INSTRUCTOR FACILITATION OF WHOLE-CLASS DISCUSSION IN TWO MATHEMATICS CLASSES FOR PRESERVICE TEACHERS

By

Sarah Helen Young

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ABSTRACT

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This study investigated how instructors engage preservice teachers (PSTs) in whole-class discussion in mathematics classes for PSTs. Participating in discussions of mathematical ideas by explaining mathematical ideas to others and responding to others' ideas and arguments might be beneficial for PSTs both to help them develop deeper knowledge of mathematical concepts and to give them experience participating in the types of discussions that teaching reforms call for them to use in their future classrooms. Unfortunately, although we know a great deal about whole-class discussion in K-12 classes, we know surprisingly little about discussions in undergraduate classes and even less about discussions in courses for future teachers.

Because of the paucity of relevant postsecondary research available, this study used research on K-12 classrooms that highlighted important factors for productive discussions as a guide to investigate whether teaching moves that are useful in K-12 contexts are used and useful in classes for PSTs. Specifically, this study focused on two teaching moves that happened in class before whole-class discussion (what activities preceded whole-class discussion and what kinds of discussion prompts instructors used to encourage discussion) and four teaching moves that occurred during whole-class discussion (how students were chosen to talk in discussion, how instructors responded to student thinking, how student thinking was connected to other student thinking, and how student thinking was explicitly connected to mathematical ideas in discussion). This study used videotaped observations of fractions lessons, collected as part of a larger research project, to investigate how two experienced instructors facilitated whole-class discussion in a mathematics class for preservice teachers. Summaries of the videotapes were examined for patterns in each practice within and between the two instructors.

The results suggest that many teaching practices that have been shown to be effective for promoting discussion in K-12 classrooms are used and useful in classes for PSTs. Before whole-class discussion began, instructors gave PSTs time to think about and work through discussion prompts (often with small groups of peers) before PSTs were expected to discuss with the whole class and instructors often chose discussion prompts that asked students to describe or explain processes rather than repeat facts from earlier in the lesson. During the whole-class discussion, instructors used many methods to choose PSTs to speak in discussion, often asked PSTs to explain their responses further, and made sure PSTs responses were audible and visible for all students. Instructors also led PSTs to correct errors in peer responses and to connect their ideas to peer responses while instructors were often responsible for explicitly connecting student thinking to mathematical ideas.

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Chapter 1: Introduction

Research on K-12 mathematics education makes clear that talking about mathematics can be an effective way to learn mathematics (e.g., Cengiz, Kline, & Grant, 2011; Franke et al., 2009; Kilpatrick, Swafford, & Findell, 2001; Saxe, Gearhart, & Seltzer, 1999; Smith & Stein, 2012a; Walshaw & Anthony, 2008). Students who explain their thinking, engage in mathematical arguments, and listen and respond to the mathematical ideas of others move toward mathematical proficiency in ways not achieved for many students in traditional mathematics classrooms (Kilpatrick et al., 2001; Saxe et al., 1999). For instructors, facilitating whole-class discussions in mathematics classrooms provides insight into student thinking (Fennema, Franke, Carpenter, & Carey, 1993; Franke, Webb, & Chan, 2010) and for students, it allows engagement in mathematical work while coming to understand that mathematics makes sense.

This study will investigate how instructors facilitate whole-class discussions in mathematics classes for preservice teachers (PSTs). As part of their preparation to become teachers, preservice teachers need to learn to engage students in talking about mathematics. Mandates for mathematics education reform in the US (e.g., National Council of Teachers of Mathematics, 1991; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) suggest K-8 teachers use a more student-centered, discussion-based approach when teaching mathematics. As Goos (1999) noted,

> these moves for curriculum reform are supported by current research in mathematics education which has re-conceptualised mathematics teaching and learning as a social and communicative activity that requires the formation of a

> classroom community of learners, where the ways of thinking, modes of inquiry, communicative conventions, values, and beliefs characteristic of the wider mathematical community can be progressively enacted and appropriated (p. 4).

The most recent of these curricular reforms, the Common Core State Standards for Mathematics (CCSS-M, 2010), does not mandate teaching practices to use in the classroom, but it does assert that students should be engaged in making sense of problems and that proficient students, "can understand the approaches of others to solving complex problems . . . identify correspondences between different approaches," and "construct viable arguments and critique the reasoning of others" (CCSS-M, 2010, pp. 6-7). Whole-class discussion can be an opportunity for students to express their reasoning, model mathematics, critique others' reasoning, and see others doing the same. Students need to be exposed to different problem solving approaches in order to make sense of these approaches, identify correspondences, and critique others' reasoning. Students also need instructor guidance, facilitation, or modeling to critique the reasoning of others.

Facilitating whole-class discussion is not simple and it is difficult to learn (Boerst, Sleep, Ball, & Bass, 2011). Compounding this difficulty, new teachers may not use more student-centered methods when they become teachers, in part, because of their "beliefs based on past teacher-centered educational experiences" (Dunn & Rakes, 2011, p. 3), including beliefs about how mathematics can and should be taught. As Crespo (2003) asserted, "teacher education courses must be designed to not only enhance pre-service teachers' knowledge of mathematics but to also provide opportunities to change and revise beliefs about subject matter and about teaching and learning'' (p. 245). Teaching mathematics to PSTs using student-centered methods, such as whole-class discussion, can influence PSTs' ideas about teaching mathematics (McNeal & Simon, 2000) and be used as a model for PSTs' future teaching practice (Nolan, 2010). Mathematics classes are often required for K-8 certification through conventional pathways (undergraduate teacher education programs), and such courses offer an opportunity to expose PSTs to mathematical discussions and ensure PSTs have experience engaging in discussion to learn mathematics.

Research has shown across many decades and many contexts that PSTs, especially those preparing to become K-8 teachers, typically have weak mathematical knowledge (e.g., Ball, 1990; Ball, Lubienski, & Mewborn, 2001; Ma, 1999; Tirosh & Graeber, 1989). Thus, in addition to learning about teaching mathematics through discussion, PSTs may also learn mathematics through whole-class discussion in their postsecondary classes (Franke et al., 2010; Nussbaum, 2008; Walshaw & Anthony, 2008; Yackel & Cobb, 1996). Using teaching practices, such as whole-class discussion, that elicit student thinking and engagement with mathematics are correlated with student achievement in studies investigating elementary mathematics classes (Saxe et al., 1999) and mathematics classes for PSTs (McCrory, 2011). Engagement in class discussions can also help students develop their reasoning (Atwood, Turnbull, & Carpendale, 2010), make connections between ideas (Boerst, Sleep, Ball, & Bass, 2010; Brookfield & Preskill, 1999), expose students to different perspectives (Boerst et al., 2010; Brookfield & Preskill, 1999). Mathematics classes required for K-8 certification offer an opportunity to strengthen mathematical knowledge by engaging future teachers in mathematical discussions.

Given the research suggesting they can *learn* mathematics through discussion and the mandates for K-8 teachers to *teach* using whole-class discussion, it makes sense to consider when and how future teachers have the opportunity to engage in mathematical discussions during their teacher preparation programs. Because there is considerable research on discussions in K-12 classrooms and much less in postsecondary classrooms, this study will consider whether the same parameters outlined for K-12 mathematics classrooms are useful in facilitating discussions in mathematics classes for future teachers. The focus here is on how whole-class discussions are facilitated and managed in PST undergraduate mathematics classes.

Even though PSTs are now expected to learn to teach in new ways, we do not know much about whole-class discussion in classes for preservice elementary teachers. This has led some, such as Atwood (2010), to call for more research on classroom talk in postsecondary contexts. Few studies have investigated the process and facilitation of whole-class discussion in preservice teacher education or postsecondary education more generally. The idea of whole-class discussion does appear in postsecondary research when excerpts from whole-class discussion are used to illustrate student thinking at a particular point in time (e.g., McNeal & Simon, 2000) or when participation in discussion is used as an outcome or proxy for student characteristics like academic self-efficacy (e.g., Galyon, Blondin, Yaw, Nalls, & Williams, 2012). Researchers also sometimes state that whole-class discussion was used in a class without elaborating on what it entailed (e.g., Harkness, D'Ambbrosio, & Morrone, 2007; Olson & Knott, 2013). Several of these studies that feature whole-class discussion in preservice teacher education or postsecondary education are self-studies, conducted by instructors trying new methods, which they dub “experimental,” in their classes. For example, Morrone, Harkness, D'Ambbrosio & Caulfield (2004) investigated classroom goal orientation in an “experimental” mathematics class for PSTs that used a social-constructivist approach, Simon & Blume (1996) and McNeal & Simon (2000) reported on PST difficulties during the implementation of a “whole class constructivist teaching experiment” by one of the authors, and Nolan (2010) reported on the difficulty of shifting PSTs away from teacher-centered methods towards inquiry-based methods in her class. These all used discussion as evidence for something else rather than investigating how instructors facilitate whole-class discussion in their classes. Studies comparing whole-class discussions across multiple classrooms are also rare.

Research on discussion in K-12 mathematics classrooms has delineated features of, and techniques for, meaningful and educative discussion (e.g., Chapin, O'Connor, & Anderson, 2009a; Franke et al., 2009; McGraw, 2002; Smith & Stein, 2012a; Stein, Engle, Smith, & Hughes, 2008). Many studies in higher education that use discussion as a variable or evidence for other constructs assume that K-12 research applies directly to postsecondary students and in some cases it very well may; however, there are many reasons why K-12 may not be the same as postsecondary education. For example, postsecondary students have chosen their course of study, and have a greater base of knowledge about and experiences with mathematics, which they are more likely to refer to than elementary students (Atwood et al., 2010; Kaartinen & Kumpulainen, 2001). In addition, they have learned mathematics in particular ways that have influenced their expectations about what happens in mathematics classes, and these expectations may be hard to amend. Few studies explicitly acknowledge these assumptions or directly test them, and those that do (e.g., Megowan-Romanowicz, 2010; Morrone et al., 2004) show mixed results. Such results suggest that postsecondary students may sometimes respond differently to classroom discussions or may require different techniques and strategies on the part of the instructor to support whole-class discussions.

Summary

In summary, classroom discussion may be an opportunity for PSTs to engage in constructing and deepening their knowledge of mathematics while at the same time learning to teach mathematics in a way they might not have previously experienced. Classroom discussions give PSTs experience participating in student-centered practices that they may not have experienced and that standards suggest they implement in their future classrooms. Although we know a great deal about discussion in classes at the elementary (e.g., Walshaw & Anthony, 2008) and secondary levels (e.g., Huang, Normandia, & Greer, 2005; Lau, Singh, & Hwa, 2009; Walshaw & Anthony, 2008), we know surprisingly little about discussions in undergraduate classes (Atwood et al., 2010) and even less about discussions in courses for future teachers.

Purpose and Research Questions

The purpose of this study is to examine how instructors engage their students in whole-class discussion in mathematics classes for PSTs, using K-12 research that has highlighted important factors for productive discussions as a guide. More specifically, this study takes the approach of an exploratory multiple case study to look at instructor teaching moves before and during whole-class discussions in two introductory mathematics courses for PSTs and to compare discussions within and between instructors. The research addresses the following questions:

Using categories from K-12 research as a guide for observation and analysis,

1. What aspects of instruction precede whole-class discussion in a mathematics class for preservice teachers?
    A. What class activities precede whole-class discussion?
    B. Using modified categories from King (1994), what kind of prompts do instructors use to encourage discussion?
2. How are instructors facilitating discussion and supporting students' engagement with the mathematics?
    A. Do instructors choose who talks in discussion? If so, how?
    B. How do instructors respond to student thinking in whole-class discussion?
    C. In what ways and how often are one student's ideas connected to another student's ideas in whole-class discussion?
    D. In what ways and how often is student thinking in discussion explicitly connected to mathematical ideas?

Chapter 2: Review of Literature

At all levels of education, instructors who are leading, facilitating, or managing whole-class discussions can structure classes and discussions in ways that encourage participation and engagement with course material (Nassaji & Wells, 2000, p. 375). Student motivation to participate in discussion is one avenue to consider to promote student participation and engagement in discussion. Jansen (2009) used an open-ended questionnaire to investigate 148 preservice elementary teachers' motivation for participating in whole-class discussion in a mathematics content course for preservice teachers (PSTs). She found five motivational profiles in these self reports from students: (a) they participated to learn mathematics and prepare for a teaching career; (b) they preferred to observe or work alone rather than participate aloud; (c) they participated to seek help when struggling to learn; (d) they tried to save face when participating; and (e) they participated to help others learn. Each of these motivational profiles was associated with certain beliefs and goals, as well as with specific participation practices. For example, participating to learn mathematics and prepare for a career was associated with seeking feedback and ideas from others, asking questions, and explaining to others. Participating to seek help was associated with asking questions and observing the strategies of others, while participating to help others was associated with providing solution strategies, reasoning, and answers. These motivation and participation profiles are pertinent to PST learning in discussions because they affect the character and quality of discussion and therefore may affect PST learning.

By the time students reach the university level, they may already have specific beliefs about what instructors can do to promote quality whole-class discussion.

Dallimore, Hertenstein, and Platt (2004) surveyed 68 students in graduate business classes on what they thought enhanced the "quality of participation and the effectiveness of discussion" in the class. Questionnaires were open coded and six categories emerged: "(1) required and graded participation, (2) incorporating instructor and students' ideas and experiences, (3) active facilitation, (4) asking effective questions, (5) creating a supportive classroom environment, and (6) affirming student contributions and providing constructive feedback" (Dallimore et al., 2004, p. 107). Within these categories, strategies that students thought increased *participation quality* included cold calling, grading participation, having the instructor or peers expand on student ideas, challenging and probing student responses, affirming student contributions, and providing constructive feedback. Strategies that increased *discussion effectiveness* included cold calling, instructors expanding on student ideas, making the material relevant to students, facilitating debate, using effective questions, reinforcing or reiterating student responses, and providing respectful and non-negative feedback. Note that many strategies, such as expanding on student ideas and providing constructive feedback, increase both participation quality and discussion effectiveness.

Studies in secondary education written from an instructor's perspective suggest other ways of facilitating whole-class discussion. Contreras (2006) investigated how a master teacher created and maintained a productive discourse community during eight sessions of a high school geometry class. Using discourse analysis, the researcher found six *discourse generating tools* and three social norms that facilitated discussion in the course. The three norms were: "all students were expected to (a) participate (b) share their reasoning when called upon, and (c) listen to, analyze, and evaluate each other's comments" (Contreras, 2006, p. 21). The six discourse generating strategies were:

> (1) using lower-order questions to engage students, (2) persisting in eliciting students' reasoning, (3) encouraging as many student participations [sic] as possible, (4) encouraging students to analyze and evaluate each other's comments, (5) encouraging students to share as many strategies as possible and (6) using a focusing discourse pattern" (Contreras, 2006, p. 22).

As in Dallimore, Hertenstein, and Platt (2004), these practices focused on encouraging student participation, helping students develop and share their reasoning, and promoting connections between student ideas.

McGraw (2002) investigated how two mathematics teachers planned and facilitated whole-class discussion in a high school Algebra 1 class as well as the supports and struggles implementing whole-class discussion. The instructors' work planning and facilitating whole-class discussion focused on four themes: "(a) engaging students with tasks, (b) engaging students in sharing and listening, (c) engaging students in questioning and clarifying, and (d) engaging students in agreeing and disagreeing" (McGraw, 2002, p. 68). When developing student engagement with tasks, they focused on "providing students with sufficient opportunities to develop opinions about the mathematics at hand prior to discussion and . . . engaging students with tasks in such a way that they would feel some need for discussion" (McGraw, 2002, p. 70). They also found that having visual representations (termed *reference objects* by the authors) took some of the student attention off the teacher and placed it on the representations and the discourse about them. To encourage students to share and listen to other students, the teachers rearranged the classroom furniture so that students could see and face each other. Instructors used activities that made student expectations for mathematics classrooms and the teachers' "view of discussion in learning" explicit, and asked students to reflect on how discussion impacted their thinking. They also tried to encourage participation from non-participating students, set norms of respectful interactions between students, and asked students to participate in alternative ways. To engage students in questioning and clarifying, the teachers asked for questions and gave adequate wait time. To engage students in agreeing and disagreeing, teachers pointed out when two students presented contradictory answers that and pushed students to take a side. In short, like the teacher investigated in Contreras (2006), these teachers used many varied strategies to facilitate student engagement in the task, and encourage students to listen to and analyze peer solutions.

K-12 research shows that without instructor guidance, students often do not engage deeply with course material or integrate new material with their previous knowledge (Huang et al., 2005; King, 1994; Nathan & Knuth, 2003). Though postsecondary students may be more apt to use prior knowledge to help them understand content (Atwood et al., 2010; Kaartinen & Kumpulainen, 2001), they, like secondary students (Baumert et al., 2010; Nilsson & Ryve, 2010), may still need instructor guidance to interact in ways that promote learning and integration of knowledge (Kerssen-Griep, Gayle, & Preiss, 2006). This may be in part related to their prior experiences in learning mathematics: most mathematics courses, and courses in many other subjects, have limited expectation of or opportunity for discussion (Cuban, 1984; Stigler & Hiebert, 1999a; Weiss, Banilower, McMahon, & Smith, 2001). Without clear structures, norms, and encouragement, students are unlikely to participate in ways that lead to productive whole-class discussions.

Next this chapter will briefly review how other research has defined *whole-class discussion*. The remainder of the chapter reviews several possible influential factors for productive whole-class discussions: activities preceding whole-class discussion, discussion prompts, choosing which students talk in discussion, instructor responses to student thinking, and connecting student reponses to each other and to mathematical ideas.

Definitions of Whole-Class Discussion in the Literature

Several definitions of whole-class discussion exist in previous literature but though most are well suited to the purposes of individual studies, the definitions are ill suited to a general definition for whole-class discussion. For example, Saxe, Gearhart, & Seltzer (1999) in a study investigating elementary students' achievement in fractions and the use of reform-based teaching practices, defined whole-class discussion as, "(a) teacher-supervised activity and interaction, (b) whose function was either to prepare students for independent or cooperative work on similar problems or to discuss work that students had completed independently or cooperatively" (p. 11). This definition, though it may have been appropriate for their focus, could be both over and under inclusive. It limits the valid functions of whole-class discussion and what is meant by "interaction" is open to the interpretation of the reader. In Nystrand and colleagues' (2003) event history investigation of discussions in eighth- and ninth-grade English and Social Studies classrooms, the authors mentioned that whole-class discussion was an "open exchange of ideas" (Nystrand et al., 2003, p. 139) then differentiated three classroom formats: *dialogic spells, discussion*, and *recitation*. Recitation was "characterized by IRE patterns and teacher test questions"[1] (Nystrand et al., 2003, p. 150), while discussion was an "open-ended

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[1] Test questions are questions with only one right answer.

conversational exchange of ideas largely absent of questions." Dialogic spells were "characterized by engaged student questions and an absence of teacher test questions" (Nystrand et al., 2003, p. 178). Again, though this categorization may have suited the purposes of their study, it does not lend itself to a general operational definition of whole-class discussion. In fact, what most teachers reported as discussion (called "question-answer discussion" by one teacher), Nystrand and colleagues termed, question-answer recitation because IRE sequences and test questions were common.

Some definitions focus on features of good or educationally productive whole-class discussion. The definition in Chapin and colleagues' (2009a) book, *Classroom Discussions: Using Math Talk to Help Student Learn*, described whole-class discussion in mathematics classes as a practice where, "the teacher . . . is attempting to get students to share their thinking, explain the steps in their reasoning, and build on one another's contributions" (p. 19). While this may describe several practices of educationally productive whole-class discussions and while it fits with their framework, it limits whole-class discussions to those that use all of these practices and constrains the functions of whole-class discussion.

For the purposes of this study, whole-class discussion is briefly defined as an episode of at least one minute in duration where a student talks to the instructor or another student about the same discussion prompt while the rest of the class is expected to attend to the exchange(s). The definition is discussed in more depth in Chapter 3.

Aspects of Instruction Preceding Whole-Class Discussion

Two aspects of instruction that may directly influence whole-class discussion are: (a) activities that directly precede whole-class discussion, and (b) the discussion prompts used to start whole-class discussion. The next two subsections discuss how each of these may influence classroom discussion.

Activities preceding whole-class discussion. Knowing what happened before a classroom interaction helps us to interpret more about what happened in that interaction (Mercer, 2008) because classrooms are, “normally a continuing, cumulative experience for the participants” (Mercer, 2010, p. 10). In other words, instructors and students have a shared history including classroom norms, previous topics covered, and previous activities they have engaged in.

Instructors may use this shared history strategically by using class activities that precede whole-class discussion to prepare PSTs to think about what will follow. Instructors may recap previous activities or lessons before discussions to elicit student discourse on that material and help students make connections between new and previously learned material (Mercer, 2000, 2008). Though researchers need to be aware of and knowledgeable about the shared history in a particular classroom to properly understand what is going on in that classroom, events that are closer to whole-class discussion in time (i.e., immediately preceding it) may be more likely to influence whether discussion occurs or influence the character of said discussion.

Nystrand and colleagues (2003) found, in a study that used event history methods to investigate features of eighth and ninth grade English and Social Studies classrooms, certain features immediately preceding classroom discourse that made dialogic spells and discussion more likely to occur. Student questions made dialogic spells more likely to occur while a “high proportions of student questions, . . . questions with high cognitive demand,” and teachers asking students follow-up questions about their responses and ideas make discussion more likely to occur (Nystrand et al., 2003, p. 187). In order for teachers to follow-up on student responses and students to have many questions, students need to engage with the problem or mathematical ideas before discussing them.

Many methods may be used to initiate student engagement with the material and to get students to think about the mathematics before sharing their ideas with the whole class (e.g., Curzan & Damour, 2003; Gayle, Preiss, & Allen, 2006; Lamberg, 2013; McKeachie, 1999). To engage students, instructors may use peer learning methods, like *Think-Pair-Share*[2] (Lyman, 1981) or *Pair-Share*, to give students a chance to collaborate with a partner or they may ask students think about or work on the prompt individually, such as asking students to write a minute paper (Cooper & Robinson, 2000b; Curzan & Damour, 2003; McKeachie, 1999; McTighe & Lyman, 1988; Mulryan-Kyne, 2010), or asking students to work on a problem for homework. Asking students to discuss with peers can have the benefit of giving students time to think about their response, exposing students to alternate explanations or thinking, giving students experience justifying their thinking to others, and giving students peer feedback (Chapin, O'Connor, & Anderson, 2009b; Cooper & Robinson, 2000a).

Asking students to work individually or in small groups on a mathematical problem during class gives instructors the opportunity to monitor student thinking. Smith and Stein (2012d) describe monitoring as,

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[2] Think-Pair-Share is a teaching strategy where,

> the teacher lectures for a period of time, then poses a question, test item, or issue for students to consider in brief individually (the Think phase). Then, individuals turn to others sitting nearby and share their responses with another person (the Pair phase). If time permits, several of the pairs share their responses with the class (the Share phase) (Cooper & Robinson, 2000b, p. 18).

Pair-Share differs from Think-Pair-Share in that students do not get time to think individually.

> not just listening in on what students are saying and observing what they are doing, but also keeping track of the approaches that they are using, identifying those that can help advance the mathematical discussion later in the lesson, and asking questions that will help students make progress on the task (2012d, p. 37).

When instructors give prompts as homework and have time to review student responses, instructors can identify responses that might advance the whole-class discussion. This may later influence how instructors choose students to talk in discussion and how instructors connect student ideas to each other and to mathematical ideas, both of which are described in later sections of this chapter.

Discussion prompts. Discussion prompts are statements or questions that are intended to elicit discussion by posing a problem, asking a question, setting a topic, etc.[3] Regardless of their form, to be effective, prompts presented to students, including PSTs, must be constructed in a way that warrants or promotes discussion (Webb, 2009). Nystrand and colleagues' (2003) event history investigation of classroom discourse, discussed in the previous section, found questions with high cognitive demand made discussion more likely to occur. In that study, higher level questions asked for new information such as making a generalization, analyzing a situation, or speculating. Lower level questions asked about known information such as reporting old information or recording an ongoing event (Nystrand et al., 2003, pp. 147-148).

More cognitively challenging tasks are also associated with higher mathematics achievement. In a study investigating connections between teacher knowledge, student

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[3] Other researchers have also called these question events because they elicit responses but may or may not have the form of a question (e.g., Nystrand et al., 2003; Soter et al., 2008).

achievement, and teaching practices in high school mathematics classes, Baumert and colleagues (2010) asserted that using "cognitively challenging and well-structured learning opportunities" was one of three classroom practices that consistently emerged from the literature, "as being crucial for initiating and sustaining insightful learning processes in mathematics lessons" (p. 145). In their sample of 194 ninth grade mathematics classes, both the cognitive level of tasks and the task alignment to the tenth grade curriculum were predictive of students' tenth grade mathematics achievement. In that study, cognitive level consisted of whether the task was "purely technical," required "computational modeling," or "conceptual modeling"; the level of mathematical argumentation required; and "translation processes within mathematics" (Baumert et al., 2010, pp. 149-150). Though the cognitive level of questions may be defined differently between Nystrand (2003) and Baumert and colleagues (2010), results from both studies point at higher level questions as being most beneficial for students.

Low level questions may be useful to initiate student participation. Contreras (2006) investigated how a master teacher used questioning in a high school Geometry class. The teacher in that study used lower level (often true/false or yes/no) questions to elicit participation in discussion. Then he asked higher level questions, which often called for explanation or justification of the student's response, to further engage students and deepen the discussion. In that study, these lower level questions preceding higher level questions were sometimes referred to as *engager questions* because they were used to engage at least one student in discussion (p. 23). There was no evidence that low level questions that were not paired with high-level questions acted to encourage discussion or learning.

Researchers have categorized prompts based on a variety of criteria including but not limited to the level of cognitive demand (e.g., Contreras, 2006; Nystrand et al., 2003; Soter et al., 2008), whether the question is open or closed (e.g., Crespo, 2003; Engle & Conant, 2002; Nystrand et al., 2003; Soter et al., 2008), and the types of responses students are expected to give (e.g., King, 1994; Kramarski & Mevarech, 2003; McKeachie, 1999; Mevarech & Kramarski, 1997). The latter is of particular interest because it focuses (more specifically than many other categorizations) on what instructors are asking students to do in answering the question. For example, the typology presented in King's (1994) investigation of peer questioning in fourth and fifth grade science classes, questions were categorized as *factual* (asking students to recall information that was explicitly covered in the lesson), *comprehension* (asking students to describe or define a process or term) or *integration* (asking students to go "beyond what was explicitly stated in the lesson by linking two ideas together in some way" [p. 351]). Factual questions can be thought of as lower demand than comprehension questions, which are lower demand than integration questions. In this categorization, factual questions are expected to elicit *knowledge restating* (a restatement of fact or knowledge from the lesson), comprehension questions are expected to elicit *knowledge assimilation* (statements that paraphrase material from the lesson "in students' own words"), and integration questions are expected to elicit *knowledge integration* (statements making "new connections" or going "beyond what was provided in the lesson") (King, 1994, p. 350). These categories also roughly fit with cognitive demand as discussed earlier in this section.

Though typologies may imply a type of student response, in practice students may not directly provide a response or the expected type of response (Carlsen, 1991; Dillon, 1982; Stein, Grover, & Henningsen, 1996). If an instructor accepts a student response that is not of the same type that the prompt was expected to elicit, instructors may be more likely to get similar responses in the future. Therefore, in practice the level of cognitive demand or type of a prompt in discussion may depend on something that many questioning studies do not take into account: how the instructor responds to student responses (Contreras, 2006; Doyle & Carter, 1984; Nystrand et al., 2003; Smith & Stein, 1998, 2012e; Stein et al., 1996). As discussed in the previous section, over time students develop a shared history with their instructor and classmates that influences how they interpret questions and decide what kinds of responses the instructor might be expecting (Doyle & Carter, 1984; Mercer, 2008; Smith & Stein, 1998). For example, when asking students, "what is multiplication?" the level of cognitive demand of the task as well as the question type in the King (1994) typology changes depending on whether the instructor expects students to give definitions directly from the textbook or to state it in their own words. Instructor responses to student thinking (discussed in the next section) give students an indication of what constitutes an acceptable response by allowing instructors to judge the appropriateness of a response or to modify student responses (through tactics such as asking for more information or revoicing) until the responses are appropriate (e.g., Contreras, 2006). These instructor responses can in turn influence future student responses.

Facilitating Whole-Class Discussion

Instructors may do many things during whole-class discussion to facilitate student engagement and learning. Instructors often choose which students share their thinking with the class, respond to student responses in various ways, and connect student responses to other students' ideas or to pertinent material in the discipline. The following sections discuss how each of these may influence classroom discussion.

Choosing which students talk in whole-class discussion. There are several methods that an instructor may use to choose which students talk in whole-class discussion. Instructors may call on students who did not volunteer (*cold calling*), call on volunteers, or allow students to participate spontaneously without being called on by the instructor (*self-selection*). Each of these methods has its own affordances and constraints.

Eliciting volunteers. Instructors may ask for volunteers, which may strengthen student agency and self-determination as well as increase student comfort in the classroom. But this practice may encourage motivation process losses, such as social loafing[4] or free riding.[5] In other words, because there is the possibility or likelihood that someone else may respond or do the work of wrestling with the discussion prompt, students may put forth less effort than they otherwise would if they were expected to be individually responsible for discussing the prompt with the class. Students who are less articulate, unsure of their answers, and/or have not put sufficient effort into the activity are unlikely to volunteer (Walshaw & Anthony, 2008). A meta-analysis of social loafing by Karau and Williams (1993) suggested several factors that may moderate this effect, including the potential for a response to be evaluated, and the uniqueness (or

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[4] Social loafing is the "reduction of individual effort exerted when people work in groups compared to when they work alone" (Forsyth, 2009, p. 294).

[5] Free riding occurs when one contributes "less to a collective task when one believes that other group members will compensate for this lack of effort" (Forsyth, 2009, p. 296). Free riding is different from social loafing: free riding is possible when "reductions of effort have a less direct impact on the chances of *group success*" while, social loafing is more likely when "reductions of effort have less direct impact on the chances of receiving *salient personal and social evaluation*" (Baron & Kerr, 2003, p. 57).

identifiability) of an individual's efforts. Similarly, free riding can be discouraged through making individual efforts "unique or essential for the group's success" (Forsyth, 2009, p. 296). Although instructors can try to minimize these motivation process losses though practices such as connecting student responses with their name (i.e., making a student response more identifiable), these problems are inherent in eliciting volunteers.

Asking for volunteers can also lessen instructor control over the content of the discussion (Chapin et al., 2009b; Smith & Stein, 2012b). Smith and Stein (2012b) called it, "either the bravest or most naive invitation that can be issued in the classroom," (p. 44) because it surrenders much of the control that the instructor has over the direction of discussion. In addition, this method of choosing students may be problematic if using discussion for formative assessment because students who volunteer may not give a representative picture of what the class as a whole knows.

Cold calling. Cold calling is calling on students who have not volunteered. Instructors can choose students either at random or intentionally (e.g. calling on students based on the expected response or calling on students who do not appear engaged with the discussion). Both types of cold calling introduce the possibility that any student's thinking could be evaluated at any time (see Dallimore et al., 2004; McDougall & Granby, 1996) and push students to participate who otherwise would not, while purposeful cold calling gives the instructor "maximum control of who talks" in discussion and what ideas they might present (Chapin et al., 2009b).

The expectation of cold calling may lead students to better prepare for class sessions. In an experimental study, McDougall and Granby (1996) investigated how 40 undergraduates' expectations of their instructor's method of questioning students (volunteer or cold call) affected their preparation for an introductory statistics class. Expectation of cold calling led students to report completing more assigned readings, recalling more information from those readings, and greater confidence in their recall compared to students who expected the instructor to ask for volunteers.

Using cold calling may increase discussion quality. Dallimore, Hertenstein, and Platt's (2004) survey of students in graduate business classes asked what they thought enhanced the "quality of participation and the effectiveness of discussion" in the class. One of the six categories that emerged from student answers was *required and graded participation*. Students cited cold calling as a way to accomplish this and they reported that it increased both the discussion effectiveness and discussion quality.

Purposefully choosing student responses based on expected content can give an instructor more control over the discussion and the mathematics presented therein. An instructor can use particular student responses to air common misconceptions, important ideas in the topic being addressed, and alternative methods or solutions (Chapin et al., 2009a; Stein et al., 2008). When an instructor chooses to air student responses based on content, that instructor has the ability to sequence those responses to further pedagogical goals (Stein et al., 2008). However, this assumes that the instructor has access to students' thinking before calling on them, which may not always be practical or even possible before whole-class discussion in some class contexts, such as large lectures.

Allowing students to select themselves. Instructors may also allow students to talk in discussion without instructor intervention. This is different from eliciting volunteers in that the instructor does not choose among volunteers, instead students talk without being called on by the instructor. This method gives instructors the least control over who is speaking and what solutions are shared with the class. Also, like eliciting volunteers, not all students are held accountable for participating in discussion so motivation process losses (like social loafing and free riding) may be more likely to occur and the sample of student responses presented may not be typical or representative for formative assessment.

Allowing students to self-select leads to classroom talk to be structured, "more like informal conversation—not the same as conversation, because there is still the large group of potential speakers and the educational necessity to stick to an agenda, but closer to it" (Cazden, 1988, p. 54). It also allows students to, "control the flow of information and affirm his or her expert status" (Atwood et al., 2010, p. 375) and may lead to cross-discussions where students discuss with each other, without the instructor as an intermediary (Cazden, 2001). An example of what this looks like in practice comes from Dixon, Egendoerfer, and Clements (2009). They observed mathematics lessons in a second grade class before and after the teacher stopped asking students to raise their hands before speaking in discussion. The teacher prepared the class for this change by emphasizing that students would "be expected . . . to explain how they came up with their solutions" and explicitly introducing several social and sociomathematical norms, such as "asking questions . . . when one does not understand, . . . explaining and justifying mathematical reasoning to students with questions, and . . . defending one's mathematical thinking to others" (Dixon et al., 2009, p. 1070). After the change, students became more engaged in discussion and began to demonstrate greater conceptual understanding.

Responses to student thinking. Discussions are co-created though the discourse of both students and instructors. Student talk influences instructor responses and instructor responses in turn shape ensuing discussion (Kerssen-Griep et al., 2006;

Nystrand et al., 2003). Instructors have many options of how to respond to student thinking. For example, instructors can ask students to explain or justify their responses, wait for the student to continue or other students to respond, give students feedback on their reasoning or participation in the class, revoice a student's responses, or ask other students to comment or give their own responses. Instructors can purposefully respond to student thinking to shape ensuing discussion, but it is a complex task (Boerst et al., 2011; Ghousseini, 2009; Smith & Stein, 2012c; Stein et al., 2008; Stein et al., 1996; Walshaw & Anthony, 2008).

Asking students for more information. Instructors can ask students for more information about their responses, including asking students to explain or justify responses in more detail and going further in their explanation (Webb, 2009). In fact, uptake, or asking others about things they previously said, "is important because it recognizes and envelops the importance of the student contribution. Following up on student responses makes the response the momentary topic of discourse" (Nystrand et al., 2003, p. 146). In addition, asking students to further explain or justify their responses has been associated with deepened engagement and learning (Franke et al., 2010; Franke et al., 2009; Walshaw & Anthony, 2008; Webb, 2009), though instructors have to judge when asking students to further explain a response advances the goal of discussion (Smith & Stein, 2012c).

How an instructor asks for more information may influence how likely students are to explain further. Franke and colleagues (2009) investigated teachers' follow up to student explanations in two second grade and one third grade classroom. In that sample, teachers questioned students about most of their explanations and when they did not, students did not explain further. The type of questions that teachers used to follow up on student explanations were related to whether students eventually gave a correct and complete explanation. In that study, the authors differentiated between three types of questions used to follow up on student explanations:

> *General questions* were not related to anything specific that a student said. *Specific questions* addressed something specific in a student's explanation. *Probing sequences of specific questions* consisted of a series of more than two related questions about something specific that a student said and included multiple teacher questions and multiple student responses. In *leading questions*, the teacher guided students toward particular answers or explanations and provided opportunities for students to respond (Franke et al., 2009, p. 384).

Whether the student explanation was initially correct and complete or not, students were most likely to develop their explanation further when instructors used a probing sequence of specific questions. Leading questions were least likely to lead students to develop on their explanations further. When student explanations were initially incorrect or incomplete, students eventually produced correct explanations when probing sequences of specific questions were used. Another student or the teacher often gave the correct solution when a general or specific question was used. When leading questions were used, often no correct and complete explanation was given. These results suggest that 1) without instructor follow up students are unlikely to elaborate further on an initial explanation and 2) single follow up questions, whether they are based on specific aspects of the student explanation or not, are less associated with a student ultimately giving a complete and correct explanation than several targeted questions about specific parts of the student explanation.

***Revoicing and elaborating*.** Repeating, rephrasing, and adding on to student responses can have many benefits in whole-class discussion. Revoicing students' responses can highlight what is relevant, make quiet student responses heard by the whole class, repackage the response for greater comprehensibility, or be used with elaboration to connect the response to mathematical ideas (Chapin, O'Connor, & Anderson, 2009c; Forman & Ansell, 2001; O'Connor & Michaels, 1993; Smith & Stein, 2012c; Walshaw & Anthony, 2008).

Instructors may have reason to use caution when revoicing student responses and determining how much to alter student responses. If instructors merely repeat the response, they may be missing an opportunity to connect the response to important mathematical ideas or highlight what is most important in the response. If instructors alter the response too much (by rephrasing, restructuring, and using more formal language), then at some point it ceases to be about the student response, shifting authority and control of discussion back to the teacher. Teachers must find balance between these conflicting demands when using revoicing (Herbel-Eisenmann, Drake, & Cirillo, 2009; Smith & Stein, 2012c).

Recording student responses in a public way (e.g. by writing student responses on the chalkboard, whiteboard, document camera, or overhead) can have many of the same benefits as revoicing, such as emphasizing student ideas and reinforcing that they are important. It may also give students a resource to help with note taking and organizing their thoughts on the lesson, lessen the cognitive load of following the discussion, and allow for comparison between student ideas (Chapin, O'Connor, & Anderson, 2009d; Stigler & Hiebert, 1999b; Yoshida, 2005). Results from the Third International Mathematics and

Science Study (TIMSS) suggest that teachers in the US are more limited in their use of overheads and chalkboards than Japanese teachers (Stigler & Hiebert, 1999b). In the US, teachers use overhead projectors (or chalkboards) to focus student attention on particular information, which is often erased or covered up when the teacher moves to talk about something else. In Japan, teachers use chalkboards, which are rarely erased before the lesson is complete, to provide a record of the lesson, "help students remember what they need to do and think about," "help students see the connection between different parts of the lesson and the progression of the lesson," "compare, contrast, and discuss ideas that students present," "help to organize student thinking and discover new ideas," and "foster organized student note-taking skills by modeling good organization" (Yoshida, 2005, p. 97).

Asking students to respond to each other. Students can be asked to respond to other students in many of the same ways instructors do. Asking students to respond to peers holds the students accountable for listening to and trying to understand responses that students present in class and it is an authentic activity that they will engage in when they are a teacher in their own classroom. For example, instructors can ask students to revoice or restate peer responses in their own words, ask students to ask questions about peer responses, or ask students to compare peer reasoning. These practices not only hold other students accountable for listening to and understanding the responses that peers present in discussion, it also emphasizes and reiterates the response (Smith & Stein, 2012c) and keeps the focus of discussion on mathematical ideas presented by students (Herbel-Eisenmann et al., 2009).

Evaluating student responses. Research suggests that evaluation of student responses may have a deleterious effect on student contributions to discussion. For example, Nassaji and Wells (2000) studied nine elementary teachers' teacher-student interactions over six years and found that there was a negative correlation between the proportion of elementary student contributions that instructors evaluated and the length and complexity of those student contributions. Results such as these seem to have led many teachers to avoid evaluating student responses or giving guidance in discussion in order to honor student contributions, despite the fact that evaluating mathematical ideas is a core part of mathematics discourse (Nathan & Knuth, 2003; Smith & Stein, 2012a; Stein et al., 2008; Walshaw & Anthony, 2008).

Postsecondary students' views on evaluation may be a bit more complicated than those of elementary students. The 68 graduate business students surveyed by Dallimore and colleagues (2004) cited constructive criticism (including correcting students and helping them understand errors) as a practice that increased the quality of participation. Students reported that positive feedback and accepting student views increased discussion effectiveness while instructor sarcasm, ridicule, and indifference decreased it. In other words, students accept that constructive criticism improves discussion as long as student ideas are honored and the instructor is civil. In addition, Weaver and Qi (2005) surveyed 1,805 college students and used path analysis to investigate how formal and informal structures, including instructor and peer evaluation, influence students' class participation. Students in that study reported that fear of instructor criticism did not significantly influence their frequency of class participation; faculty-student interaction significantly increased participation; while fear of peer disapproval had a negative effect on participation. This suggests again, that postsecondary students expect or at the very least tolerate instructor evaluation of their ideas in class.

Connectedness of student responses. Discussions that present a series of separate student responses are very different in character from discussions that present student ideas in a coherent and connected fashion (Atwood et al., 2010; Stein et al., 2008; Walshaw & Anthony, 2008). In the worst case (e.g., the furthest from a meaningful whole-class discussion), the former is characterized as "show and tell" while the latter is the goal of what classroom discussion should be (e.g., Franke et al., 2010; Smith & Stein, 2012a; Stein et al., 2008). McGraw (2002) characterized this difference for the secondary mathematics class observed as "sharing" and "discussing." During "sharing," students share their ideas or solutions without "reacting to and building upon each other's words, they often followed the separate tangents of their individual thoughts" (2002, p. 89). Whereas during "discussing," "students engaged in questioning each other, seeking additional information or clarification, and debating the merit of each other's thoughts" (McGraw, 2002, p. 89). In order to participate in "discussing," students need to listen to and understand another student's thinking before responding. In order to participate in "sharing," students need do neither.

Instructors can avoid the "show and tell" mode through a variety of strategies that encourage connections across student responses as well as between student responses and mathematical ideas (Chapin et al., 2009c; Engle & Conant, 2002; Stein et al., 2008). To encourage connections between student responses, instructors can build on, or compare and contrast student responses. Any of these practices can be used regardless of how students are chosen to speak in whole-class discussion. Instructors can tie student responses to important mathematical ideas through revoicing student responses and elaborating on them. This can highlight important content that is already present in student responses and familiarize students with mathematically acceptable language (Walshaw & Anthony, 2008).

Instructors can sequence student responses to build mathematical ideas throughout the discussion and create a “mathematically coherent story line” (Smith & Stein, 2012b, p. 44). They can do this through purposefully choosing students to speak in discussion because of their thinking (as previously discussed in the section “Choosing Which Students Talk in Discussion”). To make this feasible, instructors need to design activities that give them access to students’ thinking. For example, instructors may use a prompt in discussion that was part of the homework that they have had a chance to look over or instructors may listen in on small groups working on the discussion prompt before whole-class discussion begins. Instructors may choose to begin with the most common solution or the most common misconception, or they may choose another sequence that serves the mathematical ideas that the instructor wants students to learn. The instructor can also select students who use a variety of representations.

Students can be led to do the work of connecting their thinking to peer responses. The teacher observed in McGraw (2002) prompted these kinds of connections through explicitly asking students to build on peer responses (“Who can add to what ____ said?” (p. 90)), asking if students have questions for peers (“Does anyone have a question for _______? “(p. 90)), and engaging students in agreeing and disagreeing with one another. Instructors may also ask students to repeat, explain, or compare and contrast student solutions to make connections across student responses, or ask students to elaborate on or explain the mathematical ideas involved in a student response to connect student responses to mathematical ideas. Asking students to repeat or explain another student’s response can have multiple benefits including holding students accountable for listening to peer responses, increasing participation, and reiterating or clarifying the student response (Chapin et al., 2009c). For PSTs this activity may be particularly important because interpreting and responding to student thinking is an activity that they will have to engage in during their future teaching, especially during class discussions.

Summary

Instructors can do many things to encourage preservice teachers to participate and engage with mathematical ideas in whole-class discussions. Research on practices to promote effective whole-class discussions in postsecondary contexts is sparse but research suggests many practices that are effective in K-12 classrooms. This study used K-12 practices that promote educational discussions as a guide for initial hypotheses about factors to investigate in instructor's implementation of whole-class discussion in mathematics classes for preservice teachers. As discussed in this chapter, research literature suggests a number of key practices and important leverage points for promoting productive whole-class discussion. These are: activities preceding whole-class discussion, selection and use of discussion prompts, choice of students who talk in discussion, responses to student thinking, and connecting student reponses to each other and to mathematical ideas. Chapter 3 discusses the methodology used to investigate these practices.

Chapter 3: Methods

The purpose of this study was to examine how instructors make student thinking public in whole-class discussion in mathematics classes for preservice teachers (PSTs), using K-12 research that has highlighted important factors for productive discussions as a guide. More specifically, this study addressed the following questions:

Using categories from K-12 research as a guide for observation and analysis,

1. What aspects of instruction precede whole-class discussion in a mathematics class for preservice teachers?
    A. What class activities precede whole-class discussion?
    B. Using categories from King (1994), what kind of prompts do instructors use to encourage discussion?
2. How are instructors facilitating discussion and supporting students' engagement with the mathematics?
    A. Do instructors choose who talks in discussion? If so, how?
    B. How do instructors respond to student thinking in whole-class discussion?
    C. In what ways and how often are one student's ideas connected to another student's ideas in whole-class discussion?
    D. In what ways and how often is student thinking in discussion explicitly connected to mathematics?

To answer these questions, I used exploratory multiple case studies to compare instructor teaching moves during whole-class discussions in two introductory mathematics courses for PSTs.

Data Sources

Lessons from seven instructors in four states were videotaped as part of a large-scale research project investigating mathematics content courses taken by PSTs during their undergraduate education (see McCrory, 2009 for more information on the Mathematical Education of Elementary Teachers [ME.ET] project). Instructors were included in the video study because of their class size, location, and/or high scores on the student achievement measure used by the ME.ET project. Instructors were not chosen based on teaching methods or instructional philosophy. All instructors who were asked agreed to participate. All participating instructors were videotaped while they taught lessons on fractions, and some classrooms were videotaped for additional lessons on other topics.

In addition, instructors completed an interview about their teaching practices and 78 instructors, including all instructors who were videotaped, completed an extensive survey (available online at http://meet.educ.msu.edu/) about their class that included items on the goals and purposes of the class and allocation of time in the class, among many other topics.

Researcher involvement. I worked on the ME.ET project for over 6 years. During that time, I viewed a number of the instructor videos and observed class sessions of two courses that were videotaped for the project (one that was included in this study). Because of this, I saw a wide variation in use, quantity, and substance of discussion in these courses and it was this variation, along with my own experiences teaching undergraduates, that prompted my interest and alerted me to the need for more systematic analysis of the differences between instructors' facilitation of discussion. My experience teaching undergraduates led me to the pragmatic set of variables (what activities precede discussion, what kinds of discussion prompts are used, how instructors respond to what students say, and how instructors connect student responses to each other and to the material being taught) that I am investigating.

In addition, I was present when both instructors were interviewed about their teaching. I have watched and summarized video from class sessions in these two courses both before and after the fraction lessons that were analyzed in this paper. I have also coded and analyzed all 78 instructor surveys for the ME.ET project so I have a view of the breadth of what was taught, as well as instructor attitudes and beliefs about these courses. This gives me a broader view of shared class history and instructor intent than I would have from just viewing the lessons included in this study.

Rationale for choosing instructors. Two of the seven instructors who were videotaped, with pseudonyms of Pat and Stevie, are included in this study. Of the seven, these two instructors were chosen because they were expert instructors and they used whole-class discussion most frequently. These instructors were characterized as experts because, as shown in Table 1, they were experienced in K-12 teaching, college teaching, and teaching this course (a class addressing mathematics for elementary teachers) in particular; their students had high scores on student outcomes; and both instructors stated an interest in pedagogy and improving their teaching. The full sample of 78 instructors averaged 16 years of college teaching and had taught the class 12 times prior to the survey. Most (63%) of the 78 instructors had previously taught at the K-12 level. Data from the survey shows that both instructors had a great deal of interest in teaching this course, both before and after they had taught it.[6] In contrast, instructors from the sample of 78 had less interest in teaching the course on average both before and after teaching the course.[7]

Of the seven videotaped instructors, Pat and Stevie used whole-class discussion most frequently and consistently. In addition, members of the ME.ET project noted that during whole-class discussions both instructors had higher levels of student participation than the other videotaped instructors and their discussions focused on mathematical ideas in ways that were uncommon in the other classes. The surveyed instructors (n = 78) were not asked about their use of whole-class discussion[8] so we have no evidence about discussions across the sample. Based on the seven videotaped classrooms for this project, however, these two instructors stood out for their use of whole class discussion as a routine part of their teaching.

These two cases include an extreme class size (102 students) and a more average size class (23 students). The median class size from the 72 instructors who reported their class size was 27 students, with a range from 4-102 students.

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[6] As indicated on the instructor survey on a 4-point Likert scale from 0 to 3 with 0 indicating *no interest at all*, 1 indicating *limited interest*, 2 indicating *some interest*, and 3 indicating *a great deal of interest*. These two instructors each responded with 3.

[7] Before: Averaged 1.26 on the 4-point Likert scale, which indicated *limited* to *some interest*. After: Averaged 1.22 on the scale, which indicated *limited* to *some interest*.

[8] They were asked what percentage of class time in a typical week was spent on administrative tasks, homework review, lecture-style presentation by the instructor, instructor-guided student practice, re-teaching and clarification of content procedures, work in small groups, student independent practice, tests and quizzes, and other. In the sample, 30% of class time was spent on lecture, 22% on small groups, and 10% each on reviewing homework and instructor-guided student practice. All other practices were reported used less than 10% of the time.

Table 1. Instructor Teaching Experience and Class Size

Instructor pseudonym	Pat	Stevie
Times taught course previously	15-20	10
Years teaching at college level	10	25
Class size	23	102
Highest degree earned	PhD Mathematics Education	PhD Mathematics
Experience teaching at K-12 level	Yes	Yes
Teaching certificate	7-12 Mathematics	No

Selecting class sessions. I investigated a block of consecutive fractions lessons from each instructor. I chose to focus on the topic of fractions because both instructors were videotaped during their fractions lessons consistent with the purposes of the larger project. I viewed a block of consecutive lessons rather than lessons interspersed throughout the semester to get a more coherent view of the instructors' teaching practices since activities may be continued from previous class periods and instructors or students may refer to classroom incidents from earlier in the unit. Pat's fraction block included six consecutive class sessions that took place between 4/1/08 and 4/17/08. Though Stevie taught fractions throughout the semester, her class did have a block of five class sessions between 9/3/08 and 9/15/08 that were primarily about fractions. The topics covered during the consecutive class sessions can be seen in Table 2. Further detail of what happened on these class days can be seen in Table 19 and Table 20 in Appendix B.

Table 2. Class Dates and Topics

Instructor pseudonym	Date	Topics covered during lesson
Pat	4/1/08	Division, eastern European algorithm, classifying fraction word problems
	4/3/08	Classifying fraction problems, equal sharing, fraction addition, common denominator, modeling fractions
	4/8/08	Numerator and denominator, fraction addition, interpreting word problems
	4/10/08	Semiotics, fraction division, fraction multiplication, changing whole
	4/15/08	Fraction division (measurement and partitive)
	4/17/08	Fraction multiplication, representations of fractions, the changing whole, place value
Stevie	9/3/08	Plotting decimals on a number line, rounding, definition of fractions, rounding
	9/5/08	Place value, definition of fractions, numerator and denominator, the whole, improper fractions, K-8 student learning of fractions, equivalent fractions
	9/8/08	Fractions of objects, unit fractions, fractions as numbers, fraction on the number line, K-12 student errors about fractions on the number line, equivalent fractions
	9/10/08	Representations of fractions, ratio, changing wholes, comparing fractions, cross-multiplication
	9/15/08 [a]	Common denominators, pictorial representation of fractions, the whole, teaching K-6 students, comparing fractions, percent

[a] 9/13/08 was a test day where no teaching took place.

Definition of Whole-Class Discussion

For the purposes of this study, whole-class discussion is defined as an episode of at least one minute in duration where a student talks to the instructor or another student about the same discussion prompt while the rest of the class is expected to attend to the exchange(s). Discussion prompts are statements or questions that are intended to elicit discussion by posing a problem, asking a question, setting a topic, etc. Student responses that are in unison or a single student response that is shorter than a sentence (e.g. students giving one-word, factual answers to questions) do not count towards the minimum of student talk in discussion but may be counted as discussion if surrounded by more substantive student talk. Instructor explanations of over one minute may be included as discussion (rather than lecture), if they directly relate to previous discussion and are followed by additional student responses that are directly related to the discussion prompt.

Because this definition gives the minimum criteria for an activity being called whole-class discussion (namely that students converse in some fashion with the instructor or another student and the whole class is expected to participate or at least attend), it is inclusive, encompassing multiple modes of interaction. For example, it includes long, connected threads of responses from different students that are initiated by an instructor prompt; exchanges between an instructor and a single student that take place in front of the class; and long explanations from students to the class that are only periodically interrupted by questions or feedback. The purpose of using this inclusive definition was to capture the full breadth of whole-class discussion. This suits the purposes of this study because it allowed investigation of PST engagement and instructor teaching moves in a range of situations in these mathematics classes for PSTs rather than just looking for the types of discussion that literature suggested were paragons.

Data Reduction

I created summaries of the video observations, imported into NVivo (QSR International Pty Ltd., 2008, 2010, 2012) and then coded on the variables of interest to look for patterns or co-occurrences of variables. The sections that follow detail how the summaries were created, how they were coded, and what the variables of interest were.

Summaries. In order to code the data, I created summaries to paraphrase each class session, focusing on the variables pertinent to my research questions. During the summary process, I noted timing and duration of all activities by inserting time markers when activities started and ended. For whole-class discussions, I included additional time markers (about every 1-3 minutes) to facilitate later coding and indexing of data. I inserted a time marker when there was a change of topic, activity, or when feasible, student speaker. When multiple students interacted with each other in front of the class, which more often happened in Pat's class than Stevie's class, interactions were bounded by time markers when the conversational topic or group of speakers changed. Additional time markers were added when speakers paused in long dialogs or to bound unexpected events or events that were prototypical examples of an instructor's teaching practices for later indexing. Time markers were also used mid-activity in summaries of activities other than whole-class discussion although less frequently.

The summaries of whole-class discussions were more detailed than those of other class activities because whole-class discussion was the focus of this investigation. All student responses were transcribed word-for-word as far as they were audible. When student responses were inaudible or unable to be interpreted, this was noted. Instructor responses were often transcribed word-for-word (particularly when the instructor read a prompt or evaluated student thinking) but were sometimes summarized, particularly when instructors lectured, elaborated, or repeated student responses word-for-word. The fact that the instructor wrote on the board was recorded but the content was often summarized rather than described in detail because the video could be referenced. A new line was started when there was a change of speaker to improve readability. The method that was used to choose students to speak in discussion was indicated after the student's pseudonym the first time they spoke. Table 16 in Appendix A shows an excerpt of a summary.

Summaries rather than transcripts were used because a) they preserved the classroom processes and the temporal flow of the classroom; b) they were more readily coded since they highlighted variables of interest to the study, which also facilitated finding patterns in the data related to the variables of interest (Derry et al., 2010); c) they preserved the content of the ideas expressed in the classroom (often but not always in the same words the speakers used); d) they made my understanding of what was occurring in the classroom more explicit; and e) they saved time. Summaries were linked to the video using NVivo (QSR International Pty Ltd., 2008, 2010, 2012), which is discussed later in this section, so coding could be checked against actual words and actions in that section of video at any point during analysis. Because of this, I concluded that the additional detail that transcripts might have provided would not add information beyond the original video source nor improved my ability to index and code the video over using summaries.

Once summaries were completed, I imported them and the original video files into NVivo. Using the time markers, I then linked the video to the summary so that blocks of time on the video correspond to rows in the summaries. Using NVivo enabled me to keep the summaries and coding connected with the video.

Coding

Coding the summaries began with classroom activities. This allowed me to focus my coding and analysis on instances of whole-class discussion. Summary rows were coded for classroom activities because the start and end of activities always had a time stamp. I then coded the other variables of interest on the summary text of the whole-class discussions line-by-line or phrase-by-phrase. The other initial variables of interest were the kinds of prompts instructors used, how instructors responded to student thinking, and how instructors connected student responses to each other and to mathematical ideas. Each is discussed in detail below. Other variables emerged during the coding process and those are discussed below.

After the initial round of coding was complete, I rewatched all classroom video to revise summaries and recode, including coding for variables that emerged later. At this point I also coded what topic was being covered (see Table 17 in Appendix A) and coded what was going on in the class between each timestamp on the summary (context coding). See Table 18 in Appendix A for examples of this coding. This last round of more macro-level coding was used as context for the other coded variables and some of these codes were a way to measure the connectedness of discussions, which will be discussed in a later section. After revised coding was complete, I charted how often the variables of interest, which are discussed in more depth in the sections that follow, occurred and in what order they occurred. I then examined the data for patterns in each variable within each instructor's classroom and compared patterns between these two instructors.

Aspects of instruction preceding whole-class discussion. The following sub-sections discuss the coding of variables for aspects of instruction that precede an episode of discussion. First, I discuss how I coded classroom activities and how I defined whole-class discussion for this study. Second, I discuss how discussion prompts were coded using a modified version of King's (1994) question types to include non-interrogative expressions.

Activities preceding whole-class discussion. I coded all activities during the class session as *administrative*, *lecture*, *small group work*, *student independent work*, *test or quiz*, *whole-class discussion*, or *other*. Definitions of these codes are shown in Table 3. The only activity classified as other that occurred repeatedly in either of these two classes was *class break* (i.e., instructor announces that students have short mid-class break before they have to return to class). Since one whole-class discussion might have consisted of multiple rows of the summary and because consecutive rows of the summary might have been different whole-class discussions, after row coding was complete, I looked at the coded summary and noted the start and end of individual discussions. I then used a function in NVivo called *range coding* to code the video according to timespans and number the whole-class discussions (e.g., whole-class discussion 1, whole-class discussion 2) in each class.

After coding was complete, I examined which activities most frequently preceded whole-class discussion and how they were connected (e.g., same topic or prompt as whole-class discussion or giving students time to think about something brought up in whole-class discussion) to episodes of whole-class discussion.

Discussion prompts. In addition to coding episodes of whole-class discussion, I coded the discussion prompts that preceded the discussion phrase by phrase. I used King's (1994) question types (*factual questions*, *comprehension questions*, and *integration questions*) to code discussion prompts that were questions. Prompts that were not questions were coded for the same types of content as the questions but they were labeled *factual statements*, *comprehension statements*, and *integration statements*. For further definitions of these codes and examples of prompts coded, see Table 4. For actual text of the coded prompts, see Table 21 and Table 22 in Appendix C.

Table 3. Codes for Activities Preceding Whole-Class Discussion

Code	Definition
Administrative	An episode of at least one minute in duration where the instructor or the instructor and students discussed non-content-based topics such as future or past assignments, the timing or grading of exams, etc.
Lecture	An episode of at least one minute in duration where the instructor was the only one speaking or the only student response is in unison
Small group work	An episode of at least one minute in duration where students worked or talked in two or more groups of two or more students
Student independent work	An episode of at least one minute in duration where students worked alone on a problem, activity, or other task
Test or quiz	An episode of at least one minute in duration where students worked on a test or quiz
Whole-class discussion	An episode of at least one minute in duration where at least one student talked to the instructor or another student about the same topic or class activity in front of the rest of the class. Student responses that were in unison or a single student response that is shorter than a sentence did not count towards the minimum of student talk in discussion but may be counted as discussion if surrounded by more substantive student talk. Instructor explanations of over one minute were included as discussion if they directly related to previous discussion and were followed by additional student responses that were directly related to the class activity or if they summarized a point made in the discussion.
Other	Activities not mentioned above.

Table 4. Codes for Discussion Prompts

Code	Definition with example
Statement	Non-interrogative expressions; prompts that are not in the form of a question but that still elicit a response
Factual statement	Statement that tells students to recall facts or information that was explicitly covered in the lesson. For example, “Write each percent as a fraction in simplest form.”
Comprehension statement	Statement that tells students to describe or define a process or term. For example, “describe the whole associated with this fraction. In other words, describe what each fraction is *of*.”
Integration statement	Statement that tells students to go “beyond what was explicitly stated in the lesson by linking two ideas together in some way, asking for explanation, inference, justification, or speculation” (King, 1994, p. 351). For example, “Use the meaning of factions and the following picture to help you explain why Mireya's strategy is valid.”
Question	An interrogative
Factual question	Question that asks students to recall facts or information that was explicitly covered in the lesson. For example, “Of 7/12 and 8/12, which one is greater?”
Comprehension question	Question that asks students to describe or define a process or term. For example, “Why can you compare fractions using cross multiplication?”
Integration question	Question that asks students to go “beyond what was explicitly stated in the lesson by linking two ideas together in some way, asking for explanation, inference, justification, or speculation” (King, 1994, p. 351). For example, “Maurice says that the next picture shows that 3/6 is bigger than 2/3. The shaded portion representing 3/6 is larger than the shaded portion representing 2/3, so why is Maurice not correct?”
Compound prompt	Two or more of the above prompt types are given to students at once to work on for the same activity. Prompt types within compound prompts are called sub-prompts in this paper. For example, a factual question and a comprehension question such as, “Of 7/12 and 8/12, which one is greater and why?”

Note. The prompt types are adapted from King (1994). Prompt examples are from the current study.

I looked at the most frequent prompt types each instructor used and investigated if there was a pattern in their use, such as whether certain kinds of prompts were frequently used earlier or later in a class period, if one type of prompt typically followed another, or if instructors often combined two types of prompt into a *compound prompt*. In order to do this I created frequency tables for each class period (see Table 11 and Table 12) and a table to display the order of prompts in the classrooms (see Table 10) as well as sub-prompts of *compound prompts*.

Facilitating whole-class discussion. The following sub-sections discuss the coding of variables to address research question 2, which addresses instructor teaching moves to facilitate discussion while it is occurring. First, I discuss coding how instructors choose which students talk in discussion. Second, I discuss the coding of instructors' responses to student thinking made public in discussion. Third, I discuss coding the connections that instructors make between student responses or between student responses and disciplinary knowledge.

Choosing which students talk in whole-class discussion. When the summaries were created, the method that instructors used to choose which students spoke in class was noted. I coded whether instructors asked for volunteers (*elicited volunteer*), called on non-volunteer students (*cold calling*), or if students spoke in class without the instructor calling on them (*self-select*). When I could determine whether the instructor called on a specific student randomly or not, I coded that as well. For definitions, see Table 5. I then examined when instructors used each method, and whether instructors used one or more methods consistently or not. I created tables for frequency of use in each class period and for the sequence of methods used and looked for patterns.

Table 5. Codes for How Students Were Chosen to Talk in Whole-Class Discussion

Code	Definition	Example of summary text coded
Elicited volunteer	Students who indicate that they wish to talk in discussion and are called on by an instructor before they contribute to discussion.	(VOLUNTEER)
Self-select	Students who contribute to discussion without explicit instructor consent.	(SELF-SELECT)
Random cold call	Students who do not volunteer and are called on using a randomization method such as pulling names out of a bag.	(RANDOM)
Non-random cold call	Students who do not volunteer are called on but a randomization method is not used.	(NON-RANDOM by GROUP) (NON-RANDOM by SOLUTION)

Responses to student thinking. I coded whether the instructor *evaluated* the student response or asked the class to evaluate the response. Acknowledgment tokens, such as “yeah” or “okay,” were often used in response to student explanations (Volet, Summers, & Thurman, 2009). I did not code these as *evaluated* because, as acknowledgement tokens, they did not carry evaluative weight in this class. Instead, they seemed to indicate that a student's response was heard and/or understood. When there was evaluation, I noted what was evaluated (i.e., the student's solution, reasoning, or participation) if it was possible to discern. This was not always possible because sometimes instructors said something like, “Good job” (Pat 4/8/08 starting 92:02), without going into further detail. Similar to Hogan, et al. (1999), I looked for whether the instructor *asked for further information* (such as justifications, explanations, or clarifications), *repeated or revoiced the student response*, or *asked student(s) to apply their own reasoning to the responses*. I also looked for when instructors *asked student(s) to connect peer* *responses*, *asked student(s) to connect disciplinary knowledge to a peer's response*, *asked for questions or agreement/disagreement*, *asked for other solutions or responses*, *connected student responses*, *evaluated a student response*, *elaborated*, or *made student thinking visible*. These categories were not mutually exclusive since an instructor may make multiple pedagogical moves in response to a student. For definitions of codes, see Table 6. Again, I looked for instructor patterns in this instructional practice. To do this I looked at the visualizations that NVivo provided to see if there was a temporal pattern in responses.

Connectedness of student responses. I looked at the ways instructors connected student responses to each other and to knowledge in the discipline. I looked for instances where instructors compared or contrasted student responses, connected student responses to disciplinary knowledge, or asked students to do the same (Chapin et al., 2009c; Engle & Conant, 2002; Stein et al., 2008, p. 331). Definitions of these can be seen in Table 6, in the previous sub-section. I also looked at the context codes to see if the prompt itself asked students to connect their thinking to that of another student. Again, context codes were codes to give context to the variables of interest in this study that describe what was occurring in the class at a more macro level. See Table 18 in Appendix A for examples of context codes. These were codes were cleaned and aggregated to create four new codes: instructor connecting to mathematics, PST connecting to mathematics, instructor connecting PST response, and PST connecting to peer responses. I looked for instructor patterns in this instructional practice by looking at which of these occurred during each discussion.

Table 6. Codes for Instructor Responses to Student Thinking

Code	Definition
Asked student(s) to . . .	
Apply their own reasoning to a peer's response	Instructor asked students to connect their ideas to that of a classmate though such methods as comparing, contrasting, or combining their ideas to their peers.
Connect peer responses	Instructor asked students to connect the ideas of classmates though such methods as comparing, contrasting, or combining peer ideas.
Connect disciplinary knowledge to a peer's response	Instructor asked students to explain a student response using disciplinary knowledge, rephrase a student response using disciplinary language, or in some other way asked students to apply disciplinary knowledge to a student response.
Asked for . . .	
Questions or agreement/disagreement	Instructor asked if students have questions, if they understood what was just said, or if they agreed/disagreed with a student response.
Other solutions or responses	Instructor asked for alternative solutions from the class as a whole or another student.
Further information	Instructor asked the student for more information, additional reasoning, or justification.
Connected student responses	Instructor concatenated, compared, and/or contrasted student responses in discussion.
Evaluated a student response	Instructor expressed their judgment of a student response.
Elaborated	Instructor added to a student response.
Repeat or revoice student responses	Instructor repeated or revoiced a student response.
Made student thinking visible	Instructor made all or part of a student response visible through writing on a whiteboard/document camera/PowerPoint/etc.
Other	Instructor responses that did not fit into the above categories. These were periodically examined to determine if new categories needed to be constructed.

Table 7. Codes for Connecting Student Responses to Mathematical Ideas or Student Thinking

Connecting code	Aggregated coding
Instructor connected to mathematics	Instructor elaborated
PST connected to mathematics	Asked students to connect-apply disciplinary knowledge, PST connected with other problem, Instructor asked what mathematics are needed (C)
Instructor connected PST responses	Instructor compared strategies (C), Instructor connected student responses
PST connected peer responses	Asked students to apply their own reasoning to another student's response, Asked students to connect other students' responses, comparing strategies (C)

Note. (C) indicates context code.

Summary

In summary, the process I used included a pass through the video to create summaries and code the variables of interest in this study (see Table 16 for a summary excerpt and Table 3, Table 4, Table 5, Table 6, and Table 7 for the codes used). I used a second pass through the video to revise the summaries and coding. I then coded summaries for the topic covered and what was going on in the class to give the variables of interest more context (see Table 17 and Table 18 in Appendix A for topic and context codes). I then examined the data for patterns in each variable within each instructor's classroom and compared patterns between these two instructors. The chapters that follow present an overview of each instructor (Chapters 4 and 5), present the results of this study (Chapter 6), and discuss the conclusions, limitations, and significance of this study (Chapter 7).

Chapter 4: The Case of Pat

This chapter gives a qualitative image of how Pat taught. The case starts with a vignette of a whole-class discussion in Pat's class. Then I discuss classroom layout, instructor resources, atmosphere of the class, types of whole-class discussions, and other important aspects that characterize the class. Chapter 5 provides a qualitative image of Stevie's teaching while Chapter 6 provides a more systematic analysis of the teaching practices of interest for this study (as discussed in Chapter 3), but this chapter provides context for those practices.

Vignette of a Discussion of the Waffle Problem in Pat's Class on 4/10/08

The following whole-class discussion took place about a third of the way through the 4/10/08 class. Pat started class with a half hour lecture on the importance of semiotics for mathematics then he gave students time in small groups to discuss problems on a worksheet. This whole-class discussion lasted around 25 minutes.

Pat gave time to think about the prompt and then polled for answers. After 15-20 minutes working on a worksheet in small groups, Pat brought students out of small-group discussion and previewed that they would probably cover two of the three problems on the worksheet, saving the third for the following class period. They started with the first problem on the worksheet, the Waffle Problem. On the worksheet, the problem stated:

> For the following questions, (a) model an approach to solve following questions. Make sure you can explain all your quantities and reasoning, (b) write a mathematical record/number sentence that matches the problem you were asked to solve. Make sure you show your work and are able to explain all your mathematics.

1. A batch of waffles requires 3/4-of a cup of milk. You have two cups of milk.

Exactly how many batches of waffles could you make?

Pat cold called Student 1 who gave her group's answer: two batches with one-half a cup of milk left over. Pat asked for other solutions and students volunteered two and two-thirds batches and two and two-fourths, which a student pointed out reduced to two and one-half.

Pat asked a student to explain a (incorrect) solution at the board. Pat asked the student who gave the last solution, Harry,[9] to present this solution at the board and then reminded students, "again, pretend we're a group of 5th graders so that as you do it . . . be real clear in your representations and the meaning of your numbers and pictures." In this way all the PSTs tried to think like teachers, either trying to clearly explain a solution at a fifth grade level or trying to interpret student thinking.

Harry explained his solution at the board often pausing and seeming confused at his own thinking. He silently reproduced the diagrams that he had written on his paper: two green rectangles that were divided into four pieces and shaded three pieces of each rectangle, as seen[10] in Figure 1. He then took a moment to think because he had forgotten what these meant. He explained that each shaded portion of the bars was a batch and since he could shade one section on each rectangle, he had two batches. The remaining two one-fourths add to make two-fourths, which reduce to one-half.

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[9] All student names presented are pseudonyms.

[10] Reproductions of writing on the whiteboard were traced from printed screenshots, sometimes with slight alterations (such as increasing the size or legibility of words) to aid readability. The full whiteboard diagrams from this discussion can be seen in Figure 21 and Figure 22 in Appendix B. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

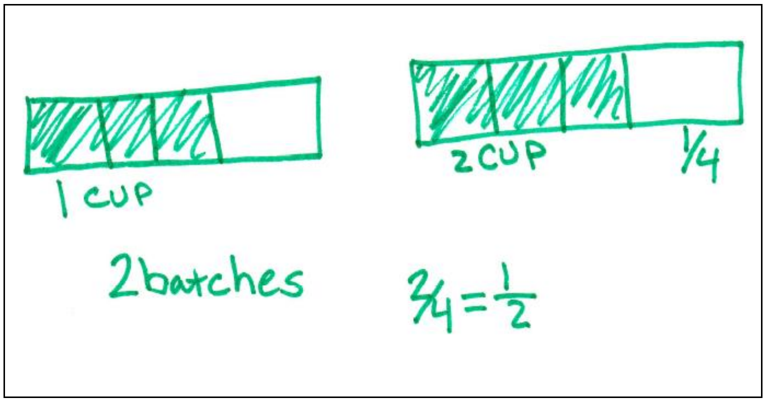

Figure 1. Harry's diagram for the Waffle Problem

Pat asked for clarification. Pat pushed Harry for clarification, outlining the rectangles used in the solution in red and asked, "What do those red rectangles represent either contextually or conceptually?"

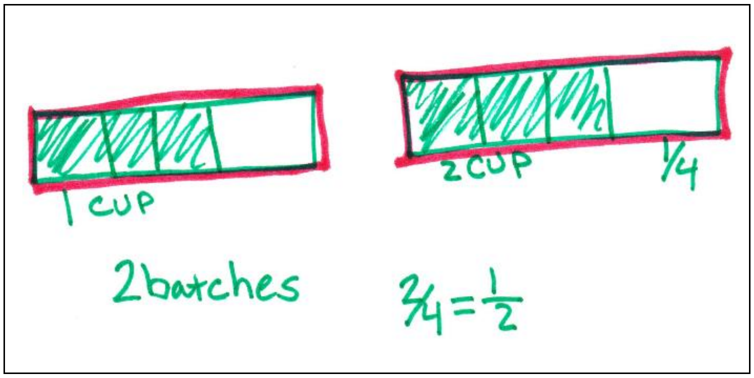

Figure 2. Pat outlines Harry's diagram to ask for explanation

Harry answered that the red rectangles were cups and his other group members add that it's "cups of milk" and "what you need." In Pat's class, small group members were free to help a student called up to present a solution. Harry went on to explain that the green shading on the first rectangle represented one batch and that he did the same to the second rectangle. The two unshaded regions were "what you have left," which is two-fourths. He said the answer was two batches and two-fourths of a cup, which is mathematically correct but not necessarily the answer to the prompt, "Exactly how many batches of waffles could you make?" At this point Harry does not give any indication that he realizes two batches and two-fourths of a cup is any different than his first answer, the unit-less two and two-fourths.

Pat asked students if students had questions. After students explained solutions, Pat always asked the other students if they had questions for the student who had just explained. In this case, Student 2 asked whether Harry was leaving the two-fourths as left over rather than making a partial batch.[11] Her question hones in on the difference between the answer being two batches and two-fourths cups or two and two-fourths. Harry confirmed he was not making a partial batch and Pat asked Student 2 to clarify what she was asking. Student 2 said that Student 2's group had said the answer was two and two-thirds because they had made a partial batch.

Pat asked the class whether the answer was two and one-half or two and two-thirds. Students discussed that the one-half was "like a remainder for the leftovers," that it was not correct to say that the answer was two and one-half batches, and that two and two-thirds batches was a right answer.

Pat asked a student to explain a (correct) solution. Student 3 began to explain how her group came up with two and two-thirds batches as an answer and Pat asked her to explain it at the board. Student 3 started using Harry's diagram to explain that three of the boxes were needed to make one batch, so they have two batches. Then she explained the remaining one-fourth by explicitly referencing something they had learned in a previous lesson. She said,

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[11] When student names could not be determined from the videotape, they were numbered.

> we have 2 of these [boxes] left over, but you only need 3 to make a batch so 3 is gonna be what we learned in the last class—how our bottom denominator, like it represented the whole. So we're saying, . . . you need three of them but we only have two left over so you have two-thirds.

Harry seemed to understand this solution at that point but another student asked Student 3 to draw a diagram, seen in Figure 3, of what she talked about to better illustrate for the rest of the class, not because she did not understand. Note that she used a blue pen, a different color than either Pat or Harry used. This allowed for each person's thinking to be visible and separate and yet superimposed to make a single representation.

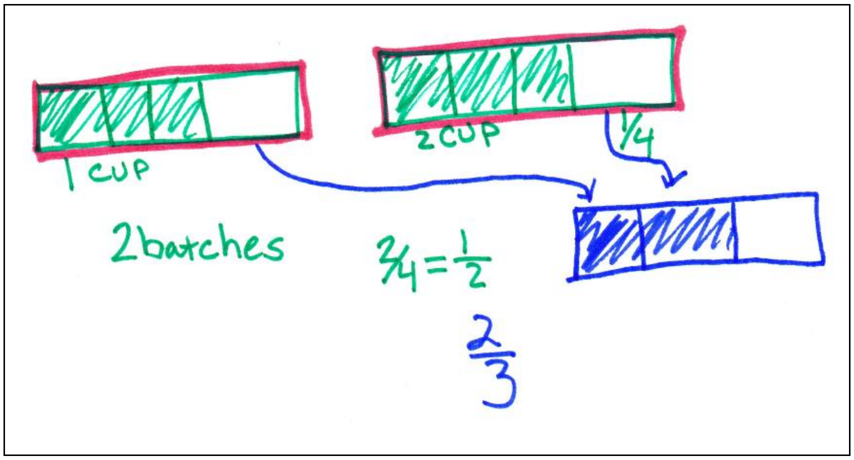

Figure 3. Student 3's additions to Harry's diagram

Pat asked, "what's the issue with this two and one-half then?" Student 3 explained, "two is the number of batches you have" and one-half is, "how much milk you have left over." Pat and the students joked about drinking the milk that was leftover when there was not enough to make a whole recipe. Several days earlier (on 9/3/08), they had discussed that when the elementary students they worked with had been confronted with remainders in division problems, they talked about real world ways to get rid of the remainders.

Pat pointed out "something goofy": PSTs explained the student error. Pat commented, "When we write two and one-half, there's something goofy about this. . . . In formal mathematics, this is not what we would write. Why not?" A student answers, "you'd have to write, like, two remainder one-half." In order to get them to answer why you would have to write it this way, Pat asked what the two and one-half represented (students answered "batches" and "milk") and whether they could add the two and one-half. Students answered that they could not "because the two stands for batches while one-half stands for leftovers. . . . In two and two-thirds, the two stands for batches and the two-thirds stands for batches."

Pat then played devil's advocate, saying "OK, but I've got a common denominator" to add two and one-half, which led students to differentiate between blindly using a procedure and doing something that is mathematically correct in the context of the problem. They commented that, "you can add it [because you have a common denominator] but it doesn't mean anything" and "It's not correct in the context that you're using."

Pat elaborated on this, "so the issue here is this two and this one-half are different things." He told them to notice that they have two different representations on the board, one that is correct, and one that is not. He uses red marker to cross out the incorrect representations [$2 + \frac{1}{2} = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$], reiterating a previous student responses that "you can't add because we have two different things." He wrote this on the board before he lectured for a couple minutes on the issue of abstraction in this problem and the problem of the changing whole.

Pat asked for a number sentence. Pat reminded them that the problem asked for a number sentence and a student volunteered $\frac{8}{4} - \frac{6}{4} = \frac{2}{4}$. Pat asked what the $\frac{6}{4}$ was and what the units on all of the numbers were. Students explained that it was the two three-fourths, which represented two batches, and the units were cups.

Then Pat asked for other solutions and a student volunteered $2 \times \frac{4}{3} = \frac{8}{3} = 2\,\frac{2}{3}$. Students questioned what the four-thirds represented and the student explained that they had used four-thirds because they were dividing by three-fourths. While the students gave their answers, Pat wrote each on the whiteboard as seen in the reproduction in Figure 4.

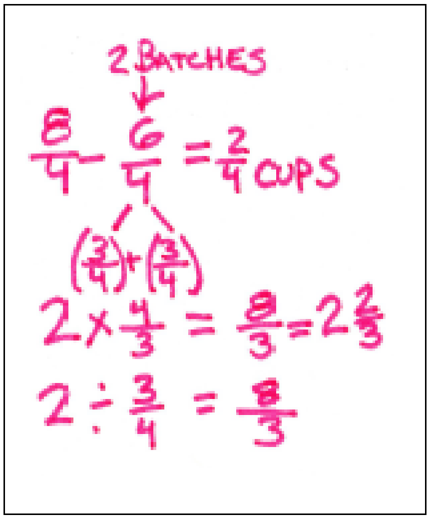

Figure 4. Number sentences that students proposed for the Waffle Problem with explication

Pat asked students to connect the number sentence to the problem. Pat asked whether they used four-thirds because it was, “the algorithm that you remembered or is it actually connected to the context in some way?” When the student admitted that it is because of the algorithm, Pat asked whether it could be tied to the context in some way. Several students then talked about the problem at once. Lauren tried to explain by drawing a diagram on the board but she did not finish, see in the reproduction in Figure 5. Other students tried to explain her reasoning but became confused in their explanations.

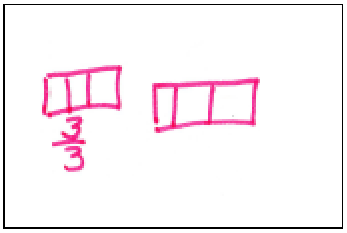

Figure 5. Unfinished diagram that Lauren drew on the board

Another student proposed, “Doesn’t the four-thirds represent 1 cup of milk, the number of batches that 1 cup of milk will make?” but Lauren and the others did not seem to hear him so they reiterated that this was the procedure they had learned.

They focused on what elementary students know. A student asked whether the “kids even know how to divide fractions,” because since she was “pretending to be the student” she did not know if she was supposed to know the “times the reciprocal” rule or not. She said students would probably come up with $2 \div \frac{3}{4}$ for the number sentence, if they did not know that rule.

Pat then reminded the class about elementary students asking, “What kind of problem is it” when they worked with them. Pat emphasized that they asked the same kind of questions here. He asked, “Some of you say it's division. If it's division, what about this problem makes it division?” A student answered,

> they all of a sudden did $\frac{2}{1}$ divided by $\frac{3}{4}$ and I was like, “OK, why is this division?” And then I was thinking, “OK, we're trying to find how many three-fourths cups.” . . . so I kind of made that in to the whole. We're trying to find out how many three-fourths are in two cups so then just trying to figure out how many are in.

Pat revoiced this and segued into the next activity, in which he asked PSTs to come up with a whole-number version of this problem to see if it was easier for them to determine what operation this problem used.

Overview of Pat's Class

Pat taught a class of 23 students in two 80-minute sessions each week. His class had a casual atmosphere. He rolled a cart in at the start of class with materials for the class and a coffee maker with coffee brewing. Students took their seats at tables of four. When the class got underway, he typically assigned a problem (or set of problems) or he asked students to review problems they worked on in the previous class or for homework. The students worked in groups at their tables, and while they worked, Pat walked around the room observing and answering questions. After several minutes, he called the class to order and selected students to share their group's work. Sometimes he picked the group based on how they solved the problem (including what representations they used, what mistakes they made, and how far they got with their solution), other times students were allowed to volunteer, or students were chosen to speak via a set of cards with student names that Pat drew from randomly. In his interview, he talked about why he chose some students to present their solutions to a problem. He said,

> They did the [Wedding] problem[12] for homework, right? And so, while they were discussing in that little 10 minute interlude I was . . . feverishly going through their papers looking for very specific representations to guide that conversation. Notice that the wrong answer went up, and I put that wrong answer knowing that over half students probably were thinking about that way. . . . But what's interesting is you

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[12] You go to a wedding. 3/5 of the men at the wedding are married to 2/3 of the women at the wedding. What fraction of all the people at the wedding are married?

> throw that wrong answer up and then a student eventually said, "you know what, you've got an extra married man," right? And that is all of part of my lesson plan in my head (Pat, Apr. 3, 2008).

A CGI (Cognitively Guided Instruction) student and teacher himself, Pat used problems with which he was very familiar and which he had used before in his teaching. He anticipated what students would do with the problems, and looked for particular solutions, errors, and explanations to get into the mix of ideas presented to the class. His familiarity with the problems, and with how his students would solve them, made it possible for him to sequence solutions purposefully to illustrate specific ideas. Also, Pat's students worked with elementary school students as part of the class and Pat sometimes connected what they were talking about in class with what the PSTs experienced with the elementary students or asked PSTs to predict how the elementary students might do a problem.

When one or more students were asked to present whole solutions to the class, Pat stayed in the back of the classroom and observed, making only occasional comments. Pat expected the presenting student(s) to field questions from the other PSTs and the other PSTs were expected to understand the presented solution well enough to explain every part of it, anticipating that they might be asked to explain all or part of the solution to the rest of the class. Sometimes Pat made this expectation explicit:

> Again, as we role play, Kennedy's the role of a teacher so your context is you're trying to explain it as clearly as you can to a group of 5th or 6th graders. Right? And then for you folks [the rest of the class], your contexts are teachers like me, sitting in the back of the room and your context is now she is an upper elementary or middle

> school student explaining her work and you're trying to figure out what she's doing and how she's thinking. So you can explain it to someone else. . . . So everyone in the room is acting like a teacher (4/15/08 starting 17:34).

Another example can be seen in the opening vignette.

Often Pat asked the PSTs to compare and contrast solutions or to connect steps in one solution to steps in another. Sometimes near the end of a class session, in the interest of time, Pat made these connections himself rather than having students do it. An example of this is a discussion on 4/8/08 where three PSTs presented solutions for the Bus Problem, which they had been given time to work on in small groups the previous day, for a majority (over 1.25 hours) of the class period. Two students had presented full solutions to the class and the rest of the students had discussed the meaning of each number and operation in each solution and connected the steps in the first solution to the steps in the second solution (for the student representations and annotations, see Figure 6). At this point, Pat asked a third student (Kayla) to present her solution with 10 minutes of class left. She presented it in about 3 minutes. While explaining the solution, Kayla drew boxes to indicate a number of people and crossed them out to indicate that people had gotten off of the bus so that when she was finished with each row, the drawing represented the number of people on the bus when the bus left the stop. Once she had finished explaining to the class, Pat took over responsibility of further explaining her solution. He said,

> Kayla did a nice job explaining her thinking but what I want to—and this is kind of the role of a teacher, is if Kayla were a 6th grade student . . . this is what I would be doing (4/8/08 starting 104:07).

At that point, he numbered the steps (rows) in the solution and added steps that represented what how many people were on the bus before people got off at each stop.

Although the types of discussion described above were in many ways typical, there were other types as well. When prompts asked for categorization or to explain a concept, the instructor took a more central role. In these instructor guided discussions, Pat often stayed at the front of the room and polled the class or each group for their responses, writing responses on the whiteboard or overhead projector. After the responses were on the board or overhead, Pat asked students to explain further and they did. At the end of discussion, he summarized student responses sometimes adding language that was more precise. If there was an incorrect answer, Pat often got students explain why the answer was incorrect.

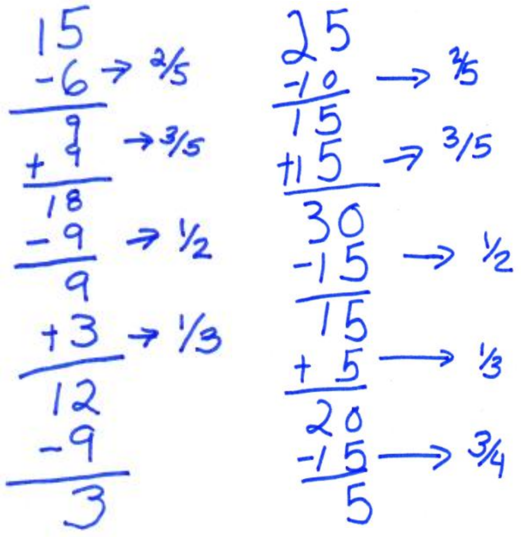
Pat also had "instructor as scribe" discussions. During these discussions, Pat served as the scribe for student ideas by typing, on a document projected for the class, what students said in response to questions or problems. His explanation to students in a class where he was the scribe was,

> [I'll] ask you share either your own opinion or your group's opinion, or what you guys discussed and again, I'll kind of be the class scribe and so you can feel free to participate in the discussion because it's not necessary to get this in your notes (4/1/08 starting 23:51).

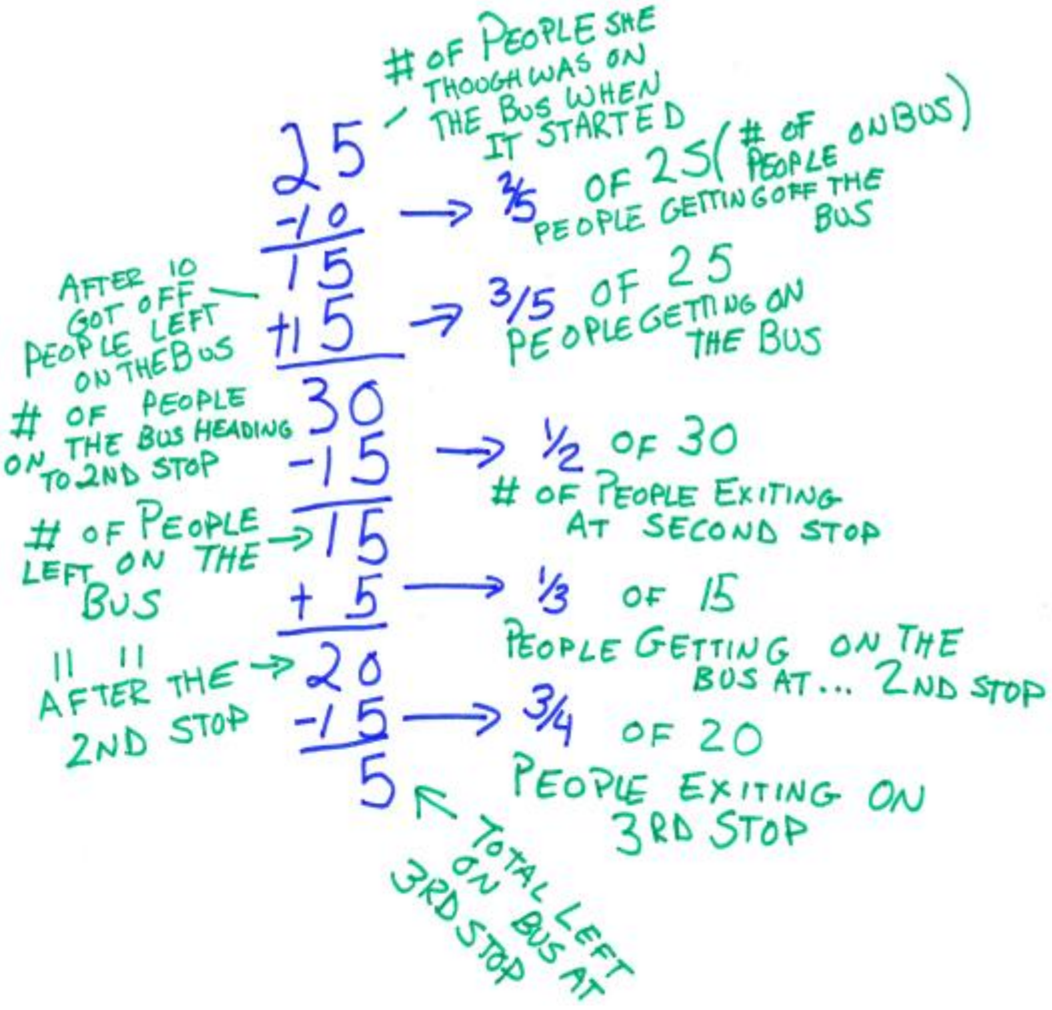
When Pat was the class scribe, he shared his notes with students after class. When asked about his purpose for this type of discussion, he said, "My purpose is to get them engaged in the conversation, and . . . just focus on what people are saying. Don't worry about writing" (Pat, Apr. 3, 2008).

First student solution: Whole-Number Arithmetic Method (starting 31:53)

Student solution:



Solution with annotations:



The student used the blue marker while Pat used the green.

Figure 6. Three annotated student solutions to the Bus Problem (4/8/09)

Figure 6 (cont'd)

Second student solution: Multiplying Fractions (starting 57:18)

Student solution:

$$1-\frac{2}{5}=\frac{3}{5}$$

$$\frac{3}{5}+\frac{3}{5}=\frac{6}{5}$$

$$\frac{6}{5}\cdot\frac{1}{2}=\frac{6}{10}=\frac{3}{5}$$

$$\frac{3}{5}\cdot\frac{4}{3}=\frac{12}{15}=\frac{4}{5}$$

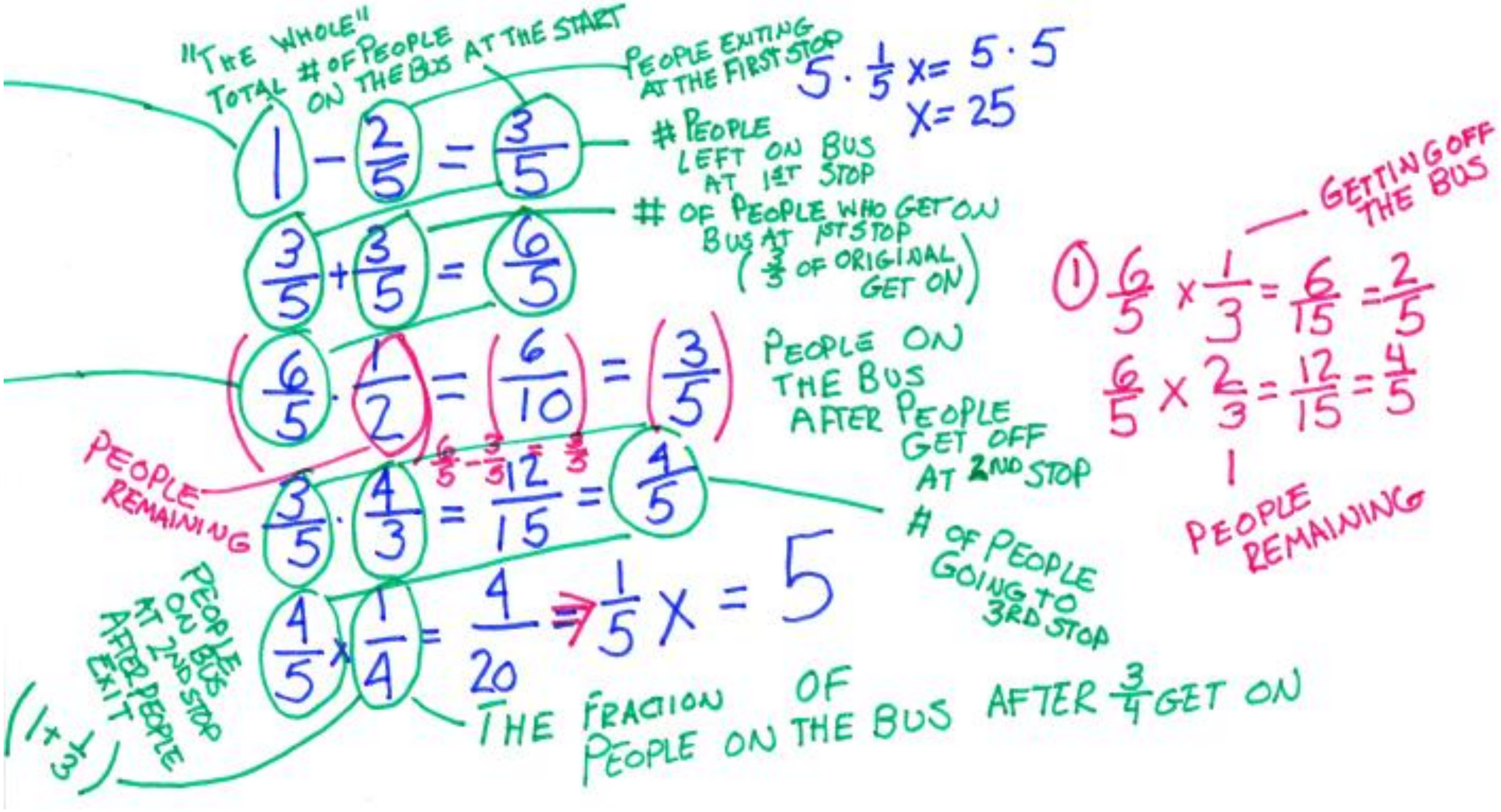
$$\frac{4}{5}\cdot\frac{1}{4}=\frac{4}{20}=\frac{1}{5}x=5$$

$$5\cdot\frac{1}{5}x=5\cdot 5$$

$$x=25$$

Figure 6 (cont'd)

Second student solution: Multiplying Fractions (starting 57:18) (continued)
Solution with annotations:

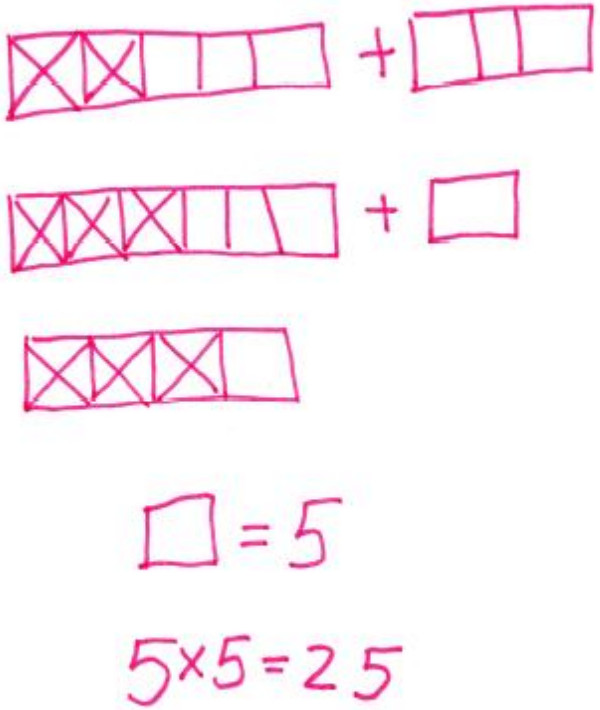


The student used the blue marker while Pat used green and red. The green lines at the left middle and top connect to the first student's solution.

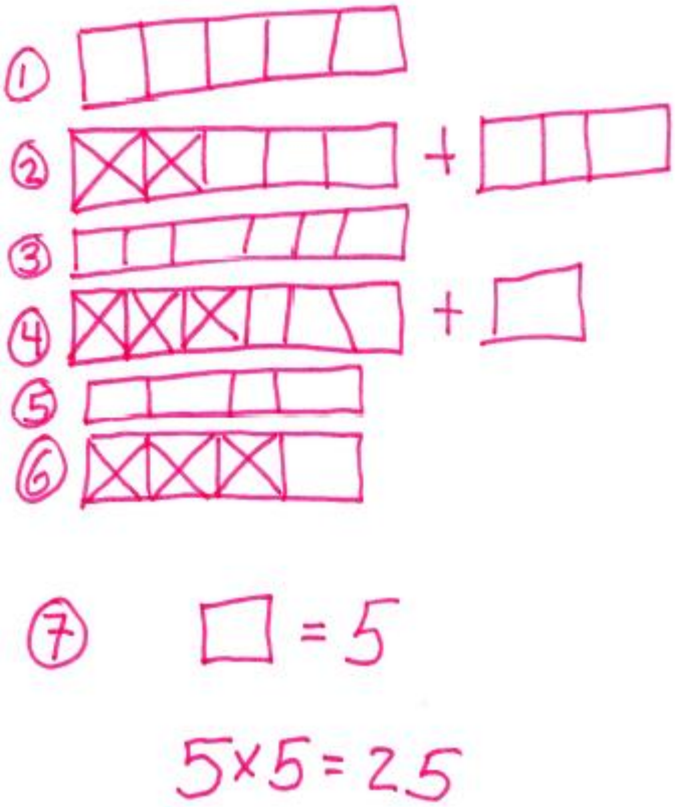
Figure 6 (cont'd)

Third student solution: Pictorial Strategy (starting 101:10)

Student solution:



Pat adds numbering and intermediate steps:



Both the student and Pat used red marker.

In Pat's class, every number, symbol, or graphic had to be connected to the context of the problem at hand, and in his review, he made sure that all such explanations were explicit. For example, in the Bus Problem,[13] there were many instances of the fraction three-fifths but it applied to different things: Was it three-fifths of the people on the bus at the last stop? Three-fifths of the people who got *on* the bus? Three-fifths of the people who got *off* the bus? The opening vignette shows other instances where students had to explain diagrams and number sentences in the context of the problem. It was not acceptable to simply put up a string of calculations or graphics that came to the correct final answer. Each step had to be meaningfully connected to the problem.[14] In an interview, Pat explained that he did not just want students to be able to explain why. He said,

> I do think about [it] much more specifically. It is not just why. What do the operations and numbers mean: meaningfulness. . . . This issue of sense making, which one of these answers makes sense. Can you follow the representation and use that as the point of evaluation. . . . I think those are big ideas (Pat, Apr. 3, 2008).

Often students made this work of making meaning explicit in their presentations, but if details were missing, Pat made sure to include them through asking students to explain peer solutions. This sometimes led to additional discussions, for example, during the discussion of the Bus Problem there was some confusion about whether a number in a student's solution referred to people getting on the bus or getting off the bus at the second

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[13] Some people were on a bus. At the first stop, 2/5 of the people got off and 3/5 of the original number got on. At the second stop, 1/2 of the people got off and 1/3 of the number that was left on the bus got on. At the last stop, 3/4 of the people got off, leaving 5 people on the bus. How many people were on the bus before the bus reached the first stop? Solve this problem in at least two different ways.

[14] Student solutions and Pat's annotations to student solutions can be seen in Figure 6.

stop. This evoked a tangential discussion about whether the same method that a student had used would work given different numbers in the problem.

A hallmark of Pat's class was that students listened to each other, asked questions, and seemed to try to make sense of and understand each other. Pat rarely intervened, instead leaving plenty of time for students to find and correct errors themselves. After each student finished presenting their group's work and taking questions from the other students, Pat gave students time to consolidate their ideas with their small group before they were expected to re-explain the solution to the class. Pat sometimes asked students to explain the solution again in its entirety but more often Pat moved to the front of the class and asked what each number or step in the problem meant, annotating the solution with student explanations. If multiple solutions were presented, he often asked students to make connections between the numbers and steps in these solutions, which he also wrote on the board. Though Pat sometimes pointed out key ideas and made corrections to errors that had not been noticed or adequately addressed, students did the bulk of the explaining. Pat used techniques like randomly calling on students to explain other students' work, with the result that students expected to be held accountable for what other students said, and for being able to explain themselves.

One important aspect of Pat’s class was his use expert use of the whiteboard. The classroom had a 3-panel whiteboard at the front that he used to make his student’s work visible and to show connections across different solutions. What went up on the board—whether written by Pat or by a student—stayed there until the problem was done, even if there were errors. Pat added to solutions, crossed out errors, and drew connecting arrows between similar parts. At the end of a discussion, the board was filled with explanations of the numbers and graphic representations in the students' solutions. Figure 6 and Figure 21 show some examples of Pat making student thinking visible in annotated solutions but the next chapter goes into further detail.

One feature of each of Pat's classes was variety. Every class included small group work, whole-class discussions, and lecture but each of these varied widely in duration and practice from class to class. For example, lecture sometimes included PowerPoint and lasted over 30 minutes and sometimes it consisted of Pat working out an example on the overhead projector for several minutes. Small group work lasted from around 1 minute to over 20 minutes. In small groups, student work might be discussing a solution that a peer just presented or completing a worksheet with several problems that Pat distributed. In order to have this variety, Pat explicitly told students what he wanted them to do, often both before they talked in small groups and in the transition between small group and whole-class discussion.

This description of Pat's class is intended to give the reader an overview of Pat's class. Next, I describe some typical aspects of Stevie's class after I present a vignette of a whole-class discussion in that class.

Chapter 5: The Case of Stevie

The following case gives a qualitative image of how Stevie taught. The case starts with a vignette of a whole-class discussion in Stevie's class. Then I discuss classroom layout, instructor resources, atmosphere of the class, types of whole-class discussions, and other important aspects that characterize the class. Chapter 6 provides a more systematic analysis of the teaching practices of interest for this study (as discussed in Chapter 3), but this chapter provides context for those practices.

Vignette of a Discussion of a Piece of Paper as a Fraction on 9/3/08

The following whole-class discussion took place during the last third of class on 9/3/08. Stevie had gone over quiz and homework problems, and rounding and plotting on a number line before turning to fractions. This was the second whole-class discussion of the class period and it lasted around 10 minutes.

Stevie introduced concepts with a lecture. Stevie introduced fractions by lecturing about unit fractions and the definition of fractions. She cautioned students that they would often go into more detail in this class than they should with their future third or fourth graders.

Stevie gave students time to think about the prompt. Stevie asked students to think individually before talking to a neighbor about the following prompt:

> Take a blank piece of paper and imagine that it is $\frac{4}{5}$ of some larger piece of paper. Fold your piece of paper to show $\frac{3}{5}$ of the larger (imagined) piece of paper. Do this as carefully and precisely as possible without using a ruler or doing any measuring. Explain why your answer is correct. Could two people have different-looking solutions that are both correct?

Stevie asked students to discuss what was tricky or confusing. After about 5 minutes, Stevie asked students to talk about what was tricky or confusing about the problem, emphasizing that she was not looking for a solution at this point. Student 1 volunteered that the denominator being five, in other words four-fifths of three-fifths was confusing. Stevie revoiced this and asked for other answers.

Student 2 volunteered that it was confusing to think about the whole piece of paper as four-fifths. Stevie asked if it was confusing thinking about that piece of paper as four-fifths of something rather than a whole. "It's a whole piece of paper, right?" Stevie asked for other things that people found hard but no one volunteers.

When there were no more volunteers, Stevie began cold calling. Stevie started choosing student names from a bag to call on students randomly. She reiterated that students do not have to give a full solution, they can just comment on their thoughts on the problem.

Student 4 talked about her group thinking,

> about it as the piece of paper was four-fifths, so four of some five pieces . . . so we folded it into four pieces. So this was our four [the four areas between the folds]. The whole piece of paper would be our four-fifths.

Stevie revoiced this so the rest of the class could hear.

Student 5 talked about,

> being confused about considering the whole instead of the four-fifths so because we kind of divided it into fifths but then we thought that we had to, I have an extra fifth to make a whole. But that didn't work so we had to go back to divide it into fourths.

Stevie revoiced and elaborated while checking her understanding,

> let me see if I understood this, you thought about you thought about dividing it into five parts because . . . it's that big piece of paper and you want to think about it as the denominator. The denominator is five and you want to think about five parts and then, then you realized there was a problem there, you wanted one more piece.

Stevie directed students to practice close listening and build on peer responses. Stevie introduced,

> one of the things I wanted us to work on . . . is listening carefully to each other so I'm going to ask this person to make some comments and then I'm going to pick another person's name and either rephrase what the person said or ask a question about what the person said and so on. And that's one of the skills that you are going to have to develop as a teacher. So I think that, I see that as part of your training.

Stevie asked another student to talk about her thinking about the problem. At this point Stevie gave students a microphone to pass around so that their responses could be heard by everyone. After Student 6 explained that her thoughts were the same as Student 1's, Stevie asked students to rephrase or ask a question about what Student 6 just said. Student 7 rephrased the previous student response and Stevie complemented her close listening.

Stevie asks the next student to “continue, perhaps along the lines of what somebody else said.” Student 8 talked about how her group folded the sheet of paper into 4 pieces and imagined the fifth piece being on the top of the piece of paper. To get three-fifths, they folded one of the four pieces under. Another student volunteered a comparison between this problem and a problem that the class had previously worked on.

Stevie asked students to connect to the definition of fractions. Stevie asked where students used the definition of fraction. A student responded that the definition of fraction came into play when they talked about the pieces as being part of the larger piece of paper.

Overview of Stevie's Class

Stevie's class met for one hour three times a week and was the largest (102 students) that was videotaped by the ME.ET Project, where the mean class size excluding Stevie's class was 27. Although she had taught the class many times before, this was the first time with such a large group, and she was doing it experimentally to see if the large class format could be successful (Stevie, Oct. 22, 2008). Stevie's class was more formal than Pat's class. The students sat in steep stadium-style seating in a lecture hall that included a three-panel chalkboard with moveable sections, a document camera, a computer projector, and a projection screen above the chalkboard. Stevie used a handheld or clip-on microphone for most of the class periods. During classes early in the semester, Stevie tried having students pass around a microphone so that all students could be heard in discussion but this proved cumbersome so Stevie more often repeated or revoiced what PSTs said so that everyone could hear. Stevie had two TAs who aided in administrative tasks (like collecting homework), graded assignments, and occasionally discussed homework problems at the start of class. Stevie and, to a much lesser extent, her TAs were the only ones to write on the board, use the document camera, or PowerPoint. All visual representations of PST ideas were filtered through them.

Before the start of each class, Stevie projected announcements and assignments on the overhead so they were visible as students entered the classroom. Stevie opened each class by going over announcements and bringing attention to important due dates or changes in the schedule. Then, on all but one of the five observed days, she or one of her TAs went over homework or test problems that many students had difficulty with or that demonstrated a particular common misconception. These problem sessions lasted from two to six minutes. The rest of the class session consisted of several activities on between two and four topics. If there was an unfinished activity from the previous lesson, that activity was continued. Otherwise, Stevie tended to start with a lecture, using prepared PowerPoint slides that introduced a new topic. During these lectures, which lasted from three to five minutes, she introduced definitions, explained new concepts, and presented common elementary student misconceptions. She sometimes demonstrated a problem, technique, or representation using the PowerPoint slides, chalkboards, or document camera.

These lectures were regularly followed by students working on a problem or question (prompt) that Stevie posed. It was routine for students to work on the prompts in pairs, sometimes starting with individual work then pairing up. After a few minutes, Stevie brought the class back together to discuss their responses in whole-class discussion. In these discussions, Stevie organized the discussion in one of several ways: one or more students gave complete answers, several students gave partial answers, or several students talked about their thinking without a final answer or solution. If there was confusion or students needed additional time to think, Stevie would send them back to small group to discuss the issue before asking them to talk again in whole-class discussion.

Stevie often ended discussions with a summary or further explanation of the last student response. When the class ended before a discussion was finished (as occurred on two of the days included in this study), Stevie gave students additional time in small groups in the next class to re-familiarize themselves with what they had been thinking the previous day. For example, on 9/10/08 Stevie instructed students,

> So let's go back to this problem that we were working on last time. We had just sort of started that and I want to give you another minute to look at that problem one more time. Maybe it's best if we just start over again rather than try to continue where we were, in the middle of that (starting 10:35).

After students had a chance to discuss with their partner, Stevie read the problem aloud,

> Jean has a casserole recipe that calls for one-half cup of butter. Jean only has one-third cup of butter. Assuming that Jean has enough of the other ingredients, what fraction of the casserole recipe can Jean make? Draw pictures to help you solve this problem. Explain why your answer is correct.

Then Stevie asked them to start with the "draw pictures part."

Similarly, on 9/15/08 they returned to an unfinished discussion on alternative methods for comparing fractions. During the previous class session (9/10/08) the class began discussing using "reasoning other than finding common denominators, cross-multiplying, or converting to decimal numbers to compare the sizes (=, <, or >) of" several pairs of fractions (see Table 22 in Appendix C for which pairs were in the prompt). On 9/15/08, Stevie did not restart the discussion at the beginning because the prompt involved comparing several different pairs of fractions, instead they started at a pair of fractions that they had not covered previously, and she gave students time with partners to think about the problem again.

Stevie made her expectations about classroom routines explicit to her students and encouraged students to listen to each other. She frequently told students her expectations for the discussion before instructing students to talk to a partner and she often reiterated her instructions when she started whole-class discussion. For example, as seen in the opening vignette, Stevie introduced her expectations on 9/3/08 for what she would later call "campfire discussions." In that introduction she said,

> one of the things I wanted us to work on a little bit . . . is listening carefully to each other so I'm going to ask this person to make some comments and then I'm going to pick another person's name and either rephrase what the person said or ask a question about what the person said and so on. And that's one of the skills that you are going to have to develop as a teacher. So I think that, I see that as part of your training (starting 44:58).

She called these campfire discussions and used the metaphor of a campfire story when she re-introduced this type of discussion on 9/8/08. She said,

> you go around a campfire and you tell a story by—a person tells the first part and the person continues and tells a little bit more and the next person continues and tells a little bit more. Pretty soon, you wind up with this wacky story, but that's not where we're going with this one, but let's just give it a try (starting 49:00).

Stevie's prompts for students to "ask a question, comment, or continue" became common thereafter and students did all three at various points in the class. In order to participate in this type of discussion, a student had to pay attention to the previous student's response and try to understand it in order to comment on it, question it, or continue the line of thought.

Stevie expertly used the document camera in her teaching. During discussions, Stevie wrote out student responses (or summaries of student responses) on the document camera while she repeated or revoiced them. In this way, nearly every student response could be heard and seen by all students. Stevie left these responses or summaries of responses on the document camera throughout the whole discussion as long as space did not become an issue. Because she used the document camera rather than the board, she could also re-post responses on the document camera if she was talking about a student strategy that she had to reference or she could refocus students by taking the solutions off the document camera, replacing them with the problem text in the activity manual, discussing the problem text, and then replacing the solutions on the document camera.

This description of Stevie's teaching is intended to give the reader an overview of Stevie's class. As seen from these descriptions of Pat and Stevie, these are both instructors who make an effort to engage students in doing mathematics during the class session, and to elicit verbal responses from students about their work. Although the classes were different in size and in many other respects, both had a clear focus on helping students understand the mathematics they were working on, and on encouraging students to express their understanding in words. The descriptions in this chapter provide a context for the more systematic analysis in the next chapter of how Pat and Stevie organized these classes.

Chapter 6: Cross-Case Results

The purpose of this study is to examine how instructors facilitated whole-class discussion in mathematics classes for preservice teachers (PSTs). This chapter compares the two instructors in the case studies to look for similarities and differences in their approaches and to contrast their teaching strategies with what K-12 research has highlighted as important factors for productive discussions. As explained in Chapter 3, a whole-class discussion is defined as an episode of at least one minute in duration where one or more students talks to the instructor or another student about the same topic or class activity while the rest of the class is expected to attend to the exchange(s).

Whole-Class Discussion

During the class periods observed, there were 18 whole-class discussions in Pat's class and 16 whole-class discussions in Stevie's class. The number and duration of whole-class discussions in each classroom for each session observed are reported in Table 8. Whole-class discussions took up on average 52% of the class time in Pat's classroom and 35% of class time in Stevie's classroom. Whole-class discussions in Pat's class lasted between 2:20 (2 minutes and 20 seconds) and 57:29. Whole-class discussions in Stevie's class lasted between 1:12 and 20:27. In general, whole-class discussions in Pat's class were longer than in Stevie's class with the median discussion length being 11:12 and 4:12 respectively.

Table 8. Frequency and Duration of Whole-Class Discussion in Each Class Session

Instructor	Date	Duration (% class time)	Frequency	Range
Pat	4/1/08	38:33 (53%)	3	9:56-17:27
	4/3/08	55:02 (59%)	2	16:38-38:24
	4/8/08	72:35 (65%)	3	4:42-57:29
	4/10/08	27:49 (36%)	2	2:46-25:03
	4/15/08	51:53 (63%)	3	7:10-30:56
	4/17/08	40:56 (39%)	5	2:52-12:45
Stevie	9/3/08	15:01 (29%)	2	4:28-10:33
	9/5/08	23:50 (39%)	4	1:48-10:43
	9/8/08	13:39 (25%)	4	2:00-4:47
	9/10/08	25:50 (47%)	4	1:12-20:27
	9/15/08	17:06 (30%)	2	1:07-12:45

Note. Discussion time was calculated from the instructor's initiation of whole-class discussion to the transition to a new class activity. Interruptions were not counted towards whole-class discussion time.

Aspects of Instruction Preceding Discussion

The following sections report on activities that preceded whole-class discussion and the types of discussion prompts used in Pat and Stevie's classrooms. Results will be organized by research question: 1A) activities preceding whole-class discussion, and 1B) prompts used by the instructor to encourage discussion.

Activities preceding whole-class discussion. The most frequent activity to precede whole-class discussion in both classrooms was small group discussion. Twelve (67%) of the discussions in Pat's class and 14 (88%) of the discussions in Stevie's class were directly preceded by small group work that was focused on the same discussion prompt as the whole-class discussion. Other activities preceding whole-class discussions, seen in Table 9, include lecture, and whole-class discussion.

Table 9. Activities Preceding Discussion

Activities preceding whole-class discussion	Pat	Stevie
Lecture	1	1
Small group discussion	12	14
Whole-class discussion	2	1
Mid-class break	2	0
Start of class	1	0

Small group discussion. In both classes, small-group discussion was often used to give students time to think before they shared their thinking in whole-class discussion. In Pat's class, a typical small group discussion preceding whole-class discussion began when Pat gave students a prompt, such as directions for a worksheet of problems or a question delivered orally. Pat used small group discussions to interrupt whole-class discussions when PSTs seemed to be stuck or when he wanted to give them time to talk over a peer solution that had just been presented before they were asked to explain or critique it. In Pat's class, small group discussions lasted between 1:18 and 24:25, the median being around 3 minutes. Nine small groups did not change composition for the whole semester as students were assigned to tables of four and those tables constituted the working groups. In the sessions observed in Pat’s class, only six of the 21 small group discussions were not followed by whole-class discussion. Three of those were small group discussions interrupting whole-class discussion to give students time to think about the problem they were discussing in whole-class discussion.

In Stevie's class, a typical small group discussion preceding a whole-class discussion began when Stevie announced the prompt and then instructed students to talk with a partner. While she was doing this she projected the prompt (if it was in the activity manual, she used the document camera; if not, she projected a PowerPoint slide) as she read the prompt aloud. After 1:30 to 5:55 minutes, she announced that they would discuss the prompt in whole-class discussion, often re-reading the prompt, clarifying her expectations for student responses, and/or giving procedural instructions on how student responses would be shared or chosen. Since seats were not assigned, PSTs might work with a different partner from day to day. Of all 19 small group discussions in Stevie's class, only five were not followed by a whole-class discussion. Three of those were small group discussions interrupting whole-class discussion to give students time to think about the problem they were discussing in whole-class discussion.

Whole-class discussion. In every case of whole-class discussion preceding whole-class discussions (i.e., discussions with different prompts that followed one another) the two discussions were related. Whole-class discussions could be used to start student thinking on a topic or clarify student understanding before moving forward. For example, the first discussion of Pat's 4/8/08 class was about defining numerators and denominators (see Table 21 in Appendix C for full prompt text). It was used to clarify student understanding about these concepts before returning to the discussion of a student solution from the previous class. After the first discussion, Pat was able to refocus thinking on the student solution by asking them to discuss what was wrong with the denominator in the student's solution.

The other instances of whole-class discussion preceding whole-class discussion that were occurred near the end of class periods so limited time may have been a factor. In all these cases, discussions were on related problems. An instance from Pat's class occurred in the last half hour of the last class of the year (4/17/08). After a long mid-class break, during which students worked on worksheet problems, the fourth discussion in the class period covered two student solutions to a worksheet problem involving distributing fudge and what CGI problem type the problem was. The fifth discussion followed directly when Pat chose a student to present his solution to a third problem about fudge and the problem's CGI type. Stevie's one instance of whole-class discussion following whole-class discussion occurred on 9/10/08 during the last 10 minutes of class. After discussing "Why can you compare fractions using cross multiplication?" for 1:12, they discussed "Of 7/12 and 8/12, which one is greater and why?" for 1:16. Both discussions were shorter than was typical for her class.

Lecture. In the case of lecture preceding whole-class discussion in Pat's class, lecture was used to remind students of unfinished business from previous class periods. On 4/8/08, Pat began class with a lecture recapping a discussion from the class period before that was unfinished. He then told them he wanted to "back up" and have them consider the common denominator. In the previous day's discussion, a PST had questioned another PST's use of the denominator, but they did not have time to pursue the student's question.

Lecture could be used to focus and give background on a topic for discussion. On 9/10/08, Stevie lectured on methods for comparing fractions then she asked the PSTs to discuss with the class why it makes sense to compare fractions using cross-multiplication. Before the lecture, she instructed the PSTs to, "Write down several methods that you know for comparing fractions," which they did for about 1:30. Though what PSTs discussed in small groups (i.e., identifying methods for comparing fractions) was related to the topic of whole-class discussion it was not as directly related as in other instance of whole-class discussion (e.g., the same prompt being discussed first in small group then in whole-class discussion).

Other activities. In Pat's class, whole-class discussions sometimes started the class when the discussion was based on homework students had completed. These homework problems were often at least partially worked out or discussed in small groups and were finished by PSTs in their own time. Basing discussion on homework gave PSTs a chance to think through a problem and retain a record of their own thinking (in the form of the homework that Pat handed back) as a reference for discussion. These homework-based discussions were often used when Pat asked several PSTs to present different solutions to the class that the PSTs were then asked to compare.

Discussions in Pat's class also sometimes followed the mid-class break. In these cases, Pat gave them a longer mid-class break and an assignment to tackle during the break with their small group. These breaks were not coded as small group time because it was under the students' discretion how much time they spent working and how long their break was. Again, like discussion preceded by small group or homework-based discussions, students had time to think and discuss the prompt with others before they were expected to share their thinking with the class.

Summary. Most of the activities that preceded discussion were related to or used the same prompt that was discussed in the whole-class discussion. These activities gave PSTs time to think about prompts on their own or more often with a group before they were expected to share their thinking about the prompt with the whole class.

Discussion Prompts. Pat used prompts from several sources. The Wedding Problem was

> from Frank Lester. . . . Al Otto and Cheryl Lubinski have created a class called Dimensions of Mathematics. And, basically to develop of a conceptual

> understanding of numbers and fractions, . . . those problems, I brought along with me. . . . Frank Lester problem, the school bus problem. . . . [T]he next problem set that I am going to give them are basically just CGI questions with fractional quantities in it. So, I will write those myself. Nancy Mack, . . . I have taken problems from her. . . . Beckmann has this class called teaching and learning middle grade of mathematics. And she's got a chapter on rational numbers and has really nice problems that I am going to use. In fact, I would say, I really like them. I think it's about as good collection . . . for my class where I teach fractions (Pat, Apr. 3, 2008).

Most of Stevie's prompts came from the activity manual that was required for the course (Beckmann, 2007). Some prompts in Pat's class were exclusively oral while others were given to students on handouts and reiterated or expanded orally during class. Stevie often read the prompt aloud while it was visible on the PowerPoint or while she showed an image of the problem in the activity manual on the document camera. Full prompt text and coding can be found in Table 21 and Table 22. Definitions of prompt types can be found in Table 4 in Chapter 3.

On half of the observed class days, Pat started the class with a compound prompt consisting of an integration statement, comprehension statement, and a comprehension question (i.e., the Waffle Problem, the Cookie Problem, and the Brownie Problem). On 4/15/08 and 4/17/08, he followed this up with an integration statement-integration question prompt that asked students to compare student strategies that had just been presented. On 4/1/08, the second discussion was also an integration statement-integration question prompt asking students to compare and contrast, but in that case, they were comparing and contrasting the eastern European and American algorithms for division rather than student solutions.

On three of the five class days, the first prompt in Stevie's class was an integration statement (as seen in Table 10). All asked students to apply a definition or concept that Stevie had just lectured on (e.g., talking with a partner on the rule for rounding to the nearest thousand on 9/3/08, "Discuss with a neighbor how the placement of fractions on the number line fits with the way we defined fractions of objects" on 9/8/08, "Use reasoning other than finding common denominators, cross-multiplying, or converting to decimal numbers to compare the sizes (=, <, or >) of the following pairs of fractions" on 9/15/08).

Table 10. Prompt Order by Discussion

		Discussion number				
Instructor	Day	1	2	3	4	5
Pat	4/1	IQ-IS	IS-IQ	IQ-IS		
	4/3	CS-CQ	CQ-IS			
	4/8	CQ	IQ	CQ-I[I]		
	4/10	IS-CS-CQ	CS			
	4/15	IS-CS-CQ	IS-IQ	IQ		
	4/17	IS-CS-CQ	IS-IQ	IQ-IS	CQ-CS	CQ-CS
Stevie	9/3	IS	IS-IQ			
	9/5	FQ-CQ	IS	IQ	IQ	
	9/8	IS	CS	IS	CQ-IQ-CS-IS	
	9/10	CQ-CS	CQ	CQ-FQ	IQ-IS	
	9/15	IS	FS-IS			

Note. IS = *integration statement*; IQ = *integration question*; I[I] = integration implicit; CS = *comprehension statement*; CQ = *comprehension question*; FS = *factual statement*; FQ = *factual question*. Dashes indicate *compound* prompts.

Both instructors frequently used compound prompts as fodder for whole-class discussion, though Pat used them more frequently (78% of discussions) than Stevie (43% of discussions). The distribution of prompt types can be seen in Table 11 (see Table 4 in the Chapter 3 for definitions). The most frequently used prompt types in Stevie's class were compound prompts (7), integration statements (5), integration questions (2), comprehension questions (1) and comprehension statements (1). Neither instructor used factual statements unless they were included in compound prompts.

Table 11. Discussion Prompt Types by Class Day

		Prompt type						
		Statement			Question			
Instructor	Day	Fac.	Com.	Int.	Fac.	Com.	Int.	Compound [a]
Pat	4/1	0	0	0	0	0	0	3
	4/3	0	0	0	0	0	0	2
	4/8	0	0	0	0	1	1	1
	4/10	0	1	0	0	0	0	1
	4/15	0	0	0	0	0	1	2
	4/17	0	0	0	0	0	0	5
	Total	0	1	0	0	1	2	14
Stevie	9/3	0	0	1	0	0	0	1
	9/5	0	0	1	0	0	2	1
	9/8	0	1	2	0	0	0	1
	9/10	0	0	0	0	1	0	3
	9/15	0	0	1	0	0	0	1
	Total	0	1	5	0	1	2	7

Note. Fac. = factual; Com. = comprehension; Int. = integration.

[a] See Table 4 for explanation of the sub-prompts that make up compound prompts.

The distribution of prompt types within compound prompts, also known as sub-prompts, is shown in Table 12. Within compound prompts, Pat frequently used integration statements (10), comprehension questions (8), comprehension statements (6), and integration questions (6). Within compound prompts, Stevie most often used integration statements (5), comprehension questions (4), and integration questions (3). Even within compound prompts, factual prompts were uncommon. Ten of Pat’s 14 compound prompts had two sub-prompts but four had three sub-prompts while six of Stevie’s seven compound prompts had two sub-prompts but she did use one compound prompt with four.

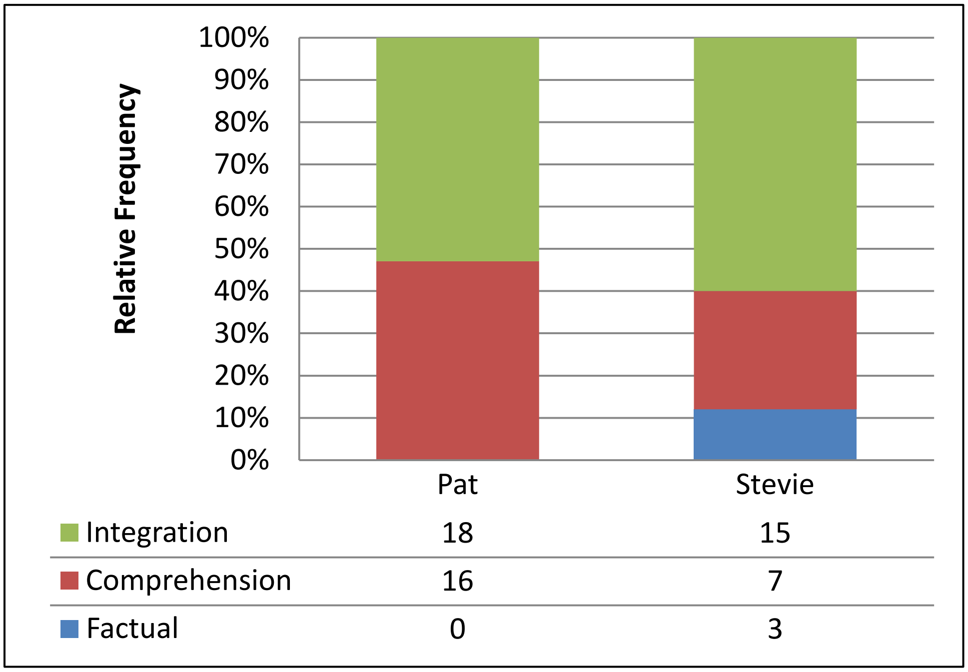
Since most prompts were part of compound prompts and most compound prompts mixed statements and questions, it is difficult to say whether students responded to statements and questions differently. There were no discernible differences in student responses to statements and questions that were not part of compound prompts. Outside of compound prompts, Pat used more questions (3) than statements (1) and Stevie used more statements (6) than questions (3) but when looking at sub-prompts, the reverse was true of both instructors. When looking at non-compound prompts and sub-prompts together, the two trends tend to cancel each other out. Pat used questions and statements equally (17 times each) while Stevie used statements (13) slightly more than questions (12).

When statement and question prompts and sub-prompts are combined (as seen in Figure 7), integration prompts accounted for over half of prompts in either class (53% in Pat's and 60% in Stevie's classroom). Stevie had three instances (12%) of factual prompts, and each was immediately followed by a request to explain how students got their answer or to explain why (see Table 13 for prompt text). All other prompts were comprehension prompts. In other words, in these classes low-level or rote prompts were rarely used and were clearly not the norm. As suggested by Contreras (2006), they seem to be used to initiate student engagement in material, or more specifically, in the sub-prompt that followed it.

Table 12. Distribution of Sub-Prompts within Compound Prompts by Date

		Compound	Statement			Question		
Instructor	Day	Prompt Number	Factual	Comp.	Integration	Factual	Comp.	Integration
Pat	4/1	1			•			•
		2			•			•
		3			•			•
	4/3	4		•			•	
		5			•		•	
	4/8	6			*		•	*
	4/10	7		•	•		•	
	4/15	8		•	•		•	
		9			•			•
	4/17	10		•	•		•	
		11			•			•
		12			•			•
		13		•			•	
		14		•			•	
	Total		0	6	10	0	8	6
Stevie	9/3	1			•			•
	9/5	2				•	•	
	9/8	3		•	•		•	•
	9/10	4			•		•	
		5				•	•	
		6			•			•
	9/15	7	•		•			
	Total		1	1	5	2	4	3

Note. Comp. = comprehension, * = implicit prompt.



	Pat	Stevie
🟩 Integration	18	15
🟥 Comprehension	16	7
🟦 Factual	0	3

Figure 7. Counts and relative frequency of integration, comprehension, and factual prompts (including sub-prompts)

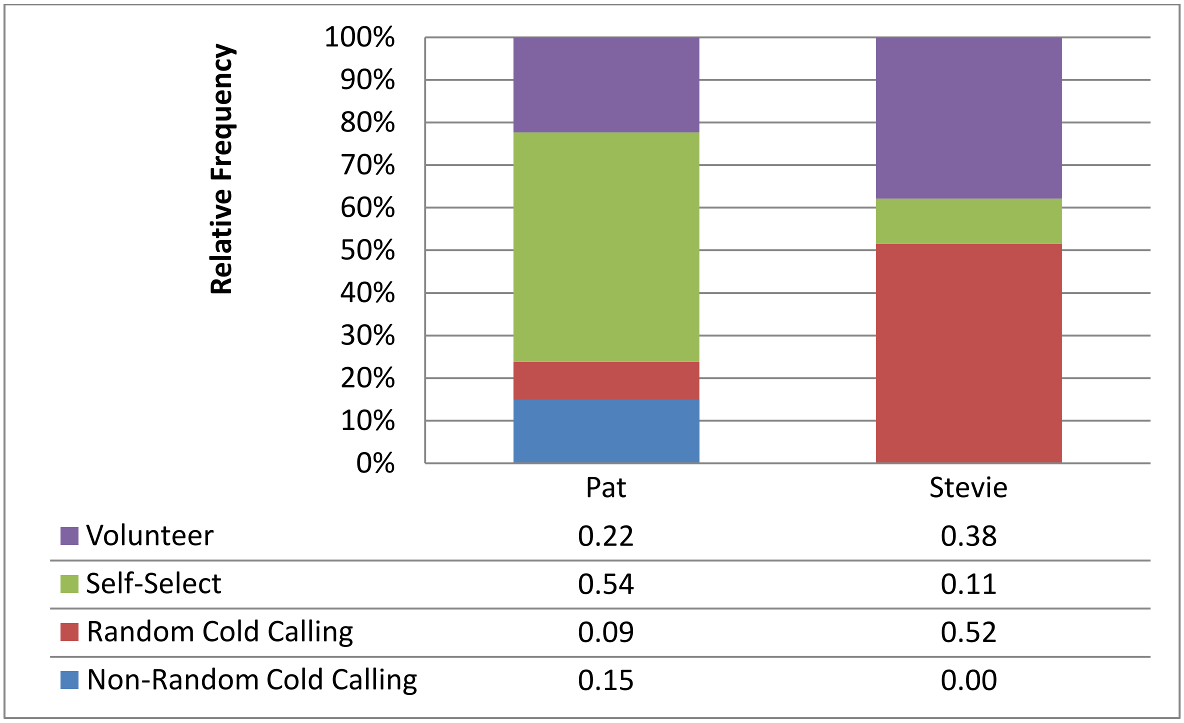
Table 13. Examples of Factual Prompts and Requests for Explanation from Stevie's Class

Date	Factual prompt	Request to explain
9/5/08	Take a blue strip of paper. Is it 1/5 of a full piece of ordinary paper?	How can you tell if it is or isn't?
9/10/08	Of 7/12 and 8/12, which one is greater?	And why?
9/15/08	Determine the percent of the diagram that is shaded	Explaining your reasoning . . . and explain how to see that this fraction of the diagram is shaded

Facilitating Discussion

The following sections report on instructor teaching moves that encourage or deepen discussion. More specifically, the following sections will discuss how Pat and Stevie chose which students spoke in discussion, responded to student thinking, and connected student responses both to other student thinking and to disciplinary knowledge.

Choosing which students talk in discussion. Instructors used several different methods to determine which PSTs spoke in whole-class discussions. Pat chose which students spoke in discussions with four methods: random cold calling, non-random cold calling, elicited volunteer, and self-selection (see Table 5 in Chapter 3 for definitions). Stevie used random cold calling and elicited volunteers primarily, although students did self-select on occasion. See Table 23 in Appendix D for each instructor's frequencies. As we can see in Figure 8, when adjusting for the total number of students chosen by each instructor (i.e., dividing the frequency of each method for each instructor by the total number of times methods were used for that instructor), Pat used self-selection much more often than Stevie did and Stevie used random cold calling much more often than Pat.



	Pat	Stevie
🟪 Volunteer	0.22	0.38
🟩 Self-Select	0.54	0.11
🟥 Random Cold Calling	0.09	0.52
🟦 Non-Random Cold Calling	0.15	0.00

Figure 8. Relative frequency of methods used to choose students to speak in discussion by instructor

In most whole-class discussions, instructors called on multiple students and used multiple methods to choose students to speak, as seen in Table 24 and Table 25 in

Appendix D. Pat had one discussion where only one student participated and Stevie had four. Other than these single-participant discussions, only four of the 16 discussions in Stevie's class used a single method of choosing students throughout. All of these had fewer than three participants.

Self-selection and volunteering. The most common method of choosing which PSTs spoke in discussion in Pat's class was self-selection, though most of these instances were limited to whole-class discussion where students were responding directly to one another (i.e., a PST had presented a solution or idea and other students were questioning the PST, helping the PST explain, or positing their own ideas). In Stevie's class, self-selection was discouraged by the class size. Students often did not talk loud enough to be heard by the whole class so if Stevie did not hear or acknowledge a student's response and revoice it, many other PSTs likely did not hear it. In addition, because of the limitations of the audio on the videotape due to camera position and the class size (100+), there were infrequent instances where it was difficult to differentiate between Stevie asking a rhetorical question and Stevie intending for students to respond without being called on. These ambiguous instances are not included in analysis. In Stevie’s class, student self-selection clearly occurred in seven unambiguous instances, and in every class session. In all these cases, responses were very short, often consisting of a phrase or less.

Both instructors often elicited volunteers, in fact it was the second most frequent method used for both instructors. Only four of the discussions in Pat's class and five of the discussions in Stevie's class did not use volunteer responses at least once. On the few occasions when volunteer responses were not forthcoming after appropriate wait time, both instructors used cold calling. Sometimes this led to students getting more time to talk about a problem in small group discussion if several cold called students reported that they needed more time to think or that they had not had time to finish the problem.

Cold calling. Both instructors frequently used cold calling to invite students to participate in discussion. Only two discussions in Pat's class and six of the discussions in Stevie's class used no cold calling. Stevie's most frequently used method for calling on students to participate in whole-class discussion was random cold call. Both instructors chose students at random by pulling student names from boxes or bags.

When Pat used non-random cold calling, it was more often based on a PST's group or position in the room (30 times) rather than their solutions (17 times). However, this may have been an artifact of needing to get a response from all 7 groups in a whole-class discussion (that is to say that each time he called on one PST based on his or her group, Pat had to call on 5 or 6 others in order to present the solutions of all class groups.). When Pat cold called students based on the content of their solutions, it was often the case that Pat called several PSTs in sequence. On most occasions Pat wrote numbers on student homework responses that he then handed back to students, he then asked students to present in sequence. Stevie did not use non-random cold calling.

Choosing which students spoke first in discussion. Neither instructor had a method that they used to start all their discussions. Pat most often started whole-class discussions by non-random cold calling (8), though random cold calling (4), student self-selection (2), and eliciting volunteers (4) were often used. Of the 16 whole-class discussions observed in Stevie's class there was a near even split in methods; seven whole-class discussions started with a volunteer and nine with random cold calling. As seen Table 24 and Table 25 in Appendix D, all but one of the seven discussions in Stevie's class that started with volunteer responses continued without cold calling being used for the duration of the instance of whole-class discussion. There were no other discernible patterns.

Responses to student thinking. Both instructors frequently responded to student thinking in ways that research in K-12 context suggests deepens student engagement and learning, makes the students' mathematical thinking the topic of discourse, and highlights what is important in student responses. The most frequent ways that Pat and Stevie responded to student thinking were by revoicing responses, asking for further information, asking for questions or agreement/disagreement, elaboration, and making student thinking visible (see Table 6 in Chapter 3 for definitions). All of these were used during discussions in all of the classes observed. Frequencies can be seen in Table 26 and Table 27 in Appendix E. These four teaching moves were present in over 72% of discussions in both classes (see Table 28 and Table 29 in Appendix E for the distribution). In every observed class period, Pat also asked for other solutions or responses (56% of discussions) and evaluated student responses (78% of discussions). Connecting student responses, asking students to apply their own reasoning to a peer's response, connect peer responses, or connect disciplinary knowledge to a peer's response were infrequently used by either instructor. They will be discussed further in the next section examining the connectedness of discussions.

After adjusting for the total number of responses that each instructor made to student thinking,[15] as seen in Figure 9, the pattern of instructor responses is similar. The

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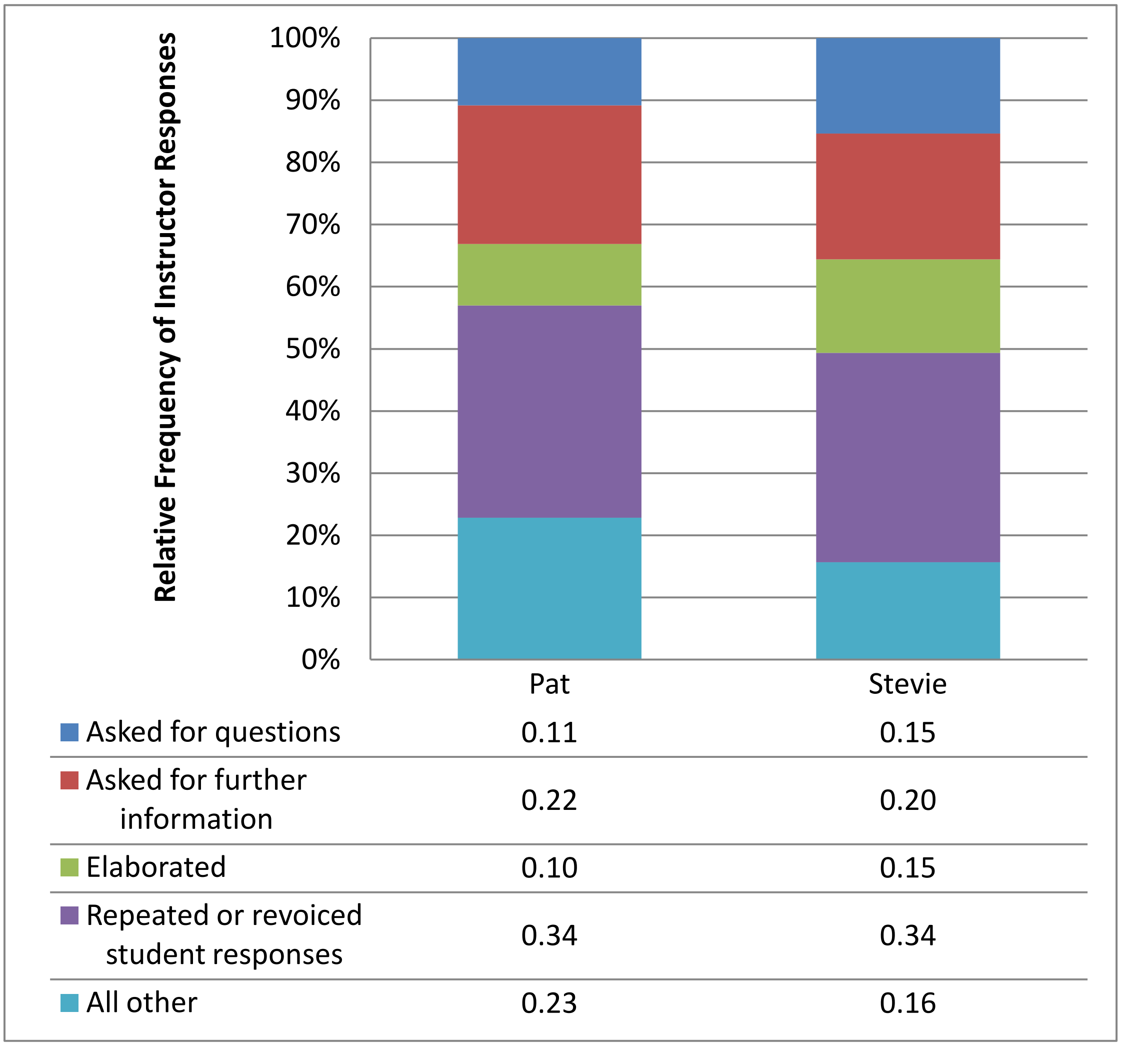
[15] Relative frequencies were calculated by dividing each type of response by the total instructor responses coded for that instructor. Infrequently used responses (i.e., responses

biggest differences (around .05) between these instructors were in asking for questions, and elaborating. All other types of responses were within 0.02 for these instructors. This suggests that the instructors had a similar routine or pattern when it came to responding to student responses in discussion.



	Pat	Stevie
🟦 Asked for questions	0.11	0.15
🟥 Asked for further information	0.22	0.20
🟩 Elaborated	0.10	0.15
🟪 Repeated or revoiced student responses	0.34	0.34
■ All other	0.23	0.16

Figure 9. Relative frequency of instructor responses to student thinking

Revoicing while making student thinking visible and elaborating. Both instructors made sure that student thinking was both audible and visible to the class.

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with a relative frequency of less than 0.10 for both instructors) were combined in the category *All other*.

Student responses were made more audible through revoicing (sometimes with elaboration to add mathematical content or language) while student responses were made visible through instructors writing on the board, overhead, or document camera. In both classes, instructors often wrote or drew the student response while they were revoicing or elaborating. An example of what using them all together looked like can be seen in Figure 10. In that figure, we see Pat's drawing of a student response on the board while he summarized the response using mathematically precise language. After discussing what each number meant in a student solution to the Bus Problem, Pat summarized the student's thinking while writing it on the board and adding the descriptive terms “separate result unknown” and “start and end result unknown.”

Pat said, “OK, so people exiting at the first stop, which is 2/5 get off, right? And boy, that's pretty clear isn't it. It says 2/5 in the problem. At the first stop two-fifths get off. Bang! 1 - $\frac{2}{5}$, right? It's a . . . separate result unknown. No start unknown, we've got some people on a bus, two-fifths get off and it's actually a start and end result unknown, all you know is the change quantity.”

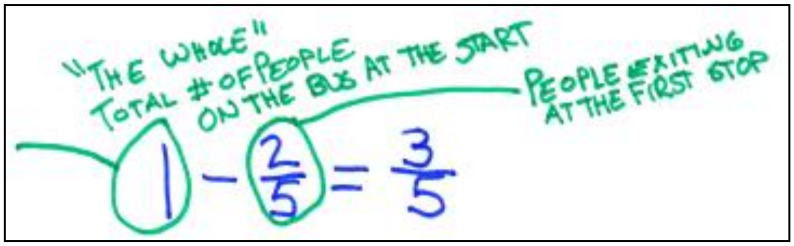


Figure 10. Pat revoiced while making student thinking visible and elaborating during the Bus Problem discussion on 4/8/08 (starting 69:02)

Both instructors revoiced student responses after nearly every student contribution to discussion. This happened both during and following student speaking turns though revoicing after a student was done speaking tended to be longer because the instructor was revoicing a whole student solution rather than a portion of a student's response. In most cases, instructors used truncated versions of student responses with the same words that students used. In a few cases instructors elaborated, replacing words that students used with language that was more mathematically correct or adding related mathematical content to student answers. Examples of revoicing can be seen in the darker text in an excerpt of the Bus Problem discussion in Figure 11. For additional context, summaries of student solutions and associated representations from the Bus Problem can be seen in Figure 24 in Appendix F. At this point in discussion, two students had already presented solutions to the Bus Problem and Pat was now asking students to explicate what the numbers mean both contextually and conceptually in the second student solution. In all the instances of revoicing in Figure 11, Pat is highlighting what was important in Amanda's responses, usually by writing it on the board. The first and fourth instances (numbered for reference and in black text in the figure) end with a request for more information while Pat elaborates on Amanda's responses in the first, second, and fifth instances by adding language that is more mathematical or additional mathematical ideas, like "separate result unknown." Repeatedly revoicing and asking for more information from a student was very common in both classes. Often this was done so that students gave a full and/or clear explanation if they had not initially.

Pat asked Amanda, “What is up with this 1? She just— I mean all of a sudden, bang! 1.”

Amanda replied, “well, she used 1 as like the whole 'cuz from before we started with the whole as 25 but because we're using the fractions we start with 1 to represent the whole thing.”

Pat revoiced (1), “So with fractions it's “the whole” [finger quotes], right and within the context it's . . .”

Amanda replied, “The total number of people that start out on the bus.”

Pat wrote on the board while he revoiced (2), “So contextually that 1 is the total number of people on the bus.”

Amanda added, “At the start.”

Pat revoiced (3) while writing, “At the start. And where does this 1 connect over there [the first student's solution]?”

Amanda replied, “It's like the 25.”

Pat repeated (4), “It's like the 25, and what else is it like?”

. . .

Amanda said, “So the two-fifths represents the people getting off the bus at the first stop.”

Pat wrote while revoicing (5), “OK, so people exiting at the first stop, which is two-fifths get off, right? And boy, that's pretty clear isn't it. It says two-fifths in the problem. At the first stop two-fifths get off. Bang! One minus two-fifths, right? It's a separate . . . start unknown, we've got some people on a bus, two-fifths get off and it's actually a start and end result unknown, all you know is the change quantity.”

Figure 11. Pat revoiced (in darker text) student thinking during the Bus Problem on 4/8/08 (starting 69:02)

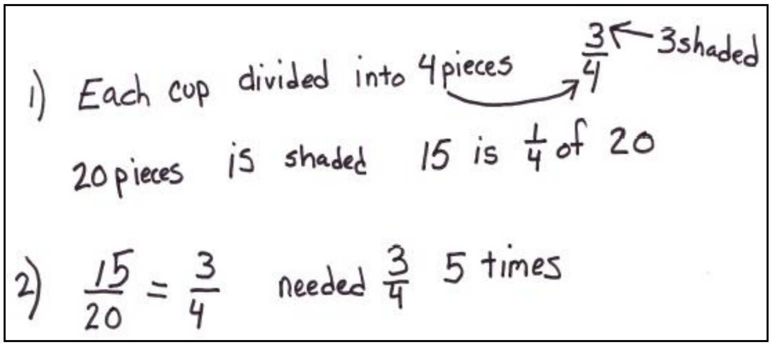
Though both Pat and Stevie made sure that student thinking was visible to the class, in practice there were differences between the instructors. When recording student thinking on the document camera or overhead, both instructors drew representations or gestured at representations of student thinking, and both demonstrated with objects (e.g., by folding pieces of paper) on the document camera or overhead under the direction of the student. However, students in Stevie's class never wrote on the board or document camera themselves so all visible student thinking was filtered through Stevie and was a response to student thinking. Figure 12 shows the notes that Stevie made on the document camera while students were talking about their thoughts on a problem about the number of cups needed to make three-fourths of a recipe (see Table 22 in Appendix C for full problem text and diagram). In this case, Stevie numbered the student responses and made sure all the responses were visible to students, serving as a record of what had taken place in discussion. Stevie did not often number responses, but she often used lists and she often left the responses of several students visible.

Mireya's Recipe Problem:

Mireya has a recipe that calls for 5 cups of flour. Mireya wants to make $\frac{3}{4}$ of the recipe. Instead of figuring out what number $\frac{3}{4}$ of 5 is, Mireya measures $\frac{3}{4}$ of a cup of flour 5 times, and uses this amount of flour for $\frac{3}{4}$ of the recipe. Using the meaning of fractions and the following picture to help you explain why Mireya's strategy is valid:

After students discussed the problem in small groups, Stevie wanted the students' initial thoughts. They could talk about how they thought about the problem, anything relevant, or even a full explanation.

Student 1 talked about each cup being divided into four pieces. The denominator was four pieces. Three were shaded. "If you count all the pieces, there's 20 of them and 15 are shaded." Then she talked about 15 being three-fourths of 20.



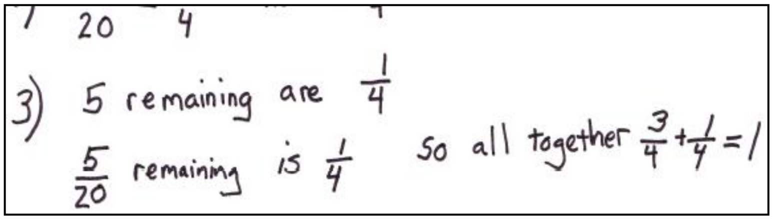
1) Each cup divided into 4 pieces → $\frac{3}{4}$ ← 3shaded

20 pieces is shaded 15 is $\frac{1}{4}$ of 20

2) $\frac{15}{20} = \frac{3}{4}$ needed $\frac{3}{4}$ 5 times

Student 2 said that she looked at the fifteen-twentieths too; you needed three-fourths five times.

Student 3 "to show that it actually adds up to a whole, you can take the five that are remaining and show that that's one-fourth. And one-fourth and three-fourths is one."



20 4

3) 5 remaining are $\frac{1}{4}$

$\frac{5}{20}$ remaining is $\frac{1}{4}$ So all together $\frac{3}{4} + \frac{1}{4} = 1$

Figure 12. Initial student thoughts on the Mireya's Recipe Problem (starting 20:10) in Stevie's 9/5/08 class with drawings reproduced from Stevie's notes on the document camera

Pat similarly recorded what students said in discussion. He sometimes called himself the class "scribe" and typed in a Word document, which he later sent to the students. This provided students with notes of the discussion and enabled them to focus their full attention on participating in discussion. More often Pat stood near the whiteboard and recorded student responses on it during the discussion. Figure 13 shows a simplified summary and pictures of the board during a discussion on 4/8/08 discussing what numerators and denominators are. Pat recorded the discussion on the board and organized student responses as to whether they referred to the numerator, N, or the denominator, D.

Pat asked, “So for the parts of fractions, how would you even describe what those numbers represent?”

Student 1 replied, “well I kind of thought of it as parts of a whole, like the denominator is the whole, like all of it together. And the numerator is how many parts of that whole.”

Pat revoiced and asked what the numerator is.

Student 1 replied, “parts of the whole.”

Pat said, “so one there's this idea of parts of the whole versus the whole, all of it together.”

Kay added, “I like her answer but I think it's that the denominator is how many parts make up the whole. You know, like if there's . . .”

Scott added, “how it's divided.”

Kay said, “yeah, like how it's divided, like how many parts it's divided into to make one whole. And then the numerator is how many parts you're adding. . . . How many pieces you're adding to it or taking away, depending on what you're doing.”

. . .

Pat asked, “you said something else about divided, how many? . . . You said something else that was quite nice. Does anyone recall?”

. . .

Kay said, “yeah, how many parts the whole is divided. I don't know.”

Pat asked her whether there was anything she would like to add.

Kay replied, “it's parts of the whole but it depends on what you're doing to it in the equation, I guess. Yeah, that's good.”

Scott added, “maybe the pieces of the whole are the parts that are being *used*.”

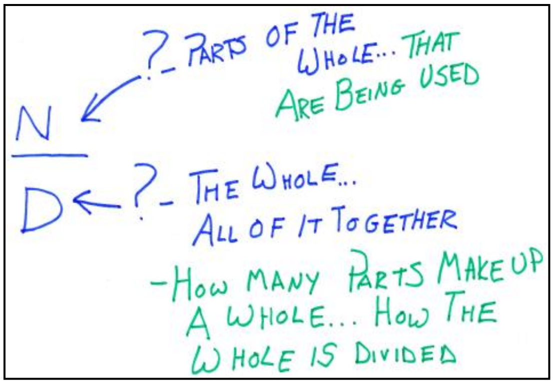
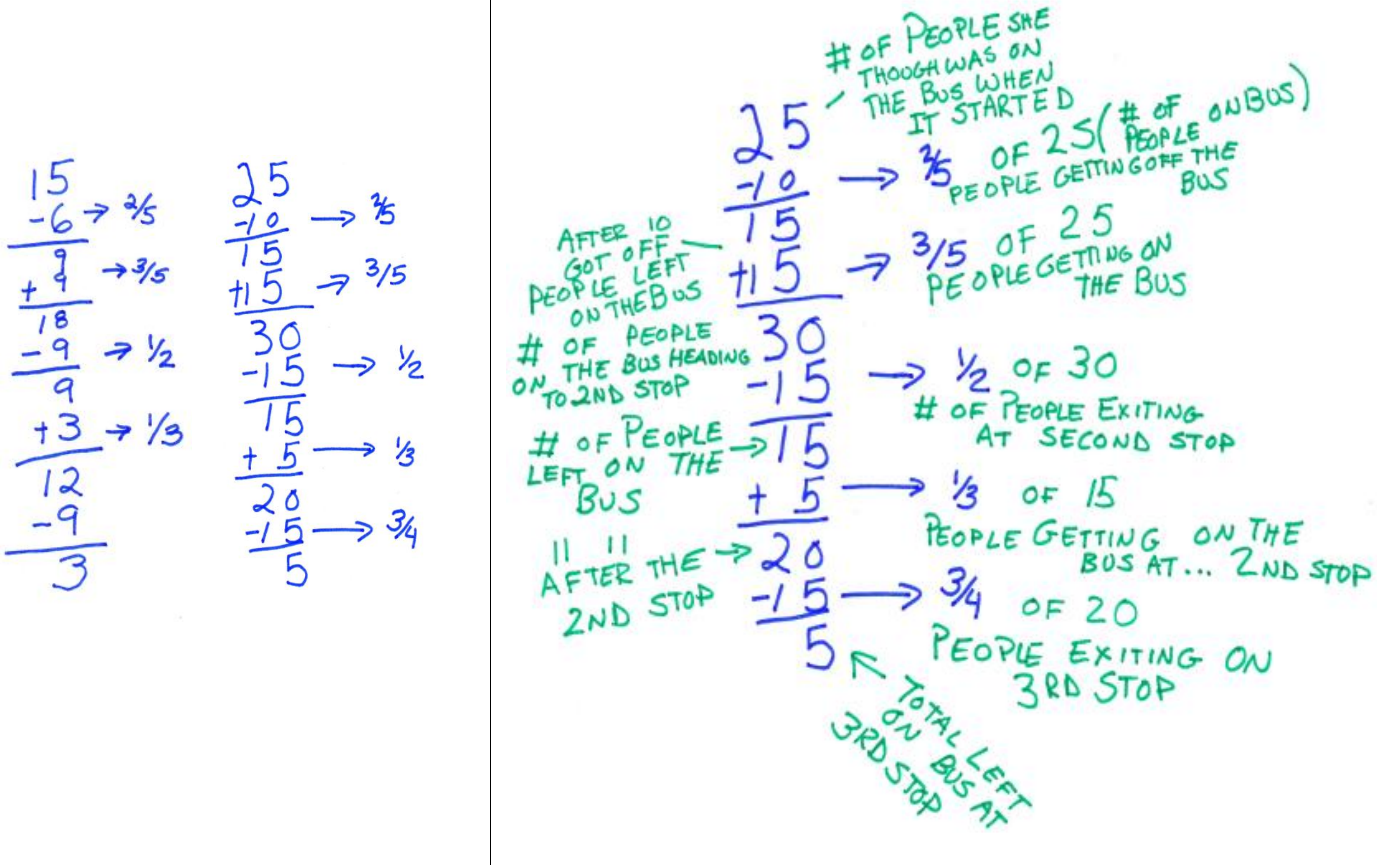


Figure 13. Excerpt from a discussion of what numerators and denominators are in Pat's 4/8/08 class (starting 5:56) with reproduced diagram of Pat’s notes on the board

Unlike Stevie's class, in Pat's class students often presented solutions at the board with diagrams or equations that they themselves drew or wrote. Though these also make student thinking visible, they were not coded here because they are not an instructor's response to student thinking. However, Pat sometimes annotated student solutions during a whole-class discussion to add clarity or precision to the explanation, which is a response to student thinking. In many cases, he asked students to provide an explanation for their peer's thinking that he wrote on the board using a different colored marker than the first student had used. We can see an example of this on 4/8/08 during a discussion on the Bus Problem (see Table 22 in Appendix C for full problem text). During the discussion three students presented different methods to solve the problem. The first was a whole-number arithmetic method where the student started with a whole number and added or subtracted a whole number of people from that number to represent the number of people on the bus. The second strategy multiplied fractions to show people leaving or boarding the bus. The third solution showed people leaving and boarding the bus pictorially. After every student solution (except the last pictorial strategy) was presented to the class, Pat asked the other students to explain "contextually and mathematically" what the numbers in the student solution represented. Several students were called on to explain the numbers used in each peer solution while Pat annotated the solution on the board. For the last solution, time ran short and Pat added in the invisible intermediate steps himself. The solutions and annotations can be seen in Figure 14.

First student solution: Whole-Number Arithmetic Method (starting 31:53)

Student solution: Solution with annotations:



Student writing is in blue while Pat's writing is in green. Diagram on right is larger to make the annotations easier to read.

Figure 14. Three annotated student solutions to the Bus Problem (4/8/08)

Figure 14 (cont’d)

Second student solution: Multiplying Fractions (starting 57:18)
Student solution:

$$1-\frac{2}{5}=\frac{3}{5}$$

$$\frac{3}{5}+\frac{3}{5}=\frac{6}{5}$$

$$\frac{6}{5}\cdot\frac{1}{2}=\frac{6}{10}=\frac{3}{5}$$

$$\frac{3}{5}\cdot\frac{4}{3}=\frac{12}{15}=\frac{4}{5}$$

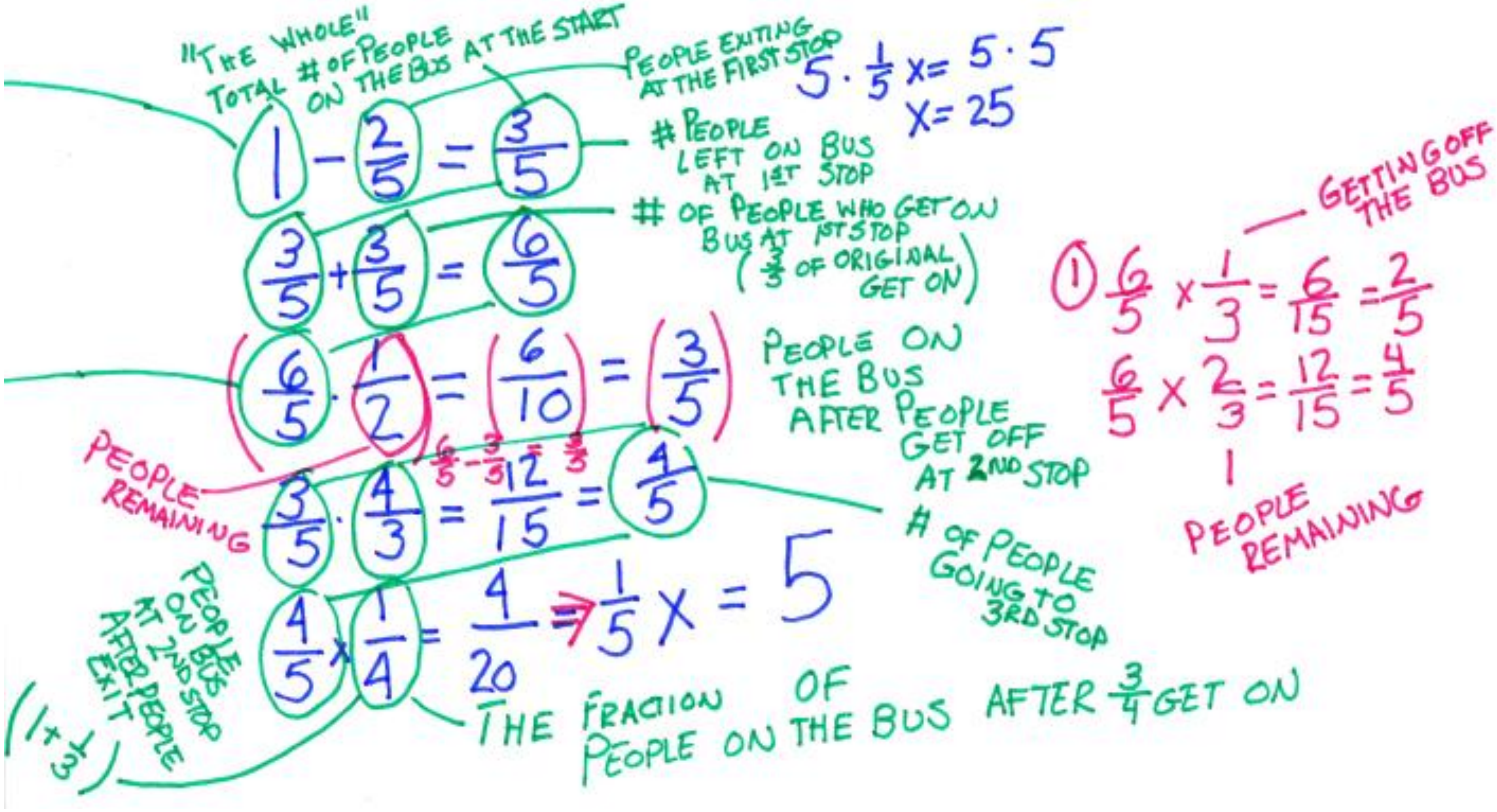
$$\frac{4}{5}\cdot\frac{1}{4}=\frac{4}{20}=\frac{1}{5}x=5$$

$$5\cdot\frac{1}{5}x=5\cdot 5$$

$$x=25$$

Figure 14 (cont'd)

Second student solution: Multiplying Fractions (starting 57:18) (continued)
Solution with annotations:

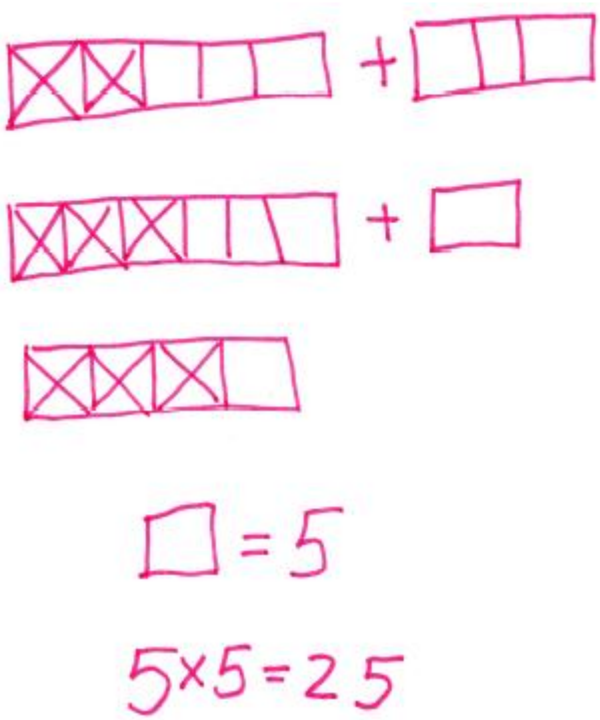


The student used the blue marker while Pat used green and red.

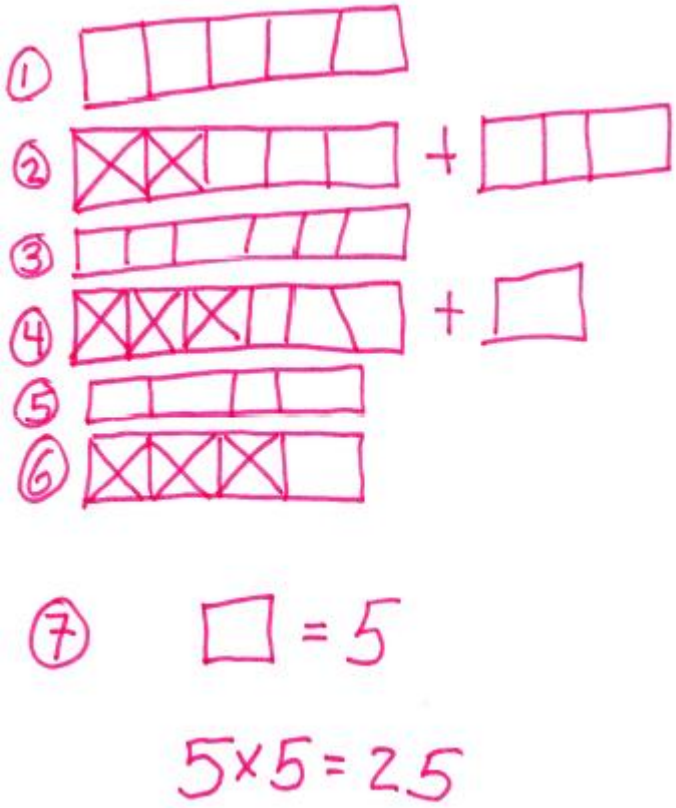
Figure 14 (cont’d)

Third student solution: Pictorial Strategy (starting 101:10)

Student solution:



Pat adds numbering and intermediate steps:



Both the student and Pat used red marker.

Asking for further information. Both instructors frequently asked students for additional information about their responses, often repeatedly during the same explanation. When instructors asked for more information multiple times during an explanation, they most often waited for the student to pause before requesting more information. These requests included probing students to clarify or go more in depth to asking students to explain why, and prompting students to continue an explanation. We can see some examples of this in the excerpt of the 4/8/08 Bus Problem discussion from Pat's class in Figure 15. Two students had presented solutions to the problem on the board and their peers were being asked to interpret all the values in those solutions, both conceptually and contextually. We see that the first request for more information (in darker text) follows Eva's response describing her interpretation of the previous student's thinking. Pat's request cued her to explain what the number $\frac{3}{5}$ meant contextually in the problem.

Eva said, “Well, the way I was thinking of it was kind of the same way she was because she's multiplying it by four-thirds because she wants to know how many people are on after adding a third to the original three-fifths.”

Pat asked what the three-fifths is. “it's the p—”

Eva replied, “Like how many people are on the bus after—are going to the third stop—at the third stop, I don't know.”

Kay suggested, “at the beginning of the third stop.'“

Pat repeated this and wrote it on the board. The three-fifths “is the beginning of the third stop so we're now we're at the beginning of the third stop. What happens at the beginning of the third stop?”

Figure 15. Excerpt of Pat asking for more information (indicated by darker text) (starting 91:18) during the Bus Problem discussion on 4/8/08

Many requests for further information followed instructors revoicing student solutions in both classes. Pat's last response in Figure 15 is an example. Pat first revoiced Kay's response then asked for further information. Sometimes the instructors used revoicing to ask students to continue to explain by ending with rising intonation. Figure 16 shows an excerpt from Pat's class on 4/3/08 during a discussion on classifying fraction problems. The class had already grouped the problems and Pat was now asking students for finer distinctions between problem classifications. Pat's response (in dark text) both summarized what the student had just said and invited additional explanation. As we can see from the summary, the student accepted the invitation.

Pat called on Eva and asked, “Any distinction between these two?”

Eva responded, “Well, the only distinction that we really saw between 5 and 6 was that in number 5 both groups of kids are splitting up the same number of objects” She was unsure and talked with her group members a minute then returned, “Yeah, in number 5 they both ended up with the same amount of clay in each group.”

Pat revoiced, “OK, both kids get the same amount of clay?”

Eva responded, “Yeah, so both groups--so the answer's the same, like instead of one group getting more clay than the other, they both get the same.”

Figure 16. Excerpt of Pat asking for more information by revoicing with rising intonation on 4/3/08 (starting 51:27)

The instructors sometimes asked for very concrete and specific information, such as “for clarification Eva, are you calling this also comparing fractions?” (Pat 4/3/08 starting at 18:09) and “so that's a whole [referring to the first square in a solution]. And what about this [referring to the second square]?” (Stevie 09/05/08 starting 49:52). Other times they asked students to tell them how a process or solution method worked, such as, “Could you maybe show us how you solved it then? There's a marker right in the back.” (Pat 4/3/08 starting 18:09), and “how does that happen?” (Stevie 09/08/08 starting 41:01). They also frequently asked about the meaning of student solutions (e.g., “Because you have more people sharing than the number of objects what does that mean?” Pat 4/3/08 starting 41:25), or why something is true (e.g., “Can somebody explain to me why two equals three-fourths x?" Pat 4/15/08 starting 24:25, and "why is that?" Stevie 09/10/08 starting 49:32).

Both instructors asked about the context of the numbers that students used in their explanations but they asked about different aspects of the context. Because semiotics and the meaning of representations were recurring ideas in Pat's class, Pat frequently asked students what numbers or representations in student responses meant. For example, during the discussion of the Bus Problem on 4/8/08 three students presented solutions at the board as described above in Figure 14. After the first and second strategies were presented to the class, Pat asked the other students to explain "contextually and mathematically" what the numbers in the student solution represented. Several students were called on to explain each number used in each peer solution.

Stevie also asked students what portions of their solutions meant but she did so less frequently and consistently than Pat. These types of questions in Stevie's class were mostly about the mathematics of the prompt rather than the context. For example, on 9/5/08 starting at 41:11 Stevie asked, "and what's the whole?" and on 9/5/08 starting 49:52, Stevie asked, "in other words, what does that mean? When you're saying don't think of it as 8 total, you're saying don't think of this whole thing [circling a diagram of 8 squares with her fingers] as the whole?" Notice that all of these examples refer to mathematical concepts rather than the context of the prompt, such as what bus stop they are at in the Bus Problem or whether the men at the wedding are married *to* the women at the wedding in the Wedding Problem.

Asking for questions or disagreement. Stevie favored asking for agreement or disagreement while Pat more often asked for questions. These two moves seemed to function similarly in these classes with agreement being equal to having no questions. Both can be used as a way of gauging student understanding or a way of asking students to apply their reasoning to another student's in a very basic and limited way. In other classes, both of these moves can function as a signal that a discussion is ending rather than an actual bid for questions or disagreement but in these classes students sometimes asked questions, stated their confusion at a peer response, or disagreed with a peer's reasoning. While both instructors asked for questions or disagreement after students were finished giving solutions or at the end of discussions, they asked at other times (like between steps in student responses) and students did in fact ask questions and disagree here too. In Pat's class, students asked questions of both Pat and of student solution givers at different points while in Stevie's class, questions were primarily addressed to Stevie.

Evaluating. Pat evaluated student responses more than Stevie did both in terms of frequency and in terms of regularity. Pat evaluated student responses in 12 of the 18 discussions in his class while Stevie evaluated student responses in three of the 16 discussions in her class. The most frequent type of evaluation made by both instructors was to make a general positive comment about the student's response. For example, "You said something else that was quite nice" (Pat 4/8/08 starting 7:01), "OK, great, great" (Pat 4/8/08 starting 90:40), "that is beautiful" (Pat 4/8/08 starting 102:28), "you've told us more" (Stevie 9/10/08 starting 19:31), or "that's a good comment" (Stevie 9/10/08 27:17). These occurred in nine of the discussions in Pat's class and three in Stevie's class. Both instructors made a positive comment about a specific aspect of a student's response but they did so in only one discussion. Pat complemented a student's representation of the problem on the board (i.e., "it's kind of a nice drawing because on the top she's thinking in terms of cups, right? . . . So her respective units are kind of divided into top and bottom, which is kind of neat" [4/15/08 starting 54:27]) while Stevie commented on a student's reasoning (i.e., "that's a nice way to reason if they can" [9/15/08 starting 30:50], "you see that the reasoning that you guys had before was totally relevant" [9/15/08 starting 45:44]). Pat confirmed that parts of student responses were mathematically correct in 4 discussions, a move that was always followed by a request for further information about the student's reasoning (e.g., "Yeah, two equals three-fourths x, but where did that [points to x] come from?" [4/15/08 starting 22:05]).

Both instructors pointed out that errors existed in student reasoning. One way they did this was by either disagreeing with a student response or pointing out that two student responses disagreed. In all cases, this focused student attention on the fact that there was an error but did not tell the students what the error was. For example, "That's proof by example . . . because numerically it works out. Now did we get lucky?" (Pat 4/8/08 starting 76:44), "When we write 2 1/2, there's something goofy about this. . . . In formal mathematics, this is not what we would write." (Pat 4/10/08 starting 52:31), "So we've got some disagreement here. Ok, interesting." (Stevie 9/15/08 starting 76:44). In all cases where instructors pointed out that an error existed but did not correct or identify it themselves, students found the error and discussed why it was an error. We can see this in Pat's class during the discussion of the Waffle Problem on 4/10/08. Two students presented solutions to the problem to the class but they got different answers.[16] One student had said that the answer was two and two-thirds. The other said it was two and

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[16] An overview of this discussion was presented in the opening vignette of the Case of Pat in Chapter 4. Also, see Figure 21 and Figure 22 in Appendix B for full diagrams of representations drawn on the board during discussion.

one-half. At this point, Pat pointed out that one of the answers was “goofy” and that it was not correct as written (see Figure 17). Without additional suggestions from the instructor, Student 1 suggested how it should be written differently. Then Pat pressed students to explain what the units of each of the numbers were to reinforce that two and one-half were different things (batches and cups of milk) while two and two-thirds were the same thing (batches) so they could be summed. Then through questioning students further, Student 2 and Kay said that two and one-half did not make sense in the context of the problem because it did not mean anything. Again, Pat pointed out that there was an error and students pointed out what the error was and why it was an error.

Pat said there was “something goofy about writing two and one-half. In formal mathematics, this is not what we would write. Why not?” . . .

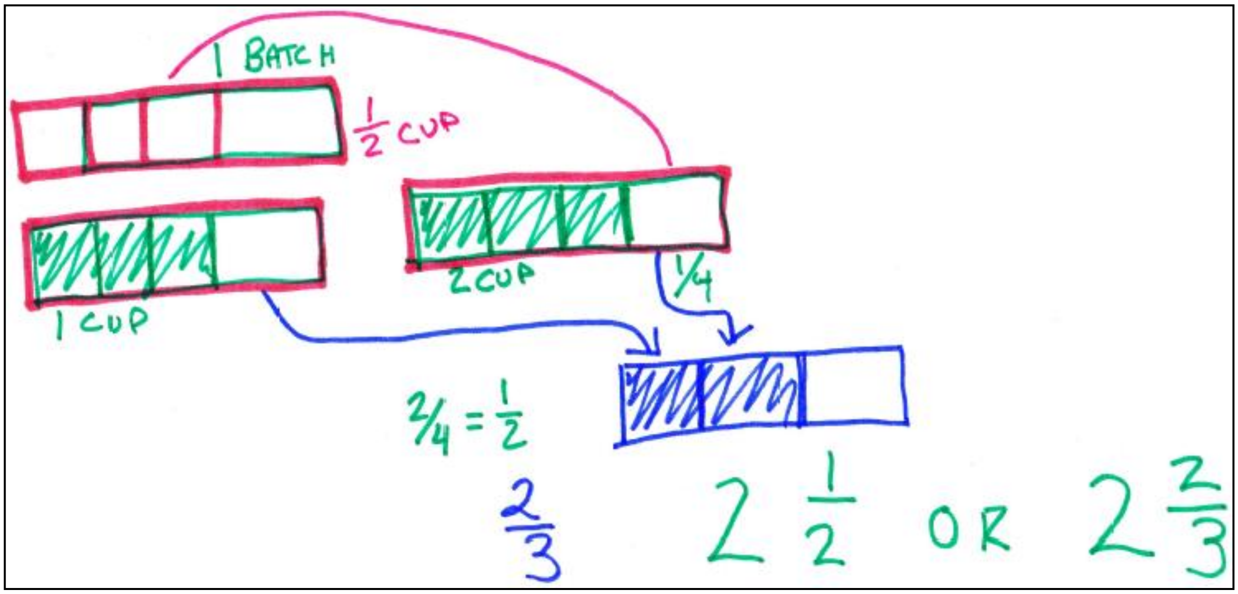


Figure 17. Excerpt from the 4/10/08 Waffle Problem discussion (starting at 52:31) where Pat pointed out “something goofy” then PSTs pointed out what the error was

Figure 17 (cont'd)

Student 1 replied, "You'd have to write, like, two remainder one-half."

Pat revoiced, "Really, what we want to write here would be two remainder one-half, right?"

Students agreed.

Pat continued, "because the two is . . ."

Students replied, "Batches,"

Pat repeated and wrote on the board, "And the one-half represents . . ."

Students replied, "Milk."

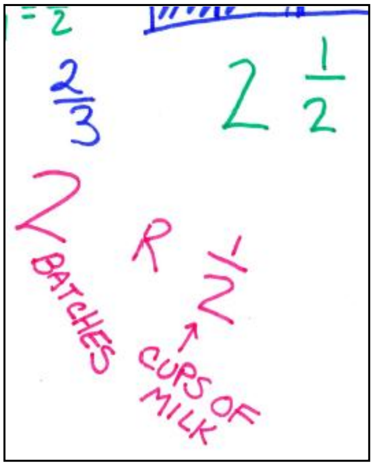
Pat revoiced while writing on the board, "milk, cups of milk, right? So it's really kind of two batches remainder one-half is another way to think too. Now, let me ask you this. Keep in mind, . . . when we say two and two-thirds, that's the same as saying two plus two-thirds, right?"

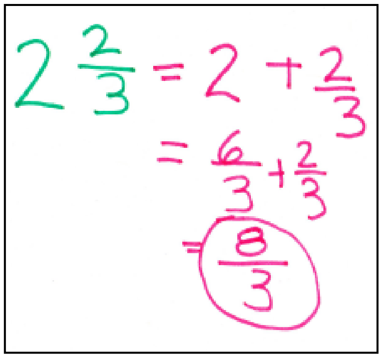
Students agreed.

Pat asked, "Does that make sense? And then in that case we can say two is the same thing as what in thirds?"

Students replied, "Six-thirds"

Pat repeated while he wrote on the board, "Sums to make eight-thirds." . . .





Pat asked whether he can add two and one-half. Meanwhile he worked through this on the board.

Students said no.

Student 2 said, "Because the two stands for batches while one-half stands for leftovers where in two and two-thirds the two stands for batches and the two-thirds stands for batches."

Pat played devil's advocate, "OK, but I've got a common denominator [to add two and one-half]."

Student 2 replied, "Well, you can add it but it doesn't mean anything."

Pat repeated, "You can add but it doesn't mean anything."

Kay replied, "It's not correct. It's not correct in the context that you're using."

Pat explained, "so the issue here is this 2 and this $\frac{1}{2}$ [the remainder] are different things." He told them to notice that they had 2 different representations on the board, one that was correct and one that was not. He used a red marker to cross out the incorrect representations. He explained that the representations were incorrect.

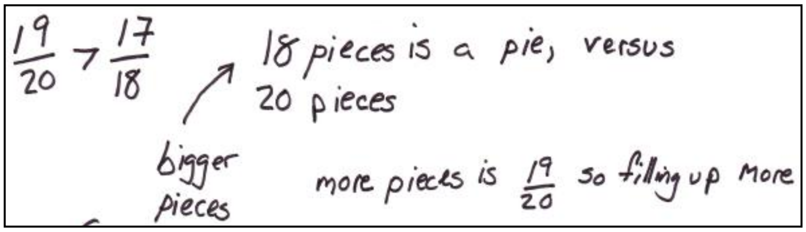
An example from Stevie's class occurred during the 9/15/08 class where students offered incorrect reasoning during a discussion on whether $\frac{19}{20}$ or $\frac{17}{18}$ was larger (shown in Figure 18). The first student said that nineteen-twentieths was bigger and compared it to pies, with the 18-piece pie having bigger pieces. The second student said that seventeen-eighteenths is bigger and compared it to a test, reasoning, "you missed less." Stevie wrote both answers and their associated reasoning on the document camera, noted that there was disagreement, and random cold called two students to share their thoughts. Student 3 shared some incorrect thinking that Stevie also wrote on the document camera. Student 4 admitted confusion and tried to clarify Student 2's test analogy. At this point Stevie summarized and then had the students talk about the disagreement with their small groups. The first student called on after small group (Student 5) gave a correct solution using reasoning from most of the previous students: if you are missing a smaller piece, the number is larger. At no time in this discussion did Stevie explicitly tell students what reasoning was incorrect. Instead, she pointed out that there were two different answers for a problem that can only have one answer, called on students to explain what was going on, gave students more time to think, and called on more students until they came to the correct solution with correct reasoning.

Stevie reminded the PSTs that the first comparison is seventeen-eighteenths versus nineteen-twentieths.

Student 1 said, “We said that nineteen-twentieths is bigger. Just because if we think about having like 18 pieces like in a pie or something versus 20 pieces. The pieces in the 18 are going to be bigger but you are going to have more so you are going to end up filling up 19 so you are going to end up filling a little bit more than seventeen-eighteenths.

Stevie wrote this on the document camera and requested, “Say that one more time. So you said the 18 pieces are bigger pieces and then what happened after that then? I lost what you said after that.”

Student 1 replied, “There are more pieces in the nineteen-twentieths so you're going to end up filling up more, even though they're really close, like in the same values.”



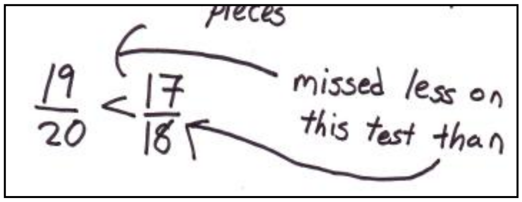
Stevie asked the class “so what about that? How do we know that, if why it would be filling up more? Does anybody, let's hear from another person.”

Student 2 said, “I said the opposite, I said that nineteen-twentieths was smaller. Because if you have a test for example there's 20 questions, it's going to count less if you miss 19 on a 20 question.”

Stevie wrote and said, “So you actually said that nineteen-twentieths was smaller than seventeen-eighteenths? So we've got some disagreement here. Ok, interesting. And, and that was because if you have a 20 question test 19 and then what did you say after that?”

Student 2 replied, “nineteen-twentieths you miss less than seventeen-eighteenths.”

Stevie wrote as she said, “so you sort of, you missed less here on this test than if you have an 18 question test and you got 17 right. Ok that's ok.”



. . .

Student 3 said, “I understand that if you got 19 right on a 20 question test you've gotten one wrong, but you miss less if you got 17 right on an 18 question test.”

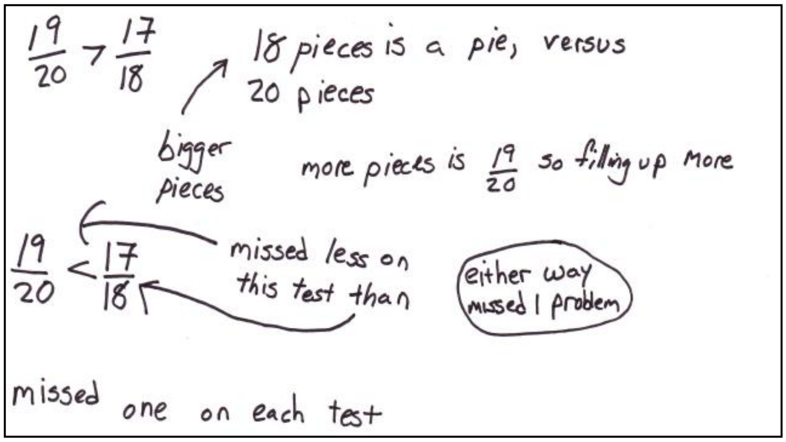
Stevie replied, “Ah, I see. You're saying in either case you've missed one problem so you've kind of missed the same amount. Ok, what do you think?” and wrote on the document camera. She wanted to hear from somebody else about this.

Figure 18. Excerpt where Stevie pointed out that two student answers “can't both be right” (starting 34:52) in a 9/15/08 discussion about comparing fractions

Figure 18 (cont'd)

Student 4 replied, "To follow what she said and follow what [Student 2] said, trying to clarify it in my brain, I think that what [Student 2] is saying is, yes, you've missed one on each test but the amount that is taken off for each problem is different."

Stevie asked, "When you say "amount taken off for each problem" are you kind of thinking percentage? . . . Because a lot of time on a test, you get so many percent out of a hundred. . . . So there can be a difference between how many points we've missed total and what that percentage is. So I think you guys are bringing up some good, good questions."



Stevie wanted to hear from another person because there are two answers up on the document camera and they "can't both be right." In addition, they have "various reasoning that is definitely relevant and appropriate." Stevie decided to give students "a minute to sort out all these ideas. And think, talk about it with your neighbor, try to sort it out and then we'll collect back together."

Students discussed.

Stevie said that she drew a line to "just to say look, let's approach it again, you can draw on any of these idea that we've discussed, but let's just kind of start over again." Stevie re-prompted the problem: "which fraction is greater and why?"

Student 5 said she, "thought of it as 2 pies and I was looking at how much was missing to make a whole. So in nineteen-twentieths, you're missing one-twentieth to make a whole pie. In seventeen-eighteenths, you're missing one-eighteenth to make a whole. And so one-twentieth, because if you divide it into 20 pieces the pieces are smaller, you're missing a smaller piece to make a whole whereas with one-eighteenth is a slightly larger piece to make the whole. So nineteen-twentieth is larger."

Pat explicitly pointed out and corrected student errors in four discussions, but in almost all cases these were changes to the symbols or imprecise words of student solutions. Some of these corrections were made immediately after students made the error. For example, Pat saying, "rather than use the term— word divide, because of circularity, you're kind of using the word to define it [operational concept of division]"

(4/15/08 starting 63:58). In other instances, Pat waited until the discussion of a solution was almost finished to make the corrections. For example, while presenting her solution to the Bus Problem (4/8/08), Renee wrote $\frac{4}{5} \times \frac{1}{4} = \frac{4}{20} = \frac{1}{5}x = 5$ on the board (61:15). More than 30 minutes later after several students had discussed what the numbers in the solution meant conceptually and contextually, while Pat summarized student reasoning, he corrected the solution. He said, "One thing that I do want to do is I want to put an arrow here [between the $\frac{4}{20}$ and the $\frac{1}{5}x$] at this point because these aren't necessarily equal quantities" (4/8/08 starting 97:47) and used a different color marker to change the response on the board to read $\frac{4}{5} \times \frac{1}{4} = \frac{4}{20} \Rightarrow \frac{1}{5}x = 5$. Stevie did not both point out *and* correct student errors in the observed class periods.

Summary. Pat and Stevie responded to student thinking in similar ways. Both frequently revoiced, asked for further information, asked for questions/agreement, elaborated, and made student thinking visible. Both frequently used several of these in sequence so that after a student response, they revoiced the response while making the response visible to the rest of the class and elaborating on it by adding language that was more mathematically precise. Evaluation was infrequent in both classes but more infrequent in Stevie's. In both classes, instructors regularly made positive comments about student responses. When students made errors and other students did not correct them, both instructors drew attention to the fact that an error was made, expecting and encouraging students to correct it themselves once it was pointed out.

Connectedness of student responses. Whole-class discussions in both observed classrooms often connected PST thinking either to peer responses or to mathematical concepts. This was done by instructors both by structuring activities so that students had to explicitly connect their thinking with specific mathematic concepts or other student solutions and by either responding to student thinking by making the connections themselves or asking students to make the connections. As we can see in Tables 6 and 7, only two whole-class discussions did not have either of these two kinds of connectedness. Instructors, and less frequently PSTs under the direction of their instructors, connected PST responses in whole-class discussion back to the mathematical ideas that they covered that day. Both instructors had a greater percentage of whole-class discussions that explicitly connected PST responses to disciplinary language or concepts than discussions that connected PST responses to each other, although the second and third most frequent form of connecting was connecting PST thinking to peer responses and instructors connecting peer responses. This pattern held in both classes.

Table 14. Connectedness of PST Responses in Pat's Class

		Connecting PST responses		Connecting to mathematics	
Date	Discussion	PST	Instructor	PST	Instructor
4/1/08	1	●			
	2	●	●		●
	3	●	●		●
4/3/08	1	●			●
	2	●			
4/8/08	1				
	2		●		●
	3	●	●		●
4/10/08	1	●	●	●	●
	2			●	●
4/15/08	1	●	●		●
	2	●	●	●	●
	3		●	●	●
4/17/08	1	●			
	2	●	●		●
	3	●	●	●	●
	4	●			●
	5	●			●
Number of discussions with connections		14	10	5	14
% of discussions with connections		78%	56%	28%	78%

Table 15. Connectedness of PST Responses in Stevie's Class

		Connecting PST responses		Connecting to mathematics	
Date	Discussion	PST	Instructor	PST	Instructor
9/3/08	1				●
	2	●	●		●
9/5/08	1				●
	2	●			●
	3				●
	4				●
9/8/08	1			●	●
	2				●
	3			●	●
	4	●			
9/10/08	1	●	●		●
	2				●
	3				●
	4		●		●
9/15/08	1	●	●		●
	2				
Number of discussions with connections		5	4	2	14
% of discussions with connections		31%	25%	13%	88%

Instructors connecting PST ideas to peers' ideas. Though both instructors more often asked students to connect peer responses rather than doing so themselves, instructors did sometimes connect student ideas to each other. One way they did this was to compare student strategies to emphasize the different mathematics or thought processes they involved. For example, on 4/17/08, during a discussion on the Brownie Problem Pat asked three students to present solutions to the class and then he added a fourth strategy from a student in another class. Pat then instructed them to compare all four strategies and describe how each strategy modeled the operational concept in the problem, a prompt that asked students to connect both to the thinking of their peers and to mathematical ideas. After students discussed comparisons for about 10 minutes, Pat summarized and added some connections of his own. Pat told them that in his other class they described one model (model D) as follows,

> you start at 15 and then you start subdividing. In this model [model A], you start subdividing and build your way up to 15. Does that make sense? And in this model [model B], you're subdividing and you build your way up to 15. So even though this [model D] looks like that [model A] the type of thinking involved are . . . very different in a sense that someone could very easily, and there's people in this room saying, “You know what, I can see A but I don't get D” or, “I see A and D but B doesn't make sense to me.” And therein lies the issue is for those of us who see A, B, C, and D simultaneously and are making that kind of semiotic connection, realize that that semiotic connection isn't as obvious to people that are just starting to learn to model this way or to think this way. Does that make sense? And therein lies the issue (starting 36:57).

In this, Pat re-emphasized the theme of semiotics in his classroom and grouped the responses by specifying that they are similar based on their strategy. After that, students continued to discuss, focusing on how the representations in the solutions constrain one’s thinking about multiplication.

During discussion, instructors can connect to responses earlier in the discussion. For example as seen Figure 19, on 9/10/08 Stevie asked students to come up with alternative ways of comparing $\frac{1}{49}$ to $\frac{1}{39}$. One student described cutting up a pie into 39 or 49 pieces, asserting that the second pie would have smaller pieces. A second student mentioned the rule that smaller denominators lead to larger fractions. Stevie connected the two by explicitly using the pie example to illustrate the rule.

Student 1 thought about drawing a picture. Then she talked about it being out of 100 then she became confused talking about 49 pieces like a pie.
Stevie repeated, “Like a pie. A pie with 49 pieces is that what . . .”
Student 1 replied that if you have two pies, one would be cut into 49 pieces and the other cut into 39, then the one-forty-ninth will be smaller.
Stevie wrote this on the document camera and then repeated that if they have on pie cut into 49 pieces and another pie cut into 39 pieces, what can they say?
Student 2 replied, “The denominator is smaller, that might be the bigger fraction.”
Stevie repeated, “The smaller denominator makes a bigger fraction”
. . .
Stevie asked if that was like “this pie idea” [Student 1's response]. Stevie then summarized the “pie idea,” saying that you have two pies, one of which is split into 39 pieces and the other of which is split into 49 pieces. Stevie asked about the slices of pie. She said that the pieces in the 49-piece pie have to be smaller than those in the 39-piece pie.

Figure 19. Stevie reminded students of a previous student response (starting 48:06) during discussion on 9/10/08

Instructors connecting PST thinking to mathematical concepts. The instructors connected PST thinking to mathematical concepts most often through elaborating on PST responses while revoicing them during the discussion or summarizing them after or near the end discussion. This could be as small as Stevie translating “big one” to “tick mark” when revoicing a PST response about rounding on a number line (9/3/08).

An example from Pat’s class occurred on 4/10/08 during the discussion of the Waffle Problem.[17] Pat elaborated by using a whole number example to illustrate the similarity between whole number and fraction problems. He changed the fractions to a whole numbers: a batch of waffles requires 3 cups of milk, and you have 8 cups of milk. Then he repurposed the diagram Harry wrote on the board to show that the problem worked with whole numbers as well as fractions by using the boxes that represented quarters to represent wholes. Pat said he then had two batches and 2 cups leftover in whole numbers. This emphasized that changing from fractions to whole numbers did not

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[17] This discussion was described in more detail in the opening vignette of Chapter 4.

change the mathematics. He then reiterated that, “the picture hasn't changed but the meaning that I'm ascribing to it” has. He emphasized that pictures help because they have meaning but if the kids do not know what the meaning is, pictures can be confusing. The meaning of diagrams and representations was a topic that Pat frequently touched on in discussion. At this point Pat asked the PSTs to talk in small groups and to come up with a problem like the Waffle Problem but using whole numbers to allow students to see that whether it is a problem involving fractions or whole numbers, the mathematics are the same.

Asking PSTs to do the connecting. Both instructors explicitly asked PSTs to connect their reasoning with that of their peers or with larger mathematical ideas, and structured activities to make such connections likely. Both Pat and Stevie asked that PSTs know any solution presented by other students well enough to present it themselves. When one or more PSTs went to the board to present full solutions to the class, the prompt for the rest of the class, which was sometimes implicit and sometimes explicit, was that they had to understand the solution well enough to re-explain it or teach it to the class. For example on 4/15/08, before a series of PSTs were going to present solutions of the Cookie Problem to the class Pat reminded his students,

> Again, as we role-play, Kennedy's the role of a teacher so your context is you're [Kennedy] trying to explain it as clearly as you can to a group of fifth or sixth graders. Right? And then for you folks [the rest of the class], your contexts are teachers like me, sitting in the back of the room and your context is now she [Kennedy] is an upper elementary or middle school student explaining her work and you're trying to figure out what she's doing and how she's thinking. So you can

> explain it to someone else. Does that make sense? . . . So everyone in the room is acting like a teacher (starting 18:23).

After Kennedy had given her solution, the exchange in Figure 20 took place. In this exchange, Pat reiterated that the other PSTs in the class would be held accountable for understanding Kennedy's solution. They were expected to be able to explain what each step means. After this exchange, students self-selected to ask questions of Kennedy.

Kennedy asked, “Pat do you have any questions?”
Pat replied, “I don't have any questions.”
Kennedy clarified, “I mean does this make sense?”
Pat replied, “I'll start asking other people to explain it in a minute. I have questions for other people but I don't have questions for you.”
Kennedy asked the rest of the class, “I mean do you guys all understand the way I figured it out?”
Student 1 (SELF-SELECT) replied, “I do but I was just wondering why you took that extra step and did 2 divided when you could have just— Up there by the 4, you could have just said four-thirds and—”

Figure 20. Excerpt from the Cookie Problem discussion (starting at 20:34) on 4/15/08 where Pat gave students the responsibility of understanding Kennedy's solution

Another way Pat and Stevie asked PSTs to connect responses was to ask them to compare and contrast solutions that had been presented by their peers. For example, during the 4/15/08 discussion of the Cookie Problem, after several students had presented their solutions, Pat asked them to “take a minute in your groups and . . . contrast Elena's strategy with Colin's. Are they indeed the same thing or are they really thinking about it differently?” To start the discussion, Pat cold called a student who talked about being confused about one strategy and not the other. Ella then volunteered, saying,

> with Colin's strategy, he starts with the whole recipe and then figures out what the three-fourths is and then figures out what he has extra. And in Elena's strategy, she figures out what three-fourths of the recipe is. So three-fourths is like her beginning

> whole. And then from that she figures out from that what she needs to add on to make the whole recipe (starting 52:30).

The volunteer student demonstrated that she had enough understanding of the two student solutions to present a comparison that focused on the precise things that made these two strategies different.

Stevie often asked PSTs to connect their responses to peer responses. On eleven occasions in her campfire discussions, students were directed to comment, rephrase, or ask a question about a previous student's response. On 9/3/08, Stevie talked to the class about her rationale for doing this. She said,

> one of the things I wanted us to work on a little bit . . . kind of lead up to it gradually is, listening carefully to each other so I'm going to ask this person to make some comments and then I'm going to pick another person's name and either rephrase what the person said or ask a question about what the person said and so on. And that's one of the skills that you are going to have to develop as a teacher. So I think that, I see that as part of your training (starting 44:58).

Both Pat and Stevie also structured activities so that PSTs had to explain their solutions in terms of definitions or concepts from the unit. For example, in Stevie's class the second discussion prompt on 9/5/08 asked students to, "use the meaning of fractions and the following picture to help you explain why Mireya's strategy is valid." Another example is the introductory prompt on 9/8/08 when she asked students to discuss "with a neighbor how the placement of fractions on the number line fits with the way we defined fractions of objects."

***Sequencing responses to build mathematical ideas*.** Pat sometimes sequenced PST responses to determine the order of mathematical ideas presented. He more often used methods of choosing PSTs to speak in discussion that did not allow him to decide what solutions were presented or in what order they were presented (i.e., elicited volunteer, random cold call, and self-selection). On occasion, however, he chose students to share responses to homework problems with the class by numbering responses when he reviewed the homework, and asking student to present in the numbered order. In some discussions, he chose one student based on the content of his or her solution while the rest were chosen by self-selection or volunteering. In other discussions, he observed students working in their groups and chose solutions for presentation in a specific order. In total, Pat's class had three discussions in which multiple students presented solutions for the same prompt in a specific order: the Wedding Problem on 4/3/08, the Bus Problem on 4/8/08, and the Cookie Problem on 4/15/08 (Figure 24 and Figure 25 in Appendix F). Algorithmic solutions were sequenced prior to pictorial strategies in all three discussions. In the Wedding problem, the two solutions essentially showed the same (incorrect) thinking but the first solution was represented by an algorithm and the second by a pictorial solution involving bar diagrams. In the discussion of the Bus Problem, the first solution was a whole-number arithmetic method that worked backwards from the number of people remaining on the bus and worked in actual numbers of people. The second solution, multiplied fractions by an implicit unknown (referred to as 1 at the start of the solution rather than 1x or x). The third solution used a diagram with boxes representing a certain number of people that was determined at the end of the solution. In the cookie problem discussion, the first solution used an algorithm while the second used a diagram with solid lines representing cups and dotted lines representing thirds of a recipe.

We see little evidence that Stevie sequenced responses. The fact that she did not use nonrandom cold calling suggests that she did not plan to call on students who had used particular solution strategies in a predetermined order. Stevie did not have as much opportunity to sequence PST responses as Pat did for two reasons. First, the size of her class made it more difficult to get more than a shallow sample of student responses when they discussed prompts in small group prior to whole-class discussion. Pat had six groups to check in on while Stevie had 30-50 groups during each activity. The time constraints of getting to each group and having a meaningful conversation while keeping the rest of the class on task were likely a limiting factor for Stevie in this case. Second, Pat based his discussions on homework problems as well as on prompts that were presented during class and discussed in small group. In contrast, Stevie did not use prompts based on homework that she had previously reviewed.

Summary. Instructors connected student reponses to other student responses and mathematical ideas both through instructor responses to student thinking (such as elaboration) and through structuring activities that explicitly ask students to connect their responses to mathematical ideas or peer responses. In both classes, students were more often explicitly asked to connect their thinking to peer responses than to mathematical ideas. Instructors were more likely to connect student thinking to mathematical ideas than to the thinking of other students.

Summary of Cross-Case Results

This chapter presented the cross-case analysis of two instructors teaching mathematics courses for PSTs. The results presented in this chapter compared instructors on what occurred before whole-class discussion, what prompts instructors used, how instructors responded to student thinking, and how instructors connected student responses to mathematical ideas as well as to other PSTs' thinking. The next chapter discusses the significance of this study's results, the limitations of this study, and further research suggested by these results.

Chapter 7: Discussion

The purpose of this study was to examine how instructors engage their students in whole-class discussion in mathematics classes for preservice teachers (PSTs), using K-12 research that has highlighted important factors for productive discussions as a guide. More specifically, this study took the approach of an exploratory multiple case study to look at PST instructor teaching moves during and adjacent to whole-class discussions in two introductory mathematics courses for PSTs and to compare discussions within and between instructors. This study investigated what students did before whole-class discussion, what kinds of discussion prompts were used, whether and how instructors chose who spoke in whole-class discussion, how instructors responded to student thinking in discussion, and how student responses were connected to mathematical ideas and to other student responses. Patterns in these variables were explored. This chapter discusses how the findings of this study address the research questions as well as the limitations and significance of this study.

Instructors Gave PSTs Time to Think about Prompts before Discussing Them

Research question 1A asked: What class activities preceded whole-class discussion? Before whole-class discussion, instructors in both classes gave PSTs time to think about discussion prompts, often with a partner or small group, before they were expected to discuss them. This was much like Pair-Share, described in Chapter 2, where students discuss with a partner before they are expected to share responses with the class. During small group time in these classes, the PSTs were assigned to work on a problem or set of problems that were expected to be discussed with the whole class later in the class period. This expectation was a class routine that was often made explicit. Pat also gave students time to work on prompts by giving them time during mid-class breaks and by assigning prompts for homework. Students were also often given time in small groups to think about prompts when whole-class discussions were continued from a previous class period.

Instructors Most Often Used Cognitively Challenging Tasks

Research question 1B asked: What kind of prompts do instructors use to encourage discussion? As suggested by K-12 research (e.g., Baumert et al., 2010; Nystrand et al., 2003; Webb, 2009), to encourage discussion both instructors frequently used cognitively challenging prompts that asked students to describe, explain, and make connections between mathematical ideas. Comprehension and integration prompts were the most common types of prompt in both classes, meaning that instructors were asking students to describe or define a process or term and asking students to go “beyond what was explicitly stated in the lesson by linking two ideas together in some way” (King, 1994, p. 351). Both instructors commonly asked PSTs to explain or explain why. Even when the discussion prompt did not explicitly ask for explanation, PSTs were often expected to be able to explain their responses or their thinking to the whole class. Both instructors asked PSTs to interpret thinking of peers or thinking of elementary school students either real or hypothetical. In Pat's class, there was always an expectation that students understand peer solutions well enough to explain them to somebody else.

Both instructors frequently used compound prompts (e.g., several related prompts presented to students at once). In Stevie's class, factual prompts were never used on their own and comprehension prompts were rarely used on their own, though both were used with integration prompts as part of a compound prompt. Pat did not use factual prompts during the period observed. The paucity of factual prompts may be because they limit discussion because they ask for simple recall, which by definition requires no explanation or elaboration. Using factual prompts with either comprehension or integration prompts may have allowed the factual prompts to function as *engager questions* to promote initial participation (Contreras, 2006) while the higher level prompts served to extend and deepen discussion on the topic.

Instructors Used Several Methods to Choose Who Spoke in Discussion

Research question 2A asked: Do instructors choose who talks in discussion? If so, how? In both classes eliciting volunteers, self-selection, and random cold calling were used but in different proportions and most discussions used two or more methods. Pat most frequently used student self-selection and eliciting volunteers while Stevie most used random cold calling and eliciting volunteers.

Students self-selected when responding to peers. Students self-selected more often to respond to peer ideas than to the instructor. Atwood et al. (2010) suggested that self-selection allows postsecondary students to, "control the flow of information and affirm his or her expert status" (p. 375). Self-selection functioned this way in these two classes. Though self-selection was used less frequently in Stevie's class, both instructors tried to position PSTs as responsible for listening to and evaluating peer responses, similar to what PSTs will have to do as teachers.

Self-selection was much more prevalent in Pat's class than Stevie's in part because of class norms and practical concerns. In Pat's class, it was well within the norms of the class for PSTs to ask questions of a peer who presented a solution on the board, for PSTs to comment on or question other peer responses in front of the class, or for PSTs to aid a peer who faltered or seemed confused while sharing his or her response. When this occurred, Pat was no longer an intermediary between PSTs and peer thinking, which Cazden (2001) asserted could be a consequence of allowing student self-selection. In Stevie's class, practical concerns made student self-selection more difficult. In that class, students were often so far away from each other that they could not hear each other's responses if Stevie did not revoice them. In an early class period, Stevie had tried circulating a microphone so students who wanted to speak could be heard without her their words going through her but this proved more cumbersome than Stevie revoicing each response so it was discontinued.

Random cold calling held students accountable while non-random cold calling structured discussions. Pat used non-random cold calling when he wanted to choose several PSTs to present their thinking to the class. Though any type of cold calling can hold students accountable for participating by introducing the possibility that they may have to share their thinking with the class at any time (e.g., Dallimore et al., 2004; McDougall & Granby, 1996), non-random cold calling allows instructors more control over the ideas presented in discussion than any other method of choosing students to speak in discussion (Chapin et al., 2009b). As suggested by K-12 research (e.g., Chapin et al., 2009a; Stein et al., 2008), non-random cold calling allowed Pat to choose and strategically sequence student responses with different approaches, graphic representations, and/or answers. Pat was able to do this because he had access to PST solutions during the small group discussions that preceded whole-class discussion and through reviewing PST homework that he used as prompts for later whole-class discussion. Stevie did not have access to most student thinking about prompts because of her class size and because she did not use homework problems as prompts for whole-class discussion. This may have been a reason why non-random cold calling was not observed in her class though random cold calling was frequently used. Pat often used random cold calling when multiple PST solutions were not presented at the board, such as in "Instructor as Scribe" discussions. In both classes, random cold calling held students accountable for paying attention by creating the possibility that anyone could be called on to respond to a prompt.

Instructors Repeatedly Asked Students to Explain Further

Research question 2B asked: How do instructors respond to student thinking in whole-class discussion? Instructors responded to PST thinking in various ways but requests for further information, revoicing, and asking for questions or disagreement were the most common.

When instructors asked for more information from students, they most often asked several specific questions about the PST's responses rather than asking a general question or asking only a single specific question. Asking several specific questions about the PST's response kept the discussion focused on the response and the pertinent mathematical ideas therein. It also allowed instructors to elicit more complete and explicit explanations from students and to highlight the mathematical ideas present. In other words, instructors honored student thinking while asking students to go further. Research in second and third grade classrooms by Franke and colleagues (2009) showed that using sequences of specific question like this led to students giving correct and complete explanations of their thinking when their explanations were initially not correct and complete.

Student Responses Were Audible and Visible for All Students

Revoicing and making PST thinking visible were often done simultaneously with the instructor writing the PST responses on the board/document camera while revoicing what they were writing. Some form (summary, diagram, etc.) of nearly every student response was written on the document camera, overhead projector, or whiteboard. In Pat's class both Pat and PSTs wrote on the whiteboard or overhead in different types of discussion while in Stevie's class she was the only one to write student thinking on the document camera. When instructors wrote responses, they often made sure that the verbal and visual representations of student thinking were accurate. When students wrote their own responses on the whiteboard, Pat required them to explain what they were writing.

Unlike many teachers in the U.S.(Stigler & Hiebert, 1999a), both Pat and Stevie often left student solutions or ideas on the whiteboard or document camera without erasing until the discussion was over, though on some occasions Stevie was limited by the space available on the document camera. The instructors used multiple visible student responses in several ways. Pat often annotated (added explicit explanation or meaning of numbers/diagrams) PST solutions under the direction of the presenting PST or several peers (see examples in Figure 14 in Chapter 6 as well as Figure 21 and Figure 22 in Appendix B). Often, by the end of the discussion multiple full (and often annotated) solutions were present on the board. During some discussions, Pat asked students to connect corresponding ideas, numbers, or steps between the student solutions on the whiteboard. This was something that would have been much more difficult if the solutions were not still visible side-by-side on the whiteboard. Stevie juxtaposed student solutions on the document camera to help illuminate student errors when students disagreed on the answer to a problem. In some discussions, Stevie asked students for partial solutions that she wrote on the document camera to make a full solution. Before the discussion was over, Stevie or a student revoiced all the students' ideas combining them into a complete and coherent solution.

Most Instructor Evaluation Was Encouraging and Students Evaluated Peer Responses

Instructors rarely evaluated PST responses; more often they asked the PST's peers to explain or questioned PST responses until the PSTs themselves disagreed with the answer and knew why it was an incorrect one. When instructors did evaluate, what they said was mostly positive and nonspecific. These functioned more as encouragement than a stamp of approval on the student response. In Nassaji and Wells' (2000) study of grade 1-8 students, these types of evaluation were not correlated, as negative evaluations were, with future student responses being less complex.

Instructors often let PSTs find and correct errors in peer responses. In both classes, student responses were critically evaluated by both instructors and other PSTs. Both instructors pointed out that errors existed in student reasoning and then let the students find, describe, and correct them. In some cases, two or more students presented solutions with differing answers, in part or in full, to the class. The classes then discussed the solutions and instructors made it clear that one was incorrect, letting students sort out why it was incorrect. Students did this without explicit help or direction from the instructor beyond being prompted that an error existed.

Pat sometimes corrected students while they explained a solution at the whiteboard but these were corrections of mathematical notation rather than evaluations of student reasoning. We did not see these types of errors in Stevie's class since she wrote the student responses on the document camera and would not likely make these sorts of errors. Most of Pat's explicit instructor corrections were suggested at the end of the discussion of a solution rather than immediately after the PST made the error, which gave the other PSTs time to catch the error. Pat often gave students the chance to question students who presented responses before he commented on them. In these question and answer periods, students asked questions about what confused them about the solutions presented and the students who presented solutions answered, which led them to give more detailed explanations or recognize problems in their explanations. Stevie also often asked students to question or comment on peer solutions.

Students Connected Student Ideas to Peers' Thinking

Research questions 2C: In what ways and how often are one student's ideas connected to another student's ideas in whole-class discussion? Though instructors often made these types of connections, they often asked students to do the connecting. Both instructors oriented students to peer responses through asking PSTs to apply their own reasoning to peer responses or to build on peer responses. Many K-12 sources suggest that instructors are the ones responsible for making these types of connections (e.g., Chapin et al., 2009c; Engle & Conant, 2002; Stein et al., 2008), but some sources suggest students can be asked to make these connections themselves (e.g., Chapin et al., 2009c; McGraw, 2002).

Both instructors asked students to connect their ideas to a peer's response frequently enough that there were routines in each class. Pat often structured discussions so that PSTs were interpreting or comparing and contrasting peer responses. In many of the observed whole-class discussions, several PSTs presented full solutions for the same problem. Though several solutions may be presented in parallel, Pat made sure that students made connections between solutions. After students explained their solutions, the rest of the class was instructed to explain the steps or reasoning in the solution, compare or contrast the solutions or thinking underlying the solutions, or ask the presenting students questions until they understood. At the end of these discussions, students understood several different methods of solving or thinking about the same problem (sometimes including common student errors), the equivalent steps in each solution, and what each number or representation meant in the context of the problem. When single solutions were presented in discussion, students were still asked to understand the solution well enough to explain it to someone else, were often encouraged to ask questions about it, and were expected to know what each number and representation meant in the context of the problem. The class did this as a whole through the connected reasoning of several students.

During campfire discussions, Stevie asked PSTs to comment, rephrase, or ask a question about a peer's response. Because PSTs only had to explain part of the response to the problem, other PSTs were expected to add to the explanation or question it, meaning that they had to consider peer responses when formulating their own responses. This made a series of student responses that created a full explanation rather than several parallel explanations. Though Stevie sometimes asked PSTs to summarize the string of responses from these discussions, more often Stevie summarized the series of PST contributions.

Instructors Connected Student Thinking to Mathematical Ideas

Research question 2D asked: In what ways and how often is student thinking in discussion explicitly connected to mathematics? Both instructors elaborated on PST responses to phrase them more mathematically or to bring in additional mathematical content. During discussion, the instructors tied student responses to more canonical terms or prompted students to revise or annotate their definitions with mathematical language that was more precise. Stevie explicitly asked PSTs to connect their solutions or thinking to definitions or mathematical concepts. Pat sometimes gave short lectures on related mathematical topics as they were brought up in discussion. Both Stevie and Pat asked students to create their own lay definitions of concepts before the instructors introduced the mathematical definition. Later in both classes, instructors started explicitly asking students what mathematics were needed to solve a problem or what mathematics were needed for a particular solution method.

Limitations

While this study adds to our understanding of what goes on in whole-class discussions for preservice teachers, it does have its limitations. Video quality was a technological limitation in some instances. Because the camera was usually in a fixed position at the back of the classroom, sometimes writing on the board could not be read and it was sometimes difficult to determine which students were speaking. In a few cases, the tripod malfunctioned and the camera was left to record the ceiling for several minutes. In these cases, while I could follow the audible portions of discussion and approximate drawings on the board from field notes, nonverbal communication and the process of drawing representations on the board was lost. Most of the video issues in Stevie's class were because of the class size. It was sometimes difficult to hear students responding or to decipher what they were saying. When student responses were unintelligible, I noted it in the summary.

Generalizability is a limitation inherent in case studies, particularly those that use atypical cases (Yin, 1994). The instructors were typical in that most of the instructors who completed the ME.ET project instructor survey were experienced in college teaching and teaching this course in particular. However, they were also more interested in teaching this class than other instructors who completed the ME.ET instructor survey and used whole-class discussion more frequently and consistently than other instructors that were videotaped for the project. Stevie's class size was also atypical. However, the purpose of these cases was not to say that whole-class discussion was typical or widely used. The purpose was to investigate what whole-class discussions in such classes was like and in order to do that I had to look at classes where there was in fact whole-class discussion to investigate.

Significance and Contributions

This study contributes to the literature through providing an example of whole-class discussion in a large postsecondary classroom, comparing whole-class discussion across classrooms, showing teaching moves and routines used to facilitate whole-class discussion in postsecondary classrooms, and by investigating teaching practices that are productive in K-12 discussions in postsecondary classrooms. Further discussion of each follows.

Providing an example of whole-class discussion in a large postsecondary classroom. Large class sizes may be especially common in introductory postsecondary classrooms but the process of using teaching practices in these classes (rather than student attitudes or outcomes when a teaching practice is or is not used (e.g., Galyon et al., 2012; Jansen, 2009)) is not often discussed. Though class size may be an important difference between the two classes investigated in this study, it did not preclude educationally beneficial participation or interactions during whole-class discussions. In both classes, instructors were able to facilitate whole-class discussion and used many of the same practices to do so. The important differences between these two classes may be the types of interaction and participation that instructors can make available to students in these settings in the time available for such interactions. The class size may have affected this since some activities, such as several students presenting full solutions at the board, may not scale up well because class time does not allow enough attention to each student. Because it may be more difficult to hold students accountable for their interactions with other students or listening to student explanations during class activities in larger classes, free riding may be more likely. However, as discussed in Chapter 2, free riding may occur in a class of any size when students feel that they can get away with it. Using strategies such as random cold calling and asking students to connect their thinking to peer responses though questioning or building-on peer responses can increase accountability even in large classes. Asking students to present partial thinking rather than asking for complete responses and asking students to talk about what they are confused about may also increase accountability because any student can give a response, they need not have the whole solution figured out.

Comparing whole-class discussion across classrooms. As stated earlier, there is very little research investigating what occurs in discussions in postsecondary classrooms, particularly in classes for PSTs or in multiple classrooms. This study contributes to the education of PSTs by comparing two cases of whole-class discussion facilitated by instructors in mathematics classes for PSTs. Though these classes may have been structured differently, had differing class sizes, and different instructors; there were similarities in these discussion-rich classes. Students were often asked to explain their thinking, further explain their thinking, and explain or connect to peer thinking. Student thinking was made audible and visible to all students and instructors made sure that student thinking was connected to the mathematical ideas that they were teaching.

Had I not investigated these two classes using the same lens, some of these practices may have been overlooked. Using the same lens or framework to analyze multiple classrooms allows comparison to see similarities in practice despite the differences in context. For example, if these classes were analyzed separately either instructor's propensity for making student thinking visible may have been overlooked because though both instructors used it consistently, the other responses to student thinking categories were primarily verbal, it was integrated in the flow of classroom discourse (since it often occurred *while* instructors were revoicing), and it was not often discussed in the K-12 research that guided my investigation. Because the practice was used frequently, consistently, and in both classrooms, it stood out as a practice that the instructors must be using purposefully and that needed further investigation.

Showing teaching moves and routines used to facilitate whole-class discussion in postsecondary classrooms. As mentioned previously, few studies focus what occurs in whole-class discussion in postsecondary classes. Systematic analysis of classroom discussions can uncover unusual teaching strategies or routines that elevate the level of discussion. One such example is the use of visual representations of student thinking, such as was seen on the whiteboard and document camera, in whole-class discussions by these instructors. Both instructors made sure that most student contributions to discussions were visible in some way on the whiteboard or document camera for the rest of the class to see. While there is some literature on the general historical uses of blackboards (Evans, 1910; Kidwell, Akerberg-Hastings, & Roberts, 2008a) and overhead devices in the classroom (Kidwell, Akerberg-Hastings, & Roberts, 2008b), and the use of gestures with blackboards in discussion and lecture (e.g., Greiffenhagen & Sharrock, 2005; Hwang & Roth, 2011), there is very little research on how and for what purposes instructors (postsecondary or otherwise) use blackboards or overheads in whole-class discussion. Research on *effective* or *strategic* use of blackboards and other display media is near absent from the research literature I was able to find through extensive searches on a wide range of search terms. What research there is that discusses instructor purposes for writing on the whiteboard or overhead mainly comes from work on Japanese lesson study (Takahashi, 2005-2006, 2011; Yoshida, 2005) or other international studies (Stigler & Hiebert, 1999a), and it does not have a postsecondary focus.

In addition, the results of this study may give instructors of postsecondary students teaching moves or routines to pay attention to or use to attempt to influence classroom climate and norms to facilitate discussion in their classrooms. Though this study focuses on mathematical discussions, this framework may be useful in non-mathematics classes as well. After all, in most classes instructors can ask challenging questions; give students time to think about them before responding; use multiple ways of choosing students to share in discussion so that students are held accountable for participating, content is logically sequenced, and students gain agency and comfort in discussion; ask students to thoroughly explain their thinking, make student responses audible and visible to all students, encourage students to connect their reasoning to that of their peers or make the connections themselves, and make sure student thinking is tied to the content being taught and discussed. Personally, constructing this study and watching videos for preliminary coding has changed my teaching. Though my classes have always had high levels of participation in whole-class discussions, consistently using some of these teaching moves or routines (such as asking students several specific questions about their responses or asking students to interpret the responses of peers) has allowed me to facilitate deeper discussions than I had previously. In addition, it has inspired conversations with other instructors about their teaching routines and assumptions about facilitating discussion.

Investigating in postsecondary classrooms, teaching practices that are productive in K-12 discussions. As discussed earlier, researchers often assume that whole-class discussion in postsecondary classrooms is similar and should be facilitated similarly to whole-class discussion in elementary and secondary classrooms. This assumption should be made explicit and tested. Postsecondary students, such as PSTs, are in many ways different than K-12 students and may view or respond to teaching practices very differently. For example, asking a PST to interpret a peer's thinking may hold more relevance for them than asking an elementary student to do the same because PSTs will have to interpret and respond to their students' thinking when they become teachers whereas elementary students are much less likely to have future students in mind. Post secondary students also have more experience with being taught than elementary or secondary students and this may hinder or help their engagement in whole-class discussions.

Using practices that are successful in K-12 classrooms as a lens to investigate multiple class sessions of multiple instructors can help us determine how similar discussions in postsecondary classrooms might be to K-12 classroom discussions. In other words, analysis provides insight into whether teaching practices that are successful in promoting discussions in K-12 classroom are useful or at least used in postsecondary classrooms. From this study, it is clear that many of the strategies that have been found useful in K-12 classrooms were also fundamental to the repertoire of these two instructors. Strategies used successfully in the two classrooms in this study are:

- giving students time to engage with the activity before discussing their ideas (Chapin et al., 2009b; Lamberg, 2013; Smith & Stein, 2012d);
- using tasks for discussion that are cognitively challenging (Baumert et al., 2010; Nystrand et al., 2003; Webb, 2009);
- allowing students to self-select (Atwood et al., 2010; Cazden, 2001);
- holding students accountable for listening and responding to each other (Chapin et al., 2009c; Herbel-Eisenmann et al., 2009; McGraw, 2002; Smith & Stein, 2012c);
- choosing and ordering student responses in a deliberate way to get specific mathematical ideas into the discussion (Chapin et al., 2009b; Smith & Stein, 2012b; Stein et al., 2008);
- asking several specific questions about student responses to elicit further explanation (Franke et al., 2009);
- making student thinking public and/or visible (Chapin et al., 2009a; Smith & Stein, 2012a; Stigler & Hiebert, 1999a);
- avoiding negative evaluation of student responses by the instructor (Nassaji & Wells, 2000);
- asking students to engage with and evaluate peer responses (Chapin et al., 2009c; McGraw, 2002); and

- connecting student reponses to mathematical ideas (Smith & Stein, 2012b; Walshaw & Anthony, 2008).

Further Research

This study's findings suggest many topics for future research. Further investigation can be done in a wider variety of postsecondary contexts to see if this lens is useful in describing similarities and differences in whole-class discussion across contexts. For example, the affordances and constraints of using whole-class discussion in large and small classes warrant further study. Though the median class size of the mathematics classes for PSTs investigated by the ME.ET project was 27 students, many postsecondary classes have much larger class sizes. While this study suggests the whole-class discussion is possible in such classes, it is important to know what limitations or opportunities such a large class size might present for student learning and participation in order to determine when to use whole-class discussion in these settings. It is also possible that strategies (such as using homework problems as prompts for discussion) can get around some of the possible constraints (such as not knowing what students are thinking about a problem) of discussion in larger classes.

How instructors use the board and other visuals during whole-class discussions is an area ripe for investigation. As discussed earlier in this chapter, there is little research on how instructors use the board, overhead, or document camera during whole-class discussion beyond historical perspectives on their general use in the classroom (e.g., Evans, 1910; Kidwell, Akerberg-Hastings, & Roberts, 2008c), the use of gestures with blackboards in discussion and lecture (e.g., Greiffenhagen & Sharrock, 2005; Hwang & Roth, 2011), or cross cultural work, such as work on Japanese lesson study (e.g., Takahashi, 2005-2006, 2011; Yoshida, 2005) or reports from TIMSS (Stigler & Hiebert, 1999b). The cross-cultural studies were the most informative in this case because they addressed teacher rationales for writing certain material on the board and differences in practice between instructors; however, the contexts of instruction are very different both culturally and in terms of student age.

Conclusion

Whole-class discussions in mathematics classes of PSTs mirror whole-class discussions in K-12 classrooms, although with some important differences. Every element of the framework proposed here has a counterpart in the literature on discussion in K-12 mathematics classrooms. The differences arise from both the vastly different context of postsecondary teacher education and the almost palpable difference in both instructor's and PSTs' recognition of the importance of what they are doing. These PSTs are young adults, on their way to being responsible for teaching children. Observing these classrooms (and the others involved in the larger study) made it clear that *mathematical discussion* helped the PSTs realize how much they had to learn, and to appreciate the seriousness of the endeavor of becoming a teacher. In discussion, the weaknesses in their own mathematical knowledge were revealed, and they could no longer hide behind the assertion that this is easy math, that they already know it. The norms established in both of these classrooms helped the PSTs put down their guard and participate even when they were uncertain of the answers. Such norms—that students are accountable for listening to and understanding each other, that they will be asked to share their work even if it is not correct, that incorrect work and confusion are not signs of stupidity—are at least as important for these classrooms as for K-12 classrooms, and perhaps even more important.

Unlike the young students they will teach in elementary school, PSTs can be set in their mathematical ways making new norms key to changing how they approach, think about, and do mathematics.

As I mentioned above, doing this study—observing a range of teachers and especially, analyzing the work of these two expert teachers—has changed the way I teach, and I hope the details of this study will be useful in helping others take a new approach to their own teaching.

APPENDICES

Table 16. Portion of a Summary of Pat's Class Exported from NVivo

Row	Start	Content
54	69:51	Pat asks them to re-write this problem in their groups putting whole numbers in it: “do this for me. Here's what I want you to do at your seat for a minute. I want you to rewrite this problem—just rewrite this problem, but I want you at your groups to put whole numbers in it. Does that make sense? Use the exact same problem but rewrite it with whole numbers, just at your groups and then see what happens when you put your own whole numbers in it. No fractions, you know, we want whole number answers in this whole thing, so we're making a lot of cookies, is basically what I'm saying. Take a minute in your groups.” After a couple seconds, he corrects that they're making waffles.
55	70:23	They immediately begin discussing. Scott asks if they have to get the same answer, Pat says no and repeats that he wants then to use the exact same problem, just use different numbers.
56	71:09	Pat asks what word problem Edna group came up with. Edna (NON-RANDOM by GROUP) “a batch of waffles requires” . . . ”2 cups of milk. You have 4 cups of milk. Exactly how many batches of waffles could you make?” Pat asks how they got their answer. Edna says, “four divided by 2.”
57	71:27	Pat asks for a word problem from someone else. Pat calls on Alan asking what numbers they put in the problem. Alan (NON-RANDOM by GROUP) says “6 and 2.” Pat asks him to read the problem. Alan incorrectly reads the problem yielding a problem with a fraction as an answer: “A batch of waffles requires 6 cups of milk, you have 2 cups of milk. How many batches of waffles can you make?” SSS murmur. Pat “OK, then what's the answer?” The rest of the class lets him know that it ends up in a fraction. Pat tells him to change it once Alan realizes his mistake. Alan reads, “A batch of waffles requires 2 cups of milk. You have 6 cups of milk.” Pat repeats, “you have six cups of milk. How many batches can you make?” and asks Alan for his answer. Alan “three” Pat says three then he asks if whole numbers help. SSSS, “yeah.”

Table 16 (cont'd)

Row	Start	Content
58	72:10	Pat asks what whole numbers another group put in the problem. Kay (NON-RANDOM by GROUP) "we did 1 and 2" Pat "OK" Class laughs. Kay "we did a batch of waffles requires 1 cup of milk, if you have 2 cups of milk, exactly how many batches could you make?" Pat, "right, and your answer would be? [Pause] two right."
59	72:33	Pat calls on Kayla who apparently has a question or comment. Kayla (VOLUNTEER) comments on something her elementary kids did (but she is really hard to hear). Pat also suggests that they could have made it into a subtraction problem. Then he explains but is turned around in the numbers a bit, finally he posits a batch that needs two cups of milk and you have 6 cups of milk. ((He writes out his chain of subtraction on the board)), voicing what he is writing. Kay "That's what I originally did, I did—" Pat continues with his example, counting up the batches. He says, that's what your kids would have done. Pat says that he thought he saw someone use an additive strategy, he concedes it might be from his other class. Student (SELF-SELECT) "that's what I did." Pat gives an example ((writing on the board)) of such a strategy ($1^{1}/_{2}$ cups is two batches so half a cup left over), saying that this is what they see their kids do with whole numbers.
60	73:55	Pat says that fractional problems are technically the same as whole number problems, "at the same time, they're a world apart." He gives the example of a batch of waffles needing 3 cups of milk and you have 8 cups of milk. Then he repurposes the diagram Harry wrote on the board, now using the boxes that represented quarters to represent wholes. He says he now has 2 batches and 2 cups leftover in whole numbers. He asks if that makes sense. He's changed the representation to a whole number representation, "the picture hasn't changed but the meaning that I'm ascribing to it" has. He says, "therein lies the issue . . . pictures are cool" and he references Kayla's bus problem solution. He says that these things help because they have meaning but if the kids aren't on board with what these things mean or you don't explain it clearly it can be confusing. He asks if that makes sense.

Note. Rows 54-55 show a small group discussion, rows 56-59 show a whole-class discussion, and row 60 shows part of a lecture. In rows 56-59, since new speakers start new lines, the speaker's name always appears on the far left of the table. Student and teacher names are pseudonyms.

Table 17. Topic Codes

Name	Classes	References
Bosnian-eastern European algorithm	1	91
Bosnian-eastern European v. American algorithm	1	20
Spontaneous student: Bosnian-eastern European v. American algorithm	1	3
Finish using Bosnian-eastern European algorithm	1	12
Explaining the actual solution	1	2
Student 1	1	1
Student 2	1	1
Teaching Bosnian-eastern European algorithm to 6th graders	1	41
How to teach - student 1	1	10
How to teach - student 2	1	7
Spontaneous student - how to teach Bosnian-eastern European algorithm	1	1
Brownie Problem	1	47
Comparing student 1 and student 2s strategies	1	10
Comparing student 1, student 2, student 3	1	23
Student question - why start with 15	1	6
Student 1	1	6
Student 2 - confusing team solution	1	5
Student 3 - other class uses 15 dots	1	1
Bus Problem	1	64
1st strategy - the whole-number arithmetic method	1	21
2nd strategy - multiplying fractions	1	31
Thinking about how many people remain	1	5
Hypothetical problem and the hidden step	1	7
3rd strategy - pictorial	1	3
Comparing fractions	2	34
Alternate methods for comparing fractions	1	20
Cross multiplication	1	3
Same denominator, same numerator	1	5
Cookie Problem	1	139
Comparing to numerically same problem	1	9
Comparing student 3 strategy to student 4's	1	16
Student 1 - algorithmic solution	1	10
Student 2 - flipped version of algorithm	1	7
Student 3 - pictorial strategy	1	31
Student 4 - another pictorial strategy	1	19
What mathematics do you need and write a number sentence	1	11

Table 17 (cont'd)

Name	Classes	References
Definition of fractions	2	39
Mireya's recipe and the definition of fractions	1	17
Piece of paper as fraction	1	18
PST connects to juice box problem	1	1
What is confusing or what did you think about	1	11
Elementary student errors about the whole	1	5
Equivalent fractions	1	21
Explaining equivalent fractions to elementary students	1	3
Fraction of what	2	62
What fraction of the casserole can Jean make	2	44
Drawing a representation	1	12
What's the whole	1	6
Fractions as objects-numbers	1	13
Fractions on a number line	1	9
Student errors with fractions on the number line	1	5
Improper fractions	1	12
Take a blue strip of paper. Is it one-fifth of a full piece of ordinary paper?	1	4
ME.ET post-test	1	2
Percent	1	1
What percent of the diagram is shaded	1	3
Solving percent problems	1	1
Definition of percent	1	2
Place value	1	16
Fudge Problem 1 - student 1 - base 10 blocks	1	4
Fudge Problem 1 - student 2	1	1
Fudge Problem 2	1	1
Fudge Problem 3 - student 1	1	1
Fudge Problems - determine what CGI problem type it is	1	7
Quiz & homework problem review	4	29
Exam question review	1	7
Homework review - PST errors	1	4
PST homework errors - common denominator	1	2
When to drop a zero	1	2
Rounding and plotting on the number line	1	20
Explaining rule for rounding	1	5
Plotting 34,617 on a number line and how it should be rounded	1	13
Semiotics	2	29

Table 17 (cont'd)

Name	Classes	References
Sorting word problems	2	56
Comparing and contrasting comparison problems	1	8
Comparing equal sharing problems	1	13
Partitive division problems	1	2
Pat proposes CGI problem type	1	1
Waffle Problem	2	67
1st solution	1	11
Number sentence	1	8
Re-writing problem with whole numbers	1	7
Wedding Problem	2	85
Clarifying whether men are married to the women	1	2
Student 1 - adding the denominator	1	3
Student 2 - adding denominators pictorially	1	1
Students bring up that you can't add denominators	1	6
Understanding the common denominator	1	23
What's a numerator-denominator	1	9
What's wrong with the common denominator in the Wedding Problem solution	1	11

Table 18. Context Codes

Name	Classes	References
Clarification of expectations	1	2
Comparing strategies	5	85
Connecting definition to example	1	2
Connecting to experiences with elementary students	4	8
Feedback on classroom procedure	1	1
How to teach/use with elementary students	2	29
Instructor asked for further explanation	5	21
Instructor asked for other solutions	2	5
Instructor asked PSTs to sort out opposing reasoning	1	1
Instructor pointed out where an error is	1	2
Instructor proposed another student method or algorithm	3	5
Instructor ran through solution again	1	5
Instructor summarized and elaborated	9	42
Instructor summarized remaining problem	1	1
Instructor transcribed to board	1	2
Interpret PST thinking	2	16
Joked	1	1
Peer comments	5	17
Peer re-explained solution	2	44
Polled class for answers	3	10
PST connected with other problem	1	1
Explained	6	34
Question and related answer	2	2
Rephrased peer thought/answer	1	1
Ran through the whole thing again	1	1
PSTs checked their understanding	1	2
PSTs continued peer answers/thinking	1	2
PSTs gave answers and explain	9	74
PSTs proposed definitions-explanations	1	3
PSTs responded/answered	1	3
PSTs shared thinking-partial solutions	1	6
PSTs worked on problem(s)	5	12
Question, comment, continue	2	15
Questions for student solution giver	5	23
Reading the problem	2	3
Re-approach a problem	1	2
Related tangent	4	12
Review of what they talked about last class	3	11
What mathematics	2	32

Appendix B. Classroom overviews and summaries

Table 19. Summaries of Topics and Activity Types in Pat's Class

Day	Topic	Type	Duration	Description
4/1	Eastern European algorithm	SG	3:39	Pat asked students to find 10488/23 without using a calculator.
		LC	4:07	After polling the class for answers, Pat introduced the Bosnian-eastern European algorithm.
		SG	2:54	Pat asked students to finish the problem using the Bosnian-eastern European algorithm.
		WC	9:56	Pat called on Kennedy who explained her group's solution while Pat writes it on the board. Pat asked if anyone solved it differently and Alan is called up to present his group's solution. Pat then presented the way the problem would be finished using the algorithm and asked students to explain. Students began spontaneously comparing the Bosnian-eastern European algorithm with the American algorithm. Students continue to interpret the algorithm.
		SG	3:15	Pat asked students to compare and contrast the eastern European algorithm with the American algorithm with their small groups.
		WC	11:10	Students compared the eastern European and American algorithms. Pat summarized student responses and a student connected this to problems an elementary student she worked with had. They then discussed how to teach this to elementary students.
		SG	7:03	Pat asked students to discuss how to teach the eastern European algorithm to 6th graders.
		WC	17:09	Pat called on Eva to explain teaching the eastern European Algorithm to 6th graders. Pat directed the students to ask Eva questions about her response. Pat asked, as a sixth grader, about a step in their solution. Pat asked for other solutions and called on Alan who presents his solution. Students asked him questions.
		AD	2:59	Pat asked students to answer several questions related to the discussion for homework.
		LC	2:25	Pat lectured on the value of the eastern European algorithm.

Table 19 (cont'd)

Day	Topic	Type	Duration	Description
4/1	Class break	BK	5:59	Class break
	Sorting 9 word problems	SG	2:08+	Pat gave students 9 fraction problems to sort in small groups.
4/3	Administrative	AD	2:44	Pat asked students to turn in their writing on the Wedding and Bus Problems.
	Sorting 9 word problems (continued)	SG	2:26	Pat asked students to re-familiarize themselves with the word problems that they sorted in small groups during the last class.
		WC	22:38	Pat asked groups how they sorted the word problems then he asked students to explain their groupings.
		LC	4:26	Pat explained how they could be grouped into CGI problem types.
		SG	4:01	Pat asked students to discuss how the equal sharing problems are different.
		WC	9:01	Pat asked groups to discuss how the equal sharing problems were different then Pat summarized and connected to their experiences with elementary students.
		LC	1:44	Pat reminded students that they have to be careful of context when using problems with elementary students.
		SG	2:54	Pat asked students to contrast problems 4, 5, and 6.
		WC	4:25	They discussed how 4, 5, and 6 are different.
		LC	5:15	Pat summarized the discussion then gave CGI for problem 7.
		WC	2:20	Pat asked students what CGI question type 8 and 9 are.
		LC	3:07	Pat summarized the discussion and talked about using these types of problems with elementary students.
		AD	2:51	Pat gave a credit/no credit assignment writing problems for elementary students.
	Class break	BK	5:38	Class break

Table 19 (cont'd)

Day	Topic	Type	Duration	Description
4/3	Wedding Problem (continued)	WC	10:39	Pat asked Alan to explain his solution to the Wedding Problem at the board and then other students ask Alan questions. The class then discussed whether the men at the wedding are married to the women at the wedding in the context of the problem. Pat called Blair to present another way to solve the problem.
		SG	3:15	Pat asked students to understand these 2 strategies well enough to explain students.
		WC	5:58+	Pat called on a student to explain the solutions. Then students began to question whether you could add the denominators like the students did.
4/8	Wedding Problem (continued)	LC	5:10	Pat reviewed the discussion they had online about the Wedding Problem.
		WC	2:35	Pat asked students to discuss what the numerator and denominator mean.
		SG	2:15	Pat asked students to discuss what the numerator and denominator mean in their groups.
		WC	2:07	Students further explain what the numerator and denominator mean.
		WC	10:24	Pat asked students what's wrong with the common denominator in the Wedding problem. Pat summarized the discussion, students discussed further, and Pat summarized again.
		AD	2:51	Pat gave students an assignment: write a word problem for which the answer nineteen-fifteenths makes sense.
	Class break	BK	5:44	Class break
	Bus Problem	WC	9:58	They re-read the Bus Problem and then Kay explained a whole-number arithmetic method for solving it.
		SG	5:30	Pat asked students to be able to explain Kay's method.
		WC	13:55	Several students re-explained the numbers in Kay's solution. Pat summarized their responses and then Renee explained her solution that multiplies fractions.
		SG	7:02	Pat asked students to be able to explain Renee's method and connect it with Kay's solution.

Table 19 (cont'd)

Day	Topic	Type	Duration	Description
4/8	Bus Problem (continued)	WC	30:39	Several students explained the Renee's solution and connected it to Kay's solution. There was some confusion and Pat asked students whether the thinking in the solution is correct. A student proposed that it would not be if the numbers in the problem were different and another student pointed out an implicit step. Pat summarized the discussion and they discussed whether Renee was thinking about the number of students remaining on the bus. Students continued to explain the numbers in the solution then they talked about why she used multiplication in her strategy.
		LC	2:14	Pat warned about using word cues.
		WC	2:57	Kayla explained a third, pictorial, solution to the Bus Problem.
		LC	2:07	Pat added in the intermediate steps in Kayla's solution.
	Administrative	AD	3:23	Pat assigned homework for the next class period.
4/10	Semiotics	LC	30:09	Pat lectured on Semiotics.
	Waffle Problem	SG	11:43+	Pat gave students a worksheet to work on in their groups. He said that he was going to give students 15-20 minutes to work through 3 problems.
		WC	13:53	Pat polled the class for answers and found that students came up with 2 answers. Harry explained an answer, Pat asked for clarification, students asked Harry questions, and group members helped him answer. Pat pointed out that there was an error.
		LC	2:56	Pat lectured on the issue of abstraction and changing wholes.
		WC	11:10	Pat asked for a number sentence for the problem. Students proposed several number sentences. Pat asked what type of problem it is.
		SG	1:18	Pat asked students to rewrite this problem using whole numbers.
		WC	2:46	Pat polled the class for several answers. A student compared it to something that the elementary student she worked with did.

Table 19 (cont'd)

Day	Topic	Type	Duration	Description
4/10	Waffle Problem (continued)	LC	2:35	Pat lectured that fractional problems are the same as whole number problems.
	Administrative	AD	1:20	Pat assigned homework for the next class period.
4/15	Waffle Problem (continued)	LC	10:26	Pat recapped what they talked about the Waffle Problem last class. He talked about a strategy that a student from his other class thought of.
	Cookie Problem	SG	17:34	Pat asked students to re-familiarize themselves with the word problems that they worked on last class.
		WC	21:00	Kennedy talked about her algorithmic solution, students ask her questions, and a student proposes a flipped version of her solution. Pat asked for further information and they compare the 2 methods. Pat very briefly talked about the semantic to syntactic translation then Colin presented a pictorial solution. Students asked him questions and Pat asked him to go through it again step by step. Pat summarized the discussion.
		SG	1:26	Pat asked students why Colin labeled that 1 cup.
		WC	9:56	The class discussed why Colin labeled something 1 cup. Then Elena presented another pictorial strategy and students asked her questions.
		SG	1:30	Pat asked students to compare the 2 pictorial strategies.
		WC	4:42	The class compared the 2 pictorial strategies.
		SG	4:00	Pat asked what mathematics you need to know in order to solve Colin and Elena's strategies.
		WC	9:05	The class discussed what mathematics is needed for those two strategies. Pat summarized the responses.
	Comparing numerically similar problems	SG	3:33	Pat asked students to compare 2 numerically similar problems.
		WC	7:10	The class discussed their comparisons of 2 numerically similar problems. Pat summarized responses.
	Pictorial strategies	LC	1:55	Pat talked about the insight that pictorial strategies can give students.

Table 19 (cont'd)

Day	Topic	Type	Duration	Description
4/15	Student evaluations	AD	0:31	Students completed student evaluations.
4/17	Brownie Problem	WC	12:45	Stacy talked about her solution for the brownie problem and then Pat asked for further information about her strategy. Amanda and her group then presented another solution.
		SG	1:52	Pat asked students to compare these 2 strategies.
		WC	6:03	The class compared the 2 strategies.
		LC	2:53	Pat summarized the discussion and proposed another student method using 15 dots.
		SG	3:40	Pat asked students to compare the 3 student strategies.
		WC	11:14	The class discussed the 3 student strategies, Pat elaborated. A student asked why they started with 15 and Pat elaborated.
		LC	7:07	Pat lectured about what mathematics is needed for these strategies.
	Place value	SG	24:25	Pat asked students to do a Place Value worksheet.
		WC	1:27	A student explained a solution for Fudge Problem 1 and Pat asked for further explanation.
		LC	2:44	Pat lectured on the semiotics of talking about decimals.
		WC	9:27	The student continued to explain her solution. Kay presented her solution to the second fudge problem. Pat asked students to discuss in small group what CGI problem type it was and then he asked students to give answers and explain in whole-class discussion. A student presented a solution to the 3rd fudge problem and then they discussed what CGI problem type is was.
	ME.ET	TQ	20:37	ME.ET post test

Note. AD = administrative, LC = lecture, SG = small group, TQ = tests and quizzes, and WC = whole-class discussion. Times ending in + indicate instances where the tape either stopped or skipped. In these cases, the time indicates the duration on video while the plus indicates that it continued while the videotape was not running.

Table 20. Summaries of Topics and Activity Types in Stevie's Class

Day	Topics	Type	Duration	Description
9/3	Administrative	AD	3:00	Stevie asked students to pick up their quizzes.
	Quiz and homework problem review	LC	4:50	Stevie reviewed "Plot the point 1.011on number lines and show how to zoom in on that point." from the quiz then a problem from the homework that asked students to plot 7.148 and a number 1 apart.
	Rounding and plotting on the number line	LC	3:22	Stevie lectured about rounding numbers on the number line, talking about zooming in and out.
		SG	4:30	Stevie asked students to talk about a prompt on page 26 that asked them to plot 34,617 on 3 number lines. Students discussed.
		LC	7:32	Stevie clarified what the diagram in the problem means because there was confusion. Stevie talked about zooming out and precision then she moved to talk about the other number lines.
		SG	2:45	Stevie asked students to talk to their neighbor about the rule for rounding to the nearest thousand.
		WG	4:28	Stevie asked them to explain, "How would you explain why this rule makes sense in light of thinking of numbers on a number line?" Students gave answers and explained. Stevie joked then summarized and elaborated, talking about rounding to 100s. Stevie asked for questions and a student asked when students learn rounding and number lines. Stevie replied that it is 4th or 5th grade.
	Definition of fractions	LC	5:34	Stevie lectured on the definition of fractions.
	Piece of paper as a fraction	IP	2:23	Stevie asked students to do an activity from the activity manual: Take a blank piece of paper and imagine that it is 4/5 of some larger piece of paper. Fold your piece of paper to show 3/5 of the larger (imagined) piece of paper. Do this as carefully and precisely as possible without using a ruler or doing any measuring. Explain why your answer is correct. Could two people have different-looking solutions that are both correct?

Table 20 (cont'd)

Day	Topic	Type	Duration	Description
9/3	Piece of paper as a fraction (continued)	SG	2:38	Stevie told them they could talk to their neighbor if they are ready.
		WC	10:33	Stevie asked students what is tricky or confusing in the problem. They discussed. She reiterated that they should share their thinking and not worry about presenting a full solution. Later Stevie asked students to rephrase or ask a question about what a student said. At the end of discussion, Stevie revoiced a student solution.
9/5	Administrative	AD	3:25	PowerPoint slide with announcements. There was a problem at the bottom: Take a blue strip of paper. Is it 1/5 of a full piece of ordinary paper? How can you tell if it is or isn't?
	Review of homework: dropping zeroes	LC	5:26	Stevie asked them if they picked up their homework from the front of the room then deferred to a TA who talked about the homework. She saw a common theme: when can you drop the zero in decimals?
	Is the strip of paper 1/5 of a full piece of paper?	SG	10:35	Stevie asked them to talk about the problem on the PowerPoint with a partner.
		WC	1:48	Stevie asked them how they decided whether it was or was not a fifth of a regular sheet of paper. A student talked about repeatedly putting the strip of paper on a regular sheet of paper to see if it covered five times.
		LC	1:37	Stevie talked about how that fits with their definition of 1/5 from last class: a piece is 1/B of an object if B copies of the piece make the whole object.
	Mireya's recipe and the definition of fraction	LC	1:41	Stevie lectured about definitions of the numerator and denominator.
		IP	2:36	Stevie asked them to Use the definition of fraction in solving this problem: page 32 number. She told them to think about it quietly and then they will share with their neighbor.
		SG	1:53	Stevie told them they could talk to their neighbor if they were ready.

Table 20 (cont'd)

Day	Topic	Type	Duration	Description
9/5	Mireya's recipe and the definition of fraction (continued)	WC	4:09	Stevie said she wanted the students' initial thoughts. They could talk about how they thought about the problem, anything relevant, or even a full explanation if they wanted. Students talked about staring with fifteen-twentieths or needing three-fourths five times, "to show that it actually adds up to a whole, you can take the 5 that are remaining and show that that's one-fourth. And one-fourth and three-fourths is one," fifteen-twentieths plus everything remaining is the whole.
		SG	2:27	Stevie asked students how they know that what she was getting is three-fourths of what you need for a full recipe? What does three-fourths of something mean? She wanted them to talk about it in small groups.
		LC	3:01	Stevie told the students that a collection of things is the object in this problem.
		WC	6:34	Stevie asked them how they can "see the shaded part as 3 pieces, each of which is a fourth of that whole amount?" Students talked about the recipe as one whole thing and each cup as one-fifth of the recipe. Stevie asked another student to go through the explanation again. Stevie summarized the discussion and emphasized certain points including the "issue of the whole."
	Maurice, Kayla, and the whole	SG	1:55	Stevie directed them to page 33 numbers 1 and 2 and asked students to talk about them in small groups.
		WC	5:09	PSTs interpreted Kayla and Maurice's thinking focusing on the common denominator and what the whole is.
	Improper fractions	LC	2:23	Stevie talked about the importance of the whole in proper fractions.
		SG	1:30	Students worked in small groups on problem 3 on page 38 in small groups.

Table 20 (cont'd)

Day	Topic	Type	Duration	Description
9/5	Improper fractions (continued)	WC	6:10	PSTs discussed how the picture could be confusing to elementary students. They talked about needing more than 1 whole to represent the quantity, making sure that each section has equal parts, not thinking about it as 8 parts total, and whether you can outline the whole.
9/8	Administrative	AD	4:48	Class announcements
	Fractions of objects and fractions as numbers	LC	5:09	Stevie lectured on fractions of object and fractions as numbers then segues into fractions on a number line.
		SG	2:39	Stevie asked students to do an activity involving strips of paper folded as number lines that asked students to discuss how the placement of fractions on the number line fits with the way they defined fractions as objects.
		WC	4:15	Students discussed how the placement of fractions on the number line fits with the definition of fractions as objects then they talked about how to use this with elementary students.
	Errors with fractions on number lines	LC	5:32	Stevie transitioned to student errors with fractions on number lines.
	Equivalent fractions	LC	2:15	Stevie talked about how whole numbers and decimals can be written only one way. Then she turned to fractions.
		SG	0:58	Stevie asked them to talk to their neighbor about "Why is 2/3 = 4/6? How can we explain why these are equal?"
		LC	6:24	Stevie talked about answers that people may have had. Stevie gave students instruction on how to use paper strips in an activity.
		SG	1:59	Students talk to their neighbors about the problem on page 41, focusing on the part having to do with the part.
		WC	2:00	Stevie asked "briefly explain that first part. How is two-thirds equal to four-sixths? How can we see that?" PSTs discussed the same amount of paper being shaded and that they have just subdivided, they have not made anything bigger or smaller.

Table 20 (cont'd)

Day	Topic	Type	Duration	Description
9/8	Equivalent fractions (continued)	SG	4:00	She directed students to page 42, #2 and asked students to talk in small groups.
		WC	4:47	Stevie asked students to talk about why they can multiply the numerator and denominator by the same number and get the same thing. Students said that though there were more parts, each part was smaller. Stevie asked what the use of equivalent fractions is and a student talked about the use of common denominators in addition.
	Fraction *of* what?	SG	4:05	Stevie directed them to page 46 activity 3J, #3 and asked students to talk about it in small groups.
		WC	2:56	Stevie introduced "campfire discussions," where students build on each other's responses through asking a question, commenting, or adding to the previous student's thinking. Students talked about one-half being the whole and multiplying one-half by 2/1. The next student questioned why they did that, the first student answered. Another student talked about making the denominators equal.
9/10	Administrative	AD	4:42	Stevie talked about homework that students have to turn in.
	Homework review	LC	5:53	A TA talked about a common student error from the homework.
	Fraction *of* what? (continued)	SG	5:55	Stevie gave them more time in small groups to discuss page 46 activity 3J, #3 to remember what they did last time.

Table 20 (cont'd)

Day	Topic	Type	Duration	Description
9/10	Fraction *of* what? (continued)	WC	20:27	Stevie asked students to use the "campfire" format. A student said she has no idea and Stevie asked questions to help the student talk about what she was confused about. Other students talked about how they drew the representation, the relationship between one-third and one-half, equations that they had used, etc. Stevie asked them to think in terms of both cups and the amount they needed for the recipe. Stevie ran through the whole solution again, asking what of the wholes were in relation to some of the quantities in the solution. Stevie asked if anyone solved it algebraically and a student explained her algebraic solution.
	Comparing fractions	LC	1:08	Stevie talked about comparing numbers and asked how you can tell one of two fractions is greater.
		SG	1:30	Stevie asked students to "write down several methods that you know for comparing fractions."
		AD	0:44	Stevie talked about posting the PowerPoint slides on the web.
		LC	3:01	Stevie talked about methods that they may have come up with for comparing fractions focusing on cross-multiplication.
		WC	1:13	Stevie asked why they could compare fractions using cross-multiplication.
		WC	1:17	Stevie asked students "if they have the same denominator why can't you just compare the numerators?"
		SG	2:16	Stevie directed students to 3O on p. 51 and asked students to talk with their neighbor.
		WC	2:55	Students talked about the pieces being like pieces of a pie, "the denominator is smaller, that might be the bigger fraction." Stevie summarized student solutions.
		AD	10:26	Stevie answered a student question about the exam.

Table 20 (cont'd)

Day	Topic	Type	Duration	Description
9/15	Homework errors	LC	1:47	One of the teaching assistants went over a homework problem about finding a common denominator with fractions.
	Administrative	AD	3:00	Stevie encouraged student to do the homework problems that they did not have to hand in.
	Exam question review	LC	8:58	Stevie lectured about some problems form the exam.
	Comparing fractions (continued)	SG	4:59	Stevie directed students to look at page 51, activity 3P and work on it with their partner or work alone and then talk to a partner.
		WC	4:43	Students gave several alternative ways to compare fractions and Stevie summarized.
		SG	2:50	Stevie gave students more time to work on the remaining problems with a partner.
		WC	5:20	Students talked about the one of the problems, Stevie summarized the discussion, and a student admitted she was confused with the previous student's thinking.
		SG	2:05	Stevie gave students time to think about the two students' reasoning, because "they can't both be right."
		WC	5:43	Stevie asked them to start over with the comparison. A student explained while Stevie drew a representation. Stevie summarized and revoiced, students checked their understanding, Stevie summarized again, and they moved to the next problem in the set.
	Percent	LC	3:49	Stevie introduced the definition of percent.
		SG	1:39	Stevie directed them to Class Activity 3R #1 on page and asked students to talk to their neighbor.
		WC	1:20	Stevie asked students to explain what percent is shaded and the student explained.
	Solving percent problems	LC	2:04	Stevie talked about solving percent problems.

Note. AD = administrative, LC = lecture, SG = small group, TQ = tests and quizzes, and WC = whole-class discussion.

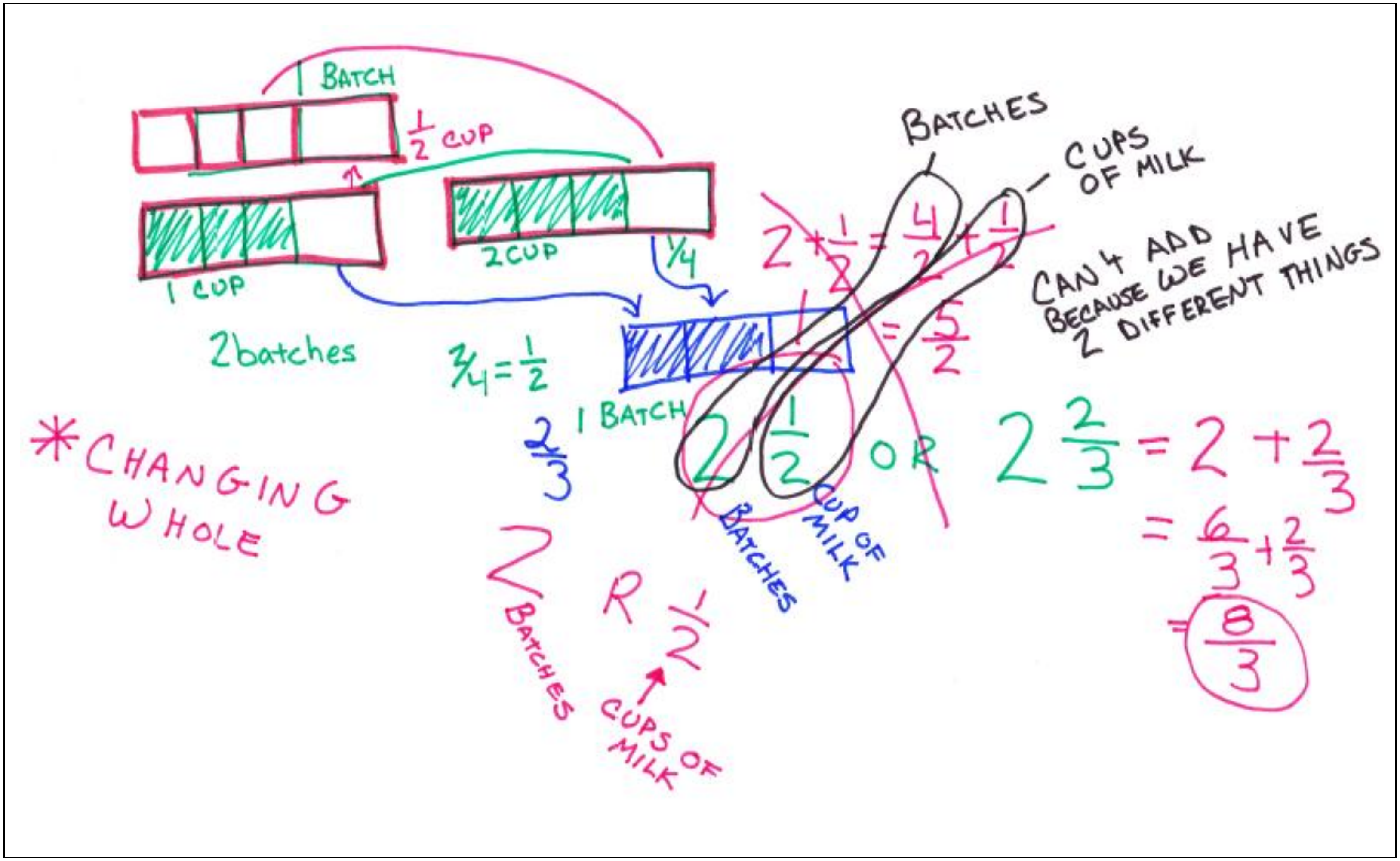


Figure 21. Reproduction of writing on center whiteboard at the end of the 4/10/08 Waffle Problem discussion in Pat's class

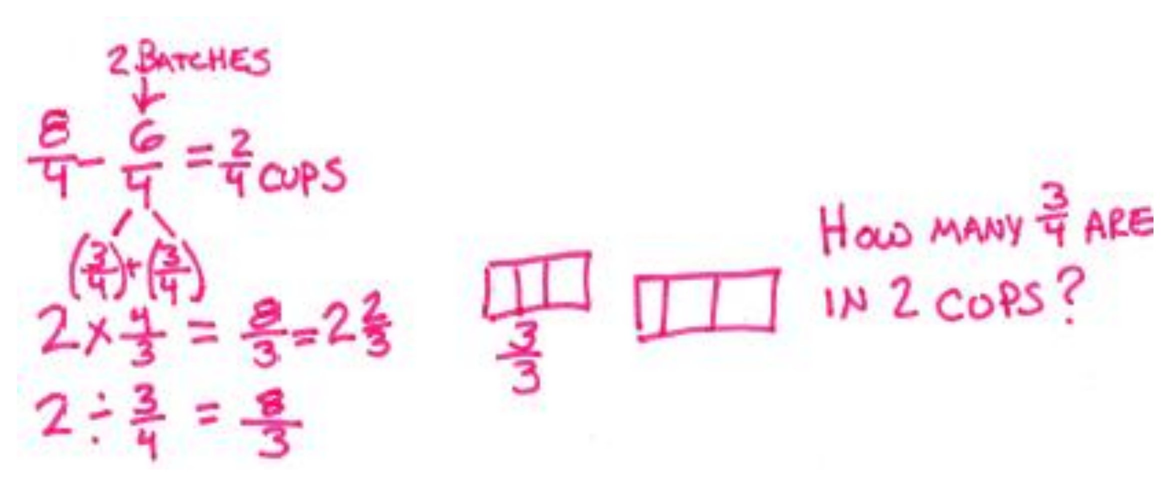


Figure 22. Reproduction of writing on right whiteboard at the end of the 4/10/08 Waffle Problem discussion in Pat's class

Appendix C. Prompts

Table 21. Discussion Prompts in Pat's Class by Day

Day	Discussion	Prompt text	Prompt types
4/1	1	(7:46) “what I'd like you to do in your groups is using, what you've seen this boy do, his strategy; I'd like you to finish this problem. Continuing to use this method or strategy that he's modeled. So in your groups finish this strategy. How do you think the rest of the problem works its way out?”	Integration question, integration statement
	2	(20:36) “In your groups, I want you to take a minute and I'd like you to compare and contrast . . . this eastern European algorithm to the American algorithm. What do you think of it? Do you like it? Do you not like it? What are the pros? What are the cons? Compare and contrast it and I want you to talk about it in your groups and then we'll talk about it as a class. So take a minute and kind of talk about it in your groups.”	Integration statement, integration question
	3	(35:01) Pat asks them to discuss in their groups for 2 minutes, “How would you explain the Eastern European algorithm and we're going to use this very example and I'm going to have someone—I'll draw a card—have someone come up and explain for us. Hey, you're going to be a teacher to a group of sixth graders and we're going to be sixth graders. How would you teach this algorithm to a group of sixth graders right? What would you say, what would you do? And in a minute, I'll have someone talk so take a minute in your groups, kind of talk about that process and then we'll have someone.”	Integration question, integration statement

Table 21 (cont'd)

Day	Discussion	Prompt text	Prompt types
4/3	1	(H) For discussion, reflect on the fraction problem types we discussed in class, equal sharing, comparison and ??? For the equal sharing and comparison questions, while they were grouped together, within each group, how are the different? How would you categorize/describe the ??? problems? (5:10) Pat says he's going to "have groups share how they chose to sort them [the fraction problems] out and as the groups share how to sort them out, I'd like you guys to kind of compare and contrast their sorting to your sorting." (32:14) "How are these equal sharing problems different from each other? And how are these comparing problems different from each other? . . . And I'm going to give you a few minutes in your groups and I'm going to say we have two groups, equal sharing and comparing problems, but then I want to suggest to you that we have sub-groups within them. The question is . . . although these all involve equal sharing of objects by kids, equal sharing is the action, they are somehow different, and I'd like you to kind of think through, how are they different? So take a few minutes in your groups and kind of look at the candy bar questions, the equal sharing, and look at the comparison and what do you observe is the difference here?" (47:00) Pat asks students to utilize the same idea to look at 4, 5, and 6, all have clay and all of them involved comparison. What's the difference between them? Asked them to think about the ideas of context and number sizes. "So I'm going to give you a couple minutes to reflect on 4, 5, and 6. OK? How are they different than each other?" (59:34) What type of question is that? When I say 4 children want to share 8 chocolate bars, what question type is it?"	Comprehension statement, comprehension Question

Table 21 (cont'd)

Day	Discussion	Prompt text	Prompt types
4/3	2	(H) For credit/no credit check, solve the following two word problems: b. You go to a wedding. 3/5 of the men at the wedding are married to 2/3 of the women at the wedding. What fraction of all the people at the wedding are married? (84:09) "OK, what I'd like you to do is in your groups I want you to look at both these strategies and evaluate them very carefully so that you understand them well enough so that if I were to arbitrarily call on any one of you, you should be able to come up and explain or re-teach, you know, go through and say, "OK, this is how you solve it." Kind of explaining each step, maybe even more so than what they did it. So take a minute and kind of go through and figure out what they did and kind of explain it. I'm going to ask you to try and explain it so that it makes sense. So I'll give you a few minutes in your groups to discuss."	Comprehension question, integration statement
4/8	1	(5:56) "So for the parts of fractions, how would you even describe what those numbers [numerator and denominator] represent?"	Comprehension question
	2	(12:53) "What the problem is what's wrong with the common denominator here [the student solution from the previous class]? Now we know contextually that we have one more male than female, does that make sense? . . . So the problem with the common denominator is you get one more male than female, right? Well, we know that, that's a contextual issue, right? . . . Getting the common denominator correctly? Yeah, we talked about that on Tuesday, we're doing it correctly. What's the problem with this [adding the denominator in the student solution from the previous class]?"	Integration question

Table 21 (cont'd)

Day	Discussion	Prompt text	Prompt types
4/8 (Cont.)	3	The Bus Problem (H) For credit/no credit check, solve the following two word problems: a. Some people were on a bus. At the first stop, 2/5 of the people got off and 3/5 of the original number got on. At the second stop, 1/2 of the people got off and 1/3 of the number that was left on the bus got on. At the last stop, 3/4 of the people got off, leaving 5 people on the bus. How many people were on the bus before the bus reached the first stop? Solve this problem in at least two different ways. (31:52) Pat says he wants them to take out their notes and he calls Kay to go up to the board to share her solution to the Bus Problem. He previews that they are going to cover 3 solutions, they should write about this in their notes, and there will be a question about this on the final. [Implicit that they should understand the solution well enough to explain it.]	Comprehension question, integration [implicit]
4/10	1	The Waffle Problem (H) For the following questions, (a) model an approach to solve following questions. Make sure you can explain all your quantities and reasoning, (b) write a mathematical record/number sentence that matches the problem you were asked to solve. Make sure you show your work and are able to explain all your mathematics. I. A batch of waffles requires 3/4 of a cup of milk. You have two cups of milk. Exactly how many batches of waffles could you make?	Integration statement, comprehension statement, comprehension question
	2	(69:51) "Here's what I want you to do at your seat for a minute. . . . Use the exact same problem but rewrite it with whole numbers, just at your groups and then see what happens when you put your own whole numbers in it. No fractions, you know, we want whole number answers in this whole thing, so we're making a lot of cookies, is basically what I'm saying. Take a minute in your groups." After a couple seconds, he corrects that they are making waffles.	Comprehension statement

Table 21 (cont'd)

Day	Discussion	Prompt text	Prompt types
4/15	1	The Cookie Problem (H) For the following questions, (a) model an approach to solve following questions. Make sure you can explain all your quantities and reasoning, (b) write a mathematical record/number sentence that matches the problem you were asked to solve. Make sure you show your work and are able to explain all your mathematics. You have 2 cups of flour to makes [sic] some cookies. This is 3/4 of what you need for one full recipe. How many cups of flour are needed for a full recipe? (10:47) Pat wanted them to take a couple minutes in small groups to re-familiarize themselves with question 3 and come to a solution within their groups. . . . (17:34)"Again, as we role play, Kennedy's the role of a teacher so your context is you're trying to explain it as clearly as you can to a group of 5th or 6th graders. Right? And then for you folks [the rest of the class], your contexts are teachers like me, sitting in the back of the room and your context is now she is an upper elementary or middle school student explaining her work and you're trying to figure out what she's doing and how she's thinking. So you can explain it to someone else. . . . So everyone in the room is acting like a teacher."	Integration statement, comprehension statement, comprehension question
	2	(49:56) "take a minute in your groups and . . . contrast Elena's strategy with Colin's. Are they indeed the same thing or are they really thinking about it differently?" (56:08) Pat asks "what mathematics do you need to know in order to solve this [Colin's] strategy compared to what mathematics do you need to know to solve this [Elena's] strategy?" (57:48) Pat adds another question, "for this problem [Elena's] write a number sentence."	Integration statement, integration question

Table 21 (cont'd)

Day	Discussion	Prompt text	Prompt types
4/15	3	(69:13) "This problem [the Cookie Problem], if you were going to type it into a calculator, is 2 ÷ 3/4 = 2 2/3. . . . That this is the number sentence that models it and similarly, this problem that we did the other day, it's also 2 ÷ 3/4 = 2 2/3. . . . Now here is my question: What's different about this division question in contrast to this division question? Does that make sense? Here [the previous problem] you were able to solve it, probably more comfortably. You were able to see it was division when I asked you to write a number sentence. . . . So my question is why? What's different about that word problem versus this word problem? Take a look at them and I'll give you a couple minutes in your groups and we'll talk about them together."	Integration question
4/17	1	The Brownie Problem (H)For the following questions, (a) model an approach to solve following questions. Make sure you can explain all your quantities and reasoning, (b) write a mathematical record/number sentence that matches the problem you were asked to solve. Make sure you show your work and are able to explain all your mathematics. You have 2/3 a pan of brownies. You give away 4/5 of what you have. What fraction of the whole pan have you given away?	Integration statement, comprehension statement, comprehension question
	2	(12:45) "I want you to compare and contrast them [Kennedy's and Amanda's strategies]. How are they different? And the interesting thing is, you know, Amanda said, you know, "we started out the same way." Yeah really did. In fact they even used the same color right. How are they're similar and how are they different? So take a few minutes in your groups to kind of compare and contrast those two strategies. OK? And then we'll talk about it with the class."	Integration statement, integration question

Table 21 (cont'd)

Day	Discussion	Prompt text	Prompt types
4/17	3	(23:33) "so here's what I'd like you to consider in your groups: what do you think about this model [D] as you compare and contrast it to A, B, and C? . . . How does D compare to these over here? Right? So kind of what's the difference between the pictorial representations is the one thing I want you to discuss in your groups. The second thing to consider in your groups is how do A, B, and C—any of these really—help us see why this is the multiplication of two fractions? . . . In what way do those pictures, if at all, do any of these pictures model that operational concept? Which ones of these helps us see that. OK? So what I'd like you to do is kind of, in your groups, kind of discuss that for a little bit and then we'll talk about that as a class."	Integration question, integration statement
	4	(H) You have one and three tenths pounds of fudge, when you eat a quarter pound of it, how much fudge would you have left? What type of problem type is it? (69:59) Pat says they are going to quickly go over strategies and compare them. He says they are going to start on the first one on the back page. . . . FSS(MY) is going to show them the first question on the board, the PSTs are to follow along. . . . Pat tells her to draw the picture and show them her thinking. . . . (77:29) Pat gives them one minute in small group to look at the question and determine what CGI problem type is it.	Comprehension question, comprehension statement
	5	(80:45) Pat reads the question, "you have one and six tenths pounds of fudge and seventy-five hundredths pounds of peanut butter fudge. How much fudge would you have all together?" . . . (82:30) Pat tells them to take a minute and find the problem type for that one.	Comprehension question, comprehension statement

Note: Prompts with time codes were given verbally while those with (H) were presented in handouts.

Table 22. Discussion Prompts in Stevie's Class by Day

Day	Discussion	Prompt	Prompt type
9/3	1	(23:14) She asks them to talk to their neighbor about the rule for rounding to the nearest thousand. Then she talks about speaking to hypothetical students. She posits that they would tell the students to look at the thousands place and then to look at the number next to it and determine whether it is greater than or less than 5. If it is greater, they round up. . . . She asks them to talk to their neighbor now, even suggesting they may want to pretend their partner is their future 4th grader, if the partner will let them. They are to explain why the rule makes sense thinking about placement on number lines.	Integration statement
	2	Take a blank piece of paper and imagine that it is $\frac{4}{5}$ of some larger piece of paper. Fold your piece of paper to show $\frac{3}{5}$ of the larger (imagined) piece of paper. Do this as carefully and precisely as possible without using a ruler or doing any measuring. Explain why your answer is correct. Could two people have different-looking solutions that are both correct?	Integration statement, Integration question
9/5	1	Take a blue strip of paper. Is it 1/5 of a full piece of ordinary paper? How can you tell if it is or isn't?	Factual question, Comprehension question
	2	Mireya has a recipe that calls for 5 cups of flour. Mireya wants to make $\frac{1}{3}$ of the recipe. Instead of figuring out what number $\frac{1}{3}$ of 5 is, Mireya measures $\frac{1}{3}$ of a cup of flour 5 times, and uses this amount of flour for $\frac{1}{3}$ of the recipe. Use the meaning of fractions and the following picture to help you explain why Mireya's strategy is valid: 	Integration statement

Table 22 (cont'd)

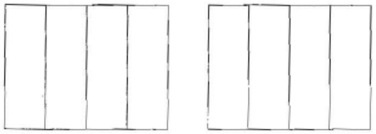
Day	Discussion	Prompt	Prompt type
9/5	3	Maurice says that the next picture shows that $\frac{3}{6}$ is bigger than $\frac{2}{3}$. The shaded portion representing $\frac{3}{6}$ *is* larger than the shaded portion representing $\frac{2}{3}$, so why is Maurice not correct? 	Integration question
		Kayla says that the shaded part of the next picture cannot represent $\frac{1}{4}$ because there are 3 shaded circles, and 3 is more than 1, but $\frac{1}{4}$ is supposed to be less than 1. What can you tell Kayla about fractions that might help her? 	
	4	Suppose you use a picture like the next one to talk about the fraction $\frac{5}{4}$. What kind of confusion could arise about this picture? What must we do in order to interpret the shaded region as $\frac{5}{4}$? 	Integration question

Table 22 (cont'd)

Day	Discussion	Prompt	Prompt type
9/8	1	Take a strip of paper. Draw a line on a piece of paper and use the ends of the strip to mark 0 to 1 on the line to make it a number line. Fold the strip to show fourths Use the folds to mark, 1/4, 2/4, 3/4, 4/4, 5/4 on your number line. Discuss with a neighbor how the placement of fractions on the number line fits with the way we defined fractions of objects.	Integration statement
9/8	2	Subdivide and label the second, third, and fourth strips and number lines in order to show that $\frac{1}{3}=\frac{2}{6}=\frac{3}{9}=\frac{4}{12}$ $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}$ $\frac{3}{3}=\frac{6}{6}=\frac{9}{9}=\frac{12}{12}$ $\frac{4}{3}=\frac{8}{6}=\frac{12}{9}=\frac{16}{12}$	Comprehension statement

Table 22 (cont’d)

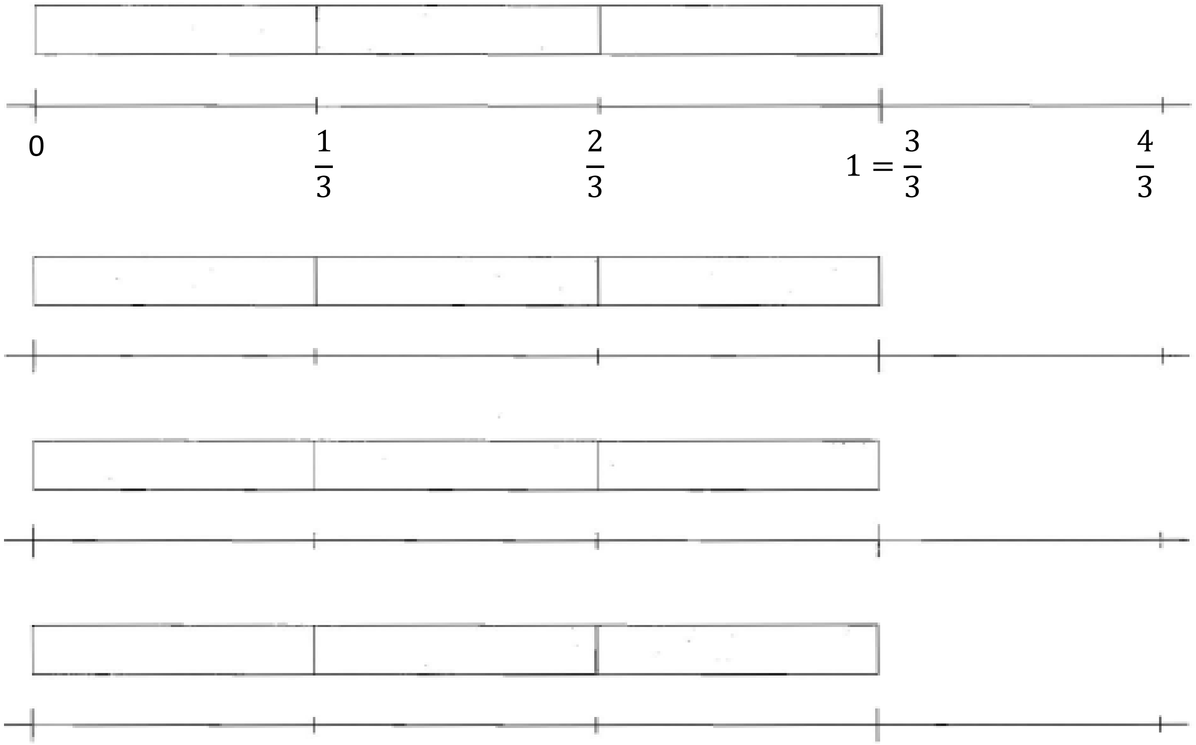
Day	Discussion	Prompt	Prompt type
9/8	2 (cont.)	 	

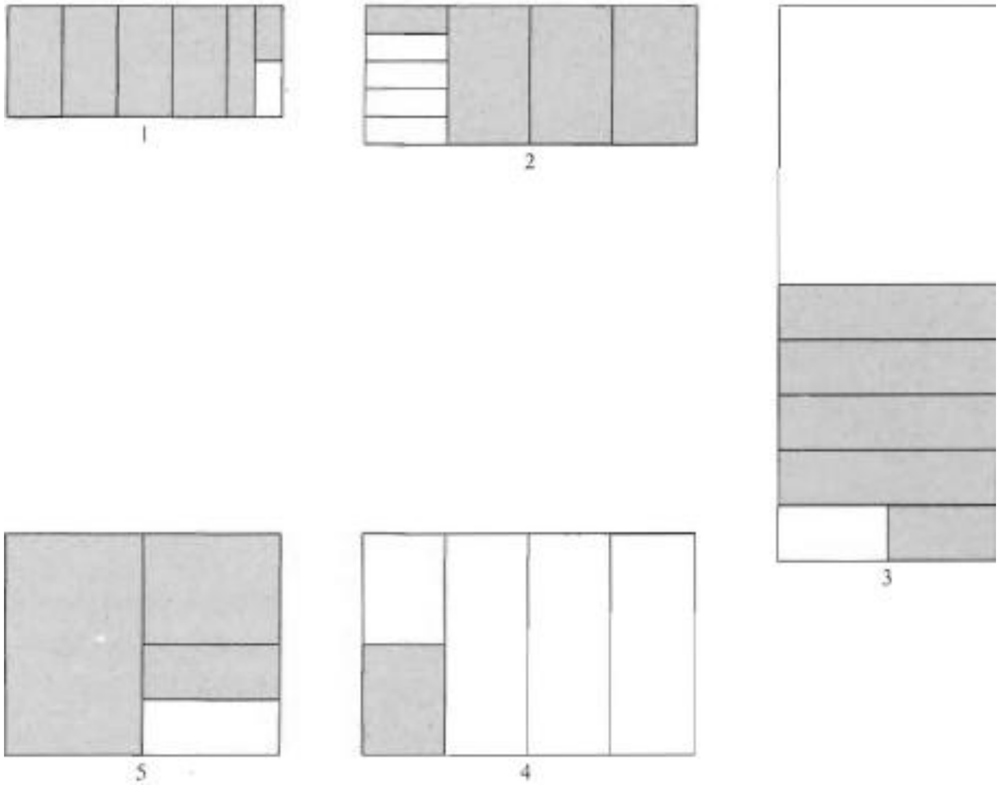
Table 22 (cont'd)

Day	Discussion	Prompt	Prompt type
9/8	3	Use the meaning of fractions of objects to give a detailed conceptual explanation for why $$\frac{2}{3}$$ of a cake is the same amount of cake as $$\frac{2 \cdot 4}{3 \cdot 4}$$ of the cake. Draw pictures to support your explanation. Discuss how your pictures show the process of multiplying both the numerator and denominator of $\frac{2}{3}$ by 4.	Integration statement
	4	Jean has a casserole recipe that calls for $\frac{1}{2}$ cup of butter. Jean only has $\frac{1}{3}$ cup of butter. Assuming that Jean has enough of the other ingredients, what fraction of the casserole recipe can Jean make? Draw pictures to help you solve the problem. Explain why your answer is correct. In solving this problem, how do $\frac{1}{2}$ and $\frac{1}{3}$ appear in different forms? For each fraction in this problem, and in your solution, describe the whole associated with this fraction. In other words, describe what each fraction is *of*.	Comprehension question, integration statement, integration question, comprehension statement
9/10	1	Jean has a casserole recipe that calls for 1/2 cup of butter. Jean only has 1/3 cup of butter. Assuming that Jean has enough of the other ingredients, what fraction of the casserole recipe can Jean make? Draw pictures to help you solve this problem. Explain why your answer is correct.	Comprehension question, integration statement

Table 22 (cont’d)

Day	Discussion	Prompt	Prompt type
9/10	2	Why can you compare fractions using cross multiplication?	Comprehension question
	3	Of 7/12 and 8/12, which one is greater and why?	Factual question, comprehension question
	4	For each of the pairs of fractions shown, determine which fraction is greater in a way other than finding common denominators or converting to decimals. Explain your reasoning. $\frac{1}{49}$ $\frac{1}{39}$ $\frac{7}{37}$ $\frac{7}{45}$	Integration question, integration statement
9/15	1	Use reasoning other than finding common denominators, cross-multiplying, or converting to decimal numbers to compare the sizes (=, <, or >) of the following pairs of fractions: $\frac{27}{43}$ $\frac{26}{45}$ $\frac{13}{25}$ $\frac{34}{70}$ $\frac{17}{18}$ $\frac{19}{20}$ $\frac{9}{40}$ $\frac{12}{44}$ $\frac{51}{53}$ $\frac{65}{67}$ $\frac{13}{25}$ $\frac{5}{8}$	Integration statement

Table 22 (cont'd)

Day	Discussion	Prompt	Prompt type
9/15	2	For each of the diagrams 1-5, determine the percent of the diagram that is shaded, explaining your reasoning. Write each percent as a fraction in simplest form, and explain how to see that this fraction of the diagram is shaded. You may assume that portions of each diagram which appear to be the same size really are the same size.  	Factual statement integration statement,

Note. All images of problems from the activity manual used in the class (Beckmann, 2007). Verbal prompts have time codes. In other cases, Stevie read the problem aloud using words not significantly different from the problem text presented here.

Appendix D. Choosing

Table 23. Frequencies of Methods of Choosing PST to Speak in Class

		Cold Call			
Instructor	Class Date	Non-Random	Random	Self-Select	Volunteer
Pat	4/1/08	1	5	14	21
	4/3/08	14	7	26	12
	4/5/08	11	9	56	15
	4/10/08	4	1	28	7
	4/15/08	11	0	31	13
	4/17/08	8	7	21	5
	Total	49	29	176	73
Stevie	9/3/08	0	5	1	7
	9/5/08	0	8	1	10
	9/8/08	0	5	1	2
	9/10/08	0	9	1	5
	9/15/08	0	7	3	1
	Total	0	34	7	25

Note. Grey region indicates the unambiguous instances of student self-selection in Stevie's class.

Table 24. Sequence of Methods of Choosing Students in Pat's Class

Day	Discussion	Prompt type	Sequence of choosing methods
4/1	1	Integration question, integration statement	N V V V S V V V S V V V V
	2	Integration statement, integration question	R R R V R R V V V S V S V
	3	Integration question, integration statement	V S S S S S S S S V V V V S S
4/3	1	Comprehension statement, comprehension question	S N V N N N S S S S S S S S N N N N S N V | S R N S N R R V V V S N N S | R S R R V V | S
	2	Comprehension question, integration statement	V N S S S S V S S N | R S S V S V V V S
4/8	1	Comprehension question	V V S V | N N S N N N N N
	2	Integration question	V V S S S S V V S S
	3	Comprehension question, integration [implicit]	N S S V V S S S S S S | R S R R R S R S S R S N S | S R R S R S S S S S S S S S S V S S V S S S S S N V S S S S V V S S S S S S S S V S S | N S
4/10	1	Integration statement, comprehension statement, comprehension question	R V S S N S S S S S S S S S S S S V S S S | S V S S S S S V S S S V S V
	2	Comprehension statement	N N N V S
4/15	1	Integration statement, comprehension statement, comprehension question	N S S S S S S S S S V S S V | N S S S S S S V | N N V N S S S V S S
	2	Integration statement, integration question	N N S S V N N V
	3	Integration question	N S V S S S S N V V
4/17	1	Integration statement, comprehension statement, comprehension question	N S S V S S
	2	Integration statement, integration question	R R R S S S S S S S S
	3	Integration question, integration statement	R R N R R S S V V S S V V
	4	Comprehension question, comprehension statement	N N S | N N S S S
	5	Comprehension question, comprehension statement	S N N

Note. N = Non-Random Cold Call; R = Random Cold Call; S = Self-Select; V = Volunteer. Vertical lines indicate an interruption in discussion.

Table 25. Sequence of Methods of Choosing Students in Stevie's Class

Day	Discussion	Prompt type	Sequence of choosing methods
9/3	1	Integration statement	V V S
	2	Integration question, integration statement	V V R R R R V R V V
9/5	1	Factual question, comprehension question	V V V
	2	Integration statement	R R R R R | V V
	3	Integration question	R V V R
	4	Integration question	R V V V S
9/8	1	Integration statement	R S
	2	Comprehension statement	R
	3	Factual statement	V V
	4	Comprehension question, integration question, comprehension statement	R R R
9/10	1	Comprehension statement, comprehension question	R R R R R R R V V S
	2	Comprehension question	V
	3	Comprehension question, factual question	V
	4	Integration question, integration statement	R R
9/15	1	Integration statement	R R | R R R R | R S S S
	2	Factual statement, integration statement	V

Note. N = Non-Random Cold Call; R = Random Cold Call; S = Self-Select; V = Volunteer. Vertical lines indicate an interruption in discussion.

Appendix E. Responding to Student Thinking

Table 26. Pat's Verbal Responses to Student Thinking

	Class day						
Instructor responses	4/1	4/3	4/8	4/10	4/15	4/17	Total
Asked students to . . .							
Apply their own reasoning to a peer's response	2	4	14	4	5	0	29
Connect peer responses	0	0	2	0	1	4	7
Connect disciplinary knowledge to a peer's response	0	0	0	1	1	0	2
Asked for questions or agreement/disagreement	9	14	23	9	27	18	100
Asked for other solutions or responses	5	5	1	3	5	2	21
Connected student responses	2	0	10	2	7	8	29
Evaluated a student response	7	2	9	4	9	10	41
Asked for further information	9	40	64	34	33	26	206
Elaborated	7	15	20	10	21	18	91
Repeated or revoiced student responses	18	83	111	24	47	32	315
Other	6	19	29	10	8	10	82

Note. Other includes classroom management comments, direct answers to student questions, checking if the instructor heard a student, instructor jokes, and telling a student that they will cover something later.

Table 27. Stevie's Verbal Responses to Student Thinking

	Class day					
Instructor responses	9/3	9/5	9/8	9/10	9/15	Total
Asked students to . . .						
Apply their own reasoning to a peer's response	2	1	3	3	4	13
Connect peer responses	0	0	0	0	0	0
Connect disciplinary knowledge to a peer's response	0	0	0	0	0	0
Asked for questions or agreement/disagreement	10	6	8	10	13	47
Asked for other solutions or responses	0	2	2	1	0	5
Connected student responses	0	0	0	6	1	7
Evaluated a student response	1	0	0	5	6	12
Asked for further information	6	14	8	29	5	62
Elaborated	6	17	7	9	7	46
Repeated or revoiced student responses	14	22	15	28	24	103
Other	3	0	4	3	1	11

Note. Other includes classroom management comments, direct answers to student questions, checking if the instructor heard a student, instructor jokes, and telling a student that they will cover something later.

Table 28. Pat's Frequent Responses to Student Thinking by Discussion

		Asked for						
Day	Discussion	Other solutions or response	Question or agreement	Further information	Connected PST responses	Evaluated a student response	Elaborated	Revoiced
4/1	1	•	•	•		•		•
	2		•	•		•	•	•
	3	•	•	•	•	•		•
4/3	1	•	•	•			•	•
	2	•	•	•		•	•	•
4/8	1	•	•	•		•		•
	2		•	•	•		•	•
	3		•	•	•	•	•	•
4/10	1	•	•	•	•	•	•	•
	2		•	•	•		•	•
4/15	1	•	•	•	•	•	•	•
	2	•	•	•	•	•	•	•
	3		•	•	•	•	•	•
4/17	1	•	•	•		•		•
	2	•	•	•	•			•
	3		•	•	•	•	•	•
	4		•	•		•	•	•
	5		•			•	•	•

Table 29. Stevie's Frequent Responses to Student Thinking by Discussion

		Asked for						
Day	Discussion	Other solutions or responses	Question or agreement	Further info.	Connected PST responses	Evaluated a student response	Elaborated	Revoiced
9/3	1		•	•			•	•
	2		•	•		•		•
9/5	1	•					•	•
	2		•	•			•	•
	3		•	•			•	•
	4		•	•			•	•
9/8	1	•	•	•			•	•
	2	•	•	•			•	•
	3		•	•			•	•
	4			•				•
9/10	1	•	•	•	•	•	•	•
	2		•	•			•	
	3		•	•			•	•
	4			•	•		•	•
9/15	1		•	•	•	•	•	•
	2		•				•	•

Appendix F. Connectedness of Discussions

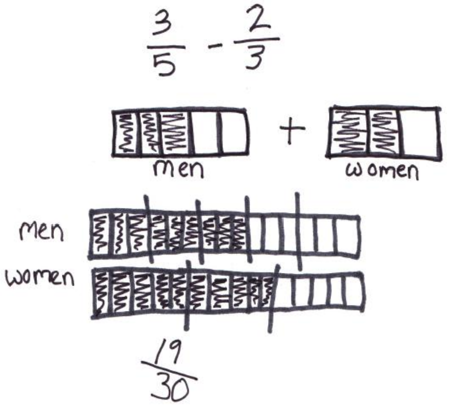
Wedding Problem: You go to a wedding and three fifths of the men are married to two thirds of the women at the wedding. What fraction of all the people at the wedding are married?		
Solution	PST explanation	Reproduction of PST solution
1	(75:03) Mary restated the prompt and said she knew she had to add the men and women together and to do that she had to have a common denominator. So she multiplied each fraction to get 15 as a common denominator. She multiplied the first one "by 3," [meaning $\frac{3}{3}$] and the second one "by 5" [meaning $\frac{5}{5}$]. "And I added them together to get 19 people."	$\frac{3}{5}\cdot\frac{3}{3}=\frac{9}{15}$ men married $\frac{2}{3}\cdot\frac{5}{5}=\frac{10}{15}$ women married $\frac{9}{15}+\frac{10}{15}=\frac{19}{30}$ 19 people at wedding are married
2	(81:17) Blair solved it the same way as the first student but she "drew it out." She spoke about needing to find a common denominator, explaining as she writes, and that she found 15 using her fact knowledge. She then divided the "men bar" into groups of 3 [because each fifth was now three-fifteenths] and shaded 3 groups of 3. She divided the "women bar" into groups of 5 [because each third was now fife-fifteenths] and shaded 2 groups of 5. She added up the shaded squares and got 19 and then she added up all the squares to get 30, so she got the same answer as the first student, $\frac{19}{30}$.	

Figure 23. Student solutions for the Wedding Problem with reproductions of student solutions

Bus Problem:

Some people were on a bus. At the first stop, 2/5 of the people got off and 3/5 of the original number got on. At the second stop, half of the people got off and 1/3 of the number that was left on the bus got on. At the last stop, 3/4 of the people got off leaving 5 people on the bus. How many people were on the bus originally?

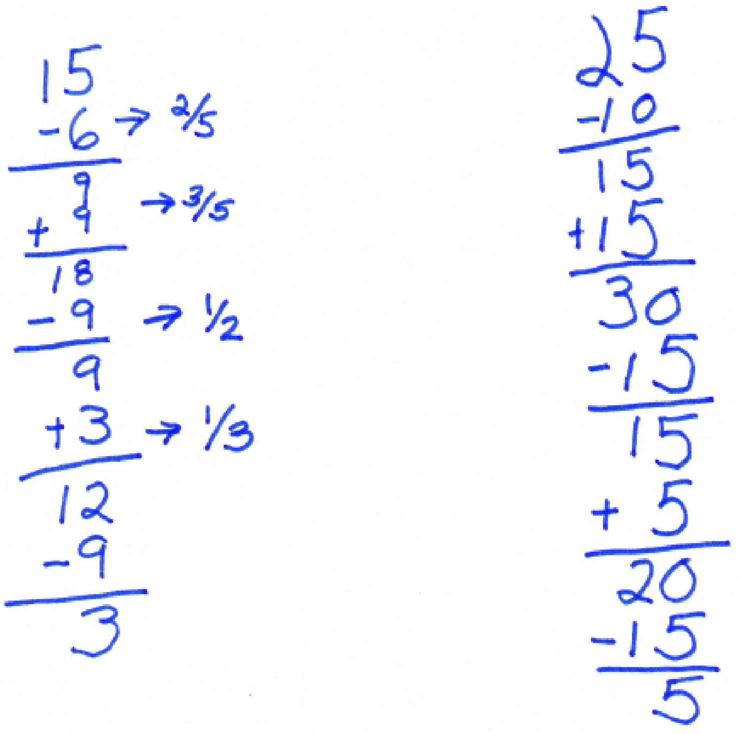
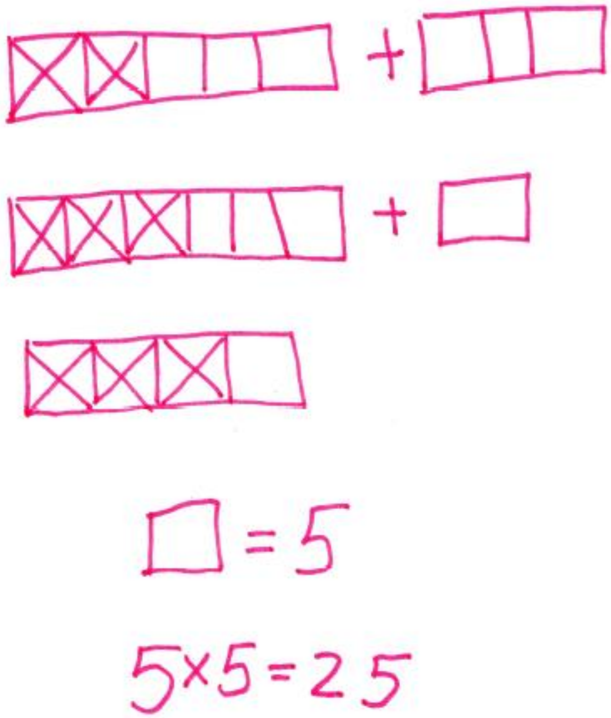
Solution	PST explanation	Reproduction of PST solution
1	(35:19) Kay started [in the left column] with a random number (15) and worked through the story problem. Two-fifths of the people getting off is -6, three-fifths of the original number is +9, one-half getting off was -9, one-third getting on was +3, and three-fourths getting off was -9. This left her with the final number of people on the bus being 3, which is the wrong number so she tried again. In the right column, she worked backwards. She started at the bottom of the column with 5 and then worked through the problem backwards to get the answer. Three-fourths of the people leaving was 15, one-third getting on was +5, half getting off was -15, three-fifths the original number getting on was +15, and two-fifths of the people getting off was -10.	

Figure 24. Student solutions to the Bus Problem with reproductions of the student solution

Figure 24 (cont'd)

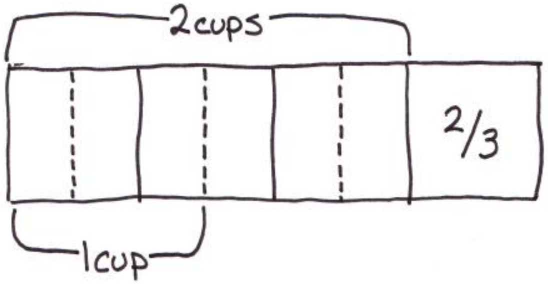
Solution	PST explanation	Reproduction of PST solution
2	(57:18) Renee assumed that the number of people at the start of the problem was 1 or "a whole." Two-fifths of the people leaving was represented by $-\frac{2}{5}$, three-fifths getting on by $+\frac{3}{5}$, half leaving by $\cdot\frac{1}{2}$, one-third that was left getting on by $\cdot\frac{4}{3}$, and three-fourths of the people leaving by $\cdot\frac{1}{4}$. The end fraction was $\frac{1}{5}$ and since we know there were 5 people at the end she added the x and made $\frac{1}{5}x = 5$. Therefore, x (and the number of people starting on the bus) was 25.	$1-\frac{2}{5}=\frac{3}{5}$ $\frac{3}{5}+\frac{3}{5}=\frac{6}{5}$ $\frac{6}{5}\cdot\frac{1}{2}=\frac{6}{10}=\frac{3}{5}$ $\frac{3}{5}\cdot\frac{4}{3}=\frac{12}{15}=\frac{4}{5}$ $\frac{4}{5}\cdot\frac{1}{4}=\frac{4}{20}=\frac{1}{5}X=5$ $5\cdot\frac{1}{5}x=5\cdot 5$ $x=25$

Figure 24 (cont'd)

Solution	PST explanation	Reproduction of PST solution
3	(101:10) Kayla represented the problem as a series of boxes. She started with 5 boxes and crossed 2 out (because 2 people got off) then she added 3 because three-fifths of the total, which was 5, got on. She continued by drawing 6 boxes, which represent the number of people on the bus before the second stop. She then crossed out 3 because one-half got off the bus and she added 1 box because one-third of the number of people on the bus got on. She "brought down [her] total to the next line," drawing 4 boxes then crossing out 3 because three-fourths got off. She was left with one box, which she brought down and since she knew that 5 people were left she knew that 1 box = 5 people. Therefore, since she had 5 boxes at the start, she had 25 people.	

Cookie Problem:

You have 2 cups of flour to make some cookies. This is 3/4 of what you need for one full recipe. Who many cups of flour are needed for a full recipe?

Solution	PST explanation	Reproduction of PST solution [a]
1	(17:34) Kennedy said the first step was writing out the first algorithm, “two divided by three-fourths and you're trying to find x.” She stated that from, “basic algebra” you have to “put x by itself and to do that you need to get rid of this [indicating $^3/_4$] so to get rid of this [indicating $^3/_4$] you have to bring it over to the other side and since it's $^3/_4$ times x, to get rid of it on this side, you need to divide it.” She explained that is what she did in the second algorithm. Finally, she stated that you need two and two-thirds cups of flour.	$2 = \frac{3}{4}X$ $2 - \frac{3}{4} = \frac{2}{1} \times \frac{4}{3} = \frac{8}{3} = 2\frac{2}{3}$
2	(28:27) Colin said, “What I knew is the 2 cups was three-fourths of what we needed for the full recipe” He drew a full block to represent the full recipe. “Then I thought of it like well, how much for 1 cup? So it's one and one-half of this block . . . so I broke it into thirds so I knew that this last block here was two-thirds left of what we needed for the full recipe. So two and two-thirds.”	

[a] The diagram reproduced for the second solution is reproduced from field notes because it was not caught on videotape.

Figure 25. PST solutions to the Cookie Problem with reproductions of PST solutions

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