ON SIGN PATTERNS OF BRANCH MATRICES AND R-GRAPH REALIZATION

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ABSTRACT

ON SIGN PATTERNS OF BRANCH MATRICES AND R-GRAPH REALIZATION

by David Paul Brown

This thesis deals with properties of sign patterns of the entries in the coefficient matrix of the branch (node-pair) equations, branch matrix, for any graph, and the realization of a given matrix as the branch matrix of an R-graph.

In the second section, properties of sign patterns are classified as to those fixed by: (1) element orientations per se; (2) element orientations determined by their being contained in subgraphs. The main result in (1) is that if the branch orientation of a tree of a part is changed, then there is a row and column sign change of the entries in the corresponding branch matrix for diagonal element matrix, and conversely. This result is based on the relationship between the s-orientation of any two f-segs and the s-orientations of the corresponding common elements. In (2), a subgraph consisting of any two branches, b_i and b_j, contained in a path-in-tree is considered. The fact that the p- and e-orientation of b_i and b_j coincide is shown to imply that the s-orientation of the elements common to the f-segs corresponding to b_i and b_j are the same, and conversely. A set of pairs of branches, each pair having coincident p- and e-orientations is shown to imply that all branches of the set are contained in a path-in-tree with coincident p- and e-orientations. The situation when the p- and e-orientations do not coincide is also considered. The specific character of a subgraph of the tree corresponding to a branch matrix containing a principal submatrix with all positive or all negative entries is then

obtained. The complete tree form is determined for the case of all positive or all negative entries in the branch matrix.

The necessary and sufficient conditions on a given matrix such that it is realizable as an R-graph consisting of the union of a complete graph and Lagrangian tree are determined in the fourth section. Formulas for corresponding element values and a process to determine the orientation of the tree are also given. It is found that the conditions for realization are fixed by the tree form associated with the branch matrix. Using the tree transformation matrix of the third section, necessary and sufficient conditions for realization are determined for an arbitrary tree. The detailed form of the conditions for realization are given for a tree in the form of a path. For the case of a five vertex complete graph, the conditions fixed by the three tree forms are given in detail.

ON SIGN PATTERNS OF BRANCH MATRICES AND R-GRAPH REALIZATION

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I. INTRODUCTION

The problem of determining restrictions on matrices with constant entries such that they are coefficient matrices of some system of equations determined from a graph has been considered by many investigators.

A method of realizing symmetric matrices with constant entries as R-networks has been given by W. Cauer [1]. Here the requirement that the given matrix be positive-semidefinite leads to networks containing ideal transformers. The well-known condition of dominance, discussed by Burington [2], is sufficient for synthesis of R-networks without ideal transformers. For networks without ideal transformers, Cederbaum [3, 4] has shown that a necessary condition for synthesis of short circuit admittance matrices is that they are paramount. This result is based on properties discussed by Talbot [5]. As a method of realizing matrices, Cederbaum [6, 7] has given a procedure to decompose a matrix into a triple product of matrices, where the center matrix, which is diagonal, is pre- and post-multiplied by a unimodular matrix and its transpose respectively. Slepian and Weinberg [8] have summarized this area of network synthesis and raised many questions. A recent discussion of synthesis of networks without ideal transformers has also been given by Guillemin [9].

The problem of characterizing the patterns of the signs of the entries of certain types of matrices has recently been considered. In particular, using matrix algebra, Cederbaum [7] has determined some elementary sign properties of the entries in a matrix triple product, the center matrix, which is diagonal, being pre- and post-multiplied by a unimodular matrix and its transpose respectively. These results have been associated with the realization of loop-resistance matrices by So [10]. Some

discussion of sign patterns relevant to synthesis has been given by Slepian and Weinberg [8]. Biorci and Civalleri [11] have recently developed a procedure to realize a restricted class of short circuit admittance matrices based on forming the graph from the signs of the entries in the given matrix.

In this thesis the matrix triple product $\mathcal{S}\mathcal{G}_e\mathcal{S}' = [G_{ij}]$ is considered, where $\mathcal{S}_f = [\mathcal{U}\ \mathcal{S}]$ is the fundamental f-seg matrix discussed by Reed [12, 13] and $\mathcal{G}_e$ is an element matrix containing constant entries. Although the properties of the f-seg matrix are the same as the cut set matrix [14], the definitions of the corresponding subgraphs, i.e. the seg [15] and the cut set [14], are logically different. Because of the clarity of the seg concept, it is used in the following discussion.

The objective of Section II is to determine the relationship between the sign pattern of the branch matrix, $\mathcal{S}\mathcal{G}_e\mathcal{S}'$, and the orientation of the elements of the corresponding connected graph P(part). It is also shown that the sign pattern of the branch matrix or principal submatrices of the branch matrix is fixed by the orientation of elements of specific types of subgraphs.

In Section IV, the necessary and sufficient conditions on a matrix such that it is a branch matrix corresponding to an R-graph are determined. Equations to calculate the element values associated with the graph are also given. The conditions for realization associated with a branch matrix are fixed by the possible tree forms of the part. Therefore, the totality of necessary and sufficient conditions can be obtained by cataloging the conditions fixed by each tree form. These results are obtained using the tree transformation matrix of Section III.

A list of symbols, which are used repeatedly, is given in the Appendix.

II. SIGN PATTERN OF BRANCH MATRIX

2.1 Introduction

The sign pattern of $\mathcal{S}\mathcal{G}_e\mathcal{S}'$ is investigated in terms of its dependence on the orientation of elements in a graph. The following theorems, which are based on the seg and f-seg matrix, indicate that the elements in the cotree of a graph do not effect the sign pattern and that the branch orientations are the controlling factors. The operation of cross-sign change, Definition 2.1, is used to describe the general pattern of signs as a function of branch orientations. Some results not directly related to the sign patterns have been included as corollaries.

The orientation of any two branches, b_i and b_j, is shown to fix the sign of the i, j entry of $\mathcal{S}\mathcal{G}_e\mathcal{S}'$ for $\mathcal{G}_e$ diagonal with non-negative entries. This result is used to determine the signs of the entries in the branch matrix as a function of the orientation of pairs of branches which are contained in a path-in-tree and conversely. For the case of a path and Lagrangian tree, the complete sign pattern is determined.

<u>Definition 2.1</u>: A <u>cross-sign change</u> of a matrix $\mathcal{A}$ is the operation of changing the sign of each (non-zero) entry in the i-row and i-column.

<u>Theorem 2.1</u>: For any matrix $\mathcal{A} = [a_{ij}]_n$, consider a sequence of K different cross-sign changes. If $a_{ij} \neq 0$, $i \neq j$, then the number of entries in $\mathcal{A}$ which change sign as a result of the K cross-sign changes is

$$2K(n-K).$$

<u>Proof</u>: Each cross-sign change changes the sign of 2(n-1) entries of $\mathcal{A}$. Entries common to two cross-sign changes are changed in sign twice, i.e. they do not change sign. The number of entries in i-row and

i-column common to K cross-sign changes is $x_1 + x_2 + \ldots + x_{K-1}$ where x_i, $i = 1, 2, \ldots, K-1$, is the number of entries in i-row and i-column common to one cross-sign change. Since $x_i = 2$,

$$\sum_{i=1}^{K-1} x_i = 2(K-1).$$

Therefore, the number of entries in the i-row and i-column which change sign as a result of K cross-sign changes is $2(n-1) - 2(K-1) = 2(n-K)$. Since there are K similar patterns, the total number of entries which change sign is the sum of $2(n-K)$ for each row and column, that is $2K(n-K)$.

Corollary 2.1: Consider any matrix $\mathcal{A} = [a_{ij}]_n$. If $a_{ij} > 0$ ($a_{ij} < 0$), $i \neq j$, then the maximum possible number of negative (positive) off-diagonal entries as a result of cross-sign changes is

$$\frac{n^2}{2} \quad \text{if n is even}$$

$$\frac{n^2-1}{2} \quad \text{if n is odd.}$$

Proof: It is only necessary to find the maximum of $2K(n-K)$, since this is the number of off-diagonal entries which are negative (positive) as a result of K cross-sign changes. Since $\frac{d}{dK}(2K[n-K]) = 2n-4K = 0$, $K_{max.} = \frac{n}{2}$ if n is even and $\frac{n\pm 1}{s}$ if n is odd. Substituting these values of K in $2K(n-K)$ gives the conclusion.

2.2 Sign Pattern Fixed by Certain Elements

For any tree T, S_i and S_j are any two f-segs defined by branches b_i and b_j respectively, and $\mathcal{S}^{j}_{f} = [\,\mathcal{U}\ \mathcal{S}\,]$ where $\mathcal{S} = [s_{ij}]$.

Theorem 2.2.1: If and only if S_i and S_j contain common elements (chords) C, then the X-vertex set of C when C is in S_i is the X- or NX-vertex set of C when C is in S_j.

Proof: Sufficiency: Suppose C contains only one chord c_1, then c_1 is an X-NX element in S_i and also in S_j and the theorem applies.

Suppose C contains two or more chords c_k. By Theorem 15 [15], b_i is in the f-circuit defined by c_k and so is b_j.

Consider the vertex segregation defined by b_j. By Theorem 14 [15], there is a path-in-tree in the X-vertex set (NX-vertex set) between any two X-vertices (any two NX-vertices) which contains only X-vertices (NX-vertices). Because S_i is an f-seg, b_j is an X-element (NX-element), hence there is a path-in-tree containing only NX-vertices (X-vertices) and does not contain b_j.

Suppose the X-vertex set of C in S_i is neither the X- nor NX-vertex set of C in S_j. Then there is a path-in-tree between a vertex in the X-set of S_i and a vertex in the NX-set of S_j and this path does not contain both b_i and b_j. By the initial argument in the proof of this theorem, there is a path-in-tree between the same pair of vertices which contains b_i and b_j. This implies a circuit in tree and contradicts the hypothesis.

Necessity: Assume one element of C, c_k, is not common to S_i and S_j, i.e., if c_k is in S_i, it is not in S_j. Hence c_k is not an X-NX-element in S_j. Since this conclusion is independent of the X or NX labeling of vertices, the theorem follows.

Corollary 2.2.1: All elements of C have the same (opposite) b_i and b_j defined S-orientation.

Proof: The X-vertices corresponding to b_i are all X- or all NX-vertices corresponding to b_j. Therefore the conclusion follows from the definition of S-orientation.

Theorem 2.2.2: If and only if the S_i and S_j orientations of all elements of C are the same (opposite), then

$$s_{ip} s_{jp} = +1 \quad (s_{ip} s_{jp} = -1)$$

where p corresponds to elements of C and $i \neq j$.

Proof: There are four and only four cases for elements in C as shown in Figure 1.

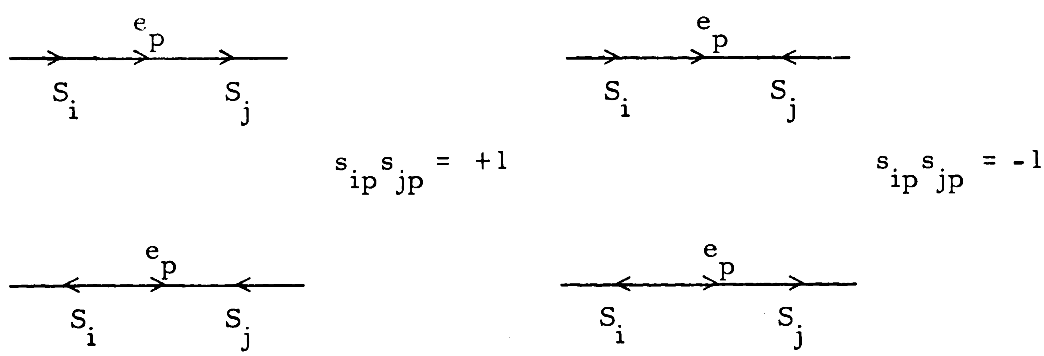


Fig. 1--Possible orientation patterns.

Hence, the theorem follows.

Corollary 2.2.2.1: Suppose S_i and S_j have at least two common elements. If

(1) $s_{ip} s_{jq} = +1$ $(=-1)$, then $s_{iq} s_{jp} = +1$ $(=-1)$

(2) $s_{ip} s_{iq} = +1$ $(=-1)$, then $s_{jp} s_{jq} = +1$ $(=-1)$

where p and q correspond to elements of C and $i \neq j$.

Proof: By Theorem 2.2.2, either $s_{ip} s_{jp} = +1$ and $s_{iq} s_{jq} = +1$ or $s_{ip} s_{jp} = -1$ and $s_{iq} s_{jq} = -1$. However, in either case $s_{ip} s_{jp} s_{iq} s_{jq} = +1$. Grouping the terms as $(s_{ip} s_{jq}) (s_{iq} s_{jp}) = +1$, (1) of corollary follows. The second part of the corollary follows from the following:

$$(s_{ip} s_{iq})(s_{jp} s_{jq}) = +1.$$

Corollary 2.2.2.2: The determinant of any submatrix of $\mathcal{S}$ of the form

$$\begin{bmatrix} s_{ip} & s_{iq} \\ s_{jp} & s_{jq} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} s_{ip} & s_{jp} \\ s_{jq} & s_{iq} \end{bmatrix} \text{ is } +1, \ -1 \text{ or } 0.$$

Proof: Either S_i and S_j have common elements or not. If they have common elements, all entries in the submatrices of hypothesis could be non-zero. Since each entry in the submatrices is +1 or -1, each product is $s_{ip}s_{jq} - s_{iq}s_{jp}$ and $s_{ip}s_{iq} - s_{jp}s_{jq}$ can have a value of +1 or -1. By Corollary 2.2.2.1, the products associated with either submatrix are equal. Hence, for this case, the value of the determinants is 0.

For all other cases one or more of the entries in the submatrices is zero. Since the remaining entries can only have the value +1 or -1, the conclusion follows.

Corollary 2.2.2.3: Let $\mathscr{G}_e$ of $\mathscr{S}\mathscr{G}_e\mathscr{S}' = [G_{ij}]$ be diagonal with positive entries.

(1) If and only if the S_i and S_j orientations of all elements of C are the same (opposite), then

$$G_{ij} > 0 \ (<0), \ i \neq j$$

(2) $G_{ii} > 0$.

Proof: Since $G_{ij} = \Sigma_p s_{ip} s_{jp} g_{pp}$, the corollary follows from Theorem 2.2.2.

Theorem 2.2.3: If b_i is a branch of any tree T of P, then corresponding to a change in the orientation of b_i there is a cross-sign change of $\mathscr{G}_T = \mathscr{S}\mathscr{G}_e\mathscr{S}'$.

Proof: If the orientation of b_i is changed, then every entry in the i-row of $\mathscr{S}$ changes sign. Therefore, every non-zero entry in the i-row

of $\mathcal{S}\mathcal{G}_e$ also changes sign. Hence, in the product $(\mathcal{S}\mathcal{G}_e)\mathcal{S}'$ every non-zero entry in the i-row and i-column changes sign.

Corollary 2.2.3: $|G_{ij}|$ is invariant through changes in orientation of branches of T.

Proof: Direct consequence of Theorem 2.2.3 and Definition 2.1.

Theorem 2.2.4: If b_i is a branch of any tree of P, then corresponding to a change in the orientation of b_i there is a cross-sign change of $\mathcal{G}_T = \mathcal{S}\mathcal{G}_e\mathcal{S}'$, $\mathcal{G}_e$ diagonal, and conversely.

Proof: The first part of the theorem follows from Theorem 2.2.3 for the case of diagonal $\mathcal{G}_e$.

For $i \neq j$, entries g_{pp} which appear in $G_{ij} = \sum_p s_{ip} s_{jp} g_{pp}$ correspond to elements which are common to S_i and S_j, and therefore by Corollary 2.2.1 all these elements have S-orientations which are the same or opposite. In addition, from Theorem 2.2.2, $s_{ip}s_{jp} = +1$ for all p or -1 for all p. Therefore, if G_{ij} is changed in sign, then each $s_{ip}s_{jp}$ in the defining sum must change sign. For this to be the case, the orientation of either b_i or b_j must have been changed. A change in the sign of the entries in the i-row and i-column of $\mathcal{G}_T$ implies a change in the orientation of b_i since s_{ip}, for all p, are the only entries of $\mathcal{S}$ common to all terms.

2.3 Sign Pattern Fixed by Certain Subgraphs

Theorem 2.3.1: Any two branches b_i and b_j of a tree are contained in some path-in-tree P_T.

Proof: Let the vertices of b_i and b_j be v_{i1}, v_{i2} and v_{j1}, v_{j2} respectively. Since the tree is connected there is a path between any pair of vertices. In particular, there is a path p with end vertices v_{i1}

and v_{j2}. Either p contains both branches, one branch or neither branch. In the first case the theorem is true. Hence assume both branches are not contained in p. Since an element is a path, there is a path with end vertices v_{i1} and v_{i2} and a path with end vertices v_{j1} and v_{j2}. These paths do not form a circuit. Therefore from properties of a path there is a path-in-tree containing both branches.

Theorem 2.3.2: If and only if for some path-in-tree the p- and e-orientations of b_i and b_j coincide, then the S_i and S_j orientations of all elements of C are the same.

Proof: Sufficiency: By the definition of f-seg orientation and hypothesis, b_i and b_j have e-, p- and s-orientations which coincide. Let K_i be the compliment of b_i in tree which contains the X-vertex of b_i. By definition of f-seg, the vertices of K_i are X-vertices of S_i. There is one and only one path-in-tree, P', between the X-vertex of b_j and the NX-vertex of b_i. This implies that every vertex of P' is an X-vertex of S_j. Therefore every X-vertex of S_i is an X-vertex of S_j. This implies the conclusion.

Necessity: By hypothesis b_i and b_j are e-oriented from X- to NX-vertex sets of S_i and S_j respectively. By Theorem 15 [15] b_i and b_j are in the f-circuit defined by any element of C. Therefore b_i and b_j are contained in a path-in-tree P'. Let the elements of P' be p-oriented opposite the f-circuit orientation. Therefore b_i and b_j are p-oriented from the X- to NX-vertex sets of S_i and S_j respectively. Hence the conclusion follows.

Corollary 2.3.2: With the same hypothesis,

$$s_{ip} s_{jp} = +1$$

where p corresponds to elements of C and $i \neq j$.

Proof: Direct consequence of Theorem 2.2.2 and Theorem 2.3.2.

Theorem 2.3.3: If b_i and b_j have coincident p- and e-orientations in some P_T, then b_i and b_j have coincident p- and e-orientations in every P_T.

Proof: By hypothesis, b_i and b_j are contained in some path-in-tree P_T. From properties of a path, there is a subpath of P_T, P_{T1}, which contains b_i and b_j as end elements. Every path-in-tree which contains b_i and b_j contains P_{T1}. The theorem follows from the method of p-orienting the elements.

Theorem 2.3.4: Consider a set of r branches B_r. If and only if each pair b_i and b_j of B_r have coincident p- and e-orientations for some path-in-tree, then the branches B_r are contained in a path-in-tree with coincident p- and e-orientations.

Proof: Sufficiency: It is only necessary to show that the branches B_r are contained in a path-in-tree since then the hypothesis fixes the p- and e-orientation of the conclusion.

By induction on r. For $r = 2$, the hypothesis and conclusion are the same. Therefore consider $r = 3$, where the branches of B_r are b_1, b_2 and b_3. Suppose b_1, b_2 and b_3 are not contained in a path-in-tree. Therefore by hypothesis and properties of paths, each branch is contained in a path-in-tree and these paths form a star with common vertex v_x. Also by hypothesis, e-orienting any branch, b_1, fixes the p- and e-orientations of the other branches. Thus the p- and e-orientation arrows of b_2 and b_3 are pointing toward or away from v_x. By Theorem 2.3.3, this is true for every path containing b_1 and b_2 and for every path containing b_1 and b_3. This implies that the p- and e-orientations of b_2 and b_3, for any path-in-tree containing b_2 and b_3 do not coincide. The conclusion follows for $r = 3$.

Suppose the theorem is true for $r = k$, and the branches of B_k are $b_1, b_2, \ldots, b_k$. If the theorem is not true for $r = k + 1$, b_{k+1} is contained in a path-in-tree which has a terminal vertex v_x that is one of the non-terminal vertices of every path-in-tree containing $b_1, b_2, \ldots, b_k$. By hypothesis, e-orienting any branch, b_1, fixes the p- and e-orientations of all branches. The p- and e-orientation of the branches, b_{k+1} and b_i, $b_{i+1}, \ldots, b_k$ are pointing toward or away from v_x for $b_1 \neq b_{k+1}$. This implies a contradiction of hypothesis for the p- and e-orientation of the branches b_{k+1} and b_j, $j = i, i + 1, \ldots, k$, do not coincide for any path-in-tree.

The necessity follows from the fact that a path-in-tree which contains the branches $b_1, b_2, \ldots, b_r$, has subpaths containing any pair of branches b_i and b_j.

Corollary 2.3.4: Consider any v-vertex part and tree T. With the same hypothesis and $r = v - 1$, T is a path whose branches have coincident p- and e-orientations.

Proof: Direct consequence of Theorem 2.3.4.

Theorem 2.3.5: Consider $\mathcal{G}_e$ diagonal with non-negative entries. If and only if $\mathcal{G}_T = [G_{ij}]$ contains a principal submatrix of order r with positive entries, then r branches of T define f-segs with common elements and are contained in a path-in-tree with coincident p- and e-orientations.

Proof: Sufficiency: Any entry in the submatrix of hypothesis is of the form $\sum_p s_{ip} s_{jp} g_{pp}$. Terms in this sum correspond to elements common to S_i and S_j, $i \neq j$, and therefore, all non-zero products $s_{ip} s_{jp}$ are +1 or all are -1. Since the sum is positive and $g_{pp} \geq 0$, $s_{ip} s_{jp} = +1$ for at least one p. In addition, Corollary 2.3.2 implies that the branches b_i and b_j

defining S_i and S_j respectively, are contained in some path-in-tree with coincident p- and e-orientations. This statement is true for all pairs of branches associated with off-diagonal entries of the submatrix of hypothesis, i.e. b_i and b_j for $1 \leq i < j \leq 2, 3, \ldots, r$. Therefore by Theorem 2,3.4, the conclusion follows.

Necessity: All entries in $\mathcal{G}_T$ associated with the r branches of hypothesis have the form $\sum_p s_{ip} s_{jp} g_{pp}$. Since all f-segs defined by the branches have common elements, each entry is non-zero. By hypothesis and Corollary 2.3.2, all non-zero $s_{ip} s_{jp} = +1$. Thus the conclusion follows since $g_{pp} \geq 0$.

Corollary 2.3.5: Consider $\mathcal{G}_e$ diagonal with non-negative entries. If and only if for $\mathcal{G}_T \doteq [G_{ij}]$, $G_{ij} > 0$, then

(1) T is a path

(2) all branches of T define f-segs with common elements and have coincident p- and e-orientations.

Proof: This follows from Theorem 2.3.5.

Theorem 2.3.6: If and only if for some path-in-tree the p- and e-orientations of b_i and b_j are not coincident, then the S_i and S_j orientations of all elements of C are opposite.

Proof: This is the contrapositive form of Theorem 2.3.2.

Corollary 2.3.6: With the same hypothesis,

$$s_{ip} s_{jp} = -1$$

where p corresponds to elements of C and $i \neq j$.

Proof: Direct consequence of Theorem 2.2.2 and Theorem 2.3.6.

Theorem 2.3.7: If b_i and b_j do not have coincident p- and e-orientations in some P_T, then b_i and b_j do not have coincident p- and e-orientations for every P_T.

Proof: Similar to Theorem 2.3.3, for P_T contains a subpath P_{T1} which has b_i and b_j as end elements. Since every path-in-tree which contains b_i and b_j contains P_{T1}, the conclusion follows.

Theorem 2.3.8: If and only if each pair b_i and b_j of B_r do not have coincident p- and e-orientations for some path-in-tree, then there exist paths-in-tree, p_i, such that

(1) each p_i contains one and only one branch of B_r

(2) some one vertex v_x is a terminal vertex of each p_i

(3) the e-orientation of the b_i are all toward or all away from v_x.

Proof: Sufficiency: It is only necessary to show that (1) and (2) of the conclusion are satisfied for then the hypothesis fixes the e-orientations of the conclusion.

By induction on r. For $r = 2$, the hypothesis and conclusion are the same. Therefore consider $r = 3$ where the branches of B_r are b_1, b_2 and b_3. Suppose the branches b_1, b_2 and b_3 do not satisfy (1) and (2) of the conclusion. The only situation that can exist is that b_1, b_2, and b_3 are contained in a path-in-tree. By hypothesis, arbitrarily e-orienting any branch, b_1, fixes the p- and e-orientation of b_2 and b_3. Since b_2 and b_3 have the same p- and therefore e-orientations, the hypothesis is contradicted. Therefore the conclusion follows for $r = 3$.

Suppose the theorem is true for $r = k$ and that the branches of B_k are $b_1, b_2, \ldots, b_k$. If the theorem is not true for $r = k + 1$, b_{k+1} is contained in one of the p_i. By hypothesis, e-orienting any branch, b_1, fixes the p- and e-orientations of all branches. The p- and e-orientation of b_i and b_{k+1} are the same for b_1 not b_i or b_{k+1}. This fact contradicts the hypothesis. Hence the conclusion follows.

Necessity: By hypothesis, any pair of branches b_i and b_j is contained in a path-in-tree containing v_x. The e-orientation of b_i and b_j are both toward or both away from v_x. Therefore from the method of p-orienting the elements of a path the conclusion follows.

Corollary 2.3.8: Consider any v-vertex part and tree T. With the same hypothesis and r = v - 1, T is a Lagrangian tree whose branches are e-oriented toward or away from the common vertex.

Proof: Direct consequence of Theorem 2.3.8.

Theorem 2.3.9: Consider $\mathcal{G}_e$ diagonal with non-negative entries. If and only if $\mathcal{G}_T = [G_{ij}]$ contains a principal submatrix or order r with negative entries, then r branches of T define f-segs with common elements and satisfy (1), (2) and (3) of Theorem 2.3.8.

Proof: Sufficiency: Similar to Theorem 2.3.5 for any entry of the submatrix of hypothesis, $\sum_p s_{ip} s_{jp} g_{pp}$, has terms which correspond to elements common to S_i and S_j, $i \neq j$. By hypothesis and the fact that all non-zero products $s_{ip} s_{jp}$ equal +1 or all equal -1, $s_{ip} s_{jp} = -1$ for at least one p. Corollary 2.3.6 implies that the branches b_i and b_j defining S_i and S_j respectively, are contained in some path-in-tree with p- and e-orientations that do not coincide. This is true for all pairs of branches b_i and b_j where $1 \leq i < j \leq 2, 3, \ldots, r$, i.e. all branches associated with off-diagonal entries of the submatrix of hypothesis. Therefore by Theorem 2.3.8, the conclusion follows.

Necessity: All entries in $\mathcal{G}_T$ associated with the r branches of hypothesis, $\sum_p s_{ip} s_{jp} g_{pp}$, are non-zero since the f-segs defined by these branches have common elements. By Corollary 2.3.6 and hypothesis all non-zero $s_{ip} s_{jp} = -1$. Since $g_{pp} \geq 0$, the conclusion follows.

Corollary 2.3.9: Consider $\mathcal{G}_e$ diagonal with non-negative entries. If and only if for $\mathcal{G}_T = [G_{ij}]$, $G_{ij} < 0$, $i \neq j$, then T is a Lagrangian tree whose branches define f-segs with common elements and are e-oriented toward or away from the common vertex.

Proof: This follows from Theorem 2.3.9.

III. TREE TRANSFORMATION MATRIX

3.1 Basic Properties

A v-vertex connected graph, a part, is the union of a tree, T, and the complement of T, a cotree, G. Any part will be designated as $P = G \cup T$ or $P_i = G \cup T_i$ when reference is made to a specific tree.

Definition 3.1: Consider a v-vertex part $P = T_i \cup T_j$ where T_i and T_j are any two v-vertex trees, and consider T_j the tree and T_i the cotree of P. Let the corresponding f-seg matrix be $[\mathcal{U}\ \mathcal{S}_{ij}]$ where the columns of $\mathcal{U}$ correspond to T_j and the columns of $\mathcal{S}_{ij}$ correspond to T_i. The submatrix $\mathcal{S}_{ij}$ is called the tree transformation matrix from T_i to T_j.

For any part $P = G \cup T$, let $\mathcal{A} = [\mathcal{A}_T\ \mathcal{A}_G]$ be the incidence matrix where the columns of $\mathcal{A}_T$ correspond to the tree T and the columns of $\mathcal{A}_G$ correspond to the cotree of P.

Lemma 3.1.1: For $P = T_i \cup T_j$,

$$\mathcal{S}_{ij} = \mathcal{A}_{Tj}^{-1}\mathcal{A}_{Ti}$$

Proof: Since the f-seg matrix $\mathcal{S}_f$ and the incidence matrix satisfy the relation [12]:

$$\mathcal{S}_f = [\mathcal{U}\ \mathcal{S}] = \mathcal{A}_T^{-1}[\mathcal{A}_T\ \mathcal{A}_G] = \mathcal{A}_T^{-1}\mathcal{A},$$

the conclusion follows.

Lemma 3.1.2: The tree transformation matrix, $\mathcal{S}_{ij}$, is non-singular, and

$$\det \mathcal{S}_{ij} = \pm 1.$$

Proof: Since the columns of $\mathcal{S}_{ij}$ correspond to a tree, $\mathcal{S}_{ij}^{-1}$ exists [14].

For any tree T, $\mathcal{a}_T$ is square, therefore by Lemma 3.1

$$\det \mathcal{S}_{ij} = \det \mathcal{a}_{Tj}^{-1} \det \mathcal{a}_{Ti}.$$

The conclusion follows since det $\mathcal{a}_T = \pm 1$ [14].

Theorem 3.1.1: For any part $P_i = GUT_i$ and $P_j = GUT_j$ with f-seg matrices $[\mathcal{U}\ \mathcal{S}_i]$ and $[\mathcal{U}\ \mathcal{S}_j]$ respectively,

$$\mathcal{S}_j = \mathcal{S}_{ij}\,\mathcal{S}_i$$

Proof: By Lemma 3.1.1, it must be shown that

$$\mathcal{a}_{Tj}^{-1}\mathcal{a}_{Gj} = (\mathcal{a}_{Tj}^{-1}\mathcal{a}_{Ti})(\mathcal{a}_{Ti}^{-1}\mathcal{a}_{Gi})$$

Employing the associative law,

$$\mathcal{a}_{Tj}^{-1}\mathcal{a}_{Gj} = \mathcal{a}_{Tj}^{-1}\mathcal{a}_{Gi}$$

which reduces to

$$\mathcal{a}_{Gj} = \mathcal{a}_{Gi}$$

That this is the case follows from the fact that the cotree of P_i and P_j are identical. Since the above process is reversible, the conclusion follows.

Corollary 3.1.1.1: For $\mathcal{G}_{Ti} = \mathcal{S}_i\,\mathcal{G}_e\,\mathcal{S}_i'$ and $\mathcal{G}_{Tj} = \mathcal{S}_j\,\mathcal{G}_e\,\mathcal{S}_j'$, $\mathcal{G}_{Tj} = \mathcal{S}_{ij}\,\mathcal{G}_{Ti}\,\mathcal{S}_{ij}'$.

Proof: By hypothesis

$$\mathcal{G}_{Tj} = \mathcal{S}_{ij}(\mathcal{S}_i \mathcal{G}_e \mathcal{S}_i')\mathcal{S}_{ij}'$$

therefore

$$\mathcal{G}_{Tj} = (\mathcal{S}_{ij}\ \mathcal{S}_i)\mathcal{G}_e(\mathcal{S}_{ij}\ \mathcal{S}_i)' .$$

Hence, by Theorem 3.1.1,

$$\mathcal{G}_{Tj} = \mathcal{S}_j \mathcal{G}_e \mathcal{S}_j'$$

Corollary 3.1.1.2: $\det \mathcal{G}_{Tj} = \det \mathcal{G}_{Ti}$.

Proof: Since

$$\det \mathcal{G}_{Tj} = \det \mathcal{S}_{ij} \cdot \det \mathcal{G}_{Ti} \cdot \det \mathcal{S}_{ij}'$$

the conclusion follows from Lemma 3.1.2.

Theorem 3.1.2: The branches of any tree T_i can be e-oriented such that the tree transformation matrix from a Lagrangian tree with all elements oriented from or toward the common vertex to T_i, $\mathcal{S}_{Li}$, has non-negative (non-positive) entries.

Proof: Let $P = T_L \cup T_i$ where T_i is the tree and T_L is the cotree of P. Consider any f-seg S_i defined by a branch b_i of T_i. Suppose the common vertex of the Lagrangian tree of hypothesis, T_L, is an X- (or NX-) vertex of S_i. Then by hypothesis all X-NX elements of S_i, branches of T_L, have e-orientation either (1) from the X- to NX-vertex sets, or (2) from the NX- to X-vertex sets. By e-orienting b_i from the X- to NX-vertex sets, the signs of all non-zero entries in the i-row of $\mathcal{S}_{Li}$ are fixed. For case (1) all non-zero entries are +1, and for

case (2) all non-zero entries are -1. Since this is true for each f-seg, the conclusion follows.

Theorem 3.1.3: The tree transformation matrix from a Lagrangian tree to any other tree T_i, $\mathcal{S}_{Li}$, can be formed so that submatrices obtained by deleting rows 1, 2, . . . , i and columns 1, 2, . . . , i, i = 1, 2, . . . , v - 2, are nonsingular.

Proof: Consider $P = T_L \cup T_i$ where T_i is the tree and T_L is the co-tree of P. Since any tree T_i has at least two end elements and all the elements of T_L are end elements, only one branch b_1 of T_i and only one chord c_1, branch of T_L, are incident to one end vertex of T_i. Let the first row of the f-seg matrix of P and therefore $\mathcal{S}_{Li}$ correspond to b_1. Since the complement of an end element of a v-vertex tree is a tree of v - 1 vertices [13], the complement of b_1 in T_i is a tree on v - 1 vertices and the complement of c_1 in T_L is a Lagrangian tree on v - 1 vertices. Therefore the complement of $b_1 \cup C_1$ in P, P_1, is the union of a Lagrangian tree and some other tree on v - 1 vertices. Form the second row of the f-seg matrix of P by applying the argument used to form the first row on the subgraph P_1. The remaining rows of the f-seg matrix of P are to be formed by applying the complementing procedure to P_1 and the resulting subgraphs of P_1. By the above construction process, deleting rows 1, 2, . . . , i and columns 1, 2, . . . , i, i = 1, 2, . . . , v - 2, results in a matrix which is a tree transformation matrix from a Lagrangian tree to some tree of v - i vertices. Therefore this matrix is nonsingular by Lemma 3.1.2.

3.2 Tree Forms

Definition 3.2: Consider a v-vertex tree $T_i = T_{i\alpha}$ with vertices labeled v_i, i = 1, 2, . . . , v. Let $T_{i\beta}$ be any tree formed from $T_{i\alpha}$ by relabeling its vertices. The two trees, $T_{i\alpha}$ and $T_{i\beta}$, are said to be of the same form - - - form -i.

Lemma 3.2: Consider $P = T_{i\alpha} \cup T_{i\beta}$ where the branches of $T_{i\alpha}$ and $T_{i\beta}$ are oriented from or toward the corresponding common vertices. The tree transformation matrix from $T_{i\alpha}$ to $T_{i\beta}$, $\mathcal{S}_{i\alpha\beta}$, is either

$$(\pm 1)\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ & & & & & & \\ & & & & & & \\ & & & \cdots & \cdots & & \\ & & & & & & \\ & & & & & & \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix} \quad \text{or } (\pm 1)\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 1 & . & . & . & 0 & 0 & 0 \\ & & & \cdots & \cdots & & & & \\ -1 & -1 & -1 & . & . & . & -1 & -1 & -1 \\ & & & \cdots & \cdots & & & & \\ 0 & 0 & 0 & . & . & . & 1 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & . & . & . & 0 & 0 & 1 \end{bmatrix}$$

Proof: This follows by forming the tree transformation matrix of hypothesis for the case that the common vertex of $T_{i\alpha}$ and the common vertex of $T_{i\beta}$ coincide, and the case that they do not coincide.

Theorem 3.2.1: For v-vertex trees $T_{i\alpha}$, $T_{i\beta}$ and T_j $(= T_{j\alpha})$, there is a v-vertex tree of the same form as T_j, $T_{j\beta}$, such that

$$\mathcal{S}_{i\alpha j\alpha} = \mathcal{S}_{i\beta j\beta}$$

where the rows correspond to the same branches.

Proof: Let P_1 and P_2 be $T_i \cup T_j$. The tree transformation matrices of P_1 and P_2 are identical if they are formed using the same sequence of branches. Label T_i of P_1 and T_i of P_2 so that $T_{i\alpha}$ and $T_{i\beta}$ of hypothesis result. The conclusion follows if $T_{j\beta}$ is the complement of $T_{i\beta}$.

Theorem 3.2.2: For $P_i = T_{i\alpha} U T_{i\beta}$ and $P_j = P_{i\alpha} U T_j$,

$$\mathcal{S}_{i\alpha j} = \mathcal{S}_{i\beta i}\, \mathcal{S}_{i\alpha i\beta}$$

Proof: This follows from Theorem 3.1.1 for the case of the parts of hypothesis.

IV. SYNTHESIS OF R-GRAPHS

4.1 Introduction

The necessary and sufficient conditions on a given matrix such that it is realizable as an R-graph are determined. This result is based on the properties of the branch matrix of a part consisting of the union of a complete graph and Lagrangian tree. The complete graph has the character of a "canonical configuration" discussed by Darlington [16] which is contrary to some procedures of R-graph synthesis [3]. The conditions associated with a Lagrangian tree are extended to trees of the same form and results are given for trees of different forms. It is shown that trees of the same form impose the same realizability conditions on a given matrix.

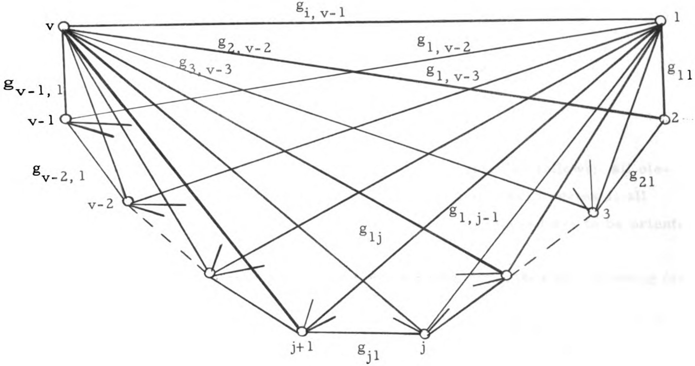
Definition 4.1.1: A complete graph, G_v, is a v-vertex graph such that there is one and only one element between every pair of vertices--Figure 2a.

Definition 4.1.2: The canonical form of a square matrix $\mathcal{G} = [G_{ij}]$ is the matrix $\mathcal{G}_c = [G_{cij}]$ which contains entries $G_{cii} = G_{ii}$ and $G_{cij} = -|Gij|$ for $i \neq j$.

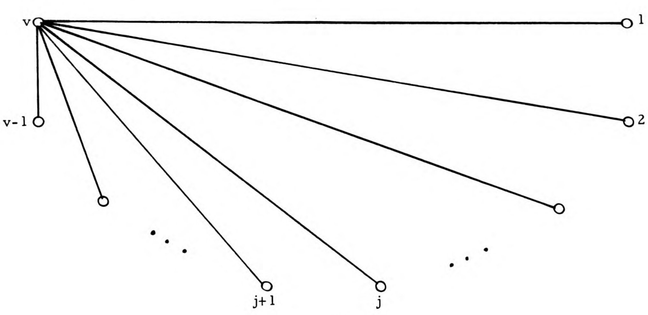
4.2 Some properties of $\mathcal{G}_L$

Theorem 4.2.1: Let $[\,\mathcal{U}\ \mathcal{S}\,]$ be the f-seg matrix for $P_L = G_v U T_L$, where T_L is defined by Figure 2b. For $\mathcal{S}\,\mathcal{G}_e(g_{ij})\,\mathcal{S}' = [G_{ij}(g_{ij})]$, $\mathcal{G}_e$ diagonal and $g_{ij} \geq 0$:

(1) $G_{ij} \leq 0$, $i \neq j$, for branches of T_L oriented from (or toward) the common vertex; all other G_{ij} sign patterns are deducible from alterations in orientation of branches of T_L.



(a)



(b)

Fig. 2--(a) V-vertex complete graph. (b) Lagrangian tree, T_L.

(2) $|G_{ji}| = |G_{ij}| = g_{i,j-i}$ for $j > i$

$$G_{ii} = \sum_{j=1}^{v-i} g_{ij} + g_{1,i-1} + g_{2,i-2} + \cdots + g_{i-1,1}$$

for $i = 1, 2, \ldots, v-1$.

<u>Proof:</u> The elements of P_L are to be oriented as follows: all elements incident to vertex No. 1 are to be oriented toward vertex 1; all elements incident to vertex No. 2 not already considered are to be oriented toward vertex 2; etc.

For this arrangement, the f-seg matrix, $\mathcal{S}_{f}$, has the following form:

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ \cdot \\ \cdot \\ \cdot \\ v-1 \end{array} \left[\begin{array}{cccccccccccccccccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & & & & & & & & & & & & & & & & \\ & \cdots\cdots & & & & & & & \cdots\cdots & & & & & & & \cdots\cdots & & & \\ & & & & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & \cdots & 1 \end{array}\right]$$

where the columns of $\mathcal{S}$ correspond to elements 11, 12, . . . , 1 v-1; 21, 22, . . . , 2v-2; . . . v-11. Since $\mathcal{G}_e$ is diagonal, $\mathcal{S}\mathcal{G}_e\mathcal{S}'$ is symmetric and has entries shown by ($g_{ij} \equiv 0$ for $i \leq 0$ or $j \leq 0$)

$$G_{ij} = -g_{i,j-i}, \quad j > i$$
$$G_{ii} = \sum_{j=1}^{v-i} g_{ij} + g_{1,i-1} + g_{2,i-2} + \cdots + g_{i-1,1} \qquad (4.2.1)$$

for $i = 1, 2, \ldots, v-1$. This result proves (2).

For the assumed orientation pattern all off-diagonal entries are non-positive. By Theorem 2.2.4, the signs of these entries are affected only by changing branch orientations by way of cross-sign changes of $\mathcal{S}\mathcal{G}_e\mathcal{S}'$ which proves (1).

Theorem 4.2.2: Let $[\,\mathcal{U}\;\mathcal{S}\,]$ be the f-seg matrix for $\mathbf{P}_L = G_v U T_L$ of Figure 2. For the matrix $\mathcal{S}\,\mathcal{G}_e(g_{ij})\,\mathcal{S}' = [G_{ij}(g_{ij})]$ written as

$$\chi_G = \mathcal{S}_s\,\chi_g$$

where $\chi_G' = [G_{11}\,G_{12}\ldots G_{1,v-1}\,G_{22}\,G_{23}\ldots G_{2,v-1}\,G_{33}\ldots G_{3,v-1}\ldots G_{v-1,v-1}]$ and $\chi_g' = [g_{11}g_{12}\ldots g_{1,v-1}\,g_{21}\,g_{22}\ldots g_{2,v-2}\,g_{3,1}\ldots g_{3,v-3}\ldots g_{v-1,1}]$, $\mathcal{S}_s$ is non-singular.

Proof: Orient the elements of P_L as in the proof of Theorem 4.2.1. The form of the entries in $\mathcal{S}\,\mathcal{G}_e(g_{ij})\,\mathcal{S}'$ is given in (4.2.1) and therefore for this orientation pattern the detailed form of $\mathcal{S}_s$ is given in (4.2.2).

$$\begin{bmatrix} \mathcal{A}_{11} & & & & & & & \\ \mathcal{A}_{21} & \mathcal{A}_{22} & & & & & \bigcirc & \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \mathcal{A}_{i1} & \mathcal{A}_{i2} & \mathcal{A}_{i3} & \cdots & \mathcal{A}_{ii} & & & \\ \cdot & \cdot & \cdot & & & & & \\ \mathcal{A}_{v-1,1} & \mathcal{A}_{v-1,2} & \mathcal{A}_{v-1,3} & \cdots & \mathcal{A}_{v-1,i} & \cdots & & \mathcal{A}_{v-1,v-1} \end{bmatrix} \tag{4.2.2}$$

The square submatrices,

$$\mathcal{A}_{ii} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}, \quad i \neq v-1,$$

are of order v-i and $\mathcal{A}_{v-1,\,v-1} = [1]$. The submatrices $\mathcal{A}_{ij}$, $i > j$, are of the order v-i x v-j and contain all zero entries except for +1 in the (1, i-j) position.

By the Laplace Expansion [17] with the first v-1 columns, the determinant of (4.2.2) is given by the product of the determinants of the square diagonal submatrices each of which is non-singular. Therefore, for the assumed orientation pattern, $\mathcal{S}_s$ is non-singular.

The form of $\mathcal{S}\mathcal{G}_e(g_{ij})\mathcal{S}'$ for any other orientation pattern of elements of P_L can be obtained through the use of cross-sign changes. The operation of cross-sign change changes only the sign of off-diagonal entries in $\mathcal{S}\mathcal{G}_e(g_{ij})\mathcal{S}'$. Therefore if $\mathcal{X}_{G1}$ corresponds to $\mathcal{S}\mathcal{G}_e(g_{ij})\mathcal{S}'$ after any number of cross-sign changes, then

$$\mathcal{X}_{G1} = \mathcal{J}\,\mathcal{X}_G$$

where $\mathcal{J}$ is diagonal with +1 and -1 entries. The transformation matrix $\mathcal{J}$ [17] is necessarily non-singular. Since $\mathcal{X}_G = \mathcal{S}_s\,\mathcal{X}_g$, $\mathcal{X}_{G1} = \mathcal{J}\mathcal{S}_s\,\mathcal{X}_g$. The matrix $\mathcal{J}\mathcal{S}_s$ is non-singular since it is the product of two non-singular matrices which proves the theorem.

<u>Corollary 4.2.2</u>: Let $[G'_{ij}(g_{ij})]$ be a matrix obtained from $\mathcal{S}\mathcal{G}_e(g_{ij})\mathcal{S}' = [G_{ij}(g_{ij})]$ by interchanging rows i and j and columns i and j. Writing $[G'_{ij}(g_{ij})]$ as

$$\mathcal{X}_{G1} = \mathcal{S}_{s1}\,\mathcal{X}_g$$

where $\mathcal{X}'_{G1} = [G'_{11}G'_{12}\cdots G'_{1,v-1}G'_{22}G'_{23}\cdots G'_{2,v-1}G'_{33}\cdots G'_{3,v-1}\cdots G'_{v-1,v-1}]$ and $\mathcal{X}'_g = [g_{11}g_{12}\cdots g_{1,v-1}g_{21}g_{22}\cdots g_{2,v-2}g_{31}\cdots g_{3,v-3}\cdots g_{v-1,1}]$, $\mathcal{S}_{s1}$ is non-singular.

<u>Proof:</u> The operation of interchanging rows i and j and columns i and j of $[G_{ij}]$ interchanges G_{pi} and G_{pj} for p = 1, 2, . . . , i-1, i+1, . . . , j-1, j+1, . . . , v-1, j > i, and G_{ii} and G_{jj}. All other entries of $[G_{ij}]$

remain fixed. Thus $\mathcal{X}_{G1}$ can be obtained from $\mathcal{X}_{G}$, defined in Theorem 4.2.2, by interchanging certain rows, that is

$$\mathcal{X}_{G1} = \mathcal{J} \mathcal{X}_{G}$$

where $\mathcal{J}$ is a non-singular transformation [17]. By Theorem 4.2.2, $\mathcal{X}_{G1} = \mathcal{J} \mathcal{S}_s \mathcal{X}_g$ and $\mathcal{S}_{s1} = \mathcal{J} \mathcal{S}_s$ is non-singular since it is the product of non-singular matrices.

4.3 Necessary and Sufficient Conditions for R-Graph Synthesis

In the following theorems, it is assumed that $\mathcal{G}_L$, of order v-1, is the coefficient matrix of the branch equations, branch matrix, for $P_L = G_v U T_L$ as shown in Figure 2 for diagonal $\mathcal{G}_e$. Branch matrices for other parts $P_i = G_v U T_i$ are designated as $\mathcal{G}_i$ and for the parts $P_{i\alpha} = G_v U T_{i\alpha}$ and $P_{i\beta} = G_v U T_{i\beta}$, as $\mathcal{G}_{i\alpha}$ and $\mathcal{G}_{i\beta}$ respectively.

Theorem 4.3.1: Consider any $\mathcal{G}_L = [G_{ij}]_{v-1}$ and the branches of T_L oriented from (or toward) the common vertex. If and only if

(1) $G_{ij} \leq 0$ for $i \neq j$

(2) $G_{ii} \geq 0$

$\sum_{j=1}^{v-1} G_{ij} \geq 0$ for $i = 1, 2, \ldots, v-1$,

then $g_{ij} \geq 0$.

Proof: Sufficiency: The $\mathcal{G}_L(G_{ij})$ of hypothesis is in terms of G_{ij} and $\mathcal{S} \mathcal{G}_e(g_{ij}) \mathcal{S}'$ of Theorem 4.2.1 is in terms of g_{ij}. Furthermore (1) of Theorem 4.2.1 permits by cross-sign changes altering $\mathcal{S}\mathcal{G}_e(g_{ij}) \times \mathcal{S}'$ into $\mathcal{S}_L \mathcal{G}_e(g_{ij}) \mathcal{S}_L'$ in correlation with T_L oriented as the T_L to which $\mathcal{G}_L$ corresponds. Since $\mathcal{G}_L$ and $\mathcal{S}_L \mathcal{G}_e(g_{ij}) \mathcal{S}_L'$ correspond to identical T_L (orientation and form), these matrices are equal. Therefore

$$\mathscr{G}_L(G_{ij}) = \mathscr{S}_L \, \mathscr{G}_e(g_{ij}) \, \mathscr{S}_L'.$$

This last equation can be written as

$$\mathcal{X}_G = \mathscr{S}_s \, \mathcal{X}_g$$

where $\mathcal{X}_G' = [G_{11}G_{12}\ldots G_{1,v-1}G_{22}G_{23}\cdots G_{2,v-1}G_{33}\cdots G_{3,v-1}\cdots G_{v-1,v-1}]$ and $\mathcal{X}_g' = [g_{11}g_{12}\cdots g_{1,v-1}g_{21}g_{22}\cdots g_{2,v-2}g_{31}\cdots g_{3,v-3}\cdots g_{v-1,1}]$. By Theorem 4.2.2, $\mathscr{S}_s$ is non-singular and therefore the solution is:

$$\begin{bmatrix} g_{11} \\ g_{12} \\ \cdot \\ \cdot \\ \cdot \\ g_{1,v-2} \\ g_{1,v-1} \\ g_{21} \\ g_{22} \\ \cdot \\ \cdot \\ \cdot \\ g_{2,v-3} \\ g_{2,v-2} \\ \cdot \\ \cdot \\ \cdot \\ g_{v-1,1} \end{bmatrix} = \begin{bmatrix} -G_{12} \\ -G_{13} \\ \cdot \\ \cdot \\ \cdot \\ -G_{1,v-1} \\ \sum_{j=1}^{v-1} G_{1j} \\ -G_{23} \\ -G_{24} \\ \cdot \\ \cdot \\ \cdot \\ -G_{2,v-1} \\ \sum_{j=1}^{v-1} G_{2j} \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=1}^{v-1} G_{v-1,j} \end{bmatrix} \qquad (4.3.1)$$

That all entries on the right side of (4.3.1) which are not sums are non-negative follows from (1) of hypothesis. All entries which are sums are non-negative by (2) of hypothesis.

Necessity: Since the equation

$$\chi_G = \mathcal{S}_s \chi_g$$

as defined in the sufficiency part of the proof is independent of the values of G_{ij} and g_{ij}, (4.3.1) applies to this proof. From (4.3.1) for $g_{ij} \geq 0$, (1) and (2) of hypothesis follow.

Corollary 4.3.1: Consider any $\mathcal{G}_L = [G_{ij}]_{v-1}$. If and only if

(1) $\mathcal{G}_L$ can be chanted to the canonical form, $\mathcal{G}_c = [G_{cij}]$, by a finite number of cross-sign changes,

(2) $G_{ii} \geq 0$

$$\sum_{j=1}^{v-1} G_{cij} \geq 0 \qquad \text{for } i = 1, 2, \ldots, v-1.$$

then $g_{ij} \geq 0$.

Proof: Sufficiency: By (1) of hypothesis change $\mathcal{G}_L$ to canonical form $\mathcal{G}_c$. By Theorem 2.2.4 and (1) of Theorem 4.2.1, $\mathcal{G}_c$ corresponds to T_L with all branches oriented from (or toward) the common vertex. Therefore Theorem 4.3.1 implies the conclusion.

Necessity: Change orientation of the branches of T_L and apply corresponding finite set of cross-sign changes to $\mathcal{G}_L$ until all branches are oriented from common vertex. This process produces a matrix satisfying Theorem 4.3.1. Therefore hypothesis of corollary follows.

Definition 4.3.1: A symmetric matrix $\mathcal{G} = [G_{ij}]_n$ is an R-matrix if

(1) $\mathcal{G}$ can be changed to the canonical form, $\mathcal{G}_c = [G_{cij}]$, by a finite number of cross-sign changes,

(2) $G_{ii} \geq 0$

for i = 1, 2, . . . , n.

$$\sum_{j=1}^{v-1} G_{cij} \geq 0$$

<u>Theorem 4.3.2:</u> Consider any $\mathcal{G}_{L\alpha} = [G_{ij}]_{v-1}$ and the branches of $T_{L\alpha}$ oriented from (or toward) the common vertex. If and only if

(1) $G_{ij} \leq 0$ for $i \neq j$

(2) $G_{ii} \geq 0$

for i = 1, 2, . . . , v-1,

$$\sum_{j=1}^{v-1} G_{ij} \geq 0$$

then $g_{ij} \geq 0$.

<u>Proof:</u> <u>Sufficiency:</u> Let $\mathcal{S}_{L\alpha\beta}$ be the tree transformation matrix from $T_{L\alpha}$ of hypothesis to $T_{L\beta}$ of Figure 2b with branches oriented from or toward the common vertex. The branch matrix for the part of Figure 1 is $\mathcal{S}_{L\alpha\beta}\,\mathcal{G}_{L\alpha}\,\mathcal{S}'_{L\alpha\beta}$ by Corollary 3.1.1.1. The possible forms of $\mathcal{S}_{L\alpha\beta}$ are given in Lemma 3.2. If $\mathcal{S}_{L\alpha\beta} = \pm\,\mathcal{U}$, the theorem follows from Theorem 4.3.1. For all other cases,

$$\mathcal{S}_{L\alpha\beta}\,\mathcal{G}_{L\alpha}\,\mathcal{S}'_{L\alpha\beta} =$$

$$\begin{bmatrix}
G_{11} & G_{12} & \cdots & G_{1,m-1} & -\sum_{j=1}^{v-1} G_{1j} & G_{1,m+1} & \cdots & G_{1,v-1} \\
G_{12} & G_{22} & \cdots & G_{2,m-1} & -\sum_{j=1}^{v-1} G_{2j} & G_{2,m+1} & \cdots & G_{2,v-1} \\
 & & & \cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot\;\cdot & & \\
-\sum_{j=1}^{v-1} G_{1j} & -\sum_{j=1}^{v-1} G_{2j} & \cdots & -\sum_{j=1}^{v-1} G_{m-1,j} & \sum_{m=1}^{v-1}\sum_{j=1}^{v-1} G_{mj} & -\sum_{j=1}^{v-1} G_{j,m+1} & \cdots & -\sum_{j=1}^{v-1} G_{j,v-1} \\
 & & & \cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot\;\cdot & & \\
G_{1,v-1} & G_{2,v-1} & \cdots & G_{m-1,v-1} & -\sum_{j=1}^{v-1} G_{j,v-1} & G_{m+1,v-1} & \cdots & G_{v-1,v-1}
\end{bmatrix}$$

From (1) and (2) of hypothesis, off-diagonal entries of this last matrix are non-positive and diagonal entries are non-negative. The sum of the entries in any row $n \neq m$ is $-G_{nm}$ and in row $n = m$, is $\sum_{j=1}^{v-1} G_{mj}$ and by hypothesis, these terms are non-negative. Therefore by Theorem 4.3.1, the conclusion follows.

Necessity: Let $\mathcal{G}_{L\beta}$ be any branch matrix for the part of Figure 2 with the branches of $T_{L\beta}$ oriented from or toward the common vertex. By hypothesis, $\mathcal{G}_{L\beta}$ satisfies the properties of Theorem 4.3.1. By Corollary 3.1.1.1, $\mathcal{S}_{L\beta a} \mathcal{G}_{L\beta} \mathcal{S}'_{L\beta a}$ is the branch matrix for T_{La} of hypothesis. Since $\mathcal{S}_{La\beta}$ of Lemma 3.2 satisfied

$$\mathcal{S}_{La\beta} \mathcal{S}_{La\beta} = \mathcal{U}$$

and in general

$$\mathcal{S}^{-1}_{La\beta} = \mathcal{S}_{L\beta A}$$

it follows that

$$\mathcal{S}_{La\beta} = \mathcal{S}_{L\beta a}.$$

The conclusion follows then by the same argument as used above.

Corollary 4.3.2: Consider any $\mathcal{G}_{La}$. If and only if $\mathcal{G}_{La}$ is an R-matrix, then $g_{ij} \geq 0$.

Proof: Any possible $\mathcal{G}_{La}$ can be obtained from a canonical $\mathcal{G}_{La}$ by cross-sign changes which correlate with altering the orientations of a T_{La} which has all orientations from or toward the common vertex, to a T_{La} with some other pattern of orientation. Therefore, a set of cross-sign changes exists which alters $\mathcal{G}_{La}$ into a canonical form and also alters the orientations of T_{La} into orientation pattern of T_{La} of Theorem 4.3.2. Therefore from Theorem 4.3.2, the conclusion follows.

Theorem 4.3.3: Consider any $\mathcal{G}_i$ and $\mathcal{S}_{iLa}$, the tree transformation matrix from T_i to T_{La}. If and only if $\mathcal{S}_{iLa}\mathcal{G}_i\mathcal{S}'_{iLa}$ is an R-matrix, then $g_{ij} \geq 0$.

Proof: Since $\mathcal{G}_i$ is the branch matrix for P_i, Corollary 3.1.1.1 implies that $\mathcal{S}_{iLa}\mathcal{G}_i\mathcal{S}'_{iLa}$ is the branch matrix for P_{La}. Hence by Corollary 4.3.2 the conclusion follows.

Corollary 4.3.3: Consider any $\mathcal{G}_i = [G_{ij}]_{v-1}$ and $\mathcal{S}_{iLa}$, the tree transformation matrix from T_i to T_{La} with branches oriented from or toward the common vertex. If and only if, for $\mathcal{S}_{iLa}\mathcal{G}_i\mathcal{S}'_{iLa} = [f_{mn}(G_{ij})]_{v-1}$,

(1) $f_{mn}(G_{ij}) \leq 0$ for $m \neq n$

(2) $f_{mn}(G_{ij}) \geq 0$

for $m = 1, 2, \ldots, v-1$,

$$\sum_{n=1}^{v-1} f_{mn}(G_{ij}) \geq 0$$

then $g_{ij} \geq 0$.

Proof: The conclusion follows from Theorem 4.3.3.

Theorem 4.3.4: Consider any $\mathcal{G}_{La}$ and $\mathcal{S}_{Lai}$, the tree transformation matrix from T_{La} to T_i. If and only if $\mathcal{G}_{La}$ is an R-matrix, then all g_{ij} associated with $\mathcal{S}_{Lai}\mathcal{G}_{La}\mathcal{S}'_{Lai}$ are non-negative.

Proof: Since $\mathcal{G}_{La}$ is the branch matrix for P_{La}, Corollary 3.1.1.1 implies that $\mathcal{S}_{Lai}\mathcal{G}_{La}\mathcal{S}'_{Lai} = \mathcal{G}_i$ is the branch matrix for P_i. Since $\mathcal{S}_{Lai} = \mathcal{S}^{-1}_{iLa}$, Theorem 4.3.3 and hypothesis implies the conclusion.

Corollary 4.3.4: Consider any $\mathcal{G}_i$ and $\mathcal{G}_j$. If and only if all g_{ij} associated with $\mathcal{G}_i$ are non-negative, then all g_{ij} associated with $\mathcal{G}_j$ are non-negative.

Proof: By Theorem 4.3.3, $\mathcal{S}_{iL\alpha}\mathcal{G}_i\mathcal{S}'_{iL\alpha}$ is an R-matrix. By Corollary 3.1.1.1 and Theorem 3.2.2, $\mathcal{S}_{iL\alpha}\mathcal{G}_i\mathcal{S}'_{iL\alpha} = \mathcal{S}_{iL\alpha}(\mathcal{S}_{ij}\mathcal{G}_j\mathcal{S}'_{ij})\mathcal{S}'_{iL\alpha} = \mathcal{S}_{jL\alpha}\mathcal{G}_j\mathcal{S}'_{jL\alpha}$. Therefore the conclusion follows from Theorem 4.3.4.

Theorem 4.3.5: If $\mathcal{G}_{i\alpha} = [G^{\alpha}_{ij}]$ and $\mathcal{G}_{i\beta} = [G^{\beta}_{ij}]$ with rows corresponding to same branches, then $\mathcal{S}_{i\alpha j\alpha}\mathcal{G}_{i\alpha}\mathcal{S}_{i\alpha j\alpha} = [f_{mn}(G^{\alpha}_{ij})]$ and $\mathcal{S}_{i\beta j\beta}\mathcal{G}_{i\beta}\mathcal{S}'_{i\beta j\beta} = [f_{mn}(G^{\beta}_{ij})]$.

Proof: This follows from Theorem 3.2.1.

4.4 Discussion of Results

The sufficiency part of Corollary 4.3.1 is actually a synthesis procedure. A given matrix must satisfy (1) and (2) of Corollary 4.3.1, that is, be an R-matrix, for it to be realized as the R-graph P_L. To test a matrix, first simply form, if possible, the canonical form $\mathcal{G}_c$. The next step is to apply (2) to $\mathcal{G}_c$. If both conditions are satisfied, the matrix can be realized as P_L and the element values can be determined from (4.3.1). The branch orientations can be determined from Theorem 4.3.1 and Theorem 2.2.4.

By definition 4.1.2, $G_{cij} = -|G_{ij}|$ for $i \neq j$, the second relation in (2) of Corollary 4.3.1 has the following form.

$$2G_{ii} \geq \sum_{j=1}^{v-1}|G_{ij}|, \quad i = 1, 2, \ldots, v-1$$

This is the well-known condition of dominance discussed by Burington [2]. That the second relation in (2) of Corollary 4.3.1 is that of dominance stems from the fact that the branch equations are identical to the incidence equations for the Lagrangian tree. It is not necessary to consider a Lagrangian tree to determine the necessary and sufficient conditions as in Theorem 4.3.1. A procedure similar to that of Theorem 4.3.1 with a different tree form leads to different expressions [18]. In addition the method implied by Theorem 4.3.3 and Corollary 4.3.3 can be used to determine the necessary and sufficient conditions for synthesis. For example, the tree transformation matrix from T_P to T_L, where T_P is a path with the e- and p-orientations of all elements coincident and the elements of T_L are all oriented toward or away from the common vertex,

$$\mathcal{S}_{PL} = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 & 0 \\ -1 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & 1 & . & . & . & 0 & 0 \\ & & & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -1 & 1 \end{bmatrix}$$

Therefore, showing only the entries above and including the main diagonal,

$$\mathcal{S}_{PL} \mathcal{G}_P \mathcal{S}'_{PL} =$$

$$\begin{bmatrix} G_{11} & G_{12}-G_{11} & G_{13}-G_{12} & \cdots & G_{1,v-1}-G_{1,v-2} \\ & G_{22}+G_{11}-2G_{12} & G_{23}-G_{13}-(G_{22}-G_{12}) & \cdots & G_{2,v-1}-G_{1,v-1}-(G_{2,v-2}-G_{1,v-2}) \\ & & & \cdots & \\ & & & & G_{v-1,v-1}+G_{v-2,v-2}-2G_{v-2v-1} \end{bmatrix}$$

By applying (1) and (2) of Corollary 4.3.3, the following conditions are obtained.

(1) $G_{ij} \geq 0$ for all (i, j)

(2) $\Delta_i^{v-1} \geq 0$

$\Delta_i^{v-2} - \Delta_i^{v-1} \geq 0$

.

. for i = 1, 2, ..., v-1

.

$\Delta_i^{i} - \Delta_i^{i+1} \geq 0$

where $\Delta_i^j = G_{ij} - G_{i-1,j}$ for $i \neq 1$, $\Delta_1^j = G_{1j}$ and $\Delta_{v-1}^v \equiv 0$.
These conditions are necessary and sufficient for the realization of a given matrix as P_P with the elements of T_P oriented so that the e- and p-orientations coincide. For an arbitrary orientation pattern the matrix must satisfy (1) and (2) after a finite number of corss-sign changes.

If a matrix $\mathcal{G}$ is to be realized as a v-vertex complete graph by means of a specified tree form T_i, the necessary and sufficient conditions which must be satisfied are implied by Theorem 4.3.3. If $\mathcal{G}$ does not satisfy these conditions, it still may be realized in the required form. Since the row order has not been specified, different arrangements of the entries corresponding to different row orders of $\mathcal{S}$ must be considered. Interchanging row i and j and then column i and j of $\mathcal{G}$ corresponds to interchanging row i and j of $\mathcal{S}$. The conditions can then be applied to the new branch matrix. If all such matrices do not satisfy the conditions, then the given matrix cannot be realized in the required form.

If only the matrix is given, the necessary and sufficient conditions for realization corresponding to each different tree form can be determined and applied to the given matrix or a matrix obtained from the original by interchanging rows i and j and columns i and j. If the matrix satisfies the conditions associated with any tree form, it can be realized with a part consisting of the union of the particular tree form and a v-vertex complete graph.

By considering only the sign pattern of the given matrix, the form of a subgraph of the corresponding tree can be determined by the results of Section 2.3. For the case that all off-diagonal entries have the same sign, the complete tree can be determined: all positive and all negative correspond to path and Lagrangian trees respectively. By applying cross-sign changes to a given matrix, if the off-diagonal entries are not all positive or all not negative, then the matrix cannot be realized as a path or Lagrangian tree.

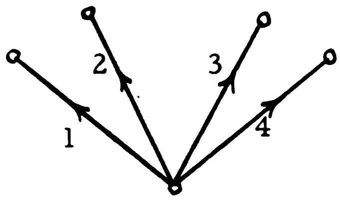
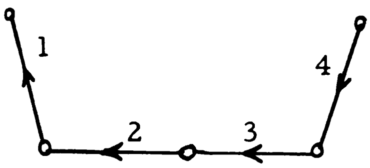
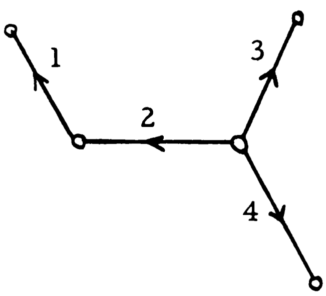
The necessary and sufficient conditions for realization with a complete graph containing five vertices in terms of the three different tree forms are given in Table 1. A necessary condition which is easily checked is the sign requirement indicated. Only one form is indicated since all other sign patterns can be obtained using cross-sign changes.

4.5 Additional Problems

Particular sign patterns of submatrices of a given matrix have been shown to fix the form of subgraphs of the corresponding tree. An extension of this result would be to determine the form of the tree (if it exists) corresponding to an arbitrary arrangement of signs in the given matrix.

The necessary and sufficient conditions for realization of a given matrix of order v-1, as a one part, v vertex graph have been determined. The point of view in the technique used to determine such conditions can be enlarged by proving Theorem 4.2.2 for an arbitrary tree. Necessary and sufficient conditions for graphs of more than v vertices and multiple part graphs are additional extensions of the above results.

Table 1.--Necessary and Sufficient Conditions for G_5.

Tree Form	Sign Pattern ($G_{ii} \geq 0$)	Conditions On $[G_{ij}]_4$
	$G_{ij} \leq 0$ $(i \neq j)$	$\sum_{j=1}^{4} G_{ij} \geq 0,$ $i = 1, 2, 3, 4.$
	$G_{ij} \geq 0$ $(i \neq j)$	1. $G_{11} \geq G_{12} \geq G_{13} \geq G_{14} \geq 0,$ $G_{44} \geq G_{34} \geq G_{24} \geq G_{14} \geq 0.$ 2. $G_{22} - G_{23} \geq G_{12} - G_{13} \geq 0,$ $G_{33} - G_{34} \geq G_{23} - G_{24} \geq$ $G_{13} - G_{14} \geq 0.$
	$G_{12} \geq 0$, and all other $G_{ij} \leq 0$, $i \neq j$	1. $G_{11} - G_{12} \geq 0,\ G_{13} - G_{23} \geq 0,$ $G_{14} - G_{24} \geq 0.$ 2. $G_{22} + G_{23} + G_{24} \geq G_{12} + G_{13} +$ $G_{14} \geq 0,\ G_{33} + G_{23} + G_{34} \geq 0,$ $G_{44} + G_{34} + G_{24} \geq 0.$

LIST OF REFERENCES

1. Cauer, W., Synthesis of Linear Communication Networks, Vol. I and II, McGraw-Hill Book Company, Inc., 173-178, 1958.

2. Burington, R. S., R-matrices and equivalent networks, J. Math. Phys., Vol. 16, No. 2, 85-103, 1937.

3. Cederbaum, I., Conditions for the impedance and admittance of N-ports without ideal transformers, Proc. I. E. E., Part C, Monograph 276R, 1958.

4. Cederbaum, I., On networks without ideal transformers, I. R. E. Trans. on Circuit Theory, Vol. CY-3, 179-182, 1956.

5. Talbot, A., Some fundamental properties of networks without mutual inductance, Proc. I. E. E., Part C, Monograph 118R, 1955.

6. Cederbaum, I., Matrices all of whose elements and subdeterminants are +1, -1, 0. J. Math. Phys. Vol. 36, 351-361, 1958.

7. Cederbaum, I., Application of matrix algebra to network theory, I. R. E. Trans. on Circuit Theory, Vol. CT-6, special suppl., 127-137, 1959.

8. Slepian, P. and Weinberg, L., Applications of paramount and dominant matrices, Proc. N. E. C., Vol. 14, 611-630, 1958.

9. Guillemin, E. A., On the analysis and synthesis of single element-kind networks, I. R. E. Trans. on Circuit Theory, Vol. CT-7, 303-312, 1960.

10. So, Hing-Cheong., Realization of loop-resistance matrices, Interim Technical Report No. 17, Contract No. DA-11-022-ORD-1983, U. of Ill., 1960.

11. Biorci, G. and Civalleri, P. P., On the synthesis of resistive N-port networks, I. R. E. Trans. on Circuit Theory, Vol. CT-8, 1961.

12. Reed, M. B., Foundation for Electric Network Theory, Prentice-Hall, Inc., 1961.

13. Reed, M. B., Class Notes for E. E. 861, 2, 3, E. E. Dept., Michigan State U., 1961.

14. Seshu, S., and Reed, M. B., Linear Graphs and Electrical Networks, Addison Wesley Company, Inc., 1961.

15. Reed, M. B., The seg: A new class of subgraphs, I. R. E. Trans. on Circuit Theory, Vol. CT-8, 1961.

16. Darlington, S., A survey of network realization techniques, I. R. E. Trans. on Circuit Theory, Vol. CT-2, 291, 1955.

17. Hohn, F. E., Elementary Matrix Algebra, The Macmillan Company, 1958.

18. Brown, D. P., and Tokad, Y., On the synthesis of R-networks, I. R. E. Trans. on Circuit Theory, Vol. CT-8, 1961.

APPENDIX

LIST OF SYMBOLS

Symbol	Description
v	Number of vertices of a graph
T, T_i, T_j	Trees of a graph
G	Cotree of a graph
P, P_i, P_j	Connected graphs (Parts)
b_i, b_j	Elements of a tree (branches)
c_i, c_k	Elements of a cotree (chords)
s_i, s_j	f-segs
C	Elements common to f-segs
$\mathcal{S}_f$	f-seg matrix
$\mathcal{U}$	Unit matrix
$\mathcal{S}$	Non unit submatrix of f-seg matrix
$\mathcal{G}_e$	Element matrix
$\mathcal{G}_T$	Branch matrix
$\mathcal{G}_c$	Canonical form of branch matrix
G_{ii}, G_{ij}	Entries of branch matrix
g_{pp}, g_{ij}	Entries of element matrix