TEMPERED FRACTIONAL BROWNIAN MOTION

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ABSTRACT

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Tempered fractional Brownian motion TFBM modifies the power law kernel in the moving average representation of a fractional Brownian motion (FBM), adding an exponential tempering. It also has a harmonizable representation. The increments of TFBM are stationary, and the autocovariance of the resulting tempered fractional Gaussian noise TFGN has semi-long range dependence, in which the autocorrelations decay like a power law over a moderate length scale, but eventually fall off more rapidly. TFBM can be represented as the linear combination of tempered fractional derivative (or tempered fractional integral) of the indicator functions. This representation and the classical Itô isometry provides to characterize the class of all deterministic functions for which the stochastic integral with respect to TFBM is well defined. Replacing the Gaussian random measure (Brownian motion) in the moving average or harmonizable representation of TFBM by a stable random measure, a linear tempered fractional stable motion (LTFSM), or a real harmonizable tempered fractional stable motion (HTFSM), respectively. Unlike the Gaussian case, LTFSM and HTFSM are two completely different processes. Existence, basic properties, sample path behavior, and dependence structure of both processes will be described.

Keywords: Fractional Brownian motion, tempered fractional derivative, harmonizable representation, long range dependence, reproducing kernel Hilbert space.

I dedicate this dissertation to my lovely wife, Sara Hazinia.

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Chapter 1

Introduction

Fractional Brownian motion (FBM) is a Gaussian stochastic process whose increments, termed fractional Gaussian noise (FGN), can exhibit long range dependence [7, 22]. FBM has become popular in applications to science and engineering, since it yields a simple tractable model that captures the correlation structure seen in many natural systems [34, 43]. Formed by convolving Brownian motion with a power law, FBM is essentially the fractional integral or derivative of that Brownian motion [48, 56]. This is also reflected in the correlation function of FGN, which falls off like a power law with lag, and the corresponding spectral density, which behaves like a power law near the origin. The increments of long range dependent FBM have a spectral density that blows up like a power law at the origin. This diverging spectral density is one of the hallmarks of long range dependence. For wind speed measurements, the spectral density follows this power law model for moderate frequencies, but the data deviates from that model at low frequencies, and the measured spectal density remains bounded [55, 28, 53].

In Chapter 2 we define a closely related process, which we call tempered fractional Brownian motion (TFBM). It is defined by exponentially tempering the power law kernel in the moving average representation of a fractional Brownian motion. TFBM is a Gaussian process with stationary increments, and we call those increments tempered fractional Gaussian noise (TFGN). When FGN is long range dependent, TFGN exhibits *semi-long range dependence*. Its autocovariance function closely resembles that of FGN on an intermediate scale, but then it eventually falls off more rapidly. Its spectral density resembles a negative power law for low frequencies, but eventually converges to zero at very low frequencies. TFBM can be a useful stochastic model for applications where the data follows FBM at some intermediate scale, but then deviates from FBM at longer scales. For example, wind speed measurements typically resemble long range dependent FBM over a range of frequencies, but deviate significantly at very low frequencies (corresponding to very long spatial scales). Since the spectal density of semi-long range dependent TFGN follows the same pattern, it can provide a useful model for such data.

In Chapter 3 we develop the theory of stochastic integration for TFBM. Our approach follows the seminal work of Pipiras and Taqqu [56] for FBM. A FBM is the fractional derivative (or integral) of a Brownian motion, in a sense made precise in [56]. A fractional derivative (or integral) is a (distributional) convolution with a power law [48, 54, 60]. Multiplying that power law by an exponential factor leads to tempered fractional derivatives and integrals. TFBM can be written in terms of tempered fractional derivatives (or integrals) of a Brownian motion. This representation and the classical *Itô Isometry* allows us to characterize the class of all deterministic functions for which the stochastic integral with respect to TFBM is well defined. We also apply the harmonizable representation of TFBM to define another type of stochastic integral of deterministic functions with respect to TFBM.

In chapter 4 we consider heavy tailed analogues to TFBM. Replacing the Gaussian random measure (Brownian motion) in the moving average or harmonizable representation of TFBM by a stable random measure, we obtain a linear tempered fractional stable motion (LTFSM), or a real harmonizable tempered fractional stable motion (HTFSM), respectively. Unlike the Gaussian case, LTFSM and HTFSM are two completely different processes. Existence, basic properties, sample path behavior, and dependence structure of both processes are described in this thesis. We also prove that LTFSM and HTFSM are locally nondeterministic on every compact interval.

Chapter 2

Tempered Fractional Brownian Motion

This Chapter has five sections. In Section 2.1, we define tempered fractional Brownian motion (TFBM) using a moving average representation, and we establish some of its basic properties. Section 2.2 develops the harmonizable representation of TFBM, and Section 2.3 discusses tempered fractional Gaussian noise (TFGN). Sample path properties of TFBM are proven in Section 2.4, and an application to wind speed is discussed in Section 2.5.

2.1 Moving average representation

Let $\{B(t)\}_{t\in\mathbb{R}}$ be a real-valued Brownian motion on the real line, a process with stationary independent increments such that $B(t)$ has a Gaussian distribution with mean zero and variance $\sigma^2|t|$ for all $t \in \mathbb{R}$, for some $\sigma > 0$. Define an independently scattered Gaussian random measure $B(dx)$ with control measure $m(dx) = \sigma^2 dx$ by setting $B[a, b] = B(b) - B(a)$ for any real numbers $a < b$, and then extending to all Borel sets. Then the stochastic integrals $I(f) := \int f(x)B(dx)$ are defined for all functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int f(x)^2 dx < \infty$, as Gaussian random variables with mean zero and covariance $\mathbb{E}[I(f)I(g)] = \sigma^2 \int f(x)g(x)dx$, see for example Chapter 3 in [61].

Definition 2.1.1. *Given an independently scattered Gaussian random measure $B(dx)$ on $\mathbb{R}$*

with control measure $\sigma^2 dx$, for any $\alpha < \frac{1}{2}$ and $\lambda \geq 0$, the stochastic integral

$$B_{\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \right] \, B(dx), \tag{2.1}$$

where $(x)_+ = xI(x > 0)$, and $0^0 = 0$, will be called a tempered fractional Brownian motion (TFBM).

It is easy to check that the function

$$g_{\alpha,\lambda,t}(x) := e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \tag{2.2}$$

is square integrable over the entire real line for any $\alpha < \frac{1}{2}$, so that TFBM is well-defined, and of course FBM is a special case of TFBM with $\lambda = 0$. Note also that

$$g_{\alpha,\lambda,ct}(cx) = c^{-\alpha} g_{\alpha,c\lambda,t}(x), \tag{2.3}$$

for all $t, x \in \mathbb{R}$ and all $c > 0$. The next result shows that TFBM has a nice scaling property, involving both the time scale and the tempering. Here the symbol $\triangleq$ indicates equality of finite dimensional distributions.

Proposition 2.1.2. TFBM (2.1) *is Gaussian stochastic process with stationary increments, such that*

$$\left\{B_{\alpha,\lambda}(ct)\right\}_{t\in\mathbb{R}} \triangleq \left\{c^H B_{\alpha,c\lambda}(t)\right\}_{t\in\mathbb{R}} \tag{2.4}$$

for any scale factor $c > 0$, where the Hurst index $H = 1/2 - \alpha$.

Proof. Since $B(dx)$ has control measure $m(dx) = \sigma^2 dx$, the random measure $B(c\,dx)$ has control measure $c^{1/2}\sigma^2 dx$. Given $t_1 < t_2 < \cdots < t_n$, a change of variable $x = cx'$ then yields

$$\begin{aligned}\big(B_{\alpha,\lambda}(ct_i) : i = 1, \ldots, n\big) &= \left(\int g_{\alpha,\lambda,ct_i}(x) B(dx) : i = 1, \ldots, n\right) \\ &\triangleq \left(\int c^{-\alpha} g_{\alpha,c\lambda,t_i}(x') c^{1/2} B(dx') : i = 1, \ldots, n\right)\end{aligned}$$

so that (2.4) holds with $H = 1/2 - \alpha$. For any $s, t \in \mathbb{R}$, the integrand (2.2) satisfies $g_{\alpha,\lambda,s+t}(s + x) - g_{\alpha,\lambda,s}(s + x) = g_{\alpha,\lambda,t}(x)$, and hence a change of variable $x = s + x'$ in the moving average representation yields

$$\big(B_{\alpha,\lambda}(s + t_i) - B_{\alpha,\lambda}(s) : i = 1, \ldots, n\big) \triangleq \left(\int g_{\alpha,\lambda,t_i}(x') B(dx') : i = 1, \ldots, n\right)$$

which shows that TFBM has stationary increments. □

Proposition 2.1.3. *The covariance function of* TFBM (2.1) *has the form*

$$\operatorname{Cov}\left[B_{\alpha,\lambda}(t), B_{\alpha,\lambda}(s)\right] = \frac{\sigma^2}{2}\left[C_t^2\,|t|^{2H} + C_s^2\,|s|^{2H} - C_{t-s}^2\,|t - s|^{2H}\right], \tag{2.5}$$

for any $s, t \in \mathbb{R}$, where $H = 1/2 - \alpha$. Here

$$C_t^2 = \frac{2\Gamma(2H)}{(2\lambda|t|)^{2H}} - \frac{2\Gamma(H + \frac{1}{2})}{\sqrt{\pi}} \frac{1}{(2\lambda|t|)^H} K_H(\lambda|t|), \tag{2.6}$$

for $t \neq 0$, $C_t^2 = 0$ when $t = 0$, where $K_\nu(z)$ is the modified Bessel function of the second kind.

Proof. Use the moving average representation (2.1) with $\sigma = 1$ to define

$$
\begin{aligned}
C_t^2 :&= \mathbb{E}[B_{\alpha,\lambda|t|}(1)^2] = \int_{-\infty}^{+\infty} \left[e^{-\lambda t(1-x)_+}(1-x)_+^{-\alpha} - e^{-\lambda t(-x)_+}(-x)_+^{-\alpha} \right]^2 dx \\
&= \int_{-\infty}^{+\infty} e^{-2\lambda t(1-x)_+}(1-x)_+^{-2\alpha} dx + \int_{-\infty}^{+\infty} e^{-2\lambda t(-x)_+}(-x)_+^{-2\alpha} dx \\
&\quad - 2\int_{-\infty}^{+\infty} e^{-\lambda t(1-x)_+}(1-x)_+^{-\alpha} e^{-\lambda t(-x)_+}(-x)_+^{-\alpha} dx.
\end{aligned}
\tag{2.7}
$$

Apply the definition of the gamma function, along with a standard integral formula from Page 344 in [24], to see that (2.6) holds. Since TFBM has stationary increments, it follows from (2.4) that $\mathbb{E}[B_{\alpha,\lambda}(t)^2] = |t|^{2H} C_t^2$ for all t real. Recall the elementary formula $ab = \frac{1}{2}[a^2 + b^2 - (a-b)^2]$, set $a = B^{\alpha,\lambda}(t)$ and $b = B^{\alpha,\lambda}(s)$, take expectations, and use the stationary increments property again, to see that (2.5) holds. □

Remark 2.1.4. The integral representation (2.1) is causal, i.e., $B_{\alpha,\lambda}(t)$ depends only on the values of $B(s)$ for $s \le t$. For applications to spatial statistics, consider

$$
\begin{aligned}
B_{\alpha,\lambda}^{p,q}(t) = p\int_{-\infty}^{+\infty} &\left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \right] B(dx) \\
+ q\int_{-\infty}^{+\infty} &\left[e^{-\lambda(x-t)_+}(x-t)_+^{-\alpha} - e^{-\lambda(x)_+}(x)_+^{-\alpha} \right] B(dx)
\end{aligned}
\tag{2.8}
$$

for $p, q \ge 0$. It is not hard to check, by mimicking the proof of Proposition 2.1.2, that this process also has stationary increments, and satisfies the scaling property

$$
\left\{ B_{\alpha,\lambda}^{p,q}(ct) \right\}_{t\in\mathbb{R}} \triangleq \left\{ c^H B_{\alpha,c\lambda}^{p,q}(t) \right\}_{t\in\mathbb{R}} \tag{2.9}
$$

for any scale factor $c > 0$, where the Hurst index $H = 1/2 - \alpha$. When $p = q > 1$, (2.8) is a *well-balanced* TFBM.

2.2 Harmonizable representation

Let $\hat{B}_1$ and $\hat{B}_2$ be independent Gaussian random measures with $\hat{B}_1(A) = \hat{B}_1(-A)$, $\hat{B}_2(A) = -\hat{B}_2(-A)$ and $\mathbb{E}[(\hat{B}_i(A))^2] = m(A)/2$, where $m(dx) = \sigma^2 dx$, and define the complex-valued Gaussian random measure $\hat{B} = \hat{B}_1 + i\hat{B}_2$. If $f(x)$ is a complex-valued function of x real such that its Fourier transform $\hat{f}(k) := (2\pi)^{-1/2} \int e^{-ikx} f(x)\, dx$ exists and $\int |\hat{f}(k)|^2 dk < \infty$, we define the stochastic integral $\hat{I}(\hat{f}) = \int \hat{f}(k)\hat{B}(dk) := \int \hat{f}_1(k)\hat{B}_1(dk) - \int \hat{f}_2(k)\hat{B}_2(dk)$, where $\hat{f} = \hat{f}_1 + i\hat{f}_2$ is separated into real and imaginary parts. Then $\hat{I}(\hat{f})$ is a Gaussian random variable with mean zero, such that $\mathbb{E}[\hat{I}(\hat{f})\hat{I}(\hat{g})] = \int \hat{f}(k)\overline{\hat{g}(k)}\, dk$. The Parseval identity $\int f(x)g(x)\, dx = \int \hat{f}(k)\overline{\hat{g}(k)}\, dk$ implies that $(\int f(x)B(dx), \int g(x)B(dx)) \triangleq (\int \hat{f}(k)\hat{B}(dk), \int \hat{g}(k)\hat{B}(dk))$, see Proposition 7.2.7 in [61].

Proposition 2.2.1. *The* TFBM (2.1) *has the harmonizable representation*

$$B^{\alpha,\lambda}(t) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-itk} - 1}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk). \tag{2.10}$$

Proof. To show that the stochastic integral (2.10) exists, note that

$$\int_{-\infty}^{+\infty} \left| \frac{e^{-itx} - 1}{(\lambda - ix)^{1-\alpha}} \right|^2 dx \leq \int_{-\infty}^{+\infty} \frac{4}{(\lambda^2 + x^2)^{1-\alpha}} dx < \infty,$$

since the last integrand is bounded and $O(x^{2\alpha-2})$ as $|x| \to \infty$, for $\alpha < \frac{1}{2}$. Observe that the function $g_{\alpha,\lambda,t}$, given by (2.2), has the Fourier transform

$$
\begin{aligned}
\widehat{g_{\alpha,\lambda,t}}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \right] dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{t} e^{-ikx} e^{-\lambda(t-x)} (t-x)^{-\alpha} dx - \int_{-\infty}^{0} e^{-ikx} e^{\lambda x} (-x)^{-\alpha} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[e^{-ikt} \int_{0}^{+\infty} e^{-u(\lambda-ik)} u^{-\alpha} du - \int_{0}^{+\infty} e^{-u(\lambda-ik)} u^{-\alpha} du \right] \\
&= \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \frac{e^{-ikt}-1}{(\lambda-ik)^{1-\alpha}}.
\end{aligned}
\tag{2.11}
$$

Hence by (2.1),

$$
\begin{aligned}
B_{\alpha,\lambda}(t) &= \int_{-\infty}^{+\infty} g_{\alpha,\lambda,t}(x) B(dx) \\
&\triangleq \int_{-\infty}^{+\infty} \widehat{g_{\alpha,\lambda,t}}(k) \hat{B}(dk) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ikt}-1}{(\lambda-ik)^{1-\alpha}} \hat{B}(dk),
\end{aligned}
$$

which is equivalent to (2.10). □

Remark 2.2.2. The spectral representation (2.10) reduces to that of causal FBM in the special case $\lambda = 0$, see for example Equation 7.2.17 in [61]. The general TFBM (2.8) has spectral representation

$$
B_{\alpha,\lambda}^{p,q}(t) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-itk}-1}{ik} \left[\frac{p\, ik}{(\lambda-ik)^{1-\alpha}} - \frac{q\, ik}{(\lambda+ik)^{1-\alpha}} \right] \widehat{B}(dk). \tag{2.12}
$$

2.3 Tempered fractional Gaussian noise

Given a TFBM (2.1), we define tempered fractional Gaussian noise (TFGN)

$$
X_j = B_{\alpha,\lambda}(j+1) - B_{\alpha,\lambda}(j), \quad \text{for integers } -\infty < j < \infty. \tag{2.13}
$$

It follows easily from (2.1) that TFGN has the moving average representation

$$X_j = \int_{-\infty}^{+\infty} \left[e^{-\lambda(j+1-x)_+}(j+1-x)_+^{-\alpha} - e^{-\lambda(j-x)_+}(j-x)_+^{-\alpha} \right] B(dx). \tag{2.14}$$

Using (2.10), it also follows that the harmonizable representation of TFGN is

$$X_j = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikj} \frac{e^{-ik}-1}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk). \tag{2.15}$$

It follows from (2.5) that TFGN is a stationary Gaussian time series with mean zero and covariance function

$$r(j) := \mathbb{E}[X_0 X_j] = \frac{\sigma^2}{2} \left[|j+1|^{2H} C_{j+1}^2 - 2|j|^{2H} C_j^2 + |j-1|^{2H} C_{j-1}^2 \right], \tag{2.16}$$

where $H = 1/2 - \alpha$, and C_j is given by (2.6).

Remark 2.3.1. Using the well-known fact that $K_\nu(x) \sim \sqrt{\pi}(2x)^{-1/2}e^{-x}$ as $x \to \infty$, it follows easily from (2.6) that

$$t^{2H} C_t^2 \to 2\Gamma(2H)(2\lambda)^{-2H}, \quad as\ t \to \infty. \tag{2.17}$$

Hence $C_j \sim C_{j+1}$ as $j \to \infty$. Then, (2.16) along with a Taylor series expansion, shows that

$$r(j) \sim \sigma^2 C_j^2 H(2H-1)|j|^{2H-2} \quad as\ j \to \infty.$$

Compare this with Proposition 7.2.10 in [61]. For $\lambda > 0$, sufficiently small, the power law terms in (2.7) dominate, C_j^2 remains almost constant, and $r(j)$ falls off like $|j|^{2H-2}$

for moderate values of $j > 0$. For larger j, the exponential terms in (2.7) dominate, and (2.17) implies that $r(j) \sim j^{-2} 2H(2H-1)\Gamma(2H)(2\lambda)^{-2H}$, as $j \to \infty$. Hence TFGN is short range dependent, but its covariance function is arbitrarily close to that of long range dependent FGN for small values of λ, and moderate lags. We call this property *semi-long range dependence*, since it is analogous to the *semi-heavy tails* of Barndorff and Nielsen [21]. Figure 2.1 shows a log-log plot of $r(j)$ in the case $H = 0.7$ and $\lambda = 0.001$, where FGN exhibits long range dependence.

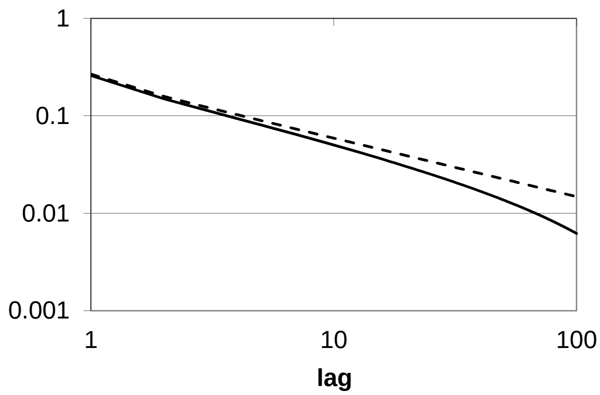

Figure 2.1: The autocovariance function (2.16) for TFGN with $\sigma = 1$, $\lambda = 0.001$ and $H = 0.7$ (solid line) and for the corresponding FGN with $\sigma = 1$, $\lambda = 0$ and $H = 0.7$ (dotted line).

Proposition 2.3.2. TFGN (2.13) *has the spectral density*

$$h(k) = \frac{\Gamma(1-\alpha)^2}{2\pi} \left| e^{-ik} - 1 \right|^2 \sum_{\ell=-\infty}^{+\infty} \frac{\sigma^2}{[\lambda^2 + (k + 2\pi\ell)^2]^{H+1/2}}. \tag{2.18}$$

Proof. Recall that the spectral density

$$h(k) = \frac{1}{2\pi} \sum_{j=-\infty}^{+\infty} e^{ikj} r(j) \quad \text{and} \quad r(j) = \int_{-\pi}^{\pi} e^{-ikj} h(k) dk. \tag{2.19}$$

Define $C = \sqrt{2\pi}/\Gamma(1-\alpha)$ and apply (2.15) to write

$$\begin{aligned} r(j) =& \frac{\sigma^2}{C^2} \int_{-\infty}^{+\infty} e^{-ikj} \frac{\left|e^{-ik} - 1\right|^2}{(\lambda^2 + k^2)^{(1-\alpha)}} dk \\ =& \frac{1}{C^2} \int_{-\pi}^{+\pi} e^{-ikj} \left|e^{-ik} - 1\right|^2 \sum_{\ell=-\infty}^{+\infty} \frac{\sigma^2}{[\lambda^2 + (k + 2\pi\ell)^2]^{(1-\alpha)}} dk \end{aligned} \tag{2.20}$$

and then it follows from (2.19) that the spectral density of TFGN is given by (2.18). □

Remark 2.3.3. Extending (2.13) to all j real, we obtain the continuous parameter TFGN

$$X_t = B_{\alpha,\lambda}(t+1) - B_{\alpha,\lambda}(t).$$

The harmonizable representation of this process is given by (2.15) with j replaced by t, and the proof of Proposition 2.3.2 implies that X_t has spectral density

$$h(\omega) = \frac{\Gamma(1-\alpha)^2}{2\pi} \left|e^{-i\omega} - 1\right|^2 \frac{\sigma^2}{[\lambda^2 + \omega^2]^{H+1/2}} \tag{2.21}$$

for all real ω. The fact that $e^{-i\omega} - 1 \sim -i\omega$ as $\omega \to 0$ yields the low frequency approximation

$$h(\omega) \approx \frac{\sigma^2\Gamma(1-\alpha)^2}{2\pi} \frac{\omega^2}{(\lambda^2 + \omega^2)^{H+1/2}}.$$

See Section 2.5 for an application to wind speed data.

2.4 Sample path properties

We say that the sample paths of a stochastic process $X(t)$ satisfy a uniform Hölder condition of order β on the compact set $K \subset \mathbb{R}$ if there exists a positive random variable A such that

$$|X(x) - X(y)| \leq A|x-y|^{\beta},$$

almost surely for all $x, y \in K$. We say that the process has Hölder critical exponent $\gamma \in (0,1)$ if the process satisfies a uniform Hölder condition of any order $\beta \in (0,\gamma)$ on any compact set $K \subset \mathbb{R}$, and fails to satisfy this condition for $\beta \in (\gamma, 1)$.

Theorem 2.4.1. *The sample paths of the TFBM* (2.1) *have Hölder critical exponent* $H = 1/2 - \alpha$*, for any* $\alpha \in (-1/2, 1/2)$*, and for any* $\lambda \geq 0$*.*

Proof. Since $B_{\alpha,\lambda}(0) = 0$, it follows from Proposition 4 in [10] that if

$$\gamma = \sup\left\{\beta > 0 : \mathbb{E}\left[B_{\alpha,\lambda}(t)^2\right] = \mathrm{o}\left(|t|^{2\beta}\right) \quad \text{as } |t| \to 0\right\}, \tag{2.22}$$

then the TFBM $B_{\alpha,\lambda}(t)$ satisfies a uniform Hölder condition of order β on any compact set for any $\beta \in (0,\gamma)$. Moreover, if we also have

$$\gamma = \inf\left\{\beta > 0 : |t|^{2\beta} = \mathrm{o}\left(\mathbb{E}\left[B_{\alpha,\lambda}(t)^2\right]\right) \quad \text{as } |t| \to 0\right\}, \tag{2.23}$$

then this TFBM has Hölder critical exponent γ. Use the harmonizable representation (2.10) to write

$$\mathbb{E}\left[B_{\alpha,\lambda}(t)^2\right] = \frac{1}{C^2}\int_{-\infty}^{+\infty} \frac{e^{-itk}-1}{(\lambda-ik)^{1-\alpha}}\overline{\left[\frac{e^{-itk}-1}{(\lambda-ik)^{1-\alpha}}\right]}dk$$
$$= \frac{2}{C^2}\int_{-\infty}^{+\infty} \left[1-\cos(tk)\right](\lambda^2+k^2)^{\alpha-1}dk,$$

where $C = \sqrt{2\pi}/\Gamma(1-\alpha)$, and apply the Tauberian theorem for Fourier transforms, Theorem 1 in [57], to see that $\mathbb{E}\left[B_{\alpha,\lambda}(t)^2\right] \sim H(1/t)$ as $t \to 0$, where

$$H(x) = \frac{2}{C^2}\int_{|k|>x} (\lambda^2+k^2)^{\alpha-1}dk.$$

Since $\lambda^2+k^2 \sim k^2$ as $k \to \infty$, for any $\varepsilon > 0$, there exists some $M > 0$ such that $(1-\varepsilon)k^{2\alpha-2} < (\lambda^2+k^2)^{\alpha-1} < (1+\varepsilon)k^{2\alpha-2}$ for all $k > M$, and hence we have

$$\frac{4(1-\varepsilon)}{(1-2\alpha)C^2}\,x^{2\alpha-1} < H(x) < \frac{4(1+\varepsilon)}{(1-2\alpha)C^2}\,x^{2\alpha-1}$$

, for all $x > M$. Substitute $t = 1/x$ to see that both (2.22) and (2.23) hold with $\gamma = 1-2\alpha = 2H$, which completes the proof. □

Remark 2.4.2. The harmonizable representation

$$X(t) = \int_{-\infty}^{+\infty} \left(e^{-itk}-1\right)\hat{f}(k)\hat{B}(dk)$$

defines a mean zero Gaussian processes with stationary increments for any Fourier filter $\hat{f}(k)$ such that $\int \left[1-\cos(tk)\right]|\hat{f}(k)|^2dk < \infty$. If $|\hat{f}(k)|^2$ is regularly varying at infinity with index $2\alpha-2$ for some $-1/2 < \alpha < 1/2$, the Karamata Theorem (e.g., see Lemma 5.3.8 (d) in [47]) implies that $H(x)$ varies regularly at infinity with index $2\alpha-1$, and then the proof

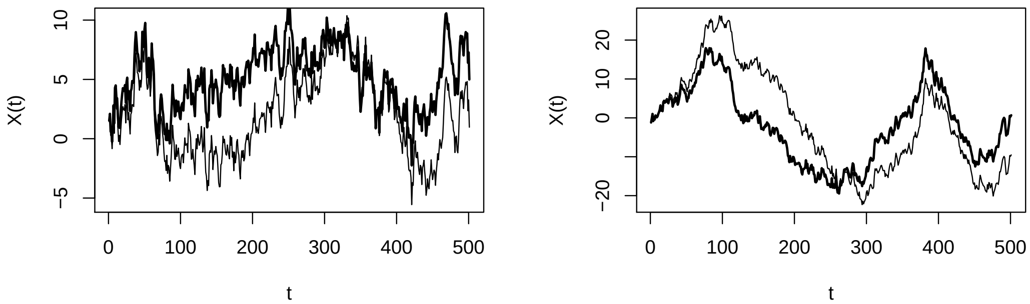

Figure 2.2: Left panel: Sample paths of TFBM (thick black line) with $\lambda = 0.03$ and $H = 0.3$, and FBM (thin black line) with $H = 0.3$. Both graphs use the same noise realization $B(t)$. The right panel shows the same plots for $\lambda = 0.01$ and $H = 0.7$.

of Theorem 2.4.1 extends to show that $X(t)$ has Hölder critical exponent $1 - 2\alpha$. Several examples of such processes are given in [10].

The sample paths of TFBM closely resemble that of FBM for small values of the tempering parameter $\lambda > 0$. The left panel in Figure 2.2 compares a typical sample path of both processes, simulated using the same white noise $B(dx)$, in a case where FBM is negative dependent. The right panel shows the corresponding sample paths in a case where FBM is long range dependent. These simulations use a discretized version of the moving average representation (2.1). It would also be interesting to develop a simulation method based on the harmonizable representation (2.10).

2.5 Discussion

Wind speed data are important for electrical power generation and structural engineering. The most popular model for wind speed near the earth surface, due to Davenport [16], see also [38], can be written in the form $s_t = \mu + X_t$, where $\mu = \mathbb{E}[s_t]$ is the average wind speed,

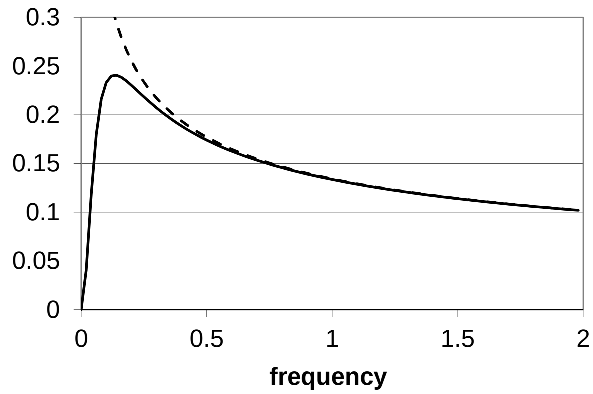

Figure 2.3: The spectral density (2.21) for TFGN with $\sigma = 1$, $\lambda = 0.06$ and $H = 0.7$ (solid line) and FGN with $\sigma = 1$, $\lambda = 0$ and $H = 0.7$ (dotted line).

and X_t has normalized spectral density

$$4800DV_{10}\frac{x^2}{(1+x^2)^{\frac{4}{3}}}, \tag{2.24}$$

where V_{10} is the mean velocity (m/sec) at an altitude of 10 meters, D is the corresponding drag coefficient, and $x = 1200\omega/V_{10}$. In view of Remark 2.3.3, it is not hard to check that (2.24) corresponds to the the spectral density of a continuous parameter TFGN with $\lambda = V_{10}/1200$ and $H = 5/6$. Hence TFGN can provide a useful stochastic process model for wind speed data. Figure 2.3 compares the spectral density of TFGN and FGN in the case where FGN is long range dependent. The spectral density of FGN blows up at the origin like a power law. The spectral density of TFGN follows the same power law at moderate frequencies, but remains bounded at very low frequencies, a behavior typically seen in wind speed data for example in [55, 16, 53, 28].

Chapter 3

Tempered Fractional Calculus

This chapter has three sections. In Section 3.1 we prove some basic results on tempered fractional calculus, which will be needed in the sequel. In Section 3.2 we apply the methods of Section 3.1 to construct a suitable theory of stochastic integration for tempered fractional Brownian motion. Finally, in Section 3.3 we discuss model extensions, related results, and some open questions.

3.1 Tempered fractional calculus

In this section, we define tempered fractional integrals and derivatives, and establish their essential properties. These results will form the foundation of the stochastic integration theory developed in Section 3.2. We begin with the definition of a tempered fractional integral.

Definition 3.1.1. *For any $f \in L^p(\mathbb{R})$ (where $1 \le p < \infty$), the positive and negative tempered fractional integrals are defined by*

$$\mathbb{I}_{+}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} f(u)(t-u)_{+}^{\alpha-1} e^{-\lambda(t-u)_{+}} du, \tag{3.1}$$

and

$$\mathbb{I}_{-}^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} f(u)(u-t)_{+}^{\alpha-1} e^{-\lambda(u-t)_{+}} du, \tag{3.2}$$

respectively, for any $\alpha > 0$ *and* $\lambda > 0$*, where* $\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}dx$ *is the Euler gamma function, and* $(x)_+ = xI(x > 0)$*.*

When $\lambda = 0$ these definitions reduce to the (positive and negative) Riemann-Liouville fractional integral [48, 54, 60], which extends the usual operation of iterated integration to a fractional order. When $\lambda = 1$, the operator (3.1) is called the Bessel fractional integral [60, Section 18.4].

Lemma 3.1.2. *For any* $\alpha > 0$*,* $\lambda > 0$*, and* $p \geq 1$*,* $\mathbb{I}_{\pm}^{\alpha,\lambda}$ *is a bounded linear operator on* $L^p(\mathbb{R})$ *such that*

$$\|\mathbb{I}_{\pm}^{\alpha,\lambda} f\|_p \leq \lambda^{-\alpha}\|f\|_p, \tag{3.3}$$

for all $f \in L^p(\mathbb{R})$*.*

Proof. Young's Theorem [60, p. 12] states that if $\phi \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ then $\phi * f \in L^p(\mathbb{R})$ and the inequality

$$\|\phi * f\|_p \leq \|\phi\|_1\|f\|_p, \tag{3.4}$$

where $*$ denotes the convolution

$$[f * \phi](t) = \int_{-\infty}^{+\infty} f(u)\phi(t-u)du = [\phi * f](t).$$

Obviously $\mathbb{I}_{\pm}^{\alpha,\lambda}$ is linear, and $\mathbb{I}_{\pm}^{\alpha,\lambda} f(t) = [f * \phi_{\alpha}^{\pm}](t)$ where

$$\begin{aligned} \phi_{\alpha}^{+}(t) &= \frac{1}{\Gamma(\alpha)}\, t^{\alpha-1}e^{-\lambda t}\mathbf{1}_{(0,\infty)}(t), \\ \phi_{\alpha}^{-}(t) &= \frac{1}{\Gamma(\alpha)}\, (-t)^{\alpha-1}e^{-\lambda(-t)}\mathbf{1}_{(-\infty,0)}(t) \end{aligned} \tag{3.5}$$

for any $\alpha, \lambda > 0$. But

$$\|\phi_\alpha^\pm\|_1 = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t} dt = \frac{1}{\Gamma(\alpha)} \left[\lambda^{-\alpha}\Gamma(\alpha)\right] = \lambda^{-\alpha},$$

using the formula for the Laplace transform (moment generating function) of the gamma probability density, and then (3.3) follows from Young's Inequality (3.4). □

Next we prove a semigroup property for tempered fractional integrals, which follows easily from the following property of the convolution kernels in the definition (3.1.1).

Lemma 3.1.3. *For any* $\lambda > 0$ *the functions* (3.5) *satisfy*

$$\phi_\alpha^\pm * \phi_\beta^\pm = \phi_{\alpha+\beta}^\pm, \tag{3.6}$$

for any $\alpha > 0$ *and* $\beta > 0$.

Proof. For $t > 0$ we have

$$\begin{aligned}\phi_\alpha^+ * \phi_\beta^+(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)}\, s^{\beta-1} e^{-\lambda s}\, ds \\ &= \frac{1}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} e^{-\lambda t} = \phi_{\alpha+\beta}^+(t)\end{aligned}$$

using the formula for the beta probability density. The proof for ϕ_α^- is similar. □

The following lemma establishes the semigroup property for tempered fractional integrals on $L^p(\mathbb{R})$. In the case $\lambda = 0$, the semigroup property for fractional integrals is well known (e.g., see Samko et al. [60, Theorem 2.5]).

Lemma 3.1.4. *For any $\lambda > 0$ we have*

$$\mathbb{I}_{\pm}^{\alpha,\lambda}\mathbb{I}_{\pm}^{\beta,\lambda}f = \mathbb{I}_{\pm}^{\alpha+\beta,\lambda}f \tag{3.7}$$

for all $\alpha, \beta > 0$ and all $f \in L^p(\mathbb{R})$.

Proof. Lemma 3.1.2 shows that both sides of (3.7) belong to $L^p(\mathbb{R})$ for any $f \in L^p(\mathbb{R})$, and then the result follows immediately from Lemma 3.1.3 along with the fact that $\mathbb{I}_{\pm}^{\alpha,\lambda}f(t) = [f * \phi_{\alpha}^{\pm}](t)$. □

The next result shows that positive and negative tempered fractional integrals are adjoint operators with respect to the inner product $\langle f, g\rangle_2 = \int f(x)g(x)\,dx$ on $L^2(\mathbb{R})$.

Lemma 3.1.5 (Integration by parts)**.** *Suppose $f, g \in L^2(\mathbb{R})$. Then*

$$\langle f, \mathbb{I}_{+}^{\alpha,\lambda}g\rangle_2 = \langle \mathbb{I}_{-}^{\alpha,\lambda}f, g\rangle_2 \tag{3.8}$$

for any $\alpha > 0$ and any $\lambda > 0$.

Proof. Write

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(x)\mathbb{I}_{+}^{\alpha,\lambda}g(x)\ dx &= \int_{-\infty}^{+\infty} f(x)\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x} g(u)(x-u)^{\alpha-1}e^{-\lambda(x-u)}\ du\ dx \\
&= \int_{-\infty}^{+\infty} \frac{g(u)}{\Gamma(\alpha)}\int_{u}^{+\infty} f(x)(x-u)^{\alpha-1}e^{-\lambda(x-u)}\ dx\ du \\
&= \int_{-\infty}^{+\infty} \mathbb{I}_{-}^{\alpha,\lambda}f(x)g(x)\ dx
\end{aligned}$$

and this completes the proof. □

Next we discuss the relationship between tempered fractional integrals and Fourier transforms. Recall that the Fourier transform

$$\mathcal{F}[f](k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx$$

, for functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be extended to an isometry (a linear onto map that preserves the inner product) on $L^2(\mathbb{R})$ such that

$$\widehat{f}(k) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ikx} f(x)\, dx, \tag{3.9}$$

for any $f \in L^2(\mathbb{R})$, see for example [31, Theorem 6.6.4].

Lemma 3.1.6. *For any $\alpha > 0$ and $\lambda > 0$ we have*

$$\mathcal{F}[\mathbb{I}_{\pm}^{\alpha,\lambda} f](k) = \hat{f}(k)(\lambda \pm ik)^{-\alpha}, \tag{3.10}$$

for all $f \in L^1(\mathbb{R})$ and all $f \in L^2(\mathbb{R})$.

Proof. The function ϕ_α^+ in (3.5) has Fourier transform

$$\mathcal{F}[\phi_\alpha^+](k) = \frac{1}{\Gamma(\alpha)\sqrt{2\pi}} \int_0^\infty e^{-ikt} t^{\alpha-1} e^{-\lambda t} dt = \frac{1}{\sqrt{2\pi}} (\lambda + ik)^{-\alpha} \tag{3.11}$$

by the formula for the Fourier transform of a gamma density. For any two functions $f, g \in L^1(\mathbb{R})$, the convolution $f * g \in L^1(\mathbb{R})$ has Fourier transform $\sqrt{2\pi}\hat{f}(k)\hat{g}(k)$ (e.g., see [48, p. 65]), and then (3.10) follows. The argument for $\mathbb{I}_{-}^{\alpha,\lambda}$ is quite similar. If $f \in L^2(\mathbb{R})$, approximate by the L^1 function $f(x)\mathbf{1}_{[-n,n]}(x)$ and let $n \to \infty$. □

Remark 3.1.7. Recall that the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$ consists of the infinitely differentiable functions $g : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} \left| x^n g^{(m)}(x) \right| < \infty,$$

where n, m are non-negative integers, and $g^{(m)}$ is the derivative of order m. The space $\mathcal{S}'(\mathbb{R})$ of continuous linear functionals on $\mathcal{S}(\mathbb{R})$ is called the space of tempered distributions. The Fourier transform, and inverse Fourier transform, can then be extended to linear continuous mappings of $\mathcal{S}'(\mathbb{R})$ into itself. If $f : \mathbb{R} \to \mathbb{R}$ is a measurable function with polynomial growth, so that $\int |f(x)|(1+|x|)^{-p} dx < \infty$ for some $p > 0$, then $T_f(\varphi) = \int f(x)\varphi(x)\, dx := \langle f, \varphi \rangle_1$ is a tempered distribution, also called a generalized function. The Fourier transform of this generalized function is defined as $\hat{T}_f(\varphi) = \langle \hat{f}, \varphi \rangle_1 = \langle f, \hat{\varphi} \rangle_1 = T_f(\hat{\varphi})$ for $\varphi \in \mathcal{S}(\mathbb{R})$. See Yosida [68, Ch.VI] for more details. If f is a tempered distribution, then the tempered fractional integrals $\mathbb{I}_{\pm}^{\alpha,\lambda} f(x)$ exist as convolutions with the tempered distributions $\phi_\alpha^{\pm}$ in (3.5). The same holds for Riemann-Liouville fractional integrals (the case $\lambda = 0$), but that case is more delicate, because the power law kernels $\phi_\alpha^{\pm}$ of (3.5) with $\lambda = 0$ are not in $L^1(\mathbb{R})$.

Next we consider the inverse operator of the tempered fractional integral, which is called a tempered fractional derivative. For our purposes, we only require derivatives of order $0 < \alpha < 1$, and this simplifies the presentation.

Definition 3.1.8. *The positive and negative tempered fractional derivatives of a function $f : \mathbb{R} \to \mathbb{R}$ are defined as*

$$\mathbb{D}_{+}^{\alpha,\lambda} f(t) = \lambda^\alpha f(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t) - f(u)}{(t-u)^{\alpha+1}}\, e^{-\lambda(t-u)} du, \tag{3.12}$$

and

$$\mathbb{D}_{-}^{\alpha,\lambda} f(t) = \lambda^{\alpha} f(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{t}^{+\infty} \frac{f(t) - f(u)}{(u-t)^{\alpha+1}}\, e^{-\lambda(u-t)} du, \tag{3.13}$$

respectively, for any $0 < \alpha < 1$ *and any* $\lambda > 0$.

If $\lambda = 0$, the definitions (3.12) and (3.13) reduce to the positive and negative Marchaud fractional derivatives [60, Section 5.4].

Note that tempered fractional derivatives cannot be defined pointwise for all functions $f \in L^p(\mathbb{R})$, since we need $|f(t) - f(u)| \to 0$ fast enough to counter the singularity of the denominator $(t-u)^{\alpha+1}$, as $u \to t$.

Next we establish the existence and compute the Fourier transform of tempered fractional derivatives on a natural domain.

Theorem 3.1.9. *Assume* f *and* f' *are in* $L^1(\mathbb{R})$. *Then the tempered fractional derivative* $\mathbb{D}_{+}^{\alpha,\lambda} f(t)$ *exists and*

$$\mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda} f](k) = \widehat{f}(k)(\lambda \pm ik)^{\alpha}, \tag{3.14}$$

for any $0 < \alpha < 1$ *and any* $\lambda > 0$.

Proof. A standard argument from functional analysis (e.g., see [50, Proposition 2.2]) shows that if $f, f' \in L^1(\mathbb{R})$, then

$$I := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(t) - f(u)|}{|t-u|^{1+\alpha}}\, dt\, du < \infty, \tag{3.15}$$

for any $0 < \alpha < 1$. To see this, write $I = I_1 + I_2$ where

$$\begin{aligned} I_1 :&= \int_{\mathbb{R}} \int_{\mathbb{R}\cap\{|t-u|<1\}} \frac{|f(t)-f(u)|}{|t-u|^{1+\alpha}}\, dt\, du \\ &= \int_{\mathbb{R}} \int_{\{|z|<1\}} \frac{|f(t)-f(z+t)|}{|z|^{1+\alpha}}\, dz\, dt \\ &\leq \int_{\mathbb{R}} \int_{\{|z|<1\}} |z|^{-\alpha} \int_0^1 |f'(t+uz)|\, du\, dz\, dt = \frac{2}{1-\alpha} \|f'\|_{L^1(\mathbb{R})} < \infty \end{aligned}$$

and

$$\begin{aligned} I_2 :&= \int_{\mathbb{R}} \int_{\mathbb{R}\cap\{|t-u|\geq 1\}} \frac{|f(t)-f(u)|}{|t-u|^{1+\alpha}}\, dt\, du \\ &\leq \int_{\mathbb{R}} \int_{\{|z|\geq 1\}} \frac{|f(t)|+|f(z+t)|}{|z|^{1+\alpha}}\, dt\, dz = \frac{2}{\alpha} \|f\|_{L^1(\mathbb{R})} < \infty. \end{aligned}$$

Now it follows easily from (3.15) that $\mathbb{D}_{\pm}^{\alpha,\lambda} f$ exists for all $f, f' \in L^1(\mathbb{R})$. Define

$$F(t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t)-f(u)}{(t-u)^{\alpha+1}} e^{-\lambda(t-u)}\, du,$$

and apply the Fubini Theorem, along with the shift property $\mathcal{F}[f(t-y)](k) = e^{-iky}\hat{f}(k)$ of the Fourier transform, to see that

$$\begin{aligned} \widehat{F}(k) &= \frac{\alpha}{\Gamma(1-\alpha)\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikt} \int_0^{\infty} \frac{f(t)-f(t-y)}{y^{\alpha+1}} e^{-\lambda y}\, dy\, dt \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} y^{-\alpha-1} e^{-\lambda y} \left(1 - e^{-iky}\right) \widehat{f}(k)\, dy = \frac{I_\lambda(\alpha)}{\Gamma(1-\alpha)} \widehat{f}(k), \end{aligned} \tag{3.16}$$

where

$$I_\lambda(\alpha) = \int_0^{+\infty} \left(e^{-\lambda y} - e^{-(\lambda+ik)y}\right) \alpha y^{-\alpha-1}\, dy.$$

Integrate by parts with $u = e^{-\lambda y} - e^{-(\lambda+ik)y}$ and use the fact that $e^{-\lambda y} - e^{-(\lambda+ik)y} = \mathrm{O}(y)$ as $y \to 0$, to obtain

$$I_\lambda(\alpha) = \left[\left(e^{-\lambda y} - e^{-(\lambda+ik)y}\right)\left(-y^{-\alpha}\right)\right]\Big|_0^\infty + \int_0^\infty y^{-\alpha}\left[-\lambda e^{-\lambda y} + (\lambda+ik)e^{-(\lambda+ik)y}\right] dy.$$

Use the definition of the gamma function, and the formula for the Fourier tranform of the gamma probability density, to obtain

$$\begin{aligned} I_\lambda(\alpha) &= -\lambda\int_0^\infty y^{-\alpha}e^{-\lambda y}\, dy + (\lambda+ik)\int_0^\infty y^{-\alpha}e^{-(\lambda+ik)y}\, dy \\ &= -\lambda^\alpha\Gamma(1-\alpha) + (\lambda+ik)\frac{\Gamma(1-\alpha)}{\lambda^{1-\alpha}}\left(1+\frac{ik}{\lambda}\right)^{\alpha-1} \\ &= \Gamma(1-\alpha)\left[(\lambda+ik)^\alpha - \lambda^\alpha\right]. \end{aligned}$$

Then $\widehat{F}(k) = \widehat{f}(k)\left[(\lambda+ik)^\alpha - \lambda^\alpha\right]$, and hence $\mathcal{F}[\mathbb{D}_+^{\alpha,\lambda}f](k) = (\lambda+ik)^\alpha\hat{f}(k)$. The proof for $\mathcal{F}[\mathbb{D}_-^{\alpha,\lambda}f](k)$ is similar. □

Remark 3.1.10. Theorem 3.1.9 can also be proven, under somewhat stronger conditions, using the generator formula for infinitely divisible semigroups [48, Theorem 3.17 and Theorem 3.23 (b)].

Next we extend the definition of tempered fractional derivatives to a suitable class of functions in $L^2(\mathbb{R})$. For any $\alpha > 0$ and $\lambda > 0$, we define the fractional Sobolev space

$$W^{\alpha,2}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (\lambda^2+k^2)^\alpha|\hat{f}(k)|^2\, dk < \infty\}, \tag{3.17}$$

which is a Banach space with norm $\|f\|_{\alpha,\lambda} = \|(\lambda^2+k^2)^{\alpha/2}\hat{f}(k)\|_2$. The space $W^{\alpha,2}(\mathbb{R})$ is the same for any $\lambda > 0$ (typically we take $\lambda = 1$) and all the norms $\|f\|_{\alpha,\lambda}$ are equivalent, since $1+k^2 \le \lambda^2+k^2 \le \lambda^2(1+k^2)$ for all $\lambda \ge 1$, and $\lambda^2+k^2 \le 1+k^2 \le \lambda^{-2}(1+k^2)$ for all $0<\lambda<1$.

Definition 3.1.11. *The positive (resp., negative) tempered fractional derivative* $\mathbb{D}_{\pm}^{\alpha,\lambda} f(t)$ *of a function* $f \in W^{\alpha,2}(\mathbb{R})$ *is defined as the unique element of* $L^2(\mathbb{R})$ *with Fourier transform* $\widehat{f}(k)(\lambda \pm ik)^{\alpha}$*, for any* $\alpha>0$ *and any* $\lambda>0$*.*

Remark 3.1.12. The pointwise definition of the tempered fractional derivative in real space is more complicated when $\alpha>1$. For example, when $1<\alpha<2$ we have

$$\mathbb{D}_{+}^{\alpha,\lambda} f(t) = \lambda^{\alpha} f(t) + \alpha\lambda^{\alpha-1} f'(x) + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(u)-f(t)+(t-u)f'(t)}{(t-u)^{\alpha+1}}\, e^{-\lambda(t-u)} du,$$

for all $f \in W^{1,2}(\mathbb{R})$, compare [48, Remark 7.11].

Lemma 3.1.13. *For any* $\alpha>0$*,* $\beta>0$ *and* $\lambda>0$ *we have*

$$\mathbb{D}_{\pm}^{\alpha,\lambda}\mathbb{D}_{\pm}^{\beta,\lambda} f(t) = \mathbb{D}_{\pm}^{\alpha+\beta,\lambda} f(t),$$

for any $f \in W^{\alpha+\beta,2}(\mathbb{R})$*.*

Proof. It is obvious from (3.17) that $W^{\alpha,2}(\mathbb{R}) \subset W^{\beta,2}(\mathbb{R})$ for $\alpha>\beta$. It is clear from Definition 3.1.11 that $\mathbb{D}_{\pm}^{\beta,\lambda} f(t)$ exists and belongs to $W^{\alpha,2}(\mathbb{R})$ for any $f \in W^{\alpha+\beta,2}(\mathbb{R})$, and likewise, $\mathbb{D}_{\pm}^{\alpha,\lambda} f(t)$ exists and belongs to $L^2(\mathbb{R})$ for any $f \in W^{\alpha,2}(\mathbb{R})$. □

Lemma 3.1.14. *For any* $\alpha>0$ *and* $\lambda>0$*, we have*

$$\mathbb{D}_{\pm}^{\alpha,\lambda}\mathbb{I}_{\pm}^{\alpha,\lambda} f(t) = f(t) \tag{3.18}$$

for any function $f \in L^2(\mathbb{R})$, *and*

$$\mathbb{I}_{\pm}^{\alpha,\lambda}\mathbb{D}_{\pm}^{\alpha,\lambda} f(t) = f(t) \tag{3.19}$$

for any $f \in W^{\alpha,2}(\mathbb{R})$.

Proof. Given $f \in L^2(\mathbb{R})$, note that $g(t) = \mathbb{I}_{\pm}^{\alpha,\lambda} f(t)$ satisfies $\hat{g}(k) = \hat{f}(k)(\lambda \pm ik)^{-\alpha}$ by Lemma 3.1.6, and then it follows easily that $g \in W^{\alpha,2}(\mathbb{R})$. Definition 3.1.11 implies that

$$\mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda}\mathbb{I}_{\pm}^{\alpha,\lambda} f](k) = \mathcal{F}[\mathbb{D}_{\pm}^{\alpha,\lambda} g](k) = \widehat{g}(k)(\lambda \pm ik)^{\alpha} = \hat{f}(k), \tag{3.20}$$

and then (3.18) follows using the uniqueness of the Fourier transform. The proof of (3.19) is similar. □

Lemma 3.1.15. *Suppose* $f, g \in W^{\alpha,2}(\mathbb{R})$. *Then*

$$\langle f, \mathbb{D}_{+}^{\alpha,\lambda} g\rangle_2 = \langle \mathbb{D}_{-}^{\alpha,\lambda} f, g\rangle_2, \tag{3.21}$$

for any $\alpha > 0$ *and any* $\lambda > 0$.

Proof. Apply the Plancherel Theorem along with Definition 3.1.11 to see that

$$\langle f, \mathbb{D}_{+}^{\alpha,\lambda} g\rangle_2 = \int f(x)\overline{\mathbb{D}_{+}^{\alpha,\lambda} g(x)}\, dx = \langle \hat{f}, (\lambda + ik)^{\alpha}\hat{g}\rangle_2 = \langle (\lambda - ik)^{\alpha}\hat{f}, \hat{g}\rangle_2 = \langle \mathbb{D}_{-}^{\alpha,\lambda} f, g\rangle_2$$

and this completes the proof. □

Remark 3.1.16. One can also prove (3.21) for $f, f', g, g' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ using integration by parts, compare [69, Appendix A.1].

A slightly different tempered fractional derivative

$$\begin{aligned}\mathbf{D}_{+}^{\alpha,\lambda} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f(t)-f(u)}{(t-u)^{\alpha+1}} e^{-\lambda(t-u)} du, \\ \mathbf{D}_{-}^{\alpha,\lambda} f(t) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{t}^{+\infty} \frac{f(t)-f(u)}{(u-t)^{\alpha+1}} e^{-\lambda(u-t)} du,\end{aligned} \tag{3.22}$$

was proposed by Cartea and del-Castillo-Negrete [12] for a problem in physics, and studied further by Baeumer and Meerschaert [4, 48] using tools from probability theory and semi-groups. When $0 < \alpha < 1$, $\mathcal{F}[\mathbf{D}_{\pm}^{\alpha,\lambda} f](k) = \widehat{f}(k)[(\lambda \pm ik)^{\alpha} - \lambda^{\alpha}]\hat{f}(k)$ for suitable functions f. The additional λ^{α} term makes the evolution equation

$$\frac{\partial}{\partial t} u(x,t) = [p\mathbf{D}_{+}^{\alpha,\lambda} + q\mathbf{D}_{-}^{\alpha,\lambda}]u(x,t), \tag{3.23}$$

for $p, q \geq 0$ mass preserving, which can easily be seen by considering the Fourier transform $\hat{u}(k,t) = \exp(t[(\lambda \pm ik)^{\alpha} - \lambda^{\alpha}])$ of point source solutions to the tempered fractional diffusion equation (3.23). Now $x \mapsto u(x,t)$ are the probability density functions of a tempered stable Lévy process, as in Rosiński [59]. That process arises as the long-time scaling limit of a random walk with exponentially tempered power law jumps, see Chakrabarty and Meerschaert [13]. The tempered fractional diffusion equation (3.23) has been applied to contaminant plumes in underground aquifers, and sediment transport in rivers [49, 70, 71].

Remark 3.1.17. Tempered fractional derivatives are a natural analogues of integer (and fractional) order derivatives. For suitable functions $f(x)$, the Fourier transform of the derivative $f'(x)$ is $(ik)\hat{f}(k)$ (e.g., see [48, p. 8]), and one can define the fractional derivative $\mathbb{D}_{\pm}^{\alpha} f(t)$ as the function with Fourier transform $(ik)^{\alpha}\hat{f}(k)$. Definition 3.1.11 extends to tempered fractional derivatives.

3.2 Stochastic Integrals

In this section, we apply tempered fractional calculus to define stochastic integrals with respect to TFBM. First we recall the moving average representation of TFBM as a stochastic integral with respect to Brownian motion.

Recall from Definition 2.1.1, the stochastic integral

$$B_{\alpha,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \right] B(dx). \tag{3.24}$$

When $\lambda = 0$ and $\alpha < -1/2$, the right-hand side of (3.24) does not exist, since the integrand is not in $L^2(\mathbb{R})$. However, TFBM with $\lambda > 0$ and $\alpha < -1/2$ is well-defined, because the exponential tempering keeps the integrand in $L^2(\mathbb{R})$. In fact, if $\alpha < -1/2$ and $\lambda > 0$, or if $\alpha = 0$ and $\lambda > 0$, we will now show that TFBM is a semimartingale, and hence one can define stochastic integrals $I(f) := \int f(x) B_{\alpha,\lambda}(dx)$ in the standard manner, via the Itô stochastic calculus (e.g., see Kallenberg [30, Chapter 15]).

Theorem 3.2.1. *A tempered fractional Brownian motion $\{B_{\alpha,\lambda}(t)\}_{t\geq 0}$ with $\alpha < -1/2$ and $\lambda > 0$ is a continuous semimartingale with the canonical decomposition*

$$B_{\alpha,\lambda}(t) = -\lambda \int_0^t M_{\alpha,\lambda}(s)\, ds - \alpha \int_0^t M_{\alpha+1,\lambda}(s)\, ds \tag{3.25}$$

where

$$M_{\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha}\, B(dx). \tag{3.26}$$

Moreover, $\{B_{\alpha,\lambda}(t)\}_{t\geq 0}$ is a finite variation process. The same is true if $\alpha = 0$ and $\lambda > 0$.

Proof. Let $\{\mathcal{F}_t^B\}_{t\geq 0}$ be the σ-algebra generated by $\{B_s : 0 \leq s \leq t\}$. Given a function

$g : \mathbb{R} \to \mathbb{R}$ such that $g(t) = 0$ for all $t < 0$, and

$$g(t) = C + \int_0^t h(s),\ ds \quad \text{for all } t > 0, \tag{3.27}$$

for some $C \in \mathbb{R}$ and some $h \in L^2(\mathbb{R})$, a result of Cheridito [14, Theorem 3.9] shows that the Gaussian stationary increment process

$$Y_t^g := \int_{\mathbb{R}} [g(t-u) - g(-u)]\ B(du),\ t \geq 0 \tag{3.28}$$

is a continuous $\{\mathcal{F}_t^B\}_{t\geq 0}$ semimartingale with canonical decomposition

$$Y_t^g = g(0)B_t + \int_0^t \int_{-\infty}^s h(s-u)B(du)ds, \tag{3.29}$$

and conversely, that if (3.28) defines a semimartingale on $[0, T]$ for some $T > 0$, then g satisfies these properties. Define $g(t) = 0$ for $t \leq 0$ and

$$g(t) := e^{-\lambda t} t^{-\alpha} \quad \text{for } t > 0. \tag{3.30}$$

It is easy to check that the function $g(t-u) - g(-u)$, which is the integrand in (3.24), is square integrable over the entire real line for any $\alpha < 1/2$ and $\lambda > 0$. Next observe that (3.27) holds with $C = 0$, $h(s) = 0$ for $s < 0$ and

$$h(s) := \frac{d}{ds}[e^{-\lambda s} s^{-\alpha}] = -\lambda e^{-\lambda s} s^{-\alpha} - \alpha e^{-\lambda s} s^{-\alpha-1} \in L^2(\mathbb{R}) \tag{3.31}$$

for any $\alpha < -1/2$ and $\lambda > 0$. Then it follows from [14, Theorem 3.9] that TFBM is a continuous semimartingale with canonical decomposition

$$B_{\alpha,\lambda} = \int_0^t \int_{-\infty}^s -\lambda e^{-\lambda(s-u)}(s-u)^{-\alpha} - \alpha e^{-\lambda(s-u)}(s-u)^{-\alpha-1} B(du)\, ds \tag{3.32}$$

which reduces to (3.25). Since $C = 0$, Theorem 3.9 in [14] implies that $\{B_{\alpha,\lambda}(t)\}$ is a finite variation process. The proof for $\alpha = 0$ is similar, using $g(t) = e^{-\lambda t}$ for $t > 0$. □

Remark 3.2.2. When $\alpha = 0$ and $\lambda > 0$, the Gaussian stochastic process (3.26) is an Ornstein-Uhlenbeck process. When $\alpha < -1/2$ and $\lambda > 0$, it is a one dimensional Matérn stochastic process [5, 23, 26], also called a "fractional Ornstein-Uhlenbeck process" in the physics literature [40]. It follows from Knight [32, Theorem 6.5] that $M_{\alpha,\lambda}(t)$ is a semimartingale in both cases.

Cheridito [14, Theorem 3.9] provides a necessary and sufficient condition for the process (3.28) to be a semimartingale, and then it is not hard to check that TFBM is *not a semimartingale* in the remaining cases when $-1/2 < \alpha < 0$ or $0 < \alpha < 1/2$. Next we will investigate the problem of stochastic integration with deterministic integrands in these two cases. Our approach follows that of Pipiras and Taqqu [56].

Next we establish a link between TFBM and tempered fractional calculus.

Lemma 3.2.3. *For a tempered fractional Brownian motion* (3.24) *with* $\lambda > 0$*, we have:*

(i) When $-1/2 < \alpha < 0$*, we can write*

$$B_{\alpha,\lambda}(t) = \Gamma(\kappa+1) \int_{-\infty}^{+\infty} \left[\mathbb{I}_{-}^{\kappa,\lambda} \mathbf{1}_{[0,t]}(x) - \lambda \mathbb{I}_{-}^{\kappa+1,\lambda} \mathbf{1}_{[0,t]}(x) \right] \, B(dx) \tag{3.33}$$

where $\kappa = -\alpha$*.*

(ii) When $0 < \alpha < 1/2$, we can write

$$B_{\alpha,\lambda}(t) = \Gamma(1-\alpha) \int_{-\infty}^{+\infty} \left[\mathbb{D}_{-}^{\alpha,\lambda} \mathbf{1}_{[0,t]}(x) - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} \mathbf{1}_{[0,t]}(x) \right] B(dx). \tag{3.34}$$

Proof. To prove part (i), write the kernel function from (3.24) in the form

$$\begin{aligned} g_{t,\lambda}(x) &:= e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(-x)_+}(-x)_+^{-\alpha} \\ &= \int_0^t \frac{d\left[e^{-\lambda(u-x)_+}(u-x)_+^{\kappa}\right]}{du} du \\ &= -\lambda \int_{-\infty}^{+\infty} \mathbf{1}_{[0,t]}(u) e^{-\lambda(u-x)_+}(u-x)_+^{(\kappa+1)-1} du \\ &\quad + \kappa \int_{-\infty}^{+\infty} \mathbf{1}_{[0,t]}(u) e^{-\lambda(u-x)_+}(u-x)_+^{\kappa-1} du \end{aligned}$$

and apply the definition (3.2) of the tempered fractional integral.

To prove part (ii), it suffices to show that the integrand

$$g_{t,\lambda}(x) = e^{-\lambda(t-x)_+}(t-x)_+^{-\alpha} - e^{-\lambda(0-x)_+}(0-x)_+^{-\alpha} =: \phi_t(x) - \phi_0(x)$$

in (3.24) equals the integrand in (3.34). We will prove this using Fourier transforms. The substitution $u = t - x$ shows that

$$\widehat{\phi_t}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-ikx} e^{-\lambda(t-x)}(t-x)^{-\alpha}\, dx = \frac{e^{-ikt}\Gamma(1-\alpha)}{\sqrt{2\pi}(\lambda - ik)^{1-\alpha}},$$

using the formula for the Fourier transform of the gamma density, and hence

$$\widehat{g_{t,\lambda}}(k) = \widehat{\phi_t}(k) - \widehat{\phi_0}(k) = \Gamma(1-\alpha) \frac{e^{-ikt} - 1}{\sqrt{2\pi}(\lambda - ik)^{1-\alpha}}. \tag{3.35}$$

On the other hand, from Lemma 3.1.6 and Theorem 3.1.9 we obtain

$$\begin{aligned}\mathcal{F}[\mathbb{D}_-^{\alpha,\lambda}\mathbf{1}_{[0,t]} - \lambda\mathbb{I}_-^{1-\alpha,\lambda}\mathbf{1}_{[0,t]}](k) &= [(\lambda - ik)^\alpha - \lambda(\lambda - ik)^{\alpha-1}] \cdot \frac{e^{-ikt}-1}{(-ik)\sqrt{2\pi}} \\ &= (\lambda - ik)^{\alpha-1} \cdot \frac{e^{-ikt}-1}{\sqrt{2\pi}},\end{aligned} \tag{3.36}$$

where we have used the formula (which is easy to verify)

$$\hat{h}(k) = \mathcal{F}[\mathbf{1}_{[a,b)}](k) = \frac{e^{-ikb} - e^{-ika}}{(-ik)\sqrt{2\pi}}. \tag{3.37}$$

The desired result now follows by the uniqueness of the Fourier transform. □

Next, we explain the connection between the fractional calculus representations (3.33) and (3.34). Substitute $\kappa = -\alpha$ into (3.33) and note that the resulting formula differs from (3.34) only in that the tempered fractional integral $\mathbb{I}_-^{-\alpha,\lambda}$ is replaced by the tempered fractional derivative $\mathbb{D}_-^{\alpha,\lambda}$. Lemma 3.1.14 shows that $\mathbb{I}_-^{\alpha,\lambda}$ and $\mathbb{D}_-^{\alpha,\lambda}$ are inverse operators, and hence it makes sense to define $\mathbb{I}_\pm^{-\alpha,\lambda} := \mathbb{D}_\pm^{\alpha,\lambda}$ when $0 < \alpha < 1$. Now equations (3.33) and (3.34) are equivalent.

Next, we discuss a general construction for stochastic integrals with respect to TFBM. For a standard Brownian motion $\{B(t)\}_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, P)$, the stochastic integral $\mathcal{I}(f) := \int f(x)B(dx)$ is defined for any $f \in L^2(\mathbb{R})$, and the mapping $f \mapsto \mathcal{I}(f)$ defines an isometry from $L^2(\mathbb{R})$ into $L^2(\Omega)$, called the *Itô isometry*:

$$\langle \mathcal{I}(f), \mathcal{I}(g)\rangle_{L^2(\Omega)} = \mathrm{Cov}[\mathcal{I}(f), \mathcal{I}(g)] = \int f(x)g(x)\,dx = \langle f, g\rangle_{L^2(\mathbb{R})}. \tag{3.38}$$

Since this isometry maps $L^2(\mathbb{R})$ onto the space $\overline{\mathrm{Sp}}(B) = \{\mathcal{I}(f) : f \in L^2(\mathbb{R})\}$, we say that these two spaces are isometric. For any elementary function (step function)

$$f(u) = \sum_{i=1}^{n} a_i \mathbf{1}_{[t_i, t_{i+1})}(u), \tag{3.39}$$

where a_i, t_i are real numbers such that $t_i < t_j$ for $i < j$, it is natural to define the stochastic integral

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) = \sum_{i=1}^{n} a_i \left[B_{\alpha,\lambda}(t_{i+1}) - B_{\alpha,\lambda}(t_i) \right], \tag{3.40}$$

and then it follows immediately from (3.33) that for $f \in \mathcal{E}$, the space of elementary functions, the stochastic integral

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) = \Gamma(\kappa+1) \int_{\mathbb{R}} \left[\mathbb{I}_{-}^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_{-}^{\kappa+1,\lambda} f(x) \right] \, B(dx)$$

is a Gaussian random variable with mean zero, such that for any $f, g \in \mathcal{E}$ we have

$$\begin{aligned} &\langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g) \rangle_{L^2(\Omega)} = \mathbb{E} \left(\int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) \int_{\mathbb{R}} g(x) B_{\alpha,\lambda}(dx) \right) \\ &= \Gamma(\kappa+1)^2 \int_{\mathbb{R}} \left[\mathbb{I}_{-}^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_{-}^{\kappa+1,\lambda} f(x) \right] \left[\mathbb{I}_{-}^{\kappa,\lambda} g(x) - \lambda \mathbb{I}_{-}^{\kappa+1,\lambda} g(x) \right] \, dx, \end{aligned} \tag{3.41}$$

in view of (3.33) and the Itô isometry (3.38). The linear space of Gaussian random variables $\left\{ \mathcal{I}^{\alpha,\lambda}(f), f \in \mathcal{E} \right\}$ is contained in the larger linear space

$$\overline{\mathrm{Sp}}(B_{\alpha,\lambda}) = \left\{ X : \mathcal{I}^{\alpha,\lambda}(f_n) \to X \text{ in } L^2(\Omega) \text{ for some sequence } (f_n) \text{ in } \mathcal{E} \right\}. \tag{3.42}$$

An element $X \in \overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ is mean zero Gaussian with variance

$$\mathrm{Var}(X) = \lim_{n \to \infty} \mathrm{Var}[\mathcal{I}^{\alpha,\lambda}(f_n)],$$

and X can be associated with an equivalence class of sequences of elementary functions (f_n) such that $\mathcal{I}^{\alpha,\lambda}(f_n) \to X$ in $L^2(\mathbb{R})$. If $[f_X]$ denotes this class, then X can be written in an integral form as

$$X = \int_{\mathbb{R}} [f_X] dB_{\alpha,\lambda} \tag{3.43}$$

and the right hand side of (3.43) is called the stochastic integral with respect to TFBM on the real line (see, for example, Huang and Cambanis [27], page 587). In the special case of a Brownian motion $\lambda = \alpha = 0$, $\mathcal{I}^{\alpha,\lambda}(f_n) \to X$ along with the Itô isometry (3.38) implies that (f_n) is a Cauchy sequence, and then since $L^2(\mathbb{R})$ is a (complete) Hilbert space, there exists a unique $f \in L^2(\mathbb{R})$ such that $f_n \to f$ in $L^2(\mathbb{R})$, and we can write $X = \int_{\mathbb{R}} f(x)B(dx)$. However, if the space of integrands is not complete, then the situation is more complicated. We begin with the case $-1/2 < \alpha < 0$, where the corresponding FBM is long range dependent.

3.2.1 Semi-long range dependence

Here we investigate stochastic integrals with respect to TFBM in the case $-1/2 < \alpha < 0$, so that $1/2 < H < 1$ in (2.4). Equation (3.41) suggests the appropriate space of integrands for TFBM, in order to obtain a nice isometry that maps into the space $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ of stochastic integrals.

Theorem 3.2.4. *Given $-1/2 < \alpha < 0$ and $\lambda > 0$, let $\kappa = -\alpha$. Then the class of functions*

$$\mathcal{A}_1 := \left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \left| \mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x) \right|^2 dx < \infty \right\}, \tag{3.44}$$

is a linear space with inner product

$$\langle f, g\rangle_{\mathcal{A}_1} := \langle F, G\rangle_{L^2(\mathbb{R})} \tag{3.45}$$

where

$$\begin{aligned} F(x) &= \Gamma(\kappa+1)[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x)], \\ G(x) &= \Gamma(\kappa+1)[\mathbb{I}_-^{\kappa,\lambda} g(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} g(x)]. \end{aligned} \tag{3.46}$$

The set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}_1$. The space $\mathcal{A}_1$ is not complete.

The proof of Theorem 3.2.4 requires one simple lemma, which shows that $\mathbb{I}_-^{\kappa,\lambda} - \lambda\mathbb{I}_-^{\kappa+1,\lambda}$ is a bounded linear operator on $L^p(\mathbb{R})$ for any $1 \le p < \infty$.

Lemma 3.2.5. *Under the assumptions of Theorem 3.2.4, suppose $1 \le p < \infty$. Then for any $f \in L^p(\mathbb{R})$ we have*

$$\|\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x)\|_p \le C\|f\|_p \tag{3.47}$$

where C is a constant depending only on α and λ.

Proof. It follows from Lemma 3.1.2 that $\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x) \in L^p(\mathbb{R})$ and that

$$\|\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x)\|_p \le \|\mathbb{I}_-^{\kappa,\lambda} f(x)\|_p + \lambda\|\mathbb{I}_-^{\kappa+1,\lambda} f(x)\|_p \le 2\lambda^{-\kappa}\|f\|_p$$

for any $f \in L^p(\mathbb{R})$. □

Remark 3.2.6. It follows from Lemma 3.2.5 that $\mathcal{A}_1$ contains every function in $L^2(\mathbb{R})$, and hence they are the same set, but endowed with a different inner product. The inner product on the space $\mathcal{A}_1$ is required to obtain a nice isometry.

Proof of Theorem 3.2.4. The proof is similar to [56, Theorem 3.2]. To show that $\mathcal{A}_1$ is an inner product space, we will check that $\langle f, f\rangle_{\mathcal{A}_1} = 0$ implies $f = 0$ almost everywhere. If $\langle f, f\rangle_{\mathcal{A}_1} = 0$, then in view of (3.45) and (3.46) we have $\langle F, F\rangle_2 = 0$, so $F(x) = \Gamma(1 + \kappa)[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x)] = 0$ for almost every $x \in \mathbb{R}$. Then

$$\mathbb{I}_-^{\kappa,\lambda} f(x) = \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x) \quad \text{for almost every } x \in \mathbb{R}. \tag{3.48}$$

Apply $\mathbb{D}_-^{\kappa,\lambda}$ to both sides of equation (3.48) and use Lemma 3.1.4 along with Lemma 3.1.14 to get

$$f(x) = \mathbb{D}_-^{\kappa,\lambda}\mathbb{I}_-^{\kappa,\lambda} f(x) = \mathbb{D}_-^{\kappa,\lambda}\lambda\mathbb{I}_-^{\kappa+1,\lambda} f(x) = \lambda\left[\mathbb{D}_-^{\kappa,\lambda}\mathbb{I}_-^{\kappa,\lambda}\right]\mathbb{I}_-^{1,\lambda} f(x) = \lambda\mathbb{I}_-^{1,\lambda} f(x)$$

for almost every $x \in \mathbb{R}$, and in view of the definition (3.1) this is equivalent to

$$f(x) = \lambda \int_x^{+\infty} f(u) e^{-\lambda(u-x)} du = \lambda e^{\lambda x} \int_x^{+\infty} f(u) e^{-\lambda u} du \tag{3.49}$$

for almost every $x \in \mathbb{R}$. Observe that the functions $f(u)$ and $e^{-\lambda u}$ are in $L^2[x, \infty)$ for any $x \in \mathbb{R}$ and then, by the Cauchy-Schwartz inequality, the function $f(u)e^{-\lambda u}$ is in $L^1[x, \infty)$. It follows that $\int_x^{+\infty} f(u)e^{-\lambda u} du$ is absolutely continuous, and so the function $f(x)$ in (3.49) is also absolutely continuous. Taking the derivative on both sides of (3.49) using the Lebesgue Differentiation Theorem (e.g., see [66, Theorem 7.16]) we get

$$f'(x) = \lambda f(x) - \lambda e^{\lambda x} f(x) e^{-\lambda x} = 0 \quad \text{for almost every } x \in \mathbb{R}.$$

Then for any $a, b \in R$ we have

$$f(b) = f(a) + \int_a^b f'(x)\, dx = f(a).$$

and so $f(x)$ is a constant function. Since $f \in L^2(\mathbb{R})$, it follows that $f(x) = 0$ for all $x \in \mathbb{R}$, and hence $\mathcal{A}_1$ is an inner product space.

Next, we want to show that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{A}_1$. For any $f \in \mathcal{A}_1$, we also have $f \in L^2(\mathbb{R})$, and hence there exists a sequence of elementary functions (f_n) in $L^2(\mathbb{R})$ such that $\|f - f_n\|_2 \to 0$. But

$$\|f - f_n\|_{\mathcal{A}_1} = \langle f - f_n, f - f_n \rangle_{\mathcal{A}_1} = \langle F - F_n, F - F_n \rangle_2 = \|F - F_n\|_2,$$

where $F_n(x) = \mathbb{I}_-^{\kappa,\lambda} f_n(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f_n(x)$ and $F(x)$ is given by (3.46). Lemma 3.2.5 implies that

$$\|f - f_n\|_{\mathcal{A}_1} = \|F - F_n\|_2 = \|\mathbb{I}_-^{\kappa,\lambda}(f - f_n) - \lambda \mathbb{I}_-^{\kappa+1,\lambda}(f - f_n)\|_2 \leq C\|f - f_n\|_2$$

for some $C > 0$, and since $\|f - f_n\|_2 \to 0$, it follows that the set of elementary functions is dense in $\mathcal{A}_1$.

Finally, we provide an example to show that $\mathcal{A}_1$ is not complete. The functions

$$\widehat{f_n}(k) = |k|^{-p} \mathbf{1}_{\{1<|k|<n\}}(k), \;\; p > 0,$$

are in $L^2(\mathbb{R})$, $\overline{\widehat{f_n}(k)} = \widehat{f_n}(-k)$, and hence they are the Fourier transforms of functions $f_n \in L^2(\mathbb{R})$. Apply Lemma 3.1.6 to see that the corresponding functions $F_n(x) = \Gamma(\kappa +$

$1)[\mathbb{I}_-^{\kappa,\lambda} f_n(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f_n(x)]$ from (3.46) have Fourier transform

$$\mathcal{F}[F_n](k) = \Gamma(1-\alpha)[(\lambda - ik)^\alpha - \lambda(\lambda - ik)^{\alpha-1}]\hat{f}_n(k) = \frac{-ik\Gamma(1-\alpha)}{(\lambda - ik)^{1-\alpha}}\hat{f}_n(k). \tag{3.50}$$

Since $\alpha < 0$, it follows that

$$\|F_n\|_2^2 = \|\hat{F}_n\|_2^2 = \Gamma(1-\alpha)^2 \int_{-\infty}^{\infty} \left|\widehat{f_n}(k)\right|^2 \frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}} \, dk < \infty$$

for each n, which shows that $f_n \in \mathcal{A}_1$. Now it is easy to check that $f_n - f_m \to 0$ in $\mathcal{A}_1$, as $n, m \to \infty$, whenever $p > 1/2 + \alpha$, so that (f_n) is a Cauchy sequence. Choose $p \in (1/2 + \alpha, 1/2)$ and suppose that there exists some $f \in \mathcal{A}_1$ such that $\|f_n - f\|_{\mathcal{A}_1} \to 0$ as $n \to \infty$. Then

$$\int_{-\infty}^{\infty} \left|\widehat{f_n}(k) - \widehat{f}(k)\right|^2 \frac{k^2}{(\lambda^2 + k^2)^{1-\alpha}} \, dk \to 0 \tag{3.51}$$

as $n \to \infty$, and since, for any given $m \geq 1$, the value of $\widehat{f_n}(k)$ does not vary with $n > m$ whenever $k \in [-m, m]$, it follows that $\hat{f}(k) = |k|^{-p} 1_{\{|k|>1\}}$ on any such interval. Since m is arbitrary, it follows that $\hat{f}(k) = |k|^{-p} 1_{\{|k|>1\}}$, but this function is not in $L^2(\mathbb{R})$, so $\hat{f}(k) \notin \mathcal{A}_1$, which is a contradiction. Hence $\mathcal{A}_1$ is not complete, and this completes the proof. □

We now define the stochastic integral with respect to TFBM for any function in $\mathcal{A}_1$ in the case where $1/2 < H < 1$ in (2.4).

Definition 3.2.7. *For any $-1/2 < \alpha < 0$ and $\lambda > 0$, we define*

$$\int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) := \Gamma(\kappa + 1) \int_{\mathbb{R}} \left[\mathbb{I}_-^{\kappa,\lambda} f(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f(x)\right] \, B(dx) \tag{3.52}$$

for any $f \in \mathcal{A}_1$, where $\kappa = -\alpha$.

Theorem 3.2.8. *For any $-1/2 < \alpha < 0$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ in (3.52) is an isometry from $\mathcal{A}_1$ into $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$. Since $\mathcal{A}_1$ is not complete, these two spaces are not isometric.*

Proof. It follows from Lemma 3.2.5 that the stochastic integral (3.52) is well-defined for any $f \in \mathcal{A}_1$. Proposition 2.1 in Pipiras and Taqqu [56] implies that, if $\mathcal{D}$ is an inner product space such that $(f,g)_{\mathcal{D}} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)}$ for all $f,g \in \mathcal{E}$, and if $\mathcal{E}$ is dense $\mathcal{D}$, then there is an isometry between $\mathcal{D}$ and a linear subspace of $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ that extends the map $f \to \mathcal{I}^{\alpha,\lambda}(f)$ for $f \in \mathcal{E}$, and furthermore, $\mathcal{D}$ is isometric to $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$ itself if and only if $\mathcal{D}$ is complete. Using the Itô isometry and the definition (3.52), it follows from (3.45) that for any $f, g \in \mathcal{A}_1$ we have

$$\langle f, g\rangle_{\mathcal{A}_1} = \langle F, G\rangle_{L^2(\mathbb{R})} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)},$$

and then the result follows from Theorem 3.2.4. □

3.2.2 Anti-persistence

Next we investigate stochastic integrals with respect to TFBM in the case $0 < \alpha < 1/2$, so that $0 < H < 1/2$ in (2.4). It follows from (3.34) that the stochastic integral (3.40) can be written in the form

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) = \Gamma(1-\alpha)\int_{-\infty}^{\infty} \left[\mathbb{D}_{-}^{\alpha,\lambda} f(x) - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} f(x)\right] \; B(dx)$$

for any $f \in \mathcal{E}$, the space of elementary functions. Then $\mathcal{I}^{\alpha,\lambda}(f)$ is a Gaussian random variable with mean zero, such that

$$\begin{aligned}
\langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)} &= \mathbb{E}\left(\int_{\mathbb{R}} f(x)B_{\alpha,\lambda}(dx)\int_{\mathbb{R}} g(x)B_{\alpha,\lambda}(dx)\right)\\
&= \Gamma(1-\alpha)^2\int_{\mathbb{R}}\left[\mathbb{D}_-^{\alpha,\lambda}f(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}f(x)\right]\left[\mathbb{D}_-^{\alpha,\lambda}g(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}g(x)\right]\,dx.
\end{aligned} \tag{3.53}$$

for any $f, g \in \mathcal{E}$, using (3.34) and the Itô isometry (3.38). Equation (3.53) suggests the following space of integrands for TFBM in the case $0 < H < 1/2$. Recall that $W^{\alpha,2}(\mathbb{R})$ is the fractional Sobolev space (3.17).

Theorem 3.2.9. *For any $0 < \alpha < 1/2$ and $\lambda > 0$, the class of functions*

$$\mathcal{A}_2 := \left\{ f \in W^{\alpha,2}(\mathbb{R}) : \varphi_f = \mathbb{D}_-^{\alpha,\lambda}f - \lambda\mathbb{I}_-^{1-\alpha,\lambda}f \text{ for some } \varphi_f \in L^2(\mathbb{R}).\right\}. \tag{3.54}$$

is a linear space with inner product

$$\langle f, g\rangle_{\mathcal{A}_2} := \langle F, G\rangle_{L^2(\mathbb{R})} \tag{3.55}$$

where

$$\begin{aligned}
F(x) &= \Gamma(1-\alpha)\left[\mathbb{D}_-^{\alpha,\lambda}f(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}f(x)\right]\\
G(x) &= \Gamma(1-\alpha)\left[\mathbb{D}_-^{\alpha,\lambda}g(x)-\lambda\mathbb{I}_-^{1-\alpha,\lambda}g(x)\right].
\end{aligned} \tag{3.56}$$

The set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}_2$. The space $\mathcal{A}_2$ is not complete.

We begin with the two lemmas. The first lemma shows that the set $\mathcal{A}_2$ contains every function in $W^{\alpha,2}(\mathbb{R})$, and hence they are the same set, but different spaces, since they have different inner products.

Lemma 3.2.10. *Under the assumptions of Theorem 3.2.9, every $f \in W^{\alpha,2}(\mathbb{R})$ is an element of $\mathcal{A}_2$.*

Proof. Given $f \in W^{\alpha,2}(\mathbb{R})$, we need to show that

$$\varphi_f = \mathbb{D}_-^{\alpha,\lambda} f - \lambda \mathbb{I}_-^{1-\alpha,\lambda} f, \tag{3.57}$$

for some $\varphi_f \in L^2(\mathbb{R})$. From the definition (3.17) we see that $\int (\lambda^2 + k^2)^\alpha |\hat{f}(k)|^2 \, dk < \infty$. Define $h_1(k) = (\lambda - ik)^\alpha \hat{f}(k)$ and note that h_1 is the Fourier transform of some function $\varphi_1 \in L^2(\mathbb{R})$. Define $h_2(k) := (\lambda - ik)^{\alpha-1} \widehat{f}(k)$, and observe that

$$\begin{aligned} \int |h_2(k)|^2 \, dk &= \int |\widehat{f}(k)|^2 (\lambda^2 + k^2)^{\alpha-1} \, dk \\ &= \int \frac{|h_1(k)|^2}{\lambda^2 + k^2} \, dk < \infty, \end{aligned}$$

since $h_1 \in L^2(\mathbb{R})$ and $1/(\lambda^2 + k^2)$ is bounded. Hence there is another function $\varphi_2 \in L^2(\mathbb{R})$ such that $h_2 = \hat{\varphi}_2$. Define $\varphi_f := \varphi_1 - \lambda \varphi_2$ so that

$$\widehat{\varphi_f}(k) = \widehat{\varphi_1}(k) - \lambda \widehat{\varphi_2}(k) = \widehat{f}(k)(\lambda - ik)^\alpha - \widehat{f}(k)\lambda(\lambda - ik)^{\alpha-1}. \tag{3.58}$$

Since $f \in W^{\alpha,2}(\mathbb{R}) \subset L^2(\mathbb{R})$, we can apply Definition 3.1.11 and Lemma 3.1.6 to see that (3.57) holds. $\square$

Lemma 3.2.11. *Under the assumptions of Theorem 3.2.9, if $f \in W^{\alpha,2}(\mathbb{R})$, then there exists* *a sequence of elementary functions* (f_n) *such that* $f_n \to f$ *in* $L^2(\mathbb{R})$*, and also*

$$\int_{-\infty}^{+\infty} |\widehat{f_n}(k) - \widehat{f}(k)|^2 |k|^{2\alpha} dk \to 0 \quad \textit{as } n \to \infty. \tag{3.59}$$

Proof. Equation (3.59) is proven in [56, Lemma 5.1]. For any $L > 0$, that proof constructs a sequence of elementary functions f_n such that $\hat{f}_n(k) \to \mathbf{1}_{[-1,1]}(k)$ almost everywhere on $-L \le x \le L$, and shows that $|\hat{f}_n(k)| \le C \min\{1, |k|^{-1}\}$ for all $k \in \mathbb{R}$ and all $n \ge 1$. In the notation of that paper, we have $\hat{f}_n(k) = k^{-1} U_n(k)$. Apply the dominated convergence theorem to see that

$$\int_{-L}^{+L} |\hat{f}_n(k) - \mathbf{1}_{[-1,1]}(k)|^2 dk \to 0$$

and note that

$$\int_{|k|>L} |\hat{f}_n(k) - \mathbf{1}_{[-1,1]}(k)|^2 dk \le 2C^2 \int_L^\infty \frac{dk}{k^2} \le \frac{2C^2}{L}.$$

Since L is arbitrary, it follows that $\hat{f}_n(k) \to \mathbf{1}_{[-1,1]}(k)$ in $L^2(\mathbb{R})$, and then the result follows as in [56, Lemma 5.1]. □

Proof of Theorem 3.2.9. For $f \in \mathcal{A}_2$ we define

$$\|f\|_{\mathcal{A}_2} = \sqrt{\langle f, f \rangle_{\mathcal{A}_2}} = \sqrt{\langle \varphi_f, \varphi_f \rangle_2} = \|\varphi_f\|_2. \tag{3.60}$$

where φ_f is given by (3.57). Next, use (3.58) to see that

$$\widehat{\varphi_f}(k) = (-ik)(\lambda - ik)^{\alpha-1} \widehat{f}(k). \tag{3.61}$$

To verify that (3.55) is an inner product, note that if $\langle f, f\rangle_{\mathcal{A}_2} = 0$ then

$$\|f\|^2_{\mathcal{A}_2} = \|\varphi_f\|_2^2 = \|\widehat{\varphi_f}\|_2^2 = \int_{-\infty}^{\infty} |\widehat{f}(k)|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}\, dk \tag{3.62}$$

equals zero, which implies that $|\widehat{f}(k)| = 0$ almost everywhere, and then $f = 0$ almost everywhere. This proves that (3.57) is an inner product.

Next we show that $\mathcal{E}$ is dense in $\mathcal{A}_2$. Apply Lemma 3.2.11 to obtain a sequence (f_n) in $\mathcal{E}$ such that $\|f_n - f\|_2 \to 0$ and (3.59) holds. It is easy to check using (3.37) that any elementary function is an element of $W^{\alpha,2}(\mathbb{R})$, and then Lemma 3.2.10 implies that it is also an element of $\mathcal{A}_2$. Now use (3.62) to write

$$\begin{aligned}\|f_n - f\|^2_{\mathcal{A}_2} = &\int_{-\infty}^{+\infty} \left|\widehat{f_n}(k) - \widehat{f}(k)\right|^2 (k^2+\lambda^2)^{\alpha}\, dk \\ &- \lambda^2 \int_{-\infty}^{+\infty} \left|\widehat{f_n}(k) - \widehat{f}(k)\right|^2 \frac{1}{(\lambda^2+k^2)^{1-\alpha}}\, dk.\end{aligned}$$

Since $1/(\lambda^2+k^2)^{1-\alpha}$ is bounded, it follows easily using (3.59) and $\|f_n - f\|_2 \to 0$ that $\|f_n - f\|_{\mathcal{A}_2} \to 0$, and hence $\mathcal{E}$ is dense in $\mathcal{A}_2$.

Finally, we want to show that $\mathcal{A}_2$ is not complete. The proof is similar to that of Theorem 3.2.4. The functions

$$\widehat{f_n}(k) = |k|^{-p}\mathbf{1}_{\{1/n<|k|<1\}}(k).$$

are the Fourier transforms of some functions $f_n \in L^2(\mathbb{R})$. Clearly $f_n \in W^{\alpha,2}(\mathbb{R})$, and then it follows from Lemmas 3.1.6 and 3.1.9 that the corresponding functions $F_n(x) = \Gamma(1-\alpha)[\mathbb{D}_-^{\alpha,\lambda} f_n(x) - \lambda \mathbb{I}_-^{1-\alpha,\lambda} f_n(x)]$ from (3.56) have Fourier transform (3.50), that is,

$$\mathcal{F}[F_n](k) = \Gamma(1-\alpha)\frac{-ik}{(\lambda - ik)^{1-\alpha}}\hat{f}_n(k).$$

Then

$$\|f_n\|_{\mathcal{A}_2}^2 = \|F_n\|_2^2 = \|\widehat{F_n}\|_2^2 = \Gamma(1-\alpha)^2 \int_{-\infty}^{\infty} \left|\widehat{f_n}(k)\right|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}\, dk < \infty$$

for any $p < 3/2$, so that $f_n \in \mathcal{A}_2$. Now it is easy to check that $f_n - f_m \to 0$ in $\mathcal{A}_2$, as $n, m \to \infty$, so that (f_n) is a Cauchy sequence. Suppose $1/2 < p < 3/2$ and that $\|f_n - f\|_{\mathcal{A}_2} \to 0$ for some $f \in \mathcal{A}_2$. Then $\hat{f}(k) = |k|^{-p} 1_{\{0<|k|<1\}}$, but this $\hat{f}$ is not in $L^2(\mathbb{R})$, so $\hat{f} \notin \mathcal{A}_2$, and hence $\mathcal{A}_2$ is not complete. □

We now define the stochastic integral with respect to TFBM for any function in $\mathcal{A}_2$ in the case where $0 < H < 1/2$ in (2.4).

Definition 3.2.12. *For any $0 < \alpha < 1/2$ and $\lambda > 0$, we define*

$$\mathcal{I}^{\alpha,\lambda}(f) = \int_{\mathbb{R}} f(x) B_{\alpha,\lambda}(dx) := \Gamma(1-\alpha) \int_{\mathbb{R}} \left[\mathbb{D}_{-}^{\alpha,\lambda} f(x) - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} f(x)\right] B(dx) \tag{3.63}$$

for any $f \in \mathcal{A}_2$.

Theorem 3.2.13. *For any $0 < \alpha < 1/2$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ is an isometry from $\mathcal{A}_2$ into $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$. Since $\mathcal{A}_2$ is not complete, these two spaces are not isometric.*

Proof. The proof is similar to that of Theorem 3.2.8. It follows from Lemma 3.2.10 that the stochastic integral (3.63) is well-defined for any $f \in \mathcal{A}_2$. Use Proposition 2.1 in Pipiras and Taqqu [56], and note that the Itô isometry, the definition (3.63), and equation (3.55) imply that for any $f, g \in \mathcal{A}_2$ we have

$$\langle f, g\rangle_{\mathcal{A}_2} = \langle F, G\rangle_{L^2(\mathbb{R})} = \langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)}.$$

Then the result follows from Theorem 3.2.9. □

3.2.3 Harmonizable representation

By now it should be clear that the Fourier transform plays an important role in the theory of stochastic integration for TFBM. Here we apply the harmonizable representation of TFBM to unify the two cases $-1/2 < \alpha < 0$ and $0 < \alpha < 1/2$.

For any $-1/2 < \alpha < 1/2$ and any $\lambda > 0$, Proposition 3.1 in [46] shows that TFBM has the harmonizable representation

$$B_{\alpha,\lambda}(t) = \frac{\Gamma(1-\alpha)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-itk}-1}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk)$$

where $\hat{B} = \hat{B}_1 + i\hat{B}_2$ is a complex-valued Gaussian random measure constructed as follows. Let $\hat{B}_1$ and $\hat{B}_2$ be two independent Brownian motions on the positive real line with $\mathbb{E}[(\hat{B}_i(t))^2] = t/2$ for $i = 1, 2$, and define two independently scattered Gaussian random measures by setting $\hat{B}_i[a, b] = \hat{B}_i(b) - \hat{B}_i(a)$, extend to Borel subsets of the positive real line, and then extend to the entire real line by setting $\hat{B}_1(A) = \hat{B}_1(-A)$, $\hat{B}_2(A) = -\hat{B}_2(-A)$.

Apply the formula (3.37) for the Fourier transform of an indicator function to write this harmonizable representation in the form

$$B_{\alpha,\lambda}(t) = \Gamma(1-\alpha) \int_{-\infty}^{+\infty} \widehat{\mathbf{1}}_{[0,t]}(k) \frac{(-ik)}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk).$$

It follows easily that for any elementary function (3.39) we may write

$$\mathcal{I}^{\alpha,\lambda}(f) = \Gamma(1-\alpha) \int_{-\infty}^{\infty} \widehat{f}(k) \frac{(-ik)}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk), \tag{3.64}$$

and then for any elementary functions f and g we have

$$\langle \mathcal{I}^{\alpha,\lambda}(f), \mathcal{I}^{\alpha,\lambda}(g)\rangle_{L^2(\Omega)} = \Gamma(1-\alpha)^2 \int_{-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}dk. \tag{3.65}$$

Theorem 3.2.14. *For any $\alpha \in (-1/2, 0) \cup (0, 1/2)$ and $\lambda > 0$, the class of functions*

$$\mathcal{A}_3 := \left\{ f \in L^2(\mathbb{R}) : \int \left|\widehat{f}(k)\right|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}\ dk < \infty \right\}. \tag{3.66}$$

is a linear space with the inner product

$$\langle f, g\rangle_{\mathcal{A}_3} = \Gamma(1-\alpha)^2 \int_{-\infty}^{+\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}dk. \tag{3.67}$$

The set of elementary functions $\mathcal{E}$ is dense in the space $\mathcal{A}_3$. The space $\mathcal{A}_3$ is not complete.

Proof. The proof combines Theorems 3.2.4 and 3.2.9 using the Plancherel Theorem. First suppose that $0 < \alpha < 1/2$ and recall that $\varphi_f = \mathbb{D}_{-}^{\alpha,\lambda} f - \lambda \mathbb{I}_{-}^{1-\alpha,\lambda} f$ is a function with Fourier transform

$$\hat{\varphi}_f = [(\lambda - ik)^{\alpha} - \lambda(\lambda - ik)^{\alpha-1}]\hat{f} = [\lambda - ik - \lambda](\lambda - ik)^{\alpha-1}\hat{f} = (-ik)(\lambda - ik)^{\alpha-1}\hat{f}.$$

Then it follows from the Plancherel Theorem that

$$\begin{aligned}\langle f, g\rangle_{\mathcal{A}_2} = \Gamma(1-\alpha)^2 \langle \varphi_f, \varphi_g\rangle_2 &= \Gamma(1-\alpha)^2 \langle \hat{\varphi}_f, \hat{\varphi}_g\rangle_2 \\ &= \Gamma(1-\alpha)^2 \int_{-\infty}^{+\infty} \widehat{f}(k)\overline{\widehat{g}(k)}\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}}dk = \langle f, g\rangle_{\mathcal{A}_3}\end{aligned}$$

and hence the two inner products are identical. If $f \in \mathcal{A}_3$, then

$$\int_{-\infty}^{+\infty} |\hat{f}(k)|^2(\lambda^2+k^2)^\alpha \, dk = \int_{-\infty}^{+\infty} |\hat{f}(k)|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} \, dk + \lambda^2 \int_{-\infty}^{+\infty} |\hat{f}(k)|^2 \frac{1}{(\lambda^2+k^2)^{1-\alpha}} \, dk. \tag{3.68}$$

The first integral on the right-hand side is finite by (3.66), and the second is finite since $1/(\lambda^2+k^2)^{1-\alpha}$ is bounded. Then it follows from the definition (3.17) that $f \in W^{\alpha,2}(\mathbb{R})$. Conversely, if $f \in W^{\alpha,2}(\mathbb{R})$ then since

$$\frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} = \frac{k^2}{\lambda^2+k^2}(\lambda^2+k^2)^\alpha \le (\lambda^2+k^2)^\alpha$$

it follows immediately that $f \in \mathcal{A}_3$, and hence $W^{\alpha,2}(\mathbb{R})$ and $\mathcal{A}_3$ are the same set of functions. Then it follows from Lemma 3.2.10 that $\mathcal{A}_2$ and $\mathcal{A}_3$ are identical when $0 < \alpha < 1/2$, and the conclusions of Theorem 3.2.14 follow from Theorem 3.2.9 in this case.

If $-1/2 < \alpha < 0$, then the function $k^2/(\lambda^2+k^2)^{1-\alpha}$ is bounded by a constant $C(\alpha, \lambda)$ that depends only on α and λ, so for any $f \in L^2(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \left|\widehat{f}(k)\right|^2 \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} \, dk \le C(\alpha,\lambda) \int_{\mathbb{R}} \left|\widehat{f}(k)\right|^2 \, dk < \infty \tag{3.69}$$

and hence $f \in \mathcal{A}_3$. Since $\mathcal{A}_3 \subset L^2(\mathbb{R})$ by definition, this proves that $L^2(\mathbb{R})$ and $\mathcal{A}_3$ are the same set of functions, and then it follows from Lemma 3.2.5 that $\mathcal{A}_1$ and $\mathcal{A}_3$ are the same set of functions in this case. Let $\kappa = -\alpha$ and note that $\varphi_f = \mathbb{I}_-^{\kappa,\lambda} f - \lambda \mathbb{I}_-^{\kappa+1,\lambda} f$ is again a function with Fourier transform

$$\hat{\varphi}_f = [(\lambda - ik)^\alpha - \lambda(\lambda - ik)^{\alpha-1}]\hat{f} = (-ik)(\lambda - ik)^{\alpha-1}\hat{f}.$$

Then it follows from the Plancherel Theorem that

$$\langle f, g\rangle_{\mathcal{A}_1} = \Gamma(\kappa+1)^2\langle \varphi_f, \varphi_g\rangle_2 = \Gamma(1-\alpha)^2\langle \hat{\varphi}_f, \hat{\varphi}_g\rangle_2$$
$$= \Gamma(1-\alpha)^2 \int_{-\infty}^{+\infty} \widehat{f}(k)\overline{\widehat{g}(k)} \frac{k^2}{(\lambda^2+k^2)^{1-\alpha}} dk = \langle f, g\rangle_{\mathcal{A}_3}$$

and hence the two inner products are identical. Then the conclusions of Theorem 3.2.14 follow from Theorem 3.2.4 in this case as well. □

Definition 3.2.15. *For any $\alpha \in (-1/2, 0) \cup (0, 1/2)$ and $\lambda > 0$, we define*

$$\mathcal{I}^{\alpha,\lambda}(f) = \Gamma(1-\alpha) \int_{-\infty}^{\infty} \widehat{f}(k) \frac{(-ik)}{(\lambda - ik)^{1-\alpha}} \hat{B}(dk) \tag{3.70}$$

for any $f \in \mathcal{A}_3$.

Theorem 3.2.16. *For any $\alpha \in (-1/2, 0) \cup (0, 1/2)$ and $\lambda > 0$, the stochastic integral $\mathcal{I}^{\alpha,\lambda}$ in* (3.70) *is an isometry from $\mathcal{A}_3$ into $\overline{\mathrm{Sp}}(B_{\alpha,\lambda})$. Since $\mathcal{A}_3$ is not complete, these two spaces are not isometric.*

Proof. The proof of Theorem 3.2.14 shows that $\mathcal{A}_1$ and $\mathcal{A}_3$ are identical when $-1/2 < \alpha < 0$, and $\mathcal{A}_2$ and $\mathcal{A}_3$ are identical when $0 < \alpha < 1/2$. Then the result follows immediately from Theorems 3.2.8 and 3.2.13. □

3.3 Discussion

In this section, we collect some remarks and extensions.

3.3.1 White noise approach

Heuristically, the TFBM (3.33) with $1/2 < H < 1$ in (2.4) can be written in terms of tempered fractional integrals of the white noise $W(x)dx = B(dx)$, since in view of (3.8) we can write

$$B_{\alpha,\lambda}(t) = \Gamma(\kappa+1) \int_{-\infty}^{+\infty} \left[\mathbb{I}_{+}^{\kappa,\lambda} W(x) - \lambda \mathbb{I}_{+}^{\kappa+1,\lambda} W(x) \right] \mathbf{1}_{[0,t]}(x)\, dx.$$

In the same way, when $0 < H < 1/2$ we can write

$$B_{\alpha,\lambda}(t) = \Gamma(1-\alpha) \int_{-\infty}^{+\infty} \left[\mathbb{D}_{+}^{\alpha,\lambda} W(x) - \lambda \mathbb{I}_{+}^{1-\alpha,\lambda} W(x) \right] \mathbf{1}_{[0,t]}(x)\, dx,$$

using Lemma 3.1.15. These ideas could be made rigorous using white noise theory [36]. Setting $\lambda = 0$, we recover the fact that FBM is the fractional integral or derivative of a Brownian motion [56, p. 261]. The white noise approach is preferred in engineering applications (e.g., see [6]).

3.3.2 Reproducing kernel Hilbert space

The reproducing kernel Hilbert space (RKHS) of TFBM provides another approach to stochastic integration that produces an isometric space of deterministic integrands. The RKHS for FBM was computed in [6, 56]. For any mean zero Gaussian process $\{X_t\}_{t\in\mathbb{R}}$ with covariance function $R(s,t) = \mathbb{E}[X_s X_t]$, the RKHS of X is the unique Hilbert space $\mathbb{H}(X)$ of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $R(\cdot,t) \in \mathbb{H}(X)$ for all $t \in \mathbb{R}$, and $\langle f, R(\cdot,t)\rangle_{\mathbb{H}(X)} = f(t)$ for all $t \in \mathbb{R}$ and $f \in \mathbb{H}(X)$ [25, 65]. As noted in [25], if there exists a measure space $(\Lambda, \mathfrak{B}, \nu)$ and a set of functions $\{f_t\} \subset L^2(\mathbb{R}, \nu)$ such that

$$R(s,t) = \int_\Lambda f_s(x) f_t(x) \nu(dx) \quad \text{for all } s, t \in \mathbb{R}, \tag{3.71}$$

Then $\mathbb{H}(X)$ consists of the functions $g(t) = \int f_t(x) g^*(x) \nu(dx)$ for $g^* \in \overline{\mathrm{Sp}}\{f_t\}$, the closure in $L^2(\mathbb{R}, \nu)$ of the set of linear combinations of functions f_t. Then $\mathbb{H}(X)$ is a Hilbert space with the inner product

$$\langle g, h \rangle_{\mathbb{H}(X)} = \int_\Lambda g^*(x) h^*(x) \nu(dx).$$

Let $\overline{\mathrm{Sp}}(X)$ denote the closure of the set of linear combinations of random variables $\{X_t\}$ in the space $L^2(\Omega)$. The mapping $\mathcal{J}$ that sends

$$\sum_{j=1}^{J} a_j R(\cdot, t_j) \mapsto \sum_{j=1}^{J} a_j X_{t_j}$$

is an isometry that maps $\mathbb{H}(X)$ onto $\overline{\mathrm{Sp}}(X)$, and hence these two Hilbert spaces are isometric. Then $\mathcal{J}(f)$ is the stochastic integral of any $f \in \mathbb{H}(X)$.

For TFBM with $-1/2 < \alpha < 0$, let $\kappa = -\alpha$. Since $B_{\alpha,\lambda}(t) = \int_\mathbb{R} \mathbf{1}_{[0,t]}(x) B_{\alpha,\lambda}(dx)$, it follows immediately from the definition (3.52) that TFBM has covariance function

$$R(s,t) = \Gamma(\kappa+1)^2 \int_\mathbb{R} \left[\mathbb{I}_-^{\kappa,\lambda} \mathbf{1}_{[0,s]}(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} \mathbf{1}_{[0,s]}\right] \left[\mathbb{I}_-^{\kappa,\lambda} \mathbf{1}_{[0,t]}(x) - \lambda \mathbb{I}_-^{\kappa+1,\lambda} \mathbf{1}_{[0,t]}\right] dx,$$

and hence the RKHS $\mathbb{H}(B_{\alpha,\lambda})$ consists of functions

$$g(t) = \Gamma(k+1) \int_\mathbb{R} \left[\mathbb{I}_-^{k,\lambda} - \lambda \mathbb{I}_-^{k+1,\lambda}\right] 1_{[0,t]}(x) g^*(x) dx$$

for $g^* \in L_2(\mathbb{R})$, with the inner product

$$\langle g, h\rangle_{\mathbb{H}(X)} = \int_{\mathbb{R}} g^*(x)h^*(x)dx = \langle g^*, h^*\rangle_{L^2(\mathbb{R})}. \tag{3.72}$$

For TFBM with $0 < \alpha < 1/2$ and $\lambda > 0$, the RKHS $\mathbb{H}(B_{\alpha,\lambda})$ consists of functions

$$g(t) = \Gamma(1-\alpha)^2 \int_{\mathbb{R}} \left[\mathbb{D}_-^{\alpha,\lambda} - \lambda \mathbb{I}_-^{1-\alpha,\lambda}\right] 1_{[0,t]}(x) g^*(x)dx$$

for $g^* \in L_2(\mathbb{R})$, with the same inner product (3.72). The proof is similar to [56, Section 6]. Complete details will be provided in the forthcoming paper [45]. Here we take $\Lambda = L^2(\mathbb{R})$, with ν the Lebesgue measure on $\mathbb{R}$. The main technical difficulty is to show that $L_2(\mathbb{R}) = \overline{\mathrm{Sp}}\{f_t\}$, where $f_t(x) = \Gamma(k+1)[\mathbb{I}_-^{k,\lambda} - \lambda \mathbb{I}_-^{k+1,\lambda}]1_{[0,t]}(x)$ in the case $-1/2 < \alpha < 0$, and $f_t(x) = \Gamma(1-\alpha)[\mathbb{D}_-^{\alpha,\lambda} - \lambda \mathbb{I}_-^{1-\alpha,\lambda}]1_{[0,t]}(x)$ for $0 < \alpha < 1/2$.

3.3.3 Tempered distributions as integrands

Jolis [29] proved that the exact domain of the Wiener integral for a fractional Brownian motion $B_H(t)$ is given by

$$\Lambda^H = \{f \in \mathcal{S}'(\mathbb{R}) = \int_{\mathbb{R}} |\widehat{f}(k)|^2 \, |k|^{1-2H} \, dk < \infty\},$$

where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions. This gives an isometry using the inner product (for a standard FBM)

$$\langle f, g\rangle = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \int \hat{f}(k)\overline{\hat{g}(k)}|k|^{1-2H} dk,$$

that makes Λ^H isometric to $\overline{\text{Sp}}(B_H)$. She also proved that this space contains distributions that cannot be represented by locally integrable functions in the case of long range dependence ($1/2 < H < 1$). Tudor [63] extended this result to subfractional Brownian motion. The distributional approach is useful in the study of partial differential equations with a Gaussian forcing term [9, 15, 64].

Following along these lines, we conjecture that the exact domain of the Wiener integral with respect to TFBM is given by the distributional fractional Sobolev space

$$\Lambda^{\alpha,\lambda} = \{f \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} |\widehat{f}(k)|^2 \, (\lambda^2 + k^2)^{\alpha} \, dk < \infty\}$$

with the inner product

$$\langle f, g \rangle = C_{\alpha,\lambda} \int \hat{f}(k)\overline{\hat{g}(k)}(\lambda^2 + k^2)^{\alpha} dk.$$

Proving this using [29, Theorem 3.5] would require computing the second derivative of the variance function (2.5) and taking the (inverse) Fourier transform of the result. This computation seems difficult, due to the Bessel function term.

Chapter 4

Tempered fractional stable motion

This chapter has four sections. In Section 4.1, we define linear tempered fractional stable motion (LTFSM) using a moving average representation, and we establish the dependence structure of its increments, which we call tempered fractional stable noise (TFSN). Section 4.2 defines tempered fractional harmonizable stable motion (HTFSM) and shows that LTFSM is different from HTFSM. Sample path properties of LTFSM and TFHSM are proven in Section 4.3. Finally, Section 4.4 investigates the local times and local nondeterminism properties for LTFSM and HTFSM .

4.1 Moving average representation

Let X be a real-valued random variable. We say that X has a symmetric α-stable, $S\alpha S$, distribution if its characteristic function has the form

$$\mathbb{E}\left[\exp\left\{i(\theta X)\right\}\right] = \exp\left\{-c|\theta|^{\alpha}\right\},$$

for some constant $c > 0$ and $0 < \alpha \leq 2$. The parameters α and σ are called the index of stability and the scale parameter respectively (see Chapter 1 in [61]). We denote the $S\alpha S$ distribution by $S_\alpha(\sigma, 0, 0)$ and write $X \simeq S_\alpha(\sigma, 0, 0)$ to indicate that that X has the stable distribution $S_\alpha(\sigma, 0, 0)$. A real-valued stochastic process $\{X(t)\}$ is called $S\alpha S$ if all linear combinations $\sum_{j=1}^{n} \theta_j X(t_j)$ are real-valued $S\alpha S$ random variables. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$, where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of $\mathbb{R}$. We say the process $\{Z_\alpha(B), B \in \mathcal{B}(\mathbb{R})\}$ is a real-valued $S\alpha S$ random measure with Lebesgue control measure dx if

$$\mathbb{E}\left[\exp\left\{i\mathrm{Re}(\overline{\theta}\widetilde{Z_\alpha}(B))\right\}\right] = \exp\left\{-|B||\theta|^\alpha\right\},$$

for any $\theta \in \mathbb{C}$ where $|B|$ denotes the Lebesgue measure of B ($B \in \mathcal{B}(\mathbb{R})$). Now, Let Z_α be a $S\alpha S$ random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with Lebesgue control measure dx. Then, we define the stochastic integral

$$I(f) := \int_{-\infty}^{+\infty} f(x)\ Z_\alpha(dx) \tag{4.1}$$

for all measurable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\int_{-\infty}^{+\infty} \Big|f(x)\Big|^\alpha\ m(dx) < \infty \tag{4.2}$$

for any $0 < \alpha < 2$, ($\alpha \neq 1$). We denote the collection of functions satisfying (4.2) by

$$L^\alpha(\mathbb{R}) = \Big\{f : f : \mathbb{R} \to \mathbb{R}\ \text{ is measurable,}\ \int_{-\infty}^{+\infty} \Big|f(x)\Big|^\alpha\ dx < \infty\Big\}. \tag{4.3}$$

Proposition 3.4.1 in [61] shows that $I(f)$, for any $f \in L^\alpha(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$, has the characteristic function

$$\mathbb{E}\Big[\exp\{i\theta I(f)\}\Big] = \exp\Big\{-\int_{-\infty}^{+\infty} \Big|\theta f(x)\Big|^\alpha\ dx\Big\}.$$

For the stochastic integral $I(f)$ in (4.1) we define

$$\begin{aligned} \Big\| I(f) \Big\|_{\alpha}^{\alpha} &= \Big\| \int_{-\infty}^{+\infty} f(x)\ Z_{\alpha}(dx) \Big\|_{\alpha}^{\alpha} \\ &:= \Big(-\log \mathbb{E}\Big[\exp\{i\ I(f)\} \Big] \Big) \\ &= \int_{-\infty}^{+\infty} \Big| f(x) \Big|^{\alpha}\ dx \end{aligned} \tag{4.4}$$

for any $0 < \alpha < 2$.

Definition 4.1.1. *A stochastic process $\{X(t), t \in \mathbb{R}\}$ is called an $S\alpha S$ Lévy motion if*

1. *$X(0) = 0$ a.s.*
2. *X has independent increments.*
3. *$X(t) - X(s) \sim S_{\alpha}(\sigma^{\alpha}|t-s|^{\frac{1}{\alpha}}, 0, 0)$ for any $-\infty < s < t < \infty$ and, for some $0 < \alpha \leq 2$, and some $\sigma > 0$.*

Note that the process X has stationary increments. It is Brownian motion when $\alpha = 2$. Also it is $\frac{1}{\alpha}$- self similar , that is, for all $c > 0$,

$$\Big\{ X(ct) \Big\}_{t \in \mathbb{R}} \triangleq \Big\{ c^{\frac{1}{\alpha}} X(t) \Big\}_{t \in \mathbb{R}},$$

where $\triangleq$ indicates equality in the sense of finite dimensional distributions.

Definition 4.1.2. *Given a $S\alpha S$ random measure $Z_{\alpha}(dx)$ on $\mathbb{R}$ with control measure $m(dx)$, for any $0 < \alpha \leq 2$ and $H > 0$ and $\lambda \geq 0$, the stochastic integral*

$$X_{H,\alpha,\lambda}(t) := \int_{-\infty}^{+\infty} \Big[e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha}} \Big]\ Z_{\alpha}(dx), \tag{4.5}$$

where $(x)_+ = \max\{x, 0\}$ *and* $0^0 = 0$ *will be called a linear tempered fractional stable motion (LTFSM).*

Remark 4.1.3. When $\alpha = 2$, $\{X_{H,\alpha,\lambda}(t)\}_{t\in\mathbb{R}}$ is TFBM defined in (2.1).

It is easy to check that the function

$$g_{\alpha,\lambda,t}(x) := e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(-x)_+}(-x)_+^{H-\frac{1}{\alpha}} \tag{4.6}$$

belongs to the linear space $L^\alpha(\mathbb{R})$ in (4.3) provided that $H > 0$, so that LTFSM is well defined and we have

$$\begin{aligned} \Big\|X_{H,\alpha,\lambda}(t)\Big\|_\alpha^\alpha &= \Big\| \int_{\mathbb{R}} g_{\alpha,\lambda,t}(x)\ Z_\alpha(dx)\Big\|_\alpha^\alpha \\ &:= \Big(- \log \mathbb{E}\Big[\exp\{i\ X_{H,\alpha,\lambda}(t)\}\Big]\Big) \\ &= \int_{\mathbb{R}} \Big|g_{\alpha,\lambda,t}(x)\Big|^\alpha\ dx, \end{aligned} \tag{4.7}$$

for any $0 < \alpha < 2$. Note also that this function has a scaling property

$$g_{\alpha,\lambda,ct}(cx) = c^{H-\frac{1}{\alpha}} g_{\alpha,c\lambda,t}(x), \tag{4.8}$$

for all $t, x \in \mathbb{R}$ and all $c > 0$. The next result shows that LTFSM has a nice scaling property, involving both the time scale and the tempering.

Proposition 4.1.4. LTFSM (4.5) *is symmetric* α*-stable stochastic process with stationary increments, such that*

$$\big\{X_{H,\alpha,\lambda}(ct)\big\}_{t\in\mathbb{R}} \triangleq \Big\{c^H X_{H,\alpha,c\lambda}(t)\Big\}_{t\in\mathbb{R}}, \tag{4.9}$$

for any scale factor $c > 0$.

Proof. Since $Z_\alpha(dx)$ has control measure $m(dx) = \sigma^\alpha dx$, the random measure $Z_\alpha(c\,dx)$ has control measure $c^{\frac{1}{\alpha}}\sigma^\alpha dx$. Given $t_1 < t_2 < \cdots < t_n$, a change of variable $x = cx'$ then yields

$$\begin{aligned}
\left(X_{H,\alpha,\lambda}(ct_i) : i = 1,\ldots,n\right) &= \left(\int g_{\alpha,\lambda,ct_i}(x)M(dx) : i = 1,\ldots,n\right)\\
&= \left(\int g_{\alpha,\lambda,ct_i}(cx')M(c\,dx') : i = 1,\ldots,n\right)\\
&\simeq \left(\int c^{H-\frac{1}{\alpha}} g_{\alpha,c\lambda,t_i}(x')c^{\frac{1}{\alpha}}M(dx') : i = 1,\ldots,n\right)\\
&= \left(c^H X_{H,\alpha,c\lambda}(t_i) : i = 1,\ldots,n\right),
\end{aligned}$$

where $\simeq$ denotes equality in distribution, so that (4.9) holds. For any $s, t \in \mathbb{R}$, the integrand (4.6) satisfies $g_{\alpha,\lambda,s+t}(s+x) - g_{\alpha,\lambda,s}(s+x) = g_{\alpha,\lambda,t}(x)$, and hence a change of variable $x = s + x'$ yields

$$\begin{aligned}
&\left(X_{H,\alpha,\lambda}(s+t_i) - X_{H,\alpha,\lambda}(s) : i = 1,\ldots,n\right)\\
&= \left(\int \left[g_{\alpha,\lambda,s+t_i}(x) - g_{\alpha,\lambda,s}(x)\right] M(dx) : i = 1,\ldots,n\right)\\
&\simeq \left(\int \left[g_{\alpha,\lambda,s+t_i}(s+x') - g_{\alpha,\lambda,s}(s+x')\right] M(dx') : i = 1,\ldots,n\right)\\
&= \left(\int g_{\alpha,\lambda,t_i}(x')M(dx') : i = 1,\ldots,n\right)\\
&= \left(X_{H,\alpha,\lambda}(t_i) : i = 1,\ldots,n\right)
\end{aligned}$$

, which shows that LTFSM has stationary increments. □

When a stochastic process $\{Y(t)\}_{t\in\mathbb{R}}$ is stationary Gaussian with mean zero, we can describe its dependence structure by its covariance function $\mathbb{E}[Y(t)Y(0)]$. However, in the non-Gaussian stable case, the covariance function does not exist. Instead, we use the following. For a stationary process $\{Y(t)\}$, let

$$\begin{aligned} r(t) :&= r(\theta_1, \theta_2, t) = \mathbb{E}\left[\exp\left\{i(\theta_1 Y(t) + \theta_2 Y(0))\right\}\right] \\ &- \mathbb{E}\left[\exp\left\{i\theta_1 Y(t)\right\}\right]\mathbb{E}\left[\exp\left\{i\theta_2 Y(0)\right\}\right], \quad \theta_1, \theta_2 \in \mathbb{R}, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} I(t) :&= I(\theta_1, \theta_2, t) \\ :&= -\log \mathbb{E}\left[\exp\left\{i(\theta_1 Y(t) + \theta_2 Y(0))\right\}\right] \\ &+ \log \mathbb{E}\left[\exp\left\{i\theta_1 Y(t)\right\}\right] + \log \mathbb{E}\left[\exp\left\{i\theta_2 Y(0)\right\}\right], \quad \theta_1, \theta_2 \in \mathbb{R}. \end{aligned} \tag{4.11}$$

The following relationship between $r(t)$ and $I(t)$ is valid:

$$r(\theta_1, \theta_2, t) = K(\theta_1, \theta_2, t)\left(e^{-I(\theta_1, \theta_2, t)} - 1\right),$$

where

$$\begin{aligned} K(\theta_1, \theta_2, t) &= \mathbb{E}\left[\exp\left\{i\theta_1 Y(t)\right\}\right]\mathbb{E}\left[\exp\left\{i\theta_2 Y(0)\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{i\theta_1 Y(0)\right\}\right]\mathbb{E}\left[\exp\left\{i\theta_2 Y(0)\right\}\right] \\ &:= K(\theta_1, \theta_2). \end{aligned} \tag{4.12}$$

Further, if $I(t) \to 0$ as $t \to \infty$, then $r(t) \sim -K(\theta_1, \theta_2)I(t)$ as $t \to \infty$ which means $r(t)$ and $I(t)$ are asymptotically equivalent. If $\{Y_t\}_{t\in\mathbb{R}}$ is Gaussian, $-I(1, -1, t)$ coincides with the covariance function and thus $r(t)$ is a natural extension. The quantity $r(t)$ was used in [2], where the authors studied the dependence structure of linear fractional stable motion.

Given a LTFSM (4.5), we define tempered fractional stable noise (TFSN)

$$Y_{H,\alpha,\lambda}(t) := X_{H,\alpha,\lambda}(t+1) - X_{H,\alpha,\lambda}(t) \quad \text{for integers } -\infty < t < \infty. \tag{4.13}$$

Remark 4.1.5. For two non-negative functions $f(t)$ and $g(t)$ on $\mathbb{R}$, we will write $f(t) \asymp g(t)$ if $C_1 \le \frac{f(t)}{g(t)} \le C_2$ for all t sufficiently large, for some $0 < C_1 < C_2 < \infty$.

Theorem 4.1.6. *Let* $0 < \alpha < 1$, $0 < H < 1$, $\lambda > 0$ *and* $Y_{H,\alpha,\lambda}(t)$ *be the tempered fractional stable noise* (4.13). *Then*

$$r(\theta_1, \theta_2, t) \asymp e^{-\lambda\alpha t} t^{H\alpha - 1},$$

as $t \to \infty$.

Proof. It follows easily from (4.5) that TFSN has the moving average representation

$$Y_{H,\alpha,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t+1-x)_+}(t+1-x)_+^{H-\frac{1}{\alpha}} - e^{-\lambda(t-x)_+}(t-x)_+^{H-\frac{1}{\alpha}} \right] Z_\alpha(dx). \tag{4.14}$$

Define $g_t(x) = (t-x)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-x)_+}$ for $t \in \mathbb{R}$ and compute $I(\theta_1, \theta_2, t)$ and $K(\theta_1, \theta_2, t)$ for TFSN $\{Y_{H,\alpha,\lambda}\}$ as follows:

$$\begin{aligned} I(\theta_1, \theta_2, t) &= -\log \mathbb{E}\left[\exp\left\{i(\theta_1 Y_{H,\alpha,\lambda}(t) + \theta_2 Y_{H,\alpha,\lambda}(0))\right\}\right] \\ &\quad + \log \mathbb{E}\left[\exp\left\{i\theta_1 Y_{H,\alpha,\lambda}(t)\right\}\right] + \log \mathbb{E}\left[\exp\left\{i\theta_2 Y_{H,\alpha,\lambda}(0)\right\}\right] \\ &= \int_{-\infty}^{+\infty} \left| \theta_1 \left[g_{t+1}(x) - g_t(x)\right] + \theta_2 \left[g_1(x) - g_0(x)\right] \right|^\alpha dx \\ &\quad - \int_{-\infty}^{+\infty} \left| \theta_1 \left[g_{t+1}(x) - g_t(x)\right] \right|^\alpha dx - \int_{-\infty}^{+\infty} \left| \theta_2 \left[g_1(x) - g_0(x)\right] \right|^\alpha dx \\ &:= I_1(\theta_1, \theta_2, t) + I_2(\theta_1, \theta_2, t), \end{aligned} \tag{4.15}$$

where

$$
\begin{aligned}
I_1(\theta_1,\theta_2,t) &= \int_{-\infty}^{0} \Big|\theta_1\left[g_{t+1}(x)-g_t(x)\right]+\theta_2\left[g_1(x)-g_0(x)\right]\Big|^{\alpha}\, dx \\
&\quad - \int_{-\infty}^{0} \Big|\theta_1\big[g_{t+1}(x)-g_t(x)\big]\Big|^{\alpha}\, dx - \int_{-\infty}^{0} \Big|\theta_2\big[g_1(x)-g_0(x)\big]\Big|^{\alpha}\, dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_2(\theta_1,\theta_2,t) &= \int_{0}^{1} \Big|\theta_1\left[g_{t+1}(x)-g_t(x)\right]+\theta_2 g_1(x)\Big|^{\alpha}\, dx \\
&\quad - \int_{0}^{1} \Big|\theta_1\big[g_{t+1}(x)-g_t(x)\big]\Big|^{\alpha}\, dx - \int_{0}^{1} \Big|\theta_2 g_1(x)\Big|^{\alpha}\, dx.
\end{aligned}
$$

Also,

$$
\begin{aligned}
K(\theta_1,\theta_2) &= \mathbb{E}\Big[e^{i\theta_1 Y(t)}\Big]\mathbb{E}\Big[e^{i\theta_2 Y(0)}\Big] \\
&= \mathbb{E}\Big[e^{i\theta_1 Y(0)}\Big]\mathbb{E}\Big[e^{i\theta_2 Y(0)}\Big] \qquad (4.16)\\
&= \exp\Big\{-(|\theta_1|^{\alpha}+|\theta_2|^{\alpha})\int_{-\infty}^{+\infty}\Big|g_1(x)-g_0(x)\Big|^{\alpha}\, dx\Big\}
\end{aligned}
$$

by stationarity. Therefore, $I(\theta_1,\theta_2,t) = K(\theta_1,\theta_2,t)(I_1(t)+I_2(t))$ and to verify the asymptotic dependence of $I(t)$ we just need to verify the asymptotic dependence of $I_1(t)$ and $I_2(t)$ as $t\to\infty$. We first verify the asymptotic dependence of $I_1(t)$. A change of variable in $I_1(t)$ for $t>1$ gives

$$
\begin{aligned}
I_1(t) \;=\; & \int_0^{\infty} \Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big] \\
+ \; & \theta_2\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(x)}(x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha} dx \\
- \; & \int_0^{\infty} \Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha} dx \\
- \; & \int_0^{\infty} \Big|\theta_2\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(x)}(x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}\, dx.
\end{aligned}
$$

Let

$$f_{t+1,t}(x) := \Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}. \tag{4.17}$$

For every $t > 1$ and $x > 0$ we get

$$\begin{aligned} e^{\alpha\lambda t}t^{-\alpha(H-\frac{1}{\alpha})}f_{t+1,t}(x) &= \Big|\theta_1\Big|^{\alpha}\Big|e^{-\lambda(1+x)}\Big(\frac{t+1+x}{t}\Big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\Big(\frac{t+x}{t}\Big)^{H-\frac{1}{\alpha}}\Big|^{\alpha} \\ &\to \Big|\theta_1\Big|^{\alpha}\, e^{-\lambda\alpha x}\,\Big|e^{-\lambda}-1\Big|^{\alpha} \text{ as } t\to\infty \end{aligned}$$

and

$$\sup_{t>1}\Big|e^{\alpha\lambda t}t^{-\alpha(H-\frac{1}{\alpha})}f_{t+1,t}(x)\Big| \le \Big|\theta_1(e^{-\lambda}-1)\Big|^{\alpha}\, e^{-\lambda\alpha x},$$

which belongs to $L^1(0,\infty)$. Now we can use the Dominated Convergence Theorem to see that

$$\begin{aligned} \int_0^\infty f_{t+1,t}(x)\;dx &\to \Big|\theta_1(e^{-\lambda}-1)\Big|^{\alpha}e^{-\lambda\alpha t}t^{\alpha(H-\frac{1}{\alpha})}\int_0^\infty e^{-\lambda\alpha x}\;dx \\ &= \frac{\Big|\theta_1(e^{-\lambda}-1)\Big|^{\alpha}e^{-\lambda\alpha t}t^{\alpha(H-\frac{1}{\alpha})}}{\lambda\alpha}, \end{aligned} \tag{4.18}$$

as $t\to\infty$. Now consider,

$$\begin{aligned} g_{t,t+1,0,1}(x) \;&:=\; \Big|\theta_1\big[e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\big] \\ &+\; \theta_2\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(x)}(x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha} \\ &-\; \Big|\theta_2\Big|^{\alpha}\Big|\big[e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x}(x)^{H-\frac{1}{\alpha}}\big]\Big|^{\alpha}. \end{aligned} \tag{4.19}$$

Then,

$$\begin{aligned}
e^{\lambda\alpha t}t^{-\alpha(H-\frac{1}{\alpha})}g_{t,t+1,0,1}(x) &= \Bigg|\theta_1\Big[e^{-\lambda(1+x)}\left(\frac{t+1+x}{t}\right)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\left(\frac{t+x}{t}\right)^{H-\frac{1}{\alpha}}\Big] \\
&+ \theta_2\Big[e^{-\lambda(1+x)}e^{\lambda t}\Big(\frac{1+x}{t}\Big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}e^{\lambda t}\Big(\frac{x}{t}\Big)^{H-\frac{1}{\alpha}}\Big]\Bigg|^{\alpha} \\
&- \Bigg|\theta_2\Big[e^{-\lambda(1+x)}e^{\lambda t}\Big(\frac{1+x}{t}\Big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}e^{\lambda t}\Big(\frac{x}{t}\Big)^{H-\frac{1}{\alpha}}\Big]\Bigg|^{\alpha} \\
&=: \Big|a_t + b_t\Big|^{\alpha} - \Big|b_t\Big|^{\alpha}
\end{aligned}$$

where

$$a_t = \theta_1\left[e^{-\lambda(1+x)}\left(\frac{t+1+x}{t}\right)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\left(\frac{t+x}{t}\right)^{H-\frac{1}{\alpha}}\right]$$

and

$$b_t = \theta_2\left[e^{-\lambda(1+x)}e^{\lambda t}\Big(\frac{1+x}{t}\Big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}e^{\lambda t}\Big(\frac{x}{t}\Big)^{H-\frac{1}{\alpha}}\right].$$

It is obvious that $a_t \to C_x := \theta_1 e^{-\lambda x}(e^{-\lambda} - 1)$ and $b_t \to \infty$ as $t \to \infty$. Then, $|a_t + b_t|^{\alpha} - |b_t|^{\alpha} \to 0$ as $t \to \infty$ since $0 < \alpha < 1$. Therefore

$$e^{\lambda\alpha t}t^{-\alpha(H-\frac{1}{\alpha})}g_{t,t+1,0,1} \to 0,$$

as $t \to \infty$. Moreover, for any $0 < \alpha < 1$, using the inequality $\Big||a|^{\alpha} - |b|^{\alpha}\Big| \leq \Big|a - b\Big|^{\alpha}$ (see [61], Page 211), we get

$$g_{t,t+1,0,1} \leq f_{t+1,t},$$

where $g_{t,t+1,0,1}$ and $g_{t,t+1,0,1}$ are defined in (4.17) and (4.19) respectively, if we let $a =$

$\theta_1(g_{t+1} - g_t) + \theta_2(g_1 - g_0)$ and $b = \theta_2(g_1 - g_0)$. Consequently

$$\begin{aligned}\sup_{t>1}\left|e^{\lambda\alpha t}t^{-\alpha(H-\frac{1}{\alpha})}g_{t,t+1,0,1}\right| &\leq \sup_{t>1}\left|e^{\alpha\lambda t}t^{-\alpha(H-\frac{1}{\alpha})}f_{t+1,t}(x)\right| \\ &\leq \left|\theta_1(e^{-\lambda}-1)\right|^{\alpha}e^{-\lambda\alpha x}\end{aligned}$$

which also belongs to $L^1(0,\infty)$. Applying the Dominated Convergence Theorem yields

$$\int_{-\infty}^{+\infty} g_{t,t+1,0,1}(x)dx \to 0 \quad \text{as } t \to \infty. \tag{4.20}$$

Therefore from (4.18) and (4.20)

$$I_1(t) \sim -\frac{|\theta_1(e^{-\lambda}-1)|^{\alpha}e^{-\lambda\alpha t}t^{\alpha(H-\frac{1}{\alpha})}}{\lambda\alpha}, \tag{4.21}$$

as $t \to \infty$. Consider now,

$$\begin{aligned}I_2(t) = &\int_0^1 \left|\theta_1[g_{t+1}(x) - g_t(x)] + \theta_2 g_1(x)\right|^{\alpha} dx \\ &- \int_0^1 \left|\theta_1[g_{t+1}(x) - g_t(x)]\right|^{\alpha} dx - \int_0^1 \left|\theta_2\ g_1(x)\right|^{\alpha} dx,\end{aligned}$$

Define,

$$u_t(x) := \theta_1\left[e^{-\lambda(t+1-x)}(t+1-x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t-x)}(t-x)^{H-\frac{1}{\alpha}}\right] \tag{4.22}$$

and

$$v(x) := \theta_2 e^{-\lambda(1-x)}(1-x)^{H-\frac{1}{\alpha}}. \tag{4.23}$$

We can rewrite

$$I_2(t) = \int_0^1 \xi(u_t(x)+v(x)) - \xi(u_t(x)) - \xi(v(x))\ dx,$$

where

$$\xi(g(x)) = \Big|g(x)\Big|^{\alpha}. \tag{4.24}$$

Lemma 3.1 in [2] implies that

$$\begin{aligned} I_2(t) &= \int_0^1 \xi(u_t(x)+v(x)) - \xi(u_t(x)) - \xi(v(x))\; dx \\ &\leq P_\alpha \int_0^1 \Big|u_t(x)\Big|^{\alpha}\; dx, \end{aligned} \tag{4.25}$$

where $P_\alpha = 2 + 4\tan(\frac{\pi\alpha}{2})$(see [2], Page 11). On the other hand

$$\begin{aligned} \Big|u_t(x)\Big| &\leqslant \Big|\theta_1\Big| \Big|(\frac{1}{\alpha} - H)(t-x)^{H-\frac{1}{\alpha}-1}e^{-\lambda(t-x)} + \lambda(t-x)^{H-\frac{1}{\alpha}}e^{-\lambda(t-x)}\Big| \\ &\leq \Big|\theta_1\Big| e^{-\lambda(t-1)}\left[(\frac{1}{\alpha} - H)\big|t-1\big|^{H-\frac{1}{\alpha}-1} + \lambda\,|t-1|^{H-\frac{1}{\alpha}}\right] \\ &\leq \Big|\theta_1\Big| e^{-\lambda(t-1)}\Big[\frac{1}{\alpha} - H + \lambda\Big]\;\Big|t-1\Big|^{H-\frac{1}{\alpha}}, \end{aligned} \tag{4.26}$$

since $0 < x < 1$. From (4.25) and (4.26) we get

$$\begin{aligned} I_2(t) &\leq P_\alpha \int_0^1 \Big|u_t(x)\Big|^{\alpha}\; dx \\ &= P_\alpha \Big|\theta_1\Big|^{\alpha} e^{-\lambda\alpha(t-1)}\Big[\frac{1}{\alpha} - H + \lambda\Big]^{\alpha}\Big|t-1\Big|^{H\alpha-1}. \end{aligned} \tag{4.27}$$

Hence from (4.21) and (4.27) we get

$$I(t) \asymp e^{-\lambda\alpha t}t^{H\alpha-1}$$

as $t \to \infty$ and this completes the proof. □

Theorem 4.1.7. *Let* $1 < \alpha < 2$, $\frac{1}{\alpha} < H < 1$, $\lambda > 0$ *and let* $Y_{H,\alpha,\lambda}(t)$ *be the tempered* *fractional stable noise* (4.13). *Then*

$$I(t) \asymp e^{-\lambda t} t^{(H-\frac{1}{\alpha})},$$

as $t \to \infty$.

Proof. The proof is similar to that of Theorem 4.1.6. Let $f_{t+1,t}(x)$ be the function which is defined by (4.17). Then

$$\begin{aligned} e^{\lambda t}\, t^{-(H-\frac{1}{\alpha})} f_{t+1,t}(x) &= \Big|\theta_1\Big|^{\alpha} e^{\lambda t} t^{-(H-\frac{1}{\alpha})} \\ &\quad \Big|e^{-\lambda(t+1+x)}(t+1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda(t+x)}(t+x)^{H-\frac{1}{\alpha}}\Big|^{\alpha} \\ &= a_t \cdot b_t, \end{aligned}$$

where

$$a_t := \Big|\theta_1\Big|^{\alpha} e^{\lambda t(\alpha-1)} t^{(H-\frac{1}{\alpha})(\alpha-1)}$$

and

$$b_t := \Big|e^{-\lambda(1+x)}\big(1+\frac{1}{t}+\frac{x}{t}\big)^{H-\frac{1}{\alpha}} - e^{-\lambda x}\big(1+\frac{x}{t}\big)^{H-\frac{1}{\alpha}}\Big|^{\alpha}.$$

Note that $a_t \to 0$ (since $1 < \alpha < 2$) and $b_t \to \Big|e^{-\lambda(1+x)} - e^{-\lambda x}\Big|^{\alpha}$ as $t \to \infty$. Now, let $h(t) = e^{-\lambda t(\alpha-1)} t^{(\alpha-1)(H-\frac{1}{\alpha})}$. Observe that $h(t)$ attains its maximum at $t = \frac{1}{\lambda}(H - \frac{1}{\alpha})$. Then

$$\begin{aligned} \sup_{t>1}\Big|e^{\lambda t}\, t^{-(H-\frac{1}{\alpha})} f_{t+1,t}(x)\Big| &= \Big|e^{-\lambda}-1\Big|^{\alpha}\Big|\theta_1\Big|^{\alpha} e^{-\lambda\alpha x} \sup_{t>1}\Big|h(t)\Big| \\ &= \Big|e^{-\lambda}-1\Big|^{\alpha}\Big|\theta_1\Big|^{\alpha} e^{-\lambda\alpha x} e^{-(H-\frac{1}{\alpha})(\alpha-1)}\Big[\frac{H-\frac{1}{\alpha}}{\lambda}\Big]^{(\alpha-1)(H-\frac{1}{\alpha})}, \end{aligned}$$

and so $f_{t+1,t}(x)$ is bounded by an $L^1(0,\infty)$ function. Therefore the Dominated Convergence

Theorem implies that

$$\int_0^\infty f_{t+1,t}(x) \to 0 \tag{4.28}$$

as $t \to \infty$. Consider now, $e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1}$ where $g_{t,t+1,0,1}$ is given by (4.19). Then

$$e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} = \Big|a_t + b_t\Big|^\alpha - \Big|b_t\Big|^\alpha$$

where

$$a_t := \theta_1 \left[e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda(1+x)} \Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} - e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda x} \Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} \right]$$

and

$$b_t := \theta_2 \left[e^{\frac{\lambda t}{\alpha}} t^{\frac{-(H-\frac{1}{\alpha})}{\alpha}} \Big[e^{-\lambda(1+x)} (1+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x} x^{(H-\frac{1}{\alpha})} \Big] \right].$$

Observe that $\lim_{t\to\infty} b_t = \infty$ and $\lim_{t\to\infty} a_t = 0$. Since $|a_t + b_t|^\alpha - |b_t|^\alpha \sim \alpha |a_t||b_t|^{\alpha-1}$, as $t \to \infty$, we get,

$$\begin{aligned} e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} \sim \alpha \Big|\theta_1\Big| & \\ & \Big| e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda(1+x)} \Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} - e^{-\lambda t(1-\frac{1}{\alpha})} e^{-\lambda x} \Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})} \Big| \\ & \Big|\theta_2\Big|^{\alpha-1} e^{\lambda t(1-\frac{1}{\alpha})} t^{-(H-\frac{1}{\alpha})(1-\frac{1}{\alpha})} \Big| e^{-\lambda(1+x)} (1+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x} x^{(H-\frac{1}{\alpha})} \Big|^{\alpha-1} \end{aligned}$$

and consequently

$$\begin{aligned} e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1} \to \alpha \Big|\theta_1\Big| \Big| e^{-\lambda(1+x)} - e^{-\lambda x} \Big| & \\ & \Big|\theta_2\Big|^{\alpha-1} \Big| e^{-\lambda(1+x)} (1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x} x^{H-\frac{1}{\alpha}} \Big|^{\alpha-1}. \end{aligned}$$

Moreover,

$$\begin{aligned}\sup_{t\geq 1}\Big|e^{\lambda t}t^{-(H-\frac{1}{\alpha})}g_{t,t+1,0,1}\Big| &= \sup_{t\geq 1}\Big|a_t+b_t\Big|^{\alpha}-\Big|b_t\Big|^{\alpha}\\ &\leq \sup_{t\geq 1}\Big|a_t\Big|^{\alpha}+\alpha\sup_{t\geq 1}\Big|a_t\Big|\Big|b_t\Big|^{\alpha-1},\end{aligned} \tag{4.29}$$

where we have used the following inequalities (Lemma 2 in [41]): $|a-b|^{\alpha}\leq a^{\alpha}+b^{\alpha}$ and $\Big||a+b|^{\alpha}-|b|^{\alpha}\Big|\leq\Big|a\Big|^{\alpha}+\alpha\Big|a\Big|\Big|b\Big|^{\alpha-1}$ valid for $a\geq 0$ and $b\geq 0$ and $\alpha\in(1,2)$. In order to find an upper bound for $\sup_{t\geq 1}|a_t|^{\alpha}$, write

$$\begin{aligned}\Big|a_t\Big|^{\alpha} &= \Big|\theta_1\Big|^{\alpha}\Big|e^{-\lambda t(1-\frac{1}{\alpha})}e^{-\lambda(1+x)}\Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})}-e^{-\lambda t(1-\frac{1}{\alpha})}e^{-\lambda x}\Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})}\Big|^{\alpha}\\ &= \Big|\theta_1\Big|^{\alpha}e^{-\lambda\alpha x}e^{-\lambda t(\alpha-1)}\Big|e^{-\lambda}\Big(\frac{t+1+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})}-\Big(\frac{t+x}{t^{\frac{1}{\alpha}}}\Big)^{(H-\frac{1}{\alpha})}\Big|^{\alpha}\\ &\leq \Big|\theta_1\Big|^{\alpha}e^{-\lambda\alpha x}\Big|e^{-\lambda}(1+1+x)^{H-\frac{1}{\alpha}}-(1+x)^{H-\frac{1}{\alpha}}\Big|^{\alpha}\\ &\leq \Big|\theta_1\Big|^{\alpha}e^{-\lambda\alpha x}\Big[e^{-\lambda\alpha}(2+x)^{H\alpha-1}+(1+x)^{H\alpha-1}\Big]\\ &\leq 2\Big|\theta_1\Big|^{\alpha}e^{-\lambda\alpha x}(2+x)^{H\alpha-1}.\end{aligned} \tag{4.30}$$

On the other hand,

$$\begin{aligned}\alpha\Big|a_t\Big|\Big|b_t\Big|^{\alpha-1} &= \alpha\Big|\theta_1\Big|\Big|\theta_2\Big|^{\alpha-1}\\ &\Big|\underbrace{e^{-\lambda(1+x)}\Big(\frac{t+1+x}{t}\Big)^{(H-\frac{1}{\alpha})}-e^{-\lambda x}\Big(\frac{t+x}{t}\Big)^{(H-\frac{1}{\alpha})}}_{:=S(t)}\Big|\times K(x)\end{aligned}$$

where

$$K(x)=\Big|e^{-\lambda(1+x)}(1+x)^{(H-\frac{1}{\alpha})}-e^{-\lambda x}(x)^{(H-\frac{1}{\alpha})}\Big|^{\alpha-1}. \tag{4.31}$$

Note that $S(t)$ is a decreasing function and hence

$$\sup_{t\geq 1} \alpha \Big|a_t\Big|\Big|b_t\Big|^{\alpha-1} = \alpha\Big|\theta_1\Big|\Big|\theta_2\Big|^{\alpha-1} \Big|e^{-\lambda(1+x)}(2+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x}(1+x)^{(H-\frac{1}{\alpha})}\Big| \times K(x) \tag{4.32}$$

where $K(x)$ is given by (4.31). From (4.29), (4.30) and (4.32)

$$\sup_{t\geq 1} \Big|e^{\lambda t} t^{-(H-\frac{1}{\alpha})} g_{t,t+1,0,1}\Big| \leq 2\Big|\theta_1\Big|^{\alpha} e^{-\lambda\alpha x}(2+x)^{H\alpha-1} + \alpha\Big|\theta_1\Big|\Big|\theta_2\Big|^{\alpha-1} \Big|e^{-\lambda(1+x)}(2+x)^{(H-\frac{1}{\alpha})} - e^{-\lambda x}(1+x)^{(H-\frac{1}{\alpha})}\Big| \times K(x) \tag{4.33}$$

which belongs to $L^1(0,\infty)$, since $H\alpha > 1$. Then, the Dominated Convergence Theorem implies that

$$\begin{aligned}\int_0^\infty g_{t,t+1,0,1}(x)\, dx \to{}& \alpha\theta_1\Big|\theta_2\Big|^{\alpha-1} e^{-\lambda t} t^{(H-\frac{1}{\alpha})} \\ & \int_0^\infty \Big|e^{-\lambda(1+x)} - e^{-\lambda x}\Big|\Big|e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x} x^{H-\frac{1}{\alpha}}\Big|^{\alpha-1} dx \\ &= C_2(\alpha,\lambda,\theta_1,\theta_2) e^{-\lambda t} t^{(H-\frac{1}{\alpha})},\end{aligned} \tag{4.34}$$

as $t \to \infty$, where

$$C_2(\alpha,\lambda,\theta_1,\theta_2) = \alpha\theta_1\Big|\theta_2\Big|^{\alpha-1} \int_0^\infty \Big|e^{-\lambda(1+x)} - e^{-\lambda x}\Big|\Big|e^{-\lambda(1+x)}(1+x)^{H-\frac{1}{\alpha}} - e^{-\lambda x} x^{H-\frac{1}{\alpha}}\Big|^{\alpha-1} dx \tag{4.35}$$

is a constant that is independent of t. Therefore from (4.28) and (4.34)

$$I_1(t) \sim C_2(\alpha,\lambda,\theta_1,\theta_2) e^{-\lambda t} t^{(H-\frac{1}{\alpha})}, \tag{4.36}$$

as $t \to \infty$. Finally, recall that

$$
\begin{aligned}
I_2(t) = & \int_0^1 \Big|\theta_1[g_{t+1}(x) - g_t(x)] + \theta_2 g_1(x)\Big|^\alpha \, dx \\
& - \int_0^1 \Big|\theta_1[g_{t+1}(x) - g_t(x)]\Big|^\alpha \, dx - \int_0^1 \Big|\theta_2 \, g_1(x)\Big|^\alpha \, dx,
\end{aligned}
$$

and that $u_t(x)$ and $v(x)$ are given by (4.22) and (4.23) respectively. Then

$$
I_2(t) = \int_0^1 \xi(u_t(x) + v(x)) - \xi(u_t(x)) - \xi(v(x)) \, dx
$$

where $\xi(g(x))$ is given by (4.24). Lemma 3.1 in [2] implies that, using argument similar to (4.26), we have

$$
\begin{aligned}
I_2(t) = & \int_0^1 \xi(u_t(x) + v(x)) - \xi(u_t(x)) - \xi(v(x)) \, dx \\
\leq & \int_0^1 R_\alpha \Big|u_t(x)\Big|\Big|v(x)\Big|^{\alpha-1} \, dx + S_\alpha \int_0^1 \Big|u_t(x)\Big|^\alpha \, dx \\
\leq & \, R_\alpha \Big|\theta_1\Big| \int_0^1 \Big[H - \frac{1}{\alpha} + \lambda\Big]\Big|t-1\Big|^{H-\frac{1}{\alpha}} e^{-\lambda(t-1)} \Big|v(x)\Big|^{\alpha-1} \, dx \\
& + S_\alpha \Big|\theta_1\Big|^\alpha \Big[H - \frac{1}{\alpha} + \lambda\Big]^\alpha \Big|t-1\Big|^{H\alpha-1} e^{-\lambda\alpha(t-1)} \\
= & \, R_\alpha \Big|\theta_1\Big| \Big[H - \frac{1}{\alpha} + \lambda\Big] \Big|t-1\Big|^{H-\frac{1}{\alpha}} e^{-\lambda(t-1)} \\
& \cdot \int_0^1 \Big|\theta_2 e^{-\lambda(1-x)}(1-x)^{H-\frac{1}{\alpha}}\Big|^{\alpha-1} \, dx \\
& + S_\alpha \Big|\theta_1\Big|^\alpha \Big[H - \frac{1}{\alpha} + \lambda\Big]^\alpha \Big|t-1\Big|^{H\alpha-1} e^{-\lambda\alpha(t-1)} \\
:= & \, C_3(\alpha, \lambda, \theta_1) \Big|t-1\Big|^{H-\frac{1}{\alpha}} e^{-\lambda(t-1)} \\
& + S_\alpha \Big|\theta_1\Big|^\alpha \Big[H - \frac{1}{\alpha} + \lambda\Big]^\alpha \Big|t-1\Big|^{H\alpha-1} e^{-\lambda\alpha(t-1)},
\end{aligned}
\tag{4.37}
$$

where

$$C_3(\alpha,\lambda,\theta_1) = R_\alpha\Big|\theta_1\Big|\Big[|H-\frac{1}{\alpha}|+\lambda\Big]$$
$$\cdot\int_0^1 \Big|\theta_2 e^{-\lambda(1-x)}(1-x)^{H-\frac{1}{\alpha}}\Big|^{\alpha-1}\,dx$$

is a constant. Recall that $R_\alpha = \alpha\big(1+\tan(\frac{\pi\alpha}{2})\big)$, and $S_\alpha = (\alpha+1)+(\alpha+3)\tan(\frac{\pi\alpha}{2})$ (see [2], Page 11) are also constants. Note that the upper bound which is obtained by (4.37) is of the same order as the upper bound for $I_1(t)$, given by (4.34). Hence

$$I(t) \asymp e^{-\lambda t}t^{(H-\frac{1}{\alpha})}$$

and this completes the proof. □

Definition 4.1.8. *A symmetric α-stable stationary process $\{Y_t\}$ has long memory if $r(\theta_1,\theta_2,t)$ defined in* (4.10) *satisfies*

$$\sum_{n=0}^{\infty}\Big|r(\theta_1,\theta_2,n)\Big| = \infty \tag{4.38}$$

Lemma 4.1.9. *The LTFSM process does not have long memory property in the sense of* (4.38)*.*

Proof. From Theorems 4.1.6 and 4.1.7 we have

$$\sum_{n=0}^{\infty}\Big|r(\theta_1,\theta_2,n)\Big| < \infty$$

which proves the statement. □

Remark 4.1.10. Theorem 4.1.7 gives the fact that when the tempering parameter λ is sufficiently small, TFSN exhibits *semi-long range dependence*, with the asymptotic rate $I(t)$

that falls off like $t^{H-\frac{1}{\alpha}}$ for moderate values of $t > 1$, and eventually that rate falls off like $e^{-\lambda t}t^{H-\frac{1}{\alpha}}$ for t sufficiently large.

4.2 Tempered fractional harmonizable stable motion

Let $X = X_1 + iX_2$ be a complex-valued random variable. We say X is isotropic $S\alpha S$ if the vector (X_1, X_2) is $S\alpha S$ and for any $\theta = \theta_1 + i\theta_2$

$$\mathbb{E}\left[\exp\left\{i(\theta_1 X_1 + \theta_2 X_2)\right\}\right] = \exp\left\{-c|\theta|^{\alpha}\right\},$$

for some constant $c > 0$. A complex-valued stochastic process $\{\widetilde{X}(t)\}$ is called isotropic $S\alpha S$ if all complex linear combinations $\sum_{j=1}^{n} \theta_j \widetilde{X}(t_j)$ are complex-valued isotropic $S\alpha S$ random variables. Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx)$, where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel subsets of $\mathbb{R}$. We say the process $\{\widetilde{Z}_\alpha(B), B \in \mathcal{B}(\mathbb{R})\}$ is a complex-valued isotropic $S\alpha S$ random measure with Lebesgue control measure dx if

$$\mathbb{E}\left[\exp\left\{i\mathrm{Re}(\overline{\theta}\widetilde{Z_\alpha}(B))\right\}\right] = \exp\left\{-|B||\theta|^{\alpha}\right\},$$

for any $\theta \in \mathbb{C}$ where $|B|$ denotes the Lebesgue measure of B ($B \in \mathcal{B}(\mathbb{R})$).

Definition 4.2.1. *Let $\{\widetilde{Z}_\alpha(B), B \in \mathcal{B}(\mathbb{R})\}$ be a complex-valued isotropic $S\alpha S$ random measure with Lebesgue control measure dx. Then the stochastic integral*

$$\widetilde{I}(f) := \mathrm{Re}\int_{-\infty}^{+\infty} f(k)\widetilde{Z}_\alpha(dk),$$

where $f \in L^{\alpha}(\mathbb{R})$ is the complex-valued $S\alpha S$ random variable such that

$$\mathbb{E}\left[\exp\left\{i\mathrm{Re}\int_{-\infty}^{+\infty} f(k)\widetilde{Z_{\alpha}}(dk)\right\}\right] = \exp\left\{-\int_{-\infty}^{+\infty} \left|f(k)\right|^{\alpha}(dk)\right\}$$

and

$$\begin{aligned}\left\|\widetilde{I}(f)\right\|_{\alpha}^{\alpha} &= \left\|\mathrm{Re}\int_{-\infty}^{+\infty} f(k)\widetilde{Z}_{\alpha}(dk)\right\|_{\alpha}^{\alpha} \\ &:= \Big(-\log\mathbb{E}\Big[\exp\{i\mathrm{Re}\ \widetilde{I}(f)\}\Big]\Big) \\ &= \int_{-\infty}^{+\infty}\left|f(k)\right|^{\alpha} dk,\end{aligned} \tag{4.39}$$

for any $0 < \alpha < 2$.

Definition 4.2.2. *The real harmonizable tempered fractional stable motion (HTFSM) is the process*

$$\widetilde{X}_{H,\alpha,\lambda}(t) = \mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{-ikt}-1}{(\lambda - ik)^{H+\frac{1}{\alpha}}}\widetilde{Z}_{\alpha}(dk) \tag{4.40}$$

where $0 < \alpha < 2$, $H > 0$, $\lambda > 0$ and $\tilde{Z}_{\alpha}$ is a complex isotropic $S\alpha S$ random measure.

Remark 4.2.3. The stochastic integral in (4.40) is well defined, since

$$\int_{-\infty}^{+\infty}\left|\frac{e^{-ikt}-1}{(\lambda - ik)^{H+\frac{1}{\alpha}}}\right|^{\alpha} dk < \infty, \tag{4.41}$$

for any $H > 0$ and $0 < \alpha < 2$ (as $|k| \to \infty$, the integrand behaves like $|k|^{-H\alpha-1}$, which is integrable for any $0 < \alpha < 2$ and $H > 0$; as $|k| \to 0$, the integrand tends to zero).

Definition 4.2.4. *Given a HTFSM* (4.40)*, we define tempered fractional harmonizable sta-*

ble noise (TFHSN)

$$\widetilde{Y}_{H,\alpha,\lambda}(t) := \widetilde{X}_{H,\alpha,\lambda}(t+1) - \widetilde{X}_{H,\alpha,\lambda}(t) \quad \textit{for integers } -\infty < t < \infty. \tag{4.42}$$

It follows easily from (4.40) that TFHSN has the harmonizable representation

$$\widetilde{Y}_{H,\alpha,\lambda}(t) = \mathrm{Re} \int_{-\infty}^{+\infty} e^{ikt} \; \Psi(dk), \tag{4.43}$$

where

$$\Psi(dk) = \frac{e^{ik} - 1}{(\lambda + ik)^{H+\frac{1}{\alpha}}} \widetilde{Z}_{\alpha}(dk) \tag{4.44}$$

is a complex symmetric α-stable ($S\alpha S$) random measure with the control measure

$$m(dk) = \frac{|e^{ik} - 1|^{\alpha}}{|\lambda + ik|^{H\alpha+1}} \; dk, \tag{4.45}$$

for any $0 < \alpha < 2$, $\lambda > 0$ and $H > 0$.

Theorem 4.2.5. *The tempered fractional stable motion (LTFSM) defined in* (4.5) *and tempered fractional harmonizable stable motion (HTFSM) defined in* (4.40) *are different processes.*

Proof. To prove that the processes $\{X_{H,\alpha,\lambda}(t)\}$ and $\{\widetilde{X}_{H,\alpha,\lambda}(t)\}$ are different it is enough to show that

$$\lim_{t\to\infty} r_{Y_{H,\alpha,\lambda}}(\theta_1, \theta_2, t) = 0 \tag{4.46}$$

and

$$\lim_{t\to\infty} r_{\widetilde{Y}_{H,\alpha,\lambda}}(\theta_1, \theta_2, t) > 0, \tag{4.47}$$

where $\{Y_{H,\alpha,\lambda}(t)\}$ and $\{\widetilde{Y}_{H,\alpha,\lambda}(t)\}$ are the increments of $\{X_{H,\alpha,\lambda}(t)\}$ and $\{\widetilde{X}_{H,\alpha,\lambda}(t)\}$ respectively. Lemma 6.1 in [33] shows that if

$$Y(t) = \int_{-\infty}^{+\infty} f(t-x) Z_\alpha(dx),$$

for $f \in L^\alpha(\mathbb{R})$, $\{Z_\alpha\}$ is the $S\alpha S$ random measure on $\mathbb{R}$, then

$$\lim_{t\to\infty} r_Y(\theta_1, \theta_2, t) = 0.$$

Along the same lines, we now define $f(x) := (x+1)_+^{H-\frac{1}{\alpha}} e^{-\lambda(x+1)_+} - (x)_+^{H-\frac{1}{\alpha}} e^{-\lambda(x)_+}$. Then

$$\begin{aligned}
Y_{H,\alpha,\lambda}(t) &= X_{H,\alpha,\lambda}(t+1) - X_{H,\alpha,\lambda}(t) \\
&= \int_{-\infty}^{+\infty} \left[(t-x+1)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-x+1)_+} - (t-x)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-x)_+} \right] Z_\alpha(dx) \\
&= \int_{-\infty}^{+\infty} f(t-x)\ Z_\alpha(dx)
\end{aligned}$$

and hence by applying Lemma 6.1 [33], as described above,

$$\lim_{t\to\infty} r_{Y_{H,\alpha,\lambda}}(\theta_1, \theta_2, t) = 0$$

which is Property (4.46). Theorem 3.1 in [37] states that if

$$\widetilde{Y}(t) = \mathrm{Re} \int_{-\infty}^{+\infty} e^{ikt}\ \Psi(dk) \quad \text{for } -\infty < t < \infty$$

is a stationary real harmonizable $S\alpha S$ process, then

$$\liminf_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} r_{\widetilde{Y}}(\theta_1,\theta_2,t)\, dt \geq K(\theta_1,\theta_2)c_0\left(m\{0\}F_0+\frac{1}{2\pi}m(\mathbb{R}-\{0\})F_1\right) > 0,$$

where m is the control measure of the isotropic complex-valued random measure Ψ, $F_0 \in \mathbb{R}$ and $F_1 > 0$ are some constants depends on α, m, θ_1 and θ_1. Along the same lines, we now define

$$\begin{aligned}\widetilde{Y}_{H,\alpha,\lambda}(t) &= \widetilde{X}_{H,\alpha,\lambda}(t+1) - \widetilde{X}_{H,\alpha,\lambda}(t)\\ &= \mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{-ik(t+1)} - e^{-ikt}}{(\lambda - ik)^{H+\frac{1}{\alpha}}}\widetilde{Z}_\alpha(dk)\\ &= \mathrm{Re}\int_{-\infty}^{+\infty} e^{ikt}\Psi(dk),\end{aligned}$$

where $\Psi(dk)$ and $m(dk)$ are given by (4.44) and (4.45) respectively. Hence by applying Theorem 3.1 in [37], as explained above, we get

$$\liminf_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} r_{\widetilde{Y}_{H,\alpha,\lambda}}(\theta_1,\theta_2,t)\, dt \geq K(\theta_1,\theta_2)c_0\left(m\{0\}F_0+\frac{1}{2\pi}m(\mathbb{R}-\{0\})F_1\right) > 0$$

and consequently

$$\lim_{t\to\infty} r_{\widetilde{Y}_{H,\alpha,\lambda}}(\theta_1,\theta_2,t) > 0,$$

which is Property (4.2) and this completes the proof of the theorem. □

4.3 Sample path properties

In this section, we present some results about the sample path properties of LTFSM and HTFSM. The path behavior of the linear fractional stable motion (LFSM) process $X_{H,\alpha}$

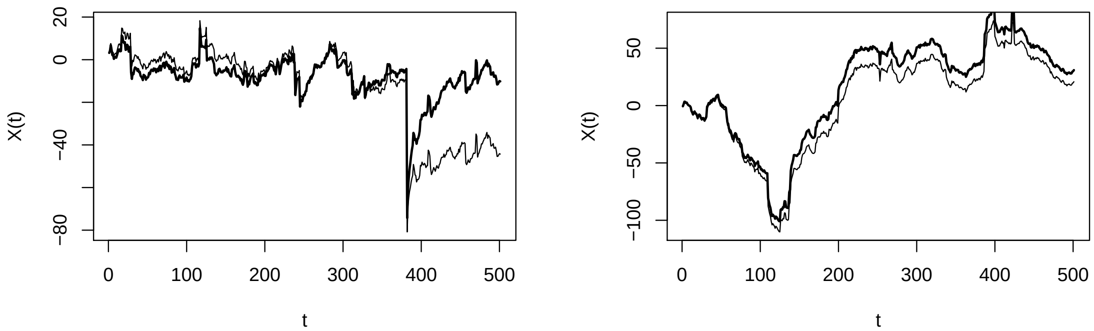

Figure 4.1: Left panel: Sample paths of LTFSM (thick black line) with $\lambda = 0.03$ and $H = 0.3$, and LFSM (thin black line) with $H = 0.3$. Both graphs use the same noise realization $Z_\alpha(t)$. The right panel shows the same plots for $\lambda = 0.001$, $H = 0.7$ and $\alpha = 1.5$.

(see Definition 7.4.1 [61]) depends on the structure of the kernel $g_{\alpha,t}(x) := (t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}$, $t, x \in \mathbb{R}$ (see [62]). When $H - \frac{1}{\alpha} < 0$, the function $g_{\alpha,t}(x)$, $x \in \mathbb{R}$, has singularities at $x = 0$ and $x = t$. By the same argument, The paths behavior of the LTFSM process $X_{H,\alpha,\lambda}$ depends on the structure of the kernel $g_{\alpha,\lambda,t}(x) := (t-x)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-x)_+} - (-x)_+^{H-\frac{1}{\alpha}} e^{-\lambda(t-x)_+}$. In fact, The function $g_{\alpha,\lambda,t}(x)$ has singularities at $x = 0$ and $x = t$ too. These singularities magnify the stable noise processes $Z_\alpha(dx)$ and cause large spikes in the paths of the LFSM and LTFSM processes. The left panel in Figure 4.1 compares a typical sample path of both processes, simulated using the same noise realization $Z_\alpha(t)$, in the case $H - \frac{1}{\alpha} < 0$. In the case $H - \frac{1}{\alpha} > 0$, (since $0 < H < 1$ it follows that $\alpha > 1$) the paths of the fractional stable motion can be made continuous with probability one (see Chapter 10 in [61]), since its kernel is bounded and positive for all $t > 0$. Similarly, the kernel of LTFSM, $g_{\alpha,\lambda,t}(x)$, is bounded and positive for all $t > 0$ and hence the paths of the process $X_{H,\alpha,\lambda}$ can be made continuous with probability one when $H - \frac{1}{\alpha} > 0$. The right panel in Figure 4.1 shows the corresponding sample paths in the case $\alpha = 1.5$. These simulations use a discretized version of the moving average representation of LTFSM (4.5).

Theorem 4.3.1. *Let $0 < \alpha < 2$, $0 < H < \frac{1}{\alpha}$ and $X = \{X_{H,\alpha,\lambda}(t)\}_{t \in \mathbb{R}}$ be the LTFSM* *process defined in* (4.5)*. Then, for any separable version* $X^* = \{X^*_{H,\alpha,\lambda}(t), t \in (a,b)\}$ *of the process* X*, we have that*

$$\mathbb{P}\Big(\{\omega : \sup_{t\in(a,b)} \big|X^*_{H,\alpha,\lambda}(t,\omega)\big| = \infty\}\Big) = 1,$$

That is, every version of the process $X = \{X_{H,\alpha,\lambda}(t)\}_{t\in\mathbb{R}}$ *has unbounded paths.*

Proof. We apply Theorem 10.2.3 in [61]. Indeed, consider the countable set $T^* := \mathbb{Q} \cap [a,b]$, where $\mathbb{Q}$ denotes the set of rational numbers. Since T^* is dense in $[a,b]$, there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \in T^*$ such that $t_n \to x$ as $n \to \infty$. Therefore

$$\begin{aligned} f^*(T^*;x) &:= \sup_{t\in T^*} \Big|g_{\alpha,\lambda,t}(x)\Big| \\ &\geq \sup_{t_n\in T^*} \Big|g_{\alpha,\lambda,t_n}(x)\Big| =: f^*_n(T^*;x) = \infty, \end{aligned}$$

as $n \to \infty$ and hence $\int_a^b f^*(T^*;x)\ dx = \infty$, and this contradicts Condition (10.2.14) of Theorem 10.2.3 in [61]. Therefore, the stochastic process $\{X_{H,\alpha,\lambda}\}$ does not have a version with bounded paths on the interval (a,b), and this completes the proof. $\square$

Lemma 4.3.2. *Let* $1 < \alpha < 2$, $\frac{1}{\alpha} < H < 1$ *and* $\lambda > 0$*. Then there exist a positive constant* C_1 *such that the LTFSM* (4.5) *satisfies*

$$\Big\|X_{H,\alpha,\lambda}(t) - X_{H,\alpha,\lambda}(s)\Big\|_\alpha^\alpha \geq C_1\Big|t-s\Big|^{H\alpha},$$

locally uniformly in $s,t \in [0,1]$.

Proof. We write

$$
\begin{aligned}
\Big\|X_{H,\alpha,\lambda}(t) - X_{H,\alpha,\lambda}(s)\Big\|_\alpha^\alpha &\geq \int_s^t |t-u|^{\alpha(H-\frac{1}{\alpha})} e^{-\lambda\alpha|t-u|}\ du \\
&\geq e^{-\lambda\alpha|t-s|} \int_s^t |t-u|^{H\alpha-1}\ du \\
&= \frac{e^{-\lambda\alpha|t-s|}}{H\alpha}|t-s|^{H\alpha} \\
&= C_1\ |t-s|^{H\alpha}
\end{aligned}
$$

and this gives the lower bound. □

Lemma 4.3.3. *Let* $1 < \alpha < 2$, $\frac{1}{\alpha} < H < 1$ *and* $\lambda > 0$*. Then there exist positive constants* C_1 *and* C_2 *such that the HTFSM* (4.40) *satisfies*

$$
C_1\Big|t-s\Big|^{H\alpha} \leq \Big\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\Big\|_\alpha^\alpha \leq C_2\Big|t-s\Big|^{H\alpha}, \tag{4.48}
$$

locally uniformly in $s, t \in [0,1]$.

Proof. To get the upper bound, write

$$
\begin{aligned}
\Big\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\Big\|_\alpha^\alpha &= \int_{-\infty}^{+\infty} \frac{|e^{-ikt} - e^{-iks}|^\alpha}{|\lambda - ik|^{H\alpha+1}}\ dk \\
&\leq C\int_{-\infty}^{+\infty} (1 \wedge |t-s|^\alpha |k|^\alpha)\,|\lambda - ik|^{-H\alpha-1}\ dk \\
&= C\Big[|t-s|^\alpha \int_{|k|<\frac{1}{|t-s|}} |k|^\alpha||\lambda - ik|^{-H\alpha-1}\ dk \\
&+ \int_{|k|>\frac{1}{|t-s|}} |\lambda - ik|^{-H\alpha-1}\ dk\Big] \\
&\leq C\Big[|t-s|^\alpha I_1 + I_2\Big]
\end{aligned} \tag{4.49}
$$

where

$$I_1 := \int_{|k|<\frac{1}{|t-s|}} \Big|k\Big|^{\alpha}\Big|\lambda - ik\Big|^{-H\alpha-1}\,dk$$

and

$$I_2 := \int_{|k|>\frac{1}{|t-s|}} \Big|\lambda - ik\Big|^{-H\alpha-1}\,dk$$

and C is a constant. Observe that

$$\begin{aligned} I_1 &= \int_{|k|<\frac{1}{|t-s|}} \Big|k\Big|^{\alpha}\Big|\lambda^2 + k^2\Big|^{\frac{-H\alpha-1}{2}}\,dk \\ &\le \int_{|k|<\frac{1}{|t-s|}} \Big|k\Big|^{\alpha}\Big|k\Big|^{-H\alpha-1}\,dk = \int_{|k|<\frac{1}{|t-s|}} \Big|k\Big|^{-H\alpha-1+\alpha}\,dk \qquad (4.50) \\ &\le \Big|t-s\Big|^{H\alpha-\alpha} \cdot \frac{2}{\alpha(1-H)} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{|k|>\frac{1}{|t-s|}} \Big|\lambda^2 + k^2\Big|^{\frac{-H\alpha-1}{2}}\,dk \\ &\le \int_{|k|>\frac{1}{|t-s|}} \Big|k^2\Big|^{\frac{-H\alpha-1}{2}}\,dk = \int_{|k|>\frac{1}{|t-s|}} \Big|k\Big|^{-H\alpha-1}\,dk \qquad (4.51) \\ &\le \Big|t-s\Big|^{H\alpha} \cdot \frac{2}{H\alpha}. \end{aligned}$$

Finally, from (4.49), (4.50) and (4.51) we get

$$\begin{aligned} \Big\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\Big\|_{\alpha}^{\alpha} &\le C\left[\Big|t-s\Big|^{\alpha} I_1 + I_2\right] \\ &\le C\left[\frac{2}{\alpha(1-H)} + \frac{2}{H\alpha}\right]\Big|t-s\Big|^{H\alpha} \\ &= C_2\,\Big|t-s\Big|^{H\alpha} \end{aligned}$$

which gives the upper bound in (4.48). In order to get the lower bound, we use the fact that there exist positive constants c_1, c_2 such that $|e^{-iy} - 1| > c_1|y|$ for $|y| < c_2$. Therefore

$$\begin{aligned}
\left\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\right\|_{\alpha}^{\alpha} &= \int_{-\infty}^{+\infty} \left|e^{-ikt} - e^{-iks}\right|^{\alpha} \left|\lambda - ik\right|^{-(H\alpha+1)} dk \\
&= \int_{-\infty}^{+\infty} \left|e^{-ik(t-s)} - 1\right|^{\alpha} \left|\lambda - ik\right|^{-(H\alpha+1)} dk \\
&\geq c_1 \int_{|k|<\frac{c_2}{|t-s|}} \left|k\right|^{\alpha}\left|t-s\right|^{\alpha}\left|\lambda - ik\right|^{-(H\alpha+1)} dk \\
&= c_1|t-s|^{\alpha} \int_{|k|<\frac{c_2}{|t-s|}} \left|k\right|^{\alpha}(\lambda^2+k^2)^{\frac{-(H\alpha+1)}{2}} dk.
\end{aligned}$$

We now use the fact that

$$\left(\lambda^2 + k^2\right)^{\frac{-(H\alpha+1)}{2}} \geq \left(1 + c_2^2\right)^{\frac{-(H\alpha+1)}{2}} \left|t-s\right|^{H\alpha+1},$$

for $\lambda < \frac{1}{|t-s|}$ and $|k| < \frac{c_2}{|t-s|}$ to continue the rest of the proof as follows:

$$\begin{aligned}
&c_1\left|t-s\right|^{\alpha} \int_{|k|<\frac{c_2}{|t-s|}} \left|k\right|^{\alpha}\left(\lambda^2+k^2\right)^{\frac{-(H\alpha+1)}{2}} dk \\
&\geq 2c_1 \left(1+c_2^2\right)^{\frac{-(H\alpha+1)}{2}} \left|t-s\right|^{\alpha}\left|t-s\right|^{H\alpha+1} \int_0^{\frac{c_2}{|t-s|}} k^{\alpha}\, dk \\
&= C_1\left|t-s\right|^{H\alpha+\alpha+1}\left|t-s\right|^{-\alpha-1} = C_1\left|t-s\right|^{H\alpha}
\end{aligned}$$

and this gives the lower bound. □

4.4 Local Times and Local nondeterminism

In this section, we prove the existence of the local time for LTFSM and HTFSM. We also show that LTFSM and HTFSM are locally nondeterministic on every compact interval. We first recall the definition of the local time (see [11] for more details). Suppose $X = \{X(t), t \geq 0\}$

is a real-valued separable random process with Borel sample functions. For any Borel set $B \subset \mathbb{R}^+$

$$\mu_B(A) = \eta(\{s \in B, X(s) \in A\})$$

is called the occupation measure of X on B, where η is the Lebesgue measure on $\mathbb{R}^+$. If μ_B is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then we say that X has a local time on B and define its local time, $L(B,.)$, to be the Radon-Nikodym derivative of μ_B with respect to Lebesgue measure. We write $L(t,x)$ instead of $L([0,t],x)$.

Proposition 4.4.1. *The LTFSM defined in* (4.5) *with* $1 < \alpha < 2$ *and* $\frac{1}{\alpha} < H < 1$ *has a square integrable local time* $L(t,x)$.

Proof. According to Theorem 3.1 in [11], a stochastic process $\{X(t), t \in [0,T]\}$ has a local time $L(t,x)$ which is continuous in t for a.e. $x \in \mathbb{R}$ and square integrable with respect to x if $\{X(t), t \in [0,T]\}$ satisfies

- Condition $(\mathcal{H})$: There exist positive numbers $(\rho_0, H) \in (0,\infty) \times (0,1)$ and a positive function $\psi \in L^1(\mathbb{R})$ such that for all $\kappa \in \mathbb{R}, t, s \in [0,T], 0 < |t-s| < \rho_0$ we have

$$\left| \mathbb{E} \exp\left(i\kappa \frac{X(t) - X(s)}{|t-s|^H} \right) \right| \leq \psi(\kappa). \tag{4.52}$$

We prove that the LTFSM $\{X_{H,\alpha,\lambda}(t)\}$ satisfies $(\mathcal{H})$. Apply (4.7) and Lemma 4.3.2 to get

$$\begin{aligned} \mathbb{E} \exp\left(i\kappa \frac{X_{H,\alpha,\lambda}(t) - X_{H,\alpha,\lambda}(s)}{|t-s|^H} \right) &= \exp\left(-|\kappa|^\alpha \frac{\|X_{H,\alpha,\lambda}(t) - X_{H,\alpha,\lambda}(s)\|_\alpha^\alpha}{|t-s|^{\alpha H}} \right) \\ &\leq \exp\left(-|\kappa|^\alpha C \right) := \psi(\kappa) \end{aligned}$$

where the function $\psi(\kappa)$ belongs to $L^1(\mathbb{R}, dk)$ which means the HTFSM satisfies $\mathcal{H}$ and this completes the proof. $\square$

Proposition 4.4.2. *The HTFSM defined in* (4.40) *with* $1 < \alpha < 2$ *and* $\frac{1}{\alpha} < H < 1$ *has a square integrable local time* $L(t, x)$.

Proof. We prove that the HTFSM $\{\widetilde{X}_{H,\alpha,\lambda}(t)\}$ satisfies $(\mathcal{H})$. Apply (4.39) and Lemma 4.3.3 to obtain

$$\mathbb{E}\exp\Big(i\kappa\frac{\widetilde{X}_{H,\alpha,\lambda}(t)-\widetilde{X}_{H,\alpha,\lambda}(s)}{|t-s|^H}\Big) = \exp\Big(-|\kappa|^\alpha\frac{\|\widetilde{X}_{H,\alpha,\lambda}(t)-\widetilde{X}_{H,\alpha,\lambda}(s)\|_\alpha^\alpha}{|t-s|^{\alpha H}}\Big)$$
$$\leq \exp\Big(-|\kappa|^\alpha C\Big) := \psi(\kappa),$$

where the function $\psi(\kappa)$ belongs to $L^1(\mathbb{R}, dk)$ which means the HTFSM satisfies $\mathcal{H}$ and this completes the proof. $\square$

We next show that HTFSM is locally nondeterministic on every compact interval $[\epsilon, T]$, for any $0 < \epsilon < T < \infty$. Recall that a stochastic process $\{X(t)\}_{t\in T}$ is *locally nondeterministic* (LND) if

1. $\|X(t)\|_\alpha > 0$ for all $t \in T$
2. $\|X(t) - X(s)\|_\alpha > 0$ for all $t, s \in T$ sufficiently close; and
3. for any $m \geq 2$,

$$\liminf_{\epsilon\downarrow 0}\frac{\|X(t_m)-\text{span}\{X(t_1),\ldots,X(t_{m-1})\}\|_\alpha}{\|X(t_m)-X(t_{m-1})\|_\alpha} > 0,$$

where the lim inf is taken over distinct, ordered $t_1 < t_2 < \ldots < t_m \in T$ with $|t_1 - t_m| < \epsilon$, $T \subset \mathbb{R}$ and $\|X\|_\alpha = \Big[-\log(\mathbb{E}\exp(iX))\Big]^{1/\alpha}$ for $0 < \alpha \leq 2$ (see [51, 52, 67] for more details).

Next, we show that HTFSM also is LND.

Proposition 4.4.3. *The HTFSM* (4.40) *with* $1 < \alpha < 2$ *and* $\frac{1}{\alpha} < H < 1$ *is LND on every interval* $[\epsilon, \kappa]$ *for* $\epsilon < \kappa < \infty$.

Proof. Theorem 3.3 in [19] shows that a harmonizable multifractional stable motion is LND $[\epsilon, \kappa]$ for $\epsilon < \kappa < \infty$. Our proof is a modification of that Theorem. We need to verify conditions (1), (2) and (3) as described above. The first and second condition follows from Lemma 4.3.3

$$\|\widetilde{X}_{H,\alpha,\lambda}(t) - \widetilde{X}_{H,\alpha,\lambda}(s)\|_\alpha^\alpha \geq C_1 |t-s|^{H\alpha},$$

where C_1 is a positive constant. To prove the third condition, observe first that the inverse Fourier transform of

$$f_{H,\alpha,\lambda}(t,k) := \frac{e^{-ikt} - 1}{(\lambda - ik)^{H+\frac{1}{\alpha}}} \tag{4.53}$$

on $L^\alpha(\mathbb{R})$ which is

$$\mathcal{F}^{-1} f_{H,\alpha,\lambda}(t,k) = \frac{\Gamma(H+\frac{1}{\alpha})}{\sqrt{2\pi}} \Big[e^{-\lambda(t-x)_+} (t-x)_+^{H-\frac{\alpha-1}{\alpha}} - e^{-\lambda(-x)_+} (-x)_+^{H-\frac{\alpha-1}{\alpha}} \Big] \tag{4.54}$$

by (2.11). In order to verify the third condition, we shall establish a lower bound for

$$\Big\| \widetilde{X}_{H,\alpha,\lambda}(t_m) - \sum_{j=1}^{m-1} u_j \widetilde{X}_{H,\alpha,\lambda}(t_j) \Big\|_\alpha = \Big\| f_{H,\alpha,\lambda}(t_m,k) - \sum_{j=1}^{m-1} u_j f_{H,\alpha,\lambda}(t_j,k) \Big\|_{L^\alpha(\mathbb{R})}$$

where $f_{H,\alpha,\lambda}(t,k)$ is defined in (4.53). Let $\beta = \frac{\alpha}{\alpha-1}$. By applying the Hausdorff-Young inequality (see Theorem 5.7 in [39]):

$$
\begin{aligned}
&\Big\| f_{H,\alpha,\lambda}(t_m,k) - \sum_{j=1}^{m-1} u_j f_{H,\alpha,\lambda}(t_j,k) \Big\|_{L^\alpha(\mathbb{R})} \\
&\geq C \Big\| \mathcal{F}^{-1} f_{H,\alpha,\lambda}(t_m,k) - \sum_{j=1}^{m-1} u_j \mathcal{F}^{-1} f_{H,\alpha,\lambda}(t_j,k) \Big\|_{L^\beta(\mathbb{R})} \\
&= C \Bigg(\int_{-\infty}^{t_{m-1}} \Big| \mathcal{F}^{-1} f_{H,\alpha,\lambda}(t_m,k) - \sum_{j=1}^{m-1} u_j \mathcal{F}^{-1} f_{H,\alpha,\lambda}(t_j,k) \Big|^\beta \\
&+ \int_{t_{m-1}}^{t_m} \Big| \mathcal{F}^{-1} f_{H,\alpha,\lambda}(t_m,k) \Big|^\beta \, dk \Bigg)^{\frac{1}{\beta}}.
\end{aligned}
\tag{4.55}
$$

From (4.54) we have

$$
\mathcal{F}^{-1} f_{H,\alpha,\lambda}(t_m,k) = \frac{\Gamma(H+\frac{1}{\alpha})}{\sqrt{2\pi}} \Big[e^{-\lambda(t_m-x)_+} (t_m-x)_+^{H-\frac{\alpha-1}{\alpha}} - e^{-\lambda(-x)_+} (-x)_+^{H-\frac{\alpha-1}{\alpha}} \Big]
$$

and the second term, $e^{-\lambda(-x)_+}(-x)_+^{H-\frac{\alpha-1}{\alpha}}$, vanishes on the interval $[t_{m-1}, t_m]$. Hence we can continue (4.55) as the following:

$$
\begin{aligned}
&\geq C \Bigg[\int_{t_{m-1}}^{t_m} (t_m - x)^{\beta(H-\frac{1}{\beta})} e^{-\lambda\beta(t_m-x)} \, dx \Bigg]^{\frac{1}{\beta}} \\
&\geq C \; e^{-\lambda(t_m - t_{m-1})} \Big| t_m - t_{m-1} \Big|^H \geq C \; e^{-\lambda(\kappa-\epsilon)} \Big\| \widetilde{X}_{H,\alpha,\lambda}(t_m) - \widetilde{X}_{H,\alpha,\lambda}(t_{m-1}) \Big\|_\alpha
\end{aligned}
\tag{4.56}
$$

for t_m and t_{m-1} close enough (and C is a constant). In the last line in (4.56), we used the fact that $|t_m - t_{m-1}| < \kappa - \epsilon$ and we also applied Lemma 4.3.3 to get the last inequality.

Therefore

$$\begin{aligned}\left\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \operatorname{span}\{\widetilde{X}_{H,\alpha,\lambda}, \ldots, \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\}\right\|_\alpha &= \left\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \sum_{j=1}^{m-1} u_j \widetilde{X}_{H,\alpha,\lambda}(t_j)\right\|_\alpha \\ &\geq C\left\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\right\|_\alpha\end{aligned}$$

and consequently

$$\liminf_{\epsilon\downarrow 0} \frac{\left\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \operatorname{span}\{\widetilde{X}_{H,\alpha,\lambda}, \ldots, \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\}\right\|_\alpha}{\left\|\widetilde{X}_{H,\alpha,\lambda}(t_m) - \widetilde{X}_{H,\alpha,\lambda}(t_{m-1})\right\|_\alpha} > C,$$

where C is a positive constant and this verifies Condition (3) of the LND property and this completes the proof. □

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