta)$ we need to choose r large enough and δ small enough. To make this precise, let $(b, \phi) \in B(\delta)$. Using the bounds in (62) we find that

$$||SW(a, \psi)||_2 \leq \frac{C}{\sqrt{r}}$$

and so together with the results of theorem 5.7 and lemma 5.8 we get

$$||Y(b, \phi)||_{1,2} \leq \frac{C}{\sqrt{r}} + C \cdot \delta^2$$

Choosing $\delta < 1/2C$ and $r > 4C^2/\delta^2$ ensures that Y is well defined.

Lemma 5.9 *The map $Y : B(\delta) \to B(\delta)$ as defined by* (87) *is a contraction mapping for r large enough and δ small enough.*

Proof. Let x, $y \in B(\delta)$, then using (86) we find

$$||Y(x) - Y(y)||_{1,2} \leq C\,||Q(x) - Q(y)||_2 \leq C\,||x + y||_{1,2}\,||x - y||_{1,2} \tag{88}$$

Choosing $\delta < 1/2C$ makes $C\,||x + y||_{1,2} \leq 2C\delta$ less than 1. ∎

We summarize in the following:

Theorem 5.10 *Let (a, ψ) be constructed from (a_i, ψ_i) as in (56). Suppose that the (a_i, ψ_i) meet assumption (59) and that J has been chosen from the Baire set $\mathcal{J}_0$ of compatible almost complex structures.*

Then there exists a $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ there exists an $r_\delta \geq 1$ such that for every $r \geq r_\delta$ there exists a unique solution $(a, \psi) + (a', \psi')$ of the Seiberg-Witten equations (with perturbation parameter r) with $(a', \psi') \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$ satisfying the bound $||(a', \psi')||_{1,2} \leq \delta$.

6 Comparison with product formulas

Before proceeding further, we would like to take a moment to point out the similarities and differences between our construction of (A, ψ) from (A_i, ψ_i) on one hand and product formulas for the Seiberg-Witten invariants on manifolds that are fiber sums of simpler manifolds. We begin by briefly (and with few details) recalling the scenario of the latter.

Let X_i, $i = 0, 1$ be two compact smooth 4-manifolds and $\Sigma_i \hookrightarrow X_i$ embedded surfaces of the same genus and with $\Sigma_0 \cdot \Sigma_0 = -\Sigma_1 \cdot \Sigma_1$. In this setup one can construct the fiber sum

$$X = X_0 \#_{\Sigma_i} X_1$$

by cutting out tubular neighborhoods $N(\Sigma_i)$ in X_i and gluing the manifolds $X_i' = \overline{X_i \backslash N(\Sigma_i)}$ along their diffeomorphic boundaries.

Under certain conditions one can calculate some of the Seiberg-Witten invariants of X in terms of the Seiberg-Witten invariants of the building blocks X_i (see e.g. [8]). One accomplishes this by showing that from solutions (B_i, Φ_i), $i = 0, 1$ on X_i one can construct a solution (B, Φ) on X (this isn't possible for any pair of solutions (B_i, Φ_i) but the details are not relevant to the present discussion). This is done by inserting a "neck" of length $r \geq 1$ between the X_i' so as to identify X with

$$X = X_0' \cup ([0, r] \times Y) \cup X_1'$$

with $Y = \partial N(\Sigma_0) \cong \partial N(\Sigma_1)$. A partition of unity $\{\varphi_0, \varphi_1\}$ is chosen for each value of $r \geq 1$ subject to the conditions

$$\varphi_i = 1 \text{ on } X_i'$$

$$\varphi_i = 0 \text{ outside of } X_i' \cup [0,r] \times Y$$

$$|\varphi_i'| \leq \frac{C}{r} \text{ on } [0,r] \times Y$$

An approximation Φ' of Φ is then defined to be $\Phi' = \varphi_0 \Phi_0 + \varphi_1 \Phi_1$ (similarly for B', a first approximation for B). The measure of the failure of (B', Φ') to solve the Seiberg-Witten equations can be made as small as desired by making r large. The honest solution (B, Φ) is then sought in the form $(B', \Phi') + (b, \phi)$ with (b, ϕ) small. The correction term (b, ϕ) is found as a fixed point of the map

$$(b, \phi) \mapsto Z(b, \phi) = -L^{-1}_{(B', \Phi')} (Q(b, \phi) + \text{err})$$

Here "err" is the size of $SW(B', \Phi')$ and L and Q are as in the previous section. Choosing r large enough and $||(b, \phi)||$ small enough makes Z a contraction mapping and so the familiar fixed point theorem for Banach spaces guarantees the existence of a unique fixed point.

In the case of fiber sums there are product formulas that allow one to calculate the Seiberg-Witten invariants of X in terms of the invariants of the manifolds X_i. The formulas typically have the form:

$$SW_X(W_E) = \sum_{E_0 + E_1 = E} SW_{X_0}(W_{E_0}) \cdot SW_{X_1}(W_{E_1}) \tag{89}$$

Due to the similarity of our construction of grafting monopoles to the one used to construct (B, Φ) from (B_i, Φ_i), it is natural to ask if such or similar formulas exist for the present case, that is, can one calculate $SW_X(W_{E_0 \otimes E_1})$ in terms of $SW_X(W_{E_0})$ and $SW_X(W_{E_1})$? The author doesn't know the answer. However, if they do exist, they can't be expected to be as simple as (89). The reason for this can be understood by trying to take the analogy between our setup and that for fiber sums further.

In the case of fiber sums, once one has established that the two solutions (B_i, Φ_i) on X_i can be used to construct a solution (B, Φ) on X, one needs to establish a converse of sorts. That is, one needs to show that every solution (B, Φ) on X is of that form. It is at this point where the analogy between the two situations breaks down. It is conceivable in our setup, that there will be solutions for the $Spin^c$-structure $(E_0 \otimes E_1) \otimes W_0^+$ that can not be obtained as products of solutions for the $Spin^c$-structures $E_i \otimes W_0^+$. Worse even, there might be monopoles that can not be obtained as products of solutions for any $Spin^c$-structures $F_j \otimes W_0^+$ with the choice of F_j, $j = 0, 1$ such that $E = F_0 \otimes F_1$ and $F_j \neq 0$. Those are the monopoles where $\alpha^{-1}(0)$ is connected. Thus if a product formula for our situation exists, it must in addition to a term similar to the right hand side of (89) also contain terms which count these "undecomposable" solutions. But then again, they might not exist.

The next section describes which solutions of the Seiberg-Witten equations for the $Spin^c$-structure $(E_0 \otimes E_1) \otimes W_0^+$ are obtained as products of solutions for the $Spin^c$-structures $E_i \otimes W_0^+$, $E = E_0 \otimes E_1$.

7 The image of the multiplication map

This section describes a partial converse to theorem 5.10. Recall that

$$\Theta : \mathcal{M}_X^{SW}(W_E) \to \mathcal{M}_X^{Gr}(E)$$

is the map assigning a J-holomorphic curve to a Seiberg-Witten monopole.

Theorem 7.1 *Let $E = E_0 \otimes E_1$ and let (A, ψ) be a solution of the Seiberg-Witten equations in the Spin^c-structure W_E with perturbation term $\mu = F_{A_0}^+ - ir\omega/8$ and with $\psi = \sqrt{r}\,(\alpha \otimes u_0, \beta)$. Assume further that J has been chosen from the Baire set $\mathcal{J}_0$ and that $\Theta(A, \psi)$ contains no multiply covered components. If there exists an r_0 such that for all $r \geq r_0$, $\alpha^{-1}(0)$ splits into a disjoint union $\alpha^{-1}(0) = \Sigma_0 \sqcup \Sigma_1$ with $[\Sigma_i] =$P.D.(E_i) then (A, ψ) lies in the image of the multiplication map*

$$\mathcal{M}_X^{SW}(E_0) \times \mathcal{M}_X^{SW}(E_1) \dot{\to} \mathcal{M}_X^{SW}(E_0 \otimes E_1)$$

The proof of theorem 7.1 is divided into 3 sections. In section 7.1 we give the definition of (A_i', ψ_i') - first approximations of Seiberg-Witten monopoles (A_i, ψ_i) for the *Spin^c*-structure W_{E_i} which when multiplied give the monopole (A, ψ) from theorem 7.1. Section 7.2 shows that for large values of r, (A_i', ψ_i') come close to solving the Seiberg-Witten equations. In the final section 7.3 we show that $L_{(A_i', \psi_i')}$ is surjective with inverse bounded independently of r. The contraction mapping principle is then used to deform the approximate solutions (A_i', ψ_i') to honest solutions (A_i, ψ_i). Section 7.3 also explains why $(A_0, \psi_0) \cdot (A_1, \psi_1) = (A, \psi)$.

We tacitly carry the assumptions of the theorem until the end of the section.

7.1 Defining (A_i', ψ_i')

The basic idea behind the definition of (A_i', ψ_i') is again that of grafting existing solutions. For example, one would like ψ_0' to be defined as the restriction of ψ to a neighborhood of Σ_0 (under an appropriate bundle isomorphism trivializing E_1 over that neighborhood) and to be the restriction of $\sqrt{r}\, u_0$ outside that neighborhood. This is essentially how the construction goes even though a bit more care is required, especially in splitting the connection A into A_0' and A_1'.

To begin with, choose regular neighborhoods V_0 and V_1 of Σ_0 and Σ_1. Once r is large enough, these choices don't need to be readjusted for larger values of r. Choose, as in section 5.3, on open set U such that

$$X = V_0 \cup U \cup V_1$$

$$U \cap \Sigma_i = \emptyset$$

Also, just as in section 5.3, arrange the choices so that $U \cap V_i$ contains a collar $\partial V_i \times [0, d]$ (with ∂V_i corresponding to $\partial V_i \times \{d\}$) and choose $\delta > 0$ smaller than $d/1000$. Assume that the curves Σ_i are connected, the general case goes through with little difficulty but with a bit more complexity of notation.

Over $U \cup V_1$, choose a section $\gamma_0 \in \Gamma(E_0; U \cup V_1)$ with $|\gamma_0| = 1$. Choose a connection B_0 on E_0 with respect to which γ_0 is covariantly constant over $U \cup V_1$, i.e.

$$B_0(\gamma_0(x)) = 0 \qquad \forall x \in U \cup V_1 \tag{90}$$

Notice that such a connection is automatically flat over $U \cup V_1$. Choose a connection B_1 on E_1 such that $B_0 \otimes B_1 = A$ over X. Now define $\tilde{\alpha_1}' \in \Gamma(E_1; U \cup V_1)$ and

$\tilde{\beta_1}' \in \Gamma(E_1 \otimes K^{-1}; U \cup V_1)$ by

$$\alpha = \gamma_0 \otimes \tilde{\alpha_1}' \tag{91}$$

$$\beta = \gamma_0 \otimes \tilde{\beta_1}' \tag{92}$$

Proceed similarly over V_0. However, since some of the data is now already defined, more caution is required. Choose a section $\gamma_1 \in \Gamma(E_1; V_0)$ with

$$\begin{aligned} \gamma_1 &= \tilde{\alpha_1}' \quad \text{on } (U \cap V_0)\backslash(\partial V_0 \times [0, 4\delta\rangle) \\ |\gamma_1| &= 1 \quad \text{on } (V_0 \backslash U) \cup (\partial V_0 \times [0, 2\delta\rangle) \end{aligned} \tag{93}$$

We continue by defining $\tilde{\alpha_0}'$ and $\tilde{\beta_0}'$ over V_0 by

$$\alpha = \tilde{\alpha_0}' \otimes \gamma_1 \tag{94}$$

$$\beta = \tilde{\beta_0}' \otimes \gamma_1 \tag{95}$$

Choose one forms a_0 and a_1 such that over V_0 the following two relations hold:

$$(B_1 + i\, a_1)\, \gamma_1 = 0 \tag{96}$$

$$(B_0 + i\, a_0) \otimes (B_1 + i\, a_1) = A \tag{97}$$

With these preliminaries in place, we are now ready to define (A_i', ψ_i'):

$$\begin{array}{ll} \tilde{\alpha_0} = \chi_{4\delta,0}\, \tilde{\alpha_0}' + (1 - \chi_{4\delta,0})\gamma_0 & \tilde{\beta_0} = \chi_{4\delta,0}\, \tilde{\beta_0}' \\ \tilde{\alpha_1} = (1 - \chi_{4\delta,0})\, \tilde{\alpha_1}' + \chi_{4\delta,0}\gamma_1 & \tilde{\beta_1} = (1 - \chi_{4\delta,0})\, \tilde{\beta_1}' \\ A_0' = B_0 + i\chi_{4\delta,0}\, a_0 & A_1' = B_1 + i\chi_{4\delta,0}\, a_1 \end{array} \tag{98}$$

Lemma 7.2 *The (A_i', ψ_i') defined above, satisfy the following properties:*

a) $A_0' \otimes A_1' = A$ on all of X.

b) $\tilde{\alpha_0} = \gamma_0$ on $(U \cap V_0)\backslash(\partial V_0 \times [0, 4\delta\rangle)$.

c) $F_{B_0} = 0$ *on* $U \cup V_1$ *and* $F_{B_1 + i\tilde{a_1}} = 0$ *on* V_0.

d) On $(U \cap V_0) \backslash (\partial V_0 \times [0, 4\delta\rangle)$, $|\tilde{a}_i|$ *and* $|d\,\tilde{a}_i|$ *converge exponentially fast to zero as* $r \to \infty$.

Proof. a) This is trivially true everywhere except possibly on the support of $d\chi_{4\delta,0}$ which is contained in $U \cap V_0$. However, on $U \cap V_0$ we have $A = B_0 \otimes B_1$ and $A = (B_0 + ia_0) \otimes (B_1 + ia_1)$ and thus $a_0 + a_1 = 0$. In particular, $A_0' \otimes A_1' = B_0 \otimes B_1 + i\chi_{4\delta,0}\,(a_0 + a_1) = B_0 \otimes B_1 = A$

b) Notice that on $(U \cap V_0) \backslash (\partial V_0 \times [0, 4\delta\rangle)$, $\gamma_1 = \tilde{\alpha_1'}$. Thus, $\alpha = \gamma_0 \otimes \gamma_1$ and $\alpha = \tilde{\alpha_0'} \otimes \gamma_1$ imply that $\gamma_0 = \tilde{\alpha_0'}$. The claim now follows from the definition of α_0.

c) Follows from the fact that both connections annihilate nowhere vanishing sections on the said regions.

d) On $(U \cap V_0) \backslash (\partial V_0 \times [0, 4\delta\rangle)$ we have $\alpha = \gamma_0 \otimes \gamma_1$ and $\nabla^A = \nabla^{B_0 + ia_0} \otimes \nabla^{B_1 + ia_1}$. Also, recall that $\nabla^{B_0}\gamma_0 = 0$ and $\nabla^{B_1 + ia_1}\gamma_1 = 0$. Thus

$$\nabla^a \alpha = (\nabla^{B_0 + ia_0} \otimes \nabla^{B_1 + ia_1})(\gamma_0 \otimes \gamma_1) = i\,a_0\gamma_0 \otimes \gamma_1$$

This equation yields

$$|a_0| = \frac{|\nabla^A \alpha|}{|\alpha|} \tag{99}$$

The claim follows now for a_0 by evoking the bounds (51). The same result holds for a_1 by the proof of part (a) where it is shown that $a_0 + a_1 = 0$ on $U \cap V_0$. The statement for da_i follows from part (c), the equation $F_A = F_{B_0 + ia_0} + F_{B_1 + ia_1}$ and the bounds (51) for $|F_A|$. ∎

7.2 Pointwise bounds on $SW(A_i', \psi_i')$

Proposition 7.3 *Let (A_i', ψ_i') be defined as above, then there exists a constant C and an $r_0 \geq 1$ such that for all $r \geq r_0$ the inequality*

$$|SW(A_i', \psi_i')|_x \leq \frac{C}{\sqrt{r}}$$

holds for all $x \in X$.

Proof. We calculate the size of the contribution of each of the three Seiberg-Witten equations separately. The only nontrivial part of the calculation is in the region of X which contains the support of $d\chi_{4\delta,0}$ i.e. in $\partial V_0 \times [4\delta, 8\delta]$. We will tacitly use the results of lemma 7.2 in the calculations below.

a) The Dirac equation

To begin with, we calculate the expression $D_A((\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0}) \otimes \gamma_1)$ in two different ways. On one hand we have:

$$D_A((\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0}) \otimes \gamma_1) = D_A(\alpha + \chi_{4\delta,0}\beta) = (1 - \chi_{4\delta,0})D_A\alpha + d\chi_{4\delta,0}.\beta$$

On the other hand we get:

$$D_A((\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0}) \otimes \gamma_1) = \tag{100}$$

$$= \gamma_1 \otimes D_{A_0'}(\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0}) + e^i.(\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0}) \otimes A_1'(\gamma_1)$$

$$= \gamma_1 \otimes D_{A_0'}(\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0}) + i(\chi_{4\delta,0} - 1)a_1\gamma_1$$

Equating the results of the two calculations we obtain:

$$|\alpha| \cdot |D_{A_0'}(\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0})| = |\gamma_1 \otimes D_{A_0'}(\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0})| \leq$$

$$\leq C\left(|\alpha|\,|a_1| + |D_A\alpha| + |\beta|\right) \leq \frac{C}{\sqrt{r}}$$

Since over $\partial V_0 \times [4\delta, 8\delta]$, $|\alpha| \longrightarrow 1$ exponentially fast as $r \longrightarrow \infty$ we obtain that

$$|D_{A_0'}(\tilde{\alpha_0} \otimes u_0 + \tilde{\beta_0})| \leq \frac{C}{\sqrt{r}} \tag{101}$$

b) The $(1,1)$-component of the curvature equation

Again, we only calculate for $x \in \partial V_0 \times [4\delta, 8\delta]$:

$$\begin{aligned} F_{A_0'}^{(1,1)} - F_{A_0}^{(1,1)} - \frac{ir}{8}\left(|\tilde{\alpha_0}|^2 - 1 - |\tilde{\beta_0}|^2\right)\omega &= \chi_{4\delta,0}\,(da_0)^{(1,1)} + \frac{ir}{8}|\tilde{\beta_0}|^2\omega \\ &= \chi_{4\delta,0}\,(da_0)^{(1,1)} + \frac{ir}{8\,|\alpha|^2}|\chi_{4\delta,0}|^2\,|\beta|^2\omega \end{aligned}$$

Both terms in the last line converge in norm exponentially fast to zero on $\partial V_0 \times [4\delta, 8\delta]$ as $r \longrightarrow \infty$.

c) The $(0,2)$-component of the curvature equation

Similar to the calculation for the $(1,1)$-component of the curvature equation on $\partial V_0 \times [4\delta, 8\delta]$, we have for the $(0,2)$-component of the same equation:

$$F_{A_0'}^{(0,2)} - F_{A_0}^{(0,2)} - \frac{ir}{4}\overline{\tilde{\alpha_0}}\,\tilde{\beta_0} = \chi_{4\delta,0}\,(da_0)^{(0,2)} - \frac{ir}{4\,|\alpha|^2}\chi_{4\delta,0}\,\overline{\alpha}\,\beta$$

Once again, both terms on the right-hand side of the above equation converge in norm exponentially fast to zero as r converges to infinity. The proofs for the case of (A_1', ψ_1') are similar and are left to the reader.

7.3 Surjectivity of $L_{(A_i',\psi_i')}$ and deforming (A_i',ψ_i') to an exact solution

The strategy employed here is very similar to the one used in section 5.4 and we only spell out part of the details. We start by showing that $L_{(A_0',\psi_0')}$ is surjective, the case $L_{(A_1',\psi_1')}$ is identical.

We begin by asking ourselves when the equation

$$L_{(A_0',\psi_0')}\xi_0 = g_0 \tag{102}$$

has a solution $\xi_0 \in L^{1,2}(i\Lambda^1 \oplus W_{E_0}^+)$ for a given $g_0 \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_{E_0}^-)$. Define the analogues of the isomorphisms Υ_i from section 5.3 to be

$$\Upsilon_0 : \mathbb{C}\times(U \cup V_1) \to \Gamma(E_0; U \cup V_1) \text{ given by } \Upsilon_0(\lambda, x) = \lambda \cdot \gamma_0(x) \text{ and}$$

$$\Upsilon_1 : \mathbb{C} \times V_0 \to \Gamma(E_1; V_0) \text{ given by } \Upsilon_1(\lambda, x) = \lambda \cdot \gamma_1(x)$$

Let $\gamma \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_0^-; U \cup V_1)$ be determined by the equation $\chi_{25\delta,0}\, g_0 = \Upsilon_0(\gamma)$ on $U \cup V_1$ and $\varsigma \in L^2(i\Lambda^0 \oplus i\Lambda^{2,+} \oplus W_E^-; V_0)$ be given by the equation $\Upsilon_1^{-1}(\varsigma) = (1 - \chi_{25\delta,0})\, g_0$ on V_0. Thus we can write g_0 as

$$g_0 = \chi_{100\delta,0}\, \Upsilon_0(\gamma) + (1 - \chi_{4\delta,0})\, \Upsilon_1^{-1}(\varsigma) \tag{103}$$

This last form suggests that, in order to split equation (102) into two components involving $L_{(A,\psi)}$ and S, one should search for ξ_0 in the form

$$\xi_0 = \chi_{100\delta,0}\, \Upsilon_0(\eta) + (1 - \chi_{4\delta,0})\, \Upsilon_1^{-1}(\kappa) \tag{104}$$

with $\eta \in L^{1,2}(i\Lambda^1 \otimes W_0^+; U \cup V_1)$ and $\kappa \in L^{1,2}(i\Lambda^1 \oplus W_E^+; V_0)$. Using relations (103) and (104) in (102) one obtains the analogue of equation (70):

$$\chi_{100\delta,0}\Upsilon_0(T(\eta) - \Upsilon_0^{-1}\Upsilon_1^{-1}\mathcal{P}(d\chi_{4\delta,0},\kappa) - \gamma) +$$

$$+ (1-\chi_{4\delta,0})\Upsilon_1^{-1}(M(\kappa) + \Upsilon_1\Upsilon_0\mathcal{P}(d\chi_{100\delta,0},\eta) - \varsigma) = 0 \qquad (105)$$

The operators T' and M' are defined over $U \cup V_1$ and V_0 respectively, through the relations

$$L_{(A_0',\psi_0')}\Upsilon_0 = \Upsilon_0 T$$

$$L_{(A_0',\psi_0')}\Upsilon_1^{-1} = \Upsilon_1^{-1} M$$

We use these operators, defined only over portions of X, to define the operators T' and M' defined on all of X by

$$T' = (1-\chi_{\delta,0})T + \chi_{\delta,0}S$$

$$M' = \chi_{200\delta,0}M + (1-\chi_{200\delta,0})L_{(A,\psi)}$$

Split equation (105) into the following two equations:

$$T'(\eta) - \Upsilon_0^{-1}\Upsilon_1^{-1}\mathcal{P}(d\chi_{4\delta,0},\kappa) = \gamma$$

$$M'(\kappa) + \Upsilon_1\Upsilon_0\mathcal{P}(d\chi_{100\delta,0},\eta) = \varsigma \qquad (106)$$

It is easy to see that solutions to the system of equations (106) provide solutions to (105) by multiplying the two lines with $\chi_{100\delta,0}\Upsilon_0$ and $(1-\chi_{4\delta,0}\Upsilon_1^{-1})$ respectively and adding them.

The following lemma is the analogue of lemma 3.6, its proof is identical to that of lemma 3.6 and will be skipped here.

Lemma 7.4 *For every $\epsilon > 0$ there exists an $r_\epsilon \geq 1$ such that for $r \geq r_\epsilon$ the following hold:*

$$||(M' - L_{(A,\psi)})\, x||_2 \leq \epsilon\, ||x||_2$$

$$||(T' - S)\, y||_2 \leq \epsilon\, ||y||_2$$

Here $x \in L^{1,2}(i\Lambda^1 \oplus W_E^+)$ and $y \in L^{1,2}(i\Lambda^1 \oplus W_0^+)$.

The lemma allows us to replace the system (106) by the system

$$S(\eta) - \Upsilon_0^{-1}\Upsilon_1^{-1}\mathcal{P}(d\,\chi_{4\delta,0}, \kappa) = \gamma$$

$$L_{(A,\psi)}(\kappa) + \Upsilon_1\,\Upsilon_0\mathcal{P}(d\,\chi_{100\delta,0}, \eta) = \varsigma \tag{107}$$

The process of solving (107) is now step by step the analogue of solving (74). In particular, we solve the first of the two equations in (107) for η in terms of κ:

$$\eta = \eta(\kappa) = S^{-1}(\Upsilon_0^{-1}\Upsilon_1^{-1}\mathcal{P}(d\,\chi_{4\delta,0}, \kappa) + \gamma)$$

Use this in the second equation of (107) and rewrite it as

$$\kappa = L_{(A,\psi)}^{-1}\,(\varsigma - \Upsilon_1\,\Upsilon_0\mathcal{P}(d\,\chi_{100\delta,0}, \eta(\kappa)))$$

To solve this last equation is the same as to find a fixed point of the map Y : $L^2(i\Lambda^1 \oplus W_E^+) \to L^2(i\Lambda^1 \oplus W_E^+)$ (the analogue of the map described by (78)) given by:

$$Y(\kappa) = L_{(A,\psi)}^{-1}\,(\varsigma - \Upsilon_1\,\Upsilon_0\mathcal{P}(d\,\chi_{100\delta,0}, \eta(\kappa)))$$

The proof of the existence of a unique fixed point of Y follows from a word by word analogue of the proof of theorem 5.7 together with the discussion preceding the theorem.

With the surjectivity of $L_{(A'_i,\psi'_i)}$ proved, the process of deforming (A'_i, ψ'_i) to an honest solution (A_i, ψ_i) is accomplished by the same method as used in section 5.4 and will be skipped here.

To finish the proof theorem 7.1, we need to show that

$$(A_0, \psi_0) \cdot (A_1, \psi_1) = (A, \psi)$$

This follows from the fact that as $r \to \infty$, the distance $\mathrm{dist}((A_i, \psi_i), (A'_i \psi'_i))$ converges to zero, together with the following relations which follow directly from the definitions:

$$\tilde{\alpha_0} \otimes \tilde{\alpha_1} = \alpha$$

$$\tilde{\alpha_0} \otimes \tilde{\beta_1} + \tilde{\alpha_1} \otimes \tilde{\beta_0} = \beta$$

$$A'_0 \otimes A'_1 = A$$

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