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GRADED LOCAL COHOMOLOGY AND ITS ASSOCIATED PRIMES

By

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ABSTRACT

GRADED LOCAL COHOMOLOGY AND ITS ASSOCIATED PRIMES

By

Chia Sien Lim

It is known that the i-th local cohomology of a finitely generated R-module M over a positively graded commutative Noetherian ring R, with respect to the irrelevant ideal R_+ is graded. Furthermore, for every integer n, the n-th component $H^i_{R_+}(M)_n$ of this local cohomology module $H^i_{R_+}(M)$ is finitely generated over R_0 and vanishes for $n >> 0$. We want to understand the behavior of $H^i_{R_+}(M)_n$ for $n << 0$.

We will specialise the study of $H^i_{R_+}(M)$ to the case where R is a Cohen-Macaulay ring and M is a Cohen-Macaulay $R-$module. When dim $R_0 = 1$, we will show that $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ becomes constant when n becomes negatively large. When R_0 is local, equidimensional and dim $R_0 = 2$, we will show that there exists an integer N such that either $H^i_{R_+}(M)_n = (0)$ for all $n < N$ or, $H^i_{R_+}(M)_n \neq (0)$ for all $n < N$. These results will extend the work of Brodmann and Hellus in [2].

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Introduction

Let R be a commutative Noetherian ring with unit, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. We want to study $H^i_{\mathfrak{a}}(M)$, the i-th order local cohomology of M with respect to the ideal $\mathfrak{a}$. Since $H^i_{\mathfrak{a}}(M)$ is not finitely generated as a R-module in general, a great effort has been expanded to understand its finiteness properties (cf. [2], [4], [8]). A question raised by C. Huneke regarding the finiteness of the set of associated primes of $H^i_{\mathfrak{a}}(M)$ has prompted many researchers to work in this very active area of mathematics. The most notable finiteness result in this direction is that if R is a regular local ring containing a field, then for every integer i, $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(R))$ is a finite set for every ideal, $\mathfrak{a}$, of R. The characteristic p case was proved by C. Huneke and R. Sharp (cf. [11]). G. Lyubeznik settled the characteristc 0 case using D-modules (cf. [13], [14] and [15]).

However, there are examples of local cohomology modules whose set of associated primes is not finite. The first example is provided by A. Singh in [18]. In this example, M is a graded hypersurface over R, R is a positively graded algebra over R_0 and R_0 is a polynomial ring over the integers. Subsequently, M. Katzman gave another example where R_0 is a polynomial ring over any field (cf. [12]). This result effectively shows that the set of associated primes of a local cohomology module over a local ring can be infinite.

In this paper, we wish to investigate the local cohomology module $H^i_{\mathfrak{a}}(M)$ in the graded case. Throughout the paper, we will assume that $R = \bigoplus_{n\geq 0}^{\infty} R_n$ is a positively graded homogeneous Noetherian ring of finite Krull dimension; $\mathfrak{a}$ is the irrelevant ideal, $\bigoplus_{n>0}^{\infty} R_n$, denoted by R_+; M is a finitely generated graded R-module. The R-module $H^i_{R_+}(M)$ has a natural grading and $H^i_{R_+}(M)_n$, the n-th component of $H^i_{R_+}(M)$, is a finitely generated R_0-module. Moreover, for $n >> 0$, $H^i_{R_+}(M)_n = (0)$.

This raises the following question. What about $H^i_{R_+}(M)_n$ and $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ for $n << 0$ and their relations with $\mathrm{Ass}_R(H^i_{R_+}(M))$? This work is motivated by a recent paper of Brodmann and Hellus in [2]. They showed the following:

Proposition 1. (cf. [2], 4.2) *If R_0 is semi-local and dim $R_0 \leq 1$, then for every $i \in \mathbb{N}_0$, $H^i_{R_+}(M)$ is asymptotically gap free.*

$H^i_{R_+}(M)$ is called asymptotically gap free if there exists an integer N such that either $H^i_{R_+}(M)_n = (0)$ for all $n < N$ or, $H^i_{R_+}(M)_n \neq (0)$ for all $n < N$.

Proposition 2. (cf. [2], 5.6) *Let $t \in \mathbb{N}_0$. If $H^i_{R_+}(M)$ is finitely generated over R for all $i < t$, then $\{\mathrm{Ass}_{R_0}(H^t_{R_+}(M)_n)\}_{n\in\mathbf{Z}}$ is asymptotically stable.*

$\{\mathrm{Ass}_{R_0}(H^t_{R_+}(M)_n)\}_{n\in\mathbf{Z}}$ is called asymptotically stable if there exists an integer N such that $\mathrm{Ass}_{R_0}(H^t_{R_+}(M)_n)$ becomes constant for all $n < N$.

It was noted by Brodmann and Hellus that asymptotic stability implies that $H^i_{R_+}(M)$ is asymptotically gap free and $\mathrm{Ass}_R(H^i_{R_+}(M))$ is finite. However, the converse is false according to the joint paper by M. Brodmann, M. Katzman and R. Sharp (cf. [3]).

Brodmann and Hellus also showed that for a local base ring (R_0, m_0), m_0 is the maximal ideal of R_0, if $H^i_{R_+}(M) \neq (0)$, then i is bounded above by dim $\frac{M}{m_0 M}$. More precisely,

Proposition 3. (cf. [2], 3.4) *Suppose that (R_0, m_0) is local and M is finitely generated graded R-module with* dim $\frac{M}{m_0M}$ $= d$. *If $M \neq \Gamma_{R_+}(M)$, then $H^d_{R_+}(M) \neq (0)$ and $H^i_{R_+}(M)$ for all $i > d$.*

In this project, we will apply and extend Brodmann and Hellus results to the case where R is a Cohen-Macaulay ring, M is a Cohen-Macaulay R module and dim R_0 is either 1 or 2. In this situation, the grade of M with respect to R_+ provides a lower bound for the order in which the local cohomology does not vanish, while Proposition (3) of Brodmann and Hellus gives an upper bound. One of our main ideas is to localise R at a prime ideal of R_0 so that our base ring can be local. This allows us to use Proposition (3) of Brodmann and Hellus to get a better handle on the support of $H^i_{R_+}(M)$. More precisely, we will prove:

Theorem 2.0.1. *Suppose that R is a positively graded homogeneous Cohen-Macaulay ring and M is a finitely generated graded Cohen-Macaulay R-module. Assume also that* $\dim R_0 = 1$. *Then, for all i,*

$$\{\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbb{Z}}$$

is asymptotically stable.

Theorem 4.0.1. *Suppose that R is a positively graded homogeneous Cohen-Macaulay ring and M is a finitely generated graded Cohen-Macaulay R-module. Assume also that R_0 is local, equidimensional and* $\dim R_0 = 2$. *Then for all i, $H^i_{R_+}(M)$ is asymptotically gap free.*

Due to the known results of Brodmann and Hellus in Propositions (1) and (2), the interest of our work will be in the case, $H^i_{R_+}(M)$ for $i >$ grade(R_+, M). In fact, for Theorem (4.0.1), we can simply focus on the situation where ht $(I' \cap R_0) = 0$ ($I' = \mathrm{ann}_R(M)$), because if ht $(I' \cap R_0) \geq 1$, $H^i_{R_+}(M)$ is a $\frac{R}{I'}$-module and dim $\left(\frac{R}{I'}\right)_0 \leq 1$; this situation is dealt with by Brodmann and Hellus's Proposition (1) above.

Here is an outline of our project: In Chapter One, we will gather the tools that will be applicable in the proof of Theorems (2.0.1) and (4.0.1); in Chapter Two, we will give a proof of Theorem (2.0.1) and its Corollaries; in Chapter Three, we will give some sufficient condition for $H^i_{R_+}(M)$ to be asymptotically gap free if dim $R_0 = 2$; in the final Chapter, we will apply the results in Chapter Two and Three to prove Theorem (4.0.1) and its Corollaries.

1 Auxiliary tools

We begin by recalling some definitions and observations made in [2]. Then, we will state three results that we will constantly utilised from the paper, [2], by Brodmann and Hellus. Then, we will prove several Lemmas which will be applicable in the cases where dim $R_0 = 1$ and dim $R_0 = 2$.

For any unexplained terminology, the reader can refer to [6] and [16]. The reference we use for background information on local cohomology is the textbook by Brodmann and Sharp (cf. [5])

We will assume throughout this paper that $R = \bigoplus_{n\geq 0}^{\infty} R_n$ is a positively graded homogeneous Noetherian ring of finite Krull dimension. This means that R is generated by degree 1 elements. Assume also that M is a finitely generated graded R-module. We will refer to these two assumptions as the standard hypotheses on R and M. We will denote $I = \sqrt{\mathrm{ann}_R(M)}$ and $R_+ = \bigoplus_{n>0}^{\infty} R_n$, the irrelevant ideal of R. We note that I is graded.

For a $q \in \mathrm{Spec}(R_0)$, M_q (resp. R_q) is the localisation of M (resp. R) at the multiplicative set, $R_0 - q$. We note that M_q and R_q are both graded and R_q is graded local. Furthermore, $(R_q)_0 \cong (R_0)_q$ as rings.

Definitions. Let T be a graded R-module. Then,

(a) T is asymptotically gap free if the set,

$$\{n \in \mathbb{Z}_{<0} \,|\, T_n \neq (0) \text{ and } T_{n+1} = (0)\}$$

is finite.

(b) $\{\mathrm{Ass}_{R_0}(T_n)\}_{n\in\mathbb{Z}}$ is asymptotically stable if there exists $N \in \mathbb{Z}$ such that for all $n \leq N$,

$$\mathrm{Ass}_{R_0}(T_n) = \mathrm{Ass}_{R_0}(T_N).$$

We note that (b) implies (a) because any R_0-module, L, is (0) if and only if $\mathrm{Ass}_{R_0}(L) = \emptyset$ (cf. [2]).

The interest in (b) is its relations to $\mathrm{Ass}_R(H^i_{R_+}(M))$. There is a bijection of sets:

$$\begin{aligned} \mathrm{Ass}_R(H^i_{R_+}(M)) &\longleftrightarrow \bigcup_{n\in\mathbf{Z}} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n) \\ p + R_+ &\longleftrightarrow p \end{aligned}$$

(cf. [2], 5.5). Consequently, $\mathrm{Ass}_R(H^i_{R_+}(M))$ is a finite set if and only if

$$\bigcup_{n\in\mathbf{Z}} \mathrm{Ass}_{R_0} H^i_{R_+}(M)_n$$

is a finite set. Furthermore, since $H^i_{R_+}(M)_n = (0)$ for $n >> 0$ and for each n, $\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$ is a finite set, so if we have asymptotic stability, then

$$\bigcup_{n\in\mathbf{Z}} \mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)$$

is finite. Hence, asymptotic stability of $\{\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbf{Z}}$ implies the finiteness of $\mathrm{Ass}_R(H^i_{R_+}(M))$.

We also note that the same correspondence gives us a bijection of sets between the support of $H^i_{R_+}(M)$ and the support of $H^i_{R_+}(M)_n$:

$$\begin{aligned} \mathrm{Supp}_R(H^i_{R_+}(M)) &\longleftrightarrow \bigcup_{n\in\mathbf{Z}} \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n) \\ p + R_+ &\longleftrightarrow p \end{aligned}$$

1.1 Results from Brodmann-Hellus in [2]

Proposition 1.1.1. *(Brodmann-Hellus) Suppose that* $\dim R_0 \leq 1$*, and* R_0 *is semi-local. Fix* $i \in \mathbb{N}_0$*. Then, for every finitely generated graded R-module* M*,* $H^i_{R_+}(M)$ *is asymptotically gap free.*

Proof. (cf. [2], 4.2) □

Proposition 1.1.2. *(Brodmann-Hellus) Suppose that* (R_0, m_0) *is local and* M *is a finitely generated graded* R*-module with* $\dim \frac{M}{m_0 M} = d$*. If* $M \neq \Gamma_{R_+}(M)$*, then*

(a) $H^d_{R_+}(M) \neq (0)$ *and*

(b) $H^i_{R_+}(M)$ *for all* $i > d$*.*

Proof. (cf. [2], 3.4) □

Proposition 1.1.3. *Suppose that* $R_+M \neq M$*. Let* $g = \mathrm{grade}(R_+, M)$*. Then:*

(a) For all $i < g$*,* $H^i_{R_+}(M) = (0)$ *;*

(b) for $i = g$*,* $H^i_{R_+}(M) \neq (0)$*;*

(c) (Brodmann-Hellus) if $H^i_{R_+}(M)$ *is finitely generated over* R *for all* $i < t$*, then*

$$\{\mathrm{Ass}_{R_0}(H^t_{R_+}(M)_n)\}_{n \in \mathbf{Z}}$$

is asymptotically stable.

Proof. For (a) and (b), (cf. [5], 6.2.7). For (c), (cf. [2], 5.6). □

We will very often use the preceding two Propositions in the following way. Given a $q \in \mathrm{Spec}(R_0)$. Suppose that $M_q \neq \Gamma_{R_+R_q}(M_q)$ and $R_+M \neq M$. Then, if $q \in \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)$, then

$$\mathrm{grade}(R_+R_q, M_q) \leq i \leq \dim \frac{M_q}{qM_q}.$$

We will give an upper bound for $\dim \frac{M_q}{qM_q}$ in terms of $\mathrm{grade}(R_+, M)$. This is because

$$\mathrm{grade}(R_+, M) \leq \mathrm{grade}(R_+R_q, M_q).$$

This basic setup can help us understand the support of $H^i_{R_+}(M)$.

1.2 Preliminary Lemmas in the Cohen-Macaulay case

Given a $q \in \mathrm{Spec}(R_0)$. This subsection is devoted to the understanding of $\dim \frac{M_q}{qM_q}$ and $\mathrm{grade}(R_+, M)$, when both R and M are Cohen-Macaulay. Our first goal is to give an upper bound of $\dim\frac{M_q}{qM_q}$ in terms of ht $(q + R_+)$ and ht $(q + I)R_q$ where, $I = \sqrt{\mathrm{ann}_R(M)}$. This is the content of Lemma (1.2.6). Then, we will use the fact that R_0 is equidimensional to write ht $(q + R_+)$ in terms of ht (R_+). This will be accompanied by a calculation of ht $(q + I)R_q$ in terms of ht (I). Finally, in Lemmas (1.2.9) and (1.2.10), we will relate grade(R_+, M) to ht (I) and ht (R_+). This will give us a more concrete link between $\dim\frac{M_q}{qM_q}$ and grade(R_+, M).

Some notations and conventions: Min(R_0) will represent the set of minimal primes of R_0. Max(R_0) will represent the set of maximal primes of R_0. For $i \in \mathbb{N}$, Spec$^i(R_0)$ will represent the set of primes of R_0 whose height equals i. For an ideal J in R, Min(J) will represent the set of minimal primes over J.

Lemma 1.2.1. *Suppose that* (R_0, m_0) *is local. Let* $m = m_0 + R_+$. *Then,*

$$\dim M = \dim M_m$$

Proof. We may assume that $M \neq (0)$. Otherwise, dim M is undefined. Let $P \in$ Supp (M) and P^* be the ideal generated by all the homogeneous elements in P. By ([6], 1.5.6), $P^* \in$ Supp (M). This implies that $M_m \neq (0)$ and hence, dim M_m is defined.

If $P = P^*$, then $\dim M_P \leq \dim M_m$, by the definition of dimension. Suppose that $P \neq P^*$. Then,

$$\dim M_P = \dim (M_{P^*}) + 1$$

(cf. [6], 1.5.8). Since m is the graded maximal ideal of R, so $P^* \subset m$ and thus $\dim (M_{P^*}) < \dim M_m$. Therefore, $\dim M_P \leq \dim M_m$.

Since $\dim M = \dim M_Q$ for some $Q \in$ Supp (M) and $\dim M \geq \dim M_m$, we have our contention. □

Lemma 1.2.2. *Suppose that both R and M are Cohen-Macaulay. Assume also that (R_0, m_0) is local. As before, $I = \sqrt{\operatorname{ann}_R(M)}$. If $P, Q \in \operatorname{Ass}_R(M)$, then*

$$ht\ P = ht\ Q.$$

In particular, all minimal primes of I have the same height.

Proof. Let $m = m_0 + R_+$. Let P and $Q \in \operatorname{Ass}_R(M)$. They are both graded and hence, contained in m ([6], 1.5.6(b)(ii)). Since M_m is Cohen-Macaulay, by ([6], 2.1.2),

$$\dim \frac{R_m}{PR_m} = \dim M_m. \tag{1.1}$$

Since R_m is also Cohen-Macaulay and $P \subseteq m$,

$$\text{ht } PR_m + \dim \frac{R_m}{PR_m} = \dim R_m. \tag{1.2}$$

(cf. [6], 2.1.4). Hence, combining equations (1.1) and (1.2), we get

$$\text{ht } PR_m + \dim M_m = \dim R_m.$$

Similar equation holds for Q implies that

$$\text{ht } PR_m = \text{ht } QR_m.$$

Since P and Q are prime ideals, and are contained in m,

$$\text{ht } P = \text{ht } Q.$$

Since M is a finitely generated R-module, $V(I) = \operatorname{Supp}_R(M)$ and $\operatorname{Ass}_R(M)$ has the same minimal elements as $\operatorname{Supp}_R(M)$. Therefore, all the minimal primes of I are in $\operatorname{Ass}_R(M)$. More importantly, by the argument above, they all have the same height. $\square$

Lemma 1.2.3. *Suppose that both R and M are Cohen-Macaulay. Assume also that* (R_0, m_0) *is local.* $I = \sqrt{\mathrm{ann}_R(M)}$. *Let* $p \in \mathrm{Spec}(R_0)$ *and* $p \supseteq (I \cap R_0)$.
If there exists $W \in \mathrm{Min}(I)$ *such that* $W \cap R_0 = p$, *then*

$$\mathrm{ht}\ (p + I)R_p = \mathrm{ht}\ I.$$

Proof. Since all minimal primes of I are graded ([6], 1.5.6), $W \subseteq (p + R_+)$. Note that $(p + R_+) \in \mathrm{Spec}(R)$ and $(p + R_+) \cap (R_0 - p) = \emptyset$. Therefore, ht $WR_p =$ ht W.
Since all minimal primes of I have the same height (cf. Lemma (1.2.2)), ht $W =$ ht I.
By the definition of height,

$$\mathrm{ht}\ WR_p \geq \mathrm{ht}\ (p + I)R_p \geq \mathrm{ht}\ I.$$

Hence, ht $(p + I)R_p =$ ht I. □

The following will only be used in Section (2.4). It is essentially Lemma (1.2.2) without assuming that R_0 is local.

Lemma 1.2.4. *Suppose that both R and M are Cohen-Macaulay.* $I = \sqrt{\mathrm{ann}_R(M)}$. *Assume also that* $\dim R_0 = 1$, $\mathrm{Min}(R_0) = \{p\}$ *and for all* $W \in \mathrm{Min}(I), W \cap R_0 = p$. *Then,*
(a) all minimal primes of R_+ *have the same height;*
(b) all minimal primes of I *have the same height.*

Proof. For (a), note that R_+ has only 1 minimal prime over it, namely $p + R_+$.
For (b), the hypothesis implies that ht $(I \cap R_0) = 0$ and $I \subseteq (p + R_+)$. Let $Q \in \mathrm{Min}(I)$. Then, $Q \cap R_0 = p$ by assumption. By ([6], 1.5.6(b)), Q is graded and hence, $Q \subseteq (p + R_+)$. In particular, ht $Q =$ ht QR_p.
Note that, $(R_p)_0$ is local as well as equidimensional, and both R_p and M_p are Cohen-Macaulay. Therefore, by Lemma (1.2.2), all minimal primes of IR_p have the same

height. Since $QR_p \in \operatorname{Min}(IR_p)$,

$$\text{ht } QR_p = \text{ht } IR_p.$$

More importantly,

$$\text{ht } Q = \text{ht } IR_p$$

and the latter is independent of Q. Therefore, all minimal primes of I have the same height. □

The fact that all the minimal primes of R_+ have the same height will make the calculations of grade(R_+, M) easier in Lemmas (1.2.9) and (1.2.10).

Lemma 1.2.5. *Suppose that R is Cohen-Macaulay and (R_0, m_0) is local. $I = \sqrt{\operatorname{ann}_R(M)}$. If $q \in \operatorname{Spec}(R_0)$ and $q \supseteq (I \cap R_0)$, then*

$$\dim \frac{R_q}{(q+I)R_q} + \text{ht } (q+I)R_q = \text{ht } (q+R_+).$$

In particular,

$$\dim \frac{R}{(m_0+I)} + \text{ht } (m_0+I) = \text{ht } (m_0+R_+).$$

Proof. Let $q \in \operatorname{Spec}(R_0)$ and $q \supseteq (I \cap R_0)$. Put $Q = q + R_+$. Note that, $Q \in \operatorname{Spec}(R)$. Since I is also graded, $(q+I) \subseteq Q$ and hence, $(q+I)R_Q \neq R_Q$. Then, by ([6], 2.1.4),

$$\dim \frac{R_Q}{(q+I)R_Q} + \text{ht } (q+I)R_Q = \dim R_Q. \tag{1.3}$$

We will make three observations:

(i)

$$\dim \frac{R_Q}{(q+I)R_Q} = \dim \frac{R_q}{(q+I)R_q}$$

because

$$\frac{R_Q}{(q+I)R_Q} \cong \left(\frac{R}{q+I}\right)_Q \cong \left(\frac{R_q}{(q+I)R_q}\right)_{QR_q}$$

as R_Q-module and since QR_q is the graded maximal ideal of R_q,

$$\dim \left(\frac{R_q}{(q+I)R_q}\right)_{QR_q} = \dim \frac{R_q}{(q+I)R_q}$$

(cf. Lemma (1.2.1)). Note that, the dimension is defined here because every minimal prime of $q+I$ is contained in Q.

(ii) ht $Q = \dim R_q$ because

$$R_Q \cong (R_q)_{QR_q}$$

as rings and since QR_q is the graded maximal ideal of R_q,

$$\dim\ (R_q)_{QR_q} = \dim\ R_q$$

(cf. Lemma (1.2.1)); $\dim R_Q = \text{ht } Q$ since $Q \in \text{Spec}(R)$.

(iii) ht $(q+I)R_q = \text{ht } (q+I)R_Q$ because $(q+I)R_q$ is graded implies that all the minimal primes of $(q+I)R_q$ are graded ([6], 1.5.6) and hence, they all have the form PR_q, where P is a graded prime ideal of R and $P \subseteq Q$.

From these observations, the following can be deduced from equation (1.3),

$$\dim \frac{R_q}{(q+I)R_q} + \text{ht } (q+I)R_q = \text{ht } (q+R_+). \tag{1.4}$$

To show that

$$\dim \frac{R}{(m_0+I)} + \text{ht } (m_0+I) = \text{ht } (m_0+R_+),$$

we simply replace q and Q by m_0 and m respectively, where $m = m_0 + R_+$, in equation (1.4). Then, we have

$$\dim \frac{R_{m_0}}{(m_0+I)R_{m_0}} + \text{ht } (m_0+I)R_{m_0} = \text{ht } (m_0+R_+).$$

Since $(R_0 - m_0)$ consists of units of R, we can identify R_{m_0} with R. In particular, the preceding equation becomes

$$\dim \frac{R}{(m_0+I)} + \text{ht } (m_0+I) = \text{ht } (m_0+R_+).$$

□

For $q \in \operatorname{Spec}(R_0)$, we will use the nice formula in Lemma (1.2.5) to give a bound on $\dim \frac{M_q}{qM_q}$.

Lemma 1.2.6. *Suppose that both R and M are Cohen-Macaulay and (R_0, m_0) is local. $I = \sqrt{\operatorname{ann}_R(M)}$. If $q \in \operatorname{Spec}(R_0)$ and $q \supseteq (I \cap R_0)$, then*

$$\dim \frac{M_q}{qM_q} \leq \text{ht } (q + R_+) - \text{ht } (q + I)R_q.$$

In particular,

$$\dim \frac{M}{m_0 M} \leq \text{ht } (m_0 + R_+) - \text{ht } (m_0 + I).$$

Proof. Let $q \in \operatorname{Spec}(R_0)$ and $q \supseteq (I \cap R_0)$. Then, $qM_q \neq M_q$. Otherwise, $M_q = (0)$ (cf. [6], 1.5.24(a)).

By the definition of dimension,

$$\dim \frac{M_q}{qM_q} \leq \dim \frac{R_q}{(q + I)R_q}. \tag{1.5}$$

By Lemma (1.2.5),

$$\dim \frac{R_q}{(q + I)R_q} + \text{ht } (q + I)R_q = \text{ht } (q + R_+). \tag{1.6}$$

Combining inequality (1.5) and equation (1.6), we get

$$\dim \frac{M_q}{qM_q} \leq \text{ht } (q + R_+) - \text{ht } (q + I)R_q. \tag{1.7}$$

Since $(R_0 - m_0)$ consists of units of R, we can identify M_{m_0} with M and R_{m_0} with R. In particular, inequality (1.7) becomes

$$\dim \frac{M}{m_0 M} \leq \text{ht } (m_0 + R_+) - \text{ht } (m_0 + I),$$

after we substitute m_0 for q. □

The object of the next few Lemmas is to facilitate the calculations of grade(R_+, M).

Lemma 1.2.7. *Suppose that R is Cohen-Macaulay and (R_0, m_0) is local.*

Then, R_0 is equidimensional if and only if all the minimal prime ideals of R_+ have the same height.

Proof. Let $P \in \text{Min}(R_+)$ and $m = m_0 + R_+$. Then, $P = p + R_+$ for some $p \in \text{Min}(R_0)$. Furthermore, for every $q \in \text{Min}(R_0)$, $(q + R_+) \in \text{Min}(R_+)$.

Since R_m is Cohen-Macaulay, by ([6], 2.1.4),

$$\text{ht } PR_m + \dim \frac{R_m}{PR_m} = \dim R_m.$$

Then, $P \in \text{Spec}(R)$ and $P \subseteq m$ implies that ht $PR_m =$ ht P.

Since

$$\dim \frac{R_m}{PR_m} = \text{ht } \frac{mR_m}{PR_m},$$

we have

$$\text{ht } P + \text{ht } \frac{m}{P} = \text{ht } m.$$

Note that

$$\text{ht } \frac{m}{P} = \text{ht } \frac{m_0}{p}$$

because, every prime containing P is of the form $q + R_+$ where $q \in \text{Spec}(R_0)$. More importantly,

$$\dim \frac{R_0}{p} = \text{ht } \frac{m_0}{p}.$$

Hence, dim $\frac{R_0}{p}$ is the same for all $p \in \text{Min}(R_0)$ if and only if all minimal primes of R_+ have the same height. □

Lemma 1.2.8. *Suppose that R is Cohen-Macaulay. Assume also that (R_0, m_0) is local and equidimensional. Let $i \in \mathbb{N}_0$.*

If $q \in \text{Spec}(R_0)$ and ht $q = i$, *then $(q + R_+) \in \text{Spec}(R)$ and*

$$\text{ht } (q + R_+) = \text{ht } (R_+) + i.$$

Proof. Let $q \in \text{Spec}(R_0)$ and ht $q = i$. Then, $(q + R_+) \in \text{Spec}(R)$ because

$$\frac{R}{q + R_+} \cong \frac{R_0}{q}$$

as rings and the latter is a domain. There also exists $p \in \text{Min}(R_0)$ such that $p \subseteq q$ and ht $\frac{q}{p}$ = i. Let $P = p + R_+$ and $Q = q + R_+$.

Since R_Q is Cohen-Macaulay,

$$\dim R_Q = \text{ht } (PR_Q) + \dim \frac{R_Q}{PR_Q} \tag{1.8}$$

(cf. [6], 2.1.4). Since $P \in \text{Spec}(R)$, equation (1.8) can be written as

$$\text{ht } Q = \text{ht } P + \text{ht } \frac{Q}{P}. \tag{1.9}$$

Since all primes of R containing R_+ must be of the form, $u + R_+$, where $u \in \text{Spec}(R_0)$,

$$\text{ht } \frac{Q}{P} = \text{ht } \frac{q}{p}. \tag{1.10}$$

Since all the minimal primes of R_+ have the same height (cf. Lemma (1.2.7)),

$$\text{ht } (P) = \text{ht } R_+. \tag{1.11}$$

Hence, combining equations (1.9), (1.10) and (1.11), we have

$$\text{ht } (q + R_+) = \text{ht } (R_+) + i.$$

□

Lemma 1.2.9. *Suppose that both R and M are Cohen-Macaulay. Assume also that (R_0, m_0) is local and equidimensional. $I = \sqrt{\text{ann}_R(M)}$. Let $i \in \mathbb{N}_0$. If* ht $(I \cap R_0)$ $= i$, *then*

$$\text{grade } (R_+, M) = \text{ht } (R_+) + i - \text{ht } I.$$

Proof. By ([6], 1.2.10(a)),

$$\text{grade}\,(R_+,\, M) = \inf\{\text{depth}\ M_P \,|\, P \in V(R_+)\,\}.$$

Since depth $(0) := \infty$, we have

$$\text{grade}\,(R_+,\, M) = \inf\{\text{depth}\ M_P \,|\, P \in V(R_+ + I)\,\}.$$

Since M is Cohen-Macaulay,

$$\text{grade}\,(R_+,\, M) = \inf\{\dim\ M_P \,|\, P \in V(R_+ + I)\,\}. \tag{1.12}$$

For any $P \in V(R_+ + I)$,

$$\dim\ M_P = \dim\ \frac{R_P}{IR_P}. \tag{1.13}$$

Since R_P is Cohen-Macaulay,

$$\dim\ \frac{R_P}{IR_P} = \dim\,(R_P) - \text{ht}\ IR_P \tag{1.14}$$

(cf. [6], 2.1.4). By Lemma (1.2.2), all minimal primes of I have the same height. Hence, combining (1.12), (1.13) and (1.14), we have

$$\text{grade}\,(R_+,\, M) = \dim\ R_Q - \text{ht}\ I \tag{1.15}$$

for some $Q \in V(R_+ + I)$ such that dim R_Q is the smallest. Since ht $(I \cap R_0) = i$, Q will be of the form $q + R_+$ such that $q \in \text{Spec}(R_0)$ and ht $q = i$. By Lemma (1.2.8),

$$\text{ht}\,(q + R_+) = \text{ht}\,(R_+) + i. \tag{1.16}$$

Therefore, combining equations (1.15) and (1.16), we get

$$\text{grade}\,(R_+,\, M) = \text{ht}\,(R_+) + i - \text{ht}\ I.$$

□

The following will only be used in Section (2.4). It is essentially Lemma (1.2.9) in the case where R_0 is not necessarily local.

Lemma 1.2.10. *Suppose that both R and M are Cohen-Macaulay.* $I = \sqrt{\operatorname{ann}_R(M)}$. *Assume also that* $\dim R_0 = 1$, $\operatorname{Min}(R_0) = \{p\}$ *and for all* $W \in \operatorname{Min}(I), W \cap R_0 = p$. *Then,*

$$\operatorname{grade}(R_+, M) = \operatorname{ht}(R_+) - \operatorname{ht} I.$$

Proof. By ([6], 1.2.10(a)),

$$\operatorname{grade}(R_+, M) = \inf\{\operatorname{depth} M_Q \,|\, Q \in V(R_+)\}.$$

Since depth $(0) := \infty$, we have

$$\operatorname{grade}(R_+, M) = \inf\{\operatorname{depth} M_Q \,|\, Q \in V(R_+ + I)\}.$$

Since M is Cohen-Macaulay,

$$\operatorname{grade}(R_+, M) = \inf\{\dim M_Q \,|\, Q \in V(R_+ + I)\}. \tag{1.17}$$

For any $Q \in V(R_+ + I)$,

$$\dim M_Q = \dim \frac{R_Q}{IR_Q}. \tag{1.18}$$

Since R_Q is Cohen-Macaulay,

$$\dim \frac{R_Q}{IR_Q} + \operatorname{ht} IR_Q = \dim R_Q \tag{1.19}$$

(cf. [6], 2.1.4). By Lemma (1.2.4)(b), all minimal primes of I have the same height. Hence, putting the equations (1.17), (1.18) and (1.19), we have

$$\operatorname{grade}(R_+, M) = \dim R_Q - \operatorname{ht} I \tag{1.20}$$

for some $Q \in V(R_+ + I)$ such that dim R_Q is the smallest. Since ht $(I \cap R_0) = 0$, $Q = p + R_+$. ht $(I \cap R_0) = 0$ because $\mathrm{Min}(R_0) = \{p\}$ and $(I \cap R_0) \subseteq p$. Now, by Lemma (1.2.4)(a),

$$\text{ht } Q = \text{ht } R_+. \tag{1.21}$$

Note that, dim R_Q = ht Q. Therefore, combining equations (1.20) and (1.21), we get

$$\text{grade } (R_+,\, M) = \text{ht } R_+ - \text{ht } I.$$

□

Lemma 1.2.11. *Suppose that both R and M are Cohen-Macaulay. Assume also that $(R_0,\, m_0)$ is local and equidimensional. $I = \sqrt{\mathrm{ann}_R(M)}$.*
If $q \in \mathrm{Spec}(R_0)$ and $q \supseteq (I \cap R_0)$, then

$$\text{grade}(R_+,\, M) \leq \text{grade}(R_+R_q,\, M_q).$$

Proof. Given $q \in \mathrm{Spec}(R_0)$ such that $q \supseteq (I \cap R_0)$. By ([6], 1.2.10(a)),

$$\text{grade } (R_+,\, M) = \inf\,\{\text{depth } M_P \,|\, P \in V(R_+)\,\}$$

and

$$\text{grade } (R_+R_q,\, M_q) = \inf\,\{\text{depth } (M_q)_{PR_q} \,|\, PR_q \in V(R_+R_q)\,\}.$$

Note that, for $PR_q \in V(R_+R_q)$,

$$(M_q)_{PR_q} \cong M_P$$

as R_P-modules where, P has the form $p + R_+$ for some $p \in \mathrm{Spec}(R_0)$ and $q \subseteq p$. Consequently,

$$\text{depth}_{(R_q)_{PR_q}}\,(M_q)_{PR_q} = \text{depth}_{R_P}\,(M_P)$$

In particular,

$$\text{grade}(R_+,\, M) \leq \text{grade}(R_+R_q,\, M_q).$$

□

2 Asymptotic stability when dim $R_0 = 1$

The object of this chapter is to show the following Theorem.

Theorem 2.0.1. *Suppose that both R and M are Cohen-Macaulay. Assume also that* dim $R_0 = 1$. *Then, for all i,*

$$\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

We will first prove Theorem (2.0.1) in two special cases namely,

(a^*) Suppose that R and M are Cohen-Macaulay. Assume also that R_0 is local and dim $R_0 \leq 1$.

(b^*) Suppose that R and M are Cohen-Macaulay. Assume also that $|\mathrm{Min}(R_0)| = 1$ and for all $W \in \mathrm{Min}(I)$, $W \cap R_0 \in \mathrm{Min}(R_0)$.

In Section (2.5), we will prove Theorem (2.0.1) by first decomposing Spec(R_0) into a finite union of localisation, where each localisation of R_0 will give us the hypothesis of Theorem (2.0.1)(a^*) and Theorem (2.0.1)(b^*).

We begin by proving the following Proposition that deals with the case ht$(I \cap R_0) = 1$, where $I = \sqrt{\mathrm{ann}_R(M)}$. This is essentially due to Brodmann and Hellus's result, Proposition (1.1.1).

Proposition 2.0.1. *Assume the standard hypotheses on R and M. Suppose that* dim $R_0 = 1$. $I = \sqrt{\mathrm{ann}_R(M)}$. *If* ht $(I \cap R_0) = 1$, *then for all i,*

$$\left\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Proof. Let $i \in \mathbb{N}_0$. First, we show that for all $n \in \mathbb{Z}$,

$$\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right).$$

The containment, " $\subseteq$ ", and both sets having the same minimal elements are common knowledge. For the other containment, let $p \in \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$ for some n. Then, $p \supseteq (I \cap R_0)$. Hence, $p \in \mathrm{Max}(R_0)$ because, ht $(I \cap R_0) = 1$. Therefore, all the elements of $\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$ are maximal. In particular, for all $n \in \mathbb{Z}$,

$$\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right).$$

Finally, we will show that

$$\left\{\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)\right\}_{n \in \mathbf{Z}}$$

is asymptotically stable. We note that $\mathrm{Supp}_R\left(H^i_{R_+}(M)\right)$ is a finite set because

$$\mathrm{Supp}_R\left(H^i_{R_+}(M)\right) \subseteq \mathrm{Supp}_R(M) \cap V(R_+)$$

and the latter is a finite set. Since, ht $(I \cap R_0) = 1$ and dim $R_0 = 1$.

Let $q + R_+ \in \mathrm{Supp}_R\left(H^i_{R_+}(M)\right)$ and $q \in \mathrm{Spec}^1(R_0)$. Since dim $(R_0)_q = 1$, $H^i_{R_+R_q}(M_q)$ is asymptotically gap free (cf. Proposition (1.1.1)). Therefore,

$$q \in \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)$$

for $n << 0$ or,

$$q \notin \mathrm{Supp}_{R_0}(H^i_{R_+}(M)_n)$$

for $n << 0$. Since $\mathrm{Supp}_R\left(H^i_{R_+}(M)\right)$ is also a finite set, there exists N such that for all $n \leq N$,

$$\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right) = \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_N\right).$$

□

Remark:

(i) During the preparation of this paper, we realise that Theorem (2.0.1)(a^*) is proven by M. Brodmann, ST. Fumasoli and R. Tajarod without assuming either R or M is

Cohen-Macaulay (cf. [1], Theorem 3.5(e)). Therefore, if we wish to prove asymptotic stability for dim $R_0 = 1$, the case (b^*) is at the heart of the matter.

(ii) Our proof of Theorem (2.0.1)(a^*) is different from the one in [1].

2.1 Auxiliary Lemmas for the special case, (a^*): R_0 is local

We recall the hypothesis of Theorem (2.0.1)(a^*) .

Suppose that both R and M are Cohen-Macaulay. Assume also that (R_0, m_0) is local and dim $R_0 \leq 1$.

Note that the assumption on R_0 imply that R_0 is equidimensional. Hence, all the minimal prime divisors of R_+ have the same height (cf. Lemma (1.2.7)).

Lemma 2.1.1. *Assumptions as in Theorem (2.0.1)(a^*). $I = \sqrt{\mathrm{ann}_R(M)}$. Then, all minimal primes of I have the same height.*

Proof. We refer to Lemma (1.2.2). □

Lemma 2.1.2. *Assumptions as in Theorem (2.0.1)(a^*). $I = \sqrt{\mathrm{ann}_R(M)}$. Let $p \in \mathrm{Spec}(R_0)$ and $p \supseteq (I \cap R_0)$. If there exists $W \in \mathrm{Min}(I)$ such that $W \cap R_0 = p$, then*

$$\mathrm{ht}\ (p + I)R_p = \mathrm{ht}\ I.$$

Proof. We refer to Lemma (1.2.3). □

Lemma 2.1.3. *Assumptions as in Theorem (2.0.1)(a^*). As before, m_0 is the maximal ideal of R_0 and $I = \sqrt{\mathrm{ann}_R(M)}$.*

If $q \in \mathrm{Spec}(R_0)$ and $q \supseteq (I \cap R_0)$, then

$$\dim \frac{M_q}{qM_q} \leq \mathrm{ht}\ (q + R_+) - \mathrm{ht}\ (q + I)R_q.$$

In particular,

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (m_0 + R_+) - \mathrm{ht}\ (m_0 + I).$$

Proof. We refer to Lemma (1.2.6). □

Lemma 2.1.4. *Assumptions as in Theorem (2.0.1)(a^*) except that* $\dim R_0 = 1$. *Then,*

$$\text{ht } (m_0 + R_+) = \text{ht } (R_+) + 1.$$

Proof. (R_0, m_0) is equidimensional and $\text{ht}_{R_0}(m_0) = 1$. By Lemma (1.2.8),

$$\text{ht } (m_0 + R_+) = \text{ht } (R_+) + 1.$$

□

Lemma 2.1.5. *Assumptions as in Theorem (2.0.1)(a^*).* $I = \sqrt{\text{ann}_R(M)}$. *If* $\text{ht } (I \cap R_0) = 0$, *then*

$$\text{grade } (R_+, M) = \text{ht } R_+ - \text{ht } I.$$

Proof. We refer to Lemma (1.2.9). □

Lemma 2.1.6. *Assumptions as in Theorem (2.0.1)(a^*).* $I = \sqrt{\text{ann}_R(M)}$. *If* $q \in \text{Spec}(R_0)$ *and* $q \supseteq (I \cap R_0)$, *then*

$$\text{grade}(R_+, M) \leq \text{grade}(R_+R_q, M_q).$$

Proof. We refer to Lemma (1.2.11). □

2.2 R_0 is local

The main object of this section is to show the following.

Theorem (2.0.1)(a^*) *Suppose that both* R *and* M *are Cohen-Macaulay. Assume also that* (R_0, m_0) *is local and* $\dim R_0 \leq 1$. *Then, for all* i,

$$\{\text{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}_{n \in \mathbf{Z}}$$

is asymptotically stable.

We begin by dealing with the case dim $R_0 = 0$. This is the content of Proposition (2.2.1). The proof is essentially a Corollary of Proposition (1.1.1).

After that, in light of Proposition (2.0.1), we will focus on the case ht $(I \cap R_0) = 0$. We will use Lemmas (2.2.2) and (2.2.3) to describe the elements in $\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$. For $p \in \mathrm{Spec}(R_0)$, we will bound dim $\frac{M_p}{pM_p}$ in terms of grade(R_+, M). We want to get a handle of dim $\frac{M_p}{pM_p}$ because by Proposition (1.1.2), it tells us that for all $i >$ dim $\frac{M_p}{pM_p}$, $H^i_{R_+R_p}(M_p) = (0)$. These effort will be culminated in Proposition (2.2.4).

Finally, we will prove Theorem (2.0.1)(a^*).

Proposition 2.2.1. *Assumptions as in Theorem (2.0.1)(a^*) except that* dim $R_0 = 0$. *Then, for all i,*

$$\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Proof. Fix $i \in \mathbb{N}_0$. Then, by Proposition (1.1.1), $H^i_{R_+}(M)$ is asymptotically gap free. Since R_0 is also local and Artinian, there exists N such that either

$$\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \emptyset$$

for all $n \leq N$ or,

$$\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \{m_0\}.$$

for all $n \leq N$. □

Lemma 2.2.2. *Assumptions as in Theorem (2.0.1)(a^*) except that* dim $R_0 = 1$. $I = \sqrt{\mathrm{ann}_R(M)}$. *Let $q \in \mathrm{Min}(R_0)$. Then:*

(a) If for all $W \in \mathrm{Min}(I)$, $W \cap R_0 \neq q$, then $M_q = (0)$.

(b) If there exists $W \in \text{Min}(I)$ *such that* $W \cap R_0 = q$*, then*

$$\dim \frac{M_q}{qM_q} \leq \text{grade}(R_+, M).$$

Proof. For (a), suppose that for all $W \in \text{Min}(I)$, $(W \cap R_0) \neq q$. Then, $q \not\supseteq (I \cap R_0)$. Otherwise, $(q + R_+)$ contains a prime $W \in \text{Min}(I)$ such that $W \cap R_0 = q$ because $q \in \text{Min}(R_0)$. This is contrary to assumption. In particular, $(I \cap R_0) \cap (R_0 - q) \neq \emptyset$. Therefore, $M_q = (0)$.

For (b), suppose that there exists $W \in \text{Min}(I)$ such that $(W \cap R_0) = q$. Then, $q \supseteq (I \cap R_0)$. By Lemma (2.1.3),

$$\dim \frac{M_q}{qM_q} \leq \text{ht } (q + R_+) - \text{ht } (q + I)R_q. \tag{2.1}$$

By Lemma (2.1.2),

$$\text{ht } (q + I)R_q = \text{ht } I. \tag{2.2}$$

By the equidimensionality of R_0,

$$\text{ht } (q + R_+) = \text{ht } R_+. \tag{2.3}$$

Putting expressions (2.1), (2.2) and (2.3), we get

$$\dim \frac{M_q}{qM_q} \leq \text{ht } R_+ - \text{ht } I,$$

which is grade(R_+, M) by lemma (2.1.5). Note that our assumption in (b) implies that ht $(I \cap R_0) = 0$. □

Lemma 2.2.3. *Assumptions as in Theorem (2.0.1)(a*) except that* $\dim R_0 = 1$*. As before,* m_0 *is the maximal ideal of* R_0 *and* $I = \sqrt{\text{ann}_R(M)}$.

Suppose that ht $(I \cap R_0) = 0$*. Then:*

(a) If for all $W \in \text{Min}(R_0)$, $W \cap R_0 \neq m_0$*, then*

$$\dim \frac{M}{m_0 M} \leq \text{grade}(R_+, M).$$

(b) If there exists $W \in \mathrm{Min}(I)$ such that $W \cap R_0 = m_0$, then

$$\dim \frac{M}{m_0 M} \leq \mathrm{grade}(R_+, M) + 1.$$

Proof. For (a), by Lemma (2.1.5), it suffices to show that

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (R_+) - \mathrm{ht}\ I.$$

By Lemma (2.1.3),

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (m_0 + R_+) - \mathrm{ht}\ (m_0 + I). \tag{2.4}$$

By Lemma (2.1.4),

$$\mathrm{ht}\ (m_0 + R_+) = \mathrm{ht}\ (R_+) + 1. \tag{2.5}$$

By assumption,

$$\mathrm{ht}\ (m_0 + I) \geq \mathrm{ht}\ (I) + 1. \tag{2.6}$$

Putting the expressions (2.4), (2.5) and (2.6), we have

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (R_+) - \mathrm{ht}\ I.$$

For (b), we still have expressions (2.4) and (2.5). By Lemma (2.1.2),

$$\mathrm{ht}\ (m_0 + I) = \mathrm{ht}\ I$$

after we identify R_{m_0} with R. Therefore, combining the three expressions together, we get

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (R_+) - \mathrm{ht}\ (I) + 1.$$

Since $\mathrm{grade}(R_+, M) = \mathrm{ht}\ (R_+) - \mathrm{ht}\ (I)$, we have our contention. □

Proposition 2.2.4. *Assumptions as in Theorem (2.0.1)(a*) except that* $\dim R_0 = 1$. *As before,* m_0 *is the maximal ideal of* R_0 *and* $I = \sqrt{\text{ann}_R(M)}$. *Let* $g = \text{grade}(R_+, M)$. *If* $\text{ht}\,(I \cap R_0) = 0$, *then*

(a) $H^i_{R_+}(M) = (0)$ *for all* $i > g + 1$;

(b) $\text{Supp}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) \subseteq \{m_0\}$.

Proof. Note that $g \in \mathbb{N}_0$ since M is Cohen-Macaulay. We may assume that $\Gamma_{R_+}(M) \neq M$. Otherwise, $H^i_{R_+}(M) = (0)$ for all $i > 0$.

For (a), let $d = \dim \frac{M}{m_0 M}$. By Proposition (1.1.2), for all $i > d$, $H^i_{R_+}(M) = (0)$. Futhermore, by Lemma (2.2.3), $d \leq g+1$. Therefore, $H^i_{R_+}(M) = (0)$ for all $i > g+1$.

For (b), let $p \in \text{Supp}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right)$ for some n. Then, by the Flat Base Change Theorem ([5], 13.1.8, 15.2.2(iv)), $H^{g+1}_{R_+R_p}(M_p) \neq (0)$. Obviously, $M_p \neq (0)$. By Proposition (1.1.2),

$$\dim \frac{M_p}{pM_p} \geq g + 1.$$

By Lemma (2.2.2), $p \notin \text{Min}(R_0)$; otherwise, $\dim \frac{M_p}{pM_p} \leq g$. This shows that for all $n \in \mathbb{Z}$,

$$\text{Supp}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) \cap \text{Min}(R_0) = \emptyset.$$

Since (R_0, m_0) is local and $\dim R_0 = 1$,

$$\text{Supp}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) \subseteq \{m_0\}.$$

□

We will now prove the main result of this section.

Theorem (2.0.1)(a^*) *Suppose that both* R *and* M *are Cohen-Macaulay. Assume also that* R_0 *is local and* $\dim R_0 \leq 1$. *Then, for all* i,

$$\left\{\text{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\right\}_{n \in \mathbb{Z}}$$

is asymptotically stable.

Proof. **Case 1:** dim $R_0 = 0$. This is proven in Proposition (2.2.1).

Case 2: dim $R_0 = 1$. We will consider the following 2 situations seperately.

Case 2α: ht $(I \cap R_0) = 1$. This is proven in Proposition (2.0.1).

Case 2β: ht $(I \cap R_0) = 0$. Let $g = \text{grade}(R_+, M)$. We may assume that $M \neq \Gamma_{R_+}(M)$. Otherwise, $H^i_{R_+}(M)$ vanishes for all $i > 0$ and $H^0_{R_+}(M) = \Gamma_{R_+}(M)$ is finitely generated. By Proposition (1.1.3)(c), for $i = g$,

$$\left\{\text{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable. By Proposition (2.2.4)(a), for $i > g+1$, $H^i_{R_+}(M) = (0)$. Therefore, it remains to check for $i = g+1$.

By Proposition (1.1.1), $H^{g+1}_{R_+}(M)$ is asymptotically gap free. The nontrivial case is when $H^{g+1}_{R_+}(M)_n \neq (0)$ for $n << 0$. Then, for $n << 0$,

$$\text{Ass}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) = \{m_0\}.$$

This is because:

By Proposition (2.2.4)(b), for all $n \in \mathbb{Z}$,

$$\text{Supp}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) \subseteq \{m_0\}$$

and

$$\text{Ass}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) \subseteq \text{Supp}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right).$$

In fact, we have equality in the last two expressions because any R_0-module L is (0) if and only if $\text{Ass}_{R_0}(L) = \emptyset$. □

2.3 Auxiliary Lemmas for the special case, (b^*): $|\text{Min}(R_0)| = 1$

We will state the hypothesis of Theorem (2.0.1)(b^*).

Suppose that both R and M are Cohen-Macaulay. $I = \sqrt{\text{ann}_R(M)}$. Assume also that dim $R_0 = 1$, $\text{Min}(R_0) = \{p\}$ *and for all* $W \in \text{Min}(I), W \cap R_0 = p$.

We will collect some Lemmas from Chapter (1) that will be applicable in the context of Theorem (2.0.1)(b^*).

Lemma 2.3.1. *Assumptions as in Theorem (2.0.1)(b^*). Then,*

(a) all minimal primes of R_+ have the same height;

(b) all minimal primes of I have the same height.

Proof. We refer to Lemma (1.2.4). □

Lemma 2.3.2. *Assumptions as in Theorem (2.0.1)(b^*). $I = \sqrt{\mathrm{ann}_R(M)}$.*
If $q \in \mathrm{Max}(R_0)$, then

$$\dim \frac{M_q}{qM_q} \leq \mathrm{ht}\ (q + R_+) - \mathrm{ht}\ (q + I)R_q.$$

Proof. Let $q \in \mathrm{Max}(R_0)$. Then, R_q is graded local and $(R_q)_0$ is equidimensional. Now, we can apply Lemma (1.2.6) to M_q and R_q to get

$$\dim \frac{M_q}{qM_q} \leq \mathrm{ht}\ (q + R_+)R_q - \mathrm{ht}\ (q + I)R_q.$$

Note that, $q(R_q)_0$ is the maximal ideal of $(R_q)_0$. Since $(q + R_+) \in \mathrm{Spec}(R)$,

$$\mathrm{ht}\ (q + R_+)R_q = \mathrm{ht}\ (q + R_+)$$

□

Lemma 2.3.3. *Assumptions as in Theorem (2.0.1)(b^*).*
If $q \in \mathrm{Max}(R_0)$, then $(q + R_+) \in \mathrm{Spec}(R)$ and

$$\mathrm{ht}\ (q + R_+) = \mathrm{ht}\ (R_+) + 1.$$

Proof. Let $q \in \mathrm{Max}(R_0)$. Then, R_q is graded local. $(R_q)_0$ is local, equidimensional and $\dim\ (R_q)_0 = 1$. Then, by Lemma (1.2.8),

$$\mathrm{ht}\ (q + R_+)R_q = \mathrm{ht}\ (R_+R_q) + 1. \tag{2.7}$$

Since $(q + R_+) \in \operatorname{Spec}(R)$,

$$\text{ht } (q + R_+)R_q = \text{ht } (q + R_+). \tag{2.8}$$

By Lemma (2.3.1)(a),

$$\text{ht } (R_+R_q) = \text{ht } R_+ \tag{2.9}$$

Therefore, combining equations (2.7), (2.8) and (2.9), we have our goal. □

Lemma 2.3.4. *Assume the hypothesis of Theorem (2.0.1)(b^*). Then,*

$$\operatorname{grade}(R_+,\, M) = \text{ht } R_+ - \text{ht } I.$$

Proof. We refer to Lemma (1.2.10) □

2.4 Proof of Theorem (2.0.1) special case, (b^*) : $|\operatorname{Min}(R_0)| = 1$

The goal here is to prove the following.

Theorem (2.0.1)(b^*) *Suppose that both R and M are Cohen-Macaulay. $I = \sqrt{\operatorname{ann}_R(M)}$. Assume also that* $\dim R_0 = 1$, $\operatorname{Min}(R_0) = \{p\}$ *and for all* $W \in \operatorname{Min}(I), W \cap R_0 = p$. *Then,*

$$H^i_{R_+}(M) = (0) \text{ if and only if } i \neq \operatorname{grade}(R_+,\, M).$$

In particular, by Proposition (1.1.3)(c), for all i,

$$\{\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Lemma 2.4.1. *Assumptions as in Theorem (2.0.1)(b^*).*

Then, for all $q \in \operatorname{Max}(R_0)$,

$$\dim \frac{M_q}{qM_q} \leq \operatorname{grade}(R_+,\, M).$$

Proof. Let $q \in \operatorname{Max}(R_0)$. By Lemma (2.3.4), it suffices to show that

$$\dim \frac{M_q}{qM_q} \leq \text{ht } R_+ - \text{ht } I.$$

We note that both M_q and R_q are Cohen Macaulay. $(R_q)_0$ ($\cong (R_0)_q$) is local and equidimensional. Therefore, by Lemma (2.3.2),

$$\dim \frac{M_q}{qM_q} \leq \text{ht } (q + R_+) - \text{ht } (q + I)R_q. \tag{2.10}$$

Since no minimal prime of IR_q contains $(q + I)R_q$

$$\text{ht } (q + I)R_q \geq \text{ht } (I) + 1. \tag{2.11}$$

By Lemma (2.3.3),

$$\text{ht } (q + R_+) = \text{ht } (R_+) + 1. \tag{2.12}$$

Putting the expressions (2.10), (2.11) and (2.12) together, we get,

$$\dim \frac{M_q}{qM_q} \leq \text{ht } R_+ - \text{ht } I.$$

□

Theorem 2.0.1(b^*) *Suppose that both R and M are Cohen-Macaulay.* $I = \sqrt{\operatorname{ann}_R(M)}$. *Assume also that* $\dim R_0 = 1$, $\operatorname{Min}(R_0) = \{p\}$ *and for all* $W \in \operatorname{Min}(I), W \cap R_0 = p$. *Then,*

$$H^i_{R_+}(M) = (0) \text{ if and only if } i \neq \operatorname{grade}(R_+, M).$$

In particular, by Proposition (1.1.3)(c), for all i,

$$\{\operatorname{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Proof. Note that, $R_+M \neq M$ because $(p+R_+) \in \mathrm{Supp}_R(M)$. Let $g = \mathrm{grade}(R_+, M)$. By Proposition (1.1.3)(a), it remains to show that for all $i > g$, $H^i_{R_+}(M) = (0)$.

Fix $i > g$. Suppose that $H^i_{R_+}(M) \neq (0)$. Then, there exists $q \in \mathrm{Max}(R_0)$ and $n \in \mathbb{Z}$ such that

$$H^i_{R_+}(M)_n \otimes_{R_0} (R_0)_q \neq (0).$$

By the Flat Base Change Theorem ([5], 13.1.8, 15.2.2(iv)), this implies that

$$H^i_{R_+R_q}(M_q) \neq (0).$$

Then, by Proposition (1.1.3),

$$\dim \frac{M_q}{qM_q} \geq i,$$

contrary to Lemma (2.4.1). □

2.5 Proof of Theorem (2.0.1)

The object here is to prove Theorem (2.0.1).

In Proposition (2.5.1), we will show that if we can break up $\mathrm{Spec}(R_0)$ into smaller pieces and on each piece, we can answer the questions on asymptotic stability and aymptotic "gap freeness", then we will have a better handle with the same questions on the whole $\mathrm{Spec}(R_0)$.

In Propositions (2.5.2) and (2.5.3), we will show that if $\dim R_0 = 1$, then $\mathrm{Spec}(R_0)$ can be decomposed into pieces where either the hypothesis of Theorem (2.0.1)(a^*) or (2.0.1)(b^*) is fulfilled. To this end, we recall the hypotheses of the two special cases of Theorem (2.0.1).

(a^*) *Suppose that both R and M are Cohen Macaulay. Assume also that R_0 is local and* $\dim R_0 \leq 1$.

(b^*) *Suppose that both R and M are Cohen Macaulay. Assume also that* $\dim R_0 = 1$, $|\mathrm{Min}(R_0)| = 1$ *and for all* $W \in \mathrm{Min}(I)$, $W \cap R_0 \in \mathrm{Min}(R_0)$.

Finally, we will prove Theorem (2.0.1).

Notation: For a closed multiplicative subset, S, of R_0, we will identify $\mathrm{Spec}(S^{-1}R_0)$ with the set,

$$\{q \in \mathrm{Spec}(R_0) \mid q \cap S = \emptyset\}.$$

Proposition 2.5.1. *Suppose that R and M satisfy the standard hypotheses. Let $S_1, \ldots, S_k$ be closed multiplicative subsets of R_0 with*

$$\mathrm{Spec}\ (R_0) = \bigcup_{j=1}^{k} \mathrm{Spec}\ (S_j^{-1}R_0).$$

Fix $i \in \mathbb{N}_0$. Then,

(a) If for all j,

$$\left\{\mathrm{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable, then

$$\left\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

(b) If for all j,

$$H^i_{S_j^{-1}R_+}(S_j^{-1}M)$$

is asymptotically gap free as a $S_j^{-1}R$-module, then $H^i_{R_+}(M)$ is asymptotically gap free.

Proof. Suppose that

$$\left\{\mathrm{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable for every j, $1 \leq j \leq k$. Then, there exists N such that for all $n < N$, and all j,

$$\mathrm{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n\right) = \mathrm{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M)_N\right).$$

Fix $n < N$ and let $p \in \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$. Then,

$$\operatorname{Spec}(R_0) = \bigcup_{j=1}^{k} \operatorname{Spec}(S_j^{-1}R_0)$$

implies that $S_j^{-1}p \in \operatorname{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n\right)$ for some j. Fix this j. Therefore, $S_j^{-1}p \in \operatorname{Ass}_{S_j^{-1}R_0}\left(H^i_{S_j^{-1}R_+}(S_j^{-1}M)_N\right)$ so that, $p \in \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_N\right)$. Consequently,

$$\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) \subseteq \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_N\right).$$

For the other inclusion, let $q \in \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_N\right)$. Then,

$$\operatorname{Spec}(R_0) = \bigcup_{j=1}^{k} \operatorname{Spec}(S_j^{-1}R_0)$$

implies that $S_t^{-1}q \in \operatorname{Ass}_{S_t^{-1}R_0}\left(H^i_{S_t^{-1}R_+}(S_t^{-1}M)_N\right)$ for some t. Fix this t. Therefore, $S_t^{-1}q \in \operatorname{Ass}_{S_t^{-1}R_0}\left(H^i_{S_t^{-1}R_+}(S_t^{-1}M)_n\right)$ for all $n < N$ so that, $q \in \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ for all $n < N$. Consequently,

$$\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_N\right) \subseteq \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$$

for all $n < N$.

For (b), suppose that

$$H^i_{S_j^{-1}R_+}(S_j^{-1}M)$$

is asymptotically gap free for every j, $1 \leq j \leq k$.

Case 1: There exists j such that

$$H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n \neq (0)$$

for $n << 0$. Then, by the Flat Base Change Theorem ([5], 13.1.8, 15.2.2(iv)), $H^i_{R_+}(M)_n \neq (0)$ for $n << 0$.

Case 2: For all j,

$$H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n = (0)$$

for $n << 0$. Then, there exists N such that for all $n < N$,

$$H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n = (0)$$

for all j. This is because we only have finitely many j.

Suppose that $H^i_{R_+}(M)_n \neq (0)$ for some $n < N$. Fix this n. Then, there exists $p \in \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$. Since

$$\mathrm{Spec}\,(R_0) = \bigcup_{j=1}^{k} \mathrm{Spec}\,(S_j^{-1}R_0),$$

$p \in \mathrm{Spec}\,(S_j^{-1}R_0)$ for some j. Fix this j. Then,

$$H^i_{R_+}(M)_n \otimes_{R_0} S_j^{-1}R_0 \neq (0)$$

since $p \cap S_j = \emptyset$. By the Flat Base Change Theorem ([5], 13.1.8, 15.2.2(iv)),

$$H^i_{S_j^{-1}R_+}(S_j^{-1}M)_n \neq (0),$$

a contradiction to assumption. □

Proposition 2.5.2. *Suppose that* $\dim R_0 = 1$. *Then, there are closed multiplicative subsets* $S_i, T_j \subseteq R$ $(1 \leq i \leq s,\ 1 \leq j \leq t)$ *such that*

$$\mathrm{Spec}(R_0) = \bigcup_{i=1}^{s} \mathrm{Spec}(S_i^{-1}R_0) \cup \bigcup_{j=1}^{t} \mathrm{Spec}(T_j^{-1}R_0) \tag{2.13}$$

where,

1. *for all* i, $|\mathrm{Min}(S_i^{-1}R_0)| = 1$ *and* $\dim S_i^{-1}R_0 = 1$;
2. *for all* j, $T_j^{-1}R_0$ *is local and* $\dim T_j^{-1}R_0 \leq 1$.

Proof. Let

$$\Omega_0 = \mathrm{Min}(R_0) \cap \mathrm{Max}(R_0)$$

and

$$\Omega_1 = \{m \in \mathrm{Max}(R_0) \mid m \text{ contains at least two distinct elements of } \mathrm{Min}(R_0)\}.$$

$\mathrm{Min}(R_0)$ is a finite set since R_0 is Noetherian. Since we also have dim $R_0 = 1$, for any two distinct minimal primes of R_0, there are at most finitely many $m \in \mathrm{Max}(R_0)$ that contains both. Consequently, $\Omega_0 \cup \Omega_1$ is a finite set.

Put

$$\mathrm{Min}(R_0) - \Omega_0 := \{p_1, \ldots, p_s\}.$$

$\mathrm{Min}(R_0) - \Omega_0 \neq \emptyset$ because dim $R_0 = 1$. For every i, $1 \leq i \leq s$, we choose

$$x_i \in \left(\bigcap_{p_j \neq p_i} p_j \right) - p_i$$

and define $S_i := \{x_i^n \mid n \in \mathbb{N}_0\}$. Then, for all i, $|\mathrm{Min}(S_i^{-1} R_0)| = 1$ and dim $S_i^{-1} R_0 = 1$.

Furthermore, for each i, we put

$$\widetilde{M_i} := \{q \in \mathrm{Spec}^1(R_0) \mid \ < x_i, p_i > \subseteq q\}$$

Since $\mathrm{ht}_{R_0}(< x_i, p_i >) = 1$ or $< x_i, p_i > = R_0$, so $\widetilde{M_i}$ is finite for all i. Therefore, we can write

$$\bigcup_{i=1}^{s} \widetilde{M_i} \cup \Omega_0 \cup \Omega_1 = \{m_1, \ldots, m_t\}.$$

Now, for $1 \leq j \leq t$, put $T_j = R_0 - m_j$. Then, $T_j^{-1} R_0$ is local and dim $T_j^{-1} R_0 \leq 1$.

With these choices of S_i's and T_j's, we would like to show that the equation (2.13) holds.

It is clear that

$$\mathrm{Min}(R_0) \subset \bigcup_{i=1}^{s} \mathrm{Spec}(S_i^{-1} R_0) \cup \bigcup_{j=1}^{t} \mathrm{Spec}(T_j^{-1} R_0).$$

Let $q \in \mathrm{Spec}^1(R_0)$. Suppose that $q \notin \mathrm{Spec}(T_j^{-1} R_0)$ for all j. Then, q contains exactly 1 minimal prime, p_i, for some $p_i \in (\mathrm{Min}(R_0) - \Omega_0)$ and does not contain x_i. Therefore, $q \in \mathrm{Spec}(S_i^{-1} R_0)$. □

Proposition 2.5.3. *Suppose that* $\dim R_0 = 1$ *and* $\mathrm{Min}(R_0) = \{ p \}$. *Let J be a graded ideal of R with* $\mathrm{ht}\ (J \cap R_0) = 0$. *Assume that*

$$\Sigma := \{q \in \mathrm{Spec}^1(R_0) \,|\, q = W \cap R_0 \text{ for some } W \in \mathrm{Min}(J)\} \neq \emptyset.$$

Then, there exist closed multiplicative subsets, $\Omega_k \subset R_0$ $(1 \le k \le l)$, *and* $x \in (R_0 - p)$ *such that*

$$\mathrm{Spec}(R_0) = \mathrm{Spec}(R_0)_x \cup \bigcup_{k=1}^{l} \mathrm{Spec}(\Omega_k^{-1} R_0)$$

where

1. *for all* k, $\Omega_k^{-1} R_0$ *is local and* $\dim \Omega_k^{-1} R_0 = 1$.
2. $\dim (R_0)_x \le 1$ *and for all* $W \in \mathrm{Min}(JR_x)$, $(W \cap (R_0)_x) = p(R_0)_x$.

Proof. Note that all minimal primes of J are graded ([6], 1.5.6). Since $\mathrm{Min}(I)$ is a finite set, Σ is a finite set. We choose

$$x \in \left(\bigcap_{m \in \Sigma} m \right) - p.$$

Then, $\dim (R_0)_x \le 1$ and for all $W \in \mathrm{Min}(JR_x)$, $(W \cap (R_0)_x) = p(R_0)_x$. We can (and will) identify $(R_x)_0$ with $(R_0)_x$.

Let $\{q_1, \ldots, q_l\}$ consists of all the prime ideals of R_0 that contains x. For all k, $1 \le k \le l$, we put

$$\Omega_k = R_0 - q_k.$$

Then, $\dim \Omega_k^{-1} R_0 = 1$ and $\Omega_k^{-1} R_0$ is local for all k. It is now clear that

$$\mathrm{Spec}(R_0) = \mathrm{Spec}(R_0)_x \cup \bigcup_{k=1}^{l} \mathrm{Spec}(\Omega_k^{-1} R_0)$$

because if a prime, q, in R_0 contains x, then $q \in \mathrm{Spec}(\Omega_k^{-1} R_0)$ for some k; if q does not contain x, then $q \in \mathrm{Spec}(R_0)_x$. □

We recall Theorem (2.0.1), Theorem (2.0.1)(a^*) and Theorem (2.0.1)(b^*).

Theorem 2.0.1 *Suppose that both R and M are Cohen-Macaulay.*

Assume also that $\dim R_0 = 1$. *Then, for all i,*

$$\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Theorem 2.0.1(a^*) *Suppose that R and M are Cohen-Macaulay. Assume also that R_0 is local and* $\dim R_0 \leq 1$. *Then, for all i,*

$$\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Theorem 2.0.1(b^*) *Suppose that R and M are Cohen-Macaulay. Assume also that* $\dim R_0 = 1$, $\mathrm{Min}(R_0) = \{p\}$ *and for all $W \in \mathrm{Min}(I)$, $W \cap R_0 = p$. Then, for all i,*

$$\{\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

We will now prove Theorem (2.0.1).

Proof. Recall that $I = \sqrt{\mathrm{ann}_R(M)}$. Fix $t \in \mathbb{N}_0$. We want to show that

$$\left\{\mathrm{Ass}_{R_0}\left(H^t_{R_+}(M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Case 1: ht $(I \cap R_0) = 1$. This is done in Proposition (2.0.1).

Case 2: ht $(I \cap R_0) = 0$. By Proposition (2.5.2), there are closed multiplicative subsets $S_i, T_j \subseteq R_0$ ($1 \leq i \leq s$, $1 \leq j \leq \tilde{t}$) such that

$$\mathrm{Spec}(R_0) = \bigcup_{i=1}^{s} \mathrm{Spec}(S_i^{-1}R_0) \cup \bigcup_{j=1}^{\tilde{t}} \mathrm{Spec}(T_j^{-1}R_0)$$

where

(1) for all i, $|\mathrm{Min}(S_i^{-1}R_0)| = 1$ and $\dim S_i^{-1}R_0 = 1$;

(2) for all j, $T_j^{-1}R_0$ is local and $\dim T_j^{-1}R_0 \leq 1$.

We wish to show that for all i, j, both

$$\left\{\mathrm{Ass}_{S_i^{-1}R_0}\left(H^t_{S_i^{-1}R_+}(S_i^{-1}M)_n\right)\right\}_{n\in\mathbf{Z}} \text{ and } \left\{\mathrm{Ass}_{T_j^{-1}R_0}\left(H^t_{T_j^{-1}R_+}(T_j^{-1}M)_n\right)\right\}_{n\in\mathbf{Z}}$$

are asymptotically stable. Then, by Proposition (2.5.1), we have our contention.

By Theorem (2.0.1)(a^*), for all j,

$$\left\{\mathrm{Ass}_{T_j^{-1}R_0}\left(H^t_{T_j^{-1}R_+}(T_j^{-1}M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable. Note that the Cohen-Macaulay condition on R and M is preserved under localisation. Hence, it remains to consider the case,

$$\dim R_0 = 1 \text{ and } \mathrm{Min}(R_0) = \{p\}.$$

Let

$$\Sigma := \{q \in \mathrm{Spec}^1(R_0) \,|\, q = W \cap R_0 \text{ for some prime } W \in \mathrm{Min}(I)\}.$$

Case 2α: $\Sigma = \emptyset$. Then, for all $W \in \mathrm{Min}(I)$, $W \cap R_0 = p$. Then, by Theorem (2.0.1)(b^*),

$$\left\{\mathrm{Ass}_{R_0}\left(H^t_{R_+}(M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Case 2β: $\Sigma \neq \emptyset$. By Proposition (2.5.3), there exist multiplicative closed sets $\{\Omega_k\}_{k=1}^l$ in R_0 and $x \in (R_0 - p)$ such that

$$\mathrm{Spec}(R_0) = \mathrm{Spec}(R_0)_x \cup \bigcup_{k=1}^{l} \mathrm{Spec}(\Omega_k^{-1}R_0)$$

where

(1) for all k, $\Omega_k^{-1}R_0$ is local and $\dim \Omega_k^{-1}R_0 = 1$;

(2) for all $W \in \mathrm{Min}(IR_x)$, $(W \cap (R_0)_x) = p(R_0)_x$ and dim $(R_0)_x \leq 1$.

Recall that, $(R_x)_0$ is identified with $(R_0)_x$. Once again, by Theorem (2.0.1)(a^*) (for the case where dim $(R_0)_x = 0$) and Theorem (2.0.1)(b^*),

$$\left\{\mathrm{Ass}_{(R_0)_x}\left(H^t_{R_+R_x}(M_x)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable. Note that $\mathrm{Min}((R_0)_x) = \{p(R_0)_x\}$.

By Theorem (2.0.1)(a^*), for all $1 \leq k \leq l$,

$$\left\{\mathrm{Ass}_{\Omega_k^{-1}R_0}\left(H^t_{\Omega_k^{-1}R_+}(\Omega_k^{-1}M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable. Therefore, by Proposition (2.5.1),

$$\left\{\mathrm{Ass}_{R_0}\left(H^t_{R_+}(M)_n\right)\right\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

□

2.6 Corollaries to Theorem (2.0.1)

We recall

Theorem 2.0.1 *Suppose that both R and M are Cohen-Macaulay. Assume also that* dim $R_0 = 1$. *Then, for all i,*

$$\{\mathrm{Ass}_{R_0}(H^i_{R_+}(M)_n)\}_{n\in\mathbf{Z}}$$

is asymptotically stable.

Corollary 2.6.1. *Suppose that both R and M are Cohen-Macaulay. Assume also that* dim $R_0 = 1$.

Then:

(a) For all i, $H^i_{R_+}(M)$ is asymptotically gap free;

(b) for all i, $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ *is a finite set.*

Proof. For $i = 0$, $H^i_{R_+}(M) = \Gamma_{R_+}(M)$ which is a submodule of M. It is finitely generated over R and hence both (a) and (b) are satisfied.

Fix $i \in \mathbb{N}$. Then, by Theorem (2.0.1), there exists N such that for all $n \leq N$,

$$\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_N\right).$$

Note that,

$$H^i_{R_+}(M)_n = (0) \text{ if and only if } \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) = \emptyset.$$

Therefore, $H^i_{R_+}(M)$ is asymptotically gap free.

As for (b), we recall the bijection between

$$\operatorname{Ass}_R(H^i_{R_+}(M)) \leftrightarrow \bigcup_{n\in\mathbf{Z}} \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right).$$

Since $H^i_{R_+}(M)_n = (0)$ for $n >> 0$ (cf. [5], 15.1.5), so by Theorem (2.0.1),

$$\bigcup_{n\in\mathbf{Z}} \operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) \text{ is finite.}$$

Therefore, $\operatorname{Ass}_{R_0}\left(H^i_{R_+}(M)\right)$ is finite. □

Corollary 2.6.2. *Suppose that both R and M are Cohen-Macaulay. Assume also that R_0 is semi-local and* dim $R_0 = 2$. *Then, for all i,* $\operatorname{Ass}_R\left(H^i_{R_+}(M)\right)$ *is a finite set.*

Proof. Fix $i \in \mathbb{N}_0$. Let W consists of all height 2 maximal ideals of R_0. We choose

$$x \in \bigcap_{m\in W} m - \bigcup_{p_i\in \operatorname{Min}(R_0)} p_i.$$

Let $T = \{p + R_+ \,|\, p \in \operatorname{Spec}(R_0) \text{ and } x \in p\}$. T is finite because R_0 is semilocal, Noetherian, dim $R_0 = 2$, and $\operatorname{ht}_{R_0}(x) = 1$.

Let $\Omega = \{x^n \mid n \in \mathbb{N}_0\}$. Now, we put

$$R'_0 = \Omega^{-1} R_0 \qquad R' = R'_0[R_+]$$

$$R'_+ = \Omega^{-1} R_+ \qquad M' = M \otimes_R R'$$

Then, $\dim R'_0 = 1$. Both M' and R' are (graded) Cohen-Macaulay. By Corollary (2.6.1)(b), $\mathrm{Ass}_{R'}(H^i_{R'_+}(M'))$ is a finite set. Since

$$\mathrm{Ass}_{R'}(H^i_{R'_+}(M')) = \{\Omega^{-1}P \mid P \in \mathrm{Ass}_R(H^i_{R_+}(M)) \text{ and } P \cap \Omega = \emptyset\},$$

and

$$\mathrm{Ass}_R(H^i_{R_+}(M)) \subseteq \{P \mid P \in \mathrm{Ass}_R(H^i_{R_+}(M)) \text{ and } P \cap \Omega = \emptyset\} \cup T$$

so, $\mathrm{Ass}_R(H^i_{R_+}(M))$ is a finite set. □

3 Asymptotically gap free if dim $R_0 = 2$

Throughout this chapter, in addition to the standard hypotheses on R and M, we will also assume that (R_0, m_0) is local and dim $R_0 = 2$. The object of this chapter is to show that under some conditions, $H^i_{R_+}(M)$ is asymptotically gap free in the case where dim $R_0 = 2$.

The first section will show that under certain conditions, $\Gamma_{m_0R}\left(H^i_{R_+}(M)\right)$ is Artinian. The final section will show that under some restrictions on $\mathrm{Ass}_R\left(H^{i-1}_{R_+}(M)\right)$, $H^i_{R_+}(M)$ is asymptotically gap free. This will be an extension of a result of Brodmann and Hellus, Proposition (1.1.1), to the case dim $R_0 = 2$.

We would like to begin by proving the following Lemma. The Lemma relates the notion of a R-module being Artinian to it being asymptotically gap free. This observation was used in the paper by Brodmann and Hellus ([2], 4.1(ii)). We include a proof here since it will be used in some of our arguments later.

Lemma 3.0.1. *Assume the standard hypothesis on R in this paper. Let T be a graded R-module. Then, if T is either Artinian or Noetherian, then T is asymptotically gap free.*

Proof. If T is Noetherian, then there are homogeneous elements $x_i's$ in T such that $T = < x_1, \ldots, x_k >$ over R; since R is positively graded, $T_n = (0)$ for all $n < \min\{\text{degree } (x_i)\}_{i=1}^{i=k}$.

Assume that T is Artinian. Suppose that T is not asymptotically gap free. Then, the set

$$\Omega := \{n \in \mathbb{Z}_{<0} \,|\, T_n \neq (0) \text{ and } T_{n+1} = (0)\}$$

is infinite.

Since R is positively graded and R_+ is generated by degree 1 elements, T_n is a graded

R-module $(R_+T_n = (0))$, for all $n \in \Omega$. For $N \in \mathbb{Z}$, we put

$$\widetilde{T}(N) := \bigoplus_{n \in \Omega \text{ and } n<N} T_n$$

Then, $\widetilde{T}(N) \supseteq \widetilde{T}(N-1)$ for all $N \in \mathbb{Z}$, gives us a decreasing chain of graded R-submodules in T that will not stablise, contrary to T being Artinian. □

3.1 When is $\Gamma_{m_0R}\left(H^i_{R_+}(M)\right)$ Artinian?

In this subsection, we would like to give some sufficient condition to guarantee that

$$\Gamma_{m_0R}\left(H^i_{R_+}(M)\right)$$

is Artinian. This has some interest in its own right. Since, there are R-modules that are only supported at a maximal ideal, but are not Artinian (cf. [?]).

The following lemma will be useful in the proof of the main proposition in this subsection.

Lemma 3.1.1. *Assume the standard hypotheses on R, R_0 and M in this chapter. Let x_0, y_0 be a system of parameters for R_0. Then,*

$$\Gamma_{y_0R}\left(H^i_{(R_+,x_0)}(M)\right)$$

is an Artinian graded R-module for all $i \in \mathbb{N}_0$.

Proof. Fix $i \in \mathbb{N}_0$. We want to show that

$$\Gamma_{y_0R}\left(H^i_{(R_+,x_0)}(M)\right)$$

is a homomorphic image of an Artinian graded R-module.

Consider the following exact sequence (cf. [5], 13.1.12),

$$H^i_{(R_+,x_0,y_0)}(M) \longrightarrow H^i_{(R_+,x_0)}(M) \longrightarrow H^i_{(R_+,x_0)}(M)_{y_0}.$$

This induces the exact sequence of graded R-modules,

$$H^i_{(R_+,x_0,y_0)}(M) \longrightarrow \Gamma_{y_0R}\,(H^i_{(R_+,x_0)}(M)) \longrightarrow 0.$$

By ([5], 7.1.4),

$$\frac{R}{(R_+,\, x_0,\, y_0)}$$

is Artinian implies that

$$H^i_{(R_+,x_0,y_0)}(M)$$

is Artinian. Therefore,

$$\Gamma_{y_0R}\,(H^i_{(R_+,x_0)}(M))$$

is a homormorphic image of an Artinian R-module.

It is clear that the module is graded. □

Proposition 3.1.2. *Assume the standard hypotheses on R, R_0 and M in this chapter. Fix $i \in \mathbb{N}_0$. Suppose that*

(a) $\mathrm{Ass}_R\left(H^{i-1}_{R_+}(M)\right)$ *is finite and*

(b) $\mathrm{Ass}_{R_0}\left(H^{i-1}_{R_+}(M)_n\right) \cap \mathrm{Min}(R_0) = \emptyset$ *for $n << 0$.*

Then,

$$\Gamma_{m_0R}\,(H^i_{R_+}(M))$$

is an Artinian graded R-module.

Proof. The idea is to show that $\Gamma_{m_0R}\,(H^i_{R_+}(M))$ has a submodule $\mathcal{A}$ such that both $\mathcal{A}$ and $\frac{\Gamma_{m_0R}\,(H^i_{R_+}(M))}{\mathcal{A}}$ are Artinian R-modules.

First, we show that we can choose 2 elements x_0, y_0 in R_0 such that

$$\Gamma_{m_0R}(H^i_{R_+}(M)) = \Gamma_{y_0R}\left(\Gamma_{x_0R}(H^i_{R_+}(M))\right).$$

Define

$$U := \{\, p_t \,|\, p_t \in \mathrm{Ass}_{R_0}\,(H^{i-1}_{R_+}(M)_n) \text{ for some } n \text{ and } \mathrm{ht}_{R_0}\,(p_t) = 1\} \cup \{m_0\}. \tag{3.1}$$

Note that U is a finite set and we choose

$$x_0 \in \bigcap_{p_j \in U} p_j - \bigcup_{p_l \in \mathrm{Min}(R_0)} p_l. \tag{3.2}$$

Then, $\mathrm{ht}_{R_0}\,(x_0) = 1$. Since dim $R_0 = 2$, we can find $y_0 \in m_0$ such that

$$\sqrt{(x_0,\, y_0)R_0} = m_0. \tag{3.3}$$

We note that for all R-module T, and any ideal $\mathfrak{a}$ of R,

$$\Gamma_{\sqrt{\mathfrak{a}}}(T) = \Gamma_{\mathfrak{a}}(T).$$

Therefore,

$$\Gamma_{m_0R}(H^i_{R_+}(M)) = \Gamma_{\sqrt{(x_0,\,y_0)R}}(H^i_{R_+}(M)).$$

Since

$$\Gamma_{\sqrt{(x_0,\,y_0)R}}(H^i_{R_+}(M)) = \Gamma_{y_0R}\left(\Gamma_{x_0R}(H^i_{R_+}(M))\right),$$

we have

$$\Gamma_{m_0R}(H^i_{R_+}(M)) = \Gamma_{y_0R}\left(\Gamma_{x_0R}(H^i_{R_+}(M))\right). \tag{3.4}$$

Consider the exact sequence of graded R-modules (cf. [5], 13.1.12),

$$H^{i-1}_{R_+}(M)_{x_0} \xrightarrow{\ \Phi\ } H^i_{(R_+,\,x_0)}(M) \longrightarrow H^i_{R_+}(M) \longrightarrow H^i_{R_+}(M)_{x_0}. \tag{3.5}$$

The sequence (3.5) induces the exact sequence of graded R-modules,

$$0 \longrightarrow \text{Image } \Phi \longrightarrow H^i_{(R_+,\,x_0)}(M) \longrightarrow \Gamma_{x_0R}(H^i_{R_+}(M)) \longrightarrow 0. \tag{3.6}$$

We can then apply the left exact functor $\Gamma_{y_0R}(\bullet)$ to sequence (3.6) to get another exact sequence of graded R-modules (cf. [5], 1.2.2, 12.3.3),

$$\Gamma_{y_0R}\left(H^i_{(R_+,x_0)}(M)\right) \xrightarrow{\Psi} \Gamma_{y_0R}\left(\Gamma_{x_0R}(H^i_{R_+}(M))\right) \longrightarrow H^1_{y_0R}\left(\text{Image } \Phi\right). \quad (3.7)$$

By equation (3.4), sequence (3.7) can be written as

$$\Gamma_{y_0R}\left(H^i_{(R_+,x_0)}(M)\right) \xrightarrow{\Psi} \Gamma_{m_0R}(H^i_{R_+}(M)) \longrightarrow H^1_{y_0R}\left(\text{Image } \Phi\right). \quad (3.8)$$

We wish to show that both Image Ψ and Coker Ψ are Artinian.

First, we will show that Image Ψ is Artinian. With sequence (3.7),

$$\text{Image } \Psi \cong \frac{\Gamma_{y_0R}\left(H^i_{(R_+,x_0)}(M)\right)}{\text{Ker } \Psi}.$$

Moreover, equation (3.3) tells us that x_0, y_0 is a system of parameters for R_0, which implies that $\Gamma_{y_0R}\left(H^i_{(R_+,x_0)}(M)\right)$ is Artinian (cf. Lemma (3.1.1)). Therefore, Image Ψ is Artinian.

Finally, we will show that Coker Ψ is Artinian. We begin by proving that it is sufficient to prove that $(H^{i-1}_{R_+}(M)_{x_0})_n = (0)$ for all $n << 0$.

Suppose that $(H^{i-1}_{R_+}(M)_{x_0})_n = (0)$ for $n << 0$.

Claim: (Coker $\Psi)_n = (0)$ for $n << 0$.

Proof. By sequence (3.5), (Image $\Phi)_n = (0)$ for $n << 0$. Therefore, for $n << 0$, $\left(H^1_{y_0R}\left(\text{Image } \Phi\right)\right)_n = (0)$. This is because

$$\left(H^1_{y_0R}\left(\text{Image } \Phi\right)\right)_n \cong H^1_{y_0R_0}\left((\text{Image } \Phi)_n\right)$$

as R_0-modules (cf. [5], 13.1.10). Then, with sequence (3.8), (Coker $\Psi)_n = (0)$ for $n << 0$. □

Claim: Coker Ψ is Artinian.

Proof. Since $H^i_{R_+}(M)_n$ is a finitely generated R_0-module and $H^i_{R_+}(M)_n = (0)$ for $n >> 0$ (cf. [5], 15.1.5), $(\text{Coker } \Psi)_n = (0)$ for $n >> 0$. This is because Coker Ψ is a graded quotient of $\Gamma_{m_0R}\left(H^i_{R_+}(M)\right)$ (cf. sequence (3.8)).
With this and the preceding *Claim*, $(\text{Coker } \Psi)_n \neq (0)$ for only finitely many n. Since for each n, $(\text{Coker } \Psi)_n$ is a finitely generated R_0-module, we conclude that Coker Ψ is a Noetherian R-module. Furthermore,

$$\text{Supp}_R\ (\text{Coker } \Psi) \subseteq V(m_0 + R_+)$$

Since, Coker Ψ is a quotient of $\Gamma_{m_0R}(H^i_{R_+}(M))$. Consequently, Coker Ψ is Artinian. □

This shows that it is sufficient to prove that $(H^{i-1}_{R_+}(M)_{x_0})_n = (0)$ for all $n << 0$.

We will check that this condition is actually satisfied.

By the definition of U (cf. expression (3.1)), for $n << 0$,

$$\bigcap_{p_j \in U} p_j \subseteq \sqrt{\text{ann}_{R_0}\left(H^{i-1}_{R_+}(M)_n\right)}.$$

Note that U is a finite set by assumption (a) in the hypothesis. Then, for $n << 0$,

$$x_0 \in \sqrt{\text{ann}_{R_0}\left(H^{i-1}_{R_+}(M)_n\right)}$$

(cf. expression (3.2)). To see this, we consider the Graded Independence Theorem ([5], 13.1.6), it says that for all j,

$$H^j_{R_+}(M)_{x_0} \cong H^j_{R_+R_{x_0}}(M)_{x_0}$$

as graded R-modules and the isomorphism is homogeneous. Furthermore, by ([5], 15.2.2(iii)), for all $n \in \mathbb{Z}$ and j,

$$\left(H^j_{R_+R_{x_0}}(M)_{x_0}\right)_n \cong H^j_{R_+}(M)_n \otimes_{R_0} R_{x_0} \tag{3.9}$$

as $(R_0)_{x_0}$ (and hence R_0)-modules. By the choice of x_0,

$$H^{i-1}_{R_+}(M)_n \otimes_{R_0} R_{x_0} = (0)$$

as a R_0-module for $n << 0$. Hence, for $n << 0$,

$$(H^{i-1}_{R_+}(M)_{x_0})_n = (0)$$

(cf. isomorphism (3.9)). $\square$

3.2 When is $H^i_{R_+}(M)$ asymptotically gap free?

We will give some sufficiency conditions for $H^i_{R_+}(M)$ to be asymptotically gap free.

Proposition 3.2.1. *Assume the standard hypothesis on R, R_0 and M in this chapter. Fix $i \in \mathbb{N}_0$. Suppose that*

(a) $\operatorname{Ass}_R \left(H^{i-1}_{R_+}(M)\right)$ *is finite;*

(b) $\operatorname{Ass}_{R_0} \left(H^{i-1}_{R_+}(M)_n\right) \cap \operatorname{Min}(R_0) = \emptyset$ *for $n << 0$;*

(c) $\operatorname{Ass}_{R_0} \left(H^i_{R_+}(M)_n\right) \subseteq \{m_0\}$ *for $n << 0$.*

Then,

$$H^i_{R_+}(M) = \mathcal{A} + \mathcal{N}$$

where $\mathcal{A}$ and $\mathcal{N}$ are both graded R-submodules of $H^i_{R_+}(M)$; $\mathcal{A}$ is Artinian and $\mathcal{N}$ is Noetherian as R-modules.

In particular, by Lemma (3.0.1), $H^i_{R_+}(M)$ is asymptotically gap free.

Proof. Define

$$\mathcal{A} := \Gamma_{m_0 R}\left(H^i_{R_+}(M)\right).$$

By Proposition (3.1.2), $\mathcal{A}$ is an Artinian graded R-module. According to ([5], 13.1.10), for all $n \in \mathbb{Z}$,

$$\mathcal{A}_n = \Gamma_{m_0 R_0}\left(H^i_{R_+}(M)_n\right).$$

Then, condition (c) in the hypothesis and the maximality of m_0 in R_0 implies that for $n << 0$,

$$\Gamma_{m_0 R_0}\left(H^i_{R_+}(M)_n\right) = H^i_{R_+}(M)_n.$$

Therefore, there exists $\Omega \in \mathbb{Z}$ such that for all $n < \Omega$,

$$\mathcal{A}_n = H^i_{R_+}(M)_n.$$

Define

$$\mathcal{N} := \bigoplus_{n \geq \Omega} H^i_{R_+}(M)_n$$

Since $H^i_{R_+}(M)_n = (0)$ for $n >> 0$ and for all $n \in \mathbb{Z}$, $H^i_{R_+}(M)_n$ is a finitely generated R_0-module (cf. [5], 15.1.5), $\mathcal{N}$ is a graded Noetherian R-module (R is positively graded).

It is now clear that we can write $H^i_{R_+}(M)$ as $\mathcal{A} + \mathcal{N}$. □

We will use Proposition (3.2.1) to prove the last Proposition of this chapter.

Proposition 3.2.2. *Assume the standard hypothesis on R, R_0 and M in this chapter. Let $x \in R$ be a homogeneous element of degree 1 which is also M-regular. Fix $i \in \mathbb{N}_0$. Suppose that*

(a) $\mathrm{Ass}_R\left(H^{i-1}_{R_+}(\frac{M}{xM})\right)$ *is a finite set;*

(b) $\mathrm{Ass}_{R_0}\left(H^{i-1}_{R_+}(\frac{M}{xM})_n\right) \cap \mathrm{Min}(R_0) = \emptyset$ *for $n << 0$;*

(c) $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ *is a finite set.*

Then, $H^i_{R_+}(M)$ is asymptotically gap free.

Proof. We may assume that $H^i_{R_+}(M)_n \neq (0)$ for infinitely many $n < 0$. Otherwise, $H^i_{R_+}(M)$ is Noetherian (cf. [5], 15.1.5) and hence, by Lemma (3.0.1), it is asymptotically gap free.

We may also assume that $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) \subseteq \{m_0\}$ for $n << 0$. Otherwise, there exists a $p \in \mathrm{Spec}(R_0)$ of height 0 or 1, such that $p \in \mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ for infinitely many $n < 0$; by Proposition (1.1.1), $H^i_{R_+}(M)_n \otimes_{R_0} (R_0)_p \neq 0$ for $n << 0$ since dimension of $(R_0)_p$ is 1. Hence, $H^i_{R_+}(M)$ is asymptotically gap free.

Since for all $n \in \mathbb{Z}$, $H^i_{R_+}(M)_n$ is a finitely generated R_0-module (cf. [5], 15.1.5);

$$\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right) \subseteq \mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$$

and m_0 is the maximal ideal of R_0, we are in fact assuming

$$\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right) \subseteq \{m_0\}. \tag{3.10}$$

We will continue with the proof with this additional assumption.

By our hypothesis, we have the exact sequence of graded R-modules,

$$0 \longrightarrow M(-1) \xrightarrow{\ x\ } M \longrightarrow \frac{M}{xM} \longrightarrow 0.$$

Then, the sequence above induces another exact sequence of graded R-modules,

$$H^i_{R_+}(M)(-1) \xrightarrow{\ x\ } H^i_{R_+}(M) \longrightarrow H^i_{R_+}\left(\tfrac{M}{xM}\right). \tag{3.11}$$

Put

$$\mathcal{C} = \frac{H^i_{R_+}(M)}{xH^i_{R_+}(M)}.$$

Claim: It suffices to show that $\mathcal{C}$ is asymptotically gap free.

Proof. If $\mathcal{C}_n \neq (0)$ for $n << 0$: Then $H^i_{R_+}(M)_n \neq (0)$ for $n << 0$.

If $\mathcal{C}_n = (0)$ for $n << 0$: Then for $n << 0$, we have an exact sequence of R_0-modules,

$$H^i_{R_+}(M)_n \xrightarrow{\ x\ } H^i_{R_+}(M)_{n+1} \longrightarrow 0$$

Hence, $H^i_{R_+}(M)_{n+1} \neq (0)$ implies that $H^i_{R_+}(M)_n \neq (0)$ for all $n << 0$. □

We will now show that $\mathcal{C}$ is asymptotically gap free.

Define

$$\mathcal{A} := H^i_{R_+}\left(\frac{M}{xM}\right)$$

Note that with sequence (3.11), $\mathcal{C}$ can (and will) be identified as a graded submodule of $\mathcal{A}$. We also note that $\operatorname{Supp}_{R_0}(\mathcal{C}_n) \subseteq \{m_0\}$ for $n << 0$. This is because

$$\operatorname{Supp}_{R_0}(\mathcal{C}_n) \subseteq \operatorname{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)$$

and expression (3.10). Hence, there exists $N \in \mathbb{Z}$ such that for all $n < N$,

$$\mathcal{C}_n \subseteq \left(\Gamma_{m_0R}(\mathcal{A})\right)_n. \tag{3.12}$$

For this, we recall that for all $n \in \mathbb{Z}$,

$$\left(\Gamma_{m_0R}(\mathcal{A})\right)_n = \Gamma_{m_0R_0}(\mathcal{A}_n).$$

With expression (3.12), we can write

$$\mathcal{C} = \left(\mathcal{C} \cap \Gamma_{m_0R}(\mathcal{A})\right) + \bigoplus_{n \geq N} \mathcal{C}_n. \tag{3.13}$$

By Proposition (3.1.2), $\Gamma_{m_0R}(\mathcal{A})$ is Artinian. Therefore, its submodule, $\mathcal{C} \cap \Gamma_{m_0R}(\mathcal{A})$, is also Artinian. By Lemma (3.0.1), it is asymptotically gap free. More importantly, by equation (3.13), for all $n < N$,

$$\mathcal{C}_n = \left(\mathcal{C} \cap \Gamma_{m_0R}(\mathcal{A})\right)_n$$

Therefore, $\mathcal{C}$ is asymptotically gap free. □

It will be more meaningful to see that the hypotheses of the two propositions that we just proved are satisfied by some concrete rings and modules. To accomplish that, we will apply the propositions to prove one of our main Theorems in the following chapter.

4 Applications

We will appply our results in the previous chapter to prove the second main result in this project:

Theorem 4.0.1. *Suppose that both R and M are Cohen-Macaulay. Assume also that (R_0, m_0) is local, equidimensional and* $\dim R_0 = 2$. *Then, for all i, $H^i_{R_+}(M)$ is asymptotically gap free.*

In Section (4.1), we will show that to prove Theorem (4.0.1), it suffices to prove the Theorem with the additional hypothesis that the residue field, $\frac{R_0}{m_0}$, is infinite. This condition will guarantee us a homogeneous M-regular element, x, of degree 1 whenever $\text{grade}(R_+, M) \geq 1$. In the context of Proposition (3.2.2), such an element will be useful. We will not be needing this additional assumption till we actually give a proof of Theorem (4.0.1) in Section (4.4).

In Section (4.2), we will state most of the Lemmas we proved in Chapter (1) under the hypotheses of Theorem (4.0.1). We will need these Lemmas to perform some of the calculations in Section (4.3). We will also include the three main results of Brodmann and Hellus, which we have constantly utilised. The reader can refer to Chapter (1) and [2] for all the proofs.

In Section (4.3), we will analyze the support of $H^i_{R_+}(M)$. From Corollary (2.6.2), we know that $\text{Ass}_R\left(H^i_{R_+}(M)\right)$ is a finite set under the assumptions of Theorem (4.0.1). In fact, we will show that for all $i > \text{grade}(R_+, M)$ and $n \in \mathbb{Z}$,

$$p \in \text{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right) \text{ implies that } \text{ht}_{R_0}(p) \geq 1.$$

This will give a situation where condition (b) of Proposition (3.2.1) is satisfied.

In Section (4.4), we will prove Theorem (4.0.1) using the results in the preceding two sections.

In Section (4.5), we will give a criterion for the vanishing of $H^i_{R_+}(M)$ using the analysis made in Section (4.3).

We begin by proving the following Proposition that deals with the case $\mathrm{ht}(I \cap R_0) \geq 1$, where $I = \sqrt{\mathrm{ann}_R(M)}$. This is essentially due to Brodmann and Hellus's result, Proposition (1.1.1).

Proposition 4.0.1. *Assume the standard hypothesis on R and M. Suppose that R_0 is local and* $\dim R_0 = 2$. *Let $I = \sqrt{\mathrm{ann}_R(M)}$. If* $\mathrm{ht}\ (I \cap R_0) \geq 1$, *then for all i, $H^i_{R_+}(M)$ is asymptotically gap free.*

Proof. Fix $i \in \mathbb{N}_0$. Let $I' = \mathrm{ann}_R(M)$ and $R' = \frac{R}{I'}$. Then, M is a finitely generated graded R'-module. R' is a positively graded homogeneous Noetherian ring. R'_0 is local and $\dim R'_0 \leq 1$. By Proposition (1.1.1), $H^i_{R'_+}(M)$ is asymptotically gap free as a R' (and hence, R)-module. By the Graded Independence Theorem ([5], 13.1.6), $H^i_{R_+}(M)$ is asymptotically gap free. □

With that, the more interesting part of our work will be in the case, $\mathrm{ht}\ (I \cap R_0) = 0$. This will be the main focus in section 4.3.

4.1 Reduction to that case: $|\frac{R_0}{m_0}|$ is infinite

We would like to use a standard construction (cf. [5], 15.2.4) to show that to prove Theorem (4.0.1), we may assume additionally that the residue field, $\frac{R_0}{m_0}$, is infinite. This construction does not use the Cohen-Macaulay condition on R or M, and the equidimensionality of R_0. We do assume that (R_0, m_0) is local. Put

$$R'_0 = R_0[X]_{m_0 R_0[X]}$$

where X is an indeterminate over R. Then, R'_0 is a faithfully flat local R_0-algebra with an infinite residue field. Furthermore, both rings have the same Krull dimension.

Put

$$R' = R \otimes_{R_0} R'_0 \qquad M' = M \otimes_R R'$$

$$m = m_0 + R_+ \qquad m' = m_0 R_0[x] + (R_+ \otimes_{R_0} R'_0).$$

Then, (R', m') is a positively graded homogeneous and graded local Noetherian ring with R'_0 as its 0-th component. M' is a finitely generated graded R'-module. By ([5], 15.2.2(iv)), for all $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$,

$$H^i_{R_+}(M)_n \otimes_{R_0} R'_0 \cong H^i_{R_+R'}(M')_n \tag{4.1}$$

as R'_0-modules. Furthermore, R'_0 being faithfully flat over R_0 implies that for every n, $H^i_{R_+}(M)_n = (0)$ if and only if $H^i_{R_+R'}(M')_n = (0)$. Therefore, $H^i_{R_+}(M)$ is asymptotically gap free if and only if $H^i_{R_+R'}(M')$ is asymptotically gap free.

Lemma 4.1.1. *Assume that we have R, R_0, M, R', R'_0, M' as above. Then*

$$\text{grade}(R_+, M) = \text{grade}\,(R'_+, M').$$

Proof. let $g = \text{grade}(R_+, M)$ and $g' = \text{grade}\,(R'_+, M')$. If $R_+M = M$, then by a graded version of Nakayama's Lemma (cf. [6], 1.5.24(a)), $M = (0)$. Then, $M' = (0)$. Hence, by definition, $g = \infty$ and $g' = \infty$.

Consider the case that $R_+M \neq M$. Then, $g \in \mathbb{N}_0$. By Proposition (1.1.3)(a) and (b), for all $i < g$, $H^i_{R_+}(M) = (0)$ and $H^g_{R_+}(M) \neq (0)$. Hence, by the isomorphism in expression (4.1), for all $i < g$, $H^i_{R'_+}(M') = (0)$ and $H^g_{R'_+}(M') \neq (0)$. Then, $R'_+M' \neq M'$. Another application of Proposition (1.1.3)(a) and (b) to R' and M', we get $g = g'$. □

Proposition 4.1.2. *If both R and M are Cohen-Macaulay, then both R' and M' are Cohen-Macaulay.*

Proof. Since R' is graded local, by ([6], 2.1.27), to show that both R' and M' are Cohen-Macaulay, it suffices to show that both $R'_{m'}$ and $M'_{m'}$ are Cohen-Macaulay. Consider the ring extension $R_m \longrightarrow R'_{m'}$. The extension is local, flat and its fibre

$$\frac{R'_{m'}}{mR'_{m'}}$$

is a field. Then by ([6]. 1.2.16), for every finitely generated R_m-module, N, we have

$$\operatorname{depth}_{R'_{m'}} N \otimes_{R_m} R'_{m'} = \operatorname{depth}_{R_m} N + \operatorname{depth}_{R'_{m'}} \frac{R'_{m'}}{mR'_{m'}}$$

and

$$\dim_{R'_{m'}} N \otimes_{R_m} R'_{m'} = \dim_{R_m} N + \dim_{R'_{m'}} \frac{R'_{m'}}{mR'_{m'}}$$

Since $\frac{R'_{m'}}{mR'_{m'}}$ is a field, $\operatorname{depth}_{R'_{m'}} \frac{R'_{m'}}{mR'_{m'}}$ and $\dim_{R'_{m'}} \frac{R'_{m'}}{mR'_{m'}}$ are both zero. If we substitute R_m (resp. M_m) for N, we will have that $R'_{m'}$ (resp. $M'_{m'}$) is Cohen-Macaulay.

□

Finally, we would like to show that if R is Cohen-Macaulay and R_0 is local, then R_0 is equidimensional if and only if R'_0 is equidimensional. By Lemma (1.2.7), this is equivalent to the following Proposition.

Proposition 4.1.3. *If R is Cohen-Macaulay and R_0 is local, then all minimal primes of R_+ have the same height if and only if all minimal primes of R'_+ have the same height.*

Proof. Let $p' \in \operatorname{Min}(R'_0)$ and $p' \cap R_0 := p$. By going-down (flat extension), $p \in \operatorname{Min}(R_0)$. In our case, $pR'_0 = p'$ because pR'_0 is a prime ideal in R'_0. Note that, if $q \in \operatorname{Min}(R_0)$, then $qR'_0 := q'$ is in $\operatorname{Min}(R'_0)$. Hence, there is one-to-one correspondence between $\operatorname{Min}(R_0)$ and $\operatorname{Min}(R'_0)$, via contraction and extension.

We put $P = p + R_+$ and $P' = p' + R'_+$. Note that P and P' are in $\operatorname{Min}(R_+)$ and $\operatorname{Min}(R'_+)$ respectively.

Now, consider the extension

$$R_P \longrightarrow R'_{P'}$$

This is a local and flat extension whose fibre $\frac{R'_{P'}}{PR'_{P'}}$ is a field. By ([6], 1.2.16),

$$\dim R'_{P'} = \dim R_P + \dim \frac{R'_{P'}}{PR'_{P'}}$$

Therefore,

$$\text{ht } P' = \text{ht } P.$$

□

4.2 Auxiliary tools for Theorem (4.0.1)

We collect all the preliminary tools we need in order to do the analysis in Section (4.3). The proof of these Lemmas and Propositions can be found in Chapter (1). We recall the assumptions in Theorem (4.0.1):

Suppose that both R and M are Cohen-Macaulay. Assume also that (R_0, m_0) is local, equidimensional and $\dim R_0 = 2$.

Lemma 4.2.1. *Assumptions as in Theorem (4.0.1). $I = \sqrt{\text{ann}_R(M)}$. Then, all minimal primes of I have the same height.*

Proof. We refer to Lemma (1.2.2). □

Lemma 4.2.2. *Assumptions as in Theorem (4.0.1). $I = \sqrt{\text{ann}_R(M)}$. Let $p \in$* $\text{Spec}(R_0)$ *and $p \supseteq (I \cap R_0)$. If there exists $W \in$* $\text{Min}(I)$ *such that $W \cap R_0 = p$, then*

$$\text{ht } (p + I)R_p = \text{ht } I.$$

Proof. We refer to Lemma (1.2.3). □

Lemma 4.2.3. *Assumptions as in Theorem (4.0.1).* $I = \sqrt{\operatorname{ann}_R(M)}$.

Then, if $q \in \operatorname{Spec}(R_0)$ *and* $q \supseteq (I \cap R_0)$, *then*

$$\dim \frac{M_q}{qM_q} \leq \text{ht } (q + R_+) - \text{ht } (q + I)R_q.$$

In particular,

$$\dim \frac{M}{m_0 M} \leq \text{ht } (m_0 + R_+) - \text{ht } (m_0 + I).$$

Proof. We refer to Lemma (1.2.6). □

Lemma 4.2.4. *Assumptions as in Theorem (4.0.1).*

Then, all minimal primes of R_+ *have the same height.*

Proof. R_0 is equidimensional implies that all minimal primes of R_+ have the same height (cf. Lemma (1.2.7)). □

Lemma 4.2.5. *Assumptions as in Theorem (4.0.1).*

Then, if $q \in \operatorname{Spec}(R_0)$ *and* ht $q = i$, *then*

$$\text{ht } (q + R_+) = \text{ht } (R_+) + i.$$

Proof. We refer to Lemma (1.2.8). □

Lemma 4.2.6. *Assumptions as in Theorem (4.0.1).* $I = \sqrt{\operatorname{ann}_R(M)}$.

If ht $(I \cap R_0) = i$, *then*

$$\text{grade } (R_+, M) = \text{ht } (R_+) + i - \text{ht } I.$$

Proof. We refer to Lemma (1.2.9). □

Lemma 4.2.7. *Assumptions as in Theorem (4.0.1).* $I = \sqrt{\operatorname{ann}_R(M)}$.

If $q \in \operatorname{Spec}(R_0)$ *and* $q \supseteq (I \cap R_0)$, *then*

$$\text{grade}(R_+, M) \leq \text{grade}(R_+R_q, M_q).$$

Proof. We refer to Lemma (1.2.11). □

Proposition 4.2.8. *(Brodmann-Hellus) Suppose that* $\dim R_0 \leq 1$*, and* R_0 *is semilocal. Then, for every finitely generated graded R-module* M*,* $H^i_{R_+}(M)$ *is asymptotically gap free.*

Proof. (cf. [2], 4.2) □

Proposition 4.2.9. *(Brodmann-Hellus) Suppose that* (R_0, m_0) *is local and* M *is finitely generated graded R-module with* $\dim \frac{M}{m_0 M} = d$*. If* $M \neq \Gamma_{R_+}(M)$*, then*

(a) $H^d_{R_+}(M) \neq (0)$ *and*

(b) $H^i_{R_+}(M)$ *for all* $i > d$*.*

Proof. (cf. [2], 3.4) □

Proposition 4.2.10. *Suppose that* $R_+M \neq M$*. Let* $g = \text{grade}(R_+, M)$*. Then:*

(a) For all $i < g$*,* $H^i_{R_+}(M) = (0)$ *;*

(b) for $i = g$*,* $H^i_{R_+}(M) \neq (0)$*;*

(c) (Brodmann-Hellus) if $H^i_{R_+}(M)$ *is finitely generated over* R *for all* $i < t$*, then*

$$\{\text{Ass}_{R_0}(H^t_{R_+}(M)_n)\}_{n \in \mathbf{Z}}$$

is asymptotically stable.

Proof. For (a) and (b), (cf. [5], 6.2.7). For (c), (cf. [2], 5.6). □

4.3 The support of $H^i_{R_+}(M)$

We recall the assumptions in Theorem (4.0.1).

Suppose that both R *and* M *are Cohen-Macaulay. Assume also that* (R_0, m_0) *is local, equidimensional and* $\dim R_0 = 2$.

Let $I = \sqrt{\text{ann}_R(M)}$ as before. For a prime, $q \in \text{Spec}(R_0)$, containing $(I \cap R_0)$, we begin by analyzing the two dichotomy:

(a) $q \neq (W \cap R_0)$ for all $W \in \mathrm{Min}(I)$;

(b) $q = (W \cap R_0)$ for some $W \in \mathrm{Min}(I)$.

We do this by case study in terms of the height of q. These findings are summarized in Proposition (4.3.4).

In view of Proposition (4.0.1), the case ht $(I \cap R_0) = 0$ is at the heart of the issue here.

Lemma 4.3.1. *Assumptions as in Theorem (4.0.1).* $I = \sqrt{\mathrm{ann}_R(M)}$.
Let $q \in \mathrm{Min}(R_0)$.

(a) If for all $W \in \mathrm{Min}(I)$, $W \cap R_0 \neq q$, *then* $M_q = (0)$.

(b) If there exists $W \in \mathrm{Min}(I)$ *such that* $W \cap R_0 = q$, *then*

$$\dim \frac{M_q}{qM_q} \leq \mathrm{grade}(R_+,\, M).$$

Proof. For (a), suppose that for all $W \in \mathrm{Min}(I)$, $W \cap R_0 \neq q$. Then, $q \not\supseteq (I \cap R_0)$. Otherwise, $(q + R_+)$ contains a $W \in \mathrm{Min}(I)$ such that $W \cap R_0 = q$ because $q \in \mathrm{Min}(R_0)$. This is contrary to assumption. In particular, $(I \cap R_0) \cap (R_0 - q) \neq \emptyset$. Therefore, $M_q = (0)$.

For (b), suppose that there exists $W \in \mathrm{Min}(I)$ such that $W \cap R_0 = q$. Then, ht $(I \cap R_0) = 0$. By Lemma (4.2.6),

$$\mathrm{grade}(R_+,\, M) = \mathrm{ht}\ (R_+) - \mathrm{ht}\ I.$$

According to Lemma (4.2.3),

$$\dim \frac{M_q}{qM_q} \leq \mathrm{ht}\ (q + R_+) - \mathrm{ht}\ (q + I)R_q.$$

By Lemma (4.2.2),

$$\mathrm{ht}\ (q + I)R_q = \mathrm{ht}\ I.$$

By Lemma (4.2.5) and $q \in \mathrm{Min}(R_0)$,

$$\mathrm{ht}\ (q + R_+) = \mathrm{ht}\ R_+.$$

Therefore, putting the preceding four expressions together, we get

$$\dim \frac{M_q}{qM_q} \leq \mathrm{grade}(R_+,\ M).$$

□

Lemma 4.3.2. *Assumptions as in Theorem (4.0.1).* $I = \sqrt{\mathrm{ann}_R(M)}$. *Assume also that* $\mathrm{ht}\ (I \cap R_0) = 0$. *Let* $p \in \mathrm{Spec}^1(R_0)$ *and* $p \supset (I \cap R_0)$.

(a) If for all $W \in \mathrm{Min}(I)$, $W \cap R_0 \neq p$, *then*

$$\dim \frac{M_p}{pM_p} \leq \mathrm{grade}(R_+,\ M).$$

(b) If there exists $W \in \mathrm{Min}(I)$ *with* $W \cap R_0 = p$, *then*

$$\dim \frac{M_p}{pM_p} \leq \mathrm{grade}(R_+,\ M) + 1.$$

Proof. For (a), by lemma (4.2.6), it suffices to show that

$$\dim \frac{M_p}{pM_p} \leq \mathrm{ht}\ (R_+) - \mathrm{ht}\ I.$$

According to Lemma (4.2.3),

$$\dim \frac{M_p}{pM_p} \leq \mathrm{ht}\ (p + R_+) - \mathrm{ht}\ (p + I)R_p. \tag{4.2}$$

By assumptions,

$$\mathrm{ht}\ (p + I)R_p \geq \mathrm{ht}\ (I) + 1.$$

Since $p \in \mathrm{Spec}^1(R_0)$,

$$\mathrm{ht}\ (p + R_+) = \mathrm{ht}\ (R_+) + 1. \tag{4.3}$$

(cf. Lemma (4.2.5)). Therefore, putting the preceding three expressions together, we get

$$\dim \frac{M_p}{pM_p} \leq \text{ht } (R_+) - \text{ht } I.$$

For (b), we note that we still have inequality (4.2) and equation (4.3). By Lemma (4.2.2),

$$\text{ht } (p + I)R_p = \text{ht } (I).$$

Putting these three expressions together, we get

$$\dim \frac{M_p}{pM_p} \leq \text{ht } (R_+) - \text{ht } (I) + 1$$

Since grade$(R_+, M) = \text{ht } (R_+) - \text{ht } I$ (cf. Lemma (4.2.6)), we have our contention.

□

Lemma 4.3.3. *Assumptions as in Theorem (4.0.1).* $I = \sqrt{\text{ann}_R(M)}$. *Assume also that* $\text{ht } (I \cap R_0) = 0$.

(a) If for all $W \in \text{Min}(I)$, $W \cap R_0 \neq m_0$, *then*

$$\dim \frac{M}{m_0 M} \leq \text{grade}(R_+, M) + 1.$$

(b) If there exists $W \in \text{Min}(I)$ *with* $W \cap R_0 = m_0$, *then*

$$\dim \frac{M}{m_0 M} \leq \text{grade}(R_+, M) + 2.$$

Proof. For (a), by lemma (4.2.6), it suffices to show that

$$\dim \frac{M}{m_0 M} \leq \text{ht } (R_+) - \text{ht } (I) + 1.$$

According to Lemma (4.2.3),

$$\dim \frac{M}{m_0 M} \leq \text{ht } (m_0 + R_+) - \text{ht } (m_0 + I). \tag{4.4}$$

By assumptions,

$$\text{ht } (m_0 + I) \geq \text{ht } (I) + 1. \tag{4.5}$$

Since $m_0 \in \text{Spec}^2(R_0)$,

$$\text{ht } (m_0 + R_+) = \text{ht } (R_+) + 2. \tag{4.6}$$

(cf. Lemma (4.2.5)). Therefore, putting inequality (4.4), equations (4.5) and (4.6) together, we have

$$\dim \frac{M}{m_0 M} \leq \text{ht } (R_+) - \text{ht } (I) + 1.$$

For (b), we note that we still have inequality (4.4) and equation (4.6). By Lemma (4.2.2),

$$\text{ht } (m_0 + I)R_{m_0} = \text{ht } (I). \tag{4.7}$$

Putting these three expressions together, we get

$$\dim \frac{M}{m_0 M} \leq (\text{ht } R_+) - \text{ht } (I) + 2$$

Since $\text{grade}(R_+,\, M) = \text{ht } (R_+) - \text{ht } I$ (cf. Lemma (4.2.6)), we have our contention.

□

Assumptions as in Theorem (4.0.1). As before, $I = \sqrt{\text{ann}_R (M)}$ and $\text{ht } (I \cap R_0) = 0$. Let $i \in \mathbb{N}_0$. According to the three Lemmas just shown, for a prime p to be in $\text{Supp}_{R_0} \left(H^i_{R_+}(M)_n\right)$, it is important to know whether there exists $W \in \text{Min}(I)$ such that $W \cap R_0 = p$. This condition determines an upper bound on $\dim \frac{M_p}{pM_p}$ in terms of $\text{grade}(R_+,\, M)$. For its relevance, we recall that if $H^i_{R_+R_p}(M_p) \neq (0)$, then

$$\text{grade}(R_+R_p,\, M_p) \leq i \leq \dim \frac{M_p}{pM_p}$$

(cf. Proposition (4.2.9)). Moreover, by Lemma (4.2.7),

$$\operatorname{grade}(R_+, M) \leq \operatorname{grade}(R_+R_p, M_p).$$

Hence, if $H^i_{R_+R_p}(M_p) \neq (0)$, then

$$\operatorname{grade}(R_+, M) \leq i \leq \dim \frac{M_p}{pM_p}.$$

Define

$$\Delta := \{p \in \operatorname{Spec}^1(R_0) \mid \text{there exists } W \in \operatorname{Min}(I) \text{ with } W \cap R_0 = p\}.$$

If $\Delta \neq \emptyset$, we put $\Delta := \{p_1 \dots p_s\}$. We further define

$$\widetilde{\Delta} := \begin{cases} \Delta & \text{if for all } W \in \operatorname{Min}(I),\ W \cap R_0 \neq m_0 \\ \Delta \cup \{m_0\} & \text{if there exists } W \in \operatorname{Min}(I) \text{ with } W \cap R_0 = m_0. \end{cases}$$

Notation: For $i \in \mathbb{N}_0$, we denote

$$\mathcal{S}_i := \bigcup_{n \in \mathbf{Z}} \operatorname{Supp}_{R_0} \left(H^i_{R_+}(M)_n \right).$$

Proposition 4.3.4. *Assumptions as in Theorem (4.0.1). Let $g = \operatorname{grade}(R_+, M)$. If $\operatorname{ht}(I \cap R_0) = 0$ then,*

(a) $H^i_{R_+}(M) = (0)$ for all $i > g + 2$ and $i < g$.

(b)

$$\mathcal{S}_{g+1} \subseteq \begin{cases} V(p_1 \cap \dots \cap p_s) & \text{if } \Delta \neq \emptyset; \\ V(m_0) = \{m_0\} & \text{if } \widetilde{\Delta} \subseteq \{m_0\}. \end{cases}$$

(c)

$$\mathcal{S}_{g+2} \subseteq \begin{cases} \{m_0\} & \text{if } m_0 \in \widetilde{\Delta}; \\ \emptyset & \text{if } m_0 \notin \widetilde{\Delta}. \end{cases}$$

In particular, for $i = g + 1,\ g + 2$, and for all $n \in \mathbb{Z}$,

$$\operatorname{Supp}_{R_0} \left(H^i_{R_+}(M)_n \right) \cap \operatorname{Min}(R_0) = \emptyset;$$

$$\left| \bigcup_{n \in \mathbf{Z}} \operatorname{Supp}_{R_0} \left(H^i_{R_+}(M)_n \right) \right| < \infty.$$

Proof. We note that $g \in \mathbb{N}_0$ because M is Cohen-Macaulay implies that $M \neq (0)$ and hence, by a graded version of Nakayama Lemma ([6], 1.5.24(a)), $R_+M \neq M$. We may also assume that $M \neq \Gamma_{R_+}(M)$. Otherwise $H^i_{R_+}(M) = (0)$ for all $i > 0$.
We note that $H^i_{R_+}(M) = (0)$ if and only if $\mathcal{S}_i = \emptyset$.

For (a), by Lemma (4.3.3), $\dim \frac{M}{m_0 M} \leq g + 2$. Moreover, by Proposition (4.2.9), for all $i > \dim \frac{M}{m_0 M}$, $H^i_{R_+}(M) = (0)$. Therefore, $H^i_{R_+}(M) = (0)$ for all $i > g + 2$. For $i < g$, $H^i_{R_+}(M) = (0)$ (cf. Proposition (4.2.10)(a)).

For (b) and (c), let $q \in \operatorname{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right) \cap \operatorname{Min}(R_0)$. We want to show that $i \leq g$. Note that, this is clear if $i = 0$ because $g \in \mathbb{N}_0$. We consider the situation where $i \geq 1$. By the Flat Base Change Theorem ([5], 13.1.8), $H^i_{R_+R_q}(M_q) \neq (0)$. Then, $M_q \neq \Gamma_{R_+R_q}(M_q)$ and $M_q \neq (0)$. By the contrapositive statement of Lemma (4.3.1)(a), there exists $W \in \operatorname{Min}(I)$ such that $W \cap R_0 = q$. By Lemma (4.3.1)(b), $\dim \frac{M_q}{qM_q} \leq g$. Since we also have $i \leq \dim \frac{M_q}{qM_q}$ by Proposition (4.2.9). Hence, $i \leq g$.

For (b), we will first show that if $p \in \mathcal{S}_{g+1} \cap \operatorname{Spec}^1(R_0)$, then $p \in \Delta$.
Let $p \in \mathcal{S}_{g+1} \cap \operatorname{Spec}^1(R_0)$. Then, $M_p \neq (0)$ and hence, $p \supseteq (I \cap R_0)$. Furthermore, $g + 1 > 0$ implies that $\Gamma_{R_+R_p}(M_p) \neq M_p$. By the Flat Base Change Theorem ([5], 13.1.8) and the choice of p, $H^{g+1}_{R_+R_p}(M_p) \neq (0)$. By Proposition (4.2.9), $\dim \frac{M_p}{pM_p} \geq g + 1$. By the contrapositive statement of Lemma (4.3.2)(a), there exists $W \in \operatorname{Min}(I)$ such that $W \cap R_0 = p$. Therefore, $p \in \Delta$.
Suppose $\Delta = \emptyset$. Then, $\mathcal{S}_{g+1} \cap \operatorname{Spec}^1(R_0) = \emptyset$. If $u \in \mathcal{S}_{g+1}$, then $u = m_0$ because R_0 is local and $\dim R_0 = 2$.

For (c), we want to show that if $v \in \mathcal{S}_{g+2}$, then $v = m_0$.
Let $v \in \mathcal{S}_{g+2}$. Then, $M_v \neq (0)$ and hence, $v \supseteq (I \cap R_0)$. We also have $H^{g+2}_{R_+R_v}(M_v) \neq (0)$ by the Flat Base Change Theorem ([5], 13.1.8). By Proposition (4.2.9), $\dim \frac{M_v}{vM_v} \geq g+2$. By Lemma (4.3.1) and (4.3.2), $\operatorname{ht}_{R_0}(v) \geq 2$. Hence, $v = m_0$ because $\dim R_0 = 2$ and R_0 is local.

Moreover, there exists $W \in \text{Min}(I)$ such that $W \cap R_0 = v$. Otherwise, by Lemma (4.3.3)(a), dim $\frac{M_v}{vM_v} \leq g+1$. a contradiction to dim $\frac{M_v}{vM_v} \geq g+2$.
Therefore, if $H^{g+2}_{R_+}(M) \neq (0)$, then $m_0 \in \widetilde{\Delta}$.

Finally, to see that

$$\left|\bigcup_{n\in\mathbf{Z}} \text{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)\right| < \infty.$$

for all $i > g$. We note that both $V(p_1 \cap \ldots \cap p_s)$ and $V(m_0)$ are finite because for all $1 \leq j \leq s$, $p_j \in \text{Spec}^1(R_0)$; R_0 is local and dim $R_0 = 2$. □

There is another ramification of this analysis besides an application of Proposition (3.2.1) to the proof of Theorem (4.0.1). We will explore it in Section (4.5), after the proof of Theorem (4.0.1).

4.4 Proof of Theorem (4.0.1)

Theorem 4.0.1. *Suppose that both R and M are Cohen-Macaulay. Assume also that R_0 local, equidimensional and* dim $R_0 = 2$. *Then, for all i, $H^i_{R_+}(M)$ is asymptotically gap free.*

Proof. We may assume that $\Gamma_{R_+}(M) \neq M$. Otherwise, $H^i_{R_+}(M) = (0)$ for all $i > 0$. Let $g = \text{grade}(R_+, M)$. $M \neq (0)$ implies that $g \in \mathbb{N}_0$. As before, $I = \sqrt{\text{ann}_R(M)}$. The case where ht $(I \cap R_0) \geq 1$ is dealt with in Proposition (4.0.1). It remains to consider the case ht $(I \cap R_0) = 0$. By Proposition (4.3.4), for all $i > g$ and $n \in \mathbb{Z}$,

$$\text{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right) \cap \text{Min}(R_0) = \emptyset \tag{4.8}$$

and

$$\left|\bigcup_{n\in\mathbf{Z}} \text{Supp}_{R_0}\left(H^i_{R_+}(M)_n\right)\right| < \infty. \tag{4.9}$$

By Proposition (4.2.10)(c) and (4.3.4)(a), it remains to check for $i = g+1$ and $i = g+2$.

Case 1: $i = g+2$. $H^{g+2}_{R_+}(M)$ is asymptotically gap free because the hypotheses of Proposition (3.2.1) are satisfied:

(a) $\mathrm{Ass}_R\left(H^{g+1}_{R_+}(M)\right)$ is finite (cf. expression (4.9)).

(b) $\mathrm{Ass}_{R_0}\left(H^{g+1}_{R_+}(M)_n\right) \cap \mathrm{Min}(R_0) = \emptyset$ for $n << 0$ (cf. expression (4.8)).

(c) $\mathrm{Ass}_{R_0}\left(H^{g+2}_{R_+}(M)_n\right) \subseteq \{m_0\}$ (cf. Proposition (4.3.4)(c)).

Case 2: $i = g+1$. We will examine the cases $g = 0$ and $g \geq 1$ separately.

Case 2(α): $g = 0$. Then $g+1 = 1$. $H^1_{R_+}(M)$ is always asymptotically gap free because it is either finitely generated over R or the first non-finitely generated local cohomology of M with respect to R_+ (cf. Proposition (4.2.10)).

Case 2(β): $g \geq 1$. By Section (4.1), we may assume that the residue field, $\frac{R_0}{m_0}$, is infinite. Then, by ([5], 15.1.4), there exists $x \in R_1$ that is M-regular. Now $\frac{M}{xM}$ is still a Cohen-Macaulay graded R-module and $\mathrm{grade}(R_+, \frac{M}{xM}) = g-1$ (cf. [6], 1.2.10(d)).

Let $I' = \sqrt{\mathrm{ann}_R\,(\frac{M}{xM})}$. $H^{g+1}_{R_+}(M)$ is asymptotically gap free because the hypotheses of Proposition (3.2.2) are satisfied:

(a) $\mathrm{Ass}_R\left(H^g_{R_+}(\frac{M}{xM})\right)$ is a finite set and

(b) $\mathrm{Ass}_{R_0}\left(H^g_{R_+}(\frac{M}{xM})_n\right) \cap \mathrm{Min}(R_0) = \emptyset$ for $n << 0$ because if ht $(I' \cap R_0) = 0$, we simply apply expressions (4.8) and (4.9) to $\frac{M}{xM}$; if ht $(I' \cap R_0) \geq 1$, we recall that

$$\mathrm{Supp}_R\left(H^g_{R_+}(\frac{M}{xM})\right) \subseteq \mathrm{Supp}_R\left(\frac{M}{xM}\right) \cap V(R_+).$$

(c) $\mathrm{Ass}_R\left(H^{g+1}_{R_+}(M)\right)$ is a finite set (cf. expression (4.9)).

□

Corollary 4.4.1. *Suppose that both R and M are Cohen-Macaulay. Assume also that R_0 is semi-local,* dim $R_0 = 2$ *and for all* $p \in \operatorname{Spec}^2(R_0)$, $(R_0)_p$ *is equidimensional. Then, for all i, $H^i_{R_+}(M)$ is asymptotically gap free.*

Proof. Fix $i \in \mathbb{N}_0$. Let $\operatorname{Max}(R_0) = \{p_1, \ldots, p_k\}$. Let $p_j \in \operatorname{Max}(R_0)$ and $S_j = R_0 - p_j$. If $\operatorname{ht}_{R_0}(p_j) \leq 1$, then

$$H^i_{R_+R_p}(M_p)$$

is asymptotically gap free by Proposition (4.2.8). If $\operatorname{ht}_{R_0}(p_j) = 2$, then by Theorem (4.0.1),

$$H^i_{R_+R_{p_j}}(M_{p_j})$$

is asymptotically gap free. Since

$$\operatorname{Spec}(R_0) = \bigcup_{p_t \in \operatorname{Max}(R_0)} \operatorname{Spec}(S_t^{-1}R_0)$$

and as shown above,

$$H^i_{S_t^{-1}R_+}(S_t^{-1}M)$$

is asymptotically gap free for all t, so $H^i_{R_+}(M)$ is asymptotically gap free (cf. Proposition (2.5.1)(b)). □

Lemma 4.4.2. *Suppose that R is regular and (R_0, m_0) is local. Then, R_0 is a domain.*

Proof. Let a and b be two elements in m_0 such that $ab = 0$. Then, $\frac{ab}{1} = 0$ in R_m, where $m = m_0 + R_+$. Since R_m is regular, it is a domain. Then, either $\frac{a}{1} = 0$ or $\frac{b}{1} = 0$. If $\frac{a}{1} = 0$, then there exists $x \notin m$ such that $xa = 0$. The 0-th component of x is a unit implies that $a = 0$. Similarly, for $\frac{b}{1} = 0$. Consequently, R_0 is a domain. □

Corollary 4.4.3. *Suppose that R is regular and M is Cohen-Macaulay. Assume also that R_0 is semilocal and* dim $R_0 = 2$. *Then, for all i, $H^i_{R_+}(M)$ is asymptotically gap free.*

Proof. R is regular implies that R is Cohen-Macaulay. Furthermore, for all $p \in \mathrm{Spec}^2(R_0)$, R_p is regular. By Lemma (4.4.2), $(R_p)_0$ is a domain and thus equidimensional. By Corollary (4.4.1), we have our result. □

4.5 Criterion for $H^i_{R_+}(M)$ to be non-vanishing

Assumptions as in Theorem (4.0.1). As before $I = \sqrt{\mathrm{ann}_R(M)}$ and $g = \mathrm{grade}(R_+, M)$. We wish to give a criterion for the vanishing of $H^i_{R_+}(M)$.

We first recall the definition of Δ and $\widetilde{\Delta}$ in Section (4.3).

$$\Delta := \{p \in \mathrm{Spec}^1(R_0) \mid \text{there exists } W \in \mathrm{Min}(I) \text{ with } W \cap R_0 = p\}.$$

and

$$\widetilde{\Delta} := \begin{cases} \Delta & \text{if for all } W \in \mathrm{Min}(I),\ W \cap R_0 \neq m_0 \\ \Delta \cup \{m_0\} & \text{if there exists } W \in \mathrm{Min}(I) \text{ with } W \cap R_0 = m_0. \end{cases}$$

Our goal in this section is to prove Proposition (4.5.2). Before we do that, we need a lemma.

Lemma 4.5.1. *Suppose that R satisfy the standard hypothesis.Let $P := (p + P_+)$ be a graded prime ideal of R, where $p = P \cap R_0$ and $P_+ = P \cap R_+$. If $q \supseteq p$ and $q \in \mathrm{Spec}(R_0)$, then $q + P_+ \in \mathrm{Spec}(R)$.*

Proof. Let $q \supseteq p$ and $q \in \mathrm{Spec}(R_0)$. Put $Q := q + P_+$. Let $a, b \in R$ such that $ab \in Q$. We wish to show that either $a \in Q$ or $b \in Q$. Since Q is graded, it suffices to show it for the case where both a and b are homogeneous. Note that $P \subseteq Q$.

Case 1: deg $(a) > 0$ and deg $(b) > 0$. Then, deg $(ab) > 0$ and $ab \in P_+$. Since P is a prime ideal, $a \in P$ or $b \in P$. In other words, $a \in Q$ or $b \in Q$.

Case 2: deg $(a) = 0$ and deg $(b) = 0$. Then, deg $(ab) = 0$ and $ab \in q$. Since q is a prime ideal, $a \in q$ or $b \in q$.

Case 3: deg $(a) = 0$ and deg $(b) > 0$. Then, deg $(ab) > 0$, implies that $ab \in P_+$. P

is a prime ideal implies that $a \in P$ or $b \in P$. Since $P \subset Q$, we are done.

Case 4: deg $(a) > 0$ and deg $(b) = 0$. This is symmetrical to the previous case. □

Proposition 4.5.2. *Assumptions as in theorem (4.0.1).* $I = \sqrt{\mathrm{ann}_R(M)}$ *and* $g =$ $\mathrm{grade}(R_+, M)$. *If* $\widetilde{\Delta} = \emptyset$ *and* $\mathrm{ht}\ (I \cap R_0) = 0$, *then*

$$H^i_{R_+}(M) = (0) \text{ if and only if } i \neq g.$$

Proof. $M \neq (0)$ because M is Cohen-Macaulay. If $\Gamma_{R_+}(M) = M$, then $g = 0$ and $H^i_{R_+}(M) = (0)$ for all $i > 0$. Therefore, we will consider the case $\Gamma_{R_+}(M) \neq M$. Let $\dim \frac{M}{m_0 M} = d$ and $m = m_0 + R_+$. By Propositions (4.2.9) and (4.2.10), it suffices to show that

$$\dim \frac{M}{m_0 M} \leq g.$$

By Lemma (4.2.6), it suffices to show that

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (R_+) - \mathrm{ht}\ I.$$

By Lemma (4.2.3),

$$\dim \frac{M}{m_0 M} \leq \mathrm{ht}\ (m_0 + R_+) - \mathrm{ht}\ (m_0 + I).$$

By Lemma (4.2.5),

$$\mathrm{ht}\ (m_0 + R_+) = \mathrm{ht}\ (R_+) + 2.$$

All we need to do next is to show that

$$\mathrm{ht}\ (m_0 + I) \geq \mathrm{ht}\ (I) + 2.$$

Let Q be a (graded) minimal prime ideal of $(m_0 + I)$ such that $\mathrm{ht}\ (m_0 + I) = \mathrm{ht}\ Q$. Note that $m_0 = (Q \cap R_0)$. Then, $Q \supseteq P$, a minimal prime over I. Since $\widetilde{\Delta} = \emptyset$ and $\mathrm{ht}\ (I \cap R_0) = 0$, so $p := P \cap R_0 \in \mathrm{Min}(R_0)$. Now, put $P_+ := (P \cap R_+)$. Since R_0 is equidimensional, there exists $u \in \mathrm{Spec}^1(R_0)$ such that

$$p \subset u \subset m_0.$$

By Lemma (4.5.1), we have a chain of primes in Spec(R):

$$(p + P_+) \subset (u + P_+) \subset (m_0 + P_+).$$

Note that, $Q = (m_0 + P_+)$ because $(m_0 + P_+)$ is a prime ideal that contains $(m_0 + I)$, $(m_0 + P_+) \subseteq Q$, and Q is minimal over $(m_0 + I)$. By Lemma (4.2.1), all minimal primes of I have the same height. In particular,

$$\text{ht } (p + P_+) = \text{ht } I.$$

Therefore,

$$\text{ht } Q = \text{ht } (m_0 + P_+) \geq \text{ht } (I) + 2.$$

□

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